

A free boundary approach to spectral optimization problems: Extremal Neumann eigenvalues

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Abstract. This is an extended version of the invited talk given at the 9th European Congress of Mathematics. It is concerned with the geometric optimization of Neumann eigenvalues of the Laplace operator on domains of $\mathbb{R}^N$ and on spheres. The main question is to maximize the k -th eigenvalue of the Neumann Laplacian under a volume constraint; we approach it in the context of free boundary problems with no energy interface. Together with a discussion about existence/non-existence of optimal geometries and possible relaxation on densities, we point out how numerical approximations play a key role in understanding the problem. A particular attention is given to the lowest two non-trivial eigenvalues for which a full answer is available in any dimension of the Euclidean space and in the hyperbolic space, but not on spheres, where a surprising phenomenon occurs: while a complete answer can be given for the second eigenvalue, a series of open problems remain unsolved for the first one. The results presented in the talk have been obtained in collaborations with A. Henrot, R. Laugesen, E. Martinet, M. Nahon and E. Oudet.

1. Introduction and statement of the result

To start, let us fix $N \geq 1$ and denote by $\mathbb{R}^N$ the Euclidean space of dimension N . Let $\rho : \mathbb{R}^N \rightarrow [0, 1]$ be a measurable function (throughout the paper called density), $\rho \not\equiv 0$. We set $\mu_0(\rho) = 0$, and for $k \geq 1$ we define

$$\mu_k(\rho) = \inf_{S \in \mathcal{S}_k} \max_{u \in S} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \rho dx}{\int_{\mathbb{R}^N} u^2 \rho dx}, \quad (1.1)$$

where $\mathcal{S}_k$ is the family of all subspaces of dimension k in

$$\left\{ u \cdot 1_{\{\rho(x) > 0\}} : u \in C_c^\infty(\mathbb{R}^N), \int_{\mathbb{R}^N} \rho u dx = 0 \right\}.$$

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We note that, in general, $\forall k \in \mathbb{N}$, $0 \leq \mu_k(\rho) \leq \mu_{k+1}(\rho)$. In the absence of any further information on ρ , the variational quantities $\mu_k(\rho)$ are well defined (and called, by abuse of language, eigenvalues), but they are not necessarily related to a well-posed eigenvalue problem; it is even possible that all of them vanish.

Assume now that $\Omega \subseteq \mathbb{R}^N$ is a bounded, open, Lipschitz set, not necessarily connected. Taking $\rho = 1_\Omega$, the characteristic function of the set Ω , the values $\mu_k(\rho)$ are precisely the eigenvalues of the Laplace operator with Neumann boundary conditions on the set Ω . Following (1.1), we denote these eigenvalues, counting the multiplicity

$$0 = \mu_0(\Omega) \leq \mu_1(\Omega) \leq \mu_2(\Omega) \leq \cdots \rightarrow +\infty.$$

If Ω is connected, then $\mu_1(\Omega) > 0$; otherwise, $\mu_1(\Omega) = 0$.

Moreover, there exists $u_k \in H^1(\Omega) \setminus \{0\}$ such that the equation

$$\begin{cases} -\Delta u_k = \mu_k(\Omega) u_k & \text{in } \Omega, \\ \frac{\partial u_k}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

is satisfied in a weak sense, n being the outward normal at $\partial\Omega$.

The generic problem we are dealing with is the following: given $m > 0$, find $\Omega \subseteq \mathbb{R}^N$ solution of the following shape optimization problem

$$\max\{\mu_k(\Omega) : \Omega \subseteq \mathbb{R}^N, |\Omega| = m\}. \quad (1.2)$$

Equivalently, taking advantage of the homogeneity of the eigenvalues to rescalings in $\mathbb{R}^N$, this is equivalent to

$$\max_{\Omega \subseteq \mathbb{R}^N} |\Omega|^{\frac{2}{N}} \mu_k(\Omega).$$

The main questions can be phrased as follows:

- (1) Does problem (1.2) have a solution?
- (2a) If so, what are the qualitative properties of an optimal domain: is the boundary smooth, does the domain have symmetries, can the exact solution be identified?
- (2b) If no existence, describe the behavior of a maximizing sequence and understand possible relaxed limits (Γ -convergence, homogenization, etc.).
- (3) If an analytic identification of the solution is not possible, then approximate it numerically.

