Chapter 3

Patterns for Bellman candidates

3.1 Preliminaries

The purpose of this section is to construct Bellman candidates (see Definition 2.2.4) on various domains. The global foliation for the Bellman function may be rather complicated, but its local structure is easy to describe. We give some heuristics to classify local Bellman candidates.

Consider a minimal locally concave function and its foliation provided by Theorem 2.2.3. We recall that this foliation consists of extremal segments and linearity domains.

The extremal segments are of two types: those that connect two points on the fixed boundary, called *chords*, and those that connect a point on the fixed boundary with a point on the free boundary, called *tangents*. We note that a chord can be tangent to the free boundary. Such a chord is called a *long* one.

It is convenient to classify linearity domains by the number of their points on the fixed boundary. We distinguish the linearity domains that have one point on the fixed boundary, the ones that have two points on the fixed boundary, and all the others. A more detailed classification will be provided later in Section 3.7.

A global foliation is glued from local ones. We explain the informal meaning of the word "glue" we use. Consider two subdomains Ω^1 and Ω^2 of Ω_ε. Let B_1 be a Bellman candidate on Ω^1 and let B_2 be a Bellman candidate on Ω^2. Suppose that $B_1 = B_2$ on $\Omega^1 \cap \Omega^2$. Consider the function B defined on the union domain $\Omega = \Omega^1 \cup \Omega^2$ as a concatenation of B_1 and B_2 (i.e., $B = B_1$ on Ω^1 and $B = B_2$ on Ω^2). Suppose that this function B is C^1-smooth. In such a case, it is locally concave, provided the functions B_1 and B_2 are, see Proposition 3.1.2 below. Thus, it is a Bellman candidate on Ω. Its foliation coincides with the foliation for B_1 on Ω^1 and with the foliation for B_2 on Ω^2. We say that the foliation for B is glued from the foliations for B_1 and B_2.

We have used the following fact in the explanation: a C^1-concatenation of two locally concave functions is locally concave. To formulate this claim rigorously, we need several new notions.

Definition 3.1.1. Suppose that Ω is a subdomain of Ω_ε. We call Ω an *induced convex* set if for every segment $l \subset \Omega_\varepsilon$, the set $\Omega \cap l$ is convex. As usual, for any set $\omega \subset \Omega_\varepsilon$, we define its *induced convex hull* $\mathrm{ind\,conv}(\omega)$ as the minimal induced convex set which contains ω.

All the domains we use for building Bellman candidates are induced convex.

Proposition 3.1.2. *Suppose that the domains Ω^1 and Ω^2 are induced convex in Ω_ε. Suppose that a C^1-smooth function B is locally concave on each of the domains Ω^i, $i = 1, 2$. Then it is locally concave on $\Omega^1 \cup \Omega^2$.*

Proof. To prove the claim we establish that the restriction of B to every segment $l \subset \Omega^1 \cup \Omega^2$ is concave. We have $l = (l \cap \Omega^1) \cup (l \cap \Omega^2)$. Each of the sets $l \cap \Omega^1$ and $l \cap \Omega^2$ is convex, i.e., they are either segments or empty sets (the latter case is trivial). By the hypothesis, B is concave on each of these segments. Using C^1smoothness of B at a common point of these segments, we get that $B|_l$ is concave. ∎

Now we can state that a C^1-smooth concatenation of two Bellman candidates is a Bellman candidate provided their domains are induced convex.

We turn to building Bellman candidates. Usually, we will give only sufficient conditions for a foliation and a function f that generate a Bellman candidate. However, to be ready to construct the Bellman function, we have to examine all possible local Bellman candidates. So, the conditions we provide are usually also necessary. To make the story shorter, sometimes we will not prove this necessity, because we do not need it.

To describe combinatorial properties of foliations, we associate a special oriented graph with each foliation. Generally, its vertices correspond to the linearity domains, whereas its edges correspond to the domains of extremal segments. A vertex is incident to an edge if the corresponding two domains are adjacent. We postpone a more detailed description of the graph to Section 3.8.2.

3.2 Fence

We start by investigating the properties of a Bellman candidate defined on a family of extremal segments with an endpoint on the fixed boundary.

Recall that $g = (g_1, g_2)\colon \mathbb{R} \to \partial_{\mathrm{fixed}}\Omega$ is the parametrization of the fixed boundary of Ω such that $g_1' > 0$ and $\kappa_2' = (g_2'/g_1')' > 0$. Let I be some interval in $\mathbb{R}$. Suppose that there exists a family of segments $S(t) = [g(t), w(t)]$, $t \in I$, which foliate a subdomain $\Omega(I) \subset \Omega$. We assume that the foliation is sufficiently smooth: $w \in C^1(I)$. We can consider t as a function from $\Omega(I)$ to I such that $x \in S(t(x))$ for $x \in \Omega(I)$. We assume the function t to be C^1-smooth as well, and also $\frac{\partial}{\partial x_2} t \neq 0$ (which implies that the segments $S(t)$ are not vertical, i.e., $w_1(t) \neq g_1(t)$). We denote the slope of the segment $S(t)$ by $\kappa(t)$:

$$\kappa(t) \stackrel{\mathrm{def}}{=} \frac{w_2(t) - g_2(t)}{w_1(t) - g_1(t)}. \tag{3.2.1}$$

Proposition 3.2.1. *Suppose that B is a C^1-smooth function on $\Omega(I)$ that is affine on each segment $S(t)$, and its gradient $\beta = (\beta_1, \beta_2) = \nabla B$ is constant on $S(t)$, $t \in I$.*

Then B is C^2-smooth. Moreover, there is a representation

$$B(x) = f(t) + (x_1 - g_1(t))[\kappa_3(t) + (\kappa(t) - \kappa_2(t))\beta_2(t)], \quad t = t(x), \tag{3.2.2}$$

where β_2 is given by

$$\beta_2(t) = \exp\Big(-\int_{t_0}^{t} \frac{\kappa_2'}{\kappa_2 - \kappa}\Big) \times \Big(\int_{t_0}^{t} \exp\Big(\int_{t_0}^{\tau} \frac{\kappa_2'}{\kappa_2 - \kappa}\Big) \frac{\kappa_3'(\tau)}{\kappa_2(\tau) - \kappa(\tau)}\, d\tau + \beta_2(t_0)\Big), \quad t, t_0 \in I, \tag{3.2.3}$$

and κ_2, κ_3 are defined in (2.1.11); here $\beta_2(t_0)$ is an arbitrary number.

Proof. We have

$$B(x) = f(t) + \beta_1(t)(x_1 - g_1(t)) + \beta_2(t)(x_2 - g_2(t)), \tag{3.2.4}$$

since the function B is affine on $S(t)$. Now, by differentiating the boundary equality $B(g(t)) = f(t)$, we obtain

$$\langle \beta(t), g'(t)\rangle = f'(t), \tag{3.2.5}$$

which is equivalent to (divide by g_1')

$$\beta_1 + \kappa_2\beta_2 = \kappa_3. \tag{3.2.6}$$

Plugging this into (3.2.4), we get

$$B(x) = f(t) + (\kappa_3(t) - \kappa_2(t)\beta_2(t))(x_1 - g_1(t)) + \beta_2(t)(x_2 - g_2(t)). \tag{3.2.7}$$

We use the formula

$$\kappa(t) = \frac{x_2 - g_2(t)}{x_1 - g_1(t)}, \quad x \in S(t), \tag{3.2.8}$$

to rewrite (3.2.7) in the form (3.2.2).

It remains to prove (3.2.3). Let us note that (3.2.2) at the point $x = w(t)$ implies C^1-smoothness of β_2, which in its turn implies C^2-smoothness of B because β_1 is C^1-smooth as well due to (3.2.6). Differentiating (3.2.7) with respect to x_2 and using the relations $\kappa_2 g_1' = g_2'$, $\kappa_3 g_1' = f'$, we get

$$\big((\kappa_3' - \kappa_2'\beta_2 - \kappa_2\beta_2')(x_1 - g_1(t)) + \beta_2'(x_2 - g_2(t))\big) \cdot \frac{\partial}{\partial x_2} t = 0.$$

We use the fact that $\frac{\partial}{\partial x_2} t \neq 0$ and (3.2.8) to obtain

$$(\kappa_2 - \kappa)\beta_2' + \kappa_2'\beta_2 = \kappa_3'. \tag{3.2.9}$$

The solution y of the equation $y'(t) + K_1(t)y(t) = K_2(t)$ is given by the formula

$$y(t) = \exp\Big(-\int_{t_0}^{t} K_1(\tau)\,d\tau\Big)\Big(\int_{t_0}^{t} \exp\Big(\int_{t_0}^{\tau} K_1(s)\,ds\Big) K_2(\tau)\,d\tau + y(t_0)\Big), \quad (3.2.10)$$

where $y(t_0)$ is an arbitrary parameter. Applying this to (3.2.9) with $y = \beta_2$, we obtain formula (3.2.3). ∎

In what follows we will need a slightly different representation for β_2 that can be obtained using integration by parts.

Corollary 3.2.2. *Under the conditions of Proposition 3.2.1, for $t, t_0 \in I$, we have*

$$\beta_2(t) = \mathfrak{K}(t) - \exp\Big(-\int_{t_0}^{t} \frac{\kappa_2'}{\kappa_2 - \kappa}\Big) \times \Big(\int_{t_0}^{t} \exp\Big(\int_{t_0}^{\tau} \frac{\kappa_2'}{\kappa_2 - \kappa}\Big)\mathfrak{K}'(\tau)\,d\tau + \mathrm{Const}\Big), \quad (3.2.11)$$

$$\beta_2'(t) = \frac{\kappa_2'(t)}{\kappa_2(t) - \kappa(t)} \exp\Big(-\int_{t_0}^{t} \frac{\kappa_2'}{\kappa_2 - \kappa}\Big) \times \Big(\int_{t_0}^{t} \exp\Big(\int_{t_0}^{\tau} \frac{\kappa_2'}{\kappa_2 - \kappa}\Big)\mathfrak{K}'(\tau)\,d\tau + \mathrm{Const}\Big), \quad (3.2.12)$$

where Const $= \beta_2'(t_0)\frac{\kappa_2(t_0)-\kappa(t_0)}{\kappa_2'(t_0)} = \mathfrak{K}(t_0) - \beta_2(t_0)$.

Proof. Let us integrate by parts in the right-hand side of (3.2.3), using the fact that

$$\exp\Big(\int_{t_0}^{\tau} \frac{\kappa_2'}{\kappa_2 - \kappa}\Big)\frac{\kappa_3'(\tau)}{\kappa_2(\tau) - \kappa(\tau)} = \Big(\exp\Big(\int_{t_0}^{\tau} \frac{\kappa_2'}{\kappa_2 - \kappa}\Big)\Big)' \cdot \mathfrak{K}(\tau).$$

Note that $\mathfrak{K}'$ is a signed measure due to Condition 2.1.11, and the integrals in (3.2.11) and (3.2.12) are considered as the integrals with respect to this measure. Now formula (3.2.12) immediately follows from (3.2.11) and (3.2.9), this also proves that the constants in (3.2.12) and (3.2.11) are the same. ∎

We formulate the converse of Proposition 3.2.1.

Proposition 3.2.3. *Suppose that the function β_2 is given by (3.2.3) and the function B is defined by (3.2.2) on the domain $\Omega(I)$. Then the function B is C^2-smooth on $\Omega(I)$ and affine on each segment $S(t)$. Its gradient coincides with $(\beta_1(t), \beta_2(t))$ on $S(t)$; here β_1 is given by (3.2.6).*

Proof. If we prove $\nabla B = \beta$, then we automatically get that B is C^2-smooth, since both β and t are C^1-smooth.

Using (3.2.6) and (3.2.8) we rewrite (3.2.2) in the form (3.2.4). Differentiating formula (3.2.4) with respect to x, we obtain, for $t \in I$, $x \in S(t)$,

$$\nabla B = f'(t)\nabla t + \beta(t) + (\langle \beta'(t), x - g(t)\rangle - \langle \beta(t), g'(t)\rangle)\nabla t. \quad (3.2.13)$$

We recall that (3.2.6) is equivalent to (3.2.5), and therefore $\langle\beta(t), g'(t)\rangle = f'(t)$. Thus, to prove that $\nabla B = \beta$, we only need to show that $\langle\beta'(t), x - g(t)\rangle = 0$. The function β_2 given by (3.2.3) satisfies (3.2.9). Again, using (3.2.6) and (3.2.8), we obtain

$$(\kappa_3' - \kappa_2'\beta_2 - \kappa_2\beta_2')(x_1 - g_1(t)) + \beta_2'(x_2 - g_2(t)) = 0,$$

which is equivalent to

$$\langle\beta'(t), x - g(t)\rangle = 0, \quad x \in S(t). \tag{3.2.14}$$

This completes the proof. ∎

Remark 3.2.4. Dividing (3.2.14) by $x_1 - g_1(t)$, we obtain the following relation:

$$\beta_1' + \kappa\beta_2' = 0. \tag{3.2.15}$$

Concluding this subsection, we name the domains $\Omega(I)$ as defined above by the term *fences*. We will need two types of fences: the points $w(t)$ are either on $\partial_{\mathrm{free}}\Omega$ or on $\partial_{\mathrm{fixed}}\Omega$. In the first case, we have a family of tangents to $\partial_{\mathrm{free}}\Omega$, and $\Omega(I)$ is called a tangent domain, and in the second case, we have a family of chords, and $\Omega(I)$ is called a chordal domain. We will use all the relations from this subsection for both types of fences in what follows. Now we consider these two cases separately and show how to find the vector-valued function β and determine the candidate B.

3.3 Tangent domains

As mentioned in Section 3.1, the extremal segments are of two types: the chords and the tangents. This section provides a study of *tangent domains*, i.e., fences that consist of the segments $S(t)$ tangent to $\partial_{\mathrm{free}}\Omega$, see Figure 3.1.

Definition 3.3.1. A fence $\Omega(I)$ with the foliation $S(t) = [g(t), w(t)]$, $t \in I$, is called a tangent domain if $w(t) \in \partial_{\mathrm{free}}\Omega$ and the segment $S(t)$ is tangent to $\partial_{\mathrm{free}}\Omega$ for $t \in I$.

We note that in general there are two possibilities: either $S(t) = S_{\mathrm{R}}(t)$ (right tangents) or $S(t) = S_{\mathrm{L}}(t)$ (left ones), see Section 2.1.1. If we consider one of these two cases, we use the notation $\Omega_{\mathrm{R}}(I)$ and $\Omega_{\mathrm{L}}(I)$, respectively. The segment $S(t)$ is tangent to $\partial_{\mathrm{free}}\Omega$ if and only if the function κ defined in (3.2.1) satisfies the following equation:

$$\kappa(t) = \frac{w_2'(t)}{w_1'(t)}. \tag{3.3.1}$$

Differentiating (3.2.1) and using (3.3.1), one can easily check that

$$\kappa' = \frac{g_1'\kappa - g_2'}{w_1 - g_1}. \tag{3.3.2}$$

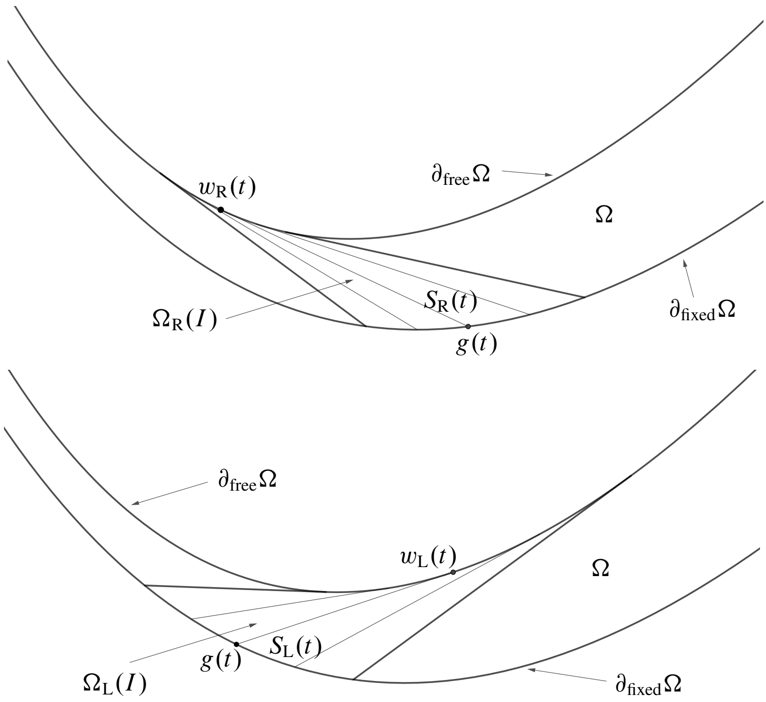

Figure 3.1. Domains Ω_R and Ω_L with right and left tangents.

Differentiating (3.2.8) with respect to x_2, using (3.3.2) and (3.2.1) once more, we find the partial derivative of the function $t = t(x)$:

$$t_{x_2} \stackrel{\rm def}{=} \frac{\partial t}{\partial x_2} = \frac{1}{g_2' + (x_1 - g_1)\kappa' - g_1'\kappa}$$
$$= \frac{w_1 - g_1}{(\kappa g_1' - g_2')(x_1 - w_1)} = \frac{w_1 - g_1}{g_1'(\kappa - \kappa_2)(x_1 - w_1)}.$$

Remark 3.3.2. Note that x_1 lies between g_1 and w_1, $\kappa > \kappa_2$ for $\Omega_L(I)$ and $\kappa < \kappa_2$ for $\Omega_R(I)$. Therefore, $t_{x_2} > 0$ for $\Omega_R(I)$ and $t_{x_2} < 0$ for $\Omega_L(I)$. Recall that B is defined by (3.2.2) and its Hessian is degenerate, therefore the concavity of B follows from $B_{x_2x_2} < 0$. The sign of $B_{x_2x_2} = \beta_2'(t)t_{x_2}$ is determined by the sign of β_2', and we obtain the following propositions.

Proposition 3.3.3. *Consider a tangent domain $\Omega_R(I)$ foliated by the family of segments $S_R(t) = [g(t), w(t)]$ such that $w(t) \in \partial_{\rm free}\Omega$ and $S_R(t)$ is a right tangent to $\partial_{\rm free}\Omega$ for $t \in I$. Suppose that the function β_2 given by (3.2.3) satisfies the inequality $\beta_2'(t) \leqslant 0$ for $t \in I$. Then the function B given by (3.2.2) is a Bellman candidate on $\Omega_R(I)$.*

Proof. The functions β_2 given by (3.2.3) and β_1 defined by (3.2.6) are the components of the gradient ∇B by Proposition 3.2.3. Thus, we only need to verify the concavity of B. If $\beta_2'(t) = 0$, then $\beta_1'(t) = 0$ by (3.2.15), and the Hessian of B is the zero matrix. If $\beta_2'(t) < 0$, then $B_{x_2x_2} = \beta_2'(t)t_{x_2} < 0$, and the Hessian is non-positive definite by Sylvester's criterion. Thus, in both cases the proposition is proved. ∎

Remark 3.3.4. If $I = [t_-, t_+]$, $\beta_2'(t_-) \leqslant 0$, and the function $\mathbf{T}$ corresponding to γ (defined in (2.1.10)) is non-positive on I, then $\beta_2' \leqslant 0$ on the entire interval I because of (3.2.12) and (2.1.12).

We state the following symmetrical proposition.

Proposition 3.3.5. *Consider a tangent domain $\Omega_{\mathrm{L}}(I)$ foliated by the family of segments $S_{\mathrm{L}}(t) = [g(t), w(t)]$ such that $w(t) \in \partial_{\mathrm{free}}\Omega$ and $S_{\mathrm{L}}(t)$ is a left tangent to $\partial_{\mathrm{free}}\Omega$ for $t \in I$. Suppose that the function β_2 given by (3.2.3) satisfies the inequality $\beta_2'(t) \geqslant 0$ for $t \in I$. Then the function B given by (3.2.2) is a Bellman candidate on $\Omega_{\mathrm{L}}(I)$.*

In the final part of this subsection we consider the case of infinite tangent domains.

Proposition 3.3.6. *Let $I = (-\infty, t_2)$, $t_2 \in \mathbb{R} \cup \{+\infty\}$. Consider a tangent domain $\Omega_{\mathrm{R}}(I)$ foliated by the family of segments $S_{\mathrm{R}}(t) = [g(t), w(t)]$ such that $w(t) \in \partial_{\mathrm{free}}\Omega$ and $S_{\mathrm{R}}(t)$ is a right tangent to $\partial_{\mathrm{free}}\Omega$ for $t \in I$. Suppose that the function β_2, given by (3.2.11) with* Const $= 0$ *and $t_0 = -\infty$, i.e.,*

$$\beta_2(t) = \mathfrak{K}(t) - \int_{-\infty}^{t} \exp\Big(-\int_{\tau}^{t} \frac{\kappa_2'}{\kappa_2 - \kappa}\Big)\mathfrak{K}'(\tau)\,d\tau, \quad t \in I, \tag{3.3.3}$$

is finite and satisfies the inequality

$$\beta_2'(t) = \frac{\kappa_2'(t)}{\kappa_2(t) - \kappa(t)} \int_{-\infty}^{t} \exp\Big(-\int_{\tau}^{t} \frac{\kappa_2'}{\kappa_2 - \kappa}\Big)\mathfrak{K}'(\tau)\,d\tau \leqslant 0, \quad t \in I. \tag{3.3.4}$$

Then the function B given by (3.2.2) is a Bellman candidate on $\Omega_{\mathrm{R}}(I)$.

