

Dualities in convex geometric analysis and beyond

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Abstract. After a short introduction to asymptotic geometric analysis, we will discuss duality theory in the context of asymptotic geometric analysis and describe recent results with roots in optimal transport theory, as well as their connections to some of the older open problems in the theory.

1. Short introduction: Asymptotic geometric analysis

Asymptotic geometric analysis (AGA) lies in the meeting point of analysis and geometry. This field, “born” in the 1980s, originally sought to understand infinite-dimensional Banach spaces by studying their finite-dimensional sections and projections. One starting point of the field is the pioneering work of Vitali Milman [68], who provided a quantitative proof for Dvoretzky’s theorem [36], a cornerstone result which states that every infinite-dimensional Banach space has subspaces of dimension n which are ε -close to Euclidean, of any dimension n and any ε -closeness. Milman showed that this result holds not only for infinite-dimensional spaces but also for spaces of sufficiently large dimension, depending on ε and on the dimension of the subspace required. Furthermore, he showed that in this regime, with the correct choice of a Euclidean structure, this holds for a large measure of subspaces. He introduced one of the first important tools in this field, which is the concentration of measure phenomenon (see [9, Chapter 3]), roughly saying that a 1-Lipschitz function on the sphere S^{n-1} has to be close to its median on a set of very large measure. This phenomenon, that he used to give a probabilistic proof of Dvoretzky’s theorem, continues to play a major role not only in the study of high-dimensional convex geometry but in many other areas as well, such as analysis, combinatorics, probability and computer science (see, e.g., [58, 83]).

Since a normed space is defined by its unit ball, which is a centrally symmetric convex body, the study of AGA naturally involves convexity theory. Within AGA, the

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focus is usually on asymptotic questions, particularly as the dimension becomes large. This shifted the types of questions considered from those in classical convexity theory. Nowadays, AGA has applications in many fields, a good example being questions in so-called “big data”, where large dimensions are ubiquitous. An exciting aspect of the field is that it combines methods from many different areas in mathematics, ranging from topology to advanced analysis, geometry, probability and combinatorics. At the same time, due to the very basic nature of the topics, tools developed in AGA are applicable in wide areas of mathematics as well.

Let us list shortly the basic concepts which will be relevant in this short note. We work in $\mathbb{R}^n$ where we will be concerned with closed convex sets, or bodies (when the sets are assumed to be compact and have non-empty interior). Since originally the sets considered were unit balls of norms, central symmetry is sometimes assumed, but usually we shall consider convex bodies without requiring central symmetry, and then it is sometimes useful to assume that their center of mass is at the origin. The notion of volume, namely n -dimensional Lebesgue measure in $\mathbb{R}^n$, will be our way of measuring the size of a set (and we denote it by $\text{Vol}(\cdot)$). There are various other ways to measure the size of convex bodies, ranging from classical geometric measures such as surface area or width, to more specialized ones like various capacities and eigenvalues. For the purpose of this note, the classical Lebesgue measure will be sufficient. A basic operation on subsets of $\mathbb{R}^n$ which combines two sets is that of Minkowski addition. For $A, B \subseteq \mathbb{R}^n$, it is defined by $A + B = \{x + y : x \in A, y \in B\}$. One easily checks that if A and B are convex, then so is $A + B$, and that $A + A = 2A$ when A is convex. This operation appears naturally and plays an important role in many different scientific fields. For example, in robotics, when an object is to traverse some region with obstacles, the permissible locations for the center of mass of the robot are the complement of the Minkowski sum of the obstacles with (the minus of) the robot.

We have not yet described the main object we address in this note, which is duality or polarity. But even before introducing dualities, we can with the above concepts already state two very fundamental theorems from AGA which will also play a role later in this text and can serve to complete this short introduction of the reader to this field. The first is the well-known Brunn–Minkowski inequality from [34, 70]; see also [80].

