Chapter 4

Evolution of Bellman candidates

In this chapter we provide an algorithm for building a special Bellman candidate on Ω_ε for each ε, with $0 < \varepsilon < \varepsilon_{\max}$. In Chapter 5 we will prove that this candidate coincides with the Bellman function $\boldsymbol{B}_\varepsilon(\,\cdot\,; f)$ using optimizers.

The algorithm starts with sufficiently small ε. In such a case, the foliation for the Bellman candidate can be composed of cups (multicups), angles, and tangent domains. Then we increase ε in order to construct the Bellman candidates for larger ε. Formally, there will be statements of two kinds (they can be called "induction steps of the first and second kind"). The first ones state that the set of ε, for which there is a Bellman candidate of a given structure, is open. They are of the form: "if for some η there is a Bellman candidate with the graph Γ, then there is some positive δ such that for all ε in $[\eta, \eta + \delta]$, the foliation with the graph Γ and perturbed numerical parameters provides a Bellman candidate for f in Ω_ε". The second ones state that the set of those ε, for which there are a graph Γ and a collection of numerical parameters that provide a Bellman candidate for f and ε, is closed. They are of the form: "if for each ε_n there is a Bellman candidate with the graph Γ, $\varepsilon_n \nearrow \varepsilon$, and the numerical parameters converge to some limits as $\varepsilon_n \to \varepsilon$, then Γ with the limiting parameters provide a foliation for f in Ω_ε". We note that the limits of numerical parameters may be degenerate in a sense (for example, a trolleybus may become a fictitious vertex of the fifth type), so Γ changes after passing to the limit. Each such induction step, in its turn, can be reduced to similar local statements, i.e., statements about the evolution behavior of lonely figures, e.g., cups, angles, etc.

The main law that rules the evolution of the foliation is "the forces decrease (grow in absolute value) as ε grows". As a result, long chords and multicups grow (Propositions 4.3.1 and 4.3.2), trolleybuses decrease (Propositions 4.3.4 and 4.3.5), multitrolleybuses, birdies, and multibirdies disintegrate (Propositions 4.3.10, 4.3.11, and 4.3.13). What is more, single figures can collide; formally this happens in the induction steps of the second kind when one of the edges has "zero length" at the limit. In the case of a collision, we use formulas from Section 3.8.1 to continue the evolution.

4.1 Simple picture

Definition 4.1.1. Let Γ be a foliation graph. We call it (and the foliation itself) *simple* if it has no oriented paths longer than one and no closed multicups.

Simple foliations consist of alternating cups (or multicups on single arcs; by a multicup on a single arc we mean $\Omega_{\mathrm{Mcup}}(\{\mathfrak{a}\})$, where $\mathfrak{a}$ is an interval) and angles connected by tangent domains. If Γ is a graph of a simple foliation consisting of N edges, then there are either $\frac{N}{2}$, or $\frac{N-1}{2}$, or $\frac{N-2}{2}$ angles in the foliation. In Γ^{free}, the vertices corresponding to angles alternate the vertices representing multicups and long chords. Each multicup is sitting on an arc whose convex hull is not contained in Ω_ε. Each long chord has a cup below it. See Figure 4.1 for the visualization.

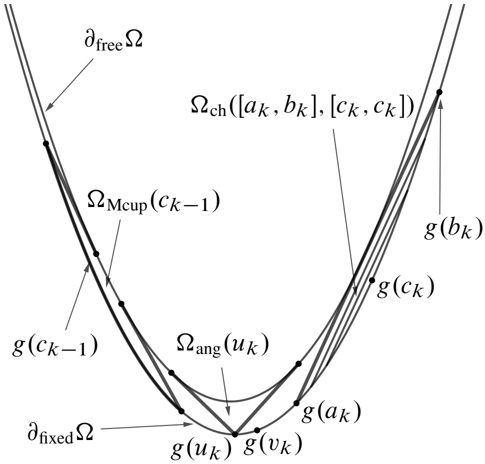

Figure 4.1. An example of simple picture.

For didactic reasons, we explain how a simple graph generates a Bellman candidate B (similar essence for general graphs will be explained in Section 4.4). Suppose Γ is a simple graph. First, we consider its roots, which are long chords, multicups on solid roots, and vertices at infinities. For long chords and multicups, we build the standard candidate by linearity (see (3.4.1)). Second, consider the edges of Γ. For the edges corresponding to chordal domains, we construct standard candidates again by linearity on chords (see (3.4.1)). For each edge corresponding to a tangent domain, we continuously glue a standard candidate in this domain to the already built standard candidate corresponding to the source of the edge. This is done by choosing an appropriate $\beta_2(t_0)$ in (3.2.3). For tangent domains whose source is infinity, we do not have to glue anything, we simply consider the standard candidates on them; such standard candidates are uniquely defined, see Definition 3.3.8. In the angles, we choose the standard candidates by Proposition 3.6.4. The constructed function B is C^1-smooth, and thus, by Proposition 3.1.2, it is locally concave.

In the theorem below, we use the notation for the essential roots, see Definition 2.1.13.

Theorem 4.1.2. *For any function f satisfying Conditions 2.1.11 and 2.1.14, there exists $\varepsilon_1 > 0$ such that for any $\varepsilon < \varepsilon_1$, there exist a simple graph and a collection of numerical parameters such that the function B constructed from this graph, f, and ε as described above is a C^1-smooth Bellman candidate. Moreover, its foliation satisfies the following properties: the origins of the cups coincide with those c_i that are single points; the multicups are sitting on those c_i that are intervals; for any $k = 1, 2, \ldots, n$, the parameter of the vertex u_k of the k-th angle in Γ^{free} tends to v_k as $\varepsilon \to 0$.*

The proof of this theorem will be presented at the end of this section, because it requires some preparation.

Consider two neighbor points c_k and v_{k+1}, where the torsion $\mathbf{T}$ of the boundary curve γ changes its sign. If c_k is a solid root, then for sufficiently small ε, we can build a multicup on c_k and define the standard candidate there. If c_k is a single point, for small ε, we use Proposition 3.5.9 to build a full cup on it. Let its upper chord be $[g(a_k), g(b_k)] = [g(a_k(\varepsilon)), g(b_k(\varepsilon))]$ (for a cup, we take its upper chord, whereas for a multicup, we consider the chord connecting its endpoints). Then, by (3.6.4), we have $F_{\mathrm{R}}(u; a_k, b_k; \varepsilon) < 0$ when $u \in (b_k, v_{k+1}]$, because $\mathrm{sign}(\mathbf{T}) = \mathrm{sign}(\mathfrak{K}')$ (see (2.1.12)). Thus, the right tail of the cup or multicup built on c_k always contains v_{k+1}. Similarly, the left tail of the cup or multicup built over c_{k+1} contains v_{k+1}. The following lemma says that the ends of these tails tend to v_{k+1} as $\varepsilon \to 0$.

Lemma 4.1.3. *Let $(g(a_k(\varepsilon)), g(b_k(\varepsilon)))$ be the upper chord of a cup or of a multicup built over c_k, and let $t_k^{\mathrm{R}} = t_k^{\mathrm{R}}(\varepsilon)$ be the endpoint of its right tail. Then $t_k^{\mathrm{R}} \to v_{k+1}$ as $\varepsilon \to 0$. Similarly, the endpoint $t_k^{\mathrm{L}}(\varepsilon)$ of the left tail tends to v_k. A similar convergence statement holds for the forces coming from the infinities.*

Proof. We will deal with the case of the right tail. We will write a for a_k and b for b_k. It suffices to prove that for each point w_+ such that $v_{k+1} < w_+$ (we also assume that w_+ is not far from v_{k+1}; we want $\mathfrak{K}$ to increase on (v_{k+1}, w_+)), the inequality $t_k^{\mathrm{R}} < w_+$ holds eventually as $\varepsilon \to 0$. We use (3.6.4) for the force $F_{\mathrm{R}}(t) = F_{\mathrm{R}}(t; a(\varepsilon), b(\varepsilon); \varepsilon)$ outside the chordal domain to obtain

$$F_{\mathrm{R}}(w_+) = \int_b^{w_+} \exp\Big(-\int_\tau^{w_+} \frac{\kappa_2'}{\kappa_2 - \kappa}\Big)\mathfrak{K}'(\tau)\,d\tau + \exp\Big(-\int_b^{w_+} \frac{\kappa_2'}{\kappa_2 - \kappa}\Big) F_{\mathrm{R}}(b). \quad (4.1.1)$$

It suffices to prove $F_{\mathrm{R}}(w_+) > 0$ for ε sufficiently small. We first deal with the first summand, which will be split into two integrals $\int_b^{w_+} = \int_b^{v_{k+1}^{\mathrm{r}}} + \int_{v_{k+1}^{\mathrm{r}}}^{w_+}$ (as usual, by v_{k+1}^{r} we denote the right endpoint of v_{k+1}). First,

$$\Big|\int_b^{v_{k+1}^{\mathrm{r}}} \exp\Big(-\int_\tau^{w_+} \frac{\kappa_2'}{\kappa_2 - \kappa}\Big)\mathfrak{K}'(\tau)\,d\tau\Big| \leqslant \exp\Big(-\int_{v_{k+1}^{\mathrm{r}}}^{w_+} \frac{\kappa_2'}{\kappa_2 - \kappa}\Big) \cdot \int_b^{v_{k+1}^{\mathrm{r}}} |\mathfrak{K}'(\tau)|\,d\tau. \quad (4.1.2)$$

Take $w \in (v^{\mathrm{r}}_{k+1}, w_+)$. Then

$$\begin{aligned}\int_{v^{\mathrm{r}}_{k+1}}^{w_+} \exp\Big(-\int_\tau^{w_+} \frac{\kappa_2'}{\kappa_2-\kappa}\Big)\mathfrak{K}'(\tau)\,d\tau &\geqslant \int_w^{w_+} \exp\Big(-\int_\tau^{w_+} \frac{\kappa_2'}{\kappa_2-\kappa}\Big)\mathfrak{K}'(\tau)\,d\tau\\ &\geqslant \exp\Big(-\int_w^{w_+} \frac{\kappa_2'}{\kappa_2-\kappa}\Big)\cdot \int_w^{w_+} \mathfrak{K}'(\tau)\,d\tau\\ &= \exp\Big(-\int_w^{w_+} \frac{\kappa_2'}{\kappa_2-\kappa}\Big)\cdot (\mathfrak{K}(w_+)-\mathfrak{K}(w)). \qquad (4.1.3)\end{aligned}$$

We multiply both sides of (4.1.1) by $\exp(\int_w^{w_+} \frac{\kappa_2'}{\kappa_2-\kappa})$, use (4.1.2) and (4.1.3), and obtain

$$\begin{aligned}\exp\Big(\int_w^{w_+} \frac{\kappa_2'}{\kappa_2-\kappa}\Big)F_{\mathrm{R}}(w_+) \geqslant \mathfrak{K}(w_+)-\mathfrak{K}(w) - \exp\Big(-\int_{v^{\mathrm{r}}_{k+1}}^{w} \frac{\kappa_2'}{\kappa_2-\kappa}\Big)\\ \times \int_b^{v^{\mathrm{r}}_{k+1}} |\mathfrak{K}'(\tau)|\,d\tau + \exp\Big(-\int_b^{w} \frac{\kappa_2'}{\kappa_2-\kappa}\Big)F_{\mathrm{R}}(b). \qquad (4.1.4)\end{aligned}$$

When $\varepsilon \to 0$, κ tends to κ_2 from below pointwise, therefore both exponents on the right-hand side of (4.1.4) tend to zero. These exponents are multiplied by bounded factors. Indeed,

$$\int_b^{v^{\mathrm{r}}_{k+1}} |\mathfrak{K}'(\tau)|\,d\tau \leqslant \int_{c_k}^{v^{\mathrm{r}}_{k+1}} |\mathfrak{K}'(\tau)|\,d\tau.$$

Boundedness of the second factor is more cumbersome. If $(g(a), g(b))$ is the upper chord of a multicup, then $F_{\mathrm{R}}(b) = 0$ and there is nothing to do. Consider the case of a cup. In this case we use (3.6.4):

$$F_{\mathrm{R}}(b) = \frac{D_{\mathrm{R}}(a,b)}{g_1'(b)\kappa_2'(b)}. \qquad (4.1.5)$$

Let us rewrite formula (3.5.14). After applying (3.5.2) and (3.5.1), its left-hand side takes the following form:

$$\begin{aligned}\det\begin{pmatrix}\gamma''(b)\\ \gamma'(a)\\ \gamma(b)-\gamma(a)\end{pmatrix} &= \det\begin{pmatrix}\gamma''(b)\\ \gamma'(a)\\ C_{\mathrm{R}}\gamma'(b)-C_{\mathrm{L}}\gamma'(a)\end{pmatrix}\\ &= C_{\mathrm{R}}\det\begin{pmatrix}\gamma''(b)\\ \gamma'(a)\\ \gamma'(b)\end{pmatrix} = -C_{\mathrm{R}}D_{\mathrm{R}}\det\begin{pmatrix}g'(b)\\ g'(a)\end{pmatrix}.\end{aligned}$$

By (3.5.2) again, the non-integral term in the right-hand side of (3.5.14) vanishes. Thus, we obtain

$$C_{\mathrm{R}}D_{\mathrm{R}}\det\begin{pmatrix}g'(b)\\ g'(a)\end{pmatrix} = g_1'(b)g_1'(a)\kappa_2'(b)\int_a^b (\mathfrak{K}(u)-\mathfrak{K}(b))\kappa_2'(u)(g_1(b)-g_1(u))\,du.$$

The function $W(u) = \kappa_2'(u)(g_1(b) - g_1(u))$ is non-negative and its integral over $[a,b]$ is equal to

$$\begin{aligned}\int_a^b W(u)\,du &= \kappa_2(u)(g_1(b) - g_1(u))\Big|_a^b + \int_a^b \kappa_2(u) g_1'(u)\,du \\ &= -\kappa_2(a)\big(g_1(b) - g_1(a)\big) + g_2(b) - g_2(a) \\ &= -\frac{1}{g_1'(a)} \det\begin{pmatrix} g(b) - g(a) \\ g'(a) \end{pmatrix} = -\frac{1}{g_1'(a)} \det\begin{pmatrix} g'(b) \\ g'(a) \end{pmatrix} C_{\mathrm{R}}.\end{aligned}$$

Therefore, (4.1.5) takes the form

$$F_{\mathrm{R}}(b) = \frac{\int_a^b (\mathfrak{K}(b) - \mathfrak{K}(u)) W(u)\,du}{\int_a^b W(u)\,du}.$$

By the mean value theorem, there exists $u \in [a,b]$ such that $F_{\mathrm{R}}(b) = \mathfrak{K}(b) - \mathfrak{K}(u)$, which is bounded and, moreover, tends to zero when $\varepsilon \to 0$. Hence, the right-hand side of (4.1.4) tends to $\mathfrak{K}(w_+) - \mathfrak{K}(w)$ which is positive. Thus, the left-hand side of (4.1.4) is positive for sufficiently small ε. ■

The notation in the following lemma is the same as in the previous one.

Lemma 4.1.4. *The difference of forces,* $F_{\mathrm{R}}(u; a_k, b_k) - F_{\mathrm{L}}(u; a_{k+1}, b_{k+1})$*, is strictly increasing (as a function of* u*) on the interval* $(t_{k+1}^{\mathrm{L}}, t_k^{\mathrm{R}}) \cap (b_k, a_{k+1})$.

