Chapter 5

Optimizers

In the previous chapter, we constructed a Bellman candidate B of a special form (Theorem 4.4.8 and Remark 4.4.6). We claim that it coincides with the Bellman function $\boldsymbol{B}_\varepsilon$. Section 2.2.2 suggests a method to prove the claim. We have to construct an optimizer φ_x for each $x \in \Omega_\varepsilon$ (see Definition 2.2.6). Here we will follow the same strategy as when we were constructing Bellman candidates: we will first study the local behavior of the optimizers (i.e., how do optimizers vary when x runs through one figure), this is done in Section 5.2, and then "glue" these local scenarios together in Section 5.3. The optimizers for the BMO case were built in [17], and here we will follow a similar strategy for the general case. In Sections 5.1 and 5.2 we will not use evolution, so ε is fixed and we omit the subscript ε till Section 5.3. In particular, we will write Ξ instead of Ξ_ε and $\Omega = \Xi_0 \setminus \Xi$ instead of Ω_ε, see Section 2.1 for the definitions of these objects. In Section 5.4 we will consider the cases when conditions (2.1.7) and (2.1.8) are violated.

5.1 Abstract theory

We begin with an abstract description of what optimizers look like. First, as mentioned in Section 2.2.2, it is natural to construct monotone optimizers. It is not difficult to build a monotone function φ_x such that $\langle \varphi_x \rangle = x$ and $B(x) = \langle \mathrm{f}(\varphi_x) \rangle$. The main difficulty is to verify that $\varphi_x \in \boldsymbol{A}_\Omega$. It was noticed in [16] that it is more natural to argue geometrically. The notion of a *delivery curve* is useful in this context.

Definition 5.1.1. Let B be a Bellman candidate on the domain Ω. Suppose $\varphi\colon [l,r] \to \partial_{\mathrm{fixed}}\Omega$ is an integrable function. The curve $\gamma\colon (l,r] \to \Omega$, given by the formula

$$\gamma(\tau) \stackrel{\mathrm{def}}{=} \langle \varphi \rangle_{[l,\tau]}, \quad \tau \in (l,r], \tag{5.1.1}$$

is called a *delivery curve* if $B(\gamma(\tau)) = \langle \mathrm{f}(\varphi) \rangle_{[l,\tau]}$ for any $\tau \in (l,r]$ (in particular, $\gamma(\tau) \in \Omega$). The function φ is called the *generating function* for γ.

In other words, γ is a curve that "delivers" optimizers to the point. The word "curve" here means a parametrized curve, because the definition depends on the parametrization. The advantage of considering such a curve is that it allows to verify the condition that φ is a test function (i.e., $\varphi \in \boldsymbol{A}_\Omega$).

The main feature we will use is the formula

$$\gamma(\tau) + (\tau - l)\gamma'(\tau) = \varphi(\tau), \tag{5.1.2}$$

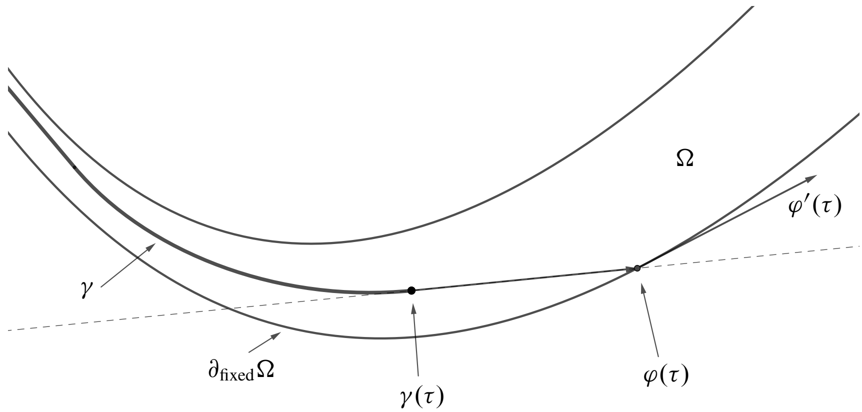

Figure 5.1. Illustration to the proof of Lemma 5.1.2.

which can be obtained by differentiation of (5.1.1). In particular, this formula shows that the tangent to γ at τ points in the direction of $\varphi(\tau)$. Thus, one can reconstruct the values of φ by looking at the points on the fixed boundary that "are indicated" by the tangents of the corresponding delivery curve. We will use this principle very often. Moreover, equation (5.1.2) allows us to reconstruct φ from γ.

Lemma 5.1.2. *A curve given by formula* (5.1.1) *is convex if its generating function is monotone.*

Proof. We will give a proof for the case of an increasing generating function φ. The case of decreasing function is symmetric.

Let us assume for a while that the function φ is C^2-smooth. In such a case, we may differentiate (5.1.2) and get

$$(\tau - l)\gamma''(\tau) = -2\gamma'(\tau) + \varphi'(\tau).$$

Thus, the curvature of γ, which is $(\gamma_1')^{-3}\det\binom{\gamma'}{\gamma''}$, has the same sign as $\det\binom{\gamma'}{\varphi'}$, because $\gamma_1' > 0$. We use (5.1.2) once again to express γ' and rewrite the determinant in the following form:

$$\det\begin{pmatrix}\gamma'\\ \varphi'\end{pmatrix} = \frac{1}{\tau - l}\det\begin{pmatrix}\varphi - \gamma\\ \varphi'\end{pmatrix}.$$

This expression is positive, because φ' is a tangent vector to $\partial_{\mathrm{fixed}}\Omega$ at $\varphi(\tau)$, with $\varphi_1' > 0$, and $\gamma(\tau)$ belongs to Ω, see Figure 5.1.

We only have to get rid of the smoothness assumption. One can approximate φ by smooth increasing functions $\varphi_n\colon (l, r] \to \partial_{\mathrm{fixed}}\Omega$ in such a way that the curves γ_n

generated by φ_n converge to γ pointwise. Each γ_n is a convex curve (in the sense that these curves are the graphs of convex functions in the standard coordinates) and $\gamma_n \to \gamma$ pointwise. Therefore, γ is a convex curve itself. ■

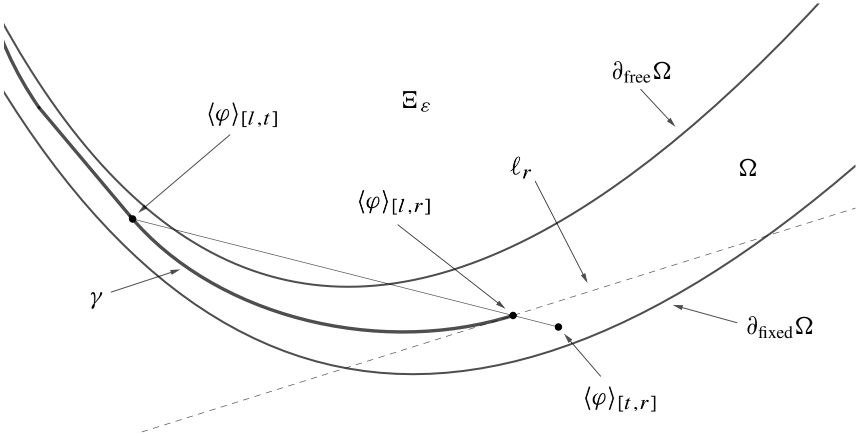

Figure 5.2. Illustration to the proof of Lemma 5.1.3.

The following lemma links the condition that γ is convex with the condition that $\varphi \in \boldsymbol{A}_\Omega$ (see Figure 5.2 for a visualization of the proof). The symbol $\gamma'(r)$ in the lemma below means the left derivative, which always exists due to convexity.

Lemma 5.1.3. *Suppose that γ is a curve parametrized by the interval $(l, r]$ and given by (5.1.1). Let it be convex in the sense that it is the graph of a convex function in the standard coordinates. Suppose also that the tangent line*

$$\ell_r = \{\gamma(r) + s\gamma'(r) \mid s \in \mathbb{R}\}$$

does not cross the domain Ξ (see Definition 2.1.1). Then, for any $t \in [l, r)$, we have $\langle\varphi\rangle_{[t,r]} \in \Omega$.

Proof. Since the curve γ is convex, it lies above the line ℓ_r. The domain Ξ also lies above the line ℓ_r. We note that $\langle\varphi\rangle_{[l,r]}$ is a convex combination of $\langle\varphi\rangle_{[l,t]}$ and $\langle\varphi\rangle_{[t,r]}$. Thus, $\langle\varphi\rangle_{[t,r]}$ is separated from Ξ by ℓ_r. On the other hand, $\langle\varphi\rangle_{[t,r]}$ surely belongs to Ξ_0, so it lies inside Ω. ■

In the following corollary, we write $\gamma'(\tau)$, $\tau \in (l, r]$, meaning any of the one-sided derivatives.

Corollary 5.1.4. *Suppose that γ is a delivery curve on $(l, r]$. Let it be convex in the sense that it is the graph of a convex function in the standard coordinates. Suppose* *also that the tangent line*

$$\ell_\tau = \{\gamma(\tau) + s\gamma'(\tau) \mid s \in \mathbb{R}\}$$

does not cross the domain Ξ *for any* $\tau \in (l, r]$. *Then the function* φ *that generates* γ *belongs to* $\boldsymbol{A}_\Omega$.

Before we pass to constructing specific delivery curves, we should postulate a heuristic principle that will help us to guess them. Since a delivery curve "consists of optimizers", it has to avoid the directions in which the Bellman candidate is non-linear. Thus, we guess that delivery curves should go either along the extremal segments or along the free boundary.

In Section 5.2 we will construct delivery curves for each elementary figure of the foliation. In the general case, a figure will have special points on the free boundary; we call them *incoming* and *outgoing nodes*. The idea is as follows: if we have special delivery curves for the incoming nodes, then we can construct a delivery curve for any point in the domain. Moreover, if the domain has outgoing nodes, then we construct special delivery curves for them. Every outgoing node for some domain is at the same time an incoming node for its neighbor domain. Continuing delivery curves along these special nodes allows us to construct optimizers for all the points in Ω.

5.2 Local behavior of optimizers

5.2.1 Optimizers for tangent domains

Consider a tangent domain $\Omega_{\mathrm{R}}(u_1, u_2)$ foliated by the segments $S(u) = [g(u), w(u)]$, $u \in [u_1, u_2]$, that are tangent to the free boundary of Ω. Let B be a standard candidate on $\Omega_{\mathrm{R}}(u_1, u_2)$. This domain has two linear parts of the boundary, namely, $S(u_1)$ and $S(u_2)$. The tangency points $w(u_1)$ and $w(u_2)$ on the free boundary are the *incoming* and *outgoing nodes* of $\Omega_{\mathrm{R}}(u_1, u_2)$, respectively. Recall the positive-valued function λ,

$$\lambda(u)(g(u) - w(u)) = w'(u), \quad u \in [u_1, u_2], \tag{5.2.1}$$

introduced in (2.1.6).

Suppose ψ is an optimizer for the incoming node $w(u_1)$ (see Figure 5.3 below) defined on the interval $(l, l_1]$. Our aim is to build the optimizers for all the points inside $\Omega_{\mathrm{R}}(u_1, u_2)$. We start with the points $w(u)$, $u \in [u_1, u_2]$, lying on the free boundary. We look for a function φ on $(l, r]$ for some $r > l_1$ such that $\varphi = \psi$ on $(l, l_1]$ and its delivery curve γ goes along the free boundary from $w(u_1)$ to $w(u_2)$ on $[l_1, r]$. We will find a monotone function $u\colon [l_1, r] \to [u_1, u_2]$ such that the function

$$\varphi(t) = g(u(t)) \tag{5.2.2}$$

generates the required delivery curve $\gamma(t) = \langle\varphi\rangle_{[l,t]}$ that will coincide with $w(u(t))$, $t \in [l_1, r]$. This is equivalent to the equation

$$(t-l)w(u(t)) = \int_l^t \varphi(\tau)\,d\tau.$$

We differentiate this identity with respect to t and obtain

$$w(u(t)) + (t-l)w'(u(t))u'(t) = \varphi(t) = g(u(t)).$$

Using (5.2.1), we obtain $(t-l)\lambda(u(t))u'(t) = 1$. We solve this differential equation with the boundary condition $u(l_1) = u_1$ and get

$$\log\frac{t-l}{l_1-l} = \int_{u_1}^{u(t)} \lambda(v)\,dv. \tag{5.2.3}$$

Since the function λ is positive and C^1-smooth (see Section 2.1.4), equality (5.2.3) defines the required C^2-smooth increasing function u on the interval $[l_1, r]$ (and the function φ defined by (5.2.2)), where

$$r = l + (l_1 - l)\exp\Big(\int_{u_1}^{u_2} \lambda(v)\,dv\Big).$$

We want to use Lemma 5.1.3 to verify that the function φ belongs to $\boldsymbol{A}_\Omega$. By construction, the curve $\gamma(t)$, $t \in (l_1, r]$, coincides with a part of the free boundary $\partial_{\mathrm{free}}\Omega$. Therefore, this part of the curve is convex and its tangents do not cross the domain Ξ. In order to use Lemma 5.1.3, we need the convexity of the curve γ on the whole interval (l, r). This consideration leads to the proposition below.

Proposition 5.2.1. *Let B be a candidate on $\Omega_{\mathrm{R}}(u_1, u_2)$. Suppose that there exists a non-decreasing optimizer ψ for B at the point $w(u_1)$. Let also $\psi \preceq g(u_1)$. Then there exists a non-decreasing optimizer φ_x for B at every point $x \in \Omega_{\mathrm{R}}(u_1, u_2)$; moreover, $\varphi_x \preceq g(u_2)$.*

Recall that the ordering $\preceq$ on $\partial_{\mathrm{fixed}}\Omega$ was introduced in Remark 2.1.2.

