



Chasing finite shadows of infinite groups through geometry

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Abstract. There are many situations in geometry and group theory where it is natural, convenient or necessary to explore infinite groups via their actions on finite objects – that is, via the finite quotients of the group. But how much understanding can one really gain about an infinite group by examining its finite images? Which properties of the group can one recognise, and when does the set of finite images determine the group completely? How hard is it to decide what the finite images of a given infinite group are?

These notes follow my plenary lecture at the ECM in Sevilla, July 2024. The goal of the lecture was to sketch some of the rich history of the preceding problems and to present results that illustrate how the field surrounding these questions has been transformed in recent years by input from low-dimensional topology and the study of non-positively curved spaces.

1. Introduction

To what extent can one understand an infinite group by studying only its finite images? This compelling question has re-emerged with different emphases throughout the history of group theory, and in recent years it has been animated by a rich interplay with geometry and low-dimensional topology. My purpose in the lecture on which these notes are based was to report on some highlights of this interplay.

Before engaging directly with the challenge of identifying and exploring the finite images of groups, we will step back and consider the following more primitive questions: how should we describe the groups that arise in nature, and how might we go about the task of understanding a group that is presented to us? The first question leads us to a discussion of group presentations and the unlimited difficulty of extracting information from such presentations. A classical and insightful response to the second question is that since a group is, by definition, the mathematical object that captures the notion of symmetry, one ought to explore groups by seeking objects they are symmetries of, that is, one should study *actions* on various objects. But what

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objects? There are many things that are worth trying, but in the absence of further information, the most obvious objects to try first are finite sets, and that brings us back to the basic question that frames our study: to what extent can one understand a group by studying only its finite images?

Our focus will be on finitely generated and finitely presented groups. We shall discuss how hard it is to identify the finite quotients of a given group, and we shall explore which properties of a group can be detected from its set of finite quotients. We will be particularly concerned with questions of *profinite rigidity*, that is, situations where an infinite group is completely determined by its finite quotients. Groups that arise in low-dimensional geometry and topology will play a central role in our discussion. For example, one might ask to what extent the group of symmetries of the paving of the hyperbolic plane shown in Figure 1 is distinguished from all other groups by its finite quotients.

The narrative of the lecture on which these notes are based was built around the following sequence of results. The first serves to illustrate the difficulty of identifying the finite quotients of a finitely presented group.

Theorem A ([42]). *There does not exist an algorithm that, given an arbitrary finite presentation, can determine whether or not the group presented has a non-trivial finite quotient.*

Following [28], we shall discuss refinements of this result that describe the (non-)existence of algorithms that can identify quotients in particular families of finite

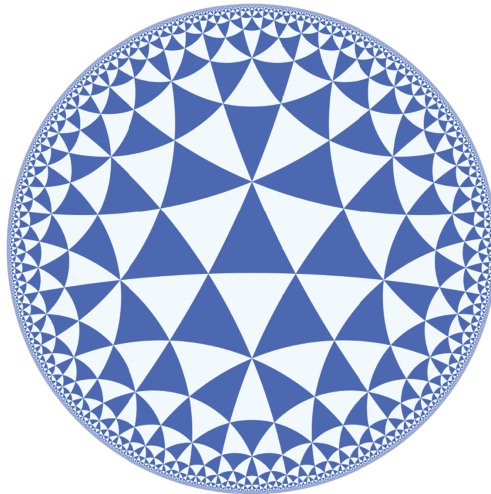


Figure 1. $\Delta(3, 3, 4) = \langle A, B, C \mid A^3 = B^3 = C^4 = 1 = ABC \rangle$

simple groups, for example, the groups $\mathrm{PSL}_n(q)$ with either n fixed and q varying, or vice versa.

The finite images of a group Γ form an inverse system indexed by the finite index normal subgroups $N \trianglelefteq \Gamma$: if $N < M$, then $\Gamma/N \twoheadrightarrow \Gamma/M$. The *profinite completion* $\hat{\Gamma}$ of Γ is the inverse limit of this system; it is a compact, totally disconnected topological group. There is a natural map $\iota : \Gamma \rightarrow \hat{\Gamma}$ with dense image. Every homomorphism from Γ to a finite group Q extends, via ι , to a map $\hat{\Gamma} \rightarrow Q$ with the same image, and if Γ is finitely generated, every finite quotient of $\hat{\Gamma}$ arises in this way, so Γ and $\hat{\Gamma}$ have the same set of finite quotients. For finitely generated groups Γ and Λ , the set of finite images of Γ will be the same as the set of finite images of Λ if and only if $\hat{\Lambda}$ and $\hat{\Gamma}$ are isomorphic (as abstract or topological groups) [49, 87]; thus, for finitely generated groups, the reader can read the statement $\hat{\Lambda} \cong \hat{\Gamma}$ as “ Γ and Λ have the same set of finite quotients”.

When discussing profinite rigidity, it is natural to restrict attention to groups Γ that are *residually finite*, that is, every $\gamma \in \Gamma \setminus \{1\}$ has non-trivial image in some finite quotient of Γ . We shall consider three versions of profinite rigidity: *absolute* profinite rigidity, in which comparisons to all finitely generated, residually finite groups are made; *relative* profinite rigidity, whereby one compares groups within a class – for example, 3-manifold groups, nilpotent groups, or lattices in Lie groups; and *Grothendieck* rigidity, where one is concerned with the existence of pairs of non-isomorphic groups $P \hookrightarrow \Gamma$ such that the inclusion induces an isomorphism of profinite completions. A central result in this last setting is the following.

Theorem B ([29]). *There exist residually finite (Gromov) hyperbolic groups H and finitely presented subgroups $P \hookrightarrow \Gamma := H \times H$ of infinite index, with $P \not\cong \Gamma$, such that the inclusion $P \hookrightarrow \Gamma$ induces an isomorphism $\hat{P} \rightarrow \hat{\Gamma}$.*

We shall see in Section 7 that in this theorem, one can assume $H < \mathrm{SL}(n, \mathbb{Z})$.

Easy examples of residually finite groups that are profinitely rigid in the absolute sense include $\mathbb{Z}^d$, but it is much more difficult to find *full-sized* examples, that is, groups that contain a non-abelian free subgroup and hence do not satisfy any group law. This is where input from low-dimensional geometry and topology comes to the fore. The reader will recall that $\mathrm{PSL}(2, \mathbb{R})$ and $\mathrm{PSL}(2, \mathbb{C})$ are the groups of orientation-preserving isometries of the hyperbolic plane $\mathbb{H}^2$ and hyperbolic 3-space $\mathbb{H}^3$, respectively.

Theorem C ([33]). *There are arithmetic lattices in $\mathrm{PSL}(2, \mathbb{C})$ and $\mathrm{PSL}(2, \mathbb{R})$ that are profinitely rigid in the absolute sense.*

Examples of such lattices in $\mathrm{PSL}(2, \mathbb{C})$ include the Bianchi group $\mathrm{PSL}(2, \mathbb{Z}[\omega])$, where $1 + \omega + \omega^2 = 1$, as well as certain cocompact lattices of small covolume. Examples in $\mathrm{PSL}(2, \mathbb{R})$ include the triangle group $\Delta(3, 3, 4)$ portrayed in Figure 1.

The remarkable nature of 3-manifold groups and the beauty and utility of hyperbolic geometry and its generalisations have been major themes in the study of profinite rigidity over recent years. Another important theme that has emerged is the importance of finiteness properties: there can be a stark difference in the behaviour of finitely generated and finitely presented groups in this context. The following recent result is perhaps the best illustration of that theme. A more refined version of this theorem, which again relies on extensive input concerning 3-dimensional manifolds, will be described in the penultimate section of these notes.

Theorem D ([37]). *There are finitely presented, residually finite groups Γ that are profinitely rigid among finitely presented groups but contain infinitely many pairwise non-isomorphic, finitely generated subgroups $H \hookrightarrow \Gamma$ such that $\hat{H} \cong \hat{\Gamma}$*

2. Presenting groups and understanding them

In a moment, we will decipher the caption of Figure 1 by recalling what it means to *present* a group. But first let me provoke the reader with the assertion that group theory is not really a branch of algebra – it belongs to the whole of mathematics, just as numbers do.

2.1. Groups everywhere

What kind of mathematics do you want to do? When we start doing formal mathematics, settling down to lend precision to an intuition gleaned from examples, we make precise definitions that encapsulate the essence of the objects that we want to study. As we start to manipulate and compare our objects, we have to decide what sort of maps $X \rightarrow Y$ we will allow. If we are studying sets, then any maps will do, but if we are interested in a linear problem and are studying vector spaces, we are likely to restrict to linear or affine maps; if we are doing topology, we will probably consider continuous maps; if we are modelling flexible geometric objects, we might study Lipschitz maps, etc. But no matter what the context, it is likely that sooner or later we will want to understand the *automorphisms* of our objects X – that is, the set of invertible maps (of our chosen type) that preserve the defining features of X . And, no matter what the context, these automorphisms form a *group*, $\mathrm{Aut}(X)$. According to one's nature, one might then be drawn to the study of groups themselves.

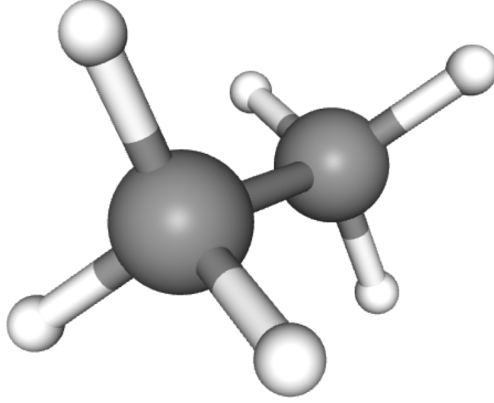


Figure 2. $\langle T, F \mid T^3 = 1 = F^2, FTF = T^2 \rangle$

2.2. How should we capture groups?

Maintaining the spirit of doing mathematics in the wild, let us consider how we might go about describing the symmetries of objects we encounter in nature. For an ethane molecule, we might start by noting that it has six (rigid, orientation-preserving) symmetries, but more information is contained in the observation that we need to combine two basic operations – T (twist) and F (flip) – in order to get everything. We also note that doing T three times or F twice amounts to doing nothing, and, after some more thought, we might also observe that doing F then T then F has the same effect as doing T twice. In summary, $\langle T, F \mid T^3 = 1 = F^2, FTF = T^2 \rangle$. We want to formalise the fact that these notes describe the group completely (see Figure 2).

2.3. Presentations of groups

A *presentation* of a group G is an expression of the form

$$G = \langle a_1, a_2, \dots \mid r_1 = 1, r_2 = 1, \dots \rangle,$$

where the r_j are finite words in the letters a_i and their inverses. The letters a_i are called *generators*, and the r_j are called *relators* (or “defining relations”). We will be mostly concerned with the case where the lists (a_i) and (r_j) are finite, in which case the presentation is said to be finite and G is said to be *finitely presented*.

If r_j is a concatenation of words $u_j v_j$, it is sometimes convenient to write $u_j = v_j^{-1}$ instead of $r_j = 1$. It is also common practice to abbreviate the list of relators by writing $r_1, r_2, \dots$ instead of $r_1 = 1, r_2 = 1, \dots$.

When we give a presentation, we are asserting three things. First, there is a choice of elements $a_1, a_2, \dots \in G$ (which need not be distinct) such that every element of G can be obtained by repeatedly performing the operations a_i and their inverses.

