

Recent progress on sparse equidistribution

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Abstract. Given a dynamical system (X, T) and a sequence of natural numbers (a_n) , one can study the distribution of the sequence $\{T^{a_n}x\}$ for a given $x \in X$. In this note, we survey recent progress on this topic. We also discuss some open problems.

1. Introduction

Classical ergodic theory studies long-term evolution of a measure-preserving system (X, T, μ) . That is, given a point $x \in X$ (or a subset $A \in X$), one is interested in the behavior of the orbit $\{T^n x\}_{n \in \mathbb{N}}$ (or the sets $\{T^n A\}_{n \in \mathbb{N}}$). Two of the most fundamental theorems in the theory are the ergodic theorems by von Neumann [50] (mean ergodic theorem) and Birkhoff [3] (pointwise ergodic theorem). They give a qualitative asymptotics for the behavior of orbits of typical points in the system. Recall that Birkhoff ergodic theorem asserts in particular that for all points $x \in X$ (outside of a measure 0 set) and for every $f \in L^1(X, \mu)$, the ergodic averages $1/N \sum_{n \leq N} f(T^n x)$ converge to a limit $f^*(x)$. Moreover, if the measure μ is ergodic, then $f^*(x) = \int_X f d\mu$. Often, one is also interested in studying a fixed initial condition, i.e., given a topological dynamical system (X, T) , $x \in X$ and $f \in C(X)$, one is interested in asymptotic behavior of $1/N \sum_{n \leq N} f(T^n x)$. In general, one cannot expect asymptotics for all points in the system; however, there are some additional conditions under which one can guarantee convergence for all points (unique ergodicity is the best-known such condition).

Birkhoff's ergodic theorem has an analogue for averaging along some sparse (structured) subsets of the integers. In his fundamental work, Bourgain [4] has shown that if $P(\cdot)$ is a non-constant polynomial with integer coefficients, then for every measure-theoretic dynamical system (X, T, μ) and every $f \in L^2(X, \mu)$, the averages given by $1/N \sum_{n \leq N} f(T^{P(n)}x)$ converge almost surely to a limit. This has been generalized by Wierdl [52] to averages along primes. In this note, we will be

interested in the everywhere counterpart of classical averages. More precisely, given a sequence of natural numbers (a_n) , a topological dynamical system (X, T) , a point $x \in X$ and $f \in C(X)$, one is interested in (topological) conditions under which averages $1/N \sum_{n \leq N} f(T^{a_n} x)$ have a limit. Let us mention a result from [26] where the authors show that if (a_n) is an almost sparse¹ sequence, then there exists a minimal, uniquely ergodic, entropy zero system (X, T) , a point $x \in X$ and $g \in C(X)$ such that the averages $1/N \sum_{n \leq N} g(T^{a_n} x)$ do not have a limit. Therefore, it seems some stronger (quantitative) conditions on the system (X, T) are needed to guarantee convergence. Recently, there has been significant progress in understanding sparse equidistribution in dynamical systems. In this note, we will highlight some classical results, discuss recent developments and formulate some open problems. We will focus on the cases where a_n is (1) a polynomial sequence (this will be discussed in Section 2), (2) a sequence of primes and k -almost primes (this will be discussed in Section 3) and (3) an arithmetic function, like the prime counting function $\Omega(\cdot)$ (this will be discussed in Section 4). In Section 5, we will formulate some open problems.

2. Polynomial sequences

In this section, we will consider the case $a_n = P(n)$, where $P \in \mathbb{Z}[x]$ is a polynomial with integer coefficients. We will also discuss some special cases of generalized polynomials, i.e., $a_n = n^r$, where $r \in \mathbb{R}_+$. It turns out that there are not too many methods to study polynomial sequences, especially for degree ≥ 2 . One classical way of studying polynomial orbits is the so-called van der Corput inequality, which we now recall.

Lemma 2.1 (van der Corput). *Let (b_n) be a sequence of complex numbers bounded in absolute value by 1. Then for every $1 \leq H \leq N$, we have*

$$\left| \frac{1}{N} \sum_{n \leq N} b_n \right| \ll \left(\frac{1}{H} \sum_{h \leq H} \left| \frac{1}{N} \sum_{n \leq N} b_{n+h} \overline{b_n} \right| \right)^{1/2} + \frac{1}{H^{1/2}} + \frac{H^{1/2}}{N^{1/2}}.$$

Given a dynamical system (X, T) , a point $x \in X$ and $f \in C(X)$, one can apply the above inequality to the sequence $b_n = f(T^n x)$. Then using the above inequality, one replaces the original problem of studying $1/N \sum_{n \leq N} f(T^{P(n)} x)$ by

$$\frac{1}{N} \sum_{n \leq N} f(T^{P(n)} x) \overline{f(T^{P(n+h)} x)}. \quad (2.1)$$

¹This means that the average gap in the sequence goes to ∞ . Primes and polynomials of degree > 1 form an almost sparse sequence.

Usually, the last expression is more complicated than the original one (as now one has a bilinear polynomial average); however, in some very special cases, it actually simplifies the problem. Below, we provide a classical example that shows how this simplification works.

Example 2.2. Let $X = \mathbb{T} = \mathbb{R}/\mathbb{Z}$, $T = R_\alpha(x) = x + \alpha \bmod 1$, $\alpha \notin \mathbb{Q}$, $f_k(x) = e^{2\pi i k x}$, and let $P(z) = z^2$. Note that the original expression in this case reduces to studying $\sum_{n \leq N} e^{2\pi i k n^2 \alpha}$, and the one in (2.1) becomes $\sum_{n \leq N} e^{2\pi i k (n^2 - (n+h)^2) \alpha}$ and $n^2 - (n+h)^2 = -h^2 - 2nh$ becomes a linear expression in n which is much easier to study.

In fact, the above example is the main idea behind Weyl equidistribution theorem [51].

Theorem 2.3. For $(R_\alpha, \mathbb{T})$ with $\alpha \notin \mathbb{Q}$, polynomial orbits equidistribute for every initial condition $x \in \mathbb{T}$.