Proof. We have constructed the desired function φ_x for the points $x \in \Omega_{\mathrm{R}}(u_1, u_2) \cap \partial_{\mathrm{free}}\Omega$. For any point $x \in \Omega_{\mathrm{R}}(u_1, u_2)$, there exists a unique $u \in [u_1, u_2]$ such that $x \in S(u)$. We already know the optimizers at the endpoints $w(u)$ and $g(u)$ of this extremal segment. Namely, they are $\varphi_{w(u)}$ on an interval $(l, r]$ and the constant function $\varphi_{g(u)} \equiv g(u)$. We will obtain the desired optimizer φ_x extending the function $\varphi_{w(u)}$ by the constant $g(u)$ to some interval $(r, r_1]$ of appropriate length.

To prove the proposition, we need to show two things. First, we need to verify that the function φ_x lies in $\boldsymbol{A}_\Omega$. Second, we need to prove the equality $B(x) = \langle \mathrm{f}(\varphi_x)\rangle$ for the candidate B.

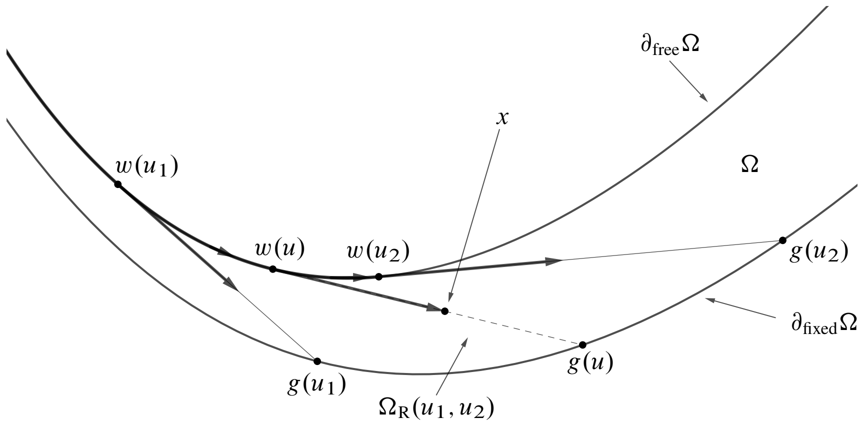

Figure 5.3. Delivery curves inside a tangent domain.

We start with the verification of convexity of the curve $\gamma = \gamma_x$ generated by $\varphi = \varphi_x$. By Lemma 5.1.2, the curve generated by ψ, i.e., $\gamma|_{(l,l_1]}$, is convex and its tangent line at the point $w(u_1)$ coincides with the tangent line to the free boundary (this is a consequence of the condition $\psi \preceq g(u_1)$ and (5.1.2)). Then the curve γ goes along the free boundary, which is convex, and then it goes along the tangent line $S(u)$. Therefore, this curve γ is convex. Let $l < t < \tau \leqslant r_1$. If $\tau \leqslant l_1$, then the point $\langle\varphi\rangle_{[t,\tau]}$ lies on the curve generated by ψ, and thus it lies in Ω by the assumption. If $\tau > l_1$, then the tangent line to γ at $\gamma(\tau)$ is the tangent line to the free boundary by construction. Therefore, this tangent line do not cross the domain Ξ, and we use Lemma 5.1.3 to conclude that $\langle\varphi\rangle_{[t,\tau]}$ lies in Ω. This proves that $\varphi \in \boldsymbol{A}_\Omega$.

It suffices to show that $B(\gamma(s)) = \langle \mathrm{f}(\varphi)\rangle_{[l,s]}$ for $s \in [l_1, r_1]$, because for $s \in (l, l_1]$, this follows from the fact that ψ is an optimizer. Thus, we need to check the following identity:

$$(s-l)B(\gamma(s)) - (l_1 - l)B(\gamma(l_1)) = \int_{l_1}^{s} \mathrm{f}(\varphi(\tau))\,d\tau. \tag{5.2.4}$$

After differentiating with respect to s, we obtain the following equivalent identity (because (5.2.4) holds for $s = l_1$):

$$B(\gamma(s)) + (s-l)\langle \nabla B(\gamma(s)), \gamma'(s)\rangle = \mathrm{f}(\varphi(s)) = B(\varphi(s)).$$

Since B is a Bellman candidate (see Definition 2.2.4), the gradient ∇B is constant on the extremal segment containing $\gamma(s)$ and $\varphi(s)$, therefore

$$B(\varphi(s)) - B(\gamma(s)) = \langle \nabla B, (\varphi(s) - \gamma(s))\rangle \overset{(5.1.2)}{=} (s-l)\langle \nabla B, \gamma'(s)\rangle. \qquad \blacksquare$$

We briefly state a symmetric proposition.

Proposition 5.2.2. *Let B be a candidate on $\Omega_{\mathrm{L}}(u_1, u_2)$. Suppose that there exists a non-increasing optimizer ψ for B at the point $w(u_2)$. Let also $g(u_2) \preceq \psi$. Then there exists a non-increasing optimizer φ_x for B at every point $x \in \Omega_{\mathrm{L}}(u_1, u_2)$; moreover, $g(u_1) \preceq \varphi_x$.*

The proof of similar propositions for infinite domains is slightly more complicated and requires the additional assumptions (2.1.7) and (2.1.8). We will use these assumptions together with the following technical lemma.

Lemma 5.2.3. *Let $\sigma \in \mathbb{R}$. Let ϑ and Y be two piecewise monotone continuous functions on $[\sigma, +\infty)$ with a finite number of intervals of monotonicity. Suppose that $\lim_{+\infty} \vartheta = 0$. Then the integration by parts formula is valid*:

$$\int_\sigma^{+\infty} \vartheta\, dY = \lim_{\nu \to +\infty} \int_\sigma^\nu \vartheta\, dY = -Y(\sigma)\vartheta(\sigma) - \lim_{\nu \to +\infty} \int_\sigma^\nu Y\, d\vartheta$$
$$= -Y(\sigma)\vartheta(\sigma) - \int_\sigma^{+\infty} Y\, d\vartheta, \tag{5.2.5}$$

where both limits exist (finite or infinite).

Symmetrically, for piecewise monotone continuous functions ϑ and Y on $(-\infty, \sigma]$ with a finite number of intervals of monotonicity, if $\lim_{-\infty} \vartheta = 0$, then

$$\int_{-\infty}^\sigma \vartheta\, dY = \lim_{\nu \to -\infty} \int_\nu^\sigma \vartheta\, dY = Y(\sigma)\vartheta(\sigma) - \lim_{\nu \to -\infty} \int_\nu^\sigma Y\, d\vartheta$$
$$= Y(\sigma)\vartheta(\sigma) - \int_{-\infty}^\sigma Y\, d\vartheta, \tag{5.2.6}$$

where both limits exist (finite or infinite).

Proof. We give the proof of (5.2.5). The proof of (5.2.6) is similar.

Since the integration by parts formula is valid on any finite interval, without loss of generality, we may assume that both Y and ϑ are monotone and do not change the sign on $[\sigma, +\infty)$. Then both integrals in (5.2.5) are monotone with respect to ν, therefore both limits exist, finite or infinite.

First, assume that $\int_\sigma^{+\infty} \vartheta\, dY$ converges. Take any $\nu > \sigma$ and write the integration by parts formula on the interval $[\sigma, \nu]$:

$$\int_\sigma^\nu \vartheta\, dY + Y(\sigma)\vartheta(\sigma) = Y(\nu)\vartheta(\nu) - \int_\sigma^\nu Y\, d\vartheta. \tag{5.2.7}$$

The left-hand side of (5.2.7) has a finite limit when $\nu \to +\infty$. The first summand on the right-hand side is of the same sign as the second one (because $\operatorname{sign}(\vartheta') = -\operatorname{sign}(\vartheta)$), therefore both summands have finite limits. If the limit of the first summand was nonzero, the limit of the second one would be infinite. Indeed,

$$|Y(\tau)| > \frac{\mathrm{Const}}{|\vartheta(\tau)|}, \quad \tau \to +\infty, \tag{5.2.8}$$

therefore

$$\left|\int^{+\infty} Y(\tau)\,d\vartheta(\tau)\right| > \mathrm{Const}\cdot\left|\int^{+\infty}\frac{d\vartheta(\tau)}{\vartheta(\tau)}\right| = +\infty, \tag{5.2.9}$$

because $\vartheta(+\infty) = 0$. Thus, the first summand on the right-hand side of (5.2.7) vanishes at $+\infty$. Formula (5.2.5) is proved.

Now, let us assume that $\int_\sigma^{+\infty} Y\,d\vartheta$ converges. The integral on the left-hand side of (5.2.7) is monotone with respect to ν, thus it has a finite or infinite limit when ν tends to $+\infty$. Therefore, the first term on the right-hand side of (5.2.7) also has a finite or infinite limit. If this limit is nonzero, then we have (5.2.8) and (5.2.9), which contradicts our assumption. Thus, the first summand on the right-hand side of (5.2.7) vanishes, the left-hand side of (5.2.7) has a finite limit when ν tends to $+\infty$, and (5.2.5) is proved.

Finally, if both limits in (5.2.5) are infinite, we need to show that the infinities on both sides have the same sign. We may assume that ϑ is positive and decreasing. We may also assume that Y is monotone and non-negative at infinity. If Y were bounded at infinity, then the limit on the left-hand side of (5.2.5) would be finite. Therefore, Y increases at infinity, and both sides of (5.2.5) are equal $+\infty$. ■

Lemma 5.2.4. *Suppose that condition* (2.1.7) *is satisfied. Then condition* (2.1.13) *is equivalent to the fact that* β_2 *given by* (3.3.3)*, i.e.,*

$$\beta_2(v) = \mathfrak{K}(v) - \int_{-\infty}^{v}\exp\Big(-\int_\tau^v \frac{\kappa_2'}{\kappa_2-\kappa}\Big)\mathfrak{K}'(\tau)\,d\tau, \tag{5.2.10}$$

satisfies the inequality $\beta_2(v) < +\infty$ *for any* $v \in \mathbb{R}$*. Moreover,*

$$\begin{aligned}\int_{-\infty}^{v} f'(\tau)\exp\Big(-\int_\tau^v \lambda\Big)\,d\tau \\ = -(w_1(v) - g_1(v))[\kappa_3(v) + (\kappa(v) - \kappa_2(v))\beta_2(v)].\end{aligned} \tag{5.2.11}$$

Here $\lambda = \lambda_{\mathrm{R}}$, $\kappa = \kappa_{\mathrm{R}}$, *and* $w = w_{\mathrm{R}}$.

Proof. Let us modify the expression in the exponent on the right-hand side of (5.2.10). Using (3.3.2), (2.1.6), and (2.1.11), we obtain

$$\int \frac{\kappa_2'}{\kappa_2-\kappa} = \log(\kappa_2-\kappa) + \log(g_1 - w_1) + \int\lambda. \tag{5.2.12}$$

First, we claim that

$$\int_{-\infty}^{v}\frac{\kappa_2'}{\kappa_2-\kappa} = +\infty, \tag{5.2.13}$$

which is equivalent to

$$\lim_{\tau\to-\infty}(\kappa_2(\tau)-\kappa(\tau))(g_1(\tau)-w_1(\tau))\exp\Big(-\int_\tau^v\lambda\Big) = 0,$$

by (5.2.12). Recall that $(g_1 - w_1)\kappa = (g_2 - w_2)$, and therefore

$$\lim_{\tau\to-\infty} \kappa(\tau)(g_1(\tau) - w_1(\tau)) \exp\Big(-\int_\tau^v \lambda\Big)$$
$$= \lim_{\tau\to-\infty} (g_2(\tau) - w_2(\tau)) \exp\Big(-\int_\tau^v \lambda\Big) = 0, \tag{5.2.14}$$

due to condition (2.1.7). The function $\kappa_2 = g_2'/g_1'$ is increasing by the convexity of the curve g. There are two cases. If κ_2 is bounded on $-\infty$, then

$$\lim_{\tau\to-\infty} \kappa_2(\tau)(g_1(\tau) - w_1(\tau)) \exp\Big(-\int_\tau^v \lambda\Big) = 0, \tag{5.2.15}$$

due to the same condition (2.1.7). If κ_2 is not bounded on $-\infty$, then $\kappa_2(\tau) \to -\infty$ as $\tau \to -\infty$, and then we use the fact that $0 > \kappa_2 > \kappa$, provided τ is sufficiently close to $-\infty$. Therefore, $|\kappa_2| < |\kappa|$ and (5.2.15) follows from (5.2.14). Thus, the claimed divergence in (5.2.13) is proved.

