



Generalized Riemann–Hilbert correspondence, wall-crossing and resurgence

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Abstract. Together with Yan Soibelman, we propose a new version of homological mirror symmetry, a hypothetical equivalence between two categories associated with a complex algebraic symplectic variety. In the case of cotangent bundles, it gives the classical Riemann–Hilbert correspondence, which is the equivalence between the category of holonomic D -modules with regular singularities on a complex algebraic variety and the category of perverse constructible sheaves. In general, the de Rham side is replaced by the algebraic deformation quantization, whereas the Betti side is replaced by the Fukaya category. New theory is applicable not only to D -modules, but also to q -difference and elliptic difference equations. The key role is played by the theory of wall-crossing, hypothetically leading to resurgence properties of WKB expansions.

This note is an extended version of my talk at ECM 2024, Seville, where I tried to give an introduction to the general program we (Yan Soibelman and I) have been working on for the last 10 years, which we have dubbed “Holomorphic Floer Theory”. Most of the text focuses on the foundational issues; the wall-crossing and resurgence only appear at the end. I apologize in advance to the reader for not always being precise, and often quite vague. One can also consult our recent preprint [21] where we considered a more manageable part of the general program related to closed 1-forms and also made several speculative predictions concerning resurgence and path integrals. Also, in the case of complex symplectic surfaces, most of the hypothetical constructions can be made precise; a detailed exposition will be given in [22] which we are preparing for publication.

The emerging picture is magnificent: it unifies many branches of mathematics into a new “algebraic analysis”.

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1. Main conjecture

Let (M, ω) be a smooth algebraic *symplectic* variety over $\mathbb{C}$. Assume that it is *A-good* and *B-good* at infinity (or (A, B) -good in short), meaning

A-good: (a complex version of *geometrically bounded real symplectic manifolds*, cf. [8]) $\exists$ *complete* Hermitian metric g on $M(\mathbb{C})$ such that the curvature tensor of g , (injectivity radius of g) $^{-1}$, 2-form $\omega \in \Gamma(M, \wedge^2 T_M^*)$ and Poisson bivector $\omega^{-1} \in \Gamma(M, \wedge^2 T_M)$ are all *uniformly* bounded.

B-good: $\exists$ a smooth compact algebraic Poisson variety P over $\mathbb{C}$ such that (M, ω) is its open Zariski-dense symplectic leaf (see Section 2.5 for a criticism).

Examples. For *any* smooth algebraic variety X over $\mathbb{C}$ (not necessarily compact), its cotangent bundle $M = T^*X$ endowed with the standard symplectic form is (A, B) -good. Also, the complex algebraic torus $(\mathbb{C}^\times)^{2n}$ endowed with the symplectic form

$$\omega = \sum_{i=1}^n \frac{dx_i}{x_i} \wedge \frac{dy_i}{y_i} \quad (1.1)$$

is (A, B) -good. Any compact algebraic symplectic manifold (M, ω) (e.g., a K3 surface) is automatically (A, B) -good at infinity. An example of a symplectic variety which is neither *A-* nor *B-good* is $M = \mathbb{C}^2 - 0, \omega = dx \wedge dy$. It looks plausible that the condition of being *B-good* implies that of being *A-good*.

In what follows, when I speak about categories, I mean $(\infty, 1)$ -categories associated with A_∞ - or dg -categories, linear over some base ring R .

Informal conjecture (not yet properly formulated. . .).

- (*A-side* $\Leftrightarrow$ *Betti side*) we have an analytic family of Fukaya categories (twisted by the so-called *B-field* $\beta_\hbar$) over $\mathbb{C}_\hbar^\times = \mathbb{C}_\hbar - 0$:

$$C_{A,\hbar}(M, \omega) = \mathcal{F}\left(M, \Re\left(\frac{\omega}{\hbar}\right), \beta_\hbar\right), \quad \beta_\hbar := i \cdot \left[\Im\left(\frac{\omega}{\hbar}\right)\right] \in H^2(M, i\mathbb{R}/2\pi i\mathbb{Z}).$$

- (*B-side* $\Leftrightarrow$ *de Rham side*) we have an analytic family of categories of deformation quantization modules over $\mathbb{C}_\hbar$:

$$C_{B,\hbar}(M, \omega), \quad \text{fiber at } \hbar = 0 \text{ is } C_{B,0}(M, \omega) \xrightarrow{\text{full}} \text{Perf}_c(M).$$

- *Generalized Riemann–Hilbert correspondence (GRH in short)*:

$$C_{A,\hbar}(M, \omega) \simeq C_{B,\hbar}(M, \omega) \quad \text{for } \hbar \neq 0,$$

with the equivalence holomorphically dependent on $\hbar$. Moreover, both categories are endowed with bounded t -structures preserved by GRH, $\Rightarrow$ equivalence of abelian hearts $C_{A,\hbar}^{\text{heart}}(M, \omega) \simeq C_{B,\hbar}^{\text{heart}}(M, \omega)$.

The notations will be explained later, in both cases we deal implicitly with objects whose “support” is compact Lagrangian (whatever it means. . .). The important point is that for any given complex parameter $\hbar \neq 0$, the notion of Fukaya category belongs to the realm of symplectic topology for real C^∞ -manifolds. The B -side categories are (hypothetically) defined in terms of algebraic/analytic deformation quantization. The main challenge is to find a satisfactory general definition for both the A and B cases.

I did not explain here what I mean speaking above about an “analytic family” of categories. The full definition is based on a quite technical notion of higher derived analytic stacks. The basic example which one should have in mind is the following. Consider an analytic family $(Y_s)_{s \in S}$ of compact complex *not necessarily algebraic* manifolds parametrized by a complex-analytic space S . Then the categories of perfect complexes $\text{Perf}(Y_s)$ should form an analytic family over S . As a part of the structure, for any analytic map $\tilde{S} \rightarrow S$, one gets a *sheaf* of A_∞ -categories on $\tilde{S}$ enriched over the sheaf $\text{Perf}_{\tilde{S}}$ of symmetric monoidal $(\infty, 1)$ -categories of perfect complexes.

Finally, I have to explain the origin of the names A and B for the sides of GRH correspondence. One can view the generalized Riemann–Hilbert correspondence as a cousin of *homological mirror symmetry* (see [14]). The difference is that in HMS the A -model on one variety is equivalent to the B -model on *another* variety, the mirror dual to the original one, whereas in the GRH correspondence we deal on both sides with the *same* variety. Also, the B -side in GRH correspondence is not the derived category of coherent sheaves (or rather, a certain full subcategory of it), but a deformation (non-perturbative quantization).

1.1. Relation to the classical RH correspondence

Let $M = T^*X$ where X is a smooth *compact* algebraic variety over $\mathbb{C}$, with the standard algebraic symplectic form ω . Assume that $c_1(X) \in H^2(X, \mathbb{Z})$ is divisible by 2, and a square root line bundle $\sqrt{K_X}$ is chosen.

- $C_{A,\hbar}(M, \omega)$ does not depend on $\hbar \neq 0$, and its abelian heart $C_{A,\hbar}^{\text{heart}}(M, \omega)$ coincides with category of $\mathbb{C}$ -local systems of finite rank on X .
- For $\hbar \neq 0$ category, $C_{B,\hbar}^{\text{heart}}(M, \omega)$ consists of holonomic D_X -modules which are algebraic vector bundles with algebraic flat connections on X (i.e., in local coordinates, modules over the algebra generated by operators x_i and $\hbar \partial_{x_i}$). If X is projective, the fiber at $\hbar = 0$ is the category of semistable Higgs bundles $(\mathcal{E}, \theta)$ where $c_1(\mathcal{E}) = 0$ and $\theta \in \Gamma(X, \text{End } \mathcal{E} \otimes \Omega_X^1)$, $\theta \wedge \theta = 0$. It is the same as the category of coherent sheaves with compact support on M which are semistable of

slope 0 for some (asymptotic) Bridgeland stability condition on the triangulated category $\text{Perf}_c(M)$ of compactly supported perfect complexes on M .

One can omit the somewhat annoying condition that $c_1(T_X)$ is divisible by 2. In general, for both categories $C_{A,\hbar}(M, \omega)$ and $C_{B,\hbar}(M, \omega)$ (and their hearts), one can define *twists* by any class $\beta \in H^2(M, i\mathbb{R}/2\pi i\mathbb{Z})$. The convenient twist for $M = T^*X$ is 2-torsion and is given by the pullback to M of the image of $c_1(T_X) \in H^2(X, \mathbb{Z})$ via the map on coefficients $1 \in \mathbb{Z} \mapsto \pi i \pmod{2\pi i\mathbb{Z}}$.

1.2. Hodge-theoretic invariant

With each of the categories $C_{A,\hbar}(M, \omega)$, $C_{B,\hbar}(M, \omega)$, we expect to be able to associate a $\mathbb{Z}/2$ -graded finitely generated abelian group, its *topological K-theory*, which in both A and B cases is canonically identified (by its own reasons in A and B cases) with the compactly supported topological K-theory $K_c^{\text{top}}(M) = K_{c,\text{even}}^{\text{top}} \oplus K_{c,\text{odd}}^{\text{top}}(M)$. The equivalence is compatible with this identification. Moreover, we should have a cycle map

$$\text{cl} : K_0(C_{A \text{ or } B, \hbar}(M, \omega)) \rightarrow K_{c,\text{even}}^{\text{top}}(M).$$

Also, on $K_{c,\text{even}}^{\text{top}}(M) \otimes \mathbb{C} \simeq H_c^{\text{even}}(M, \mathbb{C})$ (the isomorphism is given by $[\mathcal{E}] \mapsto \sqrt{\text{Todd}(T_M)} \cdot \text{ch}(\mathcal{E})$), we have *Hodge filtration* (cf. [17, point 1.36]):

$$F^i := e^{\omega/\hbar} (H_c^0(M, \mathbb{C}) \oplus H_c^2(M, \mathbb{C}) \oplus \dots \oplus H_c^{2n-2i}(M, \mathbb{C})).$$

The image of the cycle map cl is expected to lie in F^0 .

