

Unstable vortices and non-uniqueness for 2D Euler and α -gSQG

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Abstract. In the sixties, Yudovich proved global existence and uniqueness of solutions for the 2D Euler incompressible equation in $L^1 \cap L^\infty$. This result extends to the case with a force in $L^1_t(L^1 \cap L^\infty)$. Global existence in $L^1 \cap L^p$, $1 < p < \infty$, was obtained by R. DiPerna and A. Majda in 1987, but the problem of uniqueness, in this range of p , has remained open until Vishik's work in 2018. Indeed, M. Vishik proved that there are forces in $L^1_t(L^1 \cap L^p)$, $1 < p < \infty$, such that at least two different solutions exist for 2D Euler starting in the same initial datum. A nice exposition of the proof of this result can be found in D. Albritton et al. (2024). In this paper, we will present an alternative approach to prove this theorem. More precisely, we will show a different way to construct unstable radial vortices with compact support which also give rise to non-uniqueness. This strategy also works for α -gSQG, $0 \leq \alpha \leq 1$. This family of transport equations, with velocity $\mathbf{u} = -\nabla^\perp(-\Delta)^{-1+\frac{\alpha}{2}}\theta$, for the scalar θ , interpolates from 2D Euler ($\alpha = 0$) to the surface quasi-geostrophic equation ($\alpha = 1$).

1. Introduction

This article is a written version of the talk “Unstable vortices and non-uniqueness for 2D Euler and α -gSQG” in the 9ECM. This talk consisted of a review of some results concerning non-uniqueness in 2D and 3D Euler, Navier–Stokes and generalized surface quasi-geostrophic equations and a presentation of some results by the author and his collaborators, Daniel Faraco, Francisco Mengual, and Marcos Solera-Diana. A wide part of these results is published in [19].

The main topics of this paper will be the 2D Euler equations, the surface quasi-geostrophic equation (SQG) and, an interpolation between both of them, the generalized surface quasi-geostrophic equation (α -gSQG).

The forced Navier–Stokes system models the motion of an incompressible fluid. It consists of two equations for the velocity, $\mathbf{u}(\mathbf{x}, t)$, and the pressure, $p(\mathbf{x}, t)$, and

reads

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \nu \Delta \mathbf{u} + \mathbf{g}, \quad (\mathbf{x}, t) \in \mathbb{R}^d \times \mathbb{R}^+, \quad (1.1)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (\mathbf{x}, t) \in \mathbb{R}^d \times \mathbb{R}^+, \quad (1.2)$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}^\circ(\mathbf{x}), \quad (1.3)$$

where $\mathbf{u}^\circ(\mathbf{x})$ is the given initial datum, $\mathbf{g}(\mathbf{x}, t)$ a divergence-free force, $\nu \geq 0$ is the viscosity and $d = 2$, or 3 . The forced Euler equations arise in the case without viscosity, $\nu = 0$, and can be written in terms of the vorticity, $\omega(\mathbf{x}, t)$. In dimension 2, the vorticity is given by

$$\omega = \nabla^\perp \cdot \mathbf{u},$$

and, assuming decay at infinity, one can find the following transport equation for it:

$$\partial_t \omega + \mathbf{u} \cdot \nabla \omega = f, \quad (\mathbf{x}, t) \in \mathbb{R}^2 \times \mathbb{R}^+, \quad (1.4)$$

$$\mathbf{u} = \nabla^\perp \Delta^{-1} \omega, \quad (1.5)$$

$$\omega(\mathbf{x}, 0) = \omega^\circ(\mathbf{x}), \quad (1.6)$$

where

$$\Delta^{-1} \omega(\mathbf{x}) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(\mathbf{x} - \mathbf{y})^\perp}{|\mathbf{x} - \mathbf{y}|^2} \omega(\mathbf{y}) d\mathbf{y},$$

and

$$f = \nabla^\perp \cdot \mathbf{g}.$$

The SQG equation has a geophysical origin (see [81]). Roughly speaking, it models the evolution of the temperature of the air on the Earth and comes from the general quasi-geostrophic system after some assumptions and approximations. The forced SQG has the form

$$\partial_t \theta + \mathbf{u} \cdot \nabla \theta = f, \quad (\mathbf{x}, t) \in \mathbb{R}^2 \times \mathbb{R}^+, \quad (1.7)$$

$$\mathbf{u} = -\nabla^\perp (-\Delta)^{-\frac{1}{2}} \theta, \quad (1.8)$$

$$\theta(\mathbf{x}, 0) = \theta^\circ(\mathbf{x}), \quad (1.9)$$

where

$$(-\Delta)^{-\frac{1}{2}} \theta(\mathbf{x}) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{\theta(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} d\mathbf{y}.$$

In [31], P. Constantin, A. Majda and E. Tabak proposed SQG as a scalar 2D model of the 3D Euler equation because of the similarity between their structures. Since then, there has been intense research on it to understand the behavior of its solutions.

The α -generalized SQG equation also has a geophysical origin and, mathematically, is an interpolation between 2D Euler and SQG. Indeed, forced α -SQG reads

$$\begin{aligned}\partial_t \theta + \mathbf{u} \cdot \nabla \theta &= f, & (\mathbf{x}, t) \in \mathbb{R}^2 \times \mathbb{R}^+, \\ \mathbf{u} &= -\nabla^\perp (-\Delta)^{-\frac{\alpha}{2}} \theta, \\ \theta(\mathbf{x}, 0) &= \theta^\circ(\mathbf{x}),\end{aligned}$$

where

$$(-\Delta)^{-\frac{\alpha}{2}} \theta(\mathbf{x}) = C_\alpha \int_{\mathbb{R}^2} \frac{\theta(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^\alpha} d\mathbf{y}.$$

Then for $\alpha = 0$ we obtain 2D Euler, and for $\alpha = 1$ we obtain SQG. We remark that, for α -gSQG, the velocity is an operator of degree $1 - \alpha$, and this makes a big difference between the members of this family of transport equations.

In this paper, we focus on the uniqueness problem for α -gSQG, with $0 \leq \alpha \leq 1$. One could also consider the case $1 < \alpha < 2$, which has been studied in the literature, but we will skip it here. The starting point for us will be the celebrated Yudovich's theorem.

Theorem 1.1 (Yudovich). *Let $\omega_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ and $f \in L^1([0, T], L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2))$ for all $T > 0$. Then there exists a solution*

$$\omega(\mathbf{x}, t) \in L^\infty([0, T], L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)),$$

with

$$\mathbf{u}(\mathbf{x}, t) \in L^\infty([0, T], L^2(\mathbb{R}^2)),$$

to (1.4), (1.5) and (1.6) for all $T < \infty$. In addition, this solution is unique in the class $L^\infty([0, T], L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2))$.

Actually, Yudovich proved this theorem in [91] with $f = 0$, but his proof can be extended to the case with $f \in L_t^1(L^1 \cap L^\infty)$ as shown in [2].

