Chapter 1

Introduction

The Bellman functions and their probabilistic counterparts, the Burkholder functions, play an important role in modern harmonic analysis and probability theory. In fact, these are two names for the same object. The foundations were laid in the groundbreaking papers [4] and [29] (see [28] as well); we refer the reader to the monographs [33] and [58] for a general description of the field; see shorter introductions in [30] and [43].

The field may be, roughly speaking, divided into two parts. The first part uses Bellman functions as tools for proving inequalities without concern for their sharpness. Here, one usually has to come up with a sufficiently good *Bellman supersolution*. The second part aims to provide sharp solutions to Bellman optimization problems. This significantly narrows the area of application since the exact Bellman function is often extremely difficult to describe. However, the result usually comes not only with the sharp constant for the inequality in question, but also with a good understanding of optimal functions.

The present memoir provides the proofs of the results announced in [15]. We will concentrate on a special class of optimization problems that are designed to study summability properties of functions in classes like BMO or A_p. We refer the reader to [15] and to Sections 1.1.1 and 1.1.2 below for a long list of specific problems already solved within the scope of our considerations. In a sense, our main target is to provide a theory for solving optimization problems of a very special kind; this covers only a tiny part of inequalities amenable to the Bellman function method. However, the constructions and ideas arising in this small subfield are universal and they are used in other more involved Bellman problems; see Sections 1.1.3, 1.1.4, and 1.1.5 below. In the forthcoming section, we briefly motivate our study, describe the field, and outline the main examples. We also show how many parts of the present work serve as useful tools for related problems and explain the relationship with the classical moment method and themes in differential geometry. Section 1.2 contains the plan of the memoir. In Section 1.3, we discuss some related questions that fall outside the scope of this work.

1.1 Motivation, examples, and links to classical topics

Let Ξ_0 be a strictly convex unbounded open proper subset of $\mathbb{R}^2$. Let Ξ_1 be another strictly convex unbounded open set whose closure lies in Ξ_0 entirely; we set $\Omega = \operatorname{clos}\Xi_0 \setminus \Xi_1$. Let I be an interval in $\mathbb{R}$. Consider the class $\boldsymbol{A}_\Omega(I)$ of $\mathbb{R}^2$-valued summable functions $\varphi\colon I \to \partial\Xi_0$ such that the point $\langle\varphi\rangle_J$ does not belong to Ξ_1 for any subinterval $J \subset I$; here and in what follows, the symbol $\langle\psi\rangle_E$ denotes the average of a locally summable function ψ over a set E of nonzero measure:

$$\langle\psi\rangle_E = \frac{1}{|E|}\int_E \psi(t)\,dt, \quad |E| \text{ stands for the Lebesgue measure of } E \subset \mathbb{R}.$$

We assume that both boundaries of Ξ_0 and Ξ_1 may be parametrized by the first coordinate. In other words, they are graphs of convex functions in the standard coordinate system.

Let $\mathrm{f}\colon \partial\Xi_0 \to \mathbb{R}$ be a function. Assume f is measurable and does not grow too fast at infinity. We address the question of finding sharp estimates for the quantity $\langle \mathrm{f}(\varphi)\rangle_I$ under the conditions $\varphi \in \boldsymbol{A}_\Omega(I)$ and $\langle\varphi\rangle_I = x$. This setting naturally leads to the introduction of the Bellman function

$$\boldsymbol{B}_\Omega(x;\mathrm{f}) = \sup\{\langle \mathrm{f}(\varphi)\rangle_I \mid \langle\varphi\rangle_I = x,\ \varphi \in \boldsymbol{A}_\Omega(I)\}, \quad x \in \Omega. \tag{1.1.1}$$

Using dilation invariance of the average, one may prove that the definition of $\boldsymbol{B}$ does not depend on the particular choice of I. There are two main classes of examples: those related to BMO spaces and those to the Muckenhoupt classes A_p.

