Chapter 2

Setting and sketch of proof

2.1 Setting

2.1.1 Domains

Suppose that we have a family of C^3-smooth curves $\mathbf{g}(\,\cdot\,;\varepsilon)\colon \mathbb{R} \to \mathbb{R}^2$, $\varepsilon \in [0, \varepsilon_{\max})$, and let $\mathbf{g}(t;\varepsilon) = (\mathbf{g}_1(t;\varepsilon), \mathbf{g}_2(t;\varepsilon))$, $t \in \mathbb{R}$. In particular, we assume $\frac{\partial}{\partial t}\mathbf{g}(t;\varepsilon) \neq 0$. We assume that $\mathbf{g}(t,\varepsilon)$ is $C^{3,1}$-smooth with respect to (t,ε) meaning the mixed derivative $\frac{\partial^4}{\partial t^3 \partial \varepsilon}\mathbf{g}$ is continuous. For any ε fixed, we will write $\mathbf{g}(t) = (\mathbf{g}_1(t), \mathbf{g}_2(t))$ omitting ε and considering it as a function of one variable. By $g(t)$, we will denote $\mathbf{g}(t;0)$. Suppose that

$$\mathbf{g}_1' > 0, \tag{2.1.1}$$

$$\mathbf{g}_2''\mathbf{g}_1' - \mathbf{g}_2'\mathbf{g}_1'' > 0, \tag{2.1.2}$$

in particular, $\mathbf{g}(\mathbb{R})$ is a graph of a strictly convex function.

Let Ξ_ε be the strict epigraph of $\mathbf{g}(\,\cdot\,;\varepsilon)$:

$$\Xi_\varepsilon = \{(\mathbf{g}_1(t;\varepsilon), x_2) \in \mathbb{R}^2 \mid t \in \mathbb{R}, x_2 > \mathbf{g}_2(t;\varepsilon)\}.$$

We assume that $\Xi_{\varepsilon_1} \supset \Xi_{\varepsilon_2}$ if $\varepsilon_1 < \varepsilon_2$. Moreover, we assume $\mathbf{g}(t;\varepsilon_1) \notin \operatorname{clos} \Xi_{\varepsilon_2}$ for any $t \in \mathbb{R}$ and $\varepsilon_1 < \varepsilon_2$.

For any $\varepsilon \in (0, \varepsilon_{\max})$, we assume that for any $t \in \mathbb{R}$, there are two tangent lines from the point $g(t)$ to Ξ_ε, and that each of these lines intersects with the curve $g(\cdot)$ twice. We note that this assumption slightly differs from the cone condition used in [52] and [62]. The cone condition imposed on the domain by those two papers is weaker than the present one. Let $w_{\mathrm{L}}(t;\varepsilon) = \mathbf{g}(s_{\mathrm{L}};\varepsilon)$ and $w_{\mathrm{R}}(t;\varepsilon) = \mathbf{g}(s_{\mathrm{R}};\varepsilon)$ be the tangency points with the curve $\mathbf{g}(\,\cdot\,;\varepsilon)$, here s_{L} and s_{R} are some functions of t and ε; we choose $s_{\mathrm{R}} < s_{\mathrm{L}}$. Let $S_{\mathrm{L}}(t;\varepsilon) = [g(t), w_{\mathrm{L}}(t;\varepsilon)]$ and $S_{\mathrm{R}}(t;\varepsilon) = [g(t), w_{\mathrm{R}}(t;\varepsilon)]$ be the tangent segments. Here L and R mean left and right with respect to the tangency point. We will often omit the indices L and R and write simply $S(t;\varepsilon)$ and $w(t;\varepsilon)$ or even $S(t)$ and $w(t)$.

Remark 2.1.1. For ε fixed the functions $t \mapsto s_{\mathrm{L}}$ and $t \mapsto s_{\mathrm{R}}$ are C^2-smooth and have positive derivatives: $s_{\mathrm{L}}' > 0, s_{\mathrm{R}}' > 0$.

Proof. It follows from (2.1.1) that the functions s_{L} and s_{R} are increasing.

Let either $s = s_{\mathrm{L}}$ or $s = s_{\mathrm{R}}$. This number is defined by the tangency equation

$$(\mathbf{g}_2(t;0) - \mathbf{g}_2(s;\varepsilon))\mathbf{g}_1'(s;\varepsilon) - (\mathbf{g}_1(t;0) - \mathbf{g}_1(s;\varepsilon))\mathbf{g}_2'(s;\varepsilon) = 0. \tag{2.1.3}$$

The partial derivative of the left-hand side with respect to t is

$$\mathbf{g}_2'(t;0)\mathbf{g}_1'(s;\varepsilon) - \mathbf{g}_1'(t;0)\mathbf{g}_2'(s;\varepsilon),$$

which is nonzero, because the segment connecting $\mathbf{g}(t;0)$ and $\mathbf{g}(s;\varepsilon)$ is transversal to the curve $\mathbf{g}(\,\cdot\,;0)$. The partial derivative of the left-hand side of (2.1.3) with respect to s is equal to

$$(\mathbf{g}_2(t;0) - \mathbf{g}_2(s;\varepsilon))\mathbf{g}_1''(s;\varepsilon) - (\mathbf{g}_1(t;0) - \mathbf{g}_1(s;\varepsilon))\mathbf{g}_2''(s;\varepsilon).$$

This is nonzero because $\mathbf{g}_1(t;0) - \mathbf{g}_1(s;\varepsilon) \neq 0$ and due to (2.1.3) and (2.1.2).

