

Cheeger–Müller and Bismut–Zhang theorem on singular spaces

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Abstract. In this article, we present the Cheeger–Müller/Bismut–Zhang theorem for singular spaces with isolated conical singularities proved recently by the author. The classical Cheeger–Müller theorem, the comparison of analytic and topological torsion of a smooth compact manifold, is one of the most important comparison theorems in global analysis. Its most general version for smooth compact manifolds has been achieved by Bismut and Zhang by combining local index techniques and the Witten deformation.

The question of a Cheeger–Müller theorem for spaces with isolated conical singularities has been raised by Dar in the late 1980s, very early in the development of the (global) analysis on these spaces. The Cheeger–Müller/Bismut–Zhang theorem for singular spaces with isolated conical singularities presented here answers the question by generalising Bismut and Zhang’s approach to the singular setting. An important step in this programme is hence the generalisation of the Morse–Thom–Smale complex and the Witten deformation to the singular setting, achieved by the author, and which we also review here.

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1. Introduction

An important comparison theorem in global analysis is the comparison of analytic (or Ray–Singer) and topological (or Reidemeister) torsion, aka the Cheeger–Müller theorem, for smooth compact Riemannian manifolds equipped with a unitary flat vector bundle. It has been conjectured by Ray and Singer [67] and has been independently proved by Cheeger [23] and Müller [63]. Müller [64] extended the result to unimodular bundles. At the same time, Bismut and Zhang [11, 12] achieved the most general version of the comparison theorem of torsions for smooth compact manifolds, by a very different approach, namely combining the Witten deformation and local index

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techniques. Bismut and Zhang compare the analytic torsion with the Milnor torsion, a Morse theoretic version of the topological torsion defined in [62]. If the flat vector bundle is not unitary or unimodular, the two torsions are no longer equal, and the difference between them can be expressed in terms of the Mathai–Quillen current. In this note, we refer to this most general version of the comparison theorem of torsions as the Bismut–Zhang theorem. The Cheeger–Müller/Bismut–Zhang theorem has been extended to the equivariant setting [12, 46, 47] and for manifolds with boundary [18, 19, 47, 74].

In the last decade, a lot of progress has been made in the study of the Cheeger–Müller theorem for spaces with singularities of different types. Here, we only recall results in the conical world, which is the setting of this article. The study of analytic and topological torsion on *singular spaces with conical singularities* started with the work of A. Dar in [27]. She proved that on singular spaces with isolated conical singularities, the analytic torsion is well defined. She also defined the Reidemeister torsion for stratified pseudomanifolds in the context of intersection homology (with any perversity in the sense of Goresky and MacPherson [30, 31]) and proved that for an even-dimensional oriented space with isolated conical singularities, due to a duality argument, both analytic and intersection Reidemeister torsion are trivial, hence equal.

An approach to the Cheeger–Müller theorem for spaces with isolated conical singularities, suggested in [44, Problem 5.3], is to reduce the problem via gluing formulas to a comparison of torsions on the truncated cone. The gluing formula for the analytic torsion for manifolds with boundary equipped with an arbitrary flat bundle, with and without product structure assumptions near the boundary, is due to Brüning and Ma [19]. Later, Lesch [45] established gluing formulas in the context of spaces with singularities, under product assumptions, by generalising a technique of Vishik [74]. The analytic torsion of a truncated cone has been studied in [34–36, 65, 73]. A problem faced in this approach is the missing interpretation of one of the analytic torsion-like terms appearing in these explicit computations.

In [61] Mazzeo and Vertman prove the well-definedness of analytic torsion for *wedge spaces* (also called incomplete edge spaces in the literature), i.e., spaces with a singular stratum of positive dimension and a cone-like metric near the singular stratum. The authors also prove invariance of the analytic torsion under variation of the wedge metric for an odd-dimensional space with an odd-dimensional singular stratum. A Cheeger–Müller theorem on an odd-dimensional wedge space with odd-dimensional singular stratum equipped with a unimodular bundle satisfying an additional acyclicity condition has been proved in [6]. The strategy in [6] consists in the study of analytic torsion via degeneration of smooth metrics into conical metrics, a strategy already applied successfully by the authors to the study of torsion on spaces with cuspidal singularities [4, 5]. The assumptions made in [6] exclude the

case of odd-dimensional spaces with isolated conical singularities; they also exclude the case of the trivial bundle.

In [56, 58], the author proposed yet another approach to the study of torsion on singular spaces, namely by generalising the approach of Bismut and Zhang to the singular setting. This has led to an answer to the long open question of [27], by providing a Cheeger–Müller/Bismut–Zhang formula in the following singular setting (see Section 2 for the precise definitions): Let (X, g^{TX}) be a space with isolated conical singularities; the singular set is denoted by $\text{Sing}(X)$. We equip the smooth locus $X \setminus \text{Sing}(X)$ with a Hermitian flat vector bundle (F, ∇^F, g^F) . We denote by $\theta(F, g^F)$ the closed 1-form defined in (2.2), associated with the Hermitian flat vector bundle (F, ∇^F, g^F) , whose cohomology class measures the obstruction to the existence of a flat volume form on $\det F$. In particular, $\theta(F, g^F) = 0$ in case (F, ∇^F, g^F) is a unitary bundle. For a perversity $\bar{q}$ in the sense of Goresky and MacPherson, we denote by $\det IH_{\bar{q}}^{\bullet}(X, F)$ the determinant line of the intersection cohomology of X with perversity $\bar{q}$ and values in the flat bundle F . We denote by $\bar{m}$ (resp. $\bar{n}$) the lower (resp. upper) middle perversity. We denote by $\Psi(TX, \nabla^{TX})$ the $n - 1$ -dimensional Mathai–Quillen current on TX with values in the orientation line $o(TX)$, as constructed in [60, Section 7] (see also [11, Section III]).

Theorem 1.1. *Let (X, g^{TX}) be a space with isolated conical singularities, and let $\bar{q} \in \{\bar{m}, \bar{n}\}$ denote one of the middle perversities. Let (F, ∇^F, g^F) be a Hermitian flat vector bundle over $X \setminus \text{Sing}(X)$ as in Section 2.2. Let Y be an anti-radial gradient-like vector field on X (see Section 3). Then*

$$\log \left(\frac{\| \|^{\text{RS}}_{\det IH_{\bar{q}}^{\bullet}(X, F)}}{\| \|^{\text{BZ}, Y}_{\det IH_{\bar{q}}^{\bullet}(X, F)}} \right)^2 = - \int_X \theta(F, g^F) Y^* \Psi(TX, \nabla^{TX}), \quad (1.1)$$

where $\| \|^{\text{RS}}_{\det IH_{\bar{q}}^{\bullet}(X, F)}$ (resp. $\| \|^{\text{BZ}, Y}_{\det IH_{\bar{q}}^{\bullet}(X, F)}$) denotes the Ray–Singer (resp. the Bismut–Zhang) metric (see Section 5).

If (F, ∇^F, g^F) is unitary, we have that $\theta(F, g^F) = 0$. Hence, in this case, the right-hand side of formula (1.1) vanishes, and Theorem 1.1 is a generalisation of the classical Cheeger–Müller theorem [23, 63] to spaces with isolated conical singularities. Profiting from the strength of the Bismut–Zhang approach, here we provide a comparison formula in the case of a general Hermitian flat coefficient bundle, thus also generalising the Bismut–Zhang theorem in [11] to the singular setting. Note that the integral on the right-hand side in (1.1) is well defined and corresponds to the right-hand side in the smooth Bismut–Zhang theorem.

In the case that (X, g^{TX}) is a smooth compact manifold, Theorem 1.1 recovers the smooth Cheeger–Müller/Bismut–Zhang theorem. For X an even-dimensional oriented space with isolated conical singularities equipped with a unitary vector bundle, Theorem 1.1 recovers the comparison result of torsions already known by Dar [27]. Note that for non-Witt spaces, Theorem 1.1 provides two formulas for the two middle perversities $\bar{q} \in \{\bar{m}, \bar{n}\}$, which are compatible with Poincaré duality for intersection cohomology.

The treatment of the Cheeger–Müller/Bismut–Zhang formula in [56, 58] is complete: In [58], we also describe the dependence of all three terms in formula (1.1) on the choice of metrics g^{TX} , g^F on TX , F .

Indeed, our results can be extended to a slightly more general setting: namely, to spaces with isolated asymptotically conical singularities; we can also allow metrics g^F on the flat bundle F , which are smooth up to the “boundary” but do not necessarily satisfy the product-type assumption in Section 2.2.

The main actors of the proof of the smooth Bismut–Zhang theorem in [11] are local index techniques and the Witten deformation. Hence, the first step in the programme of generalising the approach of Bismut and Zhang to the singular setting is the generalisation to singular spaces of the Morse–Thom–Smale complex in [54, 59] and of the Witten deformation in [55]. They have been achieved in a more general singular setting and will be reviewed here as well. The second main actor in the proof of Theorem 1.1 is the use of local index techniques: We combine the smooth local index techniques in [11] with a local index result on the infinite cone in [57]; we will not comment on these in this article.

This article is conceived as an overview article, and we will hence focus on conveying the main constructions and main ideas only; for the proofs, we refer the reader to the original articles. In this presentation, we restrict ourselves mostly to the case of spaces with isolated singularities. Part of the results presented here have been achieved in a much more general context, which we only mention shortly here without giving all technical details.

The article is organised as follows. In Section 2, we recall basic definitions and results on singular spaces with isolated conical singularities. In Section 3, we recall the definition of anti-radial/radial Morse functions and anti-radial/radial gradient-like vector fields as well as the generalisation of the famous Morse–Thom–Smale complex to singular spaces, anti-radial gradient-like vector fields and intersection (co-) homology. In Section 4, we review the generalisation of the Witten deformation to singular spaces with conical and iterated conical singularities. In Section 5, we recall the definition of the two torsions appearing in the Cheeger–Müller/Bismut–Zhang theorem for singular spaces with isolated conical singularities, Theorem 1.1. Moreover, we also present anomaly formulas for all three terms appearing in formula (1.1).

2. Preliminaries on singular spaces with isolated conical singularities

2.1. Singular spaces with isolated conical singularities

For a smooth manifold L and $\delta > 0$, we denote by

$$c_\delta L := ([0, \delta) \times L) / (0, x) \sim (0, y)$$

the (open) truncated cone over L .

Let X be a connected topological space, $\text{Sing}(X) \subset X$ a finite set of points such that $X_{\text{sm}} := X \setminus \text{Sing}(X)$ is a smooth manifold of dimension $n \geq 2$. We denote by TX (resp. by T^*X) the tangent bundle (resp. the cotangent bundle) of X_{sm} . Let g^{TX} be a Riemannian metric on X_{sm} . We assume that (X, g^{TX}) is a space with isolated conical singularities of dimension n , i.e.,

- (1) For $p \in \text{Sing}(X)$, there exist an open neighbourhood $B_\delta(p)$ of p , a smooth compact connected manifold L_p of dimension $\dim L_p = n - 1$ and a diffeomorphism $\varphi_p : B_\delta(p) \setminus \{p\} \simeq c_\delta L_p \setminus \{0\}$. The diffeomorphism φ_p extends to a homeomorphism, still denoted by φ_p ,

$$\varphi_p : B_\delta(p) \simeq c_\delta L_p \quad \text{and} \quad g|_{B_\delta(p) \setminus \{p\}}^{TX} = \varphi_p^*(dr^2 + r^2 g^{TL_p}), \quad (2.1)$$

where r is the radial coordinate and g^{TL_p} is a Riemannian metric on the manifold L_p (not depending on r).

- (2) The set

$$X \setminus \left(\bigcup_{p \in \text{Sing}(X)} B_\delta(p) \right)$$

is a smooth compact manifold of dimension n with boundary $\bigcup_{p \in \text{Sing}(X)} L_p$.

The set $\text{Sing}(X)$ is called the singular set of X . For $p \in \text{Sing}(X)$, the manifold L_p is called the link of X at p . The assumption on the connectivity of the link manifolds L_p , $p \in \text{Sing}(X)$, can be removed easily and is included here to simplify the presentation; it is equivalent to the topological normality of the space.

Let us emphasise that the radial coordinate r in (2.1) is fixed throughout this article.

2.2. Flat vector bundles over X

Let (F, ∇^F) be a flat vector bundle over X_{sm} with canonical flat connection ∇^F . Since the punctured neighbourhood of $p \in \text{Sing}(X)$ is contractible to the link L_p , the restriction of (F, ∇^F) to the punctured neighbourhood can be identified with the pull-back of a flat vector bundle $(F_{L_p}, \nabla^{F_{L_p}})$ on the link L_p , i.e., $(F, \nabla^F)|_{B_\delta(p) \setminus \{p\}} \simeq$

$\pi^*(F_{L_p}, \nabla^{F_{L_p}})$, where $\pi : B_\delta(p) \setminus \{p\} \simeq (0, \delta) \times L_p \rightarrow L_p$ is the canonical projection.

We equip the flat vector bundle (F, ∇^F) with a (not necessarily flat) Hermitian metric g^F such that, near $p \in \text{Sing}(X)$, g^F is the pull-back of a Hermitian metric $g^{F_{L_p}}$ on the flat vector bundle F_{L_p} on L_p .

We denote by F^* the flat bundle dual to F and by $F_{L_p}^*$ its restriction to L_p , $p \in \text{Sing}(X)$.