Proposition 3.3.7. *Let $I = (t_1, +\infty)$, $t_1 \in \mathbb{R} \cup \{-\infty\}$. Consider a tangent domain $\Omega_{\mathrm{L}}(I)$ foliated by a family of segments $S_{\mathrm{L}}(t) = [g(t), w(t)]$ such that $w(t) \in \partial_{\mathrm{free}}\Omega$ and $S_{\mathrm{L}}(t)$ is a left tangent to $\partial_{\mathrm{free}}\Omega$ for $t \in I$. Suppose that the function β_2, given by formula (3.2.11) with* Const $= 0$ *and $t_0 = +\infty$, i.e.,*

$$\beta_2(t) = \mathfrak{K}(t) + \int_{t}^{+\infty} \exp\Big(\int_{t}^{\tau} \frac{\kappa_2'}{\kappa_2 - \kappa}\Big)\mathfrak{K}'(\tau)\,d\tau, \quad t \in I, \tag{3.3.5}$$

is finite and satisfies the inequality

$$\beta_2'(t) = -\frac{\kappa_2'(t)}{\kappa_2(t) - \kappa(t)} \int_{t}^{+\infty} \exp\Big(\int_{t}^{\tau} \frac{\kappa_2'}{\kappa_2 - \kappa}\Big)\mathfrak{K}'(\tau)\,d\tau \geqslant 0, \quad t \in I. \tag{3.3.6}$$

Then the function B given by formula (3.2.2) is a Bellman candidate on $\Omega_{\mathrm{L}}(I)$.

Definition 3.3.8. The functions B constructed in Propositions 3.3.3, 3.3.5, 3.3.6, and 3.3.7 are called *standard candidates* for the corresponding tangent domains $\Omega_{\mathrm{R}}(I)$ and $\Omega_{\mathrm{L}}(I)$ if the inequalities for β_2' are strict in the interior of I, i.e., $\beta_2' < 0$ for $\Omega_{\mathrm{R}}(I)$ and $\beta_2' > 0$ for $\Omega_{\mathrm{L}}(I)$.

We comment on the convergence of the integrals in (3.3.3) and (3.3.5).

Remark 3.3.9. Convergence of the integral in (3.3.3) is equivalent to convergence of the integral in condition (2.1.13); this will be proved in Lemma 5.2.4 below. Moreover, if the integral in (2.1.13) is equal to $+\infty$, then the inequality in (3.3.4) fails, i.e., there is no standard candidate on $\Omega_{\mathrm{R}}(-\infty, t_2)$ for any $t_2 \in \mathbb{R}$.

Symmetrically, convergence of the integral in (3.3.5) is equivalent to convergence of the integral in condition (2.1.14). Moreover, if the integral in (2.1.14) is equal to $-\infty$, then the inequality in (3.3.6) fails, i.e., there is no standard candidate on $\Omega_{\mathrm{L}}(t_1, +\infty)$ for any $t_1 \in \mathbb{R}$.

3.4 Chordal domains

As mentioned in Section 3.1, the extremal segments are of two types: the chords and the tangents. In this section we study *chordal domains*, i.e., the domains that consist of chords (see Figure 3.2).

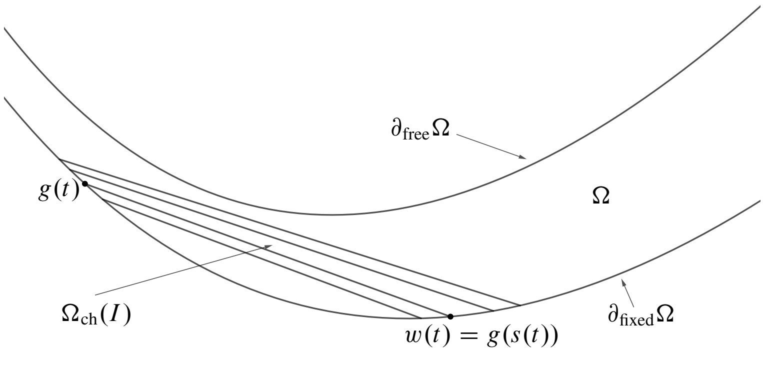

Figure 3.2. Chordal domain $\Omega_{\mathrm{ch}}(I)$.

Definition 3.4.1. A fence $\Omega(I)$ with the foliation $S(t) = [g(t), w(t)]$, $t \in I$, is called a *chordal domain* if $w(t) \in \partial_{\mathrm{fixed}}\Omega$ for $t \in I$. A chordal domain is denoted by $\Omega_{\mathrm{ch}}(I)$.

The function B is uniquely determined in a chordal domain by its boundary values and linearity on segments:

$$B(\alpha g(t) + (1-\alpha)g(s)) = \alpha f(t) + (1-\alpha) f(s), \quad \alpha \in [0,1], \tag{3.4.1}$$

where the function $s\colon I \to \mathbb{R}$ is defined by the relation $g(s(t)) = w(t)$ for $t \in I$. Let us note that the function s is decreasing. Our aim in this subsection is to provide conditions for when the function B given by (3.4.1) is a Bellman candidate.

The vectors $\gamma'(t)$, $\gamma'(s)$, and $\gamma(t)-\gamma(s)$ belong to the same tangent plane to the graph of B, therefore

$$\det\begin{pmatrix}\gamma'(t)\\ \gamma'(s)\\ \gamma(s)-\gamma(t)\end{pmatrix} = \det\begin{pmatrix} g_1'(t) & g_2'(t) & f'(t)\\ g_1'(s) & g_2'(s) & f'(s)\\ g_1(s)-g_1(t) & g_2(s)-g_2(t) & f(s)-f(t)\end{pmatrix} = 0. \quad (3.4.2)$$

We will refer to this equation as the *cup equation*.

Denote by $\theta = \theta(x_1, x_2)$ the vector orthogonal to the graph of B at the point $(x_1, x_2, B(x_1, x_2))$:

$$\theta = \nabla(B(x_1,x_2) - x_3) = (\beta(t(x_1,x_2)), -1). \quad (3.4.3)$$

We can consider θ as a function of t only: $\theta(t) = (\beta(t), -1)$. This vector is orthogonal to the tangent vector of the boundary curve γ, i.e., $\langle\theta(t), \gamma'(t)\rangle = 0$, which is equivalent to

$$\langle\beta(t), g'(t)\rangle = f'(t). \quad (3.4.4)$$

Now we are going to find β'. We differentiate equation (3.4.4) and obtain

$$\langle\beta'(t), g'(t)\rangle = f''(t) - \langle\beta(t), g''(t)\rangle = -\langle\theta(t), \gamma''(t)\rangle, \quad (3.4.5)$$

where θ is defined by (3.4.3).

We know that $\langle\theta(t), \gamma'(t)\rangle = \langle\theta(t), \gamma'(s)\rangle = 0$ and $\langle\theta(t), (0,0,1)\rangle = -1$. Therefore, for any $v \in \mathbb{R}^3$, we have $\langle\theta(t), v + \langle\theta(t), v\rangle(0,0,1)\rangle = 0$, whence

$$\det\begin{pmatrix}\gamma'(t)\\ \gamma'(s)\\ v + \langle\theta(t), v\rangle(0,0,1)\end{pmatrix} = 0 \quad \text{or} \quad \langle\theta(t), v\rangle \det\begin{pmatrix} g'(t)\\ g'(s)\end{pmatrix} = -\det\begin{pmatrix}\gamma'(t)\\ \gamma'(s)\\ v\end{pmatrix}.$$

We use it in (3.4.5) and obtain

$$\langle\beta'(t), g'(t)\rangle = -\langle\theta(t), \gamma''(t)\rangle = \frac{\det\begin{pmatrix}\gamma'(t)\\ \gamma'(s)\\ \gamma''(t)\end{pmatrix}}{\det\begin{pmatrix} g'(t)\\ g'(s)\end{pmatrix}} = \frac{\det\begin{pmatrix} g_1'(t) & g_2'(t) & f'(t)\\ g_1'(s) & g_2'(s) & f'(s)\\ g_1''(t) & g_2''(t) & f''(t)\end{pmatrix}}{\det\begin{pmatrix} g_1'(t) & g_2'(t)\\ g_1'(s) & g_2'(s)\end{pmatrix}}. \quad (3.4.6)$$

We have a system of equations (3.4.6) and (3.2.15) (with $w(t) = g(s)$) for $\beta'(t)$. We can solve it and obtain

$$\beta_2'(t) = \frac{1}{g_1'(t)} \cdot \frac{1}{\kappa_2(t) - \kappa(t)} \cdot \frac{\det\begin{pmatrix} g_1'(t) & g_2'(t) & f'(t)\\ g_1'(s) & g_2'(s) & f'(s)\\ g_1''(t) & g_2''(t) & f''(t)\end{pmatrix}}{\det\begin{pmatrix} g_1'(t) & g_2'(t)\\ g_1'(s) & g_2'(s)\end{pmatrix}}, \quad (3.4.7)$$

where $\kappa(t) = \frac{g_2(s)-g_2(t)}{g_1(s)-g_1(t)}$ was defined by (3.2.1) and $\kappa_2 = g_2'/g_1'$. We recall that $g_1' > 0$ (see (2.1.1)). It follows from the convexity of $\partial_{\mathrm{fixed}}\Omega$ that for $t < s$, the following inequality holds:

$$\frac{g_2'(t)}{g_1'(t)} = \kappa_2(t) < \kappa(t) < \kappa_2(s) = \frac{g_2'(s)}{g_1'(s)} \tag{3.4.8}$$

(and for $s < t$ both inequalities are opposite), thus, $\operatorname{sign} \beta_2'(t) = -\operatorname{sign}\det\begin{pmatrix}\gamma'(t)\\ \gamma'(s)\\ \gamma''(t)\end{pmatrix}$.

Note that $\operatorname{sign}\frac{\partial t}{\partial x_2} = \operatorname{sign}(t-s)$, whence

$$\operatorname{sign}\frac{\partial^2 B}{\partial x_2^2} = \operatorname{sign}\Big[\beta_2'(t)\frac{\partial t}{\partial x_2}\Big] = \operatorname{sign}\left[(s-t)\det\begin{pmatrix}\gamma'(t)\\ \gamma'(s)\\ \gamma''(t)\end{pmatrix}\right]. \tag{3.4.9}$$

We formulate a proposition that gives sufficient conditions for the function B defined by (3.4.1) to be a Bellman candidate on the chordal domain $\Omega_{\mathrm{ch}}(I)$.

Proposition 3.4.2. *Consider a chordal domain $\Omega_{\mathrm{ch}}(I)$ foliated by the family of segments $S(t) = [g(t), g(s(t))]$, where the function $s\colon I \to \mathbb{R}$ satisfies (3.4.2). Let B be defined on $\Omega_{\mathrm{ch}}(I)$ by the boundary conditions $B \circ g = f$ and by linearity on each segment $S(t)$, $t \in I$, i.e., by (3.4.1). Suppose that*

$$(t-s)\det\begin{pmatrix}\gamma'(t)\\ \gamma'(s)\\ \gamma''(t)\end{pmatrix} \geqslant 0, \quad t \in I. \tag{3.4.10}$$

Then the function B is a Bellman candidate on $\Omega_{\mathrm{ch}}(I)$.

Proof. We need to check that ∇B is constant on each chord. Using (3.4.2) we find a vector $\beta = \beta(t)$ such that $\theta = (\beta, -1)$ is orthogonal to $\gamma'(t)$, $\gamma'(s)$ and $\gamma(s) - \gamma(t)$. The function β chosen in such a way satisfies relations (3.4.4) and (3.2.14). Identity (3.2.4) holds for $x = g(t)$ and $x = g(s)$, therefore, by linearity it holds for any $x \in S(t)$. Then, by (3.2.13), $\nabla B = \beta(t)$, therefore ∇B is constant on $S(t)$. Under the hypothesis of the proposition, the concavity of B follows from (3.4.9). ∎

Definition 3.4.3. The function B constructed in Proposition 3.4.2 is called a *standard candidate* for the chordal domain $\Omega_{\mathrm{ch}}(I)$ if the inequality in (3.4.10) is strict for t in the interior of I.

Definition 3.4.4. Let $I = [t_-, t_+]$ and let either $s(t_-) = t_-$ or $s(t_+) = t_+$ for a function $s\colon I \to \mathbb{R}$ corresponding to the chordal domain $\Omega_{\mathrm{ch}}(I)$. Then the chordal domain $\Omega_{\mathrm{ch}}(I)$ is called a *cup* originated at $g(c)$, where $c = s(c)$ coincides with either t_- or t_+, respectively. The point $g(c)$ as well as the value of the parameter c is called the *origin of the cup.*

The origin $g(c)$ of a cup $\Omega_{\mathrm{ch}}(I)$ is a very special point on the curve g. It should satisfy the following torsion equation:

$$\det\begin{pmatrix}\gamma'(c)\\ \gamma''(c)\\ \gamma'''(c)\end{pmatrix}=0, \tag{3.4.11}$$

which can be obtained by carefully passing to the limit as $t\to c$ and $s\to c$ in (3.4.2).

Geometrically the chordal domains $\Omega_{\mathrm{ch}}(I)$ and $\Omega_{\mathrm{ch}}(s(I))$ are the same. Usually we do not need to distinguish them, so we will use the following common notation. Let $I=[t_-,t_+]$. Denote the numbers $t_\pm$ and $s(t_\pm)$ by a_0,a_1,b_1 and b_0 in the increasing order, i.e., $[g(a_0),g(b_0)]$ is the top chord of the domain and $[g(a_1),g(b_1)]$ is the bottom one. Denote such a chordal domain by $\Omega_{\mathrm{ch}}([a_0,b_0],[a_1,b_1])$. If one of these chords is unimportant, we will use $*$ instead:

$$\Omega_{\mathrm{ch}}([a_0,b_0],*)\quad\text{or}\quad\Omega_{\mathrm{ch}}(*,[a_1,b_1]).$$

3.5 Around the cup

3.5.1 Differentials

Definition 3.5.1. Suppose that a pair (s,t), $s<t$, satisfies the cup equation (3.4.2). We call the following two expressions the differentials, the left and the right one respectively:

$$D_{\mathrm{L}}(s,t)\stackrel{\mathrm{def}}{=}\frac{\det\begin{pmatrix}\gamma'(s)\\ \gamma'(t)\\ \gamma''(s)\end{pmatrix}}{\det\begin{pmatrix}g'(s)\\ g'(t)\end{pmatrix}},\qquad D_{\mathrm{R}}(s,t)\stackrel{\mathrm{def}}{=}\frac{\det\begin{pmatrix}\gamma'(t)\\ \gamma'(s)\\ \gamma''(t)\end{pmatrix}}{\det\begin{pmatrix}g'(t)\\ g'(s)\end{pmatrix}}. \tag{3.5.1}$$

It is clear that, formally, $D_{\mathrm{L}}(s,t)=D_{\mathrm{R}}(t,s)$, but it will be more convenient to keep two different symbols for these expressions.

Lemma 3.5.2. *Suppose that the pair (s,t) satisfies the cup equation (3.4.2). Then*

$$\gamma(t)-\gamma(s)=C_{\mathrm{R}}\gamma'(t)-C_{\mathrm{L}}\gamma'(s), \tag{3.5.2}$$

where

$$C_{\mathrm{L}}(s,t)=\frac{\det\begin{pmatrix}g(s)-g(t)\\ g'(t)\end{pmatrix}}{\det\begin{pmatrix}g'(s)\\ g'(t)\end{pmatrix}},\qquad C_{\mathrm{R}}(s,t)=\frac{\det\begin{pmatrix}g(t)-g(s)\\ g'(s)\end{pmatrix}}{\det\begin{pmatrix}g'(t)\\ g'(s)\end{pmatrix}}.$$

Remark 3.5.3. If $u,v,w\in\mathbb{R}^2$, then

$$u\det\begin{pmatrix}w\\ v\end{pmatrix}+v\det\begin{pmatrix}u\\ w\end{pmatrix}+w\det\begin{pmatrix}v\\ u\end{pmatrix}=0. \tag{3.5.3}$$

Proof of Lemma 3.5.2. It follows from (3.5.3) that

$$g(t) - g(s) = C_{\mathrm{R}}\, g'(t) - C_{\mathrm{L}}\, g'(s). \tag{3.5.4}$$

The vectors $\gamma(t) - \gamma(s)$, $\gamma'(s)$, $\gamma'(t)$ are linearly dependent, and the coefficients of the linear dependence are the same as for their two-dimensional projections $g(t) - g(s)$, $g'(t)$ and $g'(s)$. ∎

Remark 3.5.4. For $s \neq t$, we have $\operatorname{sign}(C_{\mathrm{R}}) = -\operatorname{sign}(C_{\mathrm{L}}) = \operatorname{sign}(t - s)$ (see (3.4.8)).

Corollary 3.5.5. *Let $\Omega_{\mathrm{ch}}(I)$ be a chordal domain with corresponding function $s = s(t)$, $t \in I$. Then*

$$C_{\mathrm{L}} \det\begin{pmatrix}\gamma'(s)\\ \gamma'(t)\\ \gamma''(s)\end{pmatrix} ds + C_{\mathrm{R}} \det\begin{pmatrix}\gamma'(t)\\ \gamma'(s)\\ \gamma''(t)\end{pmatrix} dt = 0. \tag{3.5.5}$$

Proof. We differentiate (3.4.2) and obtain

$$\begin{aligned} 0 &= \det\begin{pmatrix}\gamma''(t)\\ \gamma'(s)\\ \gamma(t)-\gamma(s)\end{pmatrix} dt + \det\begin{pmatrix}\gamma'(t)\\ \gamma''(s)\\ \gamma(t)-\gamma(s)\end{pmatrix} ds\\ &= C_{\mathrm{R}} \det\begin{pmatrix}\gamma''(t)\gamma'(s)\\ \gamma'(t)\end{pmatrix} dt - C_{\mathrm{L}} \det\begin{pmatrix}\gamma'(t)\\ \gamma''(s)\\ \gamma'(s)\end{pmatrix} ds, \end{aligned}$$

according to (3.5.2). ∎

By (3.5.1), the equality (3.5.5) can be rewritten in the following form:

$$C_{\mathrm{L}} D_{\mathrm{L}}\, ds = C_{\mathrm{R}} D_{\mathrm{R}}\, dt. \tag{3.5.6}$$

Corollary 3.5.6. *We have*

$$D_{\mathrm{R}} = \frac{\det\begin{pmatrix}\gamma'(t)\\ \gamma(t)-\gamma(s)\\ \gamma''(t)\end{pmatrix}}{\det\begin{pmatrix}g'(t)\\ g(t)-g(s)\end{pmatrix}}, \qquad D_{\mathrm{L}} = \frac{\det\begin{pmatrix}\gamma'(s)\\ \gamma(s)-\gamma(t)\\ \gamma''(s)\end{pmatrix}}{\det\begin{pmatrix}g'(s)\\ g(s)-g(t)\end{pmatrix}}. \tag{3.5.7}$$

Proof. This immediately follows from (3.5.2), (3.5.4), and (3.5.1). ∎

Lemma 3.5.7. *Let $\Omega_{\mathrm{ch}}(I)$ be a chordal domain with corresponding function $s = s(t)$, $t \in I$. Then*

$$dD_{\mathrm{R}}(s,t) = \frac{\det\begin{pmatrix}\gamma'(t)\\ \gamma''(t)\\ \gamma'''(t)\end{pmatrix}}{\det\begin{pmatrix}g'(t)\\ g''(t)\end{pmatrix}}\, dt + \left(\frac{\det\begin{pmatrix}g'(t)\\ g'''(t)\end{pmatrix}}{\det\begin{pmatrix}g'(t)\\ g''(t)\end{pmatrix}} - \frac{\det\begin{pmatrix}g''(t)\\ g(t)-g(s)\end{pmatrix}}{\det\begin{pmatrix}g'(t)\\ g(t)-g(s)\end{pmatrix}}\right) D_{\mathrm{R}}(s,t)\, dt. \tag{3.5.8}$$

Proof. Let us differentiate D_{R} using the representation (3.5.7). First, note that the partial derivative with respect to s is equal to zero:

$$\frac{\det\begin{pmatrix}\gamma'(t)\\ -\gamma'(s)\\ \gamma''(t)\end{pmatrix}}{\det\begin{pmatrix}g'(t)\\ g(t)-g(s)\end{pmatrix}} - \frac{\det\begin{pmatrix}\gamma'(t)\\ \gamma(t)-\gamma(s)\\ \gamma''(t)\end{pmatrix}}{\det\begin{pmatrix}g'(t)\\ g(t)-g(s)\end{pmatrix}^2}\det\begin{pmatrix}g'(t)\\ -g'(s)\end{pmatrix}$$
$$\overset{\substack{(3.5.7)\\(3.5.1)}}{=} -D_{\mathrm{R}}\frac{\det\begin{pmatrix}g'(t)\\ g'(s)\end{pmatrix}}{\det\begin{pmatrix}g'(t)\\ g(t)-g(s)\end{pmatrix}} + D_{\mathrm{R}}\frac{\det\begin{pmatrix}g'(t)\\ g'(s)\end{pmatrix}}{\det\begin{pmatrix}g'(t)\\ g(t)-g(s)\end{pmatrix}} = 0.$$

Second, the partial derivative with respect to t is equal to

$$\frac{\det\begin{pmatrix}\gamma'(t)\\ \gamma(t)-\gamma(s)\\ \gamma'''(t)\end{pmatrix}}{\det\begin{pmatrix}g'(t)\\ g(t)-g(s)\end{pmatrix}} - \frac{\det\begin{pmatrix}g''(t)\\ g(t)-g(s)\end{pmatrix}}{\det\begin{pmatrix}g'(t)\\ g(t)-g(s)\end{pmatrix}} D_{\mathrm{R}}. \tag{3.5.9}$$

