

The $K(\pi, 1)$ -conjecture for Artin groups

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Abstract. An Artin group G_W is the fundamental group of the space Y_W , which consists of the regular orbits of a Coxeter group W acting on the complexified Tits cone. The best-known example occurs when W is the symmetric group, in which case G_W is the classical braid group, and Y_W is the *configuration space* of n distinct points in $\mathbb{C}$. A famous long-standing conjecture, attributed to Arnol'd, Brieskorn, Pham, and Thom, asserts that the orbit space Y_W is a $K(\pi, 1)$ -space. For finite W , the conjecture was proved by Deligne (1972) in a landmark paper. Recently, in collaboration with Giovanni Paolini (2021), we proved the conjecture for the next significant class of Artin groups of *affine type*.

We discuss various aspects of our proof, providing an overview of the main methods while avoiding the numerous technical details. In particular, we focus on combinatorial methods applied to the so-called *dual approach* to Artin groups. The latter involves the development of *Garside theory* and its application to *non-crossing partition* intervals. Finally, we highlight some naturally related problems and conjectures that can be formulated for any Artin group. Delucchi, Paolini, and Salvetti (2024) have recently resolved these conjectures affirmatively for all rank-three groups.

1. Introduction

We consider Coxeter groups and their associated Artin groups (precise definitions are provided below). Since Deligne's landmark paper [14], which established (among other results) the $K(\pi, 1)$ -property for the orbit configuration space of spherical-type Artin groups, the topological study of this important class of groups has seen only sporadic progress. However, in recent years, new significant results have rekindled strong interest in this area. In collaboration with Giovanni Paolini, we proved the $K(\pi, 1)$ -conjecture for all the affine Artin groups [29]. More recently, further advancements have been achieved using different approaches, primarily based on Gromov's theory of hyperbolic groups [19, 21].

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In this paper, an extended exposition of my talk at the ECM, we focus on our approach to the $K(\pi, 1)$ -conjecture. After providing all necessary definitions and stating the conjecture in its general form, we introduce the so-called *dual approach* to the theory. Within this framework, one considers all reflections R as generators of the Coxeter group W , which in turn defines a length function ℓ_R and induces a poset structure on W . We then consider the interval $[1, c]$ in this poset, consisting of all elements less than or equal to a fixed Coxeter element c . This interval is commonly referred to as the *non-crossing partition* interval, a term originating from the symmetric group case, as we explain later.

Starting from the non-crossing partition interval, we define a *dual Artin group* and an *interval complex*. The interval complex is a CW -complex whose fundamental group is the dual Artin group. A key property in Garside theory is when the associated poset forms a *lattice*. This lattice property ensures that the dual Artin group is a Garside group and that the interval complex is a $K(\pi, 1)$ -space.

Next, we outline the main components of our proof, which partially extend the theory developed by McCammond and collaborators [26] while introducing combinatorial methods. The essential steps are as follows: first, we prove that the interval complex remains a $K(\pi, 1)$ -space even when the poset is not a lattice; second, we construct a *dual* finite model for the orbit configuration space; and finally, we apply combinatorial techniques – such as *EL*-shelling of the poset and discrete Morse theory – to simplify the interval complex, reducing it to the dual model (see also [18]).

In the last part of the paper, we consider potential generalizations of these constructions to arbitrary Coxeter groups and pose a few related questions and conjectures. Our concluding theorem provides affirmative answers to all these questions for rank-three Artin groups (see [16]).

2. Coxeter groups and Artin groups

A (finitely generated) Coxeter group is a group defined by the presentation

$$W = \left\langle S \mid s^2 = 1, \underbrace{sts \cdots}_{m(s,t) \text{ factors}} = \underbrace{tst \cdots}_{m(s,t) \text{ factors}} \text{ for } s \neq t, m(s,t) \neq \infty \right\rangle$$

(see [5, 22]), where S is a finite set and $m : S \times S \rightarrow \mathbb{N} \cup \{\infty\}$ is a function such that $m(s, s) = 1$ for all $s \in S$ and $m(s, t) = m(t, s) \in \{2, 3, 4, \dots\} \cup \{\infty\}$ for all $s \neq t$ in S .

Any Coxeter group can be represented as a group generated by *linear reflections* in the real space $V = \mathbb{R}^S$. It is convenient to consider the contragredient $\rho^* : W \rightarrow \mathrm{GL}(V^*)$ of the *Tits representation* $\rho : W \rightarrow \mathrm{GL}(V)$, which is given by

extending to all W the map

$$\rho : s \mapsto r_s : x \mapsto x - 2B(x, e_s)e_s, \quad s \in S.$$

Here, e_s denotes the canonical basis vector in V , and B is the symmetric bilinear form given by

$$B(e_s, e_{s'}) := -\cos(\pi/m(s, s'))$$

(with $B(e_s, e_{s'}) = -1$ if $m(s, s') = \infty$).

The representation ρ preserves the bilinear form B . It turns out that W is *spherical* (finite) if and only if B is positive definite; W is *affine* if and only if B is positive semidefinite (but not definite); W is *hyperbolic* if B is non-degenerate of signature $(n-1, 1)$.

The action of W on V^* , which has a canonical basis given by the dual basis $e_s^*, s \in S$, of V , is free on the *base chamber*

$$C_0 := \bigcap_{s \in S} H_s^+.$$

Here, $H_s = \text{Span}\{e_{s'}^* \mid s' \in S, s' \neq s\}$ are the *walls* of C_0 , and $H_s^+ = H_s + \mathbb{R}_+ e_s^*$ is the half-space with boundary H_s which contains C_0 . Denote by $D = \overline{C_0}$ the closure of the base chamber. The stabilizer of a face $F \subset D \setminus C_0$,

$$F = \left(\bigcap_{s \in T \subset S} H_s \right) \cap \left(\bigcap_{s \in S \setminus T} H_s^+ \right), \quad (2.1)$$

is the *parabolic subgroup* W_T generated by all $s \in T$. Notice that for an affine W , all proper parabolic subgroups are finite.

Let $R := \{r = w^{-1}sw \mid s \in S, w \in W\}$ be the set of all *reflections* in W . Each $r \in R$ is mapped by ρ^* into a linear reflection in V^* with fixed point set $H_r = w^{-1} \cdot H_s$. The collection $\mathcal{A} = \{H_r, r \in R\}$ is called the *reflection arrangement* of W . The connected components of the complement of $\bigcup_{r \in R} H_r$ are the *chambers* of the arrangement.