Two assertions are made concerning the relators: first, each of the equalities $r_i = 1$ is true in the group G (where 1 represents the identity element); second, *all* relations among the $a_i^{\pm 1}$ in G are consequences of the rules $r_j = 1$, where a “consequence” is a deduction made by repeatedly appealing to the following obvious facts and nothing else

$$[r = s \text{ and } t = u \implies rt = su], \quad [r = 1 \implies a^{-1}ra = 1].$$

These conditions can be formalised by saying that $G \cong F/R$ where F is the free group with basis $\{a_1, a_2, \dots\}$ and $R \trianglelefteq F$ is the smallest normal subgroup containing $\{r_1, r_2, \dots\}$.

With these formalities in hand, we return to the caption in Figure 1. The reader can verify (with effort) that if one takes A, B, C to be suitable rotations about the vertices of any fixed triangle in the tessellation, then the caption satisfies the three requirements of a presentation.

2.4. Groups that do not have finite presentations

A group G is termed *finitely generated* if there is a finite collection of elements $a_1, \dots, a_n \in G$ such that every element of G is a product of the a_i and their inverses. If a group is finitely generated, then it is countable, but not all countable groups are finitely generated. For example, it is easy to verify that the rational numbers under addition $(\mathbb{Q}, +)$ cannot be finitely generated.

It is less obvious how to construct finitely generated groups that cannot be finitely presented, but I want to emphasise that one does not need obscure, monstrous constructions to find such groups: the absence of a finite presentation emerges in small, concrete examples. To illustrate this, let

$$\mathbf{X} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{Y} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

and consider the 4-by-4 integer matrices

$$A = \begin{pmatrix} \mathbf{X} & 0 \\ 0 & \mathbf{I} \end{pmatrix}, \quad B = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{X} \end{pmatrix}, \quad C = \begin{pmatrix} \mathbf{Y} & 0 \\ 0 & \mathbf{Y} \end{pmatrix}.$$

Example 2.1. The subgroup of $\mathrm{SL}(4, \mathbb{Z})$ generated by $\{A, B, C\}$ has *no finite presentation*. To prove this, one first uses Klein’s ping-pong argument to prove that the subgroup of $\mathrm{SL}(2, \mathbb{Z})$ generated by X and Y is a free group F of rank 2. Next, one observes that A, B, C lie in an obvious copy of $F \times F$. In fact, $\langle A, B, C \rangle$ is normal in $F \times F$ with quotient $\mathbb{Z}$, and such a subgroup can never be finitely presented [11, 57]. This example, which was studied by Stallings [115], is the seed for a rich vein of ideas in geometric group theory involving higher-dimensional (homological) finiteness properties [12], algebraic fibering [13, 70], and residually free groups [31].

The indirect nature of the proof sketched in the preceding example points to the fact that it can be hard to decide if an explicitly described group has a finite presentation. The fact that the following question has defied our understanding for half a century underscores this point.

Question 2.2 (Serre 1973 [110]). Can every finitely generated subgroup of $\mathrm{SL}(3, \mathbb{Z})$ be finitely presented?

The following theorem points to deeper problems and introduces a theme that we will pursue in the next section. This result relies on the existence of groups with relatively small presentations where the word problem is unsolvable (e.g., [20]). It is contained in the work of Mihailova [83] and Miller [84]; for a concise proof, see [22, Lemma 2.1].

Theorem 2.3. *Let F_2 be a free group of rank 2. There is no algorithm that, given a set of 20 elements $S \subset F_2 \times F_2$, can decide if the subgroup $\langle S \rangle$ has a finite presentation or not.*

Even in situations where one knows that finite presentations of groups exist, it can be extremely hard to construct them – for example, it is far from clear how to construct an explicit finite presentation for any uniform lattice¹ $\Gamma < \mathrm{SL}(3, \mathbb{R})$. Moving beyond lattices, if one works only with groups of integer matrices given by finite subsets $S < \mathrm{SL}(d, \mathbb{Z})$, then one can prove that there is no algorithm that allows one to construct finite presentations, even in cases where one knows that such presentations exist [40].

2.5. Problem: Hard to extract information from a presentation

Notwithstanding the preceding comments on the difficulty of finding presentations, when groups arise in nature (e.g., as discrete groups of automorphisms of a nice space), there are some general procedures that one can try to use to construct presentations, with versions of the Seifert/van Kampen theorem foremost among them. These are particularly effective in low dimensions. There are also various algebraic settings in which groups come naturally equipped with a finite presentation. But it can be ferociously hard to extract information about a group from a presentation.

¹A *lattice* in a Lie group G is a discrete subgroup $\Gamma < G$ whose covolume $\mathrm{vol}(G/\Gamma)$, with respect to Haar measure, is finite. The lattice is *uniform* if G/Γ is compact, and otherwise it is non-uniform.

Rather than engaging in an abstract discussion of this difficulty, let us examine three similar-looking small presentations:

$$\begin{aligned} G_2 &= \langle A, B \mid BA = A^2B \rangle, \\ G_3 &= \langle a, b, c \mid ba = a^2b, cb = b^2c, ac = c^2a \rangle, \\ G_4 &= \langle \alpha, \beta, \gamma, \delta \mid \beta\alpha = \alpha^2\beta, \gamma\beta = \beta^2\gamma, \delta\gamma = \gamma^2\delta, \alpha\delta = \delta^2\alpha \rangle. \end{aligned} \tag{2.1}$$

The easiest of these groups to understand is G_2 : for example, by adding the relation $A = 1$, we see that G_2 maps onto $\mathbb{Z}$, so in particular it is infinite. As for G_3 and G_4 , one of these groups is infinite while the other is the trivial group – how are we to tell which is which? In my experience, asking an audience to vote on which of G_3 and G_4 is the trivial group will provide an even split of opinion, and I see no obvious reason why it should be otherwise. We will return to these groups shortly.

2.6. We need action!

One can quickly get frustrated by the task of trying to understand groups such as G_3 and G_4 simply by manipulating their presentations algebraically. Instead, one seeks to understand a group $G = \langle a_1, \dots, a_n \mid r_1, \dots, r_m \rangle$ by examining the ways in which it can be realised as a group of symmetries (automorphisms) of something. For example, one might pursue the path of classical representation theory by trying to understand linear actions of G : in finite dimensions, this amounts to searching for invertible matrices $A_1, \dots, A_n$ so that substituting each A_i for a_i in the defining relations r_j yields a product that equals the identity matrix. Most primitively of all, we can examine the ways in which G can act as a group of permutations of a finite set. Beyond this, we might look for actions on spaces that carry interesting or illuminating geometric, topological, or analytic features; this is where geometric group theory really begins.

2.7. A group springing into action: Topological models

Instead of viewing $G = \langle a_1, \dots, a_n \mid r_1, \dots, r_m \rangle$ as a recipe for expressing G as a quotient of a free group, one can regard it as a recipe for building a 2-dimensional complex K with fundamental group G : the complex has one vertex (0-cell), an edge (1-cell) for each a_i , orientated and labelled a_i , and for each relator r_j , there is 2-cell obtained by attaching a disc to the loop in the 1-skeleton that spells out the word r_j . If one does this with the standard presentation $\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle$, the complex K is a torus. One can then unwrap K to its universal cover to get the group $\mathbb{Z}^2 = \pi_1 K$ acting freely and cocompactly² (as the group of deck transformations;

²Thanks to Tim Riley for drawing this picture.

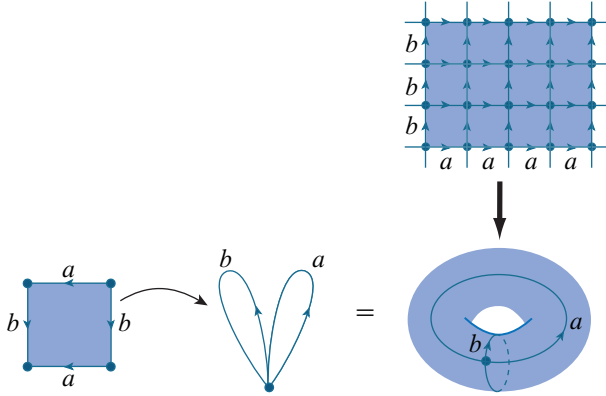


Figure 3. The 2-complex creating an action of $\mathbb{Z}^2 = \langle a, b \mid ab = ba \rangle$

see Figure 3). If we start with an arbitrary finite presentation, the 2-complex K will not be a manifold in general, but one can remedy this by embedding it in $\mathbb{R}^5$ then taking the boundary of a regular neighbourhood. This boundary can be smoothed, and with a little effort one can argue that it has the same fundamental group as the 2-complex we started with, thus proving the following proposition.

Proposition 2.4. *Every finitely presented group is the fundamental group of a smooth, compact 4-dimensional manifold.*

Classes of groups that are of special interest emerge when one asks how one might improve on this construction. The most obvious question, perhaps, is whether one can reduce to three dimensions; the answer is no, which provides us with the first hint that fundamental groups of compact 3-dimensional manifolds (“3-manifold groups” for short) have special features, a theme to which we will return.

But first let me remark that while these general constructions projecting the study of group presentations into topology are both useful and appealing, no hard work has been done, so one should expect them to preserve the hardness of most problems. Thus, for example, the task we set ourselves of deciding which of G_3 and G_4 was the trivial group is faithfully translated into the equally hard problem of deciding which of the 2-complexes built from these presentations is contractible, that is, can be continuously deformed within itself to a point. (It requires some theory to see that these problems are equivalent.)

2.8. Decision problems (Max Dehn 1911)

Combinatorial group theory is the study of groups given by generators and relations. Its origins are intertwined with the development of low-dimensional topology at the beginning of the twentieth century, particularly in the work of Max Dehn who,

in his famous paper [48], was the first to articulate clearly the algorithmic problems involved in extracting information from finite presentations $G = \langle a_1, \dots, a_n \mid r_1, \dots, r_m \rangle$, singling out three problems as the most fundamental: the *word problem* (deciding which words in the generators equal $1 \in G$), the *conjugacy problem* (deciding which words represent conjugate elements in G), and the *isomorphism problem* (which includes the triviality problem – deciding if the given group is trivial or not). One senses a certain level of frustration when he writes: “*One is led to such problems by necessity when working in geometry and topology.*” His main interest at the time was the classification of knots and related questions in 3-manifold topology.

2.9. Unsolvable problems

In the 1950s, P. S. Novikov [89] and W. W. Boone [15–19] (followed by Adyan [1], Rabin [97], and others) proved that the sought-after algorithms do not exist in general.

Theorem 2.5 (Novikov [89], Boone [15–19]). *There exist finitely presented groups $G = \langle a_1, \dots, a_n \mid r_1, \dots, r_m \rangle$ for which there is no algorithm that, given an arbitrary word w in the generators, can decide whether or not $w = 1$ in G .*

Starting with a finitely presented group G that has an unsolvable word problem, it is not difficult to construct groups that have solvable word problem but unsolvable conjugacy problem, nor is it too difficult to construct recursive sequences of finitely presented groups so that one cannot tell which are trivial. For the triviality problem, one uses the theory of HNN extensions and amalgamated free products to make a sequence of groups, indexed by words in the generators of G , so that $w = 1$ in $G \Leftrightarrow G_w = \{1\}$.

