

Introduction to inverse problems for non-linear partial differential equations

Matti Lassas

Abstract. We consider inverse problems for non-linear hyperbolic and elliptic equations and give an introduction to the method based on the multiple linearization, or on the construction of artificial sources, to solve these problems. The method is based on self-interaction of linearized waves or other solutions in the presence of non-linearities. Multiple linearization has successfully been used to solve inverse problems for non-linear equations which are still unsolved for the corresponding linear equations.

1. Introduction

In this paper, we consider inverse problems for non-linear elliptic and hyperbolic partial differential equations to demonstrate the *multiple linearization* method. In this method, the non-linearity helps one to solve inverse problems for non-linear equations which are still unsolved for the corresponding linear equations. Our aim is to give a review of recent progress on these inverse problems and discuss the main ideas of the proofs in simple cases.

In the case, where the underlying equation is linear, a standard example is the inverse problem of Calderón [15]. This problem arises in electrical impedance tomography (EIT), a method proposed for medical and industrial imaging, where the objective is to determine the electrical conductivity or impedance in a medium by making voltage-to-current measurements on its boundary. Another example of Calderón's inverse problem is the determination of an unknown potential $q(x)$ in a Schrödinger operator $\Delta + q(x)$ from boundary measurements, first solved in [63] in dimensions $n \geq 3$.

Let us next consider analogous inverse problems under non-linear settings. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^∞ -smooth boundary, and consider the equation

$$\Delta u + a(x, u) = 0 \quad \text{in } \Omega.$$

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The Dirichlet problem for this equation is related to maintaining a temperature (or concentration or population) f on the boundary. The boundary measurements for such an equation, provided that it is well posed for some class of boundary values, may be encoded by a Dirichlet-to-Neumann map Λ_a , which maps the boundary value f to the flux (i.e., the normal derivative) $\Lambda_a(f) = \partial_\nu u|_{\partial\Omega}$ of the corresponding equilibrium state across the boundary.

Since the 1990s, inverse problems for non-linear elliptic equations have been widely studied. A standard method, introduced in [29] in the parabolic case, is to show that the first linearization of the non-linear Dirichlet-to-Neumann map is actually the Dirichlet-to-Neumann map of a linear equation and to use the theory of inverse problems for linear equations. For the semilinear Schrödinger equation $\Delta u + a(x, u) = 0$, the problem of recovering the potential $a(x, u)$ was studied in [61] in dimensions $n \geq 3$ and in [26, 30] in dimension $n = 2$. In addition, inverse problems have been studied for quasilinear elliptic equations [31, 58, 62], the degenerate elliptic p -Laplace equation [13, 59], and the fractional semilinear Schrödinger equation [45]. Certain Calderón-type inverse problems for quasilinear equations on Riemannian manifolds are considered in [51]; see also references in [1, 65].

Let us introduce the mathematical setting to study canonical examples of non-linear elliptic equations. We will denote by (M, g) a compact Riemannian manifold with C^∞ boundary ∂M , where $\dim(M) = n$, $n \geq 2$. Let $q \in C^\infty(M)$. We will consider semilinear elliptic equations of the form

$$\begin{cases} \Delta_g u + qu^m = 0 & \text{in } M, \\ u = f & \text{on } \partial M, \end{cases} \quad (1.1)$$

where $m \in \mathbb{N}$ and $m \geq 2$. Here, Δ_g is the Laplace–Beltrami operator, given in local coordinates by

$$\Delta_g u = \frac{1}{\det(g)^{1/2}} \sum_{j,k=1}^n \frac{\partial}{\partial x^j} \left(\det(g)^{1/2} g^{jk} \frac{\partial u}{\partial x^k} \right),$$

where $g = (g_{jk}(x))$ and $g^{-1} = (g^{jk}(x))$. As usual, we denote the Euclidean Laplace operator, corresponding to the Euclidean metric $g^{(e)} = \delta_{jk}$ in $\mathbb{R}^n$ by Δ .

The Dirichlet problem for (1.1) with a boundary condition $u|_{\partial\Omega} = f$ has a unique small solution u for sufficiently small boundary data $f \in C^s(\partial M)$, where $s > 2$ with $s \notin \mathbb{N}$. More precisely, this means that there is $\delta > 0$ such that whenever $\|f\|_{C^s(\partial M)} \leq \delta$, there is a unique solution u to (1.1) satisfying $\|u\|_{C^s(M)} \leq c\delta$. We will call u the unique small solution and denote it by u_f to indicate the boundary value. Here, C^s is the standard Hölder space for $s > 2$ with $s \notin \mathbb{N}$ (often written

as $C^{k,\alpha}$ if $s = k + \alpha$, where $k \in \mathbb{Z}$ and $0 < \alpha < 1$). Hence, the Dirichlet-to-Neumann map is defined by using the unique small solution in the following way:

$$\Lambda_{M,g,q} : \{f \in C^s(\partial M) \mid \|f\|_{C^s(\partial M)} \leq \delta\} \rightarrow C^{s-1}(\partial M),$$

$$\Lambda_{M,g,q} : f \mapsto \partial_\nu u_f|_{\partial M},$$

where ∂_ν denotes the normal derivative on the boundary ∂M . In what follows, we denote the Dirichlet-to-Neumann map by $\Lambda_{M,g}$ to when $q = 0$. When $M = \Omega \subset \mathbb{R}^n$ and g is the identity matrix, we denote the Dirichlet-to-Neumann map by Λ_q .

Next, we consider the case when $\Omega \subset \mathbb{R}^n$ for $n \geq 2$ and Δ_g is the Euclidean Laplacian.

Theorem 1.1 ([46, Reconstruction of non-linear term]). *Let $n \geq 2$, and let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^∞ boundary $\partial\Omega$. Let $q_1, q_2 \in C^\infty(\bar{\Omega})$. Assume the Dirichlet-to-Neumann maps Λ_{q_j} for equation (1.1) with potentials q_j , $j = 1, 2$, satisfy*

$$\Lambda_{q_1}(f) = \Lambda_{q_2}(f)$$

for all $f \in C^s(\partial\Omega)$ with $\|f\|_{C^s(\partial\Omega)} < \delta$, where $\delta > 0$ is any sufficiently small number. Then $q_1 = q_2$ in Ω .

In the pioneering studies of the problem, the first- and the second-order linearizations of the non-linear Dirichlet-to-Neumann map have been used in the works [61, 62] related to non-linear equations with matrix coefficients and in [31] for a non-linear conductivity equation (see also [18]).

Next, we present the outline of the proof of Theorem 1.1. For equation (1.1) with quadratic non-linearity, that is, $m = 2$, the first linearization of the non-linear Dirichlet-to-Neumann map Λ_q , linearized at the zero boundary value, is just the Dirichlet-to-Neumann map for the standard Laplace equation

$$D|_{f=0}\Lambda_q(f) : C^s(\partial\Omega) \rightarrow C^{s-1}(\partial\Omega), \quad f \mapsto \partial_\nu v_f|_{\partial\Omega},$$

where $s > 1$ is not an integer, v_f is the unique solution of $\Delta v_f = 0$ in Ω with $v_f|_{\partial\Omega} = f$. Thus, the first linearization does not carry any information about the unknown potential q . For a quadratic non-linearity, the *second linearization* of the map $f \rightarrow \Lambda_q(f)$, denoted by $D^2|_{f=0}\Lambda_q(f)[\cdot, \cdot]$, is a symmetric bilinear form from $C^s(\partial\Omega) \times C^s(\partial\Omega)$ to $C^{s-1}(\partial\Omega)$. The second-order linearization is given by

$$D^2|_{f=0}\Lambda_q(f)[f_1, f_2] = \partial_{\varepsilon_1} \partial_{\varepsilon_2} u_{\varepsilon_1 f_1 + \varepsilon_2 f_2}|_{\varepsilon_1 = \varepsilon_2 = 0} \quad \text{on } \partial\Omega.$$

