

Courant, Bézout and persistence

Lev Buhovsky

Abstract. We give an overview of some recent applications of the theory of persistence modules and barcodes to spectral geometry and to the transcendental Bézout problem. The proofs of most of the results are given in previous works written with coauthors.

1. Introduction

Let (M, g) be a compact and connected Riemannian manifold, possibly with boundary, and consider the induced (positive) Laplace–Beltrami operator $\Delta_g = -\operatorname{div}_g \nabla_g$ on M . The spectral decomposition of Δ_g (in case of $\partial M \neq \emptyset$, we impose the Dirichlet boundary conditions) provides us with an infinite sequence $0 \leq \lambda_1 \leq \lambda_2 \leq \dots \nearrow +\infty$ of eigenvalues and a corresponding sequence $f_1, f_2, \dots$ of eigenfunctions, $\Delta_g f_k = \lambda_k f_k$, forming an orthonormal basis of $L^2(M, dV_g)$ (here dV_g is the induced Riemannian volume density on M). Since the famous Chladni experiments conducted more than two centuries ago, the study of properties of eigenfunctions has fascinated generations of mathematicians.

For a smooth (or, more generally, continuous) function on M , its nodal set is $Z_f = \{f = 0\} \subset M$, and a nodal domain of f is a connected component of the complement $M \setminus Z_f$. In the case of $M = [0, \pi]$, the k -th Dirichlet eigenfunction $f_k(x) = \sin kx$ has exactly k nodal domains. In higher dimensions, there is no such statement, but there is a fundamental upper bound.

Theorem 1.1 (R. Courant, 1923). *A Laplace–Beltrami eigenfunction f_k has at most k nodal domains.*

Note that in the above-mentioned one-dimensional example of $M = [0, \pi]$, in fact any linear combination of the first k eigenfunctions, $\sum_{\ell=1}^k a_\ell f_\ell(x) = \sum_{\ell=1}^k a_\ell \sin \ell x$, has at most $k + 1$ zeroes on $[0, \pi]$, and as a result, at most k nodal domains. Such a property of eigenfunctions is called the *extended Courant property*.

Definition 1.2. We say that the extended Courant property holds on (M, g) if for every k , any linear combination of first k eigenfunctions of the Laplace–Beltrami operator has at most k nodal domains.

It is not difficult to see that the extended Courant property holds in dimension 1. The higher-dimensional case is more complicated and interesting. In case of bounded Euclidean domains, it was claimed in the footnote of [19, p. 454] (second footnote in the German original [18, p. 394]) that “Any linear combination of the first n eigenfunctions divides the domain, by means of its nodes, into no more than n subdomains. See the Göttingen dissertation of H. Herrmann, *Beiträge zur Theorie der Eigenwerte und Eigenfunktionen*, 1932”. However, already in 1956, Pleijel [31, p. 550] mentioned that he could not find a proof of this statement in the literature.

In 1973, V. Arnold [3, 4] related the extended Courant property to Hilbert’s 16th problem, pointing out that while the property holds for $(\mathbb{R}P^2, g_0)$, it does not hold for $(\mathbb{R}P^3, g_0)$ due to counterexamples constructed by O. Viro [36]. More recent works [5–9, 23] provided many more examples where the extended Courant property fails, both in the setting of bounded Euclidean domains and in the more general setting of Riemannian manifolds.

Denote by $\mathcal{F}_\lambda$ the space of all linear combinations of eigenfunctions with eigenvalue $\leq \lambda$. Although the extended Courant property was shown to be incorrect in many cases, one could still ask whether the number of nodal domains of any linear combination of the first k eigenfunctions is bounded by Ck , where C is a constant. Or, equivalently,¹ one could ask whether for any $\lambda \geq 0$ and any $f \in \mathcal{F}_\lambda$, the number of nodal domains of f is bounded by $C(\lambda + 1)^{n/2}$, where n is the dimension of M . In [9], we show that the answer to this question is negative, and that in fact it is impossible even to have a general asymptotic upper bound.

Theorem 1.3. *There exists a Riemannian metric g on the 2-torus, a sequence f_{k_j} of eigenfunctions of the Laplacian Δ_g , and a sequence of scalars $c_j \in \mathbb{R}$, such that for each j the function $f_{k_j} - c_j$ has infinitely many nodal domains.*

The proof of the theorem combines ideas from analysis and topology. The constructed metric is of the Liouville type: $ds^2 = Q(x)(dx^2 + dy^2)$, $(x, y) \in \mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$. In particular, the variables separate, and the eigenfunctions are of the form $u(x) \cos my$ and $u(x) \sin my$, where $m \in \mathbb{Z}$, and the function u satisfies the Sturm–Liouville problem

$$u''(x) + (\lambda Q(x) - m^2)u(x) = 0. \quad (1.1)$$

¹This equivalence follows from the Weyl’s law by which λ_k grows asymptotically as $k^{2/n}$ times a constant.

For such an eigenfunction f , the difference $f - c$ will have infinitely many nodal domains provided that the function u oscillates appropriately (will have “infinitely many damped oscillations”) near the level c . A crucial observation is as follows: if the function $\lambda Q(x) - m^2$ vanishes at some point x_* and moreover oscillates near x_* in a suitable way, then a solution u of (1.1), $u'(x_*) = 0$, is forced to oscillate near x_* as well. Therefore, we construct $Q(x)$ such that the function $\lambda Q(x) - m^2$ suitably oscillates near the level 0, or, equivalently, the function $Q(x)$ oscillates near the level m^2/λ for infinitely many values of m . Here, the main difficulty is that the possible values of λ (the eigenvalues of the Sturm–Liouville problem (1.1)) depend on Q and m , and we somehow need to match the levels where Q oscillates with the values m^2/λ for infinitely many m ’s. This becomes possible by applying the Brouwer fixed point theorem in dimensions growing to infinity and by using compactness [9].

Inspired by Theorem 1.3, we ask a general question.

Question 1.4. Is it true that on any closed manifold M of dimension $n \geq 2$ there exists a Riemannian metric as in Theorem 1.3?