The question of existence in $L^1 \cap L^p$ with $1 < p < \infty$ was addressed by R. DiPerna and A. Majda [48], who gave a positive answer. However, the question of whether there can be more than one solution in the class $L_t^\infty(L^1 \cap L^p)$, with $1 < p < \infty$, for the same initial datum remains open today in most of the cases. However, very recently, there have been some remarkable advances in this question. In [86, 87], M. Vishik proved the following theorem.

Theorem 1.2 (Vishik). *For any $2 < p < \infty$, there exists a force*

$$f \in L_t^1(L^1 \cap L^p) \quad \text{with } g \in L_t^1 L^2, \quad (1.10)$$

where $g = \nabla^\perp \Delta^{-1} f$, with the property that there are two different solutions

$$\omega_0, \omega_1 \in L_t^\infty(L^1 \cap L^p) \quad \text{with } v_0, v_1 \in L_t^\infty L^2, \quad (1.11)$$

where $v_j = \nabla^\perp \Delta^{-1} \omega_j$, to the Euler equation starting from $\omega^\circ = 0$.

Thus, this result states that there exists at least a force such that uniqueness fails for 2D Euler in the class $L_t^\infty(L^1 \cap L^p)$ for any $1 < p < \infty$. A nice exposition of Vishik's proof of Theorem 1.2 can be found in [2], where the authors, apart from elaborating on some details, improve it in several aspects. In addition, in [19], the author, together with D. Faraco, F. Mengual and M. Solera-Diana, gave a proof of Theorem 1.2 which is different in several points from Vishik's proof: mainly, in the construction of the unstable vortex which gives rise to non-uniqueness. The main purpose of this paper is the presentation of this construction and its extension to α -gSQG. After Vishik's result, the main open question in this context is whether Theorem 1.2 can be proven with $f = 0$. In [9], this problem has been partially solved. Indeed, E. Bruè, M. Colombo and A. Kumar claim that there exists a small ε such that uniqueness fails in $L^1 \cap L^{1+\varepsilon}$. Despite the great interest of this last paper, its analysis exceeds the scope of this review, and we will not go into details about it.

This paper is organized as follows. In Section 2, we will review some results concerning local and global existence for 2D Euler, SQG and α -SQG. In Section 3, we will present some literature dealing with the problem of non-uniqueness in fluid mechanics without forcing. This section will consist of two parts: in the first one, we shall consider papers that approach the problem using convex integration, and in the second one, those dealing with self-similar solutions. In Section 4, we will give some more details about what is known about uniqueness in fluid mechanics when we introduce a force. Finally, in Section 5, we will sketch our proof of Theorem 1.2.

2. Local and global existence of solutions

Global existence and uniqueness of classical solutions to the 2D Euler equations was proven by W. Wolibner [89] and E. Hölder [63] with velocity $C^{k,\gamma}$ for $k \geq 1$ and $\gamma > 0$. Local well-posedness in $C^{k,\gamma}$ had been previously established by L. Lichtenstein [77] and N. Gunther [61]. Local well-posedness in H^s , $s > \frac{d}{2} + 1$, was proven by D. Ebin and J. Marsden [49] in a compact domain and by T. Kato [72] in $\mathbb{R}^d$. Local well-posedness in $W^{s,p}$, $s > \frac{d}{p} + 1$, was provided by J.P. Bourguignon and H. Brezis [6] in a compact domain and by T. Kato and G. Ponce [73] in $\mathbb{R}^d$.

The global well-posedness in 2D sharply contrasts with the 3D case due to vortex stretching, as demonstrated by T. Elgindi [50], who showed finite-time singularities with $\omega^\circ \in C^\gamma$ for some $0 < \gamma \ll 1$ in the unforced case (see also [24,39]). D. Córdoba

and L. Martínez-Zorúa [35] upgrade the regularity of the singular solutions in the forced case with $f \in L_t^\infty C^{\frac{1}{2}-\varepsilon}$. For numerical studies about singularities in $\mathbb{R}^3$, one can consult [64] and [88]. In the case with boundaries, we refer to the works [25] and [26] where J. Chen and T. Hou claimed finite-time singularities for smooth initial data and finite energy. Let us also mention the work [76] by Kiselev and Šverák for the related issue on small-scale creation in two dimensions.

J. Bourgain and D. Li [5] and, independently, T. Elgindi and N. Masmoudi [51] showed strong ill-posedness in C^k , $k \geq 1$. See [4] for ill-posedness in $H^{\frac{q}{2}+1}$ and [74] for ill-posedness in $W_p^{\frac{d}{p}+1,p}$. Instantaneous gap loss of Sobolev regularity with initial vorticity in H^β , $0 < \beta < 1$, has been proven in [37] by D. Córdoba, L. Martínez-Zorúa and W. Ożański.

For SQG, global existence of solutions in L^2 was provided by S.G. Resnick [82]. F. Marchand [78] extended this result to L^p for $p > \frac{4}{3}$. Local well-posedness in H^s , $s > 2$, can be found in [20] and [31] (see also [32] for well-posedness in a bounded domain). J. Wu [90] showed existence in $C^r \cap L^q$, $r > 1$ and $q > 1$. The problem of whether SQG presents finite-time singularities is one of the most outstanding problems in mathematical fluid mechanics. A. Kiselev and F. Nazarov [75] showed norm inflation and S. Friedlander and R. Shvydkoy [53] unstable eigenvalues in the spectrum. In addition, exponential growth of the C^2 -norm has been shown by S. He and A. Kiselev [62]. There are also examples of global solutions with non-trivial dynamics. See, for example, [3, 18, 57, 58]. Ill-posedness in C^k , $k \geq 2$, and in H^s , $s \in (3/2, 2)$, was proven by D. Córdoba and L. Martínez-Zorúa [34] (see also [51, 69]). Instantaneous loss of regularity in H^s , with $s \in (3/2, 2)$, has been recently proven in [38].

In the case of α -gSQG, in [21], D. Chae, P. Constantin, D. Córdoba and F. Gancedo extended Resnick's result, showing also global existence in L^2 . In [22], a global existence criterion is shown: solutions in C^σ , $\sigma > 1$, exist as far as the $L^1([0, T], C^\alpha)$ -norm is under control. The possible formation of singularities is still unknown. Finally, let us remark that existence and uniqueness in the spaces $H^{(1+\alpha)+}$ was proven in [23] (see Theorem 1.2 and take $\beta = 2 - \alpha - \varepsilon$ and $P(\xi) = |\xi|^{-\varepsilon}$). For the more singular case $\alpha > 1$, the interested reader can consult [21] for well-posedness and [36] for ill-posedness.

3. Uniqueness without forcing

In this section, we will take the force equal to zero and approach the problem of uniqueness into two different ways: convex integration and self-similar solutions.