We use formulas (3.5.2) and (3.5.4) to rewrite the first summand in (3.5.9):

$$\frac{\det\begin{pmatrix}\gamma'(t)\\ \gamma(t)-\gamma(s)\\ \gamma'''(t)\end{pmatrix}}{\det\begin{pmatrix}g'(t)\\ g(t)-g(s)\end{pmatrix}} = \frac{\det\begin{pmatrix}\gamma'(t)\\ \gamma'(s)\\ \gamma'''(t)\end{pmatrix}}{\det\begin{pmatrix}g'(t)\\ g'(s)\end{pmatrix}}. \tag{3.5.10}$$

It follows from (3.5.3), for the vectors $g'(t)$, $g''(t)$ and $g'''(t)$, that

$$\det\begin{pmatrix}g'(t)\\ g''(t)\end{pmatrix}\gamma'''(t) + \det\begin{pmatrix}g'''(t)\\ g'(t)\end{pmatrix}\gamma''(t) + \det\begin{pmatrix}g''(t)\\ g'''(t)\end{pmatrix}\gamma'(t) = \left(0, 0, \det\begin{pmatrix}\gamma'(t)\\ \gamma''(t)\\ \gamma'''(t)\end{pmatrix}\right).$$

We express $\gamma'''(t)$ from this identity and substitute it into (3.5.10) to obtain

$$\frac{\det\begin{pmatrix}\gamma'(t)\\ \gamma'(s)\\ \gamma'''(t)\end{pmatrix}}{\det\begin{pmatrix}g'(t)\\ g'(s)\end{pmatrix}} = \frac{\det\begin{pmatrix}\gamma'(t)\\ \gamma''(t)\\ \gamma'''(t)\end{pmatrix}}{\det\begin{pmatrix}g'(t)\\ g''(t)\end{pmatrix}} - \frac{\det\begin{pmatrix}\gamma'(t)\\ \gamma'(s)\\ \gamma''(t)\end{pmatrix}}{\det\begin{pmatrix}g'(t)\\ g'(s)\end{pmatrix}}\frac{\det\begin{pmatrix}g'''(t)\\ g'(t)\end{pmatrix}}{\det\begin{pmatrix}g'(t)\\ g''(t)\end{pmatrix}}.$$

This formula, together with (3.5.10) and (3.5.9), and definition (3.5.1), completes the proof. ■

Remark 3.5.8. Formula (3.5.8) can be rewritten in terms of the functions κ, κ_2, κ_3, and $\mathfrak{K}$ (see (2.1.5) and (2.1.11)):

$$\frac{dD_{\mathrm{R}}(s(t),t)}{dt} = \kappa_2' g_1' \mathfrak{K}' + \left[\frac{(g_1'\kappa_2')'}{g_1'\kappa_2'} - \frac{\kappa_2'}{\kappa_2 - \kappa}\right] D_{\mathrm{R}}(s(t),t). \tag{3.5.11}$$

Proof. The coincidence of the first summands of (3.5.11) and (3.5.8) follows from equation (2.1.12) and the identity

$$\det\begin{pmatrix}g'(t)\\ g''(t)\end{pmatrix} = g_1'g_2'' - g_1''g_2' = (g_1')^2\kappa_2'.$$

As for the second summand, we write the following chain of elementary equalities:

$$\det\begin{pmatrix} g'(t) \\ g(t)-g(s) \end{pmatrix} = g_1'(g_2(t)-g_2(s)) - g_2'(g_1(t)-g_1(s))$$
$$= g_1'[g_2(t)-g_2(s)-\kappa_2(g_1(t)-g_1(s))].$$

The coefficient in the big parentheses in (3.5.8) is equal to

$$\frac{\det\begin{pmatrix} g'(t) \\ g'''(t) \end{pmatrix}}{\det\begin{pmatrix} g'(t) \\ g''(t) \end{pmatrix}} - \frac{\det\begin{pmatrix} g''(t) \\ g(t)-g(s) \end{pmatrix}}{\det\begin{pmatrix} g'(t) \\ g(t)-g(s) \end{pmatrix}} = \frac{\partial}{\partial t}\log\frac{\det\begin{pmatrix} g'(t) \\ g''(t) \end{pmatrix}}{\det\begin{pmatrix} g'(t) \\ g(t)-g(s) \end{pmatrix}}$$
$$= \frac{\partial}{\partial t}\log\frac{\kappa_2' g_1'}{g_2(t)-g_2(s)-\kappa_2(g_1(t)-g_1(s))}$$
$$= \frac{(\kappa_2' g_1')'}{\kappa_2' g_1'} - \frac{g_2' - \kappa_2'(g_1(t)-g_1(s)) - \kappa_2 g_1'}{(g_1(t)-g_1(s))(\kappa-\kappa_2)}$$
$$= \frac{(\kappa_2' g_1')'}{\kappa_2' g_1'} + \frac{\kappa_2'}{\kappa-\kappa_2}. \qquad \blacksquare$$

3.5.2 Cup construction

Proposition 3.5.9. *Suppose that the torsion of γ changes its sign from $+$ to $-$ at the point t_0, i.e.,* $\mathbf{T}$ *is positive in a left neighborhood of t_0 and is negative in a right one. Then, for $\delta > 0$ sufficiently small, there exists a chordal domain $\Omega_{\mathrm{ch}}(t_0, t_0+\delta)$ with the standard Bellman candidate on it.*

Proposition 3.5.9 emphasizes the importance of Condition 2.1.11 because cups can originate only at the points $t_0 = c_i$, see Definition 2.1.13.

First, we need to obtain several useful formulas.

Lemma 3.5.10. *For any s, t the following identity holds*:

$$\det\begin{pmatrix} \gamma'(t) \\ \gamma'(s) \\ \gamma(t)-\gamma(s) \end{pmatrix}$$
$$= g_1'(t)g_1'(s)\int_s^t \mathfrak{K}'(\sigma)\Big(\int_s^\sigma\int_\sigma^t \kappa_2'(u)\kappa_2'(v)(g_1(v)-g_1(u))\,dv\,du\Big)\,d\sigma \qquad (3.5.12)$$
$$= -g_1'(t)g_1'(s)\int_s^t \mathfrak{K}(\sigma)\Big(\int_s^t \kappa_2'(\sigma)\kappa_2'(v)(g_1(v)-g_1(\sigma))\,dv\Big)\,d\sigma. \qquad (3.5.13)$$

Proof. We have

$$\det\begin{pmatrix} \gamma'(t) \\ \gamma'(s) \\ \gamma(t)-\gamma(s) \end{pmatrix} = \int_s^t \det\begin{pmatrix} \gamma'(t) \\ \gamma'(s) \\ \gamma'(\tau) \end{pmatrix}\,d\tau$$

$$= \int_s^t g_1'(t)g_1'(s)g_1'(\tau)\det\begin{pmatrix} 1 & \kappa_2(t) & \kappa_3(t) \\ 1 & \kappa_2(s) & \kappa_3(s) \\ 1 & \kappa_2(\tau) & \kappa_3(\tau) \end{pmatrix} d\tau$$

$$= -g_1'(t)g_1'(s)\int_s^t g_1'(\tau)\det\begin{pmatrix} \kappa_2(t)-\kappa_2(\tau) & \kappa_3(t)-\kappa_3(\tau) \\ \kappa_2(\tau)-\kappa_2(s) & \kappa_3(\tau)-\kappa_3(s) \end{pmatrix} d\tau$$

$$= -g_1'(t)g_1'(s)\int_s^t g_1'(\tau)\int_\tau^t\int_s^\tau \det\begin{pmatrix} \kappa_2'(v) & \kappa_3'(v) \\ \kappa_2'(u) & \kappa_3'(u) \end{pmatrix} du\,dv\,d\tau$$

$$= g_1'(t)g_1'(s)\int_s^t g_1'(\tau)\int_\tau^t\int_s^\tau \kappa_2'(v)\kappa_2'(u)\int_u^v \mathfrak{K}'(\sigma)\,d\sigma\,du\,dv\,d\tau$$

$$= g_1'(t)g_1'(s)\int_s^t \mathfrak{K}'(\sigma)\Big(\int_s^\sigma\int_\sigma^t \kappa_2'(u)\kappa_2'(v)\int_u^v g_1'(\tau)\,d\tau\,dv\,du\Big)\,d\sigma$$

$$= g_1'(t)g_1'(s)\int_s^t \mathfrak{K}'(\sigma)\Big(\int_s^\sigma\int_\sigma^t \kappa_2'(u)\kappa_2'(v)(g_1(v)-g_1(u))\,dv\,du\Big)\,d\sigma$$

$$= -g_1'(t)g_1'(s)\int_s^t \mathfrak{K}(\sigma)\Big(\int_\sigma^t \kappa_2'(\sigma)\kappa_2'(v)(g_1(v)-g_1(\sigma))\,dv$$

$$- \int_s^\sigma \kappa_2'(u)\kappa_2'(\sigma)(g_1(\sigma)-g_1(u))\,du\Big)\,d\sigma$$

$$= -g_1'(t)g_1'(s)\int_s^t \mathfrak{K}(\sigma)\Big(\int_s^t \kappa_2'(\sigma)\kappa_2'(v)(g_1(v)-g_1(\sigma))\,dv\Big)\,d\sigma,$$

where, in the penultimate line, we performed integration by parts. ■

Corollary 3.5.11. *For any s, t, the following identity holds true*:

$$\det\begin{pmatrix} \gamma''(t) \\ \gamma'(s) \\ \gamma(t)-\gamma(s) \end{pmatrix} = \frac{g_1''(t)}{g_1'(t)}\det\begin{pmatrix} \gamma'(t) \\ \gamma'(s) \\ \gamma(t)-\gamma(s) \end{pmatrix} - g_1'(t)g_1'(s)\kappa_2'(t)$$
$$\times \int_s^t (\mathfrak{K}(u)-\mathfrak{K}(t))\kappa_2'(u)(g_1(t)-g_1(u))\,du. \qquad (3.5.14)$$

Proof. The result is obtained by the direct differentiation of (3.5.13): the first summand is obtained from the differentiation of the factor $g_1'(t)$, the second on is the result of the differentiation of the integral. ■

Lemma 3.5.12. *Let $s, t, c \in \mathbb{R}$ and $s < c < t$. Suppose that a function Φ is continuous on $[s, t]$, non-decreasing on (s, c), non-increasing on (c, t), and not constant on $[s, t]$. Let the functions ϕ and ψ on $[s, t]$ be such that $\phi > 0$ and $\psi' > 0$. Suppose that*

$$\int_s^t \Phi(\sigma)\Big(\int_s^t \phi(\sigma)\phi(v)(\psi(v)-\psi(\sigma))\,dv\Big)\,d\sigma = 0. \qquad (3.5.15)$$

Then

$$\int_s^t (\Phi(u)-\Phi(t))\phi(u)(\psi(t)-\psi(u))\,du > 0. \qquad (3.5.16)$$

Proof. The inequality (3.5.16) and the equality (3.5.15) do not change if we add a constant to the function ψ. Therefore, we may assume that $\psi(t) = 0$. Under this assumption, $\psi < 0$ on (s,t).

We rewrite (3.5.15) in the following way:

$$\int_s^t \Phi(\sigma)\phi(\sigma)\,d\sigma \int_s^t \phi(v)\psi(v)\,dv = \int_s^t \Phi(\sigma)\phi(\sigma)\psi(\sigma)\,d\sigma \int_s^t \phi(v)\,dv. \quad (3.5.17)$$

We see that the problem is not sensitive to adding a constant to Φ. The integral $\int \phi\psi$ is strictly negative, so we can assume $\int_s^t \Phi\phi\psi = 0$ by adding a constant to Φ, if necessary. It now follows from (3.5.17) that $\int_s^t \Phi\phi = 0$. Under our assumptions, (3.5.16) takes the following form:

$$\Phi(t)\int \phi\psi > 0,$$

which is equivalent to $\Phi(t) < 0$. If this is not true, then $\Phi(t) \geqslant 0$ and $\Phi \geqslant 0$ on $[c,t]$. We can find a point $\sigma_0 \in (s,c)$ such that $\Phi \geqslant 0$ on (σ_0,t) and $\Phi \leqslant 0$ on (s,σ_0). However,

$$\int_s^t \Phi(\sigma)\phi(\sigma)(\psi(\sigma) - \psi(\sigma_0))\,d\sigma = 0,$$

while the integrand is non-negative. Thus, the integrand is identically zero, therefore Φ vanishes as well. This contradicts our requirement that Φ is not constant. ■

Corollary 3.5.13. *Let $s,t,t_0 \in \mathbb{R}$ and $s < t_0 < t$. Suppose that $\mathfrak{K}$ is non-decreasing on (s,t_0), non-increasing on (t_0,t), and not constant on $[s,t]$. Suppose that the pair (s,t) satisfies the cup equation (3.4.2). Then $D_{\mathrm{R}}(s,t) < 0$ and $D_{\mathrm{L}}(s,t) < 0$.*

Proof. We will prove only $D_{\mathrm{R}}(s,t) < 0$; the remaining inequality is symmetric.

We want to use Lemma 3.5.12 with $\Phi = \mathfrak{K}$, $\psi = g_1$, $\phi = \kappa_2'$, and $c = t_0$. Then (3.5.15) holds due to (3.5.13). Thus, we obtain (3.5.16), which is equivalent to

$$\det\begin{pmatrix} \gamma''(t) \\ \gamma'(s) \\ \gamma(t) - \gamma(s) \end{pmatrix} < 0,$$

by (3.5.14). We use (3.5.2) to rewrite this in terms of D_{R}:

$$0 > \det\begin{pmatrix} \gamma''(t) \\ \gamma'(s) \\ \gamma(t) - \gamma(s) \end{pmatrix} = C_{\mathrm{R}} \det\begin{pmatrix} \gamma''(t) \\ \gamma'(s) \\ \gamma'(t) \end{pmatrix} = -C_{\mathrm{R}} D_{\mathrm{R}} \det\begin{pmatrix} g'(t) \\ g'(s) \end{pmatrix}.$$

We recall that $C_{\mathrm{R}} > 0$ and $\det\begin{pmatrix} g'(t) \\ g'(s) \end{pmatrix} < 0$ when $t > s$. Thus, we obtain the claimed inequality $D_{\mathrm{R}} < 0$. ■

Proof of Proposition 3.5.9. Take $\theta_0 > 0$ to be a small number such that the torsion $\mathbf{T}$ of γ is positive on $(t_0 - \theta_0, t_0)$ and negative on $(t_0, t_0 + \theta_0)$. Recall that the sign of $\mathbf{T}$ coincides with the sign of $\mathfrak{K}'$, see (2.1.12).

Consider the following function h:

$$h(t,\theta) = \det\begin{pmatrix} \gamma'(t) \\ \gamma'(t-\theta) \\ \gamma(t)-\gamma(t-\theta) \end{pmatrix}, \quad \theta \in (0,\theta_0],\ t \in [t_0, t_0+\theta].$$

This function is continuous and differentiable on its domain. It follows from the conditions on the curve g that $\kappa_2'(u)\kappa_2'(v)(g_1(v)-g_1(u)) > 0$ for any u, v such that $u < v$. For any $\theta \in (0,\theta_0]$, the measure $\mathfrak{K}'$ is positive on $(t_0-\theta, t_0)$ and negative on $(t_0, t_0+\theta)$, therefore formula (3.5.12) implies that $h(t_0,\theta) > 0 > h(t_0+\theta,\theta)$ for any $\theta \in (0,\theta_0]$. So, for any such θ, there exists $t_\theta \in (t_0, t_0+\theta)$ such that $h(t_\theta,\theta) = 0$. We will show that such t_θ is unique for any $\theta \in (0,\theta_0]$.

Corollary 3.5.13 implies that $D_{\mathrm{R}} = D_{\mathrm{R}}(t_\theta-\theta, t_\theta)$ and $D_{\mathrm{L}} = D_{\mathrm{L}}(t_\theta-\theta, t_\theta)$ are negative. We use formula (3.5.2) to obtain

$$\begin{aligned}\frac{\partial h}{\partial t}(t_\theta,\theta) &= \det\begin{pmatrix} \gamma''(t_\theta) \\ \gamma'(t_\theta-\theta) \\ \gamma(t_\theta)-\gamma(t_\theta-\theta) \end{pmatrix} + \det\begin{pmatrix} \gamma'(t_\theta) \\ \gamma''(t_\theta-\theta) \\ \gamma(t_\theta)-\gamma(t_\theta-\theta) \end{pmatrix} \\ &= \det\begin{pmatrix} g'(t_\theta) \\ g'(t_\theta-\theta) \end{pmatrix}(C_{\mathrm{L}}D_{\mathrm{L}} - C_{\mathrm{R}}D_{\mathrm{R}}) < 0,\end{aligned}$$

because $\det\begin{pmatrix} g'(t_\theta) \\ g'(t_\theta-\theta) \end{pmatrix} < 0$, $C_{\mathrm{R}} > 0 > C_{\mathrm{L}}$, $D_{\mathrm{R}} < 0$, and $D_{\mathrm{L}} < 0$. It follows that the function $h(\cdot,\theta)$ changes its sign from plus to minus at any root on $(t_0, t_0+\theta)$. Therefore, it has only one root t_θ there.

We now calculate

$$\frac{\partial h}{\partial \theta}(t_\theta,\theta) = -\det\begin{pmatrix} \gamma'(t_\theta) \\ \gamma''(t_\theta-\theta) \\ \gamma(t_\theta)-\gamma(t_\theta-\theta) \end{pmatrix} = -\det\begin{pmatrix} g'(t_\theta) \\ g'(t_\theta-\theta) \end{pmatrix} C_{\mathrm{L}}D_{\mathrm{L}} > 0.$$

It now follows from the implicit function theorem that t_θ is a differentiable function of θ and

$$\frac{\partial t_\theta}{\partial \theta} = \frac{C_{\mathrm{L}}D_{\mathrm{L}}}{C_{\mathrm{L}}D_{\mathrm{L}} - C_{\mathrm{R}}D_{\mathrm{R}}} \in (0,1)$$

for $\theta \in (0,\theta_0)$. Thus, $t_\theta - \theta$ decreases. Moreover, $t_\theta \in (t_0, t_0+\theta)$ for any θ, therefore, $\lim_{\theta\to 0+} t_\theta = t_0$.

Let $\delta = t_{\theta_0} - t_0$, $I = [t_0, t_{\theta_0}]$. We define the function $s\colon I \to [t_{\theta_0}-\theta_0, t_0]$ by the rule $s(t_\theta) = t_\theta - \theta$ for $\theta \in (0,\theta_0]$, and $s(t_0) = t_0$. This function decreases, and the corresponding family of segments $S(t) = [g(t), g(s(t))]$, $t \in I$, forms a chordal domain. By the definition of t_θ, the pair $(s(t), t)$ satisfies the cup equation (3.4.2) for any $t \in I$, and moreover $D_{\mathrm{R}}(s,t) < 0$ and $D_{\mathrm{L}}(s,t) < 0$. Therefore, this chordal domain satisfies the conditions of Proposition 3.4.2 and the proof is complete. ■

3.5.3 How to grow a chordal domain from a single chord

Now we turn to a more general situation. In this subsection we work under the following assumption.

Condition 3.5.14. *There is a pair of numbers a_0, b_0, $a_0 < b_0$, satisfying the cup equation* (3.4.2)*, that is,*

$$\det\begin{pmatrix} \gamma'(a_0) \\ \gamma'(b_0) \\ \gamma(b_0) - \gamma(a_0) \end{pmatrix} = 0,$$

such that for some $\theta_0 > 0$, the determinant

$$L(t) = \det\begin{pmatrix} \gamma'(t) \\ \gamma'(a_0) \\ \gamma'(b_0) \end{pmatrix}$$

is negative for $t \in (b_0, b_0 + \theta_0)$ and positive for $t \in (a_0 - \theta_0, a_0)$.

Remark 3.5.15. If (a_0, b_0) satisfies the cup equation (3.4.2) and the inequalities $D_{\mathrm{R}}(a_0, b_0) < 0$ and $D_{\mathrm{L}}(a_0, b_0) < 0$, then Condition 3.5.14 holds.

Proof. It follows immediately from the fact that $L(a_0) = L(b_0) = 0$ and

$$\frac{L'(a_0)}{D_{\mathrm{L}}(a_0, b_0)} = \frac{L'(b_0)}{D_{\mathrm{R}}(a_0, b_0)} = \det\begin{pmatrix} g'(a_0) \\ g'(b_0) \end{pmatrix} > 0. \qquad \blacksquare$$

We are ready to present the main result of this subsection.

Proposition 3.5.16. *Under Condition 3.5.14, there exist $\delta > 0$ and two differentiable functions $a, b\colon [0, \delta] \to \mathbb{R}$ such that:*

(1) *$a' < 0 < b'$, $a(0) = a_0$, $b(0) = b_0$,*

(2) *$b(\theta) - a(\theta) = b_0 - a_0 + \theta$ and the pair $(a(\theta), b(\theta))$ satisfies the cup equation* (3.4.2) *for any $\theta \in [0, \delta]$,*

(3) *$D_{\mathrm{L}}(a(\theta), b(\theta)) < 0$ and $D_{\mathrm{R}}(a(\theta), b(\theta)) < 0$ for any $\theta \in (0, \delta]$.*

Proof. We can always assume that $f'(a_0) = f'(b_0) = 0$. If this is not the case, then we can replace the function f by $f + c_1 g_1 + c_2 g_2$ with appropriate $c_1, c_2 \in \mathbb{R}$. In such a case, the determinants in the cup equation and in the differentials do not depend on the choice of c_1 and c_2. Since $\det\begin{pmatrix} g'(a_0) \\ g'(b_0) \end{pmatrix} \neq 0$, we can choose the constants c_1, c_2 in such a way that $f'(a_0) = f'(b_0) = 0$ for the modified function. In this situation the sign of $L(t)$ coincides with the sign of $f'(t)$, therefore Condition 3.5.14 implies that f' is negative on the right neighborhood of b_0. From the cup equation (3.4.2), we obtain $f(a_0) = f(b_0)$.