It is not difficult to extend Theorem 1.3 from the 2-torus case to manifolds of the type $M = S^1 \times N$ (where the manifold N is closed), by using a similar reduction to the one-dimensional Sturm–Liouville problem. But for a general M , such a reduction is not possible, and new ideas are needed.

Remark 1.5. One could also try other possible ways to generalize Theorem 1.1. First, one could ask whether the theorem extends to higher-order elliptic differential operators (e.g., clamped plate problem). Second, notice that the number of nodal domains is the 0th Betti number of the complement of the nodal set. Hence one could ask whether the bound in Theorem 1.1 extends to higher-order Betti numbers (of the complement of the nodal set of an eigenfunction). However, based on Theorem 1.3, it was shown in [10] that, in general, these two questions have negative answers.

In this survey, we will describe an approach to the extended Courant property, based on the so-called coarse count of nodal domains, which leads to positive results [10, 32, 34]. Moreover, a variant of this “coarse approach” yields applications to Bézout theorems for eigenfunctions (giving a partial answer to a question of V. Arnold and L. Polterovich) [10], as well as to the transcendental Bézout problem [11]. For more details, see Sections 2 and 4.

The coarse approach is a part of a more general picture where the main focus lies on the interplay between topology and analysis of functions [10]. The topological function theory deals with invariants of functions under diffeomorphisms and, roughly speaking, enables to study oscillations of functions by looking at the topology of their sublevel sets. This theory is based on persistence modules and barcodes,

a mathematical apparatus originating from topological data analysis. On the analysis side, one considers measurements of functions based on the Sobolev scale. For more details, see Section 3.

2. Coarse Courant and Bézout

2.1. Coarse Courant

A remarkable and fruitful idea, which goes back to [34], is to disregard small oscillations while counting nodal domains (we call it *a coarse count of nodal domains*). More precisely, we have the following definition [34].

Definition 2.1. A nodal domain Ω of a function f is called δ -deep for some $\delta > 0$ if $\max_{\Omega}|f| > \delta$. Denote by $m_0(f, \delta)$ the number of δ -deep nodal domains of a function f .

In [34], L. Polterovich and M. Sodin showed that on a compact and connected Riemannian surface² (M, g) , there is an upper bound

$$m_0(f, \delta) \leq C \min(\delta^{-1}, \delta^{-2})\lambda, \quad (2.1)$$

where $f \in \mathcal{F}_{\lambda}$ is $L^2(M, V_g)$ -normalized, and $C = C(M, g)$. This result represents a version of the extended Courant property for the coarse count of nodal domains. It was deduced from a general inequality between a *generalized Banach indicatrix* of a smooth function and a quantity involving the Sobolev $W^{2,2}$ -norm of the function [34]. These and related results were later extended to the framework of the theory of *persistence modules and barcodes* [32]. In a consequent work [10], we explored this direction further and proved a version of (2.1) (*a coarse Courant theorem*) on Riemannian manifolds of any dimension.

Theorem 2.2. Let (M, g) be a compact and connected Riemannian manifold of dimension n , and let $k > n/2$ be an integer. For any $\lambda \geq 0$ and $f \in \mathcal{F}_{\lambda}$ with $\|f\|_{L^2(M, V_g)} = 1$, and any $\delta > 0$, we have

$$m_0(f, \delta) \leq C \delta^{-n/k} (\lambda + 1)^{n/2}, \quad (2.2)$$

where $C = C(M, g, k)$.

²By a surface, we mean a manifold of dimension 2.

The upper bound in (2.2) is sharp in terms of the power of $\lambda + 1$, as we show in the following theorem [10].

Theorem 2.3. *Suppose (M, g) is a closed Riemannian manifold. Then there exists $c = c(M, g) > 0$ such that for any $\lambda \geq 0$ and $\delta > 0$, one can find $f \in \mathcal{F}_\lambda$, $\|f\|_{L^2(M, V_g)} = 1$, for which we have*

$$m_0(f, \delta) \geq c \frac{(\lambda + 1)^{n/2}}{\max(1, \delta^2)} - 1.$$

2.2. Bézout, eigenfunctions and coarse count

As suggested in the foundational work [21] of Donnelly and Fefferman, eigenfunctions of the Laplace–Beltrami operator with eigenvalue λ are expected to share some common features with polynomials of degree $\sqrt{\lambda}$ when λ is sufficiently large. For instance, this principle can be clearly seen on the sphere S^n endowed with the standard spherical metric: harmonic polynomials of degree d on $\mathbb{R}^{n+1}$ correspond to eigenfunctions of the Laplace–Beltrami operator with the eigenvalue $d(d + n - 1)$. A real version of the Bézout theorem states that for a generic collection $p_1, \dots, p_n$ of real polynomials on $\mathbb{R}^n$, of degrees $d_1, \dots, d_n$, respectively, the number of common zeroes does not exceed the product $d_1 d_2 \cdots d_n$. In the 2000s, V. Arnold and L. Polterovich asked whether eigenfunctions of the Laplace–Beltrami operator satisfy a version of the Bézout theorem. The example of the round sphere S^n provides an evidence here: indeed, due to the above explicit description of the eigenfunctions, one can deduce from the Bézout theorem that for eigenfunctions $f_1, \dots, f_n$ on S^n with eigenvalues $\Lambda_1, \dots, \Lambda_n$, the number of common zeroes generically does not exceed $2\sqrt{\Lambda_1 \cdots \Lambda_n}$. Furthermore, it was proved in [22] that the expectation (in a natural probabilistic setting) of the number of common zeroes equals $2n^{-n/2} \sqrt{\Lambda_1 \cdots \Lambda_n}$. Similar bounds for certain homogeneous Riemannian manifolds have also been obtained in [1, 2].

The “coarse approach”, useful for the Courant theorem and its extensions, is also applicable to the above-mentioned question of V. Arnold and L. Polterovich. First, let us define the relevant coarse count [10].

