

Cohomology rings of character varieties

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Abstract. In this talk, I give an introduction and present some recent progress towards understanding the cohomology rings of character varieties of Riemann surfaces, such as a proof of the $P = W$ conjecture and the computation of the zero-dimensional cohomological Hall algebra (COHA). In the case of punctured sphere, I present an explicit description relating the cohomology rings to the Hilbert scheme of $\mathbb{C}^2$, refining conjectures of Hausel–Letellier–Rodriguez-Villegas and Chuang–Diaconescu–Donagi–Pantev. I explain how the general case should be related to the symplectic geometry of the Hilbert scheme.

1. Classical motivation

One way that character varieties naturally arise is through the study of linear ODEs. The monodromy theorem associates to a linear ODE a representation of the fundamental group. The famous twenty-first Hilbert problem asked if any representation arises in such a way. A historical review of this subject can be found in [2].

Referring the reader for details and references to [2, Section 3], we briefly summarize the ideas here. Suppose we have a linear ODE

$$\frac{\partial}{\partial t} \vec{\varphi}(t) = A(t) \vec{\varphi}(t) \quad (t \in \mathbb{C}). \quad (1.1)$$

Here, $A(t)$ is an $n \times n$ matrix-valued meromorphic function of one complex variable t , and we are looking for vector-valued solutions $\vec{\varphi}(t)$. Let $t_1, \dots, t_k$ be the poles of $A(t)$ (including ∞ if necessary), which we call *punctures*. Choose a basepoint $*$, away from the poles of $A(t)$. In a neighborhood of $*$, we have an n -dimensional vector space of local solutions of the ODE (1.1). For any loop γ based at $*$ avoiding the punctures, solving the ODE along γ produces a linear transformation of the space

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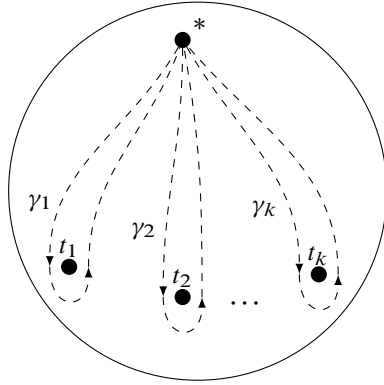


Figure 1. Fundamental group of a k -punctured sphere.

of solutions. Choosing a basis of the space of solutions, we obtain the *monodromy representation* of the fundamental group

$$\rho : \pi_1(*, \mathbb{P}^1(\mathbb{C}) \setminus \{t_1, \dots, t_k\}) \rightarrow \mathrm{GL}_n(\mathbb{C}).$$

A different choice of basis produces an isomorphic representation.

Another way to think about this is that local solutions to the ODE form a *locally constant sheaf* on $\mathbb{P}^1(\mathbb{C}) \setminus \{t_1, \dots, t_k\}$. Locally constant sheaves are also called *local systems*, and they are in a natural bijection with equivalence classes of monodromy representations; see [43, Corollary 2.6.2].

More explicitly, the fundamental group of $\mathbb{P}^1(\mathbb{C}) \setminus \{t_1, \dots, t_k\}$ is the free group on $k - 1$ generators. Drawing loops around the punctures, as shown in Figure 1, one obtains the following presentation of the fundamental group:

$$\pi_1(*, \mathbb{P}^1(\mathbb{C}) \setminus \{t_1, \dots, t_k\}) = \langle \gamma_1, \dots, \gamma_k \mid \gamma_1 \cdots \gamma_k = 1 \rangle. \quad (1.2)$$

This presentation of π_1 by k generators and one relation is better than the free group presentation for the following reason. Let us assume that $A(t)$ has simple poles. In this case, analyzing the ODE locally around each puncture and finding a basis of solutions using the *Frobenius method* (see [44, Section 4.4]), one can compute the *local monodromy matrices* $\rho(\gamma_i)$, up to conjugation. For instance, the characteristic polynomial of $\rho(\gamma_i)$ coincides with the characteristic polynomial of the matrix $\exp(2\pi\sqrt{-1}A_i)$, where A_i is the residue of $A(t)$ at $t = t_i$.

Using the presentation (1.2), any monodromy representation corresponds to a k -tuple of $n \times n$ matrices $M_1, \dots, M_k$ satisfying

$$M_1 \cdots M_k = I_n, \quad (1.3)$$

where the local monodromy computation informs us about the conjugacy class of each M_i .

To summarize, the local monodromy, which is the data of the conjugacy classes of the matrices $M_1, \dots, M_k$, can be determined by an algebraic computation. The global monodromy, i.e., the representation ρ up to isomorphism, is the data of the tuple of matrices $M_1, \dots, M_k$ satisfying (1.3), up to a *simultaneous* conjugation. Unless in special cases, we know only how to compute the global monodromy by transcendental techniques. So it is probably hopeless to look for an algebraic way to compute the global monodromy exactly. On the other hand, it still makes sense to formulate the following problem.

Problem 1.1. Given n , k , and a collection of local monodromies (i.e., conjugacy classes), determine all algebraic relations among the matrix entries of the corresponding monodromy representations that are satisfied by every representation with these local monodromies.

This is the basic idea about the character variety. Before we give a precise definition, we consider an example which can be worked out explicitly.

2. The Fricke–Klein example

2.1. Equations

This example was essentially worked out in [13]. Let $k = 4$, $n = 2$. Let $a_1, \dots, a_4 \in \mathbb{C} \setminus \{-2, 2\}$. We exclude the cases -2 and 2 because in these cases the matrices M_i do not have distinct eigenvalues, and so the Jordan forms are not completely determined by the characteristic polynomials. Let us try to parameterize all 4-tuples of 2×2 matrices $M_1, M_2, M_3, M_4 \in \mathrm{SL}_2(\mathbb{C})$, up to conjugation, satisfying

$$\prod_{i=1}^4 M_i = I_2, \quad \mathrm{tr} M_i = a_i.$$

Let $x = \mathrm{tr}(M_1 M_2)$, $y = \mathrm{tr}(M_1 M_3)$, $z = \mathrm{tr}(M_2 M_3)$. The following SAGE code computes the ideal of relations between x, y, z :

```
R.<m111,m112,m121,a1,m211,m212,m221,a2,m311,m312,m321,a3,x,y,z,a4>=QQ[]
M1=matrix(2,2,[m111,m112,m121,a1-m111])
M2=matrix(2,2,[m211,m212,m221,a2-m211])
M3=matrix(2,2,[m311,m312,m321,a3-m311])
eqns=[M1.det()-1,M2.det()-1,M3.det()-1,(M1*M2*M3).trace()-a4,(M1*M2).trace()-x, \
(M1*M3).trace()-y,(M2*M3).trace()-z]
J=R.ideal(eqns)
J.elimination_ideal([m111,m112,m121,m211,m212,m221,m311,m312,m321])
```

The computation yields the elimination ideal generated by the single cubic relation

Ideal $(a_1a_2a_3a_4 - a_1a_2x - a_1a_3y - a_2a_3z + x*y*z - a_3x*a_4 - a_2y*a_4 - a_1z*a_4 + a_1^2 + a_2^2 + a_3^2 + x^2 + y^2 + z^2 + a_4^2 - 4)$ of Multivariate Polynomial Ring in $m_{111}, m_{112}, m_{121}, a_1, m_{211}, m_{212}, m_{221}, a_2, m_{311}, m_{312}, m_{321}, a_3, x, y, z, a_4$ over Rational Field

So for each 4-tuple of matrices satisfying the conditions, the corresponding $x, y, z \in \mathbb{C}$ as above produce a solution to the so-called *Markov equation* with parameters a_1, a_2, a_3, a_4 :

$$xyz + x^2 + y^2 + z^2 - (a_1a_2 + a_3a_4)x - (a_1a_3 + a_2a_4)y - (a_1a_4 + a_2a_3)z + a_1a_2a_3a_4 + a_1^2 + a_2^2 + a_3^2 + a_4^2 - 4 = 0. \quad (2.1)$$

Conversely, suppose $x, y, z \in \mathbb{C}$ is a solution to the Markov equation. Let us see that for generically chosen values a_1, a_2, a_3, a_4 , we can always uniquely recover the tuple of matrices, up to conjugation.