2.10. 3-manifold groups are well behaved

We have already seen that 3-manifold groups are special, and this is particularly borne out in the context of decision problems, although it took the whole of the twentieth century to prove that the algorithms that Dehn sought in this context at the beginning of the century do exist.

Theorem 2.6. *If Γ is the fundamental group of a compact 3-manifold, then there are algorithms to solve the word and conjugacy problems in Γ . Moreover, there is an algorithm that, given two finite presentations of 3-manifold groups, will determine whether or not the groups presented are isomorphic.*

This theorem follows from Perelman’s resolution of the geometrisation conjecture, but to see this one requires the work of many authors; we refer the reader to [5] for a thorough discussion.

2.11. Returning to the groups G_2, G_3, G_4

We commented earlier that the general construction illustrated in Figure 3 does not make the task of deciding which of G_3 and G_4 is $\{1\}$ any easier. Since topology has not helped us, we might fall back on representation theory, or the search for finite images. But these too fail us: we can gain further insight into $G_2 = \langle A, B \mid BA = A^2B \rangle$ by noting that the defining relation is satisfied by the matrices

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix},$$

yielding a representation that one can prove is faithful; one can also construct infinitely many finite quotients, for example, the presentation we gave of the symmetries of the ethane molecule (Figure 2) provides an action on a finite object; but a search for matrices of any size over any field satisfying the defining relations of G_3 and G_4 will prove fruitless – neither group admits any non-trivial, finite-dimensional representation over any field. Likewise, a search for non-trivial finite images of these groups will prove fruitless.

In fact, G_3 is the trivial group (this is a challenging exercise) while G_4 is an infinite group that has no non-trivial finite quotients. Since G_4 has no non-trivial matrix representations and no non-trivial actions on finite sets, where are we to look for an action that will unravel the structure of the group? The answer is that we get the group to act on a *tree*: the group decomposes as a non-trivial amalgamated free product and hence (according to Bass–Serre theory [111]) acts with compact quotient on an infinite tree. This is an important insight of Higman [62], reinterpreted by Serre [111].

2.12. Really useful actions: Trees, hyperbolicity, and cube complexes

We have just had our first encounter with actions on trees, which is the base example for two classes of actions that play a central role in what follows; both originate in the seminal work of Gromov [52]. First, there is Gromov’s theory of hyperbolic groups and spaces. Whereas in a tree all triangles degenerate to tripods, hyperbolicity for geodesic metric spaces is characterised by the property that all triangles are uniformly thin: there is a constant $\delta > 0$ such that each side of each geodesic triangle is contained in the δ -neighbourhood of the union of the other two sides. A finitely generated group is (*Gromov*) *hyperbolic* if it acts properly and cocompactly by isometries on such a space. It will be helpful for the reader to have a nodding acquaintance with this theory (which can be gleaned from [30] or [52]), but no details will be needed to understand these notes.

From a different perspective, one can regard trees as 1-dimensional CAT(0) cube complexes. The rich and extensive theory of groups acting on CAT(0) cube complexes has played a central role in geometric group theory and low-dimensional topology over the last three decades. The reader unfamiliar with the subject can consult [30, 47, 107] for the basic theory and [106] for a concise introduction. Briefly, a *non-positively curved cube complex* is a space built by assembling Euclidean cubes, glued together by local isometries between faces, so that the resulting path metric on the space is non-positively curved in the sense of A. D. Alexandrov, that is, in the neighbourhood of each point, geodesic triangles are no fatter than triangles in the Euclidean plane with the same edge lengths (see [30]). This is equivalent to a purely combinatorial condition on links: there are no “empty simplices” in the link of any vertex of the complex; in dimension 2, this means that there are no circuits of length less than 4 in the link graphs (see Figure 4). If the complex is locally finite dimensional, all triangles in the universal cover (not just small ones) will satisfy Alexandrov’s comparison condition; in other words, this universal cover will be a CAT(0) space [30].

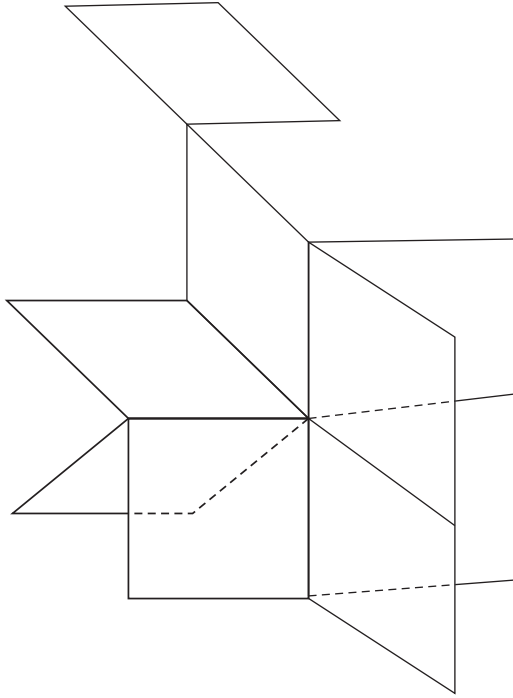


Figure 4. A 2-dimensional CAT(0) cube complex

There is a rich theory of groups that act by isometries on CAT(0) spaces and a much richer theory of groups that act properly and compactly by isometries on CAT(0) cube complexes, particularly when the complex is *special* in the sense of Haglund and Wise [59]. This last condition (which concerns the way in which cube-bisecting hyperplanes intersect) defines the class of *special* groups (sometimes called cocompact special, for clarity). Although it is defined in terms of the geometry of hyperplanes, specialness has a global characterisation that is particularly important for us: a cube complex is special if and only if there is a locally isometric embedding of it into the Salvetti complex of some right-angled Artin group [59]. For us, the important consequence of this is that any (virtually) special group can be embedded in $SL(n, \mathbb{Z})$ for some n .

The powerful theory of (virtually) special groups developed by Wise and his collaborators has had a profound influence on many aspects of geometric group theory, including the study of profinite rigidity. If the reader explores the original papers on which these notes are based, they will quickly see how useful this theory is. A fundamental reason for this is the tight connection between specialness, the behaviour of subgroups in the profinite topology, and hierarchical decompositions of groups [133].

Much of the impact of the theory of specialness in the context of profinite rigidity flows through Agol's theorem and its consequences [2]. Agol proved that if a hyperbolic group acts properly and cocompactly by isometries on any CAT(0) cube complex, then it has a subgroup of finite index that is special. Agol used this theorem to prove Thurston's virtual fibering conjecture for hyperbolic 3-manifolds. This fibering theorem, as well as various other consequences of the work of Agol and Wise, are crucial ingredients in the main results of [32, 37].

3. Hard to see if finite images exist

We want to know to what extent finitely presented groups can be understood by examining only their finite quotients. In this context, we are inevitably led to ask how hard it is to discern what the finite quotients are. There are naive processes that one can run to list all finite quotients of a finitely presented group, but if such a process has not found a non-trivial finite quotient after a certain time, how is one to know whether it is worth continuing the search? One cannot know.

Theorem 3.1 (Bridson, Wilton [42]). *There does not exist an algorithm that, given an arbitrary finite presentation, can determine whether or not the group presented has a non-trivial finite quotient.*

A non-trivial finitely generated group of matrices (i.e., a linear group) always has non-trivial finite quotients, and finite groups are linear over any field, so Theorem 3.1 implies that one also cannot decide whether a finitely presented group has a non-trivial finite-dimensional, linear representation (over any field or over a fixed field).

3.1. Sketch of proof

The proof relies on:

- Logic: Slobodskoi's proof [113] of the undecidability of the universal theory of finite groups.
- Technology related to the geometry of subgroup separability in low dimensions, particularly in the context of graphs of free and finite groups and their covering spaces, as well as CAT(0) cube complexes (ideas of Stallings [116] and Wise [129, 130]).

The *universal theory of finite groups* is the set of all one-quantifier first-order sentences that are true in all finite groups. To see why it is relevant here, consider the following sentence Ψ (remembering that $\vee$ means “or”):

$$\begin{aligned} \forall a, b, c, d : (ba^2b^{-1} \neq a^3) \vee (dc^2d^{-1} \neq c^3) \vee ([a, b] \neq d) \\ \vee ([c, d] \neq b) \vee (a = b = c = d = 1). \end{aligned}$$

The reader can convince themselves that this sentence is true in a specific group Q if and only if there is no non-trivial homomorphism $B \rightarrow Q$, where B is the group

$$B = \langle a, b, c, d \mid ba^2b^{-1}a^{-3}, dc^2d^{-1}c^{-3}, [a, b]d^{-1}, [c, d]b^{-1} \rangle. \quad (3.1)$$

The reader will then see how to write a sentence Ψ_{AR} based on any finite group-presentation $\langle A \mid R \rangle$ so that Ψ_{AR} is true in a group Q if and only if there is no non-trivial homomorphism $G \rightarrow Q$, where $G = \langle A \mid R \rangle$.

In our example, the group B (which is infinite) has no finite quotients [29], so the sentence Ψ is true in every finite group, that is, it belongs to the universal theory of finite groups. In proving that this theory is undecidable, Slobodskoi [113] constructed a finitely presented group G in which there is no algorithm to determine which elements of the group survive in any finite quotient. One can phrase this as the unsolvability of a profinite version of the word problem in G . With such a group G in hand, one might hope to prove Theorem 3.1 by using amalgamated free products and HNN extensions in the manner alluded to in Section 2.9, building a sequence of finitely presented groups G_w , indexed by words in the generators of G , so that G_w has a non-trivial finite quotient if and only if the image of w is non-trivial in some finite quotient of G : in a standard notation, $w = 1$ in $\hat{G} \Leftrightarrow \hat{G}_w = 1$.

Ultimately, this outline of strategy does succeed (see [42, Theorem C]), but the details of the implication $\Rightarrow$ are much harder than in the classical situation (see Section 2.9). This is where the technology of Stallings and Wise is used. For reasons that quickly become clear when one tries to pursue the natural strategy, a key point is that one needs to be able to control the orders of the generators of G (and related groups) in finite quotients: this requires G to be crafted carefully, and it relies on refinements of Wise's work on *omnipotence* [129], which began with his proof that, given elements a, b of a non-abelian free group that are independent in homology, one can find a finite quotient in which the orders of a and b are in any given ratio. Beyond this first stage of control, one needs further technology concerning malnormal subgroups of virtually free groups, fibre products, and theorems concerning malnormal amalgams and residual finiteness.

3.2. Refinement: Finite simple images

It follows easily from Theorem 3.1 that there is no algorithm that determines the finite simple images of a finitely presented group. Famously, every non-abelian finite simple group is either an alternating group, a group from one of 16 families of Lie type, for example, $\mathrm{PSL}_d(q)$, or one of the 26 sporadic groups. Thus the question arises: for which collections of finite simple groups is there an algorithm that determines the members of the collection that are images of a given finitely presented group, and for which families do such algorithms not exist? This question is essentially settled in [28] where it is proved that if a collection of finite simple groups contains infinitely many alternating groups, or contains classical groups of unbounded ranks, then there is no algorithm, whereas for families of simple groups of Lie type with bounded rank, the desired algorithms do exist. The following special case of this result provides a concrete illustration of this dichotomy.

