Chapter 18

Examples and basic properties

In this chapter, we perform several example computations. In Section 18.1, we compute the effect of adding a ± 1-framed meridian to a link component. We show that it corresponds to a simple bimodule. In Section 18.2, we compute the type-D module for a p/q-framed solid torus. We show that the type-D module for a p/q-framed solid torus recovers the rational surgeries formula of Ozsváth and Szabó [44]. Additionally, we show that the type-D modules for solid tori recovers the surgery exact triangle by exhibiting a homotopy equivalence

$$\mathcal{D}_\infty^{\mathcal{K}} \simeq \operatorname{Cone}(f^1 \colon \mathcal{D}_n^{\mathcal{K}} \to \mathcal{D}_{n+1}^{\mathcal{K}})$$

for all $n \in \mathbb{Z}$.

18.1 Meridional Dehn twists

In this section, we consider the effect of adding a ± 1-framed meridian to a link component. Adding a meridian may be encoded by taking the connected sum with a Hopf link, which has a predictable algebraic effect.

We first define our algebraic candidate, denoted ${}_{\mathcal{K}}\mathcal{B}_{\pm 1}^{\mathcal{K}}$. These are the bimodules for two very simple algebra endomorphisms of $\mathcal{K}$. As an $(\mathbf{I},\mathbf{I})$-module, $\mathcal{B}_{\pm 1} \cong \mathbf{I}$ with the natural $\mathbf{I}$-action. The structure map δ_1^1 vanishes. If $a \in \mathbf{I}_j \cdot \mathcal{K} \cdot \mathbf{I}_j$ for either $j = 0$ or $j = 1$, then we set $\delta_2^1(a, i) = i \otimes a$ for $i \in \mathbf{I}$. Additionally, we set

$$\delta_2^1(\sigma, i_0) = i_1 \otimes \sigma \quad \text{and} \quad \delta_2^1(\tau, i_0) = i_1 \otimes \mathscr{V}^{\pm 1}\tau.$$

We write ${}_{\mathcal{K}}\mathfrak{M}_{\pm 1}^{\mathcal{K}}$ for the type-D module

$${}_{\mathcal{K}}\mathfrak{M}_{\pm 1}^{\mathcal{K}} = \mathcal{H}_{\Lambda(\pm 1)}^{\mathcal{K}_0 \otimes \mathcal{K}_1} \mathbin{\hat{\boxtimes}} {}_{\mathcal{K}_0}\mathcal{D}_0 \mathbin{\hat{\boxtimes}} {}_{\mathcal{K}_1|\mathcal{K}}W_{\alpha\beta,\alpha}^{\mathcal{K}}, \tag{18.1}$$

where $W_{\alpha\beta,\alpha}$ is the pair-of-pants module from Chapter 15 and

$$\Lambda(\pm 1) = (\pm 1, 0).$$

In the above, $\mathcal{K}_0$ and $\mathcal{K}_1$ denote the algebras associated to different components of the Hopf link. We write $\mathcal{K}_0$ for the algebra associated to the component of the Hopf link which becomes the new meridian (and is framed by ± 1).

By the pairing theorem and Theorem 15.1, adding a ± 1-framed meridian to a component $K \subseteq L$ and tensoring ${}_{\mathcal{K}}\mathcal{D}_0$ into the factor of the meridian has the effect of tensoring with the bimodule ${}_{\mathcal{K}}\mathfrak{M}_{\pm 1}^{\mathcal{K}}$.

Proposition 18.1. *Let* ${}_{\mathcal{K}}\mathfrak{M}^{\mathcal{K}}_{\pm 1}$ *denote the* DA *bimodule in equation* (18.1) *for adding a* ± 1*-framed meridian. Then*

$${}_{\mathcal{K}}\mathfrak{M}^{\mathcal{K}}_{\pm 1} \simeq {}_{\mathcal{K}}\mathcal{B}^{\mathcal{K}}_{\mp 1}.$$

The proof is a direct computation, which we do in several steps. Firstly, we observe the very basic isomorphism

$${}_{\mathcal{K}_0}\mathcal{D}_0 \cong \mathcal{D}_0^{\mathcal{K}_2} \mathbin{\widehat{\boxtimes}} {}_{\mathcal{K}_0|\mathcal{K}_2}[\mathbb{I}^{\Supset}].$$

(The subscripts on different copies of $\mathcal{K}$ are meant to indicate only how the tensor product is formed.) Hence, we may manipulate several terms in equation (18.1) as follows:

$$\begin{aligned}\mathcal{H}^{\mathcal{K}_0 \otimes \mathcal{K}_1}_{\Lambda(\pm 1)} \mathbin{\widehat{\boxtimes}} {}_{\mathcal{K}_0}\mathcal{D}_0 &\simeq \mathcal{D}_0^{\mathcal{K}_2} \mathbin{\widehat{\boxtimes}} (\mathcal{H}^{\mathcal{K}_0 \otimes \mathcal{K}}_{\Lambda(\pm 1)} \mathbin{\widehat{\boxtimes}} {}_{\mathcal{K}_0|\mathcal{K}_2}[\mathbb{I}^{\Supset}]) \\ &:= \mathcal{D}_0^{\mathcal{K}_2} \mathbin{\widehat{\boxtimes}} {}_{\mathcal{K}_2}\mathcal{H}^{\mathcal{K}_1}_{\Lambda(\pm 1)} \\ &\simeq \mathcal{D}_0^{\mathcal{K}_2} \mathbin{\widehat{\boxtimes}} {}_{\mathcal{K}_2}\mathcal{Z}^{\mathcal{K}_1}_{\Lambda(\pm 1)}.\end{aligned} \tag{18.2}$$

Therefore, combining equations (18.1) and (18.2), we see that

$$\begin{aligned}{}_{\mathcal{K}}\mathfrak{M}^{\mathcal{K}}_{\pm 1} &= \mathcal{H}^{\mathcal{K}_0 \otimes \mathcal{K}_1}_{\Lambda(\pm 1)} \mathbin{\widehat{\boxtimes}} {}_{\mathcal{K}_0}\mathcal{D}_0 \mathbin{\widehat{\boxtimes}} {}_{\mathcal{K}_1|\mathcal{K}}W^{\mathcal{K}}_{\alpha\beta,\alpha} \\ &\simeq \mathcal{D}_0^{\mathcal{K}_2} \mathbin{\widehat{\boxtimes}} {}_{\mathcal{K}_2}\mathcal{Z}^{\mathcal{K}_1}_{\Lambda(\pm 1)} \mathbin{\widehat{\boxtimes}} {}_{\mathcal{K}_1|\mathcal{K}}W^{\mathcal{K}}_{\alpha\beta,\alpha}.\end{aligned} \tag{18.3}$$

On the other hand, we observe by direct computation that

$${}_{\mathcal{K}}\mathcal{B}^{\mathcal{K}}_{\mp 1} \simeq \mathcal{D}^{\mathcal{K}_0}_{\mp 1} \mathbin{\widehat{\boxtimes}} {}_{\mathcal{K}_0|\mathcal{K}}W^{\mathcal{K}}_{\alpha\beta,\alpha}. \tag{18.4}$$

Comparing equations (18.3) and (18.4), we observe that to prove Proposition 18.1, it suffices to show

$$\mathcal{D}_0^{\mathcal{K}_1} \mathbin{\widehat{\boxtimes}} {}_{\mathcal{K}_1}\mathcal{Z}^{\mathcal{K}}_{\Lambda(\pm 1)} \simeq \mathcal{D}^{\mathcal{K}}_{\mp 1}. \tag{18.5}$$

Let us write

$$\mathcal{Z}^{\mathcal{K}}_{\Lambda} := \mathcal{D}_0^{\mathcal{K}_1} \mathbin{\widehat{\boxtimes}} {}_{\mathcal{K}_1}\mathcal{Z}^{\mathcal{K}}_{\Lambda(\pm 1)}.$$

We note that the type-D module $\mathcal{Z}^{\mathcal{K}}_{\Lambda(\pm 1)}$ is not minimal, in the sense of Definition 17.1. Using the notation of Proposition 17.7, the complex $\mathcal{Z}^{\mathcal{K}}_{\Lambda(\pm 1)}$ may be written as follows:

$$\mathcal{Z}^{\mathcal{K}}_{\Lambda(\pm 1)} = \begin{tikzcd} \mathcal{Z}^{\mathcal{K}}_{0,0} \arrow[r, "p_2^1(\sigma_1+\tau_1{,}-)"] \arrow[d, "f_1^1"] & \mathcal{Z}^{\mathcal{K}}_{1,0} \arrow[d, "g_1^1"] \\ \mathcal{Z}^{\mathcal{K}}_{0,1} \arrow[r, "q_2^1(\sigma_1+\tau_1{,}-)"] & \mathcal{Z}^{\mathcal{K}}_{1,1} \end{tikzcd} \tag{18.6}$$

In the above, we are writing f_1^1 for the map ${}^{L_2}f_1^1 + {}^{-L_2}f_1^1$, and similarly for all of the other maps. Note that ${}^{-H}\omega_3^1$ does not make any contribution.

Our argument will go by way of viewing the above diagram as a mapping cone from the top line to the bottom line, and then applying the homological perturbation lemma. Let us write

$$\mathcal{Q}^{\mathcal{K}}_{\Lambda(\pm 1),\varepsilon} = \operatorname{Cone}\big(\mathcal{Z}^{\mathcal{K}}_{0,\varepsilon} \xrightarrow{p_2^1(\sigma_1+\tau_1,-)} \mathcal{Z}^{\mathcal{K}}_{1,\varepsilon}\big).$$

We will write

$$J^1_{\Lambda(\pm 1)} \colon \mathcal{Q}^{\mathcal{K}}_{\Lambda(\pm 1);0} \to \mathcal{Q}^{\mathcal{K}}_{\Lambda(\pm 1);1}$$

for the total morphism from the top line to the bottom line of equation (18.6) (i.e., $f_1^1 \oplus g_1^1$), so that $\mathcal{Z}^{\mathcal{K}}_{(\pm 1,0)}$ is $\operatorname{Cone}(J^1_{\Lambda(\pm 1)})$.

