

Sign patterns of the Liouville function

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Abstract. The last two decades have seen remarkable advances in our understanding of important arithmetic functions. We describe the intertwining of major developments in ergodic theory, number theory and additive combinatorics, leading to recent progress made towards the Chowla and Sarnak conjectures.

1. Introduction

The Liouville function $\lambda(n)$ is a very important arithmetic function. It takes the value -1 on the prime numbers and extends multiplicatively to a completely multiplicative function on the natural numbers. It is intimately related to the prime numbers. For example, the celebrated prime number theorem on the number of primes up to N is equivalent to a randomness statement about cancellation in the sum of the values of Liouville:

$$\sum_{n \leq N} \lambda(n) = o(N).$$

Furthermore, the Riemann hypothesis is equivalent to having square-root cancellation in the sum above:

$$\sum_{n \leq N} \lambda(n) = O_{\varepsilon}(N^{1/2+\varepsilon}).$$

One can view the above as a testament to “random behavior” of the Liouville function; indeed, square-root cancellation is exhibited by the classical random walk on the integer line.

The prime number theorem can be viewed as an instance of a general vague principle in analytic number theory concerning the randomness of the Liouville function, stating that the Liouville function should be orthogonal to any “structured”

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sequence ([16] Section 13.1). In 2010, P. Sarnak [24] suggested a formalization of this principle in the form of the following remarkable conjecture.

Conjecture 1.1 (Sarnak (2010)). *The Liouville function λ is orthogonal to any sequence arising from a topological dynamical system of zero entropy.*

A topological dynamical system is a pair (X, T) , where X is a compact metric space, $T : X \rightarrow X$ a continuous map. The topological entropy of the system $h(X, T)$ is a measure of the complexity of the system – measuring the exponential rate of growth of the number of orbits. Any continuous function $f : X \rightarrow \mathbb{C}$ and any $x \in X$ give rise to a sequence $a(n) = f(T^n x)$. Sarnak’s conjecture states that for any such sequence arising from a system of zero topological entropy, one should have

$$\sum_{n \leq N} \lambda(n) f(T^n x) = o(N).$$

We remark that if we endow the space (X, T) with an invariant Borel probability measure ν , then the above estimates hold ν -almost surely *without* the zero entropy condition; this is a classical result of Bourgain [2].

This wonderful conjecture may look very surprising, but as observed by Sarnak, it follows from a well-known old conjecture of Chowla [4] in number theory.

Conjecture 1.2 (Chowla (1965)). *For any k and any integers $0 \leq h_1 < \dots < h_k$, we have*

$$\sum_{n \leq N} \mu(n + h_1) \cdots \mu(n + h_k) = o(N).$$

One can think of Chowla’s conjecture as the multiplicative function version of a famous conjecture of Hardy and Littlewood, which predicts that given a k -tuple of integers $\mathcal{H} = \{h_1, \dots, h_k\}$, there are infinitely many k -tuples

$$x + h_1, \dots, x + h_k$$

such that all elements are simultaneously prime unless there is an obvious divisibility obstruction.

1.1. Entropy of the Liouville system

The Liouville function λ naturally gives rise to a topological dynamical system. Consider the sequence space $X = \{\pm 1\}^{\mathbb{N}}$. This is the set of all ± 1 valued sequences, with the product topology. This is a compact metric space. The topological space X is naturally endowed with a continuous map $T : X \rightarrow X$ – the left shift given by $(Tx)(n) = x(n + 1)$. We can consider the Liouville function λ as a point in X and

then consider $X_\lambda = \overline{\{T^n \lambda\}}_{n \in \mathbb{N}}$. Then (X_λ, T) is a topological dynamical system. Now λ arises naturally as a sequence in this system: indeed, consider the continuous function f_0 defined on X given by $f_0(x) = x(0)$. Then $f_0(T^n \lambda) = \lambda(n)$.

