

Positivity and non-commutativity

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Abstract. In this article, we revisit a new notion of positivity in real semisimple Lie groups that at the same time generalizes total positivity in split real Lie groups as well as positive Lie semigroups in Hermitian Lie groups of tube type. We shortly discuss the relationship with higher rank Teichmüller spaces and then focus on describing different aspects of positivity as well as open questions. In the second part, we describe a non-commutative perspective on Hermitian Lie groups of tube type that is suggested by positivity and leads to interesting applications, such as non-commutative generalizations of Markov numbers.

1. Introduction

In my plenary lecture “A beautiful world beyond hyperbolic geometry: Anosov representations and higher Teichmüller spaces,” I described aspects of hyperbolic geometry through the lens of groups acting by isometries on hyperbolic spaces and gave a selective overview of recent developments. I focused particularly on Anosov representations of higher rank Teichmüller spaces. Anosov representations generalize many aspects of convex cocompact actions on hyperbolic spaces to more general discrete subgroups of Lie groups of higher rank. This leads to interesting group actions on higher rank symmetric spaces of non-compact type, such as the space of positive definite symmetric matrices or the Siegel upper half-space. Special families of Anosov representations are given by higher rank Teichmüller spaces. Higher rank Teichmüller spaces are connected components of the representation variety of the fundamental group $\pi_1(S)$ of a closed surface into a semisimple Lie group, which consist entirely of discrete and faithful representations. For this to make sense, recall that classical Teichmüller space can be realized as the space of marked hyperbolic structures on S . The holonomy representations of marked hyperbolic structures form connected components of the representation variety of $\pi_1(S)$ into the isometry group

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of the hyperbolic plane $\mathrm{PSL}(2, \mathbb{R})$. For $\mathrm{PSL}(2, \mathbb{R})$, Anosov representations of $\pi_1(S)$ are precisely the holonomy representations of hyperbolic structures on S . For other Lie groups, the existence of such higher rank Teichmüller spaces is in fact quite surprising, and in fact, they only exist for special families of Lie groups. We refer the reader who is interested in learning more on Anosov representations to Canary's informal lecture notes on Anosov representations [12] and to Kassel's article in the proceedings of the ICM [32], and those who want to get a view into higher rank Teichmüller space to the author's article in the proceedings of the ICM [42].

Higher rank Teichmüller spaces are intimately linked to positivity in Lie groups. In the case of split real Lie groups, the connection to Lusztig's total positivity has been made by Fock and Goncharov [15]. For Hermitian Lie groups of tube type, the connection to positivity has been made in the work of Burger, Iozzi, Wienhard [11]; characterizations in terms of causal structures and bi-invariant orders appeared in the work of Ben Simon, Burger, Hartnick, Iozzi, and Wienhard [5].

In recent years, Guichard and Wienhard [28, 29] introduced a new notion of positivity in real semisimple Lie groups, which at the same time generalizes Lusztig's total positivity (defined for split real Lie groups), and positive Lie semigroups in Hermitian Lie groups of tube type. There are four families of simple real Lie groups admitting a positive structure. Besides split real Lie groups and Hermitian Lie groups of tube type, one family is formed by indefinite orthogonal groups $\mathrm{SO}(p+1, p+k)$, where $p, k > 1$. The other family consists of exceptional groups.

Positivity gives rise to a notion of positivity of configurations of flags in appropriate flag varieties associated to G . This can be used to define positive representations of surface groups $\pi_1(S)$ into G whenever G admits a positive structure. It has been conjectured by Guichard, Labourie, and Wienhard [42, Conjecture 19] that the set of positive representations always forms higher rank Teichmüller spaces. This has been proven recently by Beyrer, Guichard, Labourie, Pozzetti, and Wienhard [8, 9, 27]. Even though there are still several open questions on positive representations, this establishes that positivity provides the right underlying structure to explain the existence of higher rank Teichmüller spaces.

But positivity is much more. It opens a new perspective on the families of simple real Lie groups admitting a positive structure. In this perspective, non-commutative rings with involution play a crucial role. In this article, we want to give a glimpse of that new perspective and describe it in more detail for Hermitian Lie groups of tube type. Through the lens of positivity, Hermitian Lie groups of tube type of arbitrary rank look like Lie groups of type A_1 but defined over a non-commutative ring with involution. This opens up a fresh view, which leads to interesting new constructions, including new quantum groups, generalizations of Markov numbers, non-commutative trace identities, and many other things which are yet to be explored.

In the first part of this article, we review important aspects of positivity in Lie groups and state several open questions. In the second part, we illustrate the non-commutative perspective on Hermitian Lie groups of tube type and describe some of its consequences.

2. Positivity in Lie groups

In this section, we recall the notion of positivity in a real semisimple Lie group. It is a generalization of the notion of total positivity.

The theory of totally positive matrices arose in the beginning of the 20th century through the work of Schoenberg [41] and Gantmacher and Krein [19]. In the 1990s, the theory was generalized widely by Lusztig [34], who introduced total positivity in general split real semisimple Lie groups. Since then, total positivity plays an important role in representation theory and cluster algebras and has many relations to other areas in mathematics as well as to theoretical physics. The concept of positivity has been generalized by Guichard and Wienhard [28, 29] to more general real semisimple Lie groups which are not necessarily split. This notion of positivity is defined with respect to a subset Θ of the set of simple roots Δ . The classification of semisimple Lie groups that admit a positive structure can be reduced to simple Lie groups. There are four families of simple Lie groups that admit a positive structure. One family is the family of split real Lie groups with total positivity, which is positivity with respect to the full set Δ . The three other families carry a positive structure with respect to a proper subset Θ of Δ . As we will see below, when Θ is a proper subset of Δ , some non-commutativity enters the picture.

2.1. The four families

Let us shortly recall the notion of positivity. For more details, we refer the reader to [28, 29].

Consider a simple real Lie group G which is not necessarily split. A subset $\Theta \subset \Delta$ of the set of simple roots defines a standard parabolic subgroup P_Θ in G . (We choose the convention so that P_Δ is the minimal parabolic subgroup of G .) The group P_Θ is a semidirect product of a reductive Lie group L_Θ and its unipotent radical U_Θ . The Lie algebra $\mathfrak{u}_\Theta$ of U_Θ carries a natural action by L_Θ and can hence be decomposed into its L_Θ -irreducible pieces. For every simple root $\alpha \in \Theta$, there is an irreducible piece $\mathfrak{u}_\alpha$. When G is split and $\Delta = \Theta$, these would be precisely the root spaces $\mathfrak{g}_\alpha$. In that case, they are one dimensional. When Θ is a proper subset of Δ , then the $\mathfrak{u}_\alpha$ are in general not root spaces and not necessarily one dimensional.

The Lie group G is said to admit a positive structure with respect to Θ if it satisfies the following property.

For every $\alpha \in \Theta$, there exists a non-empty closed acute convex cone $c_\alpha \subset \mathfrak{u}_\alpha$ that is invariant under the action of L_Θ° on $\mathfrak{u}_\alpha$, where L_Θ° is the connected component of the identity.

When G is split, $\mathfrak{u}_\alpha = \mathfrak{g}_\alpha$ can be identified with $\mathbb{R}$. Then the cone c_α can be identified with $\mathbb{R}^{\geq 0} \subset \mathbb{R}$.

Theorem 2.1 ([29, Theorem 3.4]). *A simple Lie group G admits a positive structure with respect to Θ if and only if the pair (G, Θ) belongs to the following list:*

- (1) G is a split real Lie group, and $\Theta = \Delta$;
- (2) G is Hermitian of tube type and of real rank r , and $\Theta = \{\alpha_r\}$, where α_r is the long simple restricted root;
- (3) G is locally isomorphic to $\mathrm{SO}(p+1, p+k)$, $p > 1$, $k > 1$, and $\Theta = \{\alpha_1, \dots, \alpha_p\}$, where $\alpha_1, \dots, \alpha_p$ are the long simple restricted roots;
- (4) G is the real form of F_4 , E_6 , E_7 , or of E_8 whose reduced root system is of type F_4 , and $\Theta = \{\alpha_1, \alpha_2\}$, where α_1, α_2 are the long simple restricted roots.

The arguments that give the classification provide important additional information.

When Θ is a proper subset of Δ , there exists a unique $\alpha_\Theta \in \Theta$ for which $\mathfrak{u}_{\alpha_\Theta}$ is not equal to the root space $\mathfrak{g}_{\alpha_\Theta}$ and is a higher-dimensional vector space.

For all other elements α of $\Theta \setminus \{\alpha_\Theta\}$, we have that $\mathfrak{u}_\alpha$ is equal to $\mathfrak{g}_\alpha$, and this root space has to be of dimension one. Hence, in this case, if we identify $\mathfrak{u}_\alpha$ with $\mathbb{R}$, the cone c_α can be identified again with $\mathbb{R}^{\geq 0} \subset \mathbb{R}$.

2.2. A non-commutative view on the special piece $\mathfrak{u}_{\alpha_\Theta}$

For the special root α_Θ , the vector space $\mathfrak{u}_{\alpha_\Theta}$ carries an irreducible action of the reductive group L_Θ° , which preserves the convex cone $c_{\alpha_\Theta} \in \mathfrak{u}_{\alpha_\Theta}$ and acts transitively on its interior $\mathring{c}_{\alpha_\Theta}$. Such symmetric cones have been classified, and they are intimately related to Hermitian Lie groups of tube type; see [14] for more details.

Thus, let us first look at the case when G is Hermitian of tube type. In this case, there is just one special piece $\mathfrak{u}_{\alpha_\Theta}$. Then G is (up to isogeny) $\mathrm{Sp}(2n, \mathbb{R})$, $\mathrm{SU}(n, n)$, $\mathrm{SO}^*(4n)$, $\mathrm{SO}(2, q)$, or the exceptional group $E_{7(-25)}$. The first three families of classical groups can be uniformly described: Let $\mathbb{K}$ be either $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, and the space $V = \mathbb{K}^{2n}$ is a right vector space and is endowed with an anti-Hermitian form defined by

$$\omega_{\mathbb{K}}(v, v') = \sum_{i=1}^n \bar{x}_i y'_i - \bar{y}_i x'_i$$

with $v = (x_1, \dots, x_n, y_1, \dots, y_n)$ and $v' = (x'_1, \dots, x'_n, y'_1, \dots, y'_n)$. The subgroup G in $\mathrm{GL}_{\mathbb{K}}(V)$ of automorphisms of V stabilizing $\omega_{\mathbb{K}}$ is Hermitian of tube type; it is

$\mathrm{Sp}(2n, \mathbb{R})$, $\mathrm{SU}(n, n)$, or $\mathrm{SO}^*(4n)$ following that $\mathbb{K}$ is $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, respectively. The Levi factor L_Θ in this case consists of the elements

$$\ell_A = \begin{pmatrix} A & 0 \\ 0 & {}^t\overline{A}^{-1} \end{pmatrix}, \quad A \in \mathrm{GL}_n(\mathbb{K}),$$

and the unipotent algebra $\mathfrak{u}_\Theta$ is equal to $\mathfrak{u}_{\alpha_\Theta}$. It consists of the matrices

$$u_M = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}, \quad M \in M_n(\mathbb{K}) \text{ such that } {}^t\overline{M} = M;$$

thus $\mathfrak{u}_{\alpha_\Theta}$ identifies with the space $\mathrm{Herm}(n, \mathbb{K})$ of Hermitian matrices over $\mathbb{K}$. Since $\ell_A u_M \ell_A^{-1} = u_{AM {}^t\overline{A}}$, we get that $\mathrm{Herm}^{\geq 0}(n, \mathbb{K})$ is the L_Θ -invariant cone.

