

Recent progress in the study of Latin squares

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Abstract. A Latin square of order n is an n by n grid filled with n symbols so that each symbol appears exactly once in each row and each column. From their early origins in recreational mathematics, Latin squares have been formally studied for their combinatorial properties since the late 18th century. More recently, tools from extremal and probabilistic combinatorics have been applied to Latin squares to make progress on a range of old questions. This brief survey will concentrate on the topics covered by the author's talk at the 9th European Congress of Mathematics, while highlighting recent relevant surveys in the area.

1. Introduction and brief history

A Latin square of order n is an n by n grid filled with n symbols so that each symbol appears exactly once in each row and each column (see Figure 1). A key example is the Cayley table of any group G of order n , which forms a Latin square of order n : indexing both the rows and columns by G , the cell of the n by n grid indexed by (g, h) is filled with $g \circ h$. Latin squares have strong links to design theory and combinatorial designs, as well as connections to error-correcting codes, 2-dimensional permutations and finite projective planes. They have a long history in early mathematics, where often their appearance was connected to magic squares; for an overview of this history, we recommend the historical survey of Andersen [1]. More recently, Latin squares have quietly gained wide familiarity due to Sudoku puzzles, the solutions of which form a Latin square of order 9 (with an additional condition on 9 disjoint 3 by 3 subsquares).

The formal study of Latin squares for their mathematical properties began with a long paper by Euler [12] in 1782. This study was motivated by a recreational question at the time, which is now known as Euler's 36 officers problem. Given 36 officers, representing 6 different ranks held in 6 different regiments, can the officers stand in a 6 by 6 grid so that there is an officer of each rank and each regiment in each row

♠	★	♦	♥	♣	◇
◇	♥	♣	★	♦	♠
♦	◇	♥	♠	★	♣
★	♣	◇	♦	♠	♥
♣	♠	★	◇	♥	♦
♥	♦	♠	♣	◇	★

0	1	2	3	4	5	6
1	2	3	4	5	6	0
2	3	4	5	6	0	1
3	4	5	6	0	1	2
4	5	6	0	1	2	3
5	6	0	1	2	3	4
6	0	1	2	3	4	5

8	2	4	6	5	9	7	3	1
6	5	1	7	2	3	9	4	8
9	7	3	4	1	8	2	6	5
2	1	5	8	9	4	3	7	6
7	9	8	5	3	6	4	1	2
4	3	6	2	7	1	5	8	9
1	6	9	3	4	5	8	2	7
5	4	2	1	8	7	6	9	3
3	8	7	9	6	2	1	5	4

Figure 1. Latin squares of order 6, 7 and 9, respectively, where the middle square is the Cayley table for $\mathbb{Z}_7$, and the square on the right is an underlying grid for a Sudoku puzzle.

and column? For example, a solution to the corresponding problem with 5 ranks and 5 regiments is presented in Figure 2. In this solution, the ranks of the officers alone form a Latin square, while the regiments alone form another. This gives two Latin squares of order $n = 5$, where the n^2 possibilities for pairs of symbols from the two sets each occur exactly once in a pair of corresponding cells. Such pairs of Latin squares are known as *mutually orthogonal*.

Euler found examples of pairs of mutually orthogonal Latin squares of order n for each $n \not\equiv 2 \pmod{4}$, using Latin and Greek alphabets, respectively, as symbols in the pair of Latin squares, which gave rise to their name. He could, however, find no solution to the 36 officers problem and conjectured that, moreover, there are no pairs of mutually orthogonal Latin squares of order n for each $n \equiv 2 \pmod{4}$. This can be easily seen to be true for $n = 2$. Euler could not confirm his conjecture for $n = 6$, but eventually, in 1900, Tarry [47] proved that, indeed, no such pairs had eluded Euler. Perhaps surprisingly, however, Euler's conjecture is false! It was disproved first for $n = 22$ and $n = 50$ by Bose and Shrikhande [3] in 1959, before they extended this with Parker [4] to find counterexamples for every $n \equiv 2 \pmod{4}$ with $n \geq 10$. In particular, Parker's counterexample for $n = 10$ was one of the earliest counterexamples found with computer assistance.