For $\varepsilon \in \{0,1\}$, write $\mathbf{i}^{\mathcal{K}}_{\varepsilon}$ for the type-D module which is concentrated in idempotent $\mathbf{I}_{\varepsilon}$, has a single generator, and has vanishing δ^1. Equation (18.5), and consequently Proposition 18.1, follows from the subsequent two lemmas.

Lemma 18.2. *For* $\varepsilon \in \{0,1\}$, *there are maps*

$$I^1_{\varepsilon} \colon \mathbf{i}^{\mathcal{K}}_{\varepsilon} \to \mathcal{Q}^{\mathcal{K}}_{\Lambda(-1);\varepsilon}, \quad \Pi^1_{\varepsilon} \colon \mathcal{Q}^{\mathcal{K}}_{\Lambda(-1);\varepsilon} \to \mathbf{i}^{\mathcal{K}}_{\varepsilon}, \quad \text{and} \quad H^1_{\varepsilon} \colon \mathcal{Q}^{\mathcal{K}}_{\Lambda(-1);\varepsilon} \to \mathcal{Q}^{\mathcal{K}}_{\Lambda(-1);\varepsilon}$$

such that I^1_{ε} *and* Π^1_{ε} *are cycles, and* $I^1_{\varepsilon} \circ \Pi^1_{\varepsilon} = \mathrm{id} + \partial_{\mathrm{Mor}}(H^1_{\varepsilon})$ *and* $\Pi^1_{\varepsilon} \circ I^1_{\varepsilon} = \mathrm{id}$. *Furthermore,*

$$\Pi^1_1 \circ J^1_{\Lambda(-1)} \circ I^1_0 = (i_0 \mapsto i_1 \otimes (\sigma + \mathcal{V}\tau)). \tag{18.7}$$

Here, i_{ε} *denotes the generator of* $\mathbf{i}^{\mathcal{K}}_{\varepsilon}$. *In particular,* $\mathcal{Z}^{\mathcal{K}}_{\Lambda(-1)}$ *is homotopy equivalent to the solid torus module* $\mathcal{D}^{\mathcal{K}}_{+1}$ *defined in Section 8.1.*

Proof. As a first step, we record the mapping cone of $\mathcal{Q}^{\mathcal{K}}_{\Lambda(-1);\varepsilon}$. This is the following type-D module:

$$\begin{tikzcd}
\cdots & \mathcal{U}_1^3\mathbf{a} \arrow[d, "\mathcal{U}_2^3\mathcal{V}_2^4"] \arrow[dl] & \mathcal{U}_1^2\mathbf{a} \arrow[d, "\mathcal{U}_2^2\mathcal{V}_2^3"] \arrow[dl, "1"'] & \mathcal{U}_1^1\mathbf{a} \arrow[d, "\mathcal{U}_2\mathcal{V}_2^2"] \arrow[dl, "1"'] & \mathbf{a} \arrow[d, "\mathcal{V}_2"] \arrow[dl, "1"'] & \mathbf{d} \arrow[d, "1"] \arrow[dl, "\mathcal{U}_2"'] & \mathcal{V}_1\mathbf{d} \arrow[d, "1"] \arrow[dl, "\mathcal{U}_2^2\mathcal{V}_2"'] & \mathcal{V}_1^2\mathbf{d} \arrow[d, "1"] \arrow[dl, "\mathcal{U}_2^3\mathcal{V}_2^2"'] & \mathcal{V}_1^3\mathbf{d} \arrow[d, "1"] \arrow[dl, "\mathcal{U}_2^4\mathcal{V}_2^3"'] & \cdots \arrow[dl] \\
\cdots & \mathcal{V}_1^{-4}\mathbf{d} & \mathcal{V}_1^{-3}\mathbf{d} & \mathcal{V}_1^{-2}\mathbf{d} & \mathcal{V}_1^{-1}\mathbf{d} & \mathbf{d} & \mathcal{V}_1\mathbf{d} & \mathcal{V}_1^2\mathbf{d} & \mathcal{V}_1^3\mathbf{d} & \cdots
\end{tikzcd}$$

We define the type-D module $N^{\mathcal{E}}$, where $\mathcal{E}$ is the exterior algebra on one generator, θ. The module N has a single generator, and differential $\delta^1(1) = 1 \otimes \theta$. Then $\mathcal{Q}^{\mathcal{K}}_{\Lambda(-1),\varepsilon}$ is obtained by boxing $N^{\mathcal{E}}$ with the DA bimodule over $(\mathcal{E}, \mathcal{K})$ shown below:

$$\begin{tikzcd}
\cdots & \mathcal{U}_1^3\mathbf{a} \arrow[d, dashed, "\theta|\mathcal{U}_2^3\mathcal{V}_2^4"] \arrow[dl] & \mathcal{U}_1^2\mathbf{a} \arrow[d, dashed, "\theta|\mathcal{U}_2^2\mathcal{V}_2^3"] \arrow[dl, "1"'] & \mathcal{U}_1^1\mathbf{a} \arrow[d, dashed, "\theta|\mathcal{U}_2\mathcal{V}_2^2"] \arrow[dl, "1"'] & \mathbf{a} \arrow[d, dashed, "\theta|\mathcal{V}_2"] \arrow[dl, "1"'] & \mathbf{d} \arrow[d, "1"] \arrow[dl, dashed, "\theta|\mathcal{U}_2"'] & \mathcal{V}_1\mathbf{d} \arrow[d, "1"] \arrow[dl, dashed, "\theta|\mathcal{U}_2^2\mathcal{V}_2"'] & \mathcal{V}_1^2\mathbf{d} \arrow[d, "1"] \arrow[dl, dashed, "\theta|\mathcal{U}_2^3\mathcal{V}_2^2"'] & \mathcal{V}_1^3\mathbf{d} \arrow[d, "1"] \arrow[dl, dashed, "\theta|\mathcal{U}_2^4\mathcal{V}_2^3"'] & \cdots \arrow[dl] \\
\cdots & \mathcal{V}_1^{-4}\mathbf{d} & \mathcal{V}_1^{-3}\mathbf{d} & \mathcal{V}_1^{-2}\mathbf{d} & \mathcal{V}_1^{-1}\mathbf{d} & \mathbf{d} & \mathcal{V}_1\mathbf{d} & \mathcal{V}_1^2\mathbf{d} & \mathcal{V}_1^3\mathbf{d} & \cdots
\end{tikzcd}$$

In the above diagram, the solid lines denote the δ^1 action, while the dashed lines denote $\delta^1_2(\theta, -)$. A solid arrow from $\mathbf{x}$ to $\mathbf{y}$ with a 1 means that $\delta^1(\mathbf{x})$ has a summand of $\mathbf{y} \otimes 1$. If we ignore the type-A action, then the type-D module is equivalent to the one spanned by $\mathcal{V}_1^{-1}\mathbf{d}$ with vanishing δ^1. There is a canonical inclusion map i^1 and projection map π^1. A homotopy h^1 is defined by moving backwards along the solid arrows. Clearly, i^1 and π^1 are type-D homomorphisms, and

$$i^1 \circ \pi^1 = \mathrm{id} + \partial_{\mathrm{Mor}}(h^1), \quad \pi^1 \circ i^1 = \mathrm{id}.$$

Similarly, $\pi^1 \circ h^1 = 0$, $h^1 \circ h^1 = 0$, and $h^1 \circ i^1 = 0$. By the homological perturbation lemma, Lemma 4.2, i^1, π^1, and h^1 extend to a homotopy equivalence of DA bimodules, which we denote by i^1_*, π^1_*, and h^1_*. We define the type-D module morphisms in the statement by

$$I^1_\varepsilon = \mathbb{I}_N \boxtimes i^1_*, \quad \Pi^1_\varepsilon = \mathbb{I}_N \boxtimes \pi^1_*, \quad \text{and} \quad H^1_\varepsilon = \mathbb{I}_N \boxtimes h^1_*.$$

We leave it to the reader to check that the above expressions (which involve infinite sums on the completion) determine continuous morphisms.

All of the remaining claims in the statement are clear, except equation (18.7). We compute that I^1_0 maps i_0 to $\mathcal{V}_1^{-1}\mathbf{d} \otimes 1$. By the computation of Proposition 17.7, the map $J^1_{\Lambda(-1)}$ sends this to

$$\mathcal{V}_1^{-2}\mathbf{d} \otimes \tau_2 + \mathcal{V}_1^{-1}\mathbf{d} \otimes \sigma_2.$$

Then Π^1_1 maps this to

$$i_1 \otimes \mathcal{V}_2\tau_2 + i_1 \otimes \sigma_2.$$

Note that the application of Π^1_1 to $\mathcal{V}_1^{-2}\mathbf{d} \otimes \tau_2$ is slightly subtle. Indeed, the recipe from the homological perturbation lemma is to move backwards along the arrow from $\mathbf{a}$ to $\mathcal{V}_1^{-2}\mathbf{d}$ and move forward along the arrow from $\mathbf{a}$ to $\mathcal{V}_1^{-1}\mathbf{d}$, while picking up a factor of $\mathcal{V}_2$. The proof is complete. ■

We now consider the claim for the framing $\Lambda(+1)$.