For $\mathrm{SO}(2, k+1)$, let us consider the quadratic form $q_{2,k+1} = -x_1 x_{k+3} + x_2 x_{k+2} - \sum_{j=3}^{k+1} x_j^2$ on $V = \mathbb{R}^{k+3}$. The group $G = \mathrm{SO}(q_{2,k+1})$ is also a Hermitian Lie group of tube type. The unipotent algebra is composed of the matrices

$$u_v = \begin{pmatrix} 0 & {}^t v J & 0 \\ 0 & 0 & v \\ 0 & 0 & 0 \end{pmatrix}, \quad v \in \mathbb{R}^{k+1},$$

where J is the matrix of the form $q_{1,k}$. One has in this case $\ell_{\lambda,g} u_v \ell_{\lambda,g}^{-1} = u_{\lambda g v}$, and the cone defined by $x_2 > 0$, $q_{1,k} > 0$ is invariant by L_Θ° .

The exceptional Hermitian simple Lie group of tube type is $E_{7(-25)}$ (or EVII in Cartan's notation). In this case, the semisimple part of the group L_Θ is $\mathrm{EIV} = E_{6(-26)}$, and the action of L_Θ on $\mathfrak{u}_{\alpha_\Theta}$ is the action of the group of isomorphisms of $\mathrm{Herm}(3, \mathbb{O})$ in it; $\mathrm{Herm}^{\geq 0}(3, \mathbb{O})$ is a natural invariant cone (cf. [14, p. 213]).

In the general case, when $\Theta \neq \Delta$ and the cardinality of Θ is > 1 , the description of the cone c_{α_Θ} reduces to the Hermitian case. When $G = \mathrm{SO}(p+1, p+k)$, then (using standard numbering for the simple roots) $\mathfrak{u}_{\alpha_\Theta} \cong (\mathbb{R}^{1,k}, q_{1,k})$ with $c_{\alpha_\Theta} = \{v \in \mathbb{R}^{1,k} \mid v_1 \geq 0, q_{1,k}(v) \geq 0\}$ ($q_{1,k}$ denotes here the standard quadratic form of signature $(1, k)$ on $\mathbb{R}^{1+k}$).

When G is one group in the exceptional family whose system of restricted roots is of type F_4 , then $\mathfrak{u}_{\alpha_\Theta}$ and the cone c_{α_2} are given in Table 1.

There is a well-known connection between symmetric cones and Euclidean Jordan algebras (see, for example, [14]). The Lie algebra $\mathfrak{u}_{\alpha_\Theta}$ carries the structure of a Euclidean Jordan algebra, and the cone $c_{\alpha_\Theta} \in \mathfrak{u}_{\alpha_\Theta}$ that gives the positive structure is given as the cone of squares.

Here we want to stress a slightly different point of view. For the cases where $\mathfrak{u}_{\alpha_\Theta}$ identifies with the space $\mathrm{Herm}(n, \mathbb{K})$ of Hermitian matrices over $\mathbb{K}$, there is in fact an algebra $\mathcal{A}$ with an anti-involution σ such that $\mathfrak{u}_{\alpha_\Theta}$ identifies with $\mathcal{A}^\sigma$, the set of fix points of σ , and the cone is $c_{\alpha_\Theta} = \{a\sigma(a) \mid a \in \mathcal{A}\} \subset \mathcal{A}^\sigma$, and

G	$\mathfrak{u}_{\alpha_\Theta}$	$\mathfrak{c}_{\alpha_2}$
$\mathrm{FI} = \mathrm{F}_{4(4)}$	$\mathrm{Sym}(3, \mathbb{R})$	$\mathrm{Sym}^{\geq 0}(3, \mathbb{R})$
$\mathrm{EII} = \mathrm{E}_{6(2)}$	$\mathrm{Herm}(3, \mathbb{C})$	$\mathrm{Herm}^{\geq 0}(3, \mathbb{C})$
$\mathrm{EVI} = \mathrm{E}_{7(-5)}$	$\mathrm{Herm}(3, \mathbb{H})$	$\mathrm{Herm}^{\geq 0}(3, \mathbb{H})$
$\mathrm{EIX} = \mathrm{E}_{8(-24)}$	$\mathrm{Herm}(3, \mathbb{O})$	$\mathrm{Herm}^{\geq 0}(3, \mathbb{O})$

Table 1. The nontrivial cone for groups in the exceptional family.

$\mathring{\mathfrak{c}}_{\alpha_\Theta} = \{a\sigma(a) \mid a \in A \text{ invertible}\}$. In fact, the case when $\mathfrak{u}_\alpha = \mathfrak{g}_\alpha$ also fits in this setup. In this case, $\mathcal{A} = \mathbb{R}$, and σ is the identity map.

The case of indefinite orthogonal groups, when $\mathfrak{u}_{\alpha_\Theta} \simeq \mathbb{R}^{(1,q+1)}$, a slightly modified picture holds where $\mathcal{A}$ is the Clifford group of a Clifford algebra. For the case when $\mathfrak{u}_{\alpha_\Theta}$ identifies with the space $\mathrm{Herm}(3, \mathbb{O})$, there is no ambient associative algebra, but many of the other features persist in that case as well; see, for example, [24] for a detailed discussion.

2.3. The unipotent semigroup

An important role in positivity is played by the non-negative and positive semigroups of the unipotent group U_Θ . The non-negative semigroup $U_\Theta^{\geq 0}$ is the sub-semigroup of U_Θ generated by elements $\exp(v)$, with $v \in \mathfrak{c}_\alpha$ for some $\alpha \in \Theta$. The positive semigroup $U_\Theta^{> 0}$ is the interior of $U_\Theta^{\geq 0}$.

Similarly, one can define a non-negative semigroup $U_\Theta^{\mathrm{opp}, \geq 0}$ and a positive semigroup $U_\Theta^{\mathrm{opp}, > 0}$ in U_Θ^{opp} . This gives rise to the non-negative semigroup $G_\Theta^{\geq 0}$ in G , which is the semigroup generated by $U_\Theta^{\geq 0}$, $U_\Theta^{\mathrm{opp}, \geq 0}$, and L_Θ° . The positive semigroup $G_\Theta^{> 0}$ is the interior of $G_\Theta^{\geq 0}$.

In fact, for the reader who is not happy with the definition of positivity, requiring the existence of invariant convex cones, the existence of such semigroups could alternatively be used as the definition of positivity.

Theorem 2.2 ([28]). *Let G be a connected simple Lie group. Suppose that there is $U^+ \subset U_\Theta$ such that U^+ is a closed L_Θ° -invariant semigroup of non-empty interior, which contains no nontrivial invertible element. Then G admits a positive structure with respect to Θ , and the semigroup U^+ contains the semigroup $U_\Theta^{\geq 0}$.*

If we further assume that the interior of U^+ is contained in the open Bruhat cell with respect to P_Θ^{opp} , then $U^+ = U_\Theta^{\geq 0}$.

Defining the positive semigroup $U_\Theta^{> 0}$ as the interior of $U_\Theta^{\geq 0}$ is easy to state but does not provide much insight into how elements in this semigroup look.

In Lusztig's work, the positive semigroup $U_\Delta^{> 0}$ is in fact defined by giving explicit parametrizations, proving that the image of the parametrizations does not depend on

the choices made. That the thus defined set coincides with the interior of $U_{\Delta}^{\geq 0}$ will be proved afterwards. The parametrizations are associated to reduced expressions of the longest element in the Weyl group, and the change of coordinates between parametrizations associated to different reduced expressions is given by an explicit formula involving only positive rational functions.

In the case of positive structures with respect to a proper subset Θ of Δ , similar results hold. However, the Weyl group has to be replaced by another Coxeter group, called the Θ -Weyl group. Every reduced expression of the longest word in the Θ -Weyl group gives rise to a parametrization of the positive semigroup $U_{\Theta}^{>0}$, and the change of coordinates between two different reparametrizations is determined by explicit formulas [29, 30].

2.4. The Θ -Weyl group

The Θ -Weyl group is a subgroup of the Weyl group W of G . It turns out to be the Weyl group of a different root system and lies at the heart of regarding the Lie groups admitting a positive structure with respect to Θ through the lens of non-commutativity.

To introduce the Θ -Weyl group, let us recall that the Weyl group W is generated by reflections (order two elements) s_{α} , $\alpha \in \Delta$. These elements are characterized by their action on the collection of roots or root spaces. The element s_{α} sends the root space $\mathfrak{g}_{\alpha}$ to $\mathfrak{g}_{-\alpha}$ and permutes all the other root spaces associated to positive roots.

The Θ -Weyl group $W(\Theta)$ is a subgroup of W generated by elements σ_{α} for all $\alpha \in \Theta$. For all $\alpha \in \Theta$ which are not connected to $\Delta \setminus \Theta$, i.e., when u_{α} is equal to $\mathfrak{g}_{\alpha}$, σ_{α} is just the reflection s_{α} . For the unique $\alpha_{\Theta} \in \Theta$ which is connected to $\text{Weyl } \Delta \setminus \Theta$, i.e., for which $u_{\alpha_{\Theta}}$ is a higher-dimensional vector space, the element $\sigma_{\alpha_{\Theta}}$ is a specific word in the subgroup generated by the reflections associated to roots in $\{\alpha_{\Theta}\} \cup \Delta \setminus \Theta$. The precise word is not of importance to us here; the reader can consult [29]. The key point is that the action of the Θ -Weyl group on the decomposition of the Lie algebra

$$\mathfrak{g} = \bigoplus_{\alpha \in (\Theta)} u_{\alpha} \oplus \mathfrak{l}_{\Theta} \oplus \bigoplus_{\alpha \in (\Theta)} u_{-\alpha}$$

is completely analogous to the action of the classical Weyl group on the root decomposition

$$\mathfrak{g} = \bigoplus_{\alpha \in (\Delta)} \mathfrak{g}_{\alpha} \oplus \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in (\Delta)} \mathfrak{g}_{-\alpha}.$$

In other words, elements σ_{α} act on the irreducible pieces u_{α} in the following way. The element σ_{α} sends u_{α} to $u_{-\alpha}$ and permutes the u_{β} for all the other β in the positive span of Θ .

