

Large-scale behavior of out-of-equilibrium systems at and above the critical dimension

Giuseppe Cannizzaro and Fabio Toninelli

Abstract. The mathematical understanding of out-of-equilibrium, statistical physics systems at and above their critical space dimension has witnessed spectacular progress in the last few years. These systems include, for instance, driven diffusive systems (such as stochastic interacting particle systems with asymmetric jump rates) in dimension $d \geq 2$, stochastic $(d + 1)$ -dimensional growth models with $d \geq 2$, non-linear singular (super-)critical stochastic partial differential equations (SPDEs) and (self-)interacting diffusions. We give here a review of recent results on their large-scale (i.e., infrared) behavior, focusing on singular SPDEs at and above the so-called critical dimension (in particular, the stochastic Burgers equation (SBE) in space dimension $d \geq 2$ and the anisotropic KPZ (AKPZ) equation in $d = 2$) as well as on critical diffusions in random divergence-free vector field and self-repelling diffusions. A highlight of the main results includes sharp bounds on the large-time logarithmic divergence of the diffusion coefficient at the critical dimension, and Gaussian infrared scaling limits, at both the critical and super-critical dimensions.

Most of the results presented here have been obtained in collaboration with Dirk Erhard, Harry Giles, Massimiliano Gubinelli, Levi Haunschmid-Sibitz and Quentin Moulard, in different combinations.

1. Introduction

Out-of-equilibrium physical systems (driven diffusive systems, stochastic growth, etc.) are usually mathematically described by Markov processes, either of discrete type (interacting particle systems on a lattice, discrete deposition processes, ...) or of continuum nature, such as stochastic partial differential equations (SPDEs) or stochastic differential equations (SDEs). In particular, the relevant SPDEs (and SDEs) in this context turn out to be *non-linear and singular*, in that the expression derived in the physics literature is ill posed and some form of renormalization is needed in

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order to say something meaningful about them. Thanks to the groundbreaking theory of regularity structures [19], paracontrolled calculus [18] or renormalization group techniques [21], the problems of *renormalization* as well as *small-scale existence, uniqueness and regularity* have been settled for virtually any SPDE in the so-called subcritical (in the sense of scaling, as explained below) dimensions (for KPZ, such dimension is $d = 1$). In contrast, in the critical and super-critical dimensions, a local solution theory is not even expected, and the most compelling question concerning their large-scale behavior is, from a mathematical viewpoint, much less explored and still poorly understood.

This is precisely what we have been studying in a series of works in collaboration with Dirk Erhard, Massimiliano Gubinelli and Quentin Mouillard (in various combinations). For a class of singular (super-)critical SPDEs, we were able to (1) obtain the large-time asymptotics for the diffusion coefficient, (2) precisely identify the logarithmic corrections to scaling at the critical dimension which were conjectured in the physics literature and (3) derive their large-scale (infrared) behavior under the correct scaling. As detailed below, the class of equations to which our approach applies includes the two-dimensional anisotropic KPZ equation [5, 34], the stochastic Burgers equation in dimension $d \geq 2$ [32] and the incompressible stochastic Navier–Stokes equation with divergence-free noise in dimension $d = 2$. Moreover, the methods we established have proven versatile enough to apply to critical, singular SDEs, such as the diffusion in the curl of the $2d$ -Gaussian free field (a toy model for a particle in a turbulent fluid), as well as the self-repelling Brownian polymer [31]. In collaboration with Levi Haunschmid-Sibitz [11] and Harry Giles [8], respectively, substantial progress has been made in the analysis of these models, and our results have later inspired several groups of other researchers (see below for references).

The aim of this short note is to give an account of these recent developments. We keep definitions and formulations of theorems somewhat informal, and refer to the original works for precise statements and details. For the methodology and the proofs, we refer the reader to the lecture notes [13], where the mathematical ideas and key points of the arguments are explained.

Organization. In Section 2, we introduce the main mathematical models and the questions we are interested in, while in the later sections, we give a panorama of the recent mathematical results. We start the presentation of new results with the most interesting case of the critical dimension $d = 2$: large-time behavior of the diffusion matrix (Section 3), scaling limit in the weak coupling regime (Section 3.2), full scaling limit in the strong coupling regime for the stochastic Burgers equation (Section 3.3). Section 4 is instead dedicated to scaling limits of the stochastic Burgers equation in the super-critical dimensions $d \geq 3$. Finally, Section 5 discusses perspectives and open problems.

2. The models

In this section, we present the main mathematical models. Each of them corresponds to a different out-of-equilibrium irreversible physical phenomenon.

2.1. Diffusive systems with one conserved quantity: Asymmetric simple exclusion process and stochastic Burgers equation

A typical physical example of driven diffusive system is a gas or fluid of interacting particles, subject to an external field (e.g., an electric field) that induces a stationary current (e.g., an electric current).

There are several ways of modeling such system mathematically. A classical one is in terms of stochastically interacting particles on a lattice. Each vertex of $\mathbb{Z}^d$ can be either empty or occupied by a single particle. Each particle, with rate 1, attempts a jump: if the present particle position is x , it jumps to y with probability $p(y - x)$, where $z \mapsto p(z)$ is a probability measure on $\mathbb{Z}^d$; if the vertex y is already occupied, then the jump is suppressed. The stochastic process just described is a continuous-time Markov process that goes under the name of exclusion process [25], which is further dubbed “Simple” if $p(x - y)$ is non-zero only when $x - y$ has length 1 and “Symmetric” if p is even symmetric ($p(x) = p(-x)$) so that, in particular, $v := \sum_{x \in \mathbb{Z}^d} xp(x) = 0$. Here, we are interested instead in the case in which $v \neq 0$, that is, in the *asymmetric simple* exclusion process (ASEP). The vector v , which we write as $v = \lambda w$ with w a vector of norm 1 and $\lambda = |v| > 0$, quantifies the asymmetry of p and mimics the external driving field. We note that the particle number is a locally conserved quantity (particles jump but are neither created nor annihilated) and, in general, it is the only one.