Recall that a sequence $(a_n) \in X$ is equidistributed with respect to a measure κ if for every $f \in C(X)$,

$$\frac{1}{N} \sum_{n \leq N} f(a_n) \rightarrow \int f d\kappa \quad \text{as } N \rightarrow \infty.$$

The main idea is to use repeatedly the van der Corput inequality to reduce the degree of the polynomial. In fact, from the dynamical point of view, the reason why (2.1) is simpler in the case of rotations is that the characters f_k are eigenfunctions of the system, i.e., $f_k(R_\alpha x) = e^{2\pi i k \alpha} f_k(x)$. One can further explore this idea (of generalized eigenfunctions) to study polynomial orbits for nil-rotations on nil-groups. This was done by Leibman [31] and by Green and Tao [22] in a quantitative way. More precisely, we have the following.

Theorem 2.4. Let G be a nilpotent Lie group, and let Γ be a lattice in G . Let $X = G/\Gamma$, and let $g \in G$. Define $T = T_g(x\Gamma) = gx\Gamma$. Then for every $P \in \mathbb{Z}[x]$ and for any $x \in X$, the polynomial orbit $\{T^{P(n)}x\}$ is equidistributed in $\overline{\{T^n x\}_{n \in \mathbb{Z}}}$.

As mentioned above, one way to prove the above theorem is to use the existence of so-called generalized eigenfunctions for nil-translations. However, when there are no (generalized) eigenfunctions in the system, then essentially always the expression in (2.1) becomes more difficult than the initial problem (as it is still a genuine bilinear expression). In fact, the two classes above were, until recently, the only classes of systems for which equidistribution along the sequence of squares (for example) was known. In particular, there were no examples of weakly mixing systems (X, T)

(roughly those with no eigenfunctions) which would be equidistributed along, for example, the sequence $\{n^2\}$. Moreover, since usually if one can apply the van der Corput inequality once, one can apply it repeatedly, there were no systems known which would be equidistributed at $\{n^2\}$ but would fail to equidistribute at $\{n^3\}$. This has changed recently with the works of Kasprzak [28, 29]. In [28], the author showed existence of weakly mixing dynamical systems which are equidistributed at squares (and other polynomial sequences). The main new mechanism in [28] was to study systems which are rigid in a strong sense, i.e., for some sequence (n_k) , $T^{n_k} \rightarrow \text{Id}$ with a polynomial rate. This allowed us to circumvent the lack of eigenfunctions and instead rely on rigidity and arithmetic properties of polynomial sequences in arithmetic progressions and short intervals. In particular, the author worked directly with the polynomial expression and not the bilinear version (which in weakly mixing case is much harder to study). Also in [29], the author constructed symbolic dynamical systems (X, T) which are equidistributed at, for example, squares but fail to equidistribute at, say, cubes. The systems constructed in [29] belong to the class of so-called Toeplitz systems.

For polynomials of degree ≥ 2 , the van der Corput inequality and the method from [28] are essentially the only known methods to study polynomial sequences. In particular, no single mixing system (X, T) is known which would equidistribute along the sequence of squares. Also there are algebraic examples for which equidistribution is not known. This is known as the Margulis–Shah conjecture. This conjecture in the simplest case asks about distribution of polynomial orbits of the horocycle flow on finite volume quotients of $\text{SL}(2, \mathbb{R})$. Some progress toward this conjecture was made (see the following section), but it is limited to sub-quadratic polynomials. We mention more open questions related to that in Section 5.

2.1. Sub-quadratic polynomials

As mentioned above, starting with polynomials of degree 2, there are no general methods to approach, for example, mixing systems. However, there are some recently developed methods for polynomials of the form $\{n^{1+\delta}\}$, $\delta \in (0, 1)$. It will be convenient in this section to restrict our attention to flows, i.e., an $\mathbb{R}$ -parameter family (T_t) acting on a space X (since one can then talk about $\{T_{n^{1+\delta}x}\}$).

The first result of this type comes with the seminal work of Venkatesh [48], in which the author studied equidistribution properties of horocycle flows acting on quotients of $\text{SL}(2, \mathbb{R})$ at times $n^{1+\delta}$ with a small positive δ . We will now recall the needed definitions. Let $G = \text{SL}(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1 \right\}$, and let Γ be a lattice in G (a discrete subgroup such that the quotient has finite volume). Let $h_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ be the horocycle flow acting on $X = G/\Gamma$ by $h_t(x\Gamma) = (h_tx)\Gamma$. Horocycle flows are one of the fundamental objects in dynamical systems, ergodic theory and number theory.

Their dynamical and ergodic properties have been extensively studied. As mentioned above, due to their algebraic nature and tame (polynomial) behavior, it is generally believed that horocycle flows should equidistribute at polynomial times. This is in fact the content of the already mentioned Margulis–Shah conjecture (see Section 5). In [48], the following result was proved.

Theorem 2.5. *Let (h_t) act on X , where X is compact. There exists $\delta_\Gamma > 0$ such that for every $0 < \delta < \delta_\Gamma$, every $x \in X$, the orbit $\{h_{n^{1+\delta}}x\}$ is equidistributed with respect to the Haar measure on X .*

In fact, the method above gives (in many instances) $\delta_\Gamma = 1/48$; however, the author did not optimize it. This theorem was generalized by Flaminio–Forni–Tanis [16] to get rid of the dependence on Γ . More precisely, the authors showed that in the compact setting, the sequence $\{h_{n^{1+\delta}}x\}$ equidistributes for any $\delta < 1/12$. In the work of Streck [45], the author generalized the above results to the case of non-compact X (where the formulation is more delicate due to existence of periodic orbits). The approach in [48] uses so-called twisted ergodic integrals in which one twists the classical ergodic integral by an exponential phase. In fact, the two main properties of the horocycle flow that are used in the proof are polynomial mixing and polynomial equidistribution. Let us make a general definition to explain the general result.

