

Symmetries of Grothendieck rings in representation theory

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Abstract. We review certain new symmetries of Grothendieck rings that have emerged in representation theory.

Categories of representations of groups, or more generally, bialgebras and quantum groups, are endowed with monoidal structures. The Grothendieck ring of such a category has a basis consisting of classes of simple objects, with structure constants defined as multiplicities of simple representations in the decomposition of tensor products. The resulting rings together with their bases can have a very intricate structure, and many of them are fundamental with important applications (for example, leading to algorithms to compute the dimension of simple representations).

In recent years, new symmetries of Grothendieck rings have emerged, which can be used to better understand these rings. We will discuss these symmetries and illustrate some of their applications.

First, cluster symmetries give powerful tools to investigate the structure of the Grothendieck rings [19]. As an application, we have obtained the proof of long-standing conjectures on character formulas for simple finite-dimensional representations of quantum affine algebras (which are infinite-dimensional quantum groups) [14].

Second, motivated by symmetries of quantum integrable models arising in physics, we established remarkable relations in associated Grothendieck rings [8], generalizing the celebrated Baxter relations; these relations led us to the discovery of a surprising new action of the Weyl group on these Grothendieck rings [9]. These Weyl group symmetries are also closely related to the cluster symmetries [15].

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1. Grothendieck group

To start with, let us recall the general construction of Grothendieck groups.

1.1. Grothendieck group of an additive category

Let $\mathcal{C}$ be an additive category with distinguished short exact sequences

$$A \rightarrow B \rightarrow C. \quad (1.1)$$

For example, one may consider the case of an abelian category with all ordinary exact sequences.

Definition 1.1. The Grothendieck group $K(\mathcal{C})$ of $\mathcal{C}$ is the group generated by the elements $[M]$ for each isomorphism class of objects M in $\mathcal{C}$, with the relations

$$[A] + [C] = [B]$$

for each exact sequence (1.1).

We obtain the important algebraic structure $K(\mathcal{C})$ attached to the category $\mathcal{C}$. We will discuss many examples in this lecture.

The main original motivation appeared in algebraic geometry. Consider the $\mathcal{C}$ category of coherent sheaves on an algebraic variety X . The Grothendieck group

$$K(X) = K(\mathcal{C})$$

is one of the starting points of K -theory in algebraic geometry. Historically, such constructions were first studied by Grothendieck for categories of coherent and locally free sheaves on schemes, in proving the Riemann–Roch theorem.

Another basic but fundamental example is obtained for a field k : the category $\mathcal{C} = \mathcal{V}$ of finite-dimensional k -vector spaces. The isomorphism classes are parametrized by the dimension. Then $\mathcal{V}$ is semi-simple, and the direct sum descends to the sum in the Grothendieck group, that is, for V and W in $\mathcal{V}$, we have

$$[V \oplus W] = [V] + [W].$$

This implies that $K(\mathcal{V})$ is freely generated by the class $[k]$ of the one-dimensional vector space

$$K(\mathcal{V}) = \mathbb{Z}[k] \simeq \mathbb{Z}$$

as we have

$$[V] = \dim(V) \cdot [k]$$

for V in $\mathcal{V}$.

1.2. Grothendieck groups in representation theory

We now discuss the Grothendieck group of finite-dimensional representations of an algebra.

Let $\mathcal{A}$ be an algebra over a field k and $\mathcal{F}$ be the category of finite-dimensional representations of $\mathcal{A}$.

We have the following classical and fundamental result.

Theorem 1.2 (Jordan–Hölder series). *Each object M in $\mathcal{F}$ admits a series of subobjects*

$$M = M_0 \supset M_1 \supset \cdots \supset M_N = \{0\}$$

with M_i/M_{i+1} simple object.

Consider a simple object V . Then the number $n_V(M)$ of occurrences of V as a quotient M_i/M_{i+1} , depends only on M and not on the series of subobjects.

Such a series is called a composition series or a Jordan–Hölder series of the representation. A simple quotient M_i/M_{i+1} is called a composition factor of M , and its number of occurrences is called its multiplicity.

For completeness, let us briefly recall the arguments of the proof.

Proof. The existence of the series is obtained by induction on the dimension of the representation. The uniqueness of $n_V(M)$ is also proved by induction on the dimension: it is obvious for a simple representation. In general, assume that we have two series M_i and M'_i as above. If $M_1 = M'_1$, we can conclude using the induction hypothesis for this representation. Otherwise,

$$M \supset M_1 \supset M_1 \cap M'_1$$

and

$$M \supset M'_1 \supset M_1 \cap M'_1$$

are the first few terms of composition series with the same composition factors. Using the induction hypothesis, they have the same composition factors. ■

One can then define the length of a module M as the sum

$$l(M) = \sum_{V \text{ simple class in } \mathcal{F}} n_V(M).$$

As a consequence of Theorem 1.2, we obtain that

$$K(\mathcal{F}) = \bigoplus_{V \text{ simple class}} \mathbb{Z}[V]$$

has a canonical basis parametrized by simple classes. The Grothendieck group is a group with a distinguished basis.

The coefficients on the basis are the multiplicities of simple representations, that is, for M object in $\mathcal{F}$, one has

$$[M] = \sum_{V \text{ simple class}} n_V(M)[V].$$

In general, an element

$$\sum_{V \text{ simple class}} n_V[V]$$

with all $n_V \geq 0$ is called the class of an *actual* representation.

If one of the $n_V < 0$, this is the class of a *virtual* representation. The Grothendieck group allows us to treat in a uniform way actual and virtual representations.

2. Monoidal categories

We now assume that an abelian category $\mathcal{C}$ has a monoidal structure, that is, we have tensor products $M \otimes N$ for M, N objects in $\mathcal{C}$ with isomorphisms

$$(M \otimes N) \otimes P \simeq M \otimes (N \otimes P)$$

as well as a unit object $\mathbf{1}$ in $\mathcal{C}$ so that for any object M in $\mathcal{C}$,

$$M \otimes \mathbf{1} \simeq \mathbf{1} \otimes M \simeq M.$$

We have additional compatibility properties with the abelian structure (see [28]).

Remark 2.1. We do not assume that the category is necessarily a tensor category, that is, we do not assume that the category admits dualities compatible with the monoidal structure.

Let us discuss important examples of monoidal categories.