A more mesoscopic, continuum modelization of driven diffusive systems with a single conserved quantity was introduced in [32]. If for the exclusion process a state of the system consists in a function $\eta : \mathbb{Z}^d \rightarrow \{0, 1\}$ with $\eta(x)$ assessing whether vertex x is occupied by a particle or not, in the continuum model the particle density profile is described by a real-valued function u defined on $\mathbb{R}^d$ instead of $\mathbb{Z}^d$. More precisely, $u(t, x)$ represents the fluctuation at point x and time t of the locally conserved quantity (say, the mass) with respect to its average, and therefore it can be either positive or negative. The function u solves the so-called stochastic Burgers equation (SBE), that is, the stochastic partial differential equation (SPDE)

$$\partial_t u = \frac{1}{2} \Delta u + \lambda(w \cdot \nabla)u^2 + \nabla \cdot \xi, \quad (2.1)$$

where

- $u = u(t, x)$ for $t \in \mathbb{R}^+$ and $x \in \mathbb{R}^d$ (alternatively, one can define the equation on the d -dimensional torus),

- $\xi = (\xi_1, \dots, \xi_d)$ is a collection of d independent space-time white noises, that is, ξ is a centered vector-valued Gaussian field indexed by space and time, with covariance formally given by $\mathbb{E}(\xi_i(t, x)\xi_j(s, y)) = \mathbf{1}_{i=j}\delta(t-s)\delta(x-y)$, $s, t \in \mathbb{R}$, $x, y \in \mathbb{R}^d$, $1 \leq i, j \leq d$,
- w is a unit vector in $\mathbb{R}^d$ that represents the direction of the external driving field and $\lambda > 0$ is its intensity. By rotation invariance, the specific choice of w is irrelevant.

Note that u is locally conserved, as is the case for the particle number of ASEP. Local conservation means that the time derivative of the integral of u on a domain $V \subset \mathbb{R}^d$ equals the flux of a (stochastic) current j across ∂V . In formulas, this is equivalent to saying that all terms in the right-hand side of (2.1) are in divergence form: in fact, the equation can be equivalently written as

$$\partial_t u = \nabla \cdot j, \quad j = \frac{1}{2} \nabla u + \lambda w u^2 + \xi, \quad (2.2)$$

and j plays the role of the current.

Apart from local conservation, the SBE retains also other characterizing mathematical features of the interacting particle system: the Laplacian is a diffusive term that corresponds to the symmetric part of the jump kernel p , the non-linear term proportional to λ mimics the asymmetric part of the jump kernel and the noise is taken to be white both in space, because randomness in a realistic particle system is expected to have short-range correlations, and in time, as we expect fluctuations to be Markovian. The constant λ plays the role of a “coupling constant”. When $\lambda = 0$, the equation is linear and, assuming that the initial condition $u(t = 0)$ is Gaussian, the whole space-time process is Gaussian and therefore uniquely determined by its covariance (as it is easy to see that u has expectation zero).

Let us point out two other important properties of (2.1):

- The law $\mathbb{P}$ of the spatial white noise is formally stationary. That is, if the initial condition $u(t = 0)$ is the centered Gaussian field η with covariance $\mathbb{E}(\eta(x)\eta(y)) = \delta(x - y)$, then $u(t)$ has the same law $\mathbb{P}$ at every given time $t > 0$. This is the continuum analog of the well-known fact that, for ASEP, the product iid Bernoulli measure $\bigotimes_{x \in \mathbb{Z}^d} \text{Ber}(\rho)$ is stationary for any choice of $\rho \in [0, 1]$. In fact, the continuum limit (in an appropriate sense, and after centering) of $\bigotimes_{x \in \mathbb{Z}^d} \text{Ber}(\rho)$ is given by η .
- Under diffusive scaling, the equation maintains the same form, but for the coefficient of the non-linearity. That is, let $\varepsilon > 0$ and define the diffusively rescaled solution

$$u^\varepsilon(t, x) := \varepsilon^{-d/2} u\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right). \quad (2.3)$$

Then, u^ε formally satisfies

$$\partial_t u^\varepsilon = \frac{1}{2} \Delta u^\varepsilon + \lambda \varepsilon^{(d-2)/2} (\mathbf{w} \cdot \nabla) (u^\varepsilon)^2 + \nabla \cdot \xi^\varepsilon, \quad (2.4)$$

where the noise ξ^ε has the same law as ξ .

We will see later that this scaling argument is misleading in dimension $d = 2$, since it suggests that the $2d$ -SBE is diffusively scale invariant, which turns out to be false. However, an important piece of information that we gain from (2.4) is that dimension $d = 2$ should play a special (critical) role: in fact, looking at large scales, which corresponds to $\varepsilon \rightarrow 0$, the prefactor of the non-linearity diverges for $d < 2$ (subcritical) and vanishes for $d > 2$ (super-critical).

The SPDE (2.1) cannot be taken literally, since it is analytically ill posed. Indeed, regularity-wise, the noise ξ is extremely rough and thus u is at best a generalized function, in which case it is unclear what meaning should be attributed to the non-linear term u^2 . As we are interested in large scales, it is reasonable to regularize the equation thus obtaining a well-defined microscopic object. A classical way to proceed is to smooth the noise in space, for instance, by convolving it (with respect to the space variable) with a C^∞ , compactly supported function. Another option is to leave the noise unchanged and regularize the non-linearity. It turns out to be convenient to do both at the same time: that is, one replaces

$$u^2 \rightarrow \rho^{*2} * u^2 = \rho * \rho * u^2 \quad (2.5)$$

and

$$\xi \rightarrow \rho * \xi =: \xi^1, \quad (2.6)$$

where $*$ denotes convolution in space and ρ is a rotationally invariant, compactly supported C^∞ function with total mass equal to 1. The equation then becomes

$$\partial_t u = \frac{1}{2} \Delta u + \lambda \mathcal{N}^1(u) + \nabla \cdot \xi^1 \quad (2.7)$$

with

$$\mathcal{N}^1(u) := (\mathbf{w} \cdot \nabla)(\rho^{*2} * u^2). \quad (2.8)$$

Since the noise is now smooth in space, the equation admits a unique classical global-in-time solution [12], which, as we left the noise white in time, is still Markovian. The reason for this specific choice of regularization is that, as one can explicitly check, the law of the space white noise convolved with ρ is stationary. That is, if $\mathbb{P}^\rho$ denotes the law of $\rho * \eta$ with η a space white noise, then $\mathbb{P}^\rho$ is a stationary probability measure for the Markov process u .

