

# Integrable PDEs for random growth

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**Abstract.** We describe the KPZ equation and explore the KPZ fixed point, its connection to the Kadomtsev–Petviashvili equation and how the stochastic integrability arises from a microscopic model, the polynuclear growth model (PNG).

## 1. The KPZ universality class

What is the universal equation for random growth of a one-dimensional interface? Physicists will certainly answer that it is the Kardar–Parisi–Zhang (KPZ) equation [17]: A height function  $h(t, x)$ ,  $x \in \mathbb{R}$ ,  $t \geq 0$ , is growing according to the stochastic partial differential equation

$$\partial_t h = \lambda(\partial_x h)^2 + \nu \partial_x^2 h + \sqrt{D} \xi, \quad (1.1)$$

where  $\xi(t, x)$  is *space-time white noise*, i.e., the delta correlated Gaussian field,

$$\langle \xi(t, x), \xi(s, y) \rangle = \delta(t - s) \delta(x - y).$$

This equation was introduced to model stochastic growth processes such as *ballistic deposition*, a continuous time surface growth model on an integer lattice  $h(x, t) \in \mathbb{Z}$ ,  $x \in \mathbb{Z}$ , with the following stochastic rule. Attached to each  $x$  there is an independent Poisson process. At each jump time of the Poisson process, a particle falls at  $x$  and either (1) stacks on top of the particles already there or (2) sticks to one of the neighbouring stacks, whichever is larger. So the rule is that

$$h(x) \mapsto \max\{h(x - 1), h(x) + 1, h(x + 1)\}$$

independently at each site  $x$  at rate 1. The non-linear term  $(\partial_x h)^2$  in the KPZ equation is supposed to model the fact that there is a *lateral* part of the growth – the sticky particles – so the radial part of the growth should depend non-linearly on the slope of the interface. The parameter  $\lambda$  reflects the strength of this non-linearity. Similarly,  $\nu$  and  $D$  are the strengths of diffusivity and noise.

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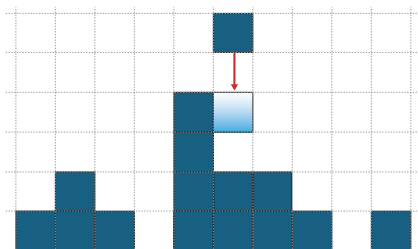


**Figure 1.** Ballistic deposition.

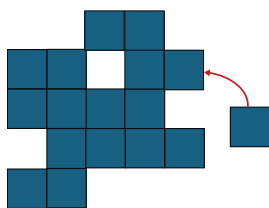


**Figure 2.** Eden model.

Ballistic deposition (see Figure 1) in turn is a toy model for the *Eden model* (see Figure 2), a growth model on  $\mathbb{Z}^2$ : Start a growing cluster with a single site. At each time step, look at all sites adjacent to the cluster and choose one uniformly at random to add to the cluster.

The Eden model can be thought of as a model for a fire front or a growing bacterial colony, etc. There is also a class of models like it, called *first passage percolation*. The same sort of thing should happen whenever we have a stable phase invading an unstable phase in the presence of randomness in two dimensions. An earlier way to model such things was through a stochastic reaction-diffusion equation for  $\varphi(t, x_1, x_2)$ :

$$\partial_t \varphi = \Delta \varphi + \xi V'(\varphi),$$

where  $\xi$  is some sort of noise and  $V$  is a potential with one stable and one unstable equilibrium.

These latter models are undirected, and their exact location of the front is a bit unclear, though one could certainly introduce some definition. If one tried to choose parameters  $\lambda, \nu, D$  to approximate the Eden model by the KPZ equation (1.1), one would find oneself stuck quite quickly. Models like Ballistic deposition (see Figure 1), which are height functions over a one-dimensional substrate, are called *directed*. All the models that we can analyse are of directed type, though for most of them, like Ballistic deposition, we cannot say much.

The phenomenon we want to study is that fluctuations are observed to be of size  $t^{1/3}$  at time  $t$  and live on spatial scales of size  $t^{2/3}$ . Furthermore, in the corresponding large space-time rescaling, one is supposed to see certain universal stochastic processes.