Theorem 1.1 (Brunn–Minkowski inequality). *For any measurable $A, B, C \subset \mathbb{R}^n$ with $A + B \subset C$, it holds that*

$$\text{Vol}(C)^{1/n} \geq \text{Vol}(A)^{1/n} + \text{Vol}(B)^{1/n}. \quad (1.1)$$

(Usually one assumes that $A + B = C$ is measurable and omits C from the statement of the theorem.) Note that no convexity was assumed, and indeed elementary

proofs for the inequality exist which can be explained to an interested middle school student. Applications of Theorem 1.1 are remarkably broad; for example, it can be used to give a one-line proof of the isoperimetric inequality. Whole books can be (and have been) written on the consequences of this inequality; see [9, 10, 43, 80].

Another fundamental fact for AGA, which can be stated using only volume and Minkowski addition, is Minkowski's polynomiality theorem [70] which we formulate next. The case of two bodies, one of which is the Euclidean ball, is due to Steiner [82].

Theorem 1.2 (Minkowski's polynomiality theorem). *Let $n \geq 1, m \geq 1$. Assume that $K_1, \dots, K_m \subset \mathbb{R}^n$ are compact convex sets, and let $(t_i)_{i=1}^m \subset \mathbb{R}^+$. The volume of the Minkowski combination $\sum_{i=1}^m t_i K_i$ is a homogeneous polynomial of degree n in the variables (t_i) and, moreover, the coefficient of any monomial depends only on the bodies corresponding to indices appearing in the monomial. One writes*

$$\text{Vol}\left(\sum_{i=1}^m t_i K_i\right) = \sum_{i_1, \dots, i_n=1}^m \left(\prod_{j=1}^n t_{i_j}\right) V(K_{i_1}, \dots, K_{i_n}), \quad t_i \geq 0, \quad (1.2)$$

and the coefficients $V(K_{i_1}, \dots, K_{i_n})$, which are symmetric with respect to permutations of the arguments, are called the mixed volumes of the bodies.

When there are only two bodies involved, the coefficients are given by the mixed volume of j copies of K_1 and $(n - j)$ copies of K_2 . We denote these quantities by $V(K_1[j], K_2[n - j])$. In such a case, we have for $s, t \geq 0$

$$\text{Vol}(tK_1 + sK_2) = \sum_{j=0}^n \binom{n}{j} t^j s^{n-j} V(K_1[j], K_2[n - j]).$$

In particular, one easily sees that $V(K[n]) = \text{Vol}(K)$ and that for the Euclidean ball B_2^n the quantity $nV(K[n - 1], B_2^n)$ is simply the surface area of K as it is the derivative of $f(t) = \text{Vol}(K + tB_2^n)$ at $t = 0$.

Observe that if we consider the Minkowski sum of K and $-K = \{-x : x \in K\}$, the Brunn–Minkowski inequality implies $2^n \text{Vol}(K) \leq \text{Vol}(K + (-K))$, with equality when K is centrally symmetric. In this case, there is a counterpart: the Rogers–Shephard inequality [77] which states that

$$\text{Vol}(K + (-K)) \leq \binom{2n}{n} \text{Vol}(K) \quad (1.3)$$

with equality attained for a simplex (the convex hull of $n + 1$ affinely independent points). This inequality is in the reverse direction of the one in Theorem 1.1, and it should be mentioned that such reverse theorems are of great interest, as we shall see in the next sections. Among such theorems are the reverse isoperimetric inequality [25]

and the reverse Brunn–Minkowski inequality [67] (see also [9, Chapter 8]), both of which require the bodies involved to be in a certain position (or, in other words, a reasonable choice of the coordinate system). The Rogers–Shephard inequality above is invariant under linear transformations and thus does not require a “position”.

Note that by Theorem 1.2, we may expand the volume of $K - K$ (and rewrite the binomial coefficient as a sum) so that inequality (1.3) takes the form

$$\sum_{j=0}^n \binom{n}{j} V(K[j], -K[n-j]) \leq \sum_{j=0}^n \binom{n}{j}^2 \text{Vol}(K).$$

This form brings us to the first conjecture to be mentioned in this note, which is due to Godbersen [47] (formulated independently in [50]). The conjecture is that for any convex body $K \subset \mathbb{R}^n$ and any $1 \leq j \leq n$, it holds that

$$V(K[j], -K[n-j]) \leq \binom{n}{j} \text{Vol}(K), \quad (1.4)$$

with equality only for K being a simplex. The conjecture is verified only in some specific cases, and the current best upper bounds stem from [6] or [19]. In the latter, we show that for any $\lambda \in [0, 1]$, it holds (with equality for simplices) that

$$\sum_{j=0}^n (1-\lambda)^j \lambda^{n-j} V(K[j], -K[n-j]) \leq \text{Vol}(K), \quad (1.5)$$

any by taking just the j th term and optimizing over λ , one gets the bound $\binom{n}{j}$ up to some factor $\sim \sqrt{n}$. This short introduction was meant to show that even with very few definitions, deep theorems can be stated as well as long-standing open problems.