- (1) The category $\mathcal{V}$ of finite-dimensional k -vector spaces has a natural monoidal structure.
- (2) For a group G , the category of finite-dimensional representations of G (that is, of its group algebra $k[G]$) has a monoidal structure. The action of $g \in G$ on a pure tensor $v \otimes w \in V \otimes W$ in a tensor product of representations is defined by

$$g \cdot (v \otimes w) = (g \cdot v) \otimes (g \cdot w).$$

- (3) For a Lie algebra $\mathfrak{g}$, the category of finite-dimensional representations of $\mathfrak{g}$ (that is, of its universal enveloping algebra $\mathcal{U}(\mathfrak{g})$) has a monoidal structure. The action of $g \in \mathfrak{g}$ on a pure tensor $v \otimes w \in V \otimes W$ in a tensor product of representations is defined by

$$g \cdot (v \otimes w) = (g \cdot v) \otimes w + v \otimes (g \cdot w).$$

This is an infinitesimal version of the group action above.

- (4) More generally: for a Hopf algebra H , that is, an algebra with a coproduct and a counit,

$$\Delta : H \rightarrow H \otimes H \quad \text{and} \quad \varepsilon : H \rightarrow k.$$

The category of finite-dimensional representations of H has a monoidal structure. This includes all examples above and new examples, in particular the Drinfeld–Jimbo *quantum groups*

$$\mathcal{U}_q(\mathfrak{g})$$

associated to $q \in \mathbb{C}^* \setminus \{-1, 1\}$ quantum parameter and $\mathfrak{g}$ simple Lie algebra. It is a q -deformation of $\mathcal{U}(\mathfrak{g})$, originated from mathematical physics. For example, $\mathcal{U}_q(\mathfrak{sl}_2)$ is the algebra by generators $E, F, K^{\pm 1}$ and relations

$$KE = q^2 EK, \quad KF = q^{-2} FK \quad \text{and} \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}.$$

- (5) Hecke algebras, and their variations such as quiver Hecke algebras (as defined by Khovanov–Lauda and Rouquier), give rise to categories of representations which have a monoidal structure via convolutions.
- (6) In various geometric contexts, convolution diagrams

$$X_1 \xleftarrow{p} X_2 \xrightarrow{q} X_3$$

of regular morphisms p, q between remarkable varieties allow us to consider convolution products and then monoidal structures on certain categories of sheaves (see, for instance, [1, 5] and references therein).

3. Grothendieck ring

3.1. Grothendieck ring of a monoidal category

Let us define the Grothendieck ring of $\mathcal{C}$ monoidal category.

Proposition 3.1. *The Grothendieck group $K(\mathcal{C})$ of $\mathcal{C}$ inherits a ring structure such that*

$$[M] \cdot [N] = [M \otimes N]$$

for M, N objects in $\mathcal{C}$ and the class $[\mathbf{1}] \in K(\mathcal{C})$ is the neutral element.

Example 3.2. For $\mathcal{C} = \mathcal{V}$ the category of finite-dimensional vector spaces on a field k , the group isomorphism

$$K(\mathcal{C}) \simeq \mathbb{Z}$$

is a ring isomorphism. This follows, for example, from the formula $\dim(V \otimes W) = \dim(V) \dim(W)$.

Remark 3.3. If the category is braided, that is,

$$M \otimes N \simeq N \otimes M$$

for all objects M, N in $\mathcal{C}$, then the Grothendieck ring is commutative. The converse statement is false: there are non-braided monoidal categories whose Grothendieck ring is commutative. We will see examples below.

3.2. Grothendieck rings in representation theory

Let $\mathcal{C}$ be the category of finite-dimensional representations of a bialgebra H . The Grothendieck ring

$$K(\mathcal{C}) = \bigoplus_{V \text{ simple}} \mathbb{Z}[V]$$

has a canonical basis $([V])_{V \text{ simple}}$, and the constant structures on this basis are positive constant structures: they are given by the multiplicities of simple modules in tensor products (sometimes called Clebsch–Gordan coefficients or fusion coefficients).

Remark 3.4. If H is a Hopf algebra, $\mathcal{C}$ is a tensor category, and dualities induce ring automorphisms of $K(\mathcal{C})$, which preserve the canonical basis.

Example 3.5. Let $\mathcal{C}$ be the category of finite-dimensional representations of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. Then we have a ring isomorphism

$$K(\mathcal{C}) \simeq \mathbb{Z}[X]$$

with the polynomial ring in one variable X . Here,

$$X = [V], \quad \text{where } V = \mathbb{C}^2 \text{ is the natural representation.}$$

It is well known that $\mathfrak{sl}_2(\mathbb{C})$ has a simple complex representation of dimension $n + 1$ for any $n \geq 0$ and that this representation is unique up to isomorphism. Let us denote by Q_n the isomorphism class of this representation in $K(\mathcal{C})$. Then the basis $(Q_n)_{n \geq 0}$ of simple classes satisfies the induction relation

$$Q_n^2 = Q_{n+1} Q_{n-1} + 1. \quad (3.1)$$

One can compute the canonical basis of simple classes

$$(Q_0, Q_1, Q_2, \dots) = (1, X, X^2 - 1, X^3 - 2X, \dots),$$

which is different from the standard monomial basis of $\mathbb{Z}[X]$.

Remark 3.6. Note that, independently of Grothendieck rings, it is not clear a priori that the inductive sequence (3.1) with initial conditions $Q_0 = 1$ and $Q_1 = X$, a priori in $\mathbb{Q}(X)$, defines a sequence in the ring $\mathbb{Z}[X]$.

3.3. Applications of Grothendieck rings in representation theory

The study of Grothendieck rings of categories of representations has led to interesting developments in various directions. Let us list some of them.

Representation theory. There is a close relation between the understanding of the global structure of Grothendieck rings and the description of the structure of simple representations. For example, if one knows the structure of certain simple representations, one may extract information on the simple constituents of their tensor products.

Example 3.7. This approach will be discussed below to compute the characters of simple representations of quantum affine algebras from the characters of fundamental representations, following [29] and then [14, 16].

Categorification. One can sometimes realize an algebra $\mathcal{A}$ as a Grothendieck ring:

$$\mathcal{A} \simeq K(\mathcal{C}).$$

The categorified structure $\mathcal{A}$ inherits a canonical basis from the canonical basis of the Grothendieck ring.

Example 3.8. The coordinate ring $\mathbb{C}[N]$, with N a unipotent subgroup of a Lie group G , can be realized from representations of quiver Hecke algebras [27, 32] or of quantum affine algebras [20], leading to a categorical interpretation of its dual canonical basis. A basic example is also explained in Example 3.5.