The KPZ equation has an advantage over the others mentioned so far, in that it has an explicit invariant measure. Two-sided Brownian motion with variance  $\frac{D}{(2\nu)}$  plus any drift  $bx$  are invariant, and there are no others. Note that it does not depend on  $\lambda$ , and for  $\lambda = 0$ , it is easy to prove. For  $\lambda \neq 0$ , it is quite miraculous. The earliest physics arguments of the invariance [22] were very non-rigorous, but it can be seen from approximating models that preserve random walk interfaces [6]. Only recently, a proof in the continuum has become available [12].

One can devise all kinds of growth models, but one has to be extremely lucky to have a tractable invariant measure. This appears to be a necessary, but not sufficient, condition to make any sort of analytical progress. For example, there is a spatial discretization of KPZ which has Gaussian random walks as invariant measure, but nobody has been able to prove much about its large-scale behaviour. On the other hand, recently highly non-trivial invariant measures for the KPZ equation in an interval or half-line with Neumann boundary conditions have been discovered [5, 9]. So there are examples where the invariant measure is not simple. And in fact, we only know them by approximation from microscopic models; there is no proof of their invariance at the continuum level yet.

If we start the KPZ equation with a nice smooth height function  $h_0(x)$ , at a very small time  $t > 0$ , it will look like a locally Brownian version of  $h_0$ . For example, the quadratic variation in an interval  $[a, b]$  will instantly jump from 0 at time 0 to  $\frac{D}{2\nu}|b - a|$  for  $t > 0$ . This means that the non-linear term  $(\partial_x h)^2$  in (1.1) cannot make classical sense, and some renormalization needs to be done to it. This was most spectacularly achieved in [14], but an easy way out [6] is to just define (1.1) via the Hopf–Cole transformation

$$h(t, x) = \frac{\nu}{\lambda} \log Z(t, x), \quad \partial_t Z = \nu \partial_x^2 Z - \beta Z, \quad (1.2)$$

with  $\beta = -\frac{\sqrt{D}\lambda}{\nu}$ . This is called the *Hopf–Cole solution*. The multiplicative stochastic heat equation (1.2) turns out to be a rare non-linear (in the noise  $\xi$ ) stochastic partial differential equation which can be handled by a naive generalization of the Itô theory of stochastic differential equations. On the other hand, for stability and convergence, one often needs a direct theory of (1.1) which has been supplied by Hairer and others [13, 15]. The multiplicative stochastic heat equation has a Feynman–Kac representation as a Brownian expectation:

$$Z(t, x) = E[e^{-\beta \int_0^t \xi(t-s, B_s+x) ds} Z(0, B_t + x)]. \quad (1.3)$$

The space-time white noise is far too irregular for this to make classical sense, but if one expands the exponential as a Taylor series and replaces each resulting multiple integral  $\int_{[0,t]^n} ds_1 \cdots ds_n$  by its ordered version  $n! \int_{0 < s_1 < \cdots < s_n < t} ds_1 \cdots ds_n$ , the result is a convergent series of multiple stochastic integrals for the Itô solution of (1.2). So we can just define the right-hand side of (1.3) to mean that. The Feynman–Kac representation (1.3) provides a link to *directed polymers in a random environment* where we assign probabilities to random walk paths  $X_n$  on  $\mathbb{Z}$  according to

$$Z_\beta^{-1} e^{-\beta \sum_{i=1}^n \xi_i \cdot x_i},$$

where  $\xi_{i,j}$  are now independent identically distributed random variables, and  $Z_\beta$  is the normalization that makes it a probability measure. The *free energy*  $h = \log Z$  can be thought of as a discrete KPZ and is expected in great generality to have the same large-scale behaviour. Moreover,  $\beta$  is thought of as an inverse temperature: As  $\beta \searrow 0$  (high temperature), the polymer is free; as  $\beta \nearrow \infty$  (low temperature), it is an energy-optimizing path, called *last passage percolation*. In  $1 + 1$  dimensions at any finite temperature, the paths on large scale are localized around favourite paths. All of them have lateral fluctuations of size  $t^{2/3}$ , as opposed to  $t^{1/2}$  at  $\beta = 0$ . But it is only in special last passage cases that one can do full computations.

