Appendix A

Splicing operations

In this chapter, we describe some topology related to splicing knot complements. Everything in this chapter is well known, and we only include it as a reference.

A.1 Splices and connected sums

Suppose that K_1 and K_2 are knots in S^3. The complements of K_1 and K_2 are manifolds with torus boundary. The following is well known, though we give a proof for the convenience of the reader. Compare [7, Section 7].

Lemma A.1. *Suppose that λ_1 and λ_2 are integral framings on knots K_1 and K_2 in S^3, and let $[\lambda_1]$ and $[\lambda_2]$ denote the induced elements of $H_1(\partial S^3 \setminus N(K_1))$ and $H_1(\partial S^3 \setminus N(K_2))$. Then $S^3_{\lambda_1+\lambda_2}(K_1 \# K_2)$ is diffeomorphic to the 3-manifold obtained by gluing $S^3 \setminus N(K_1)$ to $S^3 \setminus N(K_2)$ using the diffeomorphism which sends*

$$[\lambda_1] \mapsto -[\lambda_2] \quad \textit{and} \quad [\mu_1] \mapsto [\mu_2].$$

Proof. Consider the manifold Z obtained by gluing the complements of K_1 and K_2 as described above. We may describe Z alternatively by first attaching a 3-dimensional 1-handle to connect $S^3 \setminus \nu(K_1)$ and $S^3 \setminus \nu(K_2)$, and then attaching two 2-handles and one 3-handle. The first 2-handle is attached along $\mu_1 * \overline{\mu}_2$ (where $\overline{\mu}_2$ denotes μ_2 with orientation reverses), and the second 2-handle is attached along $\lambda_1 * \lambda_2$. In both cases, the curves are concatenated across the 1-handle.

We observe that joining $S^3 \setminus \nu(K_1)$ and $S^2 \setminus \nu(K_2)$ with the 1-handle and the 2-handle, tracing $\mu_1 * \overline{\mu}_2$ is the same as gluing $S^3 \setminus \nu(K_1)$ and $S^3 \setminus \nu(K_2)$ along meridians of K_1 and K_2. This gives $S^3 \setminus \nu(K_1 \# K_2)$. Gluing the second 2-handle and then the 3-handle is the same as performing Dehn surgery on $K_1 \# K_2$ with framing $\lambda_1 + \lambda_2$. ■

Remark A.2. The argument above also holds for arbitrary knots in 3-manifolds, as long as the framings are Morse.

Remark A.3. The same argument works for links. Suppose that L_1 and L_2 are links in S^3 with framings Λ_1 and Λ_2. Suppose that K_1 and K_2 are two chosen components of L_1 and L_2, respectively. Then

$$S^3_{\Lambda_1+\Lambda_2}(L_1 \# L_2) \cong M_{\Lambda_1}(L_1, K_1) \cup_\phi M_{\Lambda_2}(L_2, K_2),$$

where $M_{\Lambda_i}(L_i, K_i)$ is the manifold obtained by surgering along $L_i - K_i$, and removing a neighborhood of K_i. The diffeomorphism ϕ which identifies $\partial N(K_1)$ with $\partial N(K_2)$ sends λ_1 to $-\lambda_2$, and μ_1 to μ_2. The framing $\Lambda_1 + \Lambda_2$ is obtained by summing the framings for K_1 and K_2, and leaving the remaining framings unchanged.

A.2 The Hopf link and changes of parametrization

We may naturally view the Hopf link as a link which changes the boundary parametrization. There are two ways to do this:

(1) (Blow up) If K is the special component, and we connect the sum with a Hopf link, and leave K the special component. If we give the other component of the Hopf link framing ± 1, then we change the framing on $\partial N(K)$ by a Dehn twist along the meridian.

(2) (Rotating layer) Noting that the complement of the Hopf link is $\mathbb{T}^2 \times [0, 1]$, we may use the description from the previous remark to see the effect of a connect summing the Hopf link to the special component, and using the new Hopf link component as the special component. The effect is to change the boundary parametrization by composing with one of the two maps:

$$\mu \mapsto \lambda \quad \text{and} \quad \lambda \mapsto -\mu,$$

or

$$\mu \mapsto -\lambda \quad \text{and} \quad \lambda \mapsto \mu.$$

These two cases are distinguished by the sign of the Hopf link which is used in the connected sum.