The *Tits cone* $I = \bigcup_{w \in W} w \cdot D$ is the orbit of D . One has the following properties (see [22, 36]):

- $I = \mathbb{R}^S$ if and only if W is finite; when W is affine, I is a half-space (with boundary $\text{Ann}(V^\perp)$).
- D is a fundamental domain for the action of W on I .
- I is a convex cone in $\mathbb{R}^S$, which is generally neither open nor closed.
- The reflection arrangement $\mathcal{A}$ stratifies I into convex *faces*, each conjugated to a unique face of C_0 . The group W acts simply transitively on the set of codimension-0 faces (the chambers).

- Face (2.1) of C_0 has stabilizer equal to the *parabolic subgroup* W_T which is generated by the reflections $s \in T$. This face belongs to the interior $\text{int}(I)$ if and only if W_T is finite. More generally, any face F' conjugated to F (by elements in some left coset of W_T) belongs to $\text{int}(I)$ if and only if W_T is finite.

In particular, all codimension-1 faces of C_0 (the walls of C_0) belong to $\text{int}(I)$ and a codimension-2 face $F = (H_s \cap H_{s'}) \cap (\bigcap_{t \neq s, s'} H_t^+)$ is in $\text{int}(I)$ if and only if $m(s, s') \neq \infty$.

We now introduce the *configuration space* associated with W :

$$Y = (I_{\mathbb{C}}) \setminus \bigcup_{r \in R} H_{r, \mathbb{C}},$$

where $I_{\mathbb{C}} \cong I \times \mathbb{R}^S$ is the complexification of the Tits cone and $H_{r, \mathbb{C}} \cong H_r \times H_r$ is the complexification of the hyperplane H_r . (Equivalently, one can take $I \times I$ instead of $I_{\mathbb{C}}$.) The group W acts diagonally on $I_{\mathbb{C}}$, and it acts freely and properly discontinuously on Y . Thus, we have the *orbit configuration space*

$$Y_W = Y/W.$$

Example 2.1. For W the symmetric group S_n , one has $Y \simeq \mathbb{C}^n \setminus \bigcup_{i \neq j} \{z_i = z_j\}$ as the space of ordered n -tuples of points in $\mathbb{C}$ and $Y_W \simeq Y/S_n$ the space of unordered n -tuples of points in $\mathbb{C}$.

We also recall the *Salvetti complexes* $X \subset Y$ and $X_W \subset Y_W$ which are deformation retracts of the configuration space and of the orbit configuration space of W , respectively [33, 34]. In particular, X_W contains a $|T|$ -cell for each *finite* parabolic subgroup W_T . The construction of X closely follows the approach in the original paper [33], involving only the faces contained within the interior of I (so corresponding to cosets of finite parabolic subgroups; see [36, Thm. 2]). Consequently, X_W is a finite union of subcomplexes associated with the maximal finite parabolic subgroups of W .

Definition 2.2. The *Artin group* G_W , associated with a Coxeter group W , is the group presented by the generators and relations of W , except the relations $s^2 = 1$ are omitted. Specifically, it is defined by

$$G_W = \left\langle S \mid \underbrace{sts \cdots}_{m(s,t) \text{ factors}} = \underbrace{tst \cdots}_{m(s,t) \text{ factors}} \text{ for } s \neq t, m(s, t) \neq \infty \right\rangle.$$

In Example 2.1 with $W = S_n$, the Artin group G_W is the classical braid group Br_n .

Theorem 2.3 ([7] in the finite case, [35] in general). *The following holds:*

$$\pi_1(Y_W) \cong G_W. \quad \blacksquare$$

(A simple proof of this result can be obtained by examining the 2-skeleton of the associated complex X_W .)

We can now state the main conjecture.

Conjecture 2.4 ($K(\pi, 1)$ -conjecture). *The orbit configuration space Y_W is a classifying space for the corresponding Artin group G_W .*

This famous conjecture is usually attributed to Arnol'd, Thom, Brieskorn, Pham, dating back to the 1960s and 1970s in connection with topological questions in singularity theory. One consequence of the conjecture is the existence of a $K(\pi, 1)$ -space given by an explicit finite CW-complex (e.g., the abovementioned X_W).

The conjecture was established by Deligne in the 1970s for spherical Coxeter groups [14]. Recently, we found [29].

Theorem 2.5 (G. Paolini, M. Salvetti). *The $K(\pi, 1)$ -conjecture is true for all affine Artin groups.*

In previous years, the conjecture was proved in some scattered cases: types $\tilde{A}_n$, $\tilde{C}_n$ [28]; large-type Artin groups [20]; FC-type groups [11]; locally reducible type [10]; type $\tilde{B}_n$ [9]. Several additional related results were established in the meantime: a comprehensive survey up to that point can be found in [30]. Following [29], additional cases were addressed in [19, 21], including cases where the Coxeter diagram is a tree (subjected to certain constraints on the edge labelings) as well as some finite-volume hyperbolic cases.

3. Quick overview of Garside theory

Deligne's proof of the spherical case was inspired by and extended the earlier work by Garside on the braid group [17]. His work, in turn, led to the development of what is now known as Garside theory (a valuable contribution toward understanding certain key properties in the finite case was also made by [8], which was published at the same time as [14]).

A *Garside structure* on a group G is defined as a pair (M, Δ) , where M is a monoid and $\Delta \in M$ is a distinguished element called the *Garside element*. The group G is the group of fractions of M . This pair satisfies a list of properties (see [13] for a comprehensive overview and references up to that point).

The main property of interest is that M , with respect to the poset structure induced by the divisibility relation, forms a *lattice*; that is, any two elements of M have a least common multiple (*lcm*) and a greatest common divisor (*gcd*). One consequence is the

existence of a *normal form* in G , expressed as $g = \Delta^n a$, $n \in \mathbb{Z}$, $a \in M$. This normal form provides a solution to the word problem in G .

Now, consider a group G with generating set $R = R^{-1}$, $1 \notin R$. Let $\ell_R(g)$ be the length of an element $g \in G$ with respect to R (we can take $\ell_R(g)$ as the minimal length of an R -factorization of g). We can introduce a partial ordering on G as follows:

$$g \leq g' \iff \ell_R(g) + \ell_R(g^{-1}g') = \ell_R(g')$$

(equivalently, there is a reduced R -factorization of g' which begins with a reduced R -factorization of g). The Hasse diagram of $(G, \leq)$, graded by the length function ℓ_R , can be viewed as a subgraph of the Cayley graph of G . Specifically, for $g \in G$, $r \in R$, there is an oriented edge (labeled by r) from g to gr if and only if $\ell_R(g) < \ell_R(gr)$ (so $\ell_R(gr) = \ell_R(g) + 1$).