The maximization problem (1.2) is directly related to the Pólya conjecture, stating that the k -th eigenvalue is bounded from above by the main term of the Weyl

asymptotic, namely that the following inequality occurs

$$\forall k \in \mathbb{N}, \quad \mu_k(\Omega) \leq \frac{4\pi^2 k^{\frac{2}{N}}}{(\omega_N |\Omega|)^{\frac{2}{N}}}, \quad (1.3)$$

where ω_N is the volume of the unit ball of $\mathbb{R}^N$. The Pólya conjecture has been intensively studied in the last 50 years; for now, it is only known to hold for some particular classes of domains. The connection between the Pólya conjecture and problem (1.2) stems from the idea that inequality (1.3) is enough to be proved *only* for the maximizing set of (1.2), possibly taking advantage of some properties coming out from its optimality.

An incomplete collection of results concerning problem (1.2) is the following:

- Szegő 1954, [22]: the disc minimizes $\frac{1}{|\Omega|} \left(\frac{1}{\mu_1(\Omega)} + \frac{1}{\mu_2(\Omega)} \right)$, in $\mathbb{R}^2$ among smooth simply connected sets.
- Weinberger 1956, [24]: the ball maximizes $|\Omega|^{\frac{2}{N}} \mu_1(\Omega)$, in $\mathbb{R}^N$ among Lipschitz sets.
- Girouard, Nadirashvili, Polterovich 2009, [13]: the union of two equal disjoint discs maximizes $|\Omega| \mu_2(\Omega)$, in $\mathbb{R}^2$ among smooth simply connected sets.
- Poliquin and Roy-Fortin 2012, [21], μ_{22} is not maximized by a ball or union of balls. This result excludes that all maximizers in (1.2) are union of balls. The order 22 could possibly be decreased using a certified numerical approach. It is likely to be equal to 3 in $\mathbb{R}^2$.
- Bucur, Henrot 2019, [7]: the union of two equal disjoint balls maximizes $|\Omega|^{\frac{2}{N}} \mu_2(\Omega)$, in $\mathbb{R}^N$ among Lipschitz sets. Moreover, the result extends to densities $\rho : \mathbb{R}^N \rightarrow [0, 1]$, provided the volume constraint is replaced by a mass constraint (details are given in the next section).
- Freitas, Laugesen 2022, [12]: extend the maximization result for μ_2 (in the class of domains) to the hyperbolic space and give a different topological argument.

2. Main results

A recent series of results, presented below, stems from the following theorem from [7]. Although its main purpose was to find the maximum of μ_2 in the class of Lipschitz domains, the extension to densities plays a fundamental role in understanding problem (1.2).

Theorem 2.1. *Let $\rho : \mathbb{R}^N \rightarrow [0, 1]$, $\rho \in L^1(\mathbb{R}^N)$. Then, denoting $B_1 \subseteq \mathbb{R}^N$ the ball of radius 1, the following inequalities hold:*

- (1) $\mu_1(\rho) \left(\int_{\mathbb{R}^N} \rho dx \right)^{\frac{2}{N}} \leq |B_1|^{\frac{2}{N}} \mu_1(B_1).$
 (2) $\mu_2(\rho) \left(\int_{\mathbb{R}^N} \rho dx \right)^{\frac{2}{N}} \leq 2^{\frac{2}{N}} |B_1|^{\frac{2}{N}} \mu_1(B_1).$

If equality occurs in (1) or (2), then ρ equals a.e. the characteristic function of a ball or two equal disjoint balls, respectively.

For the proof of the theorem, we refer to [7] (see also [9, 12]). It is obtained by direct comparison between $\mu_1(\rho)$, $\mu_2(\rho)$ and the corresponding eigenvalues on a ball, two balls, respectively. This is done by building a suitable pack of test functions. For μ_1 , they are precisely the Weinberger functions, while for μ_2 , they are obtained by suitable gluing couples of two Weinberger functions associated to two (not necessarily disjoint) equal balls. A topological argument is necessary in order to certify the existence of a pack of valid test functions.