Definition 2.6. Let (T_t) be a C^∞ flow on a smooth compact manifold M that preserves a measure μ . We say that (T_t) is *polynomially mixing on C^k* if there exists $\eta > 0$ and $C_k > 0$ such that for any $\phi, \psi \in C^k(M)$,

$$\left| \int_X \phi \circ T_t \cdot \psi d\mu - \int_X \phi d\mu \int_X \psi d\mu \right| \leq C_k \|\phi\|_k \|\psi\|_k t^{-\eta}.$$

We say that (T_t) has *polynomially decaying ergodic averages on C^k* if there exist $\kappa > 0$ and $C'_k > 0$ such that for any $\phi \in C^k(M)$,

$$\left| \int_0^R \phi(T_t x) dt - R \int_X \phi d\mu \right| \leq C'_k \|\phi\|_k R^{1-\kappa}.$$

It is known that horocycle flows are both polynomially mixing on C^k [23, 39] and have polynomially decaying ergodic averages on C^k [15, 46]. The main idea in the method of Venkatesh is to use the above two properties. In fact, as was noted in a paper by Auer [1], this approach works in the general case. We have the following proposition.

Proposition 2.7. *Let (T_t) be a uniquely ergodic flow on M which is polynomially mixing and has polynomially decaying averages (for some $k \in \mathbb{N}$). Assume that (T_t)*

has polynomially growing derivatives.² Then there exists $0 < \delta_0 = \delta_0(\eta, \kappa) < 1$ such that $\{h_{n^{1+\delta}}x\}$ is equidistributed (with respect to the unique ergodic measure) for any $x \in X$ and any $0 < \delta < \delta_0$.

As noted above, the δ_0 depends on the mixing rate and the equidistribution rate. The above methods cannot go beyond the case $\delta_0 = 1/2$, and so other methods are needed to cover the cases $\{n^{1+\delta}\}$, $\delta \geq 1/2$. This problem remains widely open, and there are currently no new methods to prove equidistribution in this case. However, if one is interested in density of horocyclic orbits at polynomial times instead of equidistribution, one can employ a different approach to cover the cases $\delta \in [0, 1)$. In a recent preprint [27], the authors have shown that if $X = \mathrm{SL}(2, \mathbb{R})/\Gamma$ is not compact, then for every non-periodic point $x \in X$, the orbit $\{h_{n^{1+\delta}}x\}$ is dense.

Theorem 2.8. *Let Γ be a non-compact lattice in $\mathrm{SL}(2, \mathbb{R})$. Then for every $\delta \in (0, 1)$, every point $x \in X = \mathrm{SL}(2, \mathbb{R})/\Gamma$ which is not periodic for (h_t) , the orbit $\{h_{n^{1+\delta}}x\}$ is dense in X .*

Of course, the case of small $\delta > 0$ is already covered by [45] (with equidistribution not only density), so the above theorem becomes most interesting for δ close to 1. The methods of showing density are different from those that show equidistribution for small $\delta > 0$. The main idea in the proof of the above theorem is an approximation scheme, along a sequence of times, of a long enough piece of an orbit of x by a periodic orbit. For this, one uses quantitative equidistribution of horocycle flows together with non-divergence results and harmonic analysis methods on the circle to study the twist in the approximation. In particular, this method does not apply to the case of compact X (as there are no periodic points in this case). This approximation however stops working at the endpoint $\delta = 1$: one can show that the optimal piece of an orbit of $x \in X$ which is approximated by a periodic orbit at time $T > 0$ cannot in general be longer than $c \cdot T^{1/2}$ (for some small $c > 0$). In particular, this interval contains at most one square, and so one cannot use it to study square orbits (or any polynomial orbits with polynomials of degree ≥ 2). A similar approach can be used to study orbits of the horocycle flow sampled at $\Omega(n)$ (the prime factors counting function). We discuss this case in Section 4.

²This means that $\|\phi \circ T_t\| \ll_k t^{kd} \|\phi\|_k$, where $\|\cdot\|_k$ is the C^k norm. This holds for the horocycle flow.

3. Primes and almost primes

In this section, we will discuss results on so-called prime number theorems in dynamics. Given a dynamical system (X, T) and a point $x \in X$, one is interested in the behavior of the orbit $\{T^p x\}_{p \text{ is prime}}$. More generally, one also considers orbits along k -almost primes, i.e., numbers with at most k -prime factors. At first look, studying primes seems related to Sarnak's conjecture (see Section 4 for details). However, the methods used to study Sarnak's conjecture require softer input, both on the dynamical and number theory side. The number-theoretic tools (for example, [7, 8, 30, 34, 47]) that are used in the context of Sarnak's conjecture rely on the fact that most integers are composite and thus factor. These methods cannot be used for studying primes or k -almost primes. A dynamical prime number theorem has so far been established in some algebraic and symbolic examples: rotations on $\mathbb{T}$ [49], nilsystems [22], Rudin–Shapiro sequences [37], enumeration systems [5, 21], certain finite rank symbolic systems [6], automata [40]. The main (classical) tool used to establish a prime number theorem is the method of type I and type II sums. We recall one of the formulations due to Duke, Friedlander and Iwaniec [10].

Theorem 3.1 (A criterion for PNT). *Let (x_n) be a bounded sequence. If for every $\varepsilon > 0$, both of the following conditions hold:*

- *type I sums (of level $1/2$): for $D \leq N^{1/2-\varepsilon}$,*

$$\frac{1}{D} \sum_{D \leq d \leq 2D} \left| \frac{d}{N} \sum_{n \leq N/d} x_{dn} \right| \ll_{\varepsilon} \frac{1}{\log^3 N},$$

- *type II sums (of level $1/3$): for $D \leq N^{1/3-\varepsilon}$,*

$$\frac{1}{D^2} \sum_{D < d_1, d_2 < 2D} \frac{1}{\min(N/d_1, N/d_2)} \left| \sum_{n \leq \min(N/d_1, N/d_2)} x_{d_1 n} \overline{x_{d_2 n}} \right| \ll_{\varepsilon} \frac{1}{\log^{22} N},$$

then

$$\sum_{p \leq N} x_p = o(\pi(N)).$$

We should mention that there are variations in the above criterion, where one can adjust a bit between the level of distribution of type I and type II sums. However, what is important, and also makes the problem often very difficult, is that one needs quantitative information on both of these sums and the step, i.e., the numbers d, d_1, d_2 depend on N . Let us discuss the case of type II sums (type I sums are usually easier).

