

Weak mixing in polygonal billiards

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Abstract. We survey known and conjectural ergodic properties of billiards in rational and non-rational polygons. The directional flows of billiards in rational polygons were proved to be ergodic in almost all directions by Kerckhoff, Masur and Smillie in 1986. In the same paper, the authors derived the existence of a G_δ dense set of ergodic (non-rational) polygons. In recent joint work, we have proved analogous results for the weak mixing property. In joint work with Jon Chaika, we have proved the existence of a G_δ dense set of weakly mixing (non-rational) polygons and recently with Francisco Arana-Herrera and Jon Chaika, we have characterized the property of weak mixing in almost all directions of translation surfaces with applications to billiards in rational polygons. All these results are based on techniques of Teichmüller dynamics and their application to the weak mixing problem following the joint work with A. Avila from 2007. Recent advances rest on measure rigidity results of Eskin, Mirzakhani and Mohammadi and related work of Chaika and Eskin on the multiplicative ergodic theorem in Teichmüller dynamics and results of Filip and Wright on the relevant (Kontsevich–Zorich) renormalization cocycle.

1. Introduction

Dynamical systems have long been classified as *stable* or *random*, as in the title of the classical book [40], depending on whether nearby orbits tend to stay close to each other or on the contrary diverge exponentially fast with time. Systems in the first class, which are the domain of KAM theory and variational methods (weak KAM theory, Aubry–Mather theory), but also of exotic Liouvillean behavior, as in examples constructed by the Anosov–Katok approximation by conjugation [18], are often called *elliptic*, while systems in the second class are called *hyperbolic*. Hyperbolic systems are described by a well-developed theory, which includes uniformly hyperbolic (Anosov) systems, non-uniformly hyperbolic systems (Pesin’s theory), partially hyperbolic systems, dominated splitting and so on.

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Dynamical systems with intermediate behavior, characterized, for instance, by polynomial divergence with nearby orbits, often called *parabolic*, are less understood, with the exception of homogeneous unipotent flows (the domain of Ratner's theory).

The general investigation of parabolic dynamical phenomena has come to be considered an important part of the theory of dynamical systems, especially following the vision of A. Katok (see, for instance, [29]).

Planar billiards have played an important role in the development of the theory at least since the work of G. D. Birkhoff, perhaps that of a “playground”, an expression that “should be understood as ‘testing area’: various questions, conjectures, methods of solution, etc. in the theory of dynamical systems are ‘tested’ on various types of billiard problems” [34].

Thus, smooth convex (Birkhoff) billiards (Figure 1, left) are a classical example of elliptic, stable, dynamical behavior, while billiards with strictly convex scatterers (Sinai's billiards) and those with focusing-defocusing mechanism, such as the famous Bunimovich stadium, are models of hyperbolic, random, dynamics, uniform and non-uniform.

Billiards in polygons (Figure 1, right) exemplify parabolic dynamics, although the (local) behavior of orbits is elliptic with singularities [34], as splitting of nearby trajectories only occurs at vertices of the polygon.

According to A. Katok, “George D. Birkhoff was the first to consider billiards systematically as models for problems of classical mechanics. Birkhoff considered billiards only in smooth convex domains” [34].

However, R. H. Fox and R. B. Kershner took up the study of geodesics on a rational polyhedron, since “the problem may be considered as an elementary illustration of the ‘billiard ball’ problem considered by Birkhoff in Chapter VI of his Colloquium Publication *Dynamical Systems* and was suggested to us by Wintner” [26].

The topological dynamics of this problem was investigated several decades later by A. Zemlyakov and A. Katok [48] in paper which brought billiards in polygons fully into the developing theory of dynamical systems.

In this lecture, we review ergodic theoretic properties of billiards in polygons up to recent results by the author and collaborators.

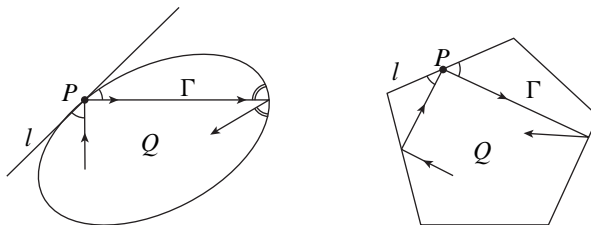


Figure 1. Convex (Birkhoff) and polygonal billiards.

2. Billiards in polygons

A *polygonal billiard* is the dynamical system given by the motion of a light ray in a polygonal (planar, Euclidean) mirror chamber P , or equivalently by the free motion of a single point mass in P with elastic collisions at the boundary. It is therefore given by straight linear motion inside the polygon and reflection at the boundary according to the law of reflection *angle of incidence = angle of reflection*.

Its (3-dimensional) phase space, after the reduction given by the (kinetic) energy integral, can be identified with the space $P \times \mathbb{T}/E$, with E expressing the reflection equivalence relation at ∂P . Indeed, the configuration space of the billiard is the polygon P , and since the kinetic energy is an integral of motion, it can be normalized. Since trajectories which hit a corner are not continued, the billiard flow is defined almost everywhere (on the complement of countably many hypersurfaces).

For background on billiards in polygons, we refer the reader to the survey by H. Masur and S. Tabachnikov [39].

2.1. Mechanical systems

The motion of two point masses m_1, m_2 on the interval $[0, 1]$ with elastic collisions between them and at the endpoints is equivalent to a billiard in a right triangle with an angle equal to $\arctan(\sqrt{m_1/m_2})$.

The motion of three point masses m_1, m_2, m_3 on a circle with elastic collisions between them is equivalent to a billiard in an acute triangle with angles

$$\arctan\left(m_i \sqrt{\frac{m_1 + m_2 + m_3}{m_1 m_2 m_3}}\right) \quad \text{for } i = 1, 2, 3.$$

This lecture reviews results and conjectures on the following fundamental question.

Question 2.1. How chaotic can a polygonal billiard be?

This question is motivated within the theory of dynamical systems by the consideration that billiards in polygons are a basic example of a parabolic system (or elliptic with singularities). The question is of interest also from the point of view of the semi-classical approach to quantum mechanics as billiards in polygons are an example of “weakly chaotic” classical dynamical system [42].

A similar, more general question concerns geodesic flow on flat compact orientable surfaces with conical singularities. In fact, the double of a Euclidean polygon is a flat sphere with conical singularities at the vertices of the polygon.

2.2. Rational versus non-rational polygons

There is a well-known fundamental dichotomy between billiards in rational and non-rational polygons.

Definition 2.2. A polygon $P \subset \mathbb{R}^2$ is called *rational* if the group $G_P \subset \mathrm{SO}(2)$ generated by reflections with respect to the edges of P is finite. Whenever G_P is infinite, the polygon P is called *non-rational*. If P is a simply connected polygon, then P is rational if and only if all of its angles are rational multiples of the round angle.

In the rational case, the map $\Theta : P \times \mathbb{T} \rightarrow \mathbb{T}/G_P$, defined as

$$\Theta : (x, v) \rightarrow [v]_{G_P} \quad \text{for all } (x, v) \in P \times \mathbb{T},$$

is a prime integral of motion, that is, a function invariant along orbits of the billiard flow, with values in the interval $\mathbb{T}/G_P$ (note that when G_P is infinite, the quotient $\mathbb{T}/G_P$ is not Hausdorff).

As a consequence, the billiard flow leaves invariant the level sets of the prime integral Θ , hence the phase space of a rational billiard is foliated by invariant surfaces. The invariant surfaces can be constructed either as an appropriate branched cover of the double of the polygon, which in the simply connected case is a flat sphere with conical points of rational angle [26] or by the so-called “unfolding construction” [48].

The invariant surfaces are *translation surfaces* since they inherit from the Euclidean plane a flat *translation* structure, that is, an atlas whose coordinate changes are given by translations of $\mathbb{R}^2$. The restrictions of the billiard flow to the invariant surfaces are linear flows called *translation flows*. Thus, the ergodic theory of rational billiards is part of that of translation flows.