Definition 3.1 (See [12, 23, 26]). For $\gamma \in G$, let $[1, \gamma]$ be the interval below γ in the poset $(G, \leq)$.

- (1) The *interval group* G_γ is the group with generators $R_\gamma = R \cap [1, \gamma]$ and relations as follows: for any pair of maximal chains c, c' of $[1, \gamma]$, with labels given by words w, w' in R , there is a relation $w = w'$. (Equivalently, there is a relation for any loop in the (undirected) Hasse diagram of $[1, \gamma]$.)
- (2) The *interval complex* K_γ is the CW-complex (delta-complex) obtained from the order complex $\Delta([1, \gamma])$ by identifying two simplices if they have the same edge labels (in particular, all 0-simplices are identified).

Recall that the order complex of a poset $\mathcal{P}$ is the simplicial complex whose q -simplices are the $(q+1)$ -chains of $\mathcal{P}$.

By examining the 2-skeleton of K_γ , it is easy to see that $\pi_1(K_\gamma) = G_\gamma$.

We say that the element $\gamma \in G$ is *balanced* if its left and right divisors coincide.

Theorem 3.2 (See [2, 12, 26, 27]). *Let γ be balanced and assume that $[1, \gamma]$ is a lattice. Then:*

- *The interval group G_γ is a Garside group.*
- *The interval complex K_γ is a $K(G_\gamma, 1)$.*

The Garside structure on G is given by the monoid G_γ^+ which has the same presentation as G_γ . The Garside element is γ itself.

3.1. Standard and dual Garside structures

When W is finite, the *standard Garside structure* is defined by taking the set of simple reflections S as the generating set and using the *longest element* $\delta \in W$ (with

respect to length function ℓ_S). In this case, the interval $[1, \delta]$ coincides with the entire group W .

The main points, following Deligne's proof within the previously introduced framework, can be summarized as follows.

Theorem 3.3. *Let W be finite. Then:*

- (1) *The interval group W_δ is the standard Artin group G_W (and the associated monoid is the standard Artin monoid as defined in [8, 14]).*
- (2) *The poset $[1, \delta]$ is a lattice. Therefore, G_W is a Garside group and K_δ is a $K(G_W, 1)$ -space.*
- (3) *K_δ is homotopy equivalent to the orbit configuration space Y_W .*

Point (1) is not too difficult. Point (2) follows from the fact that the weak Bruhat order on a finite Coxeter group is a lattice (see, e.g., [4]). Point (3) is the most delicate: one can see that K_δ is homotopy equivalent to the complex X_W mentioned earlier (see, for instance, [15]).

To introduce the *dual Garside structure* on W , we take all the reflections R as a generating set and choose any *Coxeter element* $c \in W$. Recall that a Coxeter element of W is a product of all the simple reflections in some order or, equivalently, a product of the reflections associated with the walls of a given chamber, in some order. Thus, there are many Coxeter elements, although most of them are often conjugate in W . The interval $[1, c]$ is commonly referred to as the *non-crossing partition* poset; we denote it by $\text{NC}(W, c)$. This terminology originates from the case when W is the symmetric group S_n and $c = (1, 2, \dots, n)$. Let S_n act on the n points $P_k = \exp\left(\frac{2\pi i}{k}\right)$, $k = 0, \dots, n-1$, in the unit circle. A permutation σ belongs to $[1, c]$ if and only if the convex hulls of its cycles do not intersect each other (see Figure 1 for $n = 4$) (see, e.g., [13, Ch. IX]).

Definition 3.4. The interval group $G'_W := G_c$ is called the *dual Artin group* of type W .

We also set $K_W := K_c$. The following holds.

Theorem 3.5 (See [2], generalizing [3] for the case A_n ; see also [6]). *When W is finite, one has the following:*

- (1) *There is an isomorphism $G'_W \cong G_W$ between the dual and standard Artin groups.*
- (2) *$\text{NC}(W, c)$ is a lattice (therefore, G'_W is a Garside group and $K_W = K(G'_W, 1)$).*

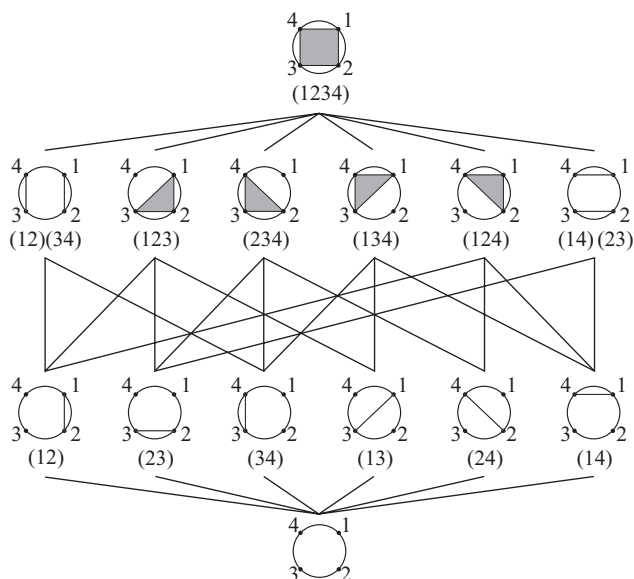


Figure 1. The non-crossing partition poset $\text{NC}(S_4; c)$, where $c = (1, 2, 3, 4)$.

(3) $K_W \cong Y_W (\cong X_W)$ (by the uniqueness of $K(\pi, 1)$ up to homotopy).

Example 3.6. We illustrate the two structures in the case where W is of type A_2 (i.e., $W = S_3$). In this case, $W = \langle S = \{s_1, s_2\} \mid s_1^2 = s_2^2 = 1, s_1 s_2 s_1 = s_2 s_1 s_2 \rangle$. The standard structure is obtained by taking generators S and using the longest element $\delta = s_1 s_2 s_1$ as the Garside element. The dual structure uses as generators all reflections $R = \{s_1, s_2, s_3\}$, where $s_3 = s_1 s_2 s_1$. The Garside element is a Coxeter element, say $c = s_1 s_2$. We depict the associated posets and the complexes X_W, K_W in Figure 2.