Definition 2.4 (Coarse count of zeroes). Let $f_1, \dots, f_n : M \rightarrow \mathbb{R}$ be a collection of continuous functions on an n -dimensional manifold M . Denote $f = (f_1, \dots, f_n) : M \rightarrow \mathbb{R}^n$. For $\delta > 0$, define $z_0(f, \delta)$ to be the number of connected components of $f^{-1}(B_\delta)$ which contain zeroes of f (i.e., which intersect $f^{-1}(0)$), where $B_\delta = \{|x| < \delta\} \subset \mathbb{R}^n$.

The following coarse Bézout theorem for linear combinations of eigenfunctions was shown in [10].

Theorem 2.5. *Let (M, g) be a compact and connected Riemannian manifold of dimension n , and let $k > n/2$ be an integer. For every $\lambda \geq 0$, and a collection $f_1, \dots, f_n \in \mathcal{F}_\lambda$ with $\|f_j\|_{L^2(M, V_g)} = 1$ ($j = 1, \dots, n$), for each $\delta > 0$ we have*

$$z_0(f, \delta) \leq C \delta^{-n/k} (\lambda + 1)^{n/2} + 1,$$

where $f = (f_1, \dots, f_n)$, and $C = C(M, g, k)$.

This theorem can be seen as a partial answer to the question of Arnold and Polterovich. Note, however, that in the theorem the functions f_j are assumed to lie in *the same* $\mathcal{F}_\lambda$, which corresponds to having the same degree bound in case of polynomials, while in the Bézout theorem the degrees can be different. Therefore, we ask the following question [10].

Question 2.6. Can we generalize the statement of Theorem 2.5 by assuming that $f_j \in \mathcal{F}_{\Lambda_j}$ for $j = 1, \dots, n$ and replacing $(\lambda + 1)^{n/2}$ by $\sqrt{\Lambda_1 \cdots \Lambda_n}$ in the upper bound?

Finally, it is natural to ask whether for the question of Arnold and Polterovich, the coarse approach is at all necessary. In contrast to the “Courant story” where we have the counterexample given by Theorem 1.3, an analogous counterexample for the question of Arnold and Polterovich (the “Bézout story”) is missing. Despite the lack of enough evidence, we would still conjecture that such a counterexample exists. See a recent work [35] of V. Stojisavljević for a possibly related counterexample in the context of harmonic functions, motivated by the transcendental Bézout problem.

2.3. Persistent Betti numbers and the Sobolev norm

As we will see later, the coarse Courant and coarse Bézout approaches are closely related, and moreover, Theorems 2.2 and 2.5 can be naturally generalized. Let us first extend the invariants m_0, z_0 to *persistent Betti numbers* of higher degree.

Definition 2.7. Let M be a smooth manifold of dimension n , and let $f : M \rightarrow \mathbb{R}^d$ be a continuous map. For any integer $0 \leq r < n$ and any $\delta > 0$, define the persistent Betti numbers

$$m_r(f, \delta) = \dim \operatorname{Im}(H_r(\{|f| > \delta\}) \rightarrow H_r(M \setminus Z_f))$$

and

$$z_r(f, \delta) = \dim \operatorname{Im}(H_r(Z_f) \rightarrow H_r(\{|f| < \delta\})).$$

Here, $|\cdot|$ stands for the Euclidean norm on $\mathbb{R}^d$, $Z_f = \{f = 0\} \subset M$, and $H_r(X)$ stands for the r -th singular homology group of a subset $X \subset M$ with coefficients in a field $\mathbb{K}$.

It is not difficult to see that in the case of $r = 0$, this definition indeed coincides with Definitions 2.1 and 2.4. The following result generalizes Theorems 2.2 and 2.5 (cf. Remark 1.5).

Theorem 2.8. *Let (M, g) be a compact Riemannian manifold of dimension n , let $\lambda \geq 0$ and let $f_1, \dots, f_d \in \mathcal{F}_\lambda$ be such that $\|f_i\|_{L^2(M, V_g)} = 1$ for each $1 \leq i \leq d$. Then for any integers $k > n/2$ and $0 \leq r < n$, and any $\delta > 0$, we have*

$$\begin{aligned} m_r(f, \delta) &\leq C \delta^{-n/k} (\lambda + 1)^{n/2}, \\ z_r(f, \delta) &\leq C \delta^{-n/k} (\lambda + 1)^{n/2} + b_r(M), \end{aligned}$$

where $f = (f_1, \dots, f_d) : M \rightarrow \mathbb{R}^d$, $b_r(M) = \dim H_r(M)$ and $C = C(M, g, k, d)$.

Theorem 2.8 appears to be a spectral geometric application of a general result which compares between m_r , z_r and a Sobolev norm. Recall that given a compact Riemannian manifold M , a real number $p \geq 1$ and an integer $k \geq 1$, for a smooth map $f : M \rightarrow \mathbb{R}^d$ the Sobolev norm $\|f\|_{W^{k,p}}$ is controlled by the L^p -norms of the derivatives of f up to order k . The Sobolev space $W^{k,p}(M; \mathbb{R}^d)$ is the completion of the space $C^\infty(M, \mathbb{R}^d)$ (consisting of all smooth functions $f : M \rightarrow \mathbb{R}^d$) with respect to the Sobolev norm $\|\cdot\|_{W^{k,p}}$. Note that if $kp > n = \dim M$, then by the Sobolev embedding theorem, the norm $\|\cdot\|_{W^{k,p}}$ dominates the uniform norm, and the Sobolev space $W^{k,p}(M; \mathbb{R}^d)$ consists of continuous maps, $W^{k,p}(M; \mathbb{R}^d) \subset C(M, \mathbb{R}^d)$. We now formulate the main result of this section [10].

Theorem 2.9. *Let (M, g) be a compact Riemannian manifold of dimension n , let $p \geq 1$ and let $k > n/p$, $0 \leq r < n$ and $d \geq 1$ be integers. Then for any $f \in W^{k,p}(M; \mathbb{R}^d)$ and any $\delta > 0$, we have*

$$m_r(f, \delta) \leq C \delta^{-n/k} \|f\|_{W^{k,p}}^{n/k}$$

and

$$z_r(f, \delta) \leq C \delta^{-n/k} \|f\|_{W^{k,p}}^{n/k} + b_r(M),$$

where $b_r(M) = \dim H_r(M)$, and $C = C(M, g, p, k, d)$.

