

Self-affine sets and measures – A survey

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Abstract. Self-affine sets can be expressed as affine copies of themselves. The dimension theory, when the defining mappings are similarities, is relatively well understood, but the understanding of the general case is far from being complete. Here, we provide an overview from the most basic definitions of the topic to the most recent results.

1. Introduction

There is no widely accepted definition of fractals. However, most authors would call a set fractal if, on infinitely many scales, its smaller parts resemble the whole. The self-affine sets provide a model for that phenomenon. This class includes several well-known fractal sets, like the triadic-Cantor set, the Sierpiński gasket, the von Koch curve, and the graph of Takagi’s function. One of the central problems of geometric measure theory is determining the “size” of these objects in several aspects, for example, finding the Hausdorff or box-counting dimension. This paper aims to introduce the most basic definitions and to survey the most recent results of the topic.

1.1. Basic definitions

Let $\mathcal{I}$ be a finite set of indices, and let $\Phi = \{f_i(x) = A_i x + t_i\}_{i \in \mathcal{I}}$ be a finite tuple of invertible contractive affine mappings on $\mathbb{R}^d$. That is, $A_i \in \text{GL}_d(\mathbb{R})$ such that $\|A_i\| < 1$ for every $i \in \mathcal{I}$ and $t_i \in \mathbb{R}^d$, where $\|\cdot\|$ denotes the induced matrix norm by the usual Euclidean norm on $\mathbb{R}^d$. Hutchinson [33] showed (in a much higher generality) that there exists a unique non-empty compact set X such that

$$X = \bigcup_{i \in \mathcal{I}} f_i(X). \quad (1.1)$$

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We call the set X *self-affine*. In the special case, when all the affine mappings of Φ are similarities, that is, $f_i(x) = \lambda_i O_i x + t_i$, where $\lambda_i \in (0, 1)$ is the scaling, $O_i \in O(d, \mathbb{R})$ is an orthogonal matrix and $t_i \in \mathbb{R}^d$, then we call the set X *self-similar*.

The construction of X relies on an iterative application of the maps in Φ , so we call Φ an *iterated function system* or IFS for short. Sometimes, we also call the set X the *attractor* of the IFS Φ . To construct X , take $R > 0$ sufficiently large such that every map in Φ maps the closed ball $B(0, R)$ to itself, that is, $f_i(B(0, R)) \subseteq B(0, R)$. For instance, the choice $R = \max_{i \in \mathcal{I}} \|t_i\| (1 - \|A_i\|)^{-1}$ is sufficient. Then for every $n \geq 1$, let

$$X_n = \bigcup_{i_1 \in \mathcal{I}} \cdots \bigcup_{i_n \in \mathcal{I}} f_{i_1} \circ \cdots \circ f_{i_n}(B(0, R)). \quad (1.2)$$

It is easy to see that the set X_n is non-empty, compact and $X_{n+1} \subseteq X_n$ for every $n \geq 1$. Hence, $X = \bigcap_{n=1}^{\infty} X_n$ is a non-empty compact set by Cantor's intersection theorem, and X clearly satisfies (1.1). The uniqueness of the set X relies on simple properties of the Hausdorff metric on compact subsets of $\mathbb{R}^d$. For details, see, for example, the book of Bárány, Simon and Solomyak [8, Theorem 1.3.6]. For some examples of self-similar and self-affine sets, see Figure 1.

Let us now consider the following Markovian random walk on X , which is also called the *chaos game*, introduced by Barnsley [9]. Let $\mathbf{p} = (p_i)_{i \in \mathcal{I}}$ be a probability vector (i.e., $p_i \geq 0$ for every $i \in \mathcal{I}$ and $\sum_{i \in \mathcal{I}} p_i = 1$). Starting from an initial point $Y_0 \in X$, let us choose a function f_{i_1} randomly from Φ according to $\mathbf{p}$ and jump to the point $Y_1 = f_{i_1}(Y_0)$. It continues iteratively generating a sequence of random points $Y_2, Y_3, \dots$, where $Y_n = f_{i_n}(Y_{n-1})$ and the indices i_n are independently and identically distributed (i.i.d.) according to $\mathbf{p}$. This process has a unique stationary measure; more precisely, there exists a unique compactly supported Borel probability measure ν such that

$$\nu = \sum_{i \in \mathcal{I}} p_i (f_i)_* \nu, \quad (1.3)$$

where $f_* \nu$ denotes the pushforward of the measure ν with respect to the map f . For a proof, see, for example, Edgar's book [14]. We call the measure ν a *self-affine measure*, and if Φ consists only of similarities, then we call ν a *self-similar measure*.

The attractor X of Φ has a strong connection with symbolic coding. Namely, every point in X can be associated with infinite sequences formed by the symbols in $\mathcal{I}$ (associated with the functions of Φ). That is, let $\Sigma = \mathcal{I}^{\mathbb{N}}$ be the set of one-sided infinite sequences, called *symbolic space*. For a finite word $\mathbf{j} \in \Sigma_* = \bigcup_{n=0}^{\infty} \mathcal{I}^n$, let us define the cylinder set $[\mathbf{j}]$ by $[\mathbf{j}] = \{\mathbf{i} \in \Sigma : i_1 = j_1, \dots, i_n = j_n\}$. The natural Borel σ -algebra on Σ is the one which is induced by the cylinder sets. For brevity, we introduce the following notation: for a finite word $\mathbf{j} = (j_1, \dots, j_n) \in \Sigma_*$, let $f_{\mathbf{j}} = f_{j_1} \circ \cdots \circ f_{j_n}$ be the iterates of the maps of Φ in the prescribed order by $\mathbf{j}$. We will also adapt this notation later for the product of matrices and real numbers.

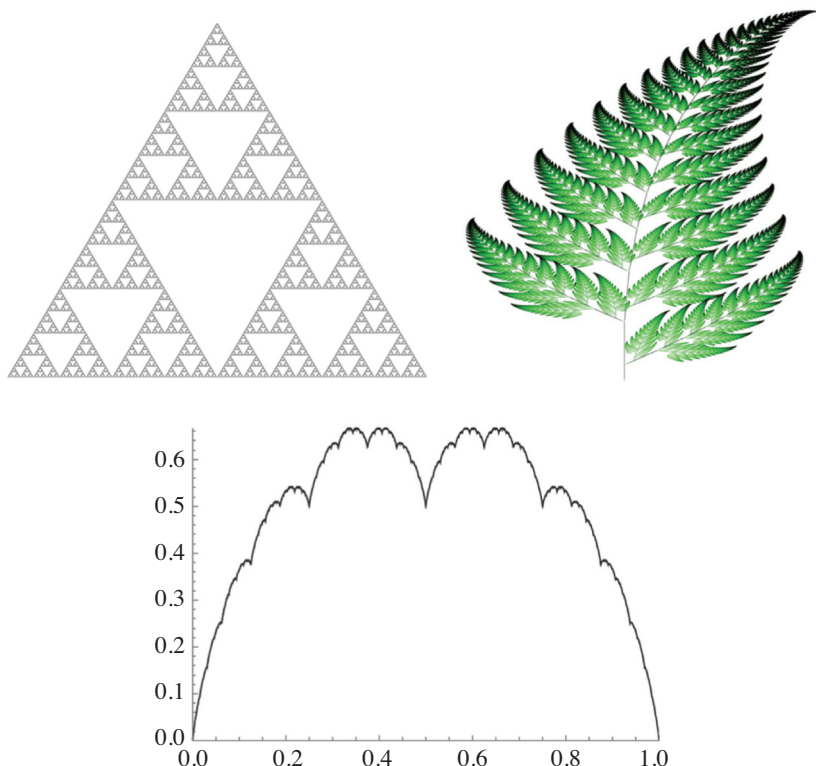


Figure 1. Examples of self-affine sets: the Sierpiński gasket, Barnsley's fern (source: Wikipedia) and Takagi's nowhere differentiable function graph.

