

Recent progress on the problem of soliton resolution

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Abstract. Dispersive partial differential equations are evolution equations (that is, involving the time variable) whose solutions preserve the energy but can still decay in large time due to the fact that various frequencies propagate with distinct velocities. In some cases, there exist special solutions called *solitons*, which do not change their shape as time passes. The soliton resolution conjecture predicts that, apart from exceptional cases, solitons are the only obstruction to the decay of solutions. More precisely, every solution eventually decomposes into a superposition of solitons and a decaying term called *radiation*.

We discuss the conjecture in the context of the critical wave maps equation, which is the analog of the wave equation for maps from $\mathbb{R}^2$ to $\mathbb{S}^2$. The solitons correspond to *harmonic maps*, which were classified by Eells and Wood in 1976. We consider *equivariant* solutions, which are solutions having a specific symmetry preserved by the flow. In a joint work with Andrew Lawrie, we prove that soliton resolution holds for these solutions. Our proof hinges on an analysis of *collisions* of solitons and an appropriate localized Lyapunov functional, which together allow us to prove a *no-return lemma* for multisoliton configurations.

Building on some of these ideas, we solve, in a joint work with Andrew Lawrie and Wilhelm Schlag, an analogous problem for the harmonic map heat flow of Eells and Sampson (1964) *without assuming any symmetry* of the initial data.

1. The phenomenon of dispersion

This section is devoted to standard introductory material. For a comprehensive introduction to the topic, the reader can consult, for instance, [37, 63].

1.1. The wave equation

Consider the *wave equation* in dimension $1 + 2$:

$$c^{-2}\partial_t^2\psi(t, x_1, x_2) = \partial_{x_1}^2\psi(t, x_1, x_2) + \partial_{x_2}^2\psi(t, x_1, x_2), \quad (t, x_1, x_2) \in \mathbb{R}^{1+2}. \quad (\text{W})$$

The positive number $c > 0$ is the wave speed. Let us assume for simplicity that ψ is real valued, but it could just as well be vector valued. We will always write $\mathbb{R}^{1+2}$

instead of $\mathbb{R}^3$ in order to stress that there is one time dimension and two space dimensions. Equation (W) is equivalent to requiring that ψ is a *critical point* of the Lagrangian

$$\mathcal{L}(\psi) := \iiint \left(\frac{1}{2c^2} (\partial_t \psi)^2 - \frac{1}{2} (\partial_{x_1} \psi)^2 - \frac{1}{2} (\partial_{x_2} \psi)^2 \right) dx_1 dx_2 dt. \quad (1.1)$$

The precise meaning of this assertion is the following. Let $\zeta : \mathbb{R}^{1+2} \rightarrow \mathbb{R}$ be a smooth compactly supported function and $\psi_\varepsilon := \psi + \varepsilon \zeta$, where ε is a small real number. We then have

$$\begin{aligned} \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \mathcal{L}(\psi_\varepsilon) &= \iiint (c^{-2} \partial_t \psi \partial_t \zeta - \partial_{x_1} \psi \partial_{x_1} \zeta - \partial_{x_2} \psi \partial_{x_2} \zeta) dx_1 dx_2 dt \\ &= - \iiint \zeta (c^{-2} \partial_t^2 \psi - \partial_{x_1}^2 \psi - \partial_{x_2}^2 \psi) dx_1 dx_2 dt, \end{aligned}$$

where the last step is integration by parts. The left-hand side can be interpreted as the directional derivative of $\mathcal{L}$ at ψ in the direction ζ . Hence, we see that all the directional derivatives vanish if and only if ψ satisfies (W).

Equation (W) appears in many physical contexts, the most familiar being the evolution in time of a small disturbance of the membrane of a drum. It should be understood that the membrane extends to infinity and occupies the whole horizontal plane, and $\psi(t, x_1, x_2)$ is the vertical displacement at time t of the element of the membrane whose horizontal coordinates are (x_1, x_2) . This is most easily seen using the Lagrangian formulation of classical mechanics. Indeed, let B be an infinitesimal ball in the horizontal plane, centered at (x_1, x_2) . Then the area of the part of the membrane located above B is given by

$$|B| \sqrt{1 + (\partial_{x_1} \psi)^2 + (\partial_{x_2} \psi)^2} \simeq |B| \left(1 + \frac{1}{2} (\partial_{x_1} \psi)^2 + \frac{1}{2} (\partial_{x_2} \psi)^2 \right),$$

where the last step uses the assumption that the disturbance is “small” in the sense that $|\nabla_{x_1, x_2} \psi| \ll 1$. Letting h be the height of the membrane and ρ its mass density, the leading term of the kinetic energy of this part of the membrane is thus

