

Fourier uniqueness pairs and interpolation formulas

Danylo Radchenko

Abstract. This is a short, informal survey of recent analytic and number-theoretic constructions of Fourier uniqueness pairs and interpolation formulas, as well as their applications in sphere packing and energy minimization problems.

1. Sphere packing and energy minimization problems

One of the main motivations for the recent works on Fourier uniqueness pairs and interpolation formulas comes from two problems in geometric optimization: sphere packing and energy minimization. We begin by briefly describing them.

Problem 1.1 (Sphere packing). How much of $\mathbb{R}^d$ can be covered by non-overlapping equal spheres?

To formulate this more precisely, we will say that a discrete closed subset $\mathcal{C} \subset \mathbb{R}^d$ is a sphere packing if $|x - y| \geq 2$ for all $x, y \in \mathcal{C}$ with $x \neq y$. If we think of $\mathcal{C}$ as the set of centers of spheres, then the condition simply says that unit spheres with centers at $\mathcal{C}$ do not overlap. For any configuration $\mathcal{C}$, we define its *center density* by

$$\rho(\mathcal{C}) = \liminf_{R \rightarrow \infty} \frac{|\mathcal{C} \cap B_R|}{\text{Vol}(B_R)}.$$

This quantity counts the average number of points of $\mathcal{C}$ per unit volume. The sphere packing problem then asks to maximize $\rho(\mathcal{C})$ over all sphere packings $\mathcal{C}$.

For simplicity, one may restrict oneself to periodic configurations $\mathcal{C}$, i.e., sets of the form $\mathcal{C} = \bigsqcup_{s \in S} s + \Lambda$ for some lattice $\Lambda \subset \mathbb{R}^d$ and some finite set of shifts S . A simple approximation argument shows that the supremum of densities remains the same if we restrict to such sphere packings, so not much is lost in making this assumption. For periodic configurations, we have $\rho(\mathcal{C}) = |S|/|\Lambda|$, where $|\Lambda| = \text{Vol}(\mathbb{R}^d/\Lambda)$ is the covolume of the lattice Λ .

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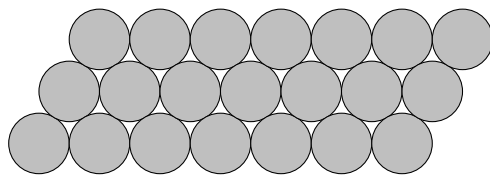


Figure 1. Part of an optimal sphere packing in $\mathbb{R}^2$.

When restricted even further to lattice packings, with $\mathcal{C} = \Lambda$ for some lattice Λ , the problem becomes equivalent to the question of determining the Hermite constant in dimension d . The Hermite constant γ_d is the maximum of the squared norm of the shortest non-zero vector of Λ taken over all lattices Λ of covolume 1. The values of γ_d for $d \leq 8$ are known from the works of Lagrange ($d = 2$; see Figure 1), Gauss ($d = 3$), Korkin–Zolotarev ($d = 4, 5$; see [18, 19]), and Blichfeldt ($d = 6, 7, 8$; see [3]). Despite the enormous increase in computing power since the 1930s, no values of γ_d for $d \geq 9$ have been computed to this day, with the exception of γ_{24} (see [8]), which requires a completely different approach.

Coming back to the general question, in [11], Conway and Sloane gave a conjectural description of what all the optimal sphere packings should be in dimensions up to 9, conjecturing among other things that for $d \leq 8$ the optimal density is attained by a root lattice of type $A_2, A_3, D_4, D_5, E_6, E_7$, or E_8 . Here, the only proof (excluding the trivial case $d = 1$) going back to the pre-computer age is the case $d = 2$ due to Thue; the proof for $d = 3$ is a very difficult computer-assisted proof due to Hales [16], and for $d = 4, 5, 6$, and 7, the problem is open.

The real surprise came from the breakthrough work of Viazovska [30], who solved the problem for $d = 8$. Shortly afterward, her work was extended in [10], solving the sphere packing problem for $d = 24$. While the proofs for $d = 2$ and $d = 3$ are based on local analysis of configurations, the proofs for $d = 8$ and $d = 24$ are based on linear programming, which connects the sphere packing problem with an optimization problem in Fourier analysis.

Before we explain the connection to Fourier analysis, let us describe a more general problem, energy minimization, where this connection figures more prominently. A potential (or potential function) is any function $p: \mathbb{R}_{>0} \rightarrow \mathbb{R}$ from positive reals to reals. The p -energy of a configuration $\mathcal{C}$, roughly speaking, is the average of pairwise interactions between particles, where the interaction between $x, y \in \mathcal{C}$ is given by $p(|x - y|)$. For a periodic configuration $\mathcal{C} = S + \Lambda$, one can define the energy by

$$E_p(\mathcal{C}) = \frac{1}{|S|} \sum_{u, v \in S} \sum_{x \in \Lambda \setminus \{v - u\}} p(|x + u - v|),$$

provided that the sum converges. We will be interested only in two classes of potentials: the Riesz potentials $p(r) = r^{-s}$, $s > d$, and the Gaussian potentials $p(r) = e^{-\alpha r^2}$, $\alpha > 0$, for which absolute convergence is guaranteed.

Problem 1.2 (Energy minimization). Given a potential function $p: (0, +\infty) \rightarrow \mathbb{R}$, what is the minimal possible p -energy of a configuration $\mathcal{C} \subset \mathbb{R}^d$ with center density ρ ?

This problem is closely related to the sphere packing problem. For example, if p is the indicator function of the interval $[0, 2)$, then the p -energy of $\mathcal{C}$ vanishes (for periodic $\mathcal{C}$) if and only if $\mathcal{C}$ is a sphere packing. Moreover, one can easily show that if $\mathcal{C}$ is optimal for $p(r) = r^{-s}$ for all $s > s_0$ for some s_0 , then $\mathcal{C}$ is an optimal sphere packing as well, and an analogous statement holds for Gaussian potentials $p(r) = e^{-\alpha r^2}$ for $\alpha > \alpha_0$.