Quantum integrable models. Certain Grothendieck rings can be identified with the ring of commuting operators of a quantum integrable model via the transfer-matrix construction. Indeed, a quantum integrable model involves a space W (the quantum space) with a family of commuting operators on this space (including quantum Hamiltonians). The transfer-matrix construction defines a ring morphism

$$T : K_0(\mathcal{C}) \rightarrow \text{End}(W),$$

where $\mathcal{C}$ is the category of representations of an algebra of symmetry of the system. Hence, any relation in $K_0(\mathcal{C})$ (a “universal relation”) gives relations between the eigenvalues of the operators on W for any related quantum integrable model.

Example 3.9. The famous XXZ model can be realized using the representation theory of the quantum affine algebra associated to $\mathfrak{sl}_2$ (see below).

The aim of this lecture is to discuss certain symmetries of Grothendieck rings and their consequences. The main examples of symmetries we present are cluster symmetries and Weyl group symmetries.

4. Cluster algebra symmetries

We discuss the first symmetries of Grothendieck rings in this lecture in the context of cluster algebras.

4.1. A quick review of cluster algebras

The theory of cluster algebras was introduced by Fomin–Zelevinsky [7]. A cluster algebra is a commutative algebra with a distinguished set of generators grouped into overlapping subsets (the clusters, obtained by an inductive process called mutation). Each element of the cluster algebra can be expressed as a rational fraction of the elements of a cluster. From a geometric point of view, one might think about a variety with various sets of local coordinates, and each element of the algebra of functions over this variety can be expressed locally in terms of the local coordinates. Then the mutations can be understood as a regular change of coordinates.

More precisely, the cluster algebra $\mathcal{A}_Q$ is a commutative algebra associated with a quiver Q (without loops or 2-cycles). It has a distinguished set of generators (the cluster variables) defined by a combinatorial process. Cluster algebras have many incarnations in various fields, and in particular in representation theory as we will discuss below.

The cluster algebra $\mathcal{A}_Q$ attached to the quiver Q (with a set of vertices Q_0) is a subring of the field

$$\mathcal{F} = \mathbb{Q}(X_i)_{i \in Q_0},$$

with free variables X_i which are called the initial cluster variables (they form the initial cluster). The cluster algebra $\mathcal{A}_Q$ is defined as the subalgebra of $\mathcal{F}$ generated by the cluster variables, obtained inductively from the initial variables by an inductive process called mutations. For example, the first-step mutated variables X_i^* are defined by the formula

$$X_i X_i^* = \prod_{j \rightarrow i} X_j + \prod_{j \leftarrow i} X_j,$$

where the arrows $\rightarrow$ and $\leftarrow$ are those in the initial quiver Q (then the quiver also gets mutated in the process). The number of cluster variables is not necessarily finite. In addition, the cluster variables are grouped into overlapping subsets called clusters, which are all in bijection with Q_0 . The cluster monomials are defined as the monomials in the cluster variables of the same cluster. In some situations, one may have additional non-mutable cluster variables: they are called frozen variables, and they belong to all clusters.

One of the fundamental properties of cluster algebras is the Laurent phenomenon: any cluster variable can be expressed as a Laurent polynomial in the cluster variables of a given seed. In the initial seed, this can be written as

$$\mathcal{A}_Q \subset \mathbb{Z}[X_i^{\pm 1}]_{i \in Q_0}.$$

Example 4.1. Consider a Q quiver of type A_2

$$\bullet \longrightarrow \bullet.$$

We have the initial cluster variables (X_1, X_2) and five cluster variables:

$$X_1, X_2, \frac{1+X_2}{X_1}, \frac{1+X_1}{X_2}, \frac{1+X_1+X_2}{X_1 X_2}.$$

There are also five clusters:

$$(X_1, X_2), \left(\frac{1+X_2}{X_1}, X_2 \right), \left(\frac{1+X_2}{X_1}, \frac{1+X_1+X_2}{X_1 X_2} \right), \\ \left(\frac{1+X_1}{X_2}, \frac{1+X_1+X_2}{X_1 X_2} \right), \left(\frac{1+X_1}{X_2}, X_1 \right).$$

Remark 4.2. This example should not be misleading: in general, the number of clusters is not equal to the number of cluster variables.

4.2. Monoidal categorification of cluster algebras

In this lecture, we focus on the relation between cluster algebras and monoidal categories (there are also important relations with additive categories; see [26] and references therein).

Definition 4.3 ([19]). A monoidal category $\mathcal{M}$ is a *monoidal categorification* of a cluster algebra $\mathcal{A}$ if there exists a ring isomorphism

$$\phi : \mathcal{A} \rightarrow K(\mathcal{M})$$

so that the cluster monomials are sent to simple classes.

In [19], a stronger version is also discussed, where it is required that ϕ induces a bijection between the cluster monomials and the real simple classes (whose tensor square is also simple). In the definition above, the simple modules corresponding to cluster monomials are called *reachable modules*. They have the property to be real, as the square of a cluster monomial is a cluster monomial.

A monoidal categorification can be regarded as a cluster symmetry of the Grothendieck ring or as a categorical realization of the cluster algebra. It gives useful information on the monoidal category: for example, it points out remarkable simple representations (corresponding to cluster variables), and it gives a factorization of simple representations corresponding to cluster monomials (in the same way as cluster monomials factorize as a product of cluster variables).

The original examples [19] were obtained from the finite-dimensional representations of quantum affine algebras. Many developments in monomial categorifications in this context have followed; one can cite, for instance, [3, 24, 30, 31] and in different contexts [4].

In all known examples, the cluster symmetry is related to the defect of symmetry of the category which produces relations in the Grothendieck ring: we have objects V, W so that

$$V \otimes W \text{ is not isomorphic to } W \otimes V$$

but

$$[V \otimes W] = [W \otimes V]$$

in the Grothendieck ring (examples will be given below).

4.3. Quantum affine algebras

Consider $\mathfrak{g}$ a complex simple Lie algebra (for example, $\mathfrak{g} = \mathfrak{sl}_2$) of rank n (that is, $n = 1$ for $\mathfrak{g} = \mathfrak{sl}_2$). Then we can construct the affine Kac–Moody algebra $\widehat{\mathfrak{g}}$, which is a central extension of the loop algebra¹

$$\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}].$$

¹We do not consider the derivation element in this lecture.