$$\rho h |B| (\partial_t \psi)^2.$$

The potential (elasticity) energy is proportional to the extension of the area and thus equals at the leading order

$$N_{\text{rr}}^* |B| \left(\frac{1}{2} (\partial_{x_1} \psi)^2 + \frac{1}{2} (\partial_{x_2} \psi)^2 \right),$$

where N_{rr}^* is the proportionality coefficient called the radial membrane resultant. In classical mechanics, the Lagrangian density is the difference of the kinetic and the

potential energy densities. Hence, up to a multiplicative constant, we obtain (1.1) with the wave speed $c := \sqrt{\frac{N_T^*}{\rho h}}$. Note that the total energy (again up to a multiplicative constant) is given by

$$E(\psi) := E_{\text{kinetic}} + E_{\text{potential}} = \iint \left(\frac{1}{2c^2} (\partial_t \psi)^2 + \frac{1}{2} (\partial_{x_1} \psi)^2 + \frac{1}{2} (\partial_{x_2} \psi)^2 \right) dx_1 dx_2$$

and is a conservation law (a quantity independent of time).

Remark 1.1. By an appropriate choice of units, one can assume that

$$c = 1,$$

which we will always do in the sequel. We will also write

$$x := (x_1, x_2), \quad \nabla := (\partial_{x_1}, \partial_{x_2}), \quad \Delta := \partial_{x_1}^2 + \partial_{x_2}^2, \quad dx := dx_1 dx_2.$$

Mechanical intuition suggests that in order to determine the evolution in time of the disturbance of the membrane, we need to specify the *initial conditions* consisting of the initial positions $\psi(0, x)$ and the initial velocities $\partial_t \psi(0, x)$ of all the elements of the drum. One can indeed prove that for any such initial conditions, equation (W) has a unique solution for all time, which moreover depends continuously on the initial conditions (in an appropriate sense which we will not make precise here), which is referred to as *global well-posedness*.

Having once more recourse to the intuition from mechanics, we can expect that if the membrane is initially perturbed only in a bounded region and flat elsewhere, then this disturbance will propagate in various directions, resulting in a decay of its amplitude, namely

$$\lim_{t \rightarrow \infty} \sup_{x \in \mathbb{R}^2} |\psi(t, x)| = 0, \quad (1.2)$$

which is referred to as *radiative behavior* and is related to the notion of the *group velocity*, which we now briefly present.

Consider *plane wave solutions* $\psi(t, x) = A \cos(k \cdot x - \omega t)$ with $k \in \mathbb{R}^2$ (the wave number), $\omega \in \mathbb{R}$ (the frequency) and $A \in \mathbb{R}$ (the amplitude). Plugging this formula into (W), we find the *dispersion relation*

$$\omega^2 = |k|^2 \Leftrightarrow \omega = \pm |k|.$$

We choose the sign “plus” and consider the so-called *slowly varying envelope ansatz*. We introduce slow time and space variables $(T, X) = (\varepsilon t, \varepsilon x)$, where $\varepsilon \ll 1$, and set

$$\psi(t, x) = A(T, X) \cos(k \cdot x - |k|t).$$

Then a quick computation using the chain rule yields

$$0 = \partial_t^2 \psi - \Delta \psi = 2\varepsilon |k| \left(\partial_T A + \frac{k}{|k|} \cdot \nabla_X A \right) \sin(k \cdot x - |k|t) + O(\varepsilon^2).$$

Hence, at main order, $A(T, X) = A_0(T - v \cdot X)$; in other words, the amplitude function is transported with the *group velocity* $v = v(k) = \frac{k}{|k|}$.

The above computation is the precise meaning of the claim that “the wave number k propagates with velocity $v(k)$ ”. We see that all the wave numbers $k \neq 0$ propagate with speed 1, but the direction in which they propagate depends on k . Hence, the energy of the wave is dispersed in various directions, leading to (1.2). We say that (W) is an example of a *dispersive* linear PDE.

1.2. A few generalities on linear dispersive PDEs

The above computation leading to the computation of the group velocity can be carried out in much greater generality, which we prefer to leave to the reader. With this preparation, we can state the following heuristic definition.

Definition 1.2. A linear PDE is called *dispersive* if

- (i) it is an evolution equation: it involves the time variable t and the space variable x ,
- (ii) various frequencies propagate with distinct velocities, in the sense that the group velocity defined analogously as above is not constant on any open ball of the momentum space.