For every $\mathbf{i} = (i_1, i_2, \dots) \in \Sigma$, we define the *symbolic coding map* as

$$\pi(\mathbf{i}) = \lim_{n \rightarrow \infty} f_{i_1} \circ \dots \circ f_{i_n}(0).$$

The set Σ is a compact metric space equipped with the natural metric $d(\mathbf{i}, \mathbf{j}) = 2^{-|\mathbf{i} \wedge \mathbf{j}|}$, where $|\mathbf{i} \wedge \mathbf{j}| = \min\{n \geq 0 : i_{n+1} \neq j_{n+1}\}$, and the map $\pi: \Sigma \rightarrow X$ is Hölder continuous and surjective but not necessarily injective.

Denote $\sigma(i_1, i_2, \dots) = (i_2, i_3, \dots)$ as the left-shift operator on Σ . As simple as this mapping is, its dynamic structure is equally rich, which plays an essential role in dynamical systems theory. It is easy to see that

$$\pi(\mathbf{i}) = f_{i_1}(\pi(\sigma \mathbf{i})). \quad (1.4)$$

Hence, the left-shift operator and its properties benefit the study of self-affine sets. We call a Borel probability measure μ on Σ left-shift invariant if $\mu = \sigma_*\mu$, that is, $\mu(A) = \mu(\sigma^{-1}A)$ for every Borel set $A \subseteq \Sigma$. We call a left-shift invariant probability

measure μ ergodic if $\mu(A) = 0$ or $\mu(A) = 1$ for every invariant set $A = \sigma^{-1}A$. One of the main advantages of ergodicity is Birkhoff's ergodic theorem, which roughly states that the time average equals the space average. That is, for every $f: \Sigma \rightarrow \mathbb{R}$ μ -integrable map, $\frac{1}{n} \sum_{k=0}^{n-1} f(\sigma^k \mathbf{i}) \rightarrow \int f(\mathbf{j}) d\mu(\mathbf{j})$ as $n \rightarrow \infty$ for μ -almost every $\mathbf{i}$. One of the most basic left-shift invariant ergodic measures is the family of Bernoulli measures. Namely, for a probability vector $\mathbf{p} = (p_i)_{i \in \mathcal{I}}$, let μ be the measure such that

$$\mu([i_1, \dots, i_n]) = p_{i_1} \cdots p_{i_n}.$$

The measure μ on Σ is σ -invariant and ergodic; furthermore, $\pi_*\mu = \nu$, where ν is the self-affine measure defined in (1.3). For an introduction to ergodic theory, see, for example, Walter's book [62].

For an ergodic left-shift invariant measure μ on Σ , let us denote the entropy of μ by h_μ . That is,

$$h_\mu = \lim_{n \rightarrow \infty} \frac{-1}{n} \sum_{\mathbf{i} \in \mathcal{I}^n} \mu([\mathbf{i}]) \log \mu([\mathbf{i}]) = \lim_{n \rightarrow \infty} \frac{-1}{n} \log \mu([\mathbf{i} | n]) \quad \text{for } \mu\text{-almost every } \mathbf{i},$$

where $\mathbf{i} | n$ denotes the first n symbols of $\mathbf{i} \in \Sigma$. The second equality follows from the Shannon–McMillan–Breiman theorem. Note that if μ is a Bernoulli measure with probability vector $\mathbf{p} = (p_i)_{i \in \mathcal{I}}$, $h_\mu = -\sum_{i \in \mathcal{I}} p_i \log p_i$.

One of the cardinal questions of fractal geometry is determining the dimension of self-affine sets and measures. Let us introduce the dimension concepts we will study.

1.2. Dimension concepts

In this note, we will be particularly interested in the Hausdorff (or covering) dimension and the Minkowski (or box-counting) dimension of sets. For a bounded set $E \subset \mathbb{R}^d$, let $N_\delta(E)$ be the minimal number of balls with radius $\delta > 0$ needed to cover E . We define the *lower and upper box-counting dimensions* by

$$\underline{\dim}_B(E) = \liminf_{\delta \rightarrow 0} \frac{\log N_\delta(E)}{-\log \delta} \quad \text{and} \quad \overline{\dim}_B(E) = \limsup_{\delta \rightarrow 0} \frac{\log N_\delta(E)}{-\log \delta}.$$

If the lower and upper Minkowski dimensions of E coincide, then we say that the Minkowski dimension of E exists, and we denote the common value by $\dim_B(E)$.

Before we define the Hausdorff dimension, we define the *s-dimensional (outer) Hausdorff measure* of a set $E \subset \mathbb{R}^d$ by

$$\mathcal{H}^s(E) = \sup_{\delta > 0} \mathcal{H}_\delta^s(E), \quad \text{where } \mathcal{H}_\delta^s(E) = \inf \left\{ \sum_i |U_i|^s : E \subset \bigcup_i U_i \text{ and } |U_i| < \delta \right\}.$$

Let $\dim_H(E)$ be the *Hausdorff dimension* of the set E defined as

$$\dim_H(E) = \inf \{s > 0 : \mathcal{H}^s(E) = 0\}.$$

For the basic properties of the Minkowski and Hausdorff dimensions, see Falconer's book [15]. Both dimension concepts measure how well a set fills the available space via effective covers with balls in some proper sense. We must mention here that there are other essential dimension concepts, like the Tricot (packing) and Assouad dimensions. Both play a significant role in geometric measure theory. For their definitions and basic properties, we direct the reader to Falconer's book [15] and Fraser's book [26].

We also introduce dimension concepts for measures via local dimension. Let ν be a Borel probability measure on $\mathbb{R}^d$. We define the *upper and lower local dimensions* of ν at the point x by

$$\overline{d}_\nu(x) = \limsup_{\delta \rightarrow 0} \frac{\log \nu(B(x, \delta))}{\log \delta} \quad \text{and} \quad \underline{d}_\nu(x) = \liminf_{\delta \rightarrow 0} \frac{\log \nu(B(x, \delta))}{\log \delta}.$$

We say that the *local dimension* exists at x if the two values above coincide, and we denote the common value by $d_\nu(x)$. We define the *lower and upper Hausdorff dimensions* of the measure ν by

$$\underline{\dim}_H(\nu) = \inf\{\dim_H(E) : \nu(E) > 0\} = \operatorname{ess\,inf}_{x \sim \nu} \underline{d}_\nu(x)$$

and

$$\overline{\dim}_H(\nu) = \inf\{\dim_H(E) : \nu(E^c) = 0\} = \operatorname{ess\,sup}_{x \sim \nu} \underline{d}_\nu(x).$$

For proof of the above equalities, see the book by Przytycki and Urbański [55, Section 7.6]. Let us note that the lower and upper Tricot dimensions can be defined similarly by using the upper local dimension

$$\underline{\dim}_P(\nu) = \operatorname{ess\,inf}_{x \sim \nu} \overline{d}_\nu(x) \quad \text{and} \quad \overline{\dim}_P(\nu) = \operatorname{ess\,sup}_{x \sim \nu} \overline{d}_\nu(x).$$

Finally, we say that the measure ν is *exact dimensional* if $\underline{\dim}_H(\nu) = \overline{\dim}_P(\nu)$. In this case, we denote the coinciding dimension values of ν by $\dim(\nu)$.