Notice that if we apply the criterion to $x_n = f(T^n x)$ for some dynamical system (X, T) , $x \in X$, $f \in C(X)$, then type II sums are expressions of the form

$$\sum_{n \leq \min(N/d_1, N/d_2)} f(T^{d_1 n} x) \overline{f(T^{d_2 n} x)},$$

where the step d_1, d_2 can be of size $N^{1/3-\varepsilon}$. This makes it impossible to apply any classical tools from ergodic theory when one fixes the transformation and then takes longer and longer averages. This also highlights the difference between Sarnak's conjecture and primes, as in the former case, the DKBSZ orthogonality criterion, allows one to fix d_1, d_2 and then let N go to ∞ . Therefore, for all the cases where a prime number theorem was established, the authors used some quantitative information about the system that would usually allow one to reduce type II sums to type I sums (as is the case for rotations) and then study type I sums (arithmetic progressions) using either the algebraic or symbolic structure of the system.

Relatively recently, the authors in [26] developed a new method to study primes that avoids studying type I and type II sums for the system directly but rather approximates the system with a sequence of periodic systems (with a polynomial twist in the phase) and then relies on results on type I and type II for polynomial exponential sums. In fact, the authors also need to use type III sums as the level for type II sums is barely sufficient. The main theorem in [26] is the following.

Theorem 3.2. *Let $X = \mathbb{T}^2$ and $g : \mathbb{T} \rightarrow \mathbb{R}$ be an analytic function. Let $T : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be given by $T(x, y) = (x + \alpha, y + g(x))$. If T is uniquely ergodic, then it satisfies a prime number theorem.*

A weaker question one might ask is density of prime orbits, i.e., is it true that for a given $x \in X$, the orbit $\{T^p x\}_{p \text{ is prime}}$ becomes dense in X ? A still weaker question is on non-concentration of primes, i.e., can $\{T^p x\}$ be trapped in a ball of small radius? In fact, to study these questions (especially non-concentration), one can use sieve methods, i.e., the upper bound sieve [24]. This was used by Sarnak and Ubis [43] to show non-concentration of prime orbits for the horocycle flow for $\Gamma = \mathrm{SL}(2, \mathbb{Z})$. The main benefit is that the upper bound sieve only uses type I sums (arithmetic progressions). These in turn can be studied using quantitative equidistribution results for horocycle flows. The main result in [43] is the following.

Theorem 3.3. *Let $\Gamma = \mathrm{SL}(2, \mathbb{Z})$. Let $x \in X = \mathrm{SL}(2, \mathbb{R})/\Gamma$ be a non-periodic point. Then $\mu_{\mathrm{Haar}}(\overline{P_x}) \geq 1/10$, where $P_x = \{h_p x\}_{p \text{ prime}}$.*

This result was generalized by Streck [44] to general non-compact lattices in $\mathrm{SL}(2, \mathbb{R})$. As mentioned in [43], the upper bound sieve and quantitative equidistribution could be used to show density of orbits sampled at k -primes for k large enough.

This was done by McAdam, who proved the following (we state the result for the horocycle flow only; the result in [38] is true for horospherical actions in $\mathrm{SL}(n, \mathbb{R})$).

Theorem 3.4. *Let Γ be co-compact. Then there exists a $k = k(\Gamma) \in \mathbb{N}$ such that the orbit of every $x \in X = \mathrm{SL}(2, \mathbb{R})/\Gamma$ is dense when sampled at k -primes (recall that these are numbers with at most k -prime factors).*

It should be mentioned that the above results use type I sums only, and so the k is large: as already mentioned in [43], the methods work for $k \geq 10$. In what follows, we will discuss the case of 2-primes for which we recently obtained equidistribution results for horocycle flows.

3.1. Equidistribution along semi-primes

In this section, we discuss equidistribution along 2-primes. Recall that these are integers of the form $p_1 \cdot p_2$ with p_i prime. It seems that to study 2-primes, type I information is not sufficient. In a recent paper [17], the authors established a criterion that allows one to study 2-primes for some classes of dynamical systems. The precise criterion is the following.

Proposition 3.5 ([17]). *Let $\varepsilon > 0$ be given. For all $N \geq 2$, let $\mathcal{P}_{\varepsilon, N}$ be the set of primes in the interval $[\exp(\log^\varepsilon N), \exp(\log^{1-\varepsilon} N)]$ with $\exp(\log^\varepsilon N) > (\log N)^{1000}$, and let $\mathcal{S}_{\varepsilon, N}$ be a subset of $\mathcal{P}_{\varepsilon, N}$ with the property that in every dyadic interval $[P, 2P]$, we discard at most $\ll P/(\log N)^{100}$ primes. Let (a_n) be a sequence with $|a_n| \leq 1$. Suppose that, for $N \geq 2$,*

$$\sum_{n \leq N} a_{nq_1} \overline{a_{nq_2}} \ll \frac{N}{(\log N)^{100}} \quad (3.1)$$

for primes $q_1, q_2 \in \mathcal{S}_{\varepsilon, N}$ with $1/5 \leq q_1/q_2 \leq 5$ and $q_1 \neq q_2$. Then,

$$\sum_{pq \leq N} a_{pq} \ll \varepsilon \cdot \sum_{pq \leq N} 1 + \frac{1}{\varepsilon^{53}} \cdot \frac{N}{(\log N)^{50}}. \quad (3.2)$$