That is, one considers boundary data $f = \varepsilon_1 f_1 + \varepsilon_2 f_2 \in C^s(\partial\Omega)$, where $\varepsilon_1, \varepsilon_2$ are sufficiently small parameters, and takes the mixed derivative

$$w(x) = \frac{\partial}{\partial \varepsilon_1} \frac{\partial}{\partial \varepsilon_2} \bigg|_{\varepsilon_1 = \varepsilon_2 = 0} u_{\varepsilon_1 f_1 + \varepsilon_2 f_2}(x)$$

of the solutions of $u_{\varepsilon_1 f_1 + \varepsilon_2 f_2}(x)$ of equation (1.1) with the coefficients $q, m = 2$ and the boundary value $f = \varepsilon_1 f_1 + \varepsilon_2 f_2$. This yields the equation

$$\Delta w = -2q v_{f_1} v_{f_2}, \quad (1.2)$$

where v_{f_k} are harmonic functions, that is, the solutions to the linearized equation $\Delta v = 0$ with the Dirichlet boundary value $v_{f_2}|_{\partial\Omega} = f_k$. Moreover,

$$D^2|_{f=0} \Lambda_q(f)[f_1, f_2] = \partial_\nu w|_{\partial\Omega}.$$

Assume next that there are two potential functions q_1 and q_2 such that

$$\Lambda_{q_1}(f) = \Lambda_{q_2}(f) \quad \text{for all } f \in C^s(\partial M), \quad \|f\|_{C^s(\partial M)} \leq \delta.$$

Taking the mixed second-order derivative of the Dirichlet-to-Neumann maps yields $\partial_\nu w_1 = \partial_\nu w_2$ on $\partial\Omega$, where $\Delta w_j = -2q_j v_{f_1} v_{f_2}$ and $w_j|_{\partial\Omega} = 0$. Subtracting equation (1.2) for q_1 and q_2 and integrating the resulting equation against the harmonic function v_{f_3} and using integration by parts yields the formula

$$\begin{aligned} \int_{\Omega} (q_1 - q_2) v_{f_1} v_{f_2} v_{f_3} dx &= -\frac{1}{2} \int_{\Omega} (\Delta w_1 - \Delta w_2) v_{f_3} dx \\ &= -\frac{1}{2} \int_{\partial\Omega} (\partial_\nu w_1 - \partial_\nu w_2) v_{f_3} dS(x) = 0. \end{aligned} \quad (1.3)$$

To apply this formula, let us denote $Q = q_1 - q_2$. When $k \in \mathbb{R}^3$, there are $\xi_1, \xi_2 \in \mathbb{C}^3$ satisfying

$$\xi_1 \cdot \xi_1 = 0, \quad \xi_2 \cdot \xi_2 = 0, \quad \xi_1 + \xi_2 = ik,$$

where $\eta \cdot \zeta = \eta_1 \zeta_1 + \eta_2 \zeta_2 + \eta_3 \zeta_3$ for $\eta = (\eta_1, \eta_2, \eta_3), \zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathbb{C}^3$. We define

$$v_1(x) = e^{\xi_1 \cdot x}, \quad v_2(x) = e^{\xi_2 \cdot x}, \quad v_3(x) = 1.$$

These functions satisfy $\Delta v_j = 0$, and we can use formula (1.3) with $f_j = v_j|_{\partial\Omega}$, that is, $v_{f_j} = f_j$. The Fourier transform $\hat{Q}(k)$ of $Q(x)$ (i.e., extended as a zero function in $\mathbb{R}^n \setminus \Omega$) satisfies

$$\hat{Q}(k) = \int_{\Omega} Q(x) e^{ik \cdot x} dx = \int_{\Omega} Q(x) e^{\xi_1 \cdot x} \cdot e^{\xi_2 \cdot x} \cdot 1 dx = \int_{\Omega} Q v_1 v_2 v_3 dx = 0.$$

Hence, $Q = 0$, that is, $q_1 = q_2$. These arguments (with some details) prove Theorem 1.1.

Analogous inverse problems for various non-linear elliptic equations using multiple linearization has been studied in [16, 21, 35, 39–41].

We move on to describe our next result. By using higher-order linearizations, we can solve the following *simultaneous recovery* on a two-dimensional Riemannian surface and coefficients of equations.

Theorem 1.2 ([46, Simultaneous recovery of metric and potential]). *Let (M_1, g_1) and (M_2, g_2) be two compact connected manifolds with mutual C^∞ smooth boundaries $\partial M_1 = \partial M_2 =: \partial M$ and $\dim(M_1) = \dim(M_2) = 2$. Let $m \geq 2$, and let Λ_{M_j, g_j, q_j} be the Dirichlet-to-Neumann maps $\Lambda_{M_j, q_j} : f \mapsto \partial_\nu u|_{\partial M}$ of the problem*

$$\Delta_{g_j} u + q_j u^m = 0 \quad \text{in } M_j$$

for $j = 1, 2$. Let $s > 2$ with $s \notin \mathbb{N}$ and assume that

$$\Lambda_{M_1, g_1, q_1} f = \Lambda_{M_2, g_2, q_2} f \quad \text{on } \partial M$$

for any $f \in C^s(\partial M)$ with $\|f\|_{C^s(\partial M)} \leq \delta$, where $\delta > 0$ is sufficiently small. Then there exist a diffeomorphism $\Psi : M_1 \rightarrow M_2$ and a positive smooth function σ on M_1 such that for $x \in M_1$, we have

$$(\sigma \Psi^* g_2)(x) = g_1(x), \quad \sigma q_1 = q_2 \circ \Psi \quad \text{in } M_1, \quad (1.4)$$

with $\Psi|_{\partial M} = \text{Id}$ and $\sigma|_{\partial M} = 1$.

For the linear equation $\Delta_g u + qu = 0$, an analogous result has been proved when M is a domain in $\mathbb{R}^2$ with a Riemannian metric [28], when M is a manifold and the potential q is zero [55], and when the conformal class of the manifold (M, g) is a priori known [24]. The recovery of properties of both the manifold and potential up to the gauge transformation (1.4) was a long-standing open question that was solved in 2024 in by Cârstea, Liimatainen, and Tzou (see [17]) using much more advanced techniques that the problem for the non-linear equation needs.

Next, we consider the *inverse obstacle problem* for elliptic equations with power-type non-linearities. Let $\Omega \subset \mathbb{R}^n$ be a bounded, connected open set with a C^∞ -smooth boundary $\partial\Omega$. Let D be an unknown cavity in Ω . More precisely, $D \subset \Omega$ is an open set with a smooth boundary ∂D such that $\overline{D} \subset \Omega$. Moreover, let $m \in \mathbb{N}$ with $m \geq 2$. Consider the following semilinear elliptic equation, with an unknown coefficient $q = q(x)$ and an unknown relatively compact domain $D \subset \Omega$:

$$\begin{cases} \Delta u + qu^m = 0 & \text{in } \Omega \setminus \overline{D}, \\ u = 0 & \text{on } \partial D, \\ u = f & \text{on } \partial\Omega. \end{cases} \quad (1.5)$$

For $s > 1$ and $s \in \mathbb{N}$, let $f \in C^s(\partial M)$ with $\|f\|_{C^s(\partial\Omega)} < \delta$, where $\delta > 0$ is any sufficiently small number. Then by the well-posedness of (1.5) [46, Proposition 2.1] again, one can define the corresponding Dirichlet-to-Neumann map $\Lambda_{D, q}$ on $\partial\Omega$ with

$$\Lambda_{D, q} : \{f \in C^s(\partial M) \mid \|f\|_{C^s(\partial M)} \leq \delta\} \rightarrow C^{s-1}(\partial\Omega), \quad \Lambda_{D, q} : f \mapsto \partial_\nu u_f|_{\partial\Omega},$$

where $s > 1$ is not an integer and u is the unique small solution of (1.5). The inverse obstacle problem is to determine both the unknown cavity D and potential q from the Dirichlet-to-Neumann map $\Lambda_{D,q}$.

Theorem 1.3 ([47, Simultaneous recovery: Cavity and potential]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain for $n \geq 2$ with a C^∞ -smooth boundary $\partial\Omega$. Let $D_1, D_2 \Subset \Omega$ be non-empty open cavities with C^∞ -smooth boundaries and $q_j \in C^\infty(\Omega \setminus \overline{D_j})$. Let $\Lambda_{D_j, q_j} : f \mapsto \partial_\nu u|_{\partial\Omega}$, $j = 1, 2$, be the Dirichlet-to-Neumann maps for equation (1.5) with $D = D_j$ and $q = q_j$, $j = 1, 2$. Assume that $\Lambda_{D_1, q_1} f = \Lambda_{D_2, q_2} f$ for all sufficiently small $f \in C^\infty(\partial\Omega)$ on $\partial\Omega$. Then $D_1 = D_2$ and $q_1 = q_2$ in $\Omega \setminus \overline{D_1}$.*

The simultaneous recovery of an obstacle and an unknown surrounding potential in dimensions $n \geq 3$ is a long-standing open problem. This problem is closely related to the *partial data* Calderón problem studied in [27, 34], where the boundary is known a priori, but the boundary measurements are done only on a subset of the boundary.