1.1.1 BMO case

The case where

$$\Omega = \{x \in \mathbb{R}^2 \mid x_1^2 \leqslant x_2 \leqslant x_1^2 + 1\},$$

corresponds to optimizing an integral functional $\langle f(\varphi)\rangle_I$ on the ball of unit radius of the BMO space. Here we have chosen

$$\Xi_0 = \{x \in \mathbb{R}^2 \mid x_1^2 < x_2\} \quad \text{and} \quad \Xi_1 = \{x \in \mathbb{R}^2 \mid x_1^2 + 1 < x_2\}.$$

Note that if $\varphi\colon I \to \partial\Xi_0$ belongs to $\boldsymbol{A}_\Omega(I)$, then its first coordinate, $\varphi_1\colon I \to \mathbb{R}$, satisfies the inequality

$$\langle\varphi_1^2\rangle_J - (\langle\varphi_1\rangle_J)^2 \leqslant 1 \quad \text{for any subinterval } J \subset I.$$

This may be rewritten as

$$\langle|\varphi_1 - \langle\varphi_1\rangle_J|^2\rangle_J \leqslant 1 \quad \text{for any subinterval } J \subset I.$$

This is equivalent to the inequality $\|\varphi_1\|_{\mathrm{BMO}} \leqslant 1$, provided that we equip the space $\mathrm{BMO}(I)$ with the quadratic seminorm. The general BMO_p seminorm is

$$\|\psi\|_{\mathrm{BMO}_p(I)} = \sup\{(\langle|\psi - \langle\psi\rangle_J|^p\rangle_J)^{1/p} \mid J \text{ is a subinterval of } I\}.$$

For our considerations, the main norm is BMO_2, so we omit the subscript 2 in this case. By the John–Nirenberg inequality, these norms define the same Banach space of functions. We refer the reader to [46] for general BMO theory.

The BMO setting suggests it might be useful to consider the intermediate domains

$$\Omega_\varepsilon = \{x \in \mathbb{R}^2 \mid x_1^2 \leqslant x_2 \leqslant x_1^2 + \varepsilon^2\}, \tag{1.1.2}$$

which are given by the set-theoretic difference between Ξ_0 and

$$\Xi_\varepsilon = \{x \in \mathbb{R}^2 \mid x_1^2 + \varepsilon^2 < x_2\}.$$

The class $\boldsymbol{A}_{\Omega_\varepsilon}(I)$ generated by Ω_ε corresponds to the ε-ball of BMO. From now on, we fix some $\varepsilon > 0$.

The choice $\mathrm{f}(t, t^2) = e^t$ and the analytic expression

$$\boldsymbol{B}_{\Omega_\varepsilon}(x; e^t) = \frac{1 - \sqrt{x_1^2 - x_2 + \varepsilon^2}}{1 - \varepsilon}\, e^{x_1 - \varepsilon + \sqrt{x_1^2 - x_2 + \varepsilon^2}}, \quad x \in \Omega_\varepsilon,\ \varepsilon < 1, \tag{1.1.3}$$

for the Bellman function (1.1.1), found in [41], lead to sharp integral forms of the John–Nirenberg inequality. This formula explains what we mean by the phrase “compute the Bellman function”. While $\boldsymbol{B}$ is completely defined by (1.1.1), one can extract almost no information from this definition. By computation of a Bellman function we mean finding a good analytic expression for it. Usually our expressions are not as brief as (1.1.3). They are composed of integrations, differentiations, and also include specific functions. What is more, usually we define functions by cases, and the splitting into cases is defined in terms of inequalities for solutions to certain analytic equations. Though sometimes such a description is implicit, we always prove that the solutions exist and are unique.

The choice $\mathrm{f}(t, t^2) = |t|^p$ leads to sharp inequalities between BMO_p norms; this case was considered in [42]. The sharp constants in the classical forms of the John–Nirenberg inequalities were found in [57]. The paper [31] derives sharp embeddings of BMO into Lorentz spaces from the Bellman functions in [57] (see [18] for similar estimates for BMO martingales) and [32] uses the case where $f(t) = \mathrm{f}(t, t^2)$ is piecewise affine to derive the sharp embedding of BMO into the weak L_∞-space. The paper [35] employs Bellman functions of the described type to find sharp constants in the embeddings of BMO into weighted BMO. The papers [47] and [59] find the sharp constants in the multiplicative inequalities $\|\varphi\|_{L_q} \leqslant c_{p,q}\|\varphi\|_{L_p}^{p/q}\|\varphi\|_{\mathrm{BMO}}^{1-p/q}$. The reasoning relies on a three-dimensional Bellman function. In the first paper, it is almost immediately constructed from two-dimensional functions from [42]. The function of the second paper is quite complicated. For the sharp symmetric form of the John–Nirenberg inequality, see the recent paper [8]. The theory for the case of a generic f was created in [17] (see the short report [14], a simpler case study in [16], and an example of application of that theory in [55]).