The billiard flow of a rational polygon cannot even be topologically transitive, not to mention ergodic, on its (3-dimensional) phase space. The simplest meaningful ergodic theoretical questions therefore concern the restriction of the billiard flow to the random invariant surface (in the random directions) with respect to the Lebesgue measure on the angle.

The genus of the invariant surface can be computed explicitly by the Euler formula (see [39]). For any rational polygon P with $\sigma \geq 3$ edges and angles of amplitudes $(m_1\pi/n_1, \dots, m_\sigma\pi/n_\sigma)$, the genus g_P of the invariant surfaces is given by the formula

$$g_P = 1 + \frac{\#G_P}{4} \left(\sigma - 2 - \sum_{i=1}^{\sigma} \frac{1}{n_i} \right).$$

From this formula, it follows that the invariant surfaces are tori ($g_P = 1$) only for the following *completely integrable polygons* (listed in terms of their angles):

$$\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2} \right), \quad \left(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4} \right), \quad \left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3} \right), \quad \left(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6} \right),$$

that is, for rectangles, isosceles right triangles, equilateral triangles and half-equilateral right triangles.

In recent decades the mathematical theory of generic translation flows (and of locally Hamiltonian flows on surfaces, which are singular time changes of translation flows) has developed tremendously, thanks to the discovery of *renormalization* dynamical systems for the dynamics of interval exchange transformation and translation flows: the *Rauzy–Veech–Zorich induction* algorithm [41, 43, 49] and H. Masur’s *Teichmüller geodesic flow* [38]. The former can be regarded as a symbolic version of the latter, for an appropriate coding. Depending on the problem, each point of view has its own advantages; however, the connection with Teichmüller theory has brought in powerful tools from Hodge theory and algebraic geometry.

However, the theory of rational billiards has lagged behind that of generic translation flows, since translation surfaces coming from rational polygons (by unfolding) are very symmetric and therefore form a “small” exceptional set in the space of all translation surfaces. The renormalization theory cannot be directly applied, but in principle dynamical properties of rational billiards can be studied, thanks to the magic wand theorem of Eskin, Mirzakhani and Mohammadi [16, 17] and related results on the Lyapunov spectrum of the relevant renormalization cocycles, the *Rauzy–Veech cocycle* over Veech’s “zippered rectangles” flow [41, 43] or its continuous time version, the *Kontsevich–Zorich cocycle* over the Teichmüller flow [13, 20–23, 36].

In contrast to the rational case, no renormalization dynamics is known for billiards in non-rational polygons. As a consequence, the mathematical theory of such billiards is much less developed. Most of the results on billiards in non-rational polygons are proved by fast approximation methods, which leverage results on translation flows and on rational polygonal billiards.

Indeed, the dichotomy between rational and non-rational billiards can be viewed as an instance of a more general dichotomy in the study of parabolic dynamical systems between “renormalizable” and “non-renormalizable”: the simplest example of a renormalizable flow is a flow along the full unstable, often called “horospherical”, foliation of a (partially) hyperbolic system which is a special property not verified by most parabolic flows, not even in some approximate sense.

Natural conjectures on the dynamics of non-renormalizable parabolic flows are often beyond the reach of current methods. For instance, in homogeneous dynamics, it is well known that the effective equidistribution of non-horospherical unipotent flows is much harder than that in the horospherical case (see, for instance, [37]). However, for the flow of a non-rational billiard, no renormalization approach is in sight, as there is no known hyperbolic system rescaling its orbits.

3. Dynamical properties of billiards in polygons

All billiards in polygons have *zero entropy*, as no hyperbolicity is present and the divergence of nearby orbits is subexponential [32]. In fact, nearby orbits can only diverge at corners, and the system is in fact elliptic (no-divergence) as long as the trajectories stay away from corners of the polygon.

3.1. Rational polygons

The billiard flow in a rational polygon has the following dynamical properties:

- it is *uniquely ergodic* on almost all invariant surfaces [35], with polynomial speed of ergodicity [4];
- it is *never mixing*, that is, it is not mixing on any invariant surface [31].

All of the above results are in fact proved for ALL translation surfaces. Additional results have been proved for generic translation surfaces; for instance,

- the translation flow in a fixed direction is *weakly mixing* for almost all translation surfaces with respect to the canonical absolutely continuous (Masur–Veech) measure on the relevant stratum of translation surfaces [6].

The unique ergodicity and weak mixing results can be refined with estimates on the Hausdorff dimension of the set of non-uniquely ergodic directions (see [15] and references therein for unique ergodicity and [8] for weak mixing).

In recent work with F. Arana-Herrera and J. Chaika, we have been able to completely classify rational polygons whose billiard flow is weakly mixing on almost all invariant surfaces.

Definition 3.1 ([27]). A polygon is *almost integrable* if its boundary is drawn on the planar grid obtained from a completely integrable polygon under the subgroup of the group of isometries of the plane generated by reflections with respect to its edges.

Our results prove in a strong form a conjecture of E. Gutkin [27].

Theorem 3.2 ([2]). *A billiard in a rational polygon is weakly mixing on almost all invariant surfaces (in almost all directions), unless the polygon is almost integrable or it is the rotation of a polygon with only vertical and horizontal sides such that the horizontal sides or the vertical sides have commensurable lengths.*

In light of A. Katok non-mixing result quoted above, our theorem completes the description of the weak mixing property in the random direction for rational polygonal billiards. For the reasons explained above, there were so far few results on weak mixing in rational polygons. A. Katok and E. Gutkin [28] proved the weak mixing

property in a given direction for topologically generic billiard tables in certain classes of polygons. More recently, A. Avila and V. Delecroix [5] proved a result for translation surfaces which implies weak mixing in almost all directions for regular polygons with at least five edges and other special tables (unfolding to non-arithmetic Veech surfaces).

The above theorem is derived by a standard argument from the following result for translation surfaces. We recall that a translation surface can be defined as a pair (X, ω) of a Riemann surface and a holomorphic differential ω on X . We have the following theorem.

Theorem 3.3 ([2]). *For any translation surface (X, ω) , the following properties are equivalent:*

- (1) *The directional flow of (X, ω) is weakly mixing in almost all directions.*
- (2) *The directional flow of (X, ω) is weakly mixing in some direction.*
- (3) *The surface (X, ω) does not have an affine circle factor.*
- (4) *The (tautological) plane $T(\omega) = \mathbb{R} \cdot \operatorname{Re}(\omega) + \mathbb{R} \cdot \operatorname{Im}(\omega) \subset H^1(X, \mathbb{R})$ has trivial intersection with the lattice $H^1(X, \mathbb{Z})$.*

The core of the theorem is (4) $\Rightarrow$ (1).

We note that, as weak mixing is equivalent to the existence of a measurable circle factor, by the above theorem, the existence of a measurable circle factor in a positive measure set of directions implies the existence of an affine circle factor.

As mentioned above, the conclusion of the theorem was known in the special case of Veech translation surfaces [5]. D. Auricino, A. Avila and V. Delecroix have announced a generalization to any surface in a rank one suborbifold, which includes countably many L-shaped billiards with a barrier.

3.2. Non-rational polygons

The mathematical theory of billiards in non-rational polygons is much less advanced, and the main results are derived by approximation:

- The set of ergodic polygons is a dense G_δ [35].
- The set of *topologically* weakly mixing polygons is a dense G_δ [9].

An effective version of the above ergodicity result of Kerckhoff, Masur and Smillie [35], with an estimate on the speed of approximation of the angles by rationals sufficient for ergodicity, has been proved by Y. Vorobets [45].

A few years ago, in a collaboration with J. Chaika, we proved that the weak mixing property of the billiard flow is topologically generic among polygons, a result first conjectured in the work of E. Gutkin and A. Katok [28].

Theorem 3.4 ([14]). *The set of weakly mixing polygonal billiards is a dense G_δ .*

The drawback of all of the above results is that ergodic properties are established for polygons with Liouvillean non-rational angles (that is, extremely well-approximable by rationals), which is a set of measure zero.