The implicit function theorem guarantees that s and t are C^2-smooth functions of each other and that $s'(t)$ is finite and nonzero. Therefore, $s'(t) > 0$. ■

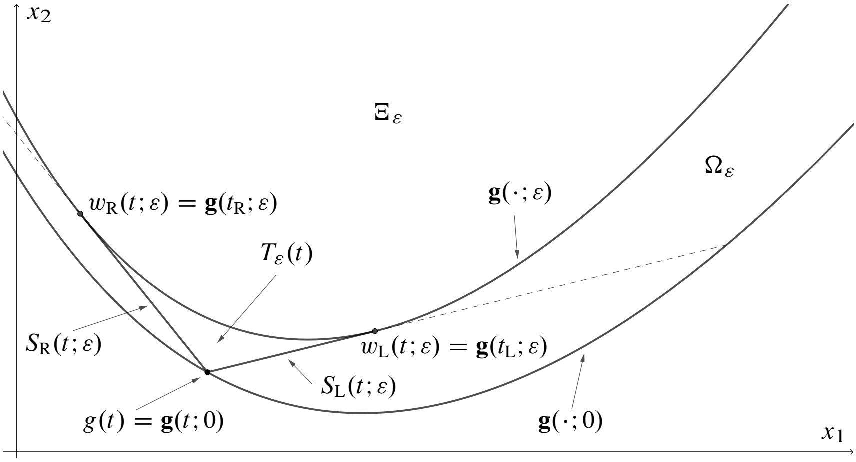

Figure 2.1. Triangle $T_\varepsilon(t)$.

Let $T_\varepsilon(t)$ be the closed curvilinear triangle with the vertex $g(t)$ whose sides are $S_L(t;\varepsilon)$, $S_R(t;\varepsilon)$, and the part of the curve $\mathbf{g}(\,\cdot\,;\varepsilon)$ between the two tangency points, see Figure 2.1. We define the domain

$$\Omega_\varepsilon = \bigcup_{t\in\mathbb{R}} T_\varepsilon(t). \tag{2.1.4}$$

One may see that $\Omega_\varepsilon = \operatorname{clos}\Xi_0 \setminus \Xi_\varepsilon$. We provide several examples of admissible domains Ω_ε in Figure 2.2, see Section 5.4.7 for more information concerning these examples. The domain on the last picture of Figure 2.2 formally does not satisfy (2.1.2), but after application of a suitable isometry of the plane, it becomes admissible for the theory. The parametrization by t running from $-\infty$ to $+\infty$ is convenient