Let $\omega(F, g^F)$ be the 1-form on X with values in the self-adjoint endomorphisms of F :

$$\omega(F, g^F) = (g^F)^{-1} \nabla^F g^F.$$

Note that $\omega(F, g^F) = 0$ in case g^F is flat. By taking the pointwise trace, denoted Tr , we get a closed 1-form $\theta(F, g^F)$ on X :

$$\theta(F, g^F) = \text{Tr}[\omega(F, g^F)]. \quad (2.2)$$

The cohomology class $[\theta(F, g^F)]$ measures the obstruction to the existence of a flat volume form on F . By our assumption, near $p \in \text{Sing}(X)$, the form $\theta(F, g^F)$ does not depend on the radial coordinate r .

2.3. Local model near $p \in \text{Sing}(X)$

Let $p \in \text{Sing}(X)$. We denote by $cL_p := ([0, \infty) \times L_p)/(0, x) \sim (0, y)$ the infinite cone over L_p , by 0 the cone tip and by

$$Z_p := cL_p \setminus \{0\} \simeq \mathbb{R}_{>0} \times L_p$$

the punctured infinite cone. We write $x \in Z_p$ in its polar coordinates $x = (r, y)$, where r is the radial coordinate and y is the coordinate on the link. We equip Z_p with the conical metric $g^{TZ_p} = dr^2 + r^2 g^{TL_p}$. The flat bundle $(F_{L_p}, \nabla^{F_{L_p}}, g^{F_{L_p}})$ can be extended in a trivial way to a flat bundle $(F_{Z_p}, \nabla^{F_{Z_p}}, g^{F_{Z_p}})$ over Z_p . If no confusion can occur, we still write F for F_{Z_p} .

2.4. Intersection (co-)homology

An important topological invariant for singular spaces is the intersection homology and cohomology introduced by Goresky and MacPherson [30, 31]. For a perversity $\bar{q}$ in the sense of Goresky and MacPherson, we denote by $IH_{\bar{q}}^{\bullet}(X, F)$ (resp. by $IH_{\bar{q}}^*(X, F)$) the intersection homology (resp. cohomology) with perversity $\bar{q}$ and coefficients in the local system associated with the flat bundle F .

For the most part of this article, we will focus on the two middle perversities: the lower middle perversity $\bar{m}$ and the upper middle perversity $\bar{n}$. A perversity $\bar{q}$ is a tuple of nonnegative natural numbers. For a space with isolated singularities, however, the

only relevant information is the last entry in this tuple. Hence, by slight abuse of notation, for a space with isolated singularities, we will identify a perversity $\bar{q}$ with this last entry; more concretely,

$$\bar{q} = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor - 1 & \text{for } \bar{q} = \bar{m}, \\ \left\lfloor \frac{n-1}{2} \right\rfloor & \text{for } \bar{q} = \bar{n}. \end{cases}$$

For an even-dimensional space with isolated singularities, lower and upper middle perversities coincide, and hence also the two intersection cohomologies are the same. More generally, if X is a Witt space, i.e., if $H^{\frac{n-1}{2}}(L_p, F_{L_p}) = 0$ for all $p \in \text{Sing}(X)$, then by [70, Theorem 3.4],

$$IH_{\bar{m}}^*(X, F) \simeq IH_{\bar{n}}^*(X, F).$$

In this paper, we will not assume the Witt condition, and hence in the odd-dimensional case, we have to distinguish two cases.

We denote by $IH_{\bar{q}}^{\bar{q}}(cL_p, F)$ (resp. $IH_{\bar{q}}^{\bar{q}}(cL_p, L_p, F)$) the absolute (resp. relative) homology with perversity $\bar{q}$ of the cone cL_p ; similarly, $IH_{\bar{q}}^*(cL_p, F)$ (resp. $IH_{\bar{q}}^*(cL_p, L_p, F)$) denotes the absolute (resp. relative) cohomology of the cone cL_p . Since it will appear frequently in this article, let us shortly recall the local calculation for the relative intersection cohomology of a cone (see [31, Section 2.4]):

$$IH_{\bar{q}}^k(cL_p, L_p, F) = \begin{cases} H^{k-1}(L_p, F_{L_p}) & \text{for } k \geq n - \bar{q}, \\ 0 & \text{else,} \end{cases} \quad (2.3)$$

and an analogous formula holds for the intersection homology.

2.5. L^2 -cohomology

Throughout this article, we use the language of Hilbert complexes as introduced by Brüning and Lesch in [16].

We denote by $\langle \cdot, \cdot \rangle$ the L^2 -inner product on the space of sections of $\Lambda(T^*X) \otimes F$ induced from the metrics g^{TX} , g^F . We denote by $\Omega_c^*(X, F)$ (resp. by $\Omega^*(X, F)$) the graded vector space of smooth compactly supported sections (resp. of smooth sections) of $\Lambda(T^*X) \otimes F$. We denote by $L^2(\Lambda(T^*X) \otimes F)$ the space of L^2 -sections. We denote by d_c the exterior derivative acting on $\Omega_c^*(X, F)$.

The de Rham complex of compactly supported forms $(\Omega_c^*(X, F), d_c, \langle \cdot, \cdot \rangle)$ admits several closed extensions (in the Hilbert space of L^2 -forms) into a Hilbert complex, a choice of which is called an ideal boundary condition (shortly *ibc*) by Cheeger. Here, we focus on the maximal (resp. minimal) extension, denoted by $(\mathcal{C}_{\bar{m}}^*, d_{\bar{m}}, \langle \cdot, \cdot \rangle) := (\mathcal{C}_{\max}^*, d_{\max}, \langle \cdot, \cdot \rangle)$ (resp. $(\mathcal{C}_{\bar{n}}^*, d_{\bar{n}}, \langle \cdot, \cdot \rangle) := (\mathcal{C}_{\min}^*, d_{\min}, \langle \cdot, \cdot \rangle)$),

where $d_{\overline{m}} := d_{\max}$ (resp. $d_{\overline{n}} := d_{\min}$) denotes the maximal (resp. the minimal) closed extension of d_c in $L^2(\Lambda(T^*X) \otimes F)$. By [17, Theorems 3.7 and 3.8],

$$\operatorname{dom}(d_{\max})/\operatorname{dom}(d_{\min}) \simeq \bigoplus_{p \in \operatorname{Sing}(X)} H^{\frac{n-1}{2}}(L_p, F_{L_p}). \quad (2.4)$$

Hence, for a Witt space and in particular for an even-dimensional space, the extension of the de Rham complex $(\Omega_c^*(X, F), d_c, \langle \cdot, \cdot \rangle)$ into a Hilbert complex is unique, $d_{\min} = d_{\max}$.

The cohomology of the maximal Hilbert complex

$$(\mathcal{C}_{\overline{m}}^\bullet, d_{\overline{m}}, \langle \cdot, \cdot \rangle) = (\mathcal{C}_{\max}^\bullet, d_{\max}, \langle \cdot, \cdot \rangle)$$

is called the L^2 -cohomology of X with values in F :

$$H_{(2)}^\bullet(X, F) := H_{(2), \overline{m}}^\bullet(X, F) := H^\bullet((\mathcal{C}_{\max}^\bullet, d_{\max}, \langle \cdot, \cdot \rangle)). \quad (2.5)$$

We can also define the cohomology of the minimal extension $(\mathcal{C}_{\overline{n}}^\bullet, d_{\overline{n}}, \langle \cdot, \cdot \rangle) = (\mathcal{C}_{\min}^\bullet, d_{\min}, \langle \cdot, \cdot \rangle)$:

$$H_{(2), \overline{n}}^\bullet(X, F) := H^\bullet((\mathcal{C}_{\min}^\bullet, d_{\min}, \langle \cdot, \cdot \rangle)). \quad (2.6)$$

The two cohomologies introduced here only depend on the quasi-isometry class of the Riemannian metric. From (2.4), we have for a Witt space $H_{(2), \overline{m}}^\bullet(X, F) = H_{(2), \overline{n}}^\bullet(X, F)$.

2.6. L^2 -Hodge–de Rham theorem

By a result of Cheeger, Goresky and MacPherson [26, Section 3.4], integration of L^2 -forms over intersection chains induces a de Rham isomorphism

$$H_{(2), \overline{q}}^\bullet(X, F) \simeq IH_{\overline{q}}^\bullet(X, F), \quad \overline{q} \in \{\overline{m}, \overline{n}\}. \quad (2.7)$$

Note that this shows, in particular, that the L^2 -cohomologies (2.5) and (2.6) are indeed topological invariants of X .

For $\overline{q} \in \{\overline{m}, \overline{n}\}$, we denote by $\Delta^{\overline{q}}$ the Laplace operator associated with the Hilbert complex $(\mathcal{C}_{\overline{q}}^\bullet, d_{\overline{q}}, \langle \cdot, \cdot \rangle)$ in the sense of [16], and by $\Delta^{\overline{q}, (i)}$ its restriction to i -forms. The Laplacian $\Delta^{\overline{q}}$ is a discrete operator. By a standard result on Hilbert complexes (see [16, Lemma 2.2 and Corollary 2.5]), the following L^2 -Hodge isomorphism holds:

$$\ker(\Delta^{\overline{q}}) =: \mathcal{H}_{(2), \overline{q}}^\bullet(X, F) \simeq H_{(2), \overline{q}}^\bullet(X, F), \quad \overline{q} \in \{\overline{m}, \overline{n}\}, \quad (2.8)$$

a result first established by Cheeger [24, Section 1 and Theorem 5.1].

3. Radial/anti-radial gradient-like vector fields, radial/anti-radial Morse functions, the “singular” Morse–Thom–Smale complex computing intersection homology

The famous Morse inequalities establish a relation between the singular cohomology of a smooth compact manifold and the number of critical points of a smooth Morse function. A way of proving the Morse inequalities is to show the existence of a combinatorial complex, which is generated by the critical points of the Morse function and whose (co-)homology is isomorphic to the singular (co-)homology of the manifold. Such a complex was first established by Thom [72] and Smale [71], using the unstable cell decomposition of the manifold for the negative gradient flow of the Morse function with respect to a generic Riemannian metric. The boundary operator in this complex is given by counting trajectories between critical points of index difference 1. The Morse–Thom–Smale complex has seen a revival in the 1990s, starting with the work of Floer [29], where the idea of counting trajectories between critical points has been exploited in an infinite-dimensional setting.

In [55, Section 6] and [54], the Morse–Thom–Smale complex has been generalised for spaces with isolated singularities equipped with an anti-radial gradient-like vector field and intersection homology with arbitrary perversity $\bar{q}$. The construction also works more generally for all mezzo-perversities in the sense of [3]. This Morse theoretic description of intersection homology plays an important role in the generalisation of the Cheeger–Müller/Bismut–Zhang theorem in Section 5: In particular, the “singular” Morse–Thom–Smale complex is one of the two ingredients used in the definition of the Bismut–Zhang torsion (see Definition 5.4), providing the “topological” side in the comparison theorem of torsions in this singular setting.

In this section, we recall from [48, 54, 55] the definition of radial/anti-radial gradient-like vector fields and Morse functions, as well as the construction and main properties of the “singular” Morse–Thom–Smale complex for spaces with isolated singularities. The construction has been extended to stratified pseudomanifolds in [59], which we shortly mention in Section 3.3.

3.1. Radial/anti-radial gradient-like vector fields and radial/anti-radial Morse functions

Radial/anti-radial gradient-like vector fields and radial/anti-radial Morse functions have been introduced in [48, 54, 55]. One inspiration stems from radial vector fields introduced by Marie-Hélène Schwartz in [68, 69], in the context of Whitney stratified spaces, in her study of characteristic classes on singular spaces via obstruction theory. Radial Morse functions increase in the direction normal to the singular stratum. This is also the idea behind Goresky’s π -fibre Morse functions in [33, Section 9].

Let us consider a vector field Y on X_{sm} satisfying the following conditions:

- (a) Near a singular point $p \in \text{Sing}(X)$, the vector field Y has the following normal form in the local coordinates (2.1):

$$Y = Y^\pm = \pm r \frac{\partial}{\partial r}. \quad (3.1)$$

- (b) To any singular point $p \in X_{\text{sm}}$ of Y (i.e., $Y(p) = 0$), there exist an open neighbourhood $U(p)$ of p and local coordinates $x_1, \dots, x_n$, as well as $k \in \{0, \dots, n\}$ such that

$$Y|_{U(p)} = -x_1 \frac{\partial}{\partial x_1} - \dots - x_k \frac{\partial}{\partial x_k} + x_{k+1} \frac{\partial}{\partial x_{k+1}} + \dots + x_n \frac{\partial}{\partial x_n}. \quad (3.2)$$

We call k the index of Y at p . We denote by $\text{Crit}(Y)$ (resp. by $\text{Crit}_k(Y)$, $k = 0, \dots, n$) the set of singular points of Y in X_{sm} (resp. the set of singular points of Y of index k).

The vector field $-Y$ induces a well-defined smooth flow on X_{sm} , which extends to a continuous flow $\Phi : X \times \mathbb{R} \rightarrow X$.

Definition 3.1. A smooth vector field Y over X_{sm} is called a standard radial (resp. anti-radial) gradient-like vector field, if it satisfies conditions (a) and (b) above as well as the following conditions:

- (c) The set of non-wandering points of the flow Φ equals $\text{Crit}(Y) \cup \text{Sing}(X)$.
 (d) The Morse–Smale transversality condition holds, i.e., all intersections of stable and unstable manifolds (w.r.t. the flow Φ) are transverse.

We call the vector field radial (resp. anti-radial) gradient-like vector field, if instead of (b), we only assume that, locally near $p \in \text{Crit}(Y)$, the vector field can be written as the gradient vector field of a Morse function with respect to some Riemannian metric.

Note that the Morse–Smale transversality condition is a generic condition (see, e.g., [37, Theorem 7.2.13]) and can always be achieved by perturbing a vector field which satisfies (a)–(c) outside a neighbourhood of $\text{Crit}(Y) \cup \text{Sing}(X)$. Existence of radial/anti-radial gradient-like vector fields (in a more general context) has been proved in [48, Proposition 6.4].