Moreover, assume that for every $M \in [N^{9/10}, N]$, for N large enough,

$$\sum_{n \leq M} a_{nq_1} \overline{a_{nq_2}} \ll \frac{M}{(\log \log M)^{10}}$$

holds for all $q_1 \neq q_2$, with $q_1, q_2 \in [e^{(\log \log \log N)^3}, e^{(\log \log N)^{10}}]$ and $1/5 \leq q_1/q_2 \leq 5$. Then for every multiplicative function v with $|v| \leq 1$, for all $N \in \mathbb{N}$ large enough,

$$\left| \sum_{n \leq N} v(n) a_n \right| \ll N(\log \log N)^{-4}.$$

The main parts of the criterion are (3.1) and (3.2). The part on multiplicative functions is a quantitative version of the orthogonality criterion already studied in [7]. Notice that (3.1) is still a bound on type II sums. However, the main difference with respect to the criterion for primes is that the level of distribution is much lower. In fact, it follows from the statement that if there exists $\gamma > 0$ such that for any two primes $q_1, q_2 \leq N^\gamma$, we have that (3.1) holds, then we can deduce equidistribution along 2-primes (recall that classically for primes, one needs $\gamma = 1/3$). Therefore, this general criterion seems to be applicable to a wider class of dynamical systems. In [17], the authors use this criterion to show equidistribution of horocycle flows along 2-primes. The main result in [17] is the following.

Theorem 3.6. *Let Γ be a co-compact arithmetic lattice in $\mathrm{SL}(2, \mathbb{R})$, or let $\Gamma = \mathrm{SL}(2, \mathbb{Z})$. Then for every non-periodic point $x \in X = \mathrm{SL}(2, \mathbb{R})/\Gamma$, the orbit $\{h_{p_1 \cdot p_2} x\}_{p_1, p_2 \text{ primes}}$ is equidistributed in X .*

The main input for the theorem is a quantitative type II sums estimate (at any polynomial level). This reduces (in the case of horocycle flows) to studying quantitatively deviation of ergodic averages for the product flow $h_t \times h_t$ on $X \times X$. Until recently, this was a major open problem in homogeneous dynamics. Qualitative results were established in the landmark work of M. Ratner in the 1980s [41]; however, no rates could be obtained from Ratner's work. In a recent breakthrough of Lindenstrauss–Mohammadi–Wang [32] (see also [53] for related work), the authors obtained a quantitative result for the square of the horocycle flow (with a polynomial error). To state the main result of [32], let $a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$ be the geodesic flow on X . Notice the following renormalization relation between the geodesic and horocycle flows:

$$a_t h_s = h_{e^t s} a_t.$$

Let H denote the diagonal embedding of $\mathrm{SL}(2, \mathbb{R})$. The main result of [32] is as follows.

Theorem 3.7. *Assume Γ is an arithmetic lattice. For every $(x, y) \in X \times X$ and large enough R (depending explicitly on X), for any $T \geq R^{A_1}$, at least one of the following holds:*

- (1) *For every $\varphi \in C_c^\infty(X \times X)$, we have*

$$\left| \frac{1}{T} \int_0^T \varphi(h_t \times h_t(x, y)) - \int \varphi d\mu \times \mu \right| \leq S(\varphi) R^{-\kappa_2},$$

where $S(\varphi)$ is a certain Sobolev norm.

- (2) *There exists a periodic $z \in X \times X$ with $\text{vol}(Hz) \leq R^{A_1}$, and for every $r \in [0, T]$, there exists $g \in \text{SL}(2, \mathbb{R})^2$ with $\|g\| \leq R^{A_1}$ so that*

$$d_{X \times X}((h_s \times h_s)(x, y), gHz) \leq R^{A_1} \left(\frac{|s - r|}{T} \right)^{1/A_2} \quad \text{for all } s \in [0, T].$$

- (3) *For every $r \in [0, T]$ and $t \in [\log R, \log T]$ the injectivity radius at $(a_{-t} \times a_{-t})((h_r \times h_r)(x, y))$ is at most $R^{A_1} e^{-t}$.*

The constants A_1, A_2 and κ are positive and depend on X but not on (x, y) .

We should point out that the meaning of periodic orbit in the above theorem is that it is a closed H -invariant subspace (not necessarily a periodic point). We will not define some of the terms in the above theorem (like volume of a periodic orbit). The above theorem is optimal in the following sense: it is known and not so hard to see that if a point satisfies (2) or (3), then it cannot be polynomially equidistributed (i.e., it cannot satisfy (1)). The significance of the above result lies in the fact that the straightforward obstructions to (1) are the only obstructions. Let us now briefly comment on how to approach Theorem 3.6. We use the criterion for 2-primes which requires studying ergodic averages of length N for the map $h_{q_1} \times h_{q_2}$, where q_i can be as large as N^ε . Notice that using renormalization (at times $\log q_i$), one can reduce this problem to studying ergodic averages for the map $h_1 \times h_1$ at a point $(a_{-\log q_1} x, a_{-\log q_2} x)$ and the function $F(y_1, y_2) = f \circ a_{\log q_1}(y_1) \cdot f \circ a_{\log q_2}(y_2)$ (where we need a polynomial gain over the trivial bound for the averages). This puts us in the position to apply Theorem 3.7 with $R = N^\kappa$ for some small κ . Using this, we get that either we have a polynomial saving in the ergodic average for $h_1 \times h_1$ (and we can apply the criterion) or there are q_1, q_2 such that the point $(a_{-\log q_1} x, a_{-\log q_2} x)$ satisfies (2) or (3). Here, the approach is different in the case of co-compact lattices and $\text{SL}(2, \mathbb{Z})$. First note that if the lattice is co-compact, then (3) can never hold since the injectivity radius is uniformly bounded below on X . Moreover, using some algebraic considerations, we show that the point $(a_{-\log q_1} x, a_{-\log q_2} x)$ never satisfies (2) for co-compact arithmetic lattices. The argument for this can be thought of as a quantitative version of the corresponding argument in [7], where the authors showed that $(a_{-\log q_1} x, a_{-\log q_2} x)$ never lies on an invariant subvariety. In our case, we need to relate the largeness of the q_i 's with the volume of the subvariety.