Let us prove that κ_3' is non-positive in some right neighborhood of b_0. The function $\mathfrak{K}$ is piecewise monotone by Condition 2.1.11, therefore we can find a right neighborhood of b_0 where it is monotone and does not change its sign. Then $\kappa_3' = \mathfrak{K}\kappa_2'$ does not change its sign in the same right neighborhood of b_0 as well. We know that $\kappa_3(b_0) = 0$ and

$$\kappa_3 = \frac{f'}{g_1'} < 0 \quad \text{on the right of } b_0, \tag{3.5.18}$$

therefore $\kappa_3' \leqslant 0$ there. Without loss of generality, in what follows we assume that

$$\kappa_3' \leqslant 0 \qquad \text{on } (b_0, b_0 + \theta_0). \tag{3.5.19}$$

We write the following inequality for $t \in (b_0, b_0 + \theta_0)$:

$$\begin{aligned} f(t) - f(a_0) &= f(t) - f(b_0) \\ &= \int_{b_0}^{t} f'(\tau)\,d\tau = \int_{b_0}^{t} g_1'(\tau)\kappa_3(\tau)\,d\tau \geqslant (g_1(t) - g_1(b_0))\kappa_3(t) \end{aligned}$$

because $g_1' > 0$ and $\kappa_3' \leqslant 0$. We use this inequality to estimate the determinant:

$$\begin{aligned} \det\begin{pmatrix} \gamma'(t) \\ \gamma'(a_0) \\ \gamma(t)-\gamma(a_0) \end{pmatrix} &\leqslant \det\begin{pmatrix} g_1'(t) & g_2'(t) & f'(t) \\ g_1'(a_0) & g_2'(a_0) & 0 \\ g_1(t)-g_1(a_0) & g_2(t)-g_2(a_0) & (g_1(t)-g_1(b_0))\kappa_3(t) \end{pmatrix} \\ &= f'(t)g_1'(a_0)\det\begin{pmatrix} 1 & \kappa_2(t) & 1 \\ 1 & \kappa_2(a_0) & 0 \\ g_1(t)-g_1(a_0) & g_2(t)-g_2(a_0) & g_1(t)-g_1(b_0) \end{pmatrix}. \end{aligned}$$

When t approaches b_0 from the right, the latter determinant tends to

$$g_2(b_0) - g_2(a_0) - \kappa_2(a_0)(g_1(b_0) - g_1(a_0)).$$

This expression is strictly positive due to convexity of the curve g. Since $f'(t) < 0$, this leads to the inequality

$$\det\begin{pmatrix} \gamma'(t) \\ \gamma'(a_0) \\ \gamma(t) - \gamma(a_0) \end{pmatrix} < 0 \tag{3.5.20}$$

for $t > b_0$ sufficiently close to b_0. Reducing θ_0 if needed, we can assume that (3.5.19) and (3.5.20) hold for all $t \in (b_0, b_0 + \theta_0)$.

Similarly, we can prove that for all t in a left neighborhood of a_0, we have that $\kappa_3'(t) \geqslant 0$ and

$$\det\begin{pmatrix} \gamma'(b_0) \\ \gamma'(t) \\ \gamma(b_0) - \gamma(t) \end{pmatrix} > 0. \tag{3.5.21}$$

We can reduce θ_0 once more, and in what follows, we assume that (3.5.21) holds for all $t \in (a_0 - \theta_0, a_0)$. Consider the function

$$h(t,\theta) = \det\begin{pmatrix} \gamma'(t) \\ \gamma'(t-(b_0-a_0)-\theta) \\ \gamma(t)-\gamma(t-(b_0-a_0)-\theta)\end{pmatrix}, \quad \theta \in (0,\theta_0],\ t \in [b_0, b_0+\theta].$$

This function is continuous and differentiable on its domain. It follows from (3.5.20) and (3.5.21) that

$$h(b_0,\theta) > 0 > h(b_0+\theta,\theta), \quad \theta \in (0,\theta_0).$$

Thus, for any such θ, there exists $b_\theta \in (b_0, b_0+\theta)$ such that $h(b_\theta,\theta) = 0$. We will show that such b_θ is unique. Let $a_\theta = b_\theta - (b_0 - a_0) - \theta$. We would like to verify that the functions $a(\theta) = a_\theta$ and $b(\theta) = b_\theta$ are the function whose existence is stated in the proposition.

The remaining part of the proof is based on the following lemma.

Lemma 3.5.17. *If $\theta \in (0,\theta_0)$ and θ is sufficiently small, then $D_{\mathrm{R}}(a_\theta,b_\theta) < 0$ and $D_{\mathrm{L}}(a_\theta,b_\theta) < 0$.*

The proof of this lemma is rather technical, and we present it after completing the proof of the proposition.

The remaining part of the proof literally repeats the proof of Proposition 3.5.9. We can assume that θ_0 is so small that the statement of Lemma 3.5.17 holds for all θ in the interval $(0,\theta_0)$. We calculate the derivative of h:

$$\frac{\partial h}{\partial t}(b_\theta,\theta) = \det\begin{pmatrix} g'(b_\theta) \\ g'(a_\theta)\end{pmatrix} \cdot [C_{\mathrm{L}}D_{\mathrm{L}} - C_{\mathrm{R}}D_{\mathrm{R}}](a_\theta,b_\theta) < 0. \tag{3.5.22}$$

Therefore, the function $h(\,\cdot\,,\theta)$ has unique root on $(b_0, b_0+\theta)$. We save the notation b_θ for this unique root. One can calculate the partial derivative with respect to θ:

$$\frac{\partial h}{\partial \theta}(b_\theta,\theta) = -\det\begin{pmatrix} g'(b_\theta) \\ g'(a_\theta)\end{pmatrix} \cdot [C_{\mathrm{L}}D_{\mathrm{L}}](a_\theta,b_\theta) > 0. \tag{3.5.23}$$

The implicit function theorem implies that b_θ is a differentiable function of θ and

$$\frac{\partial b_\theta}{\partial \theta} = \frac{C_{\mathrm{L}}D_{\mathrm{L}}}{C_{\mathrm{L}}D_{\mathrm{L}} - C_{\mathrm{R}}D_{\mathrm{R}}} \in (0,1)$$

for $\theta \in (0,\theta_0)$. It follows that $a_\theta = b_\theta - \theta - (b_0 - a_0)$ decreases and

$$\lim_{\theta\to 0+} b_\theta = b_0, \quad \lim_{\theta \to 0+} a_\theta = a_0.$$

Proposition 3.5.16 is proved. ∎

Proof of Lemma 3.5.17. We will deal with D_{R} only, the other inequality is symmetric. Since $f'(b_0) = 0$ and $f'(t) < 0$ for $t \in (b_0, b_0 + \theta_0)$, we have $f''(b_0) \leqslant 0$. Since also $f'(a_0) = 0$, this yields $D_{\mathrm{R}}(a_0, b_0) \leqslant 0$. If $D_{\mathrm{R}}(a_0, b_0) < 0$, then $D_{\mathrm{R}}(a_\theta, b_\theta) < 0$ for small θ, by continuity. The same reasoning shows that $D_{\mathrm{L}}(a_0, b_0) \leqslant 0$, and if $D_{\mathrm{L}}(a_0, b_0) < 0$, then $D_{\mathrm{L}}(a_\theta, b_\theta) < 0$ for θ sufficiently small.

Let us consider the case when one of $D_{\mathrm{L}}(a_0, b_0)$ and $D_{\mathrm{R}}(a_0, b_0)$ is strictly negative while the other one is zero. The case where both differentials vanish will be considered later. Without loss of generality assume $D_{\mathrm{L}}(a_0, b_0) < 0$ and $D_{\mathrm{R}}(a_0, b_0) = 0$. Then $f''(b_0) = 0$, and therefore

$$\kappa_3'(b_0) = \frac{f''(b_0)g_1'(b_0) - f'(b_0)g_1''(b_0)}{(g_1'(b_0))^2} = 0, \qquad \mathfrak{K}(b_0) = \frac{\kappa_3'(b_0)}{\kappa_2'(b_0)} = 0.$$

Condition 2.1.11 implies that $\mathfrak{K}'$ does not change its sign in a right neighborhood of b_0. From (3.5.19) we know that $\mathfrak{K}$ is non-positive on $(b_0, b_0 + \theta_0)$, therefore $\mathfrak{K}' \leqslant 0$ in a right neighborhood of b_0 as well. Let us note that

$$\mathfrak{K}' \text{ is not identically zero in any right neighborhood of } b_0. \tag{3.5.24}$$

Assume on the contrary that $\mathfrak{K}'$ is identically zero in a right neighborhood of b_0. Then the function $\mathfrak{K}$ is constant there and coincides with $\mathfrak{K}(b_0) = 0$. Therefore, $\kappa_3' = 0$ in the same neighborhood. Then κ_3 is constant there and equal to $\kappa_3(b_0) = 0$. This contradicts (3.5.18).

Inequalities (3.5.22) and (3.5.23) hold for $\theta = 0$, therefore, by continuity, they also hold for θ sufficiently small. Thus,

$$\frac{\partial b_\theta}{\partial \theta} = \frac{C_{\mathrm{L}} D_{\mathrm{L}}}{C_{\mathrm{L}} D_{\mathrm{L}} - C_{\mathrm{R}} D_{\mathrm{R}}} > 0.$$

Let $F(\theta) = D_{\mathrm{R}}(a_\theta, b_\theta)$. From (3.5.11), we obtain

$$F'(\theta) = \Big(\kappa_2'(b_\theta)g_1'(b_\theta)\mathfrak{K}'(b_\theta) + \Big[\frac{(g_1'\kappa_2')'}{g_1'\kappa_2'}(b_\theta) - \frac{\kappa_2'}{\kappa_2 - \kappa}(b_\theta)\Big]F(\theta)\Big)\frac{\partial b_\theta}{\partial \theta}.$$

We use the integrating factor

$$M(\theta) = \exp\Big(-\int_0^\theta \Big[\frac{(g_1'\kappa_2')'}{g_1'\kappa_2'}(b_\eta) - \frac{\kappa_2'}{\kappa_2 - \kappa}(b_\eta)\Big]\frac{\partial b_\eta}{\partial \eta}\,d\eta\Big), \qquad \theta > 0,$$

to conclude

$$\frac{d}{d\theta}(F(\theta)M(\theta)) = M(\theta)\kappa_2'(b_\theta)g_1'(b_\theta)\mathfrak{K}'(b_\theta)\frac{\partial b_\theta}{\partial \theta} \leqslant 0. \tag{3.5.25}$$

We know that $F(0) = 0$, therefore $F(\eta) \leqslant 0$ for $\eta > 0$. If $F(\eta) = 0$ for some $\eta > 0$, then the expression in (3.5.25) vanishes on $(0, \eta)$. Therefore, $\mathfrak{K}'(b_\theta) = 0$ on $(0, \eta)$ as well, which contradicts (3.5.24). Hence, we have $D_{\mathrm{R}}(a_\theta, b_\theta) = F(\theta) < 0$ for sufficiently small $\theta > 0$. The inequality $D_{\mathrm{L}}(a_\theta, b_\theta) < 0$ follows from continuity and our assumption $D_{\mathrm{L}}(a_0, b_0) < 0$.

It remains to prove the statement for the case $D_{\mathrm{L}}(a_0, b_0) = D_{\mathrm{R}}(a_0, b_0) = 0$. In this case, $f''(a_0) = f''(b_0) = 0$, $\mathfrak{K}' \leqslant 0$ in a right neighborhood of b_0, and $\mathfrak{K}' \geqslant 0$ in a left neighborhood of a_0. Instead of f consider the function

$$\tilde{f} = (1 - \chi_{[a_0,b_0]})f + \chi_{[a_0,b_0]} f(a_0),$$

and the corresponding function $\tilde{\mathfrak{K}}$. The function $\tilde{f}$ is C^2-smooth. For any θ, the pair (a_θ, b_θ) satisfies the cup equation (3.4.2) with $\tilde{f}$ instead of f. Also, outside (a_0, b_0), we have $\tilde{\mathfrak{K}} = \mathfrak{K}$. The function $\tilde{\mathfrak{K}} = \mathfrak{K}$ is not constant on (b_0, b_θ) by (3.5.24). For any $t_0 \in (a_0, b_0)$, we obtain that $\tilde{\mathfrak{K}}$ satisfies the assumptions of Corollary 3.5.13, which yields $D_{\mathrm{R}}(a_\theta, b_\theta) < 0$ and $D_{\mathrm{L}}(a_\theta, b_\theta) < 0$. ■

3.6 Forces

Definition 3.6.1. Let B be a Bellman candidate on a fence $\Omega(I)$. The function

$$\mathcal{F}_I = \frac{\kappa_2 - \kappa}{\kappa_2'} \beta_2' \tag{3.6.1}$$

on I is called its *force function*.

Remark 3.6.2. Suppose that $\Omega_{\mathrm{ch}}(I)$ is a chordal domain with corresponding function $s\colon I \to \mathbb{R}$, and that B is the standard Bellman candidate on $\Omega_{\mathrm{ch}}(I)$. Then

$$\mathcal{F}_I(t) = \frac{D}{g_1'(t)\kappa_2'(t)}, \qquad t \in I,$$

where $D = D_{\mathrm{R}}(s,t)$ if $t > s(t)$ and $D = D_{\mathrm{L}}(t,s)$ if $t < s(t)$.

Proof. This is a direct consequence of formulas (3.4.7), (3.5.1), and (3.6.1). ■

Remark 3.6.3. The force function $\mathcal{F}$ is non-positive on the domain of its definition.

Proof. If $\Omega(I)$ is a right tangent domain, then we have $\beta_2' \leqslant 0$ (see Remark 3.3.2) and $\kappa_2 - \kappa > 0$ (this follows from geometric properties of Ω). For the case of left tangents, both inequalities are opposite. Therefore, $\mathcal{F} \leqslant 0$.

For the case of a chordal domain, both differentials D_{L} and D_{R} are negative, hence $\mathcal{F} \leqslant 0$. ■

3.6.1 Gluing fences

In this part we present necessary and sufficient conditions for gluing two Bellman candidates on fences with a common point. By this we mean the following situation.

Proposition 3.6.4. *Let $t_-, t_0, t_+ \in \mathbb{R}$, $t_- \leqslant t_0 \leqslant t_+$, $I_- = [t_-, t_0]$ and $I_+ = [t_0, t_+]$. Suppose that $\Omega(I_\pm)$ are two fences with Bellman candidates $B_\pm$ on them with corresponding functions $w_\pm$, $S_\pm, \beta_\pm$. Let* $\operatorname{int}\Omega(I_-) \cap \operatorname{int}\Omega(I_+) = \varnothing$. *Suppose that*

$$\beta_+(t_0) = \beta_-(t_0) \stackrel{\text{def}}{=} \beta_0. \tag{3.6.2}$$

Then the function B defined by the formula

$$B(x) = \begin{cases} B_-(x), & x \in \Omega(I_-), \\ f(t_0) + \langle \beta_0, x - g(t_0) \rangle, & x \in \operatorname{ind\,conv}(S_-(t_0) \cup S_+(t_0)), \\ B_+(x), & x \in \Omega(I_+). \end{cases} \tag{3.6.3}$$

is a C^1-smooth Bellman candidate on its domain (*see Definition* 3.1.1 *for the induced convex hull*).

Proof. The C^1-smoothness of B follows from (3.6.2) and the C^1-smoothness of $B_\pm$. The concavity of B is then implied by Proposition 3.1.2. ∎

Corollary 3.6.5. *The conclusion of Proposition* 3.6.4 *holds if we replace* (3.6.2) *by the forces equality $\mathcal{F}_{I_+}(t_0) = \mathcal{F}_{I_-}(t_0)$.*

Proof. Since κ_2 and κ_3 do not depend on the fences, it follows from (3.2.6) and (3.2.9) applied to t_0 that $\beta_+(t_0) = \beta_-(t_0)$. ∎

Now we consider specific cases of gluing fences we have investigated in Sections 3.3 and 3.4.

3.6.2 Forces and tails

In Definition 3.6.1, forces were defined on fences. For a given chordal domain, we have a force function defined inside this domain. We wish to extend the forces outside for the purpose of continuation of a Bellman candidate from a chordal domain via tangents.

Let $I = [t_-, t_+]$ be an interval, and let $\Omega_{\mathrm{ch}}(I)$ be a chordal domain with corresponding function $s\colon I \to \mathbb{R}$ such that $s(t) \leqslant t$ for $t \in I$. We define a right force of the chordal domain $\Omega_{\mathrm{ch}}(I)$ in the following way. We define t^{R} to be the supremum of the numbers τ, $\tau \geqslant t_+$, such that there exists a standard Bellman candidate on the right tangent domain $\Omega_{\mathrm{R}}([t_+, \tau])$ satisfying

$$\mathcal{F}_{[t_-,t_+]}(t_+) = \mathcal{F}_{[t_+,\tau]}(t_+).$$

Note that t^{R} can be equal to $+\infty$. The concatenation of the functions $\mathcal{F}_I$ and $\mathcal{F}_{[t_+,t^{\mathrm{R}}]}$ will be called the right force function of the chordal domain $\Omega_{\mathrm{ch}}(I)$.

Definition 3.6.6. The function

$$F_{\mathrm{R}}(t, \Omega_{\mathrm{ch}}(I)) = \begin{cases} \mathcal{F}_I(t), & t \in I, \\ \mathcal{F}_{[t_+, t^{\mathrm{R}}]}(t), & t \in [t_+, t^{\mathrm{R}}], \end{cases}$$

is called the *right force function* of the chordal domain $\Omega_{\mathrm{ch}}(I)$. The set (t_-, t^{R}) is called the *right tail* of the chordal domain $\Omega_{\mathrm{ch}}(I)$.

Remark 3.6.7. If $t^{\mathrm{R}} < +\infty$, then $\mathcal{F}_{[t_+, t^{\mathrm{R}}]}(t^{\mathrm{R}}) = 0$.

Proof. This immediately follows from the maximality of t^{R}, and the continuity and non-positivity of the force. ∎

Remark 3.6.8. If $t_+ < t^{\mathrm{R}} < +\infty$, then there exists $\bar{t}$, $\bar{t} < t^{\mathrm{R}}$, such that the torsion of the curve γ is strictly positive on $(\bar{t}, t^{\mathrm{R}})$. If $t^{\mathrm{R}} > t_1 \geqslant t_+$ and the torsion of the curve γ is non-positive on an interval (t_1, t_2), then $t^{\mathrm{R}} > t_2$.

Proof. Due to Condition 2.1.11, we can find $\bar{t}$, $\bar{t} < t^{\mathrm{R}}$, sufficiently close to t^{R} such that either $\mathfrak{K}' > 0$ on $(\bar{t}, t^{\mathrm{R}})$ or $\mathfrak{K}' \leqslant 0$ on $(\bar{t}, t^{\mathrm{R}})$. If $\mathfrak{K}' \leqslant 0$, then we use (3.2.12) for $t_0 = \bar{t}$, $t = t^{\mathrm{R}}$, and obtain that $\mathcal{F}_{[t_+, t^{\mathrm{R}}]}(t^{\mathrm{R}}) < 0$, which contradicts Remark 3.6.7. Thus, $\mathfrak{K}' > 0$ on $(\bar{t}, t^{\mathrm{R}})$. The second claim follows immediately from the first one. ∎

Similarly, we define the left force function and the left tail of the chordal domain. Let $I = [t_-, t_+]$ be an interval, and let $\Omega_{\mathrm{ch}}(I)$ be a chordal domain with corresponding function $s\colon I \to \mathbb{R}$ such that $s(t) \geqslant t$ for $t \in I$. We define t^{L} to be the infimum of the numbers τ, $\tau \leqslant t_-$, such that there exists a standard Bellman candidate on the left tangent domain $\Omega_{\mathrm{L}}([\tau, t_-])$ satisfying

$$\mathcal{F}_{[\tau, t_-]}(t_-) = \mathcal{F}_{[t_-, t_+]}(t_-).$$

Note that t^{L} can be equal to $-\infty$.

Definition 3.6.9. The function

$$F_{\mathrm{L}}(t, \Omega_{\mathrm{ch}}(I)) = \begin{cases} \mathcal{F}_{[t^{\mathrm{L}}, t_-]}(t), & t \in [t^{\mathrm{L}}, t_-], \\ \mathcal{F}_I(t), & t \in I, \end{cases}$$

is called the *left force function* of the chordal domain $\Omega_{\mathrm{ch}}(I)$. The set (t^{L}, t_+) is called the *left tail* of the chordal domain $\Omega_{\mathrm{ch}}(I)$.

The following remark is the "left" analog of Remarks 3.6.7 and 3.6.8.