Theorem 2.8 readily follows from an application of Theorem 2.9 (with $p = 2$) and of a priori estimates for the Laplace–Beltrami operator. Therefore, it is not surprising that Theorem 2.8 generalizes to general elliptic differential operators, with Dirichlet boundary condition in the case of a non-empty boundary [10] (cf. Remark 1.5).

The statement of Theorem 2.9, involving persistent Betti numbers, is not yet an ultimate one. In Section 3.1, we discuss the language of persistence barcodes, and in Section 3.2, we explain the meaning of persistent Betti numbers in that context. This language will allow us to state Theorem 3.8 (Section 3.2), which is the main result of [10].

3. Topology and analysis of functions

3.1. Introduction to persistent modules and barcodes

The notions of a persistence module and a barcode first appeared within the framework of topological data analysis. Below, we review these notions and their relevant properties. For a detailed account on persistence theory, see [14, 30, 33].

Definition 3.1. A persistence module (V, π) over a field $\mathbb{K}$ consists of a family of vector spaces V_t , $t \in \mathbb{R}$ over $\mathbb{K}$ together with linear maps $\pi_{s,t} : V_s \rightarrow V_t$ defined for all $s \leq t$, called structure maps, which satisfy $\pi_{t,t} = \text{id}_{V_t}$ for all $t \in \mathbb{R}$, as well as $\pi_{s,t} \circ \pi_{r,s} = \pi_{r,t}$ for all $r \leq s \leq t$.

Important examples of persistence modules appear as topological invariants of functions. Let $f : X \rightarrow \mathbb{R}$ be a function on a Hausdorff topological space. Define $V(f)_t = H(\{f \leq t\})$ and $\pi_{s,t}^f = (i_{s,t})_*$, where $i_{s,t} : \{f \leq s\} \rightarrow \{f \leq t\}$ are inclusions and H denotes the singular homology with coefficients in a field $\mathbb{K}$.

Persistence modules form a category, where a morphism $\phi : (V, \pi^V) \rightarrow (W, \pi^W)$ is given by a family of linear maps $\phi_t : V_t \rightarrow W_t$, $t \in \mathbb{R}$ such that for all $s \leq t$ we have $\pi_{s,t}^W \circ \phi_s = \phi_t \circ \pi_{s,t}^V$. Direct sum of a collection of persistence modules is well defined and described by a natural (pointwise for each $t \in \mathbb{R}$) construction.

Simplest, and in some sense basic, examples of persistence modules are the following.

Definition 3.2. For an interval $I \subset \mathbb{R}$ (which can be either finite or infinite), we define the interval persistence module $\mathbb{K}_I$ as

$$(\mathbb{K}_I)_t = \begin{cases} \mathbb{K}, & \text{if } t \in I, \\ 0, & \text{otherwise,} \end{cases} \quad \pi_{s,t}^{\mathbb{K}_I} = \begin{cases} \text{id}_{\mathbb{K}}, & \text{if } s, t \in I, \\ 0, & \text{otherwise.} \end{cases}$$

General persistence modules can be quite complicated. In practice, one usually poses certain natural restrictions on persistence modules, under which it becomes possible to describe them in simple terms. For example, consider the class of pointwise finite-dimensional persistence modules, namely those (V, π) such that V_t is finite dimensional for each $t \in \mathbb{R}$. Then we have a structure theorem which states that any such persistence module is a direct sum of interval persistence modules.

Definition 3.3. A barcode $\mathcal{B}$ is a multiset of intervals with finite multiplicities. The intervals in $\mathcal{B}$ are called bars.

Theorem 3.4 (Structure theorem [20]). *To every pointwise finite-dimensional persistence module (V, π) , there corresponds a unique barcode $\mathcal{B}(V)$ such that*

$$(V, \pi) \cong \bigoplus_{I \in \mathcal{B}(V)} (\mathbb{K}_I, \pi^{\mathbb{K}_I}).$$

Now let $f : M \rightarrow \mathbb{R}$ be a Morse function on a compact manifold M . Recall that the persistence module $(V(f), \pi^f)$ is given by $V(f)_t = H(\{f \leq t\})$, and natural maps $\pi_{s,t}^f : V(f)_s \rightarrow V(f)_t$ induced by inclusions of sublevel sets. Then clearly $V(f)$ is pointwise finite dimensional, and hence the structure theorem gives us the corresponding barcode which we denote by $\mathcal{B}(f)$. The barcode $\mathcal{B}(f)$ consists of a finite number of bars whose endpoints are critical values of f , and it describes how the space $V(f)_t = H(\{f \leq t\})$ changes when t grows. For a finite bar $[s, t)$ in $\mathcal{B}(f)$, at the level s a new homology class is “born” and at the level t this homology class “dies”. An infinite bar $[s, +\infty)$ corresponds to a homology class that is born at the level s and never dies, that is, to a genuine homology class of M . The number of infinite bars in $\mathcal{B}(f)$ is equal to the total Betti number of M . By examining examples, it becomes evident that, in a certain sense, $\mathcal{B}(f)$ reflects the oscillatory behavior of the function f . See Figure 1.

The space of barcodes admits a remarkable pseudo-metric. We will use $\langle a, b \rangle$ to denote any of the intervals (a, b) , $[a, b)$, $(a, b]$, $[a, b]$. Two barcodes $\mathcal{B}, \mathcal{B}'$ are called ε -matched, $\varepsilon > 0$, if after erasing certain bars of length $< 2\varepsilon$ from each of them, there exists a bijection Φ between the remaining bars, which satisfies

$$\Phi(\langle a, b \rangle) = \langle c, d \rangle \implies |a - c|, |b - d| < \varepsilon.$$

The bottleneck distance between barcodes is defined as

$$d_{\text{bottle}}(\mathcal{B}, \mathcal{B}') = \inf\{\varepsilon \mid \mathcal{B}, \mathcal{B}' \text{ are } \varepsilon\text{-matched}\}.$$

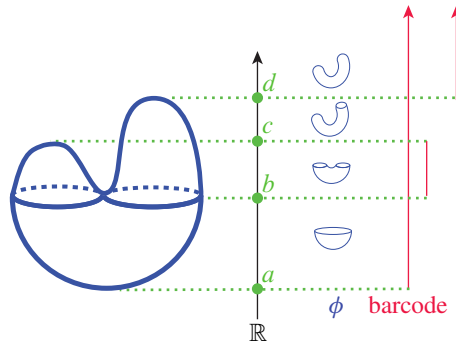


Figure 1. The barcode of the height function on a deformed sphere.

