

Applications of the renormalisation group

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Abstract. The renormalisation group provides a powerful point of view to understand random systems with strong correlations. This review discusses some applications to spin systems and related lattice models obtained in recent years.

1. What is the renormalisation group?

The general idea of the renormalisation group is to study a physical system in terms of effective models for different length scales, which roughly take the same form as the original model except that various parameters are adjusted or *renormalised*. The change of length scale defines the renormalisation group dynamics. This approach is particularly relevant for systems that are interacting and have strong correlations. Strong correlations mean that many length scales contribute to the behaviour of the system in a significant way, and interacting means that these scales have a nontrivial interaction, or dynamics. An idealised goal is given by Kadanoff's block spin picture for the Ising model, illustrated in Figure 1.

Kenneth Wilson (who received the 1982 Nobel Prize in Physics for his achievements in developing the renormalisation group picture) described Kadanoff's block spin picture as follows [38]:

In short, the Kadanoff block picture, although absurd, will be the basis for generalizations which are not absurd, and it is helpful to understand the Kadanoff picture in differential form before studying these generalizations.

The theoretical scope of the renormalisation group approach is rather general, but its roots and its most complete implementations to date remain in quantum and statistical field theory. The procedure of passing from a length scale to a larger one while

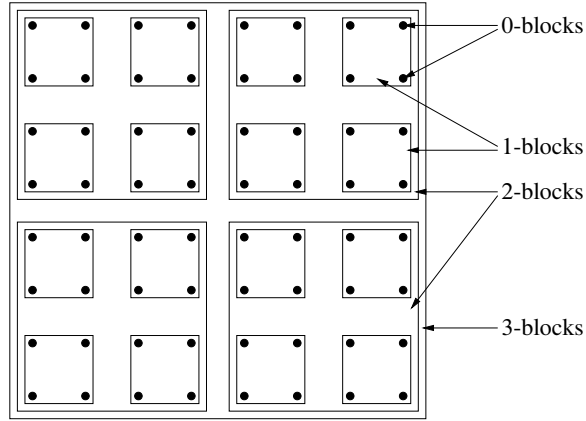


Figure 1. Block spin picture.

forgetting the microscopic details (or more precisely an implementation of this procedure) is called the *renormalisation group map*, and the point of view is to study this map as a dynamical system on the ‘space of systems’.

2. Spin systems and their strongly correlated regimes

This review focuses on applications of the renormalisation group to lattice spin models. Continuum models could be included as limits $\varepsilon \rightarrow 0$ of models on rescaled lattices of mesh size ε , but we will not focus on these. In this section, we will present a high-level overview of some of the regimes in which spin systems can develop strong correlations and where the renormalisation group approach therefore becomes relevant.

From now on, $\Lambda \subset \mathbb{Z}^d$ denotes a finite set of sites that eventually tends to $\mathbb{Z}^d$. In continuum models, one would similarly take $\Lambda \subset \varepsilon\mathbb{Z}^d$ with the understanding that $\varepsilon \rightarrow 0$. Typically, we assume that Λ is a cube with periodic boundary conditions (a torus); the interesting topic of boundary conditions is mostly undeveloped in the renormalisation group context.

To each site $x \in \Lambda$ we associate a spin φ_x . The spins can have internal degrees of freedom (i.e., they could be vector valued), and their components can be ‘bosonic’ or ‘fermionic’. The bosonic components are normally ordinary real numbers, while fermionic spins are odd elements of a Grassmann algebra.

The expectations of a spin system are roughly of the form

$$\langle F \rangle = \frac{1}{Z} e^{\frac{1}{2}(\varphi, \Delta \varphi) - \sum_{x \in \Lambda} V(\varphi_x)} \prod_{x \in \Lambda} d\varphi_x, \quad (2.1)$$

where Δ is the (negative definite) lattice Laplace operator:

$$\Delta f(x) = \sum_{y \sim x} (f(y) - f(x)), \quad (2.2)$$

and V is a local potential, and $d\varphi_x$ is an appropriate a priori measure (such as the Lebesgue measure). In several examples, we consider that the local potential is replaced by a nonlinear constraint on the spins. Constraints can be seen as degenerate potentials.

Some examples and regimes where strong correlation occur are given as follows. Precise results for representative models in all regimes are given in Section 3.

2.1. Critical temperature

Spin systems develop strong correlations over long distances near critical temperatures of continuous phase transitions.