3.1. Convex integration and non-uniqueness

In the last decades, convex integration has been applied to the uniqueness problem in incompressible fluid mechanics with great success. We can highlight, for example, the proof of Onsager's conjecture for the Euler equations. This conjecture states that there exist solutions $\mathbf{u} \in C^\alpha$, with $\alpha < \frac{1}{3}$, which dissipate kinetic energy. It was solved in 3D in [12] and [65] after a series of works which used convex integration. The fact that solutions of the Euler equations in C^α , with $\alpha > \frac{1}{3}$, do conserve energy had been previously proven in [33]. See also the work [28] by A. Cheskidov, P. Constantin, S. Friedlander and R. Shvydkoy on critical regularity.

The first proofs of non-uniqueness for the Euler equations were provided independently by V. Scheffer [83] and A. Shnirelman [85]. They proved the existence of solutions with compact support in time whose velocity $\mathbf{u} \in L^2_{t,x}$. After that, C. De Lellis and L. Székelyhidi [42] introduced a systematic way to run convex integration in the problem which produced solutions with $\mathbf{u} \in L^\infty_t L^2_x$ and compact support in space and time. In a series of papers, the same authors improved the regularity and properties of these non-unique solutions (see [43–46]). Regularity $C^{\frac{1}{5}}$ was achieved in [10] by T. Buckmaster, C. De Lellis, P. Isett and L. Székelyhidi and regularity $L^1_t C^{\frac{1}{3}-}_x$ in [11]. Finally, P. Isett proved, in [65], non-uniqueness for 3D Euler with velocity in $C^{\frac{1}{3}-}$ and T. Buckmaster, C. De Lellis, V. Vicol and L. Székelyhidi dissipation of kinetic energy also in $C^{\frac{1}{3}-}$. See also the work [80] by M. Novack and V. Vicol on an intermittent Onsager theorem, based on their joint work [13] with T. Buckmaster and N. Masmoudi, as well as the work [55] on the L^3 -based strong Onsager conjecture by V. Giri, H. Kwon and M. Novack.

The dissipative part of the Onsager conjecture in dimension 2 is harder to deal with and was solved recently in [56] by V. Giri and R.-O. Radu by means of convex integration. We refer to the recent work [47] by De Rosa and Park for the related issue on anomalous dissipation in two dimensions.

To the best of our knowledge, the problem of non-uniqueness for the unforced Euler equation in the velocity class $L^2 \cap C^\gamma$ remains open in the regime $\frac{1}{3} \leq \gamma < 1$.

In [79], F. Mengual proved the existence, for any $2 < p < \infty$, of initial data $v^\circ \in L^2$ with $\omega^\circ \in L^1 \cap L^p$ for which there are infinitely many admissible solutions $v \in C_t L^2$. This result shows the sharpness of Yudovich's proof of uniqueness, but with the drawback that the vorticity information is lost for positive times.

We emphasize that the convex integration method allowed for proving non-uniqueness of Navier–Stokes solutions in [15] by T. Buckmaster and V. Vicol, with $\mathbf{u} \in C_t H^\beta$ for small β , and sharpness of the Ladyzhenskaya–Prodi–Serrin criteria [29] by A. Cheskidov and X. Luo.

Convex integration has also been applied to study the lack of uniqueness for SQG. The first result in that direction was provided by T. Buckmaster, S. Shkoller and

V. Vicol [14]. They proved non-uniqueness with $\Lambda^{-1}\theta \in C_t^\sigma C_x^\beta$, with $\frac{1}{2} < \beta < \frac{4}{5}$ and $\sigma < \frac{\beta}{(2-\beta)}$. P. Isett and A. Ma improved this regularity in [67] and obtained an analogous result for α -gSQG in [68] (see also [27]). Very recently, what is considered the dissipative part of Onsager's conjecture for SQG has been proven: independently, M. Dai, V. Giri and R.-O. Radu [41] and S. Looi and P. Isett [66] have proven energy dissipation and non-uniqueness of solutions in $C_{t,x}^{0-}$.

3.2. Self-similar solutions and non-uniqueness

In this section, we will explain the relation between non-uniqueness and the stability of self-similar solutions. For the sake of clarity, we will take as a model the 2D Euler equation, although similar arguments can be applied to other equations including, of course, α -gSQG. We have borrowed the following argument from [71].

We will look for self-similar solutions, i.e., solutions with the form

$$\omega(\mathbf{x}, t) = t^\alpha \Omega(t^\beta \mathbf{x}),$$

where $\alpha, \beta \in \mathbb{R}$. Introducing this ansatz in (1.4), we get

$$\alpha t^{\alpha-1}(\alpha \Omega(\xi) + \beta \xi \cdot \nabla \Omega(\xi)) + t^{2\alpha} \mathbf{u}[\Omega](\xi) \cdot \nabla \Omega(\xi) = 0,$$

where

$$\mathbf{u}[\Omega](\xi) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(\xi - \bar{\xi})^\perp}{|\xi - \bar{\xi}|^2} d\bar{\xi}, \quad \text{and} \quad \xi = t^\beta \mathbf{x}.$$

Thus, taking $\alpha = -1$, we obtain the time-independent equation

$$\Omega(\xi) + (\beta \xi + \mathbf{u}[\Omega](\xi)) \cdot \nabla \Omega(\xi) = 0. \quad (3.1)$$

Let us assume that Ω satisfies (3.1) and look for solutions of the form

$$\omega(\mathbf{x}, t) = t^{-1} \Omega(t^\beta \mathbf{x}) + t^{-1} \bar{\Omega}(t^\beta \mathbf{x}, t).$$

Thus, $\bar{\Omega}$ needs to satisfy

$$\begin{aligned} t \partial_t \bar{\Omega}(\xi, t) + \bar{\Omega}(\xi, t) + (\beta \xi + \mathbf{u}[\Omega](\xi, t)) \cdot \nabla \bar{\Omega}(\xi, t) \\ + \mathbf{u}[\bar{\Omega}](\xi, t) \cdot \nabla \bar{\Omega}(\xi, t) = 0. \end{aligned} \quad (3.2)$$

At the linear level, we find that (3.2) reads

$$t \partial_t \Omega_{\text{lin}}(\xi, t) - \mathcal{L}[\Omega] \Omega_{\text{lin}} = 0, \quad (3.3)$$

where

$$\mathcal{L}_{\text{lin}} \Omega = -1 - (\beta \xi + \mathbf{u}[\Omega]) \cdot \nabla.$$

Next, let us assume that there exists W (nice) such that

$$\mathcal{L}[\Omega]W = \lambda W,$$

with $\lambda \in \mathbb{C}$ and $\operatorname{Re}(\lambda) > 0$. These properties give us the following solution for (3.3)

$$\Omega_{\text{lin}}(\xi, t) = t^\lambda W(\xi),$$

which starts from zero at time $t = 0$, i.e., $\Omega_{\text{lin}}(\xi, 0) = 0$. Thus, we have two different solutions starting from zero for (3.3), namely, Ω_{lin} and 0.