No mathematically rigorous definition exists which would cover all the interesting cases, so we will limit ourselves to the non-rigorous definition given above. Examples of linear dispersive PDEs include

- the wave equation (W) and its analogs in higher space dimensions,
- the Schrödinger equation,
- the Klein–Gordon equation,

but the list could be made longer. All these examples are *time reversible* and have a *conserved energy*, and yet smooth localized initial conditions lead to *radiative behavior* as $t \rightarrow \infty$, for reasons similar to the ones exposed above in the case of (W).

Remark 1.3. Radiative behavior crucially depends on the fact that the spatial domain is the whole Euclidean space (or in any case that it is unbounded). The notion of group velocity makes sense also for bounded domains, but we will not discuss this case here.

Remark 1.4. In some sense, for a linear dispersive PDE, the trivial solution $\{\psi \equiv 0\}$ is the *global attractor* of the flow.

2. The nonlinear setting: Dispersion, solitons, soliton resolution

For a comprehensive introduction to the topic of this section, the reader can consult the monographs [51, 58].

2.1. Wave maps

Wave maps are nonlinear, geometric analogs of linear waves in the case of maps taking values in a Riemannian manifold and not a Euclidean space. We consider here wave maps $\Psi : \mathbb{R}^{1+2} \rightarrow \mathbb{S}^2 \subset \mathbb{R}^3$. In classical mechanics, a constrained mechanical system is obtained from the same Lagrangian as for the ambient system; see [1, Chapter 4]. Following this principle and recalling (1.1), we say that a map $\Psi : \mathbb{R}^{1+2} \rightarrow \mathbb{S}^2 \subset \mathbb{R}^3$ is a *wave map* if it is a critical point of the Lagrangian

$$\mathcal{L}(\Psi) = \frac{1}{2} \iiint (|\partial_t \Psi|^2 - |\partial_{x_1} \Psi|^2 - |\partial_{x_2} \Psi|^2) dx dt, \quad (2.1)$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^3$. Similarly, as in Section 1.1, we can consider $\Psi_\varepsilon = \Psi + \varepsilon Z$, where $Z : \mathbb{R}^{1+2} \rightarrow \mathbb{R}^3$ is smooth and compactly supported. In order not to violate at main order, the condition that Ψ_ε has values in $\mathbb{S}^2$, it is necessary and sufficient to require that

$$Z(t, x) \perp \Psi(t, x) \quad \text{for all } (t, x). \quad (2.2)$$

As in Section 1.1, we have

$$\frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \mathcal{L}(\Psi_\varepsilon) = - \iiint Z \cdot (\partial_t^2 \Psi - \Delta \Psi) dx dt.$$

This quantity vanishes for all Z satisfying (2.2) if and only if

$$\partial_t^2 \Psi(t, x) - \Delta \Psi(t, x) = \mu(t, x) \Psi(t, x) \quad \text{for some } \mu(t, x) \in \mathbb{R}.$$

Differentiating twice the identity $\Psi \cdot \Psi = 1$, we obtain

$$\Psi \cdot (\partial_t^2 \Psi - \Delta \Psi) = -|\partial_t \Psi|^2 + |\partial_{x_1} \Psi|^2 + |\partial_{x_2} \Psi|^2,$$

so we can write the wave map equation $\mathbb{R}^{1+2} \rightarrow \mathbb{S}^2$ as follows:

$$\partial_t^2 \Psi - \Delta \Psi = -(|\partial_t \Psi|^2 - |\partial_{x_1} \Psi|^2 - |\partial_{x_2} \Psi|^2) \Psi. \quad (\text{WM})$$

Similarly, as in the linear case, the following *total energy* is a conservation law for wave maps:

$$E(\Psi) := \iint_{\mathbb{R}^2} \left(\frac{1}{2} |\partial_t \Psi|^2 + \frac{1}{2} |\partial_{x_1} \Psi|^2 + \frac{1}{2} |\partial_{x_2} \Psi|^2 \right) dx.$$

Equation (WM) has trivial constant in space-time solutions $\Psi(t, x) = \omega_0 \in \mathbb{S}^2$. If we linearize around such a solution, that is, if we write $\Psi = \omega_0 + \varepsilon \Phi$ with $\varepsilon \ll 1$ and plug into (WM), at main order, we obtain $\partial_t^2 \Phi - \Delta \Phi = 0$ so that each component of Φ satisfies equation (W), which indicates that small perturbations of a constant solution should exhibit a radiative behavior so that the whole wave map should converge to a constant. It was proved in the works of Tataru [60] and Tao [56, 57] that, in an appropriate sense, this is indeed the case.