We wish to use Lemma 5.2.3 with

$$\vartheta(\tau) = \exp\Big(-\int_\tau^v \frac{\kappa_2'}{\kappa_2 - \kappa}\Big) \quad \text{and} \quad Y(\tau) = \mathfrak{K}(\tau)$$

to integrate by parts and rewrite formula (5.2.10) for β_2. We verify the hypotheses of Lemma 5.2.3. It is clear that $\kappa_2' > 0$, $\kappa_2 > \kappa$, therefore the function ϑ is monotone and tends to 0 at $-\infty$ due to (5.2.13). The function Y is piecewise monotone according to Condition 2.1.11. Thus, by Lemma 5.2.3, we get

$$\beta_2(v) = \int_{-\infty}^v \frac{\kappa_2'(\tau)}{\kappa_2(\tau) - \kappa(\tau)} \exp\Big(-\int_\tau^v \frac{\kappa_2'}{\kappa_2 - \kappa}\Big)\mathfrak{K}(\tau)\, d\tau$$
$$= \int_{-\infty}^v \frac{\kappa_3'(\tau)}{\kappa_2(\tau) - \kappa(\tau)} \exp\Big(-\int_\tau^v \frac{\kappa_2'}{\kappa_2 - \kappa}\Big)\, d\tau. \tag{5.2.16}$$

Now we plug (5.2.12) into (5.2.16) and obtain

$$\beta_2(v) = \frac{1}{\kappa_2(v) - \kappa(v)} \cdot \frac{1}{g_1(v) - w_1(v)}$$
$$\times \int_{-\infty}^v \kappa_3'(\tau)\big(g_1(\tau) - w_1(\tau)\big) \exp\Big(-\int_\tau^v \lambda\Big)\, d\tau. \tag{5.2.17}$$

The next step is to integrate by parts again, using Lemma 5.2.3 once more, this time with

$$\vartheta(\tau) = (g_1(\tau) - w_1(\tau)) \exp\Big(-\int_\tau^v \lambda\Big) \quad \text{and} \quad Y(\tau) = \kappa_3(\tau).$$

We verify the hypotheses of Lemma 5.2.3. First, we have that $\vartheta \to 0$ when $\tau \to -\infty$, due to (2.1.7). Second, we calculate ϑ' using the relation $(g_1 - w_1)\lambda = w_1'$. This yields

$$\begin{aligned}\vartheta'(\tau) &= \big(\lambda(\tau)(g_1(\tau) - w_1(\tau)) + g_1'(\tau) - w_1'(\tau)\big)\exp\Big(-\int_\tau^v \lambda\Big)\\ &\overset{(2.1.6)}{=} g_1'(\tau)\exp\Big(-\int_\tau^v \lambda\Big) > 0.\end{aligned}$$

Thus, ϑ is monotone. Finally, Y is piecewise monotone by Remark 2.1.12. Thus, we may apply Lemma 5.2.3 and integrate by parts in (5.2.17):

$$\begin{aligned}\beta_2(v) = \frac{\kappa_3(v)}{\kappa_2(v) - \kappa(v)} - \frac{1}{\kappa_2(v) - \kappa(v)} \cdot \frac{1}{g_1(v) - w_1(v)}\\ \times \int_{-\infty}^{v} \kappa_3(\tau) g_1'(\tau)\exp\Big(-\int_\tau^v \lambda\Big)\, d\tau.\end{aligned}$$

Since $\kappa_3 g_1' = f'$, we get

$$\beta_2(v) = \frac{\kappa_3(v)}{\kappa_2(v) - \kappa(v)} - \frac{1}{\kappa_2(v) - \kappa(v)} \cdot \frac{1}{g_1(v) - w_1(v)} \cdot \int_{-\infty}^{v} f'(\tau)\exp\Big(-\int_\tau^v \lambda\Big)\, d\tau.$$

This completes the proof of (5.2.11).

Condition (2.1.13) states that the left-hand side of (5.2.11) is greater than $-\infty$. According to (5.2.11), this is equivalent to $\beta_2 < +\infty$. ■

Let us turn back to consideration of optimizers.

Proposition 5.2.5. *Let B be the standard candidate on $\Omega_{\mathrm{R}}(-\infty, u_2)$. There exists a non-decreasing optimizer φ_x for B at every point $x \in \Omega_{\mathrm{R}}(-\infty, u_2)$. Moreover, we have $\varphi_x \preceq g(u_2)$.*

Proof. By Definition 3.3.8 of a standard candidate in the domain $\Omega_{\mathrm{R}}(-\infty, u_2)$, the function β_2 given by (3.3.3) is finite. Thus, the integral in (5.2.10) converges.

We begin with the points on the free boundary. When this is done, we will automatically get the desired optimizer φ_x for any $x \in \Omega_{\mathrm{R}}(-\infty, u_2)$. To this end, we pick any finite u_1 such that $x \in \Omega_{\mathrm{R}}(u_1, u_2)$ and use Proposition 5.2.1. Moreover, it suffices to construct an optimizer for $x = w(u_2)$ only.

Similar to the case of a bounded domain Ω_{R}, we would like to construct a function $\varphi\colon (l, r] \to \partial_{\mathrm{fixed}}\Omega$ on some interval $(l, r]$ such that the curve $\gamma(t) = \langle \varphi \rangle_{(l,t)}$ generated by φ goes along the free boundary from the infinity to $w(u_2)$. As before, we will find a function in the form $\varphi(t) = g(u(t))$, where $u\colon (l, r] \to (-\infty, u_2]$ is a monotone function. The previous reasoning (see (5.2.3)) leads us to the following relation:

$$\int_{u(t_1)}^{u(t_2)} \lambda(v)\, dv = \log\frac{t_2 - l}{t_1 - l}, \quad l < t_1 < t_2 \leqslant r. \tag{5.2.18}$$

We want to have $u(r) = u_2$. We substitute $t_2 = r$ into (5.2.18) and obtain

$$\int_{u(t_1)}^{u_2} \lambda(v)\,dv = \log\frac{r-l}{t_1-l}, \qquad l < t_1 < r. \tag{5.2.19}$$

According to condition (2.1.7), we have

$$\int_{-\infty}^{u_2} \lambda(v)\,dv = +\infty, \tag{5.2.20}$$

therefore (5.2.19) defines the function u on (l, r).

We may choose the length of the segment (l, r) to be equal 1, namely, we may take $l = 0$, $r = 1$. We get an explicit formula for the inverse function $t = t(u)$:

$$t = \exp\Big(-\int_u^{u_2} \lambda(v)\,dv\Big). \tag{5.2.21}$$

Corollary 5.1.4 guarantees $\varphi \in \boldsymbol{A}_\Omega$, provided we know that

$$\frac{1}{t}\int_0^t \varphi = w(u(t)), \qquad t \in (0, 1].$$

Therefore, for any $t \in (0, 1]$, we need to check that

$$\int_0^t \varphi = t\,w(u(t)). \tag{5.2.22}$$

Moreover, in order to prove that φ is an optimizer for B, we also need to verify that

$$B(w(u(t))) = \langle \mathrm{f}(\varphi)\rangle_{[0,t]} = \frac{1}{t}\int_0^t f(u(s))\,ds. \tag{5.2.23}$$

We start with (5.2.22). First, we prove that the limit of the right-hand side is zero as $t \to 0$. It is equivalent to

$$\lim_{u\to-\infty} w_i(u)\exp\Big(-\int_u^{u_2}\lambda(v)\,dv\Big) = 0, \qquad i = 1, 2. \tag{5.2.24}$$

Fix $i = 1, 2$. If the function w_i is bounded on $-\infty$, then (5.2.24) follows from (5.2.20). If w_i is not bounded on $-\infty$, then we apply L'Hôpital's rule:

$$\begin{aligned}\lim_{u\to-\infty}\frac{w_i(u)}{\exp(\int_u^{u_2}\lambda)} &= -\lim_{u\to-\infty}\frac{w_i'(u)}{\lambda(u)\exp(\int_u^{u_2}\lambda)}\\ &\overset{(5.2.1)}{=} \lim_{u\to-\infty}(w_i(u)-g_i(u))\exp\Big(-\int_u^{u_2}\lambda\Big) = 0,\end{aligned}$$

due to condition (2.1.7).

We calculate the integral on the left-hand side of (5.2.22):

$$\int_0^t \varphi(\tau)\,d\tau = \int_0^t g(u(\tau))\,d\tau \overset{(5.2.21)}{=} \int_{-\infty}^{u(t)} \lambda(v)g(v)\exp\Big(-\int_v^{u_2}\lambda\Big)\,dv. \tag{5.2.25}$$

Note that

$$\int \lambda(v)g(v)\exp\Big(-\int_v^{u_2}\lambda\Big)\,dv \overset{(5.2.1)}{=} \int (w'(v)+\lambda(v)w(v))\exp\Big(-\int_v^{u_2}\lambda\Big)\,dv$$
$$= w(v)\exp\Big(-\int_v^{u_2}\lambda\Big),$$

which converges to zero as $v \to -\infty$ because of (5.2.24). This proves that the integral in (5.2.25) converges and is equal to

$$w(u(t))\exp\Big(-\int_{u(t)}^{u_2}\lambda\Big) = t\,w(u(t)).$$

Relation (5.2.22) is now proved.

We turn to the proof of (5.2.23). The right-hand side of (5.2.23) can be rewritten using (5.2.21) in the following way:

$$\begin{aligned}\frac{1}{t}\int_0^t f(u(s))\,ds &= \exp\Big(\int_{u(t)}^{u_2}\lambda\Big)\cdot\int_0^t f(u(s))\,d\Big[\exp\Big(-\int_{u(s)}^{u_2}\lambda\Big)\Big]\\ &\overset{\tau=u(s)}{=} \int_{-\infty}^{u(t)} f(\tau)\lambda(\tau)\exp\Big(-\int_\tau^{u(t)}\lambda\Big)\,d\tau \qquad (5.2.26)\\ &= f(u(t)) - \int_{-\infty}^{u(t)} f'(\tau)\exp\Big(-\int_\tau^{u(t)}\lambda\Big)\,d\tau.\end{aligned}$$

The integration by parts is guaranteed by Lemma 5.2.3, with

$$\vartheta(\tau) = \exp\Big(-\int_\tau^{u(t)}\lambda\Big) \quad\text{and}\quad Y = f.$$

Finally, using (5.2.11), we obtain (5.2.23), see representation (3.2.2) for the relation between B and β_2. ∎

Proposition 5.2.6. *If* (2.1.7) *holds but* (2.1.13) *fails, then* $\boldsymbol{B}(x;f) = +\infty$ *for any* $x \in \Omega\setminus\partial_{\mathrm{fixed}}\Omega$.

Proof. It suffices to prove $\boldsymbol{B}(w(v);f) = +\infty$ for any $v\in\mathbb{R}$. Condition (2.1.7) guarantees that the function φ constructed in the proof of Proposition 5.2.5 is a test function for the point $w(v)$. The failure of (2.1.13) means

$$\int_{-\infty}^{v} f'(\tau)\exp\Big(-\int_\tau^{v}\lambda_{\mathrm{R}}\Big)\,d\tau = -\infty,$$

whence, by Lemma 5.2.3,

$$\langle \mathrm{f}(\varphi)\rangle_{[0,1]} \overset{(5.2.26)}{=} \int_{-\infty}^{v} f(\tau)\lambda_{\mathrm{R}}(\tau)\exp\Big(-\int_\tau^{v}\lambda_{\mathrm{R}}\Big)\,d\tau = +\infty. \qquad ∎$$

We state analogous propositions for the left tangent domain at $+\infty$.

Proposition 5.2.7. *Let B be the standard candidate on $\Omega_{\mathrm{L}}(u_1, +\infty)$. There exists a non-increasing optimizer φ_x for B at every point $x \in \Omega_{\mathrm{L}}(u_1, +\infty)$. Moreover, we have $g(u_1) \preceq \varphi_x$.*

Proposition 5.2.8. *If (2.1.8) holds but (2.1.14) fails, then $\boldsymbol{B}(x; f) = +\infty$ for any $x \in \Omega \setminus \partial_{\mathrm{fixed}}\Omega$.*

5.2.2 Optimizers for all other figures

The optimizers for chordal domains are easy to construct. Indeed, the proposition below is a straightforward consequence of formula (3.4.1).

Proposition 5.2.9. *Suppose that B is the standard candidate on a chordal domain $\Omega_{\mathrm{ch}}([a_0, b_0], [a_1, b_1])$. Let $g(a)$ and $g(b)$ be the endpoints of a chord from this chordal domain, $a \in [a_0, a_1]$, $b \in [b_1, b_0]$. Let*

$$x = \alpha g(a) + (1 - \alpha) g(b),$$

where $\alpha \in [0, 1]$. Then the optimizer $\varphi_x \colon [0, 1] \to \partial_{\mathrm{fixed}}\Omega$ is given by the formula

$$\varphi_x = g(a)\chi_{[0,\alpha]} + g(b)\chi_{(\alpha,1]}.$$

Proof. It is obvious that $\langle \varphi_x \rangle = x$, and the linearity of B on the chord implies that $B(x) = \langle \mathrm{f}(\varphi_x) \rangle_{[0,1]}$. It is clear that all the averages $\langle \varphi_x \rangle_J$ lie on $[g(a), g(b)] \subset \Omega$ for $J \subset [0, 1]$, therefore, $\varphi_x \in \boldsymbol{A}_\Omega$. ■

Remark 5.2.10. We suggested a non-decreasing optimizer φ_x for a chordal domain. One can also construct a non-increasing optimizer $\varphi_x = g(b)\chi_{[0,1-\alpha]} + g(a)\chi_{(1-\alpha,1]}$.

Definition 5.2.11. The construction above works also for the special case of a long chord. In this situation the tangency point with the free boundary is called the *outgoing node* of this chord.

Let us now pass to the case of multicups.