First, we observe that knowing the traces $x, y, z, a_1, a_2, a_3, a_4$, we can compute the trace of an arbitrary product of matrices $M_1, \dots, M_4$ and their inverses. Indeed, given such a product, we first replace any occurrence of M_4 by $M_3^{-1}M_2^{-1}M_1^{-1}$. Then using

$$A^{-1} = \text{tr}(A)I_2 - A \quad (A \in \text{SL}_2(\mathbb{C})),$$

we can reduce to the situation when no inverses appear. So we have an arbitrary product of M_1, M_2, M_3 . Using

$$BA = \text{tr}(BA)I_2 - A^{-1}B^{-1} \quad (A, B \in \text{SL}_2(\mathbb{C}))$$

and getting rid of the inverses, we rearrange terms modulo products of shorter length until we obtain a linear combination of $I_2, M_1, M_2, M_3, M_1M_2, M_1M_3, M_2M_3, M_1M_2M_3 = M_4^{-1}$, all of whose traces are explicitly computed in terms of x, y, z and the parameters.

Knowing the traces of arbitrary products, let us find out if any 4 products form a basis of the space of 2×2 matrices. Indeed, because the trace pairing is non-degenerate, matrices $Q_1, \dots, Q_{n^2}$ form a basis of the space of $n \times n$ matrices if and only if the $n^2 \times n^2$ matrix of traces $\text{tr}(Q_i Q_j)$ is non-degenerate.

Suppose products of $M_1, \dots, M_4$ do span the space of matrices, i.e., the monodromy representation is irreducible, and we choose a basis $Q_1, \dots, Q_4$. Knowing the traces allows us to express any other product as a linear combination of $Q_1, \dots, Q_4$. In particular, we recover the ring structure on the 4-dimensional space of linear combinations of $Q_1, \dots, Q_4$, and any monodromy representation provides a $\mathbb{C}$ -linear isomorphism of this ring with the matrix ring. Any other monodromy representation with the same traces would produce a $\mathbb{C}$ -linear automorphism of the matrix ring. All $\mathbb{C}$ -linear automorphisms of the matrix ring are inner, i.e., related by a change-of-basis

transformation, by the Skolem–Noether theorem [28, p. 173], and so we conclude that the monodromy representations with the same traces are equivalent.

This was under the assumption that the monodromy representation is irreducible. If the representation is reducible, then in a suitable basis the matrices $M_1, \dots, M_4$ can be written in upper triangular form. In this situation, we cannot determine the matrices only from the trace information. Denoting the top-left diagonal entry of the upper triangular matrix M_i by λ_i , we must have $\prod_{i=1}^4 \lambda_i = 1$. On the other hand, λ_i is a root of the polynomial $t^2 - a_i t + 1$ which has 2 roots. So this kind of situation is possible only for special values of the parameters a_1, a_2, a_3, a_4 .

More precisely, we call a_1, a_2, a_3, a_4 *generic* if none of the 16 products of the roots of $t^2 - a_i t + 1$ (we pick one root from each quadratic equation) equals 1. We have seen that if the parameters are generic, then all monodromy representations are irreducible and in one-to-one correspondence with the solutions of the Markov equation (2.1).

2.2. Geometry of the Markov cubic

In algebraic geometry, given an algebraic variety X , i.e., a set of solutions to a system of polynomial equations, we are usually interested in its cohomology ring $H^*(X, \mathbb{C})$. Hodge theory extended by Deligne to non-compact and singular spaces [10, 11] endows the cohomology with *mixed Hodge structure*. Structure constants of the ring typically encode interesting enumerative information. Sometimes the cohomology ring as a ring itself can be an interesting object of study, perhaps making a connection with a ring constructed in some other way in a completely different field of mathematics.

Although the ring structure in the Fricke–Klein situation turns out to be trivial, the cohomology already has a non-trivial mixed Hodge structure. We explain how to compute it.

Assuming the parameters $a_1, a_2, a_3, a_4 \in \mathbb{C} \setminus \{-2, 2\}$ are generic, the cubic X given by the Markov equation (2.1) is non-singular. Let $\overline{X} \subset \mathbb{CP}^3$ be the solution set of the homogenization of (2.1). Its affine part is $X \subset \overline{X}$, and the complement $\overline{X} \setminus X$ is given in $\mathbb{CP}^2$ as the solution set of the homogeneous equation $xyz = 0$; thus, it is a union of three lines forming a triangle. Denote this triangle by Δ . We refer to Figure 2. It is straightforward to check that homogenization of any cubic of the form

$$xyz + x^2 + y^2 + z^2 + \text{lower degree terms}$$

does not have any singularities at infinity. So $\overline{X}$ is a smooth cubic surface and therefore can be obtained as a blowup of $\mathbb{CP}^2$ at 6 points (for background, see [21]). The cubic $\overline{X}$, as any other smooth cubic surface, contains 27 projective lines. Three of them form Δ . The intersection graph of the 27 lines is well known, and one can see

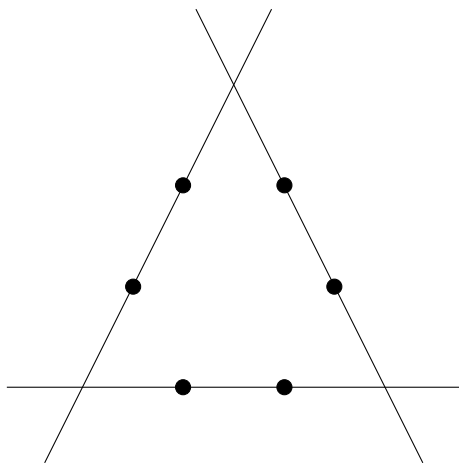


Figure 2. Markov's cubic as the complement of a triangle in the projective plane blown up at 6 points.

that for any triangle, one can choose 6 lines which do not intersect pairwise and do not contain any of the lines of the triangle. Each side of the triangle intersects 2 of the 6 lines. Blowing down these 6 lines, we obtain a map $f : \overline{X} \rightarrow \mathbb{CP}^2$. The 3 lines of Δ are isomorphically mapped to 3 lines on $\mathbb{CP}^2$. Choosing coordinates appropriately, one can assume that $f(\Delta)$ is the complement of $\mathbb{C}^* \times \mathbb{C}^* \subset \mathbb{CP}^2$. So we have

$$U = f^{-1}(\mathbb{C}^* \times \mathbb{C}^*) = \mathbb{C}^* \times \mathbb{C}^* \subset X,$$

an open subset,¹ and the complement $X \setminus U$ is a disjoint union of 6 copies of $\mathbb{C}$:

$$X = \mathbb{C}^* \times \mathbb{C}^* \sqcup \underbrace{\mathbb{C} \sqcup \cdots \sqcup \mathbb{C}}_{6 \text{ copies}}.$$

The 6 copies of $\mathbb{C}$ are divided into 3 pairs according to which of the three lines of Δ they intersect.

2.3. Cohomology

With this picture in mind, we have several ways to compute the cohomology. All cohomology groups we consider have complex coefficients. From the relative cohomology sequence for the pair $\mathbb{C}^* \times \mathbb{C}^* = U \subset X$, we obtain an exact sequence

$$\cdots \rightarrow H^1(X) \rightarrow H^1(U) \rightarrow H^2(X, U) \rightarrow H^2(X) \rightarrow H^2(U) \rightarrow \cdots \quad (2.2)$$

¹It is possible to explicitly write down the map f and correspondingly the bijection between an open subset of X and $\mathbb{C}^* \times \mathbb{C}^*$, but the resulting formulas are cumbersome and do not seem to provide any useful information for our discussion, so we postpone this to another publication.