Lemma 1.3. *Sarnak's conjecture implies that $h(X_\lambda, T) > 0$.*

The argument for this is simple: suppose $h(X_\lambda, T) = 0$ and Sarnak's conjecture holds. Then one should have that for any continuous function f on X_λ and any x , we have that

$$\sum_{n \leq N} \lambda(n) f(T^n x) = o(N).$$

In particular, this should hold for the function f_0 and $x = \lambda$. But for f_0 and $x = \lambda$, we have

$$\sum_{n \leq N} \lambda(n) f_0(T^n \lambda) = \sum_{n \leq N} \lambda(n) \lambda(n) = N,$$

a contradiction.

For closed shift-invariant subsets of X , the topological entropy measures the growth in the number of sign patterns. Let $Y \subset X$ be a closed shift-invariant subset. Denote

$$E_k(Y) = \{(y(n), \dots, y(n+k-1)) : n \in \mathbb{N}, y \in Y\}.$$

Then E_k is the set of different blocks of sign patterns exhibited in elements $y \in Y$. Denote $S_k(Y) = |E_k(Y)|$. Then the entropy of (Y, T) is given by

$$h(Y, T) = \limsup \frac{\log S_k(Y)}{k}.$$

From Lemma 1.3, we get that Sarnak's conjecture implies that $S_k(X_\lambda)$ should grow exponentially. Namely, there exists $c > 0$ such that

$$S_k(X_\lambda) \gg 2^{ck}.$$

We are still quite far from this conjecture, though recent years have seen substantial progress. The current state of the art is the following theorem [22].

Theorem 1.4 (Matomäki–Radziwiłł–Tao–Teräväinen–Ziegler (2022)). *For any constant $A \geq 1$,*

$$S_k(X_\lambda) \gg_A k^A.$$

1.2. Important instances of Sarnak's conjecture

We examine some instances of Sarnak's conjecture.

1.2.1. The trivial system. In this case, X is a point and T the trivial map. Any continuous map here is just a constant map; thus Sarnak's conjecture in this case is simply the prime number theorem:

$$\sum_{n \leq N} \lambda(n) = o(N).$$

1.2.2. Periodic systems. Here $X = \mathbf{Z}/m\mathbf{Z}$, $Tx = x + 1$. In this case, any continuous map here is periodic, and Sarnak's conjecture is the prime number theorem in arithmetic progressions:

$$\sum_{n \equiv a(m), n \leq N} \lambda(n) = o(N).$$

1.2.3. Almost periodic system. Here $X = \mathbf{R}/\mathbf{Z}$, $Tx = x + \alpha$, $\alpha \in \mathbf{R}$. Sarnak's conjecture is reduced to a statement on correlation of λ with linear exponential functions and is a theorem of Davenport [6].

Theorem 1.5 (Davenport). *For any $\alpha \in \mathbb{R}$, any $m \in \mathbb{Z}$, we have*

$$\sum_{n \leq N} \lambda(n) e^{2\pi i n m \alpha} = o(N).$$

1.2.4. Nilsystems. Here $X = G\Gamma$, G a nilpotent Lie group, Γ a lattice, $a \in G$, $Tg\Gamma = ag\Gamma$. Sarnak's conjecture in this case was proved by Green and Tao, and it played an important role in establishing asymptotics for linear equations in primes [12–14].

Theorem 1.6 (Green–Tao). *Let $F : X \rightarrow \mathbb{C}$ be a continuous function, $x \in G$, then*

$$\sum_{n \leq N} \lambda(n) F(a^n x \Gamma) = o(N).$$

1.2.5. The horocycle flow. Here $X = \mathrm{SL}_2(\mathbb{R})/\Gamma$, Γ a lattice, $Tx = ax$ where $a \in \mathrm{SL}_2(\mathbb{R})$. This was the first instance of Sarnak's conjecture where the dynamics is complex – the system is mixing but still of zero entropy. The conjecture in this case was verified by Bourgain, Sarnak, and Ziegler [3].¹

Theorem 1.7 (Bourgain–Sarnak–Ziegler). *Let (X, T) be a horocycle flow, then μ is disjoint from (X, T) , that is, given $x \in X$ and $f \in C(X)$ (if X is not compact, then take f continuous on the one-point compactification of X),*

$$\sum_{n \leq N} \mu(n) f(T^n x) = o(N).$$

¹See [25] for a related result on orbits of the horocycle flow along the primes.