The Hodge filtration for the category $C_{B,\hbar}(M, \omega)$ at $\hbar = 0$ is defined as the limit of the above filtration at $\hbar \rightarrow 0$. We expect that there exists a limiting Bridgeland stability condition such that $C_{B,\hbar}^{\text{heart}}(M, \omega)$ is an Artinian abelian full subcategory characterized as the category of semistable objects with slope 0. In particular, classes in $K_{c,\text{even}}^{\text{top}}(M)$ should belong to a certain subgroup (lying in F^0) cut out by the central charge Z .

1.3. Example related to the irregular RH correspondence

Let (M_0, ω_0) be $(\mathbb{C}^2, dx \wedge dy)$, the cotangent bundle to the affine line $\mathbb{A}_x^1$. It is an open symplectic leaf in the compact Poisson surface $P_0 = \mathbb{C}P^2$. After making several iterated blowups at points of the Poisson boundary $P_0 - M_0$, we embed M_0 as the open symplectic leaf into another Poisson surface P_1 which contains two boundary divisors D_0, D_1 , each endowed with a coordinate $D_i \simeq \mathbb{C}^1$, $i = 0, 1$, such that

$$\lim_{x \rightarrow \infty} \underbrace{(x, -2x + t/x)}_{\in \mathbb{C}^2 = M_0} = \underbrace{t}_{\in \mathbb{C} = D_0}, \quad \lim_{x \rightarrow \infty} \underbrace{(x, +2x + t/x)}_{\in \mathbb{C}^2 = M_0} = \underbrace{t}_{\in \mathbb{C} = D_1}.$$

Symplectic form ω_0 has poles of order 1 at divisors D_0, D_1 .

Let us make a *choice* of two complex numbers $\lambda_0, \lambda_1 \in \mathbb{C}$, and define Poisson surface $P = P_{(\lambda_0, \lambda_1)}$ to be the blowup of P_1 at two points in D_0, D_1 with coordinates λ_0, λ_1 , respectively.

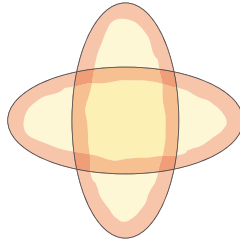
The open symplectic leaf (M, ω) of P is $(M_0 = \mathbb{C}^2) \sqcup \mathbb{C}_{(0)}^1 \sqcup \mathbb{C}_{(1)}^1$ as a set; we add two open exceptional divisors. Here M is neither affine nor projective.

1.3.1. Category $C_{A, \hbar}^{\text{heart}}(M, \omega)$ in the example. Abelian category $C_{A, \hbar}^{\text{heart}}(M, \omega)$ is the category of tuples consisting of a vector space V and 4 subspaces $V_0, V_1, V_2, V_3 \subset V$ such that V_{i+1} is complementary to V_i (here put $V_4 := V_0$), and such that two compositions of isomorphisms

$$V_0 \xrightarrow{\sim} V/V_1 \xrightarrow{\sim} V_2 \xrightarrow{\sim} V/V_3 \xrightarrow{\sim} V_0, \quad V_1 \xrightarrow{\sim} V/V_2 \xrightarrow{\sim} V_3 \xrightarrow{\sim} V/V_0 \xrightarrow{\sim} V_1$$

are equal to $\exp(2\pi i \lambda_0 / \hbar) \cdot \text{id}_{V_0}$ and $\exp(2\pi i \lambda_1 / \hbar) \cdot \text{id}_{V_1}$, respectively. More precisely, enumeration of V_i is determined up to the involution $0 \leftrightarrow 2, 1 \leftrightarrow 3$, depending on the choice $\theta/2 \in \mathbb{R}/2\pi\mathbb{Z}$ where $\theta = \arg \hbar$. Speaking in a more invariant way, subspaces should be labeled by $\theta/2, \theta/2 + \pi/2, \theta/2 + \pi, \theta/2 + 3\pi/2$ instead of $0, 1, 2, 3$.

This category coincides with the category of constructible sheaves on the plane $\mathbb{R}^2 \simeq \mathbb{C}$ with the *microlocal support* (see [12]) contained in a closed bounded domain in zero section $\mathbb{R}^2 \subset T^*\mathbb{R}^2$ together with the positive conormal bundles to two co-oriented closed curves, depicted as



1.3.2. Category $C_{B, \hbar}^{\text{heart}}(M, \omega)$ in the example. Abelian category $C_{B, \hbar}^{\text{heart}}(M, \omega)$ for $\hbar \neq 0$ is the category of holonomic D -modules on $\mathbb{A}_x^1$ which are algebraic vector bundles with connections $(\mathcal{E}, \nabla)$ such that over $\mathbb{C}((x^{-1}))$ the bundle splits into the sum of copies of two types of the trivial rank 1 bundles with connections

$$d - \hbar^{-1} \left(-2x + \frac{\lambda_0}{x} \right) dx, \quad d - \hbar^{-1} \left(+2x + \frac{\lambda_1}{x} \right) dx.$$

These two special equations have the following solutions near ∞ :

$$x^{\lambda_0/\hbar} e^{-x^2/\hbar}, \quad x^{\lambda_1/\hbar} e^{+x^2/\hbar}.$$

For $\hbar = 0$, the abelian category $C_{B,0}^{\text{heart}}(M, \omega)$ should be defined as a full subcategory of $\text{Coh}_c(M)$ consisting of coherent sheaves with compact support which are semistable with slope 0 for certain asymptotic Bridgeland stability condition. This stability condition guarantees that the support of any non-zero object is purely 1 dimensional, so that the sheaves with 0-dimensional support are excluded.

1.3.3. Equivalence of categories. The equivalence $C_{A,\hbar}(M, \omega) \simeq C_{B,\hbar}(M, \omega)$ (at least on hearts) is (after a little work) a special case of 1-dimensional *irregular Riemann–Hilbert correspondence* by P. Deligne and B. Malgrange; see [4] (the higher-dimensional generalization was worked out in [3]). For a connection $(\mathcal{E}, \nabla)$, the corresponding space V is the space of analytic flat sections of $\mathcal{E}$ over $\mathbb{C}$. The subspaces V_j (or, more precisely, $V_{\theta/2+j \cdot \pi/2}$) are subspaces of sections with the exponentially fast decay along rays with arguments $\theta/2 + j \cdot \pi/2$, $j = 0, \dots, 3$.

The dependence of (any of two equivalent) categories $C_{A,\hbar}(M, \omega) \simeq C_{B,\hbar}(M, \omega)$ on parameters $\lambda_0, \lambda_1 \in \mathbb{C}$ is quite drastic – the category vanishes unless $a\lambda_0 + b\lambda_1 = 0$ for some pair of non-negative integers $(a, b) \in \mathbb{Z}_{\geq 0}^2 - \{(0, 0)\}$. In fact, one has $a/b = \dim V_0 / \dim V_1$ for any non-zero object in the $C_{A,\hbar}^{\text{heart}}(M, \omega)$. The dependence on parameters can be promoted to an analytic family of categories over $\mathbb{C}_{\lambda_0, \lambda_1}^2 \times \mathbb{C}_{\hbar}^\times$.

1.4. Non-compact support: Log-Poisson extensions

In the previous example, we made a certain non-trivial effort in the definition of symplectic variety (M, ω) , by enlarging the more familiar cotangent bundle. One can follow another route, which is more directly related to the classical (irregular) RH correspondence.

Let us assume that M is Zariski open symplectic leaf in a compact algebraic Poisson variety (P, γ) , $\gamma \in \Gamma(P, \wedge^2 T_P)$ such that the complement $P - M$ is a divisor with *normal crossings*. Let $P_{\log} \subset P$ be the set of points such that there exist local analytic coordinates $(x_1, \dots, x_n; y_1, \dots, y_n)$ in which the Poisson structure is given by

$$\gamma = \sum_{i=1}^k x_i \partial_{x_i} \wedge \partial_{y_i} + \sum_{i=k+1}^n \partial_{x_i} \wedge \partial_{y_i}$$

for some integer k , $0 \leq k \leq n$. It follows that $P_{\log}$ is Zariski open subset in P containing M . The (possibly non-compact) Poisson variety $(P, \gamma|_P)$ is called the *log-Poisson extension* of (M, ω) .

In the case $M = T^*X$ where X is a smooth (possibly non-compact) curve, log-Poisson extensions of M correspond to arbitrary finite collections of irregular terms

for holonomic D_X -modules, that is, expressions of the type

$$\exp\left(\sum_{j=1}^{N'} c_j x^{-j/N} + O(\log|x|)\right) \quad \text{up to } c_j \leadsto e^{\frac{2\pi i}{N} j} c_j, \quad N, N' \in \mathbb{N}, c_j \in \mathbb{C},$$

where x is a local coordinate at a point of $\overline{X}$.

Different Poisson compactifications $P, P' \supset M$ can give the *same* log-Poisson extensions $P_{\log} \simeq P'_{\log}$. It looks plausible that one can define non-compactly supported versions of categories $C_{A,\hbar}(M, \omega)$ and $C_{B,\hbar}(M, \omega)$ which depend *only* on the log-Poisson extension, without the choice of a full Poisson compactification.

The non-compactly supported version of $C_{A,\hbar}(M, \omega)$ for $\hbar \neq 0$ should be an appropriately defined *partially wrapped* Fukaya category, where objects are roughly real Lagrangian submanifolds in $(M, \Re(\frac{\omega}{\hbar}))$ whose closure in $P_{\log}$ is compact and behaves well in an appropriate sense at infinity $P_{\log} - M$.