Proposition 5.2.12. *Let B be the standard candidate on $\Omega_{\mathrm{ClMcup}}(\{\mathfrak{a}_i\}_{i=1}^k)$. Then there exists a monotone optimizer φ_x for B at any point $x \in \Omega_{\mathrm{ClMcup}}(\{\mathfrak{a}_i\}_{i=1}^k)$ and, moreover, $g(\mathfrak{a}_1^{\mathrm{l}}) \preceq \varphi_x \preceq g(\mathfrak{a}_k^{\mathrm{r}})$ pointwise.*

Proof. Fix $x \in \Omega_{\mathrm{ClMcup}}(\{\mathfrak{a}_i\}_{i=1}^k)$ and represent it as a convex combination of three points $g(a_1)$, $g(a_2)$, $g(a_3)$ for some $a_1, a_2, a_3 \in \bigcup_{i=1}^k \mathfrak{a}_i$:

$$x = \alpha_1 g(a_1) + \alpha_2 g(a_2) + \alpha_3 g(a_3); \quad \alpha_1 + \alpha_2 + \alpha_3 = 1, \quad \alpha_j \geqslant 0.$$

Without loss of generality, we may assume that $a_1 \leqslant a_2 \leqslant a_3$. We put

$$\varphi_x(\tau) = g(a_1)\chi_{[0,\alpha_1]} + g(a_2)\chi_{(\alpha_1,\alpha_1+\alpha_2]} + g(a_3)\chi_{(\alpha_1+\alpha_2,1]}.$$

The equality $\langle\varphi_x\rangle = x$ is evident, and the relation $B(x) = \langle \mathrm{f}(\varphi_x)\rangle_{[0,1]}$ follows from the linearity of B on the closed multicup. The averages $\langle\varphi_x\rangle_J$ lie in the multicup for all $J \subset [0,1]$, thus, $\varphi_x \in \boldsymbol{A}_\Omega$. ∎

Proposition 5.2.13. *Let B be the standard candidate on $\Omega_{\mathrm{Mcup}}(\{\mathfrak{a}_i\}_{i=1}^k)$. Then there exists a monotone optimizer φ_x for B at any point $x \in \Omega_{\mathrm{Mcup}}(\{\mathfrak{a}_i\}_{i=1}^k)$ and, moreover, $g(\mathfrak{a}_1^{\mathrm{l}}) \preceq \varphi_x \preceq g(\mathfrak{a}_k^{\mathrm{r}})$ pointwise.*

Proof. If $x \in \partial_{\mathrm{fixed}}\Omega$, then $\varphi_x = x\chi_{[0,1]}$ is the optimizer. So, in what follows we assume $x \notin \partial_{\mathrm{fixed}}\Omega$. Consider the open convex set Ω' that is the interior of the convex hull of Ξ and the points $g(\mathfrak{a}_1^{\mathrm{l}})$ and $g(\mathfrak{a}_k^{\mathrm{r}})$. Since $x \notin \Omega'$, by the separation theorem, there exists a line $\varkappa = \varkappa(x)$ that passes through x and does not intersect Ω'. Let y and z be the leftmost and the rightmost points of the intersection of the line $\varkappa(x)$ with the boundary of $\Omega_{\mathrm{Mcup}}(\{\mathfrak{a}_i\}_{i=1}^k)$. For each crossing point (y and z) there are two possibilities: either it lies on an arc $g(\mathfrak{a}_i)$, $i \in \{1, 2, \ldots, k\}$, or on a segment $[g(\mathfrak{a}_i^{\mathrm{r}}), g(\mathfrak{a}_{i+1}^{\mathrm{l}})]$, $i \in \{1, 2, \ldots, k-1\}$.

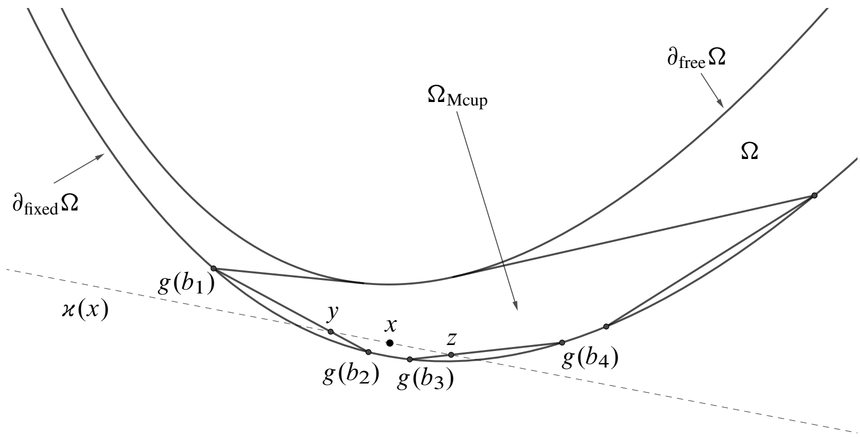

Figure 5.4. Construction of an optimizer in a multicup.

If y lies on the chord $[g(\mathfrak{a}_i^{\mathrm{r}}), g(\mathfrak{a}_{i+1}^{\mathrm{l}})]$, then we can write

$$y = \alpha_y g(\mathfrak{a}_i^{\mathrm{r}}) + (1-\alpha_y) g(\mathfrak{a}_{i+1}^{\mathrm{l}}) \quad \text{for some } \alpha_y \in [0,1].$$

Similarly, if z lies on the chord $[g(\mathfrak{a}_j^{\mathrm{r}}), g(\mathfrak{a}_{j+1}^{\mathrm{l}})]$, then

$$z = (1-\alpha_z) g(\mathfrak{a}_j^{\mathrm{r}}) + \alpha_z g(\mathfrak{a}_{j+1}^{\mathrm{l}}) \quad \text{for some } \alpha_z \in [0,1].$$

So, in any case, $y = \alpha_y g(b_1) + (1-\alpha_y) g(b_2)$, $z = (1-\alpha_z) g(b_3) + \alpha_z g(b_4)$ (see Figure 5.4), where $b_j \in \bigcup_{i=1}^k \mathfrak{a}_i$ (if y is an intersection of $\varkappa$ with an arc, then we may take $b_1 = b_2$; similarly with z) such that

$$g_1(b_1) \leqslant y_1 \leqslant g_1(b_2) \leqslant g_1(b_3) \leqslant z_1 \leqslant g_1(b_4).$$

It is convenient to define the optimizer φ_x on the segment $[y_1, z_1]$:

$$\varphi_x(t) = \begin{cases} g(b_1), & t \in J_1 = [y_1, y_1 + \alpha_y(z_1 - x_1)), \\ g(b_2), & t \in J_2 = [y_1 + \alpha_y(z_1 - x_1), z_1 - x_1 + y_1), \\ g(b_3), & t \in J_3 = [z_1 - x_1 + y_1, z_1 - \alpha_z(x_1 - y_1)), \\ g(b_4), & t \in J_4 = [z_1 - \alpha_z(x_1 - y_1), z_1]. \end{cases} \tag{5.2.27}$$

As usual, the equalities $\langle \varphi_x \rangle = x$ and $\langle \mathrm{f}(\varphi_x) \rangle_{[y_1,z_1]} = B(x)$ are evident. However, if we draw the delivery curve for φ_x, we can see that in some cases it does not fall under the scope of Lemma 5.1.3 (the tangent may cross the free boundary), so we prove directly that φ_x lies in $\boldsymbol{A}_\Omega$.

We claim that a point $\langle \varphi_x \rangle_J$, where $J \subset [y_1, z_1]$, either belongs to one of the segments $[g(b_1), g(b_2)]$ and $[g(b_3), g(b_4)]$, or it is separated from the free boundary by $\varkappa$. Once the claim is proved, we obtain $\varphi_x \in \boldsymbol{A}_\Omega$.

We will consider different cases of the position of J inside $[y_1, z_1]$. If J intersects not more than two of the intervals J_1, J_2, J_3, J_4 in (5.2.27), then the claim is obvious. If J intersects all four intervals, then we may represent x as a linear combination of $\langle \varphi_x \rangle_J$, $g(b_1)$, and $g(b_4)$. Since $g(b_1)$ and $g(b_4)$ lie above $\varkappa$ (i.e., in the same half-plane with the free boundary), the point $\langle \varphi_x \rangle_J$ lies below $\varkappa$.

So, we may suppose that J intersects three intervals from (5.2.27). Without loss of generality, we may assume that $J \cap J_4 = \varnothing$. Then $\langle \varphi_x \rangle_J$ is a convex combination of $g(b_3)$ and a point from $[y, g(b_2)]$ (since $J_2 \subset J$). Both these points lie below $\varkappa$. Therefore, $\langle \varphi_x \rangle_J$ is separated from the free boundary by $\varkappa$. ■

Remark 5.2.14. The optimizer φ_x we constructed for a multicup domain is non-decreasing. One can construct a non-increasing optimizer in a similar way.

Definition 5.2.15. The endpoints of the arc of the free boundary that is the part of the boundary of a multicup are called the *outgoing nodes* of the multicup.

Proposition 5.2.16. *Let $\Omega_{\mathrm{ang}}(u)$ be an angle with the vertex $g(u)$ and the boundary tangent lines S_{L} and S_{R} with the tangency points w_{L} and w_{R}, respectively. Let B be the standard candidate on $\Omega_{\mathrm{ang}}(u)$. Let ψ_{R} be a non-decreasing optimizer for the point w_{R}, and let ψ_{L} be a non-increasing optimizer for the point w_{L} such that $\psi_{\mathrm{R}} \preceq g(u)$ and $g(u) \preceq \psi_{\mathrm{L}}$ pointwise. Then there exists an optimizer for every point $x \in \Omega_{\mathrm{ang}}(u)$.*

Proof. The proof of this proposition is very similar to the proof of Proposition 5.2.13. First, there exist numbers $\alpha_1, \alpha_2, \alpha_3$ such that

$$x = \alpha_1 w_{\mathrm{R}} + \alpha_2 g(u) + \alpha_3 w_{\mathrm{L}}; \quad \alpha_1 + \alpha_2 + \alpha_3 = 1, \quad \alpha_j \geqslant 0.$$

Second, by Remark 2.1.5, we may model ψ_{R} and ψ_{L} on any interval. Suppose that ψ_{R} is defined on $[0,\alpha_1]$, and ψ_{L} is defined on $[0,\alpha_3]$. Define the optimizer φ_x on $[0,1]$ by the formula

$$\varphi_x(\tau) = \begin{cases} \psi_{\mathrm{R}}(\tau), & \tau \in [0,\alpha_1), \\ g(u), & \tau \in [\alpha_1,\alpha_1+\alpha_2), \\ \psi_{\mathrm{L}}(1-\tau), & \tau \in [\alpha_1+\alpha_2,1]. \end{cases}$$

As usual, the equalities $\langle \varphi_x \rangle = x$ and $\langle \mathrm{f}(\varphi_x)\rangle_{[0,1]} = B(x)$ are evident. We have to verify that $\varphi_x \in \boldsymbol{A}_\Omega$. Let $\varkappa = \varkappa(x)$ be a line passing through x that does not intersect Ξ. First, we prove that $\gamma_{\psi_{\mathrm{R}}}$ lies above $\varkappa$ and $\gamma_{\psi_{\mathrm{L}}}$ lies above $\varkappa$ as well. It suffices to prove the claim concerning $\gamma_{\psi_{\mathrm{R}}}$. Since ψ_{R} is non-decreasing, Lemma 5.1.2 says that the curve $\gamma_{\psi_{\mathrm{R}}}$ is a graph of a convex function. Therefore, it lies above its tangent line at the point w_{R}. Due to (5.1.2), this tangent line contains the segment $[w_{\mathrm{R}}, g(u)]$. Therefore, $\gamma_{\psi_{\mathrm{R}}}$ lies above the tangent line to the free boundary at the point w_{R} and on the left of this point. Thus, $\gamma_{\psi_{\mathrm{R}}}$ lies above $\varkappa$.

Let $J \subset [0,1]$ be an interval. We have to prove that $\langle \varphi_x\rangle_J \in \Omega$. Consider several cases of position of J inside $[0,1]$.

Suppose that $[\alpha_1,\alpha_1+\alpha_2] \subset J$. Then we claim that $\langle\varphi_x\rangle_J$ is separated from the free boundary by $\varkappa$. Since x is a convex combination of $\langle\varphi\rangle_{[0,\alpha_1)\setminus J}$, $\langle\varphi\rangle_{[\alpha_1+\alpha_2,1]\setminus J}$, and $\langle\varphi\rangle_J$, and the first two points lie above $\varkappa$, the point $\langle\varphi\rangle_J$ lies below $\varkappa$, indeed, see Figure 5.5.

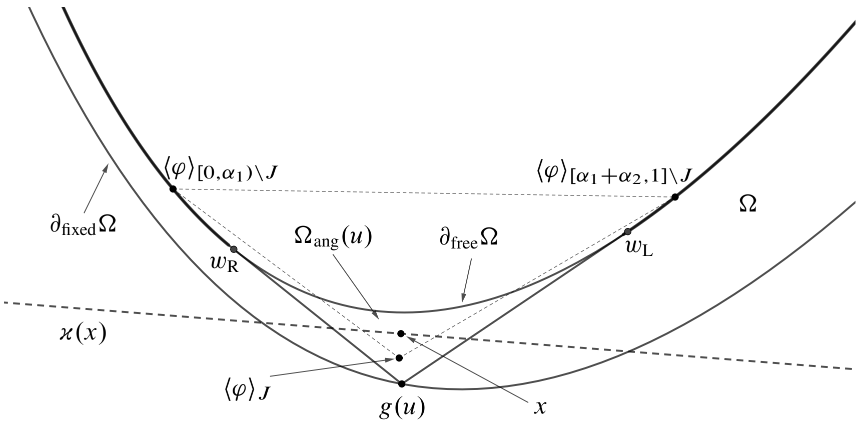

Figure 5.5. Optimizer in $\Omega_{\mathrm{ang}}(u)$.

If $[\alpha_1,\alpha_1+\alpha_2]$ is not contained in J, then we can apply Lemma 5.1.3 to the function $\varphi|_J$. Indeed, $\varphi|_J$ is a monotone function, and the tangent to the corresponding curve at the endpoint does not intersect the free boundary. ■

Definition 5.2.17. The points w_{R} and w_{L} introduced in Proposition 5.2.16 are called the *incoming nodes* of the angle.