The main observation coming out from the previous theorem is that instead of working with the shape optimization problem (1.2), one could work with the density optimization problem

$$\max \left\{ \left(\int_{\mathbb{R}^N} \rho dx \right)^{\frac{2}{N}} \mu_k(\rho) : \rho : \mathbb{R}^N \rightarrow [0, 1] \right\} := \mu_k^*. \quad (2.1)$$

The main question is to understand whether for every $k \in \mathbb{N}$ the two problems have the same solution. Note that the competitors in problem (2.1) include the competitors of problem (1.2), so that the maximum in (2.1) cannot be lower than in (1.2). The advantage to work with (2.1) comes from the flexibility offered by the space of competitors: $L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ has a much better functional structure than the class of bounded, open, Lipschitz sets.

Coming back to (2.1), the following existence result has been proved in [10].

Theorem 2.2. *Problem (2.1) admits a solution as a collection of at most k densities. Precisely, there exists $1 \leq l \leq k$ and l densities $\rho_1, \dots, \rho_l$ such that $\sum_{j=1}^l \int_{\mathbb{R}^N} \rho_j dx = 1$, together with integers $i_1, \dots, i_l \geq 0$ satisfying $k + 1 = \sum_{j=1}^l (i_j + 1)$, and such that*

$$\mu_k^* = \mu_{i_1}(\rho_1) = \dots = \mu_{i_l}(\rho_l).$$

Above, the concept of *collection* of densities plays the same role as that of connected components for an open set, with the global “spectrum” of the collection being the union of “spectra” of the individual densities. Roughly speaking, we compute the eigenvalues of each density separately and order them together, counting the multiplicities. There is no information on the display of those densities in the space, but, of course, if the supports of all densities were bounded, then one can build a unique density in $\mathbb{R}^N$ as a solution to problem (2.1).

For the proof, we refer to [10]. Here, I will only give a series of observations.

Localized version: Equibounded supports. In order to better understand the local behavior, we add an extra hypothesis and modify the problem in order to localize it. The purpose is to get some a priori compactness. Assume that all competing densities have a prescribed mass (say equal to 1) and their support is contained in a given bounded, measurable set $D \subseteq \mathbb{R}^N$. We replace (2.1) with

$$\max \left\{ \mu_k(\rho) : \rho : \mathbb{R}^N \rightarrow [0, 1], \int_{\mathbb{R}^N} \rho dx = 1, \rho|_{\mathbb{R}^N \setminus D} = 0 \right\}. \quad (2.2)$$

Of course, if $D = \mathbb{R}^N$, problems (2.2) and (2.1) are equivalent. If D is bounded, problem (2.2) is, a priori, weaker than (2.1). However, in this case, the proof of existence of a solution is immediate by the direct method. Indeed, since D is bounded, taking a maximizing sequence $(\rho_n)_n$, one subtracts a sequence (still denoted using the same index) such that

$$\rho_n \rightharpoonup \rho \text{ weakly} - * \text{ in } L^\infty(D).$$

It is not complicated to prove that

$$\mu_k(\rho) \geq \limsup_{n \rightarrow +\infty} \mu_k(\rho_n),$$

and, since $1 \in L^1(D)$, the constraint $\int_D \rho dx = 1$ is obtained by passage to the limit. The solution may depend on D . In order to recover some global information from the local one, i.e., to pass from bounded boxes D to the full space $\mathbb{R}^N$, the standard way would be to use optimality and to identify some useful properties of the best density. However, for now, there are not many results in this direction, apart from the concentration results described below.

Global optimization: Densities with support in $\mathbb{R}^N$. If one removes the bounded box, the support of ρ_n may fill the full space $\mathbb{R}^N$. A weak limit of the sequence $(\rho_n)_n$ could possibly be identified, but, for instance, if the supports of ρ_n blow away from the origin, the weak limit would be the zero density, which is not admissible since it does not satisfy the mass constraint. The key to proving the existence is to get control of the concentration of the support of ρ_n .

This is done in the next two lemmas. The fundamental tool, due to Grigor'yan, Netrusov and Yau [14], describes the concentration phenomenon of a general measure on annuli. In turn, we relate this concentration phenomenon to the magnitude of the k -th Neumann eigenvalue. The following two results are proved in [10].