It is not difficult to see that d_{bottle} is a pseudo-metric. The counterpart of this distance is a distance between persistence modules which is called the *interleaving distance*. The direct correspondence between interleaving distance and bottleneck distance is stated in the *isometry theorem* [12, 15, 27]. The following corollary of the isometry theorem is a key feature of the theory [15].

Theorem 3.5 (Stability theorem). *Assume that M is a compact manifold and $f, g : M \rightarrow \mathbb{R}$ are Morse functions, then*

$$d_{\text{bottle}}(\mathcal{B}(f), \mathcal{B}(g)) \leq d_{C^0}(f, g).$$

3.2. The bar counting function and the Sobolev norm

It will be important for us to go beyond Morse functions and to treat general continuous functions $h : M \rightarrow \mathbb{R}$ defined on a compact manifold M . This is possible since for such h the persistence module $V(h)$ is still “good enough”.

Definition 3.6. A persistence module (V, π) is called q -tame if $\pi_{s,t}$ has a finite rank for every $s < t$.

Let M be a compact manifold and $h : M \rightarrow \mathbb{R}$ be a continuous function. Approximating h by a Morse function and comparing their persistence modules, it is not difficult to deduce that the persistence module $V(h)$ is q -tame. Theorem 3.4 (the structure theorem) continues to hold also for q -tame persistence modules [13]. The barcode $\mathcal{B}(h)$ consists of a finite number of infinite bars of the form $[s, +\infty)$, and of at most countable number of finite bars of the form $[s, t)$. Moreover, by an approximation argument and by Theorem 3.5 (the stability theorem which continues to hold in the case of continuous functions), one concludes that for any $\delta > 0$ there exists at most finite number of bars of length $> \delta$ in $\mathcal{B}(h)$. This observation gives rise to an important invariant called the *bar counting function*.

Definition 3.7. For $\delta > 0$, we define $N_\delta(h)$ to be the number of bars of length $> \delta$ in the barcode $\mathcal{B}(h)$.

Let us remark that the $V(h)$ naturally decomposes into a direct sum of persistence modules $V_r(h)$, $0 \leq r \leq n$, where $V_r(h)_t = H_r(\{h \leq t\})$, and the structure maps $\pi_{s,t}^{h,r} : H_r(\{h \leq s\}) \rightarrow H_r(\{h \leq t\})$ are naturally induced by inclusions of sublevel sets ($n = \dim M$). Denote by $\mathcal{B}_r(h)$ the barcode of $V_r(h)$. As in Definition 3.6, for $\delta > 0$ we denote by $N_{r,\delta}(h)$ the number of bars of length $> \delta$ in the barcode $\mathcal{B}_r(h)$. From the isomorphism

$$(V_r(h), \pi^{h,r}) \cong \bigoplus_{I \in \mathcal{B}_r(h)} (\mathbb{K}_I, \pi^{\mathbb{K}_I}),$$

stated in the structure theorem, we get that for any $a < b$, the quantity

$$\mathrm{rk} \pi_{a,b}^{h,r} = \dim \mathrm{Im}(H_r(\{h \leq a\}) \rightarrow H_r(\{h \leq b\}))$$

equals the number of bars $[s, t)$ in $\mathcal{B}_r(h)$ such that $s \leq a < b < t$. It is also not difficult to deduce that

$$\dim \mathrm{Im}(H_r(\{h < a\}) \rightarrow H_r(\{h < b\}))$$

equals the number of bars $[s, t)$ in $\mathcal{B}_r(h)$ such that $s < a < b \leq t$, and that

$$\dim \mathrm{Im}(H_r(\{h \leq a\}) \rightarrow H_r(\{h < b\}))$$

equals the number of bars $[s, t)$ in $\mathcal{B}_r(h)$ such that $s \leq a < b \leq t$. This allows us to interpret the *persistent Betti numbers* m_r, z_r defined in Section 2.3, in the language of barcodes.

Indeed, let M be a compact manifold of dimension n , let $f : M \rightarrow \mathbb{R}^d$ be a continuous map and let $0 \leq r < n$ and $\delta > 0$. By the definition of $m_r(f, \delta)$, we have

$$m_r(f, \delta) = \dim \mathrm{Im}(H_r(\{h < -\delta\}) \rightarrow H_r(\{h < 0\})),$$

where $h = -|f|$. Hence it follows from the above that $m_r(f, \delta)$ equals the number of bars $[s, t)$ in $\mathcal{B}_r(-|f|)$ such that $s < -\delta < 0 \leq t$, that is, the number of bars in $\mathcal{B}_r(-|f|)$ whose left endpoint is less than $-\delta$ and whose right endpoint is either 0 or $+\infty$. Similarly, we conclude that $z_r(f, \delta)$ equals the number of bars in $\mathcal{B}_r(|f|)$ whose left endpoint is 0 and whose length is greater than δ . As a corollary, we get the bounds

$$m_r(f, \delta) \leq N_{r,\delta}(-|f|) \leq N_\delta(-|f|) \quad (3.1)$$

and

$$z_r(f, \delta) \leq N_{r,\delta}(|f|) \leq N_\delta(|f|). \quad (3.2)$$

Now we formulate the general theorem which compares the bar counting function and the Sobolev norm [10].