Going further, we now write

$$\omega(\mathbf{x}, t) = t^{-1}\Omega(\xi) + t^{-1}\Omega_{\text{lin}}(\xi, t) + t^{-1}\Omega_{\text{cor}}(\xi, t)$$

to try to include the nonlinear terms in (3.2) that we have neglected in (3.3). Thus,

$$\begin{aligned} t\partial_t\Omega_{\text{cor}}(\xi, t) - \mathcal{L}[\Omega]\Omega_{\text{cor}} &= \mathcal{F}[\Omega_{\text{lin}}, \Omega_{\text{cor}}](\xi, t), \\ \Omega_{\text{cor}}(\xi, 0) &= 0, \end{aligned}$$

where

$$\mathcal{F}[\Omega_{\text{lin}}, \Omega_{\text{cor}}] = -\mathbf{u}[\Omega_{\text{lin}} + \Omega_{\text{cor}}] \cdot \nabla(\Omega_{\text{lin}} + \Omega_{\text{cor}}).$$

Notice that the initial condition $\Omega_{\text{cor}}(\xi, 0) = 0$ gives us some hope to find a solution $\Omega_{\text{cor}}(\xi, t)$ of a smaller order than $\Omega_{\text{lin}}(\xi, t)$ at $t = 0$. Let us assume that we have such a solution. We have then constructed two of them, namely,

$$\begin{aligned} \omega_0(\mathbf{x}, t) &= t^{-1}\Omega(t^\beta \mathbf{x}), \\ \omega_1(\mathbf{x}, t) &= t^{-1}\Omega(t^\beta \mathbf{x}) + t^{-1}\Omega_{\text{lin}}(t^\beta \mathbf{x}, t) + t^{-1}\Omega_{\text{cor}}(t^\beta \mathbf{x}, t). \end{aligned}$$

Finally, if $\Omega, \Omega_{\text{lin}}$ and $\Omega_{\text{cor}} \in L_t^\infty L^p$, a straightforward computation shows that

$$\|\omega_i(\cdot, t)\|_{L^p} \leq C t^{-1-\frac{2\beta}{p}}$$

for both $i = 0, 1$. If $p = \infty$, we fail; however, for any $p < \infty$, we can take β such that

$$\omega_i \rightarrow 0 \quad \text{in } L^p$$

for both $i = 0, 1$.

In dimension 2, it is well known that Leray–Hopf solutions of the Navier–Stokes systems are unique. However, the same question in 3D is an open problem. Let us remind the reader of the definition of a Leray–Hopf solution. We will include in it a force, since it will be needed in the following sections.

Definition 3.1. Let $T > 0$, $\mathbf{u}^\circ \in L^2(\mathbb{R}^3)$ be a divergence-free vector field and $\mathbf{g} \in L^1([0, T]; L^2(\mathbb{R}^3))$. A Leray–Hopf solution on $\mathbb{R}^3 \times (0, T)$ with initial datum $\mathbf{u}^\circ$ and force $\mathbf{g}$ is a divergence-free vector field

$$\mathbf{u} \in L^\infty([0, T], L^2(\mathbb{R}^3)) \cap L^2([0, T], H^1(\mathbb{R}^3))$$

which

- (1) belongs to $C_w([0, T], L^2(\mathbb{R}^3))$ and attains the initial data: $\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}^\circ$,
- (2) solves the forced Navier–Stokes equations in the sense of distributions on $\mathbb{R}^3 \times (0, T)$ for some pressure $p \in L^1_{\text{loc}}(\mathbb{R}^3 \times (0, T))$ and
- (3) satisfies the energy inequality for all $t \in (0, T]$:

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}^3} |\mathbf{u}(\mathbf{x}, t)|^2 d\mathbf{x} + \int_0^t \int_{\mathbb{R}^3} |\nabla \mathbf{u}(\mathbf{x}, s)|^2 d\mathbf{x} ds \\ & \leq \frac{1}{2} \int_{\mathbb{R}^3} |\mathbf{u}^\circ(\mathbf{x})|^2 d\mathbf{x} + \int_0^t \int_{\mathbb{R}^3} \mathbf{g}(\mathbf{x}, s) \cdot \mathbf{u}(\mathbf{x}, s) d\mathbf{x} ds. \end{aligned}$$

In a couple of papers, [70, 71], H. Jia and V. Sverák explored the ideas at the beginning of this section to study the uniqueness of Leray–Hopf solutions of the 3D Navier–Stokes equations (without forcing). Indeed, in [70], they proved the existence of self-similar solutions.

Theorem 3.2 (Jia, Sverák). *Assume $\mathbf{u}^\circ$ is scale invariant and locally Hölder continuous in $\mathbb{R}^3 \setminus \{0\}$ with $\nabla \cdot \mathbf{u}^\circ = 0$ in $\mathbb{R}^3$. Then the Navier–Stokes equations have at least one scale-invariant solution $\mathbf{u}(\mathbf{x}, t)$ which is smooth in $\mathbb{R}^3 \times (0, \infty)$ and locally Hölder continuous in $\mathbb{R}^3 \times [0, \infty) \setminus \{(0, 0)\}$.*

Here, a scale-invariant solution satisfies $\lambda \mathbf{u}(\lambda \mathbf{x}, \lambda^2 t) = \mathbf{u}(\mathbf{x}, t)$, and the theorem is proven by looking at self-similar solutions of the form $\mathbf{u}(\mathbf{x}, t) = t^{-\frac{1}{2}} \mathbf{U} t'(\mathbf{x} t^{-\frac{1}{2}})$.

After this theorem, they also proved that these self-similar solutions could yield non-uniqueness of Leray–Hopf solutions in [71]. In order to do it, they need to assume that, roughly speaking, there exist unstable self-similar solutions satisfying several conditions. We shall make this statement more precise. The linearization of the Navier–Stokes equations around a family of self-similar solutions $\mathbf{U}_\sigma$, parameterized by σ , reads

$$t \partial_t \phi = \mathcal{L}_\sigma \phi,$$

where

$$\mathcal{L}_\sigma \phi \equiv \Delta \phi + \frac{\mathbf{x}}{2} \cdot \nabla \phi + \frac{1}{2} \phi - \mathbf{U}_\sigma \nabla \phi - \phi \nabla \mathbf{U}_\sigma. \quad (3.4)$$

We define the domain of $\mathcal{L}_\sigma$ as

$$\mathcal{D} = \{\phi \in X : \partial_j \phi, \partial_{ij}^2 \phi \in X, \text{ and } \mathbf{x} \cdot \nabla \phi \in X\},$$

with

$$X = \{\phi \in L^4(\mathbb{R}^3) \cap L^2(\mathbb{R}^3) : \nabla \cdot \phi = 0\}.$$

Then two different Leray–Hopf solutions exist for Navier–Stokes starting in the same initial data if the following condition (B) is satisfied:

- (B) For $\sigma < \sigma_0$, the eigenvalues of $\mathcal{L}_\sigma \subset \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda < -\delta < 0\} \cup \{\lambda_1(\sigma)\}$ for some $\delta > 0$ and $\lambda_1(\sigma) < 0$. The eigenvalue $\lambda_1(\sigma)$ is simple. Moreover, $\lim_{\sigma \rightarrow \sigma_0^-} \lambda_1(\sigma) = 0$.