Both for ASEP and for the SBE in (2.7), our goal is to identify their behavior on large space-time scales and, more precisely, to see if the process has a non-trivial scaling limit when space and time are suitably rescaled so that one looks at $t, |x| \rightarrow \infty$, which, as we will see, also corresponds to removing the regularization ρ .

To do so, we first need to understand what is the correct rate at which we should “zoom out”, that is, what is the right scaling under which the universal space-time fluctuations become visible. A convenient observable in this respect is the so-called diffusion matrix. Let S be the two-point correlation function, that is, the map on $\mathbb{R}^+ \times \mathbb{R}^2$ given by

$$S(t, x) := E(u(t, x)u(0, 0)), \quad (2.9)$$

where E denotes the expectation with respect to the noise ξ^1 and to the initial condition $u(0)$, sampled from the stationary measure $\mathbb{P}^\rho$. Then, the so-called “diffusion matrix” $D(t)$ is defined as the $d \times d$ symmetric matrix

$$D_{ij}(t) = \frac{1}{t} \int_{\mathbb{R}^d} x_i x_j (S(t, x) - S(0, x)) dx, \quad (2.10)$$

where the integral at the right-hand side converges since, as proven in [12] for $d = 2$ (but the proof works in any dimension), for every fixed time t , the function $x \mapsto S(t, x)$ decays faster than any inverse power for $|x| \rightarrow \infty$.

Intuitively, if $e \in \mathbb{R}^d$ is a unit vector, one can interpret $\langle e, D(t)e \rangle$ as the ratio $(L_e(t))^2/t$, with $L_e(t)$ the time-dependent correlation length in the direction e at time t , defined as the distance beyond which the correlations in direction e are negligible.

For the linear equation with $\lambda = 0$, it is immediately checked that

$$S(t, x)|_{\lambda=0} = (\rho^{*2} * \mathcal{P}_t)(x), \quad (2.11)$$

with $\mathcal{P}_t$ the heat semigroup, and therefore $D(t)|_{\lambda=0}$ is the $d \times d$ identity matrix. As a consequence, if for large t the diffusion matrix of the SBE (with $\lambda \neq 0$) has $O(1)$ entries, then its large-scale behavior is said to be diffusive; if it diverges with t , it is superdiffusive; and if it tends to zero, subdiffusive. The formal scaling argument (2.4) seems to suggest that the SBE is superdiffusive in dimension $d = 1$, diffusive in dimension $d > 2$ and possibly diffusive also in dimension $d = 2$. We will see that, while the first two predictions are correct, the third is not.

2.2. Stochastic $(2 + 1)$ -dimensional interface growth: The KPZ and AKPZ equations

Another intrinsically out-of-equilibrium physical phenomenon is *stochastic interface growth* [2], that occurs in many situations in nature, from crystal growth to the evolution of fire fronts. Note that, if the ambient physical space has dimension D , the

interface between two regions is a $d = (D - 1)$ -dimensional surface. We usually speak of $(d + 1)$ -dimensional growth models, where d is the geometric dimension of the interface and “+1” refers to the time variable. From the physical point of view, the most interesting cases are $d = 1$ (one-dimensional interfaces in two-dimensional physical space) or $d = 2$, but from the mathematical perspective, any dimension $d \geq 1$ has its own relevance.

As it is the case for driven diffusive systems, stochastic interface growth is usually modeled via Markov processes, either discrete (e.g., the corner growth model, ballistic deposition, polynuclear growth, ...) or continuous. Here, we focus only on the latter, whose canonical example is an SPDE, the celebrated Kardar–Parisi–Zhang (KPZ) equation [20]. In its general form, it can be (formally) written as

$$\partial_t h = \frac{1}{2} \Delta h + \lambda \langle \nabla h, Q \nabla h \rangle + \xi \quad (2.12)$$

with $h = h(t, x)$, $t \geq 0$, $x \in \mathbb{R}^d$, ξ is a space-time white noise, λ is a real parameter (coupling constant) and Q a $d \times d$ symmetric matrix. The Laplacian physically corresponds to a “surface tension” (diffusive) term, that tends to flatten out deviations of the height function from a locally linear profile. The noise is the driver of the dynamics and has to be taken (locally) uncorrelated in space and time. The quadratic non-linearity corresponds to the so-called “lateral growth effect”: in any non-trivial model, the speed of growth is a non-linear function of the local slope. Phenomenologically, the matrix λQ provides the connection between the KPZ equation and any microscopic stochastic growth model: indeed, to establish the link, one should choose Q to be proportional to $D^2 v$, the Hessian of the interface growth velocity v of the microscopic model, seen as a function of the interface slope [30]. In view of this link, while in the original KPZ equation introduced in [20] Q is the identity matrix, Q has in general no reason to be diagonal or positive definite. In fact, there are discrete stochastic $(2 + 1)$ -dimensional growth models both with $\det(D^2 v) > 0$ and with $\det(D^2 v) < 0$, such as the Gates–Westcott model [28] and certain integrable, interlaced particle systems [3] (see [30] for more references).

It is well known that in dimension $d = 1$ the KPZ equation is equivalent to the 1d SBE via the relation $\partial_x h = u$, but such correspondence fails as soon as $d \geq 2$. However, a formal scaling argument very similar to that given above for the SBE suggests that, for the KPZ equation too, at large scales (i.e., under diffusive scaling) the non-linearity grows for $d < 2$, while it vanishes for $d > 2$. On the other hand, for $\lambda = 0$, the equation is exactly scale invariant in any dimension, and it is known as the “stochastic heat equation with additive noise” or “Edwards–Wilkinson” (EW) equation in the physics literature.

Specializing to the critical dimension $d = 2$, physicists soon realized that, as the scaling argument fails, finer features of the equation, and in particular, the signature

of Q , play an important role in the large-scale behavior of the solution to (2.12). Indeed, heuristic renormalization group arguments [34] suggest that, if $\det(Q) \leq 0$, then the non-linearity should not modify the critical exponents of the equation with respect to EW, while if $\det(Q) > 0$, the large-scale fixed point should be qualitatively different from the latter. In this sense, in dimension $d = 2$ the KPZ equation(s) (2.12) can be divided into two universality classes, according to the sign of the determinant of Q .