Since these results, Latin squares have been intensively studied as a combinatorial object on their own terms and as a key part of design theory. In the last

$d \eta$	$e \alpha$	$f \beta$	$g \gamma$	$a \delta$	$b \varepsilon$	$c \zeta$
$f \gamma$	$g \delta$	$a \varepsilon$	$b \zeta$	$c \eta$	$d \alpha$	$e \beta$
$a \zeta$	$b \eta$	$c \alpha$	$d \beta$	$e \gamma$	$f \delta$	$g \varepsilon$
$c \beta$	$d \gamma$	$e \delta$	$f \varepsilon$	$g \zeta$	$a \eta$	$b \alpha$
$e \varepsilon$	$f \zeta$	$g \eta$	$a \alpha$	$b \beta$	$c \gamma$	$d \delta$
$g \alpha$	$a \beta$	$b \gamma$	$c \delta$	$d \varepsilon$	$e \zeta$	$f \eta$
$b \delta$	$c \varepsilon$	$d \zeta$	$e \eta$	$f \alpha$	$g \beta$	$a \gamma$

Figure 2. Two pairs of mutually orthogonal Latin squares. On the left is a solution for the $n = 5$ version of the (unsolvable) 36 officers problem, with chess pieces denoting 5 different ranks and symbols denoting 5 different regiments. On the right, following Euler's initial convention, a Latin square using Latin letters is entered in the grid along with a Latin square using Greek letters, where each possible combination of pairs like (d, η) appears in the grid exactly once.

decade, new progress has been made by studying them in an equivalent form using edge-coloured graphs, allowing the application of techniques and perspectives from extremal and probabilistic combinatorics. This equivalence is seen as follows. Given a Latin square L of order n , form the edge-coloured graph $H(L)$ by taking a vertex representing each row and each column and, for each entry in L , putting an edge between the row and column of the entry with the symbol of the entry as its 'colour' (see Figure 3). In $H(L)$, the conditions of the Latin square imply that an edge of every colour appears exactly once at every vertex. That no two edges of one colour share a vertex means that the graph is *properly coloured*, and so we have a proper colouring of $H(L)$ with n colours. The graph formed is a *complete bipartite graph*, i.e., its vertices split into two classes so that its edges are exactly every possible edge between these two classes. The Latin squares of order n can then be easily seen to be equivalent to the complete bipartite graphs with n vertices in each class which are properly coloured using n colours.

This survey will cover some of the recent progress made in the study of Latin squares using perspectives from graph theory under this equivalence, with a particular focus on the topics of the author's talk at the 9th European Congress of Mathematics (Seville, 2024) in conjunction with the award of an EMS prize. A recent survey by the author for a more specialist readership can be found at [34], as well as an article for a wide audience at [35]. The study of edge-coloured graphs more broadly has recently seen a great deal of activity and many additional applications. For an overview, we

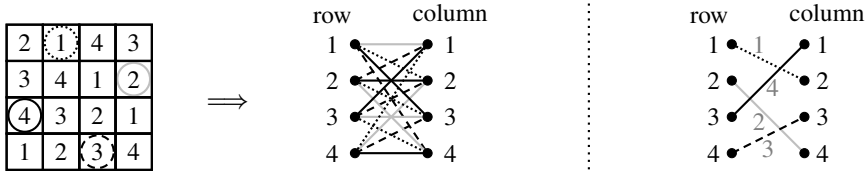


Figure 3. On the left, a Latin square of order 4 is converted into a complete bipartite properly coloured graph with 4 vertices in each class and edges with colours 1, 2, 3 and 4 drawn with dotted, grey, dashed and black lines, respectively. On the right, the perfect rainbow matching corresponding to the transversal highlighted in the Latin square on the left is shown.

recommend the survey by Sudakov [46] (Chapter 3 of the current volume), as well as the recent survey by Pokrovskiy [40], aimed at a more specialist audience.