1.3. Joinings

A main observation in [3] is that one can use the multiplicative nature of λ to convert the problem into a dynamical problem regarding *joinings of dynamical systems*.

The key to turning Sarnak's conjecture into a purely dynamical question is the following proposition [3, 5, 17].

Proposition 1.8 (Daboussi–Delange, Kátai, Bourgain–Sarnak–Ziegler). *Let $x \in X$, and let $F : X \rightarrow \mathbb{C}$ be a continuous function, $x \in X$. Suppose for (almost) all p, q distinct primes, we have*

$$\sum_{n \leq N} F(T^{p^n}x) \overline{F(T^{q^n}x)} = o(N).$$

Then

$$\sum_{n \leq N} \lambda(n) F(a^n x \Gamma) = o(N).$$

Let us give a dynamical interpretation to the above proposition: fix $x \in X$, and consider the following sequence of Borel probability measures on $X \times X$:

$$\mu_N = \frac{1}{N} \sum_{n \leq N} \delta_{(T^p \times T^q)^n(x, x)},$$

where δ_a , $a \in X$ is that δ -measure at $a \in X \times X$. The sequence μ_n has a subsequence μ_{n_k} that weakly converges to a Borel probability measure μ on $X \times X$. In particular, we have for any continuous function $F : X \rightarrow \mathbb{C}$

$$\frac{1}{N_k} \sum_{n \leq N_k} F(T^{p^n}x) \overline{F(T^{q^n}x)} \rightarrow \int F \otimes \bar{F} d\mu.$$

The measure μ is invariant under $T^p \times T^q$, and its marginals are $\pi_1\mu, \pi_2\mu$, which we can usually assume are well understood. Thus μ is a *joining* of the systems $(X, \pi_1\mu, T^p)$ and $(X, \pi_2\mu, T^q)$.

Many instances of Sarnak's conjecture have been verified via the classification of joinings of corresponding dynamical systems. In the case of the horocycle flow studies in [3] the key was the seminal work of M. Ratner classifying joinings of the horocycle flow [23].

1.4. Short averages of multiplicative functions and the entropy decrement argument

In 2016, K. Matomäki and M. Radziwiłł [21] made the first substantial progress towards Chowla's conjecture in many years. They proved the following theorem.

Theorem 1.9 (Matomäki and Radziwiłł). *For any function $H : \mathbb{N} \rightarrow \mathbb{N}$ satisfying $H \rightarrow \infty$ as $N \rightarrow \infty$, we have*

$$\sum_{n \leq N, h \leq H} \lambda(n) \lambda(n+h) = o(NH).$$

Shortly after, Tao [26] introduced a new entropy argument and combined with Theorem 1.9 was able to prove a logarithmic version of Chowla's conjecture for the case $k = 2$.

Theorem 1.10 (Tao). *For any integer $m > 0$,*

$$\sum_{n \leq N} \frac{\lambda(n) \lambda(n+m)}{n} = o(\log N).$$

Tao was also able to use the entropy argument to prove the equivalence between the logarithmic version of Chowla's conjecture and the logarithmic version of Sarnak's conjecture [27].²

Conjecture 1.11 (Log-Sarnak). *Let (X, T) be a topological dynamical system of zero entropy, and let $f : X \rightarrow \mathbb{C}$ be a continuous function. Then*

$$\sum_{n \leq N} \frac{\lambda(n) f(T^n x)}{n} = o(\log N).$$

Conjecture 1.12 (Log-Chowla). *For any k and any integers $0 \leq h_1 < \dots < h_k$, we have*

$$\sum_{n \leq N} \frac{\mu(n+h_1) \cdots \mu(n+h_k)}{n} = o(\log N).$$

Theorem 1.13 (Tao). *The log-Sarnak and log-Chowla conjectures are equivalent.*

Theorem 1.13 relies on the new entropy argument, as well as results in additive combinatorics on solutions to linear equations in primes obtained in joint work with Green and Tao [12–14].

In a major advancement in 2018, Frantzikinakis and Host [7] were able to make substantial progress on the log-Sarnak conjecture.