2. Quick overview of self-similar sets and measures

Let us first briefly consider the special case in which all the maps of the IFS $\Phi = \{f_i\}_{i \in I}$ on $\mathbb{R}^d$ are similarities. That is, $f_i(x) = \lambda_i O_i x + t_i$, where $0 < \lambda_i < 1$ is the scaling, $O_i \in O(d, \mathbb{R})$ is an orthonormal matrix and $t_i \in \mathbb{R}^d$ is the translation parameter. Iterative construction (1.2) of the self-similar set X provides a naive cover

of X and so a natural upper bound of the dimension of X . In particular, choosing $\lambda_{\max} = \max_i \lambda_i$, one can see that

$$\mathcal{H}_{\lambda_{\max}}^s(X) \leq \sum_{i \in \mathcal{I}^n} |f_i(B(0, R))|^s = R^s \left(\sum_{i \in \mathcal{I}} \lambda_i^s \right)^n \quad \text{for every } n \in \mathbb{N}. \quad (2.1)$$

In the last equality, we have used that all the maps f_i are similarities, and so $f_i(B(0, R))$ is a closed ball centred at $f_i(0)$ with radius $R \cdot \lambda_i$. Hence, $\dim_H(X) \leq s_0(\Phi)$, where $s_0(\Phi)$ is the unique solution of the equation $\sum_{i \in \mathcal{I}} \lambda_i^s = 1$. We call the quantity $s_0(\Phi)$ the *similarity dimension* of the IFS Φ . In particular, Hutchinson [33] showed that

$$\overline{\dim}_B(X) \leq \min\{d, s_0(\Phi)\}. \quad (2.2)$$

A central problem is when the lower bound $\dim_H X \geq \min\{d, s_0(\Phi)\}$ can be established. For that, we introduce some separation between the cylinder sets and the open set condition (OSC). We say that the IFS Φ satisfies the *open set condition* if there exists an open and bounded set $U \subset \mathbb{R}^d$ such that $f_i(U) \subset U$ and $f_i(U) \cap f_j(U) = \emptyset$ for every $i \neq j \in \mathcal{I}$. Many self-similar objects satisfy the OSC, for instance, the Sierpiński gasket in Figure 1. Hutchinson [33] showed the following.

Theorem 2.1. *If a self-similar IFS Φ satisfies the OSC, then*

$$\dim_H(X) = \overline{\dim}_B(X) = s_0(\Phi).$$

Falconer [19] showed that regardless of any separation condition, $\dim_H(X) = \overline{\dim}_B(X)$, and in particular, the box-counting dimension of X exists.

The idea of the proof is the following: consider the natural self-similar measure ν in (1.3) defined by the probabilities $p_i = \lambda_i^{s_0(\Phi)}$. In other words, let ν be the pushforward of the Bernoulli measure μ defined by the probability vector $(\lambda_i^{s_0(\Phi)})_{i \in \mathcal{I}}$. An easy-to-see consequence of the open set condition is that there exists a constant $C > 0$ such that for every $r > 0$ and every $x \in X$,

$$\#\{f_i(U) : f_i(U) \cap B(x, r) \neq \emptyset \text{ and } \lambda_i \leq r < \lambda_{i-}\} \leq C,$$

where $i_- = (i_1, \dots, i_{n-1})$ for $i = (i_1, \dots, i_n)$. Since $X \subseteq \overline{U}$, we get

$$\nu(B(x, r)) \leq \sum_{\substack{i \in \Sigma_* : f_i(U) \cap B(x, r) \neq \emptyset \\ \text{and } \lambda_i \leq r < \lambda_{i-}}} \lambda_i^{s_0(\Phi)} \leq C r^{s_0(\Phi)},$$

and so $\dim_H(X) \geq \underline{\dim}_H(\nu) \geq s_0(\Phi)$. In particular, we have shown the following variational principle:

$$\dim_H(X) = \max\{\dim(\pi_*\mu) : \mu \text{ is left-shift invariant and ergodic on } \Sigma\}, \quad (2.3)$$

where the dimension maximum is attained at the Bernoulli measure with probability vector $(\lambda_i^{s_0(\Phi)})_{i \in \mathcal{I}}$. Let us note that Feng and Hu [22] verified (2.3) with supremum for general self-similar sets regardless of the overlapping structure.

The dimension of self-similar measures can be studied analogously. Namely, let μ be an ergodic left-shift invariant probability measure on Σ . Let us denote by $\chi_\mu(\Phi)$ the Lyapunov exponent of μ concerning the IFS Φ . That is, $\chi_\mu(\Phi) = -\sum_{i \in \mathcal{I}} \mu([i]) \log \lambda_i$. The quantity represents the average shrinking rate of the diameter $f_i|_n(X)$ as $n \rightarrow \infty$ for a μ -typical i . One can show easily, relying on the Shannon–McMillan–Breiman theorem and Birkhoff’s ergodic theorem, that for every ergodic measure μ on Σ ,

$$\overline{\dim}_P(\pi_*\mu) \leq \min \left\{ d, \frac{h_\mu}{\chi_\mu(\Phi)} \right\},$$

where we recall that h_μ denotes the entropy of the measure μ . We call the quantity on the right-hand side the *Lyapunov dimension* of the measure μ . Furthermore, if the open set condition holds, then

$$\dim(\pi_*\mu) = \frac{h_\mu}{\chi_\mu(\Phi)}; \quad (2.4)$$

see Edgar [14, Theorem 5.2.5] or [8]. Feng and Hu [22] showed that regardless of any separation condition, the pushforward of any ergodic left-shift invariant measure is exact dimensional.

It is clear that $\dim_H(X) = \min\{d, s_0\}$ does not always hold. Consider, for example, the IFS $\Phi = \{f_1(x) = x/3, f_2(x) = x/3 + 1, f_3(x) = x/3 + 3\}$, which has similarity dimension 1. However, $f_1 \circ f_3 = f_2 \circ f_1$, and so the IFS $\Phi' = \{f_i \circ f_j\}_{(i,j) \in \{1,2,3\}^2 \setminus \{(1,3)\}}$ has the same attractor X , and the similarity dimension of Φ' is $\log_3 8 < 1$. One of the long-standing conjectures in fractal geometry claims that having a completely coinciding map is the only way on the real line to drop the dimension of the attractor strictly below the similarity dimension. This is called the *exact overlap conjecture*. More precisely, for a self-similar IFS Φ on the line with attractor X , if $\dim_H X < \min\{1, s_0(\Phi)\}$, then there exists an exact overlap, that is, there are $i \neq j \in \Sigma_*$ such that $f_i \equiv f_j$. The higher-dimensional analogue of this conjecture is false; see, for example, [8, Example 7.6].

To handle the overlapping situation (i.e., when the open set condition fails) on the real line, Hochman [29] introduced the exponential separation condition (ESC). The IFS Φ satisfies the *exponential separation condition* if there exists an $\varepsilon > 0$ such that for infinitely many $n \in \mathbb{N}$,

$$\min_{i \neq j \in \mathcal{I}^n} (|\log \lambda_i - \log \lambda_j| + |f_i(0) - f_j(0)|) > \varepsilon^n.$$

Hochman [29] showed that if a self-similar IFS Φ on the line satisfies the ESC, then for every self-similar measure $\nu = \sum_i p_i (f_i)_* \nu$,

$$\dim(\nu) = \min \left\{ 1, \frac{-\sum_i p_i \log p_i}{-\sum_i p_i \log \lambda_i} \right\},$$

and in particular,

$$\dim_H(X) = \min\{1, s_0(\Phi)\}.$$

The proof relies on an entropy growth theorem proved by Hochman [29] using methods from additive combinatorics. For some heuristics and intuition on the proof, see the survey paper of Hochman [30].

Jordan and Rapaport [35] showed that the dimension result can be extended to a more general family of measures, that is, if the self-similar IFS Φ satisfies the ESC, then for every left-shift invariant ergodic measure μ ,

$$\dim(\pi_* \mu) = \min \left\{ 1, \frac{h_\mu}{\chi_\mu(\Phi)} \right\}.$$

For more recent developments on the dimension theory of self-similar sets and measures, we direct the reader to the survey paper of Varjú [61].

3. Self-affine systems

3.1. The affinity and Lyapunov dimensions

Now, we turn to the main topic of this survey, the self-affine sets, namely, when the IFS $\Phi = \{f_i(x) = A_i x + t_i\}_{i \in \mathcal{I}}$ consists of invertible affine mappings. If one wants to cover the self-affine set X similarly to (2.1), one will immediately face the following problem: the sets $f_i(B(0, R))$ are no longer balls. To overcome this difficulty, Falconer [18] introduced the singular value function.