4. The case W affine

Let now W be an affine Coxeter group. We want to discuss some main points from [29]. In the affine case, there is no standard Garside structure because there is no longest element in W . Attempts have been made to use the dual structure by choosing a Coxeter element $c \in W$. However, the strategy that works in the finite case does not generally apply here. Specifically, we have the following result.

Theorem 4.1 ([24]). *For W affine, the non-crossing partition poset $\text{NC}(W, c)$ forms a lattice if and only if W is of type $\tilde{A}_n$ (for certain Coxeter elements c), $\tilde{C}_n$, or $\tilde{G}_2$.*

Nonetheless, several interesting properties were discovered in [26], including the following.

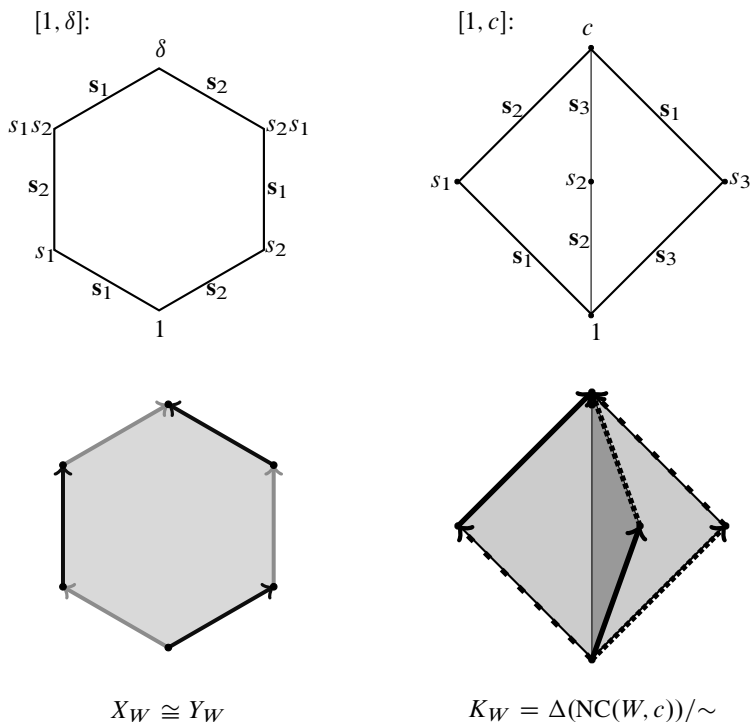


Figure 2. Standard (on the left) and dual (on the right) structures when $W = S_3$. Arrows of the same kind must be glued.

Theorem 4.2. *For W affine, there is an isomorphism between the dual and standard Artin groups: $G'_W \cong G_W$.*

We now summarize the main results proved in [29].

Theorem 4.3. *Let W be an affine Coxeter group. Then:*

- (1) *The complex K_W is always a $K(\pi, 1)$ -space, even when $\text{NC}(W, c)$ is not a lattice.*
- (2) *There exists a finite subcomplex $X'_W \subset K_W$ which is homotopy equivalent to Y_W .*
- (3) *The complex K_W deformation retracts onto X'_W .* ■

The $K(\pi, 1)$ -conjecture stated in Conjecture 2.4 follows directly from Theorem 4.3.

4.1. $K(\pi, 1)$ -property for K_W

We quickly outline the strategy to prove part (1) of Theorem 4.3.

To each element $w \in W$, acting as an isometry on the affine space, we associate the set $\text{Min}(w)$ of points which are *minimally moved* by w with respect to the euclidean distance. This set forms an affine subspace. When w is hyperbolic (i.e., it has no finite fixed points), one has $\dim(\text{Min}(w)) = \text{rk}(W) - \ell_R(w) + 1$. In particular, for a Coxeter element c , the set $\text{Min}(c)$ is an affine line, referred to as the *Coxeter axis* (relative to c).

We rely on the following facts established in [26]. A reflection $r \in R_0 = R \cap [1, c]$ is called *vertical* if $\text{Fix}(r)$ intersects the Coxeter axis; otherwise, r is called *horizontal*. When W is affine, the set of horizontal reflections R_{hor} is finite. These reflections generate a subgroup H of W , which is a direct product of affine Coxeter groups of type $\tilde{A}_i$. The poset $\text{NC}(W, c)$ forms a lattice if and only if H is irreducible, which occurs in the cases covered by Theorem 4.1. Additionally, there is a finite set of translations $T \subset [1, c]$.

In cases where the poset $\text{NC}(W, c) = [1, c]$ is not a lattice, a finite set of additional new translations $T_f \supset T$ is added, leading to a larger group $C \supseteq W$. This construction ensures that the new interval below c in C , denoted for clarity as $[1, c]^C$, becomes a lattice; furthermore, $\text{NC}(W, c)$ is a subposet of $[1, c]^C$. Let then D and F be the subgroups of C generated by $R_{\text{hor}} \cup T$ and $R_{\text{hor}} \cup T_f$, respectively. The Coxeter element c belongs to both D and F , giving rise to intervals $[1, c]^D \subset D$, $[1, c]^F \subset F$. The latter interval is also a lattice.

Using these intervals, we construct additional interval complexes K_c^C , K_c^F , and K_c^D . Furthermore, we define the subcomplex $K_c^H \subset K_c^D$ consisting of all simplices $[x_1 \mid \cdots \mid x_k]$ such that the product $x_1 \cdots x_k$ belongs to H .

The situation is summarized in Figure 3. Here, the complexes K_c^C , K_c^F are $K(\pi, 1)$ -spaces.

We prove the following in [29]:

- K_c^D is covered by $K_c^H \times \mathbb{R}$.
- K_c^H is a $K(\pi, 1)$ -space, hence K_c^D is also a $K(\pi, 1)$ -space.
- K_W is a $K(\pi, 1)$ -space (which follows by a Mayer–Vietoris argument).

4.2. The dual model

When W is finite, we just set $X'_W = K_W$. Therefore, by Theorem 3.5 (3), we have $X'_W \cong X_W \cong Y_W$ in this case.