In order to understand the  $t^{1/3}$  height fluctuations from a scaling perspective, we rescale (1.1). Let us use  $\varepsilon$  for the relation between small and large spatial scales. We want to choose exponents  $\chi$  and  $z$  in order to have an interesting limit as  $\varepsilon \searrow 0$  of

$$h_\varepsilon(t, x) = \varepsilon^\chi (h(\varepsilon^{-z}t, \varepsilon^{-1}x) - c\varepsilon^{-z}t). \quad (1.4)$$

The fact that Brownian motion is invariant forces

$$\chi = \frac{1}{2}$$

since Brownian motion is the natural fixed point of  $h(x) \mapsto \varepsilon^{1/2}h(\varepsilon^{-1}x)$ . Since white noise scales as the square root of the Dirac delta, we obtain after a little work

$$\partial_t h_\varepsilon = \varepsilon^{\frac{3}{2}-z} \lambda(\partial_x h_\varepsilon)^2 - c_\varepsilon + \varepsilon^{2-z} \nu \partial_x^2 h_\varepsilon + \varepsilon^{1-\frac{1}{2}z} \sqrt{D} \xi$$

which forces [10, 23]

$$z = \frac{3}{2}. \quad (1.5)$$

So the height fluctuations  $h \sim \varepsilon^{-\chi} = (\varepsilon^{-z})^{1/3}$ , and this is just another way of saying that the fluctuations are of size  $t^{1/3}$ . It also identifies the scale on which we expect to see non-trivial spatial correlations as  $\varepsilon^{-1} \sim t^{2/3}$ . On the other hand, to prove scaling (1.5) requires sophisticated arguments [4].

Scaling (1.4) shows that the KPZ equation is *not scaling invariant*. It is not our universal model for 1 + 1-dimensional random growth. It is more of a nice continuum model of the same sort as ballistic aggregation, but with the advantage that the invariant measure is explicit.

Let us take  $\lambda = \nu = \frac{1}{4}$  and  $D = 1$  which is a kind of standard case. So we are solving the 1:2:3 scaled KPZ equation

$$\partial_t h_\varepsilon = \frac{1}{4}(\partial_x h_\varepsilon)^2 - c_\varepsilon + \varepsilon^{\frac{1}{2}} \frac{1}{4} \partial_x^2 h_\varepsilon + \varepsilon^{\frac{1}{4}} \xi \quad (1.6)$$

and asking what the limit looks like as  $\varepsilon \rightarrow 0$  if we have initial data correspondingly rescaling as  $h_\varepsilon(0, x) \rightarrow \mathfrak{h}_0(x)$ . The constant  $c_\varepsilon$  is going to infinity, but one could simply work at the level of the derivative  $u_\varepsilon = \partial_x h_\varepsilon$  which satisfies the rescaled stochastic Burgers equation

$$\partial_t u_\varepsilon = \frac{1}{4} \partial_x u_\varepsilon^2 + \varepsilon^{\frac{1}{2}} \frac{1}{4} \partial_x^2 u_\varepsilon + \varepsilon^{\frac{1}{4}} \partial_x \xi, \quad (1.7)$$

and then the constant just is not there anymore. So one might guess the limiting evolution would be the Burgers equation

$$\partial_t u = \frac{1}{4} \partial_x u^2.$$

This equation is terribly ill-posed. Usually, in a class in partial differential equations, we are told the true solution is the inviscid solution, which means to add  $\nu \partial_x^2 u$  to the right-hand side and then let  $\nu \searrow 0$ . But this cannot be the answer for us because white noise is invariant for (1.7) all the way along. But one can compute the inviscid solution starting with white and it is not white noise. In fact, a remnant of the noise survives, and one has the following result.

**Theorem 1.1.** *The limit of (1.6) as  $\varepsilon \searrow 0$  is a Markov process of  $\mathfrak{h}(t, x)$  called the KPZ fixed point. It satisfies a Hopf–Lax-type variational formula, with a non-trivial stationary forcing noise, the Airy sheet (see (3.8)): For fixed  $t > 0$ ,*

$$\mathfrak{h}(t, x; \mathfrak{h}_0) = \sup_{y \in \mathbb{R}} \left\{ t^{\frac{1}{3}} \widehat{\mathcal{A}}(t^{-\frac{2}{3}} x, t^{-\frac{2}{3}} y) - \frac{1}{t} (x - y)^2 + \mathfrak{h}_0(y) \right\}. \quad (1.8)$$

*Its one-dimensional distributions'  $F(t, x, r) = \text{Prob}(\mathfrak{h}(t, x) \leq r)$  logarithmic derivative  $\phi = \partial_r^2 \log F$  solves the Kadomtsev–Petviashvili equation*

$$\partial_t \phi + \frac{1}{2} \partial_r \phi^2 + \frac{1}{12} \partial_r^3 \phi + \frac{1}{4} \partial_r^{-1} \partial_x^2 \phi = 0. \quad (1.9)$$