For $q \in \mathbb{C}^*$ a quantum parameter (which is assumed not to be a root of unity), we have the corresponding quantum affine algebra $\mathcal{U}_q(\widehat{\mathfrak{g}})$. It is a Hopf algebra and a q -deformation of the universal enveloping algebra $\mathcal{U}(\widehat{\mathfrak{g}})$.

Let $\mathcal{C}$ be the category of finite-dimensional representations of $\mathcal{U}_q(\widehat{\mathfrak{g}})$: it has a very rich structure. This monoidal category is neither semi-simple nor braided. The simple objects of $\mathcal{C}$ have been parametrized in terms of Drinfeld polynomials by Chari–Pressley, that is, of n -tuples $(P_i(z))_{1 \leq i \leq n}$ of monic polynomials. In particular, for $1 \leq i \leq n$ and $a \in \mathbb{C}^*$, we have the fundamental representation $V_i(a)$ which corresponds to the n -tuple $(1, \dots, 1, 1 - za, 1, \dots, 1)$ with a degree 1 polynomial in position i .

Theorem 4.4 ([12]). *The Grothendieck ring $K(\mathcal{C})$ is commutative and polynomial:*

$$K(\mathcal{C}) \simeq \mathbb{Z}[X_{i,a}]_{1 \leq i \leq n, a \in \mathbb{C}^*},$$

with $X_{i,a} = [V_{i,a}]$ the class of a fundamental representation.

4.4. Monoidal categorification – Examples

As a first example, consider the complex Lie algebra $\mathfrak{g} = \mathfrak{sl}_2$. We define $\mathcal{M}$ as the monoidal Serre subcategory of the category $\mathcal{C}$ of finite-dimensional representations of $\mathcal{U}_q(\widehat{\mathfrak{sl}_2})$ -representations generated by two fundamental representations

$$V_1(1), V_1(q^2).$$

Then we obtain a monoidal categorification

$$K(\mathcal{M}) \simeq \mathcal{A}_Q$$

with Q of type A_2 (as above) with one frozen variable. The total number of cluster variables is 3. Indeed, there is a simple representation W in $\mathcal{M}$ of dimension 3 so that we have exact sequences

$$\begin{aligned} 0 \rightarrow \mathbf{1} \rightarrow V_1(q^2) \otimes V_1(1) \rightarrow W \rightarrow 0, \\ 0 \rightarrow W \rightarrow V_1(1) \otimes V_1(q^2) \rightarrow \mathbf{1} \rightarrow 0, \end{aligned}$$

which come from degenerated braidings (which are not isomorphisms as the two tensor products are not isomorphic)

$$V_1(q^2) \otimes V_1(1) \rightarrow V_1(1) \otimes V_1(q^2) \quad \text{and} \quad V_1(1) \otimes V_1(q^2) \rightarrow V_1(q^2) \otimes V_1(1).$$

These exact sequences correspond to the unique mutation relation in the cluster algebra $\mathcal{A}_Q$:

$$[V_1(q^2)][V_1(1)] = 1 + [W].$$

The two clusters are $(V_1(q^2), W)$ and $(V_1(1), W)$. As an application, we obtain that every simple representation in $\mathcal{M}$ can be factorized into these three representations. Note that there are infinitely many simple classes in this category $\mathcal{M}$, which have the following form with $a, b \geq 0$:

$$(V_1(1))^{\otimes a} \otimes W^b \quad \text{and} \quad (V_1(q^2))^{\otimes a} \otimes W^b.$$

As another example, consider the complex Lie algebra $\mathfrak{g} = \mathfrak{sl}_3$. We define $\mathcal{M}$ as the monoidal Serre subcategory of the category $\mathcal{C}$ of finite-dimensional $\mathcal{U}_q(\widehat{\mathfrak{sl}_3})$ -representations generated by the four fundamental representations

$$V_1(1), V_1(q^2), V_2(q), V_2(q^3).$$

Then we obtain a monoidal categorification

$$K(\mathcal{M}) \simeq \mathcal{A}_Q$$

with Q of type A_2 (as above) with two additional frozen variables. The total number of cluster variables is 7.

As an application, we obtain that every simple representation in $\mathcal{M}$ can be factorized into these seven representations. Note that there are infinitely many simple classes in this category $\mathcal{M}$.

4.5. Global structures

For a general simple Lie algebra $\mathfrak{g}$, a cluster algebra structure has been obtained on $K(\mathcal{C}^-)$ for $\mathcal{C}^-$ a large subcategory² of $\mathcal{C}$ [21]. Then Kashiwara–Kim–Oh–Park established the following conjecture in [21].

Theorem 4.5 ([25]). *The category $\mathcal{C}^-$ of finite-dimensional representations of a quantum affine algebra $\mathcal{U}_q(\widehat{\mathfrak{g}})$ is a monoidal categorification of a cluster algebra.*

As an application, we could prove for simple representations corresponding to cluster monomials, a conjecture that I formulated in 2004 [16] for non-simply laced types. Using cluster algebra methods, the result extends the results obtained by Nakajima [29] in simply laced types.

Theorem 4.6 ([14]). *The dimension (and character) of simple representations corresponding to cluster monomials can be obtained from an algorithm (à la Kazhdan–Lusztig).*

²This large subcategory is sufficient to describe all simple modules by using spectral parameter shifts.

Remark 4.7. The algorithm is partly based on the existence of a quantization of the Grothendieck rings, introduced in [29, 33] for simply laced types using quiver varieties (and generalized in [16] using a different method). The deformation is then identified with a natural quantization of the cluster algebra (the quantum cluster algebra). Recently, with R. Fujita [13], we proposed a construction of monoidal Jantzen filtrations for monoidal categories, with certain degenerated braidings, in order to give a purely categorical construction of such quantum Grothendieck rings. This is based on distinguished filtrations by submodules

$$M = M_0 \supset M_1 \supset \cdots \supset M_N = \{0\}$$

which lead to deformed classes

$$[M]_t = \sum_{r \geq 0} t^r [M_r / M_{r-1}]$$

and to a deformation $*$ of the product of the Grothendieck ring by considering filtrations of tensor products.