2. Duality and polarity

Duality is a central theme in mathematics, with many manifestations across different fields. It represents a very general meta-mathematical concept with several influential realizations. Examples include duality of normed spaces (associating a normed space with the space of linear functionals on it, with the operator norm), duality in linear programming and duality for convex functions, associating to a convex function φ its Legendre transform

$$\mathcal{L}\varphi(y) = \sup_x (\langle y, x \rangle - \varphi(x)). \quad (2.1)$$

The latter is analogous to the relationship between Lagrangians and Hamiltonians in mechanics. It seems that in every area of mathematics, some special and useful

dualities appear. In the context of convex bodies, duality can be understood through polarity. The polar of a convex body K is given by

$$K^\circ = \left\{ y \in \mathbb{R}^n : \sup_{x \in K} \langle x, y \rangle \leq 1 \right\}. \quad (2.2)$$

Usually one assumes that K contains the origin in its interior, as otherwise the polar is not compact. This notion, for centrally symmetric convex bodies, coincides with the duality in (finite-dimensional) normed spaces alluded to above, under the usual identification of a space with its unit ball. We mention that with a duality at hand (we will shortly explain what we mean by a “duality”), self-dual objects are of special interest. For normed spaces self-dual objects are Hilbert spaces, for convex functions the only self-dual function is $\|x\|_2^2/2$ and for set polarity the self-polar set is B_2^n , the Euclidean unit ball.

Bringing polarity into the convex geometry picture gives rise to many exciting new possibilities. As a first step, the volume product is introduced and is defined by $\text{Vol}(K)\text{Vol}(K^\circ)$. A classical result by Blaschke and Santaló [28, 29, 79] states that the volume product is maximized when K is the Euclidean ball, namely the self-dual object in the class (or any of its linear images, a.k.a. ellipsoids). This is true when K is centrally symmetric, and when it is not, one should make sure that the origin is not too close to the boundary of K (which would cause the dual body to be very large). For example, the inequality $\text{Vol}(K)\text{Vol}(K^\circ) \leq \text{Vol}(B_2^n)^2$ holds whenever the center of mass of K is at the origin (see [64] and also [9, Chapter 1]).

A long-standing open problem is to determine the convex body which minimizes the volume product. Mahler’s conjecture [61] is that the centered simplex is yet again the solution, namely that the volume product is minimized by simplices, and that considering only centrally symmetric convex sets (we will refer to this as the “centrally symmetric Mahler conjecture”) the minimal value for the volume product is that of the cube $[-1, 1]^n$ and its dual $\text{conv}(\pm e_i)$, which is $4^n/n!$. The cube is not unique in obtaining this volume product, and there is a family of polytopes called “Hanner polytopes” all of which have the same volume product and are conjectured to be the set of minimizers. Mahler’s conjecture in dimension 2 was proved in his original manuscript [61]. The case of $n = 3$ was recently solved by Iriyeh and Shibata [52]; see also [39, 42]. In higher dimensions, the conjecture is open, but has been verified in many special cases; see, e.g., [48, 65, 75, 78].

It is worthwhile to note that taking power $1/n$ of the volumes (which is rather natural as volume is n -homogeneous, which is why these powers appear in Theorem 1.1, for example), the upper bound $(\text{Vol}(B_2^n))^{2/n}$ and the conjectured lower bound $4/(n!)^{1/n}$ are of the same order. In [32], a lower bound for the volume product of the form $c^n \text{Vol}(B_2^n)^2$ was obtained and was sufficient for many applications (it is also very intimately related to the reverse form of the Brunn–Minkowski

inequality mentioned above). Similar lower bounds in the general case were obtained in [45, 56, 72].