It is worth comparing the KP equation (1.9) with the KPZ equation (1.6). One thinks intuitively that the quadratic non-linearity in KP must have something to do with the quadratic non-linearity in KPZ. But they are not even in the same variable. As far as we know, there is no heuristic or physical argument for the appearance of (1.9). The KP equation is 1:2:3 scaling invariant, and it is a big surprise that it arises as a universal equation for random growth. The proof combines many results and is essentially a long algebraic computation combined with various asymptotic analyses. What happens is that there are certain stochastically integrable models in the class – we will describe one here, the polynuclear growth model. Scaling these, we discover the 1:2:3 scaling invariant Markov process at the centre of the class, the KPZ fixed point, which is also stochastically integrable and whose finite-dimensional distributions, given by appropriate Fredholm determinants, turn out to satisfy KP equations.

## 2. Stochastic integrability

The route to (1.9) started a quarter century ago with a groundbreaking result of Baik–Deift–Johansson [3] on the fluctuations of the longest increasing subsequence of a random permutation. Let  $\sigma$  be chosen uniformly from  $S_N$ , the set of permutations of  $\{1, \dots, N\}$ , and  $L_N$  be the length of the longest increasing subsequence  $\sigma(i_1) < \dots < \sigma(i_L)$ ,  $i_1 < \dots < i_L$ . In undergraduate combinatorics classes, we prove that once  $N$  is greater than  $n^2$ , there is always either an increasing or decreasing subsequence of length  $n + 1$ , so it is reasonable to conjecture, as Ulam did, that  $L_N \sim CN^{1/2}$ . This was proved by Hammersley in 1972. In 1976, Vershik–Kerov and Logan–Shepp independently showed that  $C = 2$ . The fluctuations are much more refined. Baik–Deift–Johansson discovered in 1999 that

$$\lim_{N \rightarrow \infty} \text{Prob} \left( \frac{L_N - 2N^{\frac{1}{2}}}{N^{\frac{1}{6}}} \leq r \right) = F_{\text{GUE}}(r), \quad \text{where } F_{\text{GUE}}(r) := e^{-\partial^{-2} q^2(r)} \quad (2.1)$$

is the Tracy–Widom GUE distribution (often called  $F_2$ ). Here,  $\partial^{-2}$  denotes the second anti-derivative, and  $q$  is the *Hastings–McLeod* [16] solution (i.e.,  $q(r) \sim 0$  as  $r \rightarrow \infty$ , and  $q(r) \sim |r/2|^{1/2}$ , implying  $D^{-2}q^2 \sim r^3/2$  as  $r \rightarrow -\infty$ ) of the Painlevé II equation

$$q'' = rq + 2q^3,$$

which already hints that there are integrable systems behind the scenes. It had been discovered a decade earlier as the limiting distribution of the top eigenvalue of a matrix from the Gaussian unitary ensemble. The connection between random matrices and longest increasing subsequences was murky at best. And what do longest increasing subsequences have to do with random growth models?

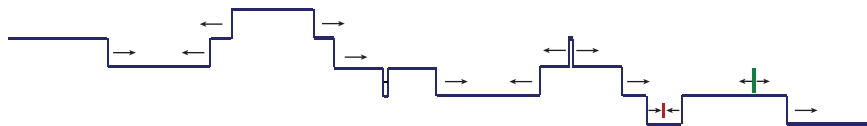


**Figure 3.** In the PNG dynamics, flat islands with integer heights expand deterministically in both directions with speed 1, merging when they collide (red), and with new islands (green) appearing above existing ones at the times of a space-time Poisson process with rate 2.