The most difficult part of the paper is studying (2) and (3) for points given by $(a_{-\log q_1} x, a_{-\log q_2} x)$ and the lattice $\text{SL}(2, \mathbb{Z})$. In this case, it can happen that both (2) and (3) are satisfied, but this gives us enough restrictions on the starting point x to complete the argument using a different approach. To skip the technical details, we will only focus on highlighting the main difficulties in both cases. The model behavior for which (3) holds is if x itself is very close to a periodic point (for (h_t)) of

length $N^{\kappa'}$. In this case (skipping lots of details), the analysis boils down to the analysis on the circle and the Hardy–Littlewood method. However, since the period might be polynomially large, we cannot use directly classical results from number theory, for example, the Siegel–Walfisz theorem (which only works for periods of the form $\log^A N$). We use stronger results on prime in arithmetic progressions of polynomial size, where the result is twisted by a quadratic character. More precisely, we prove (let $\pi_2(N)$ denote the number of semi-primes up to N) the following.

Proposition 3.8. *Let $J \subset [0, T]$ be an interval of length $|J| \geq T^{1-\kappa}$, and let $r < T^\kappa$. Then*

$$|\{pq \in J : pq \equiv a \pmod{r}\}| = 1_{(a,r)=1} \frac{\pi_2(J)}{\varphi(r)} + \chi(a) \cdot \frac{F_{J,r,T}}{\varphi(r)} + O\left(\frac{1}{\varphi(r)} \kappa \pi_2(J)\right),$$

where χ is a quadratic real character and $F_{J,r,T}$ is such that $|F_{J,r,T}| \leq \pi_2(J)$ for all large enough T .

If the period is irrational (or not too close to rational), we apply minor arc-type estimates, but there is an extra problem to overcome: we need to get polynomial saving. If the point $(a_{-\log q_1} x, a_{-\log q_2} x)$ satisfies (2), then we use Ratner’s theory to relate the periodic orbit Hx with a lattice Γ_{q_1, q_2} and the corresponding horocycle flow on $X_{q_1, q_2} = \mathrm{SL}(2, \mathbb{R})/\Gamma_{q_1, q_2}$. In this case, we need a quantitative equidistribution result for the horocycle flow with polynomial saving which would be uniform over all q_1, q_2 up to size N^ε . This requires a new input, as earlier results [15, 46] obtain bounds for a fixed lattice. Using the methods of [15], we prove the following.

Theorem 3.9. *Let $X_\Gamma = \mathrm{SL}(2, \mathbb{R})/\Gamma$. For every $s \geq 4$, there exists a constant $C_s > 0$ such that for every function $f \in W^s(X_\Gamma)$ and for all $(x, T) \in X_\Gamma \times [1, +\infty)$,*

$$\left| \frac{1}{T} \int_0^T f \circ h_t(x) dt - \int_{X_\Gamma} f d \mathrm{vol}_\Gamma \right| \leq C_s \|f\|_{W^s(X_\Gamma)} \max\{\mathrm{inj}_\Gamma^{-1}, e^{(1-\nu_\Gamma)/2 d_\Gamma(a_{\log T}(x))}\} T^{-(1-\nu_\Gamma)/2},$$

where $\nu_\Gamma = \sqrt{1 - 4\mu_\Gamma}$ and $\mu_\Gamma > 0$ is the bottom of the spectrum of the Laplace operator and $d_\Gamma(\cdot)$ denotes the distance to e in $X = \mathrm{SL}(2, \mathbb{R})/\Gamma$.

In Theorem 3.9, the volume $d \mathrm{vol}_\Gamma$ is normalized, while the Sobolev spaces are defined with respect to the constant curvature metric (whose volume is not normalized). As one can see, these estimates depend explicitly on the lattice and involve the injectivity radius of the compact part M_Γ and the number ν_Γ . We then show that the lattices Γ_{q_1, q_2} are congruence lattices, and so we have a uniform bound on ν_Γ and also on inj_Γ . We also need to use a result of [11] to get estimates on closeness of vol_Γ to the product Haar measure.

4. Arithmetic functions

In this section, we briefly discuss recent developments around correlation between dynamically defined sequences and multiplicative functions. This area has seen a big development in the last 15 years. We will focus only on main techniques that led to the progress and not so much on the particular results. For a detailed discussion, we refer the reader to [14]. In the last subsection, we will discuss the case of the $\Omega(\cdot)$ function (prime counting function). The interest in correlation of multiplicative functions with dynamical sequences was initiated by the following conjecture of Sarnak [42].

Conjecture 4.1 (Sarnak). *Let (X, T) be a dynamical system of topological entropy zero. Let $\mu(\cdot)$ be the Möbius function, i.e., $\mu(n) = 0$ if n is divisible by a square and otherwise $\mu(n) = (-1)^k$, where k is the number of different prime divisors of n . Then for any $f \in C(X)$ and any $x \in X$,*

$$\sum_{n \leq N} \mu(n) f(T^n x) = o(N). \quad (4.1)$$

Even though this conjecture is open in full generality, there has been substantial progress in verifying the conjecture for various (natural) classes of dynamical systems (see [14]). The first main tool is the so-called orthogonality criterion [7, 8, 30] which allows one to study correlations with (general) multiplicative functions by studying type II sums qualitatively, i.e., to show that (4.1) holds, it is enough to show that for two different prime numbers q_1 and q_2 ,

$$\sum_{n \leq N} f(T^{q_1 n} x) \overline{f(T^{q_2 n} x)} = o(N).$$

The fact that we only need qualitative information simplifies the situation dramatically as now one is interested in classical ergodic averages for the map $T^{q_1} \times T^{q_2}$. This is strongly connected to the theory of joinings in ergodic theory introduced by H. Furstenberg [20]. The other source of progress around Sarnak's conjecture is related to results on behavior of multiplicative functions in short intervals. Here, the breakthrough came with the work of Matomäki and Radziwiłł [37], who proved the following result.