As an illustration, in the first example of Section 4.4, we have the filtrations

$$V_1(q^2) \otimes V_1(1) \supset \mathbf{1} \supset \{0\} \quad \text{and} \quad V_1(1) \otimes V_1(q^2) \supset W \supset \{0\}$$

which lead to the deformed products

$$[V_1(q^2)] * [V_1(1)] = t[W] + 1 \quad \text{and} \quad [V_1(1)] * [V_1(q^2)] = t^{-1}[W] + 1.$$

5. Shifted quantum affine algebras

Next we handle shifted quantum groups. The shifted quantum affine algebras form a new class of quantum groups closely related to (quantized K -theoretical) Coulomb branches (in the sense of Braverman–Finkelberg–Nakajima [2]). They were introduced by Finkelberg–Tsybaliuk [6] in the study of these K -theoretical Coulomb branches.

Consider $\mathfrak{g}$ and q as above. The algebra $\mathcal{U}_q^\mu(\widehat{\mathfrak{g}})$ can be seen as a variation of the quantum affine algebra $\mathcal{U}_q(\widehat{\mathfrak{g}})$ depending on a shift parameter: a coweight μ of the Lie algebra $\mathfrak{g}$ (for example, in the $\mathfrak{sl}_2$ -case, we can see μ as an integer). For $\mu = 0$, $\mathcal{U}_q^0(\widehat{\mathfrak{g}})$ is essentially $\mathcal{U}_q(\widehat{\mathfrak{g}})$. These algebras have a very interesting representation theory.

Theorem 5.1 ([17]). *The algebra $\mathcal{U}_q^\mu(\widehat{\mathfrak{g}})$ has a non-zero finite-dimensional representation if and only if μ is codominant.*

In general, $\mathcal{U}_q^\mu(\widehat{\mathfrak{g}})$ has an abelian category $\mathcal{O}^\mu$ of representations (which might be finite dimensional or infinite dimensional). Recall that n is the rank of $\mathfrak{g}$.

Theorem 5.2 ([17]). *The simple objects in $\mathcal{O}^\mu$ are parametrized by n -tuples of rational fractions $\Psi = (\psi_i(z))_{1 \leq i \leq n}$ regular at 0, with the degree condition*

$$\deg(\psi_i(z)) = \alpha_i(\mu),$$

where α_i is the simple root of $\mathfrak{g}$ attached to the vertex i of the Dynkin diagram.

Example 5.3. For $1 \leq i \leq n$, $a \in \mathbb{C}^*$, consider Ψ defined by $\psi_j(z) = 1$ if $j \neq i$ and

$$\psi_i(z) = q^{r_i} \frac{1 - zaq^{-r_i}}{1 - zaq^{r_i}},$$

with r_i the length of the simple root α_i . Then we recover the finite-dimensional fundamental representation $V_i(a)$ of the ordinary quantum affine algebra (the rational fractions $\psi_j(z)$ should not be confused with the Drinfeld polynomials).

Example 5.4. For $1 \leq i \leq n$, $a \in \mathbb{C}^*$, consider

$$\Psi = ((1 - za)^{\delta_{i,j}})_{1 \leq j \leq n}.$$

The associated simple representation of $\mathcal{U}_q^{\omega_i^\vee}(\widehat{\mathfrak{g}})$ is called the positive prefundamental representation $L_{i,a}^+$ (here, $\omega_i^\vee$ is a fundamental coweight). It is of dimension 1 !

Example 5.5. For $1 \leq i \leq n$, $a \in \mathbb{C}^*$, consider

$$\Psi = ((1 - za)^{-\delta_{i,j}})_{1 \leq j \leq n}.$$

The associated simple representation of $\mathcal{U}_q^{-\omega_i^\vee}(\widehat{\mathfrak{g}})$ is called the negative prefundamental representation $L_{i,a}^-$. It is infinite dimensional. In addition, $\mathcal{U}_q^{-\omega_i^\vee}(\widehat{\mathfrak{g}})$ contains a copy of the ordinary quantum affine Borel algebra $\mathcal{U}_q(\widehat{\mathfrak{b}})$. Then, restricted to this subalgebra, $L_{i,a}^-$ is the simple prefundamental representation constructed in [18] as a limit of simple finite-dimensional representations.

Without loss of generality, we may assume that all roots and poles of the rational fractions are integral powers of q .

One obtains an abelian category

$$\mathcal{O}^{\text{sh}} = \bigoplus_{\mu \text{ coweight}} \mathcal{O}^\mu.$$

The (completed) sum of Grothendieck groups

$$K_0(\mathcal{O}^{\text{sh}}) = \bigoplus_{\mu} K_0(\mathcal{O}^\mu)$$

has a (topological) ring structure³ induced by the fusion product construction (which defines a representation $V * W$ in $\mathcal{O}^{\mu+\lambda}$ from simple representations V, W in $\mathcal{O}^{\mu}$ and $\mathcal{O}^{\lambda}$, respectively). This ring can be considered as an analogue of a Grothendieck ring. In addition, it is commutative.

Consider the subcategory $\mathcal{C}^{\text{sh}} \subset \mathcal{O}^{\text{sh}}$ of finite-dimensional representations of shifted quantum affine algebras. Its Grothendieck ring $K(\mathcal{C}^{\text{sh}})$ contains as a subring the Grothendieck ring $K(\mathcal{C})$ of finite-dimensional representations of the ordinary quantum affine algebra.

Theorem 5.6 ([17, 21, 25]). *The ring $K_0(\mathcal{C}^{\text{sh}})$ is isomorphic to a cluster algebra $\mathcal{A}_{\Gamma_{\infty}}$, with an explicit quiver Γ_{∞} . The initial cluster variables are classes of positive prefundamental representations. The cluster monomials correspond to certain classes of simple modules.*

Example 5.7. For $\mathfrak{g} = \mathfrak{sl}_2$, the quiver Γ_{∞} is the infinite linear quiver

$$\cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

The initial seed is formed of classes of one-dimensional positive prefundamental representations

$$\cdots \longrightarrow L_{1,q-2}^+ \longrightarrow L_{1,1}^+ \longrightarrow L_{1,q^2}^+ \longrightarrow \cdots$$

(up to factors by invertible representations, which we omit in these notes for clarity). The first-step mutations are given by certain remarkable relations called Baxter TQ -relations

$$[L_{1,a}^+][V_1(a)] = [L_{1,aq^2}^+] + [L_{1,aq^{-2}}^+],$$

where $V_1(a)$ is the two-dimensional fundamental representation. In fact, this relation is also relevant from the point of view of quantum integrable models.