Theorem 1.14 (Frantzikinakis–Host). *The log-Sarnak conjecture holds true for uniquely ergodic systems.*

²The logarithmic versions (Conjectures 1.11 and 1.12) are weaker than the original versions (Conjectures 1.1 and 1.2).

Theorems 1.13 and 1.14 are built on major advancements in ergodic theory and additive combinatorics, in particular in the development of higher-order Fourier analysis in both regimes.

1.5. Higher-order Fourier analysis

The origin of higher-order Fourier analysis can be traced back to the work of Furstenberg, providing an ergodic-theoretic proof of Szemerédi's theorem. In 1975, Szemerédi proved the following remarkable theorem.

Theorem 1.15. *Suppose $E \subset \mathbb{Z}$ has positive upper density, namely $\limsup |E \cap [-N, N]|/(2N + 1) = \delta > 0$. Then E contains arbitrarily long arithmetic progressions.*

Szemerédi's proof used tools from graph theory. In 1977, Furstenberg introduced a completely new approach via ergodic theory. In the first part of the proof, Furstenberg introduces a correspondence principle, translating the combinatorial question into a multiple recurrence question in ergodic theory. Namely, he shows that Theorem 1.15 follows from the following multiple recurrence result.

Theorem 1.16 (Furstenberg). *Let $X = (X, \mathcal{B}, \mu, T)$ be a probability measure-preserving system. Let $A \in \mathcal{B}$ be of positive measure. Then for any integer $k > 0$, there exists an integer $n > 0$ such that*

$$\mu(A \cap T^{-n}A \cap \dots \cap T^{-kn}A) > 0.$$

We remark that the case $k = 1$ is the famous Poincaré recurrence theorem, which is one of the fundamental results in ergodic theory.

To prove this theorem, Furstenberg proves a structure theorem for general ergodic probability measure-preserving systems. One of the key new innovations in his proof was the study of probability measure-preserving systems via their maps to simpler systems with additional algebraic structure. This was a revolutionary approach at the time; prior to this, systems were traditionally studied via subsystems. Furstenberg shows the existence of a sequence of factors $\pi_k : \mathbf{X} \rightarrow Z_k(\mathbf{X})$ such that the systems $Z_k(\mathbf{X})$ have additional structure, and for any bounded functions $f_1, \dots, f_k$,

$$\begin{aligned} & \frac{1}{N} \sum_{n=0}^{N-1} f_1 \circ T^n \dots f_k \circ T^{nk} \\ & - \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E}(f_1 \circ T^n \mid Z_{k-1}(\mathbf{X})) \dots \mathbb{E}(f_k \circ T^{kn} \mid Z_{k-1}(\mathbf{X})) \longrightarrow 0, \end{aligned}$$

where the convergence is in L^2 . In particular, for $f_i = 1_A$, one can approximate the average

$$\frac{1}{N} \sum_{n=0}^{N-1} \mu(A \cap T^{-n}A \cap \cdots \cap T^{-kn}A)$$

by the average

$$\frac{1}{N} \sum_{n=0}^{N-1} \int \mathbb{E}(1_A \mid Z_{k-1}(\mathbf{X})) \mathbb{E}(1_A \circ T^n \mid Z_{k-1}(\mathbf{X})) \cdots \mathbb{E}(1_A \circ T^{kn} \mid Z_{k-1}(\mathbf{X})) d\mu.$$

In the case $k = 2$, corresponding to 3-term arithmetic progressions, the system $Z_1(\mathbf{X})$ is a *Kronecker system*: namely, a system of the form $\mathbf{Z} = (Z, \mathcal{D}, \nu, S)$ where Z is a compact abelian group, $\mathcal{D}$ the Borel sigma algebra, ν the Haar measure and $S : Z \rightarrow Z$ given by $Sz = z + t$ for some $t \in Z$. For a Kronecker system, $L^2(\mathbf{Z})$ is generated by the characters of the abelian group Z , and it follows that $L^2(Z_1(\mathbf{X}))$ is generated by the eigenfunctions of the T -action.

For higher k , the factors $Z_k(\mathbf{X})$ in Furstenberg's proof are towers of *isometric extensions* (a Kronecker system is an isometric extension of the trivial system), and this structure was sufficient for Furstenberg to prove that the $\liminf$ of the sequence of averages in (*) is positive, which implies Szemerédi's via a correspondence principle [8, 9].