In fact, the longest increasing subsequence turns out to be a special case of a classical random growth model called PNG (polynuclear growth model). This is a Markov process whose state space consists of height functions  $h : \mathbb{R} \rightarrow \mathbb{Z} \cup \{-\infty\}$ . So they are piecewise constant, taking only integer jumps at discrete positions. We also (for good reasons) allow  $h$  to take the value  $-\infty$ . Like all these growth models, the natural topology on the state space is local Hausdorff convergence of hypographs, with the range of the heights compactified at  $-\infty$ . The PNG dynamics consists of two parts running simultaneously: (1) nucleations: at random times and random spatial location  $x$ ,  $h \mapsto h + 1_x$ , i.e., a tiny island of height 1 is nucleated on top of the existing interface. This occurs as a Poisson point process in space-time with rate 2 (the 2 is arbitrary, but people learned over time that this avoids strings of  $2^{1/3}$  in the exact formulas); (2) a deterministic dynamics  $h \mapsto \sup_{|y-x| \leq t} h(y)$ , i.e., islands spread out laterally at speed 1 and merge on contact (see Figure 3).

Almost equivalently, the model can be thought of as a collection of one-dimensional kinks (down steps) and antikinks (up steps) moving on  $\mathbb{R}$ . They move ballistically at speed 1: kinks to the right, antikinks to the left. They annihilate upon collision and are also produced in pairs, a kink at  $x^+$  and an antikink at  $x^-$ , according to a rate 2 space-time Poisson process. The height  $h(t, 0)$  is just its value  $h(0, 0)$  at time 0, plus the number of kinks and antikinks which have crossed the spatial point 0 up to time  $t$ .

The *narrow wedge* at  $y \in \mathbb{R}$  is the (legal) height function  $\delta_y(y) = 0$  and  $\delta_y(x) = -\infty$  for  $x \neq y$ . If we start PNG with  $\delta_y$ , we could call the resulting solution at time  $t$ , as a function of  $x$ ,  $\mathcal{A}_t(x, y)$ . The PNG dynamics, under the natural coupling where one starts with two or more different initial conditions and runs them using the same Poisson noise, preserves the max operation

$$h(t, x; \max\{h_1, h_2\}) = \max\{h(t, x; h_1), h(t, x; h_2)\},$$

where  $h(t, x; h_0)$  indicates the initial data is  $h_0$ . Hence one has the variational formula

$$h(t, x; h_0) = \sup_y \{\mathcal{A}_t(x, y) + h_0(y)\}. \quad (2.2)$$

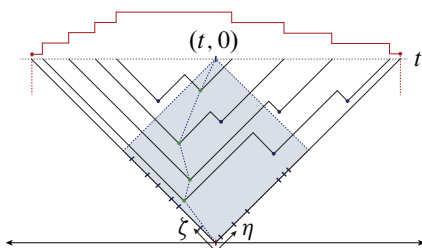


**Figure 4.** Longest increasing subsequence in PNG with narrow wedge initial data. From  $\eta = t + x$  to  $\zeta = t - x$ , order of points in box corresponds to a permutation.

Clearly, from the variational formula, the narrow wedge solutions play a special role. If we start with one of them, say,  $\delta_0$ , one can see after a little thought that the height  $h(t, 0)$  at position 0 at time  $t$  depends only on the Poisson points in the space-time  $45^\circ$  rotated square with vertices  $(0, 0)$ ,  $(\frac{t}{2}, \frac{t}{2})$ ,  $(\frac{t}{2}, -\frac{t}{2})$  and  $(t, 0)$ , and in fact the height is the largest number of such points picked up by a path from  $(0, 0)$  to  $(t, 0)$ , piecewise linear between the Poisson points, with slopes always  $\leq 1$  in absolute value. A collection of  $N$  points in a box generically give you a permutation in  $S_N$ : the way the order in one coordinate is transformed into the order in the other coordinate. The points picked up by one of our paths correspond to an increasing subsequence of the permutation. A collection of  $N$  points uniformly and independently chosen in a box correspond to a uniformly random permutation in  $S_N$ . The rate 2 Poisson points in our diamond of volume  $\frac{1}{2}t^2$  correspond to choosing  $N$  points with  $P(N = k) = \frac{t^{2k}}{k!} e^{-t^2}$  and then placing them uniformly and independently (see Figure 4). Hence  $h(t, 0; \delta_0)$  is the length of the longest increasing subsequence of a random uniformly chosen permutation from  $S_N$ , where  $N$  has the Poisson distribution with rate  $t^2$ . In this way, the longest increasing subsequence problem can be seen as a special initial data for our particular growth model.