For the study of long-time behavior of solutions of (WM), *criticality* is an important (and helpful) property. Let $\lambda > 0$ and consider

$$\Psi_\lambda(t, x) := \Psi(t/\lambda, x/\lambda).$$

It is clear from (WM) (or from the Lagrangian) that Ψ is a wave map if and only if Ψ_λ is a wave map. Moreover,

$$E(\Psi_\lambda) = E(\Psi).$$

For this reason, equation (WM) is called *energy critical* and its solutions *critical wave maps*.

Remark 2.1. In general, a problem is *subcritical* if it becomes a “small data problem” when rescaling (zooming) to a small region. It is called *supercritical* if such a zoom makes it large. It is called *critical* if the size of the data remains unchanged.

2.2. Harmonic maps

One might wonder if every, not necessarily small, solution of (WM) has radiative behavior. The answer is “no” for a simple reason: there exist non-trivial static (time-independent) solutions. Namely, inserting $\Psi(t, x) = \omega(x)$ into (WM), we obtain the *critical harmonic maps equation*:

$$-\Delta \omega = |\nabla \omega|^2 \omega, \quad \omega : \mathbb{R}^2 \rightarrow \mathbb{S}^2 \subset \mathbb{R}^3.$$

Its solutions are referred to as *harmonic maps* $\mathbb{R}^2 \rightarrow \mathbb{S}^2$. It was established by Eells and Wood [17] and Hélein [23] that harmonic maps of finite energy correspond to *rational functions* $\mathbb{S}^2 \rightarrow \mathbb{S}^2$ and their complex conjugates (we identify $\mathbb{R}^2$ with $\mathbb{S}^2$ using the stereographic projection).

Remark 2.2. It was proved by Krieger, Schlag and Tataru [34], Rodnianski and Sterbenz [47] and Raphaël and Rodnianski [46] that solutions of large energy can even cease to exist in finite time. Equation (WM) is thus locally well posed but not globally well posed.

2.3. A few generalities on nonlinear dispersive PDEs

A nonlinear PDE is called dispersive if it is related to a linear dispersive PDE. Most frequently, “related” means “obtained through linearization around trivial solutions”, like in the case of wave maps discussed above.

Nonlinear dispersive PDEs frequently appear in physics, for example, in the study of water waves and nonlinear optics; see [63, Chapters 12, 13, 16, 17]. Typical examples are Hamiltonian systems; in particular, they are *time reversible* and have a *conserved energy*.

One is often interested in the *dynamical behavior* of solutions of a given nonlinear PDE, by which we mean their asymptotic description as time becomes large (for solutions defined for all time; if they are not, one studies the limit as the time converges to the maximal time of existence of the solution). Among the most common questions of this type is the *problem of stability*, which can be formulated as follows.

Problem. Do small solutions of a nonlinear dispersive PDE exhibit radiative behavior? In other words, does the flow restricted to small solutions have a trivial attractor, like in the linear case (see Remark 1.4)?

The intuitive reasoning is that small solutions should behave in the same way as the solutions of the linearized problem, which have radiative behavior as we saw in Section 1.2.

2.4. Solitons and soliton resolution

The notion of a soliton is somewhat controversial; see [40, Section 1.5]. We adopt the following definition.

Definition 2.3. A soliton is a solution of an evolution PDE which does not change its shape in the course of time (it can, however, change its position).

Harmonic maps from Section 2.2 are examples of solitons for (WM). Solitons moving at constant velocity can be obtained using the Lorentz invariance of (WM).

Solitons do not exhibit radiative behavior. The problem of *soliton resolution* is to prove that they are the only obstruction to radiative behavior. However, it would be too naive to expect that any solution is either radiative or a soliton. Rather, one expects that any solution eventually decomposes into a superposition of solitons which interact sufficiently weakly (for example, they could travel with distinct velocities). Such superpositions are called *multisoliton configurations*.

Problem (Soliton resolution conjecture). For a given nonlinear dispersive PDE, does any solution converge in large time to a sum of a multisoliton and a radiative term?

In other words, does the flow have a simple global attractor related to multisoliton configurations?

The soliton resolution is inspired by

- numerical simulations (see Fermi, Pasta, Ulam and Tsingou [18] and Zabusky and Kruskal [64]),
- theory of *completely integrable systems* (see Segur and Ablowitz [49] and Eckhaus and Schuur [15]),
- analogous *elliptic* and *parabolic bubbling* phenomena (see the works of Sacks and Uhlenbeck [48] and Struwe [54]).

Remark 2.4. Even with such a vague formulation, the soliton resolution is not expected to hold for all nonlinear dispersive PDEs. For example, the sine-Gordon equation has so-called breather solutions, which do not fall into the regime of soliton resolution.