In view of the scarcity of results on general sphere packings, one expects even less to be known about energy minimization. For example, in dimension 2, the expectation is that A_2 should be optimal for both Gaussians and Riesz potentials. In the class of lattices, by the works of Rankin [27], Cassels [5], Diananda [12], and Ennola [14], the optimality is known for Riesz potentials and by the work of Montgomery [21] for the Gaussians. However, as of this writing, the optimality of the A_2 -lattice in the class of all periodic configurations is not known for any Riesz or Gaussian potentials.

In some dimensions, the optimal configuration for Gaussian energy $p(r) = e^{-\alpha r^2}$ appears to depend on the parameter α ; for instance, this is the expected behavior for $d = 3$. By contrast, the conjectural situation in dimension 2 is that A_2 is the optimal configuration for a wide class of potentials, including all Gaussians. Following Cohn and Kumar [7], we call a configuration $\mathcal{C}$ *universally optimal* if it minimizes p -energy for all $p: (0, \infty) \rightarrow \mathbb{R}$ that are completely monotone functions of squared distance. By a theorem of Bernstein on completely monotone functions, it is equivalent to requiring optimality of p -energy for all Gaussian potentials. In [7], the following conjecture was made.

Conjecture 1.3 (Cohn–Kumar). *In dimensions 2, 8, and 24, the configurations given by the A_2 -lattice, the E_8 -lattice, and the Leech lattice, respectively, are universally optimal.*

More precisely, Cohn and Kumar conjectured that the universal optimality in these cases can be certified via a linear programming bound, which we will explain shortly. In [7], they also prove that for $d = 1$, the set of integers $\mathbb{Z}$ is universally optimal, although this result was later discovered to be known from an earlier work of Ventevogel and Nijboer [29].

The key tool for proving lower bounds for energy and the connection to Fourier analysis is given in the following theorem.

Theorem 1.4 (Cohn–Kumar). *Let $f: \mathbb{R}^d \rightarrow \mathbb{R}$ be a Schwartz function such that*

$$\begin{aligned} f(x) &\leq p(|x|), \quad x \neq 0, \\ \widehat{f}(\xi) &\geq 0, \quad \xi \in \mathbb{R}^d. \end{aligned}$$

Then any subset $\mathcal{C} \subset \mathbb{R}^d$ of density ρ satisfies

$$E_p(\mathcal{C}) \geq \rho \widehat{f}(0) - f(0).$$

Here, we normalize Fourier transform in $\mathbb{R}^d$ by $\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \xi \cdot x} dx$. The proof of the above result, for the case when $\mathcal{C}$ is a lattice Λ , is a one-line calculation:

$$\sum_{x \in \Lambda \setminus \{0\}} p(|x|) \geq \sum_{x \in \Lambda \setminus \{0\}} f(x) = -f(0) + \frac{1}{|\Lambda|} \sum_{\xi \in \Lambda^*} \widehat{f}(\xi) \geq \frac{\widehat{f}(0)}{|\Lambda|} - f(0),$$

where we used the conditions for f and $\widehat{f}$ together with the Poisson summation formula (Λ^* denotes the lattice dual to Λ). A proof for general periodic configurations is completely analogous and is only a few lines longer.

The above result allows one to get lower bounds for energy by constructing certain auxiliary functions, but since it does not prescribe any method for constructing suitable f , it is not clear how good a bound it is. Jumping ahead, it turns out that, at least in some cases, the bound that Theorem 1.4 gives is optimal. Indeed, analyzing the above proof gives the following stronger claim: if there exists a function f such that

$$\begin{cases} f(x) \leq p(|x|), & x \neq 0, \\ f(x) = p(|x|), & x \in \Lambda \setminus \{0\}, \end{cases} \quad \text{and} \quad \begin{cases} \widehat{f}(\xi) \geq 0, & \xi \neq 0, \\ \widehat{f}(\xi) = 0, & \xi \in \Lambda^* \setminus \{0\}, \end{cases}$$

then Λ is optimal for p -energy. The precise version of the conjecture of Cohn and Kumar is that such optimal auxiliary functions f witnessing a tight bound for Gaussian energy exist in dimensions 2, 8, and 24.

Let $d = 8$ and $\Lambda = \Lambda_{E_8}$ be the E_8 -lattice. It is even and self-dual:

$$\Lambda^* = \Lambda, \quad |x|^2 \in 2\mathbb{Z} \quad \text{for all } x \in \Lambda.$$

Moreover, Λ has vectors of length $\sqrt{2n}$ for all $n \geq 0$. Assuming that the function f above is radial (which does not reduce generality, since all conditions are linear and radially symmetric, so any solution can be made radial by averaging over all rotations), we get the following rough picture; see Figure 2.

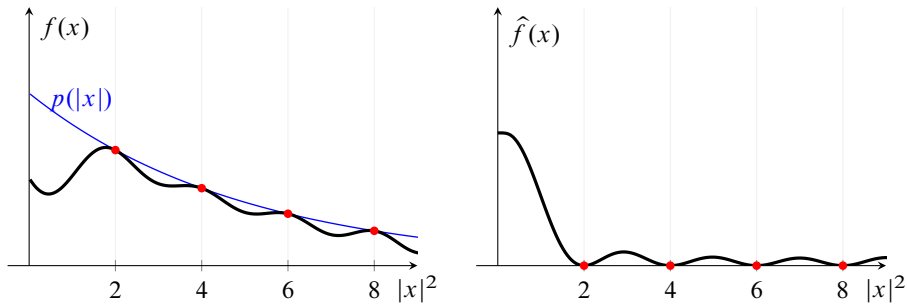


Figure 2. Behavior of any optimal auxiliary function f for $d = 8$.

