

The Poisson–Dirichlet problem in domains with Ahlfors regular boundary

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Abstract. We present an announcement of some recent results concerning well-posedness of the Poisson–Dirichlet problem with boundary data in Besov spaces with fractional smoothness. This is a far-reaching generalization as previously known theorems concerning well-posedness of the Poisson problem in such intermediate smoothness classes were mostly restricted to the context of Lipschitz domains and coefficients satisfying strong regularity assumptions.

1. Introduction: The general context

The purpose of this note is to describe some recent results concerning the Poisson problem, and the Dirichlet problem with data of fractional smoothness, for second-order elliptic operators with real coefficients in very general domains.

Poisson’s equation $-\Delta u = h$, for a given function h and an unknown function u , was historically one of the first differential equations to be studied, due in large part to its many applications in physics and beyond. A natural generalization is to replace the Laplace operator $-\Delta$ by an arbitrary divergence form elliptic operator $-\operatorname{div} A \nabla$ for some specified matrix A (which may be only bounded, measurable, and uniformly positive definite). Considering Poisson’s equation not in all of Euclidean space, but only in a given open subset Ω , requires specifying the behavior of u on the boundary $\partial\Omega$; we will impose Dirichlet boundary conditions, that is, specify the values of u on the boundary. (It is also common in the theory to specify Neumann boundary conditions, that is, to specify $\nu \cdot A \nabla u$, where ν denotes the unit outward normal vector to Ω ; we refer the interested reader to [3, 7, 26, 69] for some results on the Poisson–Neumann problem.)

We thus will discuss the Poisson–Dirichlet problem

$$-\operatorname{div} A \nabla u = h \quad \text{in } \Omega, \quad u = f \quad \text{on } \partial\Omega \quad (1.1)$$

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for specified Ω , A , f , and h .

In a few situations, the range of spaces allowing for well-posedness of the Poisson–Dirichlet boundary problem above is very well understood. Such situations include VMO and small-BMO coefficients and domains (see [51, 68], among others), and Lipschitz domains with coefficients satisfying a Dini-type continuity condition [56] or independent of the direction transverse to the boundary [7].

In recent years, endpoint results, pertaining to boundary data f lying in $L^p(\partial\Omega)$, have been established considerably beyond Lipschitz domains: first, in chord-arc domains [22, 42], then in so-called 1-sided nontangentially accessible domains with a uniformly rectifiable boundary [37], and finally, on domains with a uniformly rectifiable boundary that satisfy a weak local John condition [5]. The latter is a sharp condition, necessary and sufficient for well-posedness results for the Laplacian [5]. Finally, boundary value problems in domains with lower-dimensional boundaries were treated in [24] (a later but much simpler treatment is in [28]).

Most of these results were proven in (or may be generalized to) the generality of coefficients satisfying variations of the so-called DKP condition, which controls oscillations of the coefficients near the boundary.

Note that t -independence is meaningless in domains that lack a distinguished direction (in particular, in domains where there is no constant direction transverse to the boundary) and that all of these “endpoint” estimates are in spaces with integer smoothness. Our results, by contrast with the endpoint estimates listed above, need neither t -independence nor the DKP condition.

The results of this note (described in Sections 4 and 5) treat well-posedness of boundary problems in intermediate spaces of fractional smoothness. We will show, for the first time, that well-posedness in a wide range of Besov spaces can be established by a “blind” mechanism, in the full generality of domains with Ahlfors regular boundary (possibly of lower codimension) that satisfy the weak local John condition, and general real elliptic equations, without any additional restrictions on the coefficients. Furthermore, given endpoint (that is, $L^p(\partial\Omega)$) results such as those above, the range of Besov spaces can be expanded in, again, a “blind” fashion, with no further requirements on the domains or coefficients beyond those above and those required by the endpoint results.

The methods are new, as [7, 49] relied on layer potentials, not available in such a general setting, [41] worked with the Laplacian on Lipschitz domains and the case $p > 1$ only, and [51, 56, 68] were confined to continuous, VMO, or near-VMO coefficients and Lipschitz domains. The full proofs will appear in [9] (the case $f = 0$, that is, $u = 0$ on $\partial\Omega$) and [8] (full generality), and in this note, we only announce the upcoming results and put them into historic context.

We remark that $L^p(\partial\Omega)$ represents the “zero smoothness” endpoint; an alternative endpoint is the “smoothness 1” endpoint, often called the regularity problem. The

regularity problem, like the L^p -Dirichlet problem, has been investigated extensively. However, our results will not make use of the regularity problem, but only the L^p -Dirichlet problem and a duality argument, to expand the range of Besov spaces. For this reason, we will not discuss known results for the regularity problem.

2. Definitions

In this section, we will provide all of the definitions needed to state our main results. An experienced reader can skip them and move on to Section 3. Some additional definitions are necessary to state the corollaries to our second main result and also to discuss the history of the Poisson problem and the Dirichlet problem with boundary data in fractional smoothness spaces; these definitions will be deferred to Section 6 or to the discussion.

2.1. Geometric conditions

Definition 2.1 (Ahlfors regularity). If $\Omega \subset \mathbb{R}^n$ and $0 < \mathfrak{d} < n$, then we say that the boundary $\partial\Omega$ is $\mathfrak{d}$ -Ahlfors regular if there exists a constant $\mathcal{A} > 0$ such that

$$\frac{1}{\mathcal{A}}r^{\mathfrak{d}} \leq \sigma(B(x, r) \cap \partial\Omega) \leq \mathcal{A}r^{\mathfrak{d}}$$

for all $x \in \partial\Omega$ and all r with $0 < r < \text{diam}(\Omega)$; here, σ denotes the Hausdorff $\mathfrak{d}$ -dimensional surface measure.

Remark 2.2. Following [20], we do not require the boundary dimension $\mathfrak{d}$ to be an integer. Our main results (Theorems 4.1 and 5.1) are known only in the case $\mathfrak{d} \leq n - 1$, that is, the classical codimension 1 case and the higher codimension case.

Definition 2.3 (Interior corkscrew condition). We say that Ω satisfies the interior corkscrew condition if there is a constant $\varkappa \in (0, 1)$ such that if $x \in \partial\Omega$ and $0 < r < \text{diam} \Omega$, then there is a point $A_r(x)$ (called a corkscrew point) such that

$$A_r(x) \in B(x, r) \quad \text{and} \quad B(A_r(x), \varkappa r) \subset \Omega.$$

Definition 2.4 (Weak local John domain). Let $\Omega \subset \mathbb{R}^n$ be open. Let $x \in \partial\Omega$, and let $y \in \Omega$. We say that ζ is a λ -carrot path from x to y if ζ is a connected rectifiable path, which we require to be parameterized by arc length, such that $\zeta : [0, T] \rightarrow \Omega \cup \{x\}$ is continuous, $\zeta(0) = x$, $\zeta(T) = y$, and $\text{dist}(\zeta(t), \partial\Omega) \geq \lambda t$ for all $t \in [0, T]$.

Now, suppose in addition that $\partial\Omega$ is $\mathfrak{d}$ -Ahlfors regular for some $0 < \mathfrak{d} < n$. Let $y \in \Omega$, let $\theta, \lambda \in (0, 1]$, and let $N \geq 2$. We say that y is a (θ, λ, N) -weak

local John point if there is a Borel set $F \subset B(y, N \operatorname{dist}(y, \partial\Omega)) \cap \partial\Omega$ such that $\sigma(F) \geq \theta \sigma(B(y, N \operatorname{dist}(y, \partial\Omega)) \cap \partial\Omega)$ and such that if $x \in F$, then there is a λ -carrot path from x to y .

We say that Ω is a weak local John domain if there exist constants $\theta \in (0, 1]$, $\lambda \in (0, 1]$, and $N \geq 2$ such that every $y \in \Omega$ is a (θ, λ, N) -weak local John point.

Remark 2.5. A domain with lower-dimensional Ahlfors regular boundary (that is, with $\mathfrak{d} < n - 1$) necessarily satisfies the interior corkscrew condition (see [20, Lemma 11.6]). It also necessarily satisfies the weak local John condition: by [20, Lemma 2.1], such a domain must satisfy the Harnack chain condition (defined in many places in the literature, or see Definition 6.2 below), and by connecting corkscrew points with Harnack chains, we can construct carrot paths to a point y in the domain from *any* nearby boundary point x .

2.2. Functions and function spaces

Definition 2.6. Let $\Gamma \subseteq \mathbb{R}^n$ be a $\mathfrak{d}$ -Ahlfors regular set, let $0 < s < 1$, and let $1 \leq p \leq \infty$. The Slobodeckij space $\dot{A}^{s,p}(\Gamma)$ is defined to be the set of all equivalence classes (up to addition of constants and sets of measure zero) of locally integrable functions $f : \Gamma \rightarrow \mathbb{R}$ such that the norm

$$\|f\|_{\dot{A}^{s,p}(\Gamma)} = \left(\int_{\Gamma} \int_{\Gamma} \frac{|f(x) - f(y)|^p}{|x - y|^{\mathfrak{d} + ps}} d\sigma(y) d\sigma(x) \right)^{1/p}$$

(or the corresponding sup norm when $p = \infty$) is finite.

It is clear that the Slobodeckij spaces are Banach spaces.

Definition 2.7 ([31, Definition 2.1], [33, Definition 7]). Let $\Gamma \subset \mathbb{R}^n$ be a $\mathfrak{d}$ -Ahlfors regular set, let $z \in \Gamma$, and let $\varrho > 0$. We let $\Delta(z, \varrho) = \Gamma \cap B(z, \varrho)$. A measurable function $a : \Gamma \rightarrow \mathbb{R}$ is said to be a *block in* $\Delta(z, \varrho)$ if

- a is supported in $\Delta(z, \varrho)$,
- $|a(x)| \leq \varrho^{-\mathfrak{d}/2}$ for all $x \in \Delta(z, \varrho)$ (and thus for all $x \in \Gamma$), and
- $|a(x) - a(y)| \leq |x - y|\varrho^{-\mathfrak{d}/2-1}$ for all $x, y \in \Gamma$.

Remark 2.8. In the setting of Hardy spaces, it is customary to work with atoms, which are blocks with the additional requirement $\int_{\Gamma} a d\sigma = 0$. However, the Besov spaces are closer in character to the Hardy–Sobolev spaces, that is, functions whose gradient lies in a Hardy space. The moment condition $\int_{\mathbb{R}^{\mathfrak{d}}} \nabla a = 0$ is trivially true for any compactly supported function a , and so moment conditions are not usually required of Hardy–Sobolev atoms; thus, we do not require them of our blocks.

Definition 2.9. Let $\Gamma \subset \mathbb{R}^n$ be a δ -Ahlfors regular set, let $0 < s \leq 1$, and let $\delta/(\delta + s) < p \leq 1$ (so $s + \delta - \delta/p > 0$). We define the atomic space $\dot{A}^{s,p}(\Gamma)$ as

$$\dot{A}^{s,p}(\Gamma) = \left\{ \sum_{k=1}^{\infty} \lambda_k a_k : \sum_{k=1}^{\infty} |\lambda_k|^p \varrho_k^{\delta-\delta p/2-sp} < \infty, \right. \\ \left. \text{where each } a_k \text{ is a block in } \Delta(z_k, \varrho_k) \text{ for some } z_k \in \Gamma \right\},$$

and let the norm in $\dot{A}^{s,p}(\Gamma)$ be given by

$$\|f\|_{\dot{A}^{s,p}(\Gamma)} = \inf \left\{ \left(\sum_{k=1}^{\infty} |\lambda_k|^p \varrho_k^{\delta-\delta p/2-sp} \right)^{1/p} : f = \sum_{k=1}^{\infty} \lambda_k a_k \right. \\ \left. \text{for blocks } a_k \text{ in } \Delta(z_k, \varrho_k) \right\}.$$

It is straightforward to establish (and we will do so in the forthcoming paper [8]) that if $\sum_{k=1}^{\infty} |\lambda_k|^p \varrho_k^{\delta-\delta p/2-sp} < \infty$, then $\sum_{k=1}^{\infty} \lambda_k a_k$ converges in the Banach space $\dot{\Lambda}^{s+\delta-\delta/p,1}(\Gamma)$, and in fact $\dot{A}^{s,p}(\Gamma)$ embeds into $\dot{\Lambda}^{s+\delta-\delta/p,1}(\Gamma)$. Thus, the infinite sums above are meaningful.

Definition 2.10. Let $\Gamma \subset \mathbb{R}^n$ be a δ -Ahlfors regular set for some $0 < \delta < n$, let $0 < s \leq 1$, and let $\delta/(\delta + s) < p \leq \infty$.

We define the Besov space $\dot{B}_s^{p,p}(\Gamma)$ by

$$\dot{B}_s^{p,p}(\Gamma) = \begin{cases} \dot{\Lambda}^{s,p}(\Gamma), & 1 \leq p \leq \infty, \\ \dot{A}^{s,p}(\Gamma), & \frac{\delta}{\delta + s} < p < 1, \end{cases}$$

with the norm in $\dot{B}_s^{p,p}(\Gamma)$ given by the norm in the Slobodeckij space $\dot{\Lambda}^{s,p}(\Gamma)$ or atomic space $\dot{A}^{s,p}(\Gamma)$.