Remark 2.5. Strictly speaking, soliton resolution only provides an “upper bound” on the global attractor, in the sense that it does not say anything about the types of multisoliton configurations which can be realized by the evolution.

Our main goal is to provide an example of “natural” nonlinear dispersive PDEs for which we can prove soliton resolution. Even though we cannot provide a full description of the global attractor, we will see that it contains configurations consisting of more than one soliton.

3. Soliton resolution for equivariant energy-critical wave maps

3.1. Equivariant maps

This PDE will be obtained from (WM) by restricting the flow to a certain subclass of all the maps $\mathbb{R}^2 \rightarrow \mathbb{S}^2$ which is preserved by the flow. They are called *equivariant maps* and are defined in the following way. We fix $k \in \{1, 2, \dots\}$ and consider maps of the form

$$\Psi(t, r \cos \theta, r \sin \theta) = (\sin(\psi(t, r)) \cos(k\theta), \sin(\psi(t, r)) \sin(k\theta), \cos(\psi(t, r))), \quad (3.1)$$

where $r > 0$ and $\psi(t, r) \in \mathbb{R}$. Plugging this into (WM), we find the following *scalar equation*:

$$\partial_t^2 \psi(t, r) - \partial_r^2 \psi(t, r) - \frac{1}{r} \partial_r \psi(t, r) + \frac{k^2}{2r^2} \sin(2\psi(t, r)) = 0, \quad (\text{WM}_k)$$

where $r > 0$ is the radial coordinate. Note that ψ and $\psi + 2\pi\ell$ represent the same map Ψ for any $\ell \in \mathbb{Z}$.

Remark 3.1. In the non-geometric context, it is common to consider spherically symmetric solutions. Equivariant solutions are analogous objects in the geometric setting of (WM). More generally, whenever symmetries of a given equation lead to invariance of a certain class of states, it is a well-known technique to study the restriction of the system to this subclass.

One can check that under the substitution (3.1), the Lagrangian (2.1) becomes

$$\mathcal{L} := \pi \iint \left((\partial_t \psi)^2 - (\partial_r \psi)^2 - \frac{k^2 \sin(\psi)^2}{r^2} \right) r dr dt.$$

Its critical points are thus k -equivariant wave maps, a fact which can easily be checked directly. The kinetic energy and the potential energy are

$$\begin{aligned} E_{\text{kinetic}} &:= \pi \int_0^\infty (\partial_t \psi)^2 r dr, \\ E_{\text{potential}} &:= \pi \int_0^\infty \left((\partial_r \psi)^2 + \frac{k^2 \sin(\psi)^2}{r^2} \right) r dr. \end{aligned}$$

Their sum is the total energy. It is a conserved quantity.

We always consider strong solutions of finite energy (that is, strong limits of sequences of smooth solutions in the topology induced by the energy, locally uniformly in time). Their existence and uniqueness for any finite energy initial conditions were obtained in [20, 50]. It can be deduced from Strichartz estimates for the wave equation; see, for example, [7, Section 2] in the case $k \in \{1, 2\}$. If $k \geq 3$ is large, Strichartz estimates from [43] can be applied; see [26, Section 2]. Finite energy solutions of (WM_k) are not guaranteed to exist for all time. I denote by $(T_-, T_+) \subset \mathbb{R}$ the maximal time interval on which the solution exists.

3.2. Multibubble (multisoliton) configurations

Recalling the discussion from Section 2.2, the only k -equivariant harmonic maps correspond to rational functions $az^k, az^{-k}, a\bar{z}^k$ and $a\bar{z}^{-k}$ with $a > 0$. In order to represent these maps in the context of the scalar equation (WM_k), it is convenient to denote

$$Q_\lambda(r) := 2 \arctan\left(\frac{r^k}{\lambda^k}\right), \quad \lambda > 0.$$

Then the stationary solutions of (WM_k) are

$$Q_\lambda + 2\pi\ell, \quad -Q_\lambda + 2\pi\ell, \quad \pi + Q_\lambda + 2\pi\ell, \quad \pi - Q_\lambda + 2\pi\ell, \quad a > 0, \ell \in \mathbb{Z}.$$

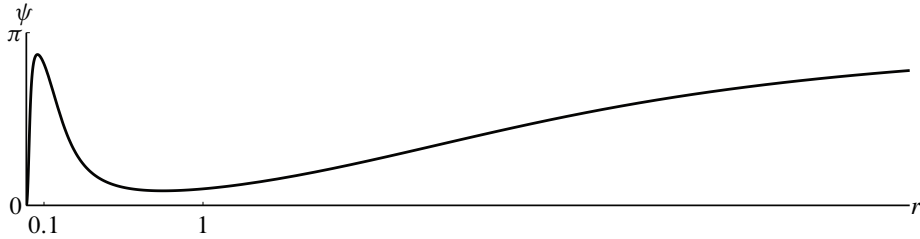


Figure 1. A multibubble configuration with three bubbles.