The Θ -Weyl group is itself an abstract Weyl group but of a different type than the Weyl group of G . Its rank is the cardinality of Θ , and it acts on the pieces $\mathfrak{u}_\alpha$ in the same way as the abstract Weyl group would act on the classical root spaces.

Here is a list of the Θ -Weyl groups:

- (1) *Split case*: If G is a split real form and $\Theta = \Delta$, then $W(\Theta) = W$.
- (2) *A_1 case*: If G is Hermitian of tube type and of real rank r and $\Theta = \{\alpha_r\}$, then the Weyl group is of type C_r , but the Θ -Weyl group is of type A_1 .
- (3) *B_p case*: If G is locally isomorphic to $\mathrm{SO}(p+1, p+k)$, $p > 1, k > 1$, and $\Theta = \{\alpha_1, \dots, \alpha_p\}$, then the Weyl group is of type B_{p+1} , but the Θ -Weyl group is of type B_p .
- (4) *G_2 case*: If G is the real form of F_4 , E_6 , E_7 , or of E_8 whose reduced root system is of type F_4 , and $\Theta = \{\alpha_1, \alpha_2\}$, then the Weyl group is of type F_4 , but the Θ -Weyl group is of type G_2 .

The labels we assigned to the different cases reflect the fresh perspective we suggest when looking at the groups G admitting a positive structure with respect to Θ . We suggest considering G as Lie groups of type $W(\Theta)$ but defined over a (partially) non-commutative ring with an involution. We will illustrate this perspective below for the case of Hermitian Lie groups of tube type. In this case, $\Theta = \{\alpha_\Theta\}$, so the Θ -Weyl group is $\mathbb{Z}/2\mathbb{Z}$. Thus, we want to consider Hermitian Lie groups of tube type as groups of type A_1 or more precisely as symplectic groups Sp_2 over some non-commutative ring. This has been made precise in [2, 40] and leads to interesting new phenomena.

Before we illustrate this non-commutative perspective, we describe how the Θ -Weyl group appears in parametrizations of the positive semigroup and state some open questions on parametrizations of the non-negative semigroups.

2.5. Parametrizing the positive semigroup

We explain now how reduced expressions of the longest work in the Θ -Weyl group give parametrizations of the positive semigroups $U_\Theta^{>0}$.

Given any reduced expression $\gamma = \sigma_{\gamma_1} \cdots \sigma_{\gamma_n}$ of the longest element $w_{\max}^\Theta \in W(\Theta)$, we define the map

$$F_\gamma: \mathfrak{u}_{\gamma_1} \times \cdots \times \mathfrak{u}_{\gamma_n} \rightarrow U_\Theta$$

as the product of the exponential map on each factor, i.e.,

$$F_\gamma(X_1, \dots, X_n) = \exp(X_1) \cdots \exp(X_n),$$

where $X_i \in \mathfrak{u}_{\gamma_i}$.

Since $\gamma_i \in \Theta$ for all i , we can restrict the map $F_{\boldsymbol{\gamma}}$ to the product of the cones $c_{\gamma_i} \subset \mathfrak{u}_{\gamma_i}$ as well as to their interiors $\overset{\circ}{c}_{\gamma_i}$.

Theorem 2.3 ([29, Section 10]). *The map $F_{\boldsymbol{\gamma}}$ enjoys the following properties:*

- (1) *The map $F_{\boldsymbol{\gamma}}$ restricted to the product of open cones $\overset{\circ}{c}_{\gamma_1} \times \cdots \times \overset{\circ}{c}_{\gamma_n}$ is injective and $F_{\boldsymbol{\gamma}}(\overset{\circ}{c}_{\gamma_1} \times \cdots \times \overset{\circ}{c}_{\gamma_n}) = U_{\Theta}^{>0}$. Thus, $F_{\boldsymbol{\gamma}}$ gives a parametrization of the positive semigroup.*
- (2) *The map $F_{\boldsymbol{\gamma}}$ restricted to the product of closed cones $c_{\gamma_1} \times \cdots \times c_{\gamma_n}$ is not injective, but proper, and $F_{\boldsymbol{\gamma}}(c_{\gamma_1} \times \cdots \times c_{\gamma_n}) = U_{\Theta}^{\geq 0}$.*

Note that in the case of split real Lie groups, all the cones $\overset{\circ}{c}_{\gamma_i} \subset c_{\gamma_i} \subset \mathfrak{u}_{\gamma_i}$ can be identified with $\mathbb{R}^{>0} \subset \mathbb{R}^{\geq 0} \subset \mathbb{R}$. Given two different reduced expressions $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}'$, the change of coordinates is given by positive rational functions [34]. Since any two different reduced expressions are related by a sequence of braid relations, it suffices to establish that for the braid relations in all split real Lie groups of rank 2. Berenstein and Zelevinsky [7] wrote explicit expressions for the change of coordinates. To derive these expressions, given a braid relation, they set up a system of polynomial equations in the universal enveloping algebra, which they then solve explicitly.

In the case when Θ is a proper subset of Δ , several of the cones in the domain of $F_{\boldsymbol{\gamma}}$ are cones in higher-dimensional vector spaces. By the above theorem, the image of $F_{\boldsymbol{\gamma}}$ is still independent of the reduced expression $\boldsymbol{\gamma}$ of the longest word in the Θ -Weyl group $W(\Theta)$. In [30], we give explicit expressions for the change of coordinates. These expressions are not positive rational functions, but surprisingly, they still preserve the positivity defined by the cones. In fact, associated to braid relations in the Θ -Weyl group, we derive an analogous system of “polynomial equations” as Berenstein and Zelevinsky, just that some of the variables are now not scalars, but take values in a vector space (or better, a Euclidean Jordan algebra), and the powers of such an element are interpreted in an appropriate way as taking a product or evaluating a linear form. For details, we refer the reader to [30].

What we would like to highlight here is that, in the split real case, the system of polynomial equations is associated to the type of the Weyl group, whereas in the general case of positive structures with respect to Θ , it is the same system of polynomial equations associated to the type of the Θ -Weyl group, where the variable associated to the special root in Θ should be thought of as elements in a Euclidean Jordan algebra.

Lusztig’s foundational work on total positivity in Lie groups has spurred a lot of further research. Many of the results that have been established for total positivity in split real Lie groups lead to natural questions for positivity with respect to Θ in more general semisimple real Lie groups. In the following subsections, we mention some open questions, but literally almost any result you come across for total positivity can

lead one to ask what the corresponding statement would be for positivity with respect to Θ .

2.6. Parametrizations of the non-negative semigroups

The map $F_{\boldsymbol{\gamma}}$ associated to a reduced expression of the longest word in the Θ -Weyl group gives a parametrization of the positive semigroup $U_{\Theta}^{>0}$. This parametrization is equivariant with respect to the action of L_{Θ}° . Since the map $F_{\boldsymbol{\gamma}}$ is not injective when restricted to the product of closed cones $c_{\gamma_1} \times \cdots \times c_{\gamma_n}$, it does not give a parametrization of the non-negative semigroup $U_{\Theta}^{\geq 0}$.

To approach the problem of parametrizing the non-negative semigroup, it is instructive to look at the Bruhat decompositions of a Lie group G . Recall that G can be decomposed under the left-right multiplication by $P_{\Delta}^{\text{opp}} \times P_{\Delta}^{\text{opp}}$, where P_{Δ}^{opp} is the minimal standard opposite parabolic subgroup. These orbits are called the Bruhat cells and are indexed by elements of the Weyl group W . One can also consider decompositions with respect to the action of $P_{\Delta}^{\text{opp}} \times P_{\Theta}^{\text{opp}}$ and $P_{\Theta}^{\text{opp}} \times P_{\Theta}^{\text{opp}}$, where Θ is a subset of Δ . For these, the orbits are indexed by elements of $W/W_{\Delta-\Theta}$ and $W_{\Delta-\Theta} \setminus W/W_{\Delta-\Theta}$, respectively, where $W_{\Delta-\Theta}$ is the Weyl group of L_{Θ} .

(1) The group G is the disjoint union

$$G = \bigcup_{w \in W} P_{\Delta}^{\text{opp}} w P_{\Delta}^{\text{opp}}.$$

(2) The group G is the disjoint union

$$G = \bigcup_{x \in W/W_{\Delta \setminus \Theta}} P_{\Delta}^{\text{opp}} x P_{\Theta}^{\text{opp}}.$$

(3) The group G is the disjoint union

$$G = \bigcup_{a \in W_{\Delta \setminus \Theta} \setminus W/W_{\Delta \setminus \Theta}} P_{\Theta}^{\text{opp}} a P_{\Theta}^{\text{opp}}.$$

In the case of total positivity, when $\Theta = \Delta$, all these decompositions agree. The intersection $U_w^{\geq 0}$ of the non-negative semigroup $U^{\geq 0}$ with the Bruhat cell $P_{\Delta}^{\text{opp}} w P_{\Delta}^{\text{opp}}$ can be parametrized in a similar way as the positive semigroup [34]. Let $\boldsymbol{\gamma} = \gamma_1 \cdots \gamma_k$ be a reduced expression of an element $w \in W$, not necessarily the longest element. Then $F_{\boldsymbol{\gamma}}$ restricted to $(c_{\gamma_1}^{\circ} \times \cdots \times c_{\gamma_k}^{\circ})$ gives a parametrization of $U_w^{\geq 0}$. Clearly, $U^{\geq 0} = \bigcup_{w \in W} U_w^{\geq 0}$ is a disjoint union. Thus, this gives a parametrization of $U^{\geq 0}$ and in fact also a cell decomposition [39].

In the case when Θ is a proper subset of Δ , the task to parametrize the non-negative semigroup becomes more difficult. This becomes already apparent in the

case of Hermitian Lie groups of tube type, which is the non-commutative A_1 -type situation, where $\Theta = \{\alpha\}$ consists of a single simple root α . Here the situation is simple; there is just one space $\mathfrak{u}_\alpha$ and cones $\mathring{c}_\alpha \subset c_\alpha \subset \mathfrak{u}_\alpha$. The exponential map $\mathfrak{u}_\alpha \rightarrow U_{\{\alpha\}}^{\geq 0}$ is an isomorphism, and therefore, the map $F : c_\alpha \rightarrow U_{\{\alpha\}}^{\geq 0}$ gives a parametrization.