Theorem 5.8 ([15]). *The Grothendieck ring $K_0(\mathcal{O}^{\text{sh}})$ is isomorphic to (a completion of) a cluster algebra $\mathcal{A}_{\Gamma'_{\infty}}$, with an explicit quiver Γ'_{∞} .*

The proof is partly based on additional symmetries of the Grothendieck ring that will be discussed in the next section.

Example 5.9. For $\mathfrak{g} = \mathfrak{sl}_2$, the new quiver Γ'_{∞} is obtained from the older quiver Γ_{∞} by just inverting the direction of one arrow:

$$\cdots \longrightarrow \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

³It is proved in [22] that the subcategory of finite length representations is stable under the fusion product, and so it gives rise to a (non-topological) ring structure on $K_0(\mathcal{O}^{\text{sh}})$.

Then the initial seed is made of positive and negative prefundamental representations

$$\cdots \longrightarrow L_{1,q^2}^+ \longrightarrow L_{1,1}^+ \longleftarrow L_{1,q^{-2}}^- \longrightarrow L_{1,q^{-4}}^- \longrightarrow \cdots .$$

The mutation at the vertex corresponding to $L_{1,1}^+$ yields

$$\cdots \longrightarrow L_{1,q^2}^+ \longleftarrow L_{1,1}^- \longrightarrow L_{1,q^{-2}}^- \longrightarrow L_{1,q^{-4}}^- \longrightarrow \cdots$$

as we have the remarkable QQ -relation

$$[L_{1,1}^+][L_{1,1}^-] = 1 + [L_{1,q^2}^+][L_{1,q^{-2}}^-]. \quad (5.1)$$

Similarly, the mutation at the vertex corresponding to $L_{1,q^{-2}}^-$ translated the seed in the other direction:

$$\cdots \longrightarrow L_{1,q^2}^+ \longrightarrow L_{1,1}^+ \longrightarrow L_{1,q^{-2}}^+ \longleftarrow L_{1,q^{-4}}^- \longrightarrow \cdots .$$

In general, we conjecture the following.

Conjecture 5.10 ([15]). *All cluster monomials in $\mathcal{A}_{\Gamma_\infty}'$ correspond to classes of simple objects in $\mathcal{O}$ through our isomorphism.*

This general conjecture is, for the moment, only known in the following case.

Theorem 5.11 ([15]). *The conjecture is true for $\mathfrak{g} = \mathfrak{sl}_2$.*

6. Symmetry of q -characters

We now discuss certain group symmetries of Grothendieck rings.

6.1. Classical theory

Let us briefly recall the well-known Weyl group symmetry of classical characters. Let $\mathfrak{g}$ be a complex finite-dimensional simple Lie algebra of rank n and $K_0(\mathfrak{g})$ be the Grothendieck ring of its finite-dimensional representations. The character morphism, which encodes the dimension of weight spaces of finite-dimensional representations, defines an injective ring morphism

$$\chi : K_0(\mathfrak{g}) \rightarrow \mathbb{Z}[y_i^{\pm 1}]_{1 \leq i \leq n}.$$

The image of χ can be characterized as an invariant subring

$$\mathrm{Im}(\chi) = (\mathbb{Z}[y_i^{\pm 1}]_{1 \leq i \leq n})^W,$$

where the Weyl group W is generated by the simple reflections s_i , $1 \leq i \leq n$, which act by the explicit formula

$$s_i(y_j) = y_j a_i^{-\delta_{ij}}, \quad \text{where } a_i = \prod_{1 \leq k \leq n} y_k^{C_{ji}}.$$

Here, a_i corresponds to a simple root, and the C_{ji} are coefficients of the Cartan matrix of $\mathfrak{g}$.

Example 6.1. For $\mathfrak{g} = \mathfrak{sl}_2$, the Cartan matrix $C = (2)$, and we have

$$\begin{aligned} a_1 &= y_1^2, \\ s_1(y_1) &= y_1 a_1^{-1} = y_1^{-1}, \\ s_1^2(y_1) &= y_1, \\ s_1(y_1 + y_1^{-1}) &= y_1 + y_1^{-1}, \\ \text{Im}(\chi) &= (\mathbb{Z}[y_1^{\pm 1}])^W = \mathbb{Z}[y_1 + y_1^{-1}]. \end{aligned}$$

6.2. q -characters

The analogue of characters for finite-dimensional representations of quantum affine algebras $\mathcal{U}_q(\widehat{\mathfrak{g}})$ are q -characters (as defined by Frenkel–Reshetikhin [12]). It defines an injective ring morphism on the Grothendieck ring $K(\mathcal{C})$ of the category $\mathcal{C}$ of finite-dimensional representations (here, we assume that the quantization parameter $q \in \mathbb{C}^*$ is not a root of unity):

$$\chi_q : K(\mathcal{C}) \rightarrow \mathcal{Y} = \mathbb{Z}[Y_{i,a}^{\pm 1}]_{1 \leq i \leq n, a \in \mathbb{C}^*}.$$

It is a refinement of characters for the action of the underlying finite-type quantum group $\mathcal{U}_q(\mathfrak{g})$ (as we can recover the ordinary by removing the spectral parameters a).

Example 6.2. For $a \in \mathbb{C}^*$, the fundamental representation $V_1(a)$ of $\mathcal{U}_q(\widehat{\mathfrak{sl}}_2)$ satisfies

$$\chi_q(V_1(a)) = Y_{1,a} + Y_{1,aq^2}^{-1}.$$

If we forget the spectral characters, we recover the ordinary character $Y_1 + Y_1^{-1}$.

There are monomial analogues of simple roots (we give here the formula for simply laced types):

$$A_{i,a} = Y_{i,aq^{-1}} Y_{i,aq} \prod_{j|C_{j,i}=-1} Y_{j,a}^{-1}.$$

For example, for $\mathfrak{g} = \mathfrak{sl}_2$, we have $A_{1,a} = Y_{1,aq^{-1}} Y_{1,aq}$.

We may wonder what the analogue of the Weyl group symmetry is in this context.

6.3. Symmetry of q -characters

We introduce in [9] the following operators:

$$\Theta_i(Y_{j,a}) = Y_{j,a} A_{i,aq^{-1}}^{-\delta_{i,j}} \frac{\Sigma_{i,aq^{-3}}^{\delta_{i,j}}}{\Sigma_{i,aq^{-1}}^{\delta_{i,j}}}$$

for $1 \leq i, j \leq n$ and $a \in \mathbb{C}^*$. Here, $\Sigma_{i,a}$ is the solution of the q -difference equation

$$\Sigma_{i,a} = 1 + A_{i,a}^{-1} \Sigma_{i,aq^{-2}}$$

in a sum

$$\Pi = \bigoplus_{w \in W} \tilde{\mathcal{Y}}^w$$

of completions $\tilde{\mathcal{Y}}^w$ of $\mathcal{Y}$, parametrized by the Weyl group.