In general, let $T \subset S$ be such that the parabolic subgroup $W_T \subset W$ is finite. Let c be the Coxeter element expressed as a product of elements $s \in S$, in some order, and let c_T be the product of the elements in T , in the same order as they appear in the

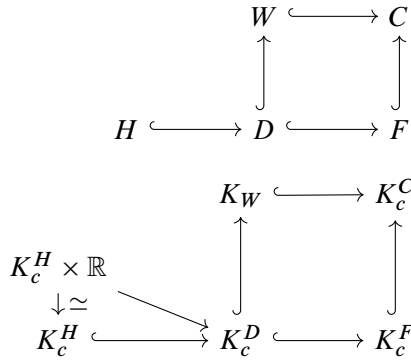


Figure 3. Diagrams of groups and complexes.

factorization of c . Then c_T is a Coxeter element of W_T , and it belongs to the interval $[1, c]$. Moreover, the interval complex K_{W_T} constructed from $[1, c_T] \subset W_T$ is a subcomplex of K_W (see [29, §7]).

We define X'_W in general as

$$X'_W = \bigcup_{W_T \subset W, W_T \text{ finite}} K_{W_T}.$$

To prove that $X'_W \simeq Y_W$, we construct a homotopy equivalence $\varphi : X_W \rightarrow X'_W$. Note that X_W can also be expressed as a union of subcomplexes X_{W_T} , corresponding to finite parabolic subgroups. The map φ is constructed by systematically gluing together homotopy equivalences between the corresponding subcomplexes.

Example 4.4. We show in Figure 4 the example of $W = \tilde{A}_2$. Here, the Coxeter element is $w = abc$. The dotted line represents the Coxeter axis of W with respect to w . In this case, $X'_W = K_{W_{a,b}} \cup K_{W_{a,c}} \cup K_{W_{b,c}}$, and $X_W = X_{W_{a,b}} \cup X_{W_{a,c}} \cup X_{W_{b,c}}$. Each block K_{W_T} is homotopy equivalent to the corresponding block X_{W_T} .

4.3. The retraction $K_W \searrow X'_W$

To prove part (3) of Theorem 4.3, we apply *discrete Morse theory* to the complex K_W . We construct a discrete flow which contracts all the cells in $K_W \setminus X'_W$ (perfect matching). To control this matching, we make use of another result, which is of independent interest.

Theorem 4.5 ([29]). *The poset $\text{NC}(W, c)$ is EL -shellable.*

(For the spherical case, an analogous result was proven in [1].) For the edge-labeled poset $\text{NC}(W, c)$, EL -shellability is equivalent to the following:

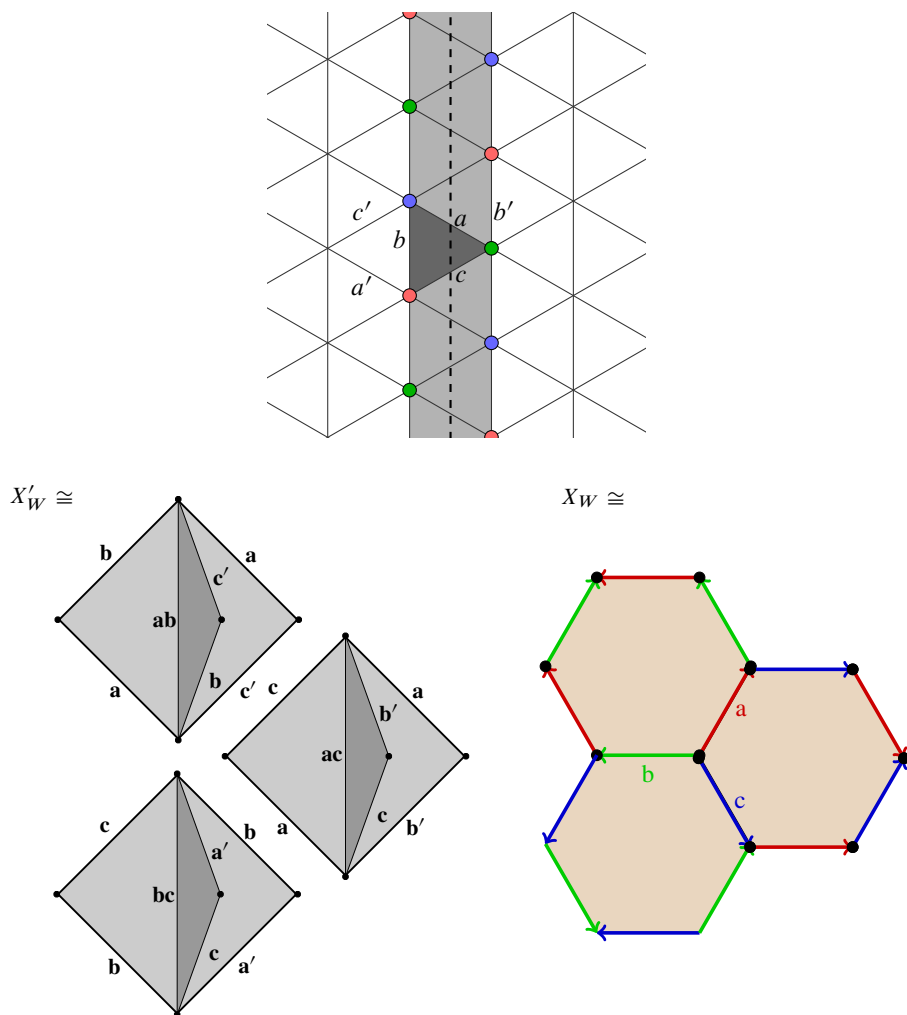


Figure 4. Case $\tilde{A}_2$. Arrows with the same labels or colors are glued together.

- There is a *total ordering* $<$ of the reflections R_0 in $\text{NC}(W, c)$ such that every $u \in \text{NC}(W, c)$ has a *unique* increasing minimal factorization $u = r_1 \cdots r_k$, where $r_1 < \cdots < r_k$.
- This factorization is the *lexicographically smallest* minimal factorization of u .

To define an ordering of the reflections R_0 , we use an *axial ordering* method. Let C_0 be the base chamber, and let ℓ_c be the Coxeter axis. The Coxeter element c acts as a translation along ℓ_c , and we orient ℓ_c accordingly. Fix a point $P_0 \in \ell_c \cap C_0$ and order R_0 as follows:

- Vertical reflections r such that $\text{Fix}(r) \cap \ell_c$ lies after P_0 in the ℓ_c -orientation: these are ordered first, following the orientation of ℓ_c .
- Horizontal reflections: these are ordered according to a specific construction, which depends on the detailed structure of each irreducible component of the horizontal root system.
- Vertical reflections r such that $\text{Fix}(r) \cap \ell_c$ lies before P_0 in the ℓ_c -orientation: these are ordered last, again respecting the orientation of ℓ_c .