The prototypical example is the Ising model, given at inverse temperature $\beta > 0$ (and assuming there is no external field) by

$$\langle F \rangle_\beta \propto \sum_{\sigma \in \{\pm 1\}^\Lambda} F(\sigma) e^{\frac{1}{2}\beta(\sigma, \Delta\sigma)}. \quad (2.3)$$

The closely related Ginzburg–Landau φ^4 model is defined for $g > 0$ and $\nu \in \mathbb{R}$ by

$$\langle F \rangle_{g,\nu} \propto \int_{\mathbb{R}^\Lambda} F(\varphi) e^{\frac{1}{2}(\varphi, \Delta\varphi) - \sum_{x \in \Lambda} \frac{1}{4}g\varphi_x^4 - \sum_{x \in \Lambda} \frac{1}{2}\nu\varphi_x^2} \prod_{x \in \Lambda} d\varphi_x. \quad (2.4)$$

For the Ginzburg–Landau φ^4 model, we consider $g > 0$ fixed and the parameter ν then takes the role of an effective temperature. Both models are expected to have essentially identical critical behaviour. In particular, both models have critical temperatures, denoted β_c and $\nu_c(g)$, respectively, which separate an ordered low-temperature phase and a disordered high-temperature phase, both of which have exponential truncated correlation decay. At and near the critical temperature, strong correlations develop. These can be quantified through various correlation functions and critical exponents. For example, in the Ising model, it is known that for $d \neq 3$ (and expected also in $d = 3$),

$$\langle \sigma_0 \sigma_x \rangle_{\beta_c} \approx |x|^{-(d-2+\eta)}. \quad (2.5)$$

The critical exponent η is 0 for $d > 4$; more delicately, it is also 0 in $d = 4$ (without logarithmic correlation, which, however, appear in closely related quantities), and it takes (or is expected to take) a nonzero value in $d = 2, 3$. The exact same exponents are expected for the φ^4 model.

The rigorous renormalisation group approach has, in particular, been used to study such models in the upper critical dimension $d = 4$. For long-range models

in which the nearest-neighbour Laplacian $-\Delta$ is replaced by a fractional power of $-\Delta$ (which then has power-law decay), models slightly below the upper critical dimension (which one may think of as a version of dimension ‘ $d = 4 - \varepsilon$ ’ but with important features missing) can also be studied; this is a version of the Wilson–Fisher ‘ ε -expansion’ [39].

The critical φ^4 type problem has been a prototypical problem considered in particular in [31, 35, 36] and more closely related to this review in [9–11, 22–25].

2.2. Low temperature

In spin systems with continuous symmetries, strong correlations are expected not only at their critical temperatures, but also in their low-temperature phases, where the continuous symmetry is spontaneously broken.

The prototypical example of a spin system with continuous symmetry is the classical Heisenberg model (or $O(n)$ model), defined by

$$\langle F \rangle_\beta \propto \int_{\sigma \in (\mathbb{S}^{n-1})^\Lambda} F(\sigma) e^{\frac{1}{2}\beta(\sigma, \Delta\sigma)} \prod_{x \in \Lambda} d\sigma_x. \quad (2.6)$$

Parallel to the situation of the Ising and φ^4 models, one can alternatively consider the n -component Ginzburg–Landau version of the $O(n)$ model, and not much is expected to change in the long-distance behaviour. At high temperatures, standard high temperature expansion methods show that correlations decay exponentially. Different from the situation of discrete spins (or discrete symmetry), the low-temperature phase is now strongly correlated as well. At low temperature, spins typically align in one direction (spontaneous symmetry breaking), but fluctuations orthogonal to the direction of broken symmetry are energetically inexpensive. These strong fluctuations orthogonal to the direction of broken symmetry are known as the Goldstone mode; see [32] for a review. For example, in $d \geq 3$ and $\beta > \beta_0$, it is the case that

$$\langle \sigma_0 \cdot \sigma_x \rangle_\beta \approx m(\beta) + \frac{c(\beta)}{\beta |x|^{d-2}}. \quad (2.7)$$

This was proven in [1–7], but a more tractable renormalisation group analysis of these models with nonabelian symmetry remains an outstanding problem. In some situation, reflection positivity and correlation inequalities can provide softer results [18–20, 33].

2.3. High temperature

In spin systems with noncompact symmetry groups, strong correlations can also occur in their high-temperature phases corresponding to unbroken symmetry.