Definition 3.2. A continuous function $f : X \rightarrow \mathbb{R}$ is called a radial (resp. an anti-radial) Morse function, if the following two conditions hold:

- (a) The restriction $f_{\text{sm}} := f|_{X_{\text{sm}}}$ is a smooth Morse function.

- (b) Near a singular point $p \in \text{Sing}(X)$, the function f has the following normal form in the local coordinates (2.1):

$$f(r, y) = f^\pm(r, y) = f(p) \pm \frac{1}{2}r^2. \quad (3.3)$$

For a radial/anti-radial Morse function f , we denote by $\text{Crit}(f_{\text{sm}})$ (resp. by $\text{Crit}_k(f_{\text{sm}})$, $k = 0, \dots, n$) the set of critical points of f_{sm} (resp. the set of critical points of f_{sm} of index k). Set

$$\text{Crit}(f) := \text{Crit}(f_{\text{sm}}) \cup \text{Sing}(X).$$

By a straightforward generalisation of a result of Smale [71, Theorem B], we have the following relation between the two notions just introduced: Let Y be a standard radial (resp. anti-radial) gradient-like vector field on X_{sm} . Then there exists a radial (resp. an anti-radial) Morse function $f : X \rightarrow \mathbb{R}$ and a Riemannian metric g'^{TX} on X_{sm} such that $Y = \nabla_{g'^{TX}} f$ and, in an open neighbourhood of $\text{Sing}(X)$, we have $g'^{TX} = g^{TX}$. Note that, to achieve (3.2) in a neighbourhood of $\text{Crit}(f_{\text{sm}}) = \text{Crit}(Y)$, we choose Morse coordinates and g'^{TX} to be the Euclidean metric in the Morse coordinates.

Let us again emphasise that the radial coordinate r near $p \in \text{Sing}(X)$ is fixed and the same in (2.1), (3.1) and (3.3).

3.2. The “singular” Morse–Thom–Smale complex computing intersection cohomology

Let us now recall the construction and main results on the “singular” Morse–Thom–Smale complex from [54, 55]: For a singular point $p \in \text{Sing}(X)$, we denote by $o(TL_p)$ the orientation bundle of L_p . Moreover, we have the following notation:

$$\Omega^\bullet(L_p, F_{L_p}^* \otimes o(TL_p))$$

denotes the space of smooth de Rham forms on L_p with values in the flat bundle $F_{L_p}^* \otimes o(TL_p)$, and $H^\bullet(L_p, F_{L_p}^* \otimes o(TL_p))$ denotes the singular cohomology of L_p . Let

$$\Xi_p^k \subset \Omega^k(L_p, F_{L_p}^* \otimes o(TL_p)) \quad (3.4)$$

be a set of closed forms on L_p , whose cohomology classes form a basis of

$$H^k(L_p, F_{L_p}^* \otimes o(TL_p)), \text{span}(\Xi_p^k) \simeq H^k(L_p, F_{L_p}^* \otimes o(TL_p)).$$

In Section 4.3, we will choose these representatives as follows: By the smooth Hodge theorem for the compact manifold (L_p, g^{TL_p}) , we can identify $H^\bullet(L_p, F_{L_p}^* \otimes o(TL_p))$ with the vector space of harmonic forms $\mathcal{H}^\bullet(L_p, F_{L_p}^* \otimes o(TL_p))$ and

choose Ξ_p^k as an ONB of $\mathcal{H}^*(L_p, F_{L_p}^* \otimes o(TL_p))$ (w.r.t. the L^2 -metric induced from $g^{TL_p}, g^{F_{L_p}^*}$).

Let Y be a standard anti-radial gradient-like vector field. For $p \in \text{Crit}(Y)$, we denote by $W^u(p)$ (resp. $W^s(p)$) the unstable (resp. stable) set of p with respect to the flow Φ ; its intersections with X_{sm} are smooth manifolds. For $p, q \in \text{Crit}(Y)$ with $\text{ind}(p) - \text{ind}(q) = 1$, by Morse–Smale transversality, the space of trajectories of the flow Φ (induced from $-Y$) starting in p and ending in q is a finite set, which we denote by $\Gamma(p, q)$. Choosing orientations on the unstable manifolds of points in $\text{Crit}(Y)$ and using the flow Φ induce orientations on $\Gamma(p, q)$ (see [40, Section (c)] for more details). With other words, to each trajectory $\gamma \in \Gamma(p, q)$ we can assign $n_\gamma(p, q) \in \{\pm 1\}$. We denote by $\tau_\gamma : F_p^* \rightarrow F_q^*$ the map induced from parallel transport with respect to the flat connection along the trajectory γ .

To a standard anti-radial gradient-like vector field Y and the sets Ξ_p^{n-k} , $p \in \text{Sing}(X)$, $k \geq n - \bar{q}$, one can associate a geometric complex, which computes the intersection homology of X with values in F^* and perversity $\bar{q}$, $IH_{\bullet}^{\bar{q}}(X, F^*)$, as follows.

Definition/Proposition 3.3. We denote by $(C_{\bullet}^{\bar{q}}(X, Y, F^*), \partial_{\bullet})$ the following complex:

$$C_k^{\bar{q}}(X, Y, F^*) = \begin{cases} \left(\bigoplus_{p \in \text{Crit}_k(Y)} \langle [W^u(p)] \rangle \otimes F_p^* \right) \oplus \left(\bigoplus_{p \in \text{Sing}(X)} \text{span}(\Xi_p^{n-k}) \right) & \text{if } k \geq n - \bar{q}, \\ \bigoplus_{p \in \text{Crit}_k(Y)} \langle [W^u(p)] \rangle \otimes F_p^* & \text{else.} \end{cases}$$

The boundary operator $\partial_{\bullet}$ is defined as follows: For $p \in \text{Crit}_k(Y)$, $h \in F_p^*$:

$$\begin{aligned} \partial_k([W^u(p)] \otimes h) &= \sum_{q \in \text{Crit}_{k-1}(Y)} \sum_{\gamma \in \Gamma(p, q)} n_\gamma(p, q) \cdot [W^u(q)] \otimes \tau_\gamma(h) \in C_{k-1}^{\bar{q}}(X, Y, F^*). \end{aligned}$$

For $p \in \text{Sing}(X)$, $k \geq n - \bar{q}$, $\xi_p^{n-k} \in \Xi_p^{n-k}$:

$$\partial_k[\xi_p^{n-k}] = \sum_{q \in \text{Crit}_{k-1}(Y)} \left(\int_{W^s(q) \cap L_p} \xi_p^{n-k} \right) \cdot [W^u(q)] \in C_{k-1}^{\bar{q}}(X, Y, F^*).$$

The complex $(C_{\bullet}^{\bar{q}}(X, Y, F^*), \partial_{\bullet})$ is well defined, i.e., $\partial_{\bullet}^2 = 0$.

The trajectories of the flow Φ induced by $-Y$ can leave the stratum only at time $t = -\infty$. This is a crucial feature used to prove the well-definedness of the singular Morse–Thom–Smale complex. The singular Morse–Thom–Smale complex does

depend on the choice of representatives $\Xi_p^\bullet$, $p \in \text{Sing}(X)$, in a mild way; see [54, Section 5.2].

The main result regarding the “singular” Morse–Thom–Smale complex is the following; see [54, Theorem 6.2].

Theorem 3.4. *The complex $(C_\bullet^\bar{q}(X, Y, F^*), \partial_\bullet)$ computes the intersection homology of X :*

$$H_\bullet(C_\bullet^\bar{q}(X, Y, F^*), \partial_\bullet) \simeq IH_\bullet^\bar{q}(X, F^*). \quad (3.5)$$

Theorem 3.4 is proved in [54] by constructing a natural embedding of the complex $(C_\bullet^\bar{q}(X, Y, F^*), \partial_\bullet)$ into the intersection chain complex and showing that this embedding is indeed a quasi-isomorphism. There are different approaches to the smooth Morse–Thom–Smale complex. In [54], we rely on the approach of Laudénbach [40], which gives an interpretation of the smooth Morse–Thom–Smale complex in terms of currents. All constructions in [54] assume that the anti-radial gradient-like vector field is standard; one can remove this latter assumption by a perturbation argument as in [39, Proposition 5.1].

From Theorem 3.4 (and using Poincaré duality), we can deduce Morse inequalities for the intersection homology of the singular space X with perversity $\bar{q}$ and values in F^* . For $k = 0, \dots, n$, we denote by $Ib_k^\bar{q}(X, F^*) := \dim IH_k^\bar{q}(X, F^*)$ the intersection Betti numbers of X . We denote by $c_k(f_{\text{sm}}) := \# \text{Crit}_k(f_{\text{sm}})$ the number of critical points of Morse index k of the Morse function f_{sm} and by

$$c_k^\bar{q}(f) := \begin{cases} \text{rk}(F) \cdot c_k(f_{\text{sm}}) + \dim IH_k^\bar{q}(cL_p, F^*) & \text{for } f \text{ radial,} \\ \text{rk}(F) \cdot c_k(f_{\text{sm}}) + \dim IH_k^\bar{q}(cL_p, L_p, F^*) & \text{for } f \text{ anti-radial.} \end{cases} \quad (3.6)$$

where $\text{rk}(F)$ is the rank of the flat vector bundle F .

Corollary 3.5 (Morse inequalities). *For a radial (resp. anti-radial) Morse function on a singular space with isolated singularities, we have*

$$\begin{aligned} \sum_{i=0}^k (-1)^{k-i} c_i^\bar{q}(f) &\geq \sum_{i=0}^k (-1)^{k-i} Ib_i^\bar{q}(X, F^*) \quad \text{for all } 0 \leq k < n, \\ \sum_{i=0}^n (-1)^i c_i^\bar{q}(f) &= \sum_{i=0}^n (-1)^i Ib_i^\bar{q}(X, F^*). \end{aligned}$$

The above Morse inequalities have long been known and can also be deduced using Goresky and MacPherson’s deformation lemmas in the framework of stratified Morse theory [32, Part I]. For the case of singular homology – for a topologically normal space, this corresponds to intersection homology with top perversity – they are

explicitly mentioned in [33, Section 9]. Let us emphasise again that the main interest of the “singular” Morse–Thom–Smale complex presented here does not lie in giving yet another proof of Morse inequalities but in its application to the comparison results in Sections 4.3 and 5.

Let $(C_{\bullet}^{\text{sm}}, \partial_{\bullet})$ denote the subcomplex of $(C_{\bullet}^{\bar{q}}(X, Y, F^*), \partial_{\bullet})$ generated by $\text{Crit}(Y)$. There is an exact sequence of complexes

$$0 \rightarrow (C_{\bullet}^{\text{sm}}, \partial_{\bullet}) \rightarrow (C_{\bullet}^{\bar{q}}(X, Y, F^*), \partial_{\bullet}) \rightarrow ((C_{\bullet}^{\bar{q}}(X, Y, F^*)/C^{\text{sm}})_{\bullet}, \partial_{\bullet}) \rightarrow 0. \quad (3.7)$$

Note that the boundary operator in the quotient complex vanishes. By the local calculation for intersection homology (2.3) and Poincaré duality on the link manifold L_p , we have

$$H_{\bullet}((C_{\bullet}^{\bar{q}}(X, Y, F^*)/C^{\text{sm}})_{\bullet}, \partial_{\bullet}) \simeq \bigoplus_{p \in \text{Sing}(X)} IH_{\bullet}^{\bar{q}}(cL_p, L_p, F^*). \quad (3.8)$$

For a complex vector space V of dimension 1, we denote by V^{-1} its dual. For a vector space V , we denote by $\det V$ the maximal exterior power of V . For a graded vector space $V^{\bullet} = \bigoplus_{k=0}^n V^k$, we denote by $\det V^{\bullet}$ the complex line

$$\det V^{\bullet} := \bigotimes_{k=0}^n (\det V^k)^{(-1)^k}.$$

From (3.5), (3.7) and (3.8), we get a natural isomorphism

$$\begin{aligned} \det H_{\bullet}(C_{\bullet}^{\text{sm}}, \partial_{\bullet}) \otimes \det \left(\bigoplus_{p \in \text{Sing}(X)} IH_{\bullet}^{\bar{q}}(cL_p, L_p, F^*) \right) \\ \simeq \det H_{\bullet}(C_{\bullet}^{\bar{q}}(X, Y, F^*), \partial_{\bullet}) \simeq \det IH_{\bullet}^{\bar{q}}(X, F^*) \\ \simeq (\det IH_{\bar{q}}^{\bullet}(X, F))^{-1}. \end{aligned} \quad (3.9)$$

We denote by $(C_{\bar{q}}^{\bullet}(X, Y, F), \partial^{\bullet})$ the dual complex of the singular Morse–Thom–Smale complex, which computes the intersection cohomology $IH_{\bar{q}}^{\bullet}(X, F)$.

3.3. Morse–Bott-type complex for stratified pseudomanifolds computing intersection cohomology

In [59], the construction of the Morse–Thom–Smale complex in Section 3.2 has been extended to the case of general compact stratified pseudomanifolds equipped with so-called *stratified anti-radial gradient-like vector fields*. In this more general singular context, we construct a cohomological complex, relying on the generalisation of the smooth Morse–Thom–Smale complex to smooth Morse–Bott functions. Several approaches to the smooth Morse–Bott complex can be found in the literature, most

of them having Floer theoretical applications in mind; see [9] for an overview and a comparison of the different approaches. The paper [59] makes use of the Morse–Bott cochain complex of Austin and Braam [8], defined in terms of the de Rham complexes of the critical submanifolds of the Morse–Bott function and the trajectory spaces between them.