Let A be a $d \times d$ invertible matrix. Let us denote the singular values of A by $\alpha_1(A) \geq \dots \geq \alpha_d(A) > 0$. Note that $\alpha_1(A) = \|A\|$, where $\|\cdot\|$ is the operator norm induced by the usual Euclidean norm. Let us recall that for finite words $\mathbf{i} = (i_1, \dots, i_n) \in \Sigma_*$, we use the notations $f_{\mathbf{i}} = f_{i_1} \circ \dots \circ f_{i_n}$ and $A_{\mathbf{i}} = A_{i_1} \cdots A_{i_n}$. Then for every $\mathbf{i} \in \Sigma_*$, the set $f_{\mathbf{i}}(B(0, R))$ is an ellipse whose main semi-axes have lengths $\alpha_1(A_{\mathbf{i}}) \cdot R, \dots, \alpha_d(A_{\mathbf{i}}) \cdot R$. One can cover the ellipse $f_{\mathbf{i}}(B(0, R))$ with one square of side length $\alpha_1(A_{\mathbf{i}}) \cdot R$ or with at most $\lceil \alpha_1(A_{\mathbf{i}})/\alpha_2(A_{\mathbf{i}}) \rceil$ -many squares of side length $\alpha_2(A_{\mathbf{i}}) \cdot R$, and so on; see Figure 2. For a given $s \in [0, d]$ and given $d \times d$ matrix A , it is easy to see that

$$\min_{k=1, \dots, d} \frac{\alpha_1(A) \cdots \alpha_{k-1}(A)}{\alpha_k(A)^{k-1}} \alpha_k(A)^s = \alpha_1(A) \cdots \alpha_{\lfloor s \rfloor}(A) \cdot \alpha_{\lfloor s \rfloor + 1}(A)^{s - \lfloor s \rfloor}.$$

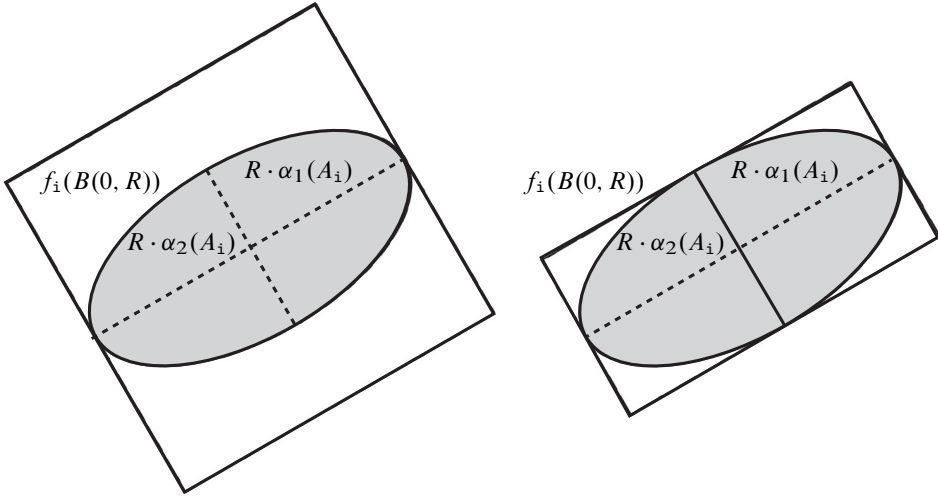


Figure 2. Two ways to cover the ellipse $f_i(B(0, R))$ with squares of side lengths $R \cdot \alpha_1(A_i)$ and $R \cdot \alpha_2(A_i)$.

Hence, the most economic cover for a given $s > 0$ using the sets $f_i(B(0, R))$ is covering $f_i(B(0, R))$ with squares of side length $\alpha_{\lfloor s \rfloor + 1}(A_i) \cdot R$. This motivates the definition of *singular value function*. Namely, for a given $s > 0$ and $d \times d$ invertible matrix A , let

$$\varphi^s(A) := \begin{cases} \alpha_1(A)^s & \text{if } s \in [0, 1], \\ \alpha_1(A) \cdots \alpha_k(A) \alpha_{k+1}(A)^{s-k} & \text{if } s \in (k, k+1] \\ & \text{for some } k \in \{1, \dots, d-1\}, \\ |\alpha_1(A) \cdots \alpha_d(A)|^{s/d} & \text{if } s > d. \end{cases} \quad (3.1)$$

And so taking $\alpha_{\max} = \max_i \alpha_1(A)$, we have

$$\mathcal{H}_{\alpha_{\max}^n R}^s(X) \leq R^s \cdot \sum_{i \in I^n} \varphi^s(A_i).$$

In particular, $\dim_H(X) \leq s$ for every $s > 0$ for which $\sum_{i \in I^n} \varphi^s(A_i) \rightarrow 0$ as $n \rightarrow \infty$. More generally, Falconer [18] showed that

$$\dim_H(X) \leq \overline{\dim}_B(X) \leq \min\{d, s_0(\Phi)\}, \quad (3.2)$$

where

$$s_0(\Phi) = \inf \left\{ s > 0 : \sum_{n=1}^{\infty} \sum_{i \in I^n} \varphi^s(A_i) < \infty \right\}.$$

We call the quantity $s_0(\Phi)$ the *affinity dimension* of Φ .

The singular value function is sub-multiplicative, see Falconer [18], and so by Fekete's lemma, the *sub-additive pressure function*

$$P_{\Phi}(s) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \left(\sum_{\mathbf{i} \in \mathcal{I}^n} \varphi^s(A_{\mathbf{i}}) \right)$$

is well defined, strictly monotone decreasing and continuous. Furthermore, $P(0) = \log \# \mathcal{I}$ and $P(s) \rightarrow -\infty$ as $s \rightarrow \infty$. Moreover, the affinity dimension $s_0(\Phi)$ is the unique root $P_{\Phi}(s_0(\Phi)) = 0$. Let us note that the affinity dimension is the natural generalisation of the similarity dimension, since $\alpha_1(\lambda O) = \dots = \alpha_d(\lambda O) = \lambda$ of every $O \in O(d, \mathbb{R})$ and $\lambda \in (0, \infty)$.

It is natural to ask whether the affinity dimension can be calculated explicitly or approximated efficiently. Unfortunately, the naive approximation $\sum_{\mathbf{i} \in \Sigma_n} \varphi^{s_n}(A_{\mathbf{i}}) = 1$ gives us the extremely inefficient $|s_0(\Phi) - s_n| = O(1/n)$ error rate. However, Pollicott and Vytnova [53] were able to introduce an approximation method with a super-exponential $O(e^{-cn^2})$ approximation rate, where n stands for the number of iterates required, for a certain class of planar matrices (like matrices with strictly positive entries) using zeta functions. This has been generalised recently by Morris [49] for higher dimensions.

The non-conformal structure of the IFS also necessitates the modification of the Lyapunov dimension compared to the self-similar situation following Jordan, Pollicott and Simon [34]. As we mentioned, the Lyapunov exponent measures the average shrinking rate of the cylinder sets at a typical point. However, the non-conformality of the system implies that this rate is different in different directions.