Conjecturally, polygons with angles of amplitudes equal to badly approximable (Diophantine) or random multiples of the round angle should exhibit stronger chaotic properties, but in the absence of a renormalization theory such conjectures remain for now out of reach.

Several conjectures can be drawn from numerical evidence in the mathematical physics literature. For instance, it is suggested that

- completely irrational acute triangles are mixing [12],
- typical right triangles may not be even ergodic [46], except in the Liouvillean case [30].

We are therefore led to formulate the following questions.

Question 3.5. The following questions are open:

- Is the non-rational billiards with random angles (with respect to the Lebesgue measure) ergodic? (see [33, Question 3])
- Does there exist a mixing polygonal billiard?
- Does there exist a relatively mixing translation surface/rational billiard?

4. Strategy and outline of the proofs

In the remainder of the paper, we outline the strategy to prove weak mixing for translation flows and interval exchange transformations (IETs) developed in [6, 8, 44] and the proofs of Theorem 3.4 from [14] and Theorem 3.3 from [2], emphasizing the novelties in the approach required in carrying out elimination of eigenvalues for an arbitrary given translation surface.

4.1. Weak mixing from eigenvalue elimination

We begin by recalling the definition of weak mixing and state some equivalent ergodic properties. A flow ϕ on (X, μ) is *weakly mixing* if

- (1) for all $f, g \in L^2(X, \mu)$ of zero average,

$$\frac{1}{T} \int_0^T |\langle f \circ \phi_t, g \rangle|^2 dt \rightarrow 0;$$

- (2) the Koopman operator on $L^2(X, \mu)$ has no non-constant eigenfunctions, no non-trivial atomic spectral measures; or
- (3) the flow $\phi \times \psi$ on $(X \times Y, \mu \otimes \nu)$ is ergodic for any ergodic flow ψ on (Y, ν) . It is enough to consider the set of all linear flows on $\mathbb{T}$; or
- (4) the square flow $\phi \times \phi$ on $(X \times X, \mu \otimes \mu)$ is ergodic.

In the proof of Theorem 3.4 on the existence of weakly mixing (non-rational) polygons, we derive *square ergodicity* (the equivalent property (4)), hence weak mixing, from the following.

Theorem 4.1 ([14]). *For every translation surface (X, ω) , for every $\alpha \neq 0$, the set of directions $\theta \in \mathbb{T}$ such that the directional flow ϕ^θ on (X, ω) has eigenvalue α has zero Lebesgue measure in $\mathbb{T}$.*

We note that it is crucial for the argument that the above result holds for all translation surfaces (indeed, it holds even for the genus 1 case of the flat torus).

We prove that for a dense G_δ set of polygons, the billiard flow is square ergodic by the following steps (in reverse order):

- (1) (Fast approximation) Square ergodicity for a G_δ -dense set of tables follows from ergodicity for rational tables on almost all products of invariant surfaces by a Baire category argument (as in [35], from a suggestion of M. Boshernitzan and A. Katok, applied in our work to the square dynamics.)
- (2) (Spectral theory of products) A product flow fails to be ergodic if and only if the factors have a common non-trivial eigenvalue. Thus, by Fubini $\phi^{\theta_1} \times \phi^{\theta_2}$ fails to be ergodic for a positive measure set of $(\theta_1, \theta_2) \in \mathbb{T}^2$ if and only if there exists α eigenvalue of ϕ^θ for a positive measure set of $\theta \in \mathbb{T}$.

In the proof of Theorem 3.2 on the weak mixing in almost all directions in translation surfaces, we prove that the Koopman operator has no non-constant eigenvalue with measurable eigenfunction (equivalent property (2)).

In summary, the proofs of Theorem 3.4 (in fact of Theorem 4.1) and of Theorem 3.3 are based on the *elimination of eigenvalues* of translation flows, more precisely of an arbitrary fixed eigenvalue in the proof of Theorem 3.4 [14] and of the full line of possible eigenvalues in the proof of Theorem 3.3.

The classical tool for the elimination of eigenvalues since W. Veech's work in the 1980s is the so-called Veech criterion [44]. This criterion has also been exploited later in [6, 8] and is sufficient for the elimination of a fixed eigenvalue as in Theorem 4.1. However, the elimination procedure presents important novelties compared with earlier works. The Veech criterion is, on the contrary, not sufficient for the elimination in the proof of Theorem 3.3. For this reason, we have introduced in [2], in the

proof of Theorem 3.3, a strengthened criterion, based on the general *partial rigidity* of translation flows [31].

4.2. The Veech criterion

We formulate the criterion in terms of the Kontsevich–Zorich (KZ) cocycle on the (real) Hodge bundle with fiber $H^1(X, \mathbb{R})$ at every translation surface (X, ω) in a stratum of the moduli space of Abelian differentials. Indeed, the moduli space of translation surfaces is stratified by fixing multiplicities of zeros of the holomorphic 1-form ω . Strata are determined by a vector $\kappa = (k_1, \dots, k_\sigma) \in \mathbb{N}^\sigma$ of multiplicities of the zeros, which satisfies the topological constraint $\sum_{i=1}^\sigma k_i = 2g - 2$ for surfaces of genus $g \geq 2$.

Let $\hat{\mathcal{H}}(\kappa)$ and $\mathcal{H}(\kappa) := \hat{\mathcal{H}}(\kappa)/\Gamma_g$ denote a stratum in the Teichmüller space, respectively, in the moduli space, of Abelian differentials on Riemann surfaces of genus $g \geq 2$ (the moduli space $\mathcal{H}(\kappa)$ is the quotient of the Teichmüller space $\hat{\mathcal{H}}(\kappa)$ by the action of the mapping class group Γ_g). There is a natural action of the group $\mathrm{SL}(2, \mathbb{R})$ on $\hat{\mathcal{H}}(\kappa)$ which commutes with that of Γ_g , hence a natural action of $\mathrm{SL}(2, \mathbb{R})$ on $\mathcal{H}(\kappa)$. This action can be defined by post-composition of charts of a translation atlas with elements of $\mathrm{SL}(2, \mathbb{R})$ as linear maps of $\mathbb{R}^2$ or equivalently for all $(X, \omega) \in \mathcal{H}_g^{(1)}$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \mathrm{Re}(\omega) \\ \mathrm{Im}(\omega) \end{pmatrix} = \begin{pmatrix} a \mathrm{Re}(\omega) + b \mathrm{Im}(\omega) \\ c \mathrm{Re}(\omega) + d \mathrm{Im}(\omega) \end{pmatrix} \quad \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}).$$

Definition 4.2. The *KZ cocycle* $\{g_t^{\mathrm{KZ}}\}$ is the restriction to the *Teichmüller flow* $\{g_t\} = \mathrm{diag}(e^t, e^{-t}) < \mathrm{SL}(2, \mathbb{R})$ of the projection to the bundle $(\hat{\mathcal{H}}(\kappa) \times H^1(\mathbb{R}))/\Gamma_g$ of the cocycle

$$\mathrm{SL}(2, \mathbb{R}) \times \mathrm{id} : \hat{\mathcal{H}}(\kappa) \times H^1(\mathbb{R}) \rightarrow \hat{\mathcal{H}}(\kappa) \times H^1(\mathbb{R}).$$

We can now state a weak mixing criterion for translation flows from [44].

Theorem 4.3 (Veech criterion). *If the real number α is an eigenvalue for the vertical flow ϕ_t on (X, ω) , then the following holds: for every compact set $K \subset \mathcal{H}(\kappa)$, there is $\ell_K > 0$ such that, if for some $\ell \geq \ell_K$ and a diverging sequence (t_n) , for all $n \in \mathbb{N}$, we have $g_{t_n-\ell}(\omega), g_{t_n}(\omega), g_{t_n+\ell}(\omega) \in K$, then*

$$\lim_{n \rightarrow +\infty} g_{t_n}^{\mathrm{KZ}}(\omega, \alpha \cdot \mathrm{Re}(\omega)) = 0 \quad \text{in } H^1(\mathbb{R})/H^1(\mathbb{Z}).$$

Recent developments concerning the Veech criterion include the proof, based on Hodge theory, of a slightly stronger version [25] and an effective Veech criterion to prove *effective weak mixing* (polynomial speed of convergence of Cesaro

averages of correlations) for translation flows [10, 11, 24] and interval exchange transformations [7].