An important property of stratified anti-radial gradient-like vector fields ξ is that they can be lifted to a smooth vector field $\tilde{\xi}$ on the smooth unfolding $\tilde{X}$ of X . Singularities of a stratified anti-radial gradient-like vector field ξ are isolated, but the smooth vector field $\tilde{\xi}$ on $\tilde{X}$ has Morse–Bott singularities. The flow associated with ξ satisfies control conditions; trajectories do not leave a stratum in finite time, and moreover, in positive infinite time, flow lines can only go from a larger into a smaller stratum. This allows us to adapt the construction in [8] and to construct, for any given perversity $\bar{q}$ in the sense of Goresky and MacPherson [30, 31], a Morse–Bott-type cochain complex $(C_{\bar{q}}^*(X, \xi), \partial^*)$ associated with the stratified anti-radial gradient-like vector field ξ on X , and which computes the intersection cohomology $IH_{\bar{q}}^*(X)$ of X with perversity $\bar{q}$. The main idea of the construction in [59] is to replace the de Rham complex of critical submanifolds of $\tilde{\xi}$ in the construction of the smooth Morse–Bott cochain complex of Austin and Braam [8] with the truncated complex of liftable intersection differential forms. The latter has been introduced by Brasselet, Hector and Saralegi in [14] and is a subcomplex of the more notorious complex of intersection differential forms of Brylinski, Goresky and MacPherson [21].

4. Witten deformation on singular spaces with conical singularities using radial and anti-radial Morse functions

In [75], Witten, inspired by ideas from quantum field theory, proposed a new analytic proof of the famous Morse inequalities for a smooth Morse function on a smooth manifold. His ideas met a lot of enthusiasm in the mathematical world. In [39], Helffer and Sjöstrand provided rigorous proofs for Witten’s method. Generalisations to manifolds with boundary [22, 38, 41, 42], to complete manifolds [1, 76] and to the holomorphic setting [28] have been studied since.

The power of the Witten deformation goes far beyond just proving Morse inequalities: For example, the holomorphic Morse inequalities of Demailly [28] provide an effective tool in complex analysis for the study of vanishing theorems. The Witten deformation combined with local index techniques is a leitmotiv in Bismut’s work on analytic torsion in its different flavours. In particular, it plays a prominent role in Bismut and Zhang’s extension of the Cheeger–Müller theorem in [11, 12].

In a series of works [49–53, 55], the author has studied generalisations of the Witten deformation to singular spaces with conical and iterated conical singularities using different classes of Morse functions on these spaces.

In Sections 4.1–4.3, we recall from [55] the generalisation of the Witten deformation to spaces with isolated conical singularities equipped with radial (resp. anti-radial) Morse functions. It is one of the key ingredients in the generalisation of the Cheeger–Müller/Bismut–Zhang theorem presented in Section 5. In Sections 4.4 and 4.5, we shortly discuss further generalisations of the Witten deformation for more general singularities and using a different class of Morse functions, respectively.

4.1. The Witten deformation and the spectral gap theorem

The Witten deformation on a smooth compact Riemannian manifold (M, g^{TM}) is a deformation of the de Rham complex $(\Omega^\bullet(M, F), d)$ using a smooth Morse function $f : M \rightarrow \mathbb{R}$. We denote by $T \geq 0$ the deformation parameter. The deformation produces a deformed complex $(\Omega^\bullet(M, F), d_T)$, where $d_T := e^{-Tf} d e^{Tf}$, and a deformed Laplacian, the famous Witten Laplacian $\tilde{\Delta}_T$. The easy part of the Witten deformation, the spectral gap theorem for the Witten Laplacian $\tilde{\Delta}_T$, states that for sufficiently large $T \gg 0$, the spectrum of $\tilde{\Delta}_T$ splits into two parts: exponentially small eigenvalues, which are in one-to-one correspondence with the critical points of the Morse function, and the large eigenvalues which – in case that $\nabla_g f$ is standard near critical points of f – grow at least linearly in T . An analytic proof of the smooth Morse inequalities can be easily deduced from the spectral gap theorem.

Let us now come to our singular setting. For a space with isolated conical singularities (X, g^{TX}) , the main idea for the generalisation of the Witten deformation is to deform the complex of L^2 -forms (instead of the de Rham complex) using an appropriate class of Morse functions. Here, we use radial (resp. anti-radial) Morse functions $f : X \rightarrow \mathbb{R}$; see Definition 3.2. As a byproduct, we get an analytic proof of the Morse inequalities for intersection (co-)homology with middle perversities $\bar{q} \in \{\bar{m}, \bar{n}\}$ in Corollary 3.5.

Let us explain this in more technical terms now: We deform the de Rham complex of compactly supported forms $(\Omega_c^\bullet(X, F), d_c, \langle \cdot, \cdot \rangle)$ by deforming the differential d_c via

$$d_{T,c} := e^{-Tf} d_c e^{Tf}.$$

The complex $(\Omega_c^\bullet(X, F), d_{T,c}, \langle \cdot, \cdot \rangle)$ admits a maximal and a minimal extension into a Hilbert complex $(\tilde{\mathcal{C}}_{T,\bar{m}/\bar{n}}^\bullet, d_{T,\bar{m}/\bar{n}}, \langle \cdot, \cdot \rangle) := (\tilde{\mathcal{C}}_{T,\max/\min}^\bullet, d_{T,\max/\min}, \langle \cdot, \cdot \rangle)$, where $d_{T,\bar{m}} := d_{T,\max}$ (resp. $d_{T,\bar{n}} := d_{T,\min}$) denotes the maximal (resp. minimal) extension of $d_{T,c}$ in $L^2(\Lambda(T^*X) \otimes F)$. The deformed Hilbert complex $(\tilde{\mathcal{C}}_{T,\bar{q}}^\bullet, d_{T,\bar{q}}, \langle \cdot, \cdot \rangle)$ still computes the L^2 -cohomology,

$$H^\bullet((\tilde{\mathcal{C}}_{T,\bar{q}}^\bullet, d_{T,\bar{q}}, \langle \cdot, \cdot \rangle)) \simeq H_{(2),\bar{q}}^\bullet(X, F), \quad \bar{q} \in \{\bar{m}, \bar{n}\}. \quad (4.1)$$

We denote by $\delta_{T,c}$ the (formal) adjoint of $d_{T,c}$ with respect to the L^2 -inner product $\langle \cdot, \cdot \rangle$; it admits a minimal (resp. maximal) extension $\delta_{T,\bar{m}} := \delta_{T,\min}$ (resp. $\delta_{T,\bar{n}} := \delta_{T,\max}$).

The Witten Laplacian

$$\tilde{\Delta}_T^{\bar{q}} := (d_{T,\bar{q}} + \delta_{T,\bar{q}})^2 \quad (4.2)$$

is the second-order operator associated with the Hilbert complex $(\tilde{\mathcal{C}}_{T,\bar{q}}^\bullet, d_{T,\bar{q}}, \langle \cdot, \cdot \rangle)$. It follows from the Hodge theorem for Hilbert complexes, see [16, Lemma 2.2 and Corollary 2.5], that

$$\ker(\tilde{\Delta}_T^{\bar{q}}) \simeq H^\bullet(\tilde{\mathcal{C}}_{T,\bar{q}}^\bullet, d_{T,\bar{q}}, \langle \cdot, \cdot \rangle). \quad (4.3)$$

We denote by $L_{\nabla f}$ the Lie derivative in direction of ∇f and by $L_{\nabla f}^*$ its adjoint with respect to the L^2 -inner product $\langle \cdot, \cdot \rangle$. The operator $L_{\nabla f} + L_{\nabla f}^*$ acting on $L^2(\Lambda(T^*X) \otimes F)$ is a bounded operator of order 0. This follows from the smooth theory combined with the local model near $\text{Sing}(X)$; see Section 4.2 for more details.

We have, see [11, Proposition 5.5], [55, Propositions 5.1],

$$\tilde{\Delta}_T^{\bar{q}} = \Delta^{\bar{q}} + T(L_{\nabla f} + L_{\nabla f}^*) + T^2|\nabla f|^2, \quad \text{dom}(\tilde{\Delta}_T^{\bar{q}}) = \text{dom}(\Delta^{\bar{q}}). \quad (4.4)$$

Let $(\tilde{\mathcal{S}}_{T,\bar{q}}^\bullet, d_{T,\bar{q}}, \langle \cdot, \cdot \rangle)$ denote the subcomplex of $(\tilde{\mathcal{C}}_{T,\bar{q}}^\bullet, d_{T,\bar{q}}, \langle \cdot, \cdot \rangle)$ generated by all eigenforms of the Witten Laplacian $\tilde{\Delta}_T^{\bar{q}}$ to eigenvalues in the interval $[0, 1]$. The following spectral gap theorem for the Witten Laplacian $\tilde{\Delta}_T^{\bar{q}}$ has been proved in [55, Theorem I], generalising Witten's smooth spectral gap theorem.

Theorem 4.1 (Spectral gap theorem). *Let (X, g^{TX}) be a space with isolated conical singularities equipped with a Hermitian flat vector bundle (F, ∇^F, g^F) as in Section 2.2. Let $\bar{q} \in \{\bar{m}, \bar{n}\}$ be one of the two middle perversities. Let $f : X \rightarrow \mathbb{R}$ be a radial (resp. an anti-radial) Morse function on X such that $\nabla_g f$ is standard near $\text{Crit}(f)$, i.e., condition (b) in Section 3.1 holds. We assume, moreover, that the Hermitian metric g^F is flat near $\text{Crit}(f_{\text{sm}})$. Then:*

- (a) *There exist constants $C_1, C_2, C_3 > 0$ and $T_0 > 0$ such that for $T \geq T_0$,*

$$\text{Spec}(\tilde{\Delta}_T^{\bar{q}}) \cap (C_1 e^{-C_2 T}, C_3 T) = \emptyset.$$

- (b) *Moreover, for $k = 0, \dots, n$, and $T \geq T_0$,*

$$\dim \tilde{\mathcal{S}}_{T,\bar{q}}^k = c_k^{\bar{q}}(f),$$

where $c_k^{\bar{q}}(f)$ is as defined in (3.6).

Combining the spectral gap theorem, (4.1), the deformed Hodge theorem (4.3) and the Cheeger–Goresky–MacPherson theorem (2.7), we get an analytic proof for

the Morse inequalities for intersection (co-)homology with lower and upper middle perversities in Corollary 3.5.

There is a second, equivalent way of describing the Witten deformation: The de Rham complex $(\Omega_c^\bullet(X, F), d_c, \langle \cdot, \cdot \rangle)$ can also be deformed by deforming the L^2 -inner product $\langle \cdot, \cdot \rangle$ via

$$\langle \alpha, \beta \rangle_T := \int_X \langle \alpha, \beta \rangle_{\Lambda(T^*X) \otimes F}(x) e^{-2Tf(x)} d \operatorname{vol}_X(x);$$

here, $d \operatorname{vol}_X$ denotes the Riemannian volume form on (X, g^{TX}) .

The deformed complex $(\Omega_c^\bullet(X, F), d_c, \langle \cdot, \cdot \rangle_T)$ also admits a maximal and minimal extension into a Hilbert complex $(\mathcal{C}_{T, \bar{q}}^\bullet, d_{\bar{q}}, \langle \cdot, \cdot \rangle_T)$, where $d_{\bar{m}} = d_{\max}$ and $d_{\bar{n}} = d_{\min}$. We denote by $\delta'_{T, c}$ the (formal) adjoint of d_c with respect to the twisted L^2 -inner product $\langle \cdot, \cdot \rangle_T$ and by $\delta'_{T, \bar{m}}$ (resp. $\delta'_{T, \bar{n}}$) its minimal (resp. its maximal) extension. We denote by $\Delta_T^{\bar{q}} := (d_{\bar{q}} + \delta'_{T, \bar{q}})^2$ the second-order self-adjoint operator associated with the Hilbert complex $(\mathcal{C}_{T, \bar{q}}^\bullet, d_{\bar{q}}, \langle \cdot, \cdot \rangle_T)$.

The two descriptions of the Witten deformation are equivalent; more precisely, we have natural isomorphisms of Hilbert complexes

$$e^{Tf} : (\tilde{\mathcal{C}}_{T, \bar{q}}^\bullet, d_{T, \bar{q}}, \langle \cdot, \cdot \rangle) \longrightarrow (\mathcal{C}_{T, \bar{q}}^\bullet, d_{\bar{q}}, \langle \cdot, \cdot \rangle_T), \quad \bar{q} \in \{\bar{m}, \bar{n}\}. \quad (4.5)$$

We denote by $(\mathcal{S}_{T, \bar{q}}^\bullet, d_{\bar{q}}, \langle \cdot, \cdot \rangle_T)$ the subcomplex of $(\mathcal{C}_{T, \bar{q}}^\bullet, d_{\bar{q}}, \langle \cdot, \cdot \rangle_T)$ generated by eigenforms of $\Delta_T^{\bar{q}}$ to eigenvalues in $[0, 1]$.

Note that the Hodge star operator $*^F$ associated with the metrics g^{TX} , g^F induces an isomorphism of the Hilbert complex associated with the Morse function f , the flat vector bundle F and the perversity $\bar{q}$, and the dual of the Hilbert complex associated with the Morse function $-f$, the flat vector bundle $F^* \otimes o(TX)$ and the dual perversity.

4.2. The model Witten Laplacian

For this section, we restrict our considerations to the case where $f : X \rightarrow \mathbb{R}$ is an anti-radial Morse function; the case of a radial Morse function can be treated analogously.