Lemma 2.3. *Let $\rho \in L^1(\mathbb{R}^N, [0, 1])$, $\int_{\mathbb{R}^N} \rho dx = 1$ such that $\mu_k(\rho) > 0$. There exist a dimensional constant c_N and a ball B_{x, R^*} with*

$$R^* = \sqrt{\frac{4(k+1)}{c_N \mu_k(\rho)}}$$

such that

$$\int_{B_{x,R^*}} \rho dx \geq \frac{c_N}{k+1}.$$

Roughly speaking, the result asserts that if $\mu_k(\rho)$ is “large” (above 0), then for some R^* , which is “low”, there is a ball of radius R^* containing a fixed proportion of the total mass of ρ . On the contrary, if $\mu_k(\rho)$ was close to 0, then R^* would be very large. For a maximizing sequence in (2.1), this gives a uniform control of the concentration of the mass on balls of fixed radius.

Since only a fraction of mass is concentrated in the balls found above, in order to explain what happens with the missing mass, we give the following.

Lemma 2.4. *Suppose $\rho \in L^1(\mathbb{R}^N, [0, 1])$, $\int_{\mathbb{R}^N} \rho dx = 1$. Assume that $\rho = \rho_0 + \rho_1 + \dots + \rho_j$ and*

$$\min_{i \neq j} \text{dist}(\{\rho_i > 0\}, \{\rho_j > 0\}) = R > 0.$$

Then, for every $l \in \{1, \dots, j\}$, there exist $R_l^* > 0$ and $x_l \in \mathbb{R}^N$ satisfying

$$\frac{1}{R_l^*} \geq \left[\frac{1}{2(k+1)} \left(\frac{\mu_k(\rho)c_N \int_{\mathbb{R}^N} \rho_l dx}{\int_{\mathbb{R}^N} \rho_l dx + \int_{\mathbb{R}^N} \rho_0 dx} \right)^{\frac{1}{2}} - \frac{3}{2R} \right]^+ \quad (2.3)$$

and

$$\int_{B_{x_l, R_l^*}} \rho_l \geq \frac{c_N}{k+1} \int_{\mathbb{R}^N} \rho_l dx,$$

where c_N is a dimensional constant.

As a consequence of these two concentration lemmas, one can prove the existence result of Theorem 2.2 by a kind of concentration-compactness argument. At this point, one may suspect that when k grows, the maximum achieved on densities in Theorem 2.2 is (much) higher than the one corresponding to classical domains (1.2). The following result proved in [10] shows that the growth is controlled by the same constants found by Kröger in [17], for domains. In other words, even if the maximal values in (2.1) are higher than the similar ones for domains in (1.2), their order of magnitude is the same.

Theorem 2.5. *Let $N \geq 2$, $\rho \in L^1(\mathbb{R}^N, [0, 1])$, $\rho \not\equiv 0$. Then*

$$\forall k \in \mathbb{N}, \quad \mu_k(\rho) \leq 4\pi^2 \left(\frac{(N+2)k}{2\omega_N} \frac{1}{\|\rho\|_1} \right)^{\frac{2}{N}}.$$

This inequality supports the possible validity of a general form of the Pólya conjecture, for densities. Previous results, obtained by Colbois, El Soufi and Savo [11], justify this inequality with a generic constant, on a Riemannian manifold.

Explicit answer to Theorem 2.2 and Pólya conjecture in dimension 1. The original problem (1.2) has no interest in dimension 1, since any open set is union of intervals, so that its Neumann eigenvalues are described exactly, ordering the eigenvalues of all intervals. However, problem (2.1) has a deep meaning on densities and is far from being trivial, even in dimension 1. The following result, from [10], not only proves the validity of the Pólya conjecture for densities in dimension 1, but also gives precisely the structure of the maximizing densities. They are the characteristic functions of unions of k equal intervals. In particular, problem (2.1) has the same solution as problem (1.2).