The prototypical example is the two-dimensional Discrete Gaussian model, which can formally be regarded as the lattice Gaussian free field conditioned to be integer valued and is a model for crystal interface in $2 + 1$ dimensions. The expectation of the Discrete Gaussian model is defined for $F : \mathbb{Z}^\Lambda \rightarrow \mathbb{R}$ and inverse temperature $\beta > 0$ by

$$\langle F \rangle_\beta \propto \sum_{\sigma \in \mathbb{Z}^\Lambda : \sigma_0 = 0} F(\sigma) e^{\frac{1}{2}\beta(\sigma, \Delta\sigma)}. \quad (2.8)$$

As for the previous models, there is a closely related Ginzburg–Landau version of the Discrete Gaussian model, known as the lattice sine-Gordon model. It is again expected to have essentially identical long-distance behaviour. It is defined for $\beta > 0$ and $z \in \mathbb{R}$ by the expectation

$$\langle F \rangle_{\beta, z} \propto \int_{\mathbb{R}^\Lambda : \varphi_0 = 0} F(\varphi) e^{\frac{1}{2}(\varphi, \Delta\varphi) - \sum_{x \in \Lambda} 2z \cos(\sqrt{\beta}\varphi)} d\varphi. \quad (2.9)$$

In our conventions, the parameter β of the Discrete Gaussian model corresponds to $1/\beta$ in the sine-Gordon model. The reason for this convention is that sine-Gordon model has another interpretation as a Coulomb gas at inverse temperature β , and one can recover the Discrete Gaussian model in an appropriate $z \rightarrow \infty$ limit (which is analogous to the $g \rightarrow \infty$ limit in which the Ising model can be recovered from the φ^4 model). In two dimensions, these models have strong fluctuations at high temperature where they are close to an ordinary Gaussian free field, and the renormalisation group approach is particularly relevant in this regime. For example, for the Discrete Gaussian model on $\mathbb{Z}^2$, $\beta \leq \beta_0$, variances behave as

$$\langle (\varphi_x - \varphi_0)^2 \rangle_\beta \approx c(\beta) \log|x|, \quad (2.10)$$

whereas $\langle (\varphi_x - \varphi_0)^2 \rangle_\beta$ is bounded for large β . This transition is the roughening transition of the crystal interface, or Kosterlitz–Thouless transition. The mathematical understanding of this transition was pioneered in [29].

2.4. Continuum limits and other remarks

Continuum limits. In continuum limits of the above systems (when they exist), strong correlations also occur at short distances. As a prototypical example, we consider the continuum φ_d^4 measure which exists if $d < 4$ as the $\varepsilon \rightarrow 0$ limit (on a suitable space of generalised functions or in the sense of correlation functions) of the regularised measure with expectation

$$\langle F \rangle_{\lambda, \mu}^\varepsilon = \int_{\mathbb{R}^{\Lambda_\varepsilon}} F(\varphi) e^{-\varepsilon^d \sum_{x \in \Lambda_\varepsilon} \varphi_x (-\Delta^\varepsilon \varphi)_x - \varepsilon^d \sum_{x \in \Lambda_\varepsilon} (\frac{1}{4} \lambda \varphi^4 + \frac{1}{2} (\mu + a_\varepsilon(\lambda)) \varphi^2)} \prod_{x \in \Lambda_\varepsilon} d\varphi_x, \quad (2.11)$$

where $a_\varepsilon(\lambda)$ is a divergent (as $\varepsilon \rightarrow 0$) counterterm needed to obtain a nontrivial continuum limit $\langle \cdot \rangle_{\lambda, \mu}$. These continuum limits satisfy, for example in $d = 3$,

$$\langle \varphi_x \varphi_y \rangle_{\lambda, \mu} \approx \frac{c}{|x - y|^{d-2}} \quad (|x - y| \rightarrow 0). \quad (2.12)$$

To connect the discussion of lattice models with continuum models, note that after rescaling the field φ by $\varepsilon^{(d-2)/2}$, the ε -regularised continuum model takes the form of a model on the unit lattice, but with coupling constants that depend on ε . In fact, the corresponding lattice coupling constant $g = \varepsilon^d \lambda$ vanishes as $\varepsilon \rightarrow 0$. This is compensated by the choice of the counterterm, which in terms of the lattice model ensures that the parameters are close to a critical point (strongly correlated), so that a macroscopic effect can survive in the limit. From the perspective of a classical continuum theory (i.e., normalising with respect to the Riemann sum $\varepsilon^d \sum_x$ instead of $\sum_x$), on the other hand, the coupling constant diverges. This is an example of the infamous infinities in quantum field theory.