However, the structure of the intersection of $U_{\{\alpha\}}^{\geq 0}$ with the different orbits in the above decompositions of G is already very interesting. This stems from the fact that the boundary of the cone $\partial c_\alpha = c_\alpha - \mathring{c}_\alpha$ is nontrivial. In the split real case, or when $\mathfrak{u}_\alpha$ is of dimension one, this boundary is just 0, but in general it has several strata, given by L_Θ° -invariant subsets. These strata are precisely the orbits $U_{\{\alpha\}}^{\geq 0} \cap P_\Theta^{\text{opp}} a P_\Theta^{\text{opp}}$ for $a \in W_{\Delta \setminus \Theta} \setminus W/W_{\Delta \setminus \Theta}$; see [29]. Note that this decomposition does not give you a cell decomposition as the individual strata have nontrivial topology.

The decompositions of $U_{\{\alpha\}}^{\geq 0}$ with respect to the other decompositions

$$G = \bigcup_{x \in W/W_{\Delta \setminus \Theta}} P_\Delta^{\text{opp}} x P_\Theta^{\text{opp}} \quad \text{and} \quad G = \bigcup_{w \in W} P_\Delta^{\text{opp}} w P_\Delta^{\text{opp}}$$

are more involved, as they are not invariant by L_Θ° . The paper [29] contains further information with respect to the first decomposition; see also [44, 45] for further work in the case of the symplectic group. In particular, for the symplectic group, [44] gives a cell decomposition of the non-negative semigroup $U_{\{\alpha\}}^{\geq 0}$.

For $\text{SO}(3, 2 + k)$, where the Θ -Weyl group is of type B_p , an explicit parametrization of the non-negative semigroup $U_\Theta^{\geq 0}$ has been provided in [29, Section 11], but even in this case, the full structure of parametrizations of the non-negative semigroup is not known.

Open Questions 2.4. Here are several open questions regarding the non-negative semigroups $U_\Theta^{\geq 0}$.

- (1) Describe the strata $U_{\Theta, a}^{\geq 0}$, $U_{\Theta, x}^{\geq 0}$, and $U_{\Theta, w}^{\geq 0}$ of the non-negative semigroup $U_\Theta^{\geq 0}$ with respect to the different decompositions of G above.
- (2) Provide parametrizations of $U_\Theta^{\geq 0}$ and of the strata $U_{\Theta, a}^{\geq 0}$, $U_{\Theta, x}^{\geq 0}$, and $U_{\Theta, w}^{\geq 0}$.
- (3) Find explicit cell decompositions of $U_\Theta^{\geq 0}$.

2.7. Positive parts of flag varieties

Totally positive and non-negative semigroups give rise to positive or non-negative parts in flag varieties. The positive Grassmannian has received a lot of interest not only in the mathematical community but also in theoretical physics; see Williams ICM talk [43] for an overview.

If G admits a positive structure with respect to Θ , the non-negative semigroup $G_\Theta^{\geq 0}$ and the positive semigroup $G_\Theta^{> 0}$ also give rise to non-negative parts

and positive parts of flag varieties of the group G . These are introduced and studied in [30].

Let us give a short description here. Let $\mathcal{F} = G/P$ be a flag variety of G . The non-negative part of the flag variety $\mathcal{F}^{\geq 0}$ with respect to Θ is

$$\mathcal{F}^{\geq 0} := \{G_{\Theta}^{\geq 0} P\} = \{uPu^{-1} \mid u \in G_{\Theta}^{\geq 0}\},$$

where P is a parabolic subgroup compatible with P_{Θ} , i.e., such that $P \cap P_{\Theta}$ is a parabolic subgroup.

The positive part $\mathcal{F}^{>0}$ of the flag variety $\mathcal{F}$ is

$$\mathcal{F}^{>0} := \{G_{\Theta}^{>0} P\} = \{uPu^{-1} \mid u \in G_{\Theta}^{>0}\},$$

where P is a parabolic subgroup compatible with P_{Θ} . The non-negative part of the flag variety $\mathcal{F}^{\geq 0}$ is the closure of $\mathcal{F}^{>0}$ in $\mathcal{F}$.

If $P = P_{\Theta}$, the positive part of $\mathcal{F}$ is what is called a diamond in [27, 29], and the non-negative part is the closure of a diamond.

In the case of the symplectic group (with the Θ -Weyl group being of type A_1), the positive and non-negative parts of the flag varieties have a nice geometric description; see [30]. The non-negative part of the Lagrangian Grassmannian is studied in detail in [44]. In general, the structure of the non-negative and positive parts of flag varieties when G is not necessarily split and the Θ -Weyl group is of rank at least 2 is not yet well understood.

Open Questions 2.5. Here are some open problems.

- (1) Provide explicit parametrizations of the positive and non-negative parts of flag varieties.
- (2) Give cell decompositions of the positive and non-negative parts of flag varieties. In the case of total positivity, for Grassmannians these are parametrized by positroid varieties.
- (3) When G is a split real form, which admits another positive structure with respect to a proper subset Θ in Δ , describe precisely the relation between the positive and non-negative parts of the flag variety, with respect to Δ and with respect to Θ .
- (4) The amplituhedron, originally presented in [4], is the image of the positive Grassmannian under a positive linear map. It plays a major role in the understanding of scattering amplitudes in gauge theories. Introduce a generalization of the amplituhedron for positive structures with respect to Θ .

In the totally positive case, the positive parts of flag varieties can be described by cluster coordinates, so we expect that the non-commutative cluster structures associated to groups admitting a positive structure, combined with a good understanding of

the situation when the Θ -Weyl group is of type A_1 , should be useful in answering the above questions.

Thus, now we turn to describe in more detail the non-commutative perspective on Lie groups admitting a positive structure with respect to Θ , with a special focus on the case when the Θ -Weyl group is of type A_1 . For this, we will first recall some aspects of the simple real Lie group of type A_1 , namely, $SL_2(\mathbb{R})$.

3. Aspects of $SL_2(\mathbb{R})$

The Lie group $SL_2(\mathbb{R})$ is the simplest real simple Lie group. It (and its Lie algebra) plays an important role in understanding the structure of any real simple Lie group. In this section, we just highlight some aspects of $SL_2(\mathbb{R})$ which will play a role later when we consider Lie groups of type SL_2 over non-commutative rings.

3.1. $SL_2(\mathbb{R})$ as $Sp_2(\mathbb{R})$

Usually, we think of $SL_2(\mathbb{R})$ as the subgroup of the general linear group $GL_2(\mathbb{R})$ of determinant 1. However, here we would like to think of $SL_2(\mathbb{R})$ as $Sp_2(\mathbb{R})$, the subgroup invertible endomorphism preserving a symplectic form ω .

The symplectic form ω is a non-degenerate skew-symmetric bilinear form

$$\omega : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R},$$

defined by $\omega((x_1, x_2), (y_1, y_2)) = x_1 y_2 - x_2 y_1$.

Identifying linear endomorphism of $\mathbb{R}^2$ with 2×2 matrices, we get

$$Sp_2(\mathbb{R}) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \det g = ad - bc = 1 \right\} = SL_2(\mathbb{R}).$$

3.2. The symmetric space

The group $Sp_2(\mathbb{R})$ acts by isometries on the hyperbolic plane $\mathbb{H}^2$, which is the symmetric space associated to $Sp_2(\mathbb{R})$. There are several different models of the hyperbolic plane, for example, the Poincaré disk model or the upper half-plane model.

The most natural model for the symmetric space of $Sp_2(\mathbb{R})$ is the space $\mathcal{J}$ of ω -compatible almost complex structures on $\mathbb{R}^2$. An almost complex structure on $\mathbb{R}^2$ is a linear endomorphism, whose square is $-\text{id}$. An almost complex structure J is ω -compatible if

$$\omega(\cdot, J\cdot) : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$$

is a positive definite symmetric bilinear form.

If we consider $\mathbb{C}^2 = \mathbb{R}^2 \otimes \mathbb{C}$ the complexification of $\mathbb{R}^2$, any almost complex structure J can be extended $\mathbb{C}$ -linearly to an endomorphism $J_{\mathbb{C}}$ of $\mathbb{C}^2$. This endomorphism now has two eigenspaces, V_J^+ and V_J^- , with eigenvalues $+i$ and $-i$, respectively. The symplectic form ω extends as well $\mathbb{C}$ -linearly to a symplectic form $\omega_{\mathbb{C}}$ and gives rise to a non-degenerate Hermitian form $h(\cdot, \cdot) = i\omega_{\mathbb{C}}(\bar{\cdot}, \cdot)$, where $\bar{\cdot}$ is the complex conjugation. The Hermitian form h is of signature $(1, 1)$, and h restricted to V_J^+ is positive definite, h restricted to V_J^- is negative definite. This provides two embeddings $i^{\pm} : \mathcal{J} \rightarrow \mathbb{C}P^1$, with $\overline{i^+(J)} = i^-(J)$ and $i^+(\mathcal{J}) \cup i^-(\mathcal{J}) = \mathbb{C}P^1 \setminus \mathbb{R}P^1$.

The Poincaré model and the upper half-plane model arise now as the image of $i^+(\mathcal{J})$ with respect to two different affine charts for $\mathbb{C}P^1$. For the Poincaré model, consider the affine chart with respect to $V_{J_0}^-$, where J_0 is the standard complex structure $J_0(x_1, x_2) = (-x_2, x_1)$. For the upper half-plane model, consider the affine chart with respect to the complex line spanned by the vector e_2 .

3.3. Trace identities

There are two invariant functions of 2×2 matrices: the trace (tr) and the determinant ($\det$). By the classical Cayley–Hamilton theorem, any 2×2 matrix satisfies the relation

$$M^2 - \text{tr}(M)M + \det(M)\text{id} = 0.$$

Restricting to $M \in \text{Sp}_2(\mathbb{R})$, where the determinant is 1, taking the trace of the Cayley–Hamilton relation, one obtains the trace identity

$$\text{tr}(M^2) = \text{tr}(M)^2 - 2.$$

This can be generalized and leads to the well-known SL_2 -trace identity, going back to Fricke and Klein’s work [18].