In this context of equation (WM_k) , solitons are also called *bubbles*. Note that, with our notational conventions, λ is the spatial scale of the bubble Q_λ . For a given number of bubbles M , an integer m and scales $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_M$, we define a *multibubble configuration* by

$$Q(\lambda_1, \dots, \lambda_M) := m\pi + \sum_{j=1}^M \pm Q_{\lambda_j} \quad (3.2)$$

(in the notation, we skip the dependence on m , which is not going to be essential here). One should think of the scales as satisfying $\lambda_1 \ll \dots \ll \lambda_M$ so that each bubble is separated in scale from all the others. Figure 1 shows a multibubble with $(\lambda_1, \lambda_2, \lambda_3) = (\frac{1}{10}, \frac{1}{2}, 5)$.

3.3. Soliton resolution

Our main result can be formulated as follows.

Theorem 3.2 (Jendrej–Lawrie [28]). *Let ψ be a solution of (WM_k) defined for all $t \in (0, \infty)$. As $t \rightarrow \infty$, $(\psi(t, \cdot), \partial_t \psi(t, \cdot))$ decomposes into a superposition of*

- *multibubble configuration,*
- *radiation, corresponding to a solution of the linear wave equation (W),*
- *remainder whose energy converges to 0.*

Remark 3.3. We also prove a similar result in the case of a finite maximal time of existence of the solution.

Remark 3.4. The case $k = 1$ was settled by Duyckaerts, Kenig, Martel and Merle [11] using a different approach (so-called channels of energy; see below).

Remark 3.5. It was proved in [25] that for all $k \geq 2$, there exists a solution containing two bubbles. For these solutions, the radiation component vanishes. It was

proved in [28] that for $k = 1$, multibubble solutions with vanishing radiation component do not exist. Existence of multibubble solutions with a non-vanishing radiation component is an open problem for all k .

The history of progress in understanding the dynamical behavior of large solutions of (WM_k) is quite long. Fundamental results were obtained in [4, 5, 52, 53], see also [51, Chapter 8], the main conclusion being the *decay of energy at the self-similar scale*, which in particular excludes self-similar blow-up, but also, as proved by Struwe [55], leads to *bubbling*: if ψ is a solution of (WM_k) which blows up in finite time T_+ , then there exist sequences $t_n \rightarrow T_+$ and $0 < \lambda_n \ll T_+ - t_n$ such that

$$(\psi(t_n, \lambda_n \cdot), \lambda_n \partial_t \psi(t_n, \lambda_n \cdot)) \rightarrow n\pi \pm Q,$$

the convergence being understood in the topology induced by the energy *locally* (on bounded sets).

The bubbling also implies that a solution whose energy is smaller than the energy of Q cannot blow up. It was proved by Côte, Kenig, Lawrie and Schlag [7] that such a solution actually has radiative behavior. Above this threshold energy, finite time blow-up can occur, as was proved in the works [34, 46, 47] already mentioned above. Important progress toward the soliton resolution conjecture was obtained in [8]. *Sequential* soliton resolution, that is, convergence to a superposition of solitons for a sequence of times, was proved by Côte [6] for $k \in \{1, 2\}$ and Jia and Kenig [30] for $k \geq 3$. Similar results without imposing equivariant symmetry assumptions, but with a less precise description of the radiation, were obtained by Grinis [21].

In [26], continuous in time resolution was proved at the minimal possible energy level allowing for existence of a two-bubble. As a relatively simple consequence, continuous in time resolution was proved in [27] under the assumption that the solution contains at most two bubbles.

For the closely related energy-critical wave equation, scattering below the ground state energy threshold was proved by Kenig and Merle [32], establishing together with [31] the so-called *Kenig–Merle route map*. In the radially symmetric case, the soliton resolution conjecture was proved by Duyckaerts, Kenig and Merle in space dimension 3 in [12], in any odd space dimension in [13] and in dimension 4 in [11] (in collaboration with Martel). All these works used the *channels of energy* introduced in [12].

In the non-radial case, sequential soliton resolution was proved by Duyckaerts, Jia, Kenig and Merle [10].