Theorem 1.17 (Furstenberg). *Let $\mathbf{X} = (X, \mathcal{B}, \mu, T)$ be a probability measure-preserving system. Let $A \in \mathcal{B}$ be of positive measure. Then*

$$\liminf \frac{1}{N} \sum_{n=0}^{N-1} \mu(A \cap T^{-n}A \cap \cdots \cap T^{-kn}A) > 0.$$

Naturally, one can then ask for a universal sequence of factors $Y_k(\mathbf{X})$, $\pi_{Y_k} : \mathbf{X} \rightarrow Y_k(\mathbf{X})$ so that for any factor $\pi_Z : \mathbf{X} \rightarrow \mathbf{Z}$ satisfying

$$\frac{1}{N} \sum_{n=0}^{N-1} f_1 \circ T^n \cdots f_k \circ T^{nk} - \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E}(f_1 \circ T^n \mid \mathbf{Z}) \cdots \mathbb{E}(f_k \circ T^{kn} \mid \mathbf{Z}) \longrightarrow_{L^2} 0, \quad (1.1)$$

such a factor is called a *characteristic factor*; the factor map $\pi_{Y_k(\mathbf{X})}$ would factor through π_Z . Namely, we would have the following commuting diagram:

$$\begin{array}{ccc} \mathbf{X} & & \\ \pi_Z \downarrow & \searrow \pi_{Y_k(\mathbf{X})} & \\ \mathbf{Z} & \xrightarrow{\exists!} & Y_k(\mathbf{X}) \end{array}$$

For $k = 1$, the universal characteristic factor $Y_1(\mathbf{X})$ is the same as $Z_1(\mathbf{X})$, the *Kronecker factor* of $\mathbf{X}$ described above, corresponding to the eigenfunctions of the T -action. A natural candidate for the universal characteristic factor $Y_k(\mathbf{X})$ would then be the factor corresponding to k -order eigenfunctions, namely functions defined inductively via the equation $Tf = \psi f$, where ψ is a $k - 1$ -order eigenfunction. However, in 1995, Furstenberg and Weiss [10] showed that this is not the case already for $k = 2$!³ They observed that there are 2-step nilsystems $\mathbf{X}$ that have no order 2 eigenfunctions but satisfy that $Y_2(\mathbf{X}) = \mathbf{X}$. This remarkable observation of Furstenberg and Weiss is one of the cornerstones of higher-order Fourier analysis.

The universal characteristic factors associated with the question of arithmetic progressions were classified and shown to have an algebraic structure associated to dynamics on nilmanifolds [15, 30].

Theorem 1.18 (Host–Kra, Ziegler). *The universal k -characteristic factor $Y_k(\mathbf{X})$ is a k -step pro-nilsystem.*

Another cornerstone is the work of Gowers [11] on the combinatorial front, providing a new quantitative proof of Szemerédi’s theorem, generalizing the Fourier analytic approach of Roth in his study of 3-term progressions. In his proof, Gowers introduces a sequence of norms, the *Gowers uniformity norms* defined as follows: let G be a finite abelian group, $f : G \rightarrow \mathbb{C}$, then

$$\|f\|_{U_k} = (\mathbb{E}_{x, h_1, \dots, h_k} \Delta_{h_1} \cdots \Delta_{h_k} f)^{1/2^k},$$

where $\Delta_h f(x) := f(x + h) \overline{f(x)}$, and for a finite set S , $\mathbb{E}_{x \in S}$ denotes $1/|S| \sum_{x \in S}$. The Gowers uniformity norms are indeed norms for $k > 1$ and are monotone in k : $\|f\|_{U_1} \leq \|f\|_{U_2} \leq \cdots$. Most importantly, they dominate the count of arithmetic progressions: if $f_i : G \rightarrow \mathbb{C}$ are 1-bounded functions, namely $|f_i| \leq 1$, then

$$\mathbb{E}_{x, y \in G} f_0(x) f_1(x + y) \cdots f_k(x + ky) \leq \min \|f_i\|_{U_k}.$$

In particular, if $A \subset G$, and $|A| = \delta|G|$, then the average number of $k + 1$ -term arithmetic progressions in A satisfies

$$\mathbb{E}_{x, y \in G} 1_A(x) 1_A(x + y) \cdots 1_A(x + ky) = \delta^{k+1} + O_k(\|1_A - \delta\|_{U_k}).$$

This naturally leads to the question: when is $\|1_A - \delta\|_{U_k}$ small? Or alternatively, what can one say about a 1-bounded function $f : G \rightarrow \mathbb{C}$ satisfying $\|f\|_{U_k} \geq \eta$?