For any $M, N \in \text{Sp}_2(\mathbb{R})$, we have

$$\text{tr}(MN) + \text{tr}(MN^{-1}) = \text{tr}(M)\text{tr}(N),$$

and for the commutator,

$$\text{tr}(MNM^{-1}N^{-1}) = \text{tr}(M)^2 + \text{tr}(N)^2 + \text{tr}(MN)^2 - \text{tr}(M)\text{tr}(N)\text{tr}(MN) - 2. \quad (3.1)$$

Trace identities have played a major role in the study of discrete subgroups of $\text{SL}_2(\mathbb{R})$ and the study of representation varieties. Since the trace is a conjugation invariant function, when S is a cylinder, $\pi_1(S) = \mathbb{Z}$, any representation $\pi_1(S) \rightarrow \text{SL}_2(\mathbb{R})$ is essentially determined by the trace of a generator. For general surfaces, the traces of generators and some products of generators of the fundamental group give explicit coordinates for the character variety [38].

3.4. Markov numbers

The trace identities allow us to connect representations $\pi_1(S_{1,1}) \rightarrow \mathrm{SL}_2(\mathbb{R})$ that arise from hyperbolic structures on the once-punctured torus $S_{1,1}$ to Markov numbers.

The fundamental group $\pi_1(S_{1,1})$ is a free group on two generators A and B . Any representation $\pi_1(S_{1,1}) \rightarrow \mathrm{SL}_2(\mathbb{R})$ (up to conjugation) is completely determined by three coordinates: $a = \mathrm{tr}(A)$, $b = \mathrm{tr}(B)$, $c = \mathrm{tr} AB$. To make the connection with Markov numbers, one considers only representations which arise as holonomy representations of complete hyperbolic structures on $S_{1,1}$ of finite volume. For such representations, the trace of the commutator $[A, B]$ is -2 .

Therefore, the trace identity becomes

$$a^2 + b^2 + c^2 - abc = 0. \quad (3.2)$$

This relates the study of hyperbolic structures on a punctured torus to the classical theory of Markov triples [37], which are of interest in number theory. Markov triples are integer solutions to

$$a^2 + b^2 + c^2 - 3abc = 0. \quad (3.3)$$

Note that these two equations are related; if $(3a, 3b, 3c)$ is a solution to the first equation, then (a, b, c) is a solution to the second equation.

The first surprising fact is that given a Markov triple (a, b, c) , the triple $(\frac{b^2+c^2}{a}, b, c)$ is also a Markov triple. As proved by Markov, every Markov triple can be obtained from the “trivial” solution $(1, 1, 1)$ by repeatedly applying this rule at each entry.

Markov numbers are well studied in number theory [1, 20].

3.5. Cluster algebras

A cluster algebra is an algebra with a rich additional combinatorial structure. Informally, the algebra can be generated from an initial “seed” via a process called “mutation” which produces new seeds from old seeds. The family of seeds is called a cluster structure of the algebra. Let us shortly recall the cluster structure on the once-punctured torus $S_{1,1}$, which will give us a link to Markov numbers:

$$x_\gamma x_{\gamma'} = x_\alpha x_\eta + x_\beta x_\delta. \quad (3.4)$$

A seed for the cluster structure of the surface $S_{1,1}$ consists of an ideal triangulation T of $S_{1,1}$ along with a choice of cluster variable x_γ for each arc γ of T . Each seed $(T, \vec{x})$ can be mutated at an arc γ to produce a new seed $\mu_\gamma(T, \vec{x}) = (T', \vec{x}')$. The triangulation T' is obtained by removing γ and replacing it with the arc γ' corresponding to the opposite quadrilateral containing γ (Figure 1). We name the edges of the bounding quadrilateral cyclically by $\alpha, \beta, \eta, \delta$. The cluster variables in $\vec{x}'$ are the same as

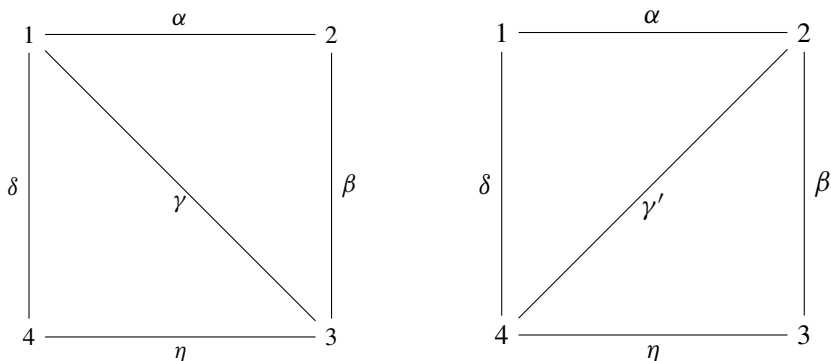


Figure 1. Mutation at the arc γ .

the variables of $\vec{x}$ except x_γ is replaced by $x_{\gamma'}$ which satisfies the exchange relation. The cluster algebra associated to the surface $S_{1,1}$ with initial seed $(T, x_1, \dots, x_{|T|})$ is the subalgebra of the ring of rational functions in the variables of the initial seed, $\mathbb{Q}(x_1, \dots, x_{|T|})$, generated by all cluster variables obtained by repeated mutation from the initial seed.

The cluster algebra associated to a triangulated surface can be geometrically realized as the coordinate ring of decorated Teichmüller space. The decorated Teichmüller space of $S_{1,1}$ is the moduli space of marked complete finite volume hyperbolic metrics on $S_{1,1}$ along with a decoration consisting of a choice of horocycle around each cusp. The cluster coordinate a_γ associated to an arc γ is the λ -length of the geodesic arc in the homotopy class of γ for the given hyperbolic metric. (See [17] for full details.) Importantly, [16] finds that Penner's decorated Teichmüller space is parametrized by the positive $\mathbb{R}$ -points of this cluster algebra.

3.6. Cluster algebra and Markov triples

Considering the cluster structure on the once-punctured torus, we can now recover the Markov equation and Markov triples again.

Each Markov triple corresponds to a seed of this cluster algebra (Figure 2). The cluster mutation rule for this seed gives that the mutated coordinate a' satisfies the relation $aa' = b^2 + c^2$. This is exactly the mutation of Markov triples described above. There is a mutation invariant function on this cluster algebra $f(a, b, c) = 2 \frac{a^2 + b^2 + c^2}{abc}$. When starting with the triple $(1, 1, 1)$, we see that $f(1, 1, 1) = 6$. Since this function is fixed by mutation, every other triple obtained by mutation also satisfies $f(a, b, c) = 6$, which is equivalent to the Markov equation. The fact that all triples obtained by mutation are positive integers is a consequence of the positivity of the cluster exchange relation.

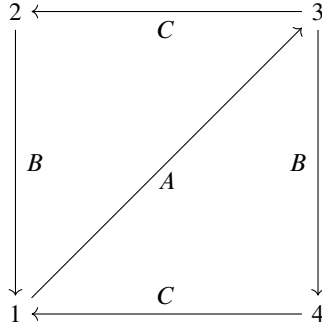


Figure 2. The Markov quiver and a seed associated to a triangulation of a punctured torus.

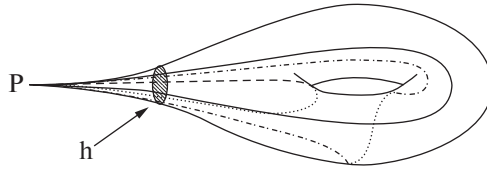


Figure 3. An ideal triangulation of a punctured torus with a horocycle. Picture taken from [13].

There is also a geometric interpretation of the mutation invariant function as the sum of “angles”

$$t_i^{jk} = \frac{a_{jk}}{a_{ij}a_{ki}}$$

in the triangulation. Each of these angles is the length of the horocycle at vertex i inside the triangle whose edges are labeled with the cluster variables a_{ij}, a_{jk}, a_{ki} (Figure 3). On the punctured torus, the total length of the horocycle is then given by the Markov function

$$f(a, b, c) = 2 \left(\frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab} \right) = 2 \frac{a^2 + b^2 + c^2}{abc}. \quad (3.5)$$

The length of the horocycle is clearly independent of the chosen triangulation. Furthermore, the mutation of the Markov quiver leaves the adjacency of the triangles fixed. Thus the exact form of the function f is invariant under mutation.

4. Symplectic groups over non-commutative rings

We saw above that the Θ -Weyl group plays a key role in understanding the positive structures with respect to Θ . In fact, a lot of the structure we see in [29, 30] suggests that if G admits a positive structure with respect to Θ , it might heuristically be seen as a group of type $W(\Theta)$ over a (partly) non-commutative ring.

For Hermitian Lie groups of tube type, the Θ -Weyl group is of type A_1 , and thus we would like to see a Hermitian Lie group of tube type as a group of type Sp_2 over a non-commutative ring. This leads to the definition of groups $\mathrm{Sp}_2(A, \sigma)$ over non-commutative rings A with involution σ in [2, 40], and in fact, many of the things we described in Section 3 can be generalized in this non-commutative setting. We first describe the construction of the groups and their symmetric space, before we discuss the relation with non-commutative cluster algebra, introduce non-commutative generalizations of Markov numbers, and construct $\mathrm{Sp}_2(A, \sigma)$ quantum groups.

We start with an involutive ring $(\mathcal{A}, \sigma)$, where $\mathcal{A}$ is a (usually) non-commutative ring with unit and where every left inverse is also a right inverse and $\sigma: \mathcal{A} \rightarrow \mathcal{A}$ an anti-involution, i.e., $\sigma(AB) = \sigma(B)\sigma(A)$, and $\sigma(\sigma(A)) = A$ for all $A, B \in \mathcal{A}$. We denote by $\mathcal{A}^\sigma$ the set of fixed points of σ . A key example of an involutive ring is $\mathrm{Mat}_n(k)$ the ring of $n \times n$ matrices over a field k with transpose as the anti-involution. Here the set of fixed points is the set of symmetric matrices. When $k = \mathbb{K} = \mathbb{C}, \mathbb{H}$, and σ is the conjugate transpose, we have $\mathcal{A}^\sigma = \mathrm{Herm}(n, \mathbb{K})$.

We denote by $\mathrm{Mat}_2(\mathcal{A}, \sigma)$ the ring of 2×2 matrices over $\mathcal{A}$.

We consider $\mathcal{A}^2 = \mathcal{A} \times \mathcal{A}$ as an $\mathcal{A}$ -module. Given

$$\Omega_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in \mathrm{Mat}_2(\mathcal{A}, \sigma),$$

we can define an $\mathcal{A}$ -valued skew-symmetric right-bilinear form $\omega: \mathcal{A}^2 \times \mathcal{A}^2 \rightarrow \mathcal{A}$ by

$$\omega(\vec{v}, \vec{w}) = \sigma(\vec{v})^T \Omega_1 \vec{w}.$$

This form is right bilinear in the sense that

$$\omega(\vec{v}B, \vec{w}A) = \sigma(B)\omega(\vec{v}, \vec{w})A$$

for $A, B \in \mathcal{A}$.