We note that the Bellman function technique described above is perfectly suited for working with BMO_2, however, it meets significant difficulties when applied to BMO_p norms with general p. All the papers cited above work with BMO_2 (the only application of the Bellman function technique to BMO_p is given in [40] and [56]). For results about sharp constants in John–Nirenberg inequalities on BMO_1, see [23, 27, 40] (the first two papers do not use Bellman function techniques).

The functions $\boldsymbol{B}_{\Omega_\varepsilon}(\,\cdot\,; f)$ for different values of ε are related to each other. The paper [17] suggested describing them all simultaneously. To be more precise, first, the function $\boldsymbol{B}$ is computed for sufficiently small ε. Then one continuously increases ε and tracks the evolution of $\boldsymbol{B}$. This provides an algorithm for constructing $\boldsymbol{B}$ for any ε. In this memoir, we will follow a similar route. This justifies the need for a continuous family of convex domains Ξ_ε that connect Ξ_0 with Ξ_1.

1.1.2 The A_{p_1,p_2} classes

Let p_1 and p_2, $p_1 > p_2$, be real numbers and let $Q > 1$. Consider the domain

$$\Omega_{p_1,p_2,Q} = \bigl\{x \in \mathbb{R}^2 \mid x_1, x_2 > 0 \text{ and } x_2^{1/p_2} \leqslant x_1^{1/p_1} \leqslant Q x_2^{1/p_2}\bigr\}. \tag{1.1.4}$$

This domain corresponds to the so-called A_{p_1,p_2} classes introduced in [54]. A weight $w\colon I \to \mathbb{R}_+$ belongs to A_{p_1,p_2} if the quantity

$$[w]_{p_1,p_2} = \sup\bigl\{\langle w^{p_1}\rangle_J^{1/p_1} \langle w^{p_2}\rangle_J^{-1/p_2} \mid J \text{ is a subinterval of } I\bigr\}$$

is finite. One may see that $[w]_{p_1,p_2} \leqslant Q$ if and only if $\varphi \in \boldsymbol{A}_{\Omega_{p_1,p_2,Q}}$, where $\varphi = (w^{p_1}, w^{p_2})$.

The paper [54] generalizes [53], where the first sharp Bellman functions for Muckenhoupt weights were found; those Bellman functions lead to sharp constants in the reverse Hölder inequality. The scale A_{p_1,p_2} includes the classical Muckenhoupt classes A_p. More precisely, $A_{p,-1/(p-1)}$ coincides with A_p when $p \in (1,\infty)$. The case $p_1 = 1$ corresponds to the so-called Gehring classes. The paper [7] is devoted to this particular case. The sharp constants for the embedding of a Gehring class into A_p are found. That paper uses the Bellman function generated by an f of the form $\mathrm{f}(x) = x_1^q$ for some q. The corresponding sharp weak-type reverse Hölder inequalities for A_{p_1,p_2}-weights were established in [37]; the latter paper uses the Bellman function for $\mathrm{f}(x) = \chi_{[1,\infty)}(x_1)$. See [36] for the limiting case of the Muckenhoupt class A_1.

We note that the definition (1.1.4) extends naturally to the limiting cases where one of the parameters p_1 and p_2 is equal to 1 and the other ∞. The case of A_∞ equipped with the so-called Hruščev's "norm"

$$[w]_\infty = \sup\{\langle w\rangle_J \exp(-\langle \log w\rangle_J) \mid J \text{ is a subinterval of } I\},$$

introduced in [12], is of particular importance. This corresponds to the domain

$$\Omega_{1,\infty,Q} = \{x \in \mathbb{R}^2 \mid e^{x_1} \leqslant x_2 \leqslant Q e^{x_1}\}, \tag{1.1.5}$$