Mahler's conjecture, surprisingly, turns out to be intimately connected with a certain conjecture in symplectic geometry [85]. The conjecture pertains to symplectic capacities, which are a symplectically invariant way of measuring the size of a set in $\mathbb{R}^{2n}$. Many capacities exist (a function is called a capacity if it is monotone, symplectic invariant and equals πr^2 for both a Euclidean ball of radius r and the complex cylinder $rB_2^2 \times \mathbb{R}^{2n-2} \subset \mathbb{R}^{2n}$), and among them there is a “largest” one and a “smallest” one, called the Gromov width and the ball capacity, respectively; see [51] for details. Viterbo's conjecture was that, among sets of fixed volume, the largest capacity is attained for the Euclidean ball

$$\frac{\text{cap}(T)}{\text{cap}(B_2^{2n})} \leq \left(\frac{\text{Vol}(T)}{\text{Vol}(B_2^{2n})} \right)^{1/n}.$$

In [11] we showed that Viterbo's conjecture for certain centrally symmetric bodies (namely, $T = K \times K^\circ$ when $\mathbb{R}^{2n}$ is identified as $\mathbb{C}^n = \mathbb{R}^n \times i\mathbb{R}^n$) is equivalent to Mahler's conjecture. (As a side note, we mention that the introduction of convexity, which is a linear notion, into symplectic geometry is quite unnatural. Nevertheless, it turns out to be quite fruitful; for instance, we proved in [18] a Brunn–Minkowski inequality for certain important capacities.) Very recently, a counterexample to Viterbo's conjecture was found by Haim-Kislev and Ostrover [49]. The example is not centrally symmetric and therefore does not directly relate to Mahler's conjecture. In the flavor of AGA, we also mention here our result from [17] where we showed that up to a universal constant C , which does not depend on the dimension nor on the bodies involved, an isoperimetric inequality of Viterbo-type does hold, namely $\text{cap}(T)/\text{cap}(B_2^{2n}) \leq C \text{Vol}(T)^{1/n}/\text{Vol}(B_2^{2n})^{1/n}$, improving the constant n which was established by Viterbo in [85].

Another direction in which questions about volume product have been addressed is that of functional versions for the geometric inequality. This is a general theme in AGA, where many definitions, theorems and conjectures have functional counterparts. These analogies have proven to be very useful, where in many cases the geometric inequalities are special cases of the functional ones, but within the realm of functions, we have many more tools at our disposal. For an overview of this direction of research, see [10, Chapter 9]. For this analogy, usually a convex body is replaced by a lower semi-continuous convex function $\varphi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$ and Minkowski addition is replaced by infimal convolution (which corresponds to Minkowski addition of the epi-graphs in $\mathbb{R}^{n+1}$ which are themselves closed convex sets). The “volume” of φ is given by $\int \exp(-\varphi)$. Under this analogy, Brunn–Minkowski inequality has a counterpart which is called the Prékopa–Leindler inequality [60, 74] (it too does not

require assumptions about convexity)

$$\int h \geq \left(\int f \right)^{1-\lambda} \left(\int g \right)^{\lambda} \quad (2.3)$$

whenever $h((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} g(y)^{\lambda}$ for all $x, y \in \mathbb{R}^n$.

To replace duality in this analogy, we will consider the Legendre transform (2.1), which is, as mentioned, the classical duality for convex functions [76]. We will readily explain in which senses this is a natural choice and what it can be replaced with. Nevertheless, under this analogy, a version of the Blaschke–Santaló inequality was proved in [24] for even functions (which are the analogues of centrally symmetric bodies):

$$\int \exp(-\varphi) \int \exp(-\mathcal{L}\varphi) \leq \left(\int \exp(-|x|^2/2) dx \right)^2 = (2\pi)^n. \quad (2.4)$$

Again we see that the maximizer is the self-dual function. In [12], this was generalized to functions which are not necessarily even, in which case “choosing the right origin” amounts, in the dual, to adding some linear function to φ . Many other proofs of the functional version of the Santaló inequality have appeared since then; see [40, 59]. Very recently an analytical approach based on heat flow was obtained, where it is proven that the functional volume product increases along the heat flow [71]. Regarding an analogue for Mahler’s conjecture, such were given in [41]. Some partial results resembling those for bodies have been obtained; see [42] for a full account.