Combinatorial models. Various more combinatorial models turn out to be very closely related to the above or similar models. Examples include self-avoiding walks (exactly described by a version of the lattice φ^4 model), random forests (exactly described by a particular lattice nonlinear sigma model), and interacting dimers (exactly described by a discrete version of the massless Thirring model). Other models such as percolation have completely parallel behaviour, but exact field theory formulations are not known.

3. Overview of some results

We discuss three models in which we exemplify the use of the renormalisation group to analyse the emergence of strong correlations in low-, high-, and critical-temperature regimes, as discussed in the previous section. These models are all studied in a framework for the renormalisation group going back to [21] and much further developed since then. We refer to [12] for an introduction and further references.

In all examples discussed in this section, $\Lambda = \Lambda_N$ denotes a d -dimensional discrete torus of side length L^N for integers $L > 1$ and $N \geq 1$, and L will always be chosen sufficiently large (but fixed). The infinite volume limit will be taken by taking $N \rightarrow \infty$.

3.1. High temperature: The Discrete Gaussian model

The two-dimensional *Discrete Gaussian model* is a model for a crystal interface (in $2 + 1$ dimensions) and has a phase transition between a smooth (localised) low-temperature phase and a rough (delocalised) high-temperature phase; a physics textbook treatment can be found in [27, Chapter 6]. The mathematical understanding of this model was pioneered by Fröhlich–Spencer [34]. Our results [15, 16] with Park and Rodriguez establish the scaling limit at high temperatures using the renormalisation group, making use of ideas previously developed by Dimock–Hurd [28] and Falco [29, 30] for the Coulomb gas.

To be precise, define the two-dimensional Discrete Gaussian model on Λ_N at inverse temperature $\beta \in (0, \infty)$ by the following expectation. For any $F : (2\pi\mathbb{Z})^{\Lambda_N} \rightarrow \mathbb{R}$ with $F(\sigma) = F(\sigma + c)$ for any constant $c \in 2\pi\mathbb{Z}$, and such that the following series converges, we define

$$\langle F \rangle_\beta^{\Lambda_N} \propto \sum_{\sigma \in \Omega^{\Lambda_N}} e^{\frac{\beta}{2}(\sigma, \Delta\sigma)} F(\sigma) = \sum_{\sigma \in \Omega^{\Lambda_N}} e^{-\frac{\beta}{4} \sum_{x \sim y} (\sigma_x - \sigma_y)^2} F(\sigma), \quad (3.1)$$

where the sum over $x \sim y$ counts every undirected edge twice and

$$\Omega^{\Lambda_N} = \{\sigma \in (2\pi\mathbb{Z})^{\Lambda_N} : \sigma_{x=0} = 0\}. \quad (3.2)$$

In [15, 16], we studied the scaling limit on the continuum torus $\mathbb{T}^2$ and on the infinite plane $\mathbb{R}^2$ for $\beta \leq \beta_0$ sufficiently small (high temperature). All results hold for general finite-range step distributions, but we simplify the statement to the nearest-neighbour interaction. We also remark that in [15, 16] we used β instead of $1/\beta$.

Theorem 3.1. *There exists $\beta_0 \in (0, \infty)$ and $\beta_{\text{eff}}(\beta) \in (0, \infty)$ for $\beta \leq \beta_0$ (high temperature), satisfying $1/\beta_{\text{eff}}(\beta) = 1/\beta + O_J(e^{-c/\beta})$ for some $c > 0$ as $\beta \rightarrow 0$, such that the following scaling limit results hold:*

- (i) *The Discrete Gaussian model $\langle \cdot \rangle_\beta^{\Lambda_N}$ on the discrete torus Λ_N converges to a continuum Gaussian free field on $\mathbb{T}^2$: For any $f \in C^\infty(\mathbb{T}^2)$ with $\int f \, dx = 0$ and f_N a suitable discretisation of f on Λ_N , as $N \rightarrow \infty$,*

$$\log \langle e^{(f_N, \sigma)_{\Lambda_N}} \rangle_\beta^{\Lambda_N} \rightarrow \frac{1}{2\beta_{\text{eff}}(\beta)} (f, (-\Delta_{\mathbb{T}^2})^{-1} f)_{\mathbb{T}^2}. \quad (3.3)$$