Theorem 3.8. *Let (M, g) be a compact Riemannian manifold of dimension n , let $p \geq 1$ and let $k > n/p$ and $d \geq 1$ be integers. Then for any $f \in W^{k,p}(M; \mathbb{R}^d)$ and any $\delta > 0$, we have*

$$N_\delta(|f|), N_\delta(-|f|) \leq C \delta^{-n/k} \|f\|_{W^{k,p}}^{n/k} + b(M), \quad (3.3)$$

where $b(M) = \dim H(M)$ is the total Betti number, and $C = C(M, g, p, k, d)$.

By (3.1) and (3.2), Theorem 3.8 implies³ Theorem 2.9 from Section 2.3.

Several results akin to Theorem 3.8 were obtained in previous works. In [38], Yomdin showed an estimate on the magnitude of the oscillations of a smooth function in terms of the uniform norm of its higher derivatives (we also refer [24, 26, 37] for earlier related results). Estimates similar to (3.3), stated in the language of persistence barcodes, were known for $p = \infty$ and $k = 1$ [16], and in the case of surfaces for $p = k = 2$ [32] (see also [34] for other estimates of this kind).

4. The transcendental Bézout problem and coarse zero count

By the classical Bézout theorem, the number of common zeroes of n polynomials in n variables is generically bounded by the product of their degrees. The *transcendental Bézout problem* asks about what can be said when we have entire functions instead of polynomials. For an n -tuple $f_1, \dots, f_n \in \text{Hol}(\mathbb{C}^n)$ of entire functions, the number of their isolated common zeroes can be infinite, and generally one can view f_j as polynomials of an infinite degree. In order to make the problem meaningful, introduce the zero counting function

$$\zeta(f, r) = \#(f^{-1}(0) \cap B_r),$$

where $f = (f_1, \dots, f_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n$, and $B_r = \{z \in \mathbb{C}^n \mid |z| \leq r\}$ is the closed Euclidean ball of radius r in $\mathbb{C}^n$ centered at the origin ($r > 0$). In addition, note that for a complex polynomial, its degree is directly related to the growth rate of the polynomial at ∞ . Namely, for $p \in \mathbb{C}[z_1, \dots, z_n]$, we have $k = \deg p$ if and only if $\max_{|z| \leq r} |p(z)| = \Theta(r^k)$ when $r \rightarrow +\infty$. Hence for extending the notion of the degree to entire maps, it is natural to consider the maximum modulus function

$$\mu(f, r) = \max_{|z| \leq r} |f(z)|.$$

Now the transcendental Bézout problem would sound as follows: is $\zeta(f, r)$ bounded in terms of $\mu(f, r)$?

Look first at the case of dimension $n = 1$. For an entire function $f \in \text{Hol}(\mathbb{C})$, by Jensen's formula, we have

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta - \log |f(0)| = \int_0^r \frac{\zeta(f, t)}{t} dt$$

(we assume $f(0) \neq 0$), which readily implies the following corollary.

³An additional argument is needed for obtaining the additive constants 0 and $b_r(M)$ in the upper bounds for m_r and z_r in Theorem 2.9.

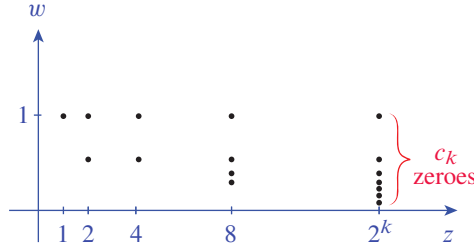


Figure 2. Zeroes of the Cornalba–Shiffman counterexample.

Corollary 4.1. *For every $a > 0$, we have*

$$\zeta(f, r) \leq C \log \left(\frac{\mu(f, ar)}{|f(0)|} \right), \quad (4.1)$$

where $C = (\log a)^{-1}$.

Note that the estimate (4.1) is sharp when $f(z) = e^z + 1$. Corollary 4.1 can be viewed as an answer to the transcendental Bézout problem in dimension 1.

It is natural to ask whether the estimate (4.1) generalizes to higher dimensions. Interestingly enough, it turns out that already in dimension 2 such an estimate cannot hold because of a remarkable counterexample due to Cornalba and Shiffman [17]. Namely, they constructed an entire map $F : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ such that:

- F is of order 0, that is, for each $\varepsilon > 0$, we have $\log \mu(F, r) \leq C_\varepsilon r^\varepsilon$ for $r \geq 1$.
- The growth of $\zeta(F, r)$ is arbitrarily fast. More precisely, we have

$$F^{-1}(0) = \left\{ \left(2^i, \frac{1}{j} \right) \mid i = 1, 2, \dots; j = 1, \dots, c_i \right\} \subset \mathbb{C}^2,$$

where $c_1 < c_2 < \dots$ can be any increasing sequence of positive integers. See Figure 2.

The counterexample of Cornalba and Shiffman implies the negative answer to the transcendental Bézout problem in dimension 2 and higher.⁴ Nevertheless, let us show that by following the “coarse approach”, we can obtain positive results. First, we define the coarse zero count in our setting which is reminiscent of Definition 2.4 from Section 2.2.

Definition 4.2. Let $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be an analytic map. For $\delta > 0$ and $r > 0$, define $\zeta(f, r, \delta)$ to be the number of connected components of $f^{-1}(B_\delta) \cap B_r$ that contain zeroes of f . See Figure 3.

⁴For any dimension $n > 2$, define the mapping $F_n : \mathbb{C}^n \rightarrow \mathbb{C}^n$ by $F_n(z_1, z_2, \dots, z_n) = (F(z_1, z_2), z_3, \dots, z_n)$. Then F_n is of order 0, and the growth of $\zeta(F_n, r)$ is arbitrarily fast.

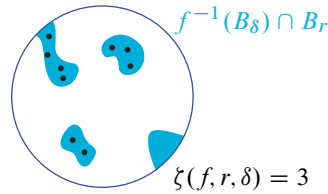


Figure 3. Coarse zero count.

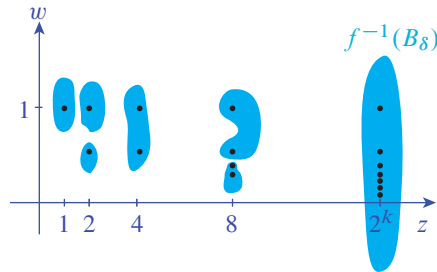


Figure 4. Coarse zero count for the Cornalba–Shiffman counterexample.