3. Characterizing duality

In the paper [40], along with providing a simple proof that the functional Santaló inequality follows from the geometric one, the authors gave a more general form of a functional inequality of “Santaló-type”, which to us served as a motivation to try and understand what is the “correct” form of a functional duality, and more generally, why are the specific forms of duality used in convex geometry, $K \mapsto K^\circ$ and $\varphi \mapsto \mathcal{L}\varphi$, so central and useful. The very basic properties of these two operations are that they reverse the natural partial order in their class (set-inclusion for convex bodies and point wise inequality for functions) and that both are bijections (in fact, they are not only bijections but actually involutions).

Motivated by this, Böröczky and Schneider [31] showed that, up to obvious linear changes, $K \mapsto K^\circ$ given in (2.2) is the only order-reversing isomorphism on the class of convex bodies. This result opened the floodgates to the idea that other classical operations are characterized by a small subset of their properties (for some such

papers, see, e.g., [1, 2, 13, 37, 44, 69]). One of the first results we proved in this direction pertained to the functional duality which we used in the functional Santaló inequality. In [15], we proved that for the class of lower semi-continuous convex functions on $\mathbb{R}^n$, the Legendre transform (2.1) is the only order-reversing isomorphism, up to obvious linear degrees of freedom coming from a choice of coordinate system.

However, it turned out that making a small modification to the class, which was still enough for all of the geometric applications we needed, the picture changes dramatically. Indeed, in [14, 16] we considered the class of *geometric convex functions*. These are non-negative lower semi-continuous convex functions which vanish at the origin. We showed that, for this subclass, there are (up to obvious linear changes) precisely two essentially different order-reversing isomorphisms. One of these is given by the Legendre transform, as the subclass is invariant under $\mathcal{L}$, but it turned out that there is yet another essentially different order-reversing isomorphism. We call this second order-reversing isomorphism “polarity for functions”, and it can be defined by

$$\mathcal{A}\psi(y) = \sup_{x: \psi(x) > 0} \left(\frac{\langle x, y \rangle - 1}{\psi(x)} \right), \quad (3.1)$$

where one should make sense of the cases where one divides by 0. This curious new player has a very geometric definition, from which the above definition can be derived. Consider the epi-graph of ψ , which is an unbounded, closed, convex set in $\mathbb{R}^{n+1}$ which lies in the upper half-space and includes the origin. The epi-graph of $\mathcal{A}\psi$ is simply the reflection (with respect to the last, special coordinate) of the set $(\text{epi}(\psi))^\circ$. So, in some sense, the functional polarity $\mathcal{A}$ is more closely related to the usual polarity for convex bodies than the Legendre transform $\mathcal{L}$. There have been several interesting developments with respect to this new operation $\mathcal{A}$. In [23], we proved Mahler- and Santaló-type inequalities revolving around $\mathcal{A}$, and more was proved in [38, 46]. In [8], we found a convincing analogue of the Prékopa–Leindler inequality, which, if we write (2.3) as

$$\text{Vol}(\mathcal{L}((1 - \lambda)\mathcal{L}\varphi + \lambda\mathcal{L}\psi)) \geq (\text{Vol}(\varphi))^{1-\lambda} (\text{Vol}(\psi))^\lambda$$

(which can be done, since $\mathcal{L}$ maps addition to infimum convolution), then our analogue can be written as

$$\text{Vol}(\mathcal{A}((1 - \lambda)\mathcal{A}\varphi + \lambda\mathcal{A}\psi)) \geq ((1 - \lambda)(\text{Vol}(\varphi))^{-1} + \lambda(\text{Vol}(\psi))^{-1})^{-1}.$$