An especially interesting case of equation (2.12) is the so-called anisotropic KPZ (or AKPZ) equation, corresponding to the case of Q diagonal with diagonal elements ± 1 (i.e., $\det(Q) = -1 < 0$), which is

$$\partial_t h = \frac{1}{2} \Delta h + \lambda((\partial_1 h)^2 - (\partial_2 h)^2) + \xi. \quad (2.13)$$

A special feature of the AKPZ equation is that, formally, the law of the two-dimensional Gaussian free field ($2d$ -GFF) is a stationary measure for its evolution [5]. We recall here that the $2d$ -GFF is a centered Gaussian distribution on $\mathbb{R}^2$, with covariance function $\mathbb{E}(\eta(x)\eta(y)) \propto -\log(|x - y|)$. Equivalently, defining $U(t, x) = (-\Delta)^{1/2} h(t, x)$, then the stationary measure for U is a space white noise. In fact, the AKPZ equation is, among the equations of the form (2.12) in dimension $d = 2$, the only one (up to rotations of the space coordinates) that admits a Gaussian stationary measure [16].

As is the case for the SBE (and for the same reasons), also the (A)KPZ equation is highly singular, and a regularization is needed. The choice adopted in [5–7], is to leave the noise ξ unchanged and regularize the non-linearity with a Fourier cutoff. That is, one replaces

$$((\partial_1 h)^2 - (\partial_2 h)^2) \rightarrow \mathcal{M}^1(h) := \Pi_1((\partial_1 \Pi_1 h)^2 - (\partial_2 \Pi_1 h)^2), \quad (2.14)$$

where Π_a is the orthogonal projection on the Fourier components with wave vector k satisfying $|k| \leq a$. The regularized AKPZ equation is then

$$\partial_t h = \frac{1}{2} \Delta h + \lambda \mathcal{M}^1(h) + \xi. \quad (2.15)$$

Looking at the equation on large diffusive scales is equivalent to removing the cutoff. In fact, $h^\varepsilon(t, x) := h(t/\varepsilon^2, x/\varepsilon)$ solves in law the SPDE

$$\partial_t h^\varepsilon = \frac{1}{2} \Delta h^\varepsilon + \lambda \mathcal{M}^\varepsilon(h^\varepsilon) + \xi, \quad (2.16)$$

with $\mathcal{M}^\varepsilon$ defined as in (2.14), but with Π_1 replaced by $\Pi_{1/\varepsilon}$, that formally tends to the identity operator as $\varepsilon \rightarrow 0$. It is a simple but important remark made in [5] that, despite the regularization, the law of the GFF is stationary for (2.15).

As we are interested in the large-scale properties of the regularized AKPZ equation (2.15), we will use the “bulk diffusion coefficient” which is the counterpart of the diffusion matrix (2.10) for SBE. (Since AKPZ has a rotation invariance property, a single coefficient captures as much information as the full matrix.) The relevant definition adopted in [6] is analogous to it,

$$D_{\text{AKPZ}}(t) = \frac{1}{t} \int_{\mathbb{R}^2} |x|^2 S(t, x) dx, \quad (2.17)$$

with $S(t, x)$ given by

$$S(t, x) = E((- \Delta)^{1/2} h(t, x) (- \Delta)^{1/2} h(0, 0)), \quad (2.18)$$

where E denotes expectation with respect to the law of the stationary process. The reason why, in contrast to (2.10), we do not subtract $S(0, x)$ is that this equals $\delta_0(x)$, that would give a vanishing contribution.

For the linear equation where $\lambda = 0$, it is immediately checked that $D_{\text{AKPZ}}(t)$ does not depend on t .

2.3. (Self-)interacting critical diffusions: The Brownian particle in the curl of the GFF and the self-repelling Brownian polymer

The so-called “Brownian particle (or diffusion) in the curl of the Gaussian free field”, in short DCGFF, was introduced in the theoretical physics literature as a toy model for the motion of a tracer particle in a turbulent two-dimensional fluid. The evolution of the particle position X_t at time t is given by the stochastic differential equation

$$dX_t = \omega(X_t) dt + dB_t, \quad X_0 = x, \quad (2.19)$$

where B_t is a standard two-dimensional Brownian motion, while the drift $\omega : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a position-dependent *random* vector field. Specifically, ω is taken to be the curl of a two-dimensional GFF η independent of B , that is, $\omega(x_1, x_2) = (\partial_{x_2} \eta(x), -\partial_{x_1} \eta(x))$. It was conjectured that the mean square displacement $\mathbb{E}(|X_t|^2)$ grows like $t \sqrt{\log t}$, both in an annealed sense [31] (where the average is with respect to both the randomness of B and of ω) and in a quenched sense [4], that is, almost surely with respect to ω .

The self-repelling Brownian polymer (SRBP), on the other hand, is a model for a Brownian particle that tries to avoid its past, or for a polymer that is repelled by itself. Mathematically, it is (formally) defined by the solution of the stochastic differential equation

$$dX_t = -\nabla L_t(X_t) dt + dB_t, \quad (2.20)$$

where X_t denotes the location of the particle at time t (or the tip of the polymer of length t), and $L_t(x)$ is its local time at x . Because of the minus sign, the drift

tends to push X_t away from locations it has already visited. Again, it was conjectured [31] (with some important caveat, see below, about the initial condition) that as $t \rightarrow \infty$, $\mathbb{E}(|X_t|^2) \asymp t\sqrt{\log t}$, that is, the self-repulsion effect makes the process logarithmically superdiffusive.

Both equations (2.19) and (2.20) are very singular and analytically ill posed, since η is a distribution and $L_t(\cdot)$ can be expected to be very rough too. To make sense of them, one usually replaces the GFF η and the local time L_t by $\eta * \rho$ and $L_t * \rho$, respectively, with ρ a smooth mollifier and $*$ denoting the spatial convolution. With such regularization in place, global existence of the solution X_t is classical for both processes, but the quantitative study of their large- t behavior remains highly non-trivial.