In particular, we see that if such a function f exists, then $f(\sqrt{2n}) = p(\sqrt{2n})$, $f'(\sqrt{2n}) = p'(\sqrt{2n})$, $\hat{f}(\sqrt{2n}) = \hat{f}'(\sqrt{2n}) = 0$, $n \geq 1$. This leads to the question of how much freedom there is in prescribing the values of f , f' , $\hat{f}$, and $\hat{f}'$ at $\sqrt{2n}$, $n \geq 1$. The following theorem from [9] gives a fairly complete answer to this question.

Theorem 1.5. *For $d \in \{8, 24\}$, there exist two sequences of radial Schwartz functions $a_n, b_n \in \mathcal{S}(\mathbb{R}^d)$, $n \geq 0$, such that for any radial Schwartz function f , we have*

$$f(x) = \sum_{n \geq n_0} (a_n(x) f(\sqrt{2n}) + b_n(x) f'(\sqrt{2n}) + \hat{a}_n(x) \hat{f}(\sqrt{2n}) + \hat{b}_n(x) \hat{f}'(\sqrt{2n})).$$

Here, $n_0 = 1$ for $d = 8$ and $n_0 = 2$ for $d = 24$. Moreover, for any four rapidly decaying sequences of numbers $f(\sqrt{2n})$, $f'(\sqrt{2n})$, $\hat{f}(\sqrt{2n})$, and $\hat{f}'(\sqrt{2n})$, the right-hand side of the above formula defines a radial Schwartz function extending these values.

Based on an explicit form of the above interpolation formula, the cases $d = 8$ and $d = 24$ of the Cohn–Kumar conjecture were resolved in [9]. The case $d = 2$ remains widely open, and it is now known that there can be no direct analogue of the interpolation formula with $\sqrt{2n}$ replaced by the lengths of vectors in the A_2 -lattice, and it is not clear how to remedy the above approach.

2. Fourier uniqueness pairs

The interpolation formula from Theorem 1.5 has some very interesting features but is rather complicated, both because it is only stated for radial functions in 8 and 24 dimensions, and it involves not only the values of f and its Fourier transform but also the values of their derivatives. The following simpler and more basic result was

inspired by Theorem 1.5 and proved a few years earlier in [23] (while Theorem 1.5 was still a conjecture), and it will serve as a better entry point for discussing Fourier uniqueness pairs.

Theorem 2.1. *There exists a sequence of even Schwartz functions $a_n: \mathbb{R} \rightarrow \mathbb{R}$ with the property that for every even Schwartz function $f: \mathbb{R} \rightarrow \mathbb{R}$, we have*

$$f(x) = \sum_{n=0}^{\infty} a_n(x) f(\sqrt{n}) + \sum_{n=0}^{\infty} \hat{a}_n(x) \hat{f}(\sqrt{n}). \quad (2.1)$$

Moreover, for any pair of rapidly decaying sequences $(x_n)_{n \geq 0}$ and $(y_n)_{n \geq 0}$ satisfying $\sum_{n \in \mathbb{Z}} x_{n^2} = \sum_{n \in \mathbb{Z}} y_{n^2}$, there exists a unique even Schwartz function f with $f(\sqrt{n}) = x_n$ and $\hat{f}(\sqrt{n}) = y_n$, and it is given by $f = \sum_{n \geq 0} x_n a_n + \sum_{n \geq 0} y_n \hat{a}_n$.

There is also an analogous statement for odd Schwartz functions that also involves the values $f'(0)$ and $\hat{f}'(0)$, but we will concentrate on the even case. The proof of the above result given in [23] is based on an explicit construction of the functions a_n , whose defining property is that they satisfy $a_n(\sqrt{m}) = \delta_{m,n}$, $\hat{a}_n(\sqrt{m}) = 0$ for $n \geq 0$, $m \geq 1$. The special functions a_n end up being closely related to weakly holomorphic modular forms for the Hecke theta group. Without going into details here, the function a_0 is given by the contour integral

$$a_0(x) = \frac{1}{4} \int_{-1}^1 \theta^3(z) e^{\pi i z x^2} dz,$$

where $\theta(z) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 z}$, and a similar formula exists for all a_n .

As an immediate corollary of the theorem, we have the following result.

Corollary 2.2. *If an even Schwartz function f satisfies $f(\sqrt{n}) = \hat{f}(\sqrt{n}) = 0$, $n \geq 0$, then $f = 0$.*

Note that we may omit the condition $\hat{f}(0) = 0$ since by Poisson summation, we have $\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n)$. With this adjustment, the above statement is tight in the sense that if we omit one of the $f(\sqrt{m}) = 0$, $m \geq 0$, or $\hat{f}(\sqrt{m}) = 0$, $m \geq 1$, then the claim fails. Indeed, for $f = a_m$, we have $f(\sqrt{n}) = 0$ for all $n \neq m$ and $\hat{f}(\sqrt{n}) = 0$ for all $n \geq 1$.

Both known proofs (the one from [23] and a more recent proof in the work of Gerbelli-Gauthier and Venkatesh [15]) of this corollary involve modular forms, whose appearance may seem quite surprising. This corollary naturally leads us to the following concept.

Definition 2.3. Let $\mathcal{H}$ be a subspace of the space of continuous absolutely integrable functions on $\mathbb{R}$. We say that a pair of closed subsets $A, B \subseteq \mathbb{R}$ is a *Fourier uniqueness pair* (for $\mathcal{H}$) if for any $f \in \mathcal{H}$ if $f|_A = 0$ and $\hat{f}|_B = 0$, then $f = 0$. If both A and B cannot be replaced by a proper closed subset while keeping the above property, we say that the uniqueness pair is *tight*.