Theorem 3.2 (Bridson, Evans, Liebeck, Segal [28]). *Both of the following statements hold:*

- (1) *There is an algorithm that, given a finitely presented group Γ , will determine whether or not Γ has infinitely many quotients $\mathrm{PSL}_n(q)$ with n fixed and q varying, and will list these quotients if there are only finitely many.*
- (2) *There does not exist an algorithm that, given a finitely presented group Γ , will determine whether or not Γ has infinitely many quotients $\mathrm{PSL}_n(q)$ with q fixed and n varying, nor does there exist an algorithm that can determine whether Γ has any quotient of this form.*

The proof of (1) is rooted in representation theory and the model theory of finite fields; the algorithm is not practical.

4. Capturing groups via finite actions: Residual finiteness

We return to our main theme: *to what extent is a finitely generated (or finitely presented) group determined by its finite quotients (equivalently, its profinite completion)?*

The information contained in the kernel of the natural map $\Gamma \rightarrow \hat{\Gamma}$ is lost when we restrict our attention to $\hat{\Gamma}$, so it is natural to focus on groups where this kernel is trivial, that is, the residually finite groups.

Definition 4.1. A group Γ is *residually finite* if for every $\gamma \in \Gamma \setminus \{1\}$ there is a homomorphism $\pi : \Gamma \rightarrow Q$ to a finite group with $\pi(\gamma) \neq 1$.

4.1. Which groups are residually finite?

There are many results in the literature establishing that groups are residually finite; the following selection is slanted towards the theme of these notes. I particularly want to emphasise the remarkable nature of items (4) and (5): they provide combinatorial criteria for verifying that a group is linear, and they have vastly extended the available reservoir of residually finite groups. The wealth of examples that these criteria provide, made available largely through the work of Dani Wise [133], underpins many advances in the field.

- (1) If a finitely generated group Γ admits a faithful linear representation over any field $\Gamma \hookrightarrow \mathrm{GL}(n, K)$, then Γ is residually finite [81].
- (2) If Γ is the fundamental group of a compact 3-manifold, then Γ is residually finite [60].
- (3) Mapping class groups of compact surfaces and (outer) automorphism groups of finitely generated free groups are residually finite [53].
- (4) Hyperbolic groups that act properly and cocompactly on $\mathrm{CAT}(0)$ complexes are residually finite.
- (5) $C'(1/6)$ small cancellation groups are residually finite.
- (6) Random finitely presented groups, in a suitable model, are residually finite.

Item (4) is a consequence of Agol's theorem, as discussed in Section 2.12. The $C'(1/6)$ small cancellation condition in (5) bounds the size of overlaps between subwords of relators in group presentations. Groups that have finite presentations satisfying this condition are hyperbolic, and Wise [132] proved that they act on $\mathrm{CAT}(0)$ cube complexes, so (4) applies. This way of phrasing things, though, obscures the fact that Wise's earlier work established residual finiteness in many cases, opening up the great reservoir of examples that (4) and (5) provide [130, 131, 133]. A popular model for a random group is Gromov's density model [86], whereby one fixes the number of generators for a presentation and chooses the relators uniformly at random

from cyclically reduced words of a given length, with the number of words determined by a density parameter; the focus is on what happens as the length gets large. At small densities, with overwhelming probability these groups satisfy the $C'(1/6)$ small cancellation condition [90].

The following challenge is one of the guiding problems of geometric group theory.

Question 4.2. Are all Gromov hyperbolic groups residually finite?

Residually finite groups harbour less pathology than arbitrary groups. For example, it is easy to see that a finitely presented, residually finite group Γ has a solvable word problem, although there do exist finitely generated, residually finite groups that have unsolvable word problem [82]. Note the recurrence of an important theme here: finitely generated groups can be much wilder than finitely presented ones.

We mention one other property of residually finite groups because it plays a surprisingly useful role in the results we shall look at later.

Lemma 4.3. *If a finitely generated group Γ is residually finite, then it has the Hopf property, that is, every epimorphism $\phi : \Gamma \rightarrow \Gamma$ is an isomorphism.*

Proof. If there were a non-trivial element $\gamma \in \Gamma$ in the kernel of an epimorphism $\phi : \Gamma \rightarrow \Gamma$, and if $\pi : \Gamma \rightarrow Q$ were a homomorphism to a finite group with $\pi(\gamma) \neq 1$, then the infinitely many maps $\pi \circ \phi^n : \Gamma \rightarrow Q$ would all be distinct, which is impossible because any homomorphism $\Gamma \rightarrow Q$ is determined by the image of a finite generating set. ■

5. Profinite properties of groups

We want to understand the extent to which properties of residually finite groups are encoded in their profinite completions.

Definition 5.1. A property $\mathfrak{P}$ of abstract groups is a *profinite property* (or a profinite invariant) if, for finitely generated, residually finite groups Γ_1 and Γ_2 , whenever Γ_1 has $\mathfrak{P}$ and $\hat{\Gamma}_1 \cong \hat{\Gamma}_2$, then Γ_2 has $\mathfrak{P}$.

5.1. Some basic facts about profinite completion

In the introduction, we defined the profinite completion $\hat{\Gamma}$ of an abstract group Γ to be the inverse limit of the system of all finite quotients Γ/N of Γ . Equivalently, one

can define $\hat{\Gamma}$ to be the closure of the image of Γ in the direct product of the finite quotients Γ/N (which is compact in the product topology)

$$\hat{\Gamma} = \{(x_N) \mid x_N M = x_M \forall M > N\} \subset \prod_N \Gamma/N.$$

Profinite completion $\Gamma \mapsto \hat{\Gamma}$ is a functor from the category of abstract groups to the category of profinite groups [103]: a homomorphism of abstract groups $u : \Gamma_1 \rightarrow \Gamma_2$ extends uniquely to a continuous homomorphism of profinite groups $\hat{u} : \hat{\Gamma}_1 \rightarrow \hat{\Gamma}_2$.

By construction, $\hat{\Gamma}$ is residually finite. And Γ is residually finite if and only if the natural map $\iota : \Gamma \rightarrow \hat{\Gamma}$ is injective. Henceforth, we shall concentrate on residually finite groups, in which setting we identify Γ with its image under ι . If Γ is finitely generated, there is a 1-1 correspondence between the finite index subgroups of Γ and the finite index subgroups³ of $\hat{\Gamma}$: this associates to $H < \Gamma$ its (topological) closure $\overline{H} < \hat{\Gamma}$, and it associates to $\mathcal{H} < \hat{\Gamma}$ the intersection $\mathcal{H} \cap \Gamma$.

Remark 5.2. An important point to keep in mind is that, for finitely generated groups Γ_1 and Γ_2 , the existence of an isomorphism $\hat{\Gamma}_1 \cong \hat{\Gamma}_2$ does not imply the existence of any non-trivial homomorphism from Γ_1 to Γ_2 . On the other hand, the correspondence between finite index subgroups of Γ_i and $\hat{\Gamma}_i$ does induce, via $\hat{\Gamma}_1 \cong \hat{\Gamma}_2$, an isomorphism between the lattices of finite index subgroups in Γ_1 and Γ_2 ; this isomorphism preserves the index of subgroups, normality, and conjugacy. It follows that any property that involves counting subgroups of a given index will be a profinite invariant, for example, subgroup growth [78]. Moreover, if $H_1 < \Gamma_1$ corresponds to $H_2 < \Gamma_2$, then $\hat{H}_1 \cong \hat{H}_2$ (an observation that can be extremely useful).

5.2. Profinite properties: Examples and counterexamples

The following list contains some properties that are easily seen to be profinite invariants, but it also points to many properties that are *not* profinite invariants. Rather than being disappointed by this, I hope that the reader will reflect that the difficulty of finding non-trivial profinite invariants makes the existence of profinite rigidity theorems all the more remarkable.

- (1) Being *abelian* is a profinite property. *Proof:* If Γ_1 is abelian, then so are all of its finite quotients. If Γ_2 is not abelian, then for some $a, b \in \Gamma_2$, the commutator $c = [a, b]$ is non-trivial. If Γ_2 is residually finite, then $p(c) = [p(a), p(b)] \neq 1$ in some finite quotient $p : \Gamma_2 \rightarrow Q$. As this quotient is not abelian, it is not a quotient of Γ_1 , and therefore $\hat{\Gamma}_1 \not\cong \hat{\Gamma}_2$.

³The simplicity of this statement relies on the Nikolov–Segal theorem [87].

- (2) By considering precisely which abelian p -groups Γ maps onto (varying the prime p), one can show that the isomorphism type of the *abelianisation* of a finitely generated group is a profinite invariant.
- (3) A slight variation on the argument in (1) shows that satisfying any *group law* is a profinite property, for example, being nilpotent of class c , or solvable of derived length d . It is considerably harder to prove that being *polycyclic* is a profinite invariant, but Sabbagh and Wilson proved that this is true [105].
- (4) The ℓ_2 -Betti numbers of groups are defined as analytic invariants, but Lück's approximation theorem [79] allows one to compute the *first ℓ_2 -Betti number* of a finitely presented, residually finite group as a limit of ordinary Betti numbers: $b_1^{(2)}(\Gamma) := \lim b_1(H_n)/[\Gamma : H_n]$, where (H_n) is any sequence of finite index normal subgroups intersecting in the identity. It follows from the facts about subgroup lattices in Remark 5.2 that the first ℓ_2 -Betti number is a profinite invariant among finitely presented, residually finite groups [27, Corollary 3.3]. But Kammeyer and Sauer show that the higher-dimensional ℓ_2 -Betti numbers are not profinite invariants [68].
- (5) Lackenby [72] proved that having a subgroup of finite index that maps onto a non-abelian free group (*largeness*) is a profinite invariant among finitely presented groups.
- (6) Aka [3] found lattices in semisimple Lie groups that have the same profinite completion even though the real rank of the Lie groups is different. He also proved that Kazhdan's *property (T)* is not a profinite invariant [4]. Work of Kassabov and Nikolov [69] shows that property (τ) is not a profinite invariant.
- (7) Property (T) can be phrased as a fixed-point property for isometric actions on Hilbert spaces. *Fixed-point properties* for actions on other CAT(0) spaces have also been studied. Cheetham-West, Lubotzky, Reid, and Spitler [45] showed that Serre's property FA (the fixed-point property for actions on trees) is not a profinite invariant, and in [26] this was extended to fixed-point properties for actions on higher-dimensional CAT(0) spaces.
- (8) Kionke and Schesler [71] proved that *amenability* is not a profinite invariant.
- (9) Being finitely presented is not a profinite invariant of finitely generated groups (see Section 7), and likewise the higher *finiteness properties* of groups [77].

- (10) Being *hyperbolic* (in the sense of Gromov) is not a profinite invariant among finitely generated groups, but it is unknown whether it is preserved under profinite equivalence of finitely presented groups.
- (11) It is unknown whether having a faithful finite-dimensional *linear* representation is a profinite property.
- (12) The theorem stated below can be used to see that the property of having a *subgroup* of a given sort is almost never a profinite invariant: for example, having a non-trivial finite subgroup, or a subgroup isomorphic to $\mathbb{Z}^3$, or a 2-generator subgroup that is neither abelian nor free. This precludes the profinite invariance of various properties that are inherited by subgroups, for example, being orderable, or locally indicable, or having cohomological dimension less than $d > 2$.