A priori, these two processes seem very different, not only phenomenologically but also mathematically as, for example, the DCGFF is a Markov process (at least in a quenched sense), while the SRBP is not, and also look very different from the SPDEs discussed above. However, if one takes a different point of view and lifts them to the *environment level*, then some similarities can be drawn. To witness, let us first look at the DCGFF and define the *environment seen by the particle* as the space-time random vector field

$$\zeta(t, \cdot) := \omega(X_t + \cdot),$$

that is nothing but the (time-independent) drift field ω , recentered at the instantaneous position of the particle. For the SRBP, one proceeds analogously and defines η as above but replacing ω with $-\nabla(L_t * \rho)(\cdot)$.

A classical and crucial fact for the DCGFF is that, since ω is ergodic and divergence-free ($\nabla \cdot \omega = 0$ pointwise), then the law of ζ is independent of time. That is, at all fixed times t , the environment seen by the particle has the law of the curl of the (smoothed) GFF. This point of view turns the DCGFF into a stationary Markov process with Gaussian stationary (non-reversible) measure. Something similar happens for the SRBP: while the process X itself is not Markovian, the environment process is, and in addition, the latter admits the gradient of the (smoothed) GFF as stationary measure. This means that if at time zero the field $-\nabla(L_0 * \rho)(\cdot)$ has the law of the gradient of the (smoothed) GFF, the same remains true at all times. We can then make the conjecture [31] about logarithmic superdiffusivity of the SRBP more precise: if the local time field is initialized at time zero so that $-\nabla(L_0 * \rho)$ is distributed according to the stationary measure, then $\mathbb{E}(|X_t|^2) \asymp t\sqrt{\log t}$, where the expectation is with respect to both the Brownian noise B and to the initial condition for the local time. In terms of particle process, starting from a drift field $-\nabla(L_0 * \rho)$ that is not zero but stationary corresponds to imagining that the particle has evolved from time $-\infty$ to time 0, so that the drift field has reached its stationary distribution. We emphasize, however, that at the moment there is no mathematical result

(or even physics conjecture we are aware of) about how stationarity is approached if the process starts from $L_0 \equiv 0$.

In other words, upon suitably initializing it, the environment seen by the particle process ζ is a stationary Markov process with stationary Gaussian measure, pretty much in the same vein as the solution to AKPZ and SBE. More than that, upon invoking Itô's formula, it can be shown that ζ solves an SPDE with quadratic non-linearity and that the diffusivity can be expressed solely as a function of it.

3. Critical dimension

In this section, we focus on the critical dimension $d = 2$ and begin by presenting the results concerning the large-time asymptotics of the diffusion matrix.

3.1. Superdiffusivity bounds

We start with the stochastic Burgers equation, for which from now on we assume, without loss of generality, that the vector w in the non-linearity is $(1, 0, \dots, 0)$, that is, the driving field is horizontal. Then, it is not hard to see that, for symmetry reasons, $D_{ij}(t) = 0$ for $i \neq j$ and $D_{ii}(t) = 1$, $i \neq 1$. That is, the only non-trivial component of the diffusion matrix is $D_{11}(t)$. In [32], it was conjectured (on the basis of mode-coupling theory) that $D_{11}(t) \asymp t^{1/3}$ as $t \rightarrow \infty$ in dimension $d = 1$, $D_{11}(t) \asymp 1$ in dimension $d \geq 3$ while, in the critical dimension $d = 2$, $D_{11}(t) \asymp (\log t)^{2/3}$. In the rest of this section, we focus exclusively on dimension $d = 2$.

Remark 3.1. The exponent $2/3$ of the logarithm for $2d$ -SBE is quite subtle and due to the (a)symmetry of the equation. Indeed, the same heuristic argument as in [32] (or an alternative one given in [31]) leads to the prediction of a $(\log t)^{1/2}$ divergence of the diffusion coefficient for the AKPZ equation and (as already mentioned above) for the DCGFF and for the self-repelling Brownian polymer. In fact, the $(\log t)^{2/3}$ divergence is typical of critical models which are *anisotropic*, in that the non-linearity acts only in one direction, while the $(\log t)^{1/2}$ divergence occurs for isotropic critical models. In this sense, *2d-SBE belongs to a universality class different from that of AKPZ, SRBP or DCGFF*.

The first rigorous result that was proven in support of the conjectured logarithmic divergence of the diffusion coefficient is due to H.-T. Yau, who in the landmark paper [36] proved that, for $2d$ -ASEP, $D_{11}(t)$ diverges indeed like $(\log t)^{2/3}$, though in a Tauberian sense (i.e., for the Laplace transform of $t \mapsto tD_{11}(t)$), and with diverging, subdominant, multiplicative corrections (see discussion after Theorem 3.3).

His work has remained an isolated result in this area for almost 20 years. In particular, progress for SPDEs (the AKPZ and SBE) is extremely recent, and the first significant step was made in [6], for the AKPZ equation.

Theorem 3.2 ([6]). *Consider the AKPZ equation, with coupling constant $\lambda > 0$. The Laplace transform $\mathcal{D}_{\text{AKPZ}}(\mu)$ of $t \mapsto tD_{\text{AKPZ}}(t)$ satisfies*

$$\frac{|\log \mu|^{1/2}}{\mu^2} (\log |\log \mu|)^{-C} \lesssim \mathcal{D}_{\text{AKPZ}}(\mu) \lesssim \frac{|\log \mu|^{1/2}}{\mu^2} (\log |\log \mu|)^C \quad (3.1)$$

as $\mu \rightarrow 0^+$ for some $C > 0$.

While these bounds do not immediately translate into upper and lower bounds on $D_{\text{AKPZ}}(t)$, morally they say that $D_{\text{AKPZ}}(t)$ diverges like $\sqrt{\log t}$, up to subdominant multiplicative corrections polynomial in $\log \log t$.

The analogous result (with a slightly different form of the subdominant corrections) has been proven around the same time in [11] for the annealed mean square displacement of the DCGFF. For the latter, [15] has subsequently obtained the sharper bound $c_- \sqrt{\log t} \leq (1/t) \mathbb{E}(|X_t|^2) \leq c_+ \sqrt{\log t}$, without the subdominant diverging corrections and without going through the Laplace transform.

As for the $2d$ -stochastic Burgers equation, the following theorem was, until very recently, the best available result on the large-time behavior of the diffusion coefficient.