We expect that in the case $M = T^*X$ and a given log-Poisson extension $P \supset M$, the non-compactly supported version of $C_{B,\hbar}^{\text{heart}}(M, \omega)$ for $\hbar \neq 0$ can be directly identified with the full category of holonomic D_X -modules consisting of objects for which the possible irregular terms of restrictions to all analytic punctured discs in X are specified, in a certain way associated with P .

An important open question here is “symplectic resolution of singularities”.

Conjecture. *For any B -good symplectic variety (M, ω) and possibly singular algebraic closed Lagrangian subset $L \subset M$, there exists a log-Poisson extension $P_{\log} \supset M$ such that the closure of L in $P_{\log}$ is compact.*

This conjecture holds by elementary means in the case $\dim M = 2$ (see, e.g., the discussion of blowups in [19, Section 8.3] where irregular terms were called “singular terms”) and, more surprisingly, for the case of an algebraic torus of *any* dimension, with the “trigonometric” symplectic structure as in (1.1).

1.5. Trigonometric case: Quantum tori

I will present a version of GRH correspondence for the algebraic torus (1.1), implicitly using some log-Poisson extensions.

Let us start with the two-dimensional case, $(M_0, \omega) = ((\mathbb{C}^\times)^2, \frac{dx}{x} \wedge \frac{dy}{y})$. Its natural algebraic quantization is algebra

$$A_\hbar = \mathbb{C}\langle X^{\pm 1}, Y^{\pm 1} \rangle / \text{relation } YX = qXY, \quad q := e^\hbar. \quad (1.2)$$

For $\hbar \notin 2\pi i\mathbb{Q}$, a finitely generated module E over $A_\hbar$ is called *holonomic* iff it admits an exhaustive increasing filtration $E_0 \subset E_1 \subset \cdots$ such that

$$X^{\pm 1} E_i, Y^{\pm 1} E_i \subset E_{i+1}, \quad \dim E_i = O(i). \quad (1.3)$$

Theorem (Can be extracted from [26]). *In the case $\hbar \notin i\mathbb{R}$, the category of holonomic $A_\hbar$ modules is equivalent to the category of triples consisting of a coherent sheaf $\mathcal{F}$ on elliptic curve $\mathbb{C}^\times/q^\mathbb{Z}$ and two exhaustive ascending filtrations of $\mathcal{F}$ by the ordered set $\mathbb{Q} \cup \{+\infty\}$ such that the associated graded sheaves are torsion sheaves for the grading $+\infty$, and semistable vector bundles with the slope $\text{rk}/\text{deg} = a/b$ for gradings $a/b \in \mathbb{Q}$.*

Notice that these filtrations are labeled in the order *opposite* to the standard Harder–Narasimhan filtration. Indices of two filtrations can be encoded geometrically as a finite collection of rays with rational slopes in $\mathbb{R}^2$.

Fukaya category perspective leads to a similar description on the *same* elliptic curve but now written as $\mathbb{C}^\times/\tilde{q}^\mathbb{Z}$ where $\tilde{q} = e^{(2\pi i)^2/\hbar}$.

One can generalize (hypothetically) this GRH correspondence to arbitrary dimension as in (1.1). The quantum algebra here is $A_\hbar^{\otimes n}$. Condition (1.3) in the definition of a holonomic module is replaced by the condition

$$\dim E_i = O(i^n).$$

On the A -side, one replaces the notion of two anti-Harder–Narasimhan filtrations by a certain mix of the local Fukaya category associated with a closed rational polyhedral Lagrangian cone in $\mathbb{R}^{2n}$, and the bounded derived category of coherent sheaves on the abelian variety $(\mathbb{C}^\times/\tilde{q}^\mathbb{Z})^n$; see Section 3.3.3 for more details.

2. B -side: Deformation quantization

2.1. Formal deformation quantization

One part of the main conjecture implies that for a B -good algebraic symplectic variety (M, ω) , we have a canonical A_∞ -category enriched over the $\text{Perf}(\mathbb{C}[[\hbar]])$. Therefore, by base change, we get a category enriched over $\text{Perf}(\mathbb{C}[[\hbar]])$.

The usual *formal deformation quantization* offers a natural guess. Namely, it follows from formality theorem (see [16]) that for any complex Poisson variety (P, γ) (here $\gamma \in \Gamma(P, \wedge^2 T_P)$, $[\gamma, \gamma] = 0 \in \Gamma(P, \wedge^3 T_P)$), without any assumption on algebraicity, one get a canonical sheaf of $\mathbb{C}[[\hbar]]$ -linear algebroids $\mathcal{O}_{q,P}$ on P (algebroids are roughly algebras up to inner automorphisms; see [15]) and the corresponding canonical sheaf of flat $\mathbb{C}[[\hbar]]$ -linear triangulated A_∞ -categories on P . In the case when γ is non-degenerate (i.e., we have a symplectic manifold (M, ω) and $\gamma = \omega^{-1}$), locally this sheaf is equivalent to the sheaf of perfect complexes over the sheaf of flat $\mathbb{C}[[\hbar]]$ -linear algebras given in local Darboux coordinates by the standard Moyal

product on $\mathcal{O}_{\mathbb{C}^{2n}}[[\hbar]]$:

$$f \star g = \sum_{i_1, \dots, i_n \geq 0} \frac{\hbar^{i_1 + \dots + i_n}}{i_1! \dots i_n!} \partial_{y_1}^{i_1} \dots \partial_{y_n}^{i_n} f \cdot \partial_{x_1}^{i_1} \dots \partial_{x_n}^{i_n} g. \quad (2.1)$$

In the polynomial case, one can associate with the function f a differential operator $f(x_1, \dots, x_n, \hbar \partial_{x_1}, \dots, \hbar \partial_{x_n})$, where one puts all derivatives on the right. Then the Moyal product gives the usual composition of operators.

We expect that the $\mathbb{C}[[\hbar]]$ -linear version of the B -model category $C_{B, \hbar}(M, \omega)$ is naturally equivalent to the category of perfect complexes of $\mathcal{O}_{q, M}$ -modules which are *holonomic* and with *compact support*. Recall that the support of any perfect complex is a coisotropic analytic subset, and the object is called holonomic iff the support is Lagrangian (possibly singular); see [13]. There is a natural bounded t -structure on this category with the heart consisting of perfect complexes concentrated in degree 0.

Alternatively, one can use arbitrary Poisson compactification $P \supset M$ such that M is an open dense symplectic leaf of P , and consider the category of perfect complexes of $\mathcal{O}_{q, P}$ -modules whose supports are compact Lagrangian subsets contained in M .

Finally, one can twist the sheaf of categories of perfect $\mathcal{O}_{q, P}$ -modules by any 2-cocycle with coefficients in $\mathbb{C}[[\hbar]]^\times$.

2.1.1. Deformation quantization of Poisson divisors. Here we present a useful result concerning formal deformation quantization.

An effective divisor $D \subset (P, \gamma)$ is called *Poisson* iff the ideal sheaf $\mathcal{I}_D = \mathcal{O}_P(-D) \subset \mathcal{O}_P$ is a sheaf of Poisson ideals: $\{\mathcal{I}_D, \mathcal{O}_P\} \subset \mathcal{I}_D$. An equivalent property is that the locus of smooth points of the associated reduced divisor D^{red} is a locally closed Poisson hypersurface.

If $M \subset P$ is an open, everywhere dense symplectic leaf, then the zero locus of $\gamma^{\dim P/2}$ (a section of the anti-canonical bundle $-K_P$) is a Poisson divisor, the associated reduced divisor is the complement $P - M$.

We claim that the canonical sheaf of algebroids $\mathcal{O}_{q, P}$ contains a canonical two-sided ideal $\mathcal{I}_{q, D}$, which is a flat deformation of $\mathcal{I}_D$. Moreover, if $f \in \mathcal{O}_{P, \text{pt}}$ is a germ at a point $\text{pt} \in P$ of an analytic function with zero locus D , then it can be lifted to a generator of $\tilde{f} \in \mathcal{I}_{q, D, \text{pt}}$ of $\mathcal{I}_{q, D, \text{pt}}$ as a free rank 1 left $\mathcal{O}_{q, P, \text{pt}}$ -module (the same is true for $\mathcal{I}_{D, \text{pt}}$ considered as a right module). As a corollary, we have a canonical sheaf of quotient algebroids $\mathcal{O}_{q, P} / \mathcal{I}_{q, D}$ supported on D and deforming $\mathcal{O}_D = \mathcal{O}_X / \mathcal{I}_D$.

The definition is easy at smooth points of D^{red} . Indeed, one can choose local coordinates $x_1, \dots, x_n$ near $\text{pt} \in P$ (here $n = \dim P$) such that $\mathcal{I}_{D, \text{pt}}$ is generated by x_1^k for some $k \geq 1$ (the multiplicity of D at pt). The condition of D being Poisson means that the bivector field γ is $\mathcal{O}_{P, \text{pt}}$ -linear combination of

$$x_1 \partial_{x_1} \wedge \partial_{x_i}, \quad \partial_{x_i} \wedge \partial_{x_j}, \quad i, j \geq 2.$$

Explicit formulas for the $\star$ -product in deformation quantization (see [15]) imply that $\mathcal{I}_{D,\text{pt}} := x_1^k \mathcal{O}_{P,\text{pt}}[[\hbar]]$ is a two-sided $\star$ -ideal, and x_1^k is its free generator as a left and as a right $\mathcal{O}_{q,P,\text{pt}}$ -module. Finally, one can extend the sheaf of ideals to the whole D by the Hartogs principle.