Figure 5 illustrates the axial ordering for $W = \tilde{A}_2$ and $W = \tilde{G}_2$, where the Coxeter element is $w = abc$.

We use the axial ordering as the main tool to construct an explicit matching of the cells in $K_W \setminus X'_W$, in order to prove part (3) of Theorem 4.3.

Example 4.6. We illustrate the matching in the smallest affine case, $W = \tilde{A}_1$. Here, $W \cong \mathbb{Z}_2 * \mathbb{Z}_2 = \langle s_0, s_1 \mid s_i^2 = 1 \rangle$ and $G_W \cong F[2] \cong \langle s_0, s_1 \rangle$. The reflections R correspond to words of odd length: $s_{-i} = s_0 s_1 s_0 \cdots$ ($2i + 1$ factors, $i \geq 0$), $s_i = s_1 s_0 s_1 \cdots$ ($2i - 1$ factors, $i \geq 1$). Thus, W is realized as the isometry group of the real line generated by reflections with respect to integer points. Let $c = s_1 s_0$ be the Coxeter element (acting on the right: c is the translation $x \mapsto x + 2$ of the real line). We have $c = s_{i+1} s_i$ for $i \in \mathbb{Z}$. The non-crossing partition lattice $\text{NC}(W, c)$ consists of the elements $1, s_i, c$ with the relations $1 < s_i < c$ for $i \in \mathbb{Z}$. The edge labels are s_i for $(1, s_i)$ and s_{i-1} for (s_i, c) (see Figure 6). Therefore, the dual Artin group is presented as $G'_W = \langle s_i \mid s_{i+1} s_i = s_{j+1} s_j \ \forall i, j \rangle$, which can be easily seen to be isomorphic to G_W .

The axial ordering here is simply

$$s_1 < s_2 < \cdots (< s_{+\infty} < s_{-\infty} <) \cdots < s_{-1} < s_0.$$

Notice that the unique increasing 2-chain in $\text{NC}(W, c)$ is $s_1 < s_0$.

The complex K_W is two-dimensional, with one 0-cell $[\]$, 1-cells $[s_i]$ for $i \in \mathbb{Z}$, and $[c]$. The 2-cells are $[s_{i+1} \mid s_i]$, $i \in \mathbb{Z}$, and their boundaries are represented by $\partial[s_{i+1} \mid s_i] = [s_i] \cup [c] \cup [s_{i+1}]$ (see Figure 7).

In Figure 8, we illustrate the Hasse diagram of the face poset $\mathcal{F}(K_W)$ of K_W . This poset consists of the simplices of K_W , ordered by face inclusion. The thick and dashed arrows indicate a proper matching on $\mathcal{F}(K_W)$. The unmatched (critical) cells are the 0-cell $[\]$ and two 1-cells $[s_0]$, $[s_1]$. These form the subcomplex $X'_W = K_{\{s_0\}} \cup K_{\{s_1\}} \cong S^1 \vee S^1 \cong X_W$. Consequently, by standard results in discrete Morse theory, K_W contracts onto X'_W .

The previous example considers the simplest affine group, $\tilde{A}_1$, and sheds light on the general procedure used to contract K_W onto X'_W . Notice that the face poset

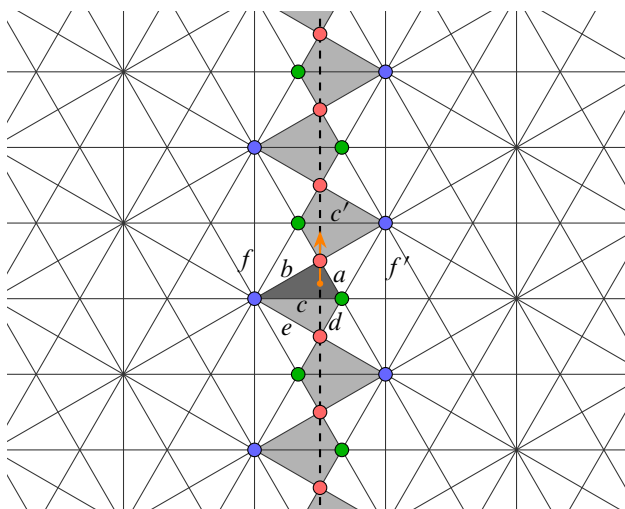
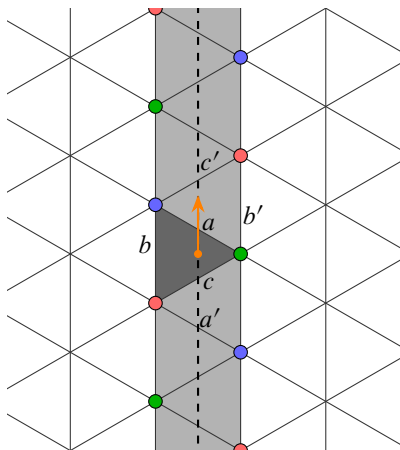


Figure 5. Type $\tilde{A}_2$: axial ordering: $a < c' < \dots < b < b' < \dots < a' < c$. Type $\tilde{G}_2$: axial ordering: $a < b < c' < \dots < f < f' < \dots < d < e < c$.

$\mathcal{F}(K_W)$ (Figure 8) can be divided into two distinct parts: the first part, $\mathcal{F}_1$, consists of the cells $[]$ and $[c]$, while the second part, $\mathcal{F}_2$, consists of the top two rows, excluding the cell $[c]$. These parts are preserved under conjugation by the Coxeter element c (with the relation $c^{-1}s_ic = s_{i-2}$).

The contraction $K_W \searrow X'_W$ can be carried out in two steps. First, one *shrinks* $\mathcal{F}_2$ onto a finite segment. In Figure 8, this is achieved using the matching indicated by

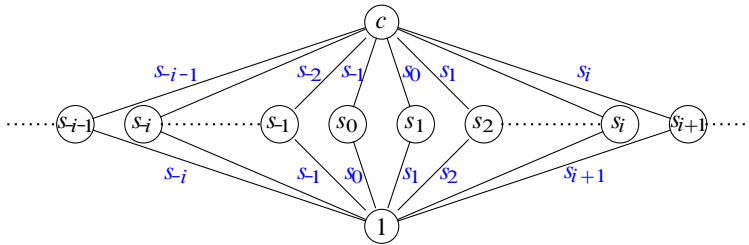


Figure 6. The non-crossing partition poset $\text{NC}(W, c)$ for $W = \tilde{A}_1$.