Lemma 18.3. *There are maps*

$$I^1_\varepsilon\colon \mathbf{i}^{\mathcal{K}}_\varepsilon \to \mathcal{Q}^{\mathcal{K}}_{\Lambda(+1);\varepsilon}, \quad \Pi^1_\varepsilon\colon \mathcal{Q}^{\mathcal{K}}_{\Lambda(+1);\varepsilon} \to \mathbf{i}^{\mathcal{K}}_\varepsilon, \quad \text{and} \quad H^1_\varepsilon\colon \mathcal{Q}^{\mathcal{K}}_{\Lambda(+1);\varepsilon} \to \mathcal{Q}^{\mathcal{K}}_{\Lambda(+1);\varepsilon}$$

*such that I^1_ε and Π^1_ε are type-*D *homomorphisms, and $I^1_\varepsilon \circ \Pi^1_\varepsilon = \mathrm{id} + \partial_{\mathrm{Mor}}(H^1_\varepsilon)$ and $\Pi^1_\varepsilon \circ I^1_\varepsilon = \mathrm{id}$. Furthermore,*

$$\Pi^1_1 \circ J^1_{\Lambda(+1)} \circ I^1_0 = (i_0 \mapsto i_1 \otimes (\sigma + \mathcal{V}^{-1}\tau)).$$

Here, i_ε denotes the generator of $\mathcal{Q}^{\mathcal{K}}_{+1,\varepsilon}$. In other words, $\mathcal{Z}^{\mathcal{K}}_{+1} \simeq \mathcal{D}^{\mathcal{K}}_{-1}$.

Proof. The proof is very much the same as the proof with framing -1. We write down the complex $\mathcal{Q}^{\mathcal{K}}_{\Lambda(+1);\varepsilon}$ below:

$$\begin{tikzcd}
\cdots \arrow[dr] & \mathcal{U}_1^3\mathbf{a} \arrow[d, "\mathcal{U}_2^3\mathcal{V}_2^4"] \arrow[dr, "1"] & \mathcal{U}_1^2\mathbf{a} \arrow[d, "\mathcal{U}_2^2\mathcal{V}_2^3"] \arrow[dr, "1"] & \mathcal{U}_1^1\mathbf{a} \arrow[d, "\mathcal{U}_2\mathcal{V}_2^2"] \arrow[dr, "1"] & \mathbf{a} \arrow[d, "\mathcal{V}_2"] \arrow[dr, "1"] & \mathbf{d} \arrow[d, "1"] \arrow[dr, "\mathcal{U}_2"] & \mathcal{V}_1\mathbf{d} \arrow[d, "1"] \arrow[dr, "\mathcal{U}_2^2\mathcal{V}_2"] & \mathcal{V}_1^2\mathbf{d} \arrow[d, "1"] \arrow[dr, "\mathcal{U}_2^3\mathcal{V}_2^2"] & \mathcal{V}_1^3\mathbf{d} \arrow[d, "1"] \arrow[dr, "\mathcal{U}_2^4\mathcal{V}_2^3"] & \cdots \\
\cdots & \mathcal{V}_1^{-4}\mathbf{d} & \mathcal{V}_1^{-3}\mathbf{d} & \mathcal{V}_1^{-2}\mathbf{d} & \mathcal{V}_1^{-1}\mathbf{d} & \mathbf{d} & \mathcal{V}_1\mathbf{d} & \mathcal{V}_1^2\mathbf{d} & \mathcal{V}_1^3\mathbf{d} & \cdots
\end{tikzcd}$$

Similar to the framing -1 case, we realize the above diagram as the box tensor product of $N^{\mathcal{E}}$ with the following DA bimodule over $(\mathcal{E}, \mathcal{K})$:

$$\begin{tikzcd}
\cdots \arrow[dr] & \mathcal{U}_1^3\mathbf{a} \arrow[d, dashed, "\theta|\mathcal{U}_2^3\mathcal{V}_2^4"] \arrow[dr, "1"] & \mathcal{U}_1^2\mathbf{a} \arrow[d, dashed, "\theta|\mathcal{U}_2^2\mathcal{V}_2^3"] \arrow[dr, "1"] & \mathcal{U}_1^1\mathbf{a} \arrow[d, dashed, "\theta|\mathcal{U}_2\mathcal{V}_2^2"] \arrow[dr, "1"] & \mathbf{a} \arrow[d, dashed, "\theta|\mathcal{V}_2"] \arrow[dr, "1"] & \mathbf{d} \arrow[d, "1"] \arrow[dr, dashed, "\theta|\mathcal{U}_2"] & \mathcal{V}_1\mathbf{d} \arrow[d, "1"] \arrow[dr, dashed, "\theta|\mathcal{U}_2^2\mathcal{V}_2"] & \mathcal{V}_1^2\mathbf{d} \arrow[d, "1"] \arrow[dr, dashed, "\theta|\mathcal{U}_2^3\mathcal{V}_2^2"] & \mathcal{V}_1^3\mathbf{d} \arrow[d, "1"] \arrow[dr, dashed, "\theta|\mathcal{U}_2^4\mathcal{V}_2^3"] & \cdots \\
\cdots & \mathcal{V}_1^{-4}\mathbf{d} & \mathcal{V}_1^{-3}\mathbf{d} & \mathcal{V}_1^{-2}\mathbf{d} & \mathcal{V}_1^{-1}\mathbf{d} & \mathbf{d} & \mathcal{V}_1\mathbf{d} & \mathcal{V}_1^2\mathbf{d} & \mathcal{V}_1^3\mathbf{d} & \cdots
\end{tikzcd}$$

We forget about the left $\mathcal{E}$-action and simplify the type-D structure. We build a homotopy equivalence with the module $\mathcal{W}^{\mathcal{K}}_{\varepsilon}$, by picking type-D module maps i^1, π^1, and h^1. The map i^1 sends i_ε to $(\mathbf{a} + \mathbf{d}) \otimes 1$. The map π^1 requires a choice. It is either $(\mathbf{a} \mapsto i_\varepsilon \otimes 1, \mathbf{d} \mapsto 0)$, or $(\mathbf{a} \mapsto 0, \mathbf{d} \mapsto i_\varepsilon \otimes 1)$. The choice of h^1 depends on our choice of π^1. If π^1 sends $\mathbf{a}$ to i_ε, then h^1 maps $\mathbf{d}$ (on the bottom row) to $\mathbf{d} \otimes 1$ (on the top row). If π^1 maps $\mathbf{d}$ to $i_\varepsilon \otimes 1$, then h^1 maps $\mathbf{d}$ (bottom row) to $\mathbf{a} \otimes 1$ (top row). Arbitrarily, pick π^1 to map $\mathbf{d}$ to $i_\varepsilon \otimes 1$. Away from the central region, the map h^1 just maps backwards along the arrows marked with 1. One easily verifies that π^1, i^1, and h^1 satisfy the assumptions of the homological perturbation lemma.

The homological perturbation lemma now gives us extensions π^1_*, i^1_* and h^1_*. Boxing these with $\mathbb{I}_N$ gives homotopy equivalences between $\mathcal{Q}^{\mathcal{K}}_{\Lambda(+1);\varepsilon}$ and $\mathbf{i}^{\mathcal{K}}_{\varepsilon}$ described in the statement. Note that the maps I^1_ε, Π^1_ε, and H^1_ε sometimes involve infinite sums, however, it is straightforward to verify that the induced maps are continuous.

It remains to perform the computation involving $J^1_{\Lambda(+1)}$. We easily compute

$$I_0^1(i_0) = \cdots + \mathcal{U}_1\mathbf{a} \otimes \mathcal{V}_2 + \mathbf{a} \otimes 1 + \mathbf{d} \otimes 1 + \mathcal{V}_1\mathbf{d} \otimes \mathcal{U}_2 + \cdots .$$

Using part (2) of Proposition 17.7, we compute that the map $J^1_{\Lambda(+1)}$ sends the above to

$$\begin{gathered}
\cdots + \mathcal{U}_1\mathbf{a} \otimes \sigma_2\mathcal{V}_2 + \mathbf{a} \otimes \sigma_2 + \mathbf{d} \otimes \sigma_2 + \mathcal{V}_1\mathbf{d} \otimes \sigma_2\mathcal{U}_2 + \cdots, \\
\cdots + \mathcal{U}_1^2\mathbf{a} \otimes \tau_2\mathcal{V}_2 + \mathcal{U}_1\mathbf{a} \otimes \tau_2 + \mathbf{a} \otimes \mathcal{V}_2^{-1}\tau_2 + \mathbf{d} \otimes \tau_2\mathcal{U}_2 + \cdots .
\end{gathered}$$

The map Π^1_1 evaluates the above $i_1 \otimes (\sigma_2 + \mathcal{V}_2^{-1}\tau_2)$ (regardless of which choice we made in the construction of π^1). Note, we are using here that $\tau_2\mathcal{U}_2 = \mathcal{V}_2^{-1}\tau_2$. The proof is complete. ■

18.2 The type-D invariants of solid tori

We now describe the type-D module for a solid torus with $p/q \in \mathbb{Q} \cup \{\infty\}$ framing. The most convenient description is to take a standard unknot complement (with standard meridian and 0-framed longitude) and perform $-q/p$ surgery to a meridian of U. See Figure 18.1. Additionally, we will consider the ∞-framed solid torus, which we view as being obtained by performing 0 surgery to a meridian of U.

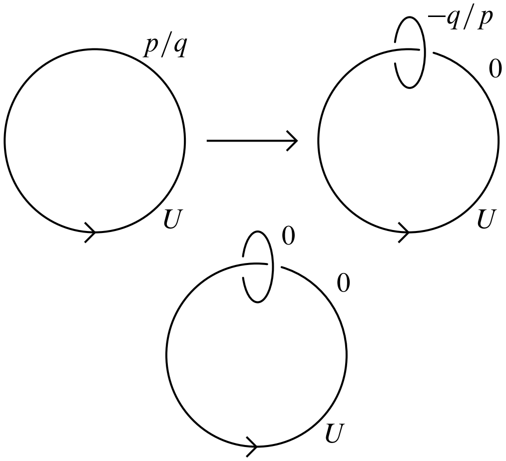

Figure 18.1. A slam dunk on a p/q-framed solid torus (top row), and an ∞-framed solid torus (bottom). The component U with an arrow on it denotes the component which we associate to $\mathcal{K}$.