³This was later shown to be true for the action of $\oplus \mathbb{F}_p$ [1].

In his landmark paper, Gowers proved a “local” structure theorem for the U_k -norms for the cyclic group $G = \mathbb{Z}/N\mathbb{Z}$ and used it to obtain the first quantitative bounds for Szemerédi’s theorem.

A global structure theorem for the U_k -norms was proved by Green, Tao and Ziegler [14].

Theorem 1.19 (Green–Tao–Ziegler). *Let $f : \mathbb{Z}/N\mathbb{Z} \rightarrow \mathbb{C}$ be with $|f| \leq 1$. Then if $\|f\|_{U_k} \geq \delta$, then $\mathbb{E} f(x)p(x) \geq c(\delta)$ for some $< k$ -step nilsequence p of complexity which is bounded in terms of δ .*

The theorems mentioned above have had a great impact on major questions in ergodic theory, number theory and additive combinatorics in the last decade. In particular, Theorem 1.19 played a pivotal role in the proof of the Hardy–Littlewood conjectures for systems of linear equations of bounded complexity in the prime numbers. A more detailed exposition of these results can be found in [31].

We finally remark that there has been remarkable progress in recent years in extending the results mentioned above to other group actions and in obtaining alternative proofs to the inverse theorems with quantitative bounds, leading in particular to improved quantitative bounds on Szemerédi’s theorem [18–20].

1.6. Local Gowers uniformity of the Liouville function

We return to the log-Chowla conjecture, Conjecture 1.12. The inverse theorem for the Gowers norms, Theorem 1.19, together with the entropy decrement argument of Tao reduces the log-Chowla conjecture to the following.

Conjecture 1.20 (Local Gowers uniformity of the Liouville function). *For any $\varepsilon > 0$ and $H \leq (\log N)^\varepsilon$, we have*

$$\sum_{n \leq N} \sup_g \left| \sum_{h \leq H} \lambda(n+h)g(n+h) \right| = o(NH),$$

where the supremum is taken over bounded complexity k -step nilsequences.

Recent work of Matomäki, Radziwiłł, Tao, Teräväinen and Ziegler [22] makes substantial progress towards Conjecture 1.20. Indeed, we obtain the conjecture in the range $H = N^\varepsilon$, and Theorem 1.4 follows as a consequence.

For $k = 1$, Conjecture 1.20 takes the following form.

Conjecture 1.21 (Local Fourier uniformity of the Liouville function). *For any $\varepsilon > 0$ and $H \geq (\log N)^\varepsilon$, we have*

$$\sum_{n \leq N} \sup_\alpha \left| \sum_{h \leq H} \lambda(n+h) \exp(2\pi i \alpha h) \right| = o(NH).$$

For this special case, it was demonstrated in [22] that one can take $H = \exp((\log N)^{3/4+\varepsilon})$. In [28], Walsh improves this to $H = \exp((\log N)^{1/2+\varepsilon})$, and in [29], Walsh shows that conditional on GRH in the range $(\log N)^{\psi(N)}$ where $\psi(N) \rightarrow \infty$ arbitrarily slow in N .

Sarnak's and Chowla's conjectures appear to be very hard and still far away. However, their logarithmic versions may be within reach, at least conditioned on the Riemann hypothesis, and, as mentioned, would in particular imply the positive entropy conjecture.

We remark that the above questions can be asked for a larger collection of arithmetic functions. Of special interest are other *multiplicative functions*. Many of the results obtained for the Liouville function hold true for general multiplicative functions that behave well on arithmetic progressions.

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