Theorem 4.2. *Let $f : \mathbb{N} \rightarrow [-1, 1]$ be a multiplicative function. Then for any $2 \leq H \leq X$,*

$$\left| \frac{1}{H} \sum_{n \in [x, x+H]} f(n) - \frac{1}{X} \sum_{n \in [X, 2X]} f(n) \right| = o_{H \rightarrow \infty}(1)$$

for all but $o(X)$ of the intervals $[x, x+H]$.

The strength of the above theorem is that the H can go to ∞ arbitrarily slowly with X . These types of results are beyond rich (or sometimes not true) for k -primes or polynomials. These results, together with related works, [35, 36] allow for many classes of dynamical systems (X, T) to reduce the problem to qualitative behavior of T on typical short intervals where one can again use tools from ergodic theory. We also mention other methods to study Sarnak's conjecture such as Furstenberg systems [18, 19], and we refer the reader to these papers for more details.

4.1. The function Ω

Here, we discuss some results on distribution of orbits in dynamical systems along the sequence $\Omega(n)$. Recall that if $n = \prod_{i=1}^k p_i^{\alpha_i}$ is the prime factorization of n , then $\Omega(n) = \sum_{i=1}^k \alpha_i$. In [2], Bergelson and Richter proved the following general result.

Theorem 4.3. *Let (X, T) be uniquely ergodic (with the unique measure μ). Then for every $f \in C(X)$, $x \in X$,*

$$\left| \frac{1}{N} \sum_{n \leq N} f(T^{\Omega(n)} x) - \int_X f d\mu \right| = o(1).$$

This theorem is general as the only assumption is unique ergodicity. It is also necessary: in [33], Loyd provided an example of a non-uniquely ergodic topological system for which the above averages do not converge for some $x_0 \in X$. More generally, for a dynamical system (X, T) and $x \in X$, one defines the empirical measures $\nu_N(x) = 1/N \sum_{n \leq N} \delta_{T^n x}$. One can then define $\text{Acc}(x)$ to be all the accumulation points of the sequence $\{\nu_N(x)\}$ (note that all measures in $\text{Acc}(x)$ are T -invariant). For a uniquely ergodic system, it follows that $\text{Acc}(x)$ is a singleton for any $x \in X$. Analogously, one can define $\nu_N^\Omega(x) = 1/N \sum_{n \leq N} \delta_{T^{\Omega(n)} x}$ and $\text{Acc}^\Omega(x)$ to be the accumulation points of the measures $\{\nu_N^\Omega(x)\}$. It follows from [2] that all the elements of $\text{Acc}^\Omega(x)$ are T -invariant. Then one can rephrase the above theorem by saying that for uniquely ergodic systems, $\text{Acc}^\Omega(x) = \text{Acc}(x)$. The relation between the sets $\text{Acc}^\Omega(x)$ and $\text{Acc}(x)$ can be studied for general dynamical systems (X, T) . In [25], the authors show that one of the inclusions persists for ergodic measures.

Theorem 4.4. *Let (X, T) be any topological dynamical system. Then for any $x \in X$,*

$$M_T^e(X) \cap \text{Acc}(x) \subset \text{Acc}^\Omega(x).$$

In other words, the theorem says that if a point $x \in X$ is quasi-generic for an ergodic measure μ , then the measure μ can be seen as a limit of a subsequence of Ω -empirical measures for x . It would be interesting to know if the ergodicity assumption can be dropped from the above theorem (see Section 5), but the current method

of proof uses ergodicity in an essential way. The main number-theoretic input behind this theorem is results on statistical behavior of Ω . It follows from Erdős and Kac [13] that the values of Ω become normally equidistributed.

Theorem 4.5 (Erdős–Kac theorem). *For $A < B \in \mathbb{R}$,*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \# \left\{ n \leq N : A \leq \frac{\Omega(n) - \log \log N}{\sqrt{\log \log N}} \leq B \right\} = \frac{1}{\sqrt{2\pi}} \int_A^B e^{-t^2/2} dt.$$

In fact, we need a stronger result (a local limit theorem). This was proven by Erdős [12, Theorem II].

Given $k \in \mathbb{N}$, let $\mathbb{P}_k$ denote the set of k -almost primes, those integers satisfying $\Omega(n) = k$. Let $\pi_N(k)$ denote the number of k -almost primes less than or equal to N . Then

$$\pi_N(k) = \#\{n \leq N : \Omega(n) = k\} = \sum_{n \in \mathbb{P}_k \cap [N]} 1.$$

The result of Erdős provides a uniform estimate for $\pi_N(k)$ for certain values of k . For $N \in \mathbb{N}$, define the interval I_N by

$$I_N := [\log \log N - C \sqrt{\log \log N}, \log \log N + C \sqrt{\log \log N}],$$

where $C \geq 1$ is some fixed constant. Define

$$R_N := \log \log N + C \sqrt{\log \log N}$$

to be the right-hand endpoint of this interval.

Lemma 4.6 ([12]). *The function $\pi_N(k)$ satisfies*

$$\pi_N(k) = \frac{N}{\log N} \frac{(\log \log N)^{k-1}}{(k-1)!} (1 + o_{N \rightarrow \infty}(1))$$

uniformly for $k \in I_N$.

Using the above number-theoretic results along with ergodic-theoretic methods, we can reduce the statement to usual ergodicity on short intervals. This can be done for a specific sequence of times (depending on x).

4.1.1. Horocycle flows at $\Omega(n)$. Recall that if $X = \mathrm{SL}(2, \mathbb{R})/\Gamma$ is compact, then the horocycle flow (and also the time 1 map) is uniquely ergodic, and therefore, the theorem of Bergelson and Richter applies. However, in the non-compact case, the flow is no longer uniquely ergodic, and therefore, it makes sense to ask what happens with the horocyclic orbits along the sequence $\Omega(n)$. To state the result, we need some

definitions. Let Γ be such that X is not compact. It is known that for some $k \geq 1$, the space X has k (inequivalent) cusps, and for every cusp C_i , there is a one-parameter family of periodic measures corresponding to C_i . Denote by $\{\nu_s^i\}_{s \in \mathbb{R}, i \leq k}$ the set of periodic measures for the horocycle flow. By the work of Dani and Smillie [9], all the ergodic invariant measures for the horocycle flow are either μ or one of the $\{\nu_s^i\}$, i.e., for every $x \in X$, $\text{Acc}(x)$ is a singleton, and it is either equal to μ or $\nu_s^{(i)}$. The main result of [25] describes the set of accumulation points for averages along $\Omega(\cdot)$.