2. Inverse problems for waves

Let us consider a linear wave equation

$$\partial_t^2 u(t, x) - c(x)^2 \Delta u(t, x) = 0 \quad \text{in } \mathbb{R} \times \Omega,$$

where $\Omega \subset \mathbb{R}^n$ and $c(x)$ is the wave speed. This equation models, e.g., acoustic waves in an isotropic medium. In inverse problems, one has access to measurements of waves (the solutions $u(t, x)$) on the boundary, or in a subset of the domain Ω , and one aims to determine unknown coefficients (e.g., $c(x)$) in the interior of the domain.

We will consider *anisotropic* materials, where the wave speed depends on the direction of propagation. This means that the scalar wave speed $c(x)$, where $x = (x^1, x^2, \dots, x^n) \in \Omega$, is replaced by a positive definite symmetric matrix $(g^{jk}(x))_{j,k=1}^n$. Anisotropic materials appear frequently in applications such as seismic imaging, where one wishes to determine the interior structure of the Earth by making various measurements of waves on its surface.

To model waves in an anisotropic medium mathematically, let (N, g) be an n -dimensional Riemannian manifold with boundary and consider the wave equation

$$\begin{aligned} \partial_t^2 u(t, x) - \Delta_g u(t, x) &= 0 \quad \text{in } (0, \infty) \times N, \\ \partial_\nu u|_{\mathbb{R}_+ \times \partial N} &= f, \quad u|_{t=0} = 0, \quad \partial_t u|_{t=0} = 0, \end{aligned} \tag{2.1}$$

where Δ_g is the Laplace–Beltrami operator corresponding to a smooth time-independent Riemannian metric g on N . We denote by SN the unit sphere bundle,

$SN = \{(y, \eta) \in TN \mid \|\eta\|_g = 1\}$, and the unit speed geodesic of (N, g) emanating from a point $(y, \eta) \in SN$ by $\alpha_{y, \eta}(t) = \exp_p(t\eta)$.

The solution of (2.1), corresponding to the boundary value f (which is interpreted as a boundary source), is denoted by $u^f = u^f(t, x)$.

Let us assume that the boundary ∂N is known. The inverse problem is to reconstruct the manifold N and the metric g when we are given the boundary ∂N , as a manifold, and the *response operator*

$$\Lambda_{N, g} : f \mapsto u^f|_{\mathbb{R}_+ \times \partial N},$$

which is also called the *Neumann-to-Dirichlet map*. Physically, $\Lambda_{N, g} f$ describes the measurement of the medium response to any applied boundary source f .

It is convenient to interpret the anisotropic wave speed $(g^{jk}(x))$ also in a domain of the Euclidean space as the inverse of a Riemannian metric, thus modeling the medium as a *Riemannian manifold*. This is due to the fact that if $\Psi : N \rightarrow N$ is a diffeomorphism such that $\Psi|_{\partial N} = \text{Id}$, then all boundary measurements for the metric g and the pullback metric Ψ^*g coincide. Thus, to prove uniqueness results for inverse problems, one has to consider properties that are invariant in diffeomorphisms and try to reconstruct those uniquely, for example, to show that an underlying manifold structure can be uniquely determined. In practice, the inverse problem in a subset of the Euclidean space is solved in two steps. The first is to reconstruct the underlying manifold structure. The second step is to find an embedding of the constructed manifold in the Euclidean space using additional a priori information; see [32]. Below, we will concentrate on the first step.

In the 1990s, the combination of the boundary control method developed by Belishev [6] and generalized for Riemannian manifolds in [7] and Tataru's unique continuation theorem [64] gave a solution to the inverse problem of determining the isometry type of a Riemannian manifold (N, g) with given boundary ∂N and the Dirichlet-to-Neumann map $\Lambda_{N, g}$.

Theorem 2.1 ([7, 64]). *Let (N_1, g_1) and (N_2, g_2) be compact Riemannian manifolds with boundary. Assume that there is a diffeomorphism $\Phi : \partial N_1 \rightarrow \partial N_2$ such that*

$$\Phi^*(\Lambda_{N_1, g_1} f) = \Lambda_{N_2, g_2}(\Phi^* f) \quad \text{for all } f \in C_0^\infty(\mathbb{R}_+ \times \partial N_1).$$

Then (N_1, g_1) and (N_2, g_2) are isometric Riemannian manifolds.

Above, $\Phi^* f(t, x) = f(t, \Phi(x))$ is the pullback of f in Φ or, more precisely, that of $\text{Id}_{\mathbb{R}} \times \Phi : \mathbb{R} \times \partial N_1 \rightarrow \mathbb{R} \times \partial N_2$.

The original method to prove Theorem 2.1 was based on combining the spectral theory for the Laplace operator and control theory for the wave equation. An

alternative way to prove Theorem 2.1 is to show that the hyperbolic Dirichlet-to-Neumann map $\Lambda_{N,g}$ can be used to determine a sequence of Dirichlet boundary values f_j for which the corresponding waves $u^{f_j}(x, T)$ focus at time $T > 2 \text{diam}(N)$; see [10, 19, 36].

Theorem 2.2 ([36, Focusing of waves]). *Let $T > 2 \text{diam}(N)$, $z \in \partial N$, $s > 0$, and $y \in N$ be a point on the normal geodesic emanating from z , $y = \alpha_{z,v}(s)$ such that $\text{dist}(y, \partial N) = s$. Using the Neumann-to-Dirichlet map $\Lambda_{N,g}$, one can construct a sequence of boundary sources $f_j \in C_0^\infty((0, T) \times \partial N)$, $j = 1, 2, \dots$, such that for some $C_y \neq 0$ at the time T*

$$\lim_{j \rightarrow \infty} \begin{pmatrix} u^{f_j}(T, x) \\ \partial_t u^{f_j}(T, x) \end{pmatrix} = \begin{pmatrix} 0 \\ C_y \delta_y(x) \end{pmatrix} \quad \text{for } x \in N,$$

where the limit takes place in the space of generalized functions, that is, in $\mathcal{D}'(N) \times \mathcal{D}'(N)$.

Let $G(t, x; y)$ be Green's function of the wave operator, that is, the wave produced by a point source at the point y at the time zero:

$$\begin{aligned} (\partial_t^2 - \Delta_g)G(t, x; y) &= \delta_{(0,y)}(t, x) \quad \text{on } (t, x) \in \mathbb{R}_+ \times N, \\ G|_{(t,x) \in \mathbb{R}_+ \times \partial N} &= 0, \quad G|_{t=0} = 0, \quad \partial_t G|_{t=0} = 0, \end{aligned}$$

where Δ_g operates in the x coordinate. Theorem 2.2 implies that the waves $u^{f_j}(t, x)$ converge in the sense of distributions to $G(t - T, x; y)$ in the domain $x \in N$, $t > T$. This means that using the Dirichlet-to-Neumann map, it is possible to construct functions $(\Lambda_{N,g} f_j)(x, t - T)$ that converge, as $j \rightarrow \infty$, to the boundary values $\partial_\nu G(\cdot, \cdot; y)|_{\mathbb{R}_+ \times \partial N}$ of the waves $G(\cdot, \cdot; y)$ produced by point sources $C_y \delta_y(x)$ at the time $t = 0$. The first time $t > 0$ when the wave $G(\cdot, \cdot; y)$ is observed to be non-vanishing at the boundary point $x \in \partial N$ is defined to be

$$\begin{aligned} \mathbf{t}_y(x) &= \sup\{t \in \mathbb{R} \mid \text{the interval } \{z\} \times (0, t) \text{ has a neighborhood} \\ &\quad U \subset \mathbb{R} \times \partial N \text{ such that } \partial_\nu G(\cdot, \cdot; y)|_U = 0\}. \end{aligned}$$

Using the finite propagation speed of waves and Tataru's unique continuation theorem [64], it turns out that $\mathbf{t}_y(x) = d_g(x, y)$; see [54]. Thus, Theorem 2.2 implies that the Dirichlet-to-Neumann map $\Lambda_{N,g}$ for the wave equation determines the functions

$$\mathbf{t}_{\alpha_{z,v}(s)}(x) = d_g(x, \alpha_{z,v}(s)) \quad \text{for } x, z \in \partial N \text{ and } s > 0 \text{ such that } \text{dist}(x, z) = s. \quad (2.2)$$

Later, we will return to these data, but before that, we consider the consequences of Theorem 2.1.