where $Q > 1$, meaning $[w]_\infty \leqslant Q$ whenever $\varphi = (\log w, w)$ belongs to $\boldsymbol{A}_{\Omega_{1,\infty,Q}}$. This domain and the study of the corresponding Bellman functions go back to [53]. See [1] for the study of this particular case and for applications of the obtained inequalities. The paper [34] delivers sharp forms of the principle that the logarithm of an A_∞ Muckenhoupt weight belongs to BMO; the reasoning relies on a Bellman function on the domain (1.1.5) with $\mathrm{f}(x) = x_1^2$. In [40] and [56], the Bellman functions for $\mathrm{f}(x) = |x_1|^p$ are computed and used to establish sharp forms of the John–Nirenberg inequalities for BMO_p, with $p \neq 2$.

1.1.3 Monotonic rearrangements and geometry of the Bellman function

One might wonder why a complicated optimization problem (1.1.1) has a transparent solution. The heuristic reason might be described in several ways. First, the notion of a monotonic rearrangement appears useful in this context. For a function $\xi\colon I \to \mathbb{R}$, define its non-increasing rearrangement $\xi^*\colon [0, |I|] \to \mathbb{R}$ by the formula

$$\xi^*(t) = \inf\bigl\{\lambda \bigm| |\{s \in I \mid \xi(s) > \lambda\}| \leqslant t\bigr\}.$$

Note that $\langle f(\xi)\rangle_I = \langle f(\xi^*)\rangle_{[0,|I|]}$ for any reasonable function f. It was proved in [21, 22] that the monotonic rearrangement operator does not increase the BMO_p norm for all $p \in [1, \infty)$. A similar principle also holds for the A_p constant: $[w^*]_p \leqslant [w]_p$ (see [2] for the case $p = 1$ and [24] for arbitrary p). Recall that we have assumed that $\partial\Xi_0$ may be parametrized with the first coordinate. A function $\varphi\colon I \to \partial\Xi_0$ is called non-increasing when its first coordinate φ_1 does not increase. The notion of monotonic rearrangement of a function $\varphi\colon I \to \partial\Xi_0$ is defined accordingly: this is a non-increasing function $\varphi^*\colon [0, |I|] \to \partial\Xi_0$ that has the same distribution. The paper [52] suggests a proof of a similar principle in the generality of $\boldsymbol{A}_\Omega$, that is, $\varphi^* \in \boldsymbol{A}_\Omega([0, |I|])$, provided $\varphi \in \boldsymbol{A}_\Omega(I)$. The reasoning is based on Bellman ideas. See [51] for dyadic classes and [45] for generalization to Campanato-type norms on VMO. Thus, since the monotonic rearrangement of a function in $\boldsymbol{A}_\Omega$ also belongs to $\boldsymbol{A}_\Omega$, we may restrict the class of functions φ in (1.1.1) to non-increasing ones. This often simplifies the investigation of the function $\boldsymbol{B}$ and gives hope for a closed formula for $\boldsymbol{B}$.

Another reason for a good formula for $\boldsymbol{B}$ is its geometric description proved in [52] (see [62] for a generalization): this function is the pointwise minimal among all locally concave functions on Ω that satisfy the boundary condition $\boldsymbol{B}(x) = \mathrm{f}(x)$, $x \in \partial\Xi_0$. The main idea of [52] is that the Bellman function (1.1.1) has a probabilistic representation in terms of certain discrete martingales; it resembles the representation formula for harmonic functions in terms of the Brownian motion. See, e.g., [26], for much more general representation formulas for solutions of a Hamilton–Jacobi–Bellman equation; the difference between [52] and [26] is that the latter paper uses continuous time martingales. The development of these ideas allowed us to transfer all the results about Bellman problems for functions on the interval to the line and the circle, see [49]. In particular, the sharp constants in various forms of the John–Nirenberg inequalities and reverse Hölder inequalities for Muckenhoupt weights are the same for functions on the interval, the circle, and the line. We note that the question of finding any reasonable sharp forms of the said inequalities for functions of several variables is widely open (see [6] for questions about dimensional dependence of constants). Some dimensional estimates are obtained with the help of monotone rearrangement estimates (see [3]) and with the semigroup approach (see [44]).