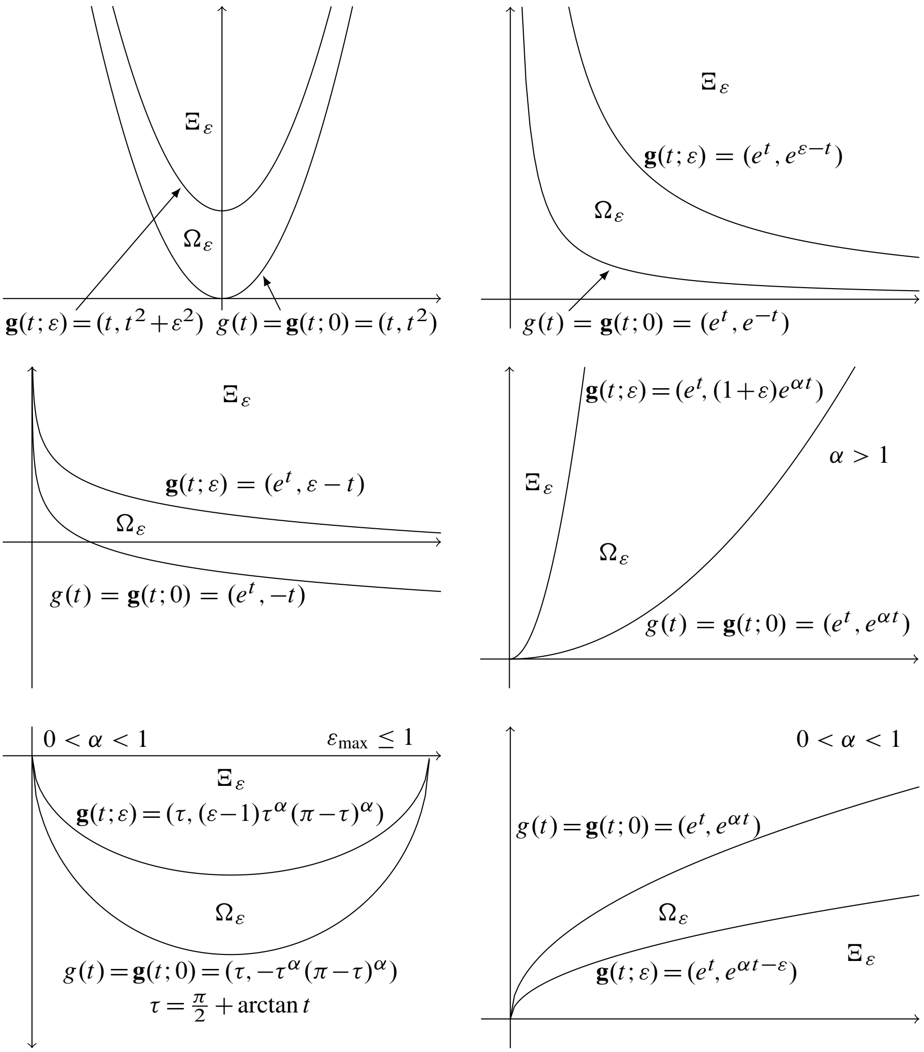

Figure 2.2. Examples of the domains Ω_ε.

for theoretical construction, but sometimes there are more natural parametrizations, as we can see in Figure 2.2.

We also make the following technical assumption. Let $\kappa_{\mathrm{L}}(t;\varepsilon)$ and $\kappa_{\mathrm{R}}(t;\varepsilon)$ be the slopes of the sides S_{L} and S_{R} of the triangle $T_\varepsilon(t)$, i.e.,

$$\kappa = \frac{w_2 - g_2}{w_1 - g_1}. \tag{2.1.5}$$

It follows from the implicit function theorem that the functions $s_{\mathrm{L}}, s_{\mathrm{R}}, w_{\mathrm{L}}, w_{\mathrm{R}}$, and κ are $C^{2,1}$-smooth with respect to (t,ε) in the same sense as before. The domains Ξ_ε

decrease by inclusion, therefore, the function κ is monotone with respect to ε: κ_{L} increases while κ_{R} decreases. We assume a bit more: $\frac{\partial}{\partial\varepsilon}\kappa(t,\varepsilon) \neq 0$ for any t and ε.

The curve $g(\mathbb{R})$ is called the *fixed boundary* of the domain Ω_ε and is denoted by $\partial_{\mathrm{fixed}}\Omega_\varepsilon$. The curve $\mathbf{g}(\mathbb{R};\varepsilon)$ is called the *free boundary* of Ω_ε and is denoted by $\partial_{\mathrm{free}}\Omega_\varepsilon$.

Remark 2.1.2. We will use the sign $\preceq$ for two points on $\partial_{\mathrm{fixed}}\Omega$ to indicate their disposition: $g(t) \preceq g(s)$ if $t \leqslant s$.

Since the segment $S(t)$ is tangent to the curve $\tau \mapsto \mathbf{g}(\tau;\varepsilon)$, there exists a scalar function λ satisfying the following vector-valued identity:

$$\lambda(t)(g(t) - w(t)) = w'(t). \tag{2.1.6}$$

There are two cases: $w = w_{\mathrm{R}}$ defines $\lambda = \lambda_{\mathrm{R}}(t;\varepsilon)$ and $w = w_{\mathrm{L}}$ defines $\lambda = \lambda_{\mathrm{L}}(t;\varepsilon)$, respectively. Let us note that $\lambda_{\mathrm{R}} > 0$ and $\lambda_{\mathrm{L}} < 0$ pointwise, the strict inequalities follow from $w'(t) = \mathbf{g}'(s;\varepsilon)s'(t) \neq 0$ that is guaranteed by Remark 2.1.1. We formulate important assumptions on the domain Ω in terms of λ and w: for all $\varepsilon \in (0, \varepsilon_{\max})$ and $t_0 \in \mathbb{R}$,

$$(1+|w_{\mathrm{R}}(t;\varepsilon)-g(t)|)\cdot\exp\Big(-\int_t^{t_0}\lambda_{\mathrm{R}}(\tau;\varepsilon)\,d\tau\Big) \to 0, \quad t \to -\infty, \tag{2.1.7}$$

$$(1+|w_{\mathrm{L}}(t;\varepsilon)-g(t)|)\cdot\exp\Big(\int_{t_0}^{t}\lambda_{\mathrm{L}}(\tau;\varepsilon)\,d\tau\Big) \to 0, \quad t \to +\infty. \tag{2.1.8}$$

These conditions are technical. If some of them fail, the general theory works, but some explanations need minor changes. The details are given in Section 5.4.

Recall the following definition from [15].

Definition 2.1.3. Let $\Omega = \Omega_\varepsilon$, let $I \subset \mathbb{R}$ be an interval and let $\varphi\colon I \to \partial_{\mathrm{fixed}}\Omega$ be a summable function. We say that the function φ belongs to the class $\boldsymbol{A}_\Omega(I)$ if $\langle\varphi\rangle_J \in \Omega$ for every subinterval $J \subset I$.