Theorem 4.7. *For any non-periodic point $x \in X$, we have the following:*

$$\text{Acc}^\Omega(x) \subset \{\mu\} \cup \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{r^2}{2}} \nu_{s_0+2\log(|1+cr|)}^{(i)} dr : 0 < c, i \leq k, s_0 \in \mathbb{R} \right\}. \quad (4.2)$$

Moreover, there exists $D' > 0$ such that for every $i \leq k$, $s_0 \in \mathbb{R}$, there exists³ $C(s_0) > 0$ and $s' \in [s_0, s_0 + 1]$ such that for some $u' \in [C(s_0), D'C(s_0)]$, we have

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{r^2}{2}} \nu_{s'+2\log(|1+u'r|)}^{(i)} dr \in \text{Acc}^\Omega(x). \quad (4.3)$$

Finally, $\mu \in \text{Acc}^\Omega(x)$.

The difference between (4.2) and (4.3) is the following: (4.2) states that any accumulation point belongs to the set on the left-hand side with *any* parameters $c > 0, i \leq k, s_0 > 0$. On the other hand, (4.3) states that given i, s_0 , one can find an accumulation point *bounded distance* from them, i.e., $s' \in [s_0, s_0 + 1]$ and $u' \in [C(s_0), 2C(s_0)]$. It is interesting to point out that for a periodic point with rational period, the corresponding measure for the time 1 map of the horocycle flow is an atomic measure on finitely many points. The above result shows that these measures never appear in the decomposition for $\text{Acc}^\Omega(x)$ for non-periodic $x \in X$, i.e., all $\{\nu_s^i\}$ appearing above are atomless. Note that the case of a periodic $x \in X$ follows from theorems on equidistribution of orbits of rotations for averages along $\Omega(\cdot)$, in particular $\text{Acc}^\Omega(x)$ is a singleton in this case.

In the proof, we use the fact that if for a time T , the Ω orbit of length T is not equidistributed in space, then it needs to spend a positive proportion of time near a periodic orbit. But then it follows that it has to spend all its time sliding along periodic orbits of varying lengths. This is the heuristic for (4.2). To show (4.3), we need to argue that along a sequence of times, the regular orbit of length T of the horocycle flow stays $\sim T^{1/2}$ close to a periodic point. This dynamical part of the argument for

³In fact, $C(\cdot)$ depends continuously on s_0 .

the above theorem is similar to results on density of orbits $\{h_{n^2-\delta}(x)\}$. We need to show that along a sequence of times, the orbit of a generic point comes sufficiently close to a given periodic orbit such that we can then use normal distribution of $\Omega(\cdot)$ to deduce (4.3).

5. Open problems

In this section, we will formulate some open problems and questions related to sparse equidistribution.

Problem 5.1. Give an example of a smooth, zero entropy, diffeomorphism (f, M) which is uniquely ergodic and for which a prime number theorem fails, i.e., there is $x \in M$ and $\phi \in C^\infty(M)$ such that the averages $1/\pi(N) \sum_{p \leq N} \phi(f^p x)$ do not have a limit. The same can be formulated for polynomials and k -almost primes.

The counterexamples in [26] are only continuous, and in the setting of [26], it seems impossible to make them smooth.

Problem 5.2. Find a uniquely ergodic system (X, T) such that it is mixing (with respect to the only invariant measure), and for every $x \in X$ and every $\phi \in C(X)$, the averages $1/N \sum_{n \leq N} \phi(T^{n^2} x)$ converge. The same can be asked for more general polynomials of degree ≥ 2 .

The only known examples of systems which equidistribute for $\{n^2\}$ are systems which have a non-trivial Kronecker factor (and so are not even weakly mixing) or systems which are rigid (and hence are not mixing).

Question 5.3. Let $\{X_n\}$ be a sequence of random variables that satisfies the local limit theorem. For a system (X, T) , $x \in X$, study the averages $1/N \sum_{n \leq N} \phi(T^{X_n(\omega)} x)$. Is the analogue of the result in [2] for $\Omega(\cdot)$ true for $X_n(\omega)$ (almost surely)?

Question 5.4. In the same setting as above, what are the (almost sure) accumulation points of $\{h_{X_n(\omega)}(x)\}_{n \in \mathbb{N}}$, where (h_t) is the horocycle flow on $\mathrm{SL}(2, \mathbb{R})/\mathrm{SL}(2, \mathbb{Z})$ (or more generally non-compact lattices)? Is the analogue of the result in [25] true in this more general setting?

It seems that for both results [2, 25], the main number-theoretic input on $\Omega(\cdot)$ is that it satisfies the local limit theorem, and so it should be possible to study the above questions in the more general setting.

Question 5.5. Let $\mathbf{u}_t$ be a horospherical flow on G/Γ (non-compact). In which generality is the analogue of [27] true in this setting? That is, if $\mathbf{u}_t$ is d -dimensional, is it true that the orbit of every $x \in X$ generic for the Haar measure is dense along $(n_1^{2-\delta}, \dots, n_d^{2-\delta})$? What is the optimal $\delta < 1$?

Question 5.6. In the same setting as above, when does $\mathbf{u}_t$ satisfy a 2-almost prime number theorem, i.e., distribution along $(p_1^1 \cdot p_2^1, \dots, p_1^d \cdot p_2^d)$, where p_i^j are primes.

Question 5.7. Does the analogue of Theorem 4.4 hold for non-ergodic measures?

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