Remark 3.6.10. If $t^{\mathrm{L}} > -\infty$, then $\mathcal{F}_{[t^{\mathrm{L}}, t_-]}(t^{\mathrm{L}}) = 0$. If $-\infty < t^{\mathrm{L}} < t_-$, then there exists $\bar{t}$, $t^{\mathrm{L}} < \bar{t}$, such that the torsion of the curve γ is strictly negative on $(t^{\mathrm{L}}, \bar{t})$. If $t^{\mathrm{L}} < t_2 \leqslant t_-$ and the torsion of the curve γ is non-negative on an interval (t_1, t_2), then $t^{\mathrm{L}} < t_1$.

Remark 3.6.11. Suppose that $\Omega_{\mathrm{ch}}(I)$ is a chordal domain, and that its upper chord is $[g(a), g(b)]$. Since the force function outside the chordal domain does not depend on the foliation inside it, we will use the notation $F_{\mathrm{R}}(t; a, b)$ instead of $F_{\mathrm{R}}(t, \Omega_{\mathrm{ch}}(I))$, and $F_{\mathrm{L}}(t; a, b)$ instead of $F_{\mathrm{L}}(t, \Omega_{\mathrm{ch}}(I))$ for t outside the chordal domain. The forces satisfy the following identities:

$$F_{\mathrm{R}}(t; a, b) = \exp\Big(-\int_b^t \frac{\kappa_2'}{\kappa_2 - \kappa}\Big)\Big(\int_b^t \exp\Big(\int_b^\tau \frac{\kappa_2'}{\kappa_2 - \kappa}\Big)\mathfrak{K}'(\tau)\,d\tau + \frac{D_{\mathrm{R}}(a, b)}{g_1'(b)\kappa_2'(b)}\Big),$$
$$t \in [b, t^{\mathrm{R}}], \qquad (3.6.4)$$

$$F_{\mathrm{L}}(t; a, b) = \exp\Big(\int_t^a \frac{\kappa_2'}{\kappa_2 - \kappa}\Big)\Big(-\int_t^a \exp\Big(-\int_\tau^a \frac{\kappa_2'}{\kappa_2 - \kappa}\Big)\mathfrak{K}'(\tau)\,d\tau + \frac{D_{\mathrm{L}}(a, b)}{g_1'(a)\kappa_2'(a)}\Big),$$
$$t \in [t^{\mathrm{L}}, a]. \qquad (3.6.5)$$

Analogously, we define the forces and tails from the infinities. Suppose $t \in \mathbb{R}$ and that there exists the standard Bellman candidate on the right tangent domain $\Omega_{\mathrm{R}}(-\infty, t)$ (see Proposition 3.3.6). Let $t^{\mathrm{R}} \leqslant +\infty$ be the supremum of such t.

Definition 3.6.12. The function

$$F_{\mathrm{R}}(t, -\infty) = \mathcal{F}_{(-\infty, t^{\mathrm{R}})}(t), \quad t \in (-\infty, t^{\mathrm{R}}),$$

is called the *right force function of* $-\infty$. The ray $(-\infty, t^{\mathrm{R}})$ is called the *right tail of* $-\infty$.

Similarly, we define the left force function and the left tail of $+\infty$. Let $t^{\mathrm{L}} \geqslant -\infty$ be the infimum of such $t \in \mathbb{R}$ that there exists the standard Bellman candidate on the left tangent domain $\Omega_{\mathrm{L}}(t^{\mathrm{L}}, +\infty)$ (see Proposition 3.3.7).

Definition 3.6.13. The function

$$F_{\mathrm{L}}(t, +\infty) = \mathcal{F}_{(t^{\mathrm{L}}, +\infty)}(t), \quad t \in (t^{\mathrm{L}}, +\infty),$$

is called the *left force function of* $+\infty$. The ray $(t^{\mathrm{L}}, +\infty)$ is called the *left tail of* $+\infty$.

3.6.3 Properties of forces

Though the expressions for the forces are well defined for arbitrary pairs of points (a_0, b_0), when we write a force concerning such a pair, we always assume that the pair (a_0, b_0) satisfies the cup equation (3.4.2). We study differential properties of forces.

Proposition 3.6.14. *Let* $\Omega_{\mathrm{ch}}([a, b], *)$ *be a chordal domain. Then its forces satisfy the following differential equation on the corresponding tails*:

$$F' = \mathfrak{K}' - \beta_2' = \mathfrak{K}' - \frac{\kappa_2'}{\kappa_2 - \kappa} F, \quad \textit{where } F \textit{ is } F_{\mathrm{R}} \textit{ or } F_{\mathrm{L}}. \qquad (3.6.6)$$

Proof. Taking into account the definition (3.6.1), this statement may be obtained via division of (3.2.9) by κ_2' and differentiation. ■

Remark 3.6.15. The functions $F_{\mathrm{R}} - \mathfrak{K}$ and $F_{\mathrm{L}} - \mathfrak{K}$ are strictly increasing and strictly decreasing on their domains, respectively.

Proof. According to (3.6.6), $F' - \mathfrak{K}' = -\frac{\kappa_2'}{\kappa_2-\kappa}F$. We always have $F < 0$ and $\kappa_2' > 0$. It remains to notice that $\kappa_2 - \kappa > 0$ in the case of the right force, and $\kappa_2 - \kappa < 0$ in the case of the left force. ■

Lemma 3.6.16. *Suppose that $\Omega_{\mathrm{ch}}(I)$ is a chordal domain, and that its upper chord is $[g(a), g(b)]$. For a fixed t, $t > b$, consider the right force function $F_{\mathrm{R}}(t, a(b), b)$ as a function of b. Then its derivative with respect to b satisfies the following equality*:

$$\frac{\partial F_{\mathrm{R}}(t,a(b),b)}{\partial b} = \frac{D_{\mathrm{R}}(a,b)}{g_1'(b)}\cdot\Big[\frac{1}{\kappa_2(b)-\kappa_{\mathrm{R}}(b)} - \frac{1}{\kappa_2(b)-\kappa_{\mathrm{chord}}(b)}\Big] \times \exp\Big(-\int_b^t \frac{\kappa_2'(\tau)}{\kappa_2(\tau)-\kappa_{\mathrm{R}}(\tau)}\,d\tau\Big), \tag{3.6.7}$$

where κ_{R} is the slope of the right tangent (see (2.1.5)), and $\kappa_{\mathrm{chord}}(b)$ is the slope of the upper chord of the chordal domain, i.e.,

$$\kappa_{\mathrm{chord}}(b) = \frac{g_2(b)-g_2(a)}{g_1(b)-g_1(a)}. \tag{3.6.8}$$

For a fixed t, $t < a$, we get a symmetric formula:

$$\frac{\partial F_{\mathrm{L}}(t,a,b(a))}{\partial a} = \frac{D_{\mathrm{L}}(a,b)}{g_1'(a)}\cdot\Big[\frac{1}{\kappa_2(a)-\kappa_{\mathrm{L}}(a)} - \frac{1}{\kappa_2(a)-\kappa_{\mathrm{chord}}(a)}\Big] \times \exp\Big(\int_t^a \frac{\kappa_2'(\tau)}{\kappa_2(\tau)-\kappa_{\mathrm{L}}(\tau)}\,d\tau\Big), \tag{3.6.9}$$

where κ_{L} is the slope of the left tangent (see (2.1.5)), and $\kappa_{\mathrm{chord}}(a) = \kappa_{\mathrm{chord}}(b)$ is given by (3.6.8).

Proof. We differentiate $F_{\mathrm{R}}(t,a,b)$ (see (3.6.4)) with respect to b, regarding $a = a(b)$:

$$\begin{aligned}\frac{\partial F_{\mathrm{R}}(t,a(b),b)}{\partial b} &= \frac{\kappa_2'(b)}{\kappa_2(b)-\kappa(b)}F_{\mathrm{R}}(t,a,b)\\ &+ \Big\{-\mathfrak{K}'(b) - \frac{\kappa_2'(b)}{\kappa_2(b)-\kappa(b)}\int_b^t \exp\Big(\int_b^\tau \frac{\kappa_2'}{\kappa_2-\kappa}\Big)\mathfrak{K}'(\tau)\,d\tau\\ &+ \frac{1}{g_1'(b)\kappa_2'(b)}\frac{dD_{\mathrm{R}}(a(b),b)}{db} - \frac{(g_1'(b)\kappa_2'(b))'}{(g_1'(b)\kappa_2'(b))^2}D_{\mathrm{R}}(a,b)\Big\}\exp\Big(-\int_b^t\frac{\kappa_2'}{\kappa_2-\kappa}\Big)\\ &= \Big\{-\mathfrak{K}' + \frac{\kappa_2'}{\kappa_2-\kappa}\cdot\frac{D_{\mathrm{R}}}{g_1'\kappa_2'} + \frac{1}{g_1'\kappa_2'}\cdot\frac{dD_{\mathrm{R}}}{db} - \frac{(g_1'\kappa_2')'}{(g_1'\kappa_2')^2}D_{\mathrm{R}}\Big\}\exp\Big(-\int_b^t\frac{\kappa_2'}{\kappa_2-\kappa}\Big).\end{aligned} \tag{3.6.10}$$

We use formula (3.5.11) to simplify the expression in the braces. We note that κ in (3.6.10) differs from the one in (3.5.11). We write κ_{R} for that which appears in (3.6.10) and κ_{chord} for the one in (3.5.11):

$$\begin{aligned}
&\Big\{-\mathfrak{K}' + \frac{\kappa_2'}{\kappa_2-\kappa_{\mathrm{R}}}\cdot\frac{D_{\mathrm{R}}}{g_1'\kappa_2'} + \frac{1}{g_1'\kappa_2'}\cdot\frac{dD_{\mathrm{R}}}{db} - \frac{(g_1'\kappa_2')'}{(g_1'\kappa_2')^2}D_{\mathrm{R}}\Big\}\\
&\quad = -\mathfrak{K}' + \frac{\kappa_2'}{\kappa_2-\kappa_{\mathrm{R}}}\cdot\frac{D_{\mathrm{R}}}{g_1'\kappa_2'}\\
&\qquad + \frac{1}{g_1'\kappa_2'}\cdot\Big[\kappa_2' g_1'\mathfrak{K}' + \Big(\frac{(g_1'\kappa_2')'}{g_1'\kappa_2'} - \frac{\kappa_2'}{\kappa_2-\kappa_{\mathrm{chord}}}\Big)D_{\mathrm{R}}\Big] - \frac{(g_1'\kappa_2')'}{(g_1'\kappa_2')^2}D_{\mathrm{R}}\\
&\quad = \frac{D_{\mathrm{R}}}{g_1'}\Big(\frac{1}{\kappa_2-\kappa_{\mathrm{R}}} - \frac{1}{\kappa_2-\kappa_{\mathrm{chord}}}\Big).
\end{aligned}$$

■

3.7 Linearity domains

As stated in Section 3.1, we classify linearity domains by the number of points on the fixed boundary. For each linearity domain $\mathfrak{L}$, we will define the corresponding force function on $\mathfrak{L} \cap \partial_{\mathrm{fixed}}\Omega$.

3.7.1 Angle

The first linearity domain we study, an *angle*, has only one point $g(u)$ on the fixed boundary. Recall that $S_{\mathrm{L}}(u)$ and $S_{\mathrm{R}}(u)$ are the left and the right tangent segments to the free boundary of Ω starting at the point $g(u)$, and $T(u)$ is the closed curvilinear triangle with the vertex $g(u)$ whose sides are $S_{\mathrm{L}}(u)$, $S_{\mathrm{R}}(u)$, and the part of $\partial_{\mathrm{free}}\Omega$ between the two tangency points. We will use the symbol $\Omega_{\mathrm{ang}}(u)$ for this domain of linearity; note that geometrically $\Omega_{\mathrm{ang}}(u) = T(u)$.

Definition 3.7.1. An affine function B on an angle $\Omega_{\mathrm{ang}}(u)$ is called a *standard candidate* on $\Omega_{\mathrm{ang}}(u)$ if the vector $\gamma'(u)$ is parallel to the graph of B on $\Omega_{\mathrm{ang}}(u)$.

An angle is the linearity domain that appeared in Proposition 3.6.4 in the case where $\Omega(I_-)$ is a right tangent domain and $\Omega(I_+)$ is a left one, see Figure 3.3. The function B defined by (3.6.3) is a standard candidate on the angle. See the graphical representation of this situation in Section 3.7.3, Figure 3.9.

3.7.2 Linearity domains with two points on the fixed boundary

Consider a linearity domain $\mathfrak{L}$ that has two points $g(a)$ and $g(b)$ on the fixed boundary $\partial_{\mathrm{fixed}}\Omega$, assuming $a < b$ and $[g(a), g(b)] \subset \Omega$. Surely, the segment $[g(a), g(b)]$ is a part of the boundary of the linearity domain. It is natural to assume that there are

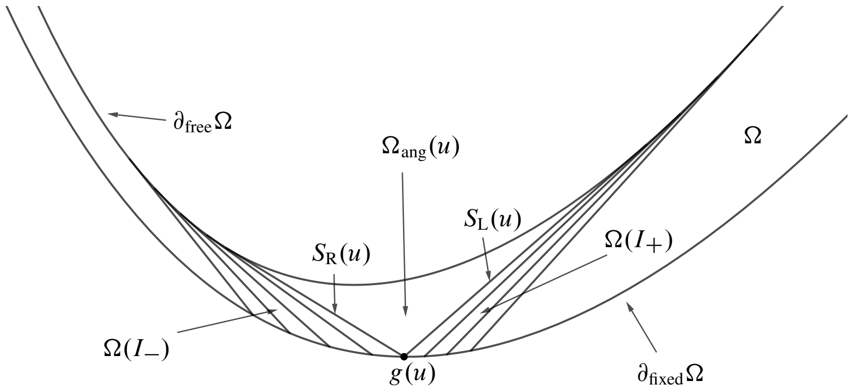

Figure 3.3. An angle $\Omega_{\text{ang}}(u)$ with adjacent domains.

two extremal segments tangent to the free boundary, $S(a)$ and $S(b)$, bounding our linearity domain from the left and right. If they have the same orientation, namely, they are either $S_{\text{L}}(a)$ and $S_{\text{L}}(b)$, or $S_{\text{R}}(a)$ and $S_{\text{R}}(b)$, then the linearity domain is called a *trolleybus*, the *left* one or the *right* one, respectively. These trolleybuses will be denoted by $\Omega_{\text{tr,L}}(a,b)$ and $\Omega_{\text{tr,R}}(a,b)$, see Figure 3.4 and Figure 3.5.

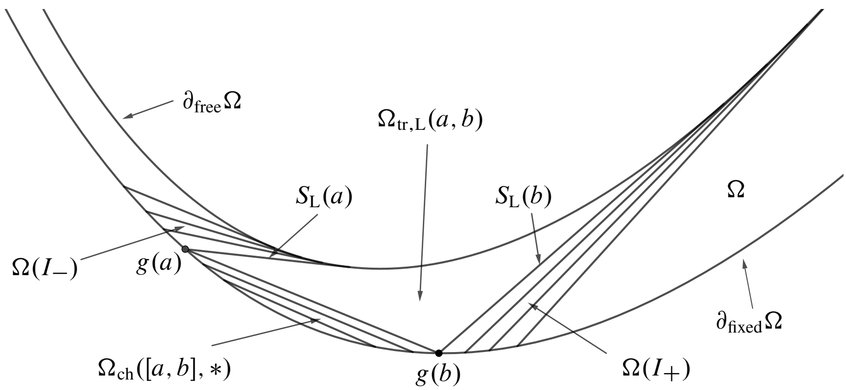

Figure 3.4. Trolleybus $\Omega_{\text{tr,L}}(a,b)$ with adjacent domains.

A linearity domain whose border tangents have different orientation, i.e., $S_{\text{R}}(a)$ and $S_{\text{L}}(b)$, is called a *birdie*, see Figure 3.6; we denote it by $\Omega_{\text{bird}}(a,b)$. It is clear that the opposite situation, $S_{\text{L}}(a)$ and $S_{\text{R}}(b)$, is impossible because these two segments intersect each other.

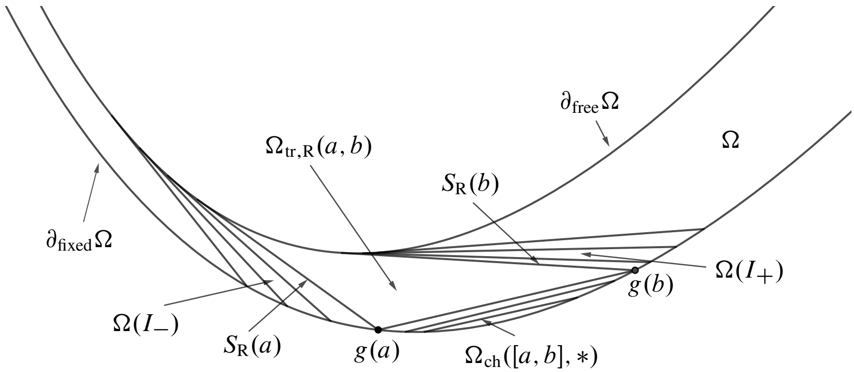

Figure 3.5. Trolleybus $\Omega_{\mathrm{tr,R}}(a,b)$ with adjacent domains.

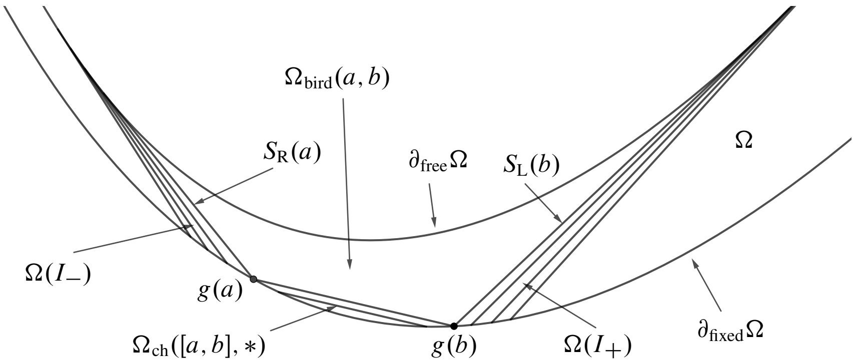

Figure 3.6. A birdie $\Omega_{\mathrm{bird}}(a,b)$ with the adjacent domains.

As in the case of an angle, we suppose that $g(a)$ and $g(b)$ are the points where we glue two different fences. So, there are three fences: a tangent domain $\Omega(I_-)$ adjacent to $\mathfrak{L}$ along the tangent $S(a)$, a tangent domain $\Omega(I_+)$ adjacent to $\mathfrak{L}$ along $S(b)$, and a chordal domain $\Omega_{\mathrm{ch}}([a,b],*)$.

Proposition 3.7.2. *Let $I_- = [t_-, a]$, $I_+ = [b, t_+]$. Suppose that $B_\pm$ are Bellman candidates on $\Omega(I_\pm)$, and $B_{\Omega_{\mathrm{ch}}([a,b],*)}$ is the Bellman candidate on $\Omega_{\mathrm{ch}}([a,b],*)$. Suppose that the gluing conditions hold for each of two pairs of fences, i.e.,*

$$\mathcal{F}_{I_-}(a) = F_{\mathrm{L}}(a, \Omega_{\mathrm{ch}}([a,b],*)), \quad \mathcal{F}_{I_+}(b) = F_{\mathrm{R}}(b, \Omega_{\mathrm{ch}}([a,b],*)).$$

Let $B_{\mathfrak{L}}(x)$ be the affine function defined by the equation

$$\det\begin{pmatrix} g_1'(a) & g_2'(a) & f'(a) \\ g_1'(b) & g_2'(b) & f'(b) \\ x_1 - g_1(a) & x_2 - g_2(a) & B_{\mathfrak{L}}(x) - f(a) \end{pmatrix} = 0, \quad x \in \mathfrak{L}. \tag{3.7.1}$$

Then the function B defined by the formula

$$B(x) = \begin{cases} B_-(x), & x \in \Omega(I_-), \\ B_{\mathfrak{L}}(x), & x \in \mathfrak{L}, \\ B_+(x), & x \in \Omega(I_+), \\ B_{\Omega_{\mathrm{ch}}([a,b],*)}, & x \in \Omega_{\mathrm{ch}}([a,b],*), \end{cases}$$

is a C^1-smooth Bellman candidate on its domain.

Proof. Condition (2.1.2) implies

$$\det\begin{pmatrix} g_1'(a) & g_2'(a) \\ g_1'(b) & g_2'(b) \end{pmatrix} \neq 0,$$

therefore (3.7.1) defines $B_{\mathfrak{L}}$ correctly. The cup equation (3.4.2) guarantees that the affine function $B_{\mathfrak{L}}$ coincides with the unique affine extension of $B_{\Omega_{\mathrm{ch}}([a,b],*)}$ to $\mathfrak{L}$. Therefore, it also coincides with the affine parts of the functions from (3.6.3) for both pairs of glued fences: for $\Omega(I_-)$ and Ω_{ch}, and for Ω_{ch} and $\Omega(I_+)$. Thus, Proposition 3.6.4 gives that the function B is a C^1-smooth Bellman candidate. ∎

Definition 3.7.3. The function $B_{\mathfrak{L}}$ defined by (3.7.1) in the linearity domain $\mathfrak{L}$ with two points on the fixed boundary is called a *standard candidate* there.