More generally, without any convexity assumptions, we proved

$$\int_{\mathbb{R}^n} h \geq \left((1 - \lambda) \left(\int_{\mathbb{R}^n} f \right)^{-1} + \lambda \left(\int_{\mathbb{R}^n} g \right)^{-1} \right)^{-1}, \quad (3.2)$$

whenever $h((1-t)x + ty) \geq \min\{f(x)^{1-t/1-\lambda}, g(y)^{t/\lambda}\}$ for all $t \in (0, 1)$ and for all $x, y \in \mathbb{R}^n$. In a different direction, we studied the differential properties of $\mathcal{A}$ in [20, 22]. We showed it can be used to linearize certain families of differential equations, and we discussed how it arises as a cost-induced functional transform from a certain non-standard cost on $\mathbb{R}^n$ which we call the polar cost. We discovered that the transform $\mathcal{A}$ has appeared in the book [76] from 1970, but received very little attention since then, and in fact was also present in some form in earlier works on duality of norms [66]. The “transportation of measure” point of view, which helped us think of $\mathcal{A}$ as induced by a cost function on $\mathbb{R}^n$, was an undercurrent that influenced what materialized next.

4. Optimal transport and cost-induced dualities

Given a symmetric cost function on $X \times X$ and two probability measures $\mu, \nu \in \mathcal{P}(X)$, a question going back to Monge asks whether there exists a “good” transport map, namely a function $T : X \rightarrow X$ satisfying $\mu(T^{-1}A) = \nu(A)$ for all measurable $A \subseteq X$. The relaxation due to Kantorovich [53, 54] has to do with transport plans. A transport plan is a probability measure $\gamma \in \mathcal{P}(X \times X)$ with marginals μ and ν , and we denote $\gamma \in \Pi(\mu, \nu)$. A transport plan always exists (e.g., $\mu \times \nu$) and the main questions regard optimal plans, where the cost of a plan γ is given by

$$C(\gamma) = \int c(x, y) d\gamma,$$

and the total cost of transporting μ to ν is given by $C(\mu, \nu) = \inf\{C(\gamma) : \gamma \in \Pi(\mu, \nu)\}$.

For an overview of transportation theory, see [84]. For the quadratic cost $c(x, y) = |x - y|^2$ in $\mathbb{R}^n$, it is well known that transport maps exist and have a special form. More precisely [33, 63].

Theorem 4.1 (Brenier–McCann). *Let μ, ν be Borel probability measures on $\mathbb{R}^n$. If μ is absolutely continuous with respect to the Lebesgue measure, then there exists a transport map between μ and ν which is given by the gradient of a convex function. Moreover, it is the optimal transport plan with respect to quadratic cost.*

Theorem 4.1 is a very important tool in AGA, and one can use it to prove the Prékopa–Leindler inequality (2.3), the Brunn–Minkowski inequality (1.1), concentration of measure inequalities and much more. It is also intimately connected with the Legendre transform; for example, one can easily check that if $\nabla\varphi$ is the optimal map from μ to ν , then $\nabla\mathcal{L}\varphi = (\nabla\varphi)^{-1}$ is the optimal from ν back to μ . This

happens, of course, since the quadratic cost in effect defines the Legendre transform (after some cosmetic manipulations).

To see more clearly how a cost function defines a functional transform, we need to recall the Kantorovich duality theorem [53, 54] which states that in the case of a lower semi-continuous cost function, the infimum in the definition of $C(\mu, \nu)$ is attained and is given by

$$C(\mu, \nu) = \sup \left\{ \int_X \varphi d\mu + \int_Y \psi d\nu : (\varphi, \psi) \text{ admissible} \right\}. \quad (4.1)$$

Here (φ, ψ) is called admissible if $\varphi \in L_1(X, d\mu)$, $\psi \in L_1(Y, d\nu)$, and for all x, y , $\varphi(x) + \psi(y) \leq c(x, y)$. This gives rise to the functional cost transform as follows. Given a function $\varphi : X \rightarrow [-\infty, \infty]$, its c -transform $\varphi^c : Y \rightarrow [-\infty, \infty]$ is given by