If (A, B) is a uniqueness pair, there exists an inverse map $(f|_A, \hat{f}|_B) \mapsto f$, which we will call a Fourier interpolation formula (there may be subtleties in determining the domain of this map, which we will largely ignore). The above definition can certainly be extended, for example, by considering general locally compact abelian or even non-abelian groups or relaxing the restrictions on $\mathcal{H}$, but it does capture most of the interesting examples of uniqueness pairs that follow.

Example 1. Our first example is related to classical results in time-frequency analysis. Namely, taking $A = \{x: |x| \geq 1/2\}$ and $B = \mathbb{Z}$ gives a (tight) uniqueness pair. The proof of this fact, together with an interpolation formula, is immediate upon noting that (for a suitable class of functions)

$$f(x) = \sum_{m \in \mathbb{Z}} \hat{f}(m) e^{2\pi i m x} - \sum_{n \neq 0} f(x + n),$$

which is a form of the Poisson summation formula. If we are in the situation when $f|_{|x| \geq 1/2} = 0$, taking the Fourier transform of the above equation would give us the famous Whittaker–Shannon formula (for $\hat{f}$ with finitely supported f):

$$\hat{f}(x) = \sum_{m \in \mathbb{Z}} \hat{f}(m) \frac{\sin(\pi(x - m))}{\pi(x - m)}.$$

Example 2. Our second example concerns discrete Fourier transform. Let p be a prime, and for $f: \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{C}$, we define its Fourier transform by

$$\hat{f}(\xi) = \frac{1}{\sqrt{p}} \sum_{x=0}^{p-1} f(x) e^{-2\pi i x \xi / p}.$$

In [28], Tao proved the following uncertainty principle: if $f: \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{C}$ is non-zero, then $|\text{supp}(f)| + |\text{supp}(\hat{f})| \geq p + 1$. This can be reformulated as saying that for any $A, B \subseteq \mathbb{Z}/p\mathbb{Z}$, (A, B) is a uniqueness pair if and only if $|A| + |B| \geq p$, and (A, B) is a tight uniqueness pair if and only if $|A| + |B| = p$. Therefore, in this case, we get a complete classification of uniqueness pairs.

Example 3. Our third example is the pair $A = \{\sqrt{n}\}_{n \geq 0}$, $B = A \setminus \{0\}$, with which we started this discussion. It is a tight uniqueness pair for the space of even Schwartz

functions by Theorem 2.1, together with an interpolation formula (2.1). This is the first example of a uniqueness pair where the sets A and B are both discrete.

Example 4. Our last example is related to zeros of L -functions. Fix $\varepsilon > 0$, and let $\mathcal{H}_1$ be the space of functions f analytic in the strip $\{z: |\operatorname{Im} z| < 1/2 + \varepsilon\}$ such that

$$\sup_{|y| < 1/2 + \varepsilon} \int_{-\infty}^{\infty} |f(x + iy)|(1 + |x|)dx < \infty.$$

In [4], the following interpolation formula was proved.

Theorem 2.4. *There exist two sequences of rapidly decaying even entire functions $U_n(z)$, $n = 1, 2, \dots$, and $V_{\rho,j}(z)$, $0 \leq j < m_\rho$, with ρ ranging over the nontrivial zeros of $\zeta(s)$ such that for every even function f in $\mathcal{H}_1$ and z with $|\operatorname{Im} z| < 1/2$,*

$$f(z) = \sum_{n=1}^{\infty} \hat{f}\left(\frac{\log n}{4\pi}\right) U_n(z) + \lim_{m \rightarrow \infty} \sum_{0 < \gamma \leq T_m} \sum_{j=0}^{m_\rho-1} f^{(j)}\left(\frac{\rho - \frac{1}{2}}{i}\right) V_{\rho,j}(z)$$

for some universal sequence of positive numbers $T_m \rightarrow \infty$.

This gives rise to a Fourier uniqueness pair.

Corollary 2.5. *If $\mathfrak{Z}$ denotes the set of nontrivial zeros of $\zeta(s)$, then*

$$A = \left\{ \frac{\rho - \frac{1}{2}}{i} : \rho \in \mathfrak{Z} \right\}, \quad B = \left\{ \pm \frac{\log n}{4\pi} : n \in \mathbb{N} \right\}$$

is a tight uniqueness pair for $\mathcal{H}_1$.

For this corollary, we need to refine the definition of a uniqueness pair to accommodate the fact that A may not lie on the real line, if Riemann hypothesis fails, or have points with multiplicities (in which case $f|_A = 0$ includes vanishing of higher derivatives, according to a multiplicity of a point in A), if the simple zeros conjecture fails, so unconditionally we only know that A is a multiset contained in a strip $|\operatorname{Im} z| < 1/2$.

Perhaps surprisingly, the proof again relies on modular forms. It also works for several other classes of L -functions: $L(\chi, s)$ for Dirichlet characters χ (with $\log n$ replaced by $\log(qn)$), $\zeta_K(s)$ for any imaginary quadratic field K (with $\log n$ replaced by $\log(4|D|n^2)$), and $L(f, s)$ for a cusp form f of weight k and level N (with $\log n$ replaced by $\log(4Nn^2)$). One could expect that there should be analogous statements for all L -functions in the Selberg class, but the strategy used in [4] cannot work for any L -functions of degree higher than 2.

The uniqueness pairs from the last two examples account for almost all of the tight uniqueness pairs (with discrete A and B) that are currently known. The proof of

Theorem 2.1 relies heavily on the arithmetic nature of the sequence $\sqrt{n}$ and does not generalize to other sequences in any direct way (although a slight modification of the argument also works for the sequence $(\sqrt{n+a})_n$ for some range of a). A perturbation argument due to Ramos and Sousa [25] shows that for sufficiently small ε_n , the sequence $(\sqrt{n+\varepsilon_n})_n$ has similar properties, but it does not allow us to get very far from the original sequence. As we will see below, the situation becomes better if we relax the tightness condition, allowing the use of more analytic techniques.