2.2. Hypothetical non-formal quantization

In [15], Questions 1 at the end of Section 2.3, I discussed a possibility to pass from $\mathbb{C}[[\hbar]]$ to germs of analytic functions $\mathbb{C}\{\hbar\}$ for a class of *algebraic* affine symplectic manifolds admitting an algebraic Poisson compactification $P \supset M$ such that $P - M$ is an ample divisor and $H^1(P, \mathcal{O}) = H^2(P, \mathcal{O}) = 0$. Examples suggest that the dependence on $\hbar$ is analytic not only in a neighborhood of 0, but is defined *everywhere* in $\mathbb{C}_\hbar$ -plane. Now I am ready to propose a more ambitious conjecture, going beyond the algebraic case.

Very optimistic conjecture. *For any compact complex Poisson manifold (P, γ) of Kähler type, and such that $H^2(P, \mathcal{O}) = 0$, one has a canonical category $\text{Perf}_{P,\gamma}$ enriched over $\text{Perf}(\mathbb{C})$, with a bounded t -structure. Moreover, the dependence on (P, γ) is analytic, and the corresponding family in the formal neighborhood of the locus of (P, γ) with $\gamma = 0$ coincides with the one given by the formal deformation quantization.*

The assumption of being of Kähler type (i.e., to admit a Kähler metric) is not really justified. It guarantees Hodge decomposition and is stable under small deformations. I included it just to be on the safe side.

One can also try to remove the constraint that the Hodge number $h^{0,2}(P)$ vanishes. In the general case of a compact Poisson manifold of Kähler type, we should also choose a 2-cocycle with values in $U(1) = i\mathbb{R}/2\pi i\mathbb{Z} \subset \mathbb{C}^\times$. The dependence on additional parameters should be real analytic (i.e., extends to a neighborhood of compact torus $H^2(P, U(1))$ in complex torus $H^2(P, \mathbb{C}^\times)$).

By adding gerbes, one gets certain redundancy of parameters. For example, at the commutative point $\gamma = 0$, the tangent space to the moduli stack of categories is (by the formality theorem)

$$H^2(P, \mathcal{O}_P \rightarrow T_P \rightarrow \wedge^2 T_P \rightarrow \dots),$$

whereas the tangent space to Poisson structures together with $U(1)$ gerbe is

$$H^2(P, i\mathbb{R} \rightarrow \mathcal{T}_P \rightarrow \wedge^2 T_P \rightarrow \dots).$$

There is a natural epimorphism from the second to the first.

The general conjecture gives an attractive way to “name” quantum categories. In concrete cases, when P is obtained by blowups from compactified cotangent bundles,

or compactified tori (trigonometric case), or, for example, Odesskii–Feigin projective spaces with elliptic brackets, one can typically make a reasonable guess of what the category $\mathrm{Perf}_{P,\gamma}$ is. For example, when P contains an open dense symplectic leaf (M, ω) , $\gamma|_M = \omega^{-1}$, the cohomology class $[\omega] \in H^2(M, \mathbb{C})$ corresponds to the so-called Deligne class of the localization $\mathrm{Perf}_{M,\gamma|_M}$ of $\mathrm{Perf}_{P,\gamma}$; see Section 1.2 and [17, points 1.30–1.37].

2.3. Objects with compact support

Now we are almost ready to propose a definition of $C_{B,\hbar}(M, \omega)$. If (M, ω^{-1}) is the open dense symplectic leaf in (P, γ) , we can try to define abelian B -model category $C_{B,\hbar}^{\mathrm{heart}}(M, \omega)$ as the full subcategory of the heart of $\mathrm{Perf}_{P,\gamma}$ consisting of objects which are “vanishing after the restriction to $P - M$ ”.

The above proposal makes perfect sense for the formal deformation quantization (see Section 2.1.1). The category over $\mathbb{C}[[\hbar]]$ can be alternatively defined as the abelian category of perfect $\mathcal{O}_{q,M}$ -modules with compact support.

One can see clearly some potential problems with such a definition. First, the limiting category at $\hbar = 0$ is $\mathrm{Coh}_c(M)$, which is “too large”, as it contains coherent sheaves with 0-dimensional support, or, in the case when M is compact, with the full $2n$ -dimensional support. One would like to exclude such objects and keep only those with n -dimensional Lagrangian support. Therefore, we like to specify a subgroup of $K_{c,\mathrm{even}}^{\mathrm{top}}(M)$ of “admissible” classes of objects. We expect that this constraint comes from a limiting stability condition, as mentioned in Section 1.2.

In the quantized case for generic $\hbar \neq 0$, by general philosophy, objects should have support of dimension $\geq \dim M/2$ (see, e.g., [11]) and can be thought of as *coisotropic* compact branes. Again, as in the case $\hbar = 0$, we would like to put a constraint via a limiting stability condition, forcing the classes to stay in a smaller subgroup of $K_{c,\mathrm{even}}^{\mathrm{top}}(M)$ corresponding roughly to compact Lagrangian cycles.

2.4. Remarks about non-compact support

It is interesting to get some understanding of quantum modules whose classical limit is a coherent sheaf on M with a *non-compact* coisotropic support.

The first example is the so-called big brane (also called *canonical coisotropic brane*), a deformation of the structure sheaf $\mathcal{O}_M$ as was suggested by physicists; see [9, 11]. In the case, for example, $M = \mathbb{A}_{\mathbb{C}}^{2n}$ with the standard symplectic form $\sum_i dx_i \wedge dy_i$, the endomorphism algebra of the big brane should be the n -th Weyl algebra, that is, the algebra of polynomial differential operators generated by $x_i, \hbar \partial_{x_i}$, $i = 1, \dots, n$. If one pursues the idea that the quantized algebra *should not depend* on the choice of Poisson compactification, one inevitably comes to the conclusion that the group of polynomial symplectomorphisms should act by automorphisms

of Weyl algebra (see [15, Question 4 in Section 4.1], main conjecture in [1] and its announced proof in [5]).

Another reason to be interested in non-compactly supported objects is GRH correspondence, where the supports are Lagrangian subsets in the classical limit $\hbar \rightarrow 0$, like, for example, in Section 1.4. This is natural if one tries to keep the contact with the traditional RH correspondence for holonomic D -modules, or its version for quantum tori (see Sections 1.5 and 3.3.3). Here one should analyze the behavior of Lagrangian subsets at infinity.

In this lecture, I will try to focus more on the approach when one deals only with compactly supported objects, which is achieved by enlarging the symplectic leaf like as, for example, in Section 1.3. The formulation of the main conjecture is a bit cleaner in the compactly supported case.

2.5. Beyond smooth Poisson compactifications

In general, the assumption that M admits a *smooth* Poisson compactification seems to be a bit unnatural. The reason is that Poisson structures behave badly under blowups, and one can envision a situation when the complement $P - M$ has singularities of high codimension which do not affect the general construction, but which cannot be resolved in the class of Poisson manifolds (see, e.g., [15, Questions 3 in Section 4.1]).

In differential geometry, it is natural to consider hyperkähler manifolds with *complete* metric, which seems to be the framework of [9]. Presumably, the “optimal” condition at infinity for B -model should go in this direction.

A more drastic change would be to allow some mild singularities even for symplectic manifolds. Both hypothetical approaches to A and B models, presented in this text, fail for non-smooth varieties.

3. A -side: Fukaya categories, real case

3.1. Almost-complex structures, conditions at infinity

Recall that for a *real* symplectic C^∞ -manifold (M, ω) , an almost-complex structure $J \in \Gamma(M, \text{Aut}(T_M))$, $J^2 = -1$ is called *compatible* with ω iff

$$\langle \omega_x, v \wedge Jv \rangle > 0 \quad \forall x \in M, \forall v \in T_x M - 0.$$

In the definition of Fukaya A_∞ -category of a *compact* symplectic manifold, one usually starts with a *choice* of compatible almost-complex structure J and then considers J -holomorphic discs with boundaries on unions of Lagrangian submanifolds. The resulting category does not depend (up to a contractible space of equivalences)

on the choice of J , and in fact can be *canonically* defined, as follows from the contractibility of the space of almost-complex structures compatible with ω .

Structure constants of higher multiplications of Fukaya categories are given by virtual fundamental classes of spaces of J -holomorphic discs of bounded area, and one needs to ensure a compactness property in order to get a well-defined count. For compact M , this follows from Gromov compactness, whereas in the non-compact case one has to impose an appropriate behavior at infinity.

A sufficient condition on (M, ω) is the following: M is separable (i.e., contains a countable dense subset), and there exists a compatible almost-complex structure J such that for any compact subset $K \Subset M$ and any real $A > 0$, there exists a larger compact $K' \subset M$ such that all J -pseudo-holomorphic discs $D \subset M$ with $\partial D \subset K$ and $\int_D \omega < A$ are contained in K' .

If this condition is satisfied, then one can define the compactly supported version of the Fukaya category (and genus zero Gromov–Witten invariants). It follows from the stronger condition of geometrical boundedness (see [8]) – the property of being A -good is its complex version – which is easier to check in practice.

3.2. Traditional approach to Fukaya categories

Let now (M, ω) be a C^∞ symplectic manifold of real dimension $2n$. Assume that (M, ω) is A -good and is Calabi–Yau in the symplectic sense, that is, $c_1(T_M) = 0 \in H^2(M, \mathbb{Z})$. Moreover, we need a trivialization of $c_1(T_X)$ on cochain level, for example, a choice of almost-complex structure J and a non-vanishing complex-valued n -form $\text{vol} \in \Gamma(M, \Omega^{n,0})$ (not necessarily closed). More generally, one can choose any non-vanishing complex-valued n -form on Lagrangian subspaces; see [10].