2. Transversals in Latin squares

A *transversal* (sometimes known as a *full transversal*) is a collection of n cells in a Latin square so that each row, column, and symbol is used exactly once (see Figure 4). As was known by Euler, not every Latin square of order n has a transversal when $n \equiv 0 \pmod 2$. The canonical example of this is the Cayley table for $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$. To see this, let G be $\mathbb{Z}_n$, or any abelian group of order n , and let T be a set of n pairs (g, h) such that $(g, h) \mapsto g$, $(g, h) \mapsto h$ and $(g, h) \mapsto g + h$ each forms a bijection from T to G . Thus, T maps out a transversal in the Latin square $L(G)$ corresponding to the Cayley table for G . Then,

$$\sum_{v \in G} v = \sum_{(g,h) \in T} (g + h) = \sum_{(g,h) \in T} g + \sum_{(g,h) \in T} h = 2 \sum_{v \in G} v,$$

and therefore, $\sum_{v \in G} v = 0$. Thus, $L(G)$ can only have a transversal if $\sum_{v \in G} v = 0$, and in particular, if $G = \mathbb{Z}_{2m}$ for any $m \in \mathbb{N}$, then $L(G)$ has no transversal.

0	1	2	3	4	5
1	2	3	4	5	0
2	3	4	5	0	1
3	4	5	0	1	2
4	5	0	1	2	3
5	0	1	2	3	4

Figure 4. Latin squares of order 6, with a transversal highlighted on the left and a partial transversal with 5 cells highlighted on the right.

In 1947, Paige [39] proved that this condition is sufficient for $L(G)$ to have a transversal for any finite abelian group G . Considering the problem for non-abelian groups G , Paige [38] proved that if $L(G)$ has a transversal, then its elements can be ordered so that their product is the identity element and conjectured that this is a sufficient condition for the existence of a transversal. In 1955, Hall and Paige [18] showed moreover that this condition is equivalent to $\sum_{x \in G} x = 0$ in the abelianisation $G^{\text{ab}} = G/G'$ of G , where G' is the commutator subgroup of G , which became known as the *Hall–Paige condition*. Hall and Paige showed that this is equivalent to the condition that all the 2-Sylow subgroups of G are trivial or non-cyclic and conjectured that it implies the existence of a transversal, as follows.

Conjecture 2.1 (The Hall–Paige conjecture). *For any finite group G , $L(G)$ has a transversal if and only if G satisfies the Hall–Paige condition.*

Conjecture 2.1 was proved in 2009 using the classification of finite simple groups, through a combination of work by Wilcox [50], Evans [13] and Bray (see [50], and [6] for the later publication). In 2022, Eberhard, Manners and Mrazović [9] gave an alternative proof of the Hall–Paige conjecture for sufficiently large groups, using tools from analytic number theory. Their proof not only yields an impressive asymptotic formula for the number of transversals in the Cayley table of a group but can also reduce some of the complexity of the original proof of the Hall–Paige conjecture by taking care of the largest examples.

The Hall–Paige conjecture was also proved for large groups using combinatorial methods by Müyesser and Pokrovskiy [37], using the equivalent form in edge-coloured graphs outlined in Section 1. Let L be a Latin square of order n , and let $H(L)$ be the corresponding edge-coloured complete bipartite n by n graph. Then the cells in a transversal T of L correspond directly to a set E of edges in $H(L)$ such that the edges in E share no vertex and each colour appears exactly once (see Figure 3). Then E is a *perfect rainbow matching* in $H(L)$.

For Latin squares more generally, the natural extremal question is: given a Latin square of order n , how close can we get to a transversal? That is, how large a *partial transversal* can we find, where a partial transversal in a Latin square is a collection of cells that share no row, column or symbol (see Figure 4). In 1967, Ryser [44] conjectured that every Latin square of odd order has a transversal, and subsequently Brualdi suggested that every Latin square of even order n should have a partial transversal with $n - 1$ cells. In 1975, Stein made some related conjectures, and in recent years, the following has become known as the Ryser–Brualdi–Stein conjecture.