Theorem 5.3 ([25]). *Given a finitely generated, recursively presented group Γ that is residually finite, one can construct a finitely generated, residually finite free-by-free group $M_\Gamma = F_\infty \rtimes F_4$ and an embedding $M_\Gamma \hookrightarrow (F_4 * \Gamma) \times F_4$ that induces an isomorphism of profinite completions.*

This theorem is proved by combining ideas described in Section 7 with the universal embedding theorem in [23], which elaborates on ideas of Higman [63] and Baumslag, Dyer, and Heller [10].

6. Profinite rigidity

When studying profinite rigidity, one is interested in theorems whose conclusion is

$$\hat{\Lambda} \cong \hat{\Gamma} \implies \Lambda \cong \Gamma.$$

6.1. Three forms of rigidity

The following definition of what it means for a group to be profinitely rigid has become standard, although it is still common to add “in the absolute sense” for clarity (as we shall do several times in this article).

Definition 6.1 (*Absolute profinite rigidity*). A finitely generated, residually finite group Γ is *profinutely rigid* if, whenever Λ is finitely generated and residually finite, $\hat{\Lambda} \cong \hat{\Gamma} \implies \Lambda \cong \Gamma$.

In the previous section, we saw that when Γ satisfies a group law, one only needs to consider groups Λ satisfying the same group law. In such cases, Definition 6.1 reduces to an instance of the following problem.

Definition 6.2 (*Relative profinite rigidity in a class $\mathcal{G}$*). If a finitely generated, residually finite group Γ belongs to a class of groups $\mathcal{G}$, then one says that Γ is *profinutely rigid among groups in $\mathcal{G}$* if whenever $\Lambda \in \mathcal{G}$ is finitely generated and residually finite, $\widehat{\Lambda} \cong \widehat{\Gamma} \Rightarrow \Lambda \cong \Gamma$.

Remark 6.3 (The absolute case is much harder). The literature contains many more theorems about relative profinite rigidity than it does about absolute profinite rigidity, for the obvious reason that even if one starts with a group Γ that has rich structure, when one finds a group Λ “lying in the gutter” and knows only that it is finitely generated, residually finite and has the same finite quotients as Γ , how is one to know that Λ shares any of the fine features of Γ ? For example, if Γ is a group of matrices, why should Λ have any interesting linear representations? If Γ is a 3-manifold group, how would one begin to argue that Λ is a 3-manifold group?

In contrast, if one is trying to argue that a group is profinitely rigid in a class $\mathcal{G}$, then one can study special features of the groups in $\mathcal{G}$ and ask if these are visible in $\widehat{\Lambda}$ when $\Lambda \in \mathcal{G}$. For example, we shall see in Section 8 that many features of compact 3-manifolds M are invariants of $\widehat{\pi_1 M}$.

The third form of profinite rigidity that is widely studied is *Grothendieck* rigidity, where one is interested in whether it is possible to obtain isomorphisms $\widehat{u} : \widehat{\Lambda} \rightarrow \widehat{\Gamma}$ induced by homomorphisms of discrete groups $u : \Lambda \rightarrow \Gamma$. This is the subject of Section 7.

6.2. Early work, sobering examples, and a guiding question

- (1) We have seen that finitely generated abelian groups are profinitely rigid. When one moves from abelian to *nilpotent* groups, one finds that some are profinitely rigid (e.g., free nilpotent groups) while others are not [101]. But in all cases, the ambiguity is limited: only finitely many finitely generated nilpotent groups can have the same profinite completion [91]. Following [56], an appealing way to express this finite ambiguity is to use the language of *profinite genus* (within a class of groups). Thus Pickel’s theorem is that the profinite genus of any finitely generated nilpotent group is finite. Grunewald, Pickel, and Segal [58] proved that *virtually polycyclic* groups also have *finite genus*.
- (2) In 1974, Baumslag [8] discovered the following striking example which underscores how quickly one can lose profinite rigidity when one moves away from the abelian setting. Let $G_1 = (\mathbb{Z}/25) \rtimes_{\alpha} \mathbb{Z}$ and $G_2 = (\mathbb{Z}/25) \rtimes_{\beta} \mathbb{Z}$, where α is the automorphism $x \mapsto x^6$ and $\beta = \alpha^2$ is the automorphism $x \mapsto x^{11}$. The reader can verify directly that $G_1 \not\cong G_2$, but G_1 and G_2 have the same finite quotients.

- (3) Baumslag drew inspiration from an important example of Serre [109]. In 1964, Serre constructed a smooth projective variety V defined over a number field K such that for the varieties V_{ϕ_1} and V_{ϕ_2} obtained by extension of scalars along different embeddings $\phi_1, \phi_2 : K \hookrightarrow \mathbb{C}$, the fundamental groups $\Gamma_i = \pi_1 V_{\phi_i} \cong \mathbb{Z}^{p-1} \rtimes (\mathbb{Z}/p)$ are not isomorphic, but $\hat{\Gamma}_1 \cong \hat{\Gamma}_2$ because each $\hat{\Gamma}_i$ is the algebraic (étale) fundamental group of the underlying scheme. Pairs of *Galois conjugate* varieties exhibiting this behaviour can also be found among complex surfaces [7]. Galois conjugation will play an important role in Section 9.2.
- (4) The examples of Stebe [117] are based on a different phenomenon. He constructed pairs of matrices in $\mathrm{SL}(2, \mathbb{Z})$ that are not conjugate in $\mathrm{GL}(2, \mathbb{Z})$ but are conjugate when one projects them to $\mathrm{GL}(2, \mathbb{Z}/d)$ for every integer $d > 2$. One such pair is

$$\phi(1) = \begin{pmatrix} 188 & 275 \\ 121 & 177 \end{pmatrix} \quad \text{and} \quad \phi(2) = \begin{pmatrix} 188 & 11 \\ 3025 & 177 \end{pmatrix}.$$

Writing $\Gamma_i = \mathbb{Z}^2 \rtimes_{\phi(i)} \mathbb{Z}$, we have $\Gamma_1 \not\cong \Gamma_2$ but $\hat{\Gamma}_1 \cong \hat{\Gamma}_2$. Every element of $\mathrm{SL}(2, \mathbb{Z})$ can be realised as an orientation-preserving homeomorphism of the torus, so Γ_i is the fundamental group of a torus bundle over the circle. In fact, it is a lattice in the Lie group Sol , which is one of the eight geometries in dimension 3 (see [108]). Thus we have examples of closed, geometric 3-manifolds that are not profinitely rigid. These types of torus-bundle examples were explored further by Funar [50]. The corresponding punctured-torus bundles are more rigid [38].

- (5) Hempel [61] produced further pairs of geometric 3-manifolds with $\pi_1 M_1 \not\cong \pi_1 M_2$ but $\widehat{\pi_1 M_1} \cong \widehat{\pi_1 M_2}$. These examples, which have the local geometry of $\mathbb{H}^2 \times \mathbb{R}$, are higher-genus surface bundles over the circle, where the holonomy has finite order. They are very close in spirit to Baumslag's finite-by-cyclic examples in (2).
- (6) Following the work of Platonov and Tavgen' [93] discussed in Section 7, one knows that the direct product of two or more non-abelian free groups is not profinitely rigid.

A noticeable absentee from the list of groups considered above is the free group F_n . The following question [88, Question 15] is a guiding challenge in the field.

Question 6.4 (Remeslennikov). Are finitely generated free groups profinitely rigid?

6.3. Relative profinite rigidity and hyperbolic geometry

There is now a large literature on relative profinite rigidity with many positive and negative results, some of which are referenced in Sections 5.2, 6.2, and 8. It is a very active field. I highlight the following result because it marked the beginning of a flowering of interest in the profinite rigidity of low-dimensional orbifold groups, and also because it underscores the point I was making in Remark 6.3: this theorem shows that free groups are profinitely rigid within lattices, but Question 6.4 is a vastly more difficult question because it is hard to extract any geometry from the hypothesis $\hat{\Lambda} \cong \hat{F}_n$.

Theorem 6.5 (Bridson, Conder, Reid [27]). *If Γ is a lattice in $\mathrm{PSL}(2, \mathbb{R})$ and Λ is a lattice in any connected Lie group, then $\hat{\Gamma} \cong \hat{\Lambda} \Rightarrow \Gamma \cong \Lambda$.*

The profinite invariance of the first ℓ_2 -Betti number plays an important role in the proof of this theorem. In particular, the work of Lott and Lück [76] allows one to reduce fairly quickly to the case where Λ is a lattice in $\mathrm{PSL}(2, \mathbb{R})$, where one understands the lattices very explicitly.

We shall discuss the profinite rigidity of lattices in $\mathrm{PSL}(2, \mathbb{C})$ in a moment. Stover [118, 119] showed that there are lattices in other rank-1 Lie groups (isometries of complex hyperbolic space) that are not profinitely rigid.

7. Grothendieck pairs

The study of Grothendieck pairs was initiated by Alexander Grothendieck in 1970 [54]. He was investigating the extent to which the topological fundamental group of a smooth complex projective variety is determined by the étale fundamental group of the underlying scheme, and he was interested in the extent to which a finitely generated, residually finite group can be recovered from a complete knowledge of its finite-dimensional representation theory.

Definition 7.1 (Grothendieck pairs). A monomorphism of residually finite groups $u : P \hookrightarrow \Gamma$ is called a *Grothendieck pair* if $\hat{u} : \hat{P} \rightarrow \hat{\Gamma}$ is an isomorphism but u is not. One says that a finitely generated, residually finite group Γ is *Grothendieck rigid*⁴ if there are no Grothendieck pairs $P \hookrightarrow \Gamma$ with $P \neq \Gamma$ finitely generated.

⁴There is also interest in fixing Γ and considering Grothendieck pairs $\Gamma \hookrightarrow \Lambda$. When pursuing this, it is natural to distinguish between the two settings with the terminology left/right (or up/down) Grothendieck rigidity [46, 67].

Grothendieck discovered a remarkably close connection between the representation theory of a finitely generated group and its profinite completion: if $A \neq 0$ is a commutative ring and $u : \Gamma_1 \rightarrow \Gamma_2$ is a homomorphism of finitely generated groups, then $\hat{u} : \hat{\Gamma}_1 \rightarrow \hat{\Gamma}_2$ is an isomorphism if and only if the restriction functor $u_A^* : \text{Rep}_A(\Gamma_2) \rightarrow \text{Rep}_A(\Gamma_1)$ is an equivalence of categories, where $\text{Rep}_A(\Gamma)$ is the category of finitely presented A -modules with a Γ -action. This led him to pose his famous question about the existence of finitely presented Grothendieck pairs.

Question 7.2 (Grothendieck 1970 [54]). If Γ_1 and Γ_2 are finitely presented and residually finite, must $u : \Gamma_1 \rightarrow \Gamma_2$ be an isomorphism if $\hat{u} : \hat{\Gamma}_1 \rightarrow \hat{\Gamma}_2$ is an isomorphism?

Platonov and Tavgen' [93] answered the corresponding question for finitely generated groups, and various other constructions followed [6, 85, 96]. Grothendieck's question was eventually settled by myself and Fritz Grunewald [29]. The following is an enhancement of what is stated in [29] (see Remark 7.9).