Proof. We differentiate the function in question with respect to u, use (3.6.6), and obtain

$$(F_{\mathrm{R}} - F_{\mathrm{L}})' = -\beta_{2,\mathrm{R}}' + \beta_{2,\mathrm{L}}' > 0,$$

because $\beta_{2,\mathrm{L}}' > 0 > \beta_{2,\mathrm{R}}'$ on $(t_{k+1}^{\mathrm{L}}, t_k^{\mathrm{R}}) \cap (b_k, a_{k+1})$ by Definition 3.3.8. The lemma is proved. ■

Corollary 4.1.5. *The balance equation*

$$F_{\mathrm{R}}(u; a_k, b_k; \varepsilon) = F_{\mathrm{L}}(u; a_{k+1}, b_{k+1}; \varepsilon) \tag{4.1.6}$$

has a unique root $u = u_{k+1}$ *in* $(t_{k+1}^{\mathrm{L}}, t_k^{\mathrm{R}})$ *for sufficiently small* ε.

Proof. First, by Lemma 4.1.3, we have $[t_{k+1}^{\mathrm{L}}, t_k^{\mathrm{R}}] \subset (b_k, a_{k+1})$ for sufficiently small ε. By the definition of tails and the continuity of forces, the force function is zero at the endpoint of its tail: $F_{\mathrm{R}}(t_k^{\mathrm{R}}; a_k, b_k; \varepsilon) = 0$ and $F_{\mathrm{L}}(t_{k+1}^{\mathrm{L}}; a_{k+1}, b_{k+1}; \varepsilon) = 0$. Therefore,

$$F_{\mathrm{R}}(t_{k+1}^{\mathrm{L}}; a_k, b_k; \varepsilon) - F_{\mathrm{L}}(t_{k+1}^{\mathrm{L}}; a_{k+1}, b_{k+1}; \varepsilon) = F_{\mathrm{R}}(t_{k+1}^{\mathrm{L}}; a_k, b_k; \varepsilon) < 0,$$

because $t_{k+1}^{\mathrm{L}} \in (b_k, t_k^{\mathrm{R}})$. Similarly,

$$F_{\mathrm{R}}(t_k^{\mathrm{R}}; a_k, b_k; \varepsilon) - F_{\mathrm{L}}(t_k^{\mathrm{R}}; a_{k+1}, b_{k+1}; \varepsilon) = -F_{\mathrm{L}}(t_k^{\mathrm{R}}; a_{k+1}, b_{k+1}; \varepsilon) > 0.$$

By the Bolzano–Weierstrass principle, there exists a root u_{k+1} of the balance equation in $(t_{k+1}^{\mathrm{L}}, t_k^{\mathrm{R}})$. By Lemma 4.1.4, this root is unique. ■

Remark 4.1.6. The results of the preceding lemma and the corollary hold true if one of the cups (or both) sit at infinity, i.e., the corresponding c_k is infinite.

Now we have all the ingredients to prove Theorem 4.1.2.

Proof of Theorem 4.1.2. First, we take ε to be a small number such that we can build a full cup around each c_k (or a multicup if c_k is a solid root) with the help of Proposition 3.5.9. Moreover, we take ε to be so small that all these figures have no intersections. This is possible because we have only a finite number of roots by our assumptions. Then, if ε satisfies the assumptions of Corollary 4.1.5 (together with Remark 4.1.6) for each k, one can paste an angle with the vertex at the unique root of the balance equation (4.1.6) between each pair of consecutive cups or multicups, see Corollary 3.6.5. The relation $u_{k+1} \to v_{k+1}$ is an immediate consequence of Lemma 4.1.3 and the inclusion $u_{k+1} \in (t_{k+1}^{\mathrm{L}}, t_k^{\mathrm{R}})$. ■

4.2 Preparation to evolution

In this section, we collect technical lemmas that are useful for the evolution. There will be three groups of lemmas. The first group consists of lemmas that describe the places where the fictitious vertices of the third type may occur, the second is about tails and forces, and the third one works with the balance equation.

4.2.1 Structural lemmas for chords

We make a convention on chordal domains: the inequalities $D_{\mathrm{L}} < 0$ and $D_{\mathrm{R}} < 0$ hold true inside the chordal domain. Note that the same inequalities are required to build the standard candidate in a chordal domain.

Lemma 4.2.1. *Let* $\Omega_{\mathrm{ch}}([a,b],*)$ *be a chordal domain. If* $D_{\mathrm{L}}(a,b) = 0$*, then* $\mathfrak{K}$ *is decreasing on the right of* a; *if* $D_{\mathrm{R}}(a,b) = 0$*, then* $\mathfrak{K}$ *is increasing on the left of* b.

Proof. We treat the case of the right differential only. The remaining case is symmetric. We will use formula (3.5.8). Let

$$U(t) = \frac{\det\begin{pmatrix} g'(t) \\ g'''(t)\end{pmatrix}}{\det\begin{pmatrix} g'(t) \\ g''(t)\end{pmatrix}} - \frac{\det\begin{pmatrix} g''(t) \\ g(t)-g(s)\end{pmatrix}}{\det\begin{pmatrix} g'(t) \\ g(t)-g(s)\end{pmatrix}},$$

where $s = s(t)$, $s < t$, is the corresponding function to the chordal domain defined in Section 3.4. The function U is in fact a measure of bounded variation in a left neighborhood of b, hence we can find an increasing function A such that $A' + U$ is a positive measure in a left neighborhood of b. Consider the function $V(t) = e^{A(t)} D_{\mathrm{R}}(s,t)$ in the left neighborhood of b. We use equations (3.5.8) and (2.1.10):

$$e^{-A(t)} V'(t) = \frac{\mathbf{T}(t)}{\det\begin{pmatrix} g'(t) \\ g''(t) \end{pmatrix}} + (A'(t) + U(t)) D_{\mathrm{R}}(s,t).$$

If $\mathfrak{K}$ does not increase to the left of b, then $\mathbf{T}(t) < 0$ in a small left neighborhood of b according to (2.1.12) and Condition 2.1.11. Since $A'(t) + U(t) > 0$ and $D_{\mathrm{R}}(s,t) < 0$, we obtain $V'(t) < 0$. Therefore, V decreases, which contradicts the fact $V(t) < 0$ for $t < b$ and $V(b) = 0$. ■

Lemma 4.2.2. *Suppose that a pair* (a,b) *satisfies the cup equation, and the chord* $[g(a), g(b)]$ *has nonzero tails (i.e.,* $t^{\mathrm{L}} < a$ *and* $b < t^{\mathrm{R}}$*). If* $D_{\mathrm{L}}(a,b) = 0$*, then* $\mathfrak{K}$ *increases on the left of* a*; if* $D_{\mathrm{R}}(a,b) = 0$*, then* $\mathfrak{K}$ *decreases on the right of* b*.*

Proof. We treat the case of the right differential only. The remaining case is symmetric. We will use formula (3.6.4). By Condition 2.1.11 for the function f, the function $\mathfrak{K}$ either increases or decreases in a right neighborhood of b. If it increases, then the force F_{R} is non-negative, which contradicts the assumption $t^{\mathrm{R}} > b$. Therefore, $\mathfrak{K}$ decreases on the right of b. ■

Combining these two lemmas we obtain the following corollary: during the evolution, the differentials can vanish only in some very special situations (we use the notation from Definition 2.1.13 in the corollary below).

Corollary 4.2.3. *Suppose that the chordal domain* $\Omega_{\mathrm{ch}}([a,b],*)$ *has nonzero tails. If* $D_{\mathrm{R}}(a,b) = 0$*, then* $b = c_i$ *for some* i*; if* $D_{\mathrm{L}}(a,b) = 0$*, then* $a = c_i$ *for some* i*.*

4.2.2 Tails growth lemmas

Lemma 4.2.4. *Let* $\Omega_{\mathrm{ch}}([a_1,b_1],[a_0,b_0])$ *be embedded into* $\Omega_{\mathrm{ch}}([a_2,b_2],[a_0,b_0])$*, i.e., the foliation of the former chordal domain coincides with some part of the foliation of the latter. Then the forces of* $\Omega_{\mathrm{ch}}([a_1,b_1],*)$ *do not exceed the corresponding forces of* $\Omega_{\mathrm{ch}}([a_2,b_2],*)$*. More precisely, they are equal on* $[a_1,a_0] \cup [b_0,b_1]$*, while outside* $[a_1,b_1]$*, the inequalities are strict.*

Proof. For the right forces, the statement follows from equation (3.6.7) and the fact that $\kappa_{\mathrm{R}}(b) < \kappa_{\mathrm{chord}}(b) < \kappa_2(b)$. Note that $D_{\mathrm{R}}(a,b) < 0$, therefore F_{R} increases as the chordal domain enlarges, i.e., b increases. Similar arguments work for the case of the left forces: we use (3.6.9) and the opposite relation between the coefficients: $\kappa_{\mathrm{L}}(a) > \kappa_{\mathrm{chord}}(a) > \kappa_2(a)$. Thus, F_{L} increases as the chordal domain enlarges, i.e., a decreases. ■

Corollary 4.2.5. *Suppose $\Omega_{\mathrm{ch}}([a_1, b_1], *)$ is embedded into $\Omega_{\mathrm{ch}}([a_2, b_2], *)$ in the sense of Lemma 4.2.4. Then the tails of the former chordal domain strictly contain the tails of the latter.*

Proof. This follows from Lemma 4.2.4 and the definition of tails. ∎

The previous statements can be reformulated informally as follows: the less the chordal domain is, the larger the tails are and the less (the larger in absolute value) the forces are inside the tails. The following lemma describes monotonicity of forces of a fixed chordal domain for a family of enlarging domains Ω_ε.

Lemma 4.2.6. *Let $\Omega_{\mathrm{ch}}([a, b], *)$ be a chordal domain and let F be its left or right force. If t belongs to the closure of the corresponding tail of $\Omega_{\mathrm{ch}}([a, b], *)$, then*

$$\frac{\partial F(t;\varepsilon)}{\partial \varepsilon} \leqslant 0,$$

and the inequality is strict outside $[a, b]$.

Proof. We treat the case of the right force only, the other one is symmetric. First, we note that inside $[a, b]$ the force does not depend on ε, therefore $\frac{\partial F_{\mathrm{R}}(t;\varepsilon)}{\partial \varepsilon} = 0$. Second, we use formula (3.6.6) and see that

$$F_{\mathrm{R}}'(t;\varepsilon) + \frac{\kappa_2'(t)}{\kappa_2(t) - \kappa(t;\varepsilon)} F_{\mathrm{R}}(t;\varepsilon) = \mathfrak{K}'(t)$$

outside this interval. We cautiously differentiate this equation with respect to ε and see that

$$\frac{\partial}{\partial \varepsilon} F_{\mathrm{R}}'(t;\varepsilon) + \frac{\kappa_2'(t)}{\kappa_2(t) - \kappa(t;\varepsilon)} \frac{\partial}{\partial \varepsilon} F_{\mathrm{R}}(t;\varepsilon) = -\frac{\kappa_2'(t) F_{\mathrm{R}}(t;\varepsilon)}{(\kappa_2(t) - \kappa(t;\varepsilon))^2} \frac{\partial}{\partial \varepsilon} \kappa(t;\varepsilon).$$

After interchanging the differentiations with respect to t and ε in the first summand, we see that $y(t) = \frac{\partial}{\partial \varepsilon} F_{\mathrm{R}}(t;\varepsilon)$ is a solution of the first-order differential equation with respect to t:

$$y'(t) + \frac{\kappa_2'(t)}{\kappa_2(t) - \kappa(t;\varepsilon)} y(t) = -\frac{\kappa_2'(t) F_{\mathrm{R}}(t;\varepsilon)}{(\kappa_2(t) - \kappa(t;\varepsilon))^2} \frac{\partial}{\partial \varepsilon} \kappa(t;\varepsilon).$$

We use (3.2.10) to express the solution:

$$\frac{\partial}{\partial \varepsilon} F_{\mathrm{R}}(t;\varepsilon) = \exp\Big(-\int_b^t \frac{\kappa_2'}{\kappa_2 - \kappa}\Big) \times \Big(\int_b^t \exp\Big(\int_b^\tau \frac{\kappa_2'}{\kappa_2 - \kappa}\Big)\Big[-\frac{\kappa_2' F_{\mathrm{R}}}{(\kappa_2 - \kappa)^2} \frac{\partial \kappa}{\partial \varepsilon}\Big](\tau)\, d\tau + \mathrm{Const}(\varepsilon)\Big),$$

were the constant is zero, because $\lim_{t \to b+} \frac{\partial}{\partial \varepsilon} F_{\mathrm{R}}(t;\varepsilon) = 0$ (this limit relation can be verified by a straightforward calculation using formula (3.6.4)). Now the result follows immediately, since $F_{\mathrm{R}} < 0$ inside the tail, $\kappa_2' > 0$, and $\frac{\partial \kappa(t;\varepsilon)}{\partial \varepsilon} < 0$ for geometrical reasons. ∎

Corollary 4.2.7. *Let $\Omega_{\mathrm{ch}}([a,b],*)$ be a chordal domain with nonzero tails. If we increase ε a little, the tails of the chordal domain strictly enlarge.*

Proof. By Lemma 4.2.6, the forces decrease (increase in absolute value) on the corresponding tails, therefore the tails cannot decrease. What is more, if we increase ε a little, the force at the end of the tail becomes negative, and the tail enlarges. ■

Remark 4.2.8. The results of Lemma 4.2.6 and Corollary 4.2.7 hold for the forces coming from the infinities and multicups as well.

For a moment, we let chordal domains fall out of Ω_ε, i.e., their chords may intersect the free boundary. Surely, when we were working with chordal domains, we did not need the upper boundary, therefore, such an assumption does not break all the results concerning chordal domains.

Lemma 4.2.9. *Let $\Omega_{\mathrm{ch}}([a_2,b_2],[a_1,b_1])$ be a chordal domain with the standard Bellman candidate there. Suppose that $\varepsilon_1 < \varepsilon_2$, and that there are two differentiable functions $a\colon[\varepsilon_1,\varepsilon_2] \to [a_2,a_1]$ and $b\colon[\varepsilon_1,\varepsilon_2] \to [b_1,b_2]$, such that the chord $[g(a(\varepsilon)),g(b(\varepsilon))]$ is tangent to the free boundary of Ω_ε for each $\varepsilon \in [\varepsilon_1,\varepsilon_2]$. Then, for any $\tilde{\varepsilon} \in [\varepsilon_1,\varepsilon_2]$, we have*

$$\frac{\partial}{\partial\varepsilon}[F(t;a(\tilde{\varepsilon}),b(\tilde{\varepsilon});\varepsilon)]\Big|_{\varepsilon=\tilde{\varepsilon}} = \frac{\partial}{\partial\varepsilon}[F(t;a(\varepsilon),b(\varepsilon);\varepsilon)]\Big|_{\varepsilon=\tilde{\varepsilon}},$$

where F stands for the right (see (3.6.4)) or the left (see (3.6.5)) force function.