3. Connection to modular forms

Before describing the recent analytic results on Fourier uniqueness pairs, it is worthwhile to give at least a very rough idea of how the proof of Theorem 2.1 involves modular forms. For a more detailed exposition, we refer to [23], [4, §3].

The first step is to note that it suffices to prove (2.1) for f from some subset of the Schwartz space with a dense linear span. A good choice turns out to be to consider all $f(x) = e^{\pi i \tau x^2}$ with $\text{Im}(\tau) > 0$, since the Fourier transform of a Gaussian is again a Gaussian. Denoting

$$F(\tau) = \sum_{n \geq 0} a_n(x) e^{\pi i n \tau}, \quad G(\tau) = \sum_{n \geq 0} \widehat{a}_n(x) e^{\pi i n \tau},$$

equation (2.1) can then be rewritten as

$$F(\tau) + \left(\frac{\tau}{i}\right)^{-1/2} G\left(\frac{-1}{\tau}\right) = e^{\pi i \tau x^2}.$$

(Here, we used the fact that $\widehat{f}(\xi) = (\tau/i)^{-1/2} e^{\pi i (-1/\tau) \xi^2}$.) Now let us fix $x \in \mathbb{R}$ and consider the above identity as a functional equation in τ . Since both F and G are 2-periodic, we get the following system of equations:

$$\begin{cases} F(\tau) + \left(\frac{\tau}{i}\right)^{-k} G\left(\frac{-1}{\tau}\right) = \varphi(\tau), \\ F(\tau) - F(\tau + 2) = 0, \\ G(\tau) - G(\tau + 2) = 0, \end{cases} \quad (3.1)$$

where $\varphi(\tau) = e^{\pi i \tau x^2}$ and $k = 1/2$. It is not hard to see that (2.1) is in fact equivalent to (3.1) if we additionally require F and G to be of moderate growth in the upper half-plane $\mathbb{H}$, meaning that they should be bounded by some polynomial in $(\text{Im } \tau)^{-1}$ and $|\tau|$ for $\tau \in \mathbb{H}$.

It turns out that (3.1) can be solved for any $k \geq 0$ and any φ that is holomorphic and of moderate growth in $\mathbb{H}$, with F and G in the same class, and moreover the solution is unique up to a finite-dimensional space of modular forms (which correspond to

solutions of the homogeneous system with $\varphi = 0$). For $k > 2$, this can be derived from the results of Knopp [17] on Eichler cohomology, but it is hard to turn his proof into an explicit formula. We instead use a contour integral construction inspired by [13]. Let us explain, ignoring all technical details, how this can be achieved. Note that symmetrizing $F_{\pm} = F \pm G$, we get equations in a single unknown function:

$$\begin{cases} F_{\pm}(\tau) \pm \left(\frac{\tau}{i}\right)^{-k} F_{\pm}\left(\frac{-1}{\tau}\right) = \varphi(\tau) \pm \left(\frac{\tau}{i}\right)^{-k} \varphi\left(\frac{-1}{\tau}\right), \\ F_{\pm}(\tau) - F_{\pm}(\tau + 2) = 0. \end{cases}$$

Let us first look at the case of F_{-} and $k = 0$:

$$\begin{cases} F_{-}(\tau) - F_{-}\left(\frac{-1}{\tau}\right) = \varphi(\tau) - \varphi\left(\frac{-1}{\tau}\right), \\ F_{-}(\tau) = F_{-}(\tau + 2). \end{cases}$$

The two fractional linear transformations $\tau \mapsto \tau + 2$ and $\tau \mapsto -1/\tau$ generate the so-called theta group Γ_{θ} , a fundamental group for which is depicted in Figure 3.

There is a conformal map J mapping the fundamental domain onto the slit plane $\mathbb{C} \setminus (-\infty, 1]$ as shown in Figure 3 (this function J is the so-called Hauptmodul for Γ_{θ}). The composition $A(t) = F_{-}(J^{-1}(t))$ is then holomorphic in $\overline{\mathbb{C}} \setminus [0, 1]$, with a jump discontinuity across $[0, 1]$ prescribed by $\varphi(J^{-1}(t))$. The solution is then given by the Sokhotski–Plemelj theorem, which says that A can be computed as an integral against the Cauchy kernel: $A(s) = \int_0^1 g(s)/(t - s)ds$, where $g(s)$ is the jump at $t = s$. Translating back into τ variable, this gives a formula for F_{-} as a contour integral over the semicircle bounding the fundamental domain.

It turns out that the general case can be derived from the special case $k = 0$ and sign “−”: first, taking instead of $\varphi(\tau)$ the function $\varphi(\tau)/\theta^{2k}(\tau)$, where $\theta(\tau) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau}$ is the Jacobi theta function, we get to the case of general k , for which one uses the modular transformation $\theta(\tau) = (\tau/i)^{-1/2} \theta(-1/\tau)$. A similar trick with division by $1 - 2\lambda(\tau)$ (where $\lambda(\tau)$ is the modular lambda invariant) allows one to get

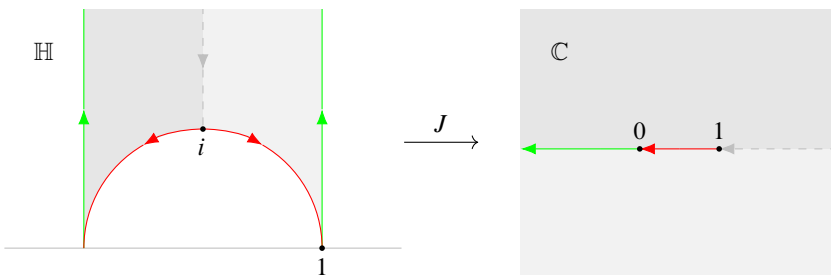


Figure 3. Fundamental domain for Γ_{θ} and conformal map J .

solutions for F_+ from the ones for F_- . After all of these manipulations, one arrives at an explicit formula

$$F_{\pm}(\tau) = \int_{-1}^1 \mathcal{K}_k^{\pm}(\tau, z) \varphi(z) dz,$$

where $\mathcal{K}_k^{\pm}(\tau, z)$ is a certain modular kernel. This formula can then be extended by analytic continuation to give a solution in all of $\mathbb{H}$, and since everything is explicit, going back to the original question, one can derive integral representations for all of the functions $a_n(x)$ appearing in (2.1).