2.1.2 Extremal problem and the Bellman function

Definition 2.1.4. Let $\mathrm{f}\colon \partial_{\mathrm{fixed}}\Omega_\varepsilon \to \mathbb{R}$ be some measurable function. We define the Bellman function

$$\boldsymbol{B}_\varepsilon(x;\mathrm{f}) \overset{\mathrm{def}}{=} \sup\{\langle \mathrm{f}(\varphi)\rangle_I \mid \langle\varphi\rangle_I = x, \varphi \in \boldsymbol{A}_{\Omega_\varepsilon}(I)\}. \tag{2.1.9}$$

In this subsection, we summarize some simple properties of the Bellman function defined by formula (2.1.9) and try to explain the choice of the Bellman function in view of the extremal problem we study. We will often use the function $f\colon \mathbb{R} \to \mathbb{R}$ associated with f given above as follows:

$$f(t) = \mathrm{f}(g(t)), \quad t \in \mathbb{R}.$$

We will also use the symbols $\boldsymbol{B}_\varepsilon(\cdot)$ and $\boldsymbol{B}_\varepsilon(\cdot\,; f)$ for the Bellman function (2.1.9) depending on what we would like to emphasize. When ε and f are fixed, we will omit them and write simply $\boldsymbol{B}(x)$. Now, we formulate the easiest properties of the Bellman function that do not need any conditions on f.

Remark 2.1.5. The Bellman function $\boldsymbol{B}_\varepsilon$ does not depend on the interval I where the class $\boldsymbol{A}_{\Omega_\varepsilon}$ is defined.

Indeed, using a linear change of variables, one can transform a function $\varphi \in \boldsymbol{A}_{\Omega_\varepsilon}(I)$ into another function $\tilde{\varphi} \in \boldsymbol{A}_{\Omega_\varepsilon}(\tilde{I})$ so that all the averages in formula (2.1.9) do not change. Thus, the supremum defined by formula (2.1.9) is taken over the same subset of the real numbers.

The next remark allows us to estimate integral functionals from below. One can consider another Bellman function,

$$\boldsymbol{B}_\varepsilon^{\min}(x;\mathrm{f}) \overset{\text{def}}{=} \inf\{\langle \mathrm{f}(\varphi)\rangle_I \mid \langle\varphi\rangle_I = x, \varphi \in \boldsymbol{A}_{\Omega_\varepsilon}(I)\}.$$

Of course, the minimal Bellman function can be easily expressed in terms of the maximal one.

Remark 2.1.6. We have $\boldsymbol{B}_\varepsilon^{\min}(x; f) = -\boldsymbol{B}_\varepsilon(x; -f)$.

Now we begin to study the domain of $\boldsymbol{B}_\varepsilon$. The following definition seems to be useful for all further reasoning.

The Bellman function (2.1.9) is defined everywhere on $\mathbb{R}^2$. However, this function is equal to $-\infty$ if the supremum is taken over the empty set. We drop such points from the domain of the Bellman function.

Definition 2.1.7. By the *Bellman domain* of the function $\boldsymbol{B}_\varepsilon$, we mean the set of points $x \in \mathbb{R}^2$ for which there exists a function $\varphi \in \boldsymbol{A}_{\Omega_\varepsilon}$ satisfying $\langle\varphi\rangle_I = x$.

Proposition 2.1.8. *The domain Ω_ε defined in* (2.1.4) *coincides with the Bellman domain of $\boldsymbol{B}_\varepsilon$.*

Proof. First, by the definition of $\boldsymbol{A}_{\Omega_\varepsilon}$, for any $\varphi \in \boldsymbol{A}_{\Omega_\varepsilon}$ we have $\langle\varphi\rangle_I \in \Omega_\varepsilon$. Second, for any given point $x \in \Omega_\varepsilon$, there exists some $t \in \mathbb{R}$ such that $x \in T_\varepsilon(t)$. We draw the tangent to $\partial_{\mathrm{free}}\Omega_\varepsilon$ passing through x. It follows from conditions on the domain that this line intersects $\partial_{\mathrm{fixed}}\Omega_\varepsilon$ twice, say, at $g(a)$ and $g(b)$. It is important that $x \in [g(a), g(b)] \subset \Omega_\varepsilon$. Let $\alpha \in [0,1]$ be such that $x = \alpha g(a) + (1-\alpha)g(b)$. Define the function φ as follows:

$$\varphi = g(a)\chi_{[0,\alpha)} + g(b)\chi_{[\alpha,1]}.$$

It is clear that $\langle\varphi\rangle_{[0,1]} = x$ and $\langle\varphi\rangle_J \in [g(a), g(b)] \subset \Omega_\varepsilon$ for any $J \subset [0,1]$. Therefore, we have $\varphi \in \boldsymbol{A}_{\Omega_\varepsilon}$. ■

This proposition does not say whether the Bellman function is finite on Ω_ε, it only cuts off those points of the plane where it is *a priori* infinite.

Proposition 2.1.9. *We have* $\boldsymbol{B}_\varepsilon(g(t); f) = f(t)$ *for all* $t \in \mathbb{R}$.