Here, since U is homotopy equivalent to the 2-torus, we have $H^0(U) = \mathbb{C}$, $H^1(U) = \mathbb{C}^2$, $H^2(U) = \mathbb{C}$. Consider the pair U, X . The complement $X \setminus U$ is the disjoint union of 6 copies of $\mathbb{C}$. Let us denote them by $Z_1, \dots, Z_6$. Since both $\mathbb{C}$ and X are non-singular, by the tubular neighborhood theorem we can find disjoint open neighborhoods $V_i \supset Z_i$, $V_i \subset X$, such that each V_i looks like a vector bundle of real rank 2 over $Z_i \cong \mathbb{C}$. By the excision theorem, we obtain

$$H^*(X, U) = \bigoplus_{i=1}^6 H^*(V_i, V_i \setminus Z_i).$$

Using the Thom isomorphism

$$H^*(V_i, V_i \setminus Z_i) \cong H^{*-2}(Z_i)$$

and the contractibility of $Z_i \cong \mathbb{C}$, we conclude that $H^2(V_i, V_i \setminus Z_i) \cong \mathbb{C}$ and all other cohomologies vanish. So we obtain $H^2(X, U) \cong \mathbb{C}^6$ and all other cohomologies vanish. So the remaining part of the long exact sequence (2.2) vanishes, except for the isomorphism $H^0(X) \cong H^0(U) \cong \mathbb{C}$. The map $\mathbb{C}^2 = H^1(U) \rightarrow H^2(X, U) = \mathbb{C}^6$ is dual to the map $\mathbb{C}^6 \cong H_2(X, U) \rightarrow H_1(U) \cong \mathbb{C}^2$. Here, a point class on $Z_i \cong \mathbb{C}$ corresponds to a disc class in $H_2(X, U)$ whose boundary is a 1-cycle in U which gets contractible in $U \cup Z_i$. From Figure 2, we see that all 1-cycles in U become contractible in X , so the map $H_2(X, U) \rightarrow H_1(U)$ is surjective, the dual map is injective, and we have $H^1(X) = 0$, while $H^2(X)$ is an extension of $H^2(U) = \mathbb{C}$ by $H^2(X, U)/H^1(U) \cong \mathbb{C}^4$.

This is reflected in the weight filtration on $H^2(X)$. As a mixed Hodge structure, the subspace $H^2(X, U)/H^1(U) \subset H^2(X)$ is pure of weight 2, while $H^2(U)$ has weight 4. Thus, we have $W_2 = H^2(X, U)/H^1(U) \subset H^2(X)$ with $\dim W_2 = 4$, $W_3 = W_2$, and $W_4 = H^2(X)$ with $\dim W_4/W_2 = 1$.

Another way to compute the cohomology is to use the compactification. Let us denote $X_0 = \overline{X}$, $X_1 = X_0 \setminus \mathbb{CP}^1$, $X_2 = X_0 \setminus (\mathbb{CP}^1 \cup \mathbb{CP}^1)$, $X_3 = X_0 \setminus \Delta$, so that we pass from X_i to X_{i+1} by removing a line. This leads to exact sequences for each i :

$$\begin{aligned} 0 \rightarrow H^1(X_{i+1}) \rightarrow H^2(X_i, X_{i+1}) \rightarrow H^2(X_i) \rightarrow H^2(X_{i+1}) \rightarrow H^3(X_i, X_{i+1}) \rightarrow 0 \\ 0 \rightarrow H^3(X_{i+1}) \rightarrow H^4(X_i, X_{i+1}) \rightarrow H^4(X_i) \rightarrow H^4(X_{i+1}) \rightarrow 0, \end{aligned}$$

as we will see that the odd cohomologies of X_i vanish at each step. At each step, we will use that $H^k(X_i, X_{i+1}) = H^{k-2}(X_i \setminus X_{i+1})$ by the combination of the tubular neighborhood theorem, excision, and the Thom isomorphism, as detailed in the previous argument. Since $X_i \setminus X_{i+1}$ is $\mathbb{CP}^1$, $\mathbb{C}$ or $\mathbb{C}^*$, the latter cohomology is non-zero only for $k = 2, 3, 4$. Initially, we have $H^0(X_0) = \mathbb{C}$, $H^2(X_0) = \mathbb{C}^7$ (represented by 7 algebraic cycles: the line in $\mathbb{CP}^2$ and the 6 exceptional curves), $H^4(X_0) = \mathbb{C}$, all other cohomologies vanish.

For $i = 0$, we have $X_0 \setminus X_1 = \mathbb{CP}^1$, which has $H^0 = \mathbb{C}$ and $H^2 = \mathbb{C}$. These coincide with $H^2(X_0, X_1)$ and $H^4(X_0, X_1)$, respectively, and we see that in $H^2(X_1)$ (resp. $H^4(X_1)$), we have one less dimension than in $H^2(X_0)$, respectively, $H^4(X_0)$. So $H^4(X_1) = 0$, $H^2(X_1) = \mathbb{C}^6$.

For $i = 1$, we have $X_1 \setminus X_2 = \mathbb{C}$, which has only $H^0 = \mathbb{C}$. So X_2 has one dimension less in $H^2(X_2)$, while nothing else changes: $H^2(X_1) = \mathbb{C}^5$.

Finally, for $i = 2$, we have $X_2 \setminus X_3 = \mathbb{C}^*$, which has $H^0 = \mathbb{C}$ and $H^1 = \mathbb{C}$, which coincide with $H^2(X_2, X_3)$ and $H^3(X_2, X_3)$ correspondingly. Thus, two things happen. First, only 4 dimensions from $H^2(X_2)$ out of 5 contribute to $H^2(X_3)$. On the other hand, one extra dimension comes from $H^3(X_3, X_2)$. The net total is $H^2(X_3) = \mathbb{C}^5$, as in X_2 , but the nature of the 5 dimensions is different, and correspondingly the weights are different. The 4 dimensions coming from $H^2(X_2)$ have weight 2, and this is reflected by the fact that they are represented by algebraic cycles. The remaining class has weight 4 as it comes from $H^3(X_3, X_2)$. The weight of $H^1(\mathbb{C}^*)$ is 2, but it gets increased by 2 when we pass to $H^3(X_3, X_2)$.

Yet another approach to describe the cohomology is by using differential forms and Chern classes. Notice that the moduli space X carries four natural line bundles $L_1, \dots, L_4$. Each point $p \in X$ corresponds to a 4-tuple of 2×2 matrices $M_1, \dots, M_4$. For each i , the matrix M_i has two one-dimensional eigenspaces, denoted E_i, E'_i . Letting p vary, one can show that the family of one-dimensional vector spaces $\text{Hom}_{\mathbb{C}}(E_i, E'_i)$ is locally trivial and therefore produces a line bundle, denoted L_i . Its Chern class $c_1(L_i)$ is an element of $H^2(X)$. These Chern classes $c_1(L_i)$ span $W_2 H^2(X)$. To obtain one more class, we use the differential form

$$\omega = \frac{dx}{x} \wedge \frac{dy}{y}$$

on $U \cong \mathbb{C}^* \times \mathbb{C}^*$, written in the natural coordinates x, y on $\mathbb{C}^* \times \mathbb{C}^*$. Let us verify that ω extends to the whole of X . It is sufficient to consider the neighborhood of one of the exceptional lines. The blowup is locally described by the equation

$$x - 1 = zy,$$

and performing the substitution, we obtain

$$\omega = \frac{dz \wedge dy}{yz + 1},$$

which extends to $y = 0$ in a neighborhood of the line $\{(y, z) \in \mathbb{C}^2 \mid y = 0\}$.

These findings can be summarized in Table 1.

H^*	$W_i H^i$	$W_{i+2} H^i / W_i H^i$
H^2	$\mathbb{C}^4 = \langle c_1(L_i) \mid i = 1, 2, 3, 4 \rangle$	$\mathbb{C} = \langle \omega = \text{symplectic form} \rangle$
H^0	$\mathbb{C} = \langle 1 \rangle$	

Table 1. Cohomology of the Markov cubic.

2.4. Mixed Hodge polynomial

The numerical information about cohomology is packaged in the *Poincaré polynomial*

$$\sum_i q^i \dim H^i(X).$$

In the case of an algebraic variety, the information about the weight filtration is encoded in the *mixed Hodge polynomial*²

$$P(X; q, t) = \sum_{i,j} q^i t^j \dim(W_j H^i(X) / W_{j-1} H^i(X)),$$

so that the Poincaré polynomial is its $t = 1$ specialization. From Figure 1, we obtain that the Poincaré polynomial of X is $1 + 5q^2$ and the mixed Hodge polynomial is $P(q, t) = 1 + 4q^2 t^2 + q^2 t^4$. We prefer to consider a slightly different version of the mixed Hodge polynomial, which encodes the same information. Let

$$\text{MH}(X; q, t) := q^{\frac{1}{2} \dim_{\mathbb{C}} X} P(X; q^{-\frac{1}{2}} t^{-\frac{1}{2}}, t^{\frac{1}{2}}).$$

For instance, in our running example, we obtain

$$\text{MH}(X; q, t) = q + t + 4.$$

We should also compare our calculations with the number of points on our variety over the finite field $\mathbb{F}_q$. Assuming all the lines are defined over the base field, we see that the first computation produces

$$(q - 1)^2 + 6q = q^2 + 4q + 1.$$

Here, $\mathbb{C}^* \times \mathbb{C}^*$ gets replaced by $\mathbb{F}_q^* \times \mathbb{F}_q^*$, which has size $(q - 1)^2$, and each of the six lines $\mathbb{C}$ becomes $\mathbb{F}_q$, which has size q . The second computation produces

$$q^2 + 7q + 1 - (q + 1) - q - (q - 1) = q^2 + 4q + 1.$$

²Normally, the mixed Hodge polynomial is defined as a polynomial in three variables. The third variable additionally records the Hodge decomposition of the pure Hodge structure $W_j H^i(X) / W_{j-1} H^i(X)$. In our situation, however, this decomposition is typically trivial, i.e., the pure Hodge structure is Tate.