4. Further results and questions

We conclude by discussing the recent analytic results on uniqueness pairs and some open questions.

4.1. Uniqueness pairs from powers of integers

In [24], Ramos and Sousa have constructed the first discrete uniqueness pairs by purely analytic techniques. Their results concern uniqueness pairs of the form $A = \{\pm cn^{\alpha}\}_{n \geq 0}$, $B = \{\pm dn^{\beta}\}_{n \geq 0}$, where $\alpha, \beta, c, d > 0$. They have proved that (A, B) is a uniqueness pair, provided that $(\alpha, \beta) \in \Omega$, where Ω is a certain open subset in $\{\alpha, \beta > 0, \alpha + \beta < 1\}$.

More recently, Kulikov, Nazarov, and Sodin [20] obtained a much more precise description:

$$\begin{aligned} \alpha + \beta < 1 &\Rightarrow (A, B) \text{ is a uniqueness pair,} \\ \alpha + \beta > 1 &\Rightarrow (A, B) \text{ is a non-uniqueness pair.} \end{aligned}$$

Moreover, in the case $\alpha + \beta = 1$, they obtained a precise threshold value for cd :

$$\alpha + \beta = 1 : \begin{aligned} 2cd < \alpha^{-\alpha} \beta^{-\beta} &\Rightarrow (A, B) \text{ is a uniqueness pair,} \\ 2cd > \alpha^{-\alpha} \beta^{-\beta} &\Rightarrow (A, B) \text{ is a non-uniqueness pair.} \end{aligned}$$

The results of [20] in fact extend to other sequences that behave roughly like n^{α} . However, they do not seem to reach the critical case, leaving the following problem.

Question. For $\alpha + \beta = 1$ and $2cd = \alpha^{-\alpha} \beta^{-\beta}$, is $A = \{\pm cn^{\alpha}\}_{n \geq 0}$, $B = \{\pm dn^{\beta}\}_{n \geq 0}$ a uniqueness pair? If so, is it a tight uniqueness pair, and is there a well-behaved interpolation formula?

Note that for $\alpha = \beta = 1/2$, the answer is provided by Theorem 2.1.

4.2. Discrete uniqueness pairs in $\mathbb{R}^d$

The situation in $\mathbb{R}^d$ for $d > 1$ is more complicated, and for discrete A, B , there seem to be no known explicit interpolation formulas akin to Theorem 2.1. Nevertheless, in [26], Ramos and Stoller proved that discrete uniqueness pairs exist by deforming interpolation formulas for the case when $A = B$ is a union of concentric spheres. More recently, Adve [1] established that for any $\alpha + \beta = 1$, the pair

$$A = \{c|n|^{-\alpha}n : n \in \mathbb{Z}^d \setminus \{0\}\}, \quad B = \{c|n|^{-\beta}n : n \in \mathbb{Z}^d \setminus \{0\}\} \quad (4.1)$$

is a uniqueness pair if $c < c_0$ and a non-uniqueness pair if $c > c_1$ for some constants $0 < c_0 < c_1$. See Figure 4 below for a picture for $\alpha = \beta = 1/2$ and $A = B$. Note that in Adve's example (for $d = 2$ and $\alpha = \beta = 1/2$), the number of points in A in a ball of radius R is approximately CR^4 for some fixed $C > 0$. In [22], a number-theoretic (based on the ring of integers in a real quadratic field) construction of a family of non-uniqueness pairs was given that also have $C'R^4$ points in a ball of radius R , and C' can be arbitrarily large. This shows that it is not just the “size” of the sets A and B that is important for (A, B) to be a uniqueness pair, but also their “regularity”.

Note that in $\mathbb{R}$, the transition from uniqueness pairs to non-uniqueness pairs in (4.1) as c goes from 0 to ∞ has a sharp threshold at certain value $c = c_{\text{crit}}$. In $\mathbb{R}^d$ for $d > 1$, there is a range of values $c \in (c_0, c_1)$ for which it is not clear whether (4.1) is a uniqueness pair or not.

Question. Is there a sharp transition value of c for all $d > 1$?

Question. Are there tight uniqueness pairs in $\mathbb{R}^d$ with discrete A, B for any $d > 1$?

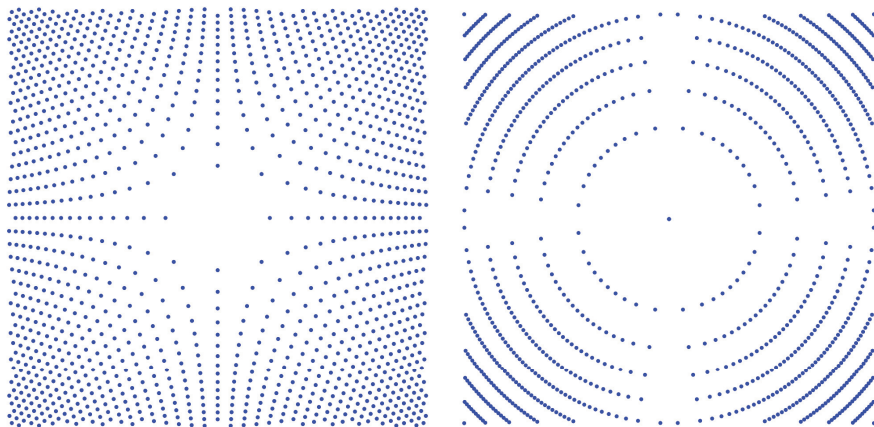


Figure 4. Left: Adve's uniqueness pair. Right: a non-uniqueness pair constructed from the ring of integers of $\mathbb{Q}(\sqrt{89})$. For the general case of $\mathbb{Q}(\sqrt{D})$ as $D \rightarrow \infty$, these non-uniqueness sets get higher and higher density in the union of concentric circles $\{x : |x|^2 \in \mathbb{Z}_{\geq 0}\}$.