In Euclidean spaces, the Besov spaces $\dot{B}_s^{p,q}(\mathbb{R}^{\delta})$ and the related Triebel–Lizorkin spaces $\dot{F}_s^{p,q}(\mathbb{R}^{\delta})$ (they are equal if $p = q$) encompass many known function spaces. See [65, Sections 2.3.5, 5.2.3, 5.2.4] and [52, Proposition 3]. For example, the Lebesgue spaces satisfy $L^p(\mathbb{R}^{\delta}) = \dot{F}_0^{p,2}(\mathbb{R}^{\delta})$ if $1 < p < \infty$; the Sobolev spaces $\dot{W}^{k,p}(\mathbb{R}^{\delta})$ (of functions f whose k th gradient $\nabla^k f$ lies in $L^p(\mathbb{R}^{\delta})$) satisfy $\dot{W}^{k,p}(\mathbb{R}^{\delta}) = \dot{F}_k^{p,2}(\mathbb{R}^{\delta})$; the Hardy spaces satisfy $H^p(\mathbb{R}^{\delta}) = \dot{F}_0^{p,2}(\mathbb{R}^{\delta})$ if $\delta/\delta + 1 < p < \infty$; the spaces $\dot{C}^{\alpha}(\mathbb{R}^{\delta})$ of Hölder continuous functions satisfy $\dot{C}^{\alpha}(\mathbb{R}^{\delta}) = \dot{B}_{\alpha}^{\infty,\infty}(\mathbb{R}^{\delta})$ for $0 < \alpha < 1$; the Slobodeckij spaces $\dot{\Lambda}^{s,p}$ satisfy the above relation $\dot{B}_s^{p,p}(\mathbb{R}^{\delta}) = \dot{\Lambda}^{s,p}(\mathbb{R}^{\delta})$ for $0 < s < 1$ and $1 \leq p \leq \infty$ (and indeed

there exists a Slobodeckij-type characterization of $\dot{B}_s^{p,q}(\mathbb{R}^{\mathfrak{d}})$ for any $0 < s < 1$, any $1 \leq p < \infty$, and any $1 \leq q \leq \infty$, not only for $q = p$), and the Bessel potential spaces $L_s^p = (I - \Delta)^{-s/2}(L^p)$ satisfy $L_s^p = F_s^{p,2}$. Even with the restrictions $p = q$, $0 < s < 1$, and $\mathfrak{d}/\mathfrak{d} + s < p \leq \infty$, there are many equivalent characterizations of the Besov spaces $\dot{B}_s^{p,p}(\mathbb{R}^{\mathfrak{d}})$.

In [34], Besov spaces were generalized from the Euclidean setting to the setting of spaces of homogeneous type. An active area of research consists of studying these spaces; we mention in particular that by [59], our Slobodeckij characterization of the Besov spaces for $p \geq 1$ coincides with the characterization in [32,34], and by [33], the *inhomogeneous* Besov spaces have an atomic characterization; it is natural to expect the homogeneous Besov spaces to have an atomic characterization (specifically, the atomic characterization used above), but we have not been able to find this result in the literature on spaces of homogeneous type.

2.3. Boundary values

Definition 2.11 (Boundary trace). Following [43], if $\Omega \subset \mathbb{R}^{\mathfrak{n}}$, if $u : \Omega \rightarrow \mathbb{R}$ is locally integrable, and if $f : \partial\Omega \rightarrow \mathbb{R}$, then we say that $u = f$ on $\partial\Omega$ if

$$\lim_{r \rightarrow 0^+} \frac{1}{r^{\mathfrak{n}}} \int_{B(x,r) \cap \Omega} |u - f(x)| = 0 \quad (2.1)$$

for σ -almost every $x \in \partial\Omega$.

In most reasonably behaved domains, this coincides with most other notions of trace. We will use this definition in Theorems 4.1 and 5.1. We mention that some of the results described in Section 3 were proven under different notions of trace (most prominently traces in the sense of Sobolev spaces); however, the different notions of trace can generally be shown to be equivalent as needed.

We observe that if Ω satisfies the interior corkscrew condition, then the value of f is uniquely determined in that, for each $x \in \partial\Omega$, there can be at most one $f(x) \in \mathbb{R}$ such that formula (2.1) is valid.

2.4. Elliptic differential operators

Definition 2.12 (Ellipticity). Let $\Omega \subset \mathbb{R}^{\mathfrak{n}}$ and suppose that $\partial\Omega$ is $\mathfrak{d}$ -Ahlfors regular. Let A be a matrix-valued function. We say that the operator $-\operatorname{div} A \nabla$ is elliptic, or, respectively, the matrix A is elliptic, if each $A(x)$ is an $\mathfrak{n} \times \mathfrak{n}$ matrix of real, not necessarily symmetric, coefficients that satisfy the following positive definiteness and boundedness conditions:

$$\xi \cdot A(x) \xi \geq \lambda \operatorname{dist}(x, \partial\Omega)^{\mathfrak{d}+1-\mathfrak{n}} \|\xi\|^2, \quad (2.2)$$

$$|\eta \cdot A(x) \xi| \leq \Lambda \operatorname{dist}(x, \partial\Omega)^{\mathfrak{d}+1-\mathfrak{n}} \|\xi\| \|\eta\| \quad (2.3)$$

for all $\eta, \xi \in \mathbb{R}^n$ and almost every $x \in \mathbb{R}^n \setminus \partial\Omega$.

The main results (Theorems 4.1 and 5.1) of the present paper hold in the full generality of real elliptic matrices, with no further restrictions on A .

Definition 2.13 (Weak solutions). We will consider solutions to $-\operatorname{div} A \nabla u = h$ in the case where h is the divergence of some vector field $\vec{G}$.

We will use the weak definition of $-\operatorname{div} A \nabla u = -\operatorname{div} \vec{G}$; that is, if $u : \Omega \rightarrow \mathbb{R}$ is a real-valued function, we say that $-\operatorname{div} A \nabla u = -\operatorname{div} \vec{G}$ in Ω in the weak sense if, for every $\varphi : \Omega \rightarrow \mathbb{R}$ that is smooth and supported in a compact subset of Ω , we have that

$$\int_{\Omega} \nabla \varphi \cdot A \nabla u = \int_{\Omega} \nabla \varphi \cdot \vec{G}.$$

Recall that A is elliptic and therefore invertible. For notational convenience, we will usually take our data to be $\vec{H} = A^{-1} \vec{G}$; thus, we consider solutions to $-\operatorname{div} A \nabla u = -\operatorname{div} A \vec{H}$, meaning functions u that satisfy

$$\int_{\Omega} \nabla \varphi \cdot A \nabla u = \int_{\Omega} \nabla \varphi \cdot A \vec{H} \quad \text{for all } \varphi \in C_0^\infty(\Omega). \quad (2.4)$$

This notational convention will considerably simplify the statement of Theorem 4.1 below.

We now state three known facts concerning solutions to $-\operatorname{div} A \nabla u = 0$ or $-\operatorname{div} A \nabla u = -\operatorname{div} A \vec{H}$ that we will reference in the statement of our main results.

First, for any domain Ω with $\mathfrak{d}$ -Ahlfors regular boundary, $0 < \mathfrak{d} \leq n - 1$, and for any real elliptic operator $-\operatorname{div} A \nabla$ as above, for every function f defined on $\partial\Omega$ that is continuous and compactly supported, there is a unique solution u to the Dirichlet problem

$$\begin{cases} -\operatorname{div} A \nabla u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \\ u \text{ is continuous and bounded on } \overline{\Omega}. \end{cases} \quad (2.5)$$

That this holds for $\mathfrak{d} = n - 1$ is the fact that Ahlfors regular boundaries satisfy the classic Wiener criterion [67]; that it holds for $\mathfrak{d} < n - 1$ is [20, Lemma 9.4].

The second result we wish to mention is Meyers's reverse Hölder estimate for gradients. This estimate states that if A satisfies conditions (2.2)–(2.3), then there exist C and $\varepsilon > 0$ depending only on the dimension n (and $\mathfrak{d}$) and on the ellipticity constants λ and Λ such that if $-\operatorname{div} A \nabla u = -\operatorname{div} A \vec{H}$ in $B(x, r) \subseteq \Omega$, and if $\vec{H} \in L^{2+\varepsilon}(B(x, r))$, then $\nabla u \in L^{2+\varepsilon}(B(x, r/2))$ and

$$\left(\int_{B(x, r/2)} |\nabla u|^{2+\varepsilon} \right)^{1/(2+\varepsilon)} \leq C \left(\int_{B(x, r)} |\nabla u|^2 \right)^{1/2} + C \left(\int_{B(x, r)} |\vec{H}|^{2+\varepsilon} \right)^{1/(2+\varepsilon)}. \quad (2.6)$$

The same of course is true for solutions to $-\operatorname{div} A^* \nabla u = -\operatorname{div} A^* \vec{H}$, possibly with a different value of ε .

Finally, we wish to state the boundary De Giorgi–Nash estimate. This estimate is well known in the case $\mathfrak{d} = n - 1$ and is still true if $0 < \mathfrak{d} < n - 1$; see [20, Lemmas 8.13 and 8.16]. It states that there exist an $\alpha > 0$ and $C > 0$ such that if $-\operatorname{div} A \nabla u = 0$ in Ω , if $u = 0$ on $\partial\Omega \cap B(\xi, r)$ for some $\xi \in \partial\Omega$ and some $r > 0$, and if $x \in B(\xi, r/2) \cap \Omega$, then

$$|u(x)| \leq \left(\frac{|x - \xi|}{r} \right)^\alpha \frac{C}{r^{\mathfrak{d}+1}} \int_{\Omega \cap B(\xi, r)} |u(y)| \operatorname{dist}(y, \partial\Omega)^{1+\mathfrak{d}-n} dy. \quad (2.7)$$

3. A brief history of the Poisson–Dirichlet problem in spaces with fractional smoothness

In this section, we will briefly review the history of the problems which we consider in this note. That is, we will review the history of the Poisson–Dirichlet problem, with one degree of smoothness for the solutions and between zero and one degree of smoothness of the boundary values, for second-order linear elliptic differential equations, with real coefficients and without lower-order terms, in domains in Euclidean space.

3.1. Higher codimension and energy solutions

Let $0 < \mathfrak{d} < n$, $n \geq 2$, let $\Omega \subset \mathbb{R}^n$ be such that $\partial\Omega$ is $\mathfrak{d}$ -Ahlfors regular for some real number $\mathfrak{d}$ with $0 < \mathfrak{d} < n$, and let A satisfy the ellipticity conditions (2.2)–(2.3). The Lax–Milgram lemma yields unique solvability of the Poisson problem

$$-\operatorname{div} A \nabla u_0 = -\operatorname{div} A \vec{H}_0 \quad \text{in } \Omega, \quad u_0 = 0 \quad \text{on } \partial\Omega \quad (3.1)$$

for all $\vec{H}_0 \in L^2(\Omega, w)$ and with the estimate

$$\|\nabla u_0\|_{L^2(\Omega, w)} \leq C \|\vec{H}_0\|_{L^2(\Omega, w)},$$

where $u_0 = 0$ on $\partial\Omega$ in the sense that u_0 lies in $\dot{W}_0^{1,2}(\Omega; w)$, the closure of $C_0^\infty(\Omega)$ in $\dot{W}^{1,2}(\Omega; w)$ (the Sobolev space of functions with gradient in $L^2(\Omega; w)$), and where w is the measure given by $dw = \operatorname{dist}(\cdot, \partial\Omega)^{\mathfrak{d}+1-n} dm$, m denoting Lebesgue measure. This method of solution is extremely general, relying only on the fact that $\dot{W}_0^{1,2}(\Omega; w)$ is a Hilbert space and on the upper and lower bounds

$$\int_{\Omega} \nabla \varphi \cdot A \nabla \psi \leq \Lambda \|\varphi\|_{\dot{W}_0^{1,2}(\Omega; w)} \|\psi\|_{\dot{W}_0^{1,2}(\Omega; w)}, \quad \int_{\Omega} \nabla u \cdot A \nabla u \geq \lambda \|u\|_{\dot{W}_0^{1,2}(\Omega; w)}^2$$

on functions $\varphi, \psi, u \in \dot{W}_0^{1,2}(\Omega; w)$. Indeed, this construction is valid in many situations beyond that of this paper, such as systems and higher-order equations.

The estimate

$$\|\nabla u_0\|_{L^p(\Omega, w)} \leq C \|\vec{H}_0\|_{L^p(\Omega, w)} \quad (3.2)$$

is also true for all p sufficiently close to 2. This can be derived by using the fact that $\dot{W}_0^{1,p}(\Omega; w)$, $1 < p < \infty$, is an interpolation scale and using Šneĭberg's lemma.

Problem (1.1), with the notation of formula (2.4), becomes

$$-\operatorname{div} A \nabla u = -\operatorname{div} A \vec{H} \quad \text{in } \Omega, \quad u = f \quad \text{on } \partial\Omega. \quad (3.3)$$

If $\partial\Omega$ is δ -Ahlfors regular, $0 < \delta < n$, then we can extend an arbitrary $f \in \dot{B}_{1-1/p}^{p,p}(\partial\Omega)$ to a function F in $\dot{W}^{1,p}(\Omega; w)$ of comparable norm. (The extension result is valid even if δ is not an integer.) See [43] if $\delta = n - 1$, [20, Chapter 7] and [21, Chapter 8] if $p = 2$, and [8] in the general case. Letting F be this extension, u_0 the solution to the homogeneous Poisson problem (3.1) with data $\vec{H}_0 = \vec{H} - \nabla F$, and $u = u_0 + F$ yields a solution u to the Poisson–Dirichlet problem (3.3) with the estimate

$$\|\nabla u\|_{L^p(\Omega, w)} \leq C \|\vec{H}\|_{L^p(\Omega, w)} + C \|f\|_{\dot{B}_{1-1/p}^{p,p}(\partial\Omega)}. \quad (3.4)$$

(Trace theorems often allow us to clarify the sense in which $u = f$ on $\partial\Omega$; see the above references.) This construction is very well known in the codimension 1 case ($\delta = n - 1$) and is explained in detail in the general case in [20, Lemma 9.1].