Remark 3.7.4. By (3.7.1) we have the following formula for the derivative of the affine function $B_{\mathfrak{L}}$:

$$\beta_2 = \frac{\partial B_{\mathfrak{L}}}{\partial x_2} = \frac{\det\begin{pmatrix} g_1'(a) & f'(a) \\ g_1'(b) & f'(b) \end{pmatrix}}{\det\begin{pmatrix} g_1'(a) & g_2'(a) \\ g_1'(b) & g_2'(b) \end{pmatrix}} = \frac{\kappa_3(b) - \kappa_3(a)}{\kappa_2(b) - \kappa_2(a)}. \tag{3.7.2}$$

Definition 3.7.5. Let $\mathfrak{L}$ be a domain of linearity with two points on the fixed boundary: $g(a), g(b) \in \mathfrak{L} \cap \partial_{\mathrm{fixed}}\Omega$, with $g(a) \prec g(b)$. Define the force function F on $\{a,b\}$ as follows:

$$F(a) = \frac{D_{\mathrm{L}}(a,b)}{g_1'(a)\kappa_2'(a)}, \qquad F(b) = \frac{D_{\mathrm{R}}(a,b)}{g_1'(b)\kappa_2'(b)}.$$

Let us note that a birdie can be regarded as a union of a trolleybus and an angle (there are two symmetric ways):

$$\begin{aligned} \Omega_{\mathrm{bird}}(a,b) &= \Omega_{\mathrm{tr,R}}(a,b) \uplus \Omega_{\mathrm{R}}(b,b) \uplus \Omega_{\mathrm{ang}}(b), \\ \Omega_{\mathrm{bird}}(a,b) &= \Omega_{\mathrm{ang}}(a) \uplus \Omega_{\mathrm{L}}(a,a) \uplus \Omega_{\mathrm{tr,L}}(a,b). \end{aligned} \tag{3.7.3}$$

Note that the right and the left sides of the equalities are equal as planar domains, provided we substitute $\cup$ for $\uplus$.

Remark 3.7.6. The symbol $\uplus$ in (3.7.3) means the following: if a function B on this domain is continuous and its restriction to each single subdomain of one side of the formula is a standard candidate, then this function B is a standard candidate for each subdomain of the other side of the formula. This easily follows from Proposition 3.6.4 and Proposition 3.7.2. Graphical representation of this equality is presented in Figure 3.14 (see Section 3.7.3 below).

It is convenient to introduce two more "linearity domains" for the purposes of formalization. First, sometimes we will treat a single chord $[g(a), g(b)]$, where (a, b) satisfies the cup equation (3.4.2), as a linearity domain $\mathfrak{L}$. The standard candidate B inside $[g(a), g(b)]$ is given by linearity. We note that Proposition 3.7.2 is valid for the following two "hidden" subcases. The first one is when the chord $[g(a), g(b)]$ is tangent to the free boundary of Ω, $\Omega(I_-) = \Omega_{\mathrm{L}}(I_-)$ is a left tangent domain, and $\Omega(I_+) = \Omega_{\mathrm{R}}(I_+)$ is a right tangent domain. We obtain the chordal domain with the glued tangent domains (see Figure 3.7).

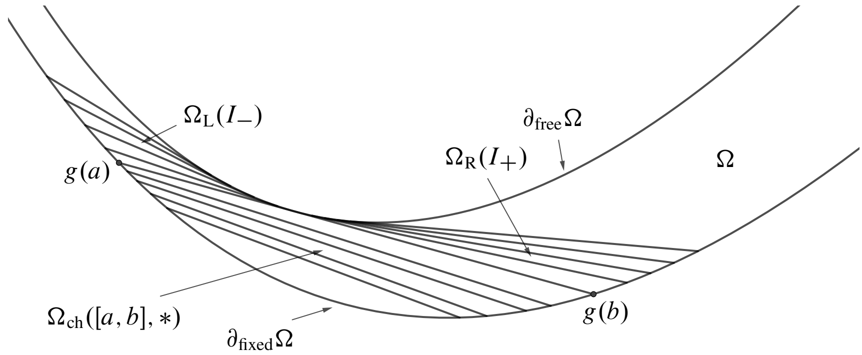

Figure 3.7. A chordal domain $\Omega_{\mathrm{ch}}([a, b], *)$ with tangent domains attached to it.

The second subcase is when the chord $[g(a), g(b)]$ is not tangent to the free boundary and $\Omega(I_\pm) = \Omega_{\mathrm{ch}}(*, [a, b])$ are the right and the left parametrization of the same chordal domain, which lies above the chord $[g(a), g(b)]$. In this subcase we obtain two glued chordal domains $\Omega_{\mathrm{ch}}(*, [a, b])$ and $\Omega_{\mathrm{ch}}([a, b], *)$, which will appear when it is impossible to consider their union as one chordal domain (if either $D_{\mathrm{R}}(a, b) = 0$ or $D_{\mathrm{L}}(a, b) = 0$), see Figure 3.8.

Second, sometimes it is useful to treat a single tangent $\Omega_{\mathrm{R}}(u, u)$ or $\Omega_{\mathrm{L}}(u, u)$ as a linearity domain $\mathfrak{L}$. Moreover, no matter how strange it seems, it is natural to think of it as a domain with two points on the fixed boundary. We treat this single tangent as a trolleybus of zero width, i.e., its base is the chord $[g(u), g(u)]$. This can be considered as a limit of a sequence of trolleybuses on the chord shrinking to

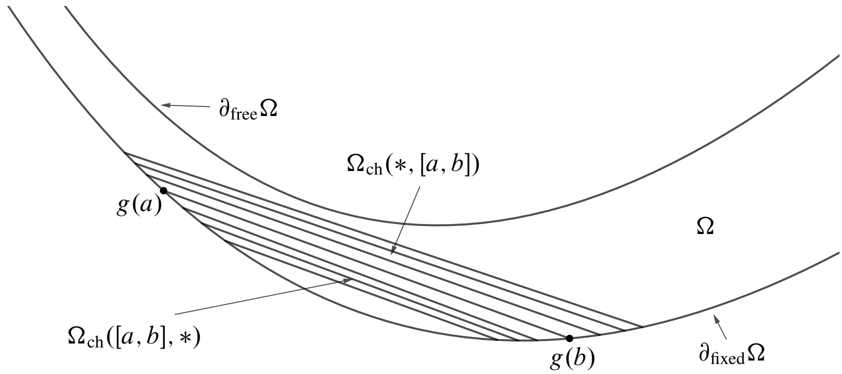

Figure 3.8. Two chordal domains $\Omega_{\mathrm{ch}}([a,b],*)$ and $\Omega_{\mathrm{ch}}(*,[a,b])$ with the chord between them.

the point u. It will appear only when $u = c_i$ for some i, where c_i is a single point root from Definition 2.1.13. The standard candidate B in this domain is obtained by passing to the limit in (3.7.1):

$$\det\begin{pmatrix} g_1'(u) & g_2'(u) & f'(u) \\ g_1''(u) & g_2''(u) & f''(u) \\ x_1 - g_1(u) & x_2 - g_2(u) & B_{\mathfrak{L}}(x) - f(u) \end{pmatrix} = 0, \quad x \in \mathfrak{L}. \tag{3.7.4}$$

This construction will appear between two tangent domains of the same direction when the standard candidates on them cannot be concatenated to be a standard candidate on the union of the domains. The obstacle of such a concatenation is the vanishing of the forces at the point u.

3.7.3 Graphical representation of the elementary domains

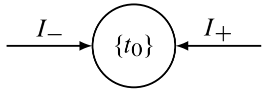

Figure 3.9. Graphical representation of an angle $\Omega_{\mathrm{ang}}(t_0)$ with adjacent tangent domains.

As it was said in Section 3.1, we give a graphical representation for a combinatorial structure of foliations. The material of this subsection essentially repeats analogous constructions in [17]. We start with the representation of the simplest local foliations: fences (tangent domains and chordal domains) and some linearity domains. In Section 3.8.2 we will give a graphical representation describing a foliation of the whole domain Ω_ε.

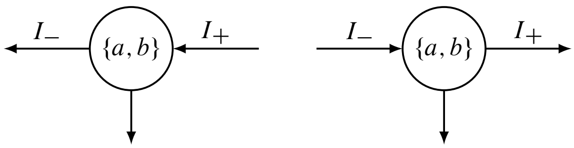

Figure 3.10. Graphical representation of trolleybuses $\Omega_{\mathrm{tr,L}}(a,b)$ and $\Omega_{\mathrm{tr,R}}(a,b)$ with adjacent tangent domains and a chordal domain.

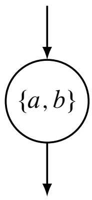

Figure 3.11. Graphical representation of two chordal domains $\Omega_{\mathrm{ch}}([a,b],*)$ and $\Omega_{\mathrm{ch}}(*,[a,b])$ glued along the chord.

Usually vertices will correspond to linearity domains, and oriented edges will always correspond to fences. We draw a vertex of the graph that corresponds to an angle, and two incoming edges that correspond to the neighbor tangent domains, see Figure 3.9. We equip the elements of the graph with numerical parameters corresponding to the points on the fixed boundary.

Now we give a graphical representation of possible variants described in Proposition 3.7.2 (trolleybus and birdie). We draw a vertex for the domain of linearity and edges for the fences – a chordal domain and two tangent domains. The edge corresponding to the chordal domain is outgoing. We will draw the edges corresponding to the tangent domains horizontally, and their directions agree with the directions of the tangents (either left or right). A trolleybus always has one incoming and two outgoing edges, and for a birdie, we have two incoming and one outgoing edge, see Figure 3.10 for the graphs corresponding to trolleybuses and the left drawing in Figure 3.12 for the birdie.

We give a graphical representation of a full chordal domain with two neighbor tangent domains, see the right drawing in Figure 3.12. The vertex corresponds to the "linearity domain", being the chord $[g(a), g(b)]$, and the edges correspond to the chordal domain and tangent domains. Figure 3.11 gives a graphical representation of two chordal domains $\Omega_{\mathrm{ch}}([a,b],*)$ and $\Omega_{\mathrm{ch}}(*,[a,b])$ glued along the chord (Figure 3.8). The vertex here corresponds to the chord $[g(a), g(b)]$, and the incoming and outgoing edges correspond to $\Omega_{\mathrm{ch}}(*,[a,b])$ and $\Omega_{\mathrm{ch}}([a,b],*)$, respectively.

Figure 3.13 gives a graphical representation of two tangent domains $\Omega_{\mathrm{L}}(t_-,u)$ and $\Omega_{\mathrm{L}}(u,t_+)$ glued along a zero-width "trolleybus" $\Omega_{\mathrm{L}}(u,u)$ (on the left), and of

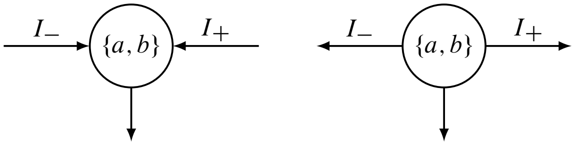

Figure 3.12. Graphical representation of a birdie $\Omega_{\mathrm{bird}}(a, b)$ and of a full chordal domain $\Omega_{\mathrm{ch}}([a, b], *)$ with adjacent domains.

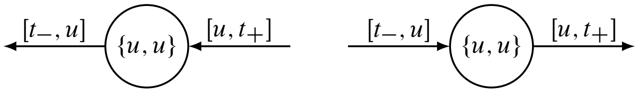

Figure 3.13. Graphical representation of two tangent domains glued along a "trolleybus" of zero width: the left and the right cases.

the symmetric case where $\Omega_{\mathrm{R}}(t_-, u)$ and $\Omega_{\mathrm{R}}(u, t_+)$ are glued along a zero-width "trolleybus" $\Omega_{\mathrm{R}}(u, u)$ (on the right). The vertex here corresponds to the "trolleybus" and the edges correspond to the tangent domains.

Finally, Figure 3.14 demonstrates a graphical representation of equality (3.7.3).

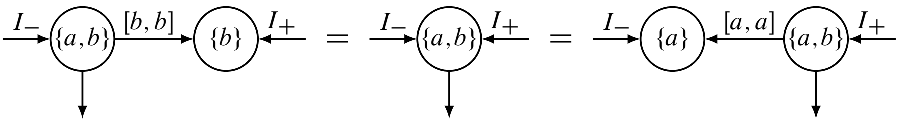

Figure 3.14. Graphical representation of the equality "birdie = angle + trolleybus".

3.7.4 Multifigures

We begin with a structural agreement. For each linearity domain $\mathfrak{L}$, we make the following finiteness assumption: the intersection $\mathfrak{L} \cap \partial_{\mathrm{fixed}}\,\Omega$ is assumed to be a union of a finite number of arcs (one or two of these arcs may be unbounded, i.e., parametrized by a ray),

$$\mathfrak{L} \cap \partial_{\mathrm{fixed}}\,\Omega = \bigcup_{i=1}^{k} \{g(t) \mid t \in \alpha_i\},$$

where $\{\alpha_i\}_{i=1}^{k}$ is a finite set of disjoint closed intervals, which can be single points. The curvilinear arc that corresponds to α_i is $g(\alpha_i)$. We remind the reader the notation introduced in Section 2.1.3: the left endpoint of α_i is α_i^{l} and the right endpoint is α_i^{r}. As we will see, due to Condition 2.1.11, all linearity domains under consideration satisfy this finiteness assumption.

Let $\mathfrak{L}$ be a linearity domain. We know that the set $\{\gamma(a) \mid g(a) \in \mathfrak{L} \cap \partial_{\mathrm{fixed}}\Omega\}$ lies in a two-dimensional plane in $\mathbb{R}^3$. Therefore, there exist a function $P_{\mathfrak{L}}$, which is a linear combination of g_1, g_2, and a constant function such that

$$f(a) = P_{\mathfrak{L}}(a), \quad g(a) \in \mathfrak{L} \cap \partial_{\mathrm{fixed}}\Omega. \tag{3.7.5}$$

Surely, the converse is also true: if there exist some linear combination $P_{\mathfrak{L}}$ of g_1, g_2, and a constant function such that (3.7.5) holds, then there exists an affine function $B_{\mathfrak{L}}$ such that $B_{\mathfrak{L}}(g(a)) = f(a)$ for all $g(a) \in \mathfrak{L} \cap \partial_{\mathrm{fixed}}\Omega$. Namely, if $P_{\mathfrak{L}} = \beta_0 + \beta_1 g_1 + \beta_2 g_2$, then

$$B_{\mathfrak{L}}(x_1, x_2) = \beta_0 + \beta_1 x_1 + \beta_2 x_2. \tag{3.7.6}$$

This function $B_{\mathfrak{L}}$ is a Bellman candidate in $\mathfrak{L}$.

Remark 3.7.7. Similar to the case where the linearity domain has only two points on the fixed boundary, identities (3.7.1) and (3.7.2) hold true for any $g(a), g(b) \in \mathfrak{L} \cap \partial_{\mathrm{fixed}}$. All the vectors $\gamma'(a)$ such that $g(a) \in \mathfrak{L} \cap \partial_{\mathrm{fixed}}\Omega$ lie in one plane orthogonal to the vector $(\beta_1, \beta_2, -1)$.

Definition 3.7.8. The function B defined by formula (3.7.1) in the linearity domain $\mathfrak{L}$, where $g(a)$ and $g(b)$ are arbitrary points from $\mathfrak{L} \cap \partial_{\mathrm{fixed}}\Omega$, is called the *standard candidate* in $\mathfrak{L}$.

As we have verified, the standard candidate in $\mathfrak{L}$ does not depend on the choice of a and b in the definition and coincides with the function given by (3.7.6).

In the following lemma, we use Definition 3.5.1.

Lemma 3.7.9. *Let a_1, a_2, and a_3 be such that $\gamma'(a_i)$, $i = 1, 2, 3$, lie in one plane. If $a_1 \leqslant a_2 \leqslant a_3$, then*

$$\begin{aligned} D_{\mathrm{L}}(a_1, a_2) &= D_{\mathrm{L}}(a_1, a_3),\\ D_{\mathrm{R}}(a_1, a_3) &= D_{\mathrm{R}}(a_2, a_3),\\ D_{\mathrm{R}}(a_1, a_2) &= D_{\mathrm{L}}(a_2, a_3). \end{aligned}$$

Proof. Let us prove the first identity; the others are similar. We find coefficients α_1 and α_2 such that $\gamma'(a_3) = \alpha_1\gamma'(a_1) + \alpha_2\gamma'(a_2)$. Then $g'(a_3) = \alpha_1 g'(a_1) + \alpha_2 g'(a_2)$. We substitute this to $D_{\mathrm{L}}(a_1, a_3)$ and obtain

$$D_{\mathrm{L}}(a_1, a_3) = \frac{\det\begin{pmatrix}\gamma'(a_1)\\ \gamma'(a_3)\\ \gamma''(a_1)\end{pmatrix}}{\det\begin{pmatrix}g'(a_1)\\ g'(a_3)\end{pmatrix}} = \frac{\alpha_2\det\begin{pmatrix}\gamma'(a_1)\\ \gamma'(a_2)\\ \gamma''(a_1)\end{pmatrix}}{\alpha_2\det\begin{pmatrix}g'(a_1)\\ g'(a_2)\end{pmatrix}} = D_{\mathrm{L}}(a_1, a_2). \qquad \blacksquare$$

Definition 3.7.10. Let $\mathfrak{L}$ be a linearity domain such that the intersection $\mathfrak{L} \cap \partial_{\mathrm{fixed}}\Omega$ contains at least two points; let $g(a) \in \mathfrak{L} \cap \partial_{\mathrm{fixed}}\Omega$. By analogy with Definition 3.7.5, we define the value of the force function F at a as

$$F(a) = \frac{D}{g_1'(a)\kappa_2'(a)},$$

where $D = D_{\mathrm{L}}(a,b)$ if $a < b$ and $D = D_{\mathrm{R}}(b,a)$ if $a > b$, and $b \neq a$ is any point such that $g(b) \in \mathfrak{L} \cap \partial_{\mathrm{fixed}}\Omega$.

Remark 3.7.11. The force function is well defined (does not depend on the choice of the point b) on the set $\{a \in \mathbb{R} \mid g(a) \in \mathfrak{L}\}$ due to Lemma 3.7.9.

Lemma 3.7.12. *Let a_1, a_2, and a_3 be such that $\gamma'(a_i)$, $i = 1,2,3$, lie in one plane. Suppose that the pairs (a_1,a_2) and (a_2,a_3) satisfy the cup equation. Then (a_1,a_3) satisfies the cup equation as well.*

Proof. The cup equations for the pairs (a_1,a_2) and (a_2,a_3) mean that the vectors $\gamma(a_2) - \gamma(a_1)$ and $\gamma(a_3) - \gamma(a_2)$ lie in the same plane as all the $\gamma'(a_i)$, hence $\gamma(a_3) - \gamma(a_1)$ also lies there. Thus, the cup equation holds for the pair (a_1,a_3). ∎

Now we are ready to describe all the remaining linearity domains. We start with the domains that are not separated from the free boundary of Ω. The boundary of such a domain, provided it is compact, consists of the arcs $g(\alpha_i)$, $i = 1,2,\ldots,k$, the chords $[g(\alpha_i^{\mathrm{r}}), g(\alpha_{i+1}^{\mathrm{l}})]$, $i = 1,2,\ldots,k-1$, two tangents $S(\alpha_1^{\mathrm{l}})$ and $S(\alpha_k^{\mathrm{r}})$, and the arc of the free boundary. We classify the multifigures with respect to the orientation of these tangents. Namely, if we have $S_{\mathrm{R}}(\alpha_1^{\mathrm{l}})$ and $S_{\mathrm{R}}(\alpha_k^{\mathrm{r}})$, then we get a *right multitrolleybus* denoted by $\Omega_{\mathrm{Mtr,R}}(\{\alpha_i\}_{i=1}^k)$, see Figure 3.15; and if we have $S_{\mathrm{L}}(\alpha_1^{\mathrm{l}})$ and $S_{\mathrm{L}}(\alpha_k^{\mathrm{r}})$, then we obtain a *left multitrolleybus* denoted by $\Omega_{\mathrm{Mtr,L}}(\{\alpha_i\}_{i=1}^k)$. If we have $S_{\mathrm{L}}(\alpha_1^{\mathrm{l}})$ and $S_{\mathrm{R}}(\alpha_k^{\mathrm{r}})$, then the linearity domain is called a *multicup* and is denoted by $\Omega_{\mathrm{Mcup}}(\{\alpha_i\}_{i=1}^k)$, see Figure 3.16.

We distinguish the case where the two border tangents $S_{\mathrm{L}}(\alpha_1^{\mathrm{l}})$ and $S_{\mathrm{R}}(\alpha_k^{\mathrm{r}})$ lie on one line and say that in this case the multicup is *full*. Finally, if we have $S_{\mathrm{R}}(\alpha_1^{\mathrm{l}})$ and $S_{\mathrm{L}}(\alpha_k^{\mathrm{r}})$, then the domain of linearity $\mathfrak{L}$ is called a *multibirdie* and is denoted by $\Omega_{\mathrm{Mbird}}(\{\alpha_i\}_{i=1}^k)$, see Figure 3.17. Graphical representation for a multifigure $\mathfrak{L}$ built over $\{\alpha_i\}_{i=1}^k$ is drawn by the following rule. The domain $\mathfrak{L}$ corresponds to a single vertex, and it has $k-1$ outgoing edges representing the chordal domains $\Omega_{\mathrm{ch}}([\alpha_i^{\mathrm{r}}, \alpha_{i+1}^{\mathrm{l}}], *)$, $i = 1,2,\ldots,k-1$. There are two more edges corresponding to two tangent domains surrounding $\mathfrak{L}$. They are both outgoing if $\mathfrak{L}$ is a multicup and both incoming in the case where $\mathfrak{L}$ is a multibirdie. If $\mathfrak{L}$ is a multitrolleybus, then it has one incoming and one outgoing edge. We provide examples of graphs for the multifigures drawn in Figures 3.15, 3.16, and 3.17.