1.1.4 Miscellaneous direct applications

We will describe two recent applications. The first one deals with sharp estimates of the distributions of martingales whose square functions are bounded. Namely, it turns out that the Bellman function (1.1.1) for the BMO case (i.e., defined on (1.1.2) with arbitrary f) yields good bounds for the following optimization problem: maximize $\mathbb{E} f(\psi)$ when ψ is the limit value of a martingale with the square function uniformly bounded by 1 and prescribed mean. Indeed, one may view ψ as a function whose BMO([0, 1]) norm is bounded by one. The paper [48] (see also the short report [50]) describes the class of f for which this principle leads to the exact solution of the initial problem; for other f, the said principle provides a fine supersolution.

The second application in [44] says that the Bellman function (1.1.1) delivers dimension-free bounds for the classes of functions such as $\mathrm{BMO}(\mathbb{R}^d)$ or $A_p(\mathbb{R}^d)$, provided that one replaces averages over balls in the definitions of these spaces and Bellman functions with averages over specific semigroup kernels and uses the associated Garsia-type norms. The main feature of these semigroup averages is that they are of martingale nature, which allows us to apply martingale representations from [52]. It is unclear whether the resulting estimates are sharp.

1.1.5 Relationship with the moment method and differential geometry

Assume for a while that Ξ_1 is an empty set. Let $(t, g(t))$ be a parametrization of $\partial\Xi_0$. Then we arrive at the following optimization problem:

$$\int_0^1 \varphi_1 = x_1, \quad \int_0^1 g(\varphi_1) = x_2, \quad \int_0^1 f(\varphi_1) \to \max.$$

Here we have set $I = [0, 1]$. This is a simple example of the moment problem. We refer the reader to [19, 20, 25] for the description of the moment method. The solution to the problem is described geometrically. Let H be the convex hull of the three-dimensional curve $\gamma(t) = (t, g(t), f(t))$. Then the desired maximum equals

$$\sup\{s \in \mathbb{R} \mid (x_1, x_2, s) \in H\}.$$

Similar to (1.1.1), we will denote this quantity by $\boldsymbol{b}(x)$. Assume for a while that Ξ_0 is bounded. Using the Carathéodory theorem on convex hulls, we see that there exists an optimal φ_1 that attains at most 3 values. One may establish a similar principle for unbounded domains. As a result of these considerations, we see that the graph of $\boldsymbol{b}$ is somehow flat: each point on this graph is a convex combination of at most 3 points of the boundary curve γ. The most common case is that $(x, \boldsymbol{b}(x))$ is a convex combination of two points of the boundary curve, A and B. Then we call the segment $[A, B]$ (as well as its projection onto the x_1x_2-plane) that connects them a *chord*. In differential geometry, the tangent plane to the graph of $\boldsymbol{b}$ at $(x, \boldsymbol{b}(x))$ is called a *bitangent plane*, because it is tangent to the boundary curve at both points A and B. For us, the condition on the points A and B for existence of the bitangent plane is expressed in the *cup equation* (see equation (3.4.2) below). The cup equation described in the present memoir has already been used by the authors in several more complicated Bellman problems, see [11, 13, 47, 59, 61].

A solution to the classical moment method as described in the books above works in the case where the boundary curve is somehow regular (say, it has positive torsion); this is related to the notion of a Chebyshev system. According to [25, Chapters III and IV], the boundary of the convex hull of a space curve $(\gamma_1(t), \dots, \gamma_{n+1}(t))$ given on an interval $[a, b]$, where $\gamma_j(t)$ are continuous functions, can be described in terms of *upper and lower principal representations*, provided that $(1, \gamma_1(t), \dots, \gamma_n(t))$ and $(1, \gamma_1(t), \dots, \gamma_{n+1}(t))$ are T_+-systems on $[a, b]$. Recall that the system of continuous functions $(\gamma_0(t), \dots, \gamma_m(t))$ is called a T_+-system if

$$\det(\{\gamma_i(t_j)\}_{i,j=0}^{m}) > 0 \quad \text{on } \Sigma = \{a \leqslant t_0 < \cdots < t_m \leqslant b\}. \tag{1.1.6}$$