We emphasize that up until the present work, in the higher codimension case, this construction of solutions to the Poisson–Dirichlet problem, restricted to boundary data in the particular Besov space $\dot{B}_{1-1/p}^{p,p}(\partial\Omega)$ for p sufficiently close to 2, was the only known result for the Dirichlet–Poisson problem with boundary data of fractional smoothness. All of the following results in this section apply only to coefficients that satisfy the ellipticity conditions (2.2)–(2.3) in the particular case $\delta = n - 1$.

3.2. The Poisson problem with Sobolev estimates

A natural line of study is to seek conditions on Ω and A such that estimate (3.2) holds for a wider range of p .

The estimate $\|\nabla u_0\|_{L^p(\Omega)} \leq C \|\vec{H}_0\|_{L^p(\Omega)}$ for all $1 < p < \infty$ has been established in the case where A satisfies (2.2)–(2.3) (with $\delta = n - 1$) and lies in the space VMO of vanishing mean oscillation and where $\partial\Omega$ is a bounded C^1 domain. See [4, 25, 47]. For any fixed $p \in (1, \infty)$, the result continues to hold if A and Ω are allowed to deviate slightly from these assumptions; see [11, 12] for the details.

3.3. The Poisson–Dirichlet problem with weighted Sobolev estimates

Recall that in Section 3.1, we used the fact that a function $f \in \dot{B}_{1-1/p}^{p,p}(\partial\Omega)$ may be extended to a function F in the Sobolev space $\dot{W}^{1,p}(\mathbb{R}^n)$. It is also possible to extend functions in $\dot{B}_s^{p,p}(\partial\Omega)$ for a range of s , and not merely the particular value $s = 1 - 1/p$, if we take our extensions to lie in a *weighted* Sobolev space with an appropriate weight. In particular, if $0 < s < 1$, $1 \leq p \leq \infty$, and $\partial\Omega$ is $(n-1)$ -Ahlfors regular, then any $f \in \dot{B}_s^{p,p}(\partial\Omega)$ may be extended to a function F with

$$\int_{\mathbb{R}^n} |\nabla F(x)|^p \operatorname{dist}(x, \partial\Omega)^{p-1-ps} dx \approx \|f\|_{\dot{B}_s^{p,p}(\partial\Omega)}^p.$$

See [8, 20, 46, 51, 56, 60, 61, 64, 66]. The case of bounded Lipschitz domains is [56, Proposition 4.1] and the $l = 1$ case of [46, Theorem 2.10], but the result was known earlier in smoother domains.

This suggests consideration of the Dirichlet–Poisson problem (3.3) with the estimate

$$\begin{aligned} & \int_{\Omega} |\nabla u(x)|^p \operatorname{dist}(x, \partial\Omega)^{p-1-ps} dx \\ & \leq C \int_{\Omega} |\vec{H}(x)|^p \operatorname{dist}(x, \partial\Omega)^{p-1-ps} dx + C \|f\|_{\dot{B}_s^{p,p}(\partial\Omega)}^p. \end{aligned} \quad (3.5)$$

This problem was studied in [56] (which in fact considered the case where L is the Hodge Laplacian in a Lipschitz domain in a compact Riemannian manifold) and [51] (which in fact studied the problem for elliptic systems of arbitrary even order, not only for the second-order equations that are the focus of the present note).

The paper [51] established a result similar to those of the previous section. Specifically, if A satisfies (2.2)–(2.3) (with $d = n-1$) and lies in VMO, and if Ω is a bounded Lipschitz domain whose unit outward normal also lies in VMO (this is a moderate generalization of the C^1 condition), then estimate (3.5) holds for any $1 < p < \infty$ and any $0 < s < 1$. Moreover, again as in the unweighted case of the previous section, for any fixed p and s in this range, if A is sufficiently close to VMO and Ω is a Lipschitz domain with sufficiently small constant, then estimate (3.5) holds.

The $p = 2$, $s = 1/2$ case of these results is the energy solutions of Section 3.1, while the $s = 1 - 1/p$ case is the previous section.

The paper [56] established estimate (3.5) in general bounded Lipschitz domains (without the VMO or near-VMO condition on the normals) but for a restricted range of s , p , and A . Specifically, they required the coefficients A to be symmetric and to satisfy the Dini-type condition

$$|A(x) - A(y)| \leq \omega(|x - y|), \quad \int_0^1 \frac{\sqrt{\omega(t)}}{t} dt < \infty$$

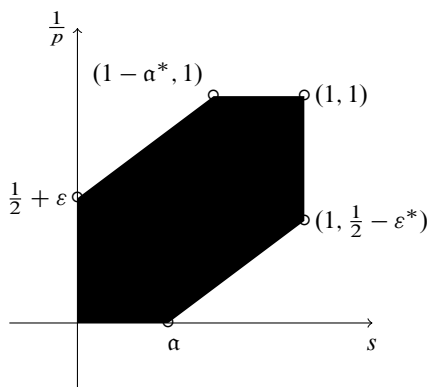


Figure 1. Values of the parameters $(s, 1/p)$ such that estimate (3.5) holds, for solutions to problem (3.3) in a Lipschitz domain, where A satisfies a Dini-type continuity condition. This region includes the edges at $p = 1$ and $p = \infty$ but does not include the edges at $s = 0$ or $s = 1$.

which is certainly stronger than lying in VMO. They were then able to establish estimate (3.5) for all s and p such that the point $(s, 1/p)$ lies in the hexagonal region illustrated in Figure 1 for some $\varepsilon, \varepsilon^*, \alpha, \alpha^* > 0$.¹

3.4. General weights

More general weights may be considered. By [1], if $1 < p < \infty$, if ω is a Muckenhoupt A_p weight, if A is symmetric and the BMO norm of A is small enough, and if Ω is a Lipschitz domain with a sufficiently small Lipschitz constant, then solutions to the Poisson problem (3.1) satisfy the estimate

$$\|\nabla u_0\|_{L^p(\Omega; \omega)} \leq C \|\tilde{H}_0\|_{L^p(\Omega; \omega)}. \quad (3.6)$$

¹The paper [56], along with its predecessors [54, 55], does not work with arbitrary second-order operators in Lipschitz domains in $\mathbb{R}^n$ but with the Hodge Laplacian Δ_g defined on a compact Riemannian manifold with metric g ; more precisely, they work with the Schrödinger-type operator $L = -\Delta_g + V$ for a nonnegative potential V .

We may return to the Euclidean situation of the present paper as follows. Given a bounded Lipschitz domain Ω in $\mathbb{R}^n$, one may easily construct a compact n -dimensional manifold M with Ω contained in one coordinate chart. In local coordinates, the Hodge Laplacian is given by $\sqrt{|g|}\Delta_g = \operatorname{div}(\sqrt{|g|}g^{-1}\nabla)$, and so we can attain any real symmetric coefficient matrix A by choosing an appropriate metric on the chart containing Ω . The technical requirements on the potential V in [54–56] may be dealt with by choosing $V = 0$ in a neighborhood of Ω and $V > 0$ in appropriate subsets of $M \setminus \Omega$.

The result was generalized to quasi-convex domains in [68, Theorem 1.10]. This builds on the earlier results [2, 62], among others, which proved the same result under stronger assumptions on ω .

As noted in [10, formula (2.5)] and the proof of Lemma 2.3 of [56], if $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain, then $\omega(x) = \text{dist}(x, \partial\Omega)^{p-1-ps}$ is a Muckenhoupt A_p weight for any $1 < p < \infty$ and any $0 < s < 1$, and so the results of the previous section fall under this umbrella.

3.5. Equivalence of norms

Inequalities (3.5) and (3.6) are in fact equivalences of norms. Observe that we may write inequality (3.5) in the form

$$\|u\|_{\dot{W}^{1,p}(\Omega;\omega_{s,p})} \leq C \|f\|_{\dot{B}_s^{p,p}(\partial\Omega)} + C \|\vec{H}\|_{L^p(\Omega;\omega_{s,p})}$$

for the weight $\omega_{s,p}(x) = \text{dist}(x, \partial\Omega)^{p-1-ps}$. If u is any function in $\dot{W}^{1,p}(\Omega;\omega_{s,p})$, where Ω is as in the previous sections, then the boundary trace of u exists and lies in the Besov space $\dot{B}_s^{p,p}(\partial\Omega)$; indeed, we have the reverse estimate

$$\|u|_{\partial\Omega}\|_{\dot{B}_s^{p,p}(\partial\Omega)} \leq C \|u\|_{\dot{W}^{1,p}(\Omega;\omega_{s,p})}.$$

This was proven in the generality of Lipschitz domains in [56, Proposition 4.1] and as the $l = 1$ case of [46, Theorem 2.10]; similar results were already known in smoother domains (see [48, 60, 61, 64, 66]) or in the unweighted case $p - 1 - ps = 0$ (see [43]). Thus, inequality (3.5) implies that

$$\frac{1}{C} \|f\|_{\dot{B}_s^{p,p}(\partial\Omega)} \leq \|u\|_{\dot{W}^{1,p}(\Omega;\omega_{s,p})} \leq C \|f\|_{\dot{B}_s^{p,p}(\partial\Omega)} + C \|\vec{H}\|_{L^p(\Omega;\omega_{s,p})}.$$

With some care, we can also get an equivalence of norms for the Poisson data $\vec{H}$ in inequality (3.5) or $\vec{H}_0$ in inequality (3.6). If $\text{div } A\vec{\Psi} = \text{div } A\vec{H}_0$, then the solution u with data $\vec{\Psi}$ must equal the solution with data $\vec{H}_0$, and so estimate (3.6) may be strengthened to

$$\|\nabla u_0\|_{L^p(\Omega;\omega)} \leq C \inf\{\|\vec{\Psi}\|_{L^p(\Omega;\omega)} : \text{div}(A\vec{\Psi} - A\vec{H}_0) = 0\}.$$

In particular, $\text{div } A\vec{H}_0 = \text{div } A\nabla u$, and so we may take $\vec{\Psi} = \nabla u_0$. Thus, the reverse estimate

$$\inf\{\|\vec{\Psi}\|_{L^p(\Omega;\omega)} : \text{div}(\vec{\Psi} - \vec{H}_0) = 0\} \leq C \|\nabla u_0\|_{L^p(\Omega;\omega)}$$

is trivially satisfied. Inequality (3.6) thus implies an equivalence of norms

$$\|\nabla u_0\|_{L^p(\Omega;\omega)} \approx \inf\{\|\vec{\Psi}\|_{L^p(\Omega;\omega)} : \text{div}(A\vec{\Psi} - A\vec{H}_0) = 0\}.$$

Similarly, inequality (3.5) implies an equivalence of norms

$$\|\nabla u\|_{L^p(\Omega;\omega)} \approx \|f\|_{\dot{B}_s^{p,p}(\partial\Omega)} + \inf\{\|\vec{\Psi}\|_{L^p(\Omega;\omega)} : \text{div}(A\vec{\Psi} - A\vec{H}) = 0\}.$$

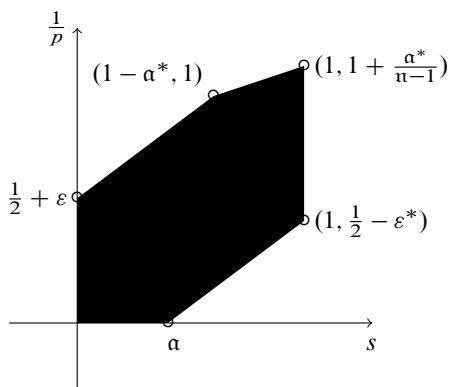


Figure 2. Values of the parameters $(s, 1/p)$ such that solutions to the Dirichlet–Poisson problem (3.3) in the domain above a Lipschitz graph, for real symmetric coefficients constant in the vertical direction, satisfy estimate (3.7).

3.6. Transversally constant coefficients and weighted averaged Sobolev estimates

Moving beyond Dini-type or VMO coefficients to another class of variable coefficient elliptic matrices, in [3, 7] the authors considered coefficients constant in the direction transverse to the boundary (but merely bounded measurable in all other directions). In [7] (written by two of the authors of the present paper), we showed that if Ω is the domain above a Lipschitz graph, and for such coefficients that in addition are real symmetric, we still have well-posedness of the Poisson–Dirichlet problem (3.3), with estimate (3.5) replaced by the estimate

$$\begin{aligned} & \int_{\Omega} \left(\int_{B(x, \text{dist}(x, \partial\Omega)/2)} |\nabla u|^{\beta} \right)^{p/\beta} \text{dist}(x, \partial\Omega)^{p-1-ps} dx \\ & \leq C \int_{\Omega} \left(\int_{B(x, \text{dist}(x, \partial\Omega)/2)} |\vec{H}|^{\beta} \right)^{p/\beta} \text{dist}(x, \partial\Omega)^{p-1-ps} dx + C \|f\|_{\dot{B}_s^{p,p}(\partial\Omega)}^p \end{aligned} \quad (3.7)$$

for β sufficiently close to 2 and for $(s, 1/p)$ in the hexagon in Figure 2. The parameters ε , ε^* , α , and α^* in that figure are all positive. (In the case where A is real, constant in the transverse direction, but not necessarily symmetric, we have well-posedness for a smaller range; namely, the range of Figure 2 for α , α^* still positive, but for ε , ε^* merely greater than $-1/2$. See the region at the bottom of Figure 4 below.)