Proposition 5.2.18. *Suppose that B is the standard candidate in a right trolleybus $\Omega_{\mathrm{tr,R}}(u_1, u_2)$. Suppose ψ is a non-decreasing optimizer for the point $w_{\mathrm{R}}(u_1)$ such that $\psi \preceq g(u_1)$ pointwise. Then, for any $x \in \Omega_{\mathrm{tr,R}}(u_1, u_2)$, there exists a non-decreasing optimizer φ_x such that $\varphi_x \preceq g(u_2)$.*

Proof. Choose any point $x \in \Omega_{\mathrm{tr,R}}(u_1, u_2)$. The trolleybus $\Omega_{\mathrm{tr,R}}(u_1, u_2)$ lies inside the triangle with the vertices $g(u_1)$, $g(u_2)$, and $w_{\mathrm{R}}(u_1)$. Thus, there exist $\alpha_0, \alpha_1, \alpha_2$ such that

$$x = \alpha_0 w_{\mathrm{R}}(u_1) + \alpha_1 g(u_1) + \alpha_2 g(u_2); \quad \alpha_0 + \alpha_1 + \alpha_2 = 1, \quad \alpha_j \geqslant 0.$$

By Remark 2.1.5, we may assume that ψ is defined on $[0, \alpha_0]$. Define the optimizer φ_x on [0, 1] by the formula

$$\varphi_x(\tau) = \begin{cases} \psi(\tau), & \tau \in [0, \alpha_0), \\ g(u_1), & \tau \in [\alpha_0, \alpha_0 + \alpha_1), \\ g(u_2), & \tau \in [\alpha_0 + \alpha_1, 1]. \end{cases}$$

It is clear that $\langle \varphi_x \rangle_{[0,1]} = x$ and $\langle \mathrm{f}(\varphi_x) \rangle_{[0,1]} = B(x)$. We only have to verify $\varphi_x \in \boldsymbol{A}_\Omega$, i.e., $\langle \varphi_x \rangle_J \in \Omega$ for any interval $J \subset [0, 1]$. Take a line $\varkappa = \varkappa(x)$ that passes through x and such that the point $g(u_2)$ together with the domain Ξ lie above this line, see Figure 5.6. As in the proof of Propositions 5.2.13 and 5.2.16, we will have to consider different cases of location of J inside [0, 1].

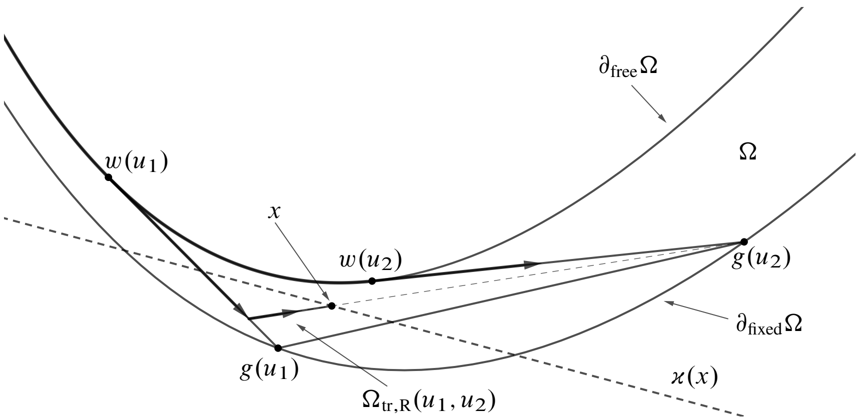

Figure 5.6. Delivery curves inside a trolleybus.

If $[\alpha_0, \alpha_0 + \alpha_1) \subset J$, then $\langle \varphi_x \rangle_J$ is separated from the free boundary by $\varkappa_x$ (this reasoning is completely analogous to that used in Propositions 5.2.13 and 5.2.16).

If $J \cap [\alpha_0 + \alpha_1, 1] = \varnothing$, then the situation falls under the scope of Lemma 5.1.3.

If $J \cap [0, \alpha_0) = \varnothing$, then $\langle \varphi_x \rangle_J \subset [g(u_1), g(u_2)]$.

We have considered all the cases and verified that $\varphi_x \in \boldsymbol{A}_\Omega$. Moreover, it follows from the construction that φ_x is non-decreasing and $\varphi_x \preceq g(u_2)$. ■

As usual, we have a symmetric proposition.

Proposition 5.2.19. *Let B be the standard candidate in a left trolleybus $\Omega_{\mathrm{tr,L}}(u_1, u_2)$. Suppose ψ is a non-increasing optimizer for the point $w_{\mathrm{L}}(u_2)$ such that $\psi \succeq g(u_2)$. Then, for any $x \in \Omega_{\mathrm{tr,L}}(u_1, u_2)$, there exists a non-increasing optimizer φ_x such that $\varphi_x \succeq g(u_1)$.*

It remains to construct the optimizers for multitrolleybuses, birdies, and multibirdies. Formulas from Section 3.8.1 will help us in this task.

Proposition 5.2.20. *Suppose that B is the standard candidate in a right multitrolleybus $\Omega_{\mathrm{Mtr,R}}(\{\mathfrak{a}_i\}_{i=1}^k)$. Suppose ψ is a non-decreasing optimizer for the point $w_{\mathrm{R}}(\mathfrak{a}_1^{\mathrm{l}})$ such that $\psi \preceq g(\mathfrak{a}_1^{\mathrm{l}})$. Then, for any $x \in \Omega_{\mathrm{Mtr,R}}(\{\mathfrak{a}_i\}_{i=1}^k)$, there exists a non-decreasing optimizer that does not exceed $g(\mathfrak{a}_k^{\mathrm{r}})$.*

Proof. We apply (3.8.11) and decompose a multitrolleybus in an alternating sequence of right trolleybuses and multitrolleybuses $\Omega_{\mathrm{Mtr,R}}(\mathfrak{a}_i)$ on solid roots. We consider each such multitrolleybus as a tangent domain with the candidate B on it (note that this is not a standard candidate for the tangent domain). Applying Proposition 5.2.18 to the trolleybuses and Proposition 5.2.1 to the tangent domains consecutively, we build optimizers for all the points inside $\Omega_{\mathrm{Mtr,R}}(\{\mathfrak{a}_i\}_{i=1}^k)$. ■

Proposition 5.2.21. *Suppose that B is the standard candidate in a left multitrolleybus $\Omega_{\mathrm{Mtr,L}}(\{\mathfrak{a}_i\}_{i=1}^k)$. Suppose ψ is a non-increasing optimizer for the point $w_{\mathrm{L}}(\mathfrak{a}_k^{\mathrm{r}})$ such that $\psi \succeq g(\mathfrak{a}_k^{\mathrm{r}})$. Then, for any $x \in \Omega_{\mathrm{Mtr,L}}(\{\mathfrak{a}_i\}_{i=1}^k)$, there exists a non-increasing optimizer that is not less than $g(\mathfrak{a}_1^{\mathrm{l}})$.*

Definition 5.2.22. Let w_- and w_+ be the left and the right endpoints of the arc of the free boundary that is the part of the boundary of a trolleybus (or a multitrolleybus). These points are called the *incoming* and *outgoing nodes* of the trolleybus (or the multitrolleybus): for the case of $\Omega_{\mathrm{tr,R}}$ (and $\Omega_{\mathrm{Mtr,R}}$), the incoming node is w_- and outgoing is w_+, and vice versa for the case of $\Omega_{\mathrm{tr,L}}$ (and $\Omega_{\mathrm{Mtr,L}}$).

Proposition 5.2.23. *Let B be the standard candidate in a multibirdie $\Omega_{\mathrm{Mbird}}(\{\mathfrak{a}_i\}_{i=1}^k)$. Let ψ_{R} be a non-decreasing optimizer for the point $w_{\mathrm{R}}(\mathfrak{a}_1^{\mathrm{l}})$ such that $\psi_{\mathrm{R}} \preceq g(\mathfrak{a}_1^{\mathrm{l}})$, let ψ_{L} be a non-increasing optimizer for the point $w_{\mathrm{L}}(\mathfrak{a}_k^{\mathrm{r}})$ such that $\psi_{\mathrm{L}} \succeq g(\mathfrak{a}_k^{\mathrm{r}})$. Then there exists an optimizer for every point $x \in \Omega_{\mathrm{Mbird}}(\{\mathfrak{a}_i\}_{i=1}^k)$.*

Proof. We apply (3.8.13) (with any choice of j) and decompose the multibirdie into two multitrolleybuses and an angle. Applying corresponding previous propositions

	outgoing	incoming
long chord	$w_- = w_+$	
$\Omega_{\mathrm{Mcup}}(\{\alpha_i\}_{i=1}^k)$	w_-, w_+	
$\Omega_{\mathrm{Mtr,R}}(\{\alpha_i\}_{i=1}^k)$ $\Omega_{\mathrm{tr,R}}(u_1, u_2)$ $\Omega_{\mathrm{R}}(u_1, u_2)$	w_+	w_-
$\Omega_{\mathrm{Mtr,L}}(\{\alpha_i\}_{i=1}^k)$ $\Omega_{\mathrm{tr,L}}(u_1, u_2)$ $\Omega_{\mathrm{L}}(u_1, u_2)$	w_-	w_+
$\Omega_{\mathrm{ang}}(w)$ $\Omega_{\mathrm{Mbird}}(\{\alpha_i\}_{i=1}^k)$		w_-, w_+

Table 5.1. Outgoing and incoming nodes of the domains of linearity.

for each of the domains, we build optimizers for all the points inside $\Omega_{\mathrm{Mbird}}(\{\alpha_i\}_{i=1}^k)$. The proof is complete. ■

Definition 5.2.24. The points $w_{\mathrm{R}}(\alpha_1^{\mathrm{l}})$ and $w_{\mathrm{L}}(\alpha_k^{\mathrm{r}})$ in Proposition 5.2.23 are called the *incoming nodes* of the multibirdie $\Omega_{\mathrm{Mbird}}(\{\alpha_i\}_{i=1}^k)$.

5.3 Global optimizers

Before passing to global optimizers, we collect the information about incoming and outgoing nodes for a vertex or an edge in the foliation graph. These nodes are points on the free boundary that are used to transfer the delivery curve from one domain to another. For some figures, the optimizers inside them are constructed by using the optimizer coming from the right or from the left (for example, to build optimizers in a trolleybus, we need an optimizer for a special point on the free boundary, see Propositions 5.2.18 and 5.2.19). The incoming node is the point we start the delivery curve from, whereas the outgoing node is the point where it leaves the figure. In Table 5.1, we give a precise description. We extend our notation from Definition 5.2.22 for other domains $\mathfrak{L}$: let w_- and w_+ be the left and the right endpoints of the arc of the free boundary $\partial_{\mathrm{free}}\Omega$ which is the part of the boundary of $\mathfrak{L}$.

Other figures (separated from the upper boundary) have no incoming or outgoing nodes at all. Now we see that the propositions of Sections 5.2.1 and 5.2.2 are of the following form: if there is a monotone optimizer for the incoming node satisfying a certain restriction (if there are incoming nodes for the figure in question), then we can build monotone optimizers satisfying a similar restriction for all the points in the domain, in particular, for the outgoing nodes (if there are any). These propositions are well suited for induction. To state the general theorem about optimizers, we need Definition 4.4.5 (see Remark 4.4.6 and the paragraph before it, where it is explained that an admissible graph generates the Bellman candidate).

Theorem 5.3.1. *Let $\varepsilon < \varepsilon_{\max}$, and let $\Gamma(\varepsilon)$ be an admissible graph for ε and f. The Bellman candidate B_ε generated by $\Gamma(\varepsilon)$ admits an optimizer at every point of Ω_ε.*

Proof. The proof of the theorem consists in inductive application of the propositions from Section 5.2.1 and 5.2.2. The details are the same as in the special case of the parabolic strip, see Theorem 5.3.1 in [17]. ■

Theorem 5.3.1 that we have just proved justifies the arguments given in Section 2.2.3. Also, the candidate B_ε constructed by the admissible graph $\Gamma(\varepsilon)$ coincides with the Bellman function $\boldsymbol{B}_\varepsilon$ and with the minimal locally concave function $\mathfrak{B}_{\Omega_\varepsilon,f}$.

Theorem 5.3.2. *For any $\varepsilon < \varepsilon_{\max}$, the constructed Bellman candidate B_ε coincides with $\boldsymbol{B}_\varepsilon$ and $\mathfrak{B}_{\Omega_\varepsilon,f}$.*

5.4 Remarks concerning global conditions

In this section we discuss conditions (2.1.7) and (2.1.8).

As the reader has seen, these technical conditions were used only during different steps of constructing optimizers. Here we discuss what happens when they fail. We note that construction of optimizers does not use any evolution arguments and deals with the fixed ε.

5.4.1 Behavior at $-\infty$

Since we have the right-left symmetry, let us consider the behavior at $-\infty$, namely, condition (2.1.7). In what follows, we omit the index R, i.e., we write λ, w, κ instead of λ_{R}, w_{R}, κ_{R}.

If (2.1.7) is not fulfilled, and the foliation for the function B contains a domain $\Omega_{\mathrm{R}}(-\infty, v)$ of the right tangents coming from $-\infty$, the statement of Proposition 5.2.5 is not valid in the sense that there are no optimizers φ_x for $x \in \Omega_{\mathrm{R}}(-\infty, v) \setminus \partial_{\mathrm{fixed}}\Omega$. However, for any such x, there is a sequence of test functions $\varphi_n \in \boldsymbol{A}_\Omega$ such that $\langle \varphi_n \rangle = x$ and $B(x) = \lim_{n\to+\infty}\langle \mathrm{f}(\varphi_n)\rangle$. We call such $\{\varphi_n\}_n$ an *optimizing sequence* for x. All the propositions from Section 5.2 remain true if we replace the word optimizer by optimizing sequence.