Theorem 2.6 (One space dimension). *In $\mathbb{R}$, for every $\rho : \mathbb{R} \rightarrow [0, 1]$, $\rho \in L^1(\mathbb{R})$, we have*

$$\forall k \in \mathbb{N}, \quad \mu_k(\rho) \leq \frac{\pi^2 k^2}{\left(\int_{\mathbb{R}} \rho dx\right)^2}.$$

For every k , equality is attained when ρ equals a.e. the characteristic function of a union of at most k open, pairwise disjoint segments each having length equal to an entire multiple of $\frac{1}{k} \int_{\mathbb{R}} \rho dx$.

As a consequence, one gets the following sharp inequality for Sturm–Liouville eigenvalues.

Corollary 2.7. *Let $\rho_1, \rho_2 \in C^1([0, 1], (0, 1])$, $\|\rho_1\|_{\infty} = \|\rho_2\|_{\infty} = 1$. We consider the (classical) eigenvalue problem*

$$\begin{cases} -(\rho_1 u')' = \mu_k \rho_2 u & \text{on } (0, 1), \\ u'(0) = u'(1) = 0. \end{cases}$$

Then

$$\forall k \geq 0, \quad \mu_k \leq \frac{\pi^2 k^2}{\min\left(\int_0^1 \rho_1, \int_0^1 \rho_2\right)^2}.$$

Numerical approximations. In dimension 2, as we do not know what the explicit solutions to problem (2.1) for $k \geq 3$ are, it is of interest to perform numerical computations in order to get an intuition on whether the optimal density corresponds to a characteristic function, or not. We refer the reader to [10] and [19] for such numerical computations in $\mathbb{R}^2$ and on the Euclidean sphere of $\mathbb{R}^3$, respectively.

In [10, Table 2], the numerics suggest that the maximum of μ_3 in $\mathbb{R}^2$ for problem (2.1) is a density, not a domain. However, the associated value is only 0.5% higher than the corresponding one on domains, obtained by Antunes and Freitas in [2].

Below, we briefly describe the numerical approach. First, one has to work with densities $\rho : D \rightarrow [0, 1]$ having support in a bounded set $D \subseteq \mathbb{R}^2$ (say a rectangle).

Second, since for an arbitrary density the eigenvalue problem is not well posed, one has to regularize it as follows. Let $\varepsilon > 0$, and consider the eigenvalue problem

$$\begin{cases} -\operatorname{div}[(\rho + \varepsilon)\nabla u] = \mu_k^\varepsilon(\rho)(\rho + \varepsilon^2)u & \text{on } D, \\ \partial_n u = 0 & \text{on } \partial D. \end{cases} \quad (2.4)$$

Clearly, this problem is well posed and, numerically, one can approximate its eigenvalues and eigenfunctions by finite elements in $H^1(D)$. Even if $\mu_k(\rho)$ has only a variational meaning and does not correspond to a classical eigenvalue, following [10] we have

$$\forall k \in \mathbb{N}, \quad \text{for } \varepsilon \rightarrow 0 : \mu_k^\varepsilon(\rho) \rightarrow \mu_k(\rho)$$

and

$$\max_{\substack{\rho \in L^1(D, [0, 1]) \\ \int \rho = m}} \mu_k^\varepsilon(\rho) \xrightarrow{\varepsilon \rightarrow 0} \max_{\substack{\rho \in L^1(D, [0, 1]) \\ \int \rho = m}} \mu_k(\rho).$$

These arguments justify the numerical approach of the original problem (2.1) via its ε -regularization.

We point out that the presence of ε^2 (instead of the more expected ε^1) on the right-hand side of (2.4) is necessary to avoid the appearance of spurious eigenvalues.

We refer the reader to [10] for numerical approximations of the solutions of (2.1) for $k = 3, \dots, 8$ in $\mathbb{R}^2$ and for comparisons with the numerical solutions of (1.2) (see also [2]).