Example 6.3. For $\mathfrak{g} = \mathfrak{sl}_2$, we have

$$\Theta_1(Y_{1,a}) = Y_{1,aq^{-2}}^{-1} \frac{\Sigma_{1,aq^{-3}}}{\Sigma_{1,aq^{-1}}}.$$

Here, $\Sigma_{1,a}$ is the couple

$$(1 + A_{1,a}^{-1}(1 + A_{1,aq^{-2}}^{-1}(1 + \cdots)), -A_{1,aq^2}(1 + A_{1,aq^4}(1 + \cdots))).$$

It belongs to

$$\Pi = \mathcal{Y}^e \oplus \mathcal{Y}^{s_1}.$$

Remark 6.4. One component of these operators is related to operators defined in [23] on a different space.

Now $\mathcal{Y}$ embeds in Π diagonally, and we have the following analogue of the classical result.

Theorem 6.5 ([9]). *The Θ_i are involutions and define a Weyl group action (with the simple reflections acting as Θ_i). Moreover, we have the ring of invariants*

$$\mathcal{Y}^W = \text{Im}(\chi_q).$$

6.4. Interpretation of the Weyl group action

Let us explain an interpretation of the Weyl group action above, which is connected to the cluster algebra symmetry.

Theorem 6.6 ([8, 17]). *The replacement of the variables*

$$Y_{i,a} = \frac{[L_{i,aq^{-1}}^+]}{[L_{i,aq}^+]}$$

in the q -character of finite-dimensional representations gives a well-defined relation in (the fraction field) of $K(\mathcal{O}^{\text{sh}})$.

Note that originally, this theorem was established in [8] in the context of representations of Borel subalgebra of quantum affine algebras (as in Example 5.5, in order to prove Frenkel–Reshetikhin conjecture on spectra of quantum integrable models), but it can also be stated for representations of shifted quantum affine algebras.

Motivated by quantum integrable models and Bethe ansatz equations, we expect that the class $[L_{i,a}^+]$ can be replaced in the theorem by the class of other simple representations that we denote by $\tilde{L}_{i,a}^+$ (see [11]). This is exactly how one can write the defining formula of the operator Θ_i , which is now interpreted as a substitution of simple classes in the Grothendieck ring.

We can now relate this picture to cluster algebra symmetry. Indeed, starting from $Q_a^{\omega_i^\vee} = [L_{i,a}^+]$, and using the Weyl group action, one can introduce for a general $w \in W$, $i \in I$,

$$Q_a^{w(\omega_i^\vee)} \in K(\mathcal{O}^{\text{sh}}).$$

Theorem 6.7 ([10] Generalized QQ -systems). *The series $Q_a^{w(\omega_i^\vee)}$ satisfy the QQ -system*

$$Q_{aq}^{(ws_i)(\omega_i^\vee)} Q_{aq^{-1}}^{w(\omega_i^\vee)} - Q_{aq^{-1}}^{(ws_i)(\omega_i^\vee)} Q_{aq}^{w(\omega_i^\vee)} = \prod_{j|C_{i,j}=-1} Q_a^{w(\omega_j^\vee)}.$$

Hence the same relation is satisfied by elements in an orbit under the Weyl group action.

Example 6.8. For $\mathfrak{g} = \mathfrak{sl}_2$, this QQ -system is the quantum Wronskian relation

$$\tilde{Q}_{aq} Q_{aq^{-1}} - \tilde{Q}_{aq^{-1}} Q_{aq} = 1, \quad (6.1)$$

where we have denoted $Q_a = Q_a^{\omega_1^\vee}$ and $\tilde{Q}_a = Q_a^{-\omega_1^\vee}$.

These QQ -system relations are crucial in the construction of the cluster algebra structure on $K(\mathcal{O}^{\text{sh}})$ in [15], as they appear in mutations of the cluster algebra (for example, compare equations (5.1) and (6.1)). Hence, cluster symmetries are related to group symmetry of Grothendieck rings in these examples, which one could find quite intriguing.

References

- [1] P. Baumann and S. Riche, [Notes on the geometric Satake equivalence](#). In V. Heiermann and D. Prasad (eds.), *Relative aspects in representation theory, Langlands functoriality and automorphic forms. CIRM Jean–Morlet Chair, Spring 2016*, pp. 1–134, Lecture Notes in Math. 2221, Springer, Cham, 2018 Zbl 1450.22009 MR 3839695
- [2] A. Braverman, M. Finkelberg, and H. Nakajima, [Towards a mathematical definition of Coulomb branches of 3-dimensional \$\mathcal{N} = 4\$ gauge theories, II](#). *Adv. Theor. Math. Phys.* **22** (2018), no. 5, 1071–1147 Zbl 1479.81043 MR 3952347
- [3] M. Brito and V. Chari, [Tensor products and \$q\$ -characters of HL-modules and monoidal categorifications](#). *J. Éc. polytech. Math.* **6** (2019), no. 3, 581–619 Zbl 1446.17026 MR 4014632
- [4] S. Cautis and H. Williams, [Cluster theory of the coherent Satake category](#). *J. Amer. Math. Soc.* **32** (2019), no. 3, 709–778 Zbl 1442.22022 MR 3981987
- [5] N. Chriss and V. Ginzburg, [Representation theory and complex geometry](#). Mod. Birkhäuser Class., Birkhäuser, Boston, MA, 1997, 495 pp. Zbl 0879.22001 MR 1433132
- [6] M. Finkelberg and A. Tsymbaliuk, [Multiplicative slices, relativistic Toda and shifted quantum affine algebras](#). In M. Gorelik, V. Hinich, and A. Melnikov (eds.), *Representations and nilpotent orbits of Lie algebraic systems*, pp. 133–304, Progr. Math. 330, Birkhäuser, Cham, 2019 Zbl 1436.17021 MR 3971731
- [7] S. Fomin and A. Zelevinsky, [Cluster algebras. I. Foundations](#). *J. Amer. Math. Soc.* **15** (2002), no. 2, 497–529 Zbl 1021.16017 MR 1887642
- [8] E. Frenkel and D. Hernandez, [Baxter’s relations and spectra of quantum integrable models](#). *Duke Math. J.* **164** (2015), no. 12, 2407–2460 Zbl 1332.82022 MR 3397389
- [9] E. Frenkel and D. Hernandez, [Spectra of quantum KdV Hamiltonians, Langlands duality, and affine opers](#). *Comm. Math. Phys.* **362** (2018), no. 2, 361–414 Zbl 1397.81060 MR 3843418
- [10] E. Frenkel and D. Hernandez, [Extended Baxter relations and QQ-systems for quantum affine algebras](#). *Comm. Math. Phys.* **405** (2024), no. 8, article no. 190, 42 pp. Zbl 1556.17028 MR 4779559
- [11] E. Frenkel and D. Hernandez, [Weyl group symmetry of \$q\$ -characters](#). *Selecta Math. (N.S.)* **31** (2025), no. 4, article no. 72, 36 pp. Zbl 08087587 MR 4939703