The β -power averages were inspired by the modified nontangential maximal function of [44] and were introduced for much the same reason: if A lacks smoothness properties, then solutions u to even the homogeneous problem $\text{div } A \nabla u = 0$

no longer have gradients with high degrees of integrability. The gradient ∇u may be locally in L^β only for β close to 2, and so these weighted averaged spaces are more appropriate to such differential equations. Even for p much larger than β , these spaces preserve the trace result

$$\|f\|_{\dot{B}_s^{p,p}(\partial\Omega)}^p \leq C \int_{\Omega} \left(\int_{B(x, \text{dist}(x, \partial\Omega)/2)} |\nabla u|^\beta \right)^{p/\beta} \text{dist}(x, \partial\Omega)^{p-1-ps} dx,$$

and so estimate (3.7) is an equivalence of norms. See [7, Theorem 6.3].

As mentioned above, we used layer potential techniques, which introduce harsh restrictions both on the coefficients and on the domains. Our results also crucially used the endpoint results (which may be viewed as the $s = 0$ and $s = 1$ boundaries of the hexagon in Figure 2); that is, with the techniques of [7], the results for Besov boundary values do not stand alone but require the results for boundary data in $L^p(\partial\Omega)$ (or in the Sobolev space $\dot{W}^{1,p}(\partial\Omega)$, that is, at the smoothness 1 endpoint) and the layer potential bounds.

The paper [3] applies the so-called “first-order approach” (developed in the endpoint theory of $L^p(\partial\Omega)$ boundary data) to the Dirichlet problem

$$-\text{div } A \nabla u = 0 \quad \text{in } \Omega, \quad u = f \quad \text{on } \partial\Omega,$$

with f in a fractional smoothness space, either the Besov spaces considered in this note or the related Hardy–Sobolev spaces.

Like [7], they considered coefficients constant in the direction transverse to the boundary. Again like [7], their direct well-posedness results for boundary data of fractional smoothness require endpoint results, that is, results for boundary data in $L^p(\partial\Omega)$ or $\dot{W}^{1,p}(\partial\Omega)$.

However, unlike [7], the methods of [3] encompass general complex coefficients, and even elliptic systems rather than equations. In particular, [7] is confined to equations and also to a restricted class of coefficients (specifically, those that satisfy the De Giorgi–Nash condition; this includes all real coefficients but misses many complex coefficients). Instead, [3] can build upon endpoint results for a much broader class of coefficients, including complex self-adjoint coefficients, and constant or triangular coefficients in the half-space $\mathbb{R}_+^n$, to produce results for Besov data.

3.7. The Poisson problem without equivalence of norms

We mention one further class of results concerning the Poisson problem with rough variable coefficients. The equivalence of norms described in Section 3.5 is a highly desirable property, in that it implies that the estimates in the preceding sections cannot be improved: the norms on the right-hand side of estimates (3.5), (3.6), and (3.7) clearly cannot be weakened without a corresponding weakening of the norms on the

left-hand side. Thus, most of the existing theory has focused on estimates with such equivalences of norms.

The homogeneous Dirichlet problem

$$-\operatorname{div} A \nabla u = 0 \quad \text{in } \Omega, \quad u = f \quad \text{on } \partial\Omega \quad (3.8)$$

has been studied extensively. The paper [3], mentioned in the previous section, considered the Dirichlet problem with boundary data in fractional smoothness spaces; however, there is much more extensive literature on the Dirichlet problem with data f in the Lebesgue space $L^p(\partial\Omega)$ or in the boundary Sobolev space $\dot{W}^{1,p}(\partial\Omega)$ and its generalizations. The most common estimates (taking the role of estimates (3.5), (3.6), and (3.7) in the above sections) are of the forms

$$\|Nu\|_{L^p(\partial\Omega)} + \|\mathcal{A}_2(\delta\nabla u)\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p(\partial\Omega)}, \quad (3.9)$$

$$\|\tilde{N}(\nabla u)\|_{L^p(\partial\Omega)} \leq C\|f\|_{\dot{W}^{1,p}(\partial\Omega)}, \quad (3.10)$$

where $\delta(y) = \operatorname{dist}(y, \partial\Omega)$, and where N , $\tilde{N}$, and $\mathcal{A}_2$ denote the nontangential maximal function, modified nontangential maximal function, and Lusin area integral given by

$$\begin{aligned} \mathcal{A}_2(\vec{H})(x) &= \left(\int_{\gamma(x)} |\vec{H}(y)|^2 \delta(y)^{-n} dy \right)^{1/2}, \\ Nu(x) &= \sup_{\gamma(x)} |u|, \quad \tilde{N}(\nabla u)(x) = \sup_{y \in \gamma(x)} \left(\int_{B(y, \delta(y)/2)} |\nabla u|^2 \right)^{1/2}, \end{aligned} \quad (3.11)$$

where $\gamma(x) = \{y \in \Omega : |x - y| < 2\delta(y)\}$. The Dirichlet problem with these estimates does generally have an equivalence of norms; that is, if $-\operatorname{div} A \nabla u = 0$ in Ω and $Nu \in L^p(\partial\Omega)$ or $\mathcal{A}_2(\delta\nabla u) \in L^p(\partial\Omega)$, then in many situations, the boundary values of u are known to exist (in the sense of nontangential limits) and lie in $L^p(\partial\Omega)$. These types of results have their origin in Fatou's celebrated result [27] that a bounded harmonic function in the unit disk has a boundary trace; for real coefficients in Lipschitz domains, these boundary results follow from [13, 17].

Major difficulties arise when we try to consider the Poisson–Dirichlet problem with Dirichlet data f in L^p and with estimates on the solution u compatible with the estimates of the homogeneous Dirichlet problem. For example, consider the function $v(x) = \eta(x) \sin(\log|\log(x_n)|)$, where η is a smooth cutoff function supported in $B(0, 1/2)$. If we let Ω be the half-space, then it is straightforward to establish that $\mathcal{A}_2(\delta\nabla v)$ is bounded and compactly supported and thus lies in $L^p(\partial\mathbb{R}_+^n)$ for all p ; however, v clearly does not even possess Dirichlet boundary values in any sense. It is thus impossible to have unique solvability of the Poisson–Dirichlet problem

$$-\operatorname{div} A \nabla u = -\operatorname{div} A \vec{H} \quad \text{in } \mathbb{R}_+^n, \quad u = f \quad \text{on } \partial\mathbb{R}_+^n,$$

with the natural estimate

$$\|\mathcal{A}_2(\delta \nabla u)\|_{L^p(\partial \mathbb{R}_+^n)} \leq C \|f\|_{L^p(\partial \mathbb{R}_+^n)} + C \|\mathcal{A}_2(\delta \vec{H})\|_{L^p(\partial \mathbb{R}_+^n)}$$

for arbitrary f and $\vec{H}$, for example, for $f = 0$, $\vec{H} = \nabla v$.

It is, however, possible to solve the Poisson–Dirichlet problem with estimates of the form

$$\|\mathcal{A}_2(\delta \nabla u)\|_{L^p(\partial \Omega)} + \|\tilde{N}u\|_{L^p(\partial \Omega)} \leq C \|f\|_{L^p(\partial \Omega)} + C \|\vec{H}\|_{\mathfrak{X}}$$

or

$$\|\tilde{N}(\nabla u)\|_{L^p(\partial \Omega)} \leq C \|f\|_{\dot{W}^{1,p}(\partial \Omega)} + C \|\vec{H}\|_{\mathfrak{Y}},$$

that is, with the same estimates on the Dirichlet data f and solutions u as are traditional in the theory of the homogeneous Dirichlet problem, but for function spaces $\mathfrak{X}$ and $\mathfrak{Y}$ that do not yield an equivalence of norms in the Poisson problem. Such estimates were found in the works [6, 40, 58], either implicitly as part of the method of attack on the L^p -Dirichlet problem [6, 40] or explicitly as a main result [58].

Specifically, [6, 40] established the estimates

$$\begin{aligned} & \|\mathcal{A}_2(\delta \nabla L^{-1}(\operatorname{div} A \vec{H}))\|_{L^p(\partial \Omega)} + \|\tilde{N}(L^{-1}(\operatorname{div} A \vec{H}))\|_{L^p(\partial \Omega)} \\ & \leq \|\tilde{\mathfrak{C}}_1(\delta \vec{H})\|_{L^p(\partial \Omega)}, \\ & \|\tilde{N}(\nabla L^{-1}(\operatorname{div} A \vec{H}))\|_{L^p(\partial \Omega)} \leq \|\mathcal{A}_2(\vec{H})\|_{L^p(\partial \Omega)}, \\ & \|\tilde{N}(\nabla L^{-1}(\operatorname{div} A \vec{H}))\|_{L^p(\partial \Omega)} \leq \|\mathfrak{C}_1(\delta \operatorname{div} A \vec{H})\|_{L^p(\partial \Omega)} \end{aligned}$$

for an appropriate range of p , for $L = -\operatorname{div} A \nabla$, and for

$$\tilde{\mathfrak{C}}_1 \vec{H}(x) = \sup_{r>0} \frac{1}{\sigma(B(x, r) \cap \partial \Omega)} \int_{B(x, r) \cap \Omega} \left(\int_{B(y, \operatorname{dist}(y, \partial \Omega)/2)} |\vec{H}|^2 \right)^{1/2} \frac{dy}{\operatorname{dist}(y, \partial \Omega)},$$

in the case where the coefficients A are transversally independent (as in Section 3.6), where the underlying domain Ω is either the upper or lower half-space, and where $\vec{H}$ is always supported in the upper half-space. (The paper [40] considered the case of complex coefficients that satisfy the local De Giorgi–Nash–Moser condition, while [6] considered arbitrary complex coefficients and higher-order operators.)

The estimates on L^{-1} were used explicitly in [6, 40], but the above estimates can be used to establish well-posedness of the Poisson–Dirichlet problem, given well-posedness of the homogeneous Dirichlet problem. If $w = L^{-1}(\operatorname{div} A \vec{H})$ satisfies $\tilde{N}(\nabla w) \in L^p(\partial \Omega)$, then w possesses boundary values (in the sense of nontangential limits) that necessarily lie in the boundary Sobolev space $\dot{W}^{1,p}(\partial \mathbb{R}_+^n) = \dot{W}^{1,p}(\mathbb{R}^{n-1})$; see [44]. If the homogeneous Dirichlet problem (3.8), with estimate (3.10), is well posed, then by taking $u = w + v$, where v is the solution to

the Dirichlet problem with boundary data $f - w|_{\partial\mathbb{R}_+^n}$, we recover a solution u to the Poisson–Dirichlet problem (3.3) that satisfies the estimate

$$\|\tilde{N}(\nabla u)\|_{L^p(\partial\mathbb{R}_+^n)} \leq C\|\mathcal{A}_2(\vec{H})\|_{L^p(\partial\mathbb{R}_+^n)} + C\|f\|_{\dot{W}^{1,p}(\partial\mathbb{R}_+^n)}.$$

A similar but more complicated argument is valid for the Dirichlet problem with $f \in L^p(\partial\mathbb{R}_+^n)$. Namely, if $\vec{H}$ is supported in $\mathbb{R}_+^n$, then $w = L^{-1}(\operatorname{div} A\vec{H})$ is a solution to $-\operatorname{div} A\nabla w = 0$ in the lower half-space, and so it has boundary values in $L^p(\partial\mathbb{R}_+^n)$ by the abovementioned Fatou-type results [13, 17]. The operator $L^{-1} \operatorname{div}$ has adequate continuity properties which implies that w takes the same values on the boundary of the upper half-space $\mathbb{R}_+^n$. Thus, $w|_{\partial\mathbb{R}_+^n}$ lies in $L^p(\partial\mathbb{R}_+^n)$, and so by the same argument as above, well-posedness of the Dirichlet problem (3.8) with estimate (3.9) immediately yields well-posedness of the Poisson–Dirichlet problem (3.3) with the estimate

$$\|\tilde{N}u\|_{L^p(\partial\mathbb{R}_+^n)} + \|\mathcal{A}_2(\delta\nabla u)\|_{L^p(\partial\mathbb{R}_+^n)} \leq C\|f\|_{\dot{W}^{1,p}(\partial\mathbb{R}_+^n)} + C\|\tilde{\mathcal{C}}_1(\delta\vec{H})\|_{L^p(\partial\mathbb{R}_+^n)}.$$

The paper [58] (like the present paper) works exclusively with real coefficients but in a much more general geometric setting than the half-space. Furthermore, their treatment of the Poisson–Dirichlet problem is much more explicit than that of [6, 40]. They show that if $\partial\Omega$ is $(n-1)$ -Ahlfors regular, if Ω satisfies the interior corkscrew condition, and if $\mathbb{R}^n \setminus \Omega$ is unbounded, then solvability of the homogeneous Dirichlet problem (3.8) with estimate (3.9) is in fact equivalent to solvability of the homogeneous Poisson problem (3.1), with the estimate

$$\|\tilde{N}u\|_{L^p(\partial\Omega)} \leq C\|\tilde{\mathcal{C}}_1(\delta\vec{H})\|_{L^p(\partial\Omega)},$$

and also to solvability of the homogeneous Poisson problem for $-\operatorname{div} A^*\nabla$, where A^* denotes the transpose of A , with the estimate

$$\|\tilde{N}(\nabla u)\|_{L^{p'}(\partial\Omega)} \leq C\|\tilde{\mathcal{C}}_1(\delta \operatorname{div} A\vec{H})\|_{L^{p'}(\partial\Omega)}$$

for p' the Hölder conjugate of p , that is, $1/p + 1/p' = 1$. Under slightly stronger conditions on the domain and coefficients, namely uniform rectifiability of the boundary and a Dahlberg–Kenig–Pipher condition on the coefficients, they establish equivalence of the above conditions to solvability of the so-called regularity problem, that is, the homogeneous Dirichlet problem (3.8) with estimate (3.10), again with A replaced by A^* and p by p' .