Theorem 2.1 can also be used to prove the uniqueness of several other inverse problems. The equivalence of inverse spectral problems and several inverse boundary value problems for wave and heat equations is considered in [33]. In particular, the above results and [33] imply the following theorem.

Theorem 2.3 ([33]). *Suppose (N, g) is a compact smooth Riemannian manifold with boundary ∂N . Then the boundary ∂N and Neumann-to-Dirichlet map $\Lambda : \partial_\nu u|_{\mathbb{R}_+ \times \partial N} \mapsto u|_{\mathbb{R}_+ \times \partial N}$, for the heat equation $(\partial_t - \Delta_g)u = 0$ or for the Schrödinger equation $(i\partial_t - \Delta_g)u = 0$, with vanishing initial data $u|_{t=0} = 0$ determine the Neumann-to-Dirichlet map for the wave equation, and therefore the manifold (N, g) up to an isometry.*

The stability of the solutions of the above inverse problems has been analyzed in [4, 5, 12, 14, 42, 52, 60].

Without making strong assumptions about the geometry of the manifold, the existing uniqueness results for the inverse problems for the linear hyperbolic equations with vanishing initial data are limited to equations whose coefficients are time independent or real-analytic in time (see, e.g., [2, 4, 7, 20, 25, 32, 37, 38, 44, 53]). The reason for this is that these results are based on Tataru's unique continuation theorem [64]. This sharp unique continuation result does not work for general wave equations whose coefficients are not real-analytic in time, as shown by Alinhac [3]. Solving the inverse problem of finding the Riemannian metric (up to an isometry), that is, proving Theorem 2.1, can be reduced to determining the metric from passive observations from sources inside the domain, and due to this, we next consider passive observations from the point sources.

2.1. Inverse problems with passive measurements

2.1.1. Lorentzian light observation sets and waves on a Riemannian manifold.

Let us consider a body in which there spontaneously appear point sources that create propagating waves. In this subsection, we assume that the velocity of the waves is independent of time and depends only on the spatial variables and model the wave speed using a Riemannian metric. We consider the inverse problem where we detect such waves either outside or at the boundary of the body and aim to determine the unknown wave speed inside the body. As an example of such a situation, one can consider the micro-earthquakes that appear very frequently near active faults. The related inverse problem is whether the surface observations of elastic waves produced by the micro-earthquakes can be used in the geophysical imaging of Earth's subsurface, that is, to determine the speed of the elastic waves in the studied volume. Below, we consider a highly idealized version of the above inverse problem. We consider the problem on an n -dimensional manifold N with a Riemannian metric $h = h_{ij}(x)$ that

corresponds to the travel time of a wave between two points (below, we will consider Lorentzian metric tensors for which we reserve the symbol g and use now the letter h for the Riemannian metric). The Riemannian distance of points $x, y \in N$ is denoted by $d_h(x, y)$. For simplicity, we assume that the manifold N is compact and has no boundary. To do this, we assume that the manifold contains a known part F and the metric is unknown outside this set, that is, in $N \setminus F$. When a spontaneous point source produces a wave at some unknown point $x \in N \setminus F$ at some unknown time $t \in \mathbb{R}$, the produced wave is observed at the point $z \in F$ at time $T_{x,t}(z) = d_h(z, x) + t$. Note that these functions are analogous to functions (2.2) (that were defined for the manifolds with boundary) with the source points $x = \alpha_{z,v}(s)$. These observation times at two points $z_1, z_2 \in F$ determine the *distance difference function*

$$D_x(z_1, z_2) = T_{x,t}(z_1) - T_{x,t}(z_2) = d_h(z_1, x) - d_h(z_2, x).$$

Physically, this function corresponds to the difference of times when the waves produced by the point source at (x, t) are observed at z_1 and z_2 . Below, we assume that there are a large number of point sources and that we are given the set F and the family of functions $\{D_x \mid x \in X\} \subset C(F \times F)$, where $X \subset N$ is either the whole set $N \setminus F$ or a dense subset of X .

Next, let (N_1, h_1) and (N_2, h_2) be compact and connected Riemannian manifolds without boundary. Let $d_{h_j}(x, y)$ denote the Riemannian distance of points $x, y \in N_j$, $j = 1, 2$. For $x \in N_j$ and $z_1, z_2 \in F_j$, we denote $D_x^j(z_1, z_2) = d_{h_j}(x, z_1) - d_{h_j}(x, z_2)$.

To prove the uniqueness of the passive imaging inverse problem, we assume the following:

- (i) The map $\phi : F_1 \rightarrow F_2$ is an isometry, that is, $d_{h_1}(z, w) = d_{h_2}(\phi(z), \phi(w))$ for all $z, w \in F_1$.
- (ii) The collections $\mathcal{D}_j(N_j) = \{D_x^j(\cdot, \cdot) \mid x \in N_j\} \subset C(F_j \times F_j)$ with $j = 1, 2$ are equivalent, that is, $\{D_x^1(\cdot, \cdot) \mid x \in N_1\} = \{D_y^2(\phi(\cdot), \phi(\cdot)) \mid y \in N_2\}$.

Theorem 2.4 ([54, Distance difference functions determine the manifold]). *Let (N_1, h_1) and (N_2, h_2) be compact and connected Riemannian manifolds of dimension $n \geq 2$ without boundary. Let $F_j \subset N_j$, $j = 1, 2$, be closures of non-empty open sets having a smooth manifold. Suppose the above conditions (i) and (ii) are valid. Then the manifolds (N_1, h_1) and (N_2, h_2) are isometric.*

We will not consider here the proof of Theorem 2.4 which is based on Riemannian geometry. Instead, we discuss next how the distance difference functions on a Riemannian manifold (N, h) determine the earliest light observation sets on the Lorentzian manifold $M = \mathbb{R} \times N$ with the product Lorentzian metric $g = -dt^2 + h$ and consider the determination of the Lorentzian metric using the knowledge of these

sets. Let (N, h) be a Riemannian manifold. We denote the unit speed geodesic of (N, h) emanating from a point $(y, \eta) \in SN$ by $\alpha_{y, \eta}^h(t) = \exp_p(t\eta)$. Here, $SN = \{(y, \eta) \in TN \mid \|\eta\|_h = 1\}$ is the unit sphere bundle of N . Using the Riemannian manifold (N, h) , we define a product manifold $M = \mathbb{R} \times N$. When $U \subset N$ is an open set having coordinates $X : U \rightarrow \mathbb{R}^n$, $X(x) = (x^1(x), x^2(x), \dots, x^n(x))$, we define a Lorentzian metric $g = -dt^2 + h(x)$ that is given in local coordinates $(t, x^1, \dots, x^n) \in \mathbb{R} \times U$ by

$$g = -dt^2 + \sum_{j,k=1}^n h_{jk}(x^1, \dots, x^n) dx^j dx^k.$$

Below, we denote the time coordinate also by $x^0 = t$. Let $p = (t_0, y) \in \mathbb{R} \times N$ and $\xi = \frac{\partial}{\partial t} + \eta$ be a vector at the point p , where $(y, \eta) \in SN$. Observe that $g(\xi, \xi) = -1 + \|\eta\|_h^2 = 0$, that is, ξ is a future-pointing light-like vector. The curve $\gamma_{p, \xi} : \mathbb{R} \rightarrow \mathbb{R} \times N$,

$$\gamma_{p, \xi}(s) = (t_0 + s, \alpha_{y, \eta}^h(s)),$$

is a geodesic of the Lorentzian manifold (M, g) . Also, let $U \subset \mathbb{R} \times F$ be an open set where we consider observations. For the point $p \in M$, we define the light observation set with the observation set $U \subset M$ by setting

$$\mathcal{P}_U(p) = \left\{ \gamma_{p, \xi}(s) \in M \mid s \geq 0, \xi = \frac{\partial}{\partial t} + \eta, \|\eta\|_h = 1 \right\} \cap U.$$

Let $\kappa : SN \rightarrow \mathbb{R} \cup \{\infty\}$ be the cut locus function on the manifold (N, h) , defined by

$$\kappa(y, \eta) = \sup\{s \geq 0 \mid \text{dist}_h(\alpha_{y, \eta}^h(s), y) = s\}.$$

Then the earliest light observation set of the point $p = (t_0, y)$ with the observation set $U \subset M$ is

$$\mathcal{E}_U(p) = \left\{ \gamma_{p, \xi}(s) \in M \mid 0 \leq s \leq \kappa(y, \eta), \xi = \frac{\partial}{\partial t} + \eta, \|\eta\|_h = 1 \right\} \cap U.$$

Next, we consider the earliest light observation sets on general Lorentzian manifolds, which may not be of the product type.