Proof. The function φ whose average $\langle\varphi\rangle_I$ lies on the fixed boundary is constant almost everywhere, because the curve $\partial_{\mathrm{fixed}}\Omega_\varepsilon$ is a graph of a strictly convex function. This constant coincides with the average, thus $\varphi = g(t)$. Consequently, the set we are taking the supremum over consists of a single number $f(t)$, therefore $\boldsymbol{B}_\varepsilon(g(t); f) = f(t)$. ■

2.1.3 Conditions on f

The function f will be subject to some restrictions. These restrictions are of two types. The first type comes from quantitative aspects. We are interested in finite Bellman functions, and we want the function f(φ) to be integrable for any function $\varphi \in \boldsymbol{A}_{\Omega_\varepsilon}$, $\varepsilon < \varepsilon_{\max}$, in order to have well-defined averages in (2.1.9). These conditions are expressed in terms of summability properties of f. The second type of conditions corresponds to regularity properties of f. These conditions make the structure of the Bellman function less complicated and thus describable.

We begin with conditions of the second type. We require $f \in C^2(\mathbb{R})$. Consider the curve

$$\gamma(t) = (g(t), f(t)), \quad t \in \mathbb{R},$$

which is the graph of the boundary condition, see Proposition 2.1.9. We introduce the following object:

$$\mathbf{T}(t) = \det\begin{pmatrix}\gamma'(t)\\ \gamma''(t)\\ \gamma'''(t)\end{pmatrix} = \det\begin{pmatrix} g_1'(t) & g_2'(t) & f'(t)\\ g_1''(t) & g_2''(t) & f''(t)\\ g_1'''(t) & g_2'''(t) & f'''(t)\end{pmatrix}, \quad t \in \mathbb{R}. \qquad (2.1.10)$$

We will impose such conditions on f that $\mathbf{T}$ will be a signed measure, see Condition 2.1.11. The sign of $\mathbf{T}$ coincides with the sign of the torsion of the curve γ. Since we assume γ to be C^2, the measure $\mathbf{T}$ does not have atoms.

Definition 2.1.10. Let μ be a signed measure on the line. The complement of its support is an open subset of the line, so it is a union of several intervals (finite or countable number of them). We call the closure of each such interval *a solid root* of μ. If μ is neither positive nor negative in every neighborhood of its solid root, then such a solid root is called *essential*.

Let $\mathbf{T} = \mathbf{T}_+ + \mathbf{T}_-$ be the Hahn decomposition of the measure $\mathbf{T}$. Note that the set $\operatorname{supp}\mathbf{T}_+ \cap \operatorname{supp}\mathbf{T}_-$ is the set of points where $\mathbf{T}$ “changes its sign”. This set is closed. The points of $\operatorname{supp}\mathbf{T}_+ \cap \operatorname{supp}\mathbf{T}_-$ are also called essential roots. Therefore, an essential root is a maximal by inclusion connected subset of the line such that $\mathbf{T}$ vanishes on it but is neither negative nor positive in any neighborhood of it. The regularity condition we will impose on $\mathbf{T}$ is that it has only a finite number of essential roots. If γ were C^3-smooth, this condition would be the same as if the function $\mathbf{T}$ had only a finite number of changes of sign.

Let us introduce the following functions:

$$\kappa_2 = \frac{g_2'}{g_1'}, \quad \kappa_3 = \frac{f'}{g_1'}, \quad \mathfrak{K} = \frac{\kappa_3'}{\kappa_2'}. \tag{2.1.11}$$

Note that κ_2 and κ_3 are the slopes of the projections of γ' onto (x_1x_2)- and (x_1x_3)-coordinate planes.

Since $g_1' > 0$, $\kappa_2' \neq 0$ (see (2.1.1) and (2.1.2)), and

$$\mathbf{T} = \mathfrak{K}' \cdot (g_1')^3 \cdot (\kappa_2')^2, \tag{2.1.12}$$

the signs of $\mathbf{T}$ and $\mathfrak{K}'$ coincide. We formulate the condition discussed above in terms of $\mathfrak{K}$.

Condition 2.1.11. *The function f is twice continuously differentiable, $\mathfrak{K}$ is piecewise monotone and has only a finite number of monotonicity intervals.*

Remark 2.1.12. Condition 2.1.11 implies that the functions f and κ_3 are piecewise monotone and change their sign only a finite number of times.

Proof. The function $\mathfrak{K}$ changes its sign only a finite number of times, $\kappa_2' > 0$, therefore the same is true for $\kappa_3' = \kappa_2'\mathfrak{K}$, and hence for κ_3. But $g_1' > 0$ and $f' = \kappa_3 g_1'$, thus f' has only a finite number of changes of sign, therefore the same is true for f. ■

The essential roots of $\mathbf{T}$ will play a significant role in what follows, therefore we fix the notation for them.

Definition 2.1.13. The essential roots of $\mathbf{T}$ (or $\mathfrak{K}'$) are closed intervals (which can be single points or rays) $c_0, c_1, \ldots, c_n$ and $v_1, v_2, \ldots, v_n$ such that $c_0 < v_1 < c_1 < v_2 < \cdots < v_n < c_n$. The measure $\mathbf{T}$ "changes sign" from "–" to "+" at v_i, from "+" to "–" at c_i.