Let μ be a left-shift invariant ergodic probability measure on Σ . By Oseledets' multiplicative ergodic theorem, see Krengel's book [43], we define the i th Lyapunov exponent $\chi_{\mu}^i(\Phi)$ by

$$\begin{aligned} \chi_{\mu}^i(\Phi) &= \lim_{n \rightarrow \infty} \frac{-1}{n} \int \log \alpha_i(A_{\mathbf{i}|_n}) d\mu(\mathbf{i}) \\ &= \lim_{n \rightarrow \infty} \frac{-1}{n} \log \alpha_i(A_{\mathbf{i}|_n}) \quad \text{for } \mu\text{-almost every } \mathbf{i} \in \Sigma \end{aligned}$$

for $i = 1, \dots, d$. Then we define the *Lyapunov dimension* of the measure μ by

$$\dim_L(\mu) = \min \left\{ d, \min_{k=1, \dots, d} \left\{ k - 1 + \frac{h_{\mu} - \sum_{n=1}^{k-1} \chi_{\mu}^n(\Phi)}{\chi_{\mu}^k(\Phi)} \right\} \right\}.$$

Rossi [59] showed that for every ergodic left-shift invariant measure μ ,

$$\underline{\dim}_H(\pi_*\mu) \leq \overline{\dim}_P(\pi_*\mu) \leq \dim_L(\mu). \quad (3.3)$$

A natural question arises: under which circumstances are the inequalities in (3.2) and (3.3) equalities? The first results to answer this question were considering “typical systems” in the sense of Lebesgue-typical choice of translation parameters

$\underline{t} = (t_i)_{i \in \mathcal{I}} \in \mathbb{R}^{d \cdot \#\mathcal{I}}$. More precisely, Falconer [18] and Solomyak [60] showed the following.

Theorem 3.1. *If $\|A_i\| < 1/2$ for every $i \in \mathcal{I}$, then for $d \cdot \#\mathcal{I}$ -dimensional Lebesgue almost every $\underline{t} = (t_i)_{i \in \mathcal{I}}$,*

$$\dim_H(X_{\underline{t}}) = \overline{\dim}_B(X_{\underline{t}}) = \min\{d, s_0(\Phi_{\underline{t}})\},$$

where we write the IFS $\Phi_{\underline{t}} = \{x \mapsto A_i x + t_i\}_{i \in \mathcal{I}}$ and the attractor $X_{\underline{t}}$ of $\Phi_{\underline{t}}$.

Solomyak [60] improved the previous bound $1/3$ of Falconer [18] to $1/2$. Let us also note that Edgar [13], based on Przytycki and Urbański [54], pointed out that the bound $1/2$ is sharp. Namely, for every $\varepsilon > 0$, there exists a planar self-affine IFS $\Phi = \{x \mapsto A_1 x + t_1, x \mapsto A_2 x + t_2\}$ such that $\max_i \|A_i\| = 1/2 + \varepsilon$ but $\dim_H(X) < \min\{2, s_0(\Phi)\}$ for any choice of (t_1, t_2) .

A similar result to Theorem 3.1 has been shown by Jordan, Pollicott and Simon [34] for the pushforward of ergodic measures.

Theorem 3.2. *If $\|A_i\| < 1/2$ for every $i \in \mathcal{I}$ and μ is an ergodic left-shift invariant measure on Σ , then for $d \cdot \#\mathcal{I}$ -dimensional Lebesgue almost every $\underline{t}$, the measure $(\pi_{\underline{t}})_* \mu$ is exact dimensional; moreover,*

$$\dim((\pi_{\underline{t}})_* \mu) = \dim_L(\mu),$$

where $\pi_{\underline{t}}: \Sigma \rightarrow X_{\underline{t}}$ denotes the natural projection with respect to the IFS $\Phi_{\underline{t}} = \{x \mapsto A_i x + t_i\}_{i \in \mathcal{I}}$ to the attractor $X_{\underline{t}}$.

The above theorems provide us with the value of dimensions for typical systems in some sense, but they do not provide simple, verifiable, sufficient conditions for particular systems. Before the recent developments on the Ledrappier–Young formula for self-affine systems (see Section 3.5) and the entropy growth theorem introduced by Hochman [29], only very few particular examples were known; see, for instance, Hueter and Lalley [32] or Falconer [20]. In the remaining part of the paper, we will give “relatively” easily verifiable geometric and algebraic conditions which ensure that equalities hold in (3.2) and (3.3), with heuristics on the necessity of them.

3.2. Self-affine carpets

First, let us make a detour now for a class of planar self-affine systems for which we see the atypical behaviour. Let us consider the following simple example:

$$\Phi = \left\{ f_1(x, y) = \left(\frac{x}{3}, \frac{y}{4} \right), f_2(x, y) = \left(\frac{x+2}{3}, \frac{y}{4} \right), \right. \\ \left. f_3(x, y) = \left(\frac{x}{3}, \frac{y+3}{4} \right), f_4(x, y) = \left(\frac{x+2}{3}, \frac{y+3}{4} \right) \right\}. \quad (3.4)$$

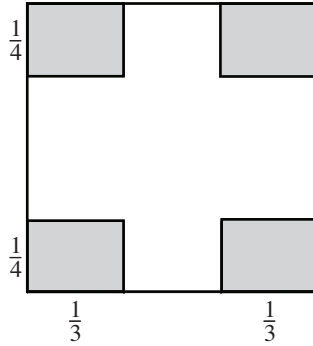


Figure 3. The maps of the IFS Φ defined in (3.4).

Every map of this IFS maps the unit square $[0, 1]^2$ to itself. For a visualisation of that, see Figure 3. Let us observe that the orthogonal projections of the attractor X of Φ are self-similar sets, and X is the Cartesian product of these projections. More precisely, the orthogonal projection to the x -axis is the triadic-Cantor set $C_{1/3}$, and let us denote the projection to the y -axis by $C_{1/4}$. Then $X = C_{1/3} \times C_{1/4}$. Using dimension formula (2.2) for self-similar sets and the properties of Hausdorff dimension regarding product sets, we get that

$$\dim_H(X) = \dim_H(C_{1/3}) + \dim_H(C_{1/4}) = \frac{\log 2}{\log 3} + \frac{\log 2}{\log 4}.$$

However, it is easy to calculate the affinity dimension in this particular case. Namely, $A_i = \begin{pmatrix} 1/3 & 0 \\ 0 & 1/4 \end{pmatrix}$ for every $i \in \{1, 2, 3, 4\}$, and so $\alpha_1(A_i) = 3^{-|i|}$ and $\alpha_2(A_i) = 4^{-|i|}$ for every $i \in \Sigma_*$. Thus,

$$s_0(\Phi) = 1 + \frac{\log 4 - \log 3}{\log 4} > \frac{\log 2}{\log 3} + \frac{\log 2}{\log 4} = \dim_H(X).$$

The drop in the dimension here is due to the alignment structure. That is, since the sets $f_i([0, 1]^2)$ for high iterates are relatively long and thin rectangles positioned in columns above each other, a square of side length 3^{-n} in an optimal cover captures many cylinders at once. This leads us to a smaller dimension value than our naive cover, which covers every cylinder set individually.

Self-affine sets of this kind are collectively called *self-affine carpets* and have been intensively studied. Bedford [10] and McMullen [48] independently studied the following class: let $a > b \geq 2$ be integers, and let $A \subseteq \{0, \dots, b-1\} \times \{0, \dots, a-1\}$ be such that it contains at least two elements. Let

$$\Phi = \left\{ f_{(i,j)}(x, y) = \left(\frac{x+i}{b}, \frac{y+j}{a} \right) \right\}_{(i,j) \in A}.$$

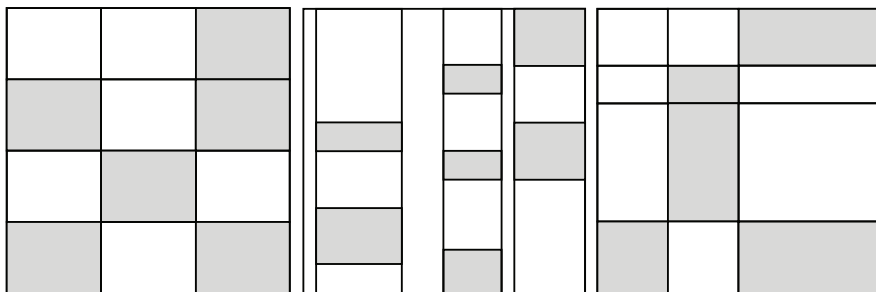


Figure 4. The images of the unit square in a Bedford–McMullen system, a Gatzouras–Lalley system and a Barański system.