2.2. Inverse problems on Lorentzian manifolds

2.2.1. Notations in a Lorentzian spacetime. Let (M, g) be a $(1 + n)$ -dimensional time-oriented Lorentzian manifold of signature $(-, +, +, \dots, +)$, where $n \geq 2$. On M , we use local coordinates $(x^0, x^1, \dots, x^n)$, where $x^0 = t$ is the time coordinate.

Let $L_p M = \{\xi \in T_p M \setminus \{0\} \mid g(\xi, \xi) = 0\}$ be the set of light-like vectors in the tangent space $T_p M$.

Let $\exp_p : T_p M \rightarrow M$ be the exponential map on (M, g) . The geodesic starting at p in the direction $\xi \in T_p M \setminus \{0\}$ is the curve $\gamma_{p,\xi}(t) = \exp_p(\xi t)$, $t \geq 0$.

Let $\mu : [-1, 1] \rightarrow M$ be a smooth future-pointing time-like geodesic and $U \subset M$ be an open neighborhood of $\mu([-1, 1])$.

Let $q \in M$. The set of future-pointing *light-like* vectors at q is denoted by

$$L_q^+ M = \{\theta \in T_q M \setminus 0 \mid g(\theta, \theta) = 0, \theta \text{ is future pointing}\}.$$

A vector $\theta \in T_q M$ is *time-like* if $g(\theta, \theta) < 0$ and *space-like* if $g(\theta, \theta) > 0$. *Causal vectors* are the collection of time-like and light-like vectors, and a curve γ is time-like (light-like, causal, future pointing) if the tangent vectors $\dot{\gamma}$ are time-like (light-like, causal, future pointing).

For $p, q \in M$, the notation $p \ll q$ means that p, q can be joined by a future-pointing time-like curve. The *chronological future* and *past* of $p \in M$ are

$$I^+(p) = \{q \in M \mid p \ll q\}, \quad I^-(p) = \{q \in M \mid q \ll p\}.$$

Similarly, for $p, q \in M$, the notation $p < q$ means that $p \neq q$ and p, q can be joined by a future-pointing causal (i.e., time-like or light-like) curve. We denote $p \leq q$ if $p < q$ or $p = q$. The *causal future* and *past* of $p \in M$ are

$$J^+(p) = \{q \in M \mid p \leq q\}, \quad J^-(p) = \{q \in M \mid q \leq p\}.$$

We will denote throughout the paper the causal diamond sets as

$$I(p, q) = I^+(p) \cap I^-(q), \quad J(p, q) = J^+(p) \cap J^-(q).$$

To emphasize the Lorentzian structure of (M, g) , we sometimes write $I_{M,g}^\pm(p) = I^\pm(p)$ and $J_{M,g}^\pm(p) = J^\pm(p)$.

A time-oriented Lorentzian manifold (M, g) is *globally hyperbolic* if there are no closed causal paths in M , and for any $p, q \in M$, the set $J(p, q)$ is compact [9]. According to [8], a globally hyperbolic manifold is isometric to the product manifold $\mathbb{R} \times N$ with the Lorentzian metric given by

$$g = -\beta(t, y)dt^2 + \kappa(t, y), \quad (2.3)$$

where $\beta : \mathbb{R} \times N \rightarrow \mathbb{R}_+$ and κ is a Riemannian metric on N depending on t .

2.3. Passive measurements on a Lorentzian manifold

2.3.1. The set of the earliest light observations. Let $s_-, s_+ \in (-1, 1)$, $s_- < s_+$, and $p^\pm = \mu(s_\pm)$. Next, define the light observation sets and consider in particular the case when $W \subset I^-(p^+) \setminus J^-(p^-)$ is a relatively compact open set; see Figure 1 (left).

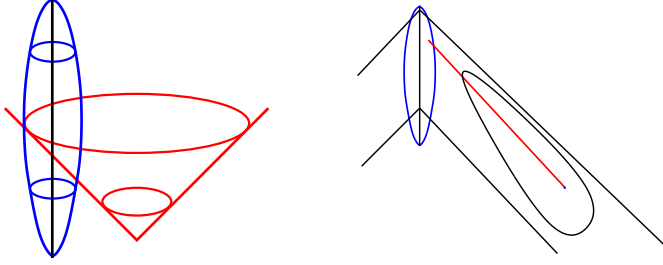


Figure 1. Left: When there are no cut points, the earliest light observation set $\mathcal{E}_U(q)$ is the intersection of the cone and the open set U . The cone is the union of future-pointing light-like geodesics from q , and the ellipsoid depicts U . Right: The unknown set $W \subset I^-(p_+) \setminus J^-(p_-)$ contains source points. Light cones emanating from these points (the red segment) are observed in the set U .

Definition 2.5. Let (M, g) be a globally hyperbolic Lorentzian manifold.

- (i) For $q \in M$, let

$$\mathcal{L}^+(q) = \exp_q(L_q^+ M) \cup \{q\} = \{\gamma_{q,\xi}(s) \in M \mid \xi \in L_q^+ M, s \geq 0\} \subset M$$

be the future-directed light cone emanating from the point q .

The light observation set of q in the observation set U is $\mathcal{P}_U(q) = \mathcal{L}^+(q) \cap U \in 2^U$.

- (ii) The earliest light observation set of $q \in M$ in U is

$$\begin{aligned} \mathcal{E}_U(q) = \{x \in \mathcal{P}_U(q) \mid \text{there exist no point } y \in \mathcal{P}_U(q) \\ \text{and a future-pointing time-like path } \alpha : [0, 1] \rightarrow U \\ \text{such that } \alpha(0) = y \text{ and } \alpha(1) = x\} \subset U. \end{aligned}$$

- (iii) Let $W \subset M$ be open. The collection of the earliest light observation sets with source points at W is

$$\mathcal{E}_U(W) = \{\mathcal{E}_U(q) \mid q \in W\} \subset 2^U.$$

Note $\mathcal{E}_U(W)$ is defined as an unindexed set, that is, for an element $\mathcal{E}_U(q) \in \mathcal{E}_U(W)$, we do not know what the corresponding point q is.

Above, $2^U = \{U' \mid U' \subset U\}$ is the power set of U . Below, when we say that the set U is given as a differentiable manifold, we mean that we are given the set U and the local coordinate charts on it for which the corresponding transition maps are C^∞ -smooth.

The set $\mathcal{P}_U(q)$ can be viewed as a model of a measurement where light emitted by a point source at q is recorded in U . As gravitational wave packets propagate at the

speed of light, $\mathcal{P}_U(q)$ could also correspond to an observation where a gravitational wave is generated at q and detected in U .

The following theorem says, roughly speaking, that observations of a large number of point sources in a region W determine the structure of the spacetime in W , up to a conformal factor.

Theorem 2.6 ([43]). *Let (M_j, g_j) , where $j = 1, 2$, be two open globally hyperbolic Lorentzian manifolds of dimension $1 + n$, $n \geq 2$. Let $\mu_j : [-1, 1] \rightarrow M_j$ be a future-pointing time-like path, let $U_j \subset M_j$ be a neighborhood of $\mu_j([-1, 1])$, and let $W_j \subset M_j$ be such that $W_j \subset I_{M_j, g_j}^-(\mu_j(1)) \setminus J_{M_j, g_j}^-(\mu_j(-1))$ are open and relatively compact sets. Assume that there is a conformal diffeomorphism $\phi : U_1 \rightarrow U_2$ such that $\phi(\mu_1(s)) = \mu_2(s)$, $s \in [0, 1]$, and*

$$\{\phi(\mathcal{E}_{U_1}(q)) \mid q \in W_1\} = \{\mathcal{E}_{U_2}(q) \mid q \in W_2\}.$$

Then there is a diffeomorphism $\Psi : W_1 \rightarrow W_2$ and a strictly positive function $\alpha \in C^\infty(W_1)$ such that $\Psi^*g_2 = \alpha g_1$ and $\Psi|_{U_1 \cap W_1} = \phi$.

Next, we consider an alternative formulation of the above inverse problem where the data are given in terms of functions, not as a collection of sets.

Assume that the open set $U \subset M$, where observations are made, is of the form $U = \bigcup_{a \in A} \mu_a(-1, 1)$, where $\mu_a : (-1, 1) \rightarrow M$, $a \in \bar{A} \subset \mathbb{R}^k$ are time-like curves that depend continuously on a .