KPZ growth models like PNG do not really have an invariant measure because there is a non-trivial height shift. But we can talk about invariant measures modulo the absolute height. For PNG, these consist of continuous time random walks  $h(x)$  which jump a unit step up at rate  $\rho$  and a unit step down at rate  $\rho^{-1}$ . The average slope  $\rho - \rho^{-1}$  can be thought of as the average density of antikinks minus kinks. It is globally preserved, since they are created and annihilated in kink-antikink pairs, and parametrizes the set of invariant measures.

We will now show that PNG is a *stochastic integrable system*, by which we mean that its Markov transition probabilities are in-principle computable in somewhat the same way that one solves classical completely integrable systems.

From the variational formula (2.2), we obtain *skew time reversal invariance*

$$P(h(t, x; g) \leq -f) = P(h(t, x; f) \leq -g).$$

It is called that because you can think of it saying that the dynamics is preserved by  $t \mapsto -t$ ,  $h \mapsto -h$ . There is another way to think about it. Let  $\mathbf{1}_{f>g}$  be the indicator that  $f(x) > g(x)$  for all  $x \in \mathbb{R}$ . Let  $L$  be the infinitesimal generator of the PNG process and  $L^*$  the infinitesimal generator of the “backwards” process  $-h$ . As the notation indicates, this is the adjoint of  $L$  with respect to the invariant measure, and if we start in such a measure, it is literally the process viewed backwards in time. Then the infinitesimal version of skew time reversal invariance is

$$L_f \mathbf{1}_{f>g} = L_g^* \mathbf{1}_{f>g},$$

where the subscripts in  $L_f$  and  $L_g^*$  indicate which variable the generators are acting on. Of course, this also says  $L_f(\mathbf{1} - \mathbf{1}_{f>g}) = L_g^*(\mathbf{1} - \mathbf{1}_{f>g})$  and

$$\mathbf{1} - \mathbf{1}_{f>g} = g \text{ hits the hypograph of } h,$$

where the hypograph of  $h = \{(x, y) \in \mathbb{R}^2 : y \leq h(x)\}$  is closed if and only if  $h$  is upper semicontinuous. So it suggests considering the *hit kernel*

$$P_{a,b}^{\text{hit}(h)}(u, v) = P(g \text{ hits the hypograph of } h \text{ on } [a, b] \text{ and } g(b) = v \mid g(a) = u),$$

where on the right-hand side, the probability measure on  $g$  is the invariant measure, i.e., the continuous time random walk  $g(x)$  which jumps up at rate  $\rho$  and down at rate  $\rho^{-1}$ . It turns out not to matter what  $\rho$  is, so we take it to be 1 to be concrete. The generator of this random walk is then the discrete Laplacian

$$\Delta f(x) = f(x+1) - 2f(x) + f(x-1).$$

Let  $E_{a,u;b,v}$  denote expectation with respect to random path  $g$  in  $[a, b]$  with  $g(a) = u$  and  $g(b) = v$ . Then,

$$L_h P_{a,b}^{\text{hit}(h)}(u, v) = E_{a,u;b,v}[L_h(\mathbf{1} - \mathbf{1}_{g>h})] = E_{a,u;b,v}[L_g^*(\mathbf{1} - \mathbf{1}_{g>h})].$$

Here,  $L^*$  has the same invariant measures as  $L$ , so if it were not for the pinning at either end, the last term would vanish. If one looks at why such models can have explicit invariant measures, it is always because an integration by parts produces a telescoping series. So it is not hard to guess that with the pinned measure, the telescoping series will produce boundary terms. For PNG, they can be computed explicitly, and the result is

$$E_{a,u;b,v}[L_g^*(\mathbf{1} - \mathbf{1}_{g>h})] = 2[P_{a,b}^{\text{hit}(h)}, \nabla].$$

Here,  $\nabla$  is the symmetric difference operator:

$$\nabla f(u) = \frac{1}{2}(f(u+1) - f(u-1)),$$

and  $[A, B] = AB - BA$  is the *commutator* of  $A$  and  $B$ . If we define, for  $a < b$ ,

$$T_{a,b}^h = e^{a\Delta} P_{a,b}^{\text{hit}(h)} e^{-b\Delta},$$

then because  $\Delta$  is the generator of the walk

$$\partial_t T_{x-t, x+t}^h = 0.$$

But this means that if we define, for  $x, r \in \mathbb{R}$  and  $t \geq 0$ , with  $\tau_r f(u) = f(u + r)$ ,

$$K(t, x, r; h) = \tau_r e^{-2t\nabla - x\Delta} T_{x-t, x+t}^h e^{2t\nabla + x\Delta} \tau_{-r}, \quad (2.3)$$

we have

$$(\partial_t - L)K = 0$$

as an operator, or we can think of them as integral kernels, and so a vast collection of solutions depending on parameters  $u_1, u_2 \in \mathbb{Z}$ .