Theorem 3.3 ([17]). *For the SBE in dimension $d = 2$, the Laplace transform $\mathcal{D}_{11}(\mu)$ of $t \mapsto tD_{11}(t)$, with $D_{11}(t)$ the first entry of the diffusion matrix $D(t)$ in (2.10), satisfies*

$$\frac{|\log \mu|^{2/3}}{\mu^2} (\log \log |\log \mu|)^{-C} \lesssim \mathcal{D}_{11}(\mu) \lesssim \frac{|\log \mu|^{2/3}}{\mu^2} (\log \log |\log \mu|)^C \quad (3.2)$$

as $\mu \rightarrow 0^+$ for some $C > 0$.

This theorem is actually the analog of the above-mentioned result [36], for $2d$ -ASEP (the theorem in [36] has worse bounds on the subdominant corrections, of order $\exp((\log \log |\log \mu|)^2)$ instead of polynomial in $\log \log |\log \mu|$).

Without entering technical details, let us mention that Theorems 3.2 and 3.3 do not actually work directly with $D_{\text{AKPZ}}(t)$ and $D_{11}(t)$, but with functions that are (at least heuristically) equivalent to them. It is only in (see [12, Proposition 2.5]) that the actual equivalence of the two definitions has been shown.

The culmination of this sequence of works is the following theorem, that represents the only instance where the conjectured logarithmic divergence of the diffusion matrix of a singular critical SPDE has been fully and sharply identified.

Theorem 3.4 ([12]). *Consider the 2d-SBE with $\lambda > 0$. Then,*

$$D_{11}(t) = C_{\text{eff}}(\lambda)(\log t)^{2/3}(1 + o(1)) \quad (3.3)$$

as $t \rightarrow \infty$, with $C_{\text{eff}}(\lambda) = (3/2\pi)^{2/3}\lambda^{4/3}$.

Remark 3.5. It is quite remarkable that Theorem 3.4 identifies the large-time behavior to the precision of the constant prefactor. This is even more true considering that the proportionality $C_{\text{eff}}(\lambda) \propto \lambda^{4/3}$ reveals even deeper universal properties of driven diffusive systems. Indeed, exactly the same holds for the *one-dimensional* stochastic Burgers equation, despite the fact that the t -dependence of the diffusion coefficients is completely different in the two cases (of order $t^{1/3}$ in one dimension and of order $(\log t)^{2/3}$ in two dimensions). As discussed in [12, Remark 1.2], the equality of the exponents of λ in $d = 1, 2$ is not a coincidence, but rather a reflection of the fact that the diffusion coefficient is asymptotically (for large t) independent of the microscopic diffusion coefficient in the direction $\mathbf{w}$ where the non-linearity acts.

3.2. Scaling limits: The weak coupling regime

In view of Theorem 3.3, in dimension $d = 2$, if we scale the solution of the stochastic Burgers equation diffusively, that is, according to (2.3), there is no hope for it to converge in the limit for $\varepsilon \rightarrow 0$. In fact, a simple change of variables argument shows that the $(\log t)^{2/3}$ divergence in Theorem 3.3 is equivalent to a $|\log \varepsilon|^{2/3}$ divergence of the diffusion coefficient for u^ε . In order to tame this divergence, two ways are possible. The most natural one is to change the diffusive space-time rescaling. This will be discussed in Section 3.3. The second one (that is referred to as the “weak coupling limit” or “intermediate disorder regime”) is to insist on diffusive scaling and tame the growth of the non-linearity by modifying the coefficients of the SBE in a scale-dependent way. More precisely, one replaces the coupling constant λ with an ε -dependent coupling constant λ_ε , tending to zero with ε in a precisely chosen way, such that the effect of the non-linearity remains of order 1 as $\varepsilon \rightarrow 0$. As it turns out, the correct choice is to take the coupling constant λ_ε inversely proportional to $\sqrt{|\log \varepsilon|}$.

Theorem 3.6 ([10]). *Let $d = 2$, $\lambda > 0$ and $T > 0$. Let u be the solution to (2.7) started from the stationary measure and with λ replaced by*

$$\lambda_\varepsilon := \frac{\lambda}{\sqrt{|\log \varepsilon|}}$$

for $\varepsilon > 0$, and u^ε be defined according to (2.3). Then, as $\varepsilon \rightarrow 0$, u^ε converges in distribution in $C([0, T], \mathcal{S}')$ to the stationary solution of

$$\partial_t \eta = \frac{1}{2}(D_{\text{SBE}}(\lambda)(\mathbf{w} \cdot \nabla)^2 + \Delta)\eta + (\sqrt{D_{\text{SBE}}(\lambda)}\mathbf{w} \cdot \nabla + \nabla \cdot)\xi, \quad (3.4)$$

where

$$D_{\text{SBE}}(\lambda) = \left(\frac{3\lambda^2}{2\pi} + 1 \right)^{2/3} - 1. \quad (3.5)$$

Here, $C([0, T], \mathcal{S}')$ denotes the space of continuous (in time) functions with values in the space of distributions (in space).

Remark 3.7. To be precise, in [10] the result is proven for the equation defined on the two-dimensional torus of size ε^{-1} rather than on the infinite Euclidean space, so that u^ε is defined on the unit two-dimensional torus $\mathbb{T}^2$ and $\mathcal{S}'$ is the space of distributions on $\mathbb{T}^2$. Also, the regularization used in [10] is slightly different from the one presented in Section 2.1: only the non-linearity is regularized, while the noise is kept white in space and time, so that for every ε the stationary measure is the law of the (non-regularized) white noise on the torus. This being said, the proof of Theorem 3.6 would work with minor modifications for the regularization of Section 2.1.