3. Maximization of Neumann eigenvalues on Euclidean spheres

Let us denote by $\mathbb{S}^N \subseteq \mathbb{R}^{N+1}$ the unit sphere of $\mathbb{R}^{N+1}$. We denote by $\Delta_{\mathbb{S}^N}$ the Laplace–Beltrami operator on $\mathbb{S}^N$, and for $\Omega \subseteq \mathbb{S}^N$ smooth, we denote by $\mu_k(\Omega)$ the k -th eigenvalue of the Laplace–Beltrami operator with Neumann boundary conditions on Ω

$$\begin{cases} -\Delta_{\mathbb{S}^N} u = \mu_k(\Omega)u & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Similarly to the Euclidean case, the eigenvalues are obtained by the min-max formula. If $\rho : \mathbb{S}^N \rightarrow [0, 1]$ is measurable, we can define the relaxed eigenvalues associated to ρ , denoted $\mu_k(\rho)$, in the same spirit as (1.1) replacing the usual gradient with the tangential one ($\mu_0(\rho) = 0$ and multiplicities are counted).

We recall the following existence result from [19].

Theorem 3.1. *For every $m \in (0, |\mathbb{S}^N|)$, for every $k \geq 1$, the problem*

$$\max \left\{ \mu_k(\rho) : \rho : \mathbb{S}^N \rightarrow [0, 1], \int_{\mathbb{S}^N} \rho d\sigma = m \right\}$$

has at least a solution.

Note that the proof of Theorem 3.1 is a straightforward consequence of the upper semi-continuity result

$$\rho_n \rightarrow \rho \text{ weakly in } * - L^\infty(\mathbb{S}^N) \implies \mu_k(\rho) \geq \limsup_{n \rightarrow +\infty} \mu_k(\rho_n).$$

With respect to the Euclidean case, the main difference which makes this existence result direct is the compactness of $\mathbb{S}^N$, which implies that $1 \in L^1(\mathbb{S}^N)$ and so the mass constraint on the density is preserved by weak $* - L^\infty$ convergence.

A result on spheres, similar to Szegő–Weinberger’s one, has been obtained by Ashbaugh and Benguria in [4]. However, it requires an extra hypothesis, namely the inclusion of the competing domains in a hemisphere.

Theorem 3.2 (Ashbaugh–Benguria). *Let $m \leq \frac{|\mathbb{S}^N|}{2}$. If $\Omega \subseteq \mathbb{S}^N \cap \{x_1 \geq 0\}$ and $|\Omega| = m$, then*

$$\mu_1(\Omega) \leq \mu_1(B^m),$$

where $B^m \subseteq \mathbb{S}^N$ is a geodesic ball of volume m .

It is a long-standing open problem to understand whether the hemisphere inclusion hypothesis is only technical or whether it is really unavoidable.

The following result from [9] is quite surprising. It is the spherical counterpart of Theorem 2.1, point (2), which completely answers the maximization question for μ_2 . A full understanding of the similar counterpart of point (1), concerning the maximization of μ_1 , is missing.

Theorem 3.3. *Let $0 < m < |\mathbb{S}^N|$. If $\rho : \mathbb{S}^N \rightarrow [0, 1]$ is measurable such that $\int_{\mathbb{S}^N} \rho d\sigma = m$. Then*

$$\mu_2(\rho) \leq \mu_1(B^{\frac{m}{2}}).$$

The original statement in [9, Theorem 2] is given for domains: if $\Omega \subseteq \mathbb{S}^N$, $|\Omega| = m$, then $\mu_2(\Omega) \leq \mu_1(B^{\frac{m}{2}})$. The proof follows the footsteps of Theorem 2.1 from [7], the construction of the test functions being adapted specifically to spheres. They are obtained by gluing two test functions of Ashbaugh–Benguria type. A topological argument justifies the existence of suitable test functions satisfying the orthogonality conditions. We point out that the zeros are counted modulo 4, correcting a sign

error in the original computation from [7]. The passage to densities is quite direct since both the topological argument and the mass transplantation procedure are not sensitive to the difference between a characteristic function and a density.

An important consequence of Theorem 3.3 is the following extension of the Ashbaugh–Benguria result (Theorem 3.2, above). This extension, even if it does not completely solve the problem, is a step forward towards better understanding it. It is a consequence of the substantial enlargement of the family of test functions (albeit for μ_2 and not for μ_1), by the gluing procedure. For the proof, we refer to [9].