We now define our candidate type-D modules. We begin by recalling that by definition the type-D module for an n-framed solid torus $\mathcal{D}_n^{\mathcal{K}}$ is the same as the surgery complex for an n-framed unknot, and hence has two generators $\mathbf{x}^0$ and $\mathbf{x}^1$, which live in idempotent 0 and 1, respectively. The structure map is given by

$$\delta^1(\mathbf{x}^0) = \mathbf{x}^1 \otimes (\sigma + \mathscr{V}^n \tau).$$

Generalizing this, we now define the candidate p/q-framed solid torus module $\mathcal{D}_{p/q}^{\mathcal{K}}$, where p, q are coprime and $q > 0$. We define $\mathcal{D}_{p/q}^{\mathcal{K}} \cdot \mathbf{I}_\varepsilon$ to be spanned by generators $\mathbf{x}_0^\varepsilon, \dots, \mathbf{x}_{q-1}^\varepsilon$, for $\varepsilon \in \{0,1\}$. The structure map is given by the following formula:

$$\delta^1(\mathbf{x}_i^0) = \mathbf{x}^1_{(i+p) \bmod q} \otimes \mathscr{V}^{\lfloor (i+p)/q \rfloor} \tau + \mathbf{x}_i^1 \otimes \sigma.$$

The examples $\mathcal{D}_{\pm 1/3}^{\mathcal{K}}$ are shown in Figure 18.2.

Remark 18.4. The bimodule $\mathcal{D}_{p/q}^{\mathcal{K}}$ recovers the rational surgery mapping cone complex $\mathbb{X}_{p/q}(K)$ of Ozsváth and Szabó [44] in the sense that if $K \subseteq S^3$ is a knot, then

$$\mathcal{D}_{p/q}^{\mathcal{K}} \mathbin{\hat{\boxtimes}} {}_{\mathcal{K}}\mathcal{X}_0(K) \cong \mathbb{X}_{p/q}(K).$$

$$\begin{tikzcd}
\mathbf{x}_0^0 \arrow[d, "\sigma"] \arrow[dr, "\tau"] & \mathbf{x}_1^0 \arrow[d, "\sigma"] \arrow[dr, "\tau"] & \mathbf{x}_2^0 \arrow[d, "\sigma"] \arrow[dll, "\mathcal{V}\tau"'] \\
\mathbf{x}_0^1 & \mathbf{x}_1^1 & \mathbf{x}_2^1
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{x}_0^0 \arrow[d, "\sigma"] \arrow[drr, "\mathcal{V}^{-1}\tau"] & \mathbf{x}_1^0 \arrow[d, "\sigma"] \arrow[dl, "\tau"'] & \mathbf{x}_2^0 \arrow[d, "\sigma"] \arrow[dl, "\tau"'] \\
\mathbf{x}_0^1 & \mathbf{x}_1^1 & \mathbf{x}_2^1
\end{tikzcd}$$

Figure 18.2. The modules $\mathcal{D}^{\mathcal{K}}_{1/3}$ (left) and $\mathcal{D}^{\mathcal{K}}_{-1/3}$ (right).

Finally, we define the type-D module $\mathcal{D}^{\mathcal{K}}_{\infty}$ for the ∞-framed solid torus. This module $\mathcal{D}^{\mathcal{K}}_{\infty} \cdot \mathbf{I}_0 = \{0\}$ and $\mathcal{D}^{\mathcal{K}}_{\infty} \cdot \mathbf{I}_1 = \operatorname{Span}_{\mathbb{F}}(\mathbf{x}, \mathbf{y})$. The structure map is given by

$$\delta^1(\mathbf{x}) = \mathbf{y} \otimes (1 + \mathcal{V}).$$

The link surgery formula naturally produces a type-D module $\mathcal{X}^{\mathcal{K}}_{p/q}$ for the p/q-framed solid torus, as follows. We may write p/q as a continued fraction expansion

$$p/q = [a_n, \dots, a_1]^- = a_n - \cfrac{1}{a_{n-1} - \frac{1}{a_{n-2} - \cdots}}.$$

We may define $\mathcal{X}^{\mathcal{K}}_{p/q}$ as the link surgery complex of a linear plumbing with weights $a_n, \dots, a_1$. This produces a type-D module over $\mathcal{L}_n$. We tensor 0-framed solid tori to the components labeled $a_{n-1}, \dots, a_1$. The remaining component, weighted a_n, corresponds to the type-D algebra action of $\mathcal{X}^{\mathcal{K}}_{p/q}$.

Proposition 18.5. *The type-*D *module for a p/q-framed solid torus $\mathcal{X}^{\mathcal{K}}_{p/q}$ is homotopy equivalent to $\mathcal{D}^{\mathcal{K}}_{p/q}$.*

Proof. The proof is by induction. By considering the continued fraction decomposition above, we will assume that the formula holds for p/q, and use this to show that it also holds for

$$m - \frac{1}{p/q} = m - \frac{q}{p},$$

when $m \in \mathbb{Z}$.

Note that increasing the framing by m may be performed by tensoring with m meridional Dehn twist bimodules ${}_{\mathcal{K}}\mathcal{B}^{\mathcal{K}}_{\pm 1}$ from Section 18.1. This tensor product is easy to understand, and indeed it is easy to check that

$$\mathcal{D}^{\mathcal{K}}_{p/q} \mathbin{\hat{\boxtimes}} {}_{\mathcal{K}}\mathcal{B}^{\mathcal{K}}_{+1} \cong \mathcal{D}^{\mathcal{K}}_{p/q+1}.$$

In particular, it suffices to show that if the claim is true for some p/q, then it is true for $-q/p$.

On the algebraic level, it is sufficient to show that

$$\mathcal{D}^{\mathcal{K}}_{-q/p} \simeq \mathcal{D}^{\mathcal{K}}_{p/q} \,\widehat{\boxtimes}\, {}_{\mathcal{K}}\mathcal{Z}^{\mathcal{K}}_{(0,0)},$$

where ${}_{\mathcal{K}}\mathcal{Z}^{\mathcal{K}}_{(0,0)}$ is the Hopf link bimodule.

In this case, we compute the type-D module in a similar manner to our computation of the meridional Dehn twist module from Section 18.1. We will only consider the case that $p < 0$ and $q > 0$. The case that $p > 0$ and $q > 0$ follows from a straightforward modification of this argument.

We compute $\mathcal{D}^{\mathcal{K}}_{p/q} \,\widehat{\boxtimes}\, {}_{\mathcal{K}}\mathcal{Z}^{\mathcal{K}}_{(0,0)} \cdot \mathbf{I}_0$ first. The tensor product has generators

$$x^i_j := \mathbf{x}^0_j \mathcal{U}^i_1 \mathbf{a} \quad \text{and} \quad y^i_j := \mathbf{x}^0_j \mathcal{V}^i_1 \mathbf{d},$$

ranging over $i \in \mathbb{N}$ and $j \in \mathbb{Z}/q$, and generators

$$z^i_j := \mathbf{x}^1_j \mathcal{V}^i_1 \mathbf{d}$$

ranging over $i \in \mathbb{Z}$ and $j \in \mathbb{Z}/q$. We display the complex in Figure 18.3.

Let us write $|p| = a \cdot q + r$, where $a \geq 0$ and $0 \leq r \leq q-1$. The minimal model of $\mathcal{D}^{\mathcal{K}}_{p/q} \,\widehat{\boxtimes}\, {}_{\mathcal{K}}\mathcal{Z}^{\mathcal{K}}_{(0,0)} \cdot \mathbf{I}_0$ is generated by the $|p|$ generators

$$\mathbf{w}^0_0 = z^{-1}_{q-1}, \mathbf{w}^0_1 = z^{-1}_{q-2}, \cdots, \mathbf{w}^0_{|p|-1} = z^{-a-1}_{q-r}.$$

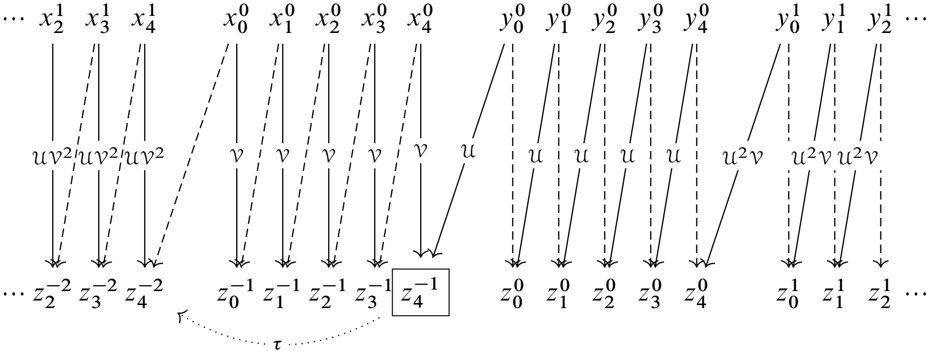

Figure 18.3. The tensor product $\mathcal{D}^{\mathcal{K}}_{-1/5} \,\widehat{\boxtimes}\, {}_{\mathcal{K}}\mathcal{Z}^{\mathcal{K}}_{(0,0)} \cdot \mathbf{I}_{\varepsilon}$, for either $\varepsilon \in \{0,1\}$. The boxed generators are the generators of the minimal model of $\mathcal{D}^{\mathcal{K}}_{p/q} \,\widehat{\boxtimes}\, {}_{\mathcal{K}}\mathcal{Z}^{\mathcal{K}}_{(0,0)} \cdot \mathbf{I}_{\varepsilon}$. The dashed arrows are weighted by 1. The dotted arrow indicates a term of δ^1 which maps from idempotent 0 to idempotent 1. (Note that technically this arrow goes from the copy of this complex in idempotent 0 to the copy in idempotent 1.)

We think of the above generators as all of the generators between z_{q-r}^{-a-1} and z_{q-1}^{-1} if we arrange the generators in a row as in the bottom row of Figure 18.3. (Recall here we are assuming that $p < 0$ and $q > 0$.) The same argument works for $\mathcal{D}_{p/q}^{\mathcal{K}} \widehat{\boxtimes} {}_{\mathcal{K}}\mathcal{Z}_{(0,0)}^{\mathcal{K}} \cdot \mathbf{I}_1$, and we write $\mathbf{w}_0^1, \dots, \mathbf{w}_{|p|-1}^1$ for these generators.