The symplectic group is given by the elements of $\mathrm{Mat}_2(\mathcal{A})$ which preserve this form.

Definition 4.1. The symplectic group $\mathrm{Sp}_2(\mathcal{A}, \sigma)$ is the set of right $\mathcal{A}$ -linear endomorphisms of $\mathcal{A}^2$ preserving ω :

$$\mathrm{Sp}_2(\mathcal{A}, \sigma) = \{f \in \mathrm{End}(\mathcal{A}^2) \mid \forall \vec{v}, \vec{w} \in \mathcal{A}^2 : \omega(f(\vec{v}), f(\vec{w})) = \omega(\vec{v}, \vec{w})\}.$$

In matrix form, this condition is equivalent to

$$\mathrm{Sp}_2(\mathcal{A}, \sigma) = \{M \in \mathrm{Mat}_2(\mathcal{A}) \mid \sigma(M)^T \Omega_1 M = \Omega_1\}.$$

When $\mathcal{A} = \mathbb{R}$ and σ is the trivial involution, then $\mathrm{Sp}_2(\mathcal{A}, \sigma) \simeq \mathrm{Sp}_2(\mathbb{R})$. More generally, when $\mathcal{A} = \mathrm{Mat}_n(k)$ is the ring of $n \times n$ matrices over a field k , and σ the matrix transpose, we have $\mathrm{Sp}_2(\mathcal{A}, \sigma) \simeq \mathrm{Sp}(2n, k)$. In fact, the classical Hermitian Lie groups of tube type can be realized essentially as $\mathrm{Sp}_2(\mathcal{A}, \sigma)$ for an appropriate choice of rings (see [2, 40]); for the indefinite orthogonal groups $\mathrm{SO}(2, n)$, a small twist is necessary. When $\mathcal{A} = \mathrm{Mat}_n(\mathbb{C})$ and σ is the conjugate transpose, then $\mathrm{Sp}_2(\mathcal{A}, \sigma) \simeq \mathrm{U}(n, n)$, and if $\mathcal{A} = \mathrm{Mat}_n(\mathbb{H})$ and σ is the conjugate transpose, then $\mathrm{Sp}_2(\mathcal{A}, \sigma) \simeq \mathrm{SO}^*_{(4n)}$. To realize the family of Hermitian Lie groups of tube type which are isogenous to $\mathrm{SO}(2, q)$, one has to adapt the construction a bit. In this case, $\mathcal{A}$ is the Clifford algebra associated to $\mathbb{R}^{1, q-1}$, and we have to consider a subgroup of $\mathrm{Sp}_2(\mathcal{A}, \sigma)$, where matrix coefficients take values in the Clifford group. For the exceptional group, one would like to take $\mathcal{A} = \mathrm{Mat}_3(\mathbb{O})$, but this is not a ring. Thus, we cannot describe the group as $\mathrm{Sp}_2(\mathcal{A}, \sigma)$, but most of the structural aspects of the group and the Lie algebra still survive and can be described nicely in terms of Euclidean Jordan algebras.

Note that the choice of anti-involution is crucial and dramatically affects the structure of $\mathrm{Sp}_2(\mathcal{A}, \sigma)$ even if the underlying algebra is commutative. Consider $\mathcal{A} = \mathbb{C}$, the complex numbers. In the case of the trivial involution, the group $\mathrm{Sp}_2(\mathcal{A}, \sigma)$ is simply $\mathrm{SL}_2(\mathbb{C})$, but in the case that the σ is complex conjugation, the group $\mathrm{Sp}_2(\mathcal{A}, \sigma)$ can be computed to be $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{U}(1)$.

Above, we described the symmetric space associated to $\mathrm{Sp}_2(\mathbb{R})$ as the space of compatible almost complex structures on $\mathbb{R}^2$ and derived several other models for the hyperbolic plane from that. Analogous constructions can be made for the groups $\mathrm{Sp}_2(\mathcal{A}, \sigma)$, starting with complex structure on $\mathcal{A}$ -modules, where a complex structure on a right $\mathcal{A}$ -module V is an $\mathcal{A}$ -linear map $J: V \rightarrow V$ such that $J^2 = -\mathrm{id}$. The space of ω -compatible almost complex structures on $\mathcal{A}^2$ then gives a model of the symmetric space associated to $\mathrm{Sp}_2(\mathcal{A}, \sigma)$. From this model, one derives several other models, a projective model, a bounded (or precompact) model, and an upper half-space model. Thus, we realize the symmetric space as a hyperbolic plane over a non-commutative ring with involution. For details, we refer the reader to [2].

When specifying to the rings so that $\mathrm{Sp}_2(\mathcal{A}, \sigma)$ is isomorphic to a classical Lie group G , this general construction provides nice models for the symmetric spaces of G . For classical Hermitian Lie groups of tube type, the projective model gives the Borel embedding, the precompact model gives the bounded symmetric domain model, and the upper half-space model gives the Siegel upper half-space model. But even in these cases, we obtain some new information. For example, the realization of the symmetric spaces as ω -compatible almost complex structures is usually only described for $\mathrm{Sp}(2n, \mathbb{R})$ but not for $\mathrm{SU}(n, n)$ or $\mathrm{SO}^*(4n)$. Also for the complexifications of Hermitian Lie groups of tube type, we get new models of the symmetric spaces that to our knowledge have not been described so far; see [2] for the details.

4.1. Trace identities

In [25], we derived generalizations of the $\mathrm{Sp}_2(\mathbb{R})$ -trace identities for the groups $\mathrm{Sp}_2(\mathcal{A}, \sigma)$.

For this, first we recall that $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is $\mathrm{Sp}_2(\mathcal{A}, \sigma)$ if it satisfies the following equations:

$$\begin{aligned}\sigma(A)C - \sigma(C)A &= 0, \\ \sigma(B)D - \sigma(D)B &= 0, \\ \sigma(A)D - \sigma(C)B &= 1 = \sigma(D)A - \sigma(B)C.\end{aligned}$$

We extend σ to a map $\sigma: \mathrm{Mat}_2(\mathcal{A}, \sigma) \rightarrow \mathrm{Mat}_2(\mathcal{A}, \sigma)$ by setting

$$\sigma\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = \begin{bmatrix} \sigma(A) & \sigma(B) \\ \sigma(C) & \sigma(D) \end{bmatrix}.$$

Note that the map σ does not respect the ring structure on $\mathrm{Mat}_2(\mathcal{A}, \sigma)$ in any way. In particular, for $M, N \in \mathrm{Mat}_2(\mathcal{A}, \sigma)$, $\sigma(MN)$ is not equal to $\sigma(N)\sigma(M)$. The involution interacts however with the matrix transpose M^T

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^T = \begin{bmatrix} A & C \\ B & D \end{bmatrix}.$$

Note that when $\mathcal{A}$ is non-commutative, the matrix transpose also does not respect the multiplicative structure on $\mathrm{Mat}_2(\mathcal{A}, \sigma)$, but the composite transform $M \mapsto \sigma(M)^T$ is an anti-homomorphism.

We introduce $\mathcal{A}$ -valued functions on $\mathrm{Mat}_2(\mathcal{A}, \sigma)$, which we call the trace and determinant. When $\mathcal{A}$ is commutative and $\sigma = \mathrm{id}$, these are precisely the trace and determinants of 2×2 matrices. However, when σ is not the identity and $\mathcal{A}$ is potentially non-commutative, they have quite different properties.

The left determinant $\det_l: \mathrm{Mat}_2(\mathcal{A}, \sigma) \rightarrow \mathcal{A}$ is given by

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \mapsto \sigma(A)D - \sigma(C)B.$$

The right determinant $\det_r: \mathrm{Mat}_2(\mathcal{A}, \sigma) \rightarrow \mathcal{A}$ is given by

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \mapsto A\sigma(D) - B\sigma(C).$$

Note that $\det_r(M) = \det_l(\sigma(M)^T)$. The trace $\mathrm{tr}: \mathrm{Mat}_2(\mathcal{A}, \sigma) \rightarrow \mathcal{A}$ is given by

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \mapsto A + D.$$

We note that the trace function satisfies some interesting properties: For all $M, N \in \text{Mat}_2(\mathcal{A}, \sigma)$, we have

$$\text{tr}(\sigma(MN)) = \text{tr}(\sigma(N) \cdot \sigma(M)).$$

Note that in general $\sigma(MN)$ is not equal to $\sigma(N)\sigma(M)$. If the involution σ is trivial and thus $\mathcal{A}$ is commutative, the above identity reduces to the classical commutativity property of the trace $\text{tr}(MN) = \text{tr}(NM)$, which further implies the conjugation invariance of the trace function $\text{tr}(NMN^{-1}) = \text{tr}(M)$. In general, the trace function is however not invariant.

If $M \in \text{Sp}_2(\mathcal{A}, \sigma)$, then $M^{-1} = \begin{bmatrix} \sigma(D) & -\sigma(B) \\ -\sigma(C) & \sigma(A) \end{bmatrix}$, and $\text{tr}(M^{-1}) = \sigma(\text{tr}(M))$. This is familiar from $\text{Sp}_2(\mathbb{R})$.

In [25], several trace identities are established.

(1) *Classical trace identity*: For $M, N \in \text{Mat}_2(\mathcal{A})$, we have

$$\text{tr}(\text{Adj}(M)N) + \text{tr}(\sigma(M)N) - \sigma(\text{tr}(M))\text{tr}(N) = 0,$$

where the adjoint matrix of $M \in \text{Mat}_2(\mathcal{A}, \sigma)$ is given by

$$\text{Adj}(M) = \begin{bmatrix} \sigma(D) & -\sigma(B) \\ -\sigma(C) & \sigma(A) \end{bmatrix}.$$

(2) *Cayley–Hamilton theorem*: An element $M \in \text{Mat}_2(\mathcal{A}, \sigma)$ lies in $\text{Sp}_2(\mathcal{A}, \sigma)^l$ if and only if it satisfies

$$\sigma(M)M - \sigma(\text{tr}(M))M + \text{id} = 0.$$

These trace identities are established in [25] by explicit computations, but we expect that there is a more natural underlying structure, which relies on considering a non-commutative algebra $\mathcal{A}$ with involution σ and modules over $\mathcal{A}$.

5. Relation to Manin matrices

There is an interesting connection to Manin matrices, introduced by Manin in [35,36]. They define some kind of SL_2 over non-commutative rings and play a role in quantizations of SL_2 .

A matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \text{Mat}_2(R)$ is Manin if it satisfies the Manin commutativity equations:

$$AC - CA = 0, \quad BD - DB = 0, \quad AD - CB = DA - BC.$$

Note that in the first two of Manin's commutativity equations, if we replace products xy with $\sigma(x)y$, we get precisely the first two conditions for a matrix to be in $\mathrm{Sp}_2(\mathcal{A}, \sigma)$. If we add the additional constraint to a Manin matrix that $AD - CB = 1$, then the third equation matches up as well.

Manin matrices can be characterized by their inverse: A matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is Manin if and only if its left inverse is given by $\frac{1}{AD-CB} \begin{bmatrix} D & -B \\ -C & A \end{bmatrix}$ and also by satisfying the Cayley–Hamilton theorem. Additionally, a matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is Manin if and only if it satisfies

$$M^2 - \mathrm{tr}(M)M + \mathrm{id} = 0.$$

If a matrix and its transpose are both Manin, then the entries are all forced to pairwise commute. In our situation, this condition simply means that $M \in \mathrm{Sp}_2(\mathcal{A}, \sigma)$. This makes $\mathrm{Sp}_2(\mathcal{A}, \sigma)$ a more natural generalization of SL_2 over non-commutative rings than Manin matrices. For more similarities, we refer the reader to [25].

6. Quantum groups

The symplectic groups $\mathrm{Sp}_2(\mathcal{A}, \sigma)$ can be quantized, and this quantization parallels the quantization of SL_2 [31, 35, 36].

Informally, the quantum SL_2 is a deformation in a parameter q of the algebra of matrix functions on SL_2 . Formally, we fix a field k . Let $k_q = k(q)$ be the extension of k by an element q . Often, q is taken to be an element of k , although this introduces additional subtlety if q is a root of unity. The coordinate ring of the quantum plane $\mathcal{A}_q$ is the algebra $k_q[x_1, x_2]$ such that x_1 and x_2 q -commute, i.e., $x_1x_2 = q^{-1}x_2x_1$. Intuitively, the quantization of $\mathrm{SL}_2(k)$ should act like the endomorphisms of the quantum plane.

The algebra of quantum matrix functions $\mathcal{O}(\mathrm{SL}_2)_q$ is the ring $k_q\{a, b, c, d\}/I$ where I is generated by the quantum SL_2 relations:

$$\begin{aligned} ac - q^{-1}ca &= 0, & ab - q^{-1}ba &= 0, \\ bd - q^{-1}db &= 0, & cd - q^{-1}dc &= 0, \\ ad - q^{-1}cb &= da - qbc, & bc &= cb, \\ ad - q^{-1}cb &= 1. \end{aligned} \tag{6.1}$$

These equations can be obtained by adding the appropriate power of q to the equations for a matrix and its transpose to be Manin. To better compare with $\mathrm{Sp}_2(\mathcal{A}, \sigma)$,

we can rewrite the quantum SL_2 relations as follows:

$$\begin{aligned}
 ac - q^{-1}ca &= 0, & ab - q^{-1}ba &= 0, \\
 bd - q^{-1}db &= 0, & cd - q^{-1}dc &= 0, \\
 ad - q^{-1}cb &= 1, & ad - q^{-1}bc &= 1, \\
 da - qbc &= 1, & da - qcb &= 1.
 \end{aligned} \tag{6.2}$$

This algebra is actually a Hopf algebra. We will not address this here but refer the reader to [25, 31] for more details.

To quantize $Sp_2(\mathcal{A}, \sigma)$, we first add a quantum parameter q to our algebra $\mathcal{A}$. Formally, let $\mathcal{A}_q = \mathcal{A}[q^{\pm}]$ with the additional assumption that q commutes with all of $\mathcal{A}$. We then extend σ to $\mathcal{A}_q$ by setting $\sigma(q) = q^{-1}$.

Following the quantization of $SL_2(\mathbb{R})$, we define a system of equations on the matrix entries of a 2×2 matrix specifying the quantum commutativity and quantum determinant of the elements.

The quantum symplectic equations for a matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \text{Mat}_2(\mathcal{A}_q)$ are

$$\begin{aligned}
 \sigma(A)C - q^{-1}\sigma(C)A &= 0, & A\sigma(B) - q^{-1}B\sigma(A) &= 0, \\
 \sigma(B)D - q^{-1}\sigma(D)B &= 0, & C\sigma(D) - q^{-1}D\sigma(C) &= 0, \\
 \sigma(A)D - q^{-1}\sigma(C)B &= 1, & A\sigma(D) - q^{-1}B\sigma(C) &= 1.
 \end{aligned} \tag{6.3}$$

Note that we consider $\sigma(A)D - q^{-1}\sigma(C)B$ to be the quantum determinant.

Let us point out that these equations correspond both to standard symplectic equations and to rephrased quantum $SL_2(\mathbb{R})$ equations (6.2). We stress that these equations do not (!) imply any commutativity between B and C .

Usually, when we define a quantum deformation of an algebra, we are constructing a non-commutative algebra out of a commutative one. Here we already start with a non-commutative algebra, so its quantization is a much less drastic change. In fact, the corresponding quantization can be seen simply as a change of symplectic form. The equations above are equivalent to M having a two-sided inverse and preserving the $\mathcal{A}_q$ -valued skew-symmetric right-bilinear form $\omega_q : \mathcal{A}_q^2 \times \mathcal{A}_q^2 \rightarrow \mathcal{A}_q$ by

$$\omega_q(\vec{v}, \vec{w}) = \sigma(\vec{v})^T \Omega_q \vec{w},$$

with

$$\Omega_q = \begin{bmatrix} 0 & 1 \\ -q^{-1} & 0 \end{bmatrix} \in \text{Mat}_2(\mathcal{A}_q).$$

The corresponding group $Sp_2(\mathcal{A}, \sigma)_q$ acts on the left quantum plane, $(\mathcal{A}_\omega^2)_q$, which is the set of generic left q -commuting elements of $\mathcal{A}_q^2$:

$$(\mathcal{A}_\omega^2)_q = \{\vec{v} = (V_1, V_2) \in \mathcal{A}_q^2 \mid \vec{v} \text{ generic and } q^{-1}\sigma(V_2)V_1 = \sigma(V_1)V_2\}.$$

In fact, and contrary to the commutative situation, the group $\mathrm{Sp}_2(\mathcal{A}, \sigma)_q$ has many elements if we assume that q has a square root $q^{1/2}$. This is very different from the usual quantization of $\mathrm{SL}_2(\mathbb{R})$ where the corresponding group is just $\left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \mid a \in \mathbb{R} \right\}$. Here $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}_2(\mathcal{A}, \sigma)$, then $\begin{bmatrix} A & q^{1/2}B \\ q^{-1/2}C & D \end{bmatrix} \in \mathrm{Sp}_2(\mathcal{A}, \sigma)_q$.

One can further introduce the algebra of quantum symplectic matrix functions and endow it with an appropriate Hopf algebra-like structure. For this construction, we refer the reader to [25]. There the reader will also find a discussion of the relation to the classical algebra of quantum matrix functions when $\mathcal{A} = k$ is a field and σ is the trivial involution.

Open Question 6.1. Is there an interpretation of this quantization in terms of double Poisson brackets?

7. Non-commutative cluster algebras

In [6], Berenstein and Retakh introduced a non-commutative generalization of the A_1 -cluster algebras associated to a surface.

For the non-commutative A_1 -type cluster algebra associated to a surface S , to construct a seed, we start again with an ideal triangulation of the surface S . Each possible orientation of an edge $i \rightarrow j$ of the triangulation is assigned a variable X_{ij} in a non-commutative algebra. There is an additional condition for each triangle cyclically labeled i, j, k . Berenstein and Retakh introduce a non-commutative angle $T_i^{jk} = X_{ji}^{-1} X_{jk} X_{ik}^{-1}$. The seed has to satisfy the triangle condition: $T_i^{jk} = T_i^{kj}$. The non-commutative cluster algebra is then again grown through mutation. Each mutable edge $i \rightarrow k$ is contained in a unique quadrilateral with vertices (i, j, k, ℓ) . The mutated seed contains the triangulation with $i \rightarrow k$ replaced with $j \rightarrow \ell$. The element assigned to $j \rightarrow \ell$ is given by the formula

$$X_{j\ell} = X_{jk} X_{ik}^{-1} X_{i\ell} + X_{ji} X_{ki}^{-1} X_{k\ell}.$$

We want to relate the variables associated to the same edge with different orientations, so we set $\sigma(X_{ij}) = X_{ji}$ and extend σ as an anti-involution. Using σ , we see that the triangle condition reads as

$$T_i^{jk} = T_i^{kj} = \sigma(T_i^{jk})$$

so that each non-commutative angle is in $C(S)^\sigma$.

The mutation rule is equivalent to the requirement that the angles are additive:

$$\begin{aligned} T_i^{j\ell} &= T_i^{jk} + T_i^{k\ell}, \\ X_{ji}^{-1} X_{j\ell} X_{i\ell}^{-1} &= X_{ji}^{-1} X_{jk} X_{ik}^{-1} + X_{ki}^{-1} X_{k\ell} X_{i\ell}^{-1}. \end{aligned}$$

The non-commutative cluster structure can also be used to parametrize the positive semigroups $U_{\Theta}^{>0}$ and $G_{\Theta}^{>0}$ for classical Hermitian Lie groups of tube type. This is discussed in [23, 25].

Recall that in the commutative A_1 -cluster algebra associated to the surface S , we had a geometric realization by looking at parametrizations of the decorated Teichmüller space by Penner's λ -lengths. The decorated Teichmüller space can be realized as the positive points of the $\mathbb{R}$ -points of the cluster algebra.

There is also a geometric realization of the non-commutative cluster algebra using positivity. In [3], the authors introduced non-commutative Λ -lengths for decorated representations $\pi_1(S) \rightarrow \mathrm{Sp}(2n, \mathbb{R})$ and show that these Λ -lengths give a geometric realization of the Berenstein–Retakh cluster algebra such that the appropriate positive points are precisely the space of decorated maximal representations, i.e., the space of positive representations with respect to the positive structure on $\mathrm{Sp}(2n, \mathbb{R})$ with respect to $\Theta = \{\alpha_n\}$. See also [25, 33, 40] for more general symplectic groups $\mathrm{Sp}_2(\mathcal{A}, \sigma)$ over non-commutative rings and [21, 33] for related work on GL_n over non-commutative rings.

These geometric realizations of the non-commutative cluster algebra of type A_1 provide non-commutative analogues of the $\mathcal{A}$ -cluster variety which was introduced by Fock and Goncharov in the setting of split real Lie groups. In [3], the authors also provide $\mathcal{X}$ -type coordinates, but they are not well adapted to the cluster structures. Goncharov and Shen [22] introduced $\mathcal{P}$ -cluster varieties, which seem to be more appropriate for a non-commutative generalization.