As noted above, this version of the Poisson problem does not have equivalence of norms in the sense of Section 3.5, but the given estimates are the best estimates known in the context of L^p boundary data, are equivalent to quite strong conditions (indeed, the main estimate) in the theory of the homogeneous Dirichlet problem, and

thus may be reasonably conjectured to be among the best estimates possible in this setting.

3.8. The Laplace operator and other estimates

Thus far, we have focused on estimates of solutions u to the Poisson problem that satisfy local Sobolev estimates, that is, solutions where the gradient ∇u satisfies an estimate that locally reduces to membership in L^2 or L^p . We have done this because the main new results of this note (to be described in Sections 4 and 5) are phrased in these terms.

However, many important results, particularly the earliest results, instead established estimates on solutions to the Poisson problem (1.1) of the form

$$\|u\|_{\dot{B}_{s+1/p}^{p,q}(\Omega)} \leq C \|h\|_{\dot{B}_{s+1/p-2}^{p,q}(\Omega)} + C \|f\|_{\dot{B}_s^{p,q}(\partial\Omega)} \quad (3.12)$$

or

$$\|u\|_{\dot{F}_{s+1/p}^{p,q}(\Omega)} \leq C \|h\|_{\dot{F}_{s+1/p-2}^{p,q}(\Omega)} + C \|f\|_{\dot{B}_s^{p,p}(\partial\Omega)}, \quad (3.13)$$

where $\dot{B}_\theta^{p,q}$ and $\dot{F}_\theta^{p,q}$ denote the Besov and Triebel–Lizorkin spaces. (The precise statement of the estimates varies from paper to paper.)

It is often possible to pass from Besov or Triebel–Lizorkin estimates on solutions to weighted Sobolev estimates such as (3.5); see, for example, [41, Theorems 4.1 and 4.2].

Estimates of the form (3.12) or (3.13) are unsuited for solutions to $-\operatorname{div} A \nabla u = -\operatorname{div} A \vec{H}$, or even to $-\operatorname{div} A \nabla u = 0$, for rough coefficients A . Such solutions u do not generally display any higher regularity and so cannot be expected to lie in $\dot{B}_{s+1/p}^{p,q}(\Omega)$ for $s + 1/p > 1$. See the discussion in [7, Chapter 10], or compare the two papers [55, 56]; both consider the Dirichlet–Poisson problem in Lipschitz domains for coefficients A satisfying the same Dini-type condition, but [55] used the traditional estimate (3.12), while [56] used estimate (3.5), and so [55] has a somewhat artificial upper bound on $s + 1/p$ which [56] does not need to impose.

However, many important results are known in the context of estimate (3.12) or (3.13).

The paper [41], concerning only the Laplacian, is one of the foundational papers in the theory of the Poisson problem and of the Dirichlet problem with boundary data of fractional smoothness; indeed, it preceded all of the results described in Sections 3.2–3.7. That paper established estimate (3.13) in Lipschitz domains for $q = 2$ and for s and p as in Figure 1. They provided examples establishing sharpness of these results (that is, of Lipschitz domains such that estimate (3.13) fails for any $(s, 1/p)$ outside of the given region). In the case of C^1 domains, they established this

estimate for all $s \in (0, 1)$ and all $p \in (1, \infty)$. (The VMO condition of [51] described in Section 3.3 is a mild generalization of the C^1 condition.)

The paper [49] removes the restriction on q in estimate (3.13) (and (3.12)) and expands the available values of p and s into the range $p < 1$, specifically to the range $1/p < 1 + s + \alpha^* - 1/n - 1$ (in Lipschitz domains) or $0 \leq 1/p < 1 + s/n - 1$ (in C^1 domains). See Figure 2. We observe that [49] precedes the papers [3, 7] described in Section 3.6 and that the paper [7] explicitly works with the same ranges of s and p as the Lipschitz case of [49].

The results of [41, 49] for C^1 domains with estimate (3.12) or (3.13) also hold in bounded convex domains; see [53].

3.9. Hölder continuous boundary data

We conclude this section with a discussion of the recent result [15]. This result considers the Dirichlet problem (2.5) with Hölder continuous boundary data; this is the $p = \infty$ (bottom) edge of the region in Figure 2 and thus falls within the ambit of boundary value problems with data of fractional smoothness.

We remark that [15] is not the only paper to treat Hölder continuous boundary values, but in the setting of real elliptic coefficients, it contains (provably) the most general possible result. In particular, they place no assumptions on the coefficients beyond the ellipticity conditions (2.2)–(2.3) (with $b = n - 1$). They showed that if $\partial\Omega$ satisfies a criterion of Wiener type, then the solution u to the continuous Dirichlet problem (2.5) satisfies the estimate

$$\|u\|_{\dot{C}^s(\Omega)} \leq C \|f\|_{\dot{C}^s(\partial\Omega)} \quad (3.14)$$

for all $s > 0$ sufficiently small, where $\dot{C}^s$ denotes spaces of Hölder continuous functions. Their result is sharp; they show that their Wiener-type criterion is a necessary, as well as a sufficient, condition for estimate (3.14) for arbitrary $f \in \dot{C}^s(\partial\Omega)$.

We remark that Ahlfors regularity of $\partial\Omega$ for $b = n - 1$ (or indeed for any $b \in (n - 2, n)$) implies their Wiener-type criterion and thus is a sufficient condition for Hölder well-posedness.

The $p = \infty$, $\vec{H} = 0$ analogue of estimate (3.7) is

$$\sup_{x \in \Omega} \left(\int_{B(x, \text{dist}(x, \partial\Omega)/2)} |\nabla u|^\beta \right)^{1/\beta} \text{dist}(x, \partial\Omega)^{1-s} \leq C \|f\|_{\dot{C}^s(\partial\Omega)},$$

and it follows immediately (for β sufficiently close to 2) from estimate (3.14) and the Caccioppoli and Meyers estimates for solutions to $-\text{div } A \nabla u = 0$.

4. The Poisson problem for very general domains and coefficients

Our first main theorem establishes well-posedness of the Poisson–Dirichlet problem for *any* second-order elliptic differential equation with real coefficients satisfying suitable upper and lower bounds and with very mild conditions on the underlying domains.

Theorem 4.1. *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 3$, be a connected open set whose boundary $\partial\Omega$ is δ -Ahlfors regular in the sense of Definition 2.1 for some real number δ with $0 < \delta \leq n - 1$. If $\delta = n - 1$, then we additionally require that Ω satisfies the weak local John condition and the interior corkscrew condition.²*

We require that $\mathbb{R}^n \setminus \Omega$ be unbounded, that is, either Ω is bounded³ or both Ω and $\partial\Omega$ are unbounded.

Let $L = -\operatorname{div} A \nabla$ be a linear differential operator associated to real, not necessarily symmetric, coefficients A that satisfy the positive definiteness and boundedness conditions (2.2)–(2.3).

There are then numbers $\alpha, \alpha^* \in (0, 1]$ and $\varepsilon, \delta > 0$ such that if $(s, 1/p)$ lies in the open pentagon indicated in Figure 3, and if $2 - \delta < \beta < 2 + \varepsilon$, then for all f in the Besov space $\dot{B}_s^{p,p}(\partial\Omega)$ defined in Definition 2.10 above, and all $\vec{H} \in L_{loc}^\beta(\Omega)$ such that the right-hand side of bound (4.2) is finite, there is a unique solution u to the Poisson–Dirichlet problem

$$\begin{cases} -\operatorname{div} A \nabla u = -\operatorname{div} A \vec{H} & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega \end{cases} \quad (4.1)$$

that satisfies

$$\begin{aligned} & \int_{\Omega} \left(\int_{B(x, \operatorname{dist}(x, \partial\Omega)/2)} |\nabla u|^\beta \right)^{p/\beta} \operatorname{dist}(x, \partial\Omega)^{\delta-n+p-ps} dx \\ & \leq C \|f\|_{\dot{B}_s^{p,p}(\partial\Omega)}^p + C \int_{\Omega} \left(\int_{B(x, \operatorname{dist}(x, \partial\Omega)/2)} |\vec{H}|^\beta \right)^{p/\beta} \operatorname{dist}(x, \partial\Omega)^{\delta-n+p-ps} dx \end{aligned} \quad (4.2)$$

²If $\delta < n - 1$, then, as noted in Remark 2.5, the weak local John and interior corkscrew conditions are automatically satisfied.

³It is straightforward to establish that if $\Omega \subset \mathbb{R}^n$ is bounded, then $\partial\Omega$ has dimension at least $n - 1$; in our circumstances, this means that $\delta = n - 1$.

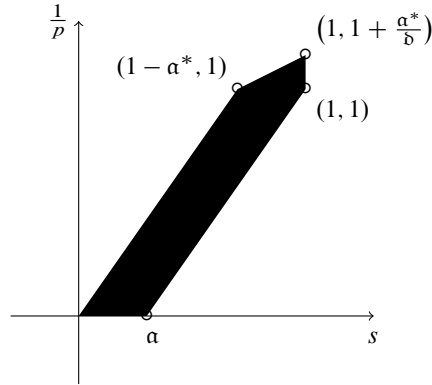


Figure 3. Values of the parameters $(s, 1/p)$ such that the $\dot{B}_s^{p,p}(\partial\Omega)$ -Dirichlet problem and associated Poisson problem for L are solvable. By Remark 4.2, the illustrated region is convex.

for some constant C depending on p, s, A , and Ω but not on f or $\tilde{H}$. If $p = \infty$ and $0 < s < \alpha$, then there is a unique solution to problem (4.1) that satisfies

$$\begin{aligned} & \sup_{x \in \Omega} \left(\int_{B(x, \text{dist}(x, \partial\Omega)/2)} |\nabla u|^\beta \right)^{1/\beta} \text{dist}(x, \partial\Omega)^{1-s} \\ & \leq C \|f\|_{\dot{B}_s^{\infty, \infty}(\partial\Omega)} + C \sup_{x \in \Omega} \left(\int_{B(x, \text{dist}(x, \partial\Omega)/2)} |\tilde{H}|^\beta \right)^{1/\beta} \text{dist}(x, \partial\Omega)^{1-s}. \end{aligned}$$

Furthermore, we have that $\alpha \geq \max(\alpha, 1 - \delta)$ and $\alpha^* \geq \max(\alpha^*, 1 - \delta)$, where α and α^* are the numbers in the boundary De Giorgi–Nash estimate (2.7).

Finally, $2 + \varepsilon$ may be taken to be the number in Meyers’s reverse Hölder estimate (2.6) for L , and $2 - \delta$ may be taken to be the Hölder conjugate of the number in Meyers’s reverse Hölder estimate for L^* .

Expressed algebraically, the pair $(s, 1/p)$ satisfies the conditions of Theorem 4.1 if either

$$\frac{\delta}{\delta + \alpha^*} < p \leq 1 \quad \text{and} \quad \frac{\delta}{p} + 1 - \delta - \alpha^* < s < 1$$

or

$$1 \leq p \leq \infty \quad \text{and} \quad \frac{1 - \alpha^*}{p} < s < \alpha + \frac{1 - \alpha}{p}.$$

We observe that, with the exception of the Sobolev estimate (3.4) for p sufficiently close to 2 in Section 3.1, the Dirichlet problem with Hölder continuous boundary data of [15] (see Section 3.9), and the Poisson–Dirichlet problem without

equivalence of norms of [58] (see Section 3.7), all of the results for the Poisson and Dirichlet problems described in Section 3 were shown exclusively in fairly regular (Lipschitz or quasi-convex) domains with boundary of codimension 1, or in domains with a higher degree of smoothness, and for coefficients satisfying some regularity assumption. Our result is the first to produce well-posedness for a broad range of $(s, 1/p)$ for arbitrary real elliptic coefficients and for domains in the above generality.

Because fundamental solutions and Green's functions display different behavior in two dimensions and in higher dimensions, certain estimates in our proof of Theorem 4.1 work only if the ambient dimension n is at least three. The case $n = 2$ is currently being investigated.

Remark 4.2. Interpolation methods can generally be used to show that if the $\dot{B}_s^{p,p}$ -Dirichlet problem is well posed for $(s, 1/p) = (s_0, 1/p_0)$ and $(s, 1/p) = (s_1, 1/p_1)$, then it is well posed whenever $(s, 1/p)$ lies on the line segment connecting the two points $(s_0, 1/p_0)$ and $(s_1, 1/p_1)$. Indeed, such techniques were used in [9] to prove Theorem 4.1. Thus, regions such as those in Figure 3 (or Figures 1–2) are generally expected to be convex.

The region of Figure 3 is indeed convex. To see this, observe that the given region is a polygon, and thus we need only establish convexity in a neighborhood of each vertex. The vertices at the origin, $(\alpha, 0)$, $(1, 1)$, and $(1, 1 + \alpha^*/\delta)$ clearly have convex neighborhoods. We are left with the vertex at the point $(1 - \alpha^*, 1)$.

The uppermost edge (connecting the points $(1 - \alpha^*, 1)$ and $(1, 1 + \alpha^*/\delta)$) has slope $1/\delta$. The adjacent edge connecting the point $(1 - \alpha^*, 1)$ to the origin has slope $1/(1 - \alpha^*)$. But because $\alpha^* \geq 1 - \delta$, we see that $1/(1 - \alpha^*) \geq 1/\delta$, and so there must be a convex neighborhood of the vertex $(1 - \alpha^*, 1)$ as well.