In [71], an additional condition (A) is given, which also yields non-uniqueness. Remarkably, in [60], J. Guillod and V. Šverák found numerical evidence of the existence of self-similar solutions satisfying condition (B).

There have been several authors who have studied the existence of self-similar solutions for 2D Euler. In [52], V. Elling showed the existence of self-similar solutions satisfying

$$\lim_{t \rightarrow 0^+} \omega(\mathbf{x}, t) = r^{-\frac{1}{\mu}} \tilde{\omega}(\theta). \quad (3.5)$$

More precisely, the result can be formulated as follows.

Theorem 3.3 (Elling). *Given $\varepsilon > 0$ and $\mu \in (\frac{2}{3}, \infty)$, there is an $N_0 \in \mathbb{N}$ so that a weak solution of 2D Euler satisfying (3.5) exists for all initial data $\tilde{\omega}$ satisfying the following conditions:*

- (1) *Periodicity: $\tilde{\omega}$ is $\frac{2\pi}{N}$ -periodic for $N \geq N_0$.*
- (2) *Dominant rotation: the Fourier coefficients satisfy*

$$|\tilde{\omega}^\wedge(0)| \geq \varepsilon \sum_{n \neq 0} |\tilde{\omega}^\wedge(n)|.$$

For α -gSQG, $0 < \alpha < 1$, introducing some new ideas, C. García and J. Gómez-Serrano [54] proved the existence of self-similar solutions.

Theorem 3.4 (García, Gómez-Serrano). *Let $\alpha \in (0, 1)$ and $\gamma \in (1, 1 + \alpha)$. Then there exists $\varepsilon > 0$ such that for any $\frac{2\pi}{N}$ -periodic $\tilde{\theta} \in B_{L^p(\mathbb{T})}(1, \varepsilon)$ and $\frac{p > 1}{(1-\alpha)}$, the initial condition $\theta^\circ = r^{-(1+\gamma-\alpha)} \tilde{\theta}$ describes a self-similar solution for α -gSQG for any $N \geq 1$.*

And, in [84], F. Shao, D. Wei and Z. Zhang improved Elling’s result for 2D Euler allowing $\mu > \frac{1}{2}$, $N_0 = 2$ and $\tilde{w} \in L^1(\mathbb{T})$.

To conclude this section, we will comment on the program by A. Bressan, R. Murray and W. Shen, collected in the papers [7, 8], to prove non-uniqueness for 2D Euler. This program is based on some numerical simulations by W. Shen which

show two different behaviors of the evolution of two different sequences of initial data which converge to the same function.

These two sequences of initial data are constructed in the following way. First, we define

$$\bar{\omega}(\mathbf{x}) = r^{-\frac{1}{\mu}} \bar{\Omega}(\theta),$$

where $\bar{\Omega} \in C^\infty$ is nonnegative periodic and such that

$$\bar{\Omega}(\theta + \pi) = \bar{\Omega}(\theta), \quad \bar{\Omega}(\theta) = 0 \quad \text{if } \frac{\pi}{4} \leq \theta \leq \pi.$$

And now we take

$$\omega_1^\varepsilon(\mathbf{x}) = \begin{cases} \varepsilon^{-\frac{1}{\mu}}, & |\mathbf{x}| < \varepsilon, \\ \bar{\omega}(\mathbf{x}), & |\mathbf{x}| > \varepsilon, \end{cases} \quad \omega_2^\varepsilon(\mathbf{x}) = \begin{cases} 0, & |\mathbf{x}| < \varepsilon, \\ \bar{\omega}(\mathbf{x}), & |\mathbf{x}| > \varepsilon. \end{cases}$$

The numerical simulations by W. Shen show that, on one hand, the sequence of ω_1^ε gives rise to a spiral-shaped structure at the point $\mathbf{x} = (0, 0)$. On the other hand, the sequence ω_2^ε yields two spiral-shaped structures whose origins move from the point $\mathbf{x} = (0, 0)$. These two different evolutions are reminiscent of the Kaden and Pullin spirals (see [30] and references therein).

4. Uniqueness with forcing

After the seminal works [86, 87] by M. Vishik, there have been some advances in the study of the Euler and Navier–Stokes equations with forcing. Apart from the papers [86, 87] and [2] dealing with the forced 2D Euler equations, here we will highlight the work [1] where D. Albritton, E. Bruè and M. Colombo prove non-uniqueness of Leray–Hopf solutions with force in $L_t^1 L^2(\mathbb{R}^3)$.

Theorem 4.1 (Albritton, Bruè, Colombo). *There exist $T > 0$, $\mathbf{g} \in L^1((0, T), L^2(\mathbb{R}^3))$ and two distinct Leray–Hopf solutions $\mathbf{u}_1, \mathbf{u}_2$ to the forced Navier–Stokes system on $\mathbb{R}^2 \times (0, t)$ with force $\mathbf{g}$ and initial condition $\mathbf{u}^\circ = 0$.*

In the next section, we will briefly sketch Vishik’s proof in [86, 87] and [2].

To conclude this section, let us comment that, in the forced case, A. Bulut, M. Huynh and S. Palasek [16], using convex integration, proved that the regularity in 3D can be upgraded to $C^{\frac{1}{2}-}$. The same authors applied a similar strategy to SQG in [17]. For other applications of convex integration to forced equations, we refer to the work [40] by M. Dai and S. Friedlander, and the references therein.

5. Sketch of the proof

In this section, we will sketch our proof of Theorem 1.2. The strategy is similar to the one in either [86, 87] or [2], except for the construction of the unstable vortex, which is totally different. In addition, the properties of our unstable vortex, which plays a fundamental role in the proof, enable us to simplify, in [19], some arguments and computations.

The proof of Theorem 1.2 divides into three steps: Eulerian, self-similar and nonlinear instability.

5.1. Eulerian instability

We start by recalling how linear stability for the Euler equation leads to the Rayleigh stability equation, which we found convenient to express in vorticity form, contrary to the usual practice.