3.4. Main ideas of the proof

Let ψ be a solution of (WM_k) defined for all $t \in (0, \infty)$. Thanks to the sequential soliton resolution results [6, 30], we know that there exists a sequence $t_n \rightarrow \infty$ such that

$(\psi(t_n, \cdot), \partial_t \psi(t_n, \cdot))$ decomposes into a superposition of a multibubble configuration, radiation and a small remainder. It thus suffices to prove a *no-return lemma*: if a multibubble configuration is destroyed (we say that a *collision* happens), it cannot recover its shape (note the analogy with non-existence of homoclinic/heteroclinic orbits). A similar idea is present in the works of Duyckaerts and Merle [14], Nakanishi and Schlag [42], Krieger, Nakanishi and Schlag [33] for a *single soliton* which is linearly unstable. In our case, interactions between solitons play a role similar to the linear instability in those works. This idea has already been used in [26] in the special case where there are only two bubbles and the radiative component vanishes.

The no-return lemma is obtained by the classical method of Lyapunov functionals. In our case, a Lyapunov functional can be obtained from the *virial identity*

$$\frac{d}{dt} \int_0^\infty \chi_R \partial_t \psi r \partial_r \psi r dr = - \int_0^\infty \chi_R (\partial_t \psi)^2 r dr + O(\mathcal{E}_R). \quad (3.3)$$

Here, χ_R is a standard cut-off function at scale R , and $\mathcal{E}_R$ can be thought of as the amount of energy in the annulus with radii R and $2R$. In order to identify the relevant spatial and temporal scales at which identity (3.3) should be applied, we introduce the notion of *collision intervals*, *inner bubbles* and *outer bubbles*. This is done as follows.

Let $\mathbf{d}(t)$ be the (appropriately defined) distance to (multibubble + radiation) at time t . We say that $[a_n, b_n]$ is a sequence of collision intervals if $\mathbf{d}(a_n) \rightarrow 0$, $\mathbf{d}(b_n) \rightarrow 0$ but $\mathbf{d}(c_n) \not\rightarrow 0$ for some $c_n \in [a_n, b_n]$. The inner bubbles are the ones which enter into a collision, and the outer bubbles are the ones which do not collide. We let K be the smallest natural number K such that there exists an infinite sequence of collision intervals with K inner bubbles.

We now claim that the duration of the collision has to be at least of order λ_K . Indeed, by assumption, at time a_n the solution is close to a multibubble configuration. At a timescale much smaller than λ_K , by finite speed of propagation, the bubble of scale λ_K cannot be destroyed. This, however, would contradict the minimality of K , and the claim follows.

We thus have an infinite sequence of collision intervals $[a_n, b_n]$ with K inner bubbles. Let us neglect for a moment the term $O(\mathcal{E}_R)$ in (3.3) and integrate the virial identity in time over a collision interval $[a_n, b_n]$ with R of order λ_K (we assume for simplicity that λ_K does not change much on any given collision interval; the possibility of important variations in time of λ_K introduces additional complications in the proof). The absolute value of the left-hand side of (3.3) can be seen to be $\ll \lambda_K$ at the endpoints of the interval (indeed, the solution being close to a multibubble configuration implies in particular that the kinetic part satisfies $\|\partial_t \psi\|_{L^2} \ll 1$, and it suffices to apply the Cauchy–Schwarz inequality). We thus conclude that the mean kinetic energy at space-timescale λ_K converges to 0 as $n \rightarrow \infty$.

Finally, we have what we call the *compactness lemma*: if the mean kinetic energy at timescale ρ vanishes, then the solution is close to a multibubble up to scale $\gg \rho$. Intuitively, vanishing of the kinetic energy implies that the solution is “almost stationary” at a given scale, and the bubbles are the only stationary solutions. Our proof of this lemma is very closely related to the already mentioned sequential resolution results. It crucially uses the *profile decomposition* method of Bahouri and Gérard [2] (see also [3, 19, 38, 39, 41] for other notable works leading to the development of the profile decomposition method).

There remains the problem of absorbing the error term $O(\mathcal{E}_R)$ in the virial identity (3.3). In order to accomplish this step, we derive an ODE system on the scales of the bubbles, resembling an n -body problem. It is easy to understand how the relevant n -body problem should look using the Lagrangian formulation again. We consider a multibubble $Q(\lambda_1, \dots, \lambda_M)$ defined by (3.2) and define the reduced Lagrangian

$$\tilde{\mathcal{L}}(\lambda_1, \dots, \lambda_M) := \mathcal{L}(Q(\lambda_1, \dots, \lambda_M)).$$

One then expects that critical points of $\tilde{\mathcal{L}}$ should describe, at least to the leading order, the evolution of the scales of the bubbles. Such a reduction goes by the name of the *modulation method*. In our case, in order to justify that the bubbles indeed evolve as predicted by the ODE, we use the refined modulation parameters introduced in [24]. The ODE system allows us to obtain upper bounds on the length of collision intervals and absorb the error term in (3.3). We note that a similar reduction to a 2-body problem was already used in [26].