Proof of the Veech criterion (outline). Let K be a fixed compact subset of the moduli space of Abelian differentials. It is enough to prove that for a given canonical basis $\mathcal{B}(t) := \{a_1(t), \dots, a_g(t), b_1(t), \dots, b_g(t)\}$ of the homology of the surface, composed of curves of bounded length for $g_t(\omega) \in K$, the periods

$$\int_{a_1(t)} \alpha \operatorname{Im}(\omega), \dots, \int_{a_g(t)} \alpha \operatorname{Im}(\omega), \int_{b_1(t)} \alpha \operatorname{Im}(\omega), \dots, \int_{b_g(t)} \alpha \operatorname{Im}(\omega)$$

are close to integers. By definition of the Teichmüller flow, there exists a constant $C_K > 0$ such that any loop $\gamma(t) \in \mathcal{B}(t)$ has horizontal length at least $C_K^{-1}e^t$ and vertical length at most $C_K e^{-t}$ on (X, ω) . Let us assume that all homology classes of loops in $\mathcal{B}(t)$ can be realized by paths composed of exactly one (long) vertical and one (short) horizontal segment. Let f denote an eigenfunction of eigenvalue $\alpha \in \mathbb{R} \setminus \{0\}$ for the vertical flow ϕ_s on (X, ω) . We can integrate the eigenvalue equation for f along the vertical segment $\gamma_v(t) = \{\phi_s(x) \mid 0 \leq t \leq L_v(t)\}$ of any path $\gamma(t) \in \mathcal{B}(t)$ to get

$$f \circ \phi_{L_v(t)}(x) = e^{2\pi i \alpha L_v(t)} f(x).$$

Since the distance $d(x, \phi_{L_v(t)}(x))$ on (X, ω) equals the length of vertical component of γ , and since f can be chosen with constant unit modulus, under the hypothesis that f is continuous, we have that, as $t \rightarrow +\infty$,

$$\left| \frac{f \circ \phi_{L_v(t)}(x)}{f(x)} - 1 \right| \rightarrow 0,$$

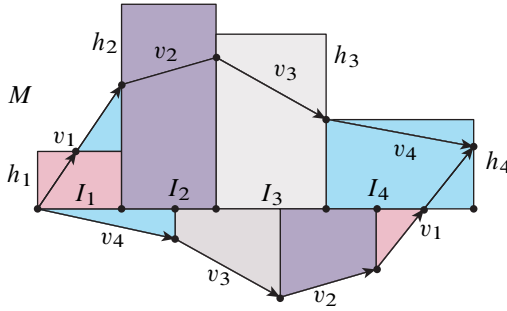
hence

$$d(\alpha L_v(t), \mathbb{Z}) = d\left(\int_{\gamma(t)} \alpha \operatorname{Im}(\omega), \mathbb{Z}\right) \rightarrow 0.$$

The argument is therefore complete in case there is a continuous eigenfunction of eigenvalue $\alpha \neq 0$. In general, the eigenfunction may be only measurable. In this case, in order to apply Luzin's theorem, which states that restrictions of the eigenfunction to sets with complements of arbitrarily small measure are continuous, we assume that the vertical segment of every loop $\gamma(t) \in \mathcal{B}(t)$ (in fact of a subset of $\mathcal{B}(t)$ which spans the homology of the surface) *belongs to a vertical rectangle $R_{\gamma(t)}$ of area uniformly bounded below*. The lower bound on the area of the rectangles ensures that Luzin sets with complement of sufficiently small measure intersect all of them, and therefore that it is possible to choose loops with vertical components starting at points of continuity of the eigenfunction. In the statement of the Veech criterion, the condition that the Teichmüller orbit has three consecutive returns (at times $t_n - \ell$, t_n and

$t_n + \ell$) to a compact set has precisely the role of enforcing a uniform equivalence of the area of all vertical rectangles $R_{\gamma(t)}$ in a decomposition of the surfaces, hence a uniform lower bound on the area of each of them.

In Figure 2, which illustrates the Veech criterion, the symbol $\mathcal{RV}$ denotes the Rauzy–Veech map, a discrete time version of the Teichmüller flow, and the sequence



Eigenfunction
equation

$$\frac{\partial u}{\partial y} = 2\pi i \alpha u$$

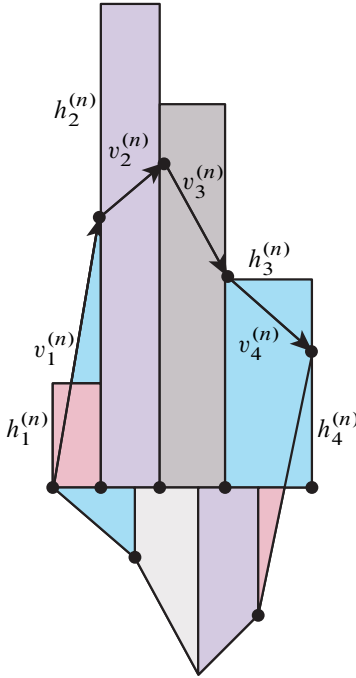
$$u \in L^2(M)$$

$$u(x, h_k) = e^{2\pi i \alpha h_k} u(x, 0) \quad \text{for } x \in I_k$$

Permutation: $(4 \ 3 \ 2 \ 1) = \sigma$

Horizontal data: $(\ell_1, \ell_2, \ell_3, \ell_4) = \ell$

Vertical data: $(h_1, h_2, h_3, h_4) = h \in H(\sigma) = \mathbb{R}^4$



Horizontal data: $(\ell_1^{(n)}, \ell_2^{(n)}, \ell_3^{(n)}, \ell_4^{(n)}) = \ell^{(n)}$

Vertical data: $(h_1^{(n)}, h_2^{(n)}, h_3^{(n)}, h_4^{(n)}) = h^{(n)}$

$(\ell^{(n)}, h^{(n)}) = (\mathcal{RV})^n(\ell, h)$

$$u(x, h_k^{(n)}) = e^{2\pi i \alpha h_k^{(n)}} u(x, 0) \quad \text{for } x \in I_k$$

$u(x, h_k^{(n)}) \approx u(x, 0)$ (Luzin's theorem)

$$\Rightarrow \alpha h^{(n)} = \alpha A_n(\ell)(h)$$

close to $\mathbb{Z}^4 \subset \mathbb{R}^4$

Figure 2. The Veech criterion.

of matrices $(A_n(\ell)(h))$ represents the Rauzy–Veech cocycle, a discrete version of the KZ cocycle, along the orbit $(\mathcal{RV}^n(\ell, h))$. We are neglecting to keep track of the permutation along the orbit of the Rauzy–Veech map. ■

4.3. Elimination of eigenvalues

The Veech criterion leads to the notion of a *weak stable space* $W_\omega \subset H^1(X, \mathbb{R})$ at almost all (X, ω) , as the set of all cohomology classes such that the conclusion of the Veech criterion holds, that is, the set of all $c \in H^1(X, \mathbb{R})$ such that $g_t^{\text{KZ}}(\omega, c)$ “converges” to $H^1(\mathbb{Z})$. The Veech criterion implies that the vertical flow on a translation surface $(X, \omega) \in \mathcal{H}$ is weakly mixing if $\mathbb{R}[\text{Re}(\omega)] \cap W_\omega = \{0\}$.