Theorem 7.3 (Bridson, Grunewald [29]). *There exist (Gromov) hyperbolic groups $\Gamma < \text{SL}(n, \mathbb{Z})$ and finitely presented subgroups $P \hookrightarrow G := \Gamma \times \Gamma$ of infinite index, with $P \not\cong G$, such that the inclusion $P \hookrightarrow G$ induces an isomorphism $\hat{P} \rightarrow \hat{G}$.*

7.1. The ideas in the proof

A natural place to begin a search for Grothendieck pairs is with short exact sequences

$$1 \rightarrow N \rightarrow \Gamma \rightarrow Q \rightarrow 1 \quad (7.1)$$

with N and Γ finitely generated and residually finite, and Q an infinite group with no non-trivial finite quotients, for example, G_4 from (2.1) or B from (3.1). If profinite completion were an exact functor, we could simply put hats on the groups in (7.1) to conclude that $\hat{N} \rightarrow \hat{\Gamma}$ was an isomorphism. But it is not an exact functor, and while $\hat{N} \rightarrow \hat{\Gamma}$ will be surjective if $\hat{Q} = 1$, it need not be injective. To understand why, the reader will need some familiarity with the (co)homology of groups [43]. Consider the 5-term exact sequence in homology associated to (7.1)

$$H_2(Q, \mathbb{Z}) \rightarrow H_0(Q, H_1(N, \mathbb{Z})) \rightarrow H_1(\Gamma, \mathbb{Z}) \rightarrow H_1(Q, \mathbb{Z}) \rightarrow 0. \quad (7.2)$$

As N is finitely generated, the automorphism group of its abelianisation $H_1(N, \mathbb{Z})$ is residually finite, so the action $Q \rightarrow \text{Aut}(H_1(N, \mathbb{Z}))$ must be trivial, since $\hat{Q} = 1$, and the group of coinvariants $H_0(Q, H_1(N, \mathbb{Z}))$ is simply $H_1(N, \mathbb{Z})$. Moreover, $\hat{Q} = 1$ implies $H_1(Q, \mathbb{Z}) = 0$, so (7.2) simplifies to an exact sequence

$$H_2(Q, \mathbb{Z}) \rightarrow H_1(N, \mathbb{Z}) \rightarrow H_1(\Gamma, \mathbb{Z}) \rightarrow 0,$$

from which we deduce that if $H_2(Q, \mathbb{Z}) \rightarrow H_1(N, \mathbb{Z})$ is not the zero map, then N will have finite abelian quotients that Γ does not have, whence $\widehat{N} \not\cong \widehat{\Gamma}$. To avoid this difficulty, we assume $H_2(Q, \mathbb{Z}) = 0$, a condition satisfied by our examples G_4 and B .

Lemma 7.4. *With $1 \rightarrow N \rightarrow \Gamma \rightarrow Q \rightarrow 1$ as above, if $H_2(Q, \mathbb{Z}) = 0$, then $N \hookrightarrow \Gamma$ is a Grothendieck pair.*

With this lemma in hand, we will have finitely generated Grothendieck pairs if we can construct short exact sequences of the desired type. To do this, we appeal to the Rips construction [104] which, given a finite presentation of a group Q , will produce a short exact sequence $1 \rightarrow N \rightarrow \Gamma \rightarrow Q \rightarrow 1$, as desired, with N finitely generated and Γ a 2-dimensional hyperbolic group that satisfies a small cancellation condition; we saw in Section 4.1 that Γ is residually finite. To obtain further examples, one can verify that in this situation, if $H < \Gamma$ is any subgroup that contains N and if $\widehat{H/N} = 1$, then $N \hookrightarrow H$ and $H \hookrightarrow \Gamma$ are both Grothendieck pairs. By taking Q to be one of the universal groups constructed in [23, Theorem A], we deduce the following result.

Theorem 7.5. *There exist residually finite, (Gromov) hyperbolic groups Γ of dimension 2 with uncountably many pairwise non-isomorphic subgroups $\iota_H : H \hookrightarrow \Gamma$ such that $\widehat{\iota_H} : \widehat{H} \rightarrow \widehat{\Gamma}$ is an isomorphism. Moreover, one can arrange for infinitely many of these subgroups to be finitely generated.*

This is not how Platonov and Tavgen' [93] produced the first examples of finitely generated Grothendieck pairs. (A version of the Rips construction that guaranteed residual finiteness was not available at that time.) Instead, they considered *fibre products*, modifying the proof of Lemma 7.4 to establish the following criterion, which they applied to $F_4 \rightarrow G_4$ to construct a Grothendieck pair $P \hookrightarrow F_4 \times F_4$ with P finitely generated.

Proposition 7.6 (Platonov, Tavgen' [93]). *Let $f : \Gamma \rightarrow Q$ be an epimorphism of groups, with Γ finitely generated and Q finitely presented. Suppose that $\widehat{Q} = 1$ and $H_2(Q, \mathbb{Z}) = 0$. Then the fibre product $P = \{(g, h) \mid f(g) = f(h)\} < \Gamma \times \Gamma$ is finitely generated and $P \hookrightarrow \Gamma \times \Gamma$ induces an isomorphism of the profinite completions $\widehat{P} \xrightarrow{\cong} \widehat{\Gamma \times \Gamma}$.*

It is easy to see that P is finitely generated: if Γ is generated by X and $Q = \langle X \mid R \rangle$, then P is generated by $\{(x, x) \mid x \in X\} \cup \{(r, 1) \mid r \in R\}$. This fact is often called the 0-1-2 lemma. But if Γ is free and Q is infinite, the fibre product P will never have a finite presentation. Indeed, while the Platonov–Tavgen' construction produces finitely generated Grothendieck pairs $P \hookrightarrow F_4 \times F_4$ (and one can replace F_4

by F_2 using [135]), the following theorem implies that there are no such finitely presented Grothendieck pairs, even if one allows the product of more free groups, or other residually free groups, such as surface groups.

Theorem 7.7 (Bridson, Wilton [39]). *If Π is finitely presented and residually free, then every finitely presented subgroup $P < \Pi$ is closed in the profinite topology; in particular, $P \hookrightarrow \Pi$ is not a Grothendieck pair.*

Remark 7.8. Once more, I want to emphasise the contrast in behaviour between finitely presented subgroups and those that are merely finitely generated.

Theorem 7.3 is proved by combining the ideas described above and appealing to results on the finite presentability of fibre products. The 1-2-3 theorem of Baumslag, Bridson, Miller, and Short [9] states that if Q has a classifying space with a finite 3-skeleton (as our examples G_4 and B do), while Γ is finitely presented and $N = \ker(\Gamma \rightarrow Q)$ is finitely generated, then the fibre product $P < \Gamma \times \Gamma$ of $\Gamma \rightarrow Q$ is finitely presented. To prove Theorem 7.3, one can take $Q = B$, use the Rips construction to generate $1 \rightarrow N \rightarrow \Gamma \rightarrow B \rightarrow 1$, and then appeal to Wise’s work [132] to see that Γ is a cubulated hyperbolic group, which by Agol’s theorem [2] can be embedded in $\mathrm{SL}(d, \mathbb{Z})$ for some integer d . The 1-2-3 theorem assures us that the associated fibre product P is finitely presented, and Proposition 7.6 assures us that $P \hookrightarrow \Gamma \times \Gamma$ is a Grothendieck pair.

Remark 7.9. When [29] was written, neither Agol’s theorem nor the virtually special Rips construction in [59] was available, so the penultimate sentence of the proof sketched above was replaced by an appeal to Wise’s residually finite version of the Rips construction [131]. This accounts for the difference between Theorems B and 7.3.

7.2. Elaborations, refinements, and Grothendieck rigidity

In the last twenty years, many papers have been written constructing Grothendieck pairs $P \hookrightarrow \Gamma$ where P and Γ have different properties. Indeed, this has been perhaps the main mechanism by which various properties have been shown not to be profinite. Typically, this is achieved by encoding additional properties into a group Q that has a finite classifying space with $\hat{Q} = 1$ and $H_2(Q, \mathbb{Z}) = 0$. One then follows the above template of proof, perhaps adjusting Γ in some way, by refining the Rips construction, or by passing to a central extension [23, 41], or replacing the hyperbolic group Γ by something entirely different [25]. Often, a more refined version of the 1-2-3 theorem is needed, as in [21] where it is shown that there exist finitely presented, residually finite groups Γ that contain infinitely many pairwise non-isomorphic, finitely presented subgroups P_n such that $P_n \hookrightarrow \Gamma$ is a Grothendieck pair.

In the opposite direction, there are notable results proving that certain groups are Grothendieck rigid. Grothendieck himself [54] proved this for groups of the form $G(A)$ where A is a commutative ring and G is an affine group scheme of finite type over A ; see Jaikin-Zapirain and Lubotzky [67] for a concise account of this and further results. In the setting of 3-manifolds, Long and Reid [75] established Grothendieck rigidity for the fundamental groups of all closed geometric 3-manifolds and all finite-volume hyperbolic 3-manifolds. Following the work of Agol and Wise, Sun was able to extend this to cover the fundamental group Γ of every compact 3-manifold [120]. In the hyperbolic case, [36, Theorem A] proves something stronger, ruling out abstract isomorphisms $\widehat{H} \cong \widehat{\Gamma}$ for subgroups $H < \Gamma$, not just those induced by inclusions $H \hookrightarrow \Gamma$. This stronger statement is false in the non-hyperbolic setting [50, 61].

8. Profinite rigidity and 3-manifolds

Thurston's geometrisation conjecture, proved by Perelman, tells us that every closed orientable 3-manifold can be cut into pieces by a system of embedded spheres and tori so that the interior of each piece is homeomorphic to a finite-volume quotient of one of the eight 3-dimensional geometries [108]. Hyperbolic geometry $\mathbb{H}^3$ is the richest and most ubiquitous of these geometries, and we encountered Sol in Section 6.2 (4). The pieces modelled on the remaining six geometries are Seifert fibre spaces, which we shall discuss in a moment.

Many beautiful results have been proved in recent years concerning the great extent to which properties of 3-manifolds are captured by the profinite completions of their fundamental groups. *As a result, we now know that specifying $\widehat{\pi_1 M}$ determines a compact 3-manifold M up to finite ambiguity, and we know that many features of the 3-manifold are captured by $\widehat{\pi_1 M}$.*

A 3-manifold is *irreducible* if every 2-sphere in it bounds a ball. Every closed, orientable 3-manifold can be written as a connected sum of irreducible manifolds and copies of $S^2 \times S^1$. In the spirit of the geometrisation conjecture, the first thing that one wants to know is whether the prime decomposition of M is visible in $\widehat{\pi_1 M}$, and Wilton and Zalesskii [128] proved that it is: if M and N are closed, orientable 3-manifolds with prime decompositions $M = M_1 \# \cdots \# M_m \# r(S^1 \times S^2)$ and $N = N_1 \# \cdots \# N_n \# s(S^1 \times S^2)$, and $\widehat{\pi_1 M} \cong \widehat{\pi_1 N}$, then $m = n$, $r = s$, and after permuting the indices, $\widehat{\pi_1 M_i} \cong \widehat{\pi_1 N_i}$. Wilkes [124] showed that this remains valid if M has an incompressible boundary. These results allow us to concentrate on irreducible manifolds, where attention falls on the process of cutting the manifold along tori. In this context, Wilton and Zalesskii [128] showed that $\widehat{\pi_1 M}$ encodes the canonical JSJ decomposition of the manifold, including the profinite completions of the

pieces into which the tori cut the manifold; again, Wilkes shows that one can allow boundary [122, 123]. Wilton and Zalesskii also proved that the profinite completion distinguishes the hyperbolic pieces in the geometric decomposition from the Seifert fibred pieces [127].