Proof. This identity follows from Lemma 3.6.16 because $\kappa_{\mathrm{R}} = \kappa_{\mathrm{L}} = \kappa_{\mathrm{chord}}$ when the chord is tangent to the free boundary. ■

Lemma 4.2.10. *Under the hypothesis of Lemma 4.2.9, the tails of $\Omega_{\mathrm{ch}}([a(\varepsilon),b(\varepsilon)],*)$ strictly enlarge in ε.*

Proof. Consider a point t that belongs to one of the tails of $\Omega_{\mathrm{ch}}([a(\varepsilon),b(\varepsilon)],*)$ for some ε. First, we need to prove that $F(t;a(\varepsilon),b(\varepsilon);\varepsilon)$ decreases in ε. The derivative of this function with respect to ε is non-positive, because, by Lemma 4.2.9, for each ε it equals the corresponding derivative taken as if the chordal domain had fixed upper chord, which is non-positive by Lemma 4.2.6. Thus, the tails do not decrease. Moreover, the derivative of the corresponding force is nonzero at the end of each tail, again by Lemma 4.2.6, therefore the tail grows. ■

Definition 4.2.11. Let $0 \leqslant \varepsilon_1 < \varepsilon_2 < \varepsilon_{\max}$. Let $\Omega_{\mathrm{ch}}([a_1,b_1],[a_0,b_0])$ be a chordal domain. Suppose that there are two continuous functions $a\colon[\varepsilon_1,\varepsilon_2] \to [a_1,a_0]$ and $b\colon[\varepsilon_1,\varepsilon_2] \to [b_0,b_1]$ such that $[g(a(\varepsilon)),g(b(\varepsilon))]$ is a chord of this chordal domain for each $\varepsilon \in [\varepsilon_1,\varepsilon_2]$. We call the family

$$\{\Omega_{\mathrm{ch}}([a(\varepsilon),b(\varepsilon)],[a_0,b_0])\}_\varepsilon$$

a *flow of chordal domains*. This flow generates the corresponding forces:

$$
\begin{aligned}
&F_{\mathrm{R}}(u;\Omega_{\mathrm{ch}}([a(\varepsilon),b(\varepsilon)],[a_0,b_0]);\varepsilon), \quad u\in[b_0,t^{\mathrm{R}}(\varepsilon)),\\
&F_{\mathrm{L}}(u;\Omega_{\mathrm{ch}}([a(\varepsilon),b(\varepsilon)],[a_0,b_0]);\varepsilon), \quad u\in(t^{\mathrm{L}}(\varepsilon),a_0],
\end{aligned} \tag{4.2.1}
$$

where $t^{\mathrm{R}}(\varepsilon)$ and $t^{\mathrm{L}}(\varepsilon)$ are the right and the left endpoints of the tails of the chordal domain $\Omega_{\mathrm{ch}}([a(\varepsilon),b(\varepsilon)],[a_0,b_0])$. We say that a flow is *decreasing* if b is decreasing (a is increasing). We say that a flow is *full* if the chord $[g(a(\varepsilon)),g(b(\varepsilon))]$ is tangent to the free boundary of Ω_ε for all $\varepsilon\in[\varepsilon_1,\varepsilon_2]$.

We gather the statements of the previous lemmas in the following corollary.

Corollary 4.2.12. *Consider a decreasing or a full flow of chordal domains. The corresponding forces* (4.2.1) *are strictly decreasing functions of ε outside the chordal domain and constant inside the chordal domain. Consequently, the tails enlarge.*

Proof. For the case of a decreasing flow, we use Lemmas 4.2.6 and 4.2.4. For the case of a full flow, we use Lemma 4.2.10. ∎

Definition 4.2.13. Let $0\leqslant\varepsilon_1<\varepsilon_2<\varepsilon_{\max}$, and let $u_-\colon[\varepsilon_1,\varepsilon_2]\to\mathbb{R}\cup\{-\infty\}$ be a non-increasing function and $u_+\colon[\varepsilon_1,\varepsilon_2]\to\mathbb{R}\cup\{+\infty\}$ a non-decreasing function such that $u_-\leqslant u_+$. Suppose that for any $\varepsilon\in[\varepsilon_1,\varepsilon_2]$, there is a fence $\Omega(I_\varepsilon)$, such that $(u_-(\varepsilon),u_+(\varepsilon))\subset I_\varepsilon$.

A continuous function $\mathfrak{F}_{\mathrm{R}}\colon\{(u,\varepsilon)\mid u\in[u_-(\varepsilon),u_+(\varepsilon)]\cap\mathbb{R},\ \varepsilon\in[\varepsilon_1,\varepsilon_2]\}\to\mathbb{R}$ is called a *right monotone force flow* if the following hold:

(1) for any $\varepsilon\in[\varepsilon_1,\varepsilon_2]$, the fence $\Omega(I_\varepsilon)$ is a right tangent domain, the function $\mathfrak{F}_{\mathrm{R}}(\,\cdot\,;\varepsilon)$ is the force of a standard candidate on $\Omega_{\mathrm{R}}(I_\varepsilon)$ introduced in Definition 3.3.8,

(2) $\mathfrak{F}_{\mathrm{R}}(u_+(\varepsilon);\varepsilon)=0$, provided that $u_+(\varepsilon)<+\infty$,

(3) for any η_1 and η_2 such that $\varepsilon_1<\eta_1<\eta_2<\varepsilon_2$, we have $\mathfrak{F}_{\mathrm{R}}(v;\eta_1)>\mathfrak{F}_{\mathrm{R}}(v;\eta_2)$ whenever $v\in(u_-(\eta_1),u_+(\eta_1)]\cap\mathbb{R}$.

A continuous function $\mathfrak{F}_{\mathrm{L}}\colon\{(u,\varepsilon)\mid u\in[u_-(\varepsilon),u_+(\varepsilon)]\cap\mathbb{R},\ \varepsilon\in[\varepsilon_1,\varepsilon_2]\}\to\mathbb{R}$ is called a *left monotone force flow* the following hold:

(1) for any $\varepsilon\in[\varepsilon_1,\varepsilon_2]$, the fence $\Omega(I_\varepsilon)$ is a left tangent domain, and the function $\mathfrak{F}_{\mathrm{L}}(\,\cdot\,;\varepsilon)$ is the force of a standard candidate on $\Omega_{\mathrm{L}}(I_\varepsilon)$,

(2) $\mathfrak{F}_{\mathrm{L}}(u_-(\varepsilon);\varepsilon)=0$, provided that $u_-(\varepsilon)>-\infty$,

(3) for any η_1 and η_2 such that $\varepsilon_1<\eta_1<\eta_2<\varepsilon_2$, we have $\mathfrak{F}_{\mathrm{L}}(v;\eta_1)>\mathfrak{F}_{\mathrm{L}}(v;\eta_2)$ whenever $v\in[u_-(\eta_1),u_+(\eta_1))\cap\mathbb{R}$.

Monotone force flows may be generated by flows of chordal domains as in (4.2.1). Then we have $u_+=t^{\mathrm{R}}$ for the right case, and $u_-=t^{\mathrm{L}}$ for the left case, see Section 3.6.2 for the definition of t^{R} and t^{L}.

Remark 4.2.14. We underline that for any fixed $\varepsilon \in [\varepsilon_1, \varepsilon_2]$, if $u_-(\varepsilon) < u_+(\varepsilon)$, then the monotone force flows $\mathfrak{F}_R(\,\cdot\,;\varepsilon)$ and $\mathfrak{F}_L(\,\cdot\,;\varepsilon)$ satisfy the differential equations

$$\mathfrak{F}_R' = \mathfrak{K}' - \frac{\kappa_2'}{\kappa_2 - \kappa_R}\mathfrak{F}_R, \quad \mathfrak{F}_L' = \mathfrak{K}' - \frac{\kappa_2'}{\kappa_2 - \kappa_L}\mathfrak{F}_L. \tag{4.2.2}$$

Therefore, the statements of Remarks 3.6.8 and 3.6.10 hold true for the forces:

- if $u_-(\varepsilon) < u_+(\varepsilon) < \infty$, then $\mathfrak{K}' > 0$ in a left neighborhood of $u_+(\varepsilon)$,
- if $u_+(\varepsilon) > u_-(\varepsilon) > -\infty$, then $\mathfrak{K}' < 0$ in a right neighborhood of $u_-(\varepsilon)$.

Remark 4.2.15. Note that the function u_+ in the definition of the right monotone force flow is strictly increasing while it is finite. Similarly, the function u_- in the definition of the left monotone force flow is strictly decreasing while it is finite.

A simple example of a monotone force flow is given by a force of a chord that does not depend on ε (the monotonicity follows from Lemma 4.2.6).

Remark 4.2.16. Let $\{\Omega_{ch}([a(\varepsilon), b(\varepsilon)], *)\}$, $\varepsilon \in [\eta_1, \eta_2]$, be either decreasing or full flow of chordal domains. Then

$$\mathfrak{F}_R(u;\varepsilon) = F_R(u;\Omega_{ch}([a(\varepsilon), b(\varepsilon)], *);\varepsilon), \quad u \in [b(\varepsilon), t^R(\varepsilon)] \cap \mathbb{R},\ \varepsilon \in [\eta_1, \eta_2],$$

is a right monotone force flow, and

$$\mathfrak{F}_L(u;\varepsilon) = F_L(u;\Omega_{ch}([a(\varepsilon), b(\varepsilon)], *);\varepsilon), \quad u \in [t^L(\varepsilon), a(\varepsilon)] \cap \mathbb{R},\ \varepsilon \in [\eta_1, \eta_2],$$

is a left monotone force flow.

4.2.3 Balance equation lemma

Definition 4.2.17. Let F_R and F_L be forces of chordal domains, infinities, or multicups (for the definitions, see Section 3.6.2 and Definition 3.7.15) such that their domains intersect. Then the *balance equation* is

$$F_R(u) = F_L(u), \tag{4.2.3}$$

where u belongs to the intersection of the domains of the forces.

We are looking for solutions of balance equations. Lemma 4.1.4 helped us to establish the existence of the solution in Corollary 4.1.5.

Lemma 4.2.18. *Let F_R and F_L be two forces of chordal domains, infinities, multicups, or simply chords such that their tails intersect. Then the function $F_R(u) - F_L(u)$ strictly increases on this intersection.*

Proof. The proof is identical to that of Lemma 4.1.4. ∎

4.3 Local evolution theorems

In this section all theorems have the following form: if for some ε we can build a Bellman candidate on a specific domain using specific formulas, then, for a slightly larger ε, we can also build a Bellman candidate on a perturbed domain using similar formulas with perturbed parameters. The statements are rather formal and somewhat bulky, so, before each statement we give a short heuristic explanation. We also recall our convention that the forces are strictly negative inside chordal domains and any tail.

The proposition below says that any full chordal domain (with nonzero differentials of the upper chord) surrounded by tangent domains enlarges as ε increases (in other words, in view of Definition 4.2.11, a full chordal domain generates a full flow of chordal domains starting from it).

Proposition 4.3.1 (Induction step for a chordal domain). *Let $\Omega_{\mathrm{ch}}([a_1,b_1],[a_0,b_0])$ be a full chordal domain, $u_1 < a_1 \leqslant a_0 \leqslant b_0 \leqslant b_1 < u_2$, and $0 < \eta_1 < \varepsilon_{\max}$. Let a continuous function B coincide with the standard candidates on $\Omega_{\mathrm{ch}}([a_1,b_1],[a_0,b_0])$, $\Omega_{\mathrm{L}}(u_1,a_1;\eta_1)$, and $\Omega_{\mathrm{R}}(b_1,u_2;\eta_1)$. If $D_{\mathrm{L}}(a_1,b_1) < 0$ and $D_{\mathrm{R}}(a_1,b_1) < 0$, then there exist η_2, $\eta_2 > \eta_1$, and a full flow $\{\Omega_{\mathrm{ch}}([a(\varepsilon),b(\varepsilon)],[a_0,b_0])\}_{\varepsilon\in[\eta_1,\eta_2]}$ of chordal domains such that $a(\eta_1)=a_1$, $b(\eta_1)=b_1$ and for each $\varepsilon\in[\eta_1,\eta_2]$, there exists a continuous function B_ε that coincides with the standard candidates on $\Omega_{\mathrm{L}}(u_1,a(\varepsilon);\varepsilon)$, $\Omega_{\mathrm{ch}}([a(\varepsilon),b(\varepsilon)],[a_0,b_0])$, and $\Omega_{\mathrm{R}}(b(\varepsilon),u_2;\varepsilon)$.*

Proof. We use Remark 3.5.15 and Proposition 3.5.16 for the pair (a_1,b_1) and find $\delta > 0$ and functions $\tilde a$ and $\tilde b$ acting from $[0,\delta]$ to $\mathbb{R}$ such that $\tilde a$ is decreasing, $\tilde a(0) = a_1$, $\tilde b$ is increasing, $\tilde b(0) = b_1$, $\tilde b(\tau) - \tilde a(\tau) = b_1 - a_1 + \tau$, and the pair $(\tilde a(\tau),\tilde b(\tau))$ satisfies (3.4.2) as well as the inequalities $D_{\mathrm{L}}(\tilde a(\tau),\tilde b(\tau)) < 0$ and $D_{\mathrm{R}}(\tilde a(\tau),\tilde b(\tau)) < 0$ for $\tau \in [0,\delta]$. We take $\eta_2 > \eta_1$ in such a way that the chord $[g(\tilde a(\delta)), g(\tilde b(\delta))]$ intersects the free boundary of Ω_{η_2}. Then, for any $\varepsilon \in [\eta_1,\eta_2]$, there exists a unique $\tau = \tau(\varepsilon) \in [0,\delta]$ such that the chord $[g(\tilde a(\tau)), g(\tilde b(\tau))]$ is tangent to the free boundary of Ω_ε. We put $a(\varepsilon) = \tilde a(\tau(\varepsilon))$, $b(\varepsilon) = \tilde b(\tau(\varepsilon))$.

By our assumptions, u_2 belongs to the right tail of $\Omega_{\mathrm{ch}}([a_1,b_1],[a_0,b_0])$ and u_1 belongs to its left tail. Due to Lemma 4.2.10, u_1 and u_2 belong to the left and the right tail of $\Omega_{\mathrm{ch}}([a(\varepsilon),b(\varepsilon)],[a_0,b_0])$, $\varepsilon \in [\eta_1,\eta_2]$, respectively. This allows us to define the required B_ε on the union of $\Omega_{\mathrm{ch}}([a(\varepsilon),b(\varepsilon)],[a_0,b_0])$, $\Omega_{\mathrm{L}}(u_1,a(\varepsilon);\varepsilon)$, and $\Omega_{\mathrm{R}}(b(\varepsilon),u_2;\varepsilon)$ for $\varepsilon \in [\eta_1,\eta_2]$. ∎

By Remark 4.2.16, the functions

$$\mathfrak{F}_{\mathrm{R}}(u;\varepsilon) = F_{\mathrm{R}}(u;[a(\varepsilon),b(\varepsilon)];\varepsilon) \quad \text{and} \quad \mathfrak{F}_{\mathrm{L}}(u;\varepsilon) = F_{\mathrm{L}}(u;[a(\varepsilon),b(\varepsilon)];\varepsilon)$$

are the right and left monotone force flows on the corresponding domains.

The following proposition describes how a non-full multicup (i.e., a multicup $\Omega_{\mathrm{Mcup}}(\{\mathfrak{a}_i\}_{i=1}^{k};\varepsilon)$ such that the chord $[g(\mathfrak{a}_1^{\mathrm{l}}), g(\mathfrak{a}_k^{\mathrm{r}})]$ does not lie in Ω_ε) evolves in ε.