We list below the basic elementary properties of the weak stable space:

- (1) W_ω is saturated by subspaces parallel to the stable space of the Kontsevich–Zorich cocycle on $H^1(\mathbb{R})$.
- (2) W_ω coincides with the union of integer translates of the stable space if the forward Teichmüller orbit is bounded.
- (3) W_ω only depends on $\text{Im}(\omega)$.

A couple of elimination schemes were introduced in [6]: the first, called *linear elimination*, is carried out on the heights parameters (first return times) of translation flows and is sufficient to prove that almost all translation flows are weakly mixing (but not for the proof of weak mixing for almost all IETs); the second, called *probabilistic elimination*, is carried out on the transverse parameters of a translation flows for a fixed vector of heights parameters, and it is adapted to the case of IETs. In fact, for IETs on d intervals, the vector of heights can be taken to be $(1, \dots, 1) \in \mathbb{R}^d$.

4.3.1. Linear elimination. The linear elimination is based on a Hausdorff dimension bound from [6], which can be stated as follows.

Lemma 4.4. *For almost all forward Oseledets regular points (X, ω) with respect to the Kontsevich–Zorich cocycle, we have the following inequality for the Hausdorff codimension of the weak stable space:*

$$\text{H-codim}(W_\omega) \geq \#\{\text{positive KZ exponents}\}.$$

The above result also holds for the effective weak stable space [11, 24].

For *Veech surfaces*, the weak stable space W_ω is the union of integer translates of the Oseledets stable space [5]. Avila, Aulicino and Delecroix have announced a generalization of this result to all rank one $\text{SL}(2, \mathbb{R})$ -invariant suborbifolds.

For any $\text{SL}(2, \mathbb{R})$ -invariant suborbifold $\mathcal{M}$, the number of positive KZ exponents in $T\mathcal{M} \subset H^1(X, \Sigma, \mathbb{R})$ is equal to the rank of $\mathcal{M}$ [19], hence when $\text{rank}(\mathcal{M}) \geq 2$ weak mixing of the vertical flow holds for almost all translation surfaces in $\mathcal{M}$ (with respect

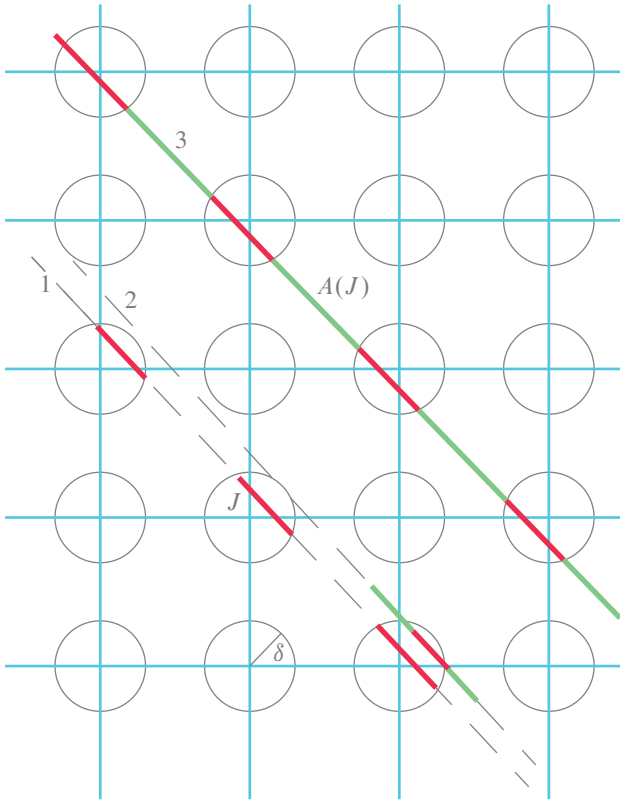


Figure 3. Repulsion and spreading.

to the canonical $\mathrm{SL}(2, \mathbb{R})$ -invariant probability measure) since W_ω only depends on $\mathrm{Re}(\omega)$ and the vector of heights (return times) of the vertical flow is $\mathrm{Im}(\omega)$ (and both vector belong to the projection of $T\mathcal{M}$ in $H^1(X, \mathbb{C})$). In fact, by dimensional reasons, we have that $\mathbb{R}[\mathrm{Im}(\omega)] \cap W_\omega = \{0\}$ for almost all $\mathrm{Im}(\omega)$.

4.3.2. Probabilistic elimination. In this elimination scheme, introduced in [6], the vector $\mathbb{R}[\mathrm{Re}(\omega)]$ is fixed. The scheme analyzes the dynamics on the bundle $H^1(\mathbb{R})$ modulo $H^1(\mathbb{Z})$ with toral fibers over strata of the moduli space of line segments approximately aligned with the top unstable Oseledets subspace.

There are two main effects (see Figure 3):

- *Repulsion:* under the hypothesis that the second Lyapunov exponent of the KZ cocycle is strictly positive ($\lambda_2 > 0$), line segments are repelled away from the nearest integer point (under the action of the KZ cocycle).

- *Spreading*: for sufficiently long excursions of the Teichmüller orbit, a line segment can get dilated (under the action of the KZ cocycle) and spread to several line segments near multiple nearby integer points.

We note that the phenomenon of “spreading” is a priori compatible with the Veech criterion, since the latter does not constrain the orbit of the cohomology class under the Kontsevich–Zorich cocycle when the Teichmüller orbit is outside of a given compact subset of a stratum.

With sufficiently high probability, a line segment is repelled “far enough” from integer points and eliminated, since by the Veech criterion its points cannot be eigenvalues (for the vertical flow). It is possible to estimate that the probability of escaping elimination by spreading goes to zero with time [6]. A version of the probabilistic elimination for effective weak mixing of IETs is carried out in [7].

The probabilistic elimination is well adapted to the proof of (effective) weak mixing for directional flows of almost all translation surfaces with respect to the Masur–Veech measure on strata [6, 7].

4.4. Weakly mixing polygonal billiards

We recall that proof of density of weakly mixing (non-rational) polygons is based on the elimination *for a given translation surface* of a single (fixed) eigenvalue, that is, on the proof of Theorem 4.1. The elimination is carried out separately for each irreducible subbundle of the Hodge (real cohomology) bundle:

- For the “tautological” plane $T_\omega = \mathbb{R}[\operatorname{Re}(\omega)] + \mathbb{R}[\operatorname{Im}(\omega)]$: an elementary transversality argument to eliminate intersections with the stable subspace $\mathbb{R}[\operatorname{Im}(\omega)]$.
- For isometric subbundles: Veech’s isometric elimination from [44].
- For “hyperbolic” (at least one positive exponent) subbundles: repulsion without spreading for a positive measure set of directions (for any given surface).

We note that, for the KZ cocycle, irreducible subbundles are either hyperbolic or isometric, never “parabolic” [19] (in the sense that either there is exponential growth of typical vectors or there is no growth at all).

4.4.1. Controlled hyperbolic elimination. The crucial novelty in the above elimination scheme is a controlled repulsion which achieves elimination (again based on the Veech criterion) before spreading occurs. The essential technical ingredient is the following large deviation result.

Theorem 4.5 ([14]). *For any translation surface (X, ω) , there exist $\lambda > 0$ and $\delta > 0$ such that (along the Teichmüller horocycle orbit segment $\{h_s\omega \mid s \in [-1, 1]\}$) for large $T > 0$ and for $c \in H^1(X, \mathbb{R})$, we have*

$$\operatorname{Leb}\{s \in [-1, 1] \mid |g_T^{\operatorname{KZ}}(h_s\omega, c)| < e^{\lambda T}\} < e^{-\delta T}.$$

This result both improves upon and complements large deviation estimates of [1, 5, 8]. It is based on the results of Eskin and Mirzakhani [16] and Eskin, Mirzakhani and Mohammadi [17], on the results of Chaika and Eskin on the Oseledets theorem in almost all directions for arbitrary surfaces [13].