We saw in Section 6.2 (4) that there are distinct quotients of Sol whose fundamental groups have the same profinite completion, but up to homeomorphism there can be only finitely many with a given completion [58]. The same is true of $\mathbb{H}^2 \times \mathbb{R}$ manifolds. Wilkes [121] extended this finite genus result to manifolds that consist entirely of Seifert pieces, giving a complete account of the ambiguities.

The first example of a hyperbolic 3-manifold that is distinguished from all other 3-manifolds by $\widehat{\pi_1 M}$ was the figure-8 knot complement (Boileau–Friedl [14] and Bridson–Reid [35]). Bridson, Reid, and Wilton later proved that every punctured-torus bundle over the circle is distinguished among 3-manifold groups by $\widehat{\pi_1 M}$. (Punctured-torus bundles account for all groups of the form $F_2 \rtimes \mathbb{Z}$. It remains unknown whether groups of the form $F_n \rtimes \mathbb{Z}$ are distinguished from each other by their profinite completions – cf. [34, 64].) These early results relied on recovering fibre-bundle structure from $\widehat{\pi_1 M}$. Subsequently, Jaikin-Zapirain [65] proved that $\widehat{\pi_1 M}$ always detects whether a 3-manifold fibres over the circle with compact fibre. Other notable results include the fact that 3-manifold groups are all good in the sense of Serre [99, 112, 125], that is, if Γ is a 3-manifold group, then for all $n > 0$ and all finite Γ -modules A , the natural map $H^n(\widehat{\Gamma}, A) \rightarrow H^n(\Gamma, A)$ from the continuous cohomology of $\widehat{\Gamma}$ to the group cohomology of Γ is an isomorphism. We refer to Reid’s ICM talk [100] for an overview of related results up to 2018 and [98, 102] for updates.

A breakthrough that deserves particular attention comes from the work of Liu [73]. He proved that only finitely many finite-volume hyperbolic 3-manifolds can have the same profinite completion. Building on much of the work described above, Xiaoyu Xu recently extended this finite genus theorem to all compact, orientable, 3-manifolds with toral or empty boundary [134].

The following discussion of Seifert fibre spaces is needed for Theorem D.

8.1. Seifert fibre spaces

A *Seifert fibre space* is a 3-manifold M that is foliated by circles. The circles of the foliation are divided into regular fibres and exceptional fibres: a circle is a regular fibre if it has a neighbourhood that admits a fibre-preserving homeomorphism to the solid torus $\mathbb{D}^2 \times S^1$ foliated by the circles $\{x\} \times S^1$; in a neighbourhood of an exceptional fibre, the foliation is modelled on the foliation of $\mathbb{D}^2 \times S^1$ that is obtained by cutting the product fibration open along a disc $\mathbb{D}^2 \times \{*\}$ and then regluing after twisting through an angle $2q\pi/p$, where $p, q \in \mathbb{Z}$ are coprime integers with $0 < q < p$.

By collapsing each circle fibre in M to a point, we obtain a 2-manifold with an orbifold structure given by marking each exceptional fibre with a cyclic group recording the order p of the twisting; this is called the base orbifold of the Seifert fibration.

We will be concerned only with Seifert fibred manifolds where $\pi_1 M$ is infinite, in which case there is a central extension

$$1 \rightarrow \mathbb{Z} \rightarrow \pi_1 M \rightarrow \Delta \rightarrow 1,$$

where the central $\mathbb{Z}$ is the fundamental group of a regular fibre and Δ is the fundamental group of the base orbifold. For example, if M has 3 exceptional fibres, with twisting data (p_1, q_1) , (p_2, q_2) , (p_3, q_3) and the underlying surface of the base orbifold is a sphere, then $\Delta = \Delta(p_1, p_2, p_3)$ is a triangle group and

$$\begin{aligned} \pi_1(M) = \langle c_1, c_2, c_3, z \mid z \text{ is central, } c_1^{p_1} z^{-q_1} = c_2^{p_2} z^{-q_2} = c_3^{p_3} z^{-q_3} = 1, \\ c_1 c_2 c_3 = z^d \rangle \end{aligned}$$

for some $d \in \mathbb{Z}$.

The examples of Hempel (Section 6.2 (5)) are surface bundles over the circle with finite holonomy. Wilkes [121] proved that all Seifert manifolds apart from these are profinitely rigid in the class of 3-manifold groups.

9. Absolute profinite rigidity

We seek finitely generated, residually finite groups Γ that are profinitely rigid in the absolute sense. We observed in Section 6.2 (3) that if Γ satisfies a group law, then this immediately reduces to a problem within the class of groups that satisfy this law, that is, it reduces to a challenge concerning relative profinite rigidity. But we also alluded (Remark 6.3) to the difficulty of knowing where to begin if there is no obvious reduction to a relative problem. That is the fundamental challenge that we want to address in this section. To avoid simple reductions, we consider only groups that are *full sized*, meaning that they contain a non-abelian free group. *Do there exist finitely generated, residually finite, full-sized groups that are profinitely rigid?* Might 3-manifolds and hyperbolic geometry⁵ help us?

9.1. Profinutely rigid Kleinian groups

We begin by describing some hyperbolic 3-orbifolds of small volume. The reader will likely be familiar with the tiling of the hyperbolic plane by ideal triangles. In the same

⁵Always remember, $\mathbb{H}$ yperbolic geometry is good for you!

way, one can tile hyperbolic 3-space $\mathbb{H}^3$ with copies of the regular ideal tetrahedron, that is, the tetrahedron with vertices at infinity where the dihedral angle between each pair of faces is $\pi/3$. Barycentric subdivision divides each tetrahedron in the tiling into 24 congruent tetrahedra, each with one vertex at infinity. The reflections in the faces of any one of these smaller tetrahedra will generate the full isometry group of the tiling. This group of isometries is

$$\Lambda_0 = \langle A, B, C, D \mid A^2, B^2, C^2, D^2, (AB)^3, (AC)^3, (AD)^2, (BC)^3, (BD)^2, (CD)^6 \rangle.$$

This is the unique non-uniform lattice of smallest covolume in $\text{Isom}(\mathbb{H}^3)$. It contains the Bianchi group $\text{PSL}(2, \mathbb{Z}[\omega])$ as a subgroup of index 4, where $\omega^2 + \omega + 1 = 0$.

Among torsion-free uniform lattices in $\text{PSL}(2, \mathbb{C})$, there is a unique one of smallest covolume, namely Γ_W , the fundamental group of the Weeks manifold [51], which can be obtained by Dehn surgery on the Whitehead link.

Theorem 9.1 (Bridson, McReynolds, Reid, Spitler [32]). *Each of the groups Γ_W , Λ_0 , $\text{PSL}(2, \mathbb{Z}[\omega])$, and $\text{PGL}(2, \mathbb{Z}[\omega])$ is profinitely rigid among all finitely generated, residually finite groups.*

The proofs in [32] cover a number of related lattices, but only finitely many. The list of lattices in $\text{PSL}(2, \mathbb{C})$ that are known to be profinitely rigid (in the absolute sense) was extended in [36, 44] but remains finite for the moment.

The following are some of the key elements of the proof of Theorem 9.1 for the given examples Γ :

- **Arithmetic:** The trace field K_Γ of Γ (defined below) has small degree, and Γ is commensurable with the group of units of a maximal order in the associated quaternion algebra.
- **Character varieties:** Γ has very few Zariski-dense representations into $\text{PSL}(2, \mathbb{C})$, up to conjugacy, and these are all accounted for by the arithmetic of K_Γ (“Galois rigidity”).
- **Mostow rigidity and volume calculations** are useful, as are various elements of 3-manifold topology, including the existence of compact cores, some knot theory, and the existence of finite covers that fibre. The consequences of cubulation, following Agol [2] and Wise [133], play a crucial role [94, 95], not least in ensuring cohomological goodness in the sense of Serre [55, 112].
- **Intricate arguments** concerning the subgroups of small finite index in each example were needed in [32], but these were superseded by the results in [36].

9.2. Trace fields and Galois rigidity

A remarkable feature of hyperbolic 3-manifolds is that their fundamental groups have *arithmetic structure* intrinsically associated to them (see Maclachlan and Reid [80]). Much of the proof of Theorem 9.1 revolves around this arithmetic structure and the way it controls representations into $\mathrm{PSL}(2, \mathbb{C})$ in our examples. A key notion in this regard is *Galois rigidity*. I will explain very briefly what this is and how it is used.

Let $\phi: \mathrm{SL}(2, \mathbb{C}) \rightarrow \mathrm{PSL}(2, \mathbb{C})$ be the quotient homomorphism, and if H is a finitely generated subgroup of $\mathrm{PSL}(2, \mathbb{C})$, set $H_1 = \phi^{-1}(H)$. It will be convenient to say that H is *Zariski dense* in $\mathrm{PSL}(2, \mathbb{C})$ when what we actually mean is that H_1 is a Zariski-dense subgroup of $\mathrm{SL}(2, \mathbb{C})$. The *trace field* of H is defined to be the field

$$K_H = \mathbb{Q}(\mathrm{tr}(\gamma) : \gamma \in H_1).$$

If K_H is a number field with ring of integers R_{K_H} , we say that H has *integral traces* if $\mathrm{tr}(\gamma) \in R_{K_H}$ for all $\gamma \in H_1$.

An important object associated to H is the subalgebra of $\mathrm{Mat}(2, \mathbb{C})$ generated by H_1 ; this is a *quaternion algebra* over K_H .

Suppose now that Λ is an abstract finitely generated group and $\rho: \Lambda \rightarrow \mathrm{PSL}(2, \mathbb{C})$ a Zariski-dense representation with $K = K_{\rho(\Lambda)}$ a number field of degree n_K . If $K = \mathbb{Q}(\theta)$ for some algebraic number θ , then the Galois conjugates of θ , say $\theta = \theta_1, \dots, \theta_{n_K}$, provide embeddings $\sigma_i: K \rightarrow \mathbb{C}$ defined by $\theta \mapsto \theta_i$. These in turn can be used to build n_K Zariski-dense non-conjugate representations $\rho_{\sigma_i}: \Lambda \rightarrow \mathrm{PSL}(2, \mathbb{C})$ with the property that $\mathrm{tr}(\rho_{\sigma_i}(\gamma)) = \sigma_i(\mathrm{tr}(\rho(\gamma)))$ for all $\gamma \in \Lambda$. One refers to these as *Galois conjugate representations*. The existence of these Galois conjugates shows that $|\mathrm{X}_{\mathrm{zar}}(\Gamma, \mathbb{C})| \geq n_{K_{\rho(\Lambda)}}$, where $\mathrm{X}_{\mathrm{zar}}(\Lambda, \mathbb{C})$ denotes the set of Zariski-dense representations $\Gamma \rightarrow \mathrm{PSL}(2, \mathbb{C})$ up to conjugacy.

Definition 9.2. Let Λ be a finitely generated group and $\rho: \Lambda \rightarrow \mathrm{PSL}(2, \mathbb{C})$ a Zariski-dense representation whose trace field $K_{\rho(\Lambda)}$ is a number field. If $|\mathrm{X}_{\mathrm{zar}}(\Lambda, \mathbb{C})| = n_{K_{\rho(\Lambda)}}$, we say that Γ is *Galois rigid* (with associated field $K_{\rho(\Lambda)}$).