This gives the stated identification of the generators of the module. It remains now to understand δ^1. This is obtained via homological perturbation (cf. Lemma 4.5) by first applying the inclusion map from $\mathcal{D}_{-q/p}^{\mathcal{K}} \cdot \mathbf{I}_0$ into $\mathcal{D}_{p/q}^{\mathcal{K}} \widehat{\boxtimes} {}_{\mathcal{K}}\mathcal{Z}_{(0,0)}^{\mathcal{K}} \cdot \mathbf{I}_0$, applying δ^1 of $\mathcal{D}_{p/q}^{\mathcal{K}} \widehat{\boxtimes} {}_{\mathcal{K}}\mathcal{Z}_{(0,0)}^{\mathcal{K}}$ and then projecting $\mathcal{D}_{p/q}^{\mathcal{K}} \widehat{\boxtimes} {}_{\mathcal{K}}\mathcal{Z}_{(0,0)}^{\mathcal{K}} \cdot \mathbf{I}_1$ to $\mathcal{D}_{-q/p}^{\mathcal{K}} \cdot \mathbf{I}_1$.

There is clearly a summand of δ^1 which sends $\mathbf{w}_i^0$ to $\mathbf{w}_i^1 \otimes \sigma$, for each $i \in \mathbb{Z}/q$. There is a remaining term of δ^1 terms which contributes τ. We recall that via homological perturbation theory, these terms are obtained by first including, applying δ^1, and then projecting. After including, the map δ^1 moves a generator of $\mathcal{D}_{p/q}^{\mathcal{K}} \widehat{\boxtimes} {}_{\mathcal{K}}\mathcal{Z}_{(0,0)}^{\mathcal{K}}$ to the left by q places. The outgoing algebra element is 1. We then apply the projection map. If the application of δ^1 sent $\mathbf{w}_i^0$ to $\mathbf{w}_j^1 \otimes \tau$ for $\mathbf{w}_j^1 \in \{\mathbf{w}_0^1, \dots, \mathbf{w}_{p-1}^1\}$, then the projection map sends $\mathbf{w}_j^1$ to $\mathbf{w}_j^1 \otimes 1$. In general, the projection map will involve traveling to the generators $\mathbf{w}_0^1, \dots, \mathbf{w}_{p-1}^1$ via a zigzag of arrows in the larger complex $\mathcal{D}_{p/q}^{\mathcal{K}} \widehat{\boxtimes} {}_{\mathcal{K}}\mathcal{Z}_{(0,0)}^{\mathcal{K}}$ (more precisely, such a zigzag may be encoded by the homological perturbation lemma, similar to our argument from Section 18.1). We pick up one power of $\mathcal{V}$ for each zigzag we have to do to get back to $\mathbf{w}_{(i+q) \bmod p}^1$. The number of zigzags that we have to do to map $\delta^1(\mathbf{w}_i^1)$ to $\mathbf{w}_{(i+q) \bmod p}^1$ is exactly $\lfloor (i+q)/(-p) \rfloor$. This coincides with $\mathcal{D}_{q/(-p)}^{\mathcal{K}}$, completing the proof. ■

Proposition 18.6. *The type-*D *module for an ∞-framed solid torus is $\mathcal{D}_{\infty}^{\mathcal{K}}$.*

Proof. We compute the tensor product

$$\mathcal{D}_{\infty}^{\mathcal{K}} := \mathcal{D}_0^{\mathcal{K}} \widehat{\boxtimes} {}_{\mathcal{K}}\mathcal{Z}_{(0,0)}^{\mathcal{K}}.$$

We note for both $\varepsilon = 0, 1$, that $\mathcal{D}_0 \widehat{\boxtimes} {}_{\mathcal{K}}\mathcal{Z}_{(0,0)}^{\mathcal{K}} \cdot \mathbf{I}_{\varepsilon}$ is isomorphic to the following complex:

$$\begin{tikzcd}
\cdots & \mathcal{U}_1\mathbf{a} \arrow[d, "1+\mathcal{U}\mathcal{V}^2"] & \mathbf{a} \arrow[d, "1+\mathcal{V}"] & \mathbf{d} \arrow[d, "1+\mathcal{U}"] & \mathcal{V}_1\mathbf{d} \arrow[d, "1+\mathcal{U}^2\mathcal{V}"] & \cdots \\
\cdots & \mathcal{V}_1^{-2}\mathbf{d} & \mathcal{V}_1^{-1}\mathbf{d} & \mathbf{d} & \mathcal{V}_1\mathbf{d} & \cdots
\end{tikzcd}$$

In idempotent $\mathbf{I}_0$, this complex is acyclic. For example, $\mathrm{Cone}(1 + \mathcal{U} : \mathbf{d} \to \mathbf{d})$ is acyclic because $1 + \mathcal{U}$ is a unit in the completion of $\mathbf{I}_0 \cdot \mathcal{K} \cdot \mathbf{I}_0$. Its inverse is given by $\sum_{i=0}^{\infty} \mathcal{U}^i$. In idempotent $\mathbf{I}_1$, each summand is acyclic except for $\mathrm{Cone}(1 + \mathcal{V} : \mathbf{a} \to \mathcal{V}_1^{-1}\mathbf{d})$. Deleting each acyclic subcomplex yields $\mathcal{D}_{\infty}^{\mathcal{K}}$, completing the proof. ■

There is a related bimodule which is useful for applications, defined by

$${}_{\mathcal{K}}\mathcal{D}_{\infty}^{\mathbb{F}[U]} := \mathcal{D}_{\infty}^{\mathcal{K}} \widehat{\boxtimes} {}_{\mathcal{K}|\mathcal{K}}[\mathbb{I}^{\supset}]^{\mathbb{F}[U]}.$$

The following lemma is helpful.

Lemma 18.7. *The* DA *bimodule* ${}_{\mathcal{K}}\mathcal{D}_{\infty}^{\mathbb{F}[U]}$ *is homotopy equivalent to the* DA *bimodule with a single generator* $\mathbf{z}^1$*, concentrated in idempotent* $\mathbf{I}_1$*, such that* $\delta_2^1(\mathcal{U}^i\mathcal{V}^j, \mathbf{z}^1) = \mathbf{z}^1 \otimes U^i$.

Proof. Write ${}_{\mathcal{K}}\mathcal{Z}_{\infty}^{\mathbb{F}[U]}$ for the small model in the statement. We view the completion of ${}_{\mathcal{K}}\mathcal{D}_{\infty}^{\mathbb{F}[U]}$ as being the vector space

$$\mathbb{F}[\![\mathcal{V},\mathcal{V}^{-1}]\!]\langle \mathbf{x}^1\rangle \oplus \mathbb{F}[\![\mathcal{V},\mathcal{V}^{-1}]\!]\langle \mathbf{y}^1\rangle.$$

We have

$$\delta_1^1(\mathcal{V}^t\mathbf{x}^1) = (1+\mathcal{V})\mathcal{V}^t\mathbf{y}^1 \otimes 1.$$

Furthermore,

$$\delta_2^1(i_1\mathcal{V}^n U^m i_1, \mathcal{V}^t\mathbf{x}^1) = \mathcal{V}^{n+t}\mathbf{x}^1 \otimes U^m,$$

and similarly for $\mathbf{y}^1$. Apply the forgetful functor to the type-A actions on ${}_{\mathcal{K}}\mathcal{D}_{\infty}^{\mathbb{F}[U]}$ and ${}_{\mathcal{K}}\mathcal{Z}_{\infty}^{\mathbb{F}[U]}$. Define type-D morphisms

$$i^1\colon \mathcal{Z}_{\infty}^{\mathbb{F}[U]} \to \mathcal{D}_{\infty}^{\mathbb{F}[U]}, \quad \pi^1\colon \mathcal{D}_{\infty}^{\mathbb{F}[U]} \to \mathcal{Z}_{\infty}^{\mathbb{F}[U]}, \quad \text{and} \quad h^1\colon \mathcal{D}_{\infty}^{\mathbb{F}[U]} \to \mathcal{D}_{\infty}^{\mathbb{F}[U]}$$

via the following formulas. We set

$$i^1(\mathbf{z}^1) = \left(\sum_{i\in\mathbb{Z}} \mathcal{V}^i\right)\mathbf{x}^1 \otimes 1, \quad \pi^1(\mathcal{V}^i\mathbf{x}^1) = \begin{cases} \mathbf{z}^1 \otimes 1 & \text{if } i = 0,\\ 0 & \text{otherwise.}\end{cases}$$

Additionally, we set

$$\pi^1(\mathcal{V}^i\mathbf{y}^1) = 0 \quad \text{and} \quad h^1(\mathcal{V}^i\mathbf{y}^1) = \begin{cases} \sum_{j<i} \mathcal{V}^j\mathbf{x}^1 \otimes 1 & \text{if } i \le 0,\\ \sum_{j\ge i} \mathcal{V}^j\mathbf{x}^1 \otimes 1 & \text{if } i > 0.\end{cases}$$

We leave it to the reader to check that the above maps are continuous, and furthermore

$$\pi^1 \circ i^1 = \mathrm{id}, \quad i^1 \circ \pi^1 = \partial_{\mathrm{Mor}}(h^1), \quad h^1 \circ h^1 = 0,$$
$$\pi^1 \circ h^1 = 0, \quad \text{and} \quad h^1 \circ i^1 = 0.$$

Applying the homological perturbation lemma from Lemma 4.2 induces a type-DA Alexander bimodule structure on $\mathcal{Z}_{\infty}^{\mathbb{F}[U]}$ which is homotopy equivalent to ${}_{\mathcal{K}}\mathcal{D}_{\infty}^{\mathbb{F}[U]}$. This bimodule structure is easily checked to coincide with our definition of ${}_{\mathcal{K}}\mathcal{Z}_{\infty}^{\mathbb{F}[U]}$, completing the proof. ∎

18.3 The surgery exact triangle

We now show that our bordered invariants recover the surgery exact triangle, in the following sense.