By the recurrence result of J. Athreya [3], for a fixed surface, the probability of an excursion outside a given compact set $\mathcal{K}$ of a stratum is exponentially small with the length of the excursion: there exists $\eta > 0$ such that, for any $S > 0$,

$$\text{Leb}\{s \in [-1, 1] \mid U_{t \in [S, S+T]} g_t(h_s \omega) \cap \mathcal{K} = \emptyset\} < e^{-\eta T}.$$

From the above estimates, we derive that if at some time we have a cohomology class $c \in H^1(X, \mathbb{R})$ such that $d(c, H^1(\mathbb{Z})) \approx \gamma$ (sufficiently small), then there exist an integer class $z \in H^1(X, \mathbb{Z})$ and a time $t := t(s)$ within $[-\log \gamma/2, -3 \log \gamma/4]$ such that, outside a set of s of measure γ^σ ($\sigma > 0$),

$$g_t(h_s \omega) \in \mathcal{K} \quad \text{and} \quad d(g_t^{\text{KZ}}(h_s \omega, c), H^1(\mathbb{Z})) \approx \gamma^{-\lambda} \|c - z\| = d(c, H^1(\mathbb{Z}))^{1-\lambda}.$$

Iteration of this argument produces controlled repulsion away from the integer lattice outside a set of “small” measure (for γ small). Indeed, the norm of the cohomology class $c - z$ (hence the distance of the cohomology class from the integer lattice) grows under the Kontsevich–Zorich cocycle like $\gamma^{(1-\lambda)^k} \rightarrow 1$ as $k \rightarrow \infty$, while the measure of the discarded set is at most

$$\sum_{j=0}^{\infty} \gamma^{\sigma(1-\lambda)^j},$$

a quantity which converges to zero as $\gamma \rightarrow 0$.

A density type argument then completes the proof of Theorem 3.4.

4.5. Weak mixing in rational polygonal billiards

The challenge in this work is to go from elimination of a single eigenvalue to the elimination of the whole line of potential eigenvalues. By the Veech criterion, it is not restrictive to focus on a single segment $J \subset \mathbb{R} \cdot \text{Im}(\omega)$ contained in a small neighborhood of a lattice point $z \in H^1(X, \mathbb{Z})$. Let

$$H^1(X, \mathbb{R}) = T_\omega \oplus T_\omega^\perp$$

denote the decomposition of the cohomology space as a direct sum of the so-called tautological subbundle $T_\omega = \mathbb{R} \text{Re}(\omega) \oplus \mathbb{R} \text{Im}(\omega)$ and of its symplectic orthogonal. Let π_ω denote the projection onto the subspace $T_\omega^\perp$. An important remark is that, since $J \subset \mathbb{R}[\text{Im}(\omega)] \subset T_\omega$, we have

$$\pi_\omega(J - z) = -\pi_\omega(z) \in T_\omega^\perp,$$

which is a single vector (for all points of J) to which the controlled repulsion argument outlined in Section 4.4.1 from [14] can be applied. However, since the projection of J belongs to the unstable subspace of the Kontsevich–Zorich cocycle, the Veech criterion cannot avoid dilation and spreading of the interval J to a collection of intervals near finitely many lattice points whenever the Teichmüller orbit makes an excursion outside a given compact set.

This difficulty is overcome in [2] by introducing a refined weak mixing criterion, which implies that, as long as the projection $\pi_\omega(z)$ remains sufficiently small under the Kontsevich–Zorich cocycle, the tautological projection of the interval $J - z$ remains confined to a neighborhood of the origin. Hence, there is no spreading in the tautological plane, and the controlled repulsion on $T_\omega^\perp$, by increasing the norm of $\pi_\omega(z)$, leads to a contradiction with the Veech criterion.

In the argument, it is important that we can restrict to subbundle on which the Kontsevich–Zorich cocycle is fully hyperbolic. Indeed, the line $\mathbb{R} \cdot \text{Im}(\omega)$ is contained in the tautological plane $T_\omega \subset H^1(X, \mathbb{R})$, which in turn is contained in the subbundle given by the sum of all of its Galois conjugate of the projection of the tangent space of the $\text{SL}(2, \mathbb{R})$ -orbit closure of (X, ω) onto the absolute cohomology. It was proved by A. Wright [47] that this Galois subbundle is defined over $\mathbb{Q}$, hence it has a well-defined integral lattice. The Veech criterion still holds for the restriction of the cocycle to the Galois subbundle, hence the argument can be carried out in this restricted set up. In addition, by a result of S. Filip [19], all the Lyapunov exponents of the Kontsevich–Zorich cocycle on the Galois subbundle are all non-zero.

4.5.1. A weak mixing criterion. The refined weak mixing criterion is based on the notion of a rigidity configuration. Roughly speaking, a rigidity configuration for a translation surface is an immersed rectangle of large vertical size and small horizontal size which contains an embedded sub-rectangle whose area is uniformly bounded from below. A *rigidity configuration* is a pair (J, R) with J a horizontal interval and R an embedded rectangle with top and bottom horizontal sides in J . It is characterized by the following parameters: the vertical height of the maximal immersed rectangle with base J , the vertical size $V > 0$ of R , the horizontal length $H > 0$ of J , the area $\sigma > 0$ of the maximal embedded sub-rectangle $R^* \subset R$ (see [2, Definition 3.2]).

We then have the following crucial existence statement.

Proposition 4.6 ([2, Proposition 4.1]). *Let $\mathcal{H}^1$ be a stratum of unit area Abelian differentials. Then, for every compact set $\mathcal{K} \subseteq \mathcal{H}^1$, there exist constants $L_0 = L_0(\mathcal{K}) \geq 1$ and $C = C(\mathcal{K}) \geq 1$ such that, for every $(X, \omega) \in \mathcal{K}$ and every $L \geq L_0$, there exists a (V, H, σ, L) -rigidity configuration (J, R) of (X, ω) with the following properties:*

- (1) $L/C \leq V \leq C \cdot L$.
- (2) $\sigma \geq 1/C$.

(3) If $0 < t < \log L$ is such that $g_t(X, \omega) \in \mathcal{K}$, then $H < C \cdot e^{-t}$.

An argument reminiscent of the proof of Veech's criterion (see [44] or [14]) then establishes the following (see [2, Definition 3.3 and Lemma 3.4]).

Theorem 4.7 (Weak mixing criterion). *Let $(X, \omega) \in \mathcal{H}$ be a translation surface. If the vertical flow is ergodic and α is an eigenvalue for the vertical flow on (X, ω) , then for every $\varepsilon > 0$ and compact $\mathcal{K} \subset \mathcal{H}^1$, there exists $t_0 := t_0(\varepsilon, \mathcal{K}, \alpha) > 0$ such that for all $t > t_0$, both of the following two conditions hold:*

(1) *For all (V, H, σ, L) -rigidity configuration such that $V \geq e^t$, at least one of the following holds:*

$$\sigma < \varepsilon \quad \text{or} \quad d_{\mathbb{R}}(\alpha V, \mathbb{Z}) < \varepsilon.$$

(2) *(Veech criterion) for all $L > t$ such that $(X_L, \omega_L) \in \mathcal{K}$,*

$$d(g_L \cdot \alpha \operatorname{Im}(\omega), H^1(X_L; \mathbb{Z})) < \varepsilon.$$

Let us recall the splitting $H^1(X, \mathbb{R}) = T_{\omega} \oplus T_{\omega}^{\perp}$ as a direct sum of the tautological plane and of its symplectic orthogonal, and let, as above, π_{ω} be the projection onto $T_{\omega}^{\perp}$ and $(I - \pi_{\omega})$ the projection onto the tautological plane. Let then $(I - \pi_{\omega})^+$ denote the composition of the projection onto T_{ω} and the projection onto the unstable (1-dimensional) subspace of T_{ω} along the (1-dimensional) stable space.