- (ii) *The Discrete Gaussian model $\langle \cdot \rangle_\beta^{\mathbb{Z}^2}$ on $\mathbb{Z}^2$ converges to a continuum Gaussian free field on $\mathbb{R}^2$: For any $f \in C_c^\infty(\mathbb{R}^2)$ with $\int f \, dx = 0$ and f_ε a suitable discretisation on $\mathbb{Z}^2$, as $\varepsilon \rightarrow 0$,*

$$\log \langle e^{(f_\varepsilon, \sigma)_{\mathbb{Z}^2}} \rangle_\beta^{\mathbb{Z}^2} \rightarrow \frac{1}{2\beta_{\text{eff}}(\beta)} (f, (-\Delta_{\mathbb{R}^2})^{-1} f)_{\mathbb{R}^2}. \quad (3.4)$$

3.2. Low temperature: The arboreal gas

The *arboreal gas* is the uniform measure on (unrooted) spanning forests of a weighted graph. It turns out to be the graphical representation of the nonlinear sigma model with $\mathrm{OSp}(1|2)$ symmetry. This is an exact analogue of the classical Heisenberg model (2.6) with the sphere $\mathbb{S}^{n-1}$ replaced by the fermionic hyperbolic plane $\mathbb{H}^{0|2}$ and the group $O(n)$ by the supergroup $\mathrm{OSp}(1|2)$. This point of view was pioneered in [26] and further developed with Crawford, Helmuth, and Swan in [14], to which we refer for an introduction and for results in $d = 2$.

In [13], with Crawford and Helmuth, we used the renormalisation group approach to prove the existence of a low-temperature phase in $d \geq 3$ by proving that the continuous symmetry of the nonlinear sigma model is spontaneously broken. This is an instance of the Goldstone mechanism [32].

The arboreal gas model is precisely defined as follows. For the graph with vertices $\Lambda = \Lambda_N$ and edges E (the nearest neighbours for us), a forest $F = (\Lambda, E(F))$ is an acyclic subgraph with the same vertex set Λ . Given an edge weight $\beta > 0$ (inverse temperature), the probability of a forest F under the arboreal gas measure is

$$\mathbb{P}_{\beta}^{\Lambda_N}[F] \equiv \frac{1}{Z_{\beta,0}^{\Lambda_N}} \beta^{|E(F)|}, \quad (3.5)$$

where $T \in F$ denotes that T is a tree in the forest, i.e., a connected component of F , and $|E(F)|$ is the number of edges in F . It is sometimes of interest (and technically useful) to add an external field $h \geq 0$ (vertex weight). The generalisation of the arboreal gas is

$$\mathbb{P}_{\beta,h}^{\Lambda_N}[F] \equiv \frac{1}{Z_{\beta,h}^{\Lambda_N}} \beta^{|E(F)|} \prod_{T \in F} (1 + h|V(T)|), \quad (3.6)$$

where $|V(T)|$ is the number of vertices in T . This generalisation with external field can also be conveniently interpreted as the standard arboreal gas on an extended graph with a ghost vertex added [13]. The arboreal gas is also known as the uniform forest model, as Bernoulli bond percolation conditioned to be acyclic, and as the $q \rightarrow 0$ limit of the q -state random-cluster model with p/q converging to β ; see [37].

Theorem 3.2. *Suppose $d \geq 3$. There exists $\beta_0 \in (0, \infty)$ such that for $\beta \geq \beta_0$ there exist functions $\theta_d(\beta) = 1 - O(1/\beta)$ and $\mathbf{c}_i(\beta) = \mathbf{c}_i + O(1/\beta)$ with $\mathbf{c} > 0$, satisfying $(\mathbf{c}_2(\beta)/\mathbf{c}_1(\beta)^2)\theta_d(\beta)^2 = 1$ and $\mathbf{c}_0(\beta) = 2\mathbf{c}_1(\beta)$, and that the following hold:*

- (i) *The torus two-point function has long-range order with free field correction (Goldstone mode):*

$$\mathbb{P}_{\beta}^{\Lambda_N}[0 \leftrightarrow x] = \theta_d(\beta)^2 + \frac{\mathbf{c}_0(\beta)}{\beta|x|^{d-2}} + O\left(\frac{1}{\beta|x|^{d-2+\kappa}}\right) + O\left(\frac{1}{\beta L^{\kappa N}}\right), \quad (3.7)$$

where $|x|$ denotes the Euclidean norm and $\kappa > 0$ a small constant.