At the heart of the proof of the spectral gap theorem lies the phenomenon of “concentration near critical points of the Morse function f ”, $\operatorname{Crit}(f) = \operatorname{Crit}(f_{\text{sm}}) \cup \operatorname{Sing}(X)$. This allows us to reduce the problem to a study of the local model situation and the model Witten Laplacian near $\operatorname{Crit}(f)$. In this section, we explain the local model situation near $\operatorname{Crit}(f)$: We will see, in particular, that indeed the spectral gap theorem holds in the model situation! Let us mention, moreover, that the model Witten Laplacian on the infinite cone (more precisely its analytic torsion) is a crucial ingredient in the definition of the Bismut–Zhang torsion (see Definition 5.4), the “topological” side in the Cheeger–Müller/Bismut–Zhang theorem.

For sufficiently small $0 < \varepsilon < \delta$, we identify a neighbourhood $B_\varepsilon(p)$ of $p \in \text{Crit}(f_{\text{sm}})$ (resp. of $p \in \text{Sing}(X)$) with a neighbourhood $B_\varepsilon(0) \subset \mathbb{R}^n$ (resp. $c_\varepsilon L_p \subset cL_p$).

Using this identification, the action of the Witten Laplacian $\tilde{\Delta}_T^q$, $T > 0$, on forms with support in a neighbourhood of $p \in \text{Crit}(f)$ can be identified with the action of the model Witten Laplacian, which we now recall.

Let $p \in \text{Crit}_k(f_{\text{sm}})$ be a smooth critical point. By Morse's lemma, we have Morse coordinates such that $f = f(p) + \frac{1}{2}(-x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2)$. We have assumed that the Riemannian metric g^{TX} is Euclidean in the Morse coordinates near $\text{Crit}(f_{\text{sm}})$. Recall that, by our assumption, the Hermitian metric g^F is also flat near p . We trivialise the flat bundle F over $B_\varepsilon(p)$ by using the connection ∇^F , so that the fibres of $F|_{B_\varepsilon(p)}$ are identified with F_p . For a multi-index $I = (i_1, \dots, i_l)$, set $dx_I := dx_{i_1} \wedge \cdots \wedge dx_{i_l}$. Denote by N^+ (resp. by N^-) the following number operators:

$$N^+ dx_I := \#\{s \mid k+1 \leq i_s \leq n\} \cdot dx_I, \quad N^- dx_I := (l - N^+) \cdot dx_I.$$

The model Witten Laplacian Δ_T^p is an operator acting on sections of $\Lambda((\mathbb{R}^n)^*) \otimes F_p$ over $\mathbb{R}^n$, which is closely related to the quantum harmonic oscillator (see [39, 75], [11, Proposition 8.2]):

$$\Delta_T^p := \sum_{j=1}^n \left(-\frac{\partial^2}{\partial x_j^2} + T^2 x_j^2 \right) + T(2k - n) + 2T(N^+ - N^-). \quad (4.6)$$

Note that we get (4.6) by evaluating formula (4.4) in our model situation.

Set

$$r := \left(\sum_{i=1}^n x_i^2 \right)^{1/2} \quad \text{and} \quad \varphi_{p,T} := \left(\frac{T}{\pi} \right)^{n/4} e^{-Tr^2/2} dx_1 \wedge \cdots \wedge dx_k.$$

We have, for $T > 0$ (see [39, 75], [11, Proposition 8.3]),

$$\text{Spec}(\Delta_T^p) = 2T\mathbb{N}_0 \quad \text{and} \quad \ker \Delta_T^p = \ker \Delta_T^{p,(k)} = \langle \varphi_{p,T} \rangle \otimes F_p.$$

Let $\chi_p : \mathbb{R}^n \rightarrow \mathbb{R}$ be a cut-off function with $\text{supp } \chi_p \subset B_\varepsilon(0)$ and $\chi_p|_{B_{\varepsilon/2}(0)} \equiv 1$. Set $\omega_{p,T} := \chi_p \varphi_{p,T}$. The forms $\omega_{p,T} \otimes h$, $h \in F_p$, can be seen as elements of $\Omega^*(X, F)$ supported in a neighbourhood of p . For $i = 0, \dots, n$, define

$$E_p^i(T) := \begin{cases} \langle \omega_{p,T,l} := \omega_{p,T} \otimes h_l \mid h_l \text{ ON basis of } F_p \rangle & \text{for } i = k, \\ 0 & \text{for } i \neq k. \end{cases} \quad (4.7)$$

Let now $p \in \text{Sing}(X)$ and $T \geq 0$. The model Witten Laplacian $\Delta_T^{p,\bar{q}}$ is an operator acting on forms on the (punctured) infinite cone Z_p and depends on the perversity

$\bar{q} \in \{\bar{m}, \bar{n}\}$. We denote by N the number operator acting by multiplication with the form degree. We denote by Δ_T^p the following operator acting on compactly supported forms on Z_p with values in the bundle F_{Z_p} :

$$\Delta_T^p := \Delta^p + T(n - 2N) + T^2 r^2; \quad (4.8)$$

here, Δ^p denotes the Laplacian acting on compactly supported forms on Z_p ; note again that we get (4.8) by using formula (4.4) for the infinite cone Z_p equipped with the model function $f^p = \frac{-r^2}{2}$. The reader is invited to compare (4.8) with (4.6) in the case of Morse index $k = n$.

The model Witten Laplacian $\Delta_T^{p,\bar{q}}, \bar{q} \in \{\bar{m}, \bar{n}\}$, is the self-adjoint closed extension of (4.8), with boundary conditions at $r = 0$ inherited from the boundary conditions near $p \in \text{Sing}(X)$ of the Witten Laplacian $\tilde{\Delta}_T^{\bar{q}}$. By completeness of the infinite cone, at $r = \infty$, we do not need to specify boundary conditions. For $T > 0$, the operator $\Delta_T^{p,\bar{q}}$ is a discrete operator. Using Cheeger's separation of variables technique, one can compute its spectrum and eigenforms explicitly in terms of the spectral data of the transversal Laplacian on the link. We denote by $b^i(L_p, F_{L_p}) := \dim H^i(L_p, F_{L_p})$, $i = 0, \dots, n-1$, the Betti numbers for the cohomology of the link L_p . For $i \geq n - \bar{q}$, let

$$\{\eta_{p,l}^{i-1} \mid l = 1, \dots, b^{i-1}(L_p, F_{L_p})\} \subset \mathcal{H}^{i-1}(L_p, F_{L_p}) \simeq H^{i-1}(L_p, F_{L_p})$$

be an orthonormal basis of harmonic forms on the link manifold, dual to the basis $\{\xi_{p,l}^{n-i} \mid \xi_{p,l}^{n-i} \in \Xi_p^{n-i}\}$ of $H^{n-i}(L_p, F_{L_p}^* \otimes o(TL_p)) \simeq \mathcal{H}^{n-i}(L_p, F_{L_p}^* \otimes o(TL_p))$ in (3.4). For $l = 1, \dots, b^{i-1}(L_p, F_{L_p})$, we define

$$\varphi_{p,T,l}^i := \left(\frac{T}{\pi}\right)^{i/2-n/4} \sqrt{\text{vol}(S^{2i-n-1})} \cdot e^{-Tr^2/2} r^{2i-n-1} \eta_{p,l}^{i-1} dr \in \Omega^i(Z_p, F).$$

One has, for $T > 0$ (see [55, Theorem 4.2]),

$$\ker \Delta_T^{p,\bar{q},(i)} = \begin{cases} \langle \varphi_{p,T,l}^i \mid l = 1, \dots, b^{i-1}(L_p, F_{L_p}) \rangle & \text{for } i \geq n - \bar{q}, \\ 0 & \text{else.} \end{cases} \quad (4.9)$$

Note that $\{\varphi_{p,T,l}^i \mid l = 1, \dots, b^{i-1}(L_p, F_{L_p})\}$ is an orthonormal basis for $\ker \Delta_T^{p,\bar{q},(i)}$, $i \geq n - \bar{q}$. From the local calculation for intersection cohomology [31, Section 2.4], recalled in (2.3) and (4.9), we have

$$\ker \Delta_T^{p,\bar{q}} \simeq IH_{\bar{q}}^\bullet(cL_p, L_p, F). \quad (4.10)$$

Let χ_p be a cut-off function $\chi_p : cL_p \rightarrow \mathbb{R}$ with $\text{supp } \chi_p \subset c_\varepsilon L_p$, $\chi_p|_{c_{\varepsilon/2} L_p} \equiv 1$. Then

$$\omega_{p,T,l}^i := \chi_p \varphi_{p,T,l}^i \in \Omega^i(X, F).$$

For $i = 0, \dots, n$, define

$$E_{p,\bar{q}}^i(T) := \begin{cases} \langle \omega_{p,T,l}^i \mid l = 1, \dots, b^{i-1}(L_p, F_{L_p}) \rangle & \text{for } i \geq n - \bar{q}, \\ 0 & \text{else.} \end{cases} \quad (4.11)$$

4.3. Comparison between the Witten complex and the “singular” Morse–Thom–Smale complex

The second step of Witten’s programme, the hard part, aims at comparing the Witten complex of eigenforms to small eigenvalues of the Witten Laplacian with the Morse–Thom–Smale complex: Witten suggested that, in case that the gradient vector field of the Morse function satisfies the Morse–Smale transversality condition, in some appropriate sense, the Witten complex should “converge” to the (dual) Morse–Thom–Smale complex as $T \rightarrow \infty$. Let us emphasise that the second part of Witten’s programme is more delicate than the first part. It has also been proved by Helffer and Sjöstrand in [39] using semi-classical analysis, but here one needs more refined estimates on the decay of eigenforms of the Witten Laplacian than in the first part. A second proof has been given by Bismut and Zhang in [12] using the interpretation due to Laudenbach [40] of the Morse–Thom–Smale complex as a complex of currents.

Let us now come back to our singular setting: Here, we will compare the Witten complex with the “singular” Morse–Thom–Smale complex constructed in Section 3.2. In this section, $f : X \rightarrow \mathbb{R}$ always denotes an anti-radial Morse function such that $\nabla_g f$ is a standard anti-radial gradient-like vector field. We moreover assume that the Hermitian metric g^F is flat near $\text{Crit}(f_{\text{sm}})$.

For $i = 0, \dots, n$ and $T > 0$, using (4.7) and (4.11), we define the vector space

$$E_{\bar{q}}^i(T) := \left(\bigoplus_{p \in \text{Crit}(f_{\text{sm}})} E_p^i(T) \right) \oplus \left(\bigoplus_{p \in \text{Sing}(X)} E_{p,\bar{q}}^i(T) \right) \subset \Omega^i(X, F).$$

We define a map of graded vector spaces $J_{T,\bar{q}} : C_{\bar{q}}^\bullet(X, \nabla_g f, F) \rightarrow E_{\bar{q}}^\bullet(T)$ as follows:

$$\begin{aligned} J_{T,\bar{q},i} : C_{\bar{q}}^i(X, \nabla_g f, F) &\longrightarrow E_{\bar{q}}^i(T), \\ [W^u(p)]^* \otimes h &\mapsto \omega_{p,T} \otimes h \quad \text{for } p \in \text{Crit}_i(f_{\text{sm}}), h \in F_p, \\ [\xi_{p,l}^{n-i}]^* &\mapsto \omega_{p,T,l}^i \quad \text{for } p \in \text{Sing}(X), i \geq n - \bar{q}, \\ &\quad l \in \{1, \dots, b^{i-1}(L_p, F_{L_p})\}. \end{aligned}$$

We denote by $\tilde{P}_{T,\bar{q}}^{[0,1]} : \tilde{\mathcal{C}}_{T,\bar{q}}^\bullet \rightarrow \tilde{\mathcal{S}}_{T,\bar{q}}^\bullet$ the orthogonal projection. The isomorphism of complexes (4.5) induces an isomorphism of complexes

$$e^{Tf} : (\tilde{\mathcal{S}}_{T,\bar{q}}^\bullet, d_{T,\bar{q}}, \langle \cdot, \cdot \rangle) \longrightarrow (\mathcal{S}_{T,\bar{q}}^\bullet, d_{\bar{q}}, \langle \cdot, \cdot \rangle_T).$$

We define the map of graded vector spaces $e_{T,\bar{q}}$ as the following composition:

$$e_{T,\bar{q}} : C_{\bar{q}}^{\bullet}(X, \nabla_g f, F) \xrightarrow{J_{T,\bar{q}}} E_{\bar{q}}^{\bullet}(T) \xrightarrow{\tilde{P}_{T,\bar{q}}^{[0,1]}} \tilde{S}_{T,\bar{q}}^{\bullet} \xrightarrow{e^{Tf}} S_{T,\bar{q}}^{\bullet}.$$

Using the embedding $(C_{\bar{q}}^{\bullet}(X, \nabla_g f, F^*), \partial_{\bullet}) \rightarrow (IC_{\bar{q}}^{\bullet}(X, F^*), \partial_{\bullet})$ into the intersection chain complex, constructed in [54, Section 6], the integration of L^2 -forms on intersection chains does induce a map of chain complexes $\int_{T,\bar{q}}$, $T \geq 0$ (see [55, Proposition 6.13] and [54, Section 6.3]):

$$\int_{T,\bar{q}} : (S_{T,\bar{q}}^{\bullet}, d, \langle \cdot, \cdot \rangle_T) \longrightarrow (C_{\bar{q}}^{\bullet}(X, \nabla_g f, F), \partial^{\bullet}).$$

The number operator acting on the graded vector space $C_{\bar{q}}^{\bullet}(X, \nabla_g f, F)$ by multiplication with the degree is denoted by N . We denote by $\mathcal{F}$ the operator which, for $p \in \text{Crit}(f_{\text{sm}})$, does act on $[W^u(p)]^* \otimes F_p$ by multiplication with $f(p)$; for $p \in \text{Sing}(X)$, the operator $\mathcal{F}$ acts by multiplication with $f(p)$ on $(\text{span}(\Xi_p^{\bullet}))^*$. The next theorem, proved in [55, Theorem II], is the analogue of the comparison theorem between Witten and Morse–Thom–Smale complex for a smooth compact manifold proved in [12, Theorem 6.11].