9.3. An outline of the proof of Theorem 9.1

An explicit understanding of the character variety of each of our examples enables us to prove that each is Galois rigid. For example, we prove that the only Zariski-dense representations of $\mathrm{PSL}(2, \mathbb{Z}[\omega])$, whose trace field is $\mathbb{Q}(\sqrt{-3})$, are the inclusion map and its complex conjugate. In each case, one also has integrality of traces. The integrality of traces is used to ensure the boundedness of the non-archimedean representations $\Gamma \rightarrow \mathrm{SL}(2, \overline{\mathbb{Q}_p})$ obtained by transporting Zariski-dense representations $\Gamma \rightarrow \mathrm{SL}(2, \mathbb{C})$ via (non-continuous) field isomorphisms $\mathbb{C} \cong \overline{\mathbb{Q}_p}$, for all primes p . This boundedness enables one to extend the representations to $\hat{\Gamma}$, then restrict to any

finitely generated Λ with $\hat{\Lambda} \cong \hat{\Gamma}$ to obtain bijections between the sets of bounded, Zariski-dense characters of Γ and Λ at every finite place [32, Lemma 4.6].

With this control established, one can argue that the abstract group Λ is Galois rigid with a Zariski-dense representation $\rho: \Lambda \rightarrow \mathrm{PSL}(2, \mathbb{C})$ whose arithmetic data match those of Γ ; see [32, Theorem 4.8] for a precise statement. In good situations (which include the low-degree number fields of our examples), this matching of data forces the number fields $K_{\rho(\Lambda)}$ and K_{Γ} to be equal, and likewise the quaternion algebras of Γ and $\rho(\Lambda)$. This in turn allows one to conclude that (up to Galois conjugation) the image of $\rho: \Lambda \rightarrow \mathrm{PSL}(2, \mathbb{C})$ is contained in Γ or a small extension of it, and in all of the examples one can force the image to lie in Γ .

At this point of the argument in [32], a host of results about subgroups of 3-manifolds in general, and the finite index subgroups of our examples in particular, were used to argue that if the image of ρ was not equal to Γ , then Λ would have a finite quotient that Γ did not have. These arguments can now be replaced by [36, Theorem A]: *if $\Pi < \mathrm{PSL}(2, \mathbb{C})$ is a lattice and $H < \Pi$ is a finitely generated, proper subgroup, then $\hat{H} \not\cong \hat{\Pi}$; if H has finite index in Π , then there is a finite group onto which H maps but Π does not.*

At this stage, we have proved that there is a surjection $\rho: \Lambda \twoheadrightarrow \Gamma$, hence $\hat{\rho}: \hat{\Lambda} \twoheadrightarrow \hat{\Gamma}$, and the Hopf property for finitely generated profinite groups (a variant of Lemma 4.3) enables us to conclude that $\hat{\rho}$ is injective and hence so is ρ . ■

9.4. Profinutely rigid Fuchsian groups

Given integers (p, q, r) with $1/p + 1/q + 1/r < 1$, one can tile the hyperbolic plane with triangles whose interior angles are $2\pi/p, 2\pi/q, 2\pi/r$. The group generated by reflections in the sides of a fixed triangle acts transitively on the set of triangles in the tiling; we denote this group $\Delta^{\pm}(p, q, r)$. The index-2 subgroup consisting of orientation-preserving isometries is $\Delta(p, q, r)$. We restrict our attention to the following triples (p, q, r) :

$$(3, 3, 4), \quad (3, 3, 5), \quad (3, 3, 6), \quad (2, 5, 5), \quad (4, 4, 4). \quad (9.1)$$

$$(2, 3, 8), \quad (2, 3, 10), \quad (2, 3, 12), \quad (2, 4, 5), \quad (2, 4, 8). \quad (9.2)$$

Theorem 9.3 (Bridson, McReynolds, Reid, Spitler [33]). *Both $\Delta(p, q, r)$ and $\Delta^{\pm}(p, q, r)$ are profinitely rigid if (p, q, r) belongs to one of these lists.*

A key fact about these groups is that they are Galois rigid (as subgroups of $\mathrm{PSL}(2, \mathbb{C})$), and in each case the trace field is a real quadratic number field where the ramification of primes is simple enough to ensure that the argument from [32] sketched above applies.

We have discovered some lattices in $\mathrm{PSL}(2, \mathbb{R})$ and $\mathrm{PSL}(2, \mathbb{C})$ that are profinitely rigid in the absolute sense, but conjecturally there should be many more.

10. Seifert fibre spaces and the importance of finiteness properties

The following theorem provides the first examples of finitely presented groups that are profinitely rigid among finitely presented groups but not among finitely generated groups. Like our earlier examples, the groups Γ in this theorem are 3-manifold groups with particular arithmetic properties. Here, $S^2(p, q, r)$ denotes the quotient $\mathbb{H}^2/\Delta(p, q, r)$ of the hyperbolic plane by the triangle group $\Delta(p, q, r)$.

Theorem 10.1 (Bridson, Reid, Spitler [37]). *There exist finitely presented, residually finite groups Γ with the following properties:*

- (1) $\Gamma \times \Gamma$ is profinitely rigid among all finitely presented, residually finite groups.
- (2) There are infinitely many pairwise distinct, finitely generated groups Λ such that $\widehat{\Lambda} \cong \widehat{\Gamma} \times \widehat{\Gamma}$.
- (3) If Λ is as in (2), then there is an embedding $\Lambda \hookrightarrow \Gamma \times \Gamma$ that induces the isomorphism $\widehat{\Lambda} \cong \widehat{\Gamma \times \Gamma}$ (in other words, $\Lambda \hookrightarrow \Gamma \times \Gamma$ is a Grothendieck pair).

If M is any Seifert fibred space with base orbifold $S^2(3, 3, 4)$ or $S^2(3, 3, 6)$ or $S^2(2, 5, 5)$, then $\Gamma = \pi_1 M$ has these properties.

10.1. An outline of the proof

Let M be a Seifert fibre space with base orbifold $S^2(p, q, r) = \mathbb{H}^2/\Delta(p, q, r)$, where (p, q, r) is one of triples from list (9.1) or (9.2). We know from the previous section that $\Delta = \Delta(p, q, r)$ is profinitely rigid. There is a central extension

$$1 \rightarrow \mathbb{Z} \rightarrow \pi_1 M \rightarrow \Delta \rightarrow 1.$$

Theorem 10.2. *Let $\Delta(p, q, r)$ be a triangle group from list (9.1) or (9.2), and let M be a Seifert fibred space with base $S^2(p, q, r)$. Then $\pi_1 M$ is profinitely rigid (in the absolute sense).*

In order to prove this theorem, one has to extend the arguments about profinite rigidity in the previous sections to cover central extensions. It is not the case that an arbitrary central extension of Δ will be profinitely rigid [92], but special arguments apply in the case where the centre is cyclic. Wilkes's work on the profinite rigidity of Seifert fibred spaces within the class of 3-manifold groups [121] is used here.

The key step in the process of reducing abstract profinite isomorphism to the study of Grothendieck pairs also requires an extension of the scope of our arguments about Galois rigidity, this time to cover direct products, fibre products, as well as central extensions.

Theorem 10.3. *Let $\Delta(p, q, r)$ be a triangle group from list (9.1) or (9.2), let M be a Seifert fibred space with base orbifold $S^2(p, q, r)$, and let $\Gamma = \pi_1 M$. Then, for every finitely generated, residually finite group Λ with $\widehat{\Lambda} \cong \widehat{\Gamma \times \Gamma}$, there is an embedding $\Lambda \hookrightarrow \Gamma \times \Gamma$ that induces the isomorphism.*

With this reduction to Grothendieck pairs in hand, the following straightforward extension of Theorem 7.7 establishes the desired profinite rigidity among finitely presented groups.

Theorem 10.4. *For every Fuchsian group F and every Seifert fibred space M with base orbifold $\mathbb{H}^2/F$, there are no Grothendieck pairs $\Lambda \hookrightarrow \pi_1 M \times \pi_1 M$ with $\Lambda \neq \pi_1 M \times \pi_1 M$ finitely presented.*

To complete the proof of Theorem 10.1, we need finitely generated Grothendieck pairs.

Theorem 10.5. *If $\Delta(p, q, r)$ is a triangle group from list (9.1), then there are infinitely many Seifert fibred spaces with base orbifold $S^2(p, q, r)$ whose fundamental group Γ has the property that there are infinitely many pairwise non-isomorphic, finitely generated groups Λ and inclusions $\Lambda \hookrightarrow \Gamma \times \Gamma$ inducing isomorphisms $\widehat{\Lambda} \cong \widehat{\Gamma \times \Gamma}$.*

Note that at this stage, not all of the groups from Theorem 10.2 are used: we need constraints on the Seifert invariants that ensure $\Gamma = \pi_1 M$ has finite index in $[G, G]$ where G is a group that maps onto a non-elementary hyperbolic group and has $H_1(G, \mathbb{Z})$ finite and $H_2(G, \mathbb{Z}) = 0$. These conditions facilitate a homological argument, adapted from an idea of Bass and Lubotzky [6], that allows us to modify the template for constructing Grothendieck pairs that was described in Section 7.

11. Open questions

The most celebrated problem concerning profinite rigidity is Remeslennikov's question (6.4), where the answer is widely believed to be positive: see [24] for a discussion of related results and recent progress, most notably [66].

Conjecture 11.1. *Finitely generated free groups are profinitely rigid.*

The consensus on whether the following conjecture is true is less uniform: it is widely believed that lattices in $\mathrm{PSL}(2, \mathbb{R})$ are profinitely rigid, but many would favour a more cautious conjecture in the Kleinian case, simply asking that a finitely generated, residually finite group Λ with the same finite quotients as a lattice in $\mathrm{PSL}(2, \mathbb{C})$ should itself be a lattice. If we knew that Λ was a lattice in $\mathrm{PSL}(2, \mathbb{C})$, Liu's finite genus theorem [73] would tell us that there are only finitely many possibilities for Λ ; it is unclear if this finite ambiguity can be removed, but recent results of [73, 74, 126] constrain it.

Conjecture 11.2. *All lattices in $\mathrm{PSL}(2, \mathbb{R})$ and $\mathrm{PSL}(2, \mathbb{C})$ are profinitely rigid (in the absolute sense).*

With Theorem 10.1 in mind, one might weaken the above conjectures and ask only for rigidity among finitely presented groups. We saw in Section 7 that the difference between finite generation and finite presentation definitely matters in the context of direct products of free groups.

Conjecture 11.3. *Let F and F' be finitely generated free groups. If a finitely presented, residually finite group Γ has the same finite quotients as $F \times F'$, then $\Gamma \cong F \times F'$.*

For groups of higher-dimensional origin, the most compelling question in the field is the following.

Question 11.4. Is $\mathrm{SL}(n, \mathbb{Z})$ profinitely rigid when $n \geq 3$?

Using results from Spitler's thesis [114] that were inspired by ideas of Lubotzky (and are close in spirit to our discussion of Galois rigidity), one can reduce this question to the challenge of deciding whether $\mathrm{SL}(n, \mathbb{Z})$ is Grothendieck rigid. But since the subgroups of $\mathrm{SL}(n, \mathbb{Z})$ are so varied and mysterious, this remains a very difficult challenge.

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