Verifying condition (1.1.6) on the set Σ may turn out to be cumbersome. If the map $t \mapsto (\gamma_0(t), \dots, \gamma_m(t))$ is of class $C^m((a, b))$, then (see [19, Chapter VIII]) the following easier condition:

$$\det(\{\gamma_i^{(j)}(t)\}_{i,j=0}^{k}) > 0 \quad \text{on } (a, b), \tag{1.1.7}$$

for all $k = 1, \dots, m$, implies (1.1.6). The system $(\gamma_0(t), \dots, \gamma_m(t))$ satisfying (1.1.7) is called an M_+-system. Clearly, if the system $(\gamma_0(t), \dots, \gamma_m(t))$ is an M_+-system, then any system $(\gamma_0(t), \dots, \gamma_k(t))$ is also an M_+-system for any k, $0 \leqslant k \leqslant m$. Notice that the system $(1, t, g(t), f(t))$ is an M_+-system if and only if $g''(t) > 0$ and $g''(t)f'''(t) - f''(t)g'''(t) > 0$ on (a, b).

If γ does not satisfy any regularity assumptions of type (1.1.6), the combinatorial structure of the boundary of the convex hull can be extremely complicated. We refer the reader to [38] and [60] for descriptions of singularities of convex hulls of generic curves. See also [9] and [39] for relationship with the four vertex theorems.

The minimal concave and locally concave functions may be treated as concave (and locally concave) solutions of the degenerate homogeneous Monge–Ampère equation (see Section 2.2.1 below). See [5] and [10] for regularity results. Note that the latter paper studies minimal locally concave functions on non-convex domains from a different point of view. In that paper, the boundary conditions are imposed on the whole boundary. We impose the boundary conditions on the convex (or *fixed*, see [52]) part of the boundary. This comes naturally from the Bellman function setting. The difference between the two settings is huge.

1.2 Description of exposition

The main body of the memoir is divided into four chapters. The general line of reasoning follows [17].

Chapter 2 contains a more formal introduction. In Section 2.1, we state the problem, list the requirements on the domains and the function f, define the Bellman function, and describe its simple properties. By simple properties we mean those that follow from the definition directly. We also provide some examples that explain the meaning of various conditions imposed on Ω and f. Section 2.2 surveys the duality theory from [52], introduces the strategy of proof, in particular, the notion of optimizers, and ends with a formal description of our results. We suggest consulting Chapter 2 for definitions while reading the further plan of the memoir or return to this section after reading Chapter 2.

We describe the possible local foliations in Chapter 3. In Section 3.1, we present a general description of what a foliation may look like. Section 3.2 introduces a *fence* – a foliation that consists of line segments attached to the fixed boundary. A fence may consist of chords, or tangents, or both of them. We study the conditions for the function constructed from this foliation to be locally concave. Section 3.3 contains the description of tangent domains and the corresponding standard candidates. In Section 3.4, we characterize chordal domains. Though the reasoning here follows the lines of [17], some non-trivial modifications are needed to make calculations simpler. Section 3.5 is devoted to the construction of a cup. Here the construction significantly differs from the one in [17], being quite lengthy and involved; Section 3.5 is split into several subsections. We introduce *forces* in Section 3.6. The forces are certain quantities that express the concavity of the constructed candidate. The main feature of these quantities is that they are negative and decrease as ε increases. The construction of the Bellman function relies heavily on these monotonicity formulas for the forces. We state and prove them in Section 3.6. We also introduce the notion of the *tail* of a force. We warn the reader that we have slightly changed the point of view concerning forces: now the domain of a force coincides with its tails by definition. In [17], the forces were defined on rays or the entire line and could attain both positive and negative values. In the present memoir, they are always negative. Section 3.7 contains the classification of linearity domains and Section 3.8 describes combinatorial properties of foliations. The material of the latter two sections is similar to the material of the corresponding sections in [17].