We now give some details on how to construct these functions φ_n for $x = w(v)$. Consider first sequences $v_n^-, v_n^+ \in \mathbb{R}$ such that $v_n^- < v_n^+ < v$, $v_n^+ \to -\infty$, and the chord $[g(v_n^-), g(v_n^+)]$ is tangent to $\partial_{\mathrm{free}}\Omega$. The tangent point is $w(v_n^+)$. Consider the domain $\Omega_{\mathrm{R}}(v_n^+, v)$ and let γ_n be the delivery curve that starts at the point $g(v_n^-)$ and goes to $w(v_n^+)$ along the chord, and then proceeds along $\partial_{\mathrm{free}}\Omega$ until the point $w(v)$. Build a test function φ_n on [0, 1] that generates this curve γ_n:

$$\varphi_n(t) = \begin{cases} g(v_n^-), & t \in [0, \alpha_1), \\ g(v_n^+), & t \in [\alpha_1, \alpha_2), \\ g(u(t)), & t \in [\alpha_2, 1]. \end{cases} \tag{5.4.1}$$

The parameters $u = u(t)$ and $t = t(u)$ are related as in (5.2.21):

$$\log t = -\int_{u(t)}^{v} \lambda, \quad t \in [\alpha_2, 1], \tag{5.4.2}$$

and $u(\alpha_2) = v_n^+$, i.e.,

$$\alpha_2 = \exp\Big(-\int_{v_n^+}^{v} \lambda\Big). \tag{5.4.3}$$

The value of α_1 is determined by the relation that represents the point $w(v_n^+)$ as a convex combination of $g(v_n^-)$ and $g(v_n^+)$:

$$\frac{\alpha_1}{\alpha_2} g(v_n^-) + \frac{\alpha_2 - \alpha_1}{\alpha_2} g(v_n^+) = w(v_n^+),$$

namely,

$$\frac{\alpha_1}{\alpha_2} = \frac{g_1(v_n^+) - w_1(v_n^+)}{g_1(v_n^+) - g_1(v_n^-)}. \tag{5.4.4}$$

Remark 5.4.1. Since φ_n defined by (5.4.1) generates the delivery curve γ_n, we have $\langle g(\varphi_n)\rangle_{[0,1]} = w(v)$. Moreover, Corollary 5.1.4 guarantees φ_n lies in $\boldsymbol{A}_\Omega$, therefore it is a test function for $w(v)$.

The following theorem generalizes Proposition 5.2.5 to the case where condition (2.1.7) fails.

Theorem 5.4.2. *Let B be the standard candidate on $\Omega_{\mathrm{R}}(-\infty, v)$. Then, for every point $x \in \Omega_{\mathrm{R}}(-\infty, v)$, there exists a sequence of non-decreasing test functions $\varphi_{n,x}$ being an optimizing sequence for B at x; moreover, $\varphi_{n,x} \preceq g(v)$.*

The proof of Theorem 5.4.2 will be given for different cases of condition (2.1.7) failure separately, namely, we consider three cases:

(A) $\exp(-\int_{-\infty}^{v} \lambda) \neq 0$, i.e., the integral $\int_{-\infty}^{v} \lambda$ converges,

(B) $\int_{-\infty}^{v} \lambda = +\infty$, and the function

$$\vartheta_1(\tau) \stackrel{\mathrm{def}}{=} (g_1(\tau) - w_1(\tau)) \exp\Big(-\int_{\tau}^{v} \lambda\Big) \tag{5.4.5}$$

does not tend to zero as $\tau \to -\infty$,

(C) $\int_{-\infty}^{v} \lambda = +\infty$, $\vartheta_1(-\infty) = 0$, and the function

$$\vartheta_2(\tau) \overset{\text{def}}{=} (g_2(\tau) - w_2(\tau)) \exp\Big(-\int_\tau^v \lambda\Big) \tag{5.4.6}$$

does not tend to zero as $\tau \to -\infty$.

Remark 5.4.3. For any function $\psi\colon \mathbb{R} \to \mathbb{R}$, we use the notation $\psi(-\infty)$ for either a finite or an infinite limit $\lim_{\tau\to-\infty} \psi(\tau)$. We will often use this notation for the functions that are monotone in a neighborhood of $-\infty$, e.g., f, κ_3, $\mathfrak{K}$ that are monotone by Condition 2.1.11 and Remark 2.1.12, as well as for ϑ_1 and ϑ_2, whose monotonicity will be proved in Lemma 5.4.4.

Lemma 5.4.4. *The functions ϑ_1 and ϑ_2, defined by* (5.4.5) *and* (5.4.6)*, are monotone in a neighborhood of $-\infty$ and have finite limits at $-\infty$. Moreover, ϑ_1 and ϑ_1' are positive, and*

$$\operatorname{sign} \vartheta_2' = \operatorname{sign} \vartheta_2 = \operatorname{sign} g_2' = \operatorname{sign} \kappa_2 = \operatorname{sign} \kappa \tag{5.4.7}$$

in a neighborhood of $-\infty$.

Proof. The function ϑ_1 is positive and increasing:

$$\vartheta_1'(\tau) \overset{(5.2.1)}{=} g_1'(\tau) \exp\Big(-\int_\tau^v \lambda\Big) > 0. \tag{5.4.8}$$

Therefore, $\vartheta_1(-\infty)$ is finite.

The similar formula for the derivative of ϑ_2,

$$\vartheta_2'(\tau) = g_2'(\tau) \exp\Big(-\int_\tau^v \lambda\Big), \tag{5.4.9}$$

implies that $\operatorname{sign} \vartheta_2'(\tau) = \operatorname{sign} \kappa_2(\tau)$, whereas $\operatorname{sign} \vartheta_2(\tau) = \operatorname{sign} \kappa(\tau)$. From the geometric assumption on the domain Ω, the point $w(\tau)$ lies on a segment connecting $g(\tau)$ and some $g(\tau_-)$, $\tau_- < \tau$. It follows from the convexity of $\partial_{\text{fixed}}\Omega$ that

$$\kappa_2(\tau_-) < \kappa(\tau) < \kappa_2(\tau). \tag{5.4.10}$$

Whence, $\operatorname{sign} \kappa_2 = \operatorname{sign} \kappa$ in a neighborhood of $-\infty$, and $\operatorname{sign} \vartheta_2' = \operatorname{sign} \vartheta_2$ as well. Therefore, the limit $\vartheta_2(-\infty)$ is finite. ∎

Preparing for the proof of Theorem 5.4.2, we calculate $\langle \mathrm{f}(\varphi_n)\rangle_{[0,1]}$:

$$\begin{aligned}\langle \mathrm{f}(\varphi_n)\rangle_{[0,1]} &= \alpha_1 f(v_n^-) + (\alpha_2 - \alpha_1) f(v_n^+) + \int_{\alpha_2}^1 f(u(t))\,dt \\ &\overset{(5.4.2)}{=} \alpha_1 f(v_n^-) + (\alpha_2 - \alpha_1) f(v_n^+) + \int_{v_n^+}^{v} f(u)\lambda(u) \exp\Big(-\int_u^v \lambda\Big)\,du\end{aligned} \tag{5.4.11}$$

$$= \alpha_1 f(v_n^-) + (\alpha_2 - \alpha_1) f(v_n^+) + f(u) \exp\Big(-\int_u^v \lambda\Big)\Big|_{u=v_n^+}^{u=v} - \int_{v_n^+}^{v} f'(u) \exp\Big(-\int_u^v \lambda\Big)\, du$$

$$= \alpha_1\big(f(v_n^-) - f(v_n^+)\big) + f(v) - \int_{v_n^+}^{v} f'(u) \exp\Big(-\int_u^v \lambda\Big)\, du \tag{5.4.12}$$

$$= -\frac{f(v_n^+) - f(v_n^-)}{g_1(v_n^+) - g_1(v_n^-)} \vartheta_1(v_n^+) + f(v) - \int_{v_n^+}^{v} f'(u) \exp\Big(-\int_u^v \lambda\Big)\, du. \tag{5.4.13}$$

In the latter equality we use (5.4.4), (5.4.3), and (5.4.5). We apply the Cauchy mean value theorem to find $v_n \in [v_n^-, v_n^+]$ such that

$$\frac{f(v_n^+) - f(v_n^-)}{g_1(v_n^+) - g_1(v_n^-)} = \frac{f'(v_n)}{g_1'(v_n)} = \kappa_3(v_n).$$

We use this to rewrite (5.4.13):

$$\langle \mathrm{f}(\varphi_n)\rangle_{[0,1]} = -\kappa_3(v_n)\vartheta_1(v_n^+) + f(v) - \int_{v_n^+}^{v} f'(u) \exp\Big(-\int_u^v \lambda\Big)\, du. \tag{5.4.14}$$

5.4.2 Case (A) : $\int_{-\infty}^{v} \lambda$ converges

Lemma 5.4.5. *If $\int_{-\infty}^{v} \lambda < +\infty$, then the limits $g(-\infty)$ and $w(-\infty)$ are finite and coincide. Thus, $\vartheta_1(-\infty) = 0$ and $\vartheta_2(-\infty) = 0$.*

Proof. We note that $\lambda > 0$, therefore the hypothesis of the lemma is equivalent to the convergence of the integral. From (5.4.8), we have

$$\vartheta_1'(\tau) \leqslant g_1'(\tau) \leqslant \vartheta_1'(\tau) \exp\Big(\int_{-\infty}^{v} \lambda\Big).$$

By Lemma 5.4.4, $\vartheta_1(-\infty)$ is finite, and therefore $\int_{-\infty}^{v} \vartheta_1' < +\infty$. So, $\int_{-\infty}^{v} g_1' < +\infty$, which means $g_1(-\infty)$ is finite.

Since the limit $g_1(-\infty)$ is finite and the curve (g_1, g_2) is convex, we see that the limit of g_2 at $-\infty$ is not $-\infty$. Therefore, $g_2(-\infty)$ is either finite or $+\infty$. If $g_2(-\infty) = +\infty$, then $g_2' < 0$ in a neighborhood of $-\infty$. From (5.4.9), we have

$$\vartheta_2'(\tau) \geqslant g_2'(\tau) \geqslant \vartheta_2'(\tau) \exp\Big(\int_{-\infty}^{v} \lambda\Big).$$

By Lemma 5.4.4, $\vartheta_2(-\infty)$ is finite, therefore $\int_{-\infty}^{v} \vartheta_2' > -\infty$. Thus, $\int_{-\infty}^{v} g_2' > -\infty$, which means g_2 has a finite limit at $-\infty$. This is a contradiction.

We have proved that $g = (g_1, g_2)$ has a finite limit at $-\infty$. From the geometric assumption on the domain Ω, any point $w(\tau)$ lies on a segment connecting $g(\tau)$ and some $g(\tau_-)$, $\tau_- < \tau$. Since both points $g(\tau)$ and $g(\tau_-)$ tend to the same limit, the limit of $w(\tau)$ as $\tau \to -\infty$ exists and $w(-\infty) = g(-\infty)$. ■

Corollary 5.4.6. *If $\int_{-\infty}^{v} \lambda < +\infty$, then $(\kappa_2 - \kappa)(g_1 - w_1)$ tends to 0 at $-\infty$.*

Proof. From Lemma 5.4.5 we know that $g_1 - w_1$ tends to 0 at $-\infty$.

If $\kappa_2 > 0$ in a neighborhood of $-\infty$, then $\kappa_2(-\infty)$ is finite because $\kappa_2' > 0$. Since any point $w(\tau)$ lies on a segment connecting $g(\tau)$ and some $g(\tau_-)$, $\tau_- < \tau$, we have an inequality $\kappa_2(\tau_-) < \kappa(\tau) < \kappa_2(\tau)$. Therefore, $\kappa(-\infty) = \kappa_2(-\infty)$, and the claim is proved.

If $\kappa_2 < 0$ in a neighborhood of $-\infty$, then $0 < \kappa_2 - \kappa < -\kappa$. Therefore, by Lemma 5.4.5, we have

$$|(\kappa_2 - \kappa)(g_1 - w_1)| \leqslant |\kappa(g_1 - w_1)| = |g_2 - w_2| \to 0,$$

as the argument tends to $-\infty$. ■

Corollary 5.4.7. *If $\int_{-\infty}^{v} \lambda < +\infty$, then $\int_{-\infty}^{v} \frac{\kappa_2'}{\kappa_2 - \kappa} = +\infty$.*

Proof. The statement follows from (5.2.12) and Corollary 5.4.6. ■

Lemma 5.4.8. *If $\int_{-\infty}^{v} \lambda < +\infty$ and $f(-\infty) = -\infty$, then $\mathfrak{K}(-\infty) = -\infty$.*

Proof. Assume on the contrary that $\mathfrak{K} > C_1$ in a neighborhood of $-\infty$ for some $C_1 \in \mathbb{R}$. Then we have $\kappa_3' > C_1\kappa_2'$ because $\kappa_2' > 0$. Integration of this inequality yields $\kappa_3 < C_1\kappa_2 + C_2$ for some $C_2 \in \mathbb{R}$. Multiply this by positive g_1' to obtain $f' < C_1 g_2' + C_2 g_1'$. Integrating this inequality, we get $f > C_1 g_2 + C_2 g_1 + C_3$ for some $C_3 \in \mathbb{R}$. This inequality contradicts the hypothesis that $f(-\infty) = -\infty$, since the functions g_1 and g_2 have finite limits at $-\infty$ by Lemma 5.4.5. ■

Proposition 5.4.9. *If $\int_{-\infty}^{v} \lambda < +\infty$ and there exists a standard candidate B on $\Omega_{\mathrm{R}}(-\infty, v)$, then $f(-\infty) > -\infty$.*

Proof. Let us assume the contrary: $f(-\infty) = -\infty$. Then, by Lemma 5.4.8, we have $\mathfrak{K}(-\infty) = -\infty$, therefore $\mathfrak{K}' > 0$ in a neighborhood of $-\infty$. By (3.3.4), $\beta_2' > 0$ in the same neighborhood of $-\infty$. This contradicts the existence of the standard candidate B on $\Omega_{\mathrm{R}}(-\infty, v)$, see Definition 3.3.8. ■

Proposition 5.4.10. *If $\int_{-\infty}^{v} \lambda < +\infty$ and $f(-\infty) = +\infty$, then $\boldsymbol{B}(x; f) = +\infty$ for any $x \in \Omega \setminus \partial_{\mathrm{fixed}}\Omega$.*

Proof. It suffices to prove $\boldsymbol{B}(w(v); f) = +\infty$. We construct a sequence of test functions φ_n for the point $w(v)$ by (5.4.1). The condition $\int_{-\infty}^{v} \lambda < +\infty$, together with (5.4.3), guarantees that α_2 is separated from 0. Therefore, from the condition $f(-\infty) = +\infty$ and (5.4.11), we obtain that $\langle \mathrm{f}(\varphi_n)\rangle_{[0,1]}$ tends to $+\infty$, and thus $\boldsymbol{B}(w(v); f) = +\infty$. ■

Proposition 5.4.11. *Suppose that* $\int_{-\infty}^{v}\lambda < +\infty$ *and the limit* $f(-\infty)$ *is finite. Let* B *be the standard candidate on* $\Omega_{\mathrm{R}}(-\infty, v)$. *Then, for every point* $x \in \Omega_{\mathrm{R}}(-\infty, v)$, *there exists a sequence of non-decreasing test functions* $\varphi_{n,x}$ *being an optimizing sequence for* B *at* x; *moreover,* $\varphi_{n,x} \preceq g(v)$.