Proposition 4.3.2 (Induction step for a multicup). *Let* $\Omega_{\mathrm{Mcup}}(\{\alpha_i\}_{i=1}^k; \eta_1)$*, with* $\eta_1 \in (0, \varepsilon_{\max})$*, be a multicup such that the chord* $[g(\alpha_1^{\mathrm{l}}), g(\alpha_k^{\mathrm{r}})]$ *does not lie in* Ω_{η_1}*, and let* $u_1 < \alpha_1^{\mathrm{l}}$ *and* $\alpha_k^{\mathrm{r}} < u_2$*. Let a continuous function* B *coincide with the standard candidates on* $\Omega_{\mathrm{Mcup}}(\{\alpha_i\}_{i=1}^k; \eta_1)$*,*$\Omega_{\mathrm{L}}(u_1, \alpha_1^{\mathrm{l}}; \eta_1)$*, and* $\Omega_{\mathrm{R}}(a_k^{\mathrm{r}}, u_2; \eta_1)$*. Then there exists* η_2*,* $\eta_1 < \eta_2$*, such that for each* $\varepsilon \in [\eta_1, \eta_2]$*, there exists a continuous function* B_ε *that coincides with the standard candidates on* $\Omega_{\mathrm{L}}(u_1, \alpha_1^{\mathrm{l}}; \varepsilon)$*,* $\Omega_{\mathrm{Mcup}}(\{\alpha_i\}_{i=1}^k; \varepsilon)$*, and* $\Omega_{\mathrm{R}}(\alpha_k^{\mathrm{r}}, u_2; \varepsilon)$*.*

Proof. We may take any $\eta_2 > \eta_1$ such that the chord $[g(\alpha_1^{\mathrm{l}}), g(\alpha_k^{\mathrm{r}})]$ intersects the free boundary of Ω_{η_2}. Take any ε from the interval prescribed. By Remark 4.2.8, the tails of the multicup enlarge with ε, therefore the points u_1 and u_2 belong to them. Thus, one can build the required function B_ε. ■

Clearly, the functions

$$F_{\mathrm{R}}(u; [\alpha_1^{\mathrm{l}}, \alpha_k^{\mathrm{r}}]; \varepsilon) \quad \text{and} \quad F_{\mathrm{L}}(u; [\alpha_1^{\mathrm{l}}, \alpha_k^{\mathrm{r}}]; \varepsilon)$$

are the right and left monotone force flows on the corresponding domains.

The following proposition says that a long chord with nonzero tails gives rise to a chordal domain. We note that this generalizes Proposition 4.3.1 (in the latter case the differentials are nonzero, and thus the tails of the upper chord are nonzero). However, for didactic reasons, we prefer to separate these two propositions. In a sense, the cases where one or both differentials are zero differ from what is described in Proposition 4.3.1. Indeed, suppose that one of the differentials is zero and we have a chordal domain below a chord $[g(a_0), g(b_0)]$ with the standard candidate on it. By Proposition 4.3.3, after we increase ε, we can build a chordal domain above $[g(a_0), g(b_0)]$ and the standard candidate there. However, we cannot glue these standard candidates into a single one, because the differentials should be strictly negative for a standard candidate. We also note that the proposition below may be applied to the upper chord of a full multicup (i.e., a multicup such that the chord $[g(\alpha_1^{\mathrm{l}}), g(\alpha_k^{\mathrm{r}})]$ is tangent to the free boundary of Ω_ε).

Proposition 4.3.3 (Induction step for a long chord). *Let* $a_0, b_0, u_1, u_2 \in \mathbb{R}$*. Suppose that* $a_0 < b_0$*, the pair* (a_0, b_0) *satisfies the cup equation* (3.4.2)*, and the chord* $[g(a_0), g(b_0)]$ *is tangent to the free boundary of* Ω_{η_1}*,* $\eta_1 \in (0, \varepsilon_{\max})$*. Assume also that* $u_1 < a_0$ *and* $u_2 > b_0$*, and that* u_1 *belongs to the left tail of the chord* $[g(a_0), g(b_0)]$*, while* u_2 *belongs to its right tail. Then there exist* η_2*,* $\eta_2 > \eta_1$*, and a full flow of chordal domains* $\{\Omega_{\mathrm{ch}}([a(\varepsilon), b(\varepsilon)], [a_0, b_0])\}_{\varepsilon \in [\eta_1, \eta_2]}$ *such that for each* $\varepsilon \in [\eta_1, \eta_2]$*, there exists a continuous function* B_ε *that coincides with the standard candidates on* $\Omega_{\mathrm{ch}}([a(\varepsilon), b(\varepsilon)], [a_0, b_0])$*,* $\Omega_{\mathrm{L}}(u_1, a(\varepsilon); \varepsilon)$*, and* $\Omega_{\mathrm{R}}(b(\varepsilon), u_2; \varepsilon)$*.*

Proof. The proof is a repetition of the proof of Proposition 4.3.1; the only difference is that here we need to verify Condition 3.5.14 directly in order to use Proposition 3.5.16.

We need to check that the quantity

$$L(t) = \det\begin{pmatrix}\gamma'(t)\\ \gamma'(a_0)\\ \gamma'(b_0)\end{pmatrix} = g_1'(t)\det\begin{pmatrix}1 & \kappa_2(t) & \kappa_3(t)\\ g_1'(a_0) & g_2'(a_0) & f'(a_0)\\ g_1'(b_0) & g_2'(b_0) & f'(b_0)\end{pmatrix}$$

is positive for t on the left of a_0 and negative on the right of b_0. We consider the right case; the left one is analogous. Since the right tail of the chord $[g(a_0), g(b_0)]$ is nonempty, we have $D_{\mathrm{R}}(a_0, b_0) \leqslant 0$. If $D_{\mathrm{R}}(a_0, b_0) < 0$, then the claim is trivial (see Remark 3.5.15). If $D_{\mathrm{R}}(a_0, b_0) = 0$, then we use Lemma 4.2.2 and conclude that $\mathfrak{K}$ decreases on $(b_0, b_0 + \delta)$ for some $\delta > 0$. Note that $L(b_0) = L'(b_0) = 0$ and for $t \in (b_0, b_0 + \delta)$, we have

$$L_1(t) \stackrel{\mathrm{def}}{=} \frac{\left(\frac{L(t)}{g_1'(t)}\right)'}{\kappa_2'(t)} = \mathfrak{K}(t)\det\begin{pmatrix}g'(a_0)\\ g'(b_0)\end{pmatrix} + \mathrm{Const}$$
$$< \mathfrak{K}(b_0)\det\begin{pmatrix}g'(a_0)\\ g'(b_0)\end{pmatrix} + \mathrm{Const} = L_1(b_0) = 0.$$

Since $\kappa_2' > 0$, the function $\frac{L}{g_1'}$ strictly decreases on $(b_0, b_0 + \delta)$, so $\frac{L}{g_1'}(t) < \frac{L}{g_1'}(b_0) = 0$ and $L(t) < 0$ for $t \in (b_0, b_0 + \delta)$ because $g_1' > 0$. ■

Now we turn to trolleybuses. The next two propositions claim that the base of a trolleybus shrinks when ε increases. On a more formal way, there exists a decreasing flow of chordal domains such that for each ε, we can build a trolleybus on the corresponding chordal domain. In what follows, we will use the notation $\bar{\mathbb{R}}$ for the extended real line $\mathbb{R} \cup \{\pm\infty\}$.

Proposition 4.3.4 (Induction step for a right trolleybus). *Let $\eta_1, \eta_3 \in \mathbb{R}$, $0 < \eta_1 < \eta_3 < \varepsilon_{\max}$. Suppose $a_1, b_1, a_2, b_2, u \in \mathbb{R}$, $a_2 < a_1 < b_1 < b_2 \leqslant u$. Let $\Omega([a_2, a_1])$ be the fence with corresponding function $s\colon [a_2, a_1] \to [b_1, b_2]$, $s(a_i) = b_i$, $i = 1, 2$. This fence coincides with $\Omega_{\mathrm{ch}}([a_2, b_2], [a_1, b_1])$. Suppose that $\mathfrak{F}_{\mathrm{R}}$ is a right monotone force flow with corresponding functions $u_\pm\colon [\eta_1, \eta_3] \to \bar{\mathbb{R}}$ such that $a_2 \in [u_-(\eta_1), u_+(\eta_1)]$. Suppose that there exists a continuous function B that coincides with the standard candidates on $\Omega_{\mathrm{R}}(u_-(\eta_1), a_2; \eta_1)$, $\Omega_{\mathrm{tr,R}}(a_2, b_2; \eta_1)$, $\Omega_{\mathrm{R}}(b_2, u; \eta_1)$, and $\Omega_{\mathrm{ch}}([a_2, a_1])$. Moreover, the force of B on $\Omega_{\mathrm{R}}(u_-(\eta_1), a_2; \eta_1)$ coincides with $\mathfrak{F}_{\mathrm{R}}(\,\cdot\,; \eta_1)$. Then there are $\eta_2 \in (\eta_1, \eta_3)$ and a strictly increasing function $a\colon [\eta_1, \eta_2] \to [a_2, a_1]$ such that $a(\eta_1) = a_2$, and for any $\varepsilon \in [\eta_1, \eta_2]$, there exists a continuous function B_ε that coincides with the standard candidates on $\Omega_{\mathrm{R}}(u_-(\varepsilon), a(\varepsilon); \varepsilon)$, $\Omega_{\mathrm{ch}}([a(\varepsilon), a_1])$, $\Omega_{\mathrm{tr,R}}(a(\varepsilon), b(\varepsilon); \varepsilon)$, and $\Omega_{\mathrm{R}}(b(\varepsilon), u; \varepsilon)$, where $b(\varepsilon) = s(a(\varepsilon))$. In addition, the force of B_ε on $\Omega_{\mathrm{R}}(u_-(\varepsilon), a(\varepsilon); \varepsilon)$ is $\mathfrak{F}_{\mathrm{R}}(\,\cdot\,; \varepsilon)$. The functions*

$$F_{\mathrm{R}}(t; a(\varepsilon), b(\varepsilon); \varepsilon) \quad \textit{and} \quad F_{\mathrm{L}}(t; a(\varepsilon), b(\varepsilon); \varepsilon) \tag{4.3.1}$$

are the right and the left monotone force flows on the corresponding domains.

We note that $\{\Omega_{\mathrm{ch}}([a(\varepsilon), a_1])\}_{\varepsilon\in[\eta_1,\eta_2]}$ is a decreasing flow of chordal domains.

Proof. Our first step is to find $\eta_2 \in (\eta_1, \eta_3)$ and a point a^- in a right neighborhood of a_2 such that $a^- < u_+(\varepsilon)$ and

$$\mathfrak{F}_{\mathrm{R}}(a^-;\varepsilon) > F_{\mathrm{L}}(a^-;\Omega_{\mathrm{ch}}([a_2,a_1])), \quad \varepsilon \in (\eta_1,\eta_2). \tag{4.3.2}$$

We consider two cases: $u_+(\eta_1) > a_2$ and $u_+(\eta_1) = a_2$.

First, if $u_+(\eta_1) > a_2$, then, by Lemma 4.2.18, $\mathfrak{F}_{\mathrm{R}}(\,\cdot\,;\eta_1) - F_{\mathrm{L}}(\,\cdot\,;\Omega_{\mathrm{ch}}([a_2,a_1]))$ is strictly increasing in some right neighborhood of a_2. Since the point a_2 is the root of the balance equation for these forces, i.e., $\mathfrak{F}_{\mathrm{R}}(a_2;\eta_1) = F_{\mathrm{L}}(a_2;\Omega_{\mathrm{ch}}([a_2,a_1]))$, we may simply take a point $a^- \in (a_2, u_+(\eta_1)) \cap (a_2, a_1)$ and obtain $\mathfrak{F}_{\mathrm{R}}(a^-;\eta_1) > F_{\mathrm{L}}(a^-;\Omega_{\mathrm{ch}}([a_2,a_1]))$. It follows from the continuity of forces that for some $\eta_2 \in (\eta_1,\eta_3)$, inequality (4.3.2) holds.

Now we assume that $u_+(\eta_1) = a_2$. Then $\mathfrak{F}_{\mathrm{R}}(a_2;\eta_1) = F_{\mathrm{L}}(a_2;\Omega_{\mathrm{ch}}([a_2,a_1])) = 0$, and, according to Lemma 4.2.1, the function $\mathfrak{K}$ decreases on an interval (a_2,a^-), with $a^- < a_1$. It follows from Remark 4.2.15 that $u_+(\varepsilon) > a_2$ for any $\varepsilon \in (\eta_1,\eta_3]$, therefore we have $u_+(\varepsilon) > a^-$ by Remark 4.2.14. Lemma 4.2.18 implies that the function $\mathfrak{F}_{\mathrm{R}}(\,\cdot\,;\varepsilon) - F_{\mathrm{L}}(\,\cdot\,;\Omega_{\mathrm{ch}}([a_2,a_1]))$ strictly increases on (a_2,a^-). If there were no required η_2, then $\mathfrak{F}_{\mathrm{R}}(a^-;\varepsilon) - F_{\mathrm{L}}(a^-;\Omega_{\mathrm{ch}}([a_2,a_1])) \leqslant 0$ for all $\varepsilon \in (\eta_1,\eta_3)$. However,

$$\mathfrak{F}_{\mathrm{R}}(a_2;\varepsilon) \to \mathfrak{F}_{\mathrm{R}}(a_2;\eta_1) = F_{\mathrm{L}}(a_2;\Omega_{\mathrm{ch}}([a_2,a_1])), \quad \varepsilon \to \eta_1+,$$

therefore $\mathfrak{F}_{\mathrm{R}}(\,\cdot\,;\varepsilon) - F_{\mathrm{L}}(\,\cdot\,;\Omega_{\mathrm{ch}}([a_2,a_1]))$ converges uniformly to zero on $[a_2,a^-]$ when $\varepsilon \to \eta_1+$, or, equivalently,

$$\mathfrak{F}_{\mathrm{R}}(\,\cdot\,;\varepsilon) - \mathfrak{K} \to F_{\mathrm{L}}(\,\cdot\,;\Omega_{\mathrm{ch}}([a_2,a_1])) - \mathfrak{K}, \quad \varepsilon \to \eta_1+, \tag{4.3.3}$$

uniformly on $[a_2,a^-]$. Due to Remark 3.6.15, the function on the right-hand side of (4.3.3) is strictly decreasing while the function on the left-hand side is strictly increasing. This contradicts the uniform convergence (4.3.3). Therefore, there exists $\eta_2 \in (\eta_1,\eta_3)$ such that (4.3.2) holds.

By the definition of the monotone force flow (see Definition 4.2.13),

$$\mathfrak{F}_{\mathrm{R}}(a_2;\varepsilon) - F_{\mathrm{L}}(a_2;\Omega_{\mathrm{ch}}([a_2,a_1])) < 0, \quad \varepsilon \in (\eta_1,\eta_2),$$

since $F_{\mathrm{L}}(a_2;\Omega_{\mathrm{ch}}([a_2,a_1]))$ does not depend on ε. Therefore, there exists a point $a = a(\varepsilon) \in [a_2,a^-]$ that solves the balance equation $\mathfrak{F}_{\mathrm{R}}(a;\varepsilon) = F_{\mathrm{L}}(a;\Omega_{\mathrm{ch}}([a_2,a_1]))$. We note that the function a is increasing. For the existence of the desired function B_ε, we only need to verify that

$$F_{\mathrm{R}}(t;a(\varepsilon),b(\varepsilon);\varepsilon) < 0, \quad \varepsilon \in [\eta_1,\eta_2],\ t \in [b(\varepsilon);u],$$

which follows from Corollary 4.2.12, because $\{\Omega_{\mathrm{ch}}([a(\varepsilon),a_1])\}$ is a decreasing flow of chordal domains.