Open Questions 7.1. We propose the following problems:

- (1) Provide appropriate non-commutative $\mathcal{P}$ -cluster varieties of type A_1 .
- (2) In [15], tropical points of cluster varieties play an important role. In the A_1 -situation, they parametrize laminations on the surface. Provide appropriate tropical non-commutative $\mathcal{A}$ -cluster varieties and $\mathcal{P}$ -cluster varieties and geometrically describe what they parametrize.
- (3) For split real Lie groups, Fock and Goncharov [15] formulated wide-ranging duality conjectures, which have been proved in [26]. It is an open question if there are any dualities for Lie groups admitting a positive structure with respect to Θ .

8. Non-commutative Markov numbers

We now use the non-commutative A_1 -cluster algebra structure to obtain non-commutative generalizations of Markov numbers. For this, we replace the real numbers $\mathbb{R}$ by an involutive algebra $(\mathcal{A}, \sigma)$. For details, we refer the reader to [25]. We start with the quiver associated to the once-punctured torus; see Figure 4.

Now we do not have a horocycle and the length of the pieces of this horocycle to build our invariant function. We considered the lengths of horocycle arcs t_i^{jk} ; they are now “upgraded” to the non-commutative angle T_i^{jk} , which appear in the triangle conditions of the non-commutative cluster algebra. We require these angles to be fixed by the anti-involution σ .

Explicitly, in A, B, C are the seed variables (see Figure 4); these angles are all cyclic permutations of

$$\sigma(A^{-1})B\sigma(C^{-1}) \in \mathcal{A}^\sigma \quad \sigma(A^{-1})C\sigma(B^{-1}) \in \mathcal{A}^\sigma.$$

The non-commutative mutation at the node A is given by the equation

$$A' = \sigma(C)A^{-1}\sigma(C) + B\sigma(A^{-1})B. \quad (8.1)$$

The generalized Markov function, which is invariant under mutations, for a seed (A, B, C) whose triangles are cyclically oriented, is

$$\begin{aligned} F(A, B, C) = & \sigma(A^{-1})B\sigma(C^{-1}) + \sigma(B^{-1})C\sigma(A^{-1}) + \sigma(C^{-1})A\sigma(B^{-1}) \\ & + \sigma(C^{-1})B\sigma(A^{-1}) + \sigma(A^{-1})C\sigma(B^{-1}) + \sigma(B^{-1})A\sigma(C^{-1}). \end{aligned} \quad (8.2)$$

The reader might check that if the involution σ is trivial and a, b, c commute, this simplifies F to $F(a, b, c) = 2 \frac{a^2+b^2+c^2}{abc}$ which is the standard Markov function.

Now we can use the generalized Markov function to consider generalizations of Markov triples. In the classical case, we get Markov triples by starting with the seed

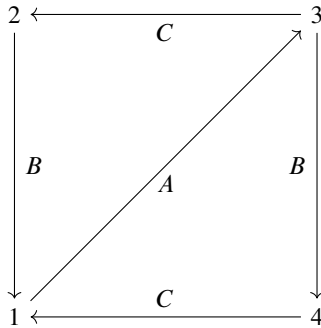


Figure 4. Seed associated to a triangulation of a punctured torus.

$(A, B, C) = (1, 1, 1)$, and by mutation, we get only integer solutions. There are two important things we have to take into account:

- (1) *Integrality*: From the cluster theory, there is nothing special about the integers as a subring of $\mathbb{R}$. But the Laurent phenomenon for cluster algebras states that every number obtained by mutation can be expressed as a Laurent polynomial in the original triple (a, b, c) . Thus, if (a, b, c) are units in any subring, all values obtained by mutation belong to that subring. Thus, for non-commutative generalizations, we are interested in starting with a seed that takes values in units of an appropriate subring of the integral elements in $(\mathcal{A}, \sigma)$. Recall that the integral elements of a ring $\mathcal{A}$ are the roots of monic polynomials with integer coefficients in $\mathcal{A}$.
- (2) *Positivity*: Another subtlety when choosing an initial triple is that we require every element of the cluster algebra to be invertible. Simply choosing a triple of invertible elements is not enough to guarantee that all elements obtained by mutation are invertible. For example, in $\mathbb{C}$ with the trivial involution, mutation at 1 in the triple $(1, 1, i)$ results in the triple $(0, 1, i)$. A sufficient solution to avoid these triples is to require that all non-commutative angles in the initial seed are “positive” in the appropriate sense.

Over $\mathbb{R}$, it is clear what positive means, but over arbitrary involutive rings, it is less clear. We define a positive structure on $(\mathcal{A}, \sigma)$ to be a subset P of $\mathcal{A}^\sigma$ containing 1 and be closed under addition, inversion, and σ -congruence ($A \mapsto \sigma(B)AB$) by invertible elements. Here we see the connection with the positive structure on Hermitian Lie groups of tube type.

Thus, putting these things together, we fix a subring R of integral elements of $\mathcal{A}$ and a positive structure P on $(\mathcal{A}, \sigma)$. An admissible seed (with respect to R and P) is then a triple $(a, b, c) \in \mathcal{A}^3$, where a, b, c are units in R and all the non-commutative angles are in P .

A simple way to get an admissible triple is to take a unit B in R , then the triple $(1, B, B^{-1})$ is admissible.

With this general setup, we now obtain several examples of non-commutative Markov triples.

8.1. Matrix Markov numbers

Consider $\mathcal{A} = \text{Mat}(n, \mathbb{R})$ with σ being the standard matrix transpose. We choose $R = \text{Mat}_n(\mathbb{Z})$ as the subring of integral elements to focus on. The positive structure P consists of the set of positive definite symmetric matrices.

There are many complicated examples of admissible triples; let us describe just the simplest one (see [25] for more examples). For this, let $B_t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$ and consider

the seed $(1, B_t, B_t^{-1})$. Then the value of the Markov function is

$$\begin{bmatrix} 6 + 2t^2 & 0 \\ 0 & 6 + 2t^2 \end{bmatrix}.$$

The matrices we obtain are matrix Markov numbers. Then $\frac{1}{2}$ times the trace of each matrix Markov number is a polynomial with integral coefficients $p(t)$, where $p(0)$ is a classical Markov number. Thus, in this case, $\frac{1}{2}$ trace of each matrix Markov number can be considered as a one-dimensional deformation of classical Markov numbers.

8.2. Complex and dual Markov numbers

We also obtain interesting examples for commutative rings with nontrivial involutions σ . One interesting example is to consider the complex numbers $\mathbb{C}$ with the involution σ which is the complex conjugation; see Figure 5. Then $\mathcal{A}^\sigma = \mathbb{R}$. For the ring of integral elements R , we take the roots of unity. The positive structure is given by the positive real numbers.

A simple admissible seed triple is a triple $z_1, z_2, z_3 \in \mathbb{C}$ of three roots of unity whose product is 1. If these roots of unity are chosen to be nontrivial, the complex Markov numbers we get differ from the real Markov numbers. In this case, the mutation rule splits into a mutation rule for the norm, which is just the classical mutation rule for Markov triples, and a mutation rule for the arguments. Therefore, the norms of these complex Markov numbers are classical real Markov numbers. It is however interesting to understand the arguments of these complex Markov numbers. In [25], the following conjecture is made.

Conjecture 8.1. *If the initial seed triple consists of three roots of unity (z_1, z_2, z_3) such that n is the minimal integer with $z_i^n = 1$, then every triple of $\frac{a}{n}, \frac{b}{n}, \frac{c}{n}$ that sums to 1 is achieved as the argument of a complex Markov triple. Furthermore, if (z_1, z_2, z_3) are irrational angles, then the image of the arguments of Markov triples is dense in the circle.*

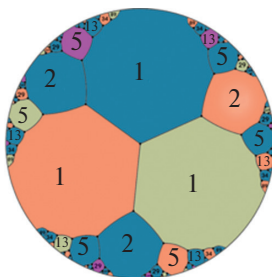


Figure 5. The Markov “tree” for complex Markov numbers.

Another interesting example is the dual numbers $\{a + \alpha\varepsilon \mid a, \alpha \in \mathbb{R}, \varepsilon^2 = 0\}$, and the subring R of integral elements of the form $a + \alpha\varepsilon$ with $a, \alpha \in \mathbb{Z}$. The units of this subring are elements of the form $1 + \alpha\varepsilon$ for $\alpha \in \mathbb{Z}$. We can write dual numbers in the form $a(1 + \alpha\varepsilon)$ with $a \neq 0$. This is analogous to the polar form of a complex number.

If one considers the trivial involution, one obtains dual Markov numbers which have also been studied in [10]. The more interesting case is again when σ is the non-trivial involution given by the conjugation map $a + \alpha\varepsilon \mapsto a - \alpha\varepsilon$. In this case, the mutation rule splits again; see [25].

9. Beyond the A_1 -case

In the above sections, we illustrated how the non-commutative perspective suggested by positivity leads to a fresh perspective on a variety of things. We mostly focused on the A_1 -case which arises from the positive structure on Hermitian Lie groups of tube type with respect to a subset Θ of Δ consisting of one element. This already extends beyond just the family of Hermitian Lie groups of tube type. When considering the groups $\mathrm{Sp}_2(\mathcal{A}, \sigma)$, the trace identities, their quantizations, or the corresponding Markov triples, the non-commutative algebra $\mathcal{A}$ does not have to be of Hermitian type – any (non-commutative) ring $\mathcal{A}$ with ant-involution σ works. And actually, one does not have to stop at skew-symmetric bilinear forms but can also consider symmetric bilinear forms over $(\mathcal{A}, \sigma)$; see [2].

The phenomena we describe in fact suggest that there is a more fundamental approach to explain some of the observations by considering σ -commutative modules over $(\mathcal{A}, \sigma)$. More on this perspective will appear in forthcoming work with Dani Kaufmann.

Going back to the families of simple Lie groups admitting a positive structure, the next family is formed by the indefinite orthogonal groups (or indefinite spin groups), whose Θ -Weyl group is of type B_p . Here the positive structure suggests looking for a partially non-commutative structure. One can in fact introduce partially non-commutative cluster structures of type B_p . This leads to the introduction of non-commutative polygonal cluster algebras [23].

In [24], building on the work of Goncharov and Shen [22], we introduce more general partially non-commutative cluster structures associated to more general Θ -Weyl groups. These cluster structures can be used to parametrize the open semigroups $U_{\Theta}^{>0}$ and $G_{\Theta}^{>0}$, to give explicit positivity tests, and provide (partially) non-commutative cluster structure on the space of decorated positive representations.

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