Proposition 18.8. *There is a type-*D *morphism* $f^1\colon \mathcal{D}_n^{\mathcal{K}} \to \mathcal{D}_{n+1}^{\mathcal{K}}$ *such that*

$$\operatorname{Cone}(f^1\colon \mathcal{D}_n^{\mathcal{K}} \to \mathcal{D}_{n+1}^{\mathcal{K}}) \simeq \mathcal{D}_\infty^{\mathcal{K}},$$

for any $n \in \mathbb{Z}$.

Remark 18.9. This is unsurprising since the knot and link surgery formulas are proven using surgery exact triangles, though it highlights the similarities with the original bordered theory, cf. [27, Section 11.2].

Remark 18.10. It is straightforward to adapt our proof to construct morphisms and homotopy equivalences

$$\mathcal{D}_{n+1}^{\mathcal{K}} \simeq \operatorname{Cone}(g^1\colon \mathcal{D}_\infty^{\mathcal{K}} \to \mathcal{D}_n^{\mathcal{K}}) \quad \text{and} \quad \mathcal{D}_n^{\mathcal{K}} \simeq \operatorname{Cone}(h^1\colon \mathcal{D}_{n+1}^{\mathcal{K}} \to \mathcal{D}_\infty^{\mathcal{K}}).$$

Proof. Write $\mathbf{x}_0, \mathbf{x}_1$ for the generators of $\mathcal{D}_n^{\mathcal{K}}$ and $\mathbf{y}_0, \mathbf{y}_1$ for the generators of $\mathcal{D}_{n+1}^{\mathcal{K}}$. We define f^1 via the formula

$$f^1(\mathbf{x}_\varepsilon) = \mathbf{y}_\varepsilon \otimes (1 + \alpha)$$

for $\varepsilon \in \{0, 1\}$, where

$$\alpha = \sum_{s \geq 1} (\mathcal{U}^s + \mathcal{V}^s) U^{s(s-1)/2} = (\mathcal{U} + \mathcal{V}) + (\mathcal{U}^2 + \mathcal{V}^2)U + (\mathcal{U}^3 + \mathcal{V}^3)U^3 + \cdots .$$

(The element α is discovered by writing $\beta = 1 + \alpha$ as $\beta = \sum_{i \in \mathbb{Z}} \beta_i$ where β_i has Alexander grading i, and then recursively solving the equation $\mathcal{V}\phi^\tau(\beta_i) = \beta_{i+1}$ starting with $\beta_0 = 1$.)

It is straightforward to see that $\partial_{\mathrm{Mor}}(f^1) = 0$. Write:

$$\operatorname{Cone}(f^1) = \begin{tikzcd} \mathbf{x}_0 \arrow[r, "1+\alpha"] \arrow[d, "\sigma+\mathcal{V}^n\tau"] & \mathbf{y}_0 \arrow[d, "\sigma+\mathcal{V}^{n+1}\tau"] \\ \mathbf{x}_1 \arrow[r, "1+\alpha"] & \mathbf{y}_1 \end{tikzcd}$$

Since $1 + \alpha$ is a unit in the completion of $\mathbf{I}_0 \cdot \mathcal{K} \cdot \mathbf{I}_0$, the above complex is homotopy equivalent to the bottom row, $\operatorname{Cone}(1 + \alpha : \mathbf{x}_1 \to \mathbf{y}_1)$. To construct a homotopy equivalence between $\operatorname{Cone}(1 + \alpha : \mathbf{x}_1 \to \mathbf{y}_1)$ and $\mathcal{D}_\infty^{\mathcal{K}}$, it is sufficient to show that $1 + \alpha = (1 + \mathcal{V})u$ for some unit u in the completion of $\mathbf{I}_1 \cdot \mathcal{K} \cdot \mathbf{I}_1$. To see this, we recall that $\mathcal{U} = U\mathcal{V}^{-1}$. We use this expression for $\mathcal{U}$ in our formula for α and then regroup terms based on the powers of U, and we obtain the following expression:

$$1 + \alpha = \sum_{s \geq 1} (\mathcal{V}^s + \mathcal{V}^{-s+1}) U^{s(s-1)/2}.$$

We observe that

$$(\mathcal{V}^s + \mathcal{V}^{-s+1}) U^{s(s-1)/2} = (1 + \mathcal{V}^{2s-1}) \mathcal{V}^{-s+1} U^{s(s-1)/2},$$

which clearly has a factor of $(1 + \mathcal{V})$, for all $s \geq 0$. Hence, $1 + \alpha = (1 + \mathcal{V})(1 + Uf)$ for some series f, and $(1 + Uf)$ is a unit in the completion of $\mathbf{I}_1 \cdot \mathcal{K} \cdot \mathbf{I}_1$, completing the proof. ■

18.4 Finite generation

In this section, we prove a basic property of the surgery modules that is useful in applications.

Proposition 18.11. *Suppose that $L \subseteq S^3$ is a link with framing Λ, and $L = K_1 \cup \cdots \cup K_\ell$. Suppose further that $0 < j < \ell$ and write $L_{1,\dots,j} = K_1 \cup \cdots \cup K_j$ and $L_{j+1,\dots,\ell} = K_{j+1} \cup \cdots \cup K_\ell$. Let $\mathcal{X}_\Lambda(L_{1,\dots,j}, L_{j+1,\dots,\ell})^{\mathcal{L}_{\ell-j}}$ denote the type-*D *module obtained by tensoring ${}_{\mathcal{K}}\mathcal{D}_0$ into each algebra component for $L_{1,\dots,j}$. Then $\mathcal{X}_\Lambda(L_{1,\dots,j}, L_{j+1,\dots,\ell})^{\mathcal{L}_{\ell-j}}$ is homotopy equivalent to a finitely generated type-*D *module.*

Our argument is similar to Manolescu and Ozsváth's truncation procedure [32, Section 10]. We will prove the claim in several steps. If $j < \ell$, we write $\Lambda_{(j)}$ for the induced framing matrix on $L_{1,\dots,j}$. We will first verify the claim when $\Lambda_{(j)}$ is positive definite. We will then use the exact triangle from Section 18.3 to verify the claim for arbitrary framings.

If $0 < j < \ell$ and we decompose L as $L = L_{1,\dots,j} \cup L_{j+1,\dots,\ell}$, then we may consider the subcube

$$\mathcal{X}_\Lambda^{(*,0)}(L_{1,\dots,j}, L_{j+1,\dots,\ell})^{\mathcal{L}_{\ell-j}}$$

generated by points of the cube $\mathbb{E}_\ell$ which have $L_{j+1,\dots,\ell}$ components 0. We will view this as a type-D module over the algebra $\mathbb{F}[\mathcal{U}_{j+1}, \mathcal{V}_{j+1}, \dots, \mathcal{U}_\ell, \mathcal{V}_\ell]$ which is continuous with respect to the topology induced by

$$\mathbf{E}_0 \cdot \mathcal{L}_{\ell-j} \cdot \mathbf{E}_0 \cong \mathbb{F}[\mathcal{U}_{j+1}, \mathcal{V}_{j+1}, \dots, \mathcal{U}_\ell, \mathcal{V}_\ell].$$

Recall from Section 8.2 that the solid torus module ${}_{\mathcal{K}}\mathcal{D}_0$ may be decomposed as a tensor product

$${}_{\mathcal{K}}[\mathcal{D}_0]^{\mathbb{F}[U]} \mathbin{\hat{\boxtimes}} {}_{\mathbb{F}[U]}\mathbb{F}[U].$$

Hence, there is a type-D module $\mathcal{X}_\Lambda(L_{1,\dots,j}, L_{j+1,\dots,\ell})^{\mathbb{F}[U_1,\dots,U_j]\otimes \mathcal{L}_{n-j}}$, from which we get $\mathcal{X}_\Lambda(L_{1,\dots,j}, L_{j+1,\dots,\ell})^{\mathcal{L}_{n-j}}$ by tensoring with ${}_{\mathbb{F}[U_1,\dots,U_j]}\mathbb{F}[U_1,\dots,U_j]$.

Lemma 18.12. *Suppose* $L = K_1 \cup \cdots \cup K_\ell$ *is a link in* S^3, *and* $0 \le j < \ell$. *Suppose that* $\Lambda_{(j)}$ *is positive definite. Then*

$$\mathcal{X}_\Lambda^{(*,0)}(L_{1,\dots,j}, L_{j+1,\dots,\ell})^{\mathbb{F}[U_1,\dots,U_j,\mathcal{U}_{j+1},\mathcal{V}_{j+1},\dots,\mathcal{U}_\ell,\mathcal{V}_\ell]}$$

*is homotopy equivalent to a finitely generated type-*D *module. Furthermore, the maps appearing in the homotopy equivalence may be taken to be continuous if we equip* $\mathcal{X}_\Lambda^{(*,0)}(L_{1,\dots,j}, L_{j+1,\dots,\ell})$ *with the product topology over a monomial basis, and we equip*

$$\mathbb{F}[U_1,\dots,U_j,\mathcal{U}_{j+1},\mathcal{V}_{j+1},\dots,\mathcal{U}_\ell,\mathcal{V}_\ell]$$

with the $(U_1,\dots,U_\ell)$*-adic topology.*

Remark 18.13. The fact that the morphisms in the homotopy equivalence may be taken to be continuous with respect to the $(U_1,\dots,U_\ell)$-adic topology, as opposed to being continuous with respect to the $(U_1,\dots,U_j,\mathcal{U}_{j+1},\mathcal{V}_{j+1},\dots,\mathcal{U}_\ell,\mathcal{V}_\ell)$-adic topology, allows us to use the homotopy equivalence constructed in the lemma for the complex at each subcube $\mathbb{E}_j \times \{\varepsilon\}$, for all $\varepsilon \in \mathbb{E}_{n-j}$.

Proof. Our proof will be by induction on j. The case that $j = 0$ is automatic since $\mathcal{X}_\Lambda^{(*,0)}(\emptyset, L_{1,\dots,\ell}) \cong \mathcal{CFL}(L)$, which is a finitely generated type-D module over $\mathbb{F}[\mathcal{U}_1,\mathcal{V}_1,\dots,\mathcal{U}_\ell,\mathcal{V}_\ell]$.