Given a background (steady) vorticity $\bar{\omega}$, a perturbation $\bar{\omega} + \varepsilon\omega$ is a second solution to the Euler equation if the deviation ω satisfies the equation

$$(\partial_t - \mathcal{L})\omega + \varepsilon(\mathbf{u} \cdot \nabla\omega) = 0.$$

Here, $\mathcal{L} = \mathcal{L}_{\bar{\omega}}$ is the linearization of the 2D Euler equation around $\bar{\omega}$. The linearization can be written as

$$\mathcal{L} = T + K, \tag{5.1}$$

where T is a transport operator, K is a compact operator, and they are defined by

$$T\omega = -\bar{\mathbf{u}} \cdot \nabla\omega, \quad K\omega = -\mathbf{u} \cdot \nabla\bar{\omega}.$$

We have to study the linear equation

$$(\partial_t - \mathcal{L})\omega^{\text{lin}} = 0 \tag{5.2}$$

and seek solutions

$$\omega_{\text{lin}}(t, \mathbf{x}) = e^{\lambda t} w(\mathbf{x}), \tag{5.3}$$

with $\text{Re}(\lambda) > 0$. This leads to the eigenvalue problem

$$\mathcal{L}w = \lambda w. \tag{5.4}$$

Since the operator $\mathcal{L}$ is real, the real-valued solutions to (5.2) will be recovered by replacing (5.3) with $\omega_{\text{lin}}(t, \mathbf{x}) = \text{Re}(e^{\lambda t} w(\mathbf{x}))$.

We restrict ourselves to radial profiles $\bar{\omega}(\mathbf{x}) = \bar{w}(r)$. Next, we choose the function space on which the eigenvalue problem will be solved. Given $0 > n \in \mathbb{N}$, we seek eigenfunctions in the subspace of L^2 formed by purely n -fold symmetric vorticities

$$U_n = \{w \in L^2 : w(\mathbf{x}) = w_n(r)e^{in\theta}\}. \tag{5.5}$$

Definition 5.1. We say that the vortex $\bar{\omega}$ is *unstable* if, for some $n \in \mathbb{N}$, there exists $0 \neq w \in U_n$ satisfying (5.4) with $\lambda \in \mathbb{C}_+$.

For $w \in U_n$, we arrive at the Rayleigh stability equation

$$\frac{\bar{v}_\theta}{r} w_n + \frac{\partial_r \bar{w}}{n} \int_0^\infty K_n\left(\frac{r}{s}\right) w_n(s) ds = z w_n, \quad (5.6)$$

where $\bar{v}_\theta$ is the angular velocity, K_n are explicit kernels which came from (1.5), and we have replaced $\lambda = -inz$. Thus, we can write

$$w_n = h \partial_r \bar{w},$$

in terms of some profile h . For that h , equation (5.6) reads as

$$\frac{\bar{v}_\theta(r)}{r} h(r) + \frac{1}{n} \int_0^\infty K_n\left(\frac{r}{s}\right) (h \partial_r \bar{w})(s) ds = z h(r), \quad r \in \text{supp}(\partial_r \bar{w}). \quad (5.7)$$

We emphasize that equation (5.7) only needs to be solved in the support of $\partial_r \bar{w}$. This fact will play a very important role in our analysis. Indeed, this is the reason why we work with the Rayleigh equation in vorticity form.

Thus, Eulerian instability reduces to finding a vortex that admits nontrivial solutions (h, z) of (5.7) with $\Im z > 0$. We will show that this is indeed the case and therefore land in our first main result.

Theorem 5.2. *There exists an unstable vortex $\bar{\Omega} \in C_c^\infty(\mathbb{R}^2)$ with zero mean.*

Our proof of Theorem 5.2 is split into two steps: the construction of a piecewise constant unstable vortex and a regularization. This strategy was already proposed by E. Grenier [59] in the context of the stability of shear flows.

5.1.1. Piecewise constant unstable vortex. We now explain how to construct an unstable vortex of the form

$$\bar{w} = c 1_{[0, r_1)} - 1_{[r_1, r_2)},$$

in terms of some parameters $0 < r_1 < r_2 < \infty$ and $c > 0$, to be determined. Firstly, we choose c so that the mean of $\bar{\omega}$ equals zero

$$\int_{\mathbb{R}^2} \bar{\omega}(\mathbf{x}) d\mathbf{x} = 2\pi \int_0^\infty \bar{w}(r) r dr = 0.$$

This condition guarantees that $\bar{v} \in L^2$. Moreover, $\bar{v}$ is Lipschitz and compactly supported with $\text{supp}(\bar{v}) = \text{supp}(\bar{\omega}) = B_{r_2}$. Secondly, since

$$\text{supp}(\partial_r \bar{w}) = \{r_1, r_2\},$$

equation (5.7) turns into two conditions for the vector $h = (h(r_1), h(r_2))$ that can be written as a linear system

$$Ah = zh, \quad (5.8)$$

where the matrix $A \in \mathbb{R}^{2 \times 2}$ depends on the parameters r_1, r_2 , and the frequency n . Thus, there is an eigenvector $h \neq 0$ if and only if z is a root of the characteristic polynomial $\det(A - z)$. Finally, we can choose r_1, r_2 such that the roots z, z^* of $\det(A - z)$ satisfy $\Im z > 0$.

5.1.2. Regularization. We now explain how to prove the existence of a smooth vortex $\bar{\omega}^\varepsilon$ which is also unstable for some small $\varepsilon > 0$. Similarly to Step 5.1.1, now we need to solve the Rayleigh stability equation (5.7) in the intervals $B_\varepsilon(r_j)$ for $j = 1, 2$. We rescale variables around r_j writing $r = r_j + \varepsilon\alpha$ with $\alpha \in I = (-1, 1)$. Next, we make the asymptotic expansions for eigenfunctions and eigenvalues. Namely, we write

$$h^\varepsilon(r) = h(r_j) + \varepsilon g_j(\alpha), \quad z^\varepsilon = z + \varepsilon y,$$

for some profiles $g = (g_1, g_2) \in L^2(I)^2$ and a constant $y \in \mathbb{C}$, to be determined. We also expand

$$A^\varepsilon g = \frac{\bar{v}_\theta^\varepsilon}{r} g + \frac{1}{n} \int_0^\infty K_n\left(\frac{r}{s}\right) (g \partial_r \bar{w}^\varepsilon)(s) ds = (A + \varepsilon B)g,$$

where B is another linear operator in $L^2(I)^2$. Therefore, the Rayleigh stability equation (5.7) can be rewritten in the rescaled variables as

$$(A^\varepsilon - z^\varepsilon)h^\varepsilon = (A - z)h + \varepsilon(A - z)g + \varepsilon(B - y)h + \varepsilon^2(B - y)g = 0.$$

Notice that the zeroth-order term vanishes by (5.8). Since $\varepsilon > 0$, we want to find (g, y) satisfying

$$(A - z)g = (y - B)h + \varepsilon(y - B)g. \quad (5.9)$$

The first step consists of “inverting” the operator $(A - z)$. Unfortunately, since z, z^* are the eigenvalues of A , the kernel and the image of $(A - z)$ are given by $\text{Ker}(A - z) = \text{span}(h)$ and $\text{Im}(A - z) = \text{span}(h^*)$. We bypass this obstacle by choosing y in such a way that the right-hand side of (5.9) becomes “parallel” to h^* . This allows us to write (5.9) in the form

$$(g, y) = H + \varepsilon R,$$

where H depends only on the (h, z) solving (5.8), while the remainder R also depends on (g, y) . Thus, there is an explicit solution for $\varepsilon = 0$, and for small enough $\varepsilon > 0$, a fixed point argument in $L^2(I)^2 \times \mathbb{C}$ yields a solution as well. Once $\varepsilon > 0$ is fixed, Theorem 5.2 holds by taking

$$\bar{\Omega} = \bar{\omega}^\varepsilon.$$

From now on, the smooth vortex $\bar{\Omega}$ is fixed, and thus we will omit ε for the sake of simplicity. The use of capital letters will become clear in the following step.