5. An extrapolation result

The result of the previous section is valid in the greatest known generality of coefficients and domains. It is essentially a consequence of the De Giorgi–Nash–Moser and Meyers inequalities which are a part of fundamental elliptic theory. As a consequence, we observe that the pentagon in Figure 3 is necessarily smaller than the hexagon in Figure 2.

The goal of our project, however, was to exploit new well-posedness results for the Dirichlet problem in L^p to obtain an enhanced range of Besov spaces. We now return to this task, and the theorem below will generalize and recover most of the results in Section 3, in particular, those in Figure 2, and will go far beyond that.

To this end, recall that the Dirichlet problem is well posed in L^p for some $1 < p < \infty$ if for any $f \in L^p(\partial\Omega)$ the solution u to the continuous Dirichlet

problem (2.5) satisfies the nontangential estimate

$$\|Nu\|_{L^q(\partial\Omega)} \leq C_q \|f\|_{L^q(\partial\Omega)} \quad (5.1)$$

and if such a solution is unique. Here, N is the nontangential maximal function of formula (3.11).

Well-posedness of the Dirichlet boundary value problem in L^p is a powerful feature. It has long been known that it is equivalent to (quantitative) absolute continuity of harmonic measure with respect to the Hausdorff measure on the boundary; see [36, Section 4] or [35] for a detailed discussion. Under suitable connectivity assumptions and assumptions on the coefficients, well-posedness of the L^p -Dirichlet problem for some p is equivalent to uniform rectifiability for $\mathfrak{d} = n - 1$; see [5, 38]. We will highlight some recent work in this context in Section 7. For now, let us state our second main result.

Theorem 5.1. *Suppose that the conditions of Theorem 4.1 are satisfied.*

Suppose furthermore that there is a $q \in (1, \infty)$ such that the L^q -Dirichlet problem is well posed. That is, suppose that there is a $q \in (1, \infty)$ and a $C_q < \infty$ such that if f is continuous and compactly supported, then the solution u to problem (2.5) also satisfies estimate (5.1).

Then there is a number $\mathfrak{b}^* \in (0, 1]$ such that the conclusions of Theorem 4.1 are valid for $(s, 1/p)$ in the region indicated on the left in Figure 4.

If instead there is a $q^* \in (1, \infty)$ such that the solution u to problem (2.5), with A replaced by A^* , satisfies estimate (5.1) (with q replaced by q^*), then there is a $\mathfrak{b} \in (0, 1]$ such that the conclusions of Theorem 4.1 are valid for $(s, 1/p)$ in the region indicated on the right in Figure 4.

Furthermore, the numbers $\mathfrak{b}^*$ and $\mathfrak{b}$ satisfy the estimates

$$\mathfrak{b} \geq \alpha, \quad \mathfrak{b}^* \geq \alpha^*, \quad \mathfrak{b} \geq 1 + \frac{\mathfrak{d}}{q^*} - \mathfrak{d}, \quad \mathfrak{b}^* \geq 1 + \frac{\mathfrak{d}}{q} - \mathfrak{d},$$

where α and α^* are the boundary De Giorgi–Nash exponents for L and L^* .

(In the event that q and q^* both exist, the conclusions of Theorem 4.1 are valid for $(s, 1/p)$ in the region indicated on the bottom in Figure 4.)

Theorem 4.1 is essentially the $q = q^* = \infty$ case of Theorem 5.1; note that estimate (5.1) for $q = \infty$ reduces to the classical maximum principle. Expressed algebraically, the pair $(s, 1/p)$ satisfies the conditions of Theorem 4.1 if

$$\begin{aligned} p = \infty \quad \text{and} \quad 0 < s < \mathfrak{b}, \\ p \leq 1 \quad \text{and} \quad \frac{\mathfrak{d}}{p} - \mathfrak{d} + 1 - \mathfrak{b}^* < s < 1, \end{aligned}$$

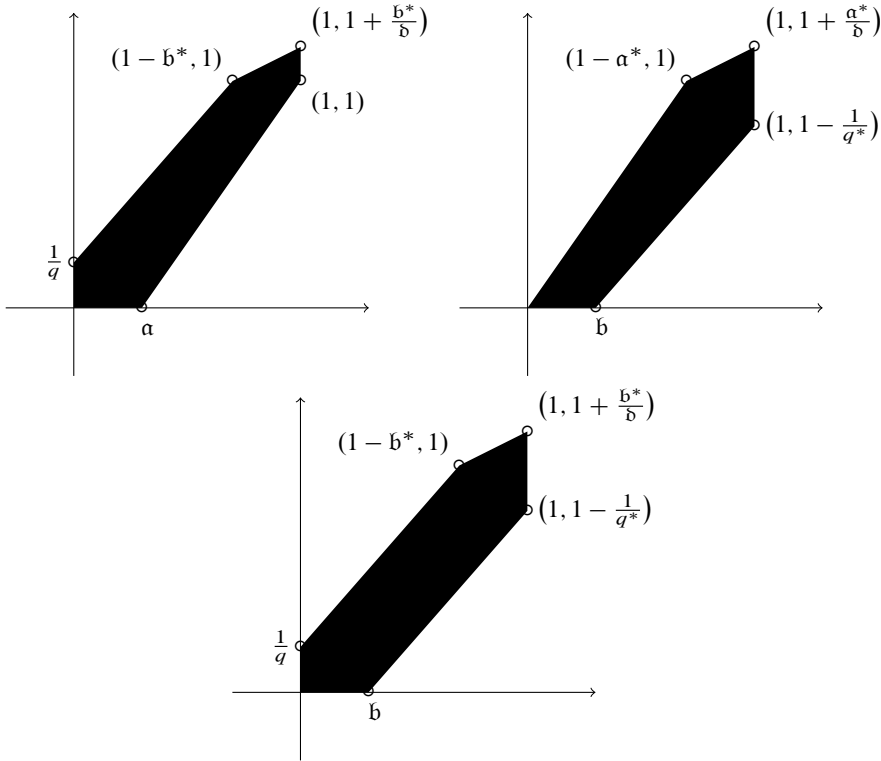


Figure 4. Values of the parameters $(s, 1/p)$ such that the $\dot{B}_s^{p,p}(\partial\Omega)$ -Dirichlet problem for L is solvable, given solvability of the L^q -Dirichlet problem for $-\operatorname{div} A\nabla$ (on the left) or the L^{q^*} -Dirichlet problem for $-\operatorname{div} A^*\nabla$ (on the right) with nontangential estimates. Because $b^* \geq 1 - b + b/q$, by an argument similar to that in Remark 4.2, the illustrated region is convex. In the common case that q and q^* both exist, the $\dot{B}_s^{p,p}(\partial\Omega)$ -Dirichlet problem for $-\operatorname{div} A\nabla$ is solvable for values of $(s, 1/p)$ as in the bottom picture.

or

$$1 < p < \infty, 0 < s < 1, \quad \text{and} \quad \frac{q-p}{qp-p}(1-b^*) < s < b + \frac{q^*}{pq^*-p}(1-b).$$

As in Remark 4.2, the condition $b^* \geq 1 + b/q - b$ implies that the regions in Figure 4 are convex in a neighborhood of the point $(1 - b^*, 1)$ and thus are convex.

Our methods of proof do not require that the number b in Theorem 4.1 or 5.1 be an integer. However, in most of the cases in which the nontangential estimate (5.1) is known to hold for a finite q , the boundary $\partial\Omega$ is of integer dimension. (Some exceptions do exist; see in particular [18].) That is, in most of the known applications of Theorem 5.1, b is an integer.

6. Further definitions

In Section 7, we will discuss some known cases in which the L^q -Dirichlet problem is well posed and thus some cases in which we may apply Theorem 5.1. It is known that the L^q -Dirichlet problem requires additional conditions on the coefficients and domains, beyond those listed in Theorem 4.1; see [14, 57]. That is, such results necessarily require more stringent conditions on the coefficients and domains. Thus, in this section, we will define some of those more stringent conditions.

6.1. Geometric conditions

Definition 6.1 (Uniformly rectifiable). Let $\mathfrak{d}, \mathfrak{n}$ be integers with $1 \leq \mathfrak{d} \leq \mathfrak{n} - 1$, and let F be a closed subset of $\mathbb{R}^{\mathfrak{n}}$. We say that F is uniformly rectifiable (or uniformly $\mathfrak{d}$ -rectifiable) if F is $\mathfrak{d}$ -Ahlfors regular and has big pieces of Lipschitz images; that is, if there exist positive constants θ and M such that if $x \in F$ and $0 < r < \text{diam } F$, then there is a Lipschitz function $\varphi : B_{\mathfrak{d}}(0, r) \rightarrow \mathbb{R}^{\mathfrak{n}}$, where $B_{\mathfrak{d}}(0, r) \subset \mathbb{R}^{\mathfrak{d}}$ is the $\mathfrak{d}$ -dimensional ball of radius r such that the Lipschitz constant of φ is at most M and such that

$$\sigma(F \cap B_{\mathfrak{n}}(x, r) \cap \varphi(B_{\mathfrak{d}}(0, r))) \geq \theta r^{\mathfrak{d}}.$$

Definition 6.2 (Harnack chains). Let $\Omega \subset \mathbb{R}^{\mathfrak{n}}$. We say that Ω satisfies the *Harnack chain condition* if there are constants $M \geq 1$ and $m \in (0, 1)$ such that if $x, y \in \Omega$, then there is an integer N , depending only on the ratio

$$\frac{|x - y|}{\min(\text{dist}(x, \partial\Omega), \text{dist}(y, \partial\Omega))}$$

such that there is a chain of open balls $\{B_k\}_{k=1}^N$ with $x \in B_1, y \in B_N, B_k \cap B_{k+1} \neq \emptyset$ for all $1 \leq k < N$, and $m \text{diam}(B_k) \leq \text{dist}(B_k, \partial\Omega) \leq M \text{diam}(B_k)$.

By [20, Lemma 2.1], and as noted in Remark 2.5, if $\partial\Omega$ is $\mathfrak{d}$ -Ahlfors regular for some $0 < \mathfrak{d} < \mathfrak{n} - 1$, then Ω automatically satisfies the Harnack chain condition. However, the higher codimension condition does not guarantee uniform rectifiability.

Definition 6.3 (Chord-arc domain). Let $\Omega \subset \mathbb{R}^{\mathfrak{n}}$. We call Ω a chord-arc domain if Ω satisfies the interior corkscrew condition (Definition 2.3) and the Harnack chain condition, if $\partial\Omega$ is $(\mathfrak{n} - 1)$ -Ahlfors regular, and if Ω , in addition, satisfies the exterior corkscrew condition, that is, if the exterior domain $\mathbb{R}^{\mathfrak{n}} \setminus \overline{\Omega}$ also satisfies the interior corkscrew condition.

Remark 6.4. By connecting corkscrew points with Harnack chains, we can construct a λ -carrot path from any point $x \in \partial\Omega$ to any point $y \in \Omega$ with $|x - y| <$

$N \operatorname{dist}(y, \partial\Omega)$, where λ depends only on N , the dimension n , and the corkscrew and Harnack chain constants. Thus, a chord-arc domain is a weak local John domain and so satisfies the conditions of Theorem 4.1.

Remark 6.5. A chord-arc domain necessarily has uniformly rectifiable boundary; this follows from [22] or [63] combined with [39].

Definition 6.6 (Chord-arc surfaces with small constant). Let $\Gamma \subset \mathbb{R}^n$ be closed and unbounded. We say that Γ is a chord-arc surface with constant ε if, for every $x \in \Gamma$ and $r > 0$, there is a Lipschitz function $\varphi_x : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ with Lipschitz constant at most ε and a rigid motion T_x such that if

$$G = \{T_x((y, \varphi(y))) : y \in \mathbb{R}^{n-1}\},$$

then $G \cap B(x, r/2) \neq \emptyset$ and

$$\sigma(\Gamma \cap B(x, r) \setminus G) + \sigma(G \cap B(x, r) \setminus \Gamma) < \varepsilon r^{n-1}.$$

Remark 6.7. By [23, Lemma 5.4] and the following remarks, if $\Omega \subset \mathbb{R}^n$ and $\partial\Omega$ is a chord-arc surface with sufficiently small constant ε , then Ω and $\mathbb{R}^n \setminus \overline{\Omega}$ are both chord-arc domains.

6.2. Conditions on coefficients

In this section, we will state one form of the Dahlberg–Kenig–Pipher (DKP) condition on coefficients A . Early forms of this condition were proposed by Dahlberg and investigated by Kenig and Pipher in [45], and many weaker forms have been studied.

In particular, for simplicity of exposition, we will state the DKP condition only in the classical codimension 1 case (that is, $d = n - 1$). However, generalizations of these notions to the higher codimensional case have been investigated (see [19, 30]) and are also a present active area of research.

Definition 6.8 (The weak DKP condition). Let $\Omega \subset \mathbb{R}^n$ be open and such that $\partial\Omega$ is $(n - 1)$ -Ahlfors regular. Let A be a matrix-valued function defined on Ω that satisfies the ellipticity conditions (2.2)–(2.3) for some $\Lambda > \lambda > 0$ with $d = n - 1$.