Proof. As before, it suffices to prove the claim for the case $x = w(v) = w_{\mathrm{R}}(v)$. In what follows, we omit the subscript x in our notation. We use the sequence of functions φ_n defined by (5.4.1). All these functions are non-decreasing and satisfy the inequality $\varphi_n \preceq g(v)$. Therefore, we only need to verify that $\langle \mathrm{f}(\varphi_n)\rangle_{[0,1]} \to B(w(v))$.

Recall (3.2.2):

$$B(w(v)) = f(v) + (w_1(v) - g_1(v))[\kappa_3(v) + (\kappa(v) - \kappa_2(v))\beta_2(v)], \qquad (5.4.15)$$

where β_2 is given by (3.3.3), i.e.,

$$\begin{aligned}\beta_2(v) &= \mathfrak{K}(v) - \int_{-\infty}^{v} \exp\Big(-\int_{\tau}^{v} \frac{\kappa_2'}{\kappa_2 - \kappa}\Big)\,\mathfrak{K}'(\tau)\,d\tau \\ &= \int_{-\infty}^{v} \exp\Big(-\int_{\tau}^{v} \frac{\kappa_2'}{\kappa_2 - \kappa}\Big)\frac{\kappa_3'(\tau)}{\kappa_2(\tau) - \kappa(\tau)}\,d\tau. \qquad (5.4.16)\end{aligned}$$

Here we have used Lemma 5.2.3 with

$$\vartheta = \exp\Big(-\int_{\tau}^{v} \frac{\kappa_2'}{\kappa_2 - \kappa}\Big) \quad \text{and} \quad Y = \mathfrak{K}.$$

Note that $\vartheta(-\infty) = 0$ by Corollary 5.4.7.

We use (5.4.12) to evaluate the limit of $\langle \mathrm{f}(\varphi_n)\rangle_{[0,1]}$:

$$\begin{aligned}\lim_{n\to\infty}\langle \mathrm{f}(\varphi_n)\rangle_{[0,1]} &= f(v) - \int_{-\infty}^{v} f'(u)\exp\Big(-\int_{u}^{v}\lambda\Big)\,du \\ &= f(v) - \int_{-\infty}^{v} \kappa_3(u)g_1'(u)\exp\Big(-\int_{u}^{v}\lambda\Big)\,du \\ &\overset{(5.4.8)}{=} f(v) - \int_{-\infty}^{v} \kappa_3(u)\vartheta_1'(u)\,du.\end{aligned}$$

Integrate by parts using Lemma 5.2.3 with $Y = \kappa_3$ and $\vartheta = \vartheta_1$ (by Lemma 5.4.5, we have $\vartheta_1(-\infty) = 0$) to obtain

$$\lim_{n\to\infty}\langle \mathrm{f}(\varphi_n)\rangle_{[0,1]} = f(v) - \kappa_3(v)(g_1(v) - w_1(v)) + \int_{-\infty}^{v} \kappa_3'(u)\vartheta_1(u)\,du.$$

Thus, to check that the limit coincides with (5.4.15), it suffices to verify the identity

$$\begin{aligned}\int_{-\infty}^{v} \kappa_3'(u)\vartheta_1(u)\,du &= (\kappa(v) - \kappa_2(v))(w_1(v) - g_1(v)) \\ &\quad\times \int_{-\infty}^{v} \exp\Big(-\int_{u}^{v} \frac{\kappa_2'}{\kappa_2 - \kappa}\Big)\frac{\kappa_3'(u)}{\kappa_2(u) - \kappa(u)}\,du, \qquad (5.4.17)\end{aligned}$$

which follows from (5.2.12):

$$\begin{aligned}\vartheta_1(u) &= (g_1(u) - w_1(u))\exp\Big(-\int_u^v \lambda\Big)\\ &= \frac{(\kappa_2(v) - \kappa(v))(g_1(v) - w_1(v))}{\kappa_2(u) - \kappa(u)} \exp\Big(-\int_u^v \frac{\kappa_2'}{\kappa_2 - \kappa}\Big).\end{aligned} \tag{5.4.18}$$

The proof is complete. ■

Propositions 5.4.9, 5.4.10, and 5.4.11 prove Theorem 5.4.2 in case (A).

5.4.3 Case (B) : $\int_{-\infty}^{v} \lambda = +\infty$, $\vartheta_1(-\infty) > 0$

Proposition 5.4.12. *If $\int_{-\infty}^{v} \lambda = +\infty$, $\vartheta_1(-\infty) > 0$, and $\kappa_3(-\infty) = -\infty$, then $\boldsymbol{B}(x; f) = +\infty$ for any $x \in \Omega \setminus \partial_{\mathrm{fixed}}\Omega$.*

Proof. It suffices to prove $\boldsymbol{B}(w(v)) = +\infty$. We consider the sequence φ_n defined in (5.4.1); they are test functions for $w(v)$, see Remark 5.4.1. Since the sign of κ_3 coincides with the sign of f', from (5.4.14), we see that $\langle \mathrm{f}(\varphi_n)\rangle_{[0,1]} \to +\infty$. ■

Lemma 5.4.13. *If $\int_{-\infty}^{v} \lambda = +\infty$ and $\vartheta_1(-\infty) > 0$, then the limit $\kappa_2(-\infty)$ is finite and*

$$\int_{-\infty}^{v} \frac{\kappa_2'}{\kappa_2 - \kappa} = +\infty.$$

Proof. By Lemma 5.4.4, both $\vartheta_1(-\infty)$ and $\vartheta_2(-\infty)$ are finite. Since $\vartheta_1(-\infty) > 0$, $\kappa(-\infty) = \frac{\vartheta_2(-\infty)}{\vartheta_1(-\infty)}$ is finite as well. From (5.4.10) and the monotonicity of κ_2, we conclude that $\kappa_2(-\infty) = \kappa(-\infty)$.

From (5.2.12), we have

$$\int_{\tau}^{v} \frac{\kappa_2'}{\kappa_2 - \kappa} = \log\frac{\kappa_2(v) - \kappa(v)}{\kappa_2(\tau) - \kappa(\tau)} + \log\frac{\vartheta_1(v)}{\vartheta_1(\tau)},$$

and the right-hand side tends to $+\infty$ as τ goes to $-\infty$. ■

Proposition 5.4.14. *If $\int_{-\infty}^{v} \lambda = +\infty$, $\vartheta_1(-\infty) > 0$, and there is a standard candidate B on $\Omega_{\mathrm{R}}(-\infty, v)$, then $\kappa_3(-\infty) < +\infty$.*

Proof. Assume on the contrary that $\kappa_3(-\infty) = +\infty$. First, we claim that $\mathfrak{K}(-\infty) = -\infty$. If not, then $\mathfrak{K} > C_1$ for some $C_1 \in \mathbb{R}$ in a neighborhood of $-\infty$, i.e., $\kappa_3' > C_1\kappa_2'$. Integration of this inequality yields $\kappa_3 < C_1\kappa_2 + C_2$ for some $C_2 \in \mathbb{R}$. This contradicts to the assumption $\kappa_3(-\infty) = +\infty$ by Lemma 5.4.13. Hence, $\mathfrak{K}(-\infty) = -\infty$ and $\mathfrak{K}' > 0$ in a neighborhood of $-\infty$. By (3.3.4), $\beta_2' > 0$ in the same neighborhood of $-\infty$, which contradicts the existence of the standard candidate B on $\Omega_{\mathrm{R}}(-\infty, v)$. We conclude that $\kappa_3(-\infty) < +\infty$. ■

Proposition 5.4.15. *Suppose that* $\int_{-\infty}^{v}\lambda = +\infty$, $\vartheta_1(-\infty) > 0$, *and* $\kappa_3(-\infty)$ *is finite. Let* B *be the standard candidate on* $\Omega_{\mathrm{R}}(-\infty, v)$. *Then for every point* $x \in \Omega_{\mathrm{R}}(-\infty, v)$, *there exists a sequence of non-decreasing test functions* $\varphi_{n,x}$ *being an optimizing sequence for* B *at* x; *moreover,* $\varphi_{n,x} \preceq g(v)$.

Proof. It suffices to prove the claim for the case $x = w(v) = w_{\mathrm{R}}(v)$. In what follows, we omit the subscript x in our notation. We use the sequence of functions φ_n defined by (5.4.1). All these functions are non-decreasing and satisfy $\varphi_n \preceq g(v)$. Therefore, we only need to verify that $\langle \mathrm{f}(\varphi_n)\rangle_{[0,1]} \to B(w(v))$.

We calculate the limit using (5.4.14):

$$\begin{aligned}\lim_{n\to\infty}\langle \mathrm{f}(\varphi_n)\rangle_{[0,1]} &= -\kappa_3(-\infty)\vartheta_1(-\infty) + f(v) - \int_{-\infty}^{v} f'(\tau)\exp\Bigl(-\int_{\tau}^{v}\lambda\Bigr)\,d\tau\\ &= -\kappa_3(-\infty)\vartheta_1(-\infty) + f(v) - \int_{-\infty}^{v}\kappa_3(\tau)\vartheta_1'(\tau)\,d\tau\\ &= f(v) - \kappa_3(v)(g_1(v) - w_1(v)) + \int_{-\infty}^{v}\kappa_3'(\tau)\vartheta_1(\tau)\,d\tau.\end{aligned}$$

Then, we use (5.4.15), (5.4.16), and (5.4.17) to conclude that $\langle \mathrm{f}(\varphi_n)\rangle_{[0,1]} \to B(w(v))$. The proof is complete. ■

Propositions 5.4.12, 5.4.14, and 5.4.15 prove Theorem 5.4.2 in case (B).

5.4.4 Case (C) : $\int_{-\infty}^{v}\lambda = +\infty$, $\vartheta_1(-\infty) = 0$, $\vartheta_2(-\infty) \neq 0$

Lemma 5.4.16. *If* $\int_{-\infty}^{v}\lambda = +\infty$, $\vartheta_1(-\infty) = 0$, *and* $\vartheta_2(-\infty) \neq 0$, *then we have* $\kappa_2(-\infty) = \kappa(-\infty) = -\infty$. *In particular,* $g_2' < 0$ *in a neighborhood of* $-\infty$ *and* $\vartheta_2(-\infty) < 0$.

Proof. We easily obtain that $\kappa(-\infty) = \frac{\vartheta_2(-\infty)}{\vartheta_1(-\infty)}$ is infinite. From (5.4.10) and the monotonicity of κ_2, we conclude that $\kappa_2(-\infty) = \kappa(-\infty)$ is infinite as well. Since the function κ_2 is increasing, both these limits are $-\infty$. Then $g_2' = g_1'\kappa_2 < 0$ in a neighborhood of $-\infty$, and from (5.4.7), we obtain that $\vartheta_2(-\infty) < 0$. ■

Proposition 5.4.17. *If* $\int_{-\infty}^{v}\lambda = +\infty$, $\vartheta_1(-\infty) = 0$, $\vartheta_2(-\infty) \neq 0$, *and* $\mathfrak{K}(-\infty) = +\infty$, *then* $\boldsymbol{B}(x; f) = +\infty$ *for any* $x \in \Omega \setminus \partial_{\mathrm{fixed}}\Omega$.