Let $p_1, p_2 \in \mu_{a_0}$, $a_0 \in A$. Also, let $W \subset J^-(p_2) \setminus J^-(p_1)$ be an open, relatively compact set. The observation time function $F_q : \bar{A} \rightarrow \mathbb{R}$ for a point $q \in W$ is

$$F_q(a) = \min\{s \in \mathbb{R} \mid \text{there exists a future-directed light-like geodesic from } q \text{ to } \mu_a(s)\}.$$

That is, $F_q(a)$ is the first time when we observe on μ_a light that comes from q .

By Definition 2.5, the set $\mathcal{E}_U(q)$ determines the function F_q in A via the formula

$$F_q(a) = s, \quad \text{where } s \in [-1, 1] \text{ is such that } \mu_a(s) \in \mathcal{E}_U(q), \quad a \in A.$$

As $F_q : \bar{A} \rightarrow \mathbb{R}$ is continuous, this determines $F_q(a)$ for all $a \in \bar{A}$. On the other hand, F_q determines $\mathcal{E}_U(q)$ via the formula

$$\mathcal{E}_U(q) = \{\mu_a(F_q(a)) \mid a \in A\}.$$

Due to this, Theorem 2.6 has the following equivalent formulation.

Theorem 2.7 ([43]). *Let (M, g) be a globally hyperbolic Lorentzian manifold of dimension $1 + n \geq 3$. Assume that $\mu_a(-1, 1) \subset M$, $a \in A \subset \mathbb{R}^m$ are time-like paths,*

$U = \bigcup_{a \in A} \mu_a$ is open, and $p_1, p_2 \in \mu_{a_0}$. Let $W \subset J^-(p_2) \setminus J^-(p_1)$ be a relatively compact open set and $W' \subset W$ be dense. Then $(U, g|_U)$ and the collection of the observation time functions,

$$\{F_q : A \rightarrow \mathbb{R} \mid q \in W'\} \subset C(A),$$

determine the topological and differentiable type of the manifold W and the conformal class of the metric g in W .

We will outline the ideas of the proof of Theorem 2.7 below in Section 3.

2.4. Active measurements: Inverse problems for non-linear wave equations

Let (M, g) be a $(1 + n)$ -dimensional globally hyperbolic Lorentzian manifold and assume, without loss of generality, that $M = \mathbb{R} \times N$ with a metric of form (2.3). Let $t_0 > 0$ and consider the semilinear wave equations

$$\square_g u(x) + a(x)u(x)^m = f(x) \quad \text{for } x \in (-\infty, t_0) \times N, \quad (2.4)$$

$$u = 0 \quad \text{in } (-\infty, 0) \times N, \quad (2.5)$$

where $m \geq 2$ is an integer. Here, $a \in C^\infty(M)$ is a nowhere vanishing function and $\square_g$ is the wave operator, or the Lorentzian Laplace operator:

$$\square_g u = \sum_{j,k=0}^n |\det(g)|^{-1/2} \frac{\partial}{\partial x^j} \left(|\det(g)|^{1/2} g^{jk} \frac{\partial}{\partial x^k} u \right).$$

Let $\mu : [-1, 1] \rightarrow M$ be a time-like curve and U be its open, relatively compact neighborhood. The solution of (2.4)–(2.5) exists when the source f is supported in U and satisfies $\|f\|_{C^k(\overline{U})} < \varepsilon$, where $k \in \mathbb{Z}_+$ is sufficiently large and $\varepsilon > 0$ is sufficiently small. For such sources f , we define the measurement operator

$$L_U : f \mapsto u|_U.$$

Theorem 2.8 ([22, 43]). *Let (M, g) be a globally hyperbolic $(n + 1)$ -dimensional Lorentzian manifold, $n \geq 2$. Let μ be a time-like path containing p^+ and p^- . Let $U \subset M$ be a neighborhood of μ , and let $a : M \rightarrow \mathbb{R}$ be a nowhere vanishing C^∞ -smooth function. Then $(U, g|_U)$ and the measurement operator L_U determine the topology, differentiable structure, and the conformal class of the metric g in the double cone $I_{M,g}(p^-, p^+)$.*

We will outline the ideas of the proof of Theorem 2.8 below in Section 3.

The proof of Theorem 2.8 uses the results on the inverse problem for passive measurements for point sources, described below, and the non-linear interaction

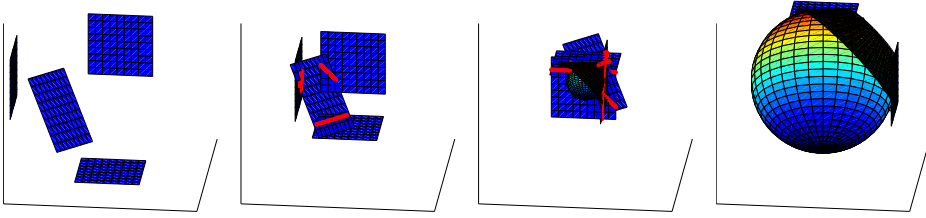


Figure 2. Four plane waves propagate in space. When the planes intersect, the non-linearity of the hyperbolic system produces new waves. Left: Plane waves before interacting. Middle left: The two-wave interactions (red line segments) appear but do not cause singularities propagating to new directions. Middle right and Right: All plane waves have intersected, and new waves have appeared. The three-wave interactions cause conic waves (the black surface). Only one such wave is shown in the figure. The interaction of four waves causes a microlocal point source that sends a spherical wave in all future light-like directions.

of waves having conormal singularities (see Figure 2). There are many results on such non-linear interaction, starting with the studies of Bony [11] and Melrose and Ritter [56].

2.4.1. Stability results and numerical reconstruction. As discussed below, any globally hyperbolic Lorentzian manifold M is isometric to a product manifold $\mathbb{R} \times N$ equipped with the product metric (2.3). Let $\Omega \subset N$ be a smooth submanifold of dimension n with smooth boundary and denote the lateral boundary of $[0, T] \times \Omega \subset M$ by $\Sigma := [0, T] \times \partial\Omega$. We denote the Sobolev space of functions having s weak derivatives on Σ by $H^s(\Sigma)$, and the closure of $C_0^\infty(\Sigma)$ in $H^s(\Sigma)$ is denoted by $H_0^s(\Sigma)$.

We consider the non-linear wave equation

$$\begin{cases} \square_g u(t, x) + q(t, x)u(t, x)^m = 0 & \text{in } [0, T] \times \Omega, \\ u = f & \text{on } [0, T] \times \partial\Omega, \\ u(0, x) = \partial_t u(0, x) = 0 & \text{on } \Omega, \end{cases} \quad (2.6)$$

where we assume that the exponent m is an integer greater than or equal to 4.

Let $\mathcal{W}$ be a compact set belonging to both the chronological future $I^+(\Sigma)$ and past of $I^-(\Sigma)$ of the lateral boundary $\Sigma = [0, T] \times \partial\Omega$:

$$\mathcal{W} \subset I^-(\Sigma) \cap I^+(\Sigma) \cap ([0, T] \times \Omega).$$

Theorem 2.9 ([50, Stability estimate]). *Suppose (M, g) , $M = \mathbb{R} \times N$, is a $(1 + n)$ -dimensional globally hyperbolic Lorentzian manifold.*

Let $T > 0$, and let $\Omega \subset N$ be an open set with smooth non-empty boundary. Let $m \geq 4$ be an integer, $s \in \mathbb{N}$ with $s + 1 > \frac{(n+1)}{2}$, and $r \in \mathbb{R}$ with $r \leq s$. Let $j = 1, 2$.