$$\varphi^c(y) = \inf_{x \in X} \{c(x, y) - \varphi(x)\}. \quad (4.2)$$

In other words, φ^c is the maximal admissible partner of φ . The c -transform reverses the pointwise order on functions and can be seen to be an involution on its image class. In other words, $\varphi^{ccc} = \varphi^c$, namely letting $\mathcal{S} = \{\varphi^c : \varphi : X \rightarrow \mathbb{R} \cup \{\pm\infty\}\}$, the mapping $\varphi \rightarrow \varphi^c$ is an order-reversing isomorphism on $\mathcal{S}$. For the “quadratic” cost function $c(x, y) = -\langle x, y \rangle$, we get that the class $\mathcal{S}$ consists of upper semi-continuous concave functions and $\psi^c = -(\mathcal{L}(\psi))$. (We write “quadratic” with quotation marks since formally the quadratic costs is $|x - y|^2$; however, the only interaction term is $-\langle x, y \rangle$, so that these two costs give rise to the same optimal plans and maps between measures.) The so-called “polar” cost $-\ln(\langle x, y \rangle - 1)$ is similarly associated, up to a logarithmic change of variables, to the class of geometric convex functions and the $\mathcal{A}$ -transform.

In dealing with the polar cost, one immediately notices that it attains the value $+\infty$. We call such costs “non-traditional”, and some of the classical facts from optimal transport become more complicated for such costs. For example, even the existence of a finite-cost plan is not guaranteed. In [21], we provide necessary conditions, and slightly stronger sufficient conditions, for the existence of such a plan. For our discussion, it is only important to note the following (easily verifiable) necessary condition on μ and ν (which are probability measures, and let us assume $X = \mathbb{R}^n$). For there to exist a finite-cost map between μ and ν , it is necessary that for every subset $A \subset \mathbb{R}^n$,

$$\mu(A) + \nu(A^c) \leq 1, \quad (4.3)$$

where $A^c = \{y \in \mathbb{R}^n : c(x, y) = \infty \ \forall x \in A\}$. Indeed, this condition can be rephrased as $\mu(A) \leq \nu(B)$ where $B = \{y : c(x, y) < \infty \text{ for some } x \in A\}$. If a plan has finite cost, then almost every $x \in A$ has to be mapped to some y with $c(x, y) < \infty$, namely to B , so clearly a necessary condition is $\mu(A) \leq \nu(B)$. We see that a non-traditional

cost function naturally defines a kind of “duality” on sets, essentially associated with its finiteness set $\{(x, y) : c(x, y) < \infty\}$.

Concentrating on subsets of $\mathbb{R}^n$, we call a map $T : P(\mathbb{R}^n) \rightarrow P(\mathbb{R}^n)$ (here use $P(X)$ to denote all subsets of X) an order-reversing quasi-involution if it reverses the partial order of inclusion, $A \subseteq B \Rightarrow TA \supseteq TB$, and when it is applied twice, we have an inclusion of the form $T(TA) \supseteq A$ for all $A \in P(\mathbb{R}^n)$. Any order-reversing quasi-involution is an involution on its image class $\{TA : A \subset \mathbb{R}^n\}$. For example, the polarity mapping $K \mapsto K^\circ$ given in (2.2) is such a mapping, and its image set consists of closed convex subsets of $\mathbb{R}^n$ which include the origin. Any symmetric subset of $S \subset \mathbb{R}^n \times \mathbb{R}^n$ (here symmetric means $(x, y) \in S \Leftrightarrow (y, x) \in S$) induces an order-reversing quasi-involution, by $A \mapsto \{y : (x, y) \in S \ \forall y \in A\}$ (as explained after (4.3)). The polarity mapping corresponds to the set $\{(x, y) : \langle x, y \rangle \leq 1\}$. The image class corresponding to S will be sets which are intersections of images of singletons $\{x\}$, namely intersections of “fibers” of S of the form $\{y : (x, y) \in S\}$. In this way, convex sets which include the origin appear as intersections of closed half-spaces which include the origin.

This point of view allows for the construction of many more set dualities, some of which seem quite natural and have not been studied in depth. Such is the “polar polarity”, which one gets by switching the sign on the inequality in the set used for usual polarity, namely taking $S = \{(x, y) : \langle x, y \rangle \geq 1\}$. The image class here will consist of unbounded convex sets, which are intersections of half-spaces which do not include the origin. These sets always have a unique point closest to the origin, and, properly rotated, can be viewed as epi-graphs of certain functions, a structure the set duality respects. For some Santaló-type inequalities for this class, see [21]. In the same paper, we also showed that all quasi-dualities on $P(\mathbb{R}^n)$ are of this form, namely to each quasi-duality there corresponds a symmetric set $S \subset \mathbb{R}^n \times \mathbb{R}^n$ such that the duality is given by $A \mapsto \{y : (x, y) \in S \ \forall y \in A\}$.