After making such a choice, we will obtain the Fukaya category $\mathcal{F}(X, \omega)$ defined over the Novikov field

$$\text{Nov} := \mathbb{Q}((e^{-\mathbb{R}/\hbar})) := \left\{ \sum_i c_i e^{-A_i/\hbar} \mid c_i \in \mathbb{Q}, A_i \in \mathbb{R}, \lim_i A_i = +\infty \right\}. \quad (3.1)$$

Usual definition (see, e.g., [6]) involves compact oriented Lagrangian submanifolds with spin structure and with the trivialized vanishing of Maslov index $\mu_L \in H^1(L, \mathbb{Z})$ associated with vol . In the lucky case when there is no J -holomorphic discs with boundary on L , such L gives a *canonical* object of $\mathcal{F}(X, \omega)$.

The complexes of morphisms are defined via (transversal) intersection points, and the differential is the usual Floer differential, counting J -holomorphic 2-gons with signs and weights $\exp(-\int_D \omega/\hbar)$.

Fukaya category can be twisted by a B -field with class $\in H^2(M, i\mathbb{R}/2\pi i\mathbb{Z})$ (or, more generally, in $H^2(M, \mathbb{C}^\times)$). In this case, the field of definition is the complex analog of the Novikov field; coefficients c_i in (3.1) are allowed to lie in $\mathbb{C}$. Also, in

the case when $[\omega]$ belongs to the image of $H^2(X, \mathbb{Z})$ in $H^2(X, \mathbb{R})$, one can replace the Novikov field by the usual Laurent series field $\mathbb{Q}((q))$ or $\mathbb{C}((q))$, where q corresponds to $e^{-1/\hbar}$ (equivalently, all areas A_i in (3.1) are integers).

In what follows, I will describe *another* potential approach to Fukaya categories, first introduced in [18]; see also [25].

3.3. Local Fukaya categories

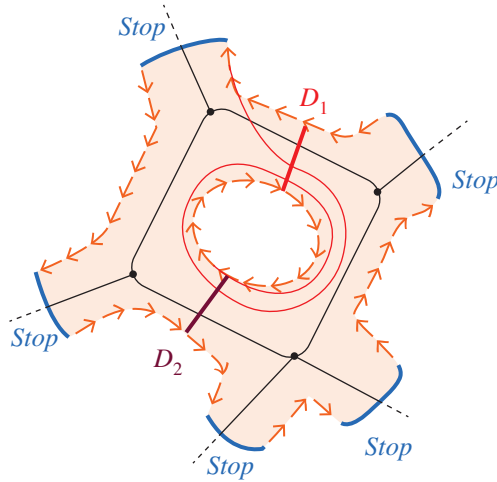
Let us assume that (M, ω) is real analytic, no condition at infinity, with the trivialization of $c_1(T_X)$ on the cochain level, as above. The following is essentially known, but not yet properly articulated.

Claim. With any subanalytic locally closed subset $L \subset M$ which is Lagrangian at smooth points, one can associate a *canonical* small A_∞ -category $\mathcal{WF}^{\text{local}}(L)$ defined over $\mathbb{Z}$ and depending *only* on the germ of (M, ω) at L such that

- set $\text{Ob}(\mathcal{WF}^{\text{local}}(L))$ consists of smooth points $x \in L$ together with a local orientation, local spin structure near X and a local Maslov lift of $\arg(\text{vol}_{T_x L})$ from $\mathbb{R}/2\pi\mathbb{Z}$ to $\mathbb{R}$,
- if L is an embedded open d -dimensional disc, then all objects are isomorphic up to cohomological shift, and the endomorphism A_∞ -algebra of each object is quasi-isomorphic to $\mathbb{Z}$,
- every open embedding $L_1 \hookrightarrow L_2$ induces a functor (tautological on objects) $\mathcal{WF}^{\text{local}}(L_1) \rightarrow \mathcal{WF}^{\text{local}}(L_2)$, so we get a co-presheaf of categories on every locally closed L ,
- this co-presheaf is a co-sheaf for every locally closed L ,
- every closed embedding $L_1 \hookrightarrow L_2$ induces a functor (obvious on objects) $\mathcal{WF}^{\text{local}}(L_2) \rightarrow \mathcal{WF}^{\text{local}}(L_1) \sqcup \{0\}$ which identifies $\mathcal{WF}^{\text{local}}(L_1)$ with the localization of $\mathcal{WF}^{\text{local}}(L_2)$ by objects corresponding to $x \in L_2^{\text{smooth}} - L_1$.

The co-sheaf of small categories on L gives a sheaf $\mathcal{WF}_L^{\text{local}, \vee}$ of large categories by associating with each open $L' \subset L$ the category of triangulated functors from $\mathcal{WF}^{\text{local}}(L')$ to complexes of $\mathbb{Z}$ -modules.

3.3.1. Heuristic picture: Wrapping with stops. Assume that L has a Liouville tubular neighborhood U such that one has a complete Liouville flow generated by vector field η , $\text{Lie}_\eta \omega = -\omega$, contracting U to L . One should interpret an object of $\mathcal{WF}^{\text{local}}(L)$ as a small Lagrangian disc D closed in U , which intersects *transversally* L at a smooth point $\text{pt} \in L$. Local orientation, local spin structure and local Maslov lift of L near x induce similar structures on D . The morphisms in $\mathcal{WF}^{\text{local}}(L)$ are defined via infinite wrapping by a Hamiltonian vector field on U with stops at “infinity” of L ; see [7] for the general theory of wrapping with stops.



The main technical trouble with this heuristic picture is that the existence of a Liouville neighborhood is not known.

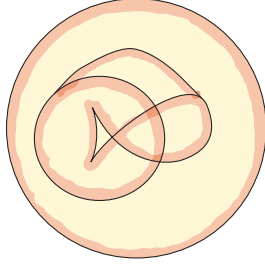
3.3.2. One can calculate $\mathcal{WF}^{\text{local}}(L)$ explicitly. Here we sketch a rigorous alternative definition based on the theory of microsupport. First, we describe sheaf $\mathcal{WF}^{\text{local}}$ locally *up to an equivalence* in the following way. For a given point $\text{pt} \in L$, first choose local Darboux coordinates $(x_i, y_i)_{1 \leq i \leq n}$, $n = \dim M/2$ such that $\omega = \sum_i dx_i \wedge dy_i$. There exists a unique germ of C^0 -function z at $\text{pt} \in L$ which vanishes at pt and whose restriction to any locally closed smooth submanifold $Y \subset L$ near pt is also smooth, and the differential of $z|_Y$ is equal to the restriction of 1-form $\sum_i y_i dx_i$ to Y . Functions $(x_i, y_i)_{1 \leq i \leq n}$ together with function z give an embedding of the germ of L at pt to $\mathbb{R}^{2n+1}$, as a singular Legendrian subset. Theory of microlocal support (see [12]) gives a canonical germ of a sheaf of categories, which in our picture is the sheaf of representations $\mathcal{WF}_{L'}^{\text{local}, \vee}$ for sufficiently small neighborhood L' of $\text{pt} \in L$. Finally, the global choice of the trivialization of $c_1(T_X)$ gives rise to the global co-sheaf of categories on L (see [2] for the discussion of ambiguities of the gluing).

The co-sheaf $\mathcal{WF}^{\text{local}}$ on L is locally constant along some Whitney stratification. The co-stalks are certain nice A_∞ -categories over $\mathbb{Z}$, of finite type in the sense of B. Toën and M. Vaquié; see [28]. Up to derived Morita equivalence, they are localizations of smooth proper categories (in fact, just finite posets) by finite collections of compact objects.

In the case $\dim M = 2$, a singular Lagrangian is just a ribbon graph. The co-stalk at vertex of degree $d \geq 1$ is Morita equivalent to Dynkin quiver A_{d-1} . In general, the co-stalk can be identified with the category of constructible sheaves on $\mathbb{R}^n$ whose

microlocal support is contained in a given closed subanalytic *conical* Lagrangian $L' \subset T^*B^n$ where $B^n \subset \mathbb{R}^n$ is the closed n -dimensional ball.

A typical picture of such a Lagrangian is given below.



3.3.3. Application: RH correspondence for quantum tori. Here we give a sketch of the description of the A -model side of the hypothetical RH correspondence for higher-dimensional quantum algebraic tori, promised in Section 1.5. Let $L \subset \mathbb{R}^{2n}$ be a closed conical subset which is a finite union of rational Lagrangian simplicial cones, for the standard symplectic form $\sum_{i=1}^n dx_i \wedge dy_i$.

Symplectic group $\mathrm{Sp}(2n, \mathbb{Z})$ acts transitively on the set of rational Lagrangian subspaces in $\mathbb{Q}^{2n}$. Similarly, a central extension of $\mathrm{Sp}(2n, \mathbb{Z})$ acts on $\mathrm{Perf}(E^n)$ where E is an elliptic curve, by generalized Fourier–Mukai transforms. With each rational Lagrangian subspace $\Lambda \subset \mathbb{Q}^{2n}$, we associate a full triangulated subcategory $C_\Lambda \subset \mathrm{Perf}(E^n)$. This association is characterized by two properties: (1) it is $\mathrm{Sp}(2n, \mathbb{Z})$ -equivariant, and (2) the category C_{Λ_0} corresponding to the standard Lagrangian subspace Λ_0 spanned by the first n base vectors has the heart given by coherent sheaves with 0-dimensional support on E^n . The latter is invariant under the linear action of $\mathrm{GL}(n, \mathbb{Z})$ on E^n , and the tensor product with line bundles, matching the stabilizer $\mathrm{GL}(n, \mathbb{Z}) \ltimes (\mathbb{Z})^{n(n+1)/2} \subset \mathrm{Sp}(2n, \mathbb{Z})$ of Λ_0 .