Conjecture 2.2 (The Ryser–Brualdi–Stein conjecture). *Every Latin square of order n has a partial transversal with $n - 1$ cells and a full transversal if n is odd.*

Progress towards the Ryser–Brualdi–Stein conjecture has concentrated on showing that increasingly large partial transversals exist in any Latin square of order n . After initial bounds by Koksma [24] and Drake [8], in 1978 Brouwer, de Vries and Wieringa [7] and Woolbright [51] showed that Conjecture 2.2 holds approximately, both showing that every Latin square of order n has a partial transversal with at least $n - \sqrt{n}$ cells. In 1982, Shor [45] gave a proof that every Latin square of order n has a partial transversal with $n - O(\log^2 n)$ cells, though this had an error later corrected by Hatami and Shor [19]. After this bound had essentially stood for almost 40 years, Keevash, Pokrovskiy, Sudakov and Yepremyan [23] made a breakthrough, working from the perspective of edge-coloured graphs. They showed that every Latin square of order n has a partial transversal with $n - O(\log n / \log \log n)$ cells. The current state of the art is a result of the current author [33] that every Latin square of order n contains a partial transversal with $n - 1$ cells when n is sufficiently large, as follows.

Theorem 2.3. *There is some n_0 such that every Latin square of order $n \geq n_0$ has a partial transversal with $n - 1$ cells.*

That Theorem 2.3 is tight when n is even is shown not only by the Cayley tables mentioned above but also by further extremal examples. In 1894, Maillet [30] showed that a wide range of extremal examples can be constructed from the Cayley tables for $\mathbb{Z}_{2m}$, where $m \in \mathbb{N}$. Rather than the direct algebraic structure of $\mathbb{Z}_{2m}$, these examples come with a partition of the symbols into $2m$ sets, where the sets have some algebraic structure between them. All the known extremal examples have some algebraic structure of this sort, but it need not hold perfectly. Indeed, in 2017, Cavenagh and Wanless made the lovely observation that you can make slight adjustments to the Cayley tables for $\mathbb{Z}_{2m}$ under some simple rule without creating a transversal. This imperfection in the algebraic structure of some extremal examples, along with the range of extremal examples known, suggests that determining the truth or otherwise of Conjecture 2.2 will be rather difficult.

The key to the progress made on Theorem 2.3 in [33] is taking a general Latin square and partitioning the symbols carefully in order to find some approximate algebraic structure, if possible, before finding a large partial transversal using a range of old and new combinatorial tools. Most of the large partial transversal is found using the semi-random method, as introduced by Rödl [41] in the 1980s. To modify this to find a partial transversal with $n - 1$ cells, the proof in [33] uses absorption, a general method for finding large structures in combinatorial objects that was codified by Rödl, Ruciński and Szemerédi [43] in 2006. In particular, it uses a form of absorption known as distributive absorption, as introduced by the current author [32], which has allowed absorption to be used in many different settings to find large rigid substructures. The proof of Theorem 2.3 uses this in conjunction with a new ‘addition structure’, which

determines which row, column and symbol is omitted from the partial transversal to achieve the right conditions for the new implementation of absorption. A further example of the graph theory tools used in [33] is that involving sublinear expansion. Sublinear expansion was introduced by Komlós and Szemerédi [25, 26] in the 1990s and has since become a critical tool that has been used in a range of results in extremal graph theory, as seen in the recent survey by Letzter [29]. In [33], sublinear expansion is used to partition the symbols into sets with some approximate algebraic structure.

3. Transversals in random Latin squares

The natural counterpart to Conjecture 2.2 from the perspective of probabilistic combinatorics is to determine whether we can expect a random Latin square to have a transversal. To make this concrete, for each $n \in \mathbb{N}$, let $\mathcal{L}(n)$ be the set of Latin squares of order n which use the symbols $1, \dots, n$, and pick $L_n \in \mathcal{L}(n)$ uniformly at random. Does the probability that L_n has a transversal tend to 1 as n tends to infinity? In other words, does a random Latin square have a transversal *with high probability* (whp)?