Remark 3.8. An analogous result was proven earlier for the AKPZ equation [7], later in [8] for the SRBP and in [35] for the DCGFF, all under weak coupling scaling. In the former, it was shown that the stationary solution of

$$\partial_t h^\varepsilon = \frac{1}{2} \Delta h^\varepsilon + \frac{\lambda}{\sqrt{|\log \varepsilon|}} \mathcal{M}^\varepsilon(h^\varepsilon) + \xi^\varepsilon \quad (3.6)$$

(to be compared with (2.16)) converges in distribution, as $\varepsilon \rightarrow 0$, to the stationary solution of the linear heat equation

$$\partial_t \eta = \frac{1}{2} (D_{\text{AKPZ}}(\lambda) + 1) \Delta \eta + \sqrt{D_{\text{AKPZ}}(\lambda) + 1} \xi, \quad (3.7)$$

with ξ a space-time white noise and

$$D_{\text{AKPZ}}(\lambda) = \sqrt{\frac{2\lambda^2}{\pi} + 1} - 1. \quad (3.8)$$

For the SRBP and the DCGFF, an invariance principle for the particle position process $(X_t)_{t \geq 0}$ holds (see [8] for the SRBP and [35] for the DCGFF). In both cases, the diffusion constant of the limiting Brownian motion is strictly larger than 1, that is, strictly larger than the diffusivity of the Brownian noise B_t that appears in the equations (2.19) and (2.20) defining the two models, implying that non-linearity has a non-trivial effect in the weak coupling limit. While at a very high level the proofs of these results have many common features (in particular, coming from the fact that they all have a Gaussian stationary measure), each has its specific technical difficulties and challenges.

What is remarkable is that, while the $2d$ -SBE and AKPZ/DCGFF/SRBP have different exponents of the log-divergence of the diffusion coefficient and therefore belong to two different universality classes (see Remark 3.1), yet the weak scaling regime in all of these cases amounts to choose the coupling constant of the same order, that is, proportional to $|\log \varepsilon|^{-1/2}$.

3.3. Scaling limit: Superdiffusive central limit theorem

We now explore the second way in which one can account for the logarithmic divergence of the diffusion matrix, namely by suitably changing the diffusive scaling to a *superdiffusive scaling*. This is the *physically relevant* approach as we are not modifying the coefficients of the equation but only zooming out at *correct scale*. Our work [12] is the first result that fully determines the large-scale behavior of a *critical, singular SPDE* (the $2d$ -SBE) beyond weak coupling and identifies its infrared fixed point under superdiffusive scaling.

Before stating the full result, let us see what the correct superdiffusive scaling should be. In view of Theorem 3.4, we should expect normal diffusion in the direction orthogonal to $\mathfrak{w} = (1, 0)$ and logarithmic superdiffusion in the direction $\mathfrak{w}$. That is, the natural scaling to consider is

$$u^\varepsilon(t, x) := \varepsilon^{-1} |\log \varepsilon|^{1/6} u\left(\frac{t}{\varepsilon^2}, R_\varepsilon^{-1/2} \frac{x}{\varepsilon}\right), \quad R_\varepsilon = \begin{pmatrix} |\log \varepsilon|^{-2/3} & 0 \\ 0 & 1 \end{pmatrix} \quad (3.9)$$

and then to take the limit $\varepsilon \rightarrow 0$. This formula requires some explanation. First of all, the matrix R_ε allows us to rescale the horizontal and the vertical space direction differently and thus take into account the anisotropy of the equation (and the shape of the diffusion matrix as stated in Theorem 3.4). Second, the prefactor $\varepsilon^{-1} |\log \varepsilon|^{1/6}$ is easily explained: it is a normalization chosen so that the stationary measure of the process u^ε converges to the spatial white noise as $\varepsilon \rightarrow 0$. At last, the scaling in (3.9) is indeed logarithmically superdiffusive in the horizontal direction as in fact, $\varepsilon^{-1} |\log \varepsilon|^{1/3} \gg \varepsilon^{-1}$.

We are now ready to state the superdiffusive central limit theorem for the $2d$ -SBE.

Theorem 3.9 ([12]). *Let $d = 2$, and $\lambda > 0$. Define u^ε as in (3.9), with u the solution of the SBE (2.7), with initial condition that is absolutely continuous with respect to the stationary measure. Then, as $\varepsilon \rightarrow 0$, u^ε converges in the sense of finite-dimensional distributions to the solution of the linear SPDE*

$$\partial_t u^{\text{eff}} = \frac{1}{2} \nabla \cdot D_{\text{eff}} \nabla u^{\text{eff}} + \nabla \cdot \sqrt{D_{\text{eff}}} \xi, \quad \text{for } D_{\text{eff}} = \begin{pmatrix} C_{\text{eff}}(\lambda) & 0 \\ 0 & 1 \end{pmatrix}, \quad (3.10)$$

where $\xi = (\xi_1, \xi_2)$ is a vector-valued space-time white noise and

$$C_{\text{eff}}(\lambda) = \left(\frac{3}{2\pi}\right)^{2/3} \lambda^{4/3}$$

is the same constant as in Theorem 3.4.

We refer the reader to [12, Theorem 1.4] for a more precise and actually stronger formulation, that (roughly speaking) states among other things that $(P_t^\varepsilon F)(u_0)$ (with P_t^ε the semigroup associated to the equation for u^ε and a large enough class of functions F) converges to $(P_t^{\text{eff}} F)(u_0)$ (with obvious notation for P_t^{eff}), in L^p with respect to stationary measure for any $p \geq 1$.

Remark 3.10. It is interesting to compare Theorems 3.6 and 3.9. In some sense, the limit equation (3.10) is the $\lambda \rightarrow \infty$ limit of (3.4). For instance, one sees immediately that $D_{\text{SBE}}(\lambda) = C_{\text{eff}}(\lambda)(1 + o(1))$. Also, it is possible to use the method of [12] to control the $2d$ -SBE in the whole crossover regime between weak and strong coupling, where the coupling constant $\lambda = \lambda_\varepsilon$ is $|\log \varepsilon|^{-1/2} \ll \lambda_\varepsilon \ll 1$ and space-time is rescaled suitably. In this whole regime, the limit equation is (3.10).

Remark 3.11. There is a fundamental reason why the study of the SBE in the strong coupling regime $\lambda = O(1)$ is much harder than in the weak coupling regime, where the non-linearity is proportional to $|\log \varepsilon|^{-1/2}$. In fact, while at the right-hand side of the SBE equation in the weak coupling scaling there is a full Laplacian, that is crucial in obtaining apriori estimates and in proving tightness of the law of the process, in the equation solved by u^ε (see [12, equation (1.9)]) the Laplacian in direction w is multiplied by the asymptotically vanishing prefactor $|\log \varepsilon|^{-2/3}$. This is highly problematic, since the regularizing effect of the Laplacian is lost. That is, the equation for u^ε is much more singular than the one corresponding to the weak coupling scaling. The fact that the Laplacian in direction w formally vanishes means that the effective diffusivity in direction w is fully produced (in the limit $\varepsilon \rightarrow 0$) by the non-linearity.