By Remark 4.2.16, the force functions (4.3.1) form monotone force flows. ■

Proposition 4.3.5 (Induction step for a left trolleybus). *Let* $\eta_1, \eta_3 \in \mathbb{R}$, $0 < \eta_1 < \eta_3 < \varepsilon_{\max}$. *Suppose that* $a_1, b_1, a_2, b_2, u \in \mathbb{R}$, $u \leqslant a_2 < a_1 < b_1 < b_2$. *Let* $\Omega_{\mathrm{ch}}([b_1, b_2])$ *be the fence with corresponding function* $s\colon [b_1, b_2] \to [a_2, a_1]$, $s(b_i) = a_i$, $i = 1, 2$. *This fence coincides with* $\Omega_{\mathrm{ch}}([a_2, b_2], [a_1, b_1])$. *Suppose that* $\mathfrak{F}_{\mathrm{L}}$ *is a left monotone force flow with corresponding functions* $u_{\pm}\colon [\eta_1, \eta_3] \to \overline{\mathbb{R}}$ *such that* $b_2 \in [u_-(\eta_1), u_+(\eta_1)]$. *Suppose that there exists a continuous function* B *that coincides with the standard candidates on* $\Omega_{\mathrm{L}}(b_2, u_+(\eta_1); \eta_1)$, $\Omega_{\mathrm{tr,L}}(a_2, b_2; \eta_1)$, $\Omega_{\mathrm{L}}(u, a_2; \eta_1)$, *and* $\Omega_{\mathrm{ch}}([b_1, b_2])$. *Moreover, the force of* B *on* $\Omega_{\mathrm{L}}(b_2, u_+(\eta_1); \eta_1)$ *is* $\mathfrak{F}_{\mathrm{L}}(\,\cdot\,; \eta_1)$. *Then there exist* η_2, *with* $\eta_1 < \eta_2 < \eta_3$, *and a strictly decreasing function* $b\colon [\eta_1, \eta_2] \to [b_1, b_2]$ *such that* $b(\eta_1) = b_2$, *and for any* $\varepsilon \in [\eta_1, \eta_2]$, *there exists a continuous function* B_ε *that coincides with the standard candidates on* $\Omega_{\mathrm{L}}(b(\varepsilon), u_+(\varepsilon); \varepsilon)$, $\Omega_{\mathrm{tr,L}}(a(\varepsilon), b(\varepsilon); \varepsilon)$, $\Omega_{\mathrm{L}}(u, a(\varepsilon); \varepsilon)$, *and* $\Omega_{\mathrm{ch}}([b_1, b(\varepsilon)])$, *where* $a(\varepsilon) = s(b(\varepsilon))$. *Moreover, the force of* B_ε *on* $\Omega_{\mathrm{L}}(b(\varepsilon), u_+(\varepsilon); \varepsilon)$ *is* $\mathfrak{F}_{\mathrm{L}}(\,\cdot\,; \varepsilon)$. *The forces* (4.3.1) *form monotone force flows.*

Remark 4.3.6. It follows from Lemma 4.2.18 that the functions a and b constructed in Propositions 4.3.4 and 4.3.5 are unique (at least when $\eta_2 - \eta_1$ is sufficiently small).

The following four propositions describe the behavior under evolution of multitrolleybuses. It appears that each multitrolleybus immediately splits into a trolleybus parade (by formulas (3.8.11) and (3.8.12)), and each of the trolleybuses decreases. We consider two simpler cases separately to make the presentation smoother.

Proposition 4.3.7 (Induction step for a right multitrolleybus on a solid root). *Let* $\eta_1, \eta_3 \in \mathbb{R}$, $0 < \eta_1 < \eta_3 < \varepsilon_{\max}$. *Consider a right multitrolleybus* $\Omega_{\mathrm{Mtr,R}}(\{\mathfrak{a}\}; \eta_1)$ *on a solid root* $\mathfrak{a} = [\mathfrak{a}^{\mathrm{l}}, \mathfrak{a}^{\mathrm{r}}]$ (*the case* $\mathfrak{a}^{\mathrm{l}} = \mathfrak{a}^{\mathrm{r}}$ *is not excluded*). *Let* $u \in \mathbb{R}$, $\mathfrak{a}^{\mathrm{r}} \leqslant u$. *Suppose that* $\mathfrak{F}_{\mathrm{R}}$ *is a right monotone force flow with corresponding functions* $u_{\pm}\colon [\eta_1, \eta_3] \to \overline{\mathbb{R}}$ *such that* $\mathfrak{a}^{\mathrm{l}} \in [u_-(\eta_1), u_+(\eta_1)]$. *Suppose that there exists a continuous function* B *that coincides with the standard candidates on* $\Omega_{\mathrm{R}}(u_-(\eta_1), \mathfrak{a}^{\mathrm{l}}; \eta_1)$, $\Omega_{\mathrm{Mtr,R}}(\{\mathfrak{a}\}; \eta_1)$, *and* $\Omega_{\mathrm{R}}(\mathfrak{a}^{\mathrm{r}}, u; \eta_1)$. *Moreover, the force of* B *on* $\Omega_{\mathrm{R}}(u_-(\eta_1), \mathfrak{a}^{\mathrm{l}}; \eta_1)$ *is* $\mathfrak{F}_{\mathrm{R}}(\,\cdot\,; \eta_1)$. *Then for any* $\varepsilon \in (\eta_1, \eta_3]$, *there exists a continuous function* B_ε *that coincides with the standard candidates on* $\Omega_{\mathrm{R}}(u_-(\varepsilon), u; \varepsilon)$, *and the force of* B_ε *on* $\Omega_{\mathrm{R}}(u_-(\varepsilon), u; \varepsilon)$ *is* $\mathfrak{F}_{\mathrm{R}}(\,\cdot\,; \varepsilon)$.

Proof. First we note that $\mathfrak{F}_{\mathrm{R}}(\mathfrak{a}^{\mathrm{l}}; \eta_1) = 0$, therefore $\mathfrak{a}^{\mathrm{l}} = u_+(\eta_1)$. The only thing we need to prove is that $u_+(\varepsilon) \geqslant u$ for any $\varepsilon \in (\eta_1, \eta_3]$. By Remark 4.2.15 we know that $u_+(\varepsilon) > u_+(\eta_1) = \mathfrak{a}^{\mathrm{l}}$. By Remark 4.2.14, $u_+(\varepsilon) \notin \mathfrak{a}$, therefore $u_+(\varepsilon) > \mathfrak{a}^{\mathrm{r}}$. From (4.2.2), for any $t \in (\mathfrak{a}^{\mathrm{r}}, \min(u_+(\varepsilon), u)]$, we have

$$\mathfrak{F}_{\mathrm{R}}(t; \varepsilon) = \int_{\mathfrak{a}^{\mathrm{r}}}^{t} \exp\Big(-\int_{\tau}^{t} \frac{\kappa_2'}{\kappa_2 - \kappa}\Big) \mathfrak{K}'(\tau)\, d\tau + \exp\Big(-\int_{\mathfrak{a}^{\mathrm{r}}}^{t} \frac{\kappa_2'}{\kappa_2 - \kappa}\Big) \mathfrak{F}_{\mathrm{R}}(\mathfrak{a}^{\mathrm{r}}; \varepsilon). \quad (4.3.4)$$

The second summand in (4.3.4) is negative because $\mathfrak{a}^{\mathrm{r}} \in [u_-(\varepsilon), u_+(\varepsilon))$. The first summand in (4.3.4) is equal to $F_{\mathrm{R}}(t; \mathfrak{a}^{\mathrm{l}}, \mathfrak{a}^{\mathrm{r}}; \varepsilon)$ (see (3.6.4), $D_{\mathrm{R}}(\mathfrak{a}^{\mathrm{l}}, \mathfrak{a}^{\mathrm{r}}) = 0$). Further, the inequality

$$F_{\mathrm{R}}(t;\mathfrak{a}^{\mathrm{l}},\mathfrak{a}^{\mathrm{r}};\varepsilon) < F_{\mathrm{R}}(t;\mathfrak{a}^{\mathrm{l}},\mathfrak{a}^{\mathrm{r}};\eta_1) \qquad (4.3.5)$$

follows from monotonicity of forces with respect to ε. The right-hand side of (4.3.5) is non-positive because $t \in (\mathfrak{a}^{\mathrm{r}}, u]$ and $F_{\mathrm{R}}(\,\cdot\,;\mathfrak{a}^{\mathrm{l}},\mathfrak{a}^{\mathrm{r}};\eta_1)$ is the force of the standard candidate on $\Omega_{\mathrm{R}}(\mathfrak{a}^{\mathrm{r}}, u;\eta_1)$. Thus, in particular, $\mathfrak{F}_{\mathrm{R}}(\min(u_+(\varepsilon),u);\varepsilon) < 0$. Since $\mathfrak{F}_{\mathrm{R}}(u_+(\varepsilon);\varepsilon) = 0$, we have $\min(u_+(\varepsilon),u) \neq u_+(\varepsilon)$, therefore $u_+(\varepsilon) > u$. ∎

Remark 4.3.8. Note that the case $\mathfrak{a}^{\mathrm{l}} = \mathfrak{a}^{\mathrm{r}}$ in Proposition 4.3.7 means that the multitrolleybus $\Omega_{\mathrm{Mtr,R}}(\{\mathfrak{a}\};\eta_1)$ is a fictitious vertex $\Omega_{\mathrm{R}}(\mathfrak{a},\mathfrak{a})$ of the fifth type. A similar statement holds for the left case.

Proposition 4.3.9 (Induction step for a right multitrolleybus with one underlying chordal domain). *Let $\eta_1,\eta_3 \in \mathbb{R}$, $0<\eta_1<\eta_3<\varepsilon_{\max}$. Suppose that $a_1,b_1,a_2,b_2,u \in \mathbb{R}$, $a_2<a_1<b_1<b_2\leqslant u$. Let $\mathfrak{a} = [\mathfrak{a}^{\mathrm{l}},\mathfrak{a}^{\mathrm{r}}]$ be a solid root, $\mathfrak{a}^{\mathrm{l}} < \mathfrak{a}^{\mathrm{r}} = a_2$. Consider a right multitrolleybus $\Omega_{\mathrm{Mtr,R}}(\{\mathfrak{a},b_2\};\eta_1)$. Let $\Omega_{\mathrm{ch}}([a_2,a_1])$ be the chordal domain with corresponding function $s\colon [a_2,a_1]\to[b_1,b_2]$, $s(a_i)=b_i$, $i=1,2$. Suppose that $\mathfrak{F}_{\mathrm{R}}$ is a right monotone force flow with corresponding functions $u_\pm\colon[\eta_1,\eta_3]\to\overline{\mathbb{R}}$ such that $\mathfrak{a}^{\mathrm{l}}\in[u_-(\eta_1),u_+(\eta_1)]$. Suppose that there exists a continuous function B that coincides with the standard candidates on $\Omega_{\mathrm{R}}(u_-(\eta_1),\mathfrak{a}^{\mathrm{l}};\eta_1)$, $\Omega_{\mathrm{Mtr,R}}(\{\mathfrak{a},b_2\};\eta_1)$, $\Omega_{\mathrm{R}}(b_2,u;\eta_1)$, and $\Omega_{\mathrm{ch}}([a_2,a_1])$. Moreover, the force of B on $\Omega_{\mathrm{R}}(u_-(\eta_1),\mathfrak{a}^{\mathrm{l}};\eta_1)$ is $\mathfrak{F}_{\mathrm{R}}(\,\cdot\,;\eta_1)$. Then there exist η_2, $\eta_1<\eta_2<\eta_3$, and a strictly increasing function $a\colon[\eta_1,\eta_2]\to[a_2,a_1]$ such that $a(\eta_1)=a_2$, and for any $\varepsilon\in(\eta_1,\eta_2]$, there exists a continuous function B_ε that coincides with the standard candidates on $\Omega_{\mathrm{R}}(u_-(\varepsilon),a(\varepsilon);\varepsilon)$, $\Omega_{\mathrm{tr,R}}(a(\varepsilon),b(\varepsilon);\varepsilon)$, $\Omega_{\mathrm{ch}}([a(\varepsilon),a_1])$, and $\Omega_{\mathrm{R}}(b(\varepsilon),u;\varepsilon)$, where $b(\varepsilon)=s(a(\varepsilon))$. Moreover, the force of B_ε on $\Omega_{\mathrm{R}}(u_-(\varepsilon),a(\varepsilon);\varepsilon)$ is $\mathfrak{F}_{\mathrm{R}}(\,\cdot\,;\varepsilon)$. The forces (4.3.1) form monotone force flows.*

Proof. The force of B on $\mathfrak{a}$ is equal to zero, in particular, $\mathfrak{F}_{\mathrm{R}}(\mathfrak{a}^{\mathrm{l}};\eta_1)=0$, therefore $u_+(\eta_1)=\mathfrak{a}^{\mathrm{l}}$. We cannot directly apply Proposition 4.3.4 in this case, but its proof works, because for any $\varepsilon\in(\eta_1,\eta_3)$, by Proposition 4.3.7, we have $u_+(\varepsilon)>\mathfrak{a}^{\mathrm{r}}$, and the arguments of the second case in the proof of Proposition 4.3.4 (where $u_+(\eta_1)=a_2$) apply verbatim. ∎

Proposition 4.3.10 (Induction step for a general right multitrolleybus). *Suppose that $\eta_1,\eta_3\in\mathbb{R}$ and $\eta_1<\eta_3<\varepsilon_{\max}$. Consider a right multitrolleybus $\Omega_{\mathrm{Mtr,R}}(\{\mathfrak{a}_i\}_{i=1}^k;\eta_1)$. Let $u\in\mathbb{R}$, $\mathfrak{a}_k^{\mathrm{r}}\leqslant u$. Suppose that $\mathfrak{F}_{\mathrm{R}}$ is a right monotone force flow with corresponding functions $u_\pm\colon[\eta_1,\eta_3]\to\overline{\mathbb{R}}$ such that $\mathfrak{a}_1^{\mathrm{l}}\in[u_-(\eta_1),u_+(\eta_1)]$. We also suppose that for each $i=1,2,\ldots,k-1$, there are chordal domains $\Omega_{\mathrm{ch}}([\mathfrak{a}_i^{\mathrm{r}},\mathfrak{a}_{i+1}^{\mathrm{l}}],*)$ with corresponding functions s_i, $s_i(\mathfrak{a}_i^{\mathrm{r}})=\mathfrak{a}_{i+1}^{\mathrm{l}}$. Suppose that there exists a continuous function B that coincides with the standard candidates on $\Omega_{\mathrm{R}}(u_-(\eta_1),\mathfrak{a}_1^{\mathrm{l}};\eta_1)$, $\Omega_{\mathrm{Mtr,R}}(\{\mathfrak{a}_i\}_{i=1}^k;\eta_1)$, $\Omega_{\mathrm{R}}(\mathfrak{a}_k^{\mathrm{r}},u;\eta_1)$, and every $\Omega_{\mathrm{ch}}([\mathfrak{a}_i^{\mathrm{r}},\mathfrak{a}_{i+1}^{\mathrm{l}}],*)$. Additionally, suppose that the force of B on $\Omega_{\mathrm{R}}(u_-(\eta_1),\mathfrak{a}_1^{\mathrm{l}};\eta_1)$ is $\mathfrak{F}_{\mathrm{R}}(\,\cdot\,;\eta_1)$. Then there exist a* *number* η_2, $\eta_2 > \eta_1$, *and a collection of strictly increasing functions* $a_i\colon [\eta_1, \eta_2] \to \mathbb{R}$, $a_i(\eta_1) = \alpha_i^{\mathrm{r}}$, $i = 1, 2, \ldots, k-1$, *such that for every* $\varepsilon \in (\eta_1, \eta_2]$, *there exists a continuous function* B_ε *defined on the domain*

$$\Omega_{\mathrm{R}}(u_-(\varepsilon), a_1(\varepsilon); \varepsilon) \cup \Big(\bigcup_{i=1}^{k-2} \Omega_{\mathrm{R}}(b_i(\varepsilon), a_{i+1}(\varepsilon); \varepsilon)\Big) \cup \Big(\bigcup_{i=1}^{k-1} \Omega_{\mathrm{tr,R}}(a_i(\varepsilon), b_i(\varepsilon); \varepsilon)\Big)$$
$$\cup \Big(\bigcup_{i=1}^{k-1} \Omega_{\mathrm{ch}}([a_i(\varepsilon), b_i(\varepsilon)], *)\Big) \cup \Omega_{\mathrm{R}}(b_{k-1}(\varepsilon), u; \varepsilon), \quad b_i(\varepsilon) = s_i(a_i(\varepsilon)),$$

that coincides with the standard candidate inside each subdomain of the partition. Moreover, the force of B_ε *on* $\Omega_{\mathrm{R}}(u_-(\varepsilon), a_1(\varepsilon); \varepsilon)$ *is* $\mathfrak{F}_{\mathrm{R}}(\,\cdot\,; \varepsilon)$. *The functions*

$$F_{\mathrm{R}}(\,\cdot\,; a_i(\varepsilon), b_i(\varepsilon); \varepsilon) \quad \text{and} \quad F_{\mathrm{L}}(\,\cdot\,; a_i(\varepsilon), b_i(\varepsilon); \varepsilon), \quad 1 \leqslant i \leqslant k-1,$$

are the right and the left monotone force flows on the corresponding domains.