Proof. It suffices to prove that $\boldsymbol{B}(w(v)) = +\infty$. We consider the sequence φ_n defined in (5.4.1); they are test functions for $w(v)$, see Remark 5.4.1. Using (5.4.6), we rewrite (5.4.12) in the form

$$\begin{aligned}\langle \mathrm{f}(\varphi_n)\rangle_{[0,1]} &= -\frac{f(v_n^+) - f(v_n^-)}{g_2(v_n^+) - g_2(v_n^-)}\vartheta_2(v_n^+) + f(v) - \int_{v_n^+}^{v} f'(u)\exp\Bigl(-\int_{u}^{v}\lambda\Bigr)\,du\\ &= -\frac{f'(v_n)}{g_2'(v_n)}\vartheta_2(v_n^+) + f(v) - \int_{v_n^+}^{v} f'(u)\exp\Bigl(-\int_{u}^{v}\lambda\Bigr)\,du, \qquad (5.4.19)\end{aligned}$$

for some $v_n \in (v_n^-, v_n^+)$. Since we have $\vartheta_2(-\infty) < 0$ and $g_2' < 0$ in a neighborhood of $-\infty$ (see Lemma 5.4.16), it suffices to prove that $\frac{f'}{g_2'}(-\infty) = +\infty$. This follows from the identity $\frac{f'}{g_2'} = \frac{\kappa_3}{\kappa_2}$ and the L'Hôpital rule: $\kappa_2(-\infty) = -\infty$, and $\frac{\kappa_3'}{\kappa_2'}(-\infty) = \mathfrak{K}(-\infty) = +\infty$. We see that $\langle \mathrm{f}(\varphi_n)\rangle_{[0,1]} \to +\infty$. ■

Proposition 5.4.18. *If $\int_{-\infty}^{v} \lambda = +\infty$, $\vartheta_1(-\infty) = 0$, $\vartheta_2(-\infty) \neq 0$, and there is a standard candidate B on $\Omega_{\mathrm{R}}(-\infty, v)$, then $\mathfrak{K}(-\infty) > -\infty$.*

Proof. Assume on the contrary that $\mathfrak{K}(-\infty) = -\infty$. Then $\mathfrak{K}' > 0$ in a neighborhood of $-\infty$. By (3.3.4), we see that $\beta_2' > 0$, which is impossible for the standard candidate B, see Definition 3.3.8. ■

Proposition 5.4.19. *Let $\int_{-\infty}^{v} \lambda = +\infty$, $\vartheta_1(-\infty) = 0$, and $\vartheta_2(-\infty) \neq 0$, and let $\mathfrak{K}(-\infty)$ be finite. If B is the standard candidate on $\Omega_{\mathrm{R}}(-\infty, v)$, then, for every point $x \in \Omega_{\mathrm{R}}(-\infty, v)$, there exists a sequence of non-decreasing test functions $\varphi_{n,x}$ being an optimizing sequence for B at x; moreover, $\varphi_{n,x} \preceq g(v)$.*

Proof. It suffices to prove the claim for the case $x = w(v) = w_{\mathrm{R}}(v)$. In what follows, we omit the subscript x in our notation. We use the sequence of functions φ_n defined by (5.4.1). All these functions are non-decreasing and satisfy $\varphi_n \preceq g(v)$. Therefore, it suffices to verify that $\langle \mathrm{f}(\varphi_n)\rangle_{[0,1]} \to B(w(v))$.

First, we note that $\frac{f'}{g_2'} = \frac{\kappa_3}{\kappa_2}$ and $\kappa_2(-\infty) = -\infty$, by Lemma 5.4.16, and therefore, by the L'Hôpital rule,

$$\frac{f'}{g_2'}(-\infty) = \frac{\kappa_3'}{\kappa_2'}(-\infty) = \mathfrak{K}(-\infty).$$

We calculate the limit using (5.4.19):

$$\begin{aligned}\lim_{n\to\infty} \langle \mathrm{f}(\varphi_n)\rangle_{[0,1]} &= -\mathfrak{K}(-\infty)\vartheta_2(-\infty) + f(v) - \int_{-\infty}^{v} f'(\tau)\exp\Big(-\int_{\tau}^{v}\lambda\Big)\,d\tau \\ &= -\mathfrak{K}(-\infty)\vartheta_2(-\infty) + f(v) - \int_{-\infty}^{v} \kappa_3(\tau)\vartheta_1'(\tau)\,d\tau \\ &= -\mathfrak{K}(-\infty)\vartheta_2(-\infty) + f(v) - \kappa_3(v)(g_1(v) - w_1(v)) \\ &\quad + \int_{-\infty}^{v} \kappa_3'(\tau)\vartheta_1(\tau)\,d\tau,\end{aligned}$$

where in the latter equality, we integrate by parts using Lemma 5.2.3 with $Y = \kappa_3$ and $\vartheta = \vartheta_1$. Then, we use (5.4.17) to rewrite this formula in the form

$$\begin{aligned}\lim_{n\to\infty} \langle \mathrm{f}(\varphi_n)\rangle_{[0,1]} &= f(v) + (w_1(v) - g_1(v))\kappa_3(v) + (w_1(v) - g_1(v))(\kappa(v) - \kappa_2(v)) \\ \times \Big(&\frac{-\mathfrak{K}(-\infty)\vartheta_2(-\infty)}{(w_1(v) - g_1(v))(\kappa(v) - \kappa_2(v))} + \int_{-\infty}^{v} \exp\Big(-\int_{\tau}^{v}\frac{\kappa_2'}{\kappa_2 - \kappa}\Big)\frac{\kappa_3'(\tau)}{\kappa_2(\tau) - \kappa(\tau)}\,d\tau\Big).\end{aligned}$$

We need to verify that the expression in the large parentheses coincides with β_2 given by (3.3.3). We have the following formula for β_2:

$$\mathfrak{K}(v) - \int_{-\infty}^{v} \exp\Big(-\int_{\tau}^{v} \frac{\kappa_2'}{\kappa_2 - \kappa}\Big)\mathfrak{K}'(\tau)\,d\tau$$
$$= \mathfrak{K}(-\infty)\exp\Big(-\int_{-\infty}^{v} \frac{\kappa_2'}{\kappa_2 - \kappa}\Big) + \int_{-\infty}^{v} \exp\Big(-\int_{\tau}^{v} \frac{\kappa_2'}{\kappa_2 - \kappa}\Big)\frac{\kappa_3'(\tau)}{\kappa_2(\tau) - \kappa(\tau)}\,d\tau.$$

Therefore, it suffices to verify that

$$\begin{aligned}
-\vartheta_2(-\infty) &\overset{?}{=} (g_1(v) - w_1(v))(\kappa_2(v) - \kappa(v))\exp\Big(-\int_{-\infty}^{v} \frac{\kappa_2'}{\kappa_2 - \kappa}\Big) \\
&= (g_1(v) - w_1(v))(\kappa_2(v) - \kappa(v)) \lim_{u\to-\infty} \exp\Big(-\int_{u}^{v} \frac{\kappa_2'}{\kappa_2 - \kappa}\Big) \\
&\overset{(5.4.18)}{=} \lim_{u\to-\infty} (\kappa_2(u) - \kappa(u))\vartheta_1(u) = \lim_{u\to-\infty} \Big(\frac{\kappa_2(u)}{\kappa(u)} - 1\Big)\vartheta_2(u),
\end{aligned} \tag{5.4.20}$$

where all the limits exist. Since the limit $\vartheta_2(-\infty)$ is finite and nonzero, it suffices to prove that $\frac{\kappa_2}{\kappa}(-\infty) = 0$; note that this limit exists. We rewrite κ_2 and κ in terms of ϑ_1 and ϑ_2:

$$\kappa = \frac{\vartheta_2}{\vartheta_1}, \qquad \kappa_2 = \frac{g_2'}{g_1'} = \frac{\vartheta_2'}{\vartheta_1'},$$

therefore

$$\frac{\kappa_2}{\kappa}(-\infty) = \frac{(\log|\vartheta_2|)'}{(\log\vartheta_1)'}(-\infty). \tag{5.4.21}$$

Since $\log\vartheta_1(-\infty) = -\infty$, and the limit (5.4.21) exists, we finish the proof by the L'Hôpital rule:

$$0 = \frac{\log|\vartheta_2|}{\log\vartheta_1}(-\infty) = \frac{(\log|\vartheta_2|)'}{(\log\vartheta_1)'}(-\infty) = \frac{\kappa_2}{\kappa}(-\infty). \qquad \blacksquare$$

Remark 5.4.20. As opposed to cases (A) and (B), in case (C), we have

$$\int_{-\infty}^{v} \frac{\kappa_2'}{\kappa_2 - \kappa} < \infty,$$

which follows from (5.4.20).

Propositions 5.4.17, 5.4.18, and 5.4.19 prove Theorem 5.4.2 in case (C). Thus, we have completed the proof of the theorem.

5.4.5 Behavior at $+\infty$

We formulate the analog of Theorem 5.4.2 for the behavior at $+\infty$ that generalizes Proposition 5.2.7 for the case when condition (2.1.8) fails.

Theorem 5.4.21. *Let B be the standard candidate on $\Omega_{\mathrm{L}}(v, +\infty)$. Then, for every point $x \in \Omega_{\mathrm{L}}(v, +\infty)$, there exists a sequence of non-increasing test functions $\varphi_{n,x}$ being an optimizing sequence for B at x; moreover, $g(v) \preceq \varphi_{n,x}$.*

Since the situation is absolutely symmetric, we omit the proof of this theorem.

5.4.6 When $\boldsymbol{B} = +\infty$?

We collect the conditions on the behavior of the function f at both infinities that guarantee $\boldsymbol{B}(x; f) = +\infty$ for any $x \in \Omega \setminus \partial_{\mathrm{fixed}}\Omega$.

Propositions 5.4.10, 5.4.12, 5.4.17, and 5.2.6 describe the conditions at $-\infty$, see Table 5.2.

condition on Ω		condition on f
(A)	$\exp(-\int_{-\infty}^{v} \lambda_{\mathrm{R}}) \neq 0$	$f(-\infty) = +\infty$
(B)	$\exp(-\int_{-\infty}^{v} \lambda_{\mathrm{R}}) = 0$, $\vartheta_1(-\infty) \neq 0$	$\kappa_3(-\infty) = -\infty$
(C)	$\exp(-\int_{-\infty}^{v} \lambda_{\mathrm{R}}) = 0$, $\vartheta_1(-\infty) = 0$, $\vartheta_2(-\infty) \neq 0$	$\mathfrak{K}(-\infty) = +\infty$
(2.1.7)	$\exp(-\int_{-\infty}^{v} \lambda_{\mathrm{R}}) = 0$, $\vartheta_1(-\infty) = 0$, $\vartheta_2(-\infty) = 0$	$\int_{-\infty}^{v} f'(\tau) \exp(-\int_{\tau}^{v} \lambda_{\mathrm{R}})\, d\tau = -\infty$

Table 5.2. Conditions at $-\infty$ for $\boldsymbol{B} = +\infty$.

Here $v \in \mathbb{R}$ is arbitrary, the conditions do not depend on v. We underline that Condition 2.1.11 is of crucial importance for these results. If Condition 2.1.11 is violated, the problem of characterizing the functions f for which $\boldsymbol{B}$ is infinite, is more involved, see [17, Lemma 6.1.4].

Remark 5.4.22. In case (A), the conditions $\int_{-\infty}^{v} f'(\tau) \exp(-\int_{\tau}^{v} \lambda_{\mathrm{R}})\, d\tau = -\infty$ and $f(-\infty) = +\infty$ are equivalent. In cases (B) and (C), the conditions on f look different, but in case (C), the condition $\mathfrak{K}(-\infty) = +\infty$ is equivalent to $\frac{f'}{g_2'}(-\infty) = +\infty$, which is symmetric to the condition $\frac{f'}{g_1'}(-\infty) = \kappa_3(-\infty) = -\infty$ that appears in case (B).

As usual, we omit the symmetrical statements for the case of $+\infty$ but collect the corresponding results in Table 5.3. Therein,

$$\vartheta^{\mathrm{L}}(\tau) = (g(\tau) - w_{\mathrm{L}}(\tau)) \exp\Big(\int_{v}^{\tau} \lambda_{\mathrm{L}}\Big), \quad \tau \in \mathbb{R},$$

where v is a fixed parameter, $v \in \mathbb{R}$.

condition on Ω		condition on f
(A)	$\exp(\int_v^{+\infty}\lambda_{\mathrm{L}}) \neq 0$	$f(+\infty) = +\infty$
(B)	$\exp(\int_v^{+\infty}\lambda_{\mathrm{L}}) = 0$, $\vartheta_1^{\mathrm{L}}(+\infty) \neq 0$	$\kappa_3(+\infty) = +\infty$
(C)	$\exp(\int_v^{+\infty}\lambda_{\mathrm{L}}) = 0$, $\vartheta_1^{\mathrm{L}}(+\infty) = 0$, $\vartheta_2^{\mathrm{L}}(+\infty) \neq 0$	$\mathfrak{K}(+\infty) = +\infty$
(2.1.8)	$\exp(\int_v^{+\infty}\lambda_{\mathrm{L}}) = 0$, $\vartheta_1^{\mathrm{L}}(+\infty) = 0$, $\vartheta_2^{\mathrm{L}}(+\infty) = 0$	$\int_v^{+\infty} f'(\tau)\exp(\int_v^{\tau}\lambda_{\mathrm{L}})\,d\tau = +\infty$

Table 5.3. Conditions at $+\infty$ for $\boldsymbol{B} = +\infty$.

Remark 5.4.23. The function $\boldsymbol{B}(\,\cdot\,; f)$ is infinite on $\Omega \setminus \partial_{\mathrm{fixed}}\Omega$ if the pair (Ω, f) satisfies one of the conditions listed in the tables presented in this subsection. If (Ω, f) satisfies none of the conditions, then the function $\boldsymbol{B}(\,\cdot\,; f)$ is finite on Ω and can be constructed using the procedure described in this memoir.

5.4.7 Examples of domains

In Section 2.1.1 we have seen several examples of the domains that satisfy conditions (2.1.7) and (2.1.8). Now we present the domains that satisfy conditions (A), (B), or (C). We provide examples of the required behavior when t tends to $-\infty$. The case of $+\infty$ is obviously symmetric.

(A) $g(t) = (e^t + e^{t/2}, e^{2t} + 2e^{3t/2})$, $w(t) = (e^t, e^{2t})$, $\lambda(t) = e^{t/2}$,

(B) $g(t) = (t, (1 + t + t^2)e^{-t^2})$, $w(t) = (-t^2, e^{-t^2})$, $\lambda(t) = -\dfrac{2}{t+1}$,

(C) $g(t) = \Big(t + \dfrac{t^2}{1+t^2}, \dfrac{e^{-t}}{1+t^2}\Big)$, $w(t) = (t, e^{-t})$, $\lambda(t) = 1 + \dfrac{1}{t^2}$.