Here, the number of points on $\overline{X}$, a smooth cubic surface, is $q^2 + 7q + 1$. We remove subsequently $\mathbb{P}^1 \mathbb{F}_q$ of size $q + 1$, $\mathbb{F}_q$ of size q , and $\mathbb{F}_q^*$ of size $q - 1$. In each case, one can track how the different powers of q contribute to the cohomology so that q^2 corresponds to H^0 and $4q + 1$ corresponds to H^2 .

In any case, for a general algebraic variety X , if the number of points is a polynomial of q , then the counting polynomial coincides with the E -polynomial, as shown by N. Katz in the appendix of [24]. Then we have

$$|X(\mathbb{F}_q)| = q^{\dim_{\mathbb{C}} X} P(X; -1, q^{-\frac{1}{2}}) = q^{\frac{1}{2} \dim_{\mathbb{C}} X} \text{MH}(X; q, q^{-1}), \quad (2.3)$$

assuming MH is indeed a polynomial in q, t , which means that all the weights are even, and the cohomology is concentrated in even degrees.

3. Setup

After reviewing in detail the basic example, we turn to the general setup and precise definitions. There is no reason to consider only genus 0 complex curves, and the classical motivation directly extends to the more general case; for instance, ODEs can be studied on complex curves of arbitrary genus. So we consider a complex curve of genus g with k punctures.

Definition 3.1. Fix $g \geq 0, n, k \geq 1$ and Jordan forms $J_1, J_2, \dots, J_k \in \text{GL}_n$. The associated genus g character variety $X = \text{Char}_{g; J_1, \dots, J_k}$ is the quotient space

$$\begin{aligned} \{ (A_1, \dots, A_g, B_1, \dots, B_g, M_1, \dots, M_k) : M_i \sim J_i, \\ A_1 B_1 A_1^{-1} B_1^{-1} \cdots A_g B_g A_g^{-1} B_g^{-1} M_1 \cdots \\ M_k = I \} / \text{PGL}_n(\mathbb{C}), \end{aligned}$$

where $\text{PGL}_n(\mathbb{C})$ acts by simultaneously conjugating all the matrices.

In general, the quotient space is an Artin stack, but we will only consider generic conjugacy classes in which case the quotient is a smooth affine algebraic variety.

Definition 3.2 ([22]). The data $J_1, \dots, J_k$ are *generic* if

- (1) $\prod_{i=1}^k \det J_i = 1$
- (2) for every $0 < n' < n$ and every block decomposition $J_i \sim \begin{pmatrix} J'_i & J''_i \\ 0 & J'''_i \end{pmatrix}$ with $J'_i \in \text{GL}_{n'}$, we have $\prod_{i=1}^k \det J'_i \neq 1$.

Theorem 3.3 ([22]). *For generic data $n, k, J_1, \dots, J_k$, the quotient space*

$$X = \text{Char}_{g; J_1, \dots, J_k}$$

has the structure of a smooth affine algebraic variety.

In fact, considering functions on X of the form

$$\text{tr}(M_{i_1} \text{ or } A_{i_1} \text{ or } B_{i_1} \cdots M_{i_m} \text{ or } A_{i_m} \text{ or } B_{i_m}),$$

we can realize X as an algebraic subset of $\mathbb{C}^N$ for sufficiently large N .

Basically, the same reasoning as in the Fricke–Klein situation can be applied in general. However, the number of traces one needs to consider is uncontrollably large, and it seems hopeless to be able to produce all equations by the elimination computation. Even if we do obtain a complete list of equations, it is not clear which interesting information we should be able to extract from such a list. For instance, equations in the case of genus 0, rank 2, five punctures are given in [42, Proposition 2.1]. We do hope, however, that computing the cohomology ring is an achievable goal.

Problem 3.4. Compute the cohomology ring $H^*(X, \mathbb{C})$.

4. What is known?

Having defined our basic object of study, we explain some important structural results.

4.1. Non-abelian Hodge correspondence

The non-abelian Hodge correspondence, pioneered by Hitchin [26] and developed by Simpson [39–41], is a construction of a remarkable bijection between local systems and stable Higgs bundles, which induces a diffeomorphism between the character variety and the moduli space that parameterizes stable Higgs bundles.

The construction of the character variety only depends on the topological type of the Riemann surface. Now we need to view the Riemann surface with punctures as a compact algebraic curve C over $\mathbb{C}$ with marked points. Then a *Higgs bundle* is a pair $(\mathcal{E}, \theta)$, where $\mathcal{E}$ is an algebraic vector bundle on C and θ a *Higgs field*, which is a twisted matrix-valued function on C of a certain kind. More precisely, $\theta : \mathcal{E} \rightarrow \mathcal{E} \otimes \omega_C(D)$, where $D = \sum_{i=1}^k [t_i]$ is the divisor on C corresponding to the marked points $t_1, \dots, t_k$. Explicitly, this means that on an open subset of C where $\mathcal{E}$ is trivialized, θ is a matrix whose entries are algebraic differential forms on C with simple residues at the marked points.

Furthermore, these Higgs bundles are required to be endowed with *parabolic structures* at the marked points. This is simply a choice of a partial flag at the fiber

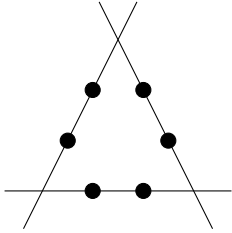
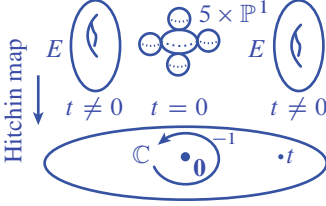
Local systems	Higgs bundles
$\pi_1(C) \rightarrow \mathrm{GL}_n(\mathbb{C})$	$(\mathcal{E}, \theta)$: $\mathcal{E}$ algebraic vector bundle on C , $\theta : \mathcal{E} \rightarrow \mathcal{E} \otimes \omega_C$ Higgs field
punctures, J_i	parabolic structure, stability
X_B “Betti” moduli space	$\cong X_{\mathrm{Dol}}$ “Dolbeault” moduli space
	
affine algebraic variety	Hitchin system $X_{\mathrm{Dol}} \rightarrow \mathbb{C}^{\mathrm{dimension}/2}$
cohomology ring $H^*(X_B, \mathbb{C})$	the same ring
weight “ W ” filtration $W_* H^*$	perverse “ P ” filtration

Figure 3. Non-abelian Hodge correspondence.

of $\mathcal{E}$ at each marked point, preserved by the residue $\mathrm{res}_{t_i} \theta$, which acts on that fiber by a linear transformation. The steps of the partial flag precisely match the dimensions of the generalized eigenspaces of the Jordan forms J_i . The data of a Higgs bundle with a parabolic structure have to satisfy a *stability condition*. The data of the eigenvalues and the Jordan blocks of J_i translate into *parabolic weights* defining the stability condition, and into the Jordan form of $\mathrm{res}_{t_i} \theta$ acting on the subquotients of the flag. Details are given in [39].

The moduli space of Higgs bundles in this context is referred to as the *Dolbeault moduli space* X_{Dol} , while the character variety is called the *Betti moduli space* X_B . The non-abelian Hodge correspondence provides a homeomorphism between X_B and X_{Dol} . In particular, their cohomology rings are isomorphic. We will use X when it is not important whether we take X_B or X_{Dol} . We summarize features appearing on both sides of this correspondence in the table in Figure 3.