Proof. We can represent $\Omega_{\mathrm{Mtr,R}}(\{\alpha_i\}_{i=1}^k; \eta_1)$ as the union of $\Omega_{\mathrm{Mtr,R}}(\{\alpha_i, \alpha_{i+1}^{\mathrm{l}}\}; \eta_1)$, where $i = 1, \ldots, k-1$, and possibly $\Omega_{\mathrm{Mtr,R}}(\{\alpha_k\}; \eta_1)$ if $\alpha_k^{\mathrm{l}} < \alpha_k^{\mathrm{r}}$. We apply Proposition 4.3.9 if $\alpha_i^{\mathrm{l}} < \alpha_i^{\mathrm{r}}$, and Proposition 4.3.4 if $\alpha_i^{\mathrm{l}} = \alpha_i^{\mathrm{r}}$ to the multitrolleybuses $\Omega_{\mathrm{Mtr,R}}(\{\alpha_i, \alpha_{i+1}^{\mathrm{l}}\}; \eta_1)$ successively. To conclude the induction, we apply Proposition 4.3.7 to $\Omega_{\mathrm{Mtr,R}}(\{\alpha_k\}; \eta_1)$ if $\alpha_k^{\mathrm{l}} < \alpha_k^{\mathrm{r}}$. ■

Proposition 4.3.11 (Induction step for a general left multitrolleybus). *Suppose that* $\eta_1, \eta_3 \in \mathbb{R}$, $\eta_1 < \eta_3 < \varepsilon_{\max}$. *Consider a left multitrolleybus* $\Omega_{\mathrm{Mtr,L}}(\{\alpha_i\}_{i=1}^k; \eta_1)$. *Let* $u \in \mathbb{R}$, $u \leqslant \alpha_1^{\mathrm{l}}$. *Suppose that* $\mathfrak{F}_{\mathrm{L}}$ *is a left monotone force flow with corresponding functions* $u_\pm\colon [\eta_1, \eta_3] \to \bar{\mathbb{R}}$ *such that* $\alpha_k^{\mathrm{r}} \in [u_-(\eta_1), u_+(\eta_1)]$. *We also suppose that for each* $i = 1, 2, \ldots, k-1$, *there are chordal domains* $\Omega_{\mathrm{ch}}([\alpha_i^{\mathrm{r}}, \alpha_{i+1}^{\mathrm{l}}], *)$ *with corresponding functions* s_i, $s_i(\alpha_{i+1}^{\mathrm{l}}) = \alpha_i^{\mathrm{r}}$. *Suppose that there exists a continuous function* B *that coincides with the standard candidates on the domains* $\Omega_{\mathrm{L}}(u, \alpha_1^{\mathrm{l}}; \eta_1)$, $\Omega_{\mathrm{Mtr,L}}(\{\alpha_i\}_{i=1}^k; \eta_1)$, $\Omega_{\mathrm{L}}(\alpha_k^{\mathrm{r}}, u_+(\eta_1); \eta_1)$, *and every* $\Omega_{\mathrm{ch}}([\alpha_i^{\mathrm{r}}, \alpha_{i+1}^{\mathrm{l}}], *)$. *Moreover, the force of* B *on* $\Omega_{\mathrm{L}}(\alpha_k^{\mathrm{r}}, u_+(\eta_1); \eta_1)$ *is* $\mathfrak{F}_{\mathrm{L}}(\,\cdot\,; \eta_1)$. *Then there exist a number* η_2, *with* $\eta_2 > \eta_1$, *and a collection of strictly decreasing functions* $b_i\colon [\eta_1, \eta_2] \to \mathbb{R}$, $b_i(\eta_1) = \alpha_{i+1}^{\mathrm{l}}$, $i = 1, 2, \ldots, k-1$, *such that for every* $\varepsilon \in (\eta_1, \eta_2]$, *there exists a continuous function* B_ε, *defined on the domain*

$$\Omega_{\mathrm{L}}(u, a_1(\varepsilon); \varepsilon) \cup \Big(\bigcup_{i=1}^{k-2} \Omega_{\mathrm{L}}(b_i(\varepsilon), a_{i+1}(\varepsilon); \varepsilon)\Big) \cup \Big(\bigcup_{i=1}^{k-1} \Omega_{\mathrm{tr,L}}(a_i(\varepsilon), b_i(\varepsilon); \varepsilon)\Big)$$
$$\cup \Big(\bigcup_{i=1}^{k-1} \Omega_{\mathrm{ch}}([a_i(\varepsilon), b_i(\varepsilon)], *)\Big) \cup \Omega_{\mathrm{L}}(b_{k-1}(\varepsilon), u_+(\varepsilon); \varepsilon), \quad a_i(\varepsilon) = s_i(b_i(\varepsilon)),$$

that coincides with the standard candidate inside each subdomain of the partition.

Moreover, the force of B_ε *on* $\Omega_{\mathrm{L}}(b_{k-1}(\varepsilon), u_+(\varepsilon); \varepsilon)$ *is* $\mathfrak{F}_{\mathrm{L}}(\,\cdot\,; \varepsilon)$. *The functions*

$$F_{\mathrm{R}}(t; a_i(\varepsilon), b_i(\varepsilon); \varepsilon) \quad \text{and} \quad F_{\mathrm{L}}(t; a_i(\varepsilon), b_i(\varepsilon); \varepsilon), \qquad 1 \leqslant i \leqslant k-1,$$

are the right and the left monotone force flows on the corresponding domains.

In the following proposition, we show that angles move continuously.

Proposition 4.3.12 (Induction step for an angle). *Let* $\eta_1, \eta_3 \in \mathbb{R}$, $0 < \eta_1 < \eta_3 < \varepsilon_{\max}$, *and let* $u_0 \in \mathbb{R}$. *Suppose that* $\mathfrak{F}_{\mathrm{L}}$ *is a left monotone force flow with corresponding functions* $u^l_\pm\colon [\eta_1, \eta_3] \to \overline{\mathbb{R}}$, *and* $\mathfrak{F}_{\mathrm{R}}$ *is a right monotone force flow with corresponding functions* $u^r_\pm\colon [\eta_1, \eta_3] \to \overline{\mathbb{R}}$ *such that* $u_0 \in (u^r_-(\eta_1), u^r_+(\eta_1)] \cap [u^l_-(\eta_1), u^l_+(\eta_1))$. *Suppose that there exists a continuous function* B *that coincides with the standard candidates on* $\Omega_{\mathrm{R}}(u^r_-(\eta_1), u_0; \eta_1)$, $\Omega_{\mathrm{ang}}(u_0; \eta_1)$, *and* $\Omega_{\mathrm{L}}(u_0, u^l_+(\eta_1); \eta_1)$. *Moreover, the forces of* B *on* $\Omega_{\mathrm{R}}(u^r_-(\eta_1), u_0; \eta_1)$ *and* $\Omega_{\mathrm{L}}(u_0, u^l_+(\eta_1); \eta_1)$ *are* $\mathfrak{F}_{\mathrm{R}}(\,\cdot\,; \eta_1)$ *and* $\mathfrak{F}_{\mathrm{L}}(\,\cdot\,; \eta_1)$, *respectively. Then there exist* η_2, $\eta_1 < \eta_2 < \eta_3$, *and a continuous function* $u\colon [\eta_1, \eta_2] \to \mathbb{R}$ *such that* $u(\eta_1) = u_0$ *and for any* $\varepsilon \in [\eta_1, \eta_2]$, *we have* $u(\varepsilon) \in [u^l_-(\varepsilon), u^l_+(\varepsilon)) \cap (u^r_-(\varepsilon), u^r_+(\varepsilon)]$, *and there exists a continuous function* B_ε *that coincides with the standard candidates on* $\Omega_{\mathrm{R}}(u^r_-(\varepsilon), u(\varepsilon); \varepsilon)$, $\Omega_{\mathrm{ang}}(u(\varepsilon); \varepsilon)$, *and* $\Omega_{\mathrm{L}}(u(\varepsilon), u^l_+(\varepsilon); \varepsilon)$. *In addition, the forces of* B_ε *on* $\Omega_{\mathrm{R}}(u^r_-(\varepsilon), u(\varepsilon); \varepsilon)$ *and* $\Omega_{\mathrm{L}}(u(\varepsilon), u^l_+(\varepsilon); \varepsilon)$ *are* $\mathfrak{F}_{\mathrm{R}}(\,\cdot\,; \varepsilon)$ *and* $\mathfrak{F}_{\mathrm{L}}(\,\cdot\,; \varepsilon)$, *respectively.*

Proof. We first note that in order to construct a desired function B_ε, it suffices to find a root $u(\varepsilon)$ of the balance equation (4.2.3) for $\mathfrak{F}_{\mathrm{R}}(\,\cdot\,; \varepsilon)$ and $\mathfrak{F}_{\mathrm{L}}(\,\cdot\,; \varepsilon)$ (see Corollary 3.6.5).

If $u^r_+(\eta_1) > u_0 > u^l_-(\eta_1)$, then the proof is simple. By Lemma 4.2.18, the function $\mathfrak{F}_{\mathrm{R}}(\cdot; \eta_1) - \mathfrak{F}_{\mathrm{L}}(\cdot; \eta_1)$ strictly increases on $(u^l_-(\eta_1), u^l_+(\eta_1)) \cap (u^r_-(\eta_1), u^r_+(\eta_1))$, therefore it is positive on some right neighborhood of u_0 and negative on a left one. By continuity with respect to ε, the function $\mathfrak{F}_{\mathrm{R}}(\cdot; \varepsilon) - \mathfrak{F}_{\mathrm{L}}(\cdot; \varepsilon)$ has a root $u(\varepsilon)$ in a fixed neighborhood of u_0 for ε sufficiently close to η_1, $\varepsilon > \eta_1$. Again, by Lemma 4.2.18, this root is unique in the intersection of the tails.

If $u^r_+(\eta_1) = u_0$ or $u_0 = u^l_-(\eta_1)$, then $\mathfrak{F}_{\mathrm{R}}(u_0; \eta_1) = \mathfrak{F}_{\mathrm{L}}(u_0; \eta_1) = 0$, because u_0 is the root of the balance equation (4.2.3) for $\mathfrak{F}_{\mathrm{R}}(\,\cdot\,; \eta_1)$ and $\mathfrak{F}_{\mathrm{L}}(\,\cdot\,; \eta_1)$. Therefore, $u^r_+(\eta_1) = u^l_-(\eta_1) = u_0$. It follows from Remark 4.2.14 that there exist $u_-, u_+ \in \mathbb{R}$ such that $u_- < u_0 < u_+$, $\mathfrak{K}' > 0$ on (u_-, u_0), and $\mathfrak{K}' < 0$ on (u_0, u_+). Thus, $u_0 = c_i$ for some i (see Definition 2.1.13). For any $\varepsilon \in (\eta_1, \eta_3)$, we have $u^l_-(\varepsilon) < u_-$, and $u_+ < u^r_+(\varepsilon)$. The only thing we need to check is that for any $\delta > 0$, the function $\mathfrak{F}_{\mathrm{R}}(\,\cdot\,; \varepsilon) - \mathfrak{F}_{\mathrm{L}}(\,\cdot\,; \varepsilon)$ has a root $u(\varepsilon) \in (u_0 - \delta, u_0 + \delta)$, provided ε sufficiently close to η_1.

If this is not the case, then there exist a positive δ and a sequence $\varepsilon_n \to \eta_1$, with $\varepsilon_n > \eta_1$, such that the functions $\Phi_n(\,\cdot\,) = \mathfrak{F}_{\mathrm{R}}(\,\cdot\,; \varepsilon_n) - \mathfrak{F}_{\mathrm{L}}(\,\cdot\,; \varepsilon_n)$ have no roots on $(u_0 - \delta, u_0 + \delta)$. Without loss of generality, we may assume that Φ_n is negative on this intersection. The function Φ_n is strictly increasing and negative on $[u_0, u_0 + \delta]$, and $\Phi_n(u_0) \to 0$, $n \to +\infty$. Thus, Φ_n converges to zero uniformly on $[u_0, u_0 + \delta]$. It follows that

$$\lim_{n\to\infty} \mathfrak{F}_{\mathrm{R}}(t;\varepsilon_n) - \mathfrak{K}(t) = \lim_{n\to\infty} \mathfrak{F}_{\mathrm{L}}(t;\varepsilon_n) - \mathfrak{K}(t) = \mathfrak{F}_{\mathrm{L}}(t;\eta_1) - \mathfrak{K}(t), \quad t \in [u_0, u_0 + \delta],$$

where the function on the right-hand side is strictly decreasing on $(u_0, u_0 + \delta)$, whereas the functions on the left-hand side are strictly increasing, by Remark 3.6.15. This leads to the contradiction and proves the claim. ■