4.3. Linear programming for energy and conformal bootstrap

To conclude the story, we return to sphere packing and energy minimization problems with which we started and present an important conjecture relating the linear programming bound of Cohn–Kumar to Fourier interpolation formulas in all dimensions ≥ 4 .

To find an optimal function for the use in Theorem 1.4, we have the following optimization problem (CK_α) : maximize the value $\hat{f}(0) - f(0)$ over all $f: \mathbb{R}^d \rightarrow \mathbb{R}$ satisfying

$$\begin{aligned} f(x) &\leq e^{-\alpha\pi|x|^2}, \quad x \in \mathbb{R}^d, \\ \hat{f}(\xi) &\geq 0, \quad \xi \in \mathbb{R}^d. \end{aligned}$$

In [9], the following conjecture was formulated.

Conjecture 4.1. *For $d \geq 4$, the solution to (CK_α) is unique and is given by a radial Schwartz function. The points where $f(r) = e^{-\alpha\pi r^2}$ is attained form a discrete set $r_1 < r_2 < \dots$ with $r_n^2 \sim 2n$, $n \rightarrow \infty$, that does not depend on α . The condition $\hat{f}(r) = 0$ holds for exactly the same values $r = r_n$. Finally, there exists a linear interpolation formula recovering any radial Schwartz function $f: \mathbb{R}^d \rightarrow \mathbb{R}$ from the values $f(r_n)$, $f'(r_n)$, $\hat{f}(r_n)$, $\hat{f}'(r_n)$, $n \geq 1$.*

This conjecture suggests a natural generalization of Theorem 1.5, directly connected to the linear programming bound in all dimensions $d \geq 4$, not just $d = 8$ and $d = 24$. It can be further refined by describing what happens to the solutions of the dual linear program. In the dual setting, one considers the space of generalized theta functions

$$\vartheta(\tau) = 1 + \sum_{n \geq 1} c_n e^{2\pi i \lambda_n \tau}$$

with $\rho_n > cn$ for some $c > 0$ and $c_n \geq 0$ such that ϑ satisfies

$$\vartheta\left(\frac{-1}{\tau}\right) = \left(\frac{\tau}{i}\right)^{d/2} \vartheta(\tau).$$

The dual optimization problem (CK_α^*) is then to minimize the value of $\vartheta(i\alpha)$ in the space of all such series for some fixed $\alpha > 0$. The above conjecture can then be extended to say that for every $d \geq 4$, there exists a *universally optimal* theta function Θ_d that minimizes the value at $i\alpha$ for all $\alpha > 0$. This function has an expansion

$$\Theta_d(\tau) = 1 + \sum_{n \geq 1} a_n e^{2\pi i \rho_n \tau},$$

with $\rho_n = \rho_n^{(d)} = r_n^2/2$ (where r_n are specified in the above conjecture) and some $a_n \geq 0$. By the results of [9], for $d = 8$ and $d = 24$, these functions are known to be

$$\begin{aligned}\Theta_8(\tau) &= 1 + 240e^{2\pi i\tau} + 2160e^{4\pi i\tau} + \dots, \\ \Theta_{24}(\tau) &= 1 + 196560e^{4\pi i\tau} + 16773120e^{6\pi i\tau} + \dots,\end{aligned}$$

the theta functions of the E_8 and the Leech lattice.

In [2], the above optimization problem is shown to be related to conformal bootstrap in conformal field theory, and extensive numerical data is presented on the values of ρ_n and a_n for a large range of dimensions. Of particular interest both in conformal bootstrap and in the sphere packing problem is the behavior of the smallest value ρ_1 . For the sphere packing, it should correspond to the optimal value of the Cohn–Elkies LP bound for sphere packing [6]. This gives further motivation for the study of the two optimization problems above, since a good understanding of their structure could give further insights into the sphere packing problem in large dimensions.

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References

- [1] A. Adve, Density criteria for Fourier uniqueness phenomena in $\mathbb{R}^d$. 2023, arXiv: [2306.07475v1](#)
- [2] N. Afkhami-Jeddi, H. Cohn, T. Hartman, D. de Laat, and A. Tajdini, [High-dimensional sphere packing and the modular bootstrap](#). *J. High Energy Phys.* **2020** (2020), no. 12, article no. 66, 45 pp. Zbl [1457.81095](#) MR [4239386](#)
- [3] H. F. Blichfeldt, [The minimum values of positive quadratic forms in six, seven and eight variables](#). *Math. Z.* **39** (1935), no. 1, 1–15 Zbl [0009.24403](#) MR [1545485](#)
- [4] A. Bondarenko, D. Radchenko, and K. Seip, [Fourier interpolation with zeros of zeta and \$L\$ -functions](#). *Constr. Approx.* **57** (2023), no. 2, 405–461 Zbl [1528.11076](#) MR [4577389](#)
- [5] J. W. S. Cassels, [On a problem of Rankin about the Epstein zeta-function](#). *Proc. Glasg. Math. Assoc.* **4** (1959), no. 2, 73–80 Zbl [0103.27602](#) MR [0117193](#)
- [6] H. Cohn and N. Elkies, [New upper bounds on sphere packings I](#). *Ann. of Math. (2)* **157** (2003), no. 2, 689–714 Zbl [1041.52011](#) MR [1973059](#)
- [7] H. Cohn and A. Kumar, [Universally optimal distribution of points on spheres](#). *J. Amer. Math. Soc.* **20** (2007), no. 1, 99–148 Zbl [1198.52009](#) MR [2257398](#)
- [8] H. Cohn and A. Kumar, [Optimality and uniqueness of the Leech lattice among lattices](#). *Ann. of Math. (2)* **170** (2009), no. 3, 1003–1050 Zbl [1213.11144](#) MR [2600869](#)