Theorem 4.2. *Let (X, g^{TX}) be a space with isolated conical singularities equipped with a Hermitian flat vector bundle (F, ∇^F, g^F) as in Section 2.2. Let $\bar{q} \in \{\bar{m}, \bar{n}\}$ be one of the two middle perversities. Let $f : X \rightarrow \mathbb{R}$ be an anti-radial Morse function on X such that $\nabla_g f$ is a standard anti-radial gradient-like vector field. We assume, moreover, that the Hermitian metric g^F is flat near $\text{Crit}(f_{\text{sm}})$. Then, as $T \rightarrow \infty$,*

$$\int_{T,\bar{q}} \circ e_{T,\bar{q}} = e^{T\mathcal{F}} \left(\frac{\pi}{T} \right)^{N/2-n/4} (\mathcal{A}_{\bar{q}} + \mathcal{O}(e^{-cT})),$$

where $\mathcal{A}_{\bar{q}}$ is the diagonal matrix with entries

$$\mathcal{A}_{\sigma} = \begin{cases} 1 & \text{for } \sigma = [W^u(p)]^* \otimes h, \ p \in \text{Crit}(f_{\text{sm}}), \ h \in F_p, \\ \frac{1}{\sqrt{\text{vol}(S^{2i-n-1})}} & \text{for } \sigma = [\xi_{p,l}^{n-i}]^*, \ p \in \text{Sing}(X), \ \xi_{p,l}^{n-i} \in \Xi_p^{n-i}. \end{cases}$$

In particular, for large T , the map $\int_{T,\bar{q}}$ is an isomorphism of complexes.

Using Theorem 4.2 and the fact that the integration map $\int_{T,\bar{q}}$ is a chain map, we get the following comparison between the boundary operators in the Witten and the Morse–Thom–Smale complex, which generalises [12, Theorem 6.12].

Theorem 4.3. *With the assumptions and notation as in Theorem 4.2, we get*

$$e_{T,\bar{q}}^{-1} \circ d \circ e_{T,\bar{q}} = \sqrt{\frac{T}{\pi}} (\mathcal{A}_{\bar{q}}^{-1} + \mathcal{O}(e^{-cT})) e^{-T\mathcal{F}} \partial^{\bullet} e^{T\mathcal{F}} (\mathcal{A}_{\bar{q}} + \mathcal{O}(e^{-cT})).$$

4.4. Witten deformation on Thom–Mather stratified pseudomanifolds using anti-radial and radial Morse functions

The easy part of the Witten deformation has been generalised in [55] in the more general context of spaces with “iterated” conical singularities. Let X be an n -dimensional compact stratified pseudomanifold; see, e.g., [55] for the exact definition. For $i = 0, \dots, n$, we denote by $X_{(i)}$ the union of all strata of X of dimension i . We denote by $\text{Sing}(X) := \bigcup_{i=0}^{n-1} X_{(i)}$ the singular set.

We equip the top stratum $X_{\text{sm}} := X_{(n)} = X \setminus \text{Sing}(X)$ with a compatible Riemannian metric. The notion of a compatible Riemannian metric on a stratified pseudomanifold X was first studied in [66, Definition 4.1] and also appeared in [15, Remark 3.3]; in [2, Definition 2.3], such metrics are called rigid edge metrics. Compatible metrics are defined inductively as follows: If $\text{depth}(X) = 0$, then X is a compact smooth manifold, and a compatible Riemannian metric g^{TX} is just a smooth Riemannian metric. Let now $\text{depth}(X) > 0$. A compatible Riemannian metric on X is a Riemannian metric g^{TX} on the top stratum $X_{\text{sm}} = X \setminus \text{Sing}(X)$ such that for each point x of the stratum $X_{(i)}$ of dimension $0 \leq i < n$, there exists a neighbourhood $U(x) \subset X$ such that in an admissible chart $\varphi : U(x) \simeq \mathbb{R}^i \times cL_x$, we have

$$g_{|U(x) \cap X_{\text{sm}}}^{TX} = g^{T\mathbb{R}^i}(z) + dr^2 + r^2 g^{TL_x}(z, y), \quad (z, (r, y)) \in \mathbb{R}^i \times cL_x, \quad (4.12)$$

where r is the radial coordinate, $g^{T\mathbb{R}^i}$ is a Riemannian metric on $\mathbb{R}^i$ and g^{TL_x} is a compatible Riemannian metric on the top stratum $L_{x, \text{sm}}$ of the link L_x – which is a stratified pseudomanifold of lower depth than X ; the metric g^{TL_x} depends smoothly on the coordinate z in $\mathbb{R}^i$.

The following definition generalises the definitions in Section 3.1 to stratified pseudomanifolds.

Definition 4.4. Let $f : X \rightarrow \mathbb{R}$ be a continuous function on X , and let g^{TX} be a compatible Riemannian metric on the compact stratified pseudomanifold X . We call the pair (f, g^{TX}) a standard *radial* (resp. *anti-radial*) *Morse pair* on the stratified space X if the following conditions are satisfied:

- (1) For each i , the restriction of $f|_{X_{(i)}}$ to $X_{(i)}$ is a smooth Morse function. We denote by $\text{Crit}_k(f|_{X_{(i)}})$ the set of its critical points of Morse index k .
- (2) For each critical point $p \in \text{Crit}_k(f|_{X_{(i)}})$, there exists an open neighbourhood $U(p)$ of p in X as well as local coordinates $z_1, \dots, z_i$ on $X_{(i)} \cap U(p)$ such that one can write

$$f|_{U(p)} = f^\pm = f(p) + \frac{1}{2}(-z_1^2 - \dots - z_k^2 + z_{k+1}^2 + \dots + z_i^2 \pm r^2),$$

and

$$g_{|U(p)}^{TX} = dz_1^2 + \dots + dz_i^2 + dr^2 + r^2 g^{TL_p}, \quad (4.13)$$

where r is the radial coordinate, g^{TL_p} is a fixed compatible Riemannian metric on the top stratum of the link L_p . (Note that in contrast to (4.12), here g^{TL_p} does not depend on the coordinates in the stratum $z_1, \dots, z_i$).

- (3) The gradient $\nabla f := \nabla_g f|_{X_{\text{sm}}}$ is bounded on $X_{\text{sm}} = X \setminus \text{Sing}(X)$. Moreover, there exists $C > 0$ such that

$$|\nabla f(x)| > C \quad \text{for all } x \in X_{\text{sm}} \setminus \left(\bigcup_{p \in \text{Crit}(f)} U(p) \right),$$

where $\text{Crit}(f) = \bigcup_{i=0}^n \text{Crit}(f|_{X_{(i)}})$.

- (4) The operator $\mathcal{L}_{\nabla f} + \mathcal{L}_{\nabla f}^*$ extends to a bounded operator on the space $L^2(\Lambda(T^*X) \otimes F)$.

A compatible Riemannian metric on a stratified pseudomanifold X can always be perturbed to achieve the product form (4.13) in a small neighbourhood of finitely many points. For a space X with isolated singularities, conditions (3) and (4) of Definition 4.4 follow already from (1) and (2), and indeed, in this case, we recover the notions defined already in Section 3.1. Note that (3) does not depend on the choice of compatible Riemannian metric. Condition (4) can be achieved by appropriate bounds on the second-order derivatives of f .

For an n -dimensional compact stratified pseudomanifold X , whose top stratum is equipped with a compatible Riemannian metric g^{TX} and a Hermitian flat vector bundle (F, ∇^F, g^F) with uniformly bounded metric g^F , we can also study the Hilbert complexes of L^2 -forms $(\mathcal{C}_{\bar{m}/\bar{n}}^\bullet, d_{\bar{m}/\bar{n}}, \langle \cdot, \cdot \rangle) := (\mathcal{C}_{\max/\min}^\bullet, d_{\max/\min}, \langle \cdot, \cdot \rangle)$, with associated Laplacian $\Delta_{\bar{m}/\bar{n}}$ and which compute the L^2 -cohomology $H_{(2), \bar{m}/\bar{n}}^\bullet(X, F)$. The Cheeger–Goresky–MacPherson theorem (2.7) and (2.8) also holds in this context. (See [24, Theorem 6.1] and [66, Theorem 4.14]; see also [10, 13] for an overview on de Rham-type theorems on singular spaces.)

The complex of L^2 -forms $(\mathcal{C}_{\bar{q}}^\bullet, d_{\bar{q}}, \langle \cdot, \cdot \rangle)$, $\bar{q} \in \{\bar{m}, \bar{n}\}$, can be deformed using radial/anti-radial Morse functions into a deformed complex $(\tilde{\mathcal{C}}_{T, \bar{q}}^\bullet, d_{T, \bar{q}}, \langle \cdot, \cdot \rangle)$. The following spectral gap theorem holds for the associated Witten Laplacian $\tilde{\Delta}_{T, \bar{q}}^\bullet$; see [55, Theorem I].

Theorem 4.5 (Spectral gap theorem). *Let X be an n -dimensional compact stratified pseudomanifold. Let (F, ∇^F, g^F) be a Hermitian flat vector bundle on the top stratum, with uniformly bounded metric g^F . Let $\bar{q} \in \{\bar{m}, \bar{n}\}$ be one of the two middle perversities. Let (f, g^{TX}) be a standard radial (resp. anti-radial) Morse pair on X . We assume that near $x \in \text{Crit}(f|_{X_{\text{sm}}})$, the metric g^F is flat, and that moreover, near $x \in \text{Crit}(f|_{\text{Sing}(X)})$, the vector bundle (F, ∇^F, g^F) is the extension of a flat vector*

bundle on the link L_x . Moreover, we assume that, for $T > 0$,

$$\operatorname{dom}(\tilde{\Delta}_T^{\bar{q}}) = \operatorname{dom}(\Delta^{\bar{q}}). \quad (4.14)$$

- (a) Then there exist constants $C_1, C_2, C_3 > 0$ and $T_0 > 0$ such that for any $T \geq T_0$ and $i = 0, \dots, n$,

$$\operatorname{Spec}(\tilde{\Delta}_T^{\bar{q}}) \cap (C_1 e^{-C_2 T}, C_3 T) = \emptyset.$$

- (b) Let us denote by $(\tilde{\mathcal{S}}_{T,\bar{q}}, d_{T,\bar{q}}, \langle \cdot, \cdot \rangle)$ the subcomplex of $(\tilde{\mathcal{C}}_{T,\bar{q}}, d_{T,\bar{q}}, \langle \cdot, \cdot \rangle)$ generated by all eigenforms of the Witten Laplacian $\tilde{\Delta}_T^{\bar{q}}$ to eigenvalues in $[0, 1]$. Then, for $T \geq T_0$,

$$\dim \tilde{\mathcal{S}}_{T,\bar{q}}^i = \operatorname{rk}(F) c_i(f|_{X \setminus \operatorname{Sing}(X)}) + \sum_{p \in \operatorname{Crit}(f) \cap \operatorname{Sing}(X)} m_{p,\bar{q}}^i =: c_i^{\bar{q}}(f),$$

where the contribution of a critical point $p \in \operatorname{Sing}(X)$ of index k is given by

$$m_{p,\bar{q}}^i := \dim IH_{\bar{q}}^{i-k}(cL_p, F) \quad (\text{resp. } m_{p,\bar{q}}^i := \dim IH_{\bar{q}}^{i-k}(cL_p, L_p, F))$$

if $f = f^+$ is radial (resp. $f = f^-$ is anti-radial) near p .

Remark 4.6. For a stratified pseudomanifold satisfying the Witt condition, one can – after possibly rescaling the compatible Riemannian metric – assume that the metric is *adapted*, i.e., that the associated de Rham operator is essentially self-adjoint. In this situation, the intersection (co-)homologies with upper middle and lower middle perversities coincide; one also has uniqueness of *ibcs*, $d_{\min} = d_{\max}$. It is mentioned in [55] that, in this situation, the technical condition (4.14) can be removed. Indeed, the technical condition (4.14) can also be removed for non-Witt spaces, with a different proof than in [55].

As a direct consequence of the spectral gap theorem, we get Morse inequalities for the L^2 -cohomology of X , or dually – in view of (2.7), which also holds in this more general setting – for the intersection (co-)homology with lower and upper middle perversities. The easy part of the Witten deformation on conical singularities using a similar class of functions has also been studied, in a slightly different manner in [7].

4.5. Witten deformation using Goresky–MacPherson’s stratified Morse theory

An important class of Morse functions for singular spaces is stratified Morse functions in the sense of the theory developed by Goresky and MacPherson in [32]. Stratified Morse functions are defined on Whitney stratified subsets of a smooth manifold. The main idea of the definition of stratified Morse functions is that critical

points in a lower stratum are not critical in normal directions. One of the main results in [32] is the stratified Morse inequalities for intersection homology for stratified Morse functions on singular complex (algebraic or analytic) varieties [32, Part II, Theorem 6.12]. In [49, 51] both the easy and the hard parts of the Witten deformation for stratified Morse functions on singular complex curves have been generalised, thus providing, in particular, an analytic proof of the famous stratified Morse inequalities of Goresky and MacPherson in the case of (complex) dimension 1. One difficulty in generalising Witten's programme in the context of Goresky and MacPherson's stratified Morse theory to higher dimensions stems from the fact that complex algebraic varieties are rarely conical in case $\dim > 1$, while the development of analytic tools for singular spaces has focused mainly on cone-like singularities. In [53], the easy part of the Witten deformation has been generalised for even-dimensional spaces with (asymptotically) cone-like singularities and so-called *admissible* Morse functions; the non-degeneracy condition for an admissible Morse function at singular points is inspired by stratified Morse theory.