Let us consider the category of functors from $\mathcal{WF}^{\mathrm{local}}(L)$ to $\mathrm{Perf}(E^n)$. This category contains a full subcategory consisting of functors F such that when F applied to an object corresponding to a smooth point x with an extra data (local orientation, local spin structure, local Maslov lift) as above, the result lies in the subcategory $C_{T_x L}$ and moreover belongs to a cohomological shift $[k]$ of the heart, where $k \in \mathbb{Z}$ is determined (in certain natural way) by the local Maslov lift of $\arg(\mathrm{vol}_{T_x L})$. The resulting category is an Artinian abelian A_∞ -category. Inclusions of rational Lagrangian conical subset give rise to inclusions of full subcategories. The inductive limit over all $L \subset \mathbb{R}^{2n}$ is (hypothetically) equivalent to the category of holonomic modules over the $2n$ -dimensional quantum torus.

3.4. Hypothetical “correct” definition of Fukaya category

For a symplectic C^ω -manifold (M, ω) geometrically bounded at infinity, with trivialized $c_1(T_M)$, with a tame choice of almost-complex structure J and a compact singular subanalytic Lagrangian $L \subset M$, we expect that J -holomorphic discs with boundary on L give rise to a *canonical* deformation of $\mathcal{WF}^{\text{local}}(L)$ over Novikov ring $\text{Nov}_+ := \mathbb{Q}[[e^{-\mathbb{R}_{\geq 0}/\hbar}]]$. To my knowledge, the general problem of the definition of this canonical deformation was not addressed, except the cases when L is smooth or $n = 2$. In the case when an appropriate Liouville neighborhood exists, it is possible to define the expected deformation using traditional tools of symplectic topology.

Let us define the global Fukaya category *supported* on L (and denoted by $\mathcal{F}_L^{\text{global}}(M, \omega)$), as the category of finite-dimensional over Nov representations of the *deformed* algebra $\mathcal{WF}^{\text{local}}(L)$, admitting a model over Nov_+ (subscript “+” indicates that the areas A_i in (3.1) are ≥ 0). The localization property implies that for $L_1 \subset L_2$ we have a fully faithful embedding $\mathcal{F}_{L_1}^{\text{global}}(M, \omega) \hookrightarrow \mathcal{F}_{L_2}^{\text{global}}(M, \omega)$. The universal Fukaya category $\mathcal{F}^{\text{global}}(M, \omega)$ can be defined as the inductive limit of $\mathcal{F}_L^{\text{global}}(M, \omega)$ over the poset of all subanalytic compact singular Lagrangians $L \subset M$.

Categories $\mathcal{F}_L^{\text{global}}(M, \omega)$ and $\mathcal{F}^{\text{global}}(M, \omega)$ can be modified by a B -field, a closed 2-cochain with coefficients in $U(1) = i\mathbb{R}/2\pi i\mathbb{Z}$. The global category does not change (up to homotopy) under deformations of geometrically bounded almost-complex structure J .

It is expected that Bridgeland stability structures on $\mathcal{F}_L^{\text{global}}(M, \omega)$ appear when one chooses a compatible almost-complex structure J and *closed* complex volume form $\text{vol} \in \Gamma(X, \Omega^{n,0})$. More generally, the constraint on vol to be of type $(n, 0)$ can be weakened; see [10].

3.5. Moduli stack of non-formal Fukaya categories

The Fukaya categories, defined either in the traditional way or via deformations of local Fukaya categories as above, still depend on the small parameter $\hbar$ in a formal way, as a series in the Novikov field, without any control of convergence. It is still not proven (although expected) that the counting of pseudo-holomorphic discs exhibits an exponential growth.

It is well known that local deformations of *compact* symplectic manifolds, up to symplectomorphisms close to identity, are governed by deformations of the cohomology class of symplectic form $[\omega] \in H^2(M, \mathbb{R})$. Adding the cohomology class of B -field $[\beta] \in H^2(M, U(1))$, one arrives to a domain in complex torus $H^2(M, \mathbb{C}^\times)$ (so-called *complexified Kähler cone*). Moreover, one can expect the same picture for *non-compact* symplectic manifolds (M, ω) when the symplectic structure is geometrically bounded at infinity and $\text{rk } H^2(M, \mathbb{Z}) < \infty$ (e.g., M is topologically cylindrical at infinity, as typically happens).

Homological mirror symmetry gives a very strong indication that there should be a “non-formal” Fukaya category associated with (M, ω) and B -field β , at least when the class $[\omega] \in H^2(M, \mathbb{R})$ is “sufficiently large”. We denote this putative category as $\mathcal{F}(M, \omega, \beta)$. It is a category enriched over $\text{Perf}(\mathbb{C})$, and not over the Novikov field.

Let us now restrict ourselves to the case of *compact* X , class $[\omega]$ is integral, and we are in the situation of mirror symmetry, which means roughly that X is fibered by Lagrangian tori, that is, endowed with the structure of a real integrable system. In this case, homological mirror symmetry identifies $\mathcal{F}^{\text{global}}(M, \omega)$ with the category of perfect complexes on the mirror family of complex Calabi–Yau varieties defined over the formal disc $\text{Spf } \mathbb{Z}[[q]]$ where q corresponds to $e^{-1/\hbar}$. Moreover, the formal disc with parameter q embedded at the cusp of the moduli space of mirror varieties is in fact defined analytically by the so-called *inverse mirror map*. For example, in the case of quintic 3-folds and $[\omega] = c_1(\mathcal{O}(1))$, the moduli space of mirrors is $\mathbb{P}_z^1 - \{0, 5^{-5}, \infty\}$. The correspondence between the A -model coordinate q and the B -model coordinate z is

$$q = \exp\left(2\pi i \frac{\int_{\gamma_1} \text{vol}}{\int_{\gamma_0} \text{vol}}\right) = z + 770z^2 + 1014275z^3 + \dots,$$

where vol is the holomorphic volume form on the dual variety $M^\vee$, and $\gamma_0, \gamma_1 \in H_3(M^\vee, \mathbb{Z})$ are certain cycles. The inverse mirror map $z(q) = q - 770q^2 + \dots$ becomes multivalued when $|q|$ is too large, that is, when the rescaled symplectic form $\hbar^{-1}\omega$ is too small. Nevertheless, all this suggests that the Fukaya category is *well defined* as an analytic family of categories over certain domain in the complexified Kähler cone.

In the case of *complex* symplectic manifolds, even non-compact ones, we expect more: the global Fukaya category is well defined even for *arbitrary small* non-zero values of $\hbar$. Usual mirror symmetry predictions do not produce singularities for the inverse mirror map, as all non-trivial Gromov–Witten invariants of complex symplectic manifolds vanish identically. The resulting moduli stack of categories is rather mysterious, as there are, for example, automorphisms which act trivially on cohomology, like squares of spherical twists in the case $\dim_{\mathbb{C}} M = 2$ (case of K3 surfaces).

It was proposed (see [11]) that the “true” A -model category should include objects with “coisotropic support”, of dimension $\in \{n, n+2, n+4, \dots\}$. One possible mathematical model of such an object would be a smooth coisotropic submanifold endowed with a Lagrangian foliation and a transversal complex structure. In the framework of the generalized RH correspondence discussed here, the heart of “true” B -model category, one should be the full subcategory of $\text{Perf}_{P, \gamma}^{\text{heart}}$ consisting of objects which are “vanishing after the restriction to $P - M$ ”, and *omitting* the constraint (see the end of Section 2.3) on classes in topological K -theory.

I have tried to make a guess (based on Hodge-theoretic invariant from Section 1.2) of what parametrizes “true” A -model categories of compact tori $(S^1)^{2n}$ corresponding to translationally invariant symplectic structure and the B -field. The expected answer is

$$\mathcal{U}_n / O(2n, 2n; \mathbb{Z}),$$

where $\mathcal{U}_n$ is the open subset of the isotropic Grassmannian of $\mathbb{C} \otimes (\mathbb{Z})^{2n, 2n}$ consisting of $2n$ -dimensional complex isotropic subspaces of $\mathbb{C}^{4n}$ intersecting trivially $\mathbb{R}^{4n}$. Remarkably, the *same* stack appears in the B -model, corresponding to translationally invariant holomorphic Poisson structures on compact tori, together with $U(1)$ gerbes, up to (potential) derived equivalences of Fourier–Mukai type.

4. A -side: Fukaya categories, complex case

4.1. Holomorphic symplectic manifolds, Lagrangian subsets

Let $M = M_{\mathbb{C}}$ be a complex-analytic manifold with a holomorphic closed non-degenerate 2-form $\omega = \omega_{\mathbb{C}}$. Then for any $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, we have a *real-analytic* symplectic manifold

$$(M, \omega_{\theta}), \quad \omega_{\theta} := \Re(e^{-i\theta} \omega_{\mathbb{C}}) = \cos \theta \cdot \Re \omega_{\mathbb{C}} + \sin \theta \cdot \Im \omega_{\mathbb{C}}.$$

Let us assume that $(M, \omega_{\mathbb{C}})$ is geometrically bounded at infinity, and let us choose a geometrically bounded almost-hyperkähler structure on M such that $\Re \omega_{\mathbb{C}}$ and $\Im \omega_{\mathbb{C}}$ are the first two standard symplectic forms among three $(\omega_I, \omega_J, \omega_K)$ (the space of such choices is contractible). Then for any θ , we get an ω_{θ} -compatible almost-complex structure J_{θ} .