We define a shifted Alexander A_j-grading on $\mathcal{X}_\Lambda^{(*,0)}(L_{1,\dots,j}, L_{j,\dots,\ell})$, as follows. We set

$$\omega(\mathbf{s}) = -(\mathrm{lk}(K_1,K_j),\dots,\mathrm{lk}(K_{j-1},K_j))\Lambda_{(j-1)}^{-1}(s_1,\dots,s_{j-1})^T. \tag{18.8}$$

In the above, $\Lambda_{(j-1)}^{-1}$ is the inverse matrix (over $\mathbb{Q}$) and T denotes transpose. Also, if $\mathbf{s} \in \mathbb{H}(L)$ we are writing $\mathbf{s} = (s_1,\dots,s_\ell)$. On the link surgery formula, we define the ω-shifted Alexander grading

$$A_j^\omega(\mathbf{s}) := s_j + \omega(\mathbf{s}).$$

We observe that if $i \neq j$, then

$$\omega(\mathbf{s} + \Lambda_{L,-K_i}) - \omega(\mathbf{s}) = -\mathrm{lk}(K_i,K_j), \tag{18.9}$$

where $\Lambda_{L,-K_i}$ (as defined in Section 6.1) consists of the vector whose j-th component is $\mathrm{lk}(K_i,K_j)$ if $i \neq j$, and whose i-th component is λ_i, the framing of K_i. Equation (18.9) follows from the fact that the i-th column of $\Lambda_{(j-1)}$ is obtained by deleting the last component of $\Lambda_{L,-K_i}$. Equation (18.9) verifies that if $\vec{M} \subseteq L_{1,\dots,j}$ is a sublink which does not contain $-K_j$, then $\Phi^{\vec{M}}$ preserves A_j^ω.

On the other hand, we compute directly from equation (18.8) and the definition of $\Lambda_{L,-K_j}$ that

$$\begin{aligned}&\lambda_j+\omega(\mathbf{s}+\Lambda_{L,-K_j})-\omega(\mathbf{s})\\ &\quad=\lambda_j-(\operatorname{lk}(K_1,K_j),\dots,\operatorname{lk}(K_{j-1},K_j))\Lambda_{(j-1)}^{-1}(\operatorname{lk}(K_1,K_j),\dots,\operatorname{lk}(K_{j-1},K_j))^T.\end{aligned}\tag{18.10}$$

Note that the above quantity, which we denote by λ_j', is the Schur complement of $\Lambda_{(j-1)}$ inside of $\Lambda_{(j)}$. In particular, $\Lambda_{(j)}$ is positive definite if and only if $\Lambda_{(j-1)}$ is positive definite and $\lambda_j'>0$. It follows from equation (18.10) that if $-K_j\subseteq\vec{M}\subseteq L_{1,\dots,j}$, then $\Phi^{\vec{M}}$ shifts A_j^ω by λ_j'.

We may thus decompose the complex $\mathcal{X}_\Lambda^{(*,0)}(L_{1,\dots,j},L_{j+1,\dots,\ell})$ as a sum of staircases:

$$\begin{tikzcd}
\cdots \arrow[dr, "F^{-K_j}"'] & \mathcal{A}^\omega_{s-\lambda_j'} \arrow[d, "F^{K_j}"'] \arrow[dr, "F^{-K_j}"'] & \mathcal{A}^\omega_{s} \arrow[d, "F^{K_j}"'] \arrow[dr, "F^{-K_j}"'] & \mathcal{A}^\omega_{s+\lambda_j'} \arrow[d, "F^{K_j}"'] \arrow[dr, "F^{-K_j}"'] & \cdots \\
\cdots & \mathcal{B}^\omega_{s-\lambda_j'} & \mathcal{B}^\omega_{s} & \mathcal{B}^\omega_{s+\lambda_j'} & \cdots
\end{tikzcd}$$

The vertical direction is the direction of K_j. Here, we are writing F^{-K_j} for the hypercube maps for all oriented sublinks $\vec{M}\subseteq L_{1,\dots,j}$ which contain $-K_j$, and we define F^{K_j} similarly. In the above, we are writing $\mathcal{A}_s^\omega\subseteq\mathcal{C}_\Lambda^{(*,0)}(L_{1,\dots,j},L_{j+1,\dots,\ell})$ for the subspace of $\mathbb{F}[U_1,\dots,U_j,\mathscr{U}_{j+1},\mathscr{V}_{j+1},\dots,\mathscr{U}_\ell,\mathscr{V}_\ell]$-module generators with K_j-component 0 and Alexander A_j^ω-grading $s\in\mathbb{Q}$. The subspace $\mathcal{B}_s^\omega$ is similar, but has K_j-component 1. Since $\mathbb{H}(L_{1,\dots,j})/\Lambda_{(j)}$ is finite, there are only finitely many such staircases. Here, the subscript s of $\mathcal{A}_s^\omega$ and $\mathcal{B}_s^\omega$ denotes their shifted Alexander grading.

Standard arguments (see [32, Lemma 10.1]) imply the following:

(h-1) If s is sufficiently large, the map F^{K_j} is a homotopy equivalence between $\mathcal{A}_s^\omega$ and $\mathcal{B}_s^\omega$.

(h-2) If s is sufficiently negative, the map F^{-K_j} is a homotopy equivalence between $\mathcal{A}_s^\omega$ and $\mathcal{B}_{s+\lambda_j'}^\omega$.

Note that we may view $\mathcal{A}_s^\omega\otimes\mathbb{F}[U_j]$ as being a subspace of $\mathcal{X}_\Lambda^{(*,0)}(L_{1,\dots,j-1},L_{j+1,\dots,\ell})\otimes\mathbb{F}[\mathscr{U}_j,\mathscr{V}_j]$ in A_j^ω-Alexander grading s. Similarly, we may view $\mathcal{B}_s^\omega\otimes\mathbb{F}[U_j]$ as being the subspace of $\mathcal{X}_\Lambda^{(*,0)}(L_{1,\dots,j-1},L_{j+1,\dots,\ell})\otimes\mathbb{F}[\mathscr{U}_j,\mathscr{V}_j,\mathscr{V}_j^{-1}]$ in A_j^ω-Alexander grading s. By induction, $\mathcal{X}_\Lambda^{(*,0)}(L_{1,\dots,j-1},L_{j+1,\dots,\ell})$ is homotopy equivalent to a finitely generated type-D module over $\mathbb{F}[U_1,\dots,U_{j-1},\mathscr{U}_j,\mathscr{V}_j,\dots,\mathscr{U}_\ell,\mathscr{V}_\ell]$ with respect to the topologies in the statement of the lemma.

In particular, we may replace $\mathcal{X}_\Lambda^{(*,0)}(L_{1,\dots,j-1},L_{j,\dots,\ell})$ with a finitely generated model $\mathcal{C}'$, and also replace the subcomplexes $\mathcal{A}_s^\omega$ and $\mathcal{B}_s^\omega$ with the finite-dimensional subspaces of $\mathcal{C}' \otimes \mathbb{F}[\mathcal{U}_j, \mathcal{V}_j]$ and $\mathcal{C}' \otimes \mathbb{F}[\mathcal{U}_j, \mathcal{V}_j, \mathcal{V}_j^{-1}]$ which form an $\mathbb{F}[U_j]$ basis in Alexander grading s. Properties (h-1) and (h-2) are preserved by homotopy equivalence, and hence still persist on the finitely generated models. Abusing notation, we write $\mathcal{A}_s^{\omega}$ and $\mathcal{B}_s^{\omega}$ also for the finitely generated models.

Using the same logic as for the ordinary knot surgery mapping cone formula [43], each such staircase above admits a finitely generated truncation. We need additionally to show that the maps appearing in a homotopy equivalence with the finite truncation are continuous with respect to the topologies in the main statement. These maps may be described easily using homological perturbation theory. Compare the proof of Lemma 18.2. Since $\mathcal{C}'$ is finitely generated, for $s \gg 0$, each generator of $\mathcal{A}_s^{\omega}$ must be a multiple of $\mathcal{V}_j^{r(s)}$ for some function $r \colon \mathbb{Q} \to \mathbb{N}$ such that $r(s) \to \infty$ as $s \to \infty$. Similarly, for $s \ll 0$, each generator of $\mathcal{A}_s^{\omega}$ must be a multiple of $\mathcal{U}_j^{l(s)}$ for some $l \colon \mathbb{Q} \to \mathbb{N}$ such that $l(s) \to \infty$ as $s \to -\infty$. Using the equivariance of the maps F^{K_j} and F^{-K_j} from Lemma 6.7, we see that for $s \ll 0$, the map F^{K_j} sends $\mathcal{A}_s^{\omega}$ into the ideal $(U_j^{l(s)})$. Similarly, for $s \gg 0$, F^{-K_j} maps $\mathcal{A}_s^{\omega}$ into the ideal $(U_j^{r(s)})$. This is sufficient to show that the maps appearing in the homotopy equivalence are continuous as stated.

Since $\mathcal{X}_{\Lambda}^{(*,0)}(L_{1,\dots,j}, L_{j+1,\dots,j})$ decomposes as a sum of finitely many staircases, each of which is homotopy equivalent to a finitely generated truncation, as above, the direct sum is homotopy equivalent to a finitely generated complex, completing the proof. ■

We now prove a basic lemma concerning homological algebra.