5.2. Self-similar instability

Let us start by writing the Euler equation in self-similar coordinates

$$\tau = \frac{1}{ab} \log t, \quad \mathbf{X} = \frac{\mathbf{x}}{(abt)^{\frac{1}{a}}},$$

in terms of two parameters $0 < a, b \leq 1$ to be determined.

It is straightforward to check that (ω, f) , given by the change of variables

$$\omega(t, \mathbf{x}) = \frac{1}{abt} \Omega(\tau, \mathbf{X}), \quad f(t, \mathbf{x}) = \frac{1}{(abt)^2} F(\tau, \mathbf{X}),$$

is a solution to the Euler equation if and only if (Ω, F) solves the *self-similar Euler equation*

$$\partial_\tau \Omega + \mathbf{V} \cdot \nabla \Omega - bS\Omega = F, \quad (5.10)$$

where $S = S_a$ is the linear operator defined by

$$S\Omega = (a + \mathbf{X} \cdot \nabla)\Omega. \quad (5.11)$$

Notice that (5.10) agrees with the 2D Euler equation but for the extra term $bS\Omega$, which vanishes (formally) as $b \rightarrow 0$.

We take the background solution as the time-dependent vortex

$$\omega_0(t, \mathbf{x}) = \frac{1}{abt} \bar{\Omega}(\mathbf{X}), \quad \mathbf{u}_0(t, \mathbf{x}) = (abt)^{\frac{1}{a}-1} \bar{\mathbf{V}}(\mathbf{X}), \quad (5.12)$$

where $\bar{\Omega}$ is the unstable vortex from Theorem 5.2. Given $2 < p < \infty$, we fix the parameter a in the regime

$$0 < ap < 2,$$

to guarantee that the integrability condition (1.11) is satisfied. Indeed, we have the scaling

$$\|\omega_0(t)\|_{L^p} = (abt)^{\frac{2}{ap}-1} \|\bar{\Omega}\|_{L^p}, \quad \|\mathbf{u}_0(t)\|_{L^2} = (abt)^{\frac{2}{a}-1} \|\bar{\mathbf{V}}\|_{L^2}. \quad (5.13)$$

The force is defined *ad hoc* in such a way that (5.12) becomes a solution. By the radial symmetry of the vortex (and defining the pressure properly), the quadratic term in the Euler equation vanishes, i.e., $\bar{\mathbf{V}} \cdot \nabla \bar{\Omega} = 0$. Thus, the force is defined by

$$\partial_t \omega_0 = f, \quad \partial_t \mathbf{u}_0 = \mathbf{g},$$

or, equivalently,

$$F(\mathbf{X}) = -bS\bar{\Omega}, \quad \mathbf{G}(\mathbf{X}) = b(a - S)\bar{\mathbf{V}}.$$

Observe that $(\bar{\Omega}, \bar{\mathbf{V}}, F, \mathbf{G})$ are supported on $\{|\mathbf{X}| \leq r_2 + \varepsilon\}$, and that the integrability condition (1.10) is also satisfied

$$\int_0^t \|f(s)\|_{L^p} ds = \frac{(abt)^{\frac{2}{ap}-1}}{ab(\frac{2}{ap}-1)} \|F\|_{L^p}, \quad \int_0^t \|\mathbf{g}(s)\|_{L^2} ds = \frac{(abt)^{\frac{2}{a}-1}}{ab(\frac{2}{a}-1)} \|G\|_{L^2}. \quad (5.14)$$

Analogously to Step 5.1, the deviation Ω of a close but different solution $\bar{\Omega} + \varepsilon\Omega$ to the self-similar Euler equation must satisfy

$$(\partial_\tau - \mathcal{L}_b)\Omega + \varepsilon(\mathbf{V} \cdot \nabla \Omega) = 0, \quad (5.15)$$

where $\mathcal{L}_b = \mathcal{L}_{b, \bar{\Omega}}$ is the linearization of the self-similar Euler equation (5.10) around $\bar{\Omega}$

$$\mathcal{L}_b = \mathcal{L} + bS \quad (5.16)$$

for S defined in (5.11). The unstable solutions in the limit $\varepsilon \rightarrow 0$ are governed by exponentially growing solutions to the linearized problem, which we denote by

$$\Omega^{\text{lin}}(\tau, \mathbf{X}) = e^{\lambda\tau} W(\mathbf{X}). \quad (5.17)$$

Hence, we need to understand the corresponding eigenvalue problem

$$\mathcal{L}_b W = \lambda W, \quad (5.18)$$

this time for the linear operator $\mathcal{L}_b = \mathcal{L}_{b, \bar{\Omega}}$.

Definition 5.3. We say that the vortex $\bar{\Omega}$ is *self-similarly unstable* if, for some $n \in \mathbb{N}$, there exists $0 \neq W \in U_n$ satisfying (5.18) with $\lambda \in \mathbb{C}_+$.

Remarkably, M. Vishik showed that the spectrum $\sigma(\mathcal{L}_b) \cap \mathbb{C}_+$ converges (in a suitable sense) to $\sigma(\mathcal{L}) \cap \mathbb{C}_+$ as $b \rightarrow 0$. To this end, it is useful to decompose

$$\mathcal{L}_b = ab + T_b + K, \quad (5.19)$$

where

$$T_b \Omega = -(\bar{\mathbf{V}} - b\mathbf{X}) \cdot \nabla \Omega, \quad K \Omega = -\mathbf{V} \cdot \nabla \bar{\Omega},$$

with $\mathbf{V} = \nabla^\perp \Delta^{-1} \Omega$. Notice that the first term only shifts the spectrum by ab . The transport operator T_b is invertible in $\mathbb{C}_+$. Since K is compact, the operator theory allows us to conclude that $\sigma(\mathcal{L}_b) \cap \mathbb{C}_+$ consists of isolated eigenvalues. Since the dependence on b is continuous, there is hope to find an eigenvalue $\lambda_b \in \sigma(\mathcal{L}_b) \cap \mathbb{C}_+$ close to $\lambda \in \sigma(\mathcal{L}) \cap \mathbb{C}_+$. Actually, the following result can be proven.