- (ii) In any possibly subsequential infinite volume limit along Λ_N , there is an infinite tree:

$$\mathbb{P}_\beta^{\mathbb{Z}^d} [|T_0| = \infty] = \theta_d(\beta). \quad (3.8)$$

- (iii) The truncated two-point functions $\tau_\beta(x)$ and $\sigma_\beta(x)$ satisfy

$$\tau_\beta(x) = \frac{c_1(\beta)}{\beta|x|^{d-2}} + O\left(\frac{1}{\beta|x|^{d-2+\kappa}}\right), \quad (3.9)$$

$$\sigma_\beta(x) = \frac{c_2(\beta)}{\beta^2|x|^{2d-4}} + O\left(\frac{1}{\beta^2|x|^{2d-4+\kappa}}\right). \quad (3.10)$$

In the sigma-model representation of the arboreal gas, (3.7) is the exact analogue of (2.7), and the truncated correlation functions (3.9) and (3.10), see [13] for their precise definition, correspond to transversal and radial correlations of the spins, respectively. The free field decay (3.7) and (3.9) represents the Goldstone mode. The decay of radial correlations (3.10) has the explanation that the radial mode is unstable and ‘decays’ into two Goldstone particles as explained in [32].

3.3. Critical temperature: The $|\varphi|^4$ model

Finally, we discuss an application of the renormalisation group to critical behaviour – which is the most well-studied context. The lattice φ^4 model is defined as in (2.4), and its n -component version is defined in the same way except that all spins φ_x become vectors in $\mathbb{R}^n$. A summary of some results for the four-dimensional model at the critical point is given in the next theorem. These results were obtained with Brydges, Slade, Tomberg, and Wallace. These results extend to $n = 0$ which can be interpreted as the weakly self-avoiding walk.

Theorem 3.3. *Let $d = 4$, $n = 0, 1, \dots$, and let $g > 0$ be small. For the n -component $|\varphi|^4$ model ($n \geq 1$), and for the weakly self-avoiding walk (corresponding to $n = 0$), there exist critical values $v_c = v_c(g, n)$ such that, as $\varepsilon \downarrow 0$ and $|x| \rightarrow \infty$, the following results hold (with constants $c_* = c_*(g, n)$ and C_p depending on p):*

- (i) For $n \geq 0$, the critical two-point function decays as [10]

$$G_{g, v_c}(x) = \lim_{N \rightarrow \infty} \langle \varphi_0 \cdot \varphi_x \rangle_{g, v_c}^{\Lambda_N} \sim c_G |x|^{-2}. \quad (3.11)$$

- (ii) For $n \geq 0$, the susceptibility obeys [8, 11]

$$\chi(g, v_c + \varepsilon) = \lim_{N \rightarrow \infty} \sum_{x \in \Lambda_N} \langle \varphi_0 \cdot \varphi_x \rangle_{g, v_c + \varepsilon}^{\Lambda_N} \sim c_\chi \varepsilon^{-1} (\log \varepsilon^{-1})^{\frac{(n+2)}{(n+8)}}. \quad (3.12)$$

(iii) For $n \geq 0$, the correlation length of order $p > 0$ obeys [17]

$$\frac{1}{\chi(g, v_c + \varepsilon)} \sum_x |x|^p G_{g, v_c + \varepsilon}(x) \sim C_p c_\chi^{-\frac{p}{2}} \varepsilon^{-\frac{p}{2}} (\log \varepsilon^{-1})^{\frac{p(n+2)}{(2n+16)}}. \quad (3.13)$$

(iv) For $n \geq 1$, the specific heat obeys [8]

$$c_H(g, v_c + \varepsilon) \sim c_H \begin{cases} (\log \varepsilon^{-1})^{\frac{(4-n)}{(n+8)}} & (n = 1, 2, 3), \\ \log \log \varepsilon^{-1} & (n = 4), \\ 1 & (n > 4). \end{cases} \quad (3.14)$$

(v) For $n \geq 1$, the spin field on Λ converges weakly to white noise if $v > v_c$, and to a massive Gaussian free field if $v \downarrow v_c$ as $N \rightarrow \infty$ appropriately; see [8].

Acknowledgements. I thank my collaborators on the various projects described here. These include D.C. Brydges, N. Crawford, T. Helmuth, J. Park, P.-F. Rodriguez, and G. Slade.

Funding. This work was partially supported by the ERC grant 851682 SPINRG and the NSF grant DMS-2348045.

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