4.2. Generators of $H^*(X)$

Generators of the cohomology ring are known. There is a general argument in [29] concerning moduli spaces of sheaves on complex surfaces. With a little modification, it can be applied to the situation of parabolic Higgs bundles. Basically speaking, certain Chern class of the complex of vector bundles on $X_{\mathrm{Dol}} \times X_{\mathrm{Dol}}$ computing the derived Hom between Higgs bundles is a multiple of the diagonal class. Moreover, there exists a compactification $\overline{X_{\mathrm{Dol}}} \supset X_{\mathrm{Dol}}$ such that the complex extends to $X_{\mathrm{Dol}} \times \overline{X_{\mathrm{Dol}}}$, and the Chern class remains a multiple of the diagonal class. Thus,

the identity map $H^*(X_{\text{Dol}}) \rightarrow H^*(X_{\text{Dol}})$ can be written in terms of the tautological classes.

Consider the tautological bundle³ $\mathcal{E} \rightarrow X_{\text{Dol}} \times C$. For each $i \in \{0, 1, 2\}$, $\xi \in H^i(C)$, and $m \in \mathbb{Z}_{\geq 0}$, define

$$e_m(\xi) := (\pi_X)_*(c_m(\mathcal{E}) \cup \xi) \in H^{2m+i-2}(X_{\text{Dol}}). \quad (4.1)$$

Here, $c_m(\mathcal{E}) \in H^{2m}(X_{\text{Dol}} \times C)$ is the Chern class and π_X is the projection $X_{\text{Dol}} \times C \rightarrow X_{\text{Dol}}$. If we have a marked point t_j , then each point of X_{Dol} gives a flag $0 = F_{j0} \subset F_{j1} \subset \cdots \subset F_{jr_j} = \mathcal{E}(t_j)$. Each subquotient of the flag produces a vector bundle on X_{Dol} , and we take its Chern classes $c_{m,j,\ell} := c_m(F_{j\ell}/F_{j\ell-1})$.

Theorem 4.1. *The cohomology ring $H^*(X_{\text{Dol}})$ is generated by the classes $e_m(\xi)$ and $c_{m,j,\ell}$, where ξ ranges over a basis of the cohomology ring $H^*(C)$.*

The cohomology ring of the Riemann surface C has a basis $1 \in H^0(C)$, $\text{pt} \in H^2(C)$, and $\sigma_1, \dots, \sigma_{2g} \in H^1(C)$ with the product structure given by $\sigma_i \sigma_{i+g} = -\sigma_{i+g} \sigma_i = \text{pt}$, and all other $\sigma_i \sigma_j$ being zero.

4.3. Explicit relations of $H^*(X_{\text{Dol}})$

Explicit relations of $H^*(X_{\text{Dol}})$ are known for $n = 2$, $k = 1$, $J_1 = -I$, and arbitrary g [25]. The cohomology is generated by certain elements $\varepsilon_1, \dots, \varepsilon_{2g} \in H^1(X_{\text{Dol}})$, $\alpha \in H^2(X_{\text{Dol}})$, $\psi_1, \dots, \psi_{2g} \in H^3(X_{\text{Dol}})$, and $\beta \in H^4(X_{\text{Dol}})$. These are certain multiples of $e_1(\sigma_i)$, $e_2(1)$, $e_2(\sigma_i)$, and $e_2(\text{pt})$. The elements ε_i generate a copy of the cohomology ring of the Jacobian of C , and we have

$$H^*(X_{\text{Dol}}) = H^*(\text{Jac}(C)) \otimes \bigoplus_{i=0}^g \wedge_0^i(\psi) \otimes \mathbb{C}[\alpha, \beta, \gamma] / J_i^{g-i},$$

where $J_i^{g-i} \subset \mathbb{C}[\alpha, \beta, \gamma]$ is the ideal generated by an explicit list of polynomials in α, β, γ . Here, $\gamma = -2 \sum_{i=1}^g \psi_i \psi_{i+g}$ and $\wedge_0^i(\psi)$ denotes the primitive part in the space of i -fold products of ψ_j -classes.

³In fact, the tautological bundle is often not well defined as a vector bundle. It is well defined as a projective bundle, and the “determinant” $\det \mathcal{E}$ is also well defined, but the determinant might have no n -th root and correspondingly $c_1(\det \mathcal{E}) \in H^2(X_{\text{Dol}}, \mathbb{Z})$ may not be divisible by n . Since we are dealing with the cohomology with complex coefficients, this does not create a problem, and all the Chern classes are still well defined.

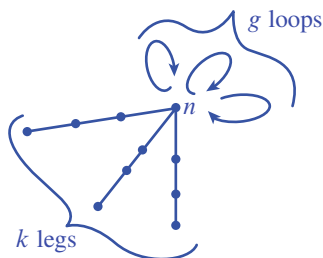


Figure 4. Comet-shaped quiver with k legs and g loops.

4.4. Weight filtration

Since X_B is an algebraic variety, its cohomology is endowed with the weight filtration constructed by Deligne in [10, 11]. The weight filtration on $H^*(X_B)$ is very easily described in terms of the tautological generators. The following was shown in [38] for the case without parabolic points and in [32] in a more general case.

Theorem 4.2. *Define the weight of the tautological generators $e_m(\xi)$ and $c_{m,j,\ell}$ to be $2m$. Then $W_m H^*(X_B)$ is spanned by products of tautological generators of total weight $\leq m$.*

4.5. The pure part

The pure part $H_{\text{pure}}^*(X_B) = \bigoplus_i W_i H^i(X_B)$ is naturally a subring of $H^*(X_B)$. In [22], it was conjectured that the Poincaré polynomial of $H_{\text{pure}}^*(X_B)$ matches that of the quiver variety associated to the corresponding comet-shaped quiver with g loops and k legs whose lengths depend on the conjugacy classes J_i ; see Figure 4.

4.6. $P = W$

Computing the characteristic polynomial of $\theta : \mathcal{E} \rightarrow \mathcal{E} \otimes \omega_C(D)$ at all points of C produces a well-defined polynomial map $X_{\text{Dol}} \rightarrow \mathbb{C}^{\dim_{\mathbb{C}} X/2}$ called the *Hitchin map*, which is known to be proper. This induces a filtration $P_* H^*(X_{\text{Dol}})$ on $H^*(X_{\text{Dol}})$ called the *perverse Leray filtration* [9]. According to the recently proved $P = W$ conjecture [8], we have the following theorem.

Theorem 4.3 ([23, 30]). *The following equality holds:*

$$P_i = W_{2i}.$$

4.7. $q - t$ symmetry

Recall the mixed Hodge polynomial $q + t + 4$ from Section 2.4. Notice that it is symmetric in the variables q and t . The mixed Hodge polynomials were conjectured

to be symmetric in q, t in general in [22]. This was called the *curious hard Lefschetz* property. Assuming $P = W$, the curious hard Lefschetz property is a consequence of the *relative hard Lefschetz theorem* on the X_{Dol} -side. We have the following theorem.

Theorem 4.4 ([32]). *The cohomology $H^*(X_B)$ satisfies curious hard Lefschetz:*

$$\omega^i : \text{Gr}_{\dim_{\mathbb{C}} X - 2i}^W H^j(X_B) \rightarrow \text{Gr}_{\dim_{\mathbb{C}} X + 2i}^W H^{j+2i}(X_B)$$

is an isomorphism, where ω is the class of a certain explicit holomorphic 2-form.

In fact, the proof of $P = W$ from [30] relies on this result. On the other hand, the proof of $P = W$ in [23] is independent of it and therefore provides another proof of the curious hard Lefschetz.

Similar curious hard Lefschetz property holds for a certain version of the Khovanov homology of knots (conjectured in [12], proved in [16]).