We will also consider three types of unbounded domains of linearity: a multicup $\Omega_{\mathrm{Mcup}}(\{\alpha_i\}_{i=1}^k)$ with at least one of the intervals α_1 or α_k being a ray, a right multitrolleybus $\Omega_{\mathrm{Mtr,R}}(\{\alpha_i\}_{i=1}^k)$, where α_k is a ray that lasts to $+\infty$, or a left multitrolleybus $\Omega_{\mathrm{Mtr,L}}(\{\alpha_i\}_{i=1}^k)$, where α_1 is a ray that lasts to $-\infty$. Such domains do not

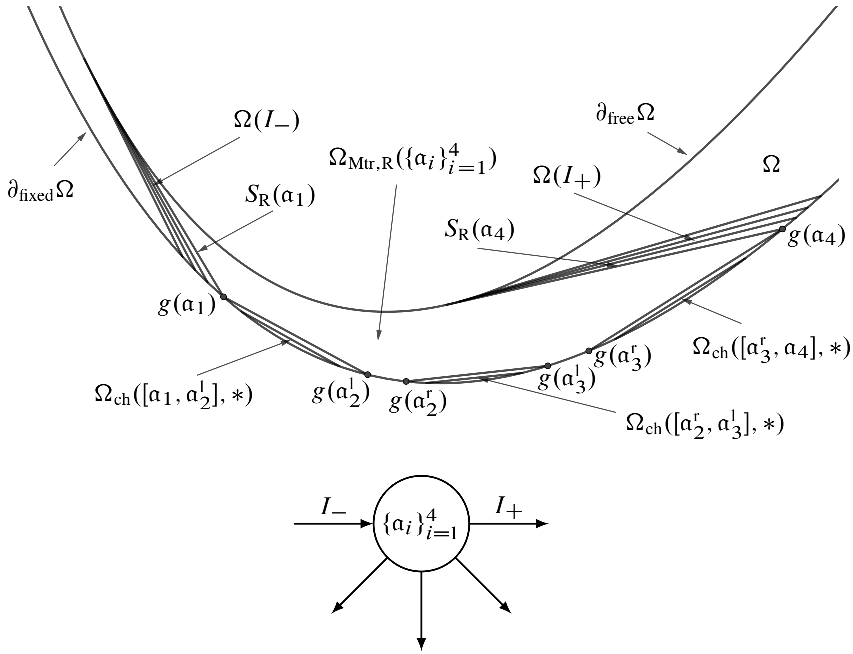

Figure 3.15. A right multitrolleybus for $k = 4$ with adjacent domains and its graphical representation.

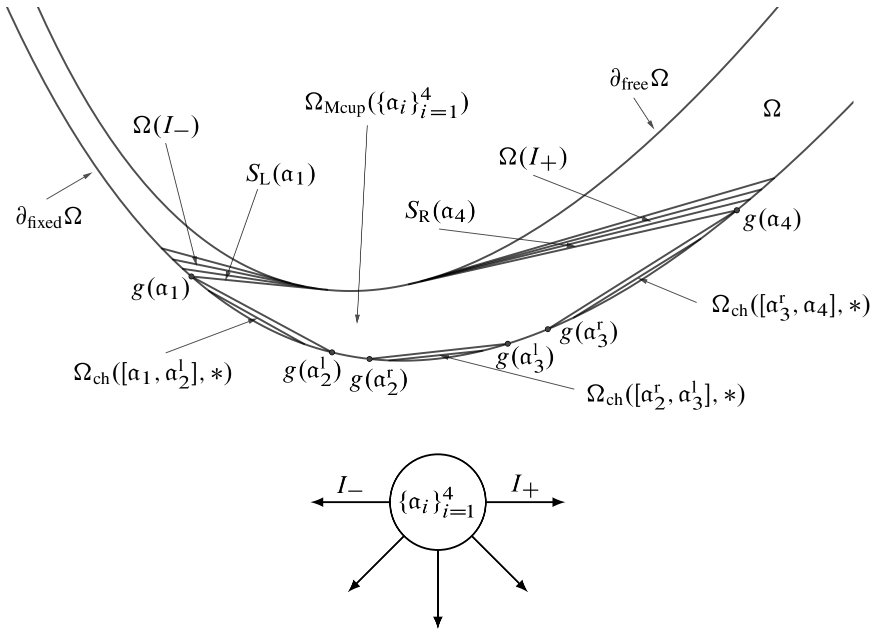

Figure 3.16. A mutlicup for $k = 4$ with adjacent domains and its graphical representation.

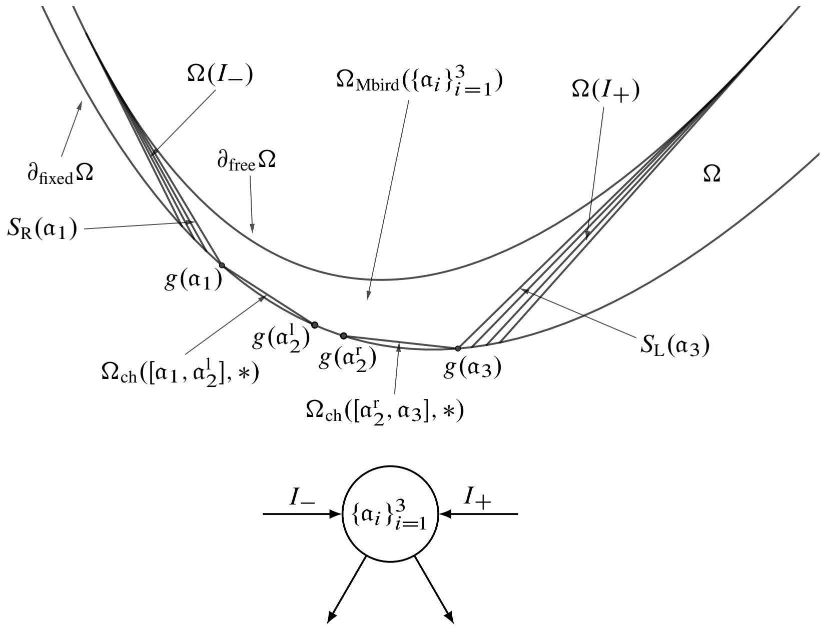

Figure 3.17. A multibirdie for $k = 3$ with adjacent domains and its graphical representation.

have one of border tangents. In such a case, the vertex representing this domain does not have the corresponding edge.

Now, consider the case of a linearity domain $\mathfrak{L}$ that is separated from the free boundary. The boundary of $\mathfrak{L}$ consists of the arcs $g(\alpha_i)$, $i = 1, 2, \ldots, k$, the chords $[g(\alpha_i^{\mathrm{r}}), g(\alpha_{i+1}^{\mathrm{l}})]$, $i = 1, 2, \ldots, k-1$, and the chord $[g(\alpha_1^l), g(\alpha_k^r)]$. Such a construction is called a *closed multicup*, denoted by $\Omega_{\mathrm{ClMcup}}(\{\alpha_i\}_{i=1}^k)$. It is represented graphically in the following way. It has one incoming edge representing $\Omega_{\mathrm{ch}}(*, [\alpha_1^{\mathrm{l}}, \alpha_k^{\mathrm{r}}])$ and several outgoing edges corresponding to the chordal domains $\Omega_{\mathrm{ch}}([\alpha_i^{\mathrm{r}}, \alpha_{i+1}^{\mathrm{l}}], *)$, $i = 1, 2, \ldots, k-1$. For example, it may look like the one in Figure 3.18.

The following proposition gives sufficient conditions for concatenation of a linearity domain with the surrounding fences (tangent domains and chordal domains). Formally, it is more general than Proposition 3.7.2, but the proof is similar.

Proposition 3.7.13. *Let $\mathfrak{L}$ be a domain of linearity surrounded by several fences. Suppose that the function B is a standard candidate on $\mathfrak{L}$ and on each surrounding fence. Let*

$$a = \inf\{t \in \mathbb{R} \mid g(t) \in \mathfrak{L}\} \quad \textit{and} \quad b = \sup\{t \in \mathbb{R} \mid g(t) \in \mathfrak{L}\}.$$

If the force function is continuous at a and b, then B is a C^1-smooth Bellman candidate on its domain.

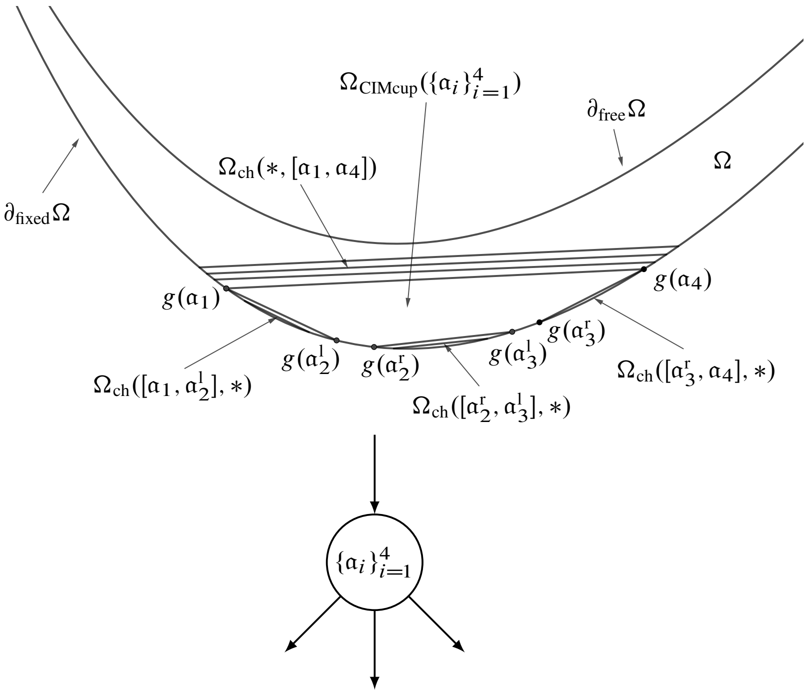

Figure 3.18. An example of the graph for a closed multicup with adjacent chordal domains.

Remark 3.7.14. In the previous proposition, if $\mathfrak{L}$ is a closed multicup, then it is surrounded by the chordal domains, and the condition of continuity of the force function always takes place.

Now is the time to define tails of a linearity domain $\mathfrak{L}$ that contains at least two points on the lower boundary. As in Proposition 3.7.13, we define $a = \inf\{t \in \mathbb{R} \mid g(t) \in \mathfrak{L}\}$ and $b = \sup\{t \in \mathbb{R} \mid g(t) \in \mathfrak{L}\}$.

Definition 3.7.15. We define the tails and the forces of a linearity domain $\mathfrak{L}$ as the tails and the forces for the chord $[g(a), g(b)]$ (which possibly does not lie in Ω). Namely, see Definitions 3.6.6, 3.6.9 for the tails and formulas (3.6.4), (3.6.5) for the corresponding forces.

3.8 Combinatorial properties of foliations

The material of this section essentially repeats that of [17, Section 3.5]. The reason for this repetition is a slight change of the notation we are forced to do since some natural parameters used in [17] do not exist in our general setting.

3.8.1 Gluing composite figures

In this subsection we present several formulas that allow us to consider a part of the foliation as a union of elementary domains in different ways. An example has already been given in (3.7.3) (see also the description of the symbol $\uplus$ in Remark 3.7.6).

We start with the formula which describes gluing of an angle $\Omega_{\mathrm{ang}}(a)$ with a long chord $[g(a), g(b)]$ (see Figure 3.19). Their union forms a trolleybus $\Omega_{\mathrm{tr}}(a, b)$:

$$\Omega_{\mathrm{ang}}(a) \uplus \Omega_{\mathrm{L}}(a, a) \uplus [g(a), g(b)] = \Omega_{\mathrm{tr,R}}(a, b). \tag{3.8.1}$$

Similarly,

$$[g(a), g(b)] \uplus \Omega_{\mathrm{R}}(b, b) \uplus \Omega_{\mathrm{ang}}(b) = \Omega_{\mathrm{tr,L}}(a, b).$$

Both these formulas can be informally named as "angle + long chord = trolleybus".

We have already considered an example of the more complicated formula (3.7.3):

$$\begin{aligned}\Omega_{\mathrm{tr,R}}(a, b) \uplus \Omega_{\mathrm{R}}(b, b) \uplus \Omega_{\mathrm{ang}}(b) &= \Omega_{\mathrm{bird}}(a, b),\\ \Omega_{\mathrm{ang}}(a) \uplus \Omega_{\mathrm{L}}(a, a) \uplus \Omega_{\mathrm{tr,L}}(a, b) &= \Omega_{\mathrm{bird}}(a, b),\end{aligned} \tag{3.8.2}$$

which can be informally named as "birdie = angle + trolleybus".

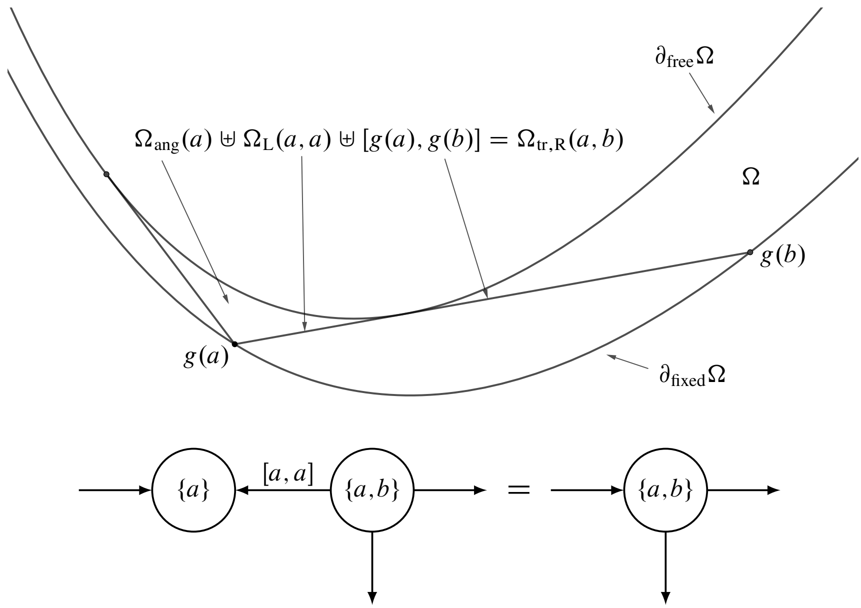

Figure 3.19. A graphical representation of formula (3.8.1).

We provide similar formulas for other domains. We leave their verification to the reader.

Angle + multicup = multitrolleybus. We have

$$\Omega_{\mathrm{ang}}(\alpha_1^{\mathrm{l}}) \uplus \Omega_{\mathrm{L}}(\alpha_1^{\mathrm{l}}, \alpha_1^{\mathrm{l}}) \uplus \Omega_{\mathrm{Mcup}}(\{\alpha_i\}_{i=1}^{k}) = \Omega_{\mathrm{Mtr,R}}(\{\alpha_i\}_{i=1}^{k}), \tag{3.8.3}$$

$$\Omega_{\mathrm{Mcup}}(\{\alpha_i\}_{i=1}^{k}) \uplus \Omega_{\mathrm{R}}(\alpha_k^{\mathrm{r}}, \alpha_k^{\mathrm{r}}) \uplus \Omega_{\mathrm{ang}}(\alpha_k^{\mathrm{r}}) = \Omega_{\mathrm{Mtr,L}}(\{\alpha_i\}_{i=1}^{k}). \tag{3.8.4}$$

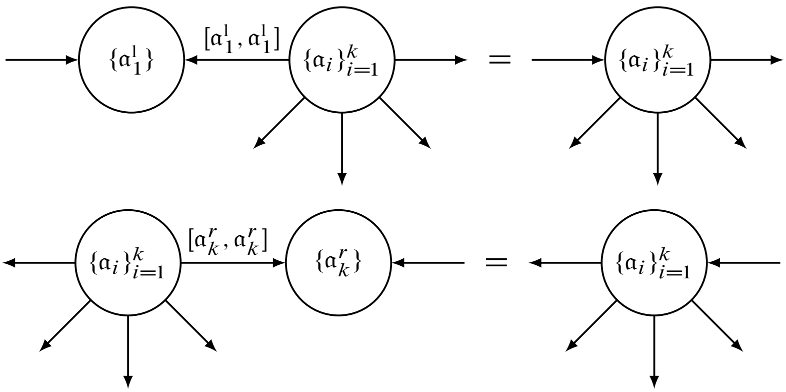

Figure 3.20. The graphs for formulas (3.8.3) and (3.8.4).

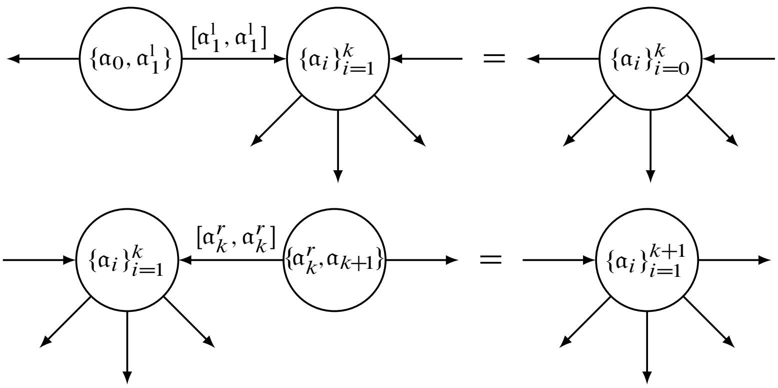

Figure 3.21. The graphs for formulas (3.8.5) and (3.8.6).

Long chord + multibirdie = multitrolleybus. We have

$$[g(a_0), g(\alpha_1^{\mathrm{l}})] \uplus \Omega_{\mathrm{R}}(\alpha_1^{\mathrm{l}}, \alpha_1^{\mathrm{l}}) \uplus \Omega_{\mathrm{Mbird}}(\{\alpha_i\}_{i=1}^{k}) = \Omega_{\mathrm{Mtr,L}}(\{a_0\} \cup \{\alpha_i\}_{i=1}^{k}), \tag{3.8.5}$$

where $[g(a_0), g(\alpha_1^{\mathrm{l}})]$ is a long chord, and

$$\begin{aligned} &\Omega_{\mathrm{Mbird}}(\{\alpha_i\}_{i=1}^{k}) \uplus \Omega_{\mathrm{L}}(\alpha_k^{\mathrm{r}}, \alpha_k^{\mathrm{r}}) \uplus [g(\alpha_k^{\mathrm{r}}), g(a_{k+1})] \\ &\quad = \Omega_{\mathrm{Mtr,R}}(\{\alpha_i\}_{i=1}^{k} \cup \{a_{k+1}\}), \end{aligned} \tag{3.8.6}$$

where $[g(\alpha_k^{\mathrm{r}}), g(a_{k+1})]$ is a long chord.

Angle + multitrolleybus = multibirdie. We have

$$\Omega_{\mathrm{Mtr,R}}(\{\alpha_i\}_{i=1}^k) \uplus \Omega_{\mathrm{R}}(\alpha^{\mathrm{r}}_k, \alpha^{\mathrm{r}}_k) \uplus \Omega_{\mathrm{ang}}(\alpha^{\mathrm{r}}_k) = \Omega_{\mathrm{Mbird}}(\{\alpha_i\}_{i=1}^k), \tag{3.8.7}$$

$$\Omega_{\mathrm{ang}}(\alpha^{\mathrm{l}}_1) \uplus \Omega_{\mathrm{L}}(\alpha^{\mathrm{l}}_1, \alpha^{\mathrm{l}}_1) \uplus \Omega_{\mathrm{Mtr,L}}(\{\alpha_i\}_{i=1}^k) = \Omega_{\mathrm{Mbird}}(\{\alpha_i\}_{i=1}^k). \tag{3.8.8}$$

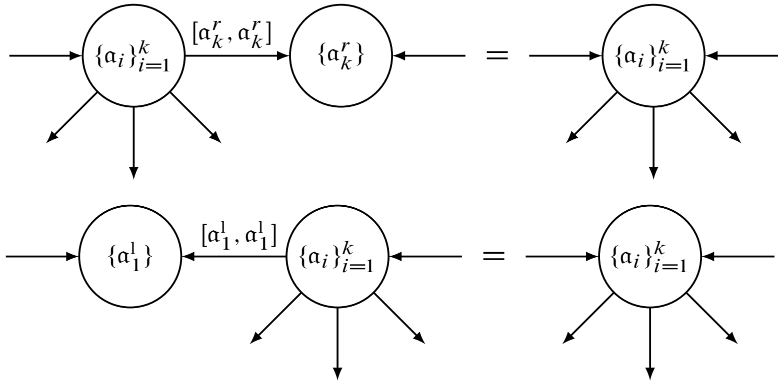

Figure 3.22. The graphs for formulas (3.8.7) and (3.8.8).

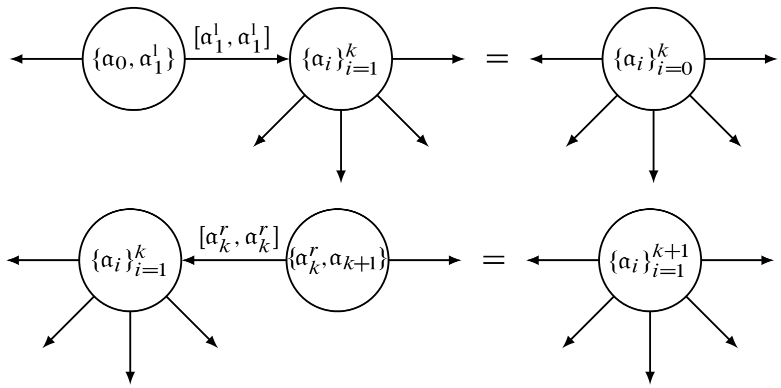

Figure 3.23. The graphs for formulas (3.8.9) and (3.8.10).