Proposition 4.3.13 (Induction step for a multibirdie). *Let $\eta_1, \eta_3 \in \mathbb{R}$, with $\eta_1 < \eta_3 < \varepsilon_{\max}$. Consider a multibirdie $\Omega_{\mathrm{Mbird}}(\{\mathfrak{a}_i\}_{i=1}^k; \eta_1)$. Suppose that $\mathfrak{F}_{\mathrm{L}}$ and $\mathfrak{F}_{\mathrm{R}}$ are left and right monotone force flows with corresponding functions $u_\pm^{\mathrm{L}}$ and $u_\pm^{\mathrm{R}}$ acting from $[\eta_1, \eta_3]$ to $\overline{\mathbb{R}}$ such that $\mathfrak{a}_1^{\mathrm{l}} \in (u_-^{\mathrm{R}}(\eta_1), u_+^{\mathrm{R}}(\eta_1)]$ and $\mathfrak{a}_k^{\mathrm{r}} \in [u_-^{\mathrm{L}}(\eta_1), u_+^{\mathrm{L}}(\eta_1))$. We also suppose that for each i, $i = 1, 2, \ldots, k-1$, there are chordal domains $\Omega_{\mathrm{ch}}([\mathfrak{a}_i^{\mathrm{r}}, \mathfrak{a}_{i+1}^{\mathrm{l}}], *)$. Suppose that there exists a continuous function B that coincides with the standard candidates on the domains $\Omega_{\mathrm{R}}(u_-^{\mathrm{R}}(\eta_1), \mathfrak{a}_1^{\mathrm{l}}; \eta_1)$, $\Omega_{\mathrm{Mbird}}(\{\mathfrak{a}_i\}_{i=1}^k; \eta_1)$, $\Omega_{\mathrm{L}}(\mathfrak{a}_k^{\mathrm{r}}, u_+^{\mathrm{L}}(\eta_1); \eta_1)$, and every $\Omega_{\mathrm{ch}}([\mathfrak{a}_i^{\mathrm{r}}, \mathfrak{a}_{i+1}^{\mathrm{l}}], *)$. In addition, the forces of B on $\Omega_{\mathrm{R}}(u_-^{\mathrm{R}}(\eta_1), \mathfrak{a}_1^{\mathrm{l}}; \eta_1)$ and $\Omega_{\mathrm{L}}(\mathfrak{a}_k^{\mathrm{r}}, u_+^{\mathrm{L}}(\eta_1); \eta_1)$ are $\mathfrak{F}_{\mathrm{R}}(\,\cdot\,; \eta_1)$ and $\mathfrak{F}_{\mathrm{L}}(\,\cdot\,; \eta_1)$, respectively. Then there exist a number η_2, with $\eta_1 < \eta_2 < \eta_3$, and a collection of strictly monotone functions a_i and b_i acting from $[\eta_1, \eta_2]$ to $\mathbb{R}$ such that the a_i are increasing and $a_i(\eta_1) = \mathfrak{a}_i^{\mathrm{r}}$, the b_i are decreasing and $b_i(\eta_1) = \mathfrak{a}_{i+1}^{\mathrm{l}}$, and $[g(a_i(\varepsilon)), g(b_i(\varepsilon))]$ is a chord of $\Omega_{\mathrm{ch}}([\mathfrak{a}_i^{\mathrm{r}}, \mathfrak{a}_{i+1}^{\mathrm{l}}], *)$. Furthermore, for every $\varepsilon \in (\eta_1, \eta_2]$, there exist an integer $j = j(\varepsilon)$, $1 \leqslant j \leqslant k$, and $u(\varepsilon) \in (b_{j-1}(\varepsilon), a_j(\varepsilon)]$ (here we put $b_0 \stackrel{\mathrm{def}}{=} u_-^{\mathrm{R}}(\varepsilon)$ and $a_k \stackrel{\mathrm{def}}{=} u_+^{\mathrm{L}}(\varepsilon)$) such that there exists a continuous function B_ε on the domain*

$$\begin{aligned}
&\Big(\bigcup_{i=0}^{j-2} \Omega_{\mathrm{R}}(b_i(\varepsilon), a_{i+1}(\varepsilon); \varepsilon)\Big) \cup \Big(\bigcup_{i=1}^{j-1} \Omega_{\mathrm{tr,R}}(a_i(\varepsilon), b_i(\varepsilon); \varepsilon)\Big) \cup \Omega_{\mathrm{R}}(b_{j-1}(\varepsilon), u(\varepsilon); \varepsilon)\\
&\quad \cup \Omega_{\mathrm{ang}}(u(\varepsilon); \varepsilon) \cup \Omega_{\mathrm{L}}(u(\varepsilon), a_j(\varepsilon); \varepsilon) \cup \Big(\bigcup_{i=j}^{k-1} \Omega_{\mathrm{tr,L}}(a_i(\varepsilon), b_i(\varepsilon); \varepsilon)\Big)\\
&\quad \cup \Big(\bigcup_{i=j}^{k-1} \Omega_{\mathrm{L}}(b_i(\varepsilon), a_{i+1}(\varepsilon); \varepsilon)\Big) \cup \Big(\bigcup_{i=1}^{k-1} \Omega_{\mathrm{ch}}([a_i(\varepsilon), b_i(\varepsilon)], *)\Big)
\end{aligned}$$

that coincides with the standard candidate inside each subdomain of the partition. Moreover, the force of B_ε coincides with $\mathfrak{F}_{\mathrm{R}}(\,\cdot\,; \varepsilon)$ in the right neighborhood of $u_-^{\mathrm{R}}(\varepsilon)$ and with $\mathfrak{F}_{\mathrm{L}}(\,\cdot\,; \varepsilon)$ in the left neighborhood of $u_+^{\mathrm{L}}(\varepsilon)$. The functions

$$\begin{aligned}
\mathfrak{F}_{\mathrm{R},i}(\,\cdot\,; \varepsilon) &= F_{\mathrm{R}}(\,\cdot\,; a_i(\varepsilon), b_i(\varepsilon); \varepsilon), \quad 1 \leqslant i \leqslant k-1,\\
\mathfrak{F}_{\mathrm{L},i}(\,\cdot\,; \varepsilon) &= F_{\mathrm{L}}(\,\cdot\,; a_i(\varepsilon), b_i(\varepsilon); \varepsilon), \quad 1 \leqslant i \leqslant k-1,
\end{aligned} \tag{4.3.6}$$

are the right and the left monotone force flows on the corresponding domains.

Proof. Consider two multitrolleybuses: $\Omega_{\mathrm{Mtr,R}}(\{\alpha_i\}_{i=1}^{k})$ and $\Omega_{\mathrm{Mtr,L}}(\{\alpha_i\}_{i=1}^{k})$. Application of Propositions 4.3.10 and 4.3.11 to these foliations gives us numbers $\eta_2^{\mathrm{R}}, \eta_2^{\mathrm{L}}$ and collections of functions $\{a_j, b_j\}_{j=1}^{k-1}$ (all of them are defined on an interval $[\eta_1, \eta_2]$, where $\eta_2 = \min(\eta_2^{\mathrm{R}}, \eta_2^{\mathrm{L}})$), which we call $\{a_j^{\mathrm{R}}, b_j^{\mathrm{R}}\}$ and $\{a_j^{\mathrm{L}}, b_j^{\mathrm{L}}\}$. Let $\varepsilon \in (\eta_1, \eta_2]$ be fixed.

We claim that if $b_j^{\mathrm{R}} \leqslant b_j^{\mathrm{L}}$ for some j, $1 < j < k$, then $b_{j-1}^{\mathrm{R}} < b_{j-1}^{\mathrm{L}}$. Indeed, we have the following chain of inequalities:

$$\begin{aligned} F_{\mathrm{R}}(b_{j-1}^{\mathrm{L}}; a_{j-1}^{\mathrm{L}}, b_{j-1}^{\mathrm{L}}; \varepsilon) &= F_{\mathrm{L}}(b_{j-1}^{\mathrm{L}}; a_j^{\mathrm{L}}, b_j^{\mathrm{L}}; \varepsilon) \\ &\geqslant F_{\mathrm{L}}(b_{j-1}^{\mathrm{L}}; a_j^{\mathrm{R}}, b_j^{\mathrm{R}}; \varepsilon) > F_{\mathrm{R}}(b_{j-1}^{\mathrm{L}}; a_{j-1}^{\mathrm{R}}, b_{j-1}^{\mathrm{R}}; \varepsilon). \end{aligned} \tag{4.3.7}$$

The first equality in (4.3.7) is simply the balance equation. The second inequality follows from Lemma 4.2.4 and our assumption that $b_j^{\mathrm{R}} \leqslant b_j^{\mathrm{L}}$. We note that the point b_{j-1}^{L} lies in the left tail of the chordal domain $\Omega_{\mathrm{ch}}([a_j^{\mathrm{R}}, b_j^{\mathrm{R}}], *)$, because it lies in the left tail of the larger chordal domain $\Omega_{\mathrm{ch}}([a_j^{\mathrm{L}}, b_j^{\mathrm{L}}], *)$ (see Corollary 4.2.5). Moreover, b_{j-1}^{L} lies on the left of the point a_j^{R}, which is the root of the balance equation of the forces $F_{\mathrm{R}}(\,\cdot\,; a_{j-1}^{\mathrm{R}}, b_{j-1}^{\mathrm{R}}; \varepsilon)$ and $F_{\mathrm{L}}(\,\cdot\,; a_j^{\mathrm{R}}, b_j^{\mathrm{R}}; \varepsilon)$, therefore the last inequality in (4.3.7) follows from Lemma 4.2.18. Inequality (4.3.7) and Lemma 4.2.4 imply that $b_{j-1}^{\mathrm{R}} < b_{j-1}^{\mathrm{L}}$. The claim is proved.

It follows that there exists $j = j(\varepsilon) \in \{1, 2, \dots, k\}$ such that for any i, with $0 < i < j$, the inequality $b_i^{\mathrm{R}} < b_i^{\mathrm{L}}$ holds true, and for any i, with $j \leqslant i < k$, one has $b_i^{\mathrm{R}} \geqslant b_i^{\mathrm{L}}$. We define $b_i(\varepsilon) = b_i^{\mathrm{R}}(\varepsilon)$ for $0 < i < j$ and $b_i(\varepsilon) = b_i^{\mathrm{L}}(\varepsilon)$ for $j \leqslant i < k$. We also put $a_i(\varepsilon)$, $1 \leqslant i \leqslant k-1$, in such a way that $[g(a_i(\varepsilon)), g(b_i(\varepsilon))]$ is a chord of $\Omega_{\mathrm{ch}}([\alpha_i^{\mathrm{r}}, \alpha_{i+1}^{\mathrm{l}}], *)$. In what follows we use the notation (4.3.6) and also put $\mathfrak{F}_{\mathrm{R},0} = \mathfrak{F}_{\mathrm{R}}$ and $\mathfrak{F}_{\mathrm{L},k} = \mathfrak{F}_{\mathrm{L}}$. We only have to prove that for ε sufficiently close to η_1, there is $u(\varepsilon) \in [b_{j(\varepsilon)-1}(\varepsilon), a_{j(\varepsilon)}(\varepsilon)]$ solving the balance equation for $\mathfrak{F}_{\mathrm{R},j(\varepsilon)-1}(\,\cdot\,;\varepsilon)$ and $\mathfrak{F}_{\mathrm{L},j(\varepsilon)}(\,\cdot\,;\varepsilon)$.

First, consider the case $1 < j(\varepsilon) < k$. Let $u_- = \max(b_{j(\varepsilon)-1}(\varepsilon), t^{\mathrm{L}})$, where t^{L} is the left end of the tail of $\mathfrak{F}_{\mathrm{L},j(\varepsilon)}(\,\cdot\,;\varepsilon)$. We claim that $\mathfrak{F}_{\mathrm{R},j(\varepsilon)-1}(u_-;\varepsilon) \leqslant \mathfrak{F}_{\mathrm{L},j(\varepsilon)}(u_-;\varepsilon)$. Indeed, if $u_- = t^{\mathrm{L}}$, then the claim is obvious:

$$\mathfrak{F}_{\mathrm{L},j(\varepsilon)}(t^{\mathrm{L}};\varepsilon) = 0 \geqslant \mathfrak{F}_{\mathrm{R},j(\varepsilon)-1}(t^{\mathrm{L}};\varepsilon),$$

because t^{L} lies in the tail of $\mathfrak{F}_{\mathrm{R},j(\varepsilon)-1}(\,\cdot\,;\varepsilon)$.

If $u_- = b_{j(\varepsilon)-1}(\varepsilon)$, then $u_- < b_{j(\varepsilon)-1}^{\mathrm{L}}(\varepsilon)$, therefore, due to Lemma 4.2.18, we have

$$\begin{aligned} &\mathfrak{F}_{\mathrm{R},j(\varepsilon)-1}(u_-;\varepsilon) - \mathfrak{F}_{\mathrm{L},j(\varepsilon)}(u_-;\varepsilon) \\ &\quad\leqslant \mathfrak{F}_{\mathrm{R},j(\varepsilon)-1}(b_{j(\varepsilon)-1}^{\mathrm{L}}(\varepsilon);\varepsilon) - \mathfrak{F}_{\mathrm{L},j(\varepsilon)}(b_{j(\varepsilon)-1}^{\mathrm{L}}(\varepsilon);\varepsilon) \\ &\quad< F_{\mathrm{R}}(b_{j(\varepsilon)-1}^{\mathrm{L}}; a_{j(\varepsilon)-1}^{\mathrm{L}}, b_{j(\varepsilon)-1}^{\mathrm{L}}; \varepsilon) - \mathfrak{F}_{\mathrm{L},j(\varepsilon)}(b_{j(\varepsilon)-1}^{\mathrm{L}}(\varepsilon);\varepsilon) = 0, \end{aligned} \tag{4.3.8}$$

where the second inequality follows from Lemma 4.2.4 and the fact that $b_{j(\varepsilon)-1}(\varepsilon) < b_{j(\varepsilon)-1}^{\mathrm{L}}(\varepsilon)$. Similarly, for $u_+ = \min(a_{j(\varepsilon)}(\varepsilon), t^{\mathrm{R}})$, where t^{R} is the right end of the tail of $\mathfrak{F}_{\mathrm{R},j(\varepsilon)-1}(\,\cdot\,;\varepsilon)$, we have $\mathfrak{F}_{\mathrm{R},j(\varepsilon)-1}(u_+;\varepsilon) \geqslant \mathfrak{F}_{\mathrm{L},j(\varepsilon)}(u_+;\varepsilon)$. Therefore, there exists a root $u(\varepsilon) \in [u_-, u_+]$ of the balance equation for the forces $\mathfrak{F}_{\mathrm{R},j(\varepsilon)-1}(\,\cdot\,;\varepsilon)$ and $\mathfrak{F}_{\mathrm{L},j(\varepsilon)}(\,\cdot\,;\varepsilon)$, which is unique due to Lemma 4.2.18.

Inequality (4.3.8) implies that $u(\varepsilon) > b_{j(\varepsilon)-1}(\varepsilon)$, therefore we have that $u(\varepsilon)$ lies in $(b_{j(\varepsilon)-1}(\varepsilon), a_{j(\varepsilon)}(\varepsilon)]$.

Now we consider the case $j(\varepsilon) = 1$ (the case $j(\varepsilon) = k$ is symmetric). Let $u_+ = \min(a_1(\varepsilon), t^{\mathrm{R}}(\varepsilon))$, where t^{R} is the right end of the tail of $\mathfrak{F}_{\mathrm{R},0}(\,\cdot\,;\varepsilon)$. We claim that $\mathfrak{F}_{\mathrm{R},0}(u_+;\varepsilon) \geqslant \mathfrak{F}_{\mathrm{L},1}(u_+;\varepsilon)$. Indeed, if $u_+ = t^{\mathrm{R}}$, then the argument is the same as before: $\mathfrak{F}_{\mathrm{R},0}(t^{\mathrm{R}};\varepsilon) = 0 \geqslant \mathfrak{F}_{\mathrm{L},1}(t^{\mathrm{R}};\varepsilon)$. If $u_+ = a_1(\varepsilon)$, then $u_+ \geqslant a_1^{\mathrm{R}}(\varepsilon)$, and from Lemma 4.2.18 and Lemma 4.2.4, we obtain

$$\begin{aligned}\mathfrak{F}_{\mathrm{R},0}(u_+;\varepsilon) - \mathfrak{F}_{\mathrm{L},1}(u_+;\varepsilon) &\geqslant \mathfrak{F}_{\mathrm{R},0}(a_1^{\mathrm{R}};\varepsilon) - \mathfrak{F}_{\mathrm{L},1}(a_1^{\mathrm{R}};\varepsilon)\\ &= \mathfrak{F}_{\mathrm{R},0}(a_1^{\mathrm{R}};\varepsilon) - F_{\mathrm{L}}(a_1^{\mathrm{R}};a_1^{\mathrm{L}},b_1^{\mathrm{L}};\varepsilon)\\ &\geqslant \mathfrak{F}_{\mathrm{R},0}(a_1^{\mathrm{R}};\varepsilon) - F_{\mathrm{L}}(a_1^{\mathrm{R}};a_1^{\mathrm{R}},b_1^{\mathrm{R}};\varepsilon) = 0.\end{aligned}$$

The claim is proved.