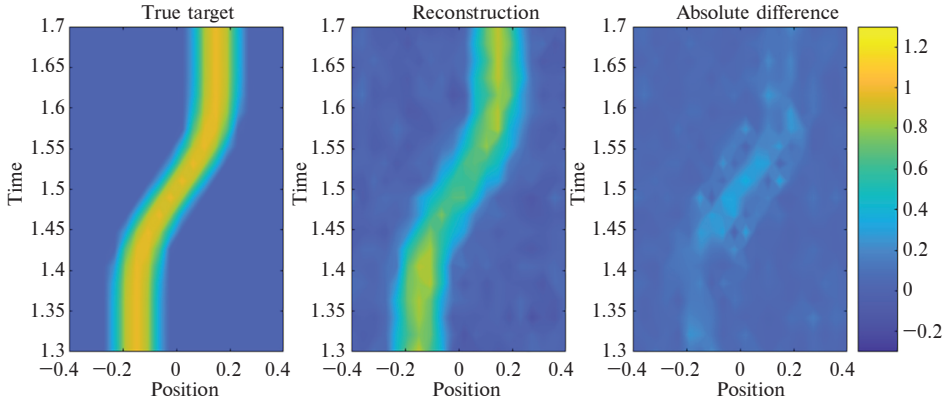


Figure 3. Reconstruction of the coefficient $q(x, t)$ in equation $\partial_t^2 u(x, t) - \partial_x^2 u(x, t) + q(x, t)u(x, t)^2 = 0$, $x \in [0, 1]$, $t \in [0, T]$, from the data $f \mapsto \langle \psi_0, \Lambda_q f \rangle_{L^2(\partial\Omega \times [0, T])}$, where Λ_q is the Dirichlet-to-Neumann map, and ψ_0 is a suitably chosen ‘device function’. Above, the data have additive Gaussian noise, and the signal-to-noise ratio is 12 dB. Left: Ground truth of the potential $q(x, t)$. Center: Numerical reconstruction. Right: Error in reconstruction. For numerical results, see [49], and for mathematical analysis, see [48, 50].

Assume that $q_j \in C^{s+1}(\mathbb{R} \times \Omega)$ satisfy $\|q_j\|_{C^{s+1}} \leq c_0$, $j = 1, 2$, for some $c_0 > 0$. Let $\Lambda_j : H_0^{s+1}(\Sigma) \rightarrow H^r(\Sigma)$ be the corresponding Dirichlet-to-Neumann maps of the non-linear wave equation (2.6).

Let $\varepsilon_0 > 0$, $L > 0$, and $\delta \in (0, L)$ be such that

$$\|\Lambda_1(f) - \Lambda_2(f)\|_{H^r(\Sigma)} \leq \delta$$

for all $f \in H_0^{s+1}(\Sigma)$ with $\|f\|_{H^{s+1}(\Sigma)} \leq \varepsilon_0$. Then there exists a constant $C > 0$, independent of q_1, q_2 and $\delta > 0$, such that

$$\|q_1 - q_2\|_{L^\infty(\mathcal{W})} \leq C\delta^{\sigma(s, m)},$$

where $\sigma(s, m) = \frac{8(m-1)}{2m(m-1)(8s-n+13)+2m-1}$.

Numerical reconstructions based on Theorem 2.9 in the $(1 + 1)$ -dimensional case are shown in Figure 3.

3. Ideas for proofs and reconstruction methods

3.1. Light observation sets and the inverse problem for passive observations

In this subsection, we outline the proof of Theorem 2.7. Below, let $\overline{W} = \text{cl}(W)$ be the closure of W in M such that $\overline{W} \subset J^-(p^+) \setminus I^-(p^-)$. The set $\overline{A} \times \overline{W}$ is compact, and the map $\mathcal{F} : \overline{W} \rightarrow C(\overline{A})$, $\mathcal{F}(q) = F_q$, is continuous.

Next, we will show that the map $\mathcal{F} : \overline{W} \rightarrow \mathcal{F}(\overline{W})$, $\mathcal{F}(q) = F_q$, is injective. To prove this, we assume the opposite: Assume that there are $q_1, q_2 \in W$ that satisfy $F_{q_1} = F_{q_2}$. Then

$$\mathcal{E}_U(q_1) = \mathcal{E}_U(q_2).$$

This implies that all light-like geodesics from q_1 to U go through q_2 , and hence, q_1 is in the past of q_2 , or vice versa. Without loss of generality, we can assume that q_1 is in the past of q_2 . Let $a_1 \in A$ and $p_1 = \mu_{a_1}(F_{q_1}^+(a_1)) \in \mu_{a_1}([-1, 1])$. Let now $\gamma_{q_1, \xi_1}([0, \ell_1])$ be a light-like geodesic from q_1 to p_1 , that is, $p_1 = \gamma_{q_1, \xi_1}(\ell_1) \in \mu_{a_1}([-1, 1])$. Next, let $a_2 \in A$ be such that $\gamma_{q_1, \xi_1} \cap \mu_{a_2}([-1, 1]) = \emptyset$. Let $p_2 = \mu_{a_2}(f_{q_1}^+(a_2)) \in \mu_{a_2}([-1, 1])$ and $\gamma_{q_1, \xi_2}([0, \ell_2])$ be a light-like geodesic from q_1 to p_2 , that is, $p_2 = \gamma_{q_1, \xi_2}(\ell_2) \in \mu_{a_2}([-1, 1])$. Then, as $\gamma_{q_1, \xi_1} \cap \mu_{a_2}([-1, 1]) = \emptyset$, ξ_2 and ξ_1 are not parallel. As all light-like geodesics from q_1 to U go through q_2 , we see that there are $t_1 \in (0, \ell_1)$ and $t_2 \in (0, \ell_2)$ such that

$$q_2 = \gamma_{q_1, \xi_1}(t_1), \quad q_2 = \gamma_{q_1, \xi_2}(t_2).$$

As the vectors ξ_2 and ξ_1 are not parallel, the geodesics γ_{q_1, ξ_1} and γ_{q_2, ξ_2} are not the same paths, and we see that the path $\beta = \gamma_{q_1, \xi_1}([0, t_1]) \cup \gamma_{q_1, \xi_2}([t_2, \ell_2])$ is a union of two causal paths from the point q_1 to q_2 and then from q_2 to p_2 . Since the geodesics γ_{q_1, ξ_1} and γ_{q_2, ξ_2} are not the same paths with different parametrizations, the path β is not a geodesic. As the manifold (M, g) is globally hyperbolic, this implies that there is a time-like path from q_1 to p_2 , and hence, it is not possible that $p_2 = \mu_{a_2}(F_{q_1}^+(a_2)) \in \mu_{a_2}([-1, 1])$, as we assumed above; see Figure 4. This contradiction implies $F_{q_1} \neq F_{q_2}$. Thus, the map $\mathcal{F} : \overline{W} \rightarrow \mathcal{F}(\overline{W})$, $\mathcal{F}(q) = F_q$, is injective.

The map $\mathcal{F} : \overline{W} \mapsto C(\overline{A})$ is continuous and injective. As $\overline{W}$ is compact, these imply that the map $\mathcal{F} : \overline{W} \mapsto C(\overline{A})$ is a homeomorphism. Thus, $\mathcal{F}(\overline{W}) \subset C(\overline{A})$ and $\overline{W}$ are homeomorphic. Furthermore, as W is an open subset of the manifold M , we can see that $\mathcal{F}(W) \subset C(\overline{A})$ and W are homeomorphic, so we can consider $\mathcal{F}(W)$

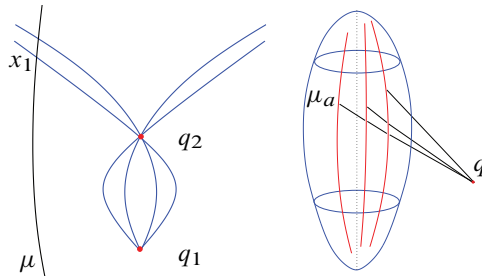


Figure 4. Left: Picture on proving that $q_1 = q_2$ by contradiction. Right: The function $X_{\vec{a}} : q \rightarrow (F_q(a_j))_{j=1}^{n+1} = (f_{a_j}^+(q))_{j=1}^{n+1} \in \mathbb{R}^{n+1}$ defines coordinates near a point q on M .

as a topological manifold which is a homeomorphic copy of the original manifold W . Thus, we have reconstructed W as a topological manifold.

Next, we consider the reconstruction of W as a differentiable manifold. Below, we will only consider how the coordinate functions on $\mathcal{F}(W)$ are constructed and omit the proof that this construction gives a differentiable structure on $\mathcal{F}(W)$ which makes it diffeomorphic to W . For details on this, we cite [43].