In [11], we show the following result.

Theorem 4.3. *For any analytic map $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$ and any $a > 1$, for each $r > 0$ and $\delta \in (0, \mu(f, ar)/2)$, we have*

$$\zeta(f, r, \delta) \leq C \left(\log \left(\frac{\mu(f, ar)}{\delta} \right) \right)^{2n-1}, \quad (4.2)$$

where $C = C(n, a)$.

Estimates analogous to (4.2) for the usual count of zeroes had been previously proven under positive lower bounds on the Jacobian of f [25, 28, 29].

It is unclear whether the estimate (4.2) is sharp in terms of the power $2n - 1$ in the upper bound. In the examples known to us, the correct power does not exceed n . For instance, for $f(z_1, \dots, z_n) = (e^{z_1} + 1, \dots, e^{z_n} + 1)$, it is not difficult to see that $\zeta(f, r, \delta) \sim r^n$ and $\log \mu(f, r) \sim r$. For the Cornalba–Shiffman counterexample $F : \mathbb{C}^2 \rightarrow \mathbb{C}^2$, one verifies [11] that $\zeta(F, r, \delta) \sim \log r$ and $\log \mu(F, r) = \Theta((\log r)^2)$. Moreover, [11] contains a careful analysis of the Cornalba–Shiffman counterexample from the coarse perspective. See Figure 4.

Acknowledgments. The main results that are explicitly mentioned in this survey were obtained in [9–11]; I cordially thank all the coauthors for many interesting discussions and fruitful collaboration. I thank Leonid Polterovich for his help in the

preparation of this note. The note is a write-up of my talk at the 9th ECM, and it is my pleasure to thank the organizers for the invitation.

Funding. The author was partially supported by the Israel Science Foundation grant 1918/23.

References

- [1] D. Akhiezer and B. Kazarnovskii, An estimate for the average number of common zeros of Laplacian eigenfunctions (in Russian). *Tr. Mosk. Mat. Obs.* **78** (2017), no. 1, 145–154. [English translation: Trans. Moscow Math. Soc.](#) **78** (2017), 123–130 Zbl [1395.53057](#) MR [3738081](#)
- [2] D. Akhiezer and B. Kazarnovskii, [On common zeros of eigenfunctions of the Laplace operator](#). *Abh. Math. Semin. Univ. Hambg.* **87** (2017), no. 1, 105–111 Zbl [1367.53034](#) MR [3623828](#)
- [3] V. I. Arnol’d, The topology of real algebraic curves (I. G. Petrovskii’s papers and developments from them) (in Russian). *Uspekhi Mat. Nauk* **28** (1973), no. 5, 260–262. English translation: [4] Zbl [0284.57001](#)
- [4] V. I. Arnold, [The topology of real algebraic curves](#). In A. Givental, B. Khesin, A. Varchenko, V. Vassilev, and O. Viro (Eds.), *Vladimir I. Arnold – Collected works. Vol. II. Hydrodynamics, bifurcation theory, and algebraic geometry 1965–1972*. Springer, Berlin, 2014, 465 pp. MR [3185029](#)
- [5] P. Bérard, P. Charron, and B. Helffer, [Non-boundedness of the number of super level domains of eigenfunctions](#). *J. Anal. Math.* **146** (2022), no. 1, 127–164 Zbl [1495.35129](#) MR [4467996](#)
- [6] P. Bérard and B. Helffer, [On Courant’s nodal domain property for linear combinations of eigenfunctions. Part I](#). *Doc. Math.* **23** (2018), 1561–1585 Zbl [1403.35192](#) MR [3890963](#)
- [7] P. Bérard and B. Helffer, [Level sets of certain Neumann eigenfunctions under deformation of Lipschitz domains application to the extended Courant property](#). *Ann. Fac. Sci. Toulouse Math. (6)* **30** (2021), no. 3, 429–462 Zbl [1479.35581](#) MR [4331305](#)
- [8] P. Bérard and B. Helffer, [On Courant’s nodal domain property for linear combinations of eigenfunctions. Part II](#). In S. Albeverio, A. Balslev, and R. Weder (Eds.), *Schrödinger operators, spectral analysis and number theory*, pp. 47–88, Springer Proc. Math. Stat. 348, Springer, Cham, 2021 Zbl [1472.35250](#) MR [4281621](#)
- [9] L. Buhovsky, A. Logunov, and M. Sodin, [Eigenfunctions with infinitely many isolated critical points](#). *Int. Math. Res. Not. IMRN* **2020** (2020), no. 24, 10100–10113 Zbl [1459.35295](#) MR [4190398](#)
- [10] L. Buhovsky, J. Payette, I. Polterovich, L. Polterovich, E. Shelukhin, and V. Stojisavljević, [Coarse nodal count and topological persistence](#). *J. Eur. Math. Soc. (JEMS)* **28** (2026), no. 7, 3131–3202 MR [5061408](#)