4. Bubble decomposition for the harmonic map heat flow

4.1. Harmonic map heat flow

We consider the gradient flow of maps $\Psi : \mathbb{R}^2 \rightarrow \mathbb{S}^2 \subset \mathbb{R}^3$ for the energy functional

$$E(\Psi) := \int_{\mathbb{R}^2} \frac{1}{2} |\nabla \Psi|^2 dx.$$

We obtain the following nonlinear heat equation called the *harmonic map heat flow*:

$$\partial_t \Psi = \mathcal{F}(\Psi) := \Delta \Psi + |\nabla \Psi|^2 \Psi, \quad \Psi : [0, T_+) \times \mathbb{R}^2 \rightarrow \mathbb{S}^2. \quad (\text{HMHF})$$

It was introduced by Eells and Sampson [16] and served as an inspiration for the Ricci flow of Hamilton [22].

Like for the wave maps equation (WM), the stationary solutions of (HMHF) are given by the harmonic maps $\mathbb{R}^2 \rightarrow \mathbb{S}^2$. Convergence to multibubble configurations for a time sequence was proved by Qing and Tian [45]; see also Struwe [54],

Qing [44], Ding and Tian [9], Wang [62], Qing and Tian [45], Lin and Wang [35] and Topping [61], as well as the book by Lin and Wang [36].

4.2. Continuous time bubble decomposition

We prove that convergence toward the set of multibubble configurations holds in continuous time.

Theorem 4.1 (Jendrej–Lawrie–Schlag [29]). *Let Ψ be a solution of (HMHF) defined for all $t \in (0, \infty)$. As $t \rightarrow \infty$, $\Psi(t)$ decomposes into a superposition of*

- *a multibubble configuration,*
- *a remainder converging to 0 in the energy space and uniformly.*

Remark 4.2. We also prove a corresponding result in the case $T_+ < \infty$.

Remark 4.3. Unlike for wave maps, here we impose no symmetry assumptions on the initial data.

4.3. Some proof ideas

Let $\mathbf{d}(t)$ be the (appropriately defined) distance to a multibubble at time t . We say that $[a_n, b_n]$ is a sequence of *collision intervals* if $\mathbf{d}(a_n) \rightarrow 0$, but $\mathbf{d}(b_n) \not\rightarrow 0$. In order to identify the relevant spatial and temporal scales, we use the *energy quantization* for harmonic maps: $E(\omega) = 4\pi|\deg(\omega)|$. We can thus let K be the smallest natural number for which there are collision intervals with colliding bubbles of total energy $4K\pi$ (and the other bubbles remaining intact throughout the collision). We then claim that the duration of the collision has to be at least of order λ^2 , where λ is the largest scale of the colliding bubbles. The reason is again the propagation speed: the heat equation cannot move a substantial portion of energy by distance λ during a timescale much smaller than λ^2 (for the linear heat equation, this follows from the well-known expression for the heat kernel; the argument is more involved in the nonlinear case, but the result remains true).

We thus have an infinite sequence of collision intervals $[a_n, b_n]$, with colliding bubbles of total energy $4K\pi$. Consider now the energy identity $\frac{d}{dt}E(t) = -\|\mathcal{T}(\Psi(t))\|_{L^2}^2$. Upon integration in time over the collision intervals $[a_n, b_n]$ and taking into account that $b_n - a_n \gtrsim \lambda_n$, where λ_n is the largest among the scales of the colliding bubbles, we obtain $\|\mathcal{T}(\Psi(t))\|_{L^2} \ll \lambda_n^{-1}$ as $n \rightarrow \infty$. However, we have the following *compactness lemma*, which is essentially a reformulation of the results of Qing and Tian [45] on almost harmonic maps: if $\Phi_n : \mathbb{R}^2 \rightarrow \mathbb{S}^2$ and $\|\mathcal{T}(\Phi_n)\|_{L^2} \ll \rho_n^{-1}$, then Φ_n converges to multibubbles in disks of radius ρ_n . Letting ρ_n be of order λ_n , we obtain a contradiction with the fact that the bubbles collide

(and thus are not close to a multibubble configuration on some part of the collision interval).

Proving uniform convergence requires additional arguments of a different kind, in particular the use of the Strichartz estimates for the heat flow proved by Tao [59].

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