For $a \in A$, let us define the evaluation function $E_a[F] = F(a)$. For $q_0 \in W$, let us choose $n + 1$ parameters $a_j \in A$, $j = 1, 2, \dots, n + 1$. When the parameters $(a_1, a_2, \dots, a_{n+1})$ are chosen in a generic way, that is, this $(n + 1)$ -tuple is in an open and dense subset of A^{n+1} , the function $X_{\vec{a}}(q) = (F_q(a_j))_{j=1}^{n+1} = (f_{a_j}^+(q))_{j=1}^{n+1} \in \mathbb{R}^{n+1}$ defines smooth coordinates on M near q_0 (here, we omit this part of the proof; see [43] on details). This implies that for any $\mathcal{F}(q_0) \in \mathcal{F}(W)$ and generic parameters $a_j \in A$, $j = 1, 2, \dots, n + 1$, there is a neighborhood $\mathcal{V} \subset \mathcal{F}(W)$ such that the function $\mathbf{X}_{\vec{a}} : \mathcal{V} \rightarrow (\mathbb{R}^{n+1})_{j=1}^{n+1} \in \mathbb{R}^{n+1}$ defines coordinates in $\mathcal{W}$ such that the map $\mathcal{F}$ is a local diffeomorphism near the point q_0 from the manifold W to $\mathcal{F}(W)$.

Next, we will consider the construction of the Lorentzian metric on $\mathcal{F}(W)$ which makes it conformal to $(W, g|_W)$. To this end, let us consider a point $f_0 = \mathcal{F}(q_0) \in \mathcal{F}(W)$ and its open neighborhood $\mathcal{W} \subset \mathcal{F}(W)$, where $\vec{a} = (a_j)_{j=1}^{n+1}$ defines coordinates $\mathbf{X}_{\vec{a}} : \mathcal{W} \rightarrow \mathbb{R}^{n+1}$ near f_0 .

When $B \subset A$ is a sufficiently small neighborhood of the point a_1 , the function $a \rightarrow f_a^+(q_0)$ is smooth for $a \in B$. Let df_a^+ denote the co-derivative of the map $f_a^+ : W \rightarrow \mathbb{R}$. We see that the co-vectors $df_a^+|_{q_0} \in T_{q_0}^*M$, $a \in B$ are light-like co-vectors at q_0 . Let $G = \mathcal{F}_*g|_W$ be the metric $g|_W$ pushed forward in the map $\mathcal{F}$ to the manifold $\mathcal{F}(W)$ and $E_a : \mathcal{F}(W) \rightarrow \mathbb{R}$ be the evaluation maps, $E_a : f \rightarrow f(a)$. Then we see that the co-derivatives

$$dE_a|_{\mathcal{F}(q_0)} \in T_{\mathcal{F}(q_0)}^*\mathcal{F}(W), \quad a \in B$$

are light-like co-vectors with respect to the metric G .

As $a \in B$ is above arbitrary, we see that the set $\{s dE_a|_{\mathcal{F}(q_0)} \in T_{\mathcal{F}(q_0)}^*\mathcal{F}(W) \mid a \in B, s > 0\}$ contains an open subset of the light cone in the space $(T_{\mathcal{F}(q_0)}^*\mathcal{F}(W), G)$. As the light cone is a quadratic variety, this set determines the whole light cone in $(T_{\mathcal{F}(q_0)}^*\mathcal{F}(W), G)$. These sets determine the conformal class of the metric G on $\mathcal{F}(W)$ at the point $\mathcal{F}(q_0)$.

3.2. Non-linear interaction of waves and inverse problems for active measurements

In this subsection, we outline the proof of Theorem 2.8. To do this, we consider the non-linear equation $\square u + u^4 = f$ in $\mathbb{R}^{1+3}$, that is, we introduce the techniques by considering the Minkowski space $\mathbb{R}^{1+3}$ and the wave operator $\square = \partial_{x_0}^2 - \sum_{j=1}^3 \partial_{x_j}^2$.

Let us consider the solutions $u_{\vec{\varepsilon}}(x)$ of the non-linear wave equation

$$\square u_{\vec{\varepsilon}} + (u_{\vec{\varepsilon}})^4 = 0$$

that depend on parameters $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \in \mathbb{R}^4$. Let us assume that $u_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0} = 0$. Then the linearized waves

$$u_j(x) = \partial_{\varepsilon_j} u_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0}, \quad j = 1, 2, 3, 4,$$

satisfy $\square u_j = 0$. Moreover, the fourth-order derivative

$$w = \partial_{\varepsilon_1} \partial_{\varepsilon_2} \partial_{\varepsilon_3} \partial_{\varepsilon_4} u_{\vec{\varepsilon}}(x)|_{\vec{\varepsilon}=0} \quad (3.1)$$

satisfies

$$\square w = S, \quad S = -24u_1u_2u_3u_4.$$

We say that S is an artificial source produced by the 4-wave interaction.

We use coordinates $x = (t, y_1, y_2, y_3) \in \mathbb{R}^{1+3}$ and consider, as an example, the linearized waves $u_j(t, y_1, y_2, y_3)$ given by

$$\begin{aligned} u_1(t, y) &= \delta(t - y_1), & u_2(t, y) &= \delta(t - y_2), \\ u_3(t, y) &= \delta(t - y_3), & u_4(t, y) &= \delta\left(t - \frac{(y_1 + y_2)}{\sqrt{2}}\right). \end{aligned}$$

Then

$$u_1u_2u_3u_4 = \delta_p(t, y), \quad \text{where } p = (0, 0, 0, 0) \in \mathbb{R}^{1+3},$$

and the 4-wave interaction, $w(x)$, is the solution of the wave equation

$$\square w = S, \quad S = -24u_1u_2u_3u_4 = -24\delta_p. \quad (3.2)$$

In physical terms, S corresponds to an (artificial) point source in the spacetime.

We now turn our attention to a general globally hyperbolic, $(1 + 3)$ -dimensional Lorentzian manifold when $m = 4$. The proof of Theorem 2.8 is based on the analysis of the interaction of four waves. Analogously to (3.1), the derivative $w = \partial_{\varepsilon_1} \partial_{\varepsilon_2} \partial_{\varepsilon_3} \partial_{\varepsilon_4} u_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0}$ of the solution $u_{\vec{\varepsilon}}$ of (2.4)–(2.5) with the source

$$f_{\vec{\varepsilon}}(x) = \sum_{j=1}^4 \varepsilon_j f_j(x), \quad \vec{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4),$$

is given by formula (3.2), where $u_j(x)$ are solutions to the wave equation $\square_g u = f_j$.

When f_j is a C^∞ -smooth function outside the point $p_j \in M$, the linearized wave $u_j = \square_g^{-1} f_j$ produced by the source f_j is microlocally similar to Green's functions (i.e., the fundamental solution of the wave equation) corresponding to a point source

at the point p_j . Let us denote the light cones in M emanating from the point p_j to future by K_j , and, for simplicity, assume that the light-like geodesics of M have no cut or conjugate points. Then, in $M \setminus \{p_j\}$, the linearized solutions u_j are conormal distributions that are non-smooth on surfaces K_j (see [23, 57]). Then the two waves u_1 and u_2 are singular on hyper-surfaces K_1 and K_2 , respectively, and these singularities interact on $K_1 \cap K_2$. This interaction creates only singularities that propagate along K_1 and K_2 on which the original singularities u_1 and u_2 propagate on. The interaction of the three waves u_1 , u_2 , and u_3 happens on the intersection $K_{123} = K_1 \cap K_2 \cap K_3$ which is a curve in the spacetime. As N^*K_{123} contains light-like directions that are not in the union $N^*K_1 \cup N^*K_2 \cup N^*K_3$, this interaction produces interesting singularities that start to propagate. These singularities correspond to the black conic wave in Figure 3 (we note that these singularities produced by 3-wave interaction can also be used to determine the light observation sets; see [22]). Finally, singularities of all four waves u_j , $j = 1, 2, 3, 4$, interact at the point $\{q\} = \bigcap_{j=1}^4 K_j$. The singularities from the point q propagate along the light cone emanating from this point, and with suitably chosen sources f_j , the wave w is singular at any given point of the light cone $\mathcal{L}(q)$. Thus, S_{1234} can be considered as a microlocal point source that sends singularities in all directions as a point source located at the point q , and these singularities are observed in the set U . In this way, the non-linear interaction of waves determines the intersection of the light cone $\mathcal{L}(q)$ and the observation domain U . Hence, we can determine the earliest light observation sets $\mathcal{E}_U(q)$ for an arbitrary point $q \in I(p^-, p^+)$. By letting q vary through the set $I(p^-, p^+)$, we can apply Theorem 2.6 to recover the topology, differentiable structure, and the conformal class of g in $I(p^-, p^+)$.

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Matti Lassas

Department of Mathematics and Statistics, University of Helsinki, P.O. Box 68 (Gustaf Hållströmin katu 2b), 00014 Helsinki, Finland; matti.lassas@helsinki.fi