For a visualisation of the images of the unit square, see the first image in Figure 4. Let A_1 be the set of $i \in \{0, \dots, b-1\}$ for which there exists j such that $(i, j) \in A$, and for every $i \in A_1$, let $T_i = \{j : (i, j) \in A\}$. Bedford [10] and McMullen [48] established the following dimension formulas:

$$\dim_B(X) = \log \#A_1 \left(\frac{1}{\log b} - \frac{1}{\log a} \right) + \frac{\log \#A}{\log a}$$

and

$$\dim_H(X) = \frac{\log(\sum_{i \in A_1} (\#T_i)^{\frac{\log b}{\log a}})}{\log b}.$$

From the formulas above, it is clear that $\dim_H(X) = \dim_B(X)$ if and only if $\#T_i = \#T_j$ for every $i, j \in A_1$. Further simple algebraic manipulations show that

$$\dim_H(X) = s_0(\Phi) \quad \text{if and only if } \#A = \#A_1 \text{ or } \#T_i = \frac{\#A}{a} \text{ for every } i,$$

and

$$\dim_B(X) = s_0(\Phi) \quad \text{if and only if } \#A = \#A_1 \text{ or } \#A_1 = b.$$

This shows that Bedford–McMullen sets fall into the set of “exceptions” in the Falconer–Solomyak theorem for the majority of cases.

Without going into the details, we mention that several generalisations of Bedford–McMullen carpets have appeared. Lalley and Gatzouras [44] studied planar systems where they kept the alignment structure and the order of the contraction rates, but the width and height of the rectangles may vary. Barański [2] also kept the alignment structure in both x - and y -directions but not the order of the contractions. (See Figure 4 for their visualisation.) Kenyon and Peres [40] generalised the result of Bedford and McMullen for higher-dimensional systems. Feng and Wang [24] gave a formula of the box-counting dimension for general planar carpet-like systems in

which the linear parts of the maps are diagonal matrices and the images of the open unit square are disjoint, relying on the contraction ratios and the dimension of the projections to the coordinate axis, which was generalised by Kolossváry [41] for higher dimensions.

Recently, considerable attention has been paid to overlapping carpet-like systems beyond Falconer's and Solomyak's "typicality" results. Fraser and Shmerkin [27] studied overlapping Bedford–McMullen carpets, in which they allowed overlaps between the columns, relying on the result on self-similar systems by Hochman [29]. Very recently, using entropy growth theorems and additive combinatorial methods, Rapaport [58] studied overlapping carpet-like sets in arbitrary dimensions. For further discussion and dimension formulas, see, for example, [8, Section 11.5].

3.3. Self-affine systems in $\mathbb{R}^2$ and $\mathbb{R}^3$ with separation and irreducibility

It is clear from the previous section that we cannot expect easily verifiable conditions to provide equalities in (3.2) when the linear parts of the maps are diagonal. More generally, similar exceptional behaviour can be observed if the maps only preserve a common subspace corresponding to the strongest contraction rate, see, for example, Kolossváry and Simon [42], or they preserve a finite collection of subspaces; see Fraser [25]. To avoid these cases, let us introduce the concept of strong irreducibility.

Let $\mathcal{A} = \{A_i\}_{i \in \mathcal{I}}$ be a finite collection of $d \times d$ invertible matrices. We say that $\mathcal{A}$ is *strongly irreducible* if there is *no* finite collection $V_1, \dots, V_N$ of proper subspaces of $\mathbb{R}^d$ such that

$$A_i V_k \subset \bigcup_{j=1}^N V_j \quad \text{for every } i \in \mathcal{I} \text{ and } k \in \{1, \dots, N\}.$$

We say that a self-affine IFS $\Phi = \{x \mapsto A_i x + t_i\}_{i \in \mathcal{I}}$ is strongly irreducible if the corresponding collection of matrices $\{A_i\}_{i \in \mathcal{I}}$ is strongly irreducible. It is clear that the systems discussed in Section 3.2 are not strongly irreducible. One would think naively that if we avoid alignments, and have separation, like an open set condition, we might achieve equality of the affinity dimension and the Hausdorff dimension of the attractor. Unfortunately, this is not the case. Let $\Phi = \{f_1(x) = A_1 x, f_2(x) = A_2 x\}$ be such that the matrices A_1, A_2 are invertible and have strictly positive entries and the images of the unit open square are disjoint $f_1((0, 1)^2) \cap f_2((0, 1)^2) = \emptyset$; see Figure 5. Clearly, the attractor $X = \{0\}$ is a single point; however, the affinity dimension of Φ is positive, since for every $\mathbf{i} \in \Sigma_*$, $\varphi^s(A_{\mathbf{i}}) \geq (\min_i \alpha_2(A_i))^{s|\mathbf{i}|}$ by the super-multiplicativity of the mini-norm. Thus, we need a stronger assumption than the open set condition.

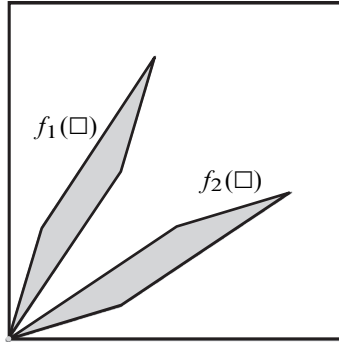


Figure 5. The IFS Φ satisfies the open set condition, but the attractor is a singleton.

We say that an IFS $\Phi = \{f_i\}_{i \in \mathcal{I}}$ satisfies the *strong open set condition (SOSC)* if there exists a non-empty, open and bounded set $U \subset \mathbb{R}^d$ such that

$$f_i(U) \subseteq U \quad \text{and} \quad f_i(U) \cap f_j(U) = \emptyset \quad \text{for every } i \neq j \in \mathcal{I} \text{ and } U \cap X \neq \emptyset,$$

where X is the attractor of Φ . Bandt and Graf [1] showed that the open set condition and the strong open set condition are equivalent for self-similar systems. This is clearly not the case for general self-affine systems.

Bárány, Hochman and Rapaport [4] showed that these two assumptions are enough for the dimension to be equal to the natural upper bounds in (3.2) and (3.3) on the plane.

Theorem 3.3. *Let $\Phi = \{f_i: x \mapsto A_i x + t_i\}_{i \in \mathcal{I}}$ be a self-affine IFS on $\mathbb{R}^2$ such that it satisfies the strong open set condition and the strong irreducibility. Then for every self-affine measure $\nu = \pi_* \mu$, where μ is the corresponding Bernoulli measure,*

$$\dim(\nu) = \dim_L(\mu) \quad \text{and} \quad \dim_H(X) = \overline{\dim}_B(X) = s_0(\Phi). \quad (3.5)$$

Note that the equation above indicates that the self-affine measure ν is exact dimensional; this and some ideas of the proof will be discussed later in Sections 3.4 and 3.5. Let us mention that in this form, the result of Bárány, Hochman and Rapaport [4] does not exclude that the IFS Φ consists only of similarities (or that the IFS Φ is conjugated to a self-similar IFS); however, that case already follows from the result of Hutchinson, Theorem 2.1. To avoid such redundancy, we introduce the concept of proximality. We say that a finite collection $\mathcal{A} = \{A_i\}_{i \in \mathcal{I}}$ of matrices is *proximal* if there is a finite product $A_{\mathbf{i}}$ for some $\mathbf{i} \in \Sigma_*$ which has maximal eigenvalue in modulus with multiplicity one. We use the same term similarly for the IFS, too.