4.8. Partition function

Perhaps the biggest problem in the field that still remains open is the conjectural formula for the generating function of the mixed Hodge polynomials of all character varieties given in [22] in terms of a partition function. Here, k (the number of marked points) and g are fixed. For each $i = 1, \dots, k$, let $\mathbf{x}_i := (x_{i1}, x_{i2}, \dots)$ be an infinite sequence of variables. Suppose we have a sequence $\boldsymbol{\mu}_i := (\mu_{i1}, \mu_{i2}, \dots)$ of non-negative integers with $\sum_j \mu_{ij} = n$. To this we associate a conjugacy class J_{μ_i} of the diagonal matrix with eigenvalues having multiplicities $\mu_{i1}, \mu_{i2}, \dots$. For a tuple of such sequences $\boldsymbol{\mu} = (\mu_1, \dots, \mu_k)$, we choose the eigenvalues arbitrarily so that the conjugacy classes satisfy the genericity condition (Definition 3.2) and consider the corresponding character variety $X_{g; J_{\mu_1}, \dots, J_{\mu_k}}$. We take its mixed Hodge polynomial (Section 2.4) with weight

$$\mathbf{x}^{\boldsymbol{\mu}} := \prod_{i=1}^k \prod_j x_{ij}^{\mu_{ij}}.$$

The result is a symmetric function in each sequence of variables $\mathbf{x}_i$ whose coefficients are polynomials in q and t :

$$\text{MH}_{g,k}(\mathbf{x}_1, \dots, \mathbf{x}_k; q, t) := \sum_{n=1}^{\infty} \sum_{\mu_1, \dots, \mu_k \vdash n} \text{MH}(X_{g; J_{\mu_1}, \dots, J_{\mu_k}}; q, t) \mathbf{x}^{\boldsymbol{\mu}}.$$

On the other hand, we define the *partition function*

$$\Omega_{g,k}(\mathbf{x}_1, \dots, \mathbf{x}_k; q, t) := \sum_{n=0}^{\infty} \sum_{\lambda \vdash n} \tilde{H}_{\lambda}(\mathbf{x}_1; q, t) \cdots \tilde{H}_{\lambda}(\mathbf{x}_k; q, t) \cdot \prod_{\square \in \lambda} \frac{(q^{a(\square)+1/2} - t^{\ell(\square)+1/2})^{2g}}{(q^{a(\square)} - t^{\ell(\square)+1})(q^{a(\square)+1} - t^{\ell(\square)})}, \quad (4.2)$$

where λ goes over all partitions (Young diagrams), $\tilde{H}_{\lambda}(\mathbf{x}_k; q, t)$ denotes the *modified Macdonald polynomial*, the product goes over the cells of the Young diagram, and a, ℓ denote the usual arms and legs statistics (see [15]).

Conjecture 4.5 ([22]). *The partition function is related to the mixed Hodge polynomials as follows:*

$$\Omega_{g,k} = \text{Exp}\left(-\frac{\text{MH}_{g,k}}{(q-1)(t-1)}\right). \quad (4.3)$$

Here, Exp denotes the *plethystic exponential*. It can be defined for any expression involving some variables $a, b, c, \dots$ as follows:

$$\text{Exp}(f(a, b, c, \dots)) := \exp\left(\sum_{n=1}^{\infty} \frac{f(a^n, b^n, c^n, \dots)}{n}\right).$$

If $a, b, c, \dots$ are monomials in some variables, then we have

$$\text{Exp}(a + b + c + \dots) = \frac{1}{1-a} \frac{1}{1-b} \frac{1}{1-c} \cdots,$$

so plethystic exponential can be thought of as the conversion between infinite sums and infinite products. There is an inverse operation Log called the *plethystic logarithm*. So if we focus on computing the mixed Hodge polynomials from (4.3), we can write

$$\text{MH}_{g,k} = -(1-q)(1-t) \text{Log } \Omega_{g,k}. \quad (4.4)$$

4.9. What is known?

Conjecture 4.5 was motivated by a computation performed in [22]. It was shown that its implication for the number of $\mathbb{F}_q$ points is correct (see (2.3)), by counting representations $\pi_1(C) \rightarrow \text{GL}_n(\mathbb{F}_q)$ using the character theory of the finite group $\text{GL}_n(\mathbb{F}_q)$.

Theorem 4.6 ([22]). *The number of points over $\mathbb{F}_q$ ($t = q^{-1}$ specialization) deduced from the conjectural equality (4.3) is correct.*

Initially, in identity (4.4), it was not at all clear that the right-hand side has coefficients which are polynomials in q, t , as they should be because the mixed Hodge polynomials on the left-hand side are polynomials by definition. Using a new identity in the combinatorics of Macdonald polynomials established in [14], the author first proved the following theorem.

Theorem 4.7 ([31]). *The coefficients of $(q - 1)(t - 1) \text{Log } \Omega$ are polynomials in q and t .*

Then, together with an approach to count Higgs bundles over $\mathbb{F}_q$ by Schiffmann [36, 37] and a new relationship between the Macdonald polynomials and the affine Springer fibers, the author proved the following theorem.

Theorem 4.8 ([33, 34]). *The Poincaré polynomials ($t = 1$ specialization) deduced from the conjectural equality (4.3) are correct.*

With the help of this theorem, there is a following line of attack on Conjecture 4.5. Suppose we managed to construct a “candidate model” M for the cohomology ring of the character variety so that it comes with a surjective map $M \rightarrow H^*(X)$. Showing surjectivity of the map is plausible because the generators of $H^*(X)$ are known. If we furthermore manage to show that the dimension of M is exactly as predicted by Conjecture 4.5, then by Theorem 4.8 the map $M \rightarrow H^*(X)$ must be an isomorphism. Partial progress in this direction is discussed in the next sections.

5. COHAs/Coulomb branches

In order to explain our approach, we need to introduce some ideas.

5.1. Cohomological Hall algebra (COHA)

Denote by $\mathcal{H}_2$ the Lie algebra of Hamiltonian vector fields on $\mathbb{C}^2$. Here, $\mathbb{C}^2$ is viewed as a symplectic variety with symplectic form $dx \wedge dy$, where x and y are the natural coordinates. Symplectic geometry (for a quick introduction, see, e.g., [1, Chapter 8]) associates to any function $f(x, y)$ the *Hamiltonian vector field*

$$H_f := \frac{\partial f}{\partial y} \frac{\partial}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial}{\partial y}.$$

The Lie bracket of vector fields is compatible with the Poisson bracket of functions:

$$[H_f, H_g] = H_{\{f, g\}}, \quad \{f, g\} = \frac{\partial f}{\partial y} \frac{\partial g}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial g}{\partial y}.$$

The vector space of functions becomes a Lie algebra with respect to the Poisson bracket, and it is this Lie algebra we consider.

Explicitly, it is the Lie algebra with basis elements $T_{r,s}$ for each pair $r, s \geq 0$, corresponding to the functions $x^r y^s$. The Lie bracket is given as follows:

$$[T_{r,s}, T_{r',s'}] = (r's - rs')T_{r+r'-1, s+s'-1} \quad (r, s \geq 0).$$

The tensor product $\mathcal{H}_2 \otimes H^*(C)$ is naturally a super-Lie algebra. Let $H^*(X)[x, y]$ be the cohomology ring with two formal variables x, y adjoined. Let n and d be the rank and the degree of the Higgs bundles parameterized by the moduli space X , and let $\chi := d + (1 - g)n$ be the Euler characteristic.

Theorem 5.1 ([23, 35]). *The Lie algebra $\mathcal{H}_2 \otimes H^*(C)$ acts on $H^*(X)[x, y]$ in such a way that $T_{0,m} \otimes \xi$ satisfies*

$$\begin{aligned} ((T_{0,m} \otimes \xi)f)(0, 0) &= \psi_m(\xi)(f) \\ &:= (p_m(\xi) + m(1 - g)p_{m-1}(\xi \cup \text{pt})) \cup f \quad (f \in H^*(X)), \end{aligned}$$

and $T_{1,0} \otimes \text{pt}$, $T_{1,0} \otimes 1$, $T_{0,1} \otimes \text{pt}$, $T_{0,1} \otimes 1$ act as x , $-n \frac{\partial}{\partial y}$, y , $n \frac{\partial}{\partial x} + \chi$, respectively.

Here, $p_m(\xi)$ is defined similarly to $e_m(\xi)$ in (4.1), except that we use the power sums of the Chern roots, i.e., $m!$ times the m -th component of the Chern character. Thus, for instance, the central operator $T_{0,0} \otimes \text{pt}$ acts via the multiplication by n , the rank of the Higgs bundles. The operator $T_{0,1} \otimes 1$ commutes with y and acts on elements of $H^*(X)[y]$ via the multiplication by $d + (1 - g)n = \chi$.