Next we use determinants to build transition probabilities out of them. The PNG generator  $L$  is a sum of two pieces, a differential operator corresponding to the deterministic part of the dynamics with the kinks going one way and the anti-kinks going the other way, mutually annihilating on contact, and a creation operator  $2 \int dy (f(h + 1_y) - f(h))$ . It is not hard to check from the definition that  $K(h + 1_y) - K(h)$  is rank one. But then by linearity and the action of differential operators and rank one perturbations on determinants,

$$(\partial_t - L) \det(I - K) = \det(I - K) \text{tr}((I - K)^{-1} (\partial_t - L)K),$$

so since  $(\partial_t - L)K = 0$ , we have

$$(\partial_t - L) \det(I - K) = 0$$

as well. In order to get the determinant to produce enough functions at  $t = 0$  to identify the Markov transition probabilities, we have to extend it in the following way. For  $x_1 < \dots < x_n$ , the *extended kernel* acting on the  $n$ -fold product of  $\ell^2(\mathbb{Z}_{>0})$  whose  $i, j$ th matrix element is, with the notation  $\vec{x} = (x_1, \dots, x_n)$  and  $\vec{r} = (r_1, \dots, r_n)$ ,

$$K_{ij}^{\text{ext}}(t, \vec{x}, \vec{r}, h) = \tau_{r_i} (-1_{x_i < x_j} e^{(x_j - x_i)\Delta} + e^{-2t\nabla - x_i\Delta} T_{x_i - t, x_j + t}^h e^{2t\nabla - x_j\Delta}) \tau_{-r_j}. \quad (2.4)$$

Here, we sometimes use  $P_{a,b}^{\text{hit}(h)}$  with  $a > b$  which we just set to 0. Now let

$$U(t, h) = \det(I - K^{\text{ext}})_{(\ell^2(\mathbb{Z}_{>0}))^n}.$$

The same argument shows we also have  $(\partial_t - L)U = 0$ , and setting  $t = 0$  in (2.4), we get an upper triangular matrix with diagonal entries  $\tau_{r_i} P_{x_i, x_i}^{\text{hit}(h)} \tau_{-r_i}(u, v) =$

$1_{u=v \leq h(x_i) - r_i}$ . The determinant is just the product of the determinants of these diagonal entries, each one on  $\ell^2(\mathbb{Z} > 0)$  giving the indicator that  $h(x_i) - r_i \geq 0$ , i.e., we get  $\prod_{i=1}^n 1_{r_i \leq h(x_i)}$ . Hence

$$U(0, h) = 1_{h(x_i) \leq r_i, i=1, \dots, n}.$$

As usual, finite-dimensional distributions determine the spatial stochastic process. In other words, we have solved the Kolmogorov equation for general  $h$ !

**Theorem 2.1.** *The finite-dimensional distributions of the transition probabilities of PNG are given by*

$$P_h(h(t, x_i) \leq r_i, i = 1, \dots, n) = \det(I - K^{\text{ext}}(t, \vec{x}, \vec{r}, h))_{(\ell^2(\mathbb{Z}_{>0}))^n}.$$

*Proof.* We have shown they satisfy the Kolmogorov backward equation with the required initial data. So all we need is uniqueness of that equation. There is no general theorem of uniqueness for the Kolmogorov backward equation, but there is a general argument that uniqueness of the backward equation follows if we have existence of the adjoint equation. But here the adjoint is just the same process upside down, so our derivation of transition probabilities gives us the existence there as well, and therefore the uniqueness of our equation. ■

We emphasize that this is a very unusual situation. Usually, in Markov process theory, the transition probabilities are defined as solutions of the Kolmogorov equation, but we do not expect to be able to write down the solutions.