We say that A satisfies the weak DKP condition with constant M in Ω if $A = B + C$, where

$$\sup_{x \in \partial\Omega} \sup_{r > 0} \frac{1}{r^{n-1}} \int_{B(x, r) \cap \Omega} |\nabla B(y)|^2 \operatorname{dist}(y, \partial\Omega) + |C(y)|^2 \frac{1}{\operatorname{dist}(y, \partial\Omega)} dy = M < \infty.$$

7. Corollaries of Theorem 5.1

The L^q -Dirichlet problem has been a subject of intense study over the past few decades, and so bound (5.1) is known to be valid in many cases beyond Lipschitz domains. A complete survey of such results is beyond the scope of this note; in this section, we will describe a few such results.

7.1. Higher codimensional results

The most elegant (albeit not the most general) of the known results for the Dirichlet problem in the higher codimensional case was established in [24] (and later by another method in [28]) and is as follows.

Let $\mathfrak{d}$ be an integer with $1 \leq \mathfrak{d} \leq \mathfrak{n} - 2$. Let $\Omega \subset \mathbb{R}^{\mathfrak{n}}$ and suppose that $\partial\Omega$ is $\mathfrak{d}$ -Ahlfors regular and uniformly rectifiable. (By Remark 2.5, because $\mathfrak{d} < \mathfrak{n} - 1$, we have that Ω satisfies the conditions of Theorem 4.1.)

If $\beta > 0$, define the function $D_\beta : \Omega \rightarrow \mathbb{R}$ by

$$D_\beta(x) = \left(\int_{\partial\Omega} |x - y|^{-\mathfrak{d}-\beta} d\sigma(y) \right)^{-1/\beta}.$$

By [19, Lemma 5.1], $D_\beta(x) \approx \text{dist}(x, \partial\Omega)$.

Let $A_\beta(x) = D_\beta(x)^{\mathfrak{d}+1-\mathfrak{n}}I$, where I is the identity matrix, and let $L_\beta = \text{div } A_\beta \nabla$ (as defined by Definition 2.13). Observe that A satisfies the ellipticity conditions (2.2)–(2.3) and is also a symmetric matrix, and hence $L_\beta = L_\beta^*$.

By [28, Theorem 1.14] or [24, Theorem 1.1], the harmonic measure associated to L is in $A_\infty(\sigma)$; by [50, Theorem 4.1], this implies that the L^q -Dirichlet problem is well posed for some $q \in (1, \infty)$.

Thus, for a uniformly rectifiable boundary, and for the operator L_β , $\beta > 0$, we have that the conditions of Theorem 5.1 are satisfied, and so the solution to the Poisson–Dirichlet problem (4.1) satisfies estimate (4.2) for all $(s, 1/p)$ in the hexagon at the bottom of Figure 4.

Especially in the higher codimensional setting, the question of coefficients A for which the L^q -Dirichlet problem is well posed is an active area of research, and we expect further results of this type to be proven soon. Some (somewhat complicated) results of this type are known; we refer the interested reader to [19, 30] for the details.

7.2. The Laplace operator and the DKP condition

We now turn to known results for the codimension 1 case, that is, the case $\mathfrak{d} = \mathfrak{n} - 1$.

As in the previous section, the most elegant known results are those for a particular operator; as is usual in the codimension 1 case, the favored particular operator is the Laplacian. By [5], if $\Omega \subset \mathbb{R}^{\mathfrak{n}}$ is a weak local John domain and satisfies the

interior corkscrew condition, and if $\partial\Omega$ is $(n-1)$ -Ahlfors regular and uniformly rectifiable, then there is a $q \in (1, \infty)$ such that the L^q -Dirichlet problem is well posed for $L = -\Delta$, that is, for coefficients $A = I$, where I is the identity matrix.

There has been considerable investigation into the weakest possible sufficient conditions (and the strongest possible necessary conditions!) for well-posedness of the L^q -Dirichlet problem. Many forms of the DKP condition have been investigated under many conditions on domains. We will now mention one known well-posedness result for operators satisfying a quite weak version of the DKP condition (defined above); this is not the only such known result, and in fact, some results are known in more general domains.

Specifically, if $\Omega \subset \mathbb{R}^n$ is a chord-arc domain, and if A satisfies the weak DKP condition of Definition 6.8 in Ω for some finite M , then there is a $q \in (1, \infty)$ such that the L^q -Dirichlet problem for $-\operatorname{div} A \nabla$ is well posed.

This is based on known results as follows. By [29, Theorem 1.21], if $\Omega \subset \mathbb{R}^n$ is open, if $\partial\Omega$ is $(n-1)$ -Ahlfors regular and uniformly rectifiable, if A satisfies the weak DKP condition, and if $-\operatorname{div} A \nabla u = 0$ and u is bounded in Ω , then

$$\int_{B(x,r) \cap \Omega} |\nabla u(y)|^2 \operatorname{dist}(y, \partial\Omega) dy \leq C \|u\|_{L^\infty}^2 \sigma(B(x,r) \cap \partial\Omega) \quad (7.1)$$

for all $x \in \partial\Omega$ and all $r > 0$.

By [16, Theorem 1.1], if Ω is a 1-sided chord-arc domain, then the Carleson condition (7.1) implies that the harmonic measure for $-\operatorname{div} A \nabla$ lies in $A_\infty(\partial\Omega)$.

Let ω^y be the harmonic measure for $-\operatorname{div} A \nabla$ with pole at y , and let k^y be the Radon–Nikodym derivative of ω^y with respect to the surface measure. As observed just following [16, Definition 2.13], their A_∞ condition implies that there exist a $C < \infty$ and a $p \in (1, \infty)$ such that if $x \in \partial\Omega$, $0 < r < \operatorname{diam} \Omega$, and $y = A_r(x)$ is the corkscrew point to x at scale r , then

$$\left(\int_{\Delta(x,r)} (k^y)^p d\sigma \right)^{1/p} \leq C \int_{\Delta(x,r)} k^y d\sigma.$$

It is well known (see, for example, [36, Proposition 4.5]) that this reverse Hölder estimate on the harmonic measure yields well-posedness of the L^q -Dirichlet problem for $q = p/(p-1)$, that is, for q the Hölder conjugate of p .

7.3. The small DKP condition

We conclude this discussion by mentioning a “small constant” result. That is, notice that the results of the previous two sections all imply the existence of a $q \in (1, \infty)$ such that the L^q -Dirichlet problem is well posed, without ever specifying precise values of q . By requiring more of our coefficients and domains, it is possible to choose a value of q and find situations in which the L^q -Dirichlet problem is well posed.

Specifically, let $q \in (1, \infty)$, let $n \geq 3$ be an integer, and let $\Lambda > \lambda > 0$. Then there is a $\delta > 0$ (depending on q , δ , Λ , and λ) such that if

- (1) $\Omega \subset \mathbb{R}^n$,
- (2) $\partial\Omega$ is a chord-arc surface with constant at most δ (as defined in Definition 6.6),
- (3) A satisfies the ellipticity conditions (2.2)–(2.3) (with $\delta = n - 1$), and
- (4) A satisfies the weak Dahlberg–Kenig–Pipher condition in Ω with constant at most δ (as defined in Definition 6.8),

then the L^q -Dirichlet problem is well posed.

This result follows from known results in the literature by a line of argument similar to the previous section. First, by the recent results of [23], the given conditions imply a quantitative local A_∞ -type condition on the harmonic measure.⁴ Second, one can show that this A_∞ -type condition implies a local reverse Hölder condition on the density (that is, the Radon–Nikodym derivative with respect to the surface measure) of the harmonic measure. Finally, as before, by [36, Proposition 4.5], this reverse Hölder estimate yields well-posedness of the L^q -Dirichlet problem.

The precise form of the A_∞ -type condition varies from paper to paper, and some work must be done to relate particular forms to the required reverse Hölder condition. Moreover, in the small constant case, we must at every step ensure that our results work for the chosen value of q and not, as in the previous section, merely for sufficiently large finite q . This means that the argument for this result is somewhat more involved than in the previous section; we will give the details in the appendix of [9].

We emphasize that all of the results mentioned in Section 7 (and many other recent advancements in this spirit) extrapolate automatically to well-posedness within the subregions in Figure 4 by virtue of Theorem 5.1. In particular, in the small constant case, the region includes arbitrarily large compact subsets of the square $(0, 1) \times (1, \infty)$. We emphasize that this is not a result of the more traditional interpolation between the Dirichlet problem and the regularity problem, as in most of these cases, the regularity problem has not been treated.

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⁴The weak DKP condition required by [23] is not quite that required by [29]. To lighten the exposition, we have chosen to state only the stronger of these two weak DKP conditions. It is straightforward to establish that the weak DKP condition of [23] implies the weak DKP condition of [29] in the setting of [23], that is, in domains Ω such that the Whitney boxes of [23, formula (1.2)] have volume comparable to r^n .

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References

- [1] K. Adimurthi, T. Mengesha, and N. C. Phuc, [Gradient weighted norm inequalities for linear elliptic equations with discontinuous coefficients](#). *Appl. Math. Optim.* **83** (2021), no. 1, 327–371 Zbl [1465.35160](#) MR [4221600](#)
- [2] K. Adimurthi and N. C. Phuc, [An end-point global gradient weighted estimate for quasilinear equations in non-smooth domains](#). *Manuscripta Math.* **150** (2016), no. 1–2, 111–135 Zbl [1342.35053](#) MR [3483172](#)
- [3] A. Amenta and P. Auscher, [Elliptic boundary value problems with fractional regularity data. The first order approach](#). CRM Monogr. Ser. 37, American Mathematical Society, Providence, RI, 2018, 152 pp. Zbl [1398.35001](#) MR [3753666](#)
- [4] P. Auscher and M. Qafsaoui, Observations on $W^{1,p}$ estimates for divergence elliptic equations with VMO coefficients. *Boll. Unione Mat. Ital. Ser. (8)* **5-B** (2002), no. 2, 487–509 Zbl [1173.35419](#) MR [1911202](#)
- [5] J. Azzam, S. Hofmann, J. M. Martell, M. Mouroglou, and X. Tolsa, [Harmonic measure and quantitative connectivity: geometric characterization of the \$L^p\$ -solvability of the Dirichlet problem](#). *Invent. Math.* **222** (2020), no. 3, 881–993 Zbl [1453.31009](#) MR [4169053](#)
- [6] A. Barton, The L^p Neumann problem for higher order elliptic equations. *Mem. Differ. Equ. Math. Phys.* **96** (2025), 29–80 Zbl [08152298](#) MR [5012550](#)
- [7] A. Barton and S. Mayboroda, [Layer potentials and boundary-value problems for second order elliptic operators with data in Besov spaces](#). *Mem. Amer. Math. Soc.* **243** (2016), no. 1149, 110 pp. Zbl [1378.35100](#) MR [3517153](#)
- [8] A. Barton, S. Mayboroda, and A. Pacati, The Dirichlet problem with boundary data of fractional smoothness in domains with Ahlfors regular boundary. 2026, in preparation
- [9] A. Barton, S. Mayboroda, and A. Pacati, The Poisson problem in domains with Ahlfors regular boundary. 2026, arXiv:[2603.22067v1](#)
- [10] K. Brewster and M. Mitrea, Boundary value problems in weighted Sobolev spaces on Lipschitz manifolds. *Mem. Differ. Equ. Math. Phys.* **60** (2013), 15–55 Zbl [1310.35089](#) MR [3288169](#)
- [11] S.-S. Byun, [Elliptic equations with BMO coefficients in Lipschitz domains](#). *Trans. Amer. Math. Soc.* **357** (2005), no. 3, 1025–1046 Zbl [1087.35027](#) MR [2110431](#)
- [12] S.-S. Byun and L. Wang, [Elliptic equations with BMO coefficients in Reifenberg domains](#). *Comm. Pure Appl. Math.* **57** (2004), no. 10, 1283–1310 Zbl [1112.35053](#) MR [2069724](#)
- [13] L. Caffarelli, E. Fabes, S. Mortola, and S. Salsa, [Boundary behavior of nonnegative solutions of elliptic operators in divergence form](#). *Indiana Univ. Math. J.* **30** (1981), no. 4, 621–640 Zbl [0512.35038](#) MR [0620271](#)