Recall that we want to prove that, for ε sufficiently close to η_1, there exists $u(\varepsilon)$ in $[b_0(\varepsilon), a_1(\varepsilon)]$ solving the balance equation for $\mathfrak{F}_{\mathrm{R},0}(\,\cdot\,;\varepsilon)$ and $\mathfrak{F}_{\mathrm{L},1}(\,\cdot\,;\varepsilon)$. Assume the contrary: for some sequence $\varepsilon_n \to \eta_1+$, the function $\Phi_n(\,\cdot\,) = \mathfrak{F}_{\mathrm{R},0}(\,\cdot\,;\varepsilon_n) - \mathfrak{F}_{\mathrm{L},1}(\,\cdot\,;\varepsilon_n)$ has no balance points on $(b_0(\varepsilon_n), a_1(\varepsilon_n)]$. Then $\Phi_n(u_+(\varepsilon_n)) > 0$ and therefore Φ_n is positive on the intersection of the tails, hence t^{L}, the left end of the tail of $\mathfrak{F}_{\mathrm{L},1}(\,\cdot\,;\varepsilon_n)$, is not greater than $b_0(\varepsilon_n)$. The function Φ_n is strictly increasing, by Lemma 4.2.18, and $\Phi_n(a_1(\eta_1))$ tends to 0 when $n \to +\infty$. Therefore, Φ_n tends to zero uniformly on $(b_0(\eta_1), a_1(\eta_1))$ (we recall that $b_0(\eta_1) \geqslant b_0(\varepsilon)$ for any $\varepsilon > \eta_1$, by the definition of a monotone force flow). Thus, for $t \in (b_0(\eta_1), a_1(\eta_1))$, we have

$$\lim_{n\to+\infty} \mathfrak{F}_{\mathrm{L},1}(t;\varepsilon_n) - \mathfrak{R}(t) = \lim_{n\to+\infty} \mathfrak{F}_{\mathrm{R},0}(t;\varepsilon_n) - \mathfrak{R}(t) = \mathfrak{F}_{\mathrm{R}}(t;\eta_1) - \mathfrak{R}(t).$$

The function on the right-hand side is strictly decreasing on $(b_0(\eta_1), a_1(\eta_1))$ and the functions on the left-hand side are strictly increasing on $(b_0(\eta_1), a_1(\eta_1))$ according to Remark 3.6.15. This leads to contradiction and proves the statement. ■

Remark 4.3.14. We note that $j(\varepsilon)$ in Proposition 4.3.13, indeed, could depend on ε, i.e., during the evolution, the angle could change its place between the trolleybuses. Moreover, the function $j(\cdot)$ could have an infinite number of jumps even on a bounded interval, see the example “Oscillating birdie” in [17, p. 123].

Remark 4.3.15. We have seen in the proof that the root $u(\varepsilon)$ of the balance equation is in the semiclosed interval $(b_{j(\varepsilon)-1}(\varepsilon), a_{j(\varepsilon)}(\varepsilon)]$. It may occur that $u(\varepsilon) = a_{j(\varepsilon)}(\varepsilon)$, and in this case, the angle $\Omega_{\mathrm{ang}}(u(\varepsilon);\varepsilon)$, the tangent domain $\Omega_{\mathrm{L}}(u(\varepsilon), u(\varepsilon);\varepsilon)$, and the trolleybus $\Omega_{\mathrm{tr,L}}(u(\varepsilon), b_{j(\varepsilon)}(\varepsilon);\varepsilon)$ glue together forming a birdie, see formula (3.8.2). Moreover, this equation could be valid for ε in some interval; the birdie can shrink without disintegrating.

4.4 Global evolution

Before passing to formal statements, we describe the rules of the evolution.

Recall that in Section 3.8.2 we constructed the graph Γ corresponding to the foliation of a Bellman candidate, and its subgraph Γ^{free} corresponding to subdomains of the foliation that are not separated from the free boundary $\partial_{\mathrm{free}}\,\Omega$. The vertices of the graph Γ are of two types: the vertices corresponding to linearity domains and fictitious vertices. The edges of Γ always correspond to fences: either chordal or tangent domains. The vertices of Γ^{free} correspond to linearity domains: multicups, angles, trolleybuses, multitrolleybuses, birdies, multibirdies, fictitious vertices of the first, third (corresponding to long chords), fourth, and fifth type.

To each edge $\mathfrak{E}$ of Γ^{free} we assign a force by the formal rule described in Table 4.1. In the first column we have the type of the vertex from where the edge starts. The numerical parameters of this vertex are placed in the second column. The force that is assigned to the tangent domain lying on the left of the figure is in the third column, and the force that is assigned to the tangent domain lying on the right of the figure is in the last one.

vertex type	parameters	left force	right force
right trolleybus	$\{a,b\}$		$F_{\mathrm{R}}(\,\cdot\,;a,b;\varepsilon)$
left trolleybus	$\{a,b\}$	$F_{\mathrm{L}}(\,\cdot\,;a,b;\varepsilon)$	
multicup	$\{\alpha_i\}_{i=1}^{k}$	$F_{\mathrm{L}}(\,\cdot\,;\alpha_1^{\mathrm{l}},\alpha_k^{\mathrm{r}};\varepsilon)$	$F_{\mathrm{R}}(\,\cdot\,;\alpha_1^{\mathrm{l}},\alpha_k^{\mathrm{r}};\varepsilon)$
right multitrolleybus	$\{\alpha_i\}_{i=1}^{k}$		$F_{\mathrm{R}}(\,\cdot\,;\alpha_1^{\mathrm{l}},\alpha_k^{\mathrm{r}};\varepsilon)$
left multitrolleybus	$\{\alpha_i\}_{i=1}^{k}$	$F_{\mathrm{L}}(\,\cdot\,;\alpha_1^{\mathrm{l}},\alpha_k^{\mathrm{r}};\varepsilon)$	
fict. vert. type 1	$\{a,b\}$	$F_{\mathrm{L}}(\,\cdot\,;a,b;\varepsilon)$	$F_{\mathrm{R}}(\,\cdot\,;a,b;\varepsilon)$
fict. vert. type 3	$\{a,b\}$	$F_{\mathrm{L}}(\,\cdot\,;a,b;\varepsilon)$	$F_{\mathrm{R}}(\,\cdot\,;a,b;\varepsilon)$
fict. vert. type 4	$-\infty$		$F_{\mathrm{R}}(\,\cdot\,;-\infty;\varepsilon)$
fict. vert. type 4	$+\infty$	$F_{\mathrm{L}}(\,\cdot\,;+\infty;\varepsilon)$	
right fict. vert. type 5	c_i		$F_{\mathrm{R}}(\,\cdot\,;c_i,c_i;\varepsilon)$
left fict. vert. type 5	c_i	$F_{\mathrm{L}}(\,\cdot\,;c_i,c_i;\varepsilon)$	

Table 4.1. The vertices of Γ^{free} with the corresponding parameters and forces.

All the foliations generated during the evolution process satisfy the following rule: if $\Omega_{\mathrm{R}}(u_1,u_2)$ or $\Omega_{\mathrm{L}}(u_1,u_2)$ is represented by the edge $\mathfrak{E}$ in Γ^{free}, then (u_1,u_2) belongs to the tail of the force corresponding to $\mathfrak{E}$. This requirement for the foliation will be called the *non-degeneracy force condition*.

Condition 4.4.1. *For each edge $\mathfrak{E}$ in Γ^{free} that corresponds to a tangent domain $\Omega_{\mathrm{R}}(u_1,u_2)$ or $\Omega_{\mathrm{L}}(u_1,u_2)$, the interval (u_1,u_2) belongs to the tail of the force assigned to $\mathfrak{E}$.*

A short inspection of definitions shows that Condition 4.4.1 holds true for all the graphs corresponding to the standard candidates constructed. In other words, all the forces in tangent domains are strictly negative. In particular, the following remark is important.

Remark 4.4.2. The Bellman candidate constructed for a simple picture in Section 4.1 fulfills the non-degeneracy force Condition 4.4.1.

As has already been said, the main rule of the evolution is that the forces decrease (grow in absolute value), see Section 4.2.2. As a consequence, the tails strictly grow (by this we mean that the t^{R} increase and the t^{L} decrease). Thus, full chordal domains grow (Proposition 4.3.1), the multicups are stable (Proposition 4.3.2),[1] the trolleybuses shrink (Propositions 4.3.4 and 4.3.5), the angles continuously wander from side to side (Proposition 4.3.12). These figures can be described as stable. If there are multitrolleybuses or multibirdies in the foliation for a fixed ε, they immediately disintegrate (Propositions 4.3.10, 4.3.11, and 4.3.13). These figures are unstable. As for the birdie, it can shrink (see Remark 4.3.15), but in general it disintegrates. Thus, it is half-stable.

There is also one useful condition that all our graphs will satisfy. It is of structural character (and thus relies on Definition 2.1.13) and concerns mostly fictitious vertices. It is called the *leaf-root condition*.

Condition 4.4.3. *Any arc of any multifigure that is not a single point coincides with one of the solid roots c_i. Numerical parameters of the fictitious vertices of the second type are some roots c_i that are single points. Each fictitious vertex of the third type corresponds to a chord $[g(a_0), g(b_0)]$ with at least one vanishing differential. If $D_{\mathrm{R}}(a_0, b_0) = 0$, then $b_0 = c_i$ for some single point root c_i; if $D_{\mathrm{L}}(a_0, b_0) = 0$, then $a_0 = c_j$, where c_j must be a single point root as well. The numeric parameter of each vertex of the fifth type is also a single point root c_i.*

Remark 4.4.4. All simple graphs constructed in Section 4.1 fulfill the leaf-root Condition 4.4.3.

Definition 4.4.5. Let $\varepsilon < \varepsilon_{\max}$. We say that a graph Γ is *admissible* for f and ε if all figures corresponding to the vertices and edges of Γ satisfy their local propositions.

By “all figures corresponding to the vertices and edges of Γ satisfy their local propositions” we mean the following: for each vertex or edge in Γ, the parameters satisfy the assumptions of the proposition indicated for this vertex or edge in Table 4.2 (in the third column).

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[1] In a sense, they also grow: the border tangents rise; however, the numerical parameters do not change.

vertex or edge type	formulas	verification	evolution rule
right tangent domain	(3.2.11), (3.2.2)	3.3.3, 3.3.6	
left tangent domain	(3.2.11), (3.2.2)	3.3.5, 3.3.7	
chordal domain	(3.4.1)	3.4.2	4.3.1
angle	(3.6.3)	3.6.4	4.3.12
right trolleybus	(3.7.1)	3.7.2	4.3.4
left trolleybus	(3.7.1)	3.7.2	4.3.5
birdie	(3.7.1)	3.7.2	4.3.13
multicup	(3.7.6), (3.7.1)	3.7.13	4.3.2
full multicup	(3.7.6), (3.7.1)	3.7.13	4.3.3
right multitrolleybus	(3.7.6), (3.7.1)	3.7.13	4.3.10
left multitrolleybus	(3.7.6), (3.7.1)	3.7.13	4.3.11
multibirdie	(3.7.6), (3.7.1)	3.7.13	4.3.13
closed multicup	(3.7.6), (3.7.1)	3.7.13	stable
fict. vert. type 1	(3.4.1)	3.6.4	4.3.1
fict. vert. type 3, long chord	(3.4.1)	3.6.4	4.3.3
fict. vert. type 5	(3.7.4)	3.6.4	4.3.7

Table 4.2. Elementary figures, corresponding Bellman candidates, and evolution rules.

In the first column, there is the type of the vertex or edge, in the second there is a reference to formulas that are used to construct the canonical function B in the corresponding figure, and in the third column the number of the proposition that guarantees that this B is a Bellman candidate is given. Finally, the last column contains the number of the proposition that describes the local evolution of the parameters for the figure. We have omitted fictitious vertices of the second and fourth types (as well as the vertices of the third type that correspond to short chords), because the value of the function B in the domains corresponding to them is defined trivially, and these figures are stable and have no evolution scenarios.

Now we describe how to construct the function B from a graph. First, one constructs this function to be the standard candidate on all the domains corresponding to vertices and edges that participate in $\Gamma \setminus \Gamma^{\mathrm{free}}$, because for their figures there is no additional information needed to construct B (no information from other figures). Second, we construct the function B to be a standard candidate on all the domains corresponding to vertices of Γ^{free} not being leaves (i.e., except angles). Third, we construct the standard candidates for the edges of Γ^{free}. For each such edge $\mathfrak{E}$, the values of B in the figure corresponding to its beginning define the force function on the domain corresponding to the edge (see Table 4.1), thus one may construct B in the tangent domain corresponding to $\mathfrak{E}$ if he knows the values of B on the domain of its source.[2] Finally, we construct B on the domains corresponding to leaves of Γ^{free} (i.e., on angles), because we know the values of B on the linear boundary of each such angle. Note that if Γ fulfills Condition 4.4.1, then the restriction of B to each figure is a standard Bellman candidate there. Admissibility of the graph guarantees that the force function (defined locally on each element of the foliation by the rules from Table 4.1) is a non-positive continuous function on $\mathbb{R}$.

Remark 4.4.6. The function B constructed from an admissible graph is a Bellman candidate.

Proof. We need to verify conditions of Definition 2.2.4 for the function B. Looking at Table 4.2, we use the corresponding verification proposition for each vertex or edge and see that the function B has the foliation on the entire Ω_ε. We also note that the function B is locally concave and C^1-smooth not only on subdomains, but globally. ∎

Since during the evolution some figures grow and angles move, several figures might collide. For example, the vertex of an angle may coincide with the right end-point of a long chord. In such a case, we look at formula (3.8.1), and see that now they form a trolleybus. Therefore, the graph of the foliation changes at this moment ε. We call such moments *the critical points of the evolution*. The idea is that if a collision occurs, then the figures involved form a new one (according to formulas from Section 3.8.1), and we can proceed the evolution. Unfortunately, there might be infinitely many critical points (see the example in [17, p. 123], where an angle flips the direction of a trolleybus infinitely many times). However, if one focuses only on those critical points, at which the structure of the graph Γ^{fixed} changes essentially, one finds only a finite number of critical points. Such points are called *essentially critical*. The following definition is also useful.

Definition 4.4.7. We say that a graph Γ is *smooth* if there are no vertices representing full multicups, multitrolleybuses, multibirdies, fictitious vertices of the third type that represent long chords, and fictitious vertices of the fifth type in Γ.

Theorem 4.4.8. *For any $\varepsilon < \varepsilon_{\max}$, there exists a graph $\Gamma(\varepsilon)$ admissible for f and ε.*

We will not give a careful proof of the theorem because it repeats literally the proof of the same theorem for the BMO case (see [17, Theorem 4.4.15]). Here we will describe the main steps of the proof.

First, we use Theorem 4.1.2 to build a smooth admissible graph $\Gamma(\varepsilon)$ for small ε. Then we use local evolution theorems from Section 4.3 collected in Table 4.2 to show

[2] There is one exception: for tangent domains coming from infinity, one does not need any boundary data.

that if there exists an admissible graph $\Gamma(\eta_1)$ for some η_1, then we can construct a smooth admissible graph $\Gamma(\varepsilon)$ for ε, $\varepsilon > \eta_1$, sufficiently close to η_1. If for some η_2 we have smooth graphs $\Gamma(\varepsilon)$, $\varepsilon \in (\eta_1, \eta_2)$, we can pass to the limit and construct a limit graph $\Gamma(\eta_2)$. It can happen that, in the limit graph $\Gamma(\eta_2)$, some edges "have zero length". In this case we modify the graph using formulas from Section 3.8.1. This modified graph can be non-smooth but it is admissible and we can continue evolution starting from it. It appears that under our assumptions there is only a finite number of essentially critical points of the evolution, when the graph $\Gamma(\varepsilon)$ is not smooth. In such a way, we obtain the graph $\Gamma(\varepsilon)$ for any $\varepsilon < \varepsilon_{\max}$.