We make an agreement that if in a neighborhood of $-\infty$ we have $\mathbf{T} < 0$, then $c_0 = -\infty$. Similarly, if in a neighborhood of $+\infty$ we have $\mathbf{T} > 0$, then $c_n = +\infty$. What is more, v_i or c_i is an interval (not a point) if and only if it is an essential solid root in the sense of Definition 2.1.10.

In light of our definition, sometimes we will have to treat the intervals as if they were points. We write $\operatorname{dist}(x, y)$ for the usual distance between subsets x and y of the real line. We will need it only to denote the distance between either two intervals or an interval and a point. Moreover, sometimes we will write, for example, $a_n \to w$, where w is a root, e.g., it can be an interval. In such situations we mean that for every neighborhood of w, all but finite number of members of $\{a_n\}_n$ lie in it. What is more, the set of intervals has an essential ordering, that is, $[a, b]$ is less than $[c, d]$ if and only if $b < c$. We have already used this ordering in Definition 2.1.13. We will also often use the notation $\mathfrak{a}^{\mathrm{r}}$ and $\mathfrak{a}^{\mathrm{l}}$ for the right and left endpoints of the interval $\mathfrak{a}$.

Let us turn to the conditions of the first type (summability conditions at infinities).

Condition 2.1.14. *For any* $\varepsilon < \varepsilon_{\max}$, *we have*

$$\int_{-\infty}^{v} f'(\tau)\exp\Bigl(-\int_{\tau}^{v}\lambda_{\mathrm{R}}(s;\varepsilon)\,ds\Bigr)\,d\tau > -\infty, \tag{2.1.13}$$

$$\int_{v}^{+\infty} f'(\tau)\exp\Bigl(\int_{v}^{\tau}\lambda_{\mathrm{L}}(s;\varepsilon)\,ds\Bigr)\,d\tau < +\infty \tag{2.1.14}$$

for any $v \in \mathbb{R}$. *Recall that* λ_{L} *and* λ_{R} *are defined in* (2.1.6).

These conditions look cumbersome, but it will be clear that they appear quite naturally, see Chapter 5, Propositions 5.2.6 and 5.2.8. If (2.1.13) fails, but (2.1.7) holds, then the function B is infinite because one can construct a special test function φ (that is, an optimizer in Proposition 5.2.5, see also Definition 2.2.6) such that $\langle \mathrm{f}(\varphi)\rangle = +\infty$. Similarly, if (2.1.14) fails, but (2.1.8) holds, then the function B is also infinite. See Section 5.4.6 for the details.

2.1.4 Smoothness conditions

Here we collect all the smoothness conditions we impose on the functions we discussed in this section.

We suppose that $\mathbf{g}$ is $C^{3,1}$-smooth and then the functions $s_{\mathrm{L}}, s_{\mathrm{R}}, w_{\mathrm{L}}, w_{\mathrm{R}}, \kappa_{\mathrm{L}}, \kappa_{\mathrm{R}}$ are $C^{2,1}$-smooth with respect to (t, ε) and κ_2 is C^2-smooth. We also have $\frac{\partial}{\partial t}\mathbf{g}(t,\varepsilon) > 0$, $\frac{\partial}{\partial \varepsilon}\kappa(t,\varepsilon) \neq 0$. The functions $\lambda_{\mathrm{R}}, \lambda_{\mathrm{L}}$ are $C^{1,1}$. The function f is C^2 with f'' being piecewise monotone with a finite number of monotonicity intervals, therefore $\kappa_3 \in C^1$ and $\mathfrak{K}$ is a piecewise monotone continuous function with a finite number of monotonicity intervals, and $\mathfrak{K}'$ (as well as $\mathbf{T}$) is a signed continuous measure.

2.2 Locally concave functions

We will use abstract results from [52] concerning the relation between the Bellman function and locally concave functions. We start with the definition of local concavity.

Definition 2.2.1. The function $G\colon \Omega \mapsto \mathbb{R}$ is said to be *locally concave* on Ω if it is concave on every convex subdomain of Ω. We introduce the set of locally concave functions on Ω majorising f on $\partial_{\mathrm{fixed}}\Omega$:

$$\Lambda_{\Omega,f} \overset{\mathrm{def}}{=} \{G \mid G \text{ is locally concave on } \Omega,\, G|_{\partial_{\mathrm{fixed}}\Omega} \geqslant \mathrm{f}\}.$$

Define the function $\mathfrak{B}_{\Omega,f}$ to be the pointwise infimum of the functions from $\Lambda_{\Omega,f}$:

$$\mathfrak{B}_{\Omega,f}(x) = \inf_{G\in\Lambda_{\Omega,f}} G(x), \quad x \in \Omega.$$

It is not difficult to see that a function is locally concave if and only if it is concave on every segment that belongs to Ω entirely.