Theorem 5.4. *The vortex $\bar{\Omega}$ described in Theorem 5.2 is self-similarly unstable for some $b > 0$. Moreover, the corresponding eigenfunction satisfies $W \in C_c^2(\mathbb{R}^2)$.*

5.3. Nonlinear instability

The last step for proving Theorem 1.2 requires controlling the nonlinear effects. We decompose the deviation into

$$\Omega = \Omega_{\text{lin}} + \varepsilon \Omega_{\text{cor}},$$

where the correcting term Ω^{cor} must satisfy the equation

$$(\partial_\tau - \mathcal{L}_b)\Omega_{\text{cor}} + \underbrace{(\mathbf{V}_{\text{lin}} + \varepsilon \mathbf{V}_{\text{cor}}) \cdot \nabla (\Omega_{\text{lin}} + \varepsilon \Omega_{\text{cor}})}_{\mathcal{F}} = 0, \quad (5.20)$$

coupled with the initial condition

$$\Omega_{\text{cor}}|_{\tau=-\infty} = 0. \quad (5.21)$$

Recall that $\Omega_{\text{lin}} = \Re(e^{\lambda\tau}W)$, while $\mathbf{V}_{\text{lin}}$, $\mathbf{V}_{\text{cor}}$ are the velocities associated to Ω_{lin} , Ω_{cor} , respectively. If we interpret $\mathcal{F}$ as a forcing term, since the linear part decays as $e^{\Re\lambda\tau}$ and the contribution of the quadratic term is expected to be negligible for short times, one can naively expect to gain a slightly faster exponential decay by exploiting the Duhamel formula. We can prove the following result.

Theorem 5.5. *There exists a solution Ω_{cor} to (5.20) and (5.21) satisfying*

$$\|\Omega_{\text{cor}}(\tau)\|_{L^p} = o(e^{\Re(\lambda)\tau}), \quad (5.22)$$

as $\tau \rightarrow -\infty$.

We conclude this section by showing how Theorems 5.4 and 5.5 allow proving Theorem 1.2. Coming back to the Eulerian coordinates, Theorems 5.4 and 5.5 imply that the vorticity

$$\omega_\varepsilon = \omega_0 + \varepsilon \omega_{\text{lin}} + \varepsilon^2 \omega_{\text{cor}}$$

satisfies that $\omega_\varepsilon|_{t=0} = 0$, and it is a solution to the Euler equation. We just need to show that for different ε 's in we have different solutions. We recall that

$$\omega_0(t, \mathbf{x}) = \frac{1}{abt} \bar{\Omega}(\mathbf{X}), \quad \omega_{\text{lin}}(t, \mathbf{x}) = \frac{1}{abt} \Omega_{\text{lin}}(\tau, \mathbf{X}), \quad \omega_{\text{cor}}(t, \mathbf{x}) = \frac{1}{abt} \Omega_{\text{cor}}(\tau, \mathbf{X}),$$

where we have applied the convention that capital letters and τ stand for self-similar variables. Thus, by applying the scaling

$$\|\omega_{\text{lin}}(t)\|_{L^p} = (abt)^{\frac{2}{ap}-1} \|\Re(e^{\lambda\tau}W)\|_{L^p}, \quad \|\omega_{\text{cor}}(t)\|_{L^p} = (abt)^{\frac{2}{ap}-1} o(e^{\Re\lambda\tau}),$$

and the reverse triangle inequality, we obtain

$$(abt)^{1-\frac{2}{ap}} e^{-\Re \lambda \tau} \|(\omega_\varepsilon - \omega_{\bar{\varepsilon}})(t)\|_{L^p} \geq |\varepsilon - \bar{\varepsilon}| (\|\Re(e^{i\Im \lambda \tau} W)\|_{L^p} - o(1)).$$

If $\Im \lambda \neq 0$, the right-hand side is positive in a sequence of times $\tau_k = \tau_0 - \frac{2\pi k}{\Im \lambda} \rightarrow -\infty$ as $k \rightarrow \infty$. If $\Im \lambda = 0$, we can assume from the beginning that W is real valued. Otherwise, it would suffice to take its imaginary part instead. This allows us to conclude that $\omega_\varepsilon \neq \omega_{\bar{\varepsilon}}$ whenever $\varepsilon \neq \bar{\varepsilon}$. Finally, it can be proven that the velocities $\mathbf{u}_\varepsilon = \nabla^\perp \Delta^{-1} \omega_\varepsilon$ belong to $L_t^\infty L^2$.

6. Extension to α -gSQG, with $0 < \alpha \leq 1$

An advantage of our construction of an unstable vortex for 2D Euler is that it can be adapted to construct unstable vortices for α -gSQG, $0 < \alpha \leq 1$. In fact, Step 5.1.1 works almost the same for the whole range $0 \leq \alpha \leq 1$. In this case, we have to consider the equation

$$\frac{1}{r} \int_0^\infty \left(I_{n,\alpha} \left(\frac{r}{s} \right) h(s) - I_{1,\alpha} \left(\frac{r}{s} \right) h(r) \right) \partial_r \bar{\theta}(s) s^{1-\alpha} ds = zh(r), \quad r \in \text{supp}(\partial_r \bar{\theta}), \quad (6.1)$$

instead of (5.7). The main difference in the analysis, when we take θ as a step function, is that the kernels $I_{n,\alpha}$ are not explicit unlike the kernel K_n in the case $\alpha = 0$. In addition, for the special case $\alpha = 1$, one has to take care of a cancellation to obtain a suitable analog of equation (5.8). In all the cases, $0 \leq \alpha \leq 1$, equation (5.8) can be analyzed to get unstable eigenvalues.

Step 5.1.2 also works in the case $0 \leq \alpha \leq 1$. Nevertheless, for $0 < \alpha \leq 1$, some extra effort is needed because of the lack of explicitness of the kernels. The main difference arises in the case $\alpha = 1$. Here, in addition to the complexity of the kernels, one has to deal with the loss of a factor $\log(\varepsilon)$ in the asymptotic expansion of equation (6.1) (with $\partial_r \bar{\theta}$ given by the regularized profile).

Step 5.2 can be adapted as well to the case $0 < \alpha < 1$. This is mainly because the operator K in (5.19) is still compact in this case. However, for Step 5.3, one needs to find new energy estimates to include the nonlinear interactions. In this issue, the degree of regularity of the velocity plays a dramatic role.

Both Steps 5.2 and 5.3 are much more involved in the case $\alpha = 1$. For example, the operator K in (5.19) is not compact anymore. In addition, the estimates we yield for $\alpha < 1$ to include the nonlinear part degenerate at $\alpha = 1$. In order to solve both problems, one needs to take into account some skew-symmetric properties of the velocity which are not needed for $\alpha < 1$.

At the end of the day, these ideas are able to produce non-uniqueness for forced α -gSQG, with $0 \leq \alpha \leq 1$, in the space H^s , $s < 1 + \alpha$. We remark again that well-posedness in H^s , with $s > 1 + \alpha$, $0 < \alpha < 1$, was proven in [23] and [31] for $\alpha = 1$. The precise statements of the theorems and proofs will be published in a forthcoming paper.

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