- [11] L. Buhovsky, I. Polterovich, L. Polterovich, E. Shelukhin, and V. Stojisavljević, [Persistent transcendental Bézout theorems](#). *Forum Math. Sigma* **12** (2024), article no. e72, 28 pp. Zbl [1550.32008](#) MR [4790778](#)
- [12] F. Chazal, D. Cohen-Steiner, M. Glisse, L. J. Guibas, and S. Y. Oudot, [Proximity of persistence modules and their diagrams](#). In *Proceedings of the 25th annual symposium on Computational geometry, SCG '09*, pp. 237–246, ACM, New York, 2009 Zbl [1380.68387](#)
- [13] F. Chazal, W. Crawley-Boevey, and V. de Silva, [The observable structure of persistence modules](#). *Homology Homotopy Appl.* **18** (2016), no. 2, 247–265 Zbl [1377.55015](#) MR [3575998](#)
- [14] F. Chazal, V. de Silva, M. Glisse, and S. Oudot, [The structure and stability of persistence modules](#). SpringerBriefs Math., Springer, Cham, 2016, 120 pp. Zbl [1362.55002](#) MR [3524869](#)
- [15] D. Cohen-Steiner, H. Edelsbrunner, and J. Harer, [Stability of persistence diagrams](#). *Discrete Comput. Geom.* **37** (2007), no. 1, 103–120 Zbl [1117.54027](#) MR [2279866](#)
- [16] D. Cohen-Steiner, H. Edelsbrunner, J. Harer, and Y. Mileyko, [Lipschitz functions have \$L_p\$ -stable persistence](#). *Found. Comput. Math.* **10** (2010), no. 2, 127–139 Zbl [1192.55007](#) MR [2594441](#)
- [17] M. Cornalba and B. Shiffman, [A counterexample to the “Transcendental Bézout Problem”](#). *Ann. of Math. (2)* **96** (1972), no. 2, 402–406 Zbl [0244.32006](#) MR [0311937](#)
- [18] R. Courant and D. Hilbert, *Methoden der mathematischen Physik*. Erster Band, Zweite verbesserte Auflage, Julius Springer, 1931 Zbl [0001.00501](#)
- [19] R. Courant and D. Hilbert, *Methods of mathematical physics. Vol. I*. 1st English edn., Interscience, New York, 1953, 561 pp. Zbl [0051.28802](#) MR [0065391](#)
- [20] W. Crawley-Boevey, [Decomposition of pointwise finite-dimensional persistence modules](#). *J. Algebra Appl.* **14** (2015), no. 5, article no. 1550066, 8 pp. Zbl [1345.16015](#) MR [3323327](#)
- [21] H. Donnelly and C. Fefferman, [Nodal sets of eigenfunctions on Riemannian manifolds](#). *Invent. Math.* **93** (1988), no. 1, 161–183 Zbl [0659.58047](#) MR [0943927](#)
- [22] V. M. Gichev, Some remarks on spherical harmonics (in Russian). *Algebra i Analiz* **20** (2008), no. 4, 64–86. [English translation](#): *St. Petersburg Math. J.* **20** (2009), no. 4, 553–567 Zbl [1206.33023](#) MR [2473744](#)
- [23] G. M. L. Gladwell and H. Zhu, [The Courant-Herrmann conjecture](#). *ZAMM Z. Angew. Math. Mech.* **83** (2003), no. 4, 275–281 Zbl [1221.35413](#) MR [1977003](#)
- [24] L. D. Ivanov, *Variations of sets and functions* (in Russian). In A. G. Vituškin (ed.), Izdat. “Nauka”, Moscow, 1975, 352 pp. Zbl [0967.26500](#) MR [0476953](#)
- [25] S. Ji, [Bézout estimate for entire holomorphic maps and their Jacobians](#). *Amer. J. Math.* **117** (1995), no. 2, 395–403 Zbl [0824.32001](#) MR [1323681](#)
- [26] A. S. Kronrod, On functions of two variables (in Russian). *Uspehi Matem. Nauk (N.S.)* **5** (1950), no. 1(35), 24–134 Zbl [0040.31603](#) MR [0034826](#)

- [27] M. Lesnick, [The theory of the interleaving distance on multidimensional persistence modules](#). *Found. Comput. Math.* **15** (2015), no. 3, 613–650 Zbl 1335.55006 MR 3348168
- [28] B. Q. Li, [On the Bézout problem and area of interpolating varieties in \$\mathbb{C}^n\$, II](#). *Amer. J. Math.* **120** (1998), no. 6, 1191–1198 Zbl 0954.58006 MR 1657162
- [29] B. Q. Li and B. A. Taylor, [On the Bézout problem and area of interpolating varieties in \$\mathbb{C}^n\$](#) . *Amer. J. Math.* **118** (1996), no. 5, 989–1010 Zbl 0863.32002 MR 1408496
- [30] S. Y. Oudot, [Persistence theory: from quiver representations to data analysis](#). Math. Surveys Monogr. 209, American Mathematical Society, Providence, RI, 2015, 218 pp. Zbl 1335.55001 MR 3408277
- [31] Å. Pleijel, [Remarks on Courant's nodal line theorem](#). *Comm. Pure Appl. Math.* **9** (1956), no. 3, 543–550 Zbl 0070.32604 MR 0080861
- [32] I. Polterovich, L. Polterovich, and V. Stojisavljević, [Persistence barcodes and Laplace eigenfunctions on surfaces](#). *Geom. Dedicata* **201** (2019), 111–138 Zbl 1420.58012 MR 3978537
- [33] L. Polterovich, D. Rosen, K. Samvelyan, and J. Zhang, [Topological persistence in geometry and analysis](#). Univ. Lecture Ser. 74, American Mathematical Society, Providence, RI, 2020, 128 pp. Zbl 1464.55003 MR 4249570
- [34] L. Polterovich and M. Sodin, [Nodal inequalities on surfaces](#). *Math. Proc. Cambridge Philos. Soc.* **143** (2007), no. 2, 459–467 Zbl 1127.58029 MR 2364662
- [35] V. Stojisavljević, Harmonic functions with highly intersecting zero sets. 2024, arXiv: [2410.03975v1](#)
- [36] O. Ya. Viro, Construction of multicomponent real algebraic surfaces (in Russian). *Sov. Math. Dokl.* **20** (1979), no. 5, 991–995. English translation: *Dokl. Akad. Nauk SSSR* **248** (1979), no. 2, 279–282 Zbl 0435.14006 MR 0553188
- [37] A. G. Vituškin, [On multi-dimensional variations](#) (in Russian). Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1955, 220 pp. Zbl 0063.09196 MR 0075267
- [38] Y. Yomdin, [Global bounds for the Betti numbers of regular fibers of differentiable mappings](#). *Topology* **24** (1985), no. 2, 145–152 Zbl 0566.57014 MR 0793180

Lev Buhovsky

School of Mathematical Sciences, Tel Aviv University, P.O. Box 39040, 69978 Tel Aviv, Israel; levbuh@tauex.tau.ac.il