Let us mention that the result of Bárány, Hochman and Rapaport [4] also remains valid in some not strongly irreducible cases, for example, for the graph of Takagi's function. Let us consider the parametrised family of maps

$$T_\lambda(x) := \sum_{n=0}^{\infty} \lambda^n d(2^n x, \mathbb{Z}),$$

where $d(\cdot, \mathbb{Z})$ denotes the distance from the closest integer. The graph of T_λ is a self-affine set invariant for the IFS

$$\Phi_\lambda = \left\{ f_1(x, y) = \left(\frac{x}{2}, \lambda y + \frac{x}{2} \right), f_2(x, y) = \left(\frac{x+1}{2}, \lambda y + \frac{1-x}{2} \right) \right\};$$

see the third image in Figure 1. It is clear that the IFS Φ is not strongly irreducible; however, the commonly preserved direction corresponds to the weaker contraction $\lambda > 1/2$, and the invariant directions of the strong contraction have a rich enough structure to avoid the occurrence of alignments. In particular, Bárány, Hochman and Rapaport [4, Corollary 7.6] showed the following: for every $\lambda \in (1/2, 1)$,

$$\dim_H(\text{graph}(T_\lambda)) = \overline{\dim}_B(\text{graph}(T_\lambda)) = s_0(\Phi_\lambda) = 2 - \frac{\log \lambda}{\log 2},$$

where $\text{graph}(T_\lambda)$ denotes the graph of T_λ over the interval $[0, 1]$.

The concept of exponential separation has been generalised for self-affine systems by Hochman and Rapaport [31], and they extended Theorem 3.3 for exponentially separated planar systems, omitting SOSC. Very recently, Rapaport [57] generalised the result on self-affine measures of Theorem 3.3 for systems in $\mathbb{R}^3$, and relying on that, Morris and Sert [51] established the result for the dimension of sets.

Theorem 3.4. *Let $\Phi = \{f_i: x \mapsto A_i x + t_i\}_{i \in \mathcal{I}}$ be a self-affine IFS on $\mathbb{R}^3$ such that it is proximal, strongly irreducible and it satisfies the strong open set condition. Then for every self-affine measure $\nu = \pi_* \mu$, where μ is the corresponding Bernoulli measure,*

$$\dim(\nu) = \dim_L(\mu) \quad \text{and} \quad \dim_H(X) = \overline{\dim}_B(X) = s_0(\Phi). \quad (3.6)$$

Rapaport [57] also gained dimension results for self-affine measures in higher dimensions, but with additional, most likely technical assumptions. So, the problem is open in $\mathbb{R}^d$ for $d \geq 4$.

Let us briefly describe here the strategy of the proofs of Theorems 3.3 and 3.4 compared to the strategy of the sketch in Section 2 in the self-similar case. First, we need a sequence of self-affine measures, whose affinity dimension approximates the affinity dimension arbitrarily well, to show the results for the dimension of sets

in (3.5) and (3.6). This is needed because the methods of proving the dimension results on measures strongly rely on the structure of self-affine measures. We will discuss the possibility of this approximation in detail in Section 3.4.

Another essential tool is the so-called Ledrappier–Young formula, which allows us to study the dimension of the planar/spatial measure via its orthogonal projections. Namely, the dimension of a self-affine measure is the Lyapunov dimension if and only if the dimension of the orthogonal projections is as large as possible. We will discuss it in Section 3.5. However, the projections of ν share a lot of common properties with self-similar measures, and by proper approximations, the entropy growth theorem of Hochman [29] is applicable, which allows us to establish the maximality of the dimension of the projections in reasonably many directions.

Let us note that it is an open problem whether under the conditions of Theorems 3.3 and 3.4 we have $\dim(\pi_*\mu) = \dim_L(\mu)$ for a general left-shift invariant ergodic measure.

Another open problem is whether the dimension of self-affine sets can be achieved via a dimension-maximising ergodic left-shift invariant measure. That is, does (2.3) hold for self-affine systems? In general, the answer is negative. Das and Simmons [12] constructed a self-affine set X in $\mathbb{R}^3$ with a special alignment structure such that

$$\dim_H(X) > \sup\{\dim_H(\pi_*\mu) : \mu \text{ is ergodic and left-shift invariant}\};$$

hence, (2.3) does not hold in general. However, we do not expect this phenomenon for sets under certain irreducibility assumptions (like under the assumptions of Theorems 3.3 and 3.4). Still, we have only partial results on the existence of dimension-maximising measures; see Bárány and Rams [7].

3.4. Thermodynamic formalism

As we mentioned at the end of Section 3.3, the results on the dimension of the measures in Theorems 3.3 and 3.4 rely strongly on the structural properties of self-affine measures, so to verify the dimension results for the sets in (3.5) and (3.6), we need to find a proper way to approximate the affinity dimension $s_0(\Phi)$ via self-affine measures, possibly by taking a high iterate of the IFS Φ .

So let $\Phi = \{x \mapsto A_i x + t_i\}_{i \in \mathcal{I}}$ be a self-affine IFS. As Falconer [18] pointed out, the singular value function φ^s , defined in (3.1), is sub-multiplicative, that is, for all A, B matrices, $\varphi^s(AB) \leq \varphi^s(A)\varphi^s(B)$. Relying on this property, Käenmäki [36] showed that for every ergodic left-shift invariant measure μ ,

$$s_0(\Phi) \geq \dim_L(\mu).$$

Moreover, there always exists a left-shift invariant ergodic measure μ such that the equality holds in the preceding formula. We call this measure *the Käenmäki measure*. Käenmäki and Vilppolainen [38] pointed out that the Käenmäki measure is not necessarily unique; however, under mild irreducibility assumptions, it is. More precisely, Feng and Käenmäki [23] showed that for planar IFS Φ , if the matrices are irreducible (there is no proper subspace V of $\mathbb{R}^2$ such that $A_i V = V$ for every $i \in \mathcal{I}$), then the Käenmäki measure is unique; furthermore, there exists a constant $C > 0$ such that for every $i \in \Sigma_*$,

$$C^{-1} \varphi^{s_0(\Phi)}(A_i) \leq \mu([i]) \leq C \varphi^{s_0(\Phi)}(A_i). \quad (3.7)$$

The uniqueness has been generalised for higher dimensions by Käenmäki and Morris [37] under certain irreducibility assumptions on the exterior product of matrices.

In general, it is not likely that the Käenmäki measure is a Bernoulli measure unless the linear parts of the IFS Φ have a very special structure. This has been recently verified by Morris and Sert [50].

Morris and Shmerkin [52] proved the following approximation result for planar self-affine IFS Φ with proximality, strong irreducibility: for every $\varepsilon > 0$, there exists $n \geq 1$ and a subset $\mathcal{G}_n \subset \Sigma_n$ such that the IFS $\Phi' = \{f_i\}_{i \in \mathcal{G}_n}$ is strongly irreducible and proximal; furthermore, the Lyapunov dimension of the self-affine measure

$$\nu' = \frac{1}{\#\mathcal{G}_n} \sum_{i \in \mathcal{G}_n} (f_i)_* \nu'$$

is at least $s_0(\Phi) - \varepsilon$.

Property (3.7) has a crucial role in this statement. A possible natural strategy is to approximate the affinity dimension with the Lyapunov dimension of Bernoulli measures of higher order; one needs to take a sufficiently large n and a sufficiently large subset $\mathcal{G}_n \subset \Sigma_n$ such that

$$\#\mathcal{G}_n \approx e^{nh_\mu}, \quad \alpha_k(A_i) \approx e^{-n\chi_\mu^k(\mu)} \quad \text{and} \quad \alpha_1(A_i A_j) \geq C \alpha_1(A_i) \alpha_1(A_j) \\ \text{for all } i, j \in \mathcal{G}_n,$$

where μ is the Käenmäki measure and C is a universal constant, independent of n . It is not trivial that such a constant $C > 0$ can be found, but in the proximal and strongly irreducible case, it is; see, for example, Bárány, Jordan, Käenmäki and Rams [5]. The approximation result for higher dimensions under certain irreducibility conditions was generalised recently by Morris and Sert [51].

3.5. The Ledrappier–Young formula

One of the main tools of the proofs of Theorems 3.3 and 3.4 is the exact dimensionality of self-affine measures and the so-called Ledrappier–Young formula. The problem

of the existence of local dimensions has a long history for self-affine measures and also in smooth dynamical systems. For an invariant measure in high-dimensional $C^{1+\alpha}$ systems, Ledrappier and Young [45, 46] proved the existence of the local dimensions along stable and unstable local manifolds and established a connection between conditional entropies, Lyapunov exponents and local dimensions. The adaptation of their method for self-affine systems remained an open question for almost three decades. The first special case was established by Feng and Hu [22] for self-affine systems, where the matrices were diagonal.