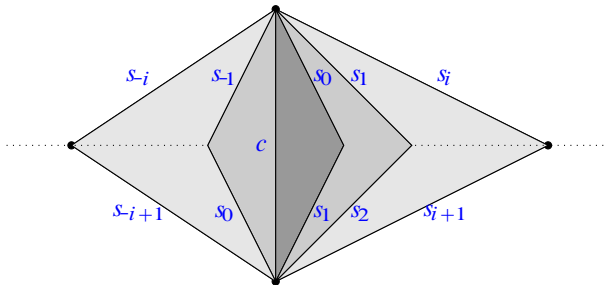


Figure 7. The interval complex K_W , $W = \tilde{A}_1$: edges with same label are identified.

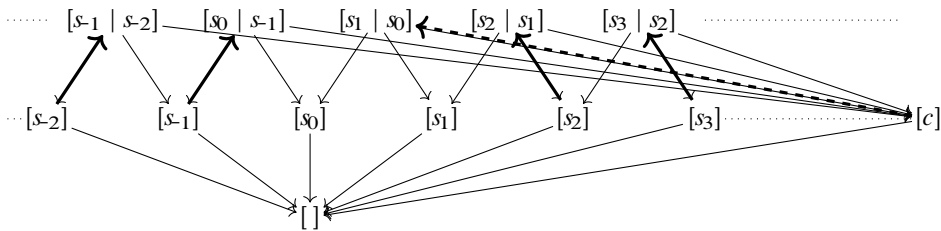


Figure 8. The face poset $\mathcal{F}(K_W)$: critical cells: $[\]$, $[s_0]$, $[s_1]$, so $K_W \simeq S^1 \vee S^1$.

the thick arrows within $\mathcal{F}_2$. This results in a contraction of K_W onto a finite subcomplex K' , which strictly contains X'_W . In this example, K' is the 2-simplex $[s_1 \mid s_0]$, together with its boundary cells (and with all vertices identified).

Next, a second matching – indicated by the dashed arrow – connects simplices in $\mathcal{F}_1$ and $\mathcal{F}_2$, further contracting K' onto X'_W . Specifically, the 1-cell $[c]$ is matched with the 2-cell $[s_1 \mid s_0]$, corresponding to the unique increasing factorization $c = s_1 s_0$.

A similar, though technically more complex, situation arises in the general case. First, the Hasse diagram of $\mathcal{F}(K_W)$ is decomposed into connected pieces, similar

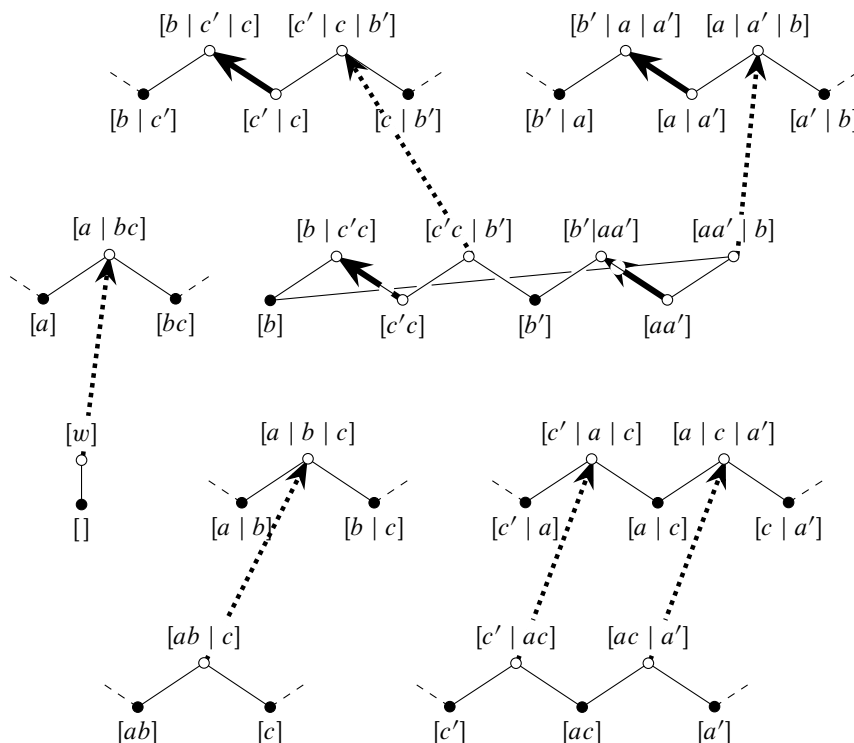


Figure 9. Components of K_W for $W = \tilde{A}_2$, with Coxeter element $w = abc$. The perfect matching in $K' \setminus X'_W$ is indicated by the thick and dotted arrows.

to $\mathcal{F}_1$ or $\mathcal{F}_2$, appearing as zigzag sequences (which can be finite or infinite) connecting simplices of adjacent dimensions. Retraction of each infinite sequence onto a finite “interval” reduces K_W to a finite subcomplex K' containing X'_W . The cells in $K' \setminus X'_W$ are matched either within their own sequence or by connecting different sequences. The latter type of matching involves splitting certain elements according to their minimal factorization.

Example 4.7. These constructions are illustrated in Figure 9 for the case where W is of type $\tilde{A}_2$.

There are nine sequences (two of which are finite). The black nodes represent the cells of X'_W . Each infinite sequence contracts onto the “convex envelope” of the cells of X'_W ; together, these cells form the finite subcomplex K' containing X'_W (comprising both black and white nodes). The matching of the cells in $K' \setminus X'_W$ is represented by the thick black arrows (connecting pairs within the same sequence) and the dotted

arrows (indicating jumps between different sequences, obtained by “splitting” certain elements according to their minimal factorization).

4.4. Some general conjectures

The constructions described in the previous sections naturally lead to some interesting questions and conjectures. Below, we provide a list of these, starting with the $K(\pi, 1)$ -conjecture as the first main conjecture.