The proposition below resembles [52, Proposition 6.1]. It follows from [52, Proposition 6.1] in the case where the limits $\lim_{t\to\pm\infty} g(t)$ are infinite. If one of the limits is finite, one should use the projective transformation trick to reduce it to the case of infinite limits, see [52, Appendix A].

Proposition 2.2.2. *Let the function f be locally bounded from below, and let the function $\mathfrak{B}_{\Omega_\varepsilon,\max(f,0)}$ be finite. Then the integral $\langle \mathrm{f}(\varphi)\rangle_I$ is well defined for all $\varphi \in \boldsymbol{A}_{\Omega_\varepsilon}$, and $\boldsymbol{B}_\varepsilon(\,\cdot\,; f) = \mathfrak{B}_{\Omega_\varepsilon,f}$.*

Proposition 2.2.2 will serve as one of the main ingredients in the proof of Theorem 2.2.7.

2.2.1 Monge–Ampère equation and ruled surfaces

Using simple convex geometry arguments, one can see that $\mathfrak{B}_{\Omega_\varepsilon,f}$ is affine in some directions. The precise statement is as follows.

Theorem 2.2.3. *Let f satisfy Conditions 2.1.11 and 2.1.14. Then the function $\mathfrak{B} = \mathfrak{B}_{\Omega_\varepsilon,f}$ satisfies the following conditions*:

(1) *For every point $x \in \operatorname{int}\Omega_\varepsilon$, there is a nonzero vector $\Theta(x)$ such that $\mathfrak{B}$ is affine along the line $\ell(x) = x + \mathbb{R}\Theta(x)$ in a neighborhood of x. If there are at least two noncollinear vectors Θ_1 and Θ_2 with this property, then $\mathfrak{B}$ is affine in a neighborhood of x. If this vector is unique, we call the maximal (by inclusion) segment of $\ell(x)$ containing x on which $\mathfrak{B}$ is affine the* extremal segment.

(2) *The function $\mathfrak{B}$ is differentiable and its gradient is constant on each extremal segment.*

(3) *The extremal segments cannot intersect the free boundary transversally, but only tangentially.*

This theorem provides a partition of Ω_ε into sets of two types. The sets of the first type are extremal segments, along which $\mathfrak{B}$ is affine. We note (and this can be easily proved) that the extremal segments cannot “stop” inside Ω_ε. One of the endpoints of an extremal segment belongs to the fixed boundary, the second one could lie either on the fixed boundary as well, or it is the point of tangency with the free boundary. The sets of the second type are the two-dimensional domains where $\mathfrak{B}$ is affine; we call them *domains of linearity*.

We should make a remark on the term “domain”. We call a domain every open connected set united with some part (or none) of its boundary.

We will not prove Theorem 2.2.3, because, from a formal point of view, we do not need it (however, it will follow from our general considerations, e.g., Theorem 5.3.2 far below). It only helps us to guess the Bellman function. It leads us to the notion of a *Bellman candidate*.

Definition 2.2.4. Let B be a continuous locally concave function on a subdomain ω of Ω_ε satisfying the boundary condition on $\omega \cap \partial_{\mathrm{fixed}}\Omega_\varepsilon$. The function B is called a *Bellman candidate* on ω if there exists a *foliation* on ω. This means that ω can be represented as a union of several domains, $\omega = \bigcup_i \omega^i$, such that $B \in C^1(\omega^i)$ and B is either affine in ω^i (thus, ω^i is a domain of linearity) or ω^i is foliated by straight line segments along which the differential of B is constant (they are the extremal segments).

If B is twice differentiable at some inner point $x \in \Omega_\varepsilon$, then $\frac{d^2B}{dx^2} \leqslant 0$, i.e., the second differential of B is negative semi-definite, because B is locally concave. This matrix has the vector $\Theta(x)$ in the kernel, thus its determinant is zero. This remark clarifies the name of the subsection, since the resulting equation is called the *homogeneous Monge–Ampère equation*:

$$B_{x_1x_1}B_{x_2x_2} - B_{x_1x_2}^2 = 0.$$

The homogeneous Monge–Ampère equation must hold almost everywhere for the function $\mathfrak{B}$, because a locally concave function is almost everywhere twice differentiable. However, it does not have to hold everywhere, because the Bellman function does not have to be C^2-smooth even for very smooth boundary values f. For example, see the function $N_{\varepsilon,p}$ from [42, Lemma 6.1] with $p > 2$; it is not C^2-smooth, however, its boundary data is C^∞ for p being an even integer greater than 2.

2.2.2 Optimizers

Let B be a Bellman candidate on the whole domain Ω_ε, i.e., let it satisfy Definition 2.2.4 with $\omega = \Omega_\varepsilon$. This subsection provides a method of verification that the candidate B coincides with the Bellman function. By Proposition 2.2.2, the inequality $\boldsymbol{B}_\varepsilon \leqslant B$ holds. To prove the reverse inequality, $B(x) \leqslant \boldsymbol{B}_\varepsilon(x)$, for a point x, $x \in \Omega_\varepsilon$, it is sufficient to find a function $\varphi \in \boldsymbol{A}_{\Omega_\varepsilon}$, with $\langle\varphi\rangle_I = x$, such that $B(x) \leqslant \langle \mathrm{f}(\varphi)\rangle_I$. Indeed, by the definition of the Bellman function, formula (2.1.9), $\langle \mathrm{f}(\varphi)\rangle_I \leqslant \boldsymbol{B}_\varepsilon(x)$, consequently, $B(x) \leqslant \boldsymbol{B}_\varepsilon(x)$. We introduce some notions.