- [14] L. A. Caffarelli, E. B. Fabes, and C. E. Kenig, [Completely singular elliptic-harmonic measures](#). *Indiana Univ. Math. J.* **30** (1981), no. 6, 917–924 Zbl 0482.35020 MR 0632860
- [15] M. Cao, P. Hidalgo-Palencia, J. M. Martell, C. Prisuelos-Arribas, and Z. Zhao, [Elliptic operators in rough sets and the Dirichlet problem with boundary data in Hölder spaces](#). *J. Funct. Anal.* **288** (2025), no. 5, article no. 110801, 68 pp. Zbl 1567.35122 MR 4842691
- [16] J. Cavero, S. Hofmann, J. M. Martell, and T. Toro, [Perturbations of elliptic operators in 1-sided chord-arc domains. Part II. Non-symmetric operators and Carleson measure estimates](#). *Trans. Amer. Math. Soc.* **373** (2020), no. 11, 7901–7935 Zbl 1453.31011 MR 4169677
- [17] B. E. J. Dahlberg, D. S. Jerison, and C. E. Kenig, [Area integral estimates for elliptic differential operators with non-smooth coefficients](#). *Ark. Mat.* **22** (1984), no. 1–2, 97–108 Zbl 0537.35025 MR 0735881
- [18] G. David, M. Engelstein, and S. Mayboroda, [Square functions, nontangential limits, and harmonic measure in codimension larger than 1](#). *Duke Math. J.* **170** (2021), no. 3, 455–501 Zbl 1471.31001 MR 4255042
- [19] G. David, J. Feneuil, and S. Mayboroda, [Dahlberg’s theorem in higher co-dimension](#). *J. Funct. Anal.* **276** (2019), no. 9, 2731–2820 Zbl 1414.42032 MR 3926132
- [20] G. David, J. Feneuil, and S. Mayboroda, [Elliptic theory for sets with higher co-dimensional boundaries](#). *Mem. Amer. Math. Soc.* **274** (2021), no. 1346, 123 pp. Zbl 1483.35099 MR 4341338
- [21] G. David, J. Feneuil, and S. Mayboroda, [Elliptic theory in domains with boundaries of mixed dimension](#). Astérisque 442, Société Mathématique de France, Paris, 2023, 139 pp. Zbl 1567.35123 MR 4619112
- [22] G. David and D. Jerison, [Lipschitz approximation to hypersurfaces, harmonic measure, and singular integrals](#). *Indiana Univ. Math. J.* **39** (1990), no. 3, 831–845 Zbl 0758.42008 MR 1078740
- [23] G. David, L. Li, and S. Mayboroda, [Small \$A_\infty\$ results for Dahlberg–Kenig–Pipher operators in sets with uniformly rectifiable boundaries](#). *Trans. Amer. Math. Soc.* **376** (2023), no. 11, 7857–7909 Zbl 1528.35038 MR 4657223
- [24] G. David and S. Mayboroda, [Harmonic measure is absolutely continuous with respect to the Hausdorff measure on all low-dimensional uniformly rectifiable sets](#). *Int. Math. Res. Not. IMRN* **2023** (2023), no. 11, 9319–9426 Zbl 1527.35149 MR 4597210
- [25] G. Di Fazio, [\$L^p\$ estimates for divergence form elliptic equations with discontinuous coefficients](#). *Boll. Unione Mat. Ital. A (7)* **10** (1996), no. 2, 409–420 Zbl 0865.35048 MR 1405255
- [26] E. Fabes, O. Mendez, and M. Mitrea, [Boundary layers on Sobolev–Besov spaces and Poisson’s equation for the Laplacian in Lipschitz domains](#). *J. Funct. Anal.* **159** (1998), no. 2, 323–368 Zbl 0930.35045 MR 1658089

- [27] P. Fatou, [Séries trigonométriques et séries de Taylor](#). *Acta Math.* **30** (1906), no. 1, 335–400 JFM [37.0283.01](#) MR [1555035](#)
- [28] J. Feneuil, [Absolute continuity of the harmonic measure on low dimensional rectifiable sets](#). *J. Geom. Anal.* **32** (2022), no. 10, article no. 247, 36 pp. Zbl [1514.42029](#) MR [4456214](#)
- [29] J. Feneuil, L. Li, and S. Mayboroda, [Green functions and smooth distances](#). *Math. Ann.* **389** (2024), no. 3, 2637–2727 Zbl [1545.35045](#) MR [4753073](#)
- [30] J. Feneuil, S. Mayboroda, and Z. Zhao, [The Dirichlet problem in domains with lower dimensional boundaries](#). *Rev. Mat. Iberoam.* **37** (2021), no. 3, 821–910 Zbl [1466.35112](#) MR [4236798](#)
- [31] Y. Han, S. Lu, and D. Yang, [Inhomogeneous Besov and Triebel–Lizorkin spaces on spaces of homogeneous type](#). *Approx. Theory Appl.* **15** (1999), no. 3, 37–65 Zbl [0957.46025](#) MR [1746473](#)
- [32] Y. Han, D. Müller, and D. Yang, [A theory of Besov and Triebel–Lizorkin spaces on metric measure spaces modeled on Carnot–Carathéodory spaces](#). *Abstr. Appl. Anal.* **2008** (2008), no. 1, article no. 893409, 250 pp. Zbl [1193.46018](#) MR [2485404](#)
- [33] Y. Han and D. Yang, [Some new spaces of Besov and Triebel–Lizorkin type on homogeneous spaces](#). *Studia Math.* **156** (2003), no. 1, 67–97 Zbl [1032.42025](#) MR [1961062](#)
- [34] Y. S. Han and E. T. Sawyer, [Littlewood–Paley theory on spaces of homogeneous type and the classical function spaces](#). *Mem. Amer. Math. Soc.* **110** (1994), no. 530, 126 pp. Zbl [0806.42013](#) MR [1214968](#)
- [35] S. Hofmann, [Quantitative absolute continuity of harmonic measure and the Dirichlet problem: a survey of recent progress](#). *Acta Math. Sin. (Engl. Ser.)* **35** (2019), no. 6, 1011–1026 Zbl [1421.31010](#) MR [3952700](#)
- [36] S. Hofmann and P. Le, [BMO solvability and absolute continuity of harmonic measure](#). *J. Geom. Anal.* **28** (2018), no. 4, 3278–3299 Zbl [1405.42032](#) MR [3881972](#)
- [37] S. Hofmann and J. M. Martell, [Uniform rectifiability and harmonic measure. I. Uniform rectifiability implies Poisson kernels in \$L^p\$](#) . *Ann. Sci. Éc. Norm. Supér. (4)* **47** (2014), no. 3, 577–654 Zbl [1302.31007](#) MR [3239100](#)
- [38] S. Hofmann, J. M. Martell, S. Mayboroda, T. Toro, and Z. Zhao, [Uniform rectifiability and elliptic operators satisfying a Carleson measure condition](#). *Geom. Funct. Anal.* **31** (2021), no. 2, 325–401 Zbl [1473.35165](#) MR [4268304](#)
- [39] S. Hofmann, J. M. Martell, and I. Uriarte-Tuero, [Uniform rectifiability and harmonic measure. II. Poisson kernels in \$L^p\$ imply uniform rectifiability](#). *Duke Math. J.* **163** (2014), no. 8, 1601–1654 Zbl [1323.31008](#) MR [3210969](#)
- [40] S. Hofmann, S. Mayboroda, and M. Mourgoglou, [Layer potentials and boundary value problems for elliptic equations with complex \$L^\infty\$ coefficients satisfying the small Carleson measure norm condition](#). *Adv. Math.* **270** (2015), 480–564 Zbl [1311.35074](#) MR [3286542](#)
- [41] D. Jerison and C. E. Kenig, [The inhomogeneous Dirichlet problem in Lipschitz domains](#). *J. Funct. Anal.* **130** (1995), no. 1, 161–219 Zbl [0832.35034](#) MR [1331981](#)

- [42] D. S. Jerison and C. E. Kenig, [Boundary behavior of harmonic functions in non-tangentially accessible domains](#). *Adv. Math.* **46** (1982), no. 1, 80–147 Zbl 0514.31003 MR 0676988
- [43] A. Jonsson and H. Wallin, Function spaces on subsets of $\mathbf{R}^n$. *Math. Rep.* **2** (1984), no. 1, 221 pp. Zbl 0875.46003 MR 0820626
- [44] C. E. Kenig and J. Pipher, [The Neumann problem for elliptic equations with non-smooth coefficients](#). *Invent. Math.* **113** (1993), no. 1, 447–509 Zbl 0807.35030 MR 1231834
- [45] C. E. Kenig and J. Pipher, [The Dirichlet problem for elliptic equations with drift terms](#). *Publ. Mat.* **45** (2001), no. 1, 199–217 Zbl 1113.35314 MR 1829584
- [46] D. Kim, [Trace theorems for Sobolev–Slobodeckij spaces with or without weights](#). *J. Funct. Spaces Appl.* **5** (2007), no. 3, 243–268 Zbl 1136.46030 MR 2352844
- [47] J. Kinnunen and S. Zhou, [A boundary estimate for nonlinear equations with discontinuous coefficients](#). *Differential Integral Equations* **14** (2001), no. 4, 475–492 Zbl 1161.35394 MR 1799417
- [48] P. I. Lizorkin, Boundary properties of functions from “weight” classes (in Russian). *Dokl. Akad. Nauk SSSR* **132** (1960), no. 3, 514–517. English translation. *Soviet Math. Dokl.* **1** (1960), 589–593 Zbl 0106.30802 MR 0123103
- [49] S. Mayboroda and M. Mitrea, [Sharp estimates for Green potentials on non-smooth domains](#). *Math. Res. Lett.* **11** (2004), no. 4, 481–492 Zbl 1073.35075 MR 2092902
- [50] S. Mayboroda and Z. Zhao, [Square function estimates, the BMO Dirichlet problem, and absolute continuity of harmonic measure on lower-dimensional sets](#). *Anal. PDE* **12** (2019), no. 7, 1843–1890 Zbl 1432.35074 MR 3986543
- [51] V. Maz’ya, M. Mitrea, and T. Shaposhnikova, [The Dirichlet problem in Lipschitz domains for higher order elliptic systems with rough coefficients](#). *J. Anal. Math.* **110** (2010), no. 1, 167–239 Zbl 1199.35080 MR 2753293
- [52] O. Mendez and M. Mitrea, [The Banach envelopes of Besov and Triebel–Lizorkin spaces and applications to partial differential equations](#). *J. Fourier Anal. Appl.* **6** (2000), no. 5, 503–531 Zbl 0972.46017 MR 1781091
- [53] D. Mitrea, M. Mitrea, and L. Yan, [Boundary value problems for the Laplacian in convex and semiconvex domains](#). *J. Funct. Anal.* **258** (2010), no. 8, 2507–2585 Zbl 1201.35095 MR 2593333
- [54] M. Mitrea and M. Taylor, [Potential theory on Lipschitz domains in Riemannian manifolds: Sobolev–Besov space results and the Poisson problem](#). *J. Funct. Anal.* **176** (2000), no. 1, 1–79 Zbl 0968.58023 MR 1781631
- [55] M. Mitrea and M. Taylor, [Sobolev and Besov space estimates for solutions to second order PDE on Lipschitz domains in manifolds with Dini or Hölder continuous metric tensors](#). *Comm. Partial Differential Equations* **30** (2005), no. 1–2, 1–37 Zbl 1068.58013 MR 2130803
- [56] M. Mitrea and M. Taylor, [The Poisson problem in weighted Sobolev spaces on Lipschitz domains](#). *Indiana Univ. Math. J.* **55** (2006), no. 3, 1063–1090 Zbl 1387.35166 MR 2244598

- [57] L. Modica and S. Mortola, [Construction of a singular elliptic-harmonic measure](#). *Manuscripta Math.* **33** (1980), no. 1, 81–98 Zbl 0505.35039 MR 0596380
- [58] M. Mourougolou, B. Poggi, and X. Tolsa, [Solvability of the Poisson–Dirichlet problem with interior data in \$L^{p'}\$ -Carleson spaces and its applications to the \$L^p\$ -regularity problem](#). *J. Eur. Math. Soc. (JEMS)* (2025), DOI 10.4171/JEMS/1660
- [59] D. Müller and D. Yang, [A difference characterization of Besov and Triebel–Lizorkin spaces on RD-spaces](#). *Forum Math.* **21** (2009), no. 2, 259–298 Zbl 1171.42013 MR 2503306
- [60] S. M. Nikol’skiĭ, [Approximation of functions of several variables and imbedding theorems](#) (in Russian). Nauka, Moscow, 1969, 420 pp. Zbl 0185.37901 MR 0310616
- [61] S. M. Nikol’skiĭ, P. I. Lizorkin, and N. V. Miroshin, [Weighted function spaces and their applications to the investigation of boundary value problems for degenerate elliptic equations](#) (in Russian). *Izv. Vyssh. Uchebn. Zaved. Mat.* (1988), no. 8, 4–30. English translation. *Soviet Math. (Iz. VUZ)* **32** (1988), no. 8, 1–40 Zbl 0679.35030 MR 0971868
- [62] N. C. Phuc, [Weighted estimates for nonhomogeneous quasilinear equations with discontinuous coefficients](#). *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **10** (2011), no. 1, 1–17 Zbl 1228.35260 MR 2829320
- [63] S. Semmes, [Analysis vs. geometry on a class of rectifiable hypersurfaces in \$\mathbf{R}^n\$](#) . *Indiana Univ. Math. J.* **39** (1990), no. 4, 1005–1035 Zbl 0796.42014 MR 1087183
- [64] V. V. Shan’kov, [The averaging operator with variable radius, and the inverse trace theorem](#) (in Russian). *Sibirsk. Mat. Zh.* **26** (1985), no. 6, 141–152. [English translation](#). *Sib. Math. J.* **26** (1985), no. 6, 891–901 Zbl 0597.46032 MR 0816512
- [65] H. Triebel, [Theory of function spaces](#). Monogr. Math. 78, Birkhäuser, Basel, 1983, 281 pp. Zbl 0546.46027 MR 0781540
- [66] S. V. Uspenskii, [Imbedding theorems for classes with weights](#) (in Russian). *Tr. Mat. Inst. Steklova* **60** (1961), 282–303. [English translation](#). *Amer. Math. Soc. Transl. Ser. II* **87** (1970), 121–145 Zbl 0198.46106 MR 0136980
- [67] N. Wiener, [The Dirichlet problem](#). *J. Math. Phys.* **3** (1924), no. 3, 127–146 JFM 51.0361.01
- [68] S. Yang, D. Yang, and W. Yuan, [Global gradient estimates for Dirichlet problems of elliptic operators with a BMO antisymmetric part](#). *Adv. Nonlinear Anal.* **11** (2022), no. 1, 1496–1530 Zbl 1498.35217 MR 4422226
- [69] D. Z. Zanger, [The inhomogeneous Neumann problem in Lipschitz domains](#). *Comm. Partial Differential Equations* **25** (2000), no. 9–10, 1771–1808 Zbl 0959.35041 MR 1778780

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