5. Cheeger–Müller/Bismut–Zhang theorem for spaces with isolated conical singularities

In this section, we present the Cheeger–Müller/Bismut–Zhang theorem for spaces with isolated conical singularities, Theorem 1.1, achieved in [56, 58]. For this section, the setting is the following: (X, g^{TX}) is a space with isolated conical singularities equipped with a Hermitian flat vector bundle (F, ∇^F, g^F) as in Section 2.2, and $\bar{q} \in \{\bar{m}, \bar{n}\}$ is one of the middle perversities.

Throughout, we adopt the point of view due to Quillen: For a smooth compact manifold, both the analytic (or Ray–Singer) torsion as well as the topological (Reidemeister and Milnor) torsion are seen as metrics on the determinant of the singular cohomology. As mentioned, here we will generalise the approach to torsion due to Bismut and Zhang: They compare the Ray–Singer metric with the Milnor metric defined in [62]; see also [11, Section I (d)]. For unitary, and more generally unimodular, vector bundles, the Milnor metric equals the Reidemeister metric and is therefore a combinatorial and topological invariant. For non-unimodular vector bundles, the Milnor metric depends on choices. Moreover, equality between Ray–Singer and Milnor metrics no longer holds: A third term appears in the Bismut–Zhang comparison theorem.

In the presence of singularities, the Ray–Singer metric and its – yet to be defined – “topological” counterpart are metrics on the determinant of the intersection cohomology with middle perversity $\bar{q} \in \{\bar{m}, \bar{n}\}$. The Cheeger–Müller/Bismut–Zhang

theorem for singular spaces with isolated conical singularities, Theorem 1.1, compares the Ray–Singer metric with the Bismut–Zhang metric, an invariant also introduced in [56, 58]. The Bismut–Zhang metric is a Milnor-like metric with an analytic correction term coming from the singularities of the space. One of the advantages of our approach is that this analytic correction can be identified: It is precisely the analytic torsion of the model Witten Laplacian on the infinite cone! While the analytic torsion of the model Witten Laplacian can be computed explicitly in terms of the spectral data of the transversal Laplacian on the link manifold (see [57, Theorem I]), the result in Theorem 1.1 does not rely on these explicit computations. The answer we give here to Dar’s question is a conceptual one!

Moreover, the treatment of the problem of a Cheeger–Müller/Bismut–Zhang theorem in [56, 58] is complete: It incorporates the case of non-unitary vector bundles and moreover also discusses how the three terms in the comparison formula vary when the pair of metrics (g^{TX}, g^F) on TX and F varies, namely, the so-called anomaly formulas for the three terms.

This section is organised as follows: In Sections 5.1 and 5.2, we recall the definitions of the Ray–Singer and the Bismut–Zhang metrics, respectively, for a space with isolated conical singularities. In Section 5.3, we present anomaly formulas for all three terms in the Cheeger–Müller/Bismut–Zhang theorem (Theorem 1.1).

5.1. The Ray–Singer metric on $\det IH_q^\bullet(X, F)$

The analytic (or Ray–Singer) torsion of a smooth compact Riemannian manifold is an important secondary spectral invariant, defined by Ray and Singer [67]; it is a positive real number associated with the spectrum of the Hodge Laplacian. The definition of the Ray–Singer torsion has been generalised to spaces with isolated conical singularities by Dar [27]. In this section, we recall its definition, rephrasing it however – in the spirit of Quillen – as a metric on the determinant line of the intersection cohomology.

Let us first recall that, as in the case of smooth compact manifolds, also for spaces with isolated conical singularities, the Laplacian $\Delta^{\bar{q}}$ is a discrete operator. Moreover, we have the following a priori estimate on the growth behaviour of its eigenvalues (see, e.g., [43, Proposition 1.4.7]): There exists $C > 0$ such that, for sufficiently large $k > k_0$ the k -th eigenvalue λ_k of $\Delta^{\bar{q}}$ satisfies

$$\lambda_k \geq C k^{\frac{1}{l}}, \quad (5.1)$$

where $l > \frac{n}{2}$.

In the following, we denote by Tr_s the supertrace of an operator. We denote by $\Delta^{\bar{q}, \perp}$ the restriction of $\Delta^{\bar{q}}$ to $(\ker \Delta^{\bar{q}})^\perp$. We denote by N the number operator acting on sections of the bundle $\Lambda(T^*X) \otimes F$ by multiplication with the form degree. For

$s \in \mathbb{C}$, $\Re(s) > \frac{n}{2}$, set

$$\zeta_{\bar{q}}(s) := -\mathrm{Tr}_s[N(\Delta^{\bar{q}, \perp})^{-s}],$$

which in view of (5.1) is a well-defined holomorphic function. By [27, Section 4], the function $\zeta_{\bar{q}}$ extends to a meromorphic function on the whole complex plane, which is holomorphic at $s = 0$. The result in [27, Section 4] has been proved in the case of unitary flat vector bundles, but can be adapted to the present more general situation; a more conceptual proof is given in [58].

The L^2 -inner product on sections of $\Lambda(T^*X) \otimes F$ restricts to an inner product on the space of L^2 harmonic forms $\mathcal{H}_{(2), \bar{q}}^\bullet(X, F)$. Using the isomorphisms (2.7) and (2.8), we get an induced metric on the line $\det IH_{\bar{q}}^\bullet(X, F)$, which we denote by $|\cdot|_{\det IH_{\bar{q}}^\bullet(X, F)}^{\mathrm{RS}}$.

Definition 5.1. The Ray–Singer metric $\|\cdot\|_{\det IH_{\bar{q}}^\bullet(X, F)}^{\mathrm{RS}}$ on the line $\det IH_{\bar{q}}^\bullet(X, F)$ is defined as

$$\|\cdot\|_{\det IH_{\bar{q}}^\bullet(X, F)}^{\mathrm{RS}} := |\cdot|_{\det IH_{\bar{q}}^\bullet(X, F)}^{\mathrm{RS}} \exp\left(\frac{1}{2} \zeta'_{\bar{q}}(0)\right).$$

In contrast to the smooth situation, where in the case of a unitary flat vector bundle, the Ray–Singer metric $\|\cdot\|_{\det IH_{\bar{q}}^\bullet(X, F)}^{\mathrm{RS}}$ is a topological invariant – this is precisely the statement of the smooth Cheeger–Müller theorem – in the presence of singularities invariance of the Ray–Singer metric on the choice of the pair of metrics (g^{TX}, g^F) no longer holds true in general. We discuss anomaly formulas for the Ray–Singer metric, i.e., its dependence on the choice of the metrics g^{TX}, g^F on TX, F , in Section 5.3.

5.2. The Bismut–Zhang metric on $\det IH_{\bar{q}}^\bullet(X, F)$

The Bismut–Zhang metric $\|\cdot\|_{\det IH_{\bar{q}}^\bullet(X, F)}^{\mathrm{BZ}, Y}$, associated with the pair of metrics (g^{TX}, g^F) and the anti-radial gradient-like vector field Y , has been defined in [56, 58], using the singular Morse–Thom–Smale complex for the perversity $\bar{q} \in \{\bar{m}, \bar{n}\}$ (recalled in Section 3.2) and the model Witten Laplacian $\Delta_T^{p, \bar{q}}$ on the infinite cone cL_p , $p \in \mathrm{Sing}(X)$ (recalled in Section 4.2). The Bismut–Zhang metric is a metric on $\det IH_{\bar{q}}^\bullet(X, F)$ with Milnor-like contributions from $\mathrm{Crit}(Y)$ and with an analytic correction term for each $p \in \mathrm{Sing}(X)$; this analytic correction is nothing else but the analytic torsion of the model Witten Laplacian $\Delta_T^{p, \bar{q}}$.

Let us now recall the definition of the Bismut–Zhang metric from [56] in more technical terms: For $T > 0$, we denote by $\Delta_T^{p, \bar{q}, \perp}$ the restriction of the model Witten Laplacian $\Delta_T^{p, \bar{q}}$ to $(\ker \Delta_T^{p, \bar{q}})^\perp$. Using the spectral properties of $\Delta_T^{p, \bar{q}}$, one can show that, for $\Re(s) \gg 0$, the zeta function

$$s \mapsto \zeta_T^{p, \bar{q}}(s) := -\mathrm{Tr}_s[N(\Delta_T^{p, \bar{q}, \perp})^{-s}]$$

is a well-defined holomorphic function. Moreover, by [56, Proposition 4.14], the zeta function $\zeta_T^{p,\bar{q}}$, $T > 0$, extends to a meromorphic function on $\mathbb{C}$, which is holomorphic at $s = 0$.

The L^2 -inner product on sections of $\Lambda(T^*Z_p) \otimes F$ restricts to an inner product on $\ker \Delta_T^{p,\bar{q}}$, and via the isomorphism (4.10), we get an induced inner product on $IH_{\bar{q}}^*(cL_p, L_p, F)$. We denote the induced metric on the line $\det IH_{\bar{q}}^*(cL_p, L_p, F)$ by $|\cdot|_{\det IH_{\bar{q}}^*(cL_p, L_p, F), T}^{\text{RS}}$. We denote by

$$\| \cdot \|_{\det IH_{\bar{q}}^*(cL_p, L_p, F), T}^{\text{RS}} := | \cdot |_{\det IH_{\bar{q}}^*(cL_p, L_p, F), T}^{\text{RS}} \exp\left(\frac{1}{2}(\zeta_T^{p,\bar{q}})'(0)\right)$$

the Ray–Singer metric of the model Witten Laplacian $\Delta_T^{p,\bar{q}}$. Set $\zeta_p^{\bar{q}} := \zeta_1^{p,\bar{q}}$.

Definition 5.2. For $p \in \text{Sing}(X)$, the Ray–Singer metric on the determinant line $\det IH_{\bar{q}}^*(cL_p, L_p, F)$ is defined as

$$\begin{aligned} \| \cdot \|_{\det IH_{\bar{q}}^*(cL_p, L_p, F)}^{\text{RS}} &:= \| \cdot \|_{\det IH_{\bar{q}}^*(cL_p, L_p, F), 1}^{\text{RS}} \\ &= | \cdot |_{\det IH_{\bar{q}}^*(cL_p, L_p, F), 1}^{\text{RS}} \exp\left(\frac{1}{2}(\zeta_p^{\bar{q}})'(0)\right). \end{aligned}$$

Remark 5.3. Using the scaling property of the model Witten Laplacian $\Delta_T^{p,\bar{q}}$, one can prove that Definition 5.2 is independent of the choice of $T > 0$ (here we took $T = 1$); see [56, Proposition 4.15].

The metrics g^{F_p} on the fibre F_p , $p \in \text{Crit}(f_{\text{sm}})$, induce a metric on $(C_{\bullet}^{\text{sm}}, \partial_{\bullet})$. We get an induced metric on $\det H^*(\text{Hom}((C_{\bullet}^{\text{sm}}, \partial_{\bullet}), \mathbb{C}))$ (see [62], [11, Section I (d)]).

Definition 5.4. We denote by $\| \cdot \|_{\det IH_{\bar{q}}^*(X, F)}^{\text{BZ}, Y}$ the metric on the determinant line $\det IH_{\bar{q}}^*(X, F)$ induced via the natural isomorphisms (3.9) from the metric on

$$\det H^*(\text{Hom}((C_{\bullet}^{\text{sm}}, \partial_{\bullet}), \mathbb{C}))$$

and the metrics $\| \cdot \|_{\det IH_{\bar{q}}^*(cL_p, L_p, F)}^{\text{RS}}$, $p \in \text{Sing}(X)$. We call this metric $\| \cdot \|_{\det IH_{\bar{q}}^*(X, F)}^{\text{BZ}, Y}$ the Bismut–Zhang metric of X which is constructed from the pair of metrics (g^{TX}, g^F) and the anti-radial gradient-like vector field Y .

For (X, g^{TX}) a smooth manifold, i.e., if for $p \in \text{Sing}(X)$, the link manifold L_p is the standard sphere, g^{TX} is Euclidean near p , and g^F is flat near p , then the Bismut–Zhang metric is equal to the Milnor metric defined in [11, Definition 1.9]; see [56, Remark 2.11 (c)]. In case that (X, g^{TX}) is an oriented even-dimensional space with isolated conical singularities and (F, ∇^F, g^F) is unitary, the Bismut–Zhang metric equals the intersection Reidemeister metric defined by Dar [27] and is

hence a combinatorial invariant; see [56, Section 8]. However, in general, in the presence of singularities, even in the case of a unitary flat vector bundle, $\| \cdot \|_{\det IH_T^*(X,F)}^{\text{BZ},Y}$ is not a purely combinatorial invariant. Anomaly formulas for the Bismut–Zhang metric will be discussed in Section 5.3.

The definition of the Bismut–Zhang metric has been generalised to the case of stratified pseudomanifolds in [59, Section 6]. Moreover, [59, Theorem II] proves that for spaces with two strata and even-codimensional singular stratum, the Bismut–Zhang metric equals the intersection Reidemeister metric of Dar and is hence a combinatorial invariant. This result is a counterpart of the invariance of analytic torsion for a wedge space proved in [61].

5.3. Anomaly formulas

For a smooth compact manifold equipped with a unitary flat vector bundle, the Ray–Singer metric is a topological invariant. This, as already mentioned, is no longer true in the presence of singularities. Also, the Bismut–Zhang metric is not a topological or combinatorial invariant in general. In this section, we study anomaly formulas for all three terms in the Cheeger–Müller/Bismut–Zhang theorem for isolated conical singularities.