Note that quadratic Hamiltonians correspond to the linear action of $\text{SL}(2, \mathbb{C})$ on $\mathbb{C}^2$, and so the elements $T_{2,0}$, $T_{1,1}$, and $T_{0,2}$ span a copy of $\mathfrak{sl}_2 \subset \mathcal{H}_2$. In fact, it is this $\mathfrak{sl}_2$ that was matched with the relative Lefschetz $\mathfrak{sl}_2$ in [23] in order to pin down the perverse filtration algebraically.

5.2. Coulomb branch

The construction described in [3] associates to a pair of a reductive group G and its representation N an affine Poisson algebraic variety called the Coulomb branch. We are interested in the situation where $G = \text{GL}_n \times \mathbb{C}^*$ acts on the space of matrices $N = \mathbb{C}^{n \times n}$: GL_n acts by conjugation and $\mathbb{C}^*$ by the scalar multiplication.

Then Higgs bundles can be seen as maps from the curve C to the stack N/G . This is because, more generally, more or less by definition, maps from some space X to some quotient stack Y/G are pairs of a principal G -bundle X' on X and a G -equivariant map $X' \rightarrow Y$. A principal G -bundle on X is a pair of a line bundle L and a rank n bundle E , and an equivariant map $X' \rightarrow N$ corresponds to a linear map

$E \otimes L \rightarrow E$. So in the case when L is the tangent bundle of C , we can view Higgs bundles as maps $C \rightarrow N/G$.

Our hope is that the Coulomb branch has something to do with the cohomology of our moduli spaces. This is the situation of the *adjoint representation with flavor*, and the Coulomb branch is related to the Hilbert scheme [4] of $\mathbb{C}^* \times \mathbb{C}$, which is a symplectic resolution of singularities of $(\mathbb{C}^* \times \mathbb{C})^n/S_n$. It is not entirely clear why one should pass from the Hilbert scheme of $\mathbb{C}^* \times \mathbb{C}$ to that of $\mathbb{C}^2$, but our construction in [23] in fact first produces an algebra of Hamiltonian fields on $\mathbb{C}^* \times \mathbb{C}$, and only then it degenerates to the corresponding algebra for $\mathbb{C}^2$. Hopefully, this passage will be clarified by experts in Coulomb branches.

Now suppose we are on $\text{Hilb}^n(\mathbb{C}^2)$. Then $\mathcal{H}_2$ acts on many natural objects such as functions, differential forms, etc. So even if we do not completely understand the Coulomb branch picture, it is a natural guess to look for our model of the cohomology ring among natural objects living on the Hilbert scheme.

5.3. Geometric engineering

An indication similar to the one above is given in [7], where a physical explanation of Conjecture 4.5 is provided via interpreting the left-hand side (the partition function) as

$$\Omega = \chi(\text{Hilb}_n(\mathbb{C}^2), P^{\otimes k} \otimes (\Omega^*)^{\otimes g} \otimes \mathcal{O}(g-1)). \quad (5.1)$$

Here, P is the Procesi bundle (S_n -equivariant bundle coming from Haiman's identification $\text{Hilb}_n(\mathbb{C}^2) = S_n - \text{Hilb}(\mathbb{C}^{2n})$), Ω^* is the bundle of differential forms, and χ denotes the equivariant Euler characteristic.

5.4. Proposal

One problem with (5.1) is that the right-hand side does not always have positive coefficients. So the vector bundle on the right-hand side of (5.1) has higher cohomologies. Thus, there seems to be no way to upgrade identity (4.3) to an isomorphism between vector spaces using (5.1). One can trace the problem to the fact that the factor $-\frac{1}{(1-q)(1-t)}$ is manifestly not positive, when expanded in q and t . To resolve this issue, we propose to expand in q^{-1} and t instead. The geometric meaning of this expansion is different from taking the global sections. Instead, we should take the local cohomology with respect to a Lagrangian subvariety.

Theorem 5.2 (A. Mellit, M. Romero). *Let $L \subset \text{Hilb}_n(\mathbb{C}^2)$ be the Lagrangian cut out by the equations $\sum_{j=1}^n x_j^r = 0$ ($r > 0$). Let $\mathcal{E}$ be any vector bundle on $\text{Hilb}_n(\mathbb{C}^2)$. Then for any $i \neq n$, we have*

$$H_L^i(\text{Hilb}_n(\mathbb{C}^2), \mathcal{E}) = 0.$$

Here, H_L^i is the functor of local cohomology (see [20]), the derived functor of taking sections supported at L . Then we expect that (4.3) has the following interpretation.

Conjecture 5.3. *The local cohomology is related to the Borel–Moore homology of a certain Higgs moduli stack as follows:*

$$H_L^n(P^{\otimes k} \otimes (\Omega^*)^{\otimes g} \otimes O(g-1))^\vee \cong H_*^{\text{BM}}(\text{Higgs moduli stack}).$$

Here, it is not clear which precise Higgs moduli stack should be taken on the right-hand side. The whole Higgs moduli stack has infinitely many irreducible components, and the expectation is that we need to take a nice open substack containing only finitely many components in each degree, possibly imposing a bound on the slope. For a bigraded vector space M , we denote by $M^\vee$ the bigraded dual vector space.

6. Main example

A special case can be worked out in full detail. We take $g = 0$, i.e., $C = \mathbb{P}^1$, and consider the $k + 1$ -twisted Higgs bundles, so that the Higgs field is a morphism $\mathcal{E} \rightarrow \mathcal{E}(k-1)$. Equivalently, we may consider $k + 1$ marked points without any parabolic structures and conditions on the residue of the Higgs field. The corresponding specialization of the partition function is given by

$$\Omega_k(z; q, t) := \sum_{n=0}^{\infty} \sum_{\lambda \vdash n} \frac{(-1)^{(k+1)|\lambda|} (q^{n(\lambda')} t^{n(\lambda)})^{k+1}}{\prod_{\square \in \lambda} (q^{a(\square)} - t^{\ell(\square)+1}) (q^{a(\square)+1} - t^{\ell(\square)})} z^n. \quad (6.1)$$

We use the variable z to keep track of the rank. In (4.2), we did not need it because the rank of each term can be extracted as the degree of the symmetric function in each set of variables. The first terms of Ω look as follows:

$$1 + \frac{(-1)^k}{(1-t)(1-q)} z + \frac{q^{k+1}(1-t^2) - t^{k+1}(1-q^2)}{(q-t)(1-q)(1-q^2)(1-t)(1-t^2)} z^2 + \dots,$$

and the first terms of its plethystic logarithm read

$$\text{Log } \Omega_k(z; q, t) = -\frac{1}{(1-q)(1-t)} \left((-1)^{k+1} z + z^2 \sum_{\substack{i \geq 0, j \geq 0, i+j \leq k-2, \\ i+j \equiv k \pmod{2}}} q^i t^j + \dots \right). \quad (6.2)$$

In (6.1), one can recognize the localization formula for the bundle $O(k)$ on the Hilbert scheme (see [18]). We have

$$\Omega_k(z; q, t) = \sum_{n=0}^{\infty} (-1)^{kn} z^n \chi(\text{Hilb}_n(\mathbb{C}^2), O(k)).$$

Let $R_x = \mathbb{C}[x_1, \dots, x_n]$, $R_{x,y} = \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]$. The symmetric group S_n acts on these, permuting x_i and y_i simultaneously. For an S_n -module M , we denote by M^+ and M^- the subspace of invariant and anti-invariant elements, respectively. Notice that $R_{x,y}^- \subset R_{x,y}$ is a module over $R_{x,y}^+$. By $(R_{x,y}^-)^k$ we denote the linear span of k -fold products of elements of $R_{x,y}^-$ inside $R_{x,y}$. This is also a module over $R_{x,y}^+$. Haiman obtained the following results about the bundles $O(k)$.

Theorem 6.1 ([18]). *For any $k \geq 0$, we have an isomorphism of $R_{x,y}^+$ -modules*

$$\Gamma(\text{Hilb}_n(\mathbb{C}^2), O(k)) \cong (R_{x,y}^-)^k \subset R_{x,y},$$

and the i -th cohomology of $O(k)$ vanishes for $i > 0$. The R_x^+ -module $(R_{x,y}^-)^k$ is free.

For instance, for $k = 0$ and $k = 1$, we have

$$\Gamma(\text{Hilb}_n(\mathbb{C}^2), O) = R_{x,y}^+, \quad \Gamma(\text{Hilb}_n(\mathbb{C}^2), O(1)) = R_{x,y}^-.$$

From this we can obtain the following presentation for the dual of the local cohomology.