Long chord + multitrolleybus = multicup. We have

$$[g(a_0), g(\alpha^{\mathrm{l}}_1)] \uplus \Omega_{\mathrm{R}}(\alpha^{\mathrm{l}}_1, \alpha^{\mathrm{l}}_1) \uplus \Omega_{\mathrm{Mtr,R}}(\{\alpha_i\}_{i=1}^k) = \Omega_{\mathrm{Mcup}}(\{a_0\} \cup \{\alpha_i\}_{i=1}^k), \tag{3.8.9}$$

where $[g(a_0), g(\alpha^{\mathrm{l}}_1)]$ is a long chord, and

$$\begin{aligned}&\Omega_{\mathrm{Mtr,L}}(\{\alpha_i\}_{i=1}^k) \uplus \Omega_{\mathrm{L}}(\alpha^{\mathrm{r}}_k, \alpha^{\mathrm{r}}_k) \uplus [g(\alpha^{\mathrm{r}}_k), g(a_{k+1})]\\ &\quad = \Omega_{\mathrm{Mcup}}(\{\alpha_i\}_{i=1}^k \cup \{a_{k+1}\}),\end{aligned} \tag{3.8.10}$$

where $[g(\alpha^{\mathrm{r}}_k), g(a_{k+1})]$ is a long chord.

Multitrolleybus = trolleybus parade. We have

$$\Omega_{\mathrm{Mtr,R}}(\{\mathfrak{a}_i\}_{i=1}^k) = \Big(\biguplus_{i=1}^{k} \Omega_{\mathrm{Mtr,R}}(\{\mathfrak{a}_i\})\Big) \uplus \Big(\biguplus_{i=1}^{k-1} \big(\Omega_{\mathrm{R}}(\mathfrak{a}_i^{\mathrm{r}},\mathfrak{a}_i^{\mathrm{r}}) \uplus \Omega_{\mathrm{tr,R}}(\mathfrak{a}_i^{\mathrm{r}},\mathfrak{a}_{i+1}^{\mathrm{l}}) \uplus \Omega_{\mathrm{R}}(\mathfrak{a}_{i+1}^{\mathrm{l}},\mathfrak{a}_{i+1}^{\mathrm{l}})\big)\Big) \quad (3.8.11)$$

$$\Omega_{\mathrm{Mtr,L}}(\{\mathfrak{a}_i\}_{i=1}^k) = \Big(\biguplus_{i=1}^{k} \Omega_{\mathrm{Mtr,L}}(\{\mathfrak{a}_i\})\Big) \uplus \Big(\biguplus_{i=1}^{k-1} \big(\Omega_{\mathrm{L}}(\mathfrak{a}_i^{\mathrm{r}},\mathfrak{a}_i^{\mathrm{r}}) \uplus \Omega_{\mathrm{tr,L}}(\mathfrak{a}_i^{\mathrm{r}},\mathfrak{a}_{i+1}^{\mathrm{l}}) \uplus \Omega_{\mathrm{L}}(\mathfrak{a}_{i+1}^{\mathrm{l}},\mathfrak{a}_{i+1}^{\mathrm{l}})\big)\Big). \quad (3.8.12)$$

In (3.8.11), if some $\mathfrak{a}_i$ is a single point, then we omit the degenerate tangent domains $\Omega_{\mathrm{R}}(\mathfrak{a}_i^{\mathrm{l}},\mathfrak{a}_i^{\mathrm{l}})$ and $\Omega_{\mathrm{R}}(\mathfrak{a}_i^{\mathrm{r}},\mathfrak{a}_i^{\mathrm{r}})$ and replace the multitrolleybus $\Omega_{\mathrm{Mtr,R}}(\{\mathfrak{a}_i\})$ by the degenerate tangent domain $\Omega_{\mathrm{R}}(\mathfrak{a}_i,\mathfrak{a}_i)$. We perform similar replacements of the left tangent domains and multitrolleybuses in (3.8.12).

Multibirdie = right multitrolleybus + angle + left multitrolleybus. We have

$$\Omega_{\mathrm{Mbird}}(\{\mathfrak{a}_i\}_{i=1}^k) = \Omega_{\mathrm{Mtr,R}}(\{\mathfrak{a}_i\}_{i=1}^{j-1} \cup \{\mathfrak{a}_j^{\mathrm{l}}\}) \uplus \Omega_{\mathrm{R}}(\mathfrak{a}_j^{\mathrm{l}},\mathfrak{a}_j^{\mathrm{l}}) \uplus \Omega_{\mathrm{Mbird}}(\{\mathfrak{a}_j\}) \uplus \Omega_{\mathrm{L}}(\mathfrak{a}_j^{\mathrm{r}},\mathfrak{a}_j^{\mathrm{r}}) \uplus \Omega_{\mathrm{Mtr,L}}(\{\mathfrak{a}_j^{\mathrm{r}}\} \cup \{\mathfrak{a}_i\}_{i=j+1}^k). \quad (3.8.13)$$

Here j is an arbitrary number, $j = 1, 2, \ldots, k$ (see a graphical example at Figure 3.24). Both multitrolleybuses can be disintegrated according to equations (3.8.11) and (3.8.12). If $\mathfrak{a}_j$ is a single point, then one should change $\Omega_{\mathrm{Mbird}}(\{\mathfrak{a}_j\})$ for $\Omega_{\mathrm{ang}}(\mathfrak{a}_j)$. If $\mathfrak{a}_j$ is a solid root, then $\Omega_{\mathrm{Mbird}}(\{\mathfrak{a}_j\})$ can be further disintegrated in two ways according to (3.8.8) and (3.8.7).

Closed multicup + trolleybus = multitrolleybus. We have

$$\Omega_{\mathrm{ClMcup}}(\{\mathfrak{a}_{\mathrm{i}}\}_{i=1}^k) \uplus \Omega_{\mathrm{ch}}([\mathfrak{a}_1^{\mathrm{l}},\mathfrak{a}_k^{\mathrm{r}}],[\mathfrak{a}_1^{\mathrm{l}},\mathfrak{a}_k^{\mathrm{r}}]) \uplus \Omega_{\mathrm{tr,R}}(\mathfrak{a}_1^{\mathrm{l}},\mathfrak{a}_k^{\mathrm{r}}) = \Omega_{\mathrm{Mtr,R}}(\{\mathfrak{a}_{\mathrm{i}}\}_{i=1}^k),$$
$$\Omega_{\mathrm{ClMcup}}(\{\mathfrak{a}_{\mathrm{i}}\}_{i=1}^k) \uplus \Omega_{\mathrm{ch}}([\mathfrak{a}_1^{\mathrm{l}},\mathfrak{a}_k^{\mathrm{r}}],[\mathfrak{a}_1^{\mathrm{l}},\mathfrak{a}_k^{\mathrm{r}}]) \uplus \Omega_{\mathrm{tr,L}}(\mathfrak{a}_1^{\mathrm{l}},\mathfrak{a}_k^{\mathrm{r}}) = \Omega_{\mathrm{Mtr,L}}(\{\mathfrak{a}_{\mathrm{i}}\}_{i=1}^k).$$

Closed multicup + birdie = multibirdie. We have

$$\Omega_{\mathrm{ClMcup}}(\{\mathfrak{a}_{\mathrm{i}}\}_{i=1}^k) \uplus \Omega_{\mathrm{ch}}([\mathfrak{a}_1^{\mathrm{l}},\mathfrak{a}_k^{\mathrm{r}}],[\mathfrak{a}_1^{\mathrm{l}},\mathfrak{a}_k^{\mathrm{r}}]) \uplus \Omega_{\mathrm{bird}}(\mathfrak{a}_1^{\mathrm{l}},\mathfrak{a}_k^{\mathrm{r}}) = \Omega_{\mathrm{Mbird}}(\{\mathfrak{a}_{\mathrm{i}}\}_{i=1}^k).$$

3.8.2 General foliations

It is natural to draw a special graph Γ corresponding to a foliation to describe its combinatorial properties. The vertices correspond to the linearity domains. Two vertices are joined with an edge if there is a fence that is their common neighbor. Such a graph

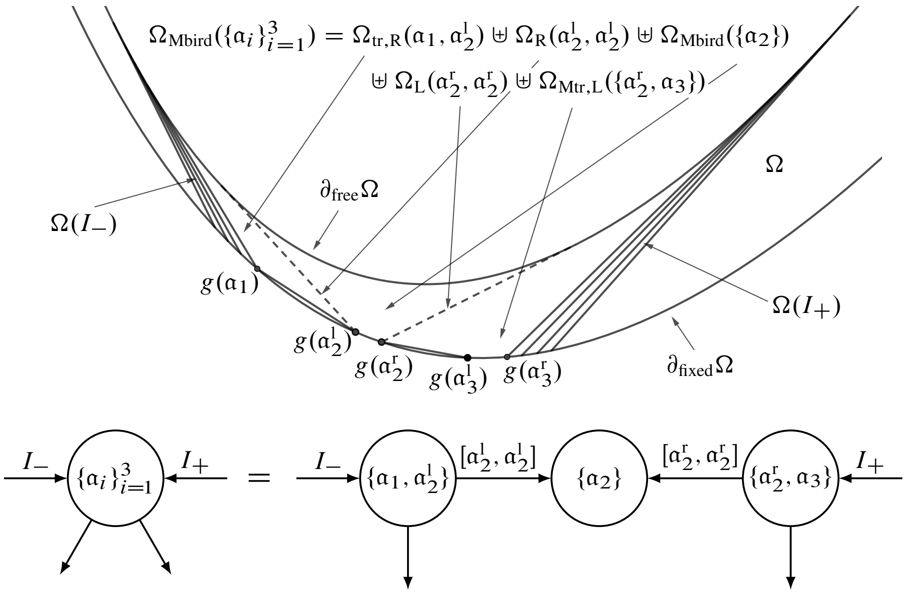

Figure 3.24. An example for formula (3.8.13) and its graphical representation.

is drawn in the plane by the mapping $\nabla B\colon \Omega_\varepsilon \to \mathbb{R}^2$. However, we need to clarify some details.

We will use a small amount of graph theory terminology. Since we study very special graphs, the use of the terminology will also be special. Our graphs are oriented trees (i.e., trees whose edges possess orientation). We call a vertex that does not have incoming edges a root, and a vertex that does not have outgoing edges a leaf (a leaf may have several incoming edges). By a path we call an oriented path, i.e., we move from the beginning of the edge to its end while exploring the path. Other terminology is clear.

The vertices of the graph will be denoted by $\{\mathfrak{L}_i\}_i$, the edges will be denoted by $\{\mathfrak{E}_i\}_i$. Edges and vertices are of different types, moreover, they are also equipped with numerical parameters to be specified later. We begin the description with the edges.

Each edge $\mathfrak{E}$ represents either a chordal domain $\Omega_{\mathrm{ch}}([a^{\mathrm{top}}, b^{\mathrm{top}}], [a^{\mathrm{bot}}, b^{\mathrm{bot}}])$ or one of the tangent domains: either $\Omega_{\mathrm{R}}(u_{\mathrm{l}}, u_{\mathrm{r}})$ or $\Omega_{\mathrm{L}}(u_{\mathrm{l}}, u_{\mathrm{r}})$. The edge representing a chordal domain $\Omega_{\mathrm{ch}}([a^{\mathrm{top}}, b^{\mathrm{top}}], [a^{\mathrm{bot}}, b^{\mathrm{bot}}])$ is oriented from its upper neighbor to its lower neighbor. We consider the functions a and b associated with a chordal domain as its numeric parameters. The edge of $\Omega_{\mathrm{R}}(u_{\mathrm{l}}, u_{\mathrm{r}})$ is oriented from the vertex of its left neighbor to the vertex of its right neighbor. The edge representing $\Omega_{\mathrm{L}}(u_{\mathrm{l}}, u_{\mathrm{r}})$ is oriented symmetrically. The closed interval $[u_{\mathrm{l}}, u_{\mathrm{r}}]$ is the numerical parameter of such an edge.

The vertices correspond to the linearity domains. For angles, trolleybuses, birdies, and multifigures, the graphical representation was given in the subsections where they were introduced (Sections 3.7.3 and 3.7.4). These vertices are of their individual types (i.e., there are several vertices of the type "angle" in the graph, several vertices of the type "birdie", etc.). Each such vertex is equipped with its numerical characteristics that are the values of the parameters corresponding to its points on the fixed boundary $\partial_{\mathrm{fixed}}\Omega$. For example, a vertex of the type "angle" has one numerical parameter u of the point $g(u)$ the angle is sitting on, whereas the collection of the intervals $\{\alpha_i\}_{i=1}^{k}$ plays the role of the numerical parameter for a vertex that has the type "multicup", or "multitrolleybus", or "multibirdie".

However, we also need some fictitious vertices, which do not correspond to any linearity domain of nonzero area. For example, in Figure 3.12 the vertex representing the long chord is fictitious. There will be five types of such vertices.

First, there will be some $\mathfrak{L}_i$ that correspond to long chords (the chords that are tangent to the free boundary of Ω_ε). Namely, suppose that we have a full chordal domain $\Omega_{\mathrm{ch}}([a^{\mathrm{top}}, b^{\mathrm{top}}], *)$ such that $D_{\mathrm{R}}(a^{\mathrm{top}}, b^{\mathrm{top}}) \neq 0$ and $D_{\mathrm{L}}(a^{\mathrm{top}}, b^{\mathrm{top}}) \neq 0$, and two tangent domains $\Omega_{\mathrm{R}}(b^{\mathrm{top}}, u_2)$ and $\Omega_{\mathrm{L}}(u_1, a^{\mathrm{top}})$. Then the vertex $\mathfrak{L}$ corresponding to the chord $[g(a^{\mathrm{top}}), g(b^{\mathrm{top}})]$ has three outgoing edges representing $\Omega_{\mathrm{ch}}([a^{\mathrm{top}}, b^{\mathrm{top}}], *)$, $\Omega_{\mathrm{R}}(b^{\mathrm{top}}, u_2)$, and $\Omega_{\mathrm{L}}(u_1, a^{\mathrm{top}})$. The set $\{a^{\mathrm{top}}, b^{\mathrm{top}}\}$ is the numerical parameter for $\mathfrak{L}$. The example is given in Figure 3.12.

Second, there will be some vertices $\mathfrak{L}_i$ that correspond to points of the fixed boundary. Suppose we have a chordal domain $\Omega_{\mathrm{ch}}(*, [a^{\mathrm{bot}}, b^{\mathrm{bot}}])$ with $a^{\mathrm{bot}} = b^{\mathrm{bot}}$ (we recall that such chordal domains are called cups). In our foliations, all such points will coincide with some c_j from Definition 2.1.13, see explanation around (3.4.11). Then the vertex $\mathfrak{L}$ corresponding to $a^{\mathrm{bot}} = b^{\mathrm{bot}} = c_j$ has one incoming edge matching $\Omega_{\mathrm{ch}}(*, [a^{\mathrm{bot}}, b^{\mathrm{bot}}])$ and one numerical parameter that equals c_j.

Third, sometimes we will need to paste a chord between two chordal domains (this will be done when one of the differentials vanishes, see Definition 3.5.1). Suppose we have two chordal domains $\Omega_{\mathrm{ch}}([a_1, b_1], [a_2, b_2])$ and $\Omega_{\mathrm{ch}}([a_2, b_2], [a_3, b_3])$. In such a case, we paste a vertex $\mathfrak{L}$ that corresponds to the chord $[g(a_2), g(b_2)]$, see Figure 3.11. It has one incoming edge and one outgoing edge and the numerical parameter $\{a_2, b_2\}$. Long chords, one or both differentials of which vanish, are also considered as fictitious vertices of the third type.

Fourth, there might be one or two vertices at infinity. If we have a tangent domain $\Omega(u_{\mathrm{l}}, u_{\mathrm{r}})$ with $u_{\mathrm{l}} = -\infty$, then there is a vertex $\mathfrak{L}$ that corresponds to $-\infty$. It has the numerical parameter $-\infty$ and one edge representing $\Omega(-\infty, u_{\mathrm{r}})$ that is outgoing for the case of Ω_{R} and incoming for the case of Ω_{L}. Similarly, if we have $u_{\mathrm{r}} = +\infty$, then we have a vertex $\mathfrak{L}$ that corresponds to $+\infty$ with the numerical parameter $+\infty$ and one edge representing $\Omega(u_{\mathrm{l}}, +\infty)$ that is incoming for the case of Ω_{R} and outgoing for the case of Ω_{L}. If $u_{\mathrm{l}} = -\infty$ and $u_{\mathrm{r}} = +\infty$ simultaneously, then we have both such vertices and one edge between them.

Fifth, there might be a vertex corresponding to a single tangent. Suppose we have a tangent domain $\Omega(u_1, u_2)$ such that $\mathcal{F} < 0$ on $[u_1, u_2]$ except for some point u, where $\mathcal{F}$ equals zero.[1] For the case of right tangent domain, it is useful to decompose $\Omega_{\mathrm{R}}(u_1, u_2)$ as

$$\Omega_{\mathrm{R}}(u_1, u) \uplus \Omega_{\mathrm{R}}(u, u) \uplus \Omega_{\mathrm{R}}(u, u_2)$$

and paste a vertex representing $\Omega_{\mathrm{R}}(u, u)$ (alternatively, one can consider it as a multitrolleybus on a single point). It has one outgoing edge $\Omega_{\mathrm{R}}(u, u_2)$ and one incoming edge $\Omega_{\mathrm{R}}(u_1, u)$, see Figure 3.13. Its numerical parameter is u. The same can be done for the case of left tangents. We note that the fictitious vertices of the fifth type may be right and left (the same as the trolleybuses).

The rules listed above define the graph of the foliation. However, we provide a further description to make its structure more transparent. It is useful to introduce a partial ordering on the set of linearity domains.

Definition 3.8.1. Let $\mathfrak{L}_1$ and $\mathfrak{L}_2$ be two linearity domains. We say that $\mathfrak{L}_2$ is subordinate to $\mathfrak{L}_1$ and write $\mathfrak{L}_2 \prec \mathfrak{L}_1$ if $\mathfrak{L}_1$ separates $\mathfrak{L}_2$ from the free boundary.

Note that if $\mathfrak{L}_2 \prec \mathfrak{L}_1$, then $\mathfrak{L}_2$ is a closed multicup. We can also let $\mathfrak{L}_1$ and $\mathfrak{L}_2$ be chords, and let $\mathfrak{L}_2$ be a point on the fixed boundary. Another point to note is that the numerical parameters of the vertices $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are sufficient to define whether the statement $\mathfrak{L}_2 \prec \mathfrak{L}_1$ is true.

We explain how to construct the graph from a foliation. Our graph is a tree if we disregard the orientation. First, we describe its subgraph Γ^{free} spanned by the edges representing tangent domains. This subgraph describes the trace of the foliation on the free boundary. Formally, we can define Γ^{free} to be the set of vertices that are not subordinated by any other vertex, and the edges between them. If we forget the orientation of edges, Γ^{free} is a path, i.e., a tree whose vertices have degree two, except, possibly for two leaves at infinity. The leaves are usually fictitious vertices of the fourth type, however, if there is a multicup or a multitrolleybus that lasts to infinity, its vertex is a leaf (in such a case there is no fictitious vertex representing the corresponding infinity). The orientation of edges has already been described. We only say that the roots of Γ^{free} are the fictitious vertices of the first and third type (the latter, of course, should belong to Γ^{free}, i.e., represent a long chord), the vertices that correspond to multicups, and possibly, the vertices at infinity. The leaves in Γ^{free} correspond to angles, birdies, multibirdies, and possibly, vertices at infinities. The necessary and sufficient condition for Γ^{free} to be a subgraph spanned by the edges corresponding to tangent domains of some foliation is that the foliation reconstructed from it covers the free boundary without intersections.

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[1] In such a situation, $u = c_j$ for some root c_j (see Definition 2.1.13).

Second, we describe the graph Γ^{fixed} spanned by the edges that correspond to chordal domains. The graph Γ^{fixed} is a forest (i.e., a finite collection of trees). Each tree of the forest is oriented from its root, being any vertex of Γ^{free} except for fictitious vertices of the fourth or fifth types, and multifigures sitting on single arcs, to its leaves. The leaves of Γ^{fixed} are the fictitious vertices of the second type (corresponding to the origins of cups) and closed multicups sitting on single arcs. All other vertices are closed multicups and fictitious vertices of the third type. We note that this graph is generated by the ordering introduced in Definition 3.8.1: each edge $\mathfrak{E}$ goes from $\mathfrak{L}_1$ to $\mathfrak{L}_2$ if and only if $\mathfrak{L}_2 \prec \mathfrak{L}_1$ and there are no vertices $\mathfrak{L}_3$ such that $\mathfrak{L}_2 \prec \mathfrak{L}_3 \prec \mathfrak{L}_1$. The necessary and sufficient condition for Γ^{fixed} to be a subgraph spanned by the edges corresponding to chordal domains of some foliation is that the linearity domains built from its vertices do not intersect; the edges are generated by the ordering from Definition 3.8.1.

So, the graph of the foliation is a finite oriented tree whose vertices and edges have types (they correspond either to some figures or to fictitious constructions described above) and several numerical characteristics regarding their type. We warn the reader that we do not write down all the numerical parameters when we draw graphs, this makes our illustrations clearer. We would like to underline that the foliation could be restored from the graph and the numerical parameters determined by this foliation.