Let us illustrate the problem for planar self-affine systems with proximality, strong irreducibility and the strong open set condition. For a proper subspace $V \subset \mathbb{R}^2$, let us denote the orthogonal projection by $\text{proj}_V: \mathbb{R}^2 \rightarrow V$. The projective circle, that is, the set of all proper subspaces of $\mathbb{R}^2$, is denoted by $\mathbb{P}\mathbb{R}^1$. A 2×2 invertible matrix A naturally defines an action on the projective circle $\mathbb{P}\mathbb{R}^1$, which is also denoted by A with a slight abuse of notation. Let $\nu = \pi_*\mu$ be a self-affine measure with the corresponding Bernoulli measure μ . Furthermore, let $\mathbf{i} \in \Sigma$ be a “ μ -typical” point. Then take the ball $B(\pi(\mathbf{i}), \alpha_2(A_{\mathbf{i}|_n}))$ centred at $\pi(\mathbf{i})$ with radius $\alpha_2(A_{\mathbf{i}|_n})$. Heuristically, it intersects only the cylinder $f_{\mathbf{i}|_n}(X)$. Let $V \in \mathbb{P}\mathbb{R}^1$ be a subspace close to the shortest main semi-axis of $A_{\mathbf{i}|_n}(B(0, 1))$. Using the invariance of the measure μ and (1.4), we get that

$$\begin{aligned} \nu(B(\pi(\mathbf{i}), \alpha_2(A_{\mathbf{i}|_n}))) &\approx \nu(f_{\mathbf{i}|_n}(X) \cap B(\pi(\mathbf{i}), \alpha_2(A_{\mathbf{i}|_n}))) \\ &\approx p_{\mathbf{i}|_n} \nu(f_{\mathbf{i}|_n}^{-1}(B(\pi(\mathbf{i}), \alpha_2(A_{\mathbf{i}|_n})))) \\ &\approx p_{\mathbf{i}|_n} \nu\left(\text{proj}_{A_{\mathbf{i}|_n}^T V}^{-1} B\left(\text{proj}_{A_{\mathbf{i}|_n}^T V} \pi(\sigma^n \mathbf{i}), \frac{\alpha_2(A_{\mathbf{i}|_n})}{\alpha_1(A_{\mathbf{i}|_n})}\right)\right). \end{aligned} \quad (3.8)$$

For a visualisation, see Figure 6.

Hence, studying the local dimension of ν requires understanding the local dimension of $(\text{proj}_V)_*\nu$. But observe that as n goes to infinity, the direction of the orthogonal projection $A_{\mathbf{i}|_n}^T V$ takes a random walk on the projective circle similarly to the chaos game defined in Section 1.1. For a proximal and strongly irreducible collection of matrices $\mathcal{A}^T = \{A_i^T\}_{i \in \mathcal{I}}$, there exists a unique stationary Borel probability measure ν_F , called the *Furstenberg–Kesten measure*, with strictly positive Hausdorff dimension such that

$$\nu_F = \sum_{i \in \mathcal{I}} p_i (A_i^T)_* \nu_F.$$

For a proof, see, for example, the book of Bougerol and Lacroix [11]. The name of the measure originates back to the paper by Furstenberg and Kesten [28], who studied the Lyapunov exponents of random matrix products and gave a formula for it by using ν_F .

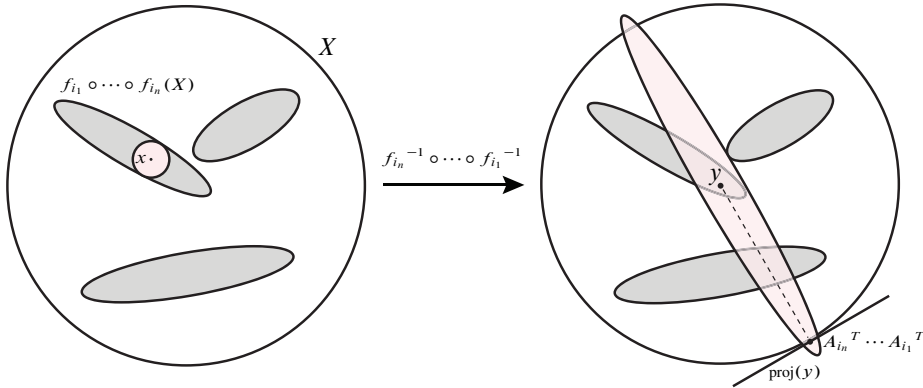


Figure 6. Applying the inverses of the maps of the IFS on a ball centred at $x = \pi(\mathbf{i})$ with radius $\alpha_2(A_{\mathbf{i}|_n})$, which is mapped to an ellipse centred at $y = \pi(\sigma^n \mathbf{i})$ comparable with $\text{proj}_{A_{\mathbf{i}|_n}^T V}^{-1} B(\text{proj}_{A_{\mathbf{i}|_n}^T V} \pi(\sigma^n \mathbf{i}), \frac{\alpha_2(A_{\mathbf{i}|_n})}{\alpha_1(A_{\mathbf{i}|_n})}) \cap X$.

Taking the logarithm on both sides of (3.8) and dividing by $\log \alpha_2(A_{\mathbf{i}|_n})$, the heuristic argument gives us the following, proven by Bárány and Käenmäki [6].

Theorem 3.5. *Let $\Phi = \{x \mapsto A_i x + t_i\}_{i \in \mathcal{I}}$ be a planar self-affine IFS such that it is proximal, strongly irreducible and satisfies the strong open set condition. Let $\nu = \pi_* \mu$ be a fully supported self-affine measure (i.e., $p_i > 0$ for every $i \in \mathcal{I}$) with Bernoulli measure μ . Then every fully supported self-affine measure ν is exact dimensional; moreover, for ν_F -almost every $V \in \mathbb{P}\mathbb{R}^1$, the projected measure $(\text{proj}_V)_* \nu$ is exact dimensional and*

$$\dim(\nu) = \frac{h_\mu}{\chi_\mu^2(\Phi)} + \left(1 - \frac{\chi_\mu^1(\Phi)}{\chi_\mu^2(\Phi)}\right) \dim((\text{proj}_V)_* \nu) \quad \text{for } \nu_F\text{-almost every } V.$$

For a more detailed discussion on the Furstenberg–Kesten measure and the Ledrappier–Young formula, see [8].

Theorem 3.5 has significant consequences. It is easy to see that

$$\dim(\nu) = \dim_L(\mu) \quad \text{if and only if} \quad \dim((\text{proj}_V)_* \nu) = \min \left\{ 1, \frac{h_\mu}{\chi_\mu^1(\Phi)} \right\} \quad \text{for } \nu_F\text{-a.e. } V.$$

Hence, to study the dimension of ν , one needs to understand the dimension of typical projections of ν . Several results appeared before Theorem 3.3 using the Ledrappier–Young formula, which were relying on the dimension estimates on exceptional sets of orthogonal projections of measures by Marstrand [47], Kaufman [39] and Falconer [17] and studying the dimension of ν_F ; see, for example, Bárány [3], Bárány and Rams [7], Falconer and Kempton [16] and Rapaport [56]. While the proof

of Theorem 3.3 relies on the Ledrappier–Young formula and the observation that the projections of ν share a lot of common properties with self-similar measures, and by proper approximations, the entropy growth theorem of Hochman [29] is applicable.

Finally, let us mention that Feng [21] generalised the Ledrappier–Young formula in a very wide generality and established the following regularity property.

Theorem 3.6. *Let $\Phi = \{x \mapsto A_i x + t_i\}_{i \in \mathcal{I}}$ be a self-affine IFS on $\mathbb{R}^d$, and let μ be an ergodic measure on Σ . Then the measure $\pi_* \mu$ is exact dimensional.*

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