Proposition 4.8. *Let $\mathcal{H}^1$ be a stratum of unit area Abelian differentials over a moduli space of Riemann surfaces. Then, for every compact set $\mathcal{K} \subseteq \mathcal{H}^1$, there exist constants $C = C(\mathcal{K}) \geq 1$ and $t_0 := t_0(\mathcal{K}) > 0$ with the following property. Let $(X, \omega) \in \mathcal{H}^1$ be any translation surface. Consider $t \geq t_0$ such that the following conditions hold:*

(1) $g_t(X, \omega) = (X_t, \omega_t) \in \mathcal{K}$.

(2) *There exists $z \in H^1(X_t, \mathbb{Z})$ such that $d(g_t^{\text{KZ}} \cdot \alpha \operatorname{Im}(\omega), z) < 1/(2C^2)$.*

Suppose $\alpha \in \mathbb{R} \setminus \{0\}$ has the property that $d_{\mathbb{R}}(V\alpha, \mathbb{Z}) < 1/C$ for all $V > e^t/C''$ whenever (X, ω) has a $(V, H, 1/C, L)$ -rigidity configuration. Let $\beta \in \mathbb{R}$ be the unique real number such that $(I - \pi_{\omega})^+(z) = g_t^{\text{KZ}} \cdot \beta \operatorname{Im}(\omega) \in H^1(X_t, \mathbb{R})$. Then,

$$d_{\mathbb{R}}(\alpha, \beta) \leq C \cdot \|\pi_{\omega}(z)\| \cdot e^{-t}.$$

The significance of the above proposition stands in the fact that as long as the norm $\|\pi_{\omega}(z)\|$ stays small, since the vector $(I - \pi_{\omega})(z - \beta \operatorname{Im}(\omega))$ belongs to the stable Oseledets space in the tautological plane, the norm of the vector $z - \alpha \operatorname{Im}(\omega)$ remains small for a time of the order $\log \|\pi_{\omega}(z)\|$. It then follows from the large deviation Theorem 4.5 as in Section 4.4.1 that if $\gamma := \|\pi_{\omega}(z)\|$ is sufficiently small, there

exists a time $t := t(s) \in [-\log \gamma/2, -3 \log \gamma/4]$ such that, outside a set of s of measure γ^σ ($\sigma > 0$),

$$g_t(h_s \omega) \in \mathcal{K} \quad \text{and} \quad \|g_t^{\text{KZ}}(\pi_\omega(z))\| \approx \gamma^{-\lambda} \|\pi_\omega(z)\| = \|\pi_\omega(z)\|^{1-\lambda}.$$

Iteration of this argument produces controlled repulsion away from the integer lattice outside a set of “small” measure (for γ small). Indeed, by Proposition 4.8, there exists $z \in H^1(\mathbb{Z}) \setminus \{0\}$ such that the norm of the vector $z - \alpha \text{Im}(\omega)$ remains under control (and less than $1/2$) up to the time $t \in [-\log \gamma/2, -3 \log \gamma/4]$, while the norm of its projection $\pi_\omega(z - \alpha \text{Im}(\omega)) = \pi_\omega(z)$ grows, under the assumption that $\pi_\omega(z) \neq 0$. The latter assumption is ensured by the hypotheses that the tautological plane T_ω contains no non-zero integer points. By a finite iteration of the procedure just described, we can prove that, outside of a set of parameters of small measure, the norm $\|g_t^{\text{KZ}}(z - \alpha \text{Im}(\omega))\| = d(g_t^{\text{KZ}}(\alpha \text{Im}(\omega)), \mathbb{Z})$ grows larger than a preassigned constant, thereby contradicting the Veech criterion for weak mixing and completing the argument.

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References

- [1] H. Al-Saqban, P. Apisa, A. Erchenko, O. Khalil, S. Mirzadeh, and C. Uyanik, [Exceptional directions for the Teichmüller geodesic flow and Hausdorff dimension](#). *J. Eur. Math. Soc. (JEMS)* **23** (2021), no. 5, 1423–1476 Zbl [1475.37038](#) MR [4244511](#)
- [2] F. Arana-Herrera, J. Chaika, and G. Forni, Weak mixing in rational billiards. 2024, arXiv: [2410.11117v1](#)
- [3] J. S. Athreya, [Quantitative recurrence and large deviations for Teichmüller geodesic flow](#). *Geom. Dedicata* **119** (2006), no. 1, 121–140 Zbl [1108.32007](#) MR [2247652](#)
- [4] J. S. Athreya and G. Forni, [Deviation of ergodic averages for rational polygonal billiards](#). *Duke Math. J.* **144** (2008), no. 2, 285–319 Zbl [1153.37001](#) MR [2437681](#)
- [5] A. Avila and V. Delecroix, [Weak mixing directions in non-arithmetic Veech surfaces](#). *J. Amer. Math. Soc.* **29** (2016), no. 4, 1167–1208 Zbl [1373.37005](#) MR [3522612](#)
- [6] A. Avila and G. Forni, [Weak mixing for interval exchange transformations and translation flows](#). *Ann. of Math. (2)* **165** (2007), no. 2, 637–664 Zbl [1136.37003](#) MR [2299743](#)
- [7] A. Avila, G. Forni, and P. Safaee, [Quantitative weak mixing for interval exchange transformations](#). *Geom. Funct. Anal.* **33** (2023), no. 1, 1–56 Zbl [1514.37012](#) MR [4561147](#)

- [8] A. Avila and M. Leguil, [Weak mixing properties of interval exchange transformations & translation flows](#). *Bull. Soc. Math. France* **146** (2018), no. 2, 391–426 Zbl 1402.37002 MR 3933880
- [9] J. Bobok and S. Troubetzkoy, [Topologically weakly mixing polygonal billiards](#). *Proc. Amer. Math. Soc.* **151** (2023), no. 8, 3391–3400 Zbl 1523.37035 MR 4591774
- [10] A. I. Bufetov and B. Solomyak, [A spectral cocycle for substitution systems and translation flows](#). *J. Anal. Math.* **141** (2020), no. 1, 165–205 Zbl 1462.37025 MR 4174040
- [11] A. I. Bufetov and B. Solomyak, [Hölder regularity for the spectrum of translation flows](#). *J. Éc. polytech. Math.* **8** (2021), 279–310 Zbl 1470.37051 MR 4218159
- [12] G. Casati and T. Prosen, [Mixing property of triangular billiards](#). *Phys. Rev. Lett.* **83** (1999), no. 23, 4729–4732
- [13] J. Chaika and A. Eskin, [Every flat surface is Birkhoff and Oseledets generic in almost every direction](#). *J. Mod. Dyn.* **9** (2015), no. 1, 1–23 Zbl 1358.37008 MR 3395258
- [14] J. Chaika and G. Forni, [Weakly mixing polygonal billiards](#). [v1] 2020, [v2] 2025, arXiv: 2003.00890v2, to appear in *Ann. of Math. (2)*
- [15] J. Chaika and H. Masur, [The set of non-uniquely ergodic \$d\$ -IETs has Hausdorff codimension \$1/2\$](#) . *Invent. Math.* **222** (2020), no. 3, 749–832 Zbl 1462.37026 MR 4169051
- [16] A. Eskin and M. Mirzakhani, [Invariant and stationary measures for the \$SL\(2, \mathbb{R}\)\$ action on moduli space](#). *Publ. Math. Inst. Hautes Études Sci.* **127** (2018), 95–324 Zbl 1478.37002 MR 3814652
- [17] A. Eskin, M. Mirzakhani, and A. Mohammadi, [Isolation, equidistribution, and orbit closures for the \$SL\(2, \mathbb{R}\)\$ action on moduli space](#). *Ann. of Math. (2)* **182** (2015), no. 2, 673–721 Zbl 1357.37040 MR 3418528
- [18] B. Fayad and A. Katok, [Constructions in elliptic dynamics](#). *Ergodic Theory Dynam. Systems* **24** (2004), no. 5, 1477–1520 Zbl 1089.37012 MR 2104594
- [19] S. Filip, [Semisimplicity and rigidity of the Kontsevich–Zorich cocycle](#). *Invent. Math.* **205** (2016), no. 3, 617–670 Zbl 1368.14013 MR 3539923
- [20] S. Filip, [Zero Lyapunov exponents and monodromy of the Kontsevich–Zorich cocycle](#). *Duke Math. J.* **166** (2017), no. 4, 657–706 Zbl 1370.37066 MR 3619303
- [21] G. Forni, [Deviation of ergodic averages for area-preserving flows on surfaces of higher genus](#). *Ann. of Math. (2)* **155** (2002), no. 1, 1–103 Zbl 1034.37003 MR 1888794
- [22] G. Forni, [On the Lyapunov exponents of the Kontsevich–Zorich cocycle](#). In B. Hasselblatt and A. Katok (eds.), *Handbook of dynamical systems*, Vol. 1B, pp. 549–580, Elsevier, Amsterdam, 2006 Zbl 1130.37302 MR 2186248
- [23] G. Forni, [A geometric criterion for the nonuniform hyperbolicity of the Kontsevich–Zorich cocycle](#). *J. Mod. Dyn.* **5** (2011), no. 2, 355–395 Zbl 1238.30031 MR 2820565
- [24] G. Forni, [Twisted translation flows and effective weak mixing](#). *J. Eur. Math. Soc. (JEMS)* **24** (2022), no. 12, 4225–4276 Zbl 1505.37009 MR 4493624