Lemma 18.14. *Write* $\mathcal{R} = \mathbb{F}[U_1, \dots, U_{j-1}, \mathcal{U}_{j+1}, \mathcal{V}_{j+1}, \dots, \mathcal{U}_{\ell}, \mathcal{V}_{\ell}]$*, where* $0 < j < \ell$*. Write* $\mathcal{R}[U_j]$ *for* $\mathcal{R} \otimes \mathbb{F}[U_j]$*. Suppose that* $\mathcal{X}^{\mathcal{R}[U_j]}$ *is a finitely generated type-*D *module which admits a relative* $\mathbb{Q} \times \mathbb{Q}$*-valued* $(\mathrm{gr}_{\mathbf{w}}, \mathrm{gr}_{\mathbf{z}})$*-bigrading. Suppose that* $U_j \simeq \mathcal{U}_{\ell}\mathcal{V}_{\ell}$ *as type-*D *morphisms (i.e., there is a morphism* J^1 *satisfying* $\partial_{\mathrm{Mor}}(J^1) = \mathrm{id} \otimes (U_j + \mathcal{U}_{\ell}\mathcal{V}_{\ell})$*). Then*

$$(\mathcal{X}^{\mathcal{R}[U_j]} \boxtimes {}_{\mathbb{F}[U_j]}\mathbb{F}[U_j])^{\mathcal{R}}$$

*is homotopy equivalent to a finitely generated type-*D *module over* $\mathcal{R}$*.*

Remark 18.15. The condition on the $(\mathrm{gr}_{\mathbf{w}}, \mathrm{gr}_{\mathbf{z}})$-bigrading is likely not absolutely necessary, though it simplifies the discussion on completions.

Proof. Our proof uses homological perturbation theory. We view $\mathcal{X}^{\mathcal{R}[U_j]}$ as a being obtained from a DA bimodule ${}_{\mathcal{E}}\mathcal{W}^{\mathcal{R}[U_j]}$ where $\mathcal{E}$ denotes the exterior algebra on one generator. We define $\delta_{k+1}^1(\theta, \dots, \theta, \mathbf{x})$ on $\mathcal{W}$ to consist of the components of δ^1 of $\mathcal{X}$ which are weighted by an algebra element of total $\mathcal{R}$-degree equal to k. Here, the $\mathcal{R}$-degree of a monomial denotes the sum of powers of variables from $\mathcal{R}$ appearing (but not of powers of U_j). It is straightforward to see that $\mathcal{W}$ satisfies the DA bimodule structure relations.

Note that δ^1_1 on $\mathcal{W}$ consists of exactly the terms of the differential of $\mathcal{X}$ which are weighted by elements of $\mathbb{F}[U_j]$.

Let $N^{\mathcal{E}}$ be the rank 1 type-D module with $\delta^1(1) = 1 \otimes \theta$, so that

$$N^{\mathcal{E}} \boxtimes {}_{\mathcal{E}}\mathcal{W}^{\mathcal{R}[U_j]} = \mathcal{X}^{\mathcal{R}[U_j]}.$$

We consider the type-D module $\mathcal{Y}^{\mathcal{R}[U_j]}$ obtained by applying the forgetful functor to

$${}_{\mathcal{E}}\mathcal{W}^{\mathcal{R}[U_j]}.$$

The type-D module $\mathcal{Y}^{\mathcal{R}[U_j]}$ has δ^1 with algebra elements in $\mathbb{F}[U_j]$, i.e., we may view it as a finitely generated free chain complex over $\mathbb{F}[U_j]$. Setting all variables except U_j equal to 0, the map J^1 induces a map h^1 on $\mathcal{Y}^{\mathcal{R}[U_j]}$ which satisfies $\partial_{\mathrm{Mor}}(j^1) = \mathrm{id} \otimes U_j$. The classification theorem for finitely generated chain complexes over a PID implies that we may find a basis of $\mathcal{Y}$ so that $\mathcal{Y}^{\mathcal{R}[U_j]}$ decomposes as a direct sum of 1-step complexes (i.e., generators with vanishing δ^1), and 2-step complexes (i.e., summands generated by two generators with $\delta^1(\mathbf{y}) = 0$ and $\delta^1(\mathbf{x}) = \mathbf{y} \otimes \alpha$). Our assumptions about the existence of a relative bigrading imply that α may always be taken to be a power of U_j. The fact that $\partial_{\mathrm{Mor}}(j^1) = \mathrm{id} \otimes U_j$ implies that there are no 1-step summands, and each 2-step summand is of the form $\delta^1(\mathbf{x}) = \mathbf{y} \otimes U_j$.

We define the finite-dimensional subspace $\mathcal{Q} \subseteq \mathcal{W} \boxtimes \mathbb{F}[U_j]$ as follows. Enumerate the generators of the U_j^1-weighted 2-step complexes as $\mathbf{x}_t$ and $\mathbf{y}_t$, so that $\delta^1(\mathbf{x}_t) = \mathbf{y}_t \otimes U_j$, for t in some finite set. We define $\mathcal{Q}$ to be the $\mathbb{F}$ span of $\mathbf{y}_t \otimes 1$, ranging over all t. We view $\mathcal{Q}$ as being a finitely generated type-D module over $\mathcal{R}$ with vanishing differential. There are maps forming a homotopy equivalence of type-D modules

$$\pi^1 \colon (\mathcal{W} \boxtimes \mathbb{F}[U_j])^{\mathcal{R}} \to \mathcal{Q}^{\mathcal{R}}, \quad i^1 \colon \mathcal{Q}^{\mathcal{R}} \to (\mathcal{W} \boxtimes \mathbb{F}[U_j])^{\mathcal{R}},$$

and

$$h^1 \colon (\mathcal{W} \boxtimes \mathbb{F}[U_j])^{\mathcal{R}} \to (\mathcal{W} \boxtimes \mathbb{F}[U_j])^{\mathcal{R}},$$

which satisfy the assumptions of the homological perturbation lemma stated in Lemma 4.2. The map π^1 sends $\mathbf{y}_t \otimes U_j^s$ to $\mathbf{y}_t \otimes 1$ if $s = 0$, and is zero on all generators. The map i^1 sends $\mathbf{y}_t$ to $(\mathbf{y}_t \otimes 1) \otimes 1$. The map h^1 sends $\mathbf{y}_t \otimes U_j^s$ to $(\mathbf{y}_t \otimes U_j^{s-1}) \otimes 1$ if $s > 0$, and h^1 vanishes on all other generators. Applying the homological perturbation lemma endows $\mathcal{Q}^{\mathcal{R}}$ with a left action of $\mathcal{E}$; and also yields a homotopy equivalence ${}_{\mathcal{E}}\mathcal{Q}^{\mathcal{R}} \simeq {}_{\mathcal{E}}\mathcal{W}^{\mathcal{R}[U_j]} \boxtimes {}_{\mathbb{F}[U_j]}\mathbb{F}[U_j]$. Note that the induced A_∞-module structure is operationally bounded by grading considerations. Finally, boxing with the identity morphism of $N^{\mathcal{E}}$ gives a homotopy equivalence as in the statement. ∎

We are now able to prove finite generation in general.

Proof of Proposition 18.11. Suppose that $L = K_1 \cup \cdots \cup K_\ell$ is a link in S^3, and Λ is an integral framing on L. Let $0 \leq j < \ell$. It is sufficient to show that

$$\mathcal{X}_\Lambda^{(*,0)}(L_{1,\dots,j}, L_{j+1,\dots,\ell})^{\mathbb{F}[\mathcal{U}_{j+1},\mathcal{V}_{j+1},\dots,\mathcal{U}_\ell,\mathcal{V}_\ell]}$$

is homotopy equivalent to a finitely generated type-D module over $\mathbb{F}[\mathcal{U}_{j+1}, \mathcal{V}_{j+1}, \dots, \mathcal{U}_\ell, \mathcal{V}_\ell]$. This is sufficient since we may use this model at each point of the cube $\mathbb{E}_{n-j}$ to obtain a finitely generated model of $\mathcal{X}_\Lambda(L_{1,\dots,j}, L_{j+1,\dots,\ell})^{\mathscr{L}_{n-j}}$.

By Proposition 13.2, changing the arc system on the components $K_1, \dots, K_j$ does not change the homotopy type of $\mathcal{X}_\Lambda(L_{1,\dots,j}, L_{j+1,\dots,\ell})^{\mathscr{L}_{n-j}}$. Hence, we assume all of the arcs for $K_1, \dots, K_j$ are alpha-parallel.

Lemma 18.12 verifies that if $\Lambda_{(j)}$ is positive definite, then

$$\mathcal{X}_\Lambda^{(*,0)}(L_{1,\dots,j}, L_{j+1,\dots,\ell})^{\mathbb{F}[U_1,\dots,U_j,\mathcal{U}_{j+1},\mathcal{V}_{j+1},\dots,\mathcal{U}_\ell,\mathcal{V}_\ell]}$$

is homotopy equivalent to a finitely generated type-D module. Furthermore, since $\Lambda_{(j)}$ is positive definite (in particular, non-singular), it is straightforward to see that this complex admits a relative $\mathbb{Q} \times \mathbb{Q}$-valued $(\mathrm{gr}_{\mathbf{w}}, \mathrm{gr}_{\mathbf{z}})$-bigrading (compare [32, Section 9.3]). Lemma 13.31 implies that each $U_1, \dots, U_j$ is chain homotopic to $\mathcal{U}_\ell \mathcal{V}_\ell$ as a type-D endomorphism on this subcube. (Note that on this subcube, the K_ℓ-direction is never incremented, so the arc for K_ℓ is irrelevant.) Lemma 18.14 now implies that $\mathcal{X}_\Lambda^{(*,0)}(L_{1,\dots,j}, L_{j+1,\dots,\ell})^{\mathbb{F}[\mathcal{U}_{j+1},\mathcal{V}_{j+1},\dots,\mathcal{U}_\ell,\mathcal{V}_\ell]}$ is homotopy equivalent to a finitely generated type-D module.

Since ${}_{\mathcal{K}}\mathcal{D}_\infty^{\mathbb{F}[U]}$ is homotopy equivalent to a finitely generated type-DA module by Lemma 18.7, we observe that the claim still holds if we replace any number of framings in a positive definite $\Lambda_{(j)}$ with $+\infty$. The mapping cone of two finitely generated complexes is finitely generated, so the surgery exact triangle from Proposition 18.8 (cf. Remark 18.10), implies that $\mathcal{X}_\Lambda(L_{1,\dots,j}, L_{j+1,\dots,\ell})^{\mathscr{L}_{n-j}}$ is homotopy equivalent to a finitely generated type-D module for any framing, concluding the proof. ■