- (1) $K(\pi, 1)$ -conjecture: for any Artin group, $Y_W \simeq K(G_W, 1)$.
- (2) Isomorphism of dual and standard Artin groups: $G'_W \cong G_W$ for any Coxeter group W ; known for spherical and affine Artin groups [2, 26, 29], and it is natural to conjecture that it holds for all Artin groups.
- (3) Interval complex as a $K(\pi, 1)$ -space: $K_W \simeq K(G'_W, 1)$. In the case where W is of spherical or affine type, the interval complex K_W is always a $K(\pi, 1)$ -space, even when the lattice property of $\text{NC}(W, c)$ does not hold (see Theorem 4.3 (1)). This observation suggests that Garside theory could potentially be extended. Therefore, the conjecture is that the interval complex remains a $K(\pi, 1)$ -space under certain conditions, which have yet to be characterized, and are weaker than the lattice property.
- (4) Equivalence of complexes: $K_W \simeq Y_W$ for all Artin groups. A stronger version of this conjecture is $K_W \searrow X'_W$. This follows directly from our construction in the affine case. The hope is that a similar construction can be generalized.
- (5) Shellability of non-crossing partitions: $\text{NC}(W, c)$ is always shellable. This has been proven for finite groups [1] and for affine-type groups (from Theorem 4.5). Recently, G. Paolini discovered (using computational methods) that, for certain hyperbolic groups, the axial ordering does not yield an EL -shelling. However, there might exist modifications of the axial ordering that work (we have identified such modifications in some cases).

It should be noted that some of these conjectures are not independent. For instance, (3) and (4) imply (1) (this is the strategy used in [29]). Moreover, (4) implies (2), as it can be deduced by examining the fundamental groups.

Some recent progress for conjecture (2) can be found in [31].

4.5. Rank-three case

We now ask whether the methods used in the affine case can be extended to other classes of Artin groups. A positive answer includes the class of all rank-three Artin groups. In Figure 10, we show an example of a rank-three Coxeter group with exponents m_{ij} given by (3, 3, 4).

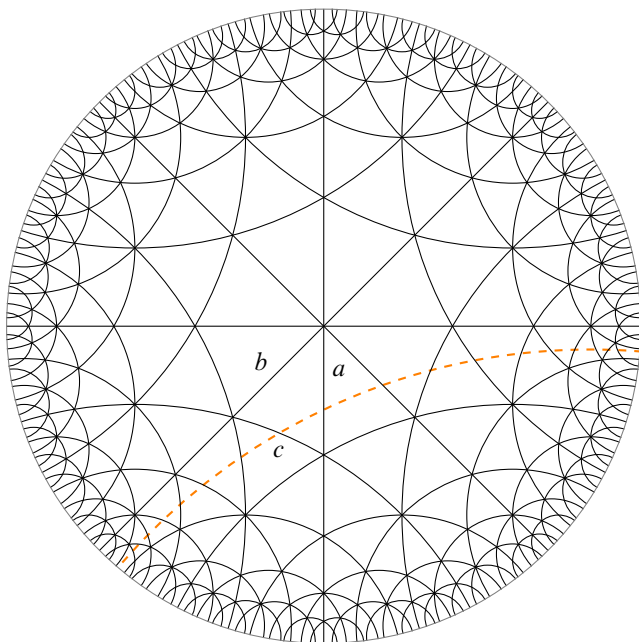


Figure 10. Example of a rank-three Coxeter group in the Poincaré model. The hyperbolic disk is tiled by triangles with angles $\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{4}$. The Coxeter element is $w = abc$, and the Coxeter axis is represented by the dotted line.

In particular, the following theorem implies that all the conjectures listed in Section 4.4 hold true for rank-three Coxeter groups.

Theorem 4.8 ([16]). *Let $\mathbf{W}$ be a Coxeter group of rank three and $c \in W$ any Coxeter element.*

- (1) $\text{NC}(W, c)$ is a lattice, so $K_W \simeq K(G'_W, 1)$ (conjecture (3)).
- (2) $\text{NC}(W, c)$ is EL-shellable (conjecture (5)).
- (3) $K_W \searrow X'_W$ (conjecture (4)).

As a consequence, we obtain the following corollary.

Corollary 4.9. *Let $\mathbf{W}$ be a Coxeter group of rank three and $c \in W$ any Coxeter element. Then:*

- (1) $Y_W \simeq K(G_W, 1)$ ($K(\pi, 1)$ -conjecture).
- (2) $G'_W \cong G_W$ (conjecture (2)).
- (3) G_W is a Garside group (in particular, the word problem is solvable, and the center of G_W is trivial unless W is finite).

For the rank-three case, the $K(\pi, 1)$ -conjecture was already known; all other properties presented here are new.

The lattice property (1) of Theorem 4.8 follows from the poset inclusion of $[1, c]$ into the poset of linear subspaces of $\mathbb{R}^3$ (ordered by inclusion; see [25]).

We remark that the EL -shelling of Theorem 4.8 (2) is also obtained by axial ordering of R_0 and involves several geometric considerations in the hyperbolic model.

The proof of part (3) of Theorem 4.8 is similar to that of part (3) of Theorem 4.3, but it is unexpectedly more challenging in certain aspects. In particular, it requires considering, in addition to the complexes $X'_W \subset K' \subset K_W$, two extra complexes:

$$X'_W \subseteq K_W^{\text{sparse}} \subseteq K_W^{\text{short}} (\cong K') \subseteq K_W^{\text{long}} \subseteq K_W.$$

Starting from the right, each complex in this chain contracts onto the one immediately to its left.

Some of these contractions can be performed in general under the presence of an axial ordering that induces an EL -shelling of $\text{NC}(W, c)$. This suggests that conjecture (4) might hold at least when $\text{NC}(W, c)$ admits an EL -shelling.

As discussed in Section 4.4, such a result would imply the isomorphism between the dual and standard Artin groups (conjecture (2)).

Remark 4.10. Several related problems are of particular interest. For example, in the finite case, Deligne proved in [14] that the complement of a simplicial arrangement of hyperplanes is a $K(\pi, 1)$ -space. These families strictly include reflection arrangements. A natural question arises: is the complement of a simplicial affine arrangement (possibly not a reflection arrangement) always a $K(\pi, 1)$ -space?

A related long-standing conjecture involves the $K(\pi, 1)$ -property for the *complements of discriminants* of the universal deformation space of non-simple singularities, such as elliptic singularities (the case of A , D , and E singularities is covered by the Deligne's result). The fundamental groups of these spaces are generally not Artin groups, and their higher homotopy groups remain unknown (see [32]).

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