This question is perhaps at first glance quite innocuous, as the known Latin squares without a transversal (as discussed in Section 2) have, or are close to having, a precise algebraic structure, while a random Latin square is likely to have no such structure. However, it turns out that a random Latin square is a difficult object to study. Effectively, this is because it is hard to make small alterations to the symbols in a Latin square and reach another Latin square. It took therefore until 2020 before this question was answered by Kwan [27], who confirmed that a random Latin square L_n has a transversal whp. To do so, Kwan approximated L_n using a modification of the random triangle removal process, using breakthrough methods of Keevash [21, 22] on the existence of designs and the analysis of the random triangle removal process due to Bohman, Frieze and Lubetzky [2].

Random Latin squares have also been approached using switching methods, as in the pioneering work of Godsil and McKay [15] and McKay and Wanless [31] in the 1990s. Using these methods, and building on their work with Kühn and Osthus [17], Gould and Kelly [16] gave in 2023 an alternative proof that a random Latin square has a transversal whp. Another, entirely different, proof of this was given by Eberhard, Manners and Mrazović [10], who moreover gave an asymptotically tight bound on the number of transversals which occur in a random Latin square whp. This used their methods from analytic number theory and gave asymptotically tight bounds on the number of transversals in Latin squares satisfying certain quasirandom conditions. These conditions are satisfied by random Latin square whp, as shown by Eberhard,

Manners and Mrazović using work of Kwan, Sah and Sawhney [28], who in particular applied the deletion method of Rödl and Ruciński [20, 42] very effectively in the setting of random Latin squares.

With these results, we now have three distinct methods to show that a random Latin square has a transversal whp, but we may be able to find much more. More specifically, are we likely to be able to find many disjoint transversals (i.e., which share no common cells) in a random Latin square whp? Most ambitiously, does the random Latin square L_n have a collection of n disjoint transversals whp? Such a collection of disjoint transversals would then use each cell of the Latin square exactly once, giving a decomposition of the Latin square into transversals. In 1990, van Rees [48] conjectured that a random Latin square does not have such a decomposition whp, but more recently, Wanless and Webb [49] observed that numerical estimates suggest it does whp.

Kwan's methods to find one transversal in a random Latin square whp can be used to find many disjoint transversals whp. Further progress in this direction, however, was not made directly on Latin squares but in the closely related setting of Steiner triple systems. Relevant definitions for this can be found, for example, in [27], but pertinent here is that, while not proved formally for Latin squares, we will state the analogous results from [28, 36] in the setting of Latin squares (particularly because these translated results now follow from Theorem 3.1 later), as, while not formally checked, the methods can be expected to translate across. In 2017, Morris [36] used Kwan's methods to show, correspondingly, that there is some $\varepsilon > 0$ such that a random Latin square contains at least εn disjoint transversals whp. Developing these methods much further and using additional tools, in 2020, Ferber and Kwan [14] then showed that, correspondingly, a random Latin square contains $(1 - o(1))n$ disjoint transversals whp.

Combining and developing many of the techniques mentioned so far, including recent implementations of the semi-random method by Ehard, Glock and Joos [11], and using an intricate absorption argument, Bowtell and the current author [5] were very recently able to confirm that a random Latin square of order n will indeed have n disjoint transversals whp, as follows.

Theorem 3.1. *The probability that the random Latin square L_n has a decomposition into disjoint transversals tends to 1 as n tends to infinity.*

Decompositions of Latin squares into disjoint transversals are very strongly linked to where this survey began. Indeed, a pair of mutually orthogonal Latin squares of order n , as studied by Euler, is exactly equivalent to a Latin square of order n which can be decomposed into disjoint transversals. In a pair of mutually orthogonal Latin squares of order n , the entries of each different symbol in the first Latin square

mark out a transversal in the second Latin square (see Figure 2), and thus, the n different symbols in the first Latin square together mark out a decomposition of the second Latin square into transversals (and vice versa). In the context of Euler's disproved conjecture that there are no Latin squares of order $n = 2 \bmod 4$ which have a decomposition into disjoint transversals, Theorem 3.1 shows that, as n gets large, counterexamples to Euler's conjecture are so common that a Latin square of order $n = 2 \bmod 4$ is much more likely to be a counterexample than not! That we now know this is thanks to a raft of techniques from extremal and probabilistic graph theory, techniques that can be expected to form the foundation of further results in the study of Latin squares.

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