Let $L \subset M$ be a compact closed $\mathbb{C}$ -analytic singular Lagrangian subset. Then L is a singular ω_{θ} -Lagrangian for any $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, with the canonical orientation and Maslov lift. Small categories $\mathcal{WF}_{\theta}^{\text{local}}(L)$ form a *local system* over $\mathbb{R}/2\pi\mathbb{Z}$, with the monodromy functor identical on objects. Let us pick a point on each connected component of L^{smooth} and consider the endomorphism algebra $\mathcal{A}_{L, \theta}$ of the formal sum of corresponding objects of $\mathcal{WF}_{\theta}^{\text{local}}(L)$. The monodromy along varying $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ gives an automorphism ψ_L of $\mathcal{A}_L := \mathcal{A}_{L, 0}$. Presumably, $\mathcal{A}_L$ is the path algebra of finite dg quiver defined over $\mathbb{Z}$ such that all the arrows have *non-positive* degrees. The category of finite-dimensional complex representations of $\mathcal{A}_L$ carries a canonical bounded t -structure, with the heart consisting of representations in degree 0. This is the perverse t -structure described in [2]. In the case when L is smooth, connected and has a trivial spin structure, in the sense $c_1(T_L) \in H^2(L, \mathbb{Z})$ is divisible by 2, algebra $\mathcal{A}_L$ is isomorphic to the algebra of $\mathbb{Z}$ -chains of the based loop monoid of P , and the automorphism ψ_L is trivial.

Algebras $\mathcal{A}_{L,\theta}$ are naturally graded by the abelian group $H_1(L, \mathbb{Z})$. The reason is that the algebraic group $H^1(L, \mathbb{G}_m)$ acts on the category of all (not necessarily finite-dimensional) representations of $\mathcal{A}_{L,\theta}$, by twisting in the sheaf-theoretic description (this remark is also applicable to local Fukaya categories in the general real case).

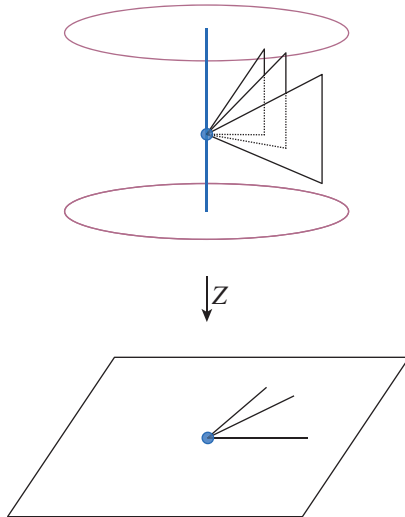
Define a finite rank lattice $\Gamma := H_2(M, L; \mathbb{Z})/\text{torsion}$, and denote by $Z : \Gamma \rightarrow \mathbb{C}$ the linear map (called central charge) given by the integration of $\omega_{\mathbb{C}}$.

Key observation. *For any J_θ -holomorphic disc D with boundary on L , one has $Z([D]) \in e^{i\theta} \mathbb{R}_{\geq 0}$.*

Indeed, the restriction of $\omega_{\mathbb{C}}$ to any non-trivial J_θ -holomorphic curve has argument θ at every point. Hence for $\theta \notin \{\arg Z(\gamma) \mid \gamma \in \Gamma, Z(\gamma) \neq 0\}$ (i.e., $e^{i\theta} \mathbb{R}_{\geq 0}$ is *not a Stokes ray*), there are no J_θ -holomorphic discs with boundary on L , and hence $\mathcal{WF}_\theta^{\text{local}}(L)$ is *not deformed*.

4.2. Complex Lagrangians and Stokes automorphisms

Staying (infinitesimally) a little bit on the left, or on the right of a given Stokes ray, we obtain *two* fully faithful embeddings of the local Fukaya category into the global one. These two embeddings coincide at the leading order (ignoring non-trivial disc corrections), hence should differ by certain *autoequivalence* $A_{\theta, \text{cat}}$ of $\mathcal{WF}_\theta^{\text{local}}(L)$ which is equal to identity in the leading order. Naturally, the coefficients for the deviation from identity should correspond to non-trivial J_θ -holomorphic discs. It follows from the uniform bounds on almost-hyperkähler structure that the possible homology classes $\gamma \in \Gamma$ of these discs belong to some strict closed convex cones $\text{Cone}_\theta \subset Z_{\mathbb{R}}^{-1} e^{i\theta} \mathbb{R}_{\geq 0} \subset \Gamma \otimes \mathbb{R}$ which are *uniformly* separated from $\ker Z_{\mathbb{R}}$.



Categories $\mathcal{WF}_\theta^{\text{local}}(L)$ and categorical Stokes autoequivalences $A_{\theta, \text{cat}}$ do not depend on the auxiliary choice of almost-complex structures J_θ as above.

The Stokes autoequivalence comes from an automorphism A_θ of the dg path algebra $\mathcal{A}_{L, \theta}$ close to the identity, and hence of the form $\exp(\delta)$ where δ is a derivation of $\mathcal{A}_{L, \theta}$ with coefficients in the non-unital semigroup algebra $\mathbb{Q}[\text{Cone}_\theta \cap \Gamma - 0]$. For each monomial $\gamma \in \text{Cone}_\theta \cap \Gamma - 0$, the γ -coefficient of δ is expected to be an element of $\text{Der}(\mathcal{A}_{L, \theta})$ which is homogeneous of degree $\partial\gamma \in H_1(L, \mathbb{Z})/\text{torsion}$ where $\partial : H_2(M, L; \mathbb{Z}) \rightarrow H_1(L, \mathbb{Z})$ is the boundary map, and $H_1(L, \mathbb{Z})$ -grading on $\text{Der}(\mathcal{A}_{L, \theta})$ is induced from the $H_1(L, \mathbb{Z})$ -grading on $\mathcal{A}_{L, \theta}$ described above in Section 4.1.

4.2.1. Analytic wall-crossing structures.

- The collection of automorphisms (A_θ) associated with Stokes rays can be interpreted as a *stability structure* (see [20]) for certain local system on S^1 of countable-dimensional Γ -graded Lie algebras:

$$\mathfrak{g}_\theta = \bigoplus_{\gamma \in \Gamma} \mathfrak{g}_{\theta, \gamma}, \quad \mathfrak{g}_{\theta, \gamma} := \text{Der}_{\partial\gamma}(\mathcal{A}_{L, \theta}).$$

- If we vary parameters (i.e., $M_{\mathbb{C}}, \omega_{\mathbb{C}}$ and L as well) preserving the equivalence class of $\mathcal{WF}_{\theta=0}^{\text{local}}(L)$ (it is a structure of a “combinatorial” nature, no continuous parameters), then the stability structure varies in a universal way (*wall-crossing* equations) controlled *only* by the varying homomorphism $Z \in \text{Hom}(\Gamma, \mathbb{C})$.
- It looks that one can “understand” all (A_θ) at once, for example, in some examples all information is encoded by finitely many non-commutative cluster transformations in free variables.

4.3. Generalization: Compatible pairs of symplectic structures

It is not even necessary to work only with complex symplectic manifolds; the whole theory makes sense in a much larger generality of *pairs* of real symplectic structures such that any non-trivial linear combination of corresponding 2-forms is non-degenerate; see [21, Section 4.6]. Compact singular subset L should be Lagrangian with respect to both structures. The importance of this generalization is that the parameter space (possible pairs of cohomology classes of symplectic forms) is locally an open domain in $H^2(M, L; \mathbb{C})$, and it is in general *strictly larger* than the image in $H^2(M, L; \mathbb{C})$ of the parameter space of complex symplectic manifolds with analytic Lagrangian subsets.

A particularly interesting class is of triples (M, ω, L) such that the cohomology class of ω belongs to $H^2(M, L; \mathbb{Q} \oplus i\mathbb{Q})$. There are only *countably many* isotopy classes of such triples, producing therefore countably many automorphisms

A_θ , for example, only for $\theta = 0$, which control *all* wall-crossing structures in GRH correspondence.

All the above deals exclusively with the A -model.

5. Local RH correspondence, and a generalization to fractional powers of $\hbar$

One can associate with the pair $(\mathcal{A}_L, \psi_L)$ a new $\mathbb{C}((\hbar))$ -linear category (see [21, Section 5.5]). Namely, introduce the universal ring with an automorphism

$$\mathbf{B} := \lim_{N \rightarrow +\infty} \mathbb{C}((\hbar^{1/N}))[\log \hbar], \quad \psi_{\mathbf{B}}(\hbar^{1/N}) = e^{2\pi i/N} \hbar^{1/N},$$

$$\psi_{\mathbf{B}}(\log \hbar) = \log \hbar + 2\pi i$$

and consider representations of $\mathcal{A}$ in $\text{Perf}(\mathbf{B})$ that admit a model over $\mathbf{B}_+$ and are equivariant with respect to $(\psi_L, \psi_{\mathbf{B}})$. The resulting category is linear over $\mathbf{B}^{\psi_{\mathbf{B}}} = \mathbb{C}((\hbar))$. In the case of smooth L and hence trivial ψ_L , we essentially obtain the category of families over $\mathbb{C}[[\hbar]]$ of finite-dimensional representations of $\mathcal{A}_L$, with inverted $\hbar$.

We have made a conjecture (*local RH correspondence*) in [21] that this category is equivalent to certain (not specified) full subcategory of the category of holonomic modules over the *formal* deformation quantization of (M, ω) , after inversion of $\hbar$. For example, in the case when L is smooth, the relevant category consists of modules which in local coordinates are isomorphic to finite sums of copies of

$$i_* \mathcal{O}_{\mathbb{C}^n}[[\hbar]], \quad \text{where } i : \mathbb{C}^n \rightarrow \mathbb{C}^{2n} \text{ is } (x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 0, \dots, 0).$$

Here the standard coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$ in $\mathbb{C}^{2n}$ endowed with the Moyal product (2.1) act as actions by operators x_i and $\hbar \partial_{x_i}$ (see [13]).