Theorem 3.4. *Let $m \leq \frac{|\mathbb{S}^N|}{2}$ and B^m be a geodesic ball of mass m . If $\rho : \mathbb{S}^N \rightarrow [0, 1]$ is measurable such that $\int_{\mathbb{S}^N} \rho d\sigma = m$, $\rho \cdot 1_{B^m} = 0$ (i.e., its support lays in the complement of B^m), then*

$$\mu_1(\rho) \leq \mu_1(B^m).$$

In particular, if $\Omega \subseteq \mathbb{S}^N \setminus B^m$ is such that $|\Omega| = m$, then

$$\mu_1(\Omega) \leq \mu_1(B^m).$$

Note that in Theorem 3.4 the room allowed for Ω is larger than a hemisphere. A domain $\Omega \subseteq \mathbb{S}^N$ which does not satisfy the inclusion constraint should intersect every geodesic ball of mass m of the sphere. If m is asymptotically vanishing, then such a domain would approximately fill the full sphere.

Some remarks are in order, all of them being suggested by the numerical computations from [9] and [19].

Remark 3.5. Numerical experiments suggest that on the unit sphere $\mathbb{S}^2 \subseteq \mathbb{R}^3$ there are values of $m \in (4, 2\pi)$ such that if the exclusion constraint above $\rho \cdot 1_{B^m} = 0$ is dropped (for every spherical cap of mass m), a relaxation phenomenon occurs. This leads to an effective construction of a density of mass m such that

$$\mu_1(\rho) > \mu_1(B^m).$$

This is only a numerical observation, not certified theoretically.

Remark 3.6. Numerical experiments based on the same strategy as in Remark 3.5, now under the exclusion constraint $\rho \cdot 1_{B^m} = 0$ for some spherical cap B^m of mass m , turn out to be very stable and produce clear evidence of the expected theoretical result from Theorem 3.4. This gives credit to the validity of the numerical approach and of the relaxation phenomenon observed in Remark 3.5.

Remark 3.7. If m is small, $m < 3$, numerics from [19] do not suggest that the relaxation phenomenon occurs in the absence of exclusion constraint. This observation

could support the thesis that the spherical cap is optimal if m is small, without any further constraint. However, these computations need to be carefully interpreted in order to understand the strength of this suggestion.

Remark 3.8. There do exist values $4\pi > m > 2\pi$ for which the spherical cap is not the optimal geometry. In other words, there exists an open set (which is not simply connected and presents several holes) satisfying the area constraint, but with a higher first eigenvalue than the corresponding geodesic ball. This is numerically observed in [19] and theoretically justified in [8].

4. Further remarks

4.1. More general functionals of eigenvalues

Ashbaugh and Benguria conjectured in [3] that for smooth domains $\Omega \subseteq \mathbb{R}^N$ of volume m , the following inequality holds

$$\sum_{k=1}^N \frac{1}{\mu_k(\Omega)} \geq \frac{N}{\mu_1(B^m)}.$$

A step towards the proof of the inequality was done by in [25], where Xia and Wang prove that

$$\sum_{k=1}^{N-1} \frac{1}{\mu_k(\Omega)} \geq \frac{N-1}{\mu_1(B^m)},$$

with equality if and only if $\Omega = B^m$. The result also holds in the hyperbolic space [25] and on Euclidean spheres [5]. On spheres, however, one required as further hypothesis that Ω is contained in a hemisphere!

By comparison to Theorem 3.3, in [9], for $\Omega \subseteq \mathbb{S}^N$, $|\Omega| = m$, without any inclusion constraint, the following is proved

$$\sum_{k=2}^N \frac{1}{\mu_k(\Omega)} \geq \frac{N-1}{\mu_1(B^{\frac{m}{2}})},$$

with equality if and only if Ω is the union of two disjoint geodesic balls of volume $\frac{m}{2}$.