In this section, we make an additional assumption on a pair of metrics (g^{TX}, g^F) on TX, F , namely: For $p \in \text{Sing}(X)$, we assume the following spectral gap condition on the co-closed spectrum $\text{Spec}(\Delta_{L_p, ccl})$ of the transversal Laplacian Δ_{L_p} on the link L_p :

$$\left\{ \begin{array}{ll} \text{Spec}(\Delta_{L_p, ccl}^{(v-1)}) \cap (0, 1) = \emptyset & \text{if } n = 2v \text{ is even,} \\ (\text{Spec}(\Delta_{L_p, ccl}^{(v-1)}) \cap (0, 3/4)) & \text{if } n = 2v + 1 \text{ is odd.} \\ \cup (\text{Spec}(\Delta_{L_p, ccl}^{(v)}) \cap (0, 3/4)) = \emptyset & \end{array} \right. \quad (5.2)$$

Condition (5.2) can be achieved by a rescaling of the metric g^{TL_p} into $c^2 g^{TL_p}$ with $c > 0$ sufficiently small.

In the following, let $\mathbb{R} \ni l \rightarrow (g_l^{TX}, g_l^F)$ be a family of metrics on TX, F depending smoothly on the parameter l . We assume, moreover, that the metrics g_l^{TX} are conical, i.e., for $p \in \text{Sing}(X)$, there is a family of Riemannian metrics $l \in \mathbb{R} \rightarrow g_l^{TL_p}$ such that $g_l^{TX} = dr^2 + r^2 g_l^{TL_p}$ near p . Similarly, we assume that near $p \in \text{Sing}(X)$, the Hermitian metric g_l^F on the flat vector bundle F is of the form explained in Section 2.2. In the following, we use a subscript or superscript l to denote the different notions and operators associated with the pair of metrics (g_l^{TX}, g_l^F) .

We denote by $*_l$ (resp. by $*_{p,l}$) the Hodge star operator associated with the metric g_l^{TX} on TX (resp. $g_l^{TZ_p} = dr^2 + r^2 g_l^{TL_p}$ on TZ_p). Moreover, for $p \in \text{Sing}(X)$,

$\bar{q} \in \{\bar{m}, \bar{n}\}$, $t > 0$, we denote by $Q_t^{p,l,\bar{q}}((r, y), (r', y'))$, $(r, y), (r', y') \in Z_p$ the heat kernel of the model operator $\Delta_l^{p,\bar{q}}$. We define the following Cheeger-type invariants:

$$\begin{aligned} c_{p,l}^{\bar{q}} &:= \frac{1}{2} \int_0^\infty \frac{du}{u} \int_{L_p} \text{Tr}_s \left[(g_l^{F_{Z_p}})^{-1} \frac{\partial g_l^{F_{Z_p}}}{\partial l} Q_u^{p,l,\bar{q}}((1, y), (1, y)) \right] d\text{vol}_{L_p}, \\ \tilde{c}_{p,l}^{\bar{q}} &:= \frac{1}{2} \int_0^\infty \frac{du}{u} \int_{L_p} \text{Tr}_s \left[(*_{p,l}^{-1} \frac{\partial *_{p,l}}{\partial l} Q_u^{p,l,\bar{q}}((1, y), (1, y)) \right] d\text{vol}_{L_p}. \end{aligned}$$

The well-definedness of the above invariants can be shown by arguments similar to those in the proof of the well-definedness of the classical Cheeger invariant, which is the contribution of the singularities of X to Cheeger's Chern–Gauss–Bonnet theorem [25, Theorem 5.1]. In [58], we deal with the Cheeger-type invariants by combining the local index techniques developed in [11] with the singular asymptotic lemma (SAL) of Brüning and Seeley [20].

For a Euclidean vector bundle (E, ∇^E) with metric connection, we denote by $e(E, \nabla^E)$ the Euler form in Chern–Weil theory. Let $\rho : X \times \mathbb{R} \rightarrow X$ be the canonical projection and ρ^*TX the pull-back bundle. We equip the vector bundle ρ^*TX with the metric $g^{TX,\text{tot}}$, where $g_{|X \times \{l\}}^{TX,\text{tot}} = g_l^{TX}$, and the compatible connection $\nabla^{TX,\text{tot}}$ (see [11, Section IV (h)]). For a form α on $X \times \mathbb{R}$, we denote by $\iota_{\partial_l} \alpha$ the interior multiplication with the vector field $\frac{\partial}{\partial l}$ (on $\mathbb{R}$). The following theorem generalises the smooth anomaly formula for the Ray–Singer metric in [11, Section IV] to spaces with isolated conical singularities.

Theorem 5.5. *Let $\mathbb{R} \ni l \rightarrow (g_l^{TX}, g_l^F)$ be a smooth family of metrics on TX , F satisfying the spectral gap condition (5.2). Then, the variation of the Ray–Singer metric is given by*

$$\begin{aligned} \partial_l \log(\| \cdot \|_{\det IH_{\bar{q}}^*(X,F),l}^{\text{RS}})^2 &= \int_X \text{Tr} \left[(g_l^F)^{-1} \frac{\partial g_l^F}{\partial l} \right] e(TX, \nabla_l^{TX}) \\ &\quad + \int_X \iota_{\partial_l} e(\rho^*TX, \nabla^{TX,\text{tot}}) \theta(F, g_l^F) \\ &\quad + \sum_{p \in \text{Sing}(X)} (c_{p,l}^{\bar{q}} + \tilde{c}_{p,l}^{\bar{q}}). \end{aligned}$$

To prove Theorem 5.5, we follow the strategy of proof for the smooth anomaly formula for the Ray–Singer metric in [11, Section IV]: First we show that the variation $\partial_l \log(\| \cdot \|_{\det IH_{\bar{q}}^*(X,F),l}^{\text{RS}})^2$ is given by the coefficient of t^0 in the asymptotic expansion of the following supertrace, as $t \searrow 0$:

$$\text{Tr}_s \left[\left((*_l^{-1} \frac{\partial *_{p,l}}{\partial l} + (g_l^F)^{-1} \frac{\partial g_l^F}{\partial l}) \exp(-\Delta_l^{\bar{q}}) \right) \right]. \quad (5.3)$$

For this, we adapt a trick explained by Cheeger in [23, Theorem 3.10] for manifolds with boundary (and absolute or relative boundary conditions at the boundary); it is here that the spectral gap assumption (5.2) comes into play.

To compute the supertrace (5.3), it is not enough to integrate the pointwise trace – obtained, as in [11], using local index techniques. In the presence of singularities, contributions of the singularities $\text{Sing}(X)$ do appear, namely the Cheeger-type invariants $c_{p,l}^{\bar{q}}, \tilde{c}_{p,l}^{\bar{q}}$. The first two terms in the formula in Theorem 5.5 are the local contributions, well known from the smooth anomaly formula for the Ray–Singer metric in [11]; they do not depend on the chosen extension of the Laplacian and vanish in the case where n is odd.

Let Y be an anti-radial gradient-like vector field. For a family of metrics $\mathbb{R} \ni l \rightarrow (g_l^{TX}, g_l^F)$ on TX and F , we use the notation $\|\cdot\|_{\det IH_{\bar{q}}^{\text{BZ},Y}(X,F),l}$ for the associated Bismut–Zhang metrics. For $p \in \text{Sing}(X)$, we denote by $\Psi(TZ_p, \nabla^{TZ_p})$ the Mathai–Quillen current on the tangent space of the infinite cone TZ_p ; it is an $(n-1)$ -dimensional current on TZ_p with values in the orientation line $o(TZ_p)$. We have $TZ_p \cong \mathbb{R} \times TL_p$, and we identify the orientation lines $o(TZ_p)$ and $o(TL_p)$ by fixing the orientation by $e_r = \frac{\partial}{\partial r}$ on the first factor. We denote by $\rho : Z_p \times \mathbb{R} \rightarrow Z_p$ the canonical projection and by $\Psi(\rho^*TZ_p, \nabla^{TZ_p, \text{tot}})$ the Mathai–Quillen form of the pull-back bundle ρ^*TZ_p . We have the model Morse function $f^p : cL_p \rightarrow \mathbb{R}$ defined by $f^p = \frac{-r^2}{2}$ its gradient vector field $\nabla f^p = -r \frac{\partial}{\partial r}$ (resp. $\nabla \widetilde{f^p}$) seen as section of TZ_p (resp. of ρ^*TZ_p).

Theorem 5.6. *Let Y be an anti-radial gradient-like vector field. Let $\mathbb{R} \ni l \rightarrow (g_l^{TX}, g_l^F)$ be a smooth family of metrics on TX , F , satisfying the spectral gap condition (5.2). Then*

$$\begin{aligned}
 & \partial_l \log \left(\left(\|\cdot\|_{\det IH_{\bar{q}}^{\text{BZ},Y}(X,F),l} \right)^2 \right) \\
 &= \sum_{p \in \text{Crit}(Y)} (-1)^{\text{ind}(p)} \partial_l \log \left(\|\cdot\|_{\det F_p, l}^2 \right) \\
 & \quad - \sum_{p \in \text{Sing}(X)} \int_{L_p} \text{Tr} \left[(g_l^{F_{Z_p}})^{-1} \frac{\partial g_l^{F_{Z_p}}}{\partial l} \right] \wedge (\nabla f^p)^* \Psi(TZ_p, \nabla^{TZ_p}) \\
 & \quad + \sum_{p \in \text{Sing}(X)} \int_{L_p} \theta(F_{Z_p}, g_l^{F_{Z_p}}) \wedge \iota_{\partial_l} (\nabla \widetilde{f^p})^* \Psi(\rho^*TZ_p, \nabla^{TZ_p, \text{tot}}) \\
 & \quad + \sum_{p \in \text{Sing}(X)} (c_{p,l}^{\bar{q}} + \tilde{c}_{p,l}^{\bar{q}}). \tag{5.4}
 \end{aligned}$$

Again the first term on the right-hand side of (5.4) is the local contribution, well known from the anomaly formula of the smooth Milnor metric; see [11, (7.3)].

The last three terms stem from the variation of the Ray–Singer metric of the model Witten Laplacian $\Delta_{T=1}^{p,\bar{q}}$ on the infinite cone cL_p . The latter is computed following the same strategy as in the computation of the variation formula for the Ray–Singer metric in Theorem 5.5: The second and the third terms on the right-hand side of (5.4) are the “local” contributions; the Cheeger-type invariants appear again as the contributions of the singular point. Similarly to [11, Theorem 16.1], one can also describe the dependence of the Bismut–Zhang metric $\|\cdot\|_{\det IH_{\bar{q}}^{BZ,Y}(X,F)}$ on the anti-radial gradient-like vector field.

In [58], we also compute variation formulas for the right-hand side of the Bismut–Zhang formula (1.1), generalising [11, Theorem 6.3] to the singular setting. Let g^{TX} , g'^{TX} be two conical metrics on X , and let g^F , g'^F be two Hermitian metrics on F satisfying the assumption explained in Section 2.2. The Levi-Civita connection associated with g'^{TX} is denoted by ∇'^{TX} , and we denote by $\nabla' f$ the gradient vector field of f with respect to the conical metric g'^{TX} . Let $\mathbb{R} \ni l \mapsto g_l^{TX}$ be a family of conical metrics on X connecting g^{TX} , g'^{TX} .

Theorem 5.7. *The following identity holds:*

$$\begin{aligned} & \int_X \theta(F, g^F)(\nabla f)^* \Psi(TX, \nabla^{TX}) - \int_X \theta(F, g'^F)(\nabla' f)^* \Psi(TX, \nabla'^{TX}) \\ &= \int_X \log\left(\frac{\|\cdot\|_{\det F}^{\prime 2}}{\|\cdot\|_{\det F}^2}\right) e(TX, \nabla^{TX}) \\ & \quad - \sum_{p \in \text{Crit}(f_{\text{sm}})} (-1)^{\text{ind}(p)} \log\left(\frac{\|\cdot\|_{\det F_p}^{\prime 2}}{\|\cdot\|_{\det F_p}^2}\right) - \int_X \theta(F, g'^F) \tilde{e}(TX, \nabla_l^{TX}) \\ & \quad + \sum_{p \in \text{Sing}(X)} \int_{L_p} \log\left(\frac{\|\cdot\|_{\det F_{Z_p}}^{\prime 2}}{\|\cdot\|_{\det F_{Z_p}}^2}\right) \wedge (\nabla f^p)^* \Psi(TZ_p, \nabla^{TZ_p}) \\ & \quad - \sum_{p \in \text{Sing}(X)} \int_{L_p} \theta(F_{Z_p}, g'^F_{Z_p}) \wedge \tilde{e}_{\Psi}(Z_p, \nabla_l^{TZ_p}), \end{aligned} \quad (5.5)$$

where

$$\tilde{e}(TX, \nabla_l^{TX}) := \int_0^1 e(\rho^* TX, \nabla^{TX, \text{tot}}) \in \Omega^{n-1}(X, o(TX))$$

and

$$\tilde{e}_{\Psi}(Z_p, \nabla_l^{TZ_p}) := \int_0^1 (\widetilde{\nabla f^p})^* \Psi(\rho^* TZ_p, \nabla^{TZ_p, \text{tot}}) \in \Omega^{n-2}(Z_p, o(TZ_p)).$$

Note that all integrals over X appearing in (5.5) are well defined, since the integrands vanish near $\text{Sing}(X)$. The only difference between (5.5) and the corresponding

smooth formula in [11, Theorem 6.3] is the last two terms on the right-hand side, which are the contributions of the singularities of X . Note that they correspond to the two boundary terms appearing on the right-hand side of (5.4).

By comparing the three anomaly formulas in Theorems 5.5, 5.6 and 5.7, one sees that they are compatible with the Bismut–Zhang formula (1.1), as it should be!

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