5. Ball duality and sets of constant width

An interesting example of a quasi-duality on $\mathbb{R}^n$ is the one associated with the set $S = \{(x, y) : |x - y| \leq 1\}$. (One could take any norm, but the Euclidean one will already be quite interesting.) The quasi-duality is thus the mapping taking a set A to the intersection of the unit balls centered at A . The image class is of course a subset of the class of convex sets (when we might want to exclude the empty set and the whole of $\mathbb{R}^n$ which arise when these balls do not intersect, respectively, when we take an empty intersection). This subclass can be characterized in many different ways. It is the class of convex bodies with all principal curvatures (at smooth points) at least 1.

At the same time, it can be described as the class of summands of B_2^n [80, Theorem 3.2.2]. In fact, the duality mapping on this class, which we denote $K \mapsto K^c$, simply switches, up to sign, between the pair of summands; that is, K is mapped to the unique body K^c satisfying $K - K^c = B_2^n$. This class has been studied in different contexts and under several names: 1-convex bodies, hyperconvex bodies, ball bodies, spindle-convex bodies, ball summands, etc. and is linked to many classical problems, including the Kneser–Poulsen conjecture [26, 55, 73] and the reverse isoperimetric problem [30, 35]. Let us briefly discuss its connection to another open problem that has seen some recent advances.

Note that the fixed points of this duality are precisely bodies K for which $K - K = B_2^n$. These are the so-called “bodies of constant width 1”, that is, bodies for which, when one considers two parallel hyperplanes supporting the body, the distance between the two is always 1, regardless of the direction of the hyperplane. Equivalently, the support function of the body, $h_K(u) = \sup_{x \in K} \langle x, u \rangle$, satisfies $h_K(u) + h_K(-u) = |u|$ (so, it is the support function of $B_2^n/2$ plus an odd function). Bodies of constant width have fascinated the scientific community for generations. One major open problem regarding bodies of constant width is called the Blaschke–Lebesgue problem. It asks, in any given dimension, which body of constant width 1 has minimal volume (the fact that the ball has maximal volume very easily follows from Urysohn’s inequality, for example). In dimension $n = 2$, the answer is known and is the Reuleaux triangle (these are the original works of Blaschke and Lebesgue [27, 57]). The problem is open for $n \geq 3$, where in $\mathbb{R}^3$ the minimizers are conjectured to be the two Meissner bodies, which are of equal volume (see, e.g., [62]).

In the spirit of AGA, Schramm [81] asked whether the minimal volume of a body of constant width 1 behaves asymptotically like the maximal value, which is $\text{Vol}(B_2^n/2)$, or is exponentially smaller than this value. He provided what is still the best lower bound for the volume of a body of constant width 1, which is $(\sqrt{3} - 1)^n \text{Vol}(B_2^n/2)$. Very recently, Arman, Bondarenko, Nazarov, Prymak and Radchenko [3] constructed a body of constant width 1 which has indeed volume exponentially smaller than $\text{Vol}(B_2^n/2)$. Nevertheless, the value of the exponent for the lower bound of the volume is still unknown, let alone exact solutions or even conjectured minimizers to the Blaschke–Lebesgue problem in dimension above 3.

It is pleasant to connect this problem to one mentioned in the introduction. Recall the Rogers–Shephard inequality (1.3), where the constant $\binom{2n}{n}$ was sharp for simplices. Since for a body of constant width 1 we have $K - K = B_2^n$, the Blaschke–Lebesgue problem is equivalent to finding the sharp constant in (1.3) for K of constant width 1. In dimension $n = 3$, the problem is also equivalent to maximizing $V(K, K, -K)$, which is called the (first) Godbersen coefficient, again for this subclass of bodies.

The study of ball duality brings some new components into the picture, and we are able to get quite a lot of insight into these problems, constructing families of bodies of constant width, understanding the boundary structure of these bodies and some of their extremal properties [4, 5, 7].

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