4.2. Quantitative versions

In recent years, a series of isoperimetric and spectral inequalities have been proved to hold in a quantitative form, i.e., with a geometric reminder measuring a distance to

the optimal set. In particular, in [6], Brasco and Pratelli proved a sharp quantitative form of the Szegő–Weinberger inequality: for every $\Omega \subseteq \mathbb{R}^N$ smooth,

$$|B|^{\frac{2}{N}} \mu_1(B) - |\Omega|^{\frac{2}{N}} \mu_1(\Omega) \geq C_N \mathcal{A}(\Omega)^2,$$

where by $\mathcal{A}(\Omega)$ one denotes the Fraenkel asymmetry of Ω

$$\mathcal{A}(\Omega) = \inf \left\{ \frac{|\Omega \Delta B|}{|\Omega|} : B \subseteq \mathbb{R}^N \text{ ball such that } |B| = |\Omega| \right\},$$

and C_N is a dimensional constant. The power 2 over $\mathcal{A}(\Omega)$ is sharp.

Concerning the second inequality of Theorem 2.1, in the class of domains, a quantitative version was proved in [23], reading

$$2^{\frac{2}{N}} |B|^{\frac{2}{N}} \mu_1(B) - |\Omega|^{\frac{2}{N}} \mu_2(\Omega) \geq C_N \mathcal{A}_2(\Omega)^{N+1},$$

where the asymmetry term $\mathcal{A}_2(\Omega)$ is of Fraenkel type and quantifies a distance of Ω to a union of two disjoint balls of half measure

$$\mathcal{A}_2(\Omega) = \inf \left\{ \frac{|\Omega \Delta (B_1 \cup B_2)|}{|\Omega|} : B_1, B_2 \subseteq \mathbb{R}^N \text{ disjoint balls such that } |B_1| = |B_2| = \frac{1}{2} |\Omega| \right\}.$$

4.3. Other constraints

Several other constraints are naturally imposed on the competing domains.

Simply connectedness. In a recent paper [18], Langford and Laugesen prove that the maximum of μ_1 is attained on a geodesic ball among simply connected sets of $\mathbb{S}^2$ with prescribed area m , provided $m \leq 4\pi \frac{16}{17}$. Note that this is not contradictory with the numerical observations of [19] and the theoretical justifications from [8] (see Remark 3.8 above), since the domains having a higher first eigenvalue than the corresponding spherical caps have multiple holes. In the class of simply connected sets, an interesting question is to understand whether the limitation of the area to $4\pi \frac{16}{17}$ is only technical, or not.

Convexity. Payne and Weinberger have proved in [20] that if $\Omega \subseteq \mathbb{R}^N$ is bounded, open and convex, then

$$\mu_1(\Omega) \geq \frac{\pi^2}{D_\Omega^2},$$

where D_Ω is the diameter of Ω . This inequality is sharp, the lower bound being asymptotically attained by a sequence of parallelepipeds of the form $\Omega_\varepsilon = (0, 1) \times$

$(0, \varepsilon)^{N-1}$. In [1], this inequality is improved by a quantitative term

$$\mu_1(\Omega) \geq \frac{\pi^2}{D_\Omega^2} + C_N \frac{a_{2,\Omega}^2}{D_\Omega^4},$$

where $a_{2,\Omega}$ is the second largest semiaxis of the John ellipsoid of Ω . This result answers, in a stronger version, a conjecture of Hang and Wang [15].

In the same class of convex subsets of $\mathbb{R}^N$, Kröger [17] found sharp upper bounds for μ_k ; precisely, he proved that

- for $N = 2$,

$$\mu_k(\Omega) \leq \frac{(2j_{0,1} + (k-1)\pi)^2}{D_\Omega^2}$$

- for $N = 3$,

$$\mu_k(\Omega) \leq \frac{(k+1)^2 \pi^2}{D_\Omega^2}$$

- for $N \geq 4$, then

$$\mu_k(\Omega) \leq \frac{\left(j_{\frac{d-2}{2}, \lfloor \frac{k+1}{2} \rfloor} + j_{\frac{d-2}{2}, \lfloor \frac{k+2}{2} \rfloor}\right)^2}{D_\Omega^2}.$$

Above, $j_{\nu,m}$ is the m -th zero of the Bessel function J_ν , and $\lfloor a \rfloor$ denotes the integer part of a . These inequalities have been revisited with a different point of view in [16]. The upper bounds are sharp and are attained asymptotically on a sequence of domains collapsing to a segment. The geometry of the collapsing domains is described in [16]. All these inequalities have a quantitative version.

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