Remark 3.12. While Theorem 3.9 is the first superdiffusive CLT for a critical, singular SPDE, an analogous result was proven in [1] for the Brownian particle in the curl of the GFF (DCGFF) introduced in Section 2.3. That is, [1] proves a quenched central limit theorem (under logarithmically superdiffusive space-time rescaling) for the trajectory of the particle.

It should be emphasized that, since the DCGFF and the $2d$ -SBE belong to two different universality classes (recall Remark 3.1), the space-time rescaling in [1] for DCGFF is different from the one in (3.9) for $2d$ -SBE. More precisely, the rescaling for the DCGFF is $(t', x') = (t/\varepsilon^2, x/(\varepsilon|\log \varepsilon|^{-1/4}))$: in particular, it is isotropic in space, since the DCGFF is rotationally invariant in distribution.

The tools exploited in [1] are completely different from those of [12], and it is unclear whether they could be adapted to the SPDE setting (and vice versa).

4. The super-critical case

The (formal) scaling argument outlined in Section 2.1 suggests that, for $d \geq 3$, driven diffusive systems behave diffusively at large scales, since the factor $\varepsilon^{(d-2)/2}$ multiplying the non-linearity tends to zero as $\varepsilon \rightarrow 0$. A series of works [14, 22, 24] has confirmed this conjecture for ASEP and, specifically, has proved the convergence of the particle density fluctuation process to a linear stochastic heat equation at large scales (under diffusive scaling). A counterpart on the SPDE side is the recent [10], where the analogous result is obtained for the SBE in dimension $d \geq 3$.

Theorem 4.1 ([10]). *Let $d \geq 3$, $\lambda > 0$ and $T > 0$. Let u be the solution of (2.7), started from the stationary measure and define u^ε according to (2.3).*

Then, as $\varepsilon \rightarrow 0$, u^ε converges in distribution in $C([0, T], \mathcal{S}')$ to the stationary solution of (3.4), for some strictly positive constant $D_{\text{SBE}}(\lambda)$.

In contrast with Theorem 3.6, the limiting diffusion constant $D_{\text{SBE}}(\lambda)$ in dimension $d \geq 3$ is not explicit, and in fact we do not expect it to have an elementary form. That said, it is strictly positive. This means that, even though the non-linearity in the equation for u^ε is multiplied by a vanishing factor (see (2.4)), it is *not irrelevant*, strictly speaking, as it has a non-trivial effect on the limit (and thus it would be dangerous to take the scaling argument of Section 2.1 too literally). Morally, it produces a new noise and a new Laplacian which though do not affect the overall Gaussian nature of the fixed point.

5. Perspectives and open problems

As briefly summarized in this note, these last few years have witnessed substantial progress on critical singular SPDEs, and more broadly on out-of-equilibrium irreversible systems at the critical dimension. In the case of the $2d$ -SBE and of the DCGFF, the conjectures of, respectively, $(\log t)^{2/3}$ [32] and $(\log t)^{1/2}$ [4, 31] divergence of the diffusion coefficient have been fully proven [12, 15], and the infrared fixed point of the processes has been identified, in the form of a superdiffusive central limit theorem [1, 12]. In a broader perspective, these works can be seen as a first step towards a more systematic understanding of out-of-equilibrium irreversible systems at the critical dimension.

This progress also raises a number of natural and intriguing new questions, such as

- How universal is the $\lambda^{4/3}$ scaling of the diffusion coefficient with respect to the coupling constant λ , for models with anisotropic driving field? (see Remark 3.5).

For instance, does the scaling hold for $2d$ -ASEP? Does it hold for the SBE in dimension $d > 2$, in the limit of large λ ?

- It is very tempting to conjecture that a superdiffusive central limit theorem of the type of Theorem 3.9 holds also for the particle fluctuation field of two-dimensional ASEP and other asymmetric, two-dimensional interacting particle systems. Can the techniques of [12] be adapted to the discrete setting?
- All the models mentioned in these notes have an explicit stationary but non-reversible measure (either for the process itself or for the associated environment process). How to go beyond cases where the stationary measure is known?
- Is it possible to combine the functional analytic methods of [12] with the stochastic homogenization methods of [1, 15]? For the moment, the two lines of research are complementary.
- There is a number of natural and mathematically non-trivial ways in which the SBE can be generalized. First of all, one would like to go beyond the case where the non-linearity is exactly quadratic. The scaling argument and the heuristics work the same for polynomial (or even more general) non-linearities, but from the technical point of view, the generalization is not at all immediate. Secondly, the SBE models driven diffusive systems with a single conserved quantity. However, systems like driven fluids have $N > 1$ conserved quantities (for a fluid they would be mass, energy and d components of momentum, for a total of $N = 2 + d$). The natural modelization suggested by the theory of fluctuating hydrodynamics [29] is then a system of coupled stochastic Burgers equations. It is known in one dimension [29] that the large-scale behavior can depend qualitatively on N (see [9] for an example). In dimension $d = 2$, the same is conjectured: in fact, while the single SBE exhibits $(\log t)^{2/3}$ superdiffusivity, two-dimensional fluids (corresponding to $N = 4$) are expected [33] to exhibit $(\log t)^{1/2}$ superdiffusivity. This problem is wide open, the only available result being a suboptimal $\log \log t$ lower bound on superdiffusivity for a discrete lattice fluid model [23]. Unfortunately, such result is too weak to distinguish between the two different powers of $\log t$.

Lastly, a very fascinating line of works [26, 27] has displayed compelling features of the DCGFF such as intermittency properties and an unexpected connection with a diffusion on the Lie group $SL(2)$, and it would be interesting to see if analogous relations can be found for the SBE, AKPZ equation or even more broadly for other out-of-equilibrium systems at the critical dimension.

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Giuseppe Cannizzaro

Department of Statistics, University of Warwick, Coventry CV4 74L, UK;
giuseppe.cannizzaro@warwick.ac.uk

Fabio Toninelli

Institut für Stochastik und Wirtschaftsmathematik, TU Wien, Wiedner Hauptstraße
8-10/105-7, 1040 Vienna, Austria; fabio.toninelli@tuwien.ac.at