Proposition 6.2. *Let $M_{n,k} := H_L^n(O(k))^\vee$. Let $\Delta_x := \prod_{i < j \leq n} (x_i - x_j)$. For $k \geq 0$, we have*

$$M_{n,k} = \left\{ f \in \frac{1}{\Delta_x^k} R_{x,y}^- \mid D(x_1, \dots, x_n, \partial_{y_1}, \dots, \partial_{y_n}) f \in R_{x,y} \ (\forall D \in (R_{x,y}^-)^k) \right\}.$$

For $k \geq 1$, we have

$$M_{n,k} = \left\{ f \in \frac{1}{\Delta_x^{k-1}} R_{x,y}^+ \mid D(x_1, \dots, x_n, \partial_{y_1}, \dots, \partial_{y_n}) f \in R_{x,y} \ (\forall D \in (R_{x,y}^-)^{k-1}) \right\}.$$

For instance, we have $M_{n,0} = R_{x,y}^-$ and $M_{n,1} = R_{x,y}^+$. Equivalence of the two presentations follows from the fact that if $f \in \frac{1}{\Delta_x} R_{x,y}^-$ satisfies $Df \in R_{x,y}$ for all $D \in R_{x,y}^-$, then we have $f \in R_{x,y}^+$.

In the case $n = 2$, the module $R_{x,y}^-$ is generated over $R_{x,y}^+$ by $x_1 - x_2$ and $y_1 - y_2$. So the condition on f is equivalent to

$$(x_1 - x_2)^i (\partial_{y_1} - \partial_{y_2})^j f \in R_{x,y} \quad (i + j = k - 1).$$

Equivalently, in the expansion

$$f = \sum_{i,j} (x_1 - x_2)^i (y_1 - y_2)^j f_{ij}(x_1 + x_2, y_1 + y_2),$$

we have $f_{ij} = 0$ unless $i \geq 0$ or $i \geq -k + 1 + j$. Note that also $f_{ij} = 0$ unless $i + j \equiv k - 1 \pmod{2}$. So $M_{2,k}$ has a direct sum decomposition $M_{2,k} = R_{x,y}^{(-1)^{k-1}} \oplus M'_{2,k}$, where $M_{2,k'}$ has the Hilbert series

$$\frac{1}{(1-q)(1-t)} \sum_{\substack{i \geq 1, j \geq 0, i+j \leq k-1 \\ i+j \equiv k-1 \pmod{2}}} q^{-i} t^j,$$

which exactly matches the z^2 -term of (6.2) after the substitution $q \rightarrow q^{-1}$.

More generally, we notice that there is a *shuffle product* $M_{n',k} \otimes M_{n'',k} \rightarrow M_{n'+n'',k}$ given by

$$f * g = \sum_{\sigma \in S_{n'+n''}/(S'_n \times S_{n''})} (-1)^{(k-1)|\sigma|} \sigma(f(x_1, \dots, x_{n'}, y_1, \dots, y_{n'})) \cdot g(x_{n'+1}, \dots, x_{n'+n''}, y_{n'+1}, \dots, y_{n'+n''}).$$

Supported by computer experiments, we formulate the following conjecture.

Conjecture 6.3. Denote by $M_{n,k}^{\text{shuffles}} \subset M_{n,k}$ the span of all shuffle products of the form $f * g$ for $f \in M_{n',k}$ and $g \in M_{n-n',k}$, $0 < n' < n$. Let the space of primitive elements be the quotient $M_{n,k}^{\text{prim}} := M_{n,k} / M_{n,k}^{\text{shuffles}}$. We have an isomorphism of bigraded vector spaces

$$H^*(X) \left[\sum_{i=1}^n x_i, \sum_{i=1}^n y_i \right] \cong M_{n,k}^{\text{prim}}.$$

The ring structure of the cohomology is not immediately evident from this description, but our COHA tells us where to look for it.

Proposition 6.4. The Lie algebra $\mathcal{H}_2 \otimes H^*(\mathbb{P}^1)$ acts on $M_{n,k}$ by the following operations:

$$(T_{r,s} \otimes \text{pt})f = \sum_{i=1}^n x_i^s \partial_{y_i}^r f,$$

$$(T_{r,s} \otimes 1)f = \sum_{i=1}^n x_i^{s-1} \partial_{y_i}^{r-1} (-s \partial_{y_i} y_i - r x_i \partial_{x_i}) f.$$

This action is compatible with the shuffle product and therefore descends to $M_{n,k}^{\text{prim}}$.

Now we describe the ring structure. First we expect that the following is true.

Conjecture 6.5. *The space $M_{n,k}^{\text{prim}}$ is spanned by products of $\frac{1}{\Delta_x^{k-1}}$ and polynomials of the form*

$$\psi_s(\text{pt}) := \sum_{i=1}^n x_i^s, \quad \psi_s(1) := s \sum_{i=1}^n x_i^{s-1} y_i.$$

This conjecture implies that $M_{n,k}^{\text{prim}}$ is isomorphic to the quotient of the ring of polynomials in the generators $\psi_s(\text{pt})$, $\psi_s(1)$ by a certain ideal. So it endows $M_{n,k}^{\text{prim}}$ with a ring structure, and it is natural to formulate the following conjecture.

Conjecture 6.6. *The isomorphism of Conjecture 6.3 is an isomorphism of rings.*

6.1. Some remarks

The Hilbert scheme has long been a source of interesting bigraded modules. For instance, there was $(n+1)^{n-1}$ conjecture and $n!$ conjecture proved by Haiman [18]. The $(n+1)^{n-1}$ conjecture is about the module of diagonal coinvariants.

Theorem 6.7 ([18]). *The following equality holds:*

$$\dim R_{x,y}/(R_{x,y}^+)_{>0} = (n+1)^{n-1}.$$

An explicit combinatorial presentation of the bigraded Frobenius character of $R_{x,y}/(R_{x,y}^+)_{>0}$ was known as a shuffle conjecture [17] and was proved in [5]. Also in this context Haiman observed that the vector space is in a certain sense generated by the differential operators $\sum_{i=1}^n x_i^{s-1} \frac{\partial}{\partial y_i}$, which led to the operator conjectures in [19]. The double coinvariants have been interpreted in terms of Borel–Moore cohomology of affine Springer fibers in [6].

Our story suggests that there are many more interesting bigraded modules to be constructed using the Hilbert scheme.

7. Poisson geometry of Coulomb branches

We would like to conclude by explaining a curious formalism which hints at how Conjectures 6.3–6.6 can be extended to surfaces of higher genus. First, observe that, in view of (5.1), we should consider the bundle of multiforms, i.e., the g -fold tensor product of the bundle of differential forms on the Hilbert scheme. How does the Lie algebra $\mathcal{H}_2 \otimes H^*(C)$ act? Quite naturally, in fact. Take a function $f \in R_{x,y}^+$,

for instance $f = \sum_{i=1}^n x_i^s y_i^r$. Symplectic geometry associates to f the Hamiltonian vector field H_f . Let

$$\begin{aligned}\psi_f(\text{pt}) &:= \text{multiplication by } f, \\ \psi_f(1) &:= \text{Lie derivative along } H_f.\end{aligned}$$

This is what we had in genus 0. But with multiforms, additional operations arise. For $i = 1, \dots, g$, set

$$\psi_f(\sigma_i) = (df)^{(i)}.$$

This is the operation of multiplication by df acting on the i -th component in the g -fold tensor product. Set

$$\psi_f(\sigma_{i+g}) = \iota_{H_f}^{(i)},$$

the operation of insertion of H_f into the i -th component. From

$$\psi_{fg}(1) = \text{Lie}_{fH_g + gH_f}, \quad \text{Lie}_{fH_g} = f \text{Lie}_{H_g} + \sum_{i=1}^g (df)^{(i)} \iota_{H_g}^{(i)},$$

we obtain the following identity:

$$\psi_{fg}(1) = \psi_f(1)\psi_g(\text{pt}) + \psi_f(\text{pt})\psi_g(1) + \sum_{i=1}^{2g} \psi_f(\sigma_i)\psi_g(\sigma_i^*).$$

This formula precisely matches the Künneth decomposition of the diagonal class in $H^2(C \times C)$, which confirms the consistency of the formalism. Okounkov has observed that this formalism essentially coincides with the Clifford algebras considered in [27].

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