Definition 2.2.5. Let $x \in \Omega_\varepsilon$. We call a function $\varphi \in \boldsymbol{A}_{\Omega_\varepsilon}$ a *test function* for x if

$$\langle\varphi\rangle_I = x.$$

Definition 2.2.6. Let $x \in \Omega_\varepsilon$ and let B be a Bellman candidate. We call a measurable function φ an *optimizer* for B at x if it satisfies two conditions:

- φ is a test function for x,
- $B(x) = \langle \mathrm{f}(\varphi)\rangle_I$.

So, in order to prove that a candidate B coincides with the Bellman function, it suffices to provide, for each point x in Ω_ε, at least one optimizer for B at x. What is the way to do this? We may consider only monotone optimizers (with respect to the ordering $\preceq$ defined in Section 2.1). This follows from the fact that the class $\boldsymbol{A}_{\Omega_\varepsilon}$ is invariant with respect to monotonic rearrangement. This fact is proved in [52, Corollary 3.12].

The following property of monotonic rearrangement is useful: for the monotone rearrangement φ^* of φ, we have $\langle \mathrm{f}(\varphi)\rangle_I = \langle \mathrm{f}(\varphi^*)\rangle_I$. Therefore, φ^* is an optimizer provided φ is. We do not need this consideration formally, but it helps us to guess the optimizers. In light of this, we consider only monotone optimizers. A more detailed discussion concerning optimizers is postponed until Chapter 5.

2.2.3 General principles and description of results

One of our main aims is to prove the following theorem.

Theorem 2.2.7. *Let f satisfy Conditions 2.1.11 and 2.1.14. Then*

$$\boldsymbol{B}_\varepsilon(x; f) = \mathfrak{B}_{\Omega_\varepsilon, f}.$$

In fact, in the whole memoir we study how to construct the minimal locally concave function $\mathfrak{B}_{\Omega_\varepsilon, f}$. In Chapters 3 and 4, for every f satisfying Conditions 2.1.11 and 2.1.14, we construct specific locally concave functions G on Ω_ε with the boundary conditions $G|_{\partial_{\mathrm{fixed}}\Omega_\varepsilon} = \mathrm{f}$. These functions are called Bellman candidates, see Definition 2.2.4. Due to Remark 2.1.12, the function f does not change its sign outside a compact set, therefore it is easy to find a function $\tilde{f}$ satisfying Conditions 2.1.11 and 2.1.14, which coincides with $\max(f, 0)$ outside a compact set. This would imply that the function $\mathfrak{B}_{\Omega_\varepsilon, \tilde{f}}$ is finite, whence $\mathfrak{B}_{\Omega_\varepsilon, \max(f,0)}$ is finite as well. Proposition 2.2.2 now gives the statement of Theorem 2.2.7. In particular, the function G constructed in our considerations satisfies the inequality $\boldsymbol{B}_\varepsilon(\,\cdot\,; f) \leqslant G$ on Ω_ε. We use optimizers to prove that G coincides with $\boldsymbol{B}_\varepsilon(\,\cdot\,; f)$; this will be done in Chapter 5, see Theorem 5.3.1.

Main results.

1. We provide an algorithm to calculate the Bellman function $\boldsymbol{B}_\varepsilon$, for f satisfying Conditions 2.1.11 and 2.1.14, and describe the evolution of this function with respect to ε; as a corollary, we obtain Theorem 2.2.7.

2. We show that for any C^1 function f not satisfying at least one of the two conditions in Condition 2.1.14, $\boldsymbol{B}_\varepsilon(\cdot\,; f) = +\infty$ everywhere except the fixed boundary.
3. We investigate the case when the technical conditions (2.1.7) and (2.1.8) for the domain do not hold. The theory still works with minor modifications: it could happen that there exists no optimizer for some points $x \in \Omega_\varepsilon$; in such a case, instead of one optimizer, we construct an optimizing sequence of functions $\varphi_n \in \boldsymbol{A}_{\Omega_\varepsilon}$ such that $\langle \varphi_n \rangle_I = x$ and $\langle \mathrm{f}(\varphi_n) \rangle_I \to \boldsymbol{B}_\varepsilon(x; f)$. We also give criteria for the function $\boldsymbol{B}_\varepsilon$ to be finite when conditions (2.1.7) and (2.1.8) fail.

Some explanation is needed. By building the Bellman function we mean mainly the construction of the corresponding foliation. This foliation evolves continuously and obeys certain monotonicity rules that are also described. In the first point we intend to provide some expression for $\boldsymbol{B}_\varepsilon$ that contains integrals, differentiation, and solution of some implicit equations. We always prove that those equations are well solvable, i.e., they do not have infinitely many solutions.