5.1. Fractional powers of $\hbar$ and enhanced Lagrangian subsets

The above discussed hypothetical local RH correspondence *does not* give the whole category of holonomic modules supported on L (up to $\hbar$ -torsion modules), even in the case of smooth L . The basic example is (in the above notations)

$$e^{\sum_{k=1}^{N-1} \hbar^{-k/N} f_k(x_1, \dots, x_n)} j_* \mathcal{O}_{\mathbb{C}^n}[[\hbar^{1/N}]],$$

where $N \geq 1$ is an integer and $f_1, \dots, f_N$ are germs at $0 \in \mathbb{C}^n$ of analytic functions (here $j : \mathbb{C}^n \rightarrow \mathbb{C}^{2n}$ is the coordinate embedding). It is a holonomic $\mathcal{O}_{q,M}$ -module, with the Lagrangian support given by $y_i = \partial_{x_i} f_N$, $i = 1, \dots, n$.

The appearance of fractional powers of $\hbar$ opens a new can of worms! We (Y. Soibelman and I) are trying now to figure out a precise definition of *enhanced singular Lagrangian subsets* controlling geometrically WKB-like modules with fractional powers of $\hbar$ as above. Heuristically, one should think of a family of singular Lagrangian subsets $L_{\hbar}$ depending on formal or small analytic parameter $\hbar$, converging at $\hbar \rightarrow 0$ to the support in the usual sheaf-theoretic sense. Hypothetically, by adding all possible enhanced singular Lagrangian subsets, both in A -model and B -model versions, we expect to get a full local RH correspondence between *all* holonomic formal $\mathcal{O}_{q,M}$ -modules after inversion of $\hbar$, and $\mathbb{C}((\hbar))$ -linear categories associated with dg algebras endowed with automorphism, as in the beginning of the current section.

Finally, we observe that the picture with wall-crossing structures should generalize to the enhanced case. The lattice $\Gamma = H_1(M, L, \mathbb{Z})/\text{torsion}$ should be replaced by the local system of lattices $\Gamma_{\theta} = H_1(M, L, \mathbb{Z})/\text{torsion}$ on S_{θ}^1 , and the central charge $Z : \Gamma \rightarrow \mathbb{C}$ should be replaced by the local system of central charges with values in $\mathbb{C}[\hbar^{1/N}]/\hbar \mathbb{C}[\hbar^{1/N}]$.

6. Reconstruction of global categories

- We have a local system on $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ of categories of finite-dimensional modules over $\mathcal{WF}_{\theta}^{\text{local}}(L)$.
- Using homotopy equivalence $S^1 \sim \mathbb{C} - 0$, we obtain a local system of categories on $\mathbb{C} - 0$.
- The putative local RH correspondence should give an extension of this family to an analytic family over $\mathbb{C}$ such that over $\mathbb{C}[[\hbar]]$ we get the usual formal deformation quantization.
- Wall-crossing formalism, together with certain assumption of *analyticity* (purely A -model statement), allows us to modify the obtained family over $\mathbb{C}$ (without changing anything over $\mathbb{C}[[\hbar]]$) by making finitely many cuts in $\mathbb{C}_{\hbar}$ (see [20]) and modifying the bundle along cuts by analytic transformations in sectors which have trivial asymptotic expansion at $\hbar = 0$ (like function $e^{-1/\hbar}$ in sector $|\arg(\hbar)| \leq \theta_0$ for any $\theta_0 \in (0, \pi)$).
- The resulting family should coincide with an open part of family $\mathcal{C}_{B,\hbar}(M, \omega)$.

Alternatively, one can formulate a version of the hypothetical global RH correspondence without mentioning local RH correspondence at all. Namely, consider the inductive limit over all enhanced compact singular Lagrangians in M of the categories enriched over $\text{Perf } R$ where $R = \mathbb{C}\{\hbar\}[\hbar^{-1}]$ is the field of germs of

meromorphic functions. The categories under consideration are the ones obtained by modifications along cuts as above. Then the resulting category should be canonically equivalent to the localization (by inverting $\hbar$) of $\mathbb{C}\{\hbar\}$ -linear category associated with the germ at $\hbar = 0$ of the analytic family $C_{B,\hbar}(M, \omega)$.

6.1. Application to resurgence

In [19], we made a Conjecture 4.2.1, which was formulated for analytic stability structure in the case of a finite rank lattice Γ with trivial automorphism, and a central charge $Z : \Gamma \rightarrow \mathbb{C}$. In the framework of GRH correspondence, it is the case when ψ_L is trivial (holds for smooth L). Here I present a slight generalization of this conjecture.

Input.

- a complex-analytic space/stack Y ,
- integer $N \geq 0$,
- a finite collection of rays $r_j = e^{i\theta_j} \cdot \mathbb{R}_{\geq 0}$ in $\mathbb{C}_{\hbar}$,
- for each ray r_j , a germ f_j at $Y \times \{0\} \subset Y \times \mathbb{C}^N$ of a map to Y , which coincides with id_Y on $Y \times 0$,
- a finite collection of numbers $z_{i,j} \in \mathbb{C} - 0$ for $1 \leq i \leq N$ such that $|\arg z_{i,j} - \theta_j| < \pi/2$.

Then one modifies the trivial bundle $Y \times \mathbb{C}_{\hbar} \rightarrow \mathbb{C}_{\hbar}$ by cutting at rays r_j and re-gluing for small $\hbar \in r_j$ by the map

$$(y, \hbar) \mapsto (f_j(y, e^{-z_{1,j}/\hbar}, \dots, e^{-z_{N,j}/\hbar}), \hbar).$$

The resulting new analytic germ of bundle is trivialized over $\mathbb{C}[[\hbar]] \Rightarrow$ for a holomorphic germ s of section of the modified bundle, and for a germ of analytic function F on Y at $s(0)$, one get a formal series $g = s^* F \in \mathbb{C}[[\hbar]]$.

Conjecture. Series $g = \sum_{n \geq 0} g_n \hbar^n$ is *resurgent*, that is, $\sum_{n \geq 0} g_n / n! \cdot \zeta^n$ admits endless analytic continuation.

This conjecture will imply resurgence of WKB series for solutions of linear differential equations depending on small parameter $\hbar$.

In the more general case of a local system of lattices and central charges with values in fractional powers of $\hbar$, one expect to get certain (not yet developed) generalization of the notion of resurgence.

6.2. Painlevé non-linear flat connection

Categories $C_{A,\hbar}(M, \omega)$ for $\hbar \neq 0$ should (locally) depend *only* on the cohomology class $\hbar^{-1}[\omega] \in H^2(M, \mathbb{C})$, as it is a general feature of Fukaya categories. Therefore,

the same should be true for categories $C_{B,\hbar}(M, \omega)$. The latter is defined via a Poisson compactification.

Assume that we have a holomorphic family of compact Poisson manifolds $(P_s, \gamma_s)_{s \in S}$ where S is an analytic space such that each P_s has an open dense symplectic leaf M_s such that topologically M_s form a locally trivial family and the cohomology class of the symplectic form $\omega_s := (\gamma_s)^{-1}_{|M_s}$ stays *constant* in s . Then the family of categories $C_{B,\hbar}(M_s, \omega_s)$ should also be analytically locally constant in s , by the GRH correspondence.

As a concrete example, consider a family of compact curves C_s , with P_s being the standard compactification of $M_s := T^*C_s$ obtained by adding a copy of C_s at infinity. Then we get a family over s of moduli stacks of vector bundles with (automatically flat) connections on C_s . Classical Riemann–Hilbert correspondence identifies these stacks analytically with the moduli stacks of $\mathbb{C}$ -local systems, which stay locally constant as we deform $s \in S$. Algebraically, we get an algebraic non-linear flat connection on the bundle of algebraic de Rham moduli stacks over S . In the case of SL_2 bundles over $\mathbb{C}P^1$ without 4 points, we get classical Painlevé equations.

6.3. Twistor family, harmonic objects

Let us recapitulate: generalized RH correspondence deals with the family over $\mathbb{C}_\hbar^\times - 0$ of Fukaya categories where [the class of real symplectic form] + [the class of B -field] gives an element of $H^2(M, \mathbb{C}/2\pi i\mathbb{Z})$ depending holomorphically on $\hbar$ in the following way:

$$\hbar \in \mathbb{C}^\times \mapsto \hbar^{-1}[\omega^{2,0}] + \beta_0, \quad \text{for a given } \beta_0 \in H^2(M, i\mathbb{R}/2\pi i\mathbb{Z}).$$

The extension of this family to $\hbar = 0$ uses local RH correspondence and formal deformation quantization. The special fiber should coincide with the category of semistable objects in $\text{Perf}_c(M)$ of slope 0 for certain stability condition (“Higgs bundles”).

It seems natural to consider another family of categories, where the dependence on $\hbar$ of the total class in $H^2(M, \mathbb{C}/2\pi i\mathbb{Z})$ is slightly different:

$$\hbar \in \mathbb{C}^\times \mapsto \hbar^{-1}[\omega^{2,0}] + \hbar[\overline{\omega^{2,0}}] + \beta_0, \quad \beta_0 \in H^2(M, i\mathbb{R}/2\pi i\mathbb{Z}).$$

In this way we get another holomorphic in $\hbar \in \mathbb{C}P^1$ family of categories $(C_\hbar)$ such that $C_{-1/\hbar} \simeq \overline{C_\hbar}^{\text{opp}}$, a *twistor family*. The categories for $\hbar \neq 0$ should be defined in general as Fukaya categories (or B -model categories for appropriately chosen complex symplectic manifolds with B -fields, up to the ambiguity of Painlevé type from the previous section).

One can try to generalize Deligne–Simpson picture of harmonic representatives to such families. Hypothetically, *all* categories from the twistor family are equivalent to each other as abstract A_∞ -categories, but the equivalence is not holomorphic in

the moduli stacks of objects. T. Mochizuki established it for holonomic D -modules in arbitrary dimensions (see the review [27]). In [22], in the case $n = \dim_{\mathbb{C}} M/2 = 1$, we worked out the relation between harmonic objects and periodic monopoles, based on Fukaya-theoretic considerations. Recently, T. Mochizuki proved encouraging results, see [23, 24], confirming our predictions.

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