- [25] G. Forni, [Effective unique ergodicity and weak mixing of translation flows](#). In C. Bonanno, A. Sorrentino, and C. Ulcigrai (eds.), *Modern aspects of dynamical systems*, pp. 161–221, Lecture Notes in Math. 2347 – CIME Found. Subser., Springer, Cham, 2024 Zbl 08107407 MR 4823777
- [26] R. H. Fox and R. B. Kershner, [Concerning the transitive properties of geodesics on a rational polyhedron](#). *Duke Math. J.* **2** (1936), no. 1, 147–150 Zbl 0014.03401 MR 1545913
- [27] E. Gutkin, [Billiards in polygons](#). *Phys. D* **19** (1986), no. 3, 311–333 Zbl 0593.58016 MR 0844706
- [28] E. Gutkin and A. Katok, [Weakly mixing billiards](#). In X. Gómez-Mont, J. A. Seade, and A. Verjovsky (eds.), *Holomorphic dynamics (Mexico, 1986)*, pp. 163–176, Lecture Notes in Math. 1345, Springer, Berlin, 1988 Zbl 0673.58028 MR 0980958
- [29] B. Hasselblatt and A. Katok, [Principal structures](#). In B. Hasselblatt and A. Katok (eds.), *Handbook of dynamical systems*, Vol. 1A, pp. 1–203, North-Holland, Amsterdam, 2002 Zbl 1047.37001 MR 1928518
- [30] Z. Hwang, C. A. Marx, J. J. Seaward, S. Jitomirskaya, and M. Olshanii, [Growth of the Wang–Casati–Prosen counter in an integrable billiard](#). *SciPost Phys.* **14** (2023), no. 2, article no. 017, 16 pp. Zbl 1551.37043 MR 4546215
- [31] A. Katok, [Interval exchange transformations and some special flows are not mixing](#). *Israel J. Math.* **35** (1980), no. 4, 301–310 Zbl 0437.28009 MR 0594335
- [32] A. Katok, [The growth rate for the number of singular and periodic orbits for a polygonal billiard](#). *Comm. Math. Phys.* **111** (1987), no. 1, 151–160 Zbl 0631.58020 MR 0896765
- [33] A. Katok, Five most resistant problems in dynamics. Clay Mathematics Institute and MSRI Conference on Recent Progress in Dynamics, 2004, <http://akatok.s3-website-us-east-1.amazonaws.com/pub/5problems-expanded.pdf> visited on 1 February 2026
- [34] A. B. Katok, [Billiard table as a playground for a mathematician](#). In V. Prasolov and Y. Ilyashenko (eds.), *Surveys in modern mathematics*, pp. 216–242, London Math. Soc. Lecture Note Ser. 321, Cambridge University Press, Cambridge, 2005 Zbl 1147.37329 MR 2166929
- [35] S. Kerckhoff, H. Masur, and J. Smillie, [Ergodicity of billiard flows and quadratic differentials](#). *Ann. of Math. (2)* **124** (1986), no. 2, 293–311 Zbl 0637.58010 MR 0855297
- [36] M. Kontsevich and A. Zorich, Lyapunov exponents and Hodge theory. In J. M. Drouffe and J. B. Zuber (eds.), *Proceedings of the conference on the mathematical beauty of physics (Saclay, 1996)*, pp. 318–332, Adv. Ser. Math. Phys. 24, World Scientific, River Edge, NJ, 1997 Zbl 1058.37508 MR 1490861
- [37] E. Lindenstrauss, A. Mohammadi, and Z. Wang, [Polynomial effective equidistribution](#). *C. R. Math. Acad. Sci. Paris* **361** (2023), 507–520 Zbl 1528.37006 MR 4548163
- [38] H. Masur, [Interval exchange transformations and measured foliations](#). *Ann. of Math. (2)* **115** (1982), no. 1, 169–200 Zbl 0497.28012 MR 0644018

- [39] H. Masur and S. Tabachnikov, [Rational billiards and flat structures](#). In B. Hasselblatt and A. Katok (eds.), *Handbook of dynamical systems*, Vol. 1A, pp. 1015–1089, North-Holland, Amsterdam, 2002 Zbl [1057.37034](#) MR [1928530](#)
- [40] J. Moser, *Stable and Random Motions in Dynamical Systems: With Special Emphasis on Celestial Mechanics (AM-77)*. REV-revised, Princeton Landmarks Math. Phys., Princeton University Press, Princeton, 1973, 216 pp. Zbl [0271.70009](#) MR [0442980](#)
- [41] G. Rauzy, [Échanges d’intervalles et transformations induites](#). *Acta Arith.* **34** (1979), no. 4, 315–328 Zbl [0414.28018](#) MR [0543205](#)
- [42] P. J. Richens and M. V. Berry, [Pseudointegrable systems in classical and quantum mechanics](#). *Phys. D* **2** (1981), no. 3, 495–512 Zbl [1194.37150](#) MR [0625449](#)
- [43] W. A. Veech, [Gauss measures for transformations on the space of interval exchange maps](#). *Ann. of Math. (2)* **115** (1982), no. 2, 201–242 Zbl [0486.28014](#) MR [0644019](#)
- [44] W. A. Veech, [The metric theory of interval exchange transformations. I. Generic spectral properties](#). *Amer. J. Math.* **106** (1984), no. 6, 1331–1359 Zbl [0631.28006](#) MR [0765582](#)
- [45] Ya. B. Vorobets, [Ergodicity of billiards in polygons](#) (in Russian). *Mat. Sb.* **188** (1997), no. 3, 65–112. [English translation](#). *Sb. Math.* **188** (1997), no. 3, 389–434 Zbl [0886.58065](#) MR [1462024](#)
- [46] J. Wang, G. Casati, and T. Prosen, [Nonergodicity and localization of invariant measure for two colliding masses](#). *Phys. Rev. E* **89** (2014), no. 4, article no. 042918, 4 pp.
- [47] A. Wright, [The field of definition of affine invariant submanifolds of the moduli space of abelian differentials](#). *Geom. Topol.* **18** (2014), no. 3, 1323–1341 Zbl [1320.32019](#) MR [3254934](#)
- [48] A. N. Zemlyakov and A. B. Katok, Topological transitivity of billiards in polygons (in Russian). *Mat. Zametki* **18** (1975), no. 2, 291–300 Zbl [0315.58014](#). [English translation](#). *Math. Notes* **18** (1975), no. 2, 760–764 Zbl [0323.58012](#) MR [0399423](#)
- [49] A. Zorich, [Finite Gauss measure on the space of interval exchange transformations. Lyapunov exponents](#). *Ann. Inst. Fourier (Grenoble)* **46** (1996), no. 2, 325–370 Zbl [0853.28007](#) MR [1393518](#)

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