Chapter 4 describes the evolution of Bellman candidates and, in particular, contains the proof that a Bellman candidate exists for any admissible f and ε. In Section 4.1, we describe what we will refer to as the simple picture case, when the foliation consists of tangents, angles, cups, and, possibly, simple multicups. We also prove that, given an admissible f, there exists ε_1 such that for any $\varepsilon < \varepsilon_1$, there exists a simple Bellman candidate for f and ε. Section 4.2 contains the local monotonicity formulas for forces, the proofs of the existence of roots of balance equations, and other lemmas that will be used in further proofs. In Section 4.3, we describe the local evolution scenarios. Roughly speaking, the cups and full multicups grow, the trolleybuses shrink, the multibirdies and multitrolleybuses disintegrate. We state and prove a rigorous proposition for each of these principles. Section 4.4 contains the combinatorial reasoning that glues together local evolution scenarios. Since this combinatorial reasoning is identical to the one in [17], we omit it, providing a detailed citation.

Chapter 5 provides the theory of optimizers. In particular, it is proved that all the Bellman candidates constructed in Chapter 4 coincide with the corresponding Bellman functions. The opening Section 5.1 contains a geometric description of optimizers: it is useful to represent them as special curves in Ω, called delivery curves (they deliver an optimizer to a given point in Ω). Section 5.2 considers all local foliations and constructs optimizers for each of them. Then these "local" optimizers are glued together in Section 5.3. The concluding Section 5.4 considers various situations when the summability conditions on f are violated. We study the question of finiteness of the Bellman function in this setting and provide simple necessary and sufficient conditions on f, assuming this function satisfies the regularity condition that the torsion of γ changes sign a finite number of times. Sometimes, the Bellman function is finite when the summability assumptions are violated. In this case, the optimizers do not exist. However, we provide an optimizing sequence for each point.

1.3 Related questions

1.3.1 Case of bounded Ω

We may consider the case where Ω_0 and Ω_1 are both bounded domains. Say, the case where

$$\Omega_0 = \big\{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 1\big\}, \quad \Omega_1 = \big\{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 1 - \delta\big\}$$

naturally corresponds to the δ-"ball" of the class of BMO mappings of $[0, 1]$ into the unit circle S^1. See the third example in [62, Section 2]. While it seems that all the constructions of the present memoir are applicable in this situation, there are two subtleties that distinguish the case where Ω_0 is bounded.

First, the boundaries of Ω_0 and Ω_1 are not graphs of functions anymore. Therefore, to adjust the methods of the current memoir to this situation, we need to find coordinate-free geometric versions of our reasoning. For some of them, this is not difficult. Say, a torsion is a geometric notion that does not depend on the choice of the coordinate system. Similarly, forces can be expressed geometrically as exterior products of certain vectors. However, we were not able to find a geometric interpretation for most of the material of Section 3.5.

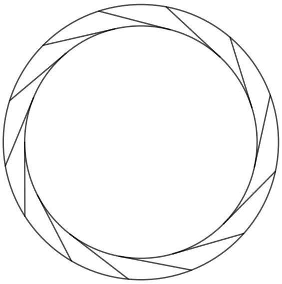

Figure 1.1. Impossible foliation.

Second, some topological effects related to the four vertex theorem (see [9]) come into play. That theorem says that a regular curve that bounds a convex surface changes the sign of its torsion at least four times. Based on the experience of the present memoir, it is natural to expect that the points where the torsion changes sign from $+$ to $-$ are origins of cups (at least for the case where δ is sufficiently small) and there are vertices of angles in neighborhoods of the points where the torsion changes sign from $-$ to $+$. In particular, the foliation of the type presented in Figure 1.1 is impossible.

1.3.2 Higher dimensional cases

One may consider the case where Ω_0 and Ω_1 are subsets of $\mathbb{R}^d$ and state a similar Bellman function problem for them. For example, such a setting appears if one wishes to work with vector-valued BMO functions (see the second example in [62]). As it was proved in [62], the assertion that the Bellman function is the minimal locally concave function holds in this generality. Its geometry for arbitrary boundary conditions might be involved and difficult to describe. However, if one assumes that the domains Ω_0 and Ω_1 possess rotational symmetry, and f is also rotationally symmetric, then, maybe, the computation of the Bellman function may be reduced to the computation of certain functions of the types described in the present memoir. In particular, the problems for vector-valued BMO functions fit into this framework. This has an analogue in the probabilistic setting: many Burkholder problems in [33] allow generalizations to martingales attaining values in a Hilbert space.