- [9] H. Cohn, A. Kumar, S. Miller, D. Radchenko, and M. Viazovska, [Universal optimality of the \$E_8\$ and Leech lattices and interpolation formulas](#). *Ann. of Math. (2)* **196** (2022), no. 3, 983–1082 Zbl [1509.11061](#) MR [4502595](#)
- [10] H. Cohn, A. Kumar, S. D. Miller, D. Radchenko, and M. Viazovska, [The sphere packing problem in dimension 24](#). *Ann. of Math. (2)* **185** (2017), no. 3, 1017–1033 Zbl [1370.52037](#) MR [3664817](#)
- [11] J. H. Conway and N. J. A. Sloane, [What are all the best sphere packings in low dimensions?](#) *Discrete Comput. Geom.* **13** (1995), no. 3–4, 383–403 Zbl [0844.52013](#) MR [1318784](#)
- [12] P. H. Diananda, [Notes on two lemmas concerning the Epstein zeta-function](#). *Proc. Glasg. Math. Assoc.* **6** (1964), no. 4, 202–204 Zbl [0128.04501](#) MR [0168537](#)
- [13] W. Duke, Ö. Imamoglu, and Á. Tóth, [Cycle integrals of the \$j\$ -function and mock modular forms](#). *Ann. of Math. (2)* **173** (2011), no. 2, 947–981 Zbl [1270.11044](#) MR [2776366](#)
- [14] V. Ennola, [On a problem about the Epstein zeta-function](#). *Proc. Cambridge Philos. Soc.* **60** (1964), no. 4, 855–875 Zbl [0146.05504](#) MR [0168535](#)
- [15] M. Gerbelli-Gauthier and A. Venkatesh, [Cohomology of Fuchsian groups and Fourier interpolation](#). *Essent. Number Theory* **4** (2025), no. 2, 217–249 Zbl [08085866](#) MR [4959855](#)
- [16] T. C. Hales, [A proof of the Kepler conjecture](#). *Ann. of Math. (2)* **162** (2005), no. 3, 1065–1185 Zbl [1096.52010](#) MR [2179728](#)
- [17] M. I. Knopp, [Some new results on the Eichler cohomology of automorphic forms](#). *Bull. Amer. Math. Soc.* **80** (1974), no. 4, 607–632 Zbl [0292.10022](#) MR [0344454](#)
- [18] A. Korkine and G. Zolotareff, [Sur les formes quadratiques positives quaternaires](#). *Math. Ann.* **5** (1872), no. 4, 581–583 JFM [04.0081.03](#) MR [1509795](#)
- [19] A. Korkine and G. Zolotareff, [Sur les formes quadratiques positives](#). *Math. Ann.* **11** (1877), no. 2, 242–292 JFM [09.0139.01](#) MR [1509914](#)
- [20] A. Kulikov, F. Nazarov, and M. Sodin, [Fourier uniqueness and non-uniqueness pairs](#). *J. Math. Phys. Anal. Geom.* **21** (2025), no. 1, 84–130 Zbl [1565.42003](#) MR [4866154](#)
- [21] H. L. Montgomery, [Minimal theta functions](#). *Glasg. Math. J.* **30** (1988), no. 1, 75–85 Zbl [0639.10017](#) MR [0925561](#)
- [22] D. Radchenko and M. Stoller, [Fourier non-uniqueness sets from totally real number fields](#). *Comment. Math. Helv.* **97** (2022), no. 3, 513–553 Zbl [1497.11259](#) MR [4468993](#)
- [23] D. Radchenko and M. Viazovska, [Fourier interpolation on the real line](#). *Publ. Math. Inst. Hautes Études Sci.* **129** (2019), no. 1, 51–81 Zbl [1455.11075](#) MR [3949027](#)
- [24] J. P. G. Ramos and M. Sousa, [Fourier uniqueness pairs of powers of integers](#). *J. Eur. Math. Soc. (JEMS)* **24** (2022), no. 12, 4327–4351 Zbl [1504.42014](#) MR [4493626](#)
- [25] J. P. G. Ramos and M. Sousa, [Perturbed interpolation formulae and applications](#). *Anal. PDE* **16** (2023), no. 10, 2327–2384 Zbl [1537.41001](#) MR [4678143](#)

- [26] J. P. G. Ramos and M. Stoller, [Perturbed Fourier uniqueness and interpolation results in higher dimensions](#). *J. Funct. Anal.* **282** (2022), no. 12, article no. 109448, 34 pp.
Zbl [1486.42001](#) MR [4403065](#)
- [27] R. A. Rankin, [A minimum problem for the Epstein zeta-function](#). *Proc. Glasg. Math. Assoc.* **1** (1953), no. 4, 149–158 Zbl [0052.28005](#) MR [0059300](#)
- [28] T. Tao, [An uncertainty principle for cyclic groups of prime order](#). *Math. Res. Lett.* **12** (2005), no. 1, 121–127 Zbl [1080.42002](#) MR [2122735](#)
- [29] W. J. Ventevogel and B. R. A. Nijboer, [On the configuration of systems of interacting particles with minimum potential energy per particle](#). *Phys. A* **98** (1979), no. 1–2, 274–288 MR [0546896](#)
- [30] M. S. Viazovska, [The sphere packing problem in dimension 8](#). *Ann. of Math. (2)* **185** (2017), no. 3, 991–1015 Zbl [1373.52025](#) MR [3664816](#)

Danylo Radchenko

Laboratoire Alexander Grothendieck, Institut des Hautes Études Scientifiques, CNRS, 35 route de Chartres, 91440 Bures-sur-Yvette, France; radchenko@ihes.fr