- [12] E. Frenkel and N. Reshetikhin, [The \$q\$ -characters of representations of quantum affine algebras and deformations of \$\mathcal{W}\$ -algebras](#). In N. Jing and K. C. Misra (eds.), *Recent developments in quantum affine algebras and related topics* (Raleigh, NC, 1998), pp. 163–205, Contemp. Math. 248, American Mathematical Society, Providence, RI, 1999
Zbl 0973.17015 MR 1745260
- [13] R. Fujita and D. Hernandez, [Monoidal Jantzen filtrations](#). *Adv. Math.* **495** (2026), article no. 110963, 81 pp. MR 5061762
- [14] R. Fujita, D. Hernandez, S.-j. Oh, and H. Oya, [Isomorphisms among quantum Grothendieck rings and cluster algebras](#). 2023, arXiv:2304.02562v2
- [15] C. Geiss, D. Hernandez, and B. Leclerc, [Representations of shifted quantum affine algebras and cluster algebras. I. The simply laced case](#). *Proc. Lond. Math. Soc. (3)* **129** (2024), no. 3, article no. e12630, 75 pp. Zbl 1564.17026 MR 4793282
- [16] D. Hernandez, [Algebraic approach to \$q, t\$ -characters](#). *Adv. Math.* **187** (2004), no. 1, 1–52
Zbl 1098.17009 MR 2074171
- [17] D. Hernandez, [Representations of shifted quantum affine algebras](#). *Int. Math. Res. Not. IMRN* **2023** (2023), no. 13, 11035–11126 Zbl 1532.17013 MR 4609778
- [18] D. Hernandez and M. Jimbo, [Asymptotic representations and Drinfeld rational fractions](#). *Compos. Math.* **148** (2012), no. 5, 1593–1623 Zbl 1266.17010 MR 2982441
- [19] D. Hernandez and B. Leclerc, [Cluster algebras and quantum affine algebras](#). *Duke Math. J.* **154** (2010), no. 2, 265–341 Zbl 1284.17010 MR 2682185
- [20] D. Hernandez and B. Leclerc, [Quantum Grothendieck rings and derived Hall algebras](#). *J. Reine Angew. Math.* **2015** (2015), no. 701, 77–126 Zbl 1315.17011 MR 3331727
- [21] D. Hernandez and B. Leclerc, [A cluster algebra approach to \$q\$ -characters of Kirillov–Reshetikhin modules](#). *J. Eur. Math. Soc. (JEMS)* **18** (2016), no. 5, 1113–1159
Zbl 1405.17028 MR 3500832
- [22] D. Hernandez and H. Zhang, [Jordan–Hölder property for shifted quantum affine algebras](#). 2025, arXiv:2501.16859v2
- [23] R. Inoue, [Cluster realization of Weyl groups and \$q\$ -characters of quantum affine algebras](#). *Lett. Math. Phys.* **111** (2021), no. 1, article no. 4, 32 pp. Zbl 1454.13033 MR 4197422
- [24] S.-J. Kang, M. Kashiwara, M. Kim, and S.-j. Oh, [Monoidal categorification of cluster algebras](#). *J. Amer. Math. Soc.* **31** (2018), no. 2, 349–426 Zbl 1460.13039 MR 3758148
- [25] M. Kashiwara, M. Kim, S.-j. Oh, and E. Park, [Monoidal categorification and quantum affine algebras II](#). *Invent. Math.* **236** (2024), no. 2, 837–924 Zbl 1551.17017
MR 4728243
- [26] B. Keller, [Cluster algebras and cluster monomials](#). In R. Latała, A. Ruciński, P. Strzelecki, J. Świątkowski, D. Wrzosek, and P. Zakrzewski (eds.), *European Congress of Mathematics*, pp. 399–413, European Mathematical Society, Zürich, 2013
Zbl 1364.13023 MR 3469134
- [27] M. Khovanov and A. D. Lauda, [A diagrammatic approach to categorification of quantum groups. I](#). *Represent. Theory* **13** (2009), 309–347 Zbl 1188.81117 MR 2525917

- [28] S. Mac Lane, *Categories for the working mathematician*. 2nd edn., Grad. Texts in Math. 5, Springer, New York, 1998, 318 pp. Zbl [0906.18001](#) MR [1712872](#)
- [29] H. Nakajima, *Quiver varieties and t -analogs of q -characters of quantum affine algebras*. *Ann. of Math. (2)* **160** (2004), no. 3, 1057–1097 Zbl [1140.17015](#) MR [2144973](#)
- [30] H. Nakajima, *Quiver varieties and cluster algebras*. *Kyoto J. Math.* **51** (2011), no. 1, 71–126 Zbl [1223.13013](#) MR [2784748](#)
- [31] F. Qin, *Triangular bases in quantum cluster algebras and monoidal categorification conjectures*. *Duke Math. J.* **166** (2017), no. 12, 2337–2442 Zbl [1454.13037](#) MR [3694569](#)
- [32] R. Rouquier, 2-Kac–Moody algebras. 2008, arXiv:[0812.5023v1](#)
- [33] M. Varagnolo and E. Vasserot, *Perverse sheaves and quantum Grothendieck rings*. In A. Joseph, A. Melnikov, and R. Rentschler (eds.), *Studies in memory of Issai Schur (Chevaleret/Rehovot, 2000)*, pp. 345–365, Progr. Math. 210, Birkhäuser, Boston, MA, 2003 Zbl [1162.17307](#) MR [1985732](#)

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