Let us take as an example the narrow wedge initial data  $\mathfrak{d}_0$ , which as we saw leads to a Poissonized version of the longest increasing subsequence. Since the random walk  $g$  can only hit the hypograph of  $\mathfrak{d}_0$  at the origin, for  $a < 0 < b$ ,

$$T_{a,b}^{\mathfrak{d}_0} = e^{a\Delta} P_{a,b}^{\text{hit}(\mathfrak{d}_0)} e^{-b\Delta} = \mathbf{1}_{\mathbb{Z}_{\leq 0}}.$$

One can readily check that

$$e^{2t\nabla + x\Delta}(u_1, u_2) = e^{-2x} \left( \frac{t-x}{t+x} \right)^{\frac{u_2 - u_1}{2}} J_{u_2 - u_1}(2\sqrt{t^2 - x^2}),$$

where  $J_n$  are Bessel functions of the first kind. If we set  $\sqrt{t^2 - x^2} = 1$  and only worry about one-point distribution, we get

$$\begin{aligned} F_r &:= P(h(1, 0; \mathfrak{d}_0) \leq r) = \det(I - \tau_r B \tau_{-r})_{\ell^2(\mathbb{Z}_{>0})} \\ &= e^{-s^2} \det(I_{i-j}(2s))_{i,j=0, \dots, r-1}, \end{aligned}$$

where

$$B(u, v) = \frac{J_{u-1}(2)J_v(2) - J_u(2)J_{v-1}(2)}{u - v}$$

is the *discrete Bessel kernel* and  $I_n$  are modified Bessel functions of the first kind. The second equality is from [8]. The last is Gessel's formula [11] which eventually led to the Tracy–Widom asymptotics for the longest increasing subsequence (2.1).

The Toeplitz determinants  $\det(I_{i-j}(2s))_{i,j=0,\dots,r-1}$  are intimately related to the orthogonal polynomials on the unit circle associated with the weight  $\varphi_s(\zeta) = e^{s(\zeta+\zeta^{-1})}$ . In the Szegő three-term recurrence for such orthogonal polynomials on the circle with respect to a general weight [27], only one coefficient  $\alpha_n$  depends on the choice of weight. Thus, the  $\alpha_n$ s, known as the *Verblunsky coefficients*, “encode” the weight and can be thought of as the analogue of reflection coefficients in this context. In the present case, they are related to  $F_r$  through

$$\alpha_r^2 = 1 - \frac{F_r F_{r+2}}{F_{r-1}^2}.$$

They are known ([7] or see [32, §3]) to satisfy the discrete Painlevé II equation

$$-s(1 - \alpha_r^2)(\alpha_{r+1} + \alpha_{r-1}) = (r + 1)\alpha_r$$

as well as the Ablowitz–Ladik lattice [1]

$$\partial_s \alpha_r = (1 - \alpha_r^2)(\alpha_{r+1} - \alpha_{r-1}).$$

The Ablowitz–Ladik hierarchy is a reduction of the 2D Toda hierarchy [28], suggesting the generalization to arbitrary initial data. Setting  $\zeta = \frac{1}{2}(t - \bar{x})$  and  $\eta = \frac{1}{2}(t + \bar{x})$ , where  $\bar{x} = x_1 + \dots + x_n$ ,  $\partial_{\bar{x}} = \partial_{x_1} + \dots + \partial_{x_n}$  and  $r \pm 1 = (r_1, \dots, r_n) \pm (1, \dots, 1)$ , it is not hard to check that the PNG kernel  $K$  from (2.4) satisfies for  $u, v > 0$ , and  $r_i > \underline{r} = \sup_{|y-x| \leq t} h_0(y)$ , the height achieved at  $(t, x)$  without nucleations, the following:

- (1)  $K_r(u, v) = K_{r-1}(u + 1, v + 1)$ .
- (2)  $\partial_\eta K_r(u, v) = K_{r-1}(u + 1, v) - K_r(u + 1, v)$ .
- (3)  $\partial_\zeta K_r(u, v) = K_{r-1}(u, v + 1) - K_r(u, v + 1)$ .

Then it is a general fact [19] that for  $r > \underline{r}$ , the  $n \times n$  matrix

$$Q_r = (I - K_r)^{-1}(1, 1)$$

satisfies the *non-Abelian 2D Toda equation*

$$\partial_\zeta(\partial_\eta Q_r Q_r^{-1}) + Q_r Q_{r-1}^{-1} - Q_{r+1} Q_r^{-1} = 0. \quad (2.5)$$

**Theorem 2.2** ([19]). *Let  $h_0$  be any deterministic initial data for PNG. The matrix function  $Q_r$  is related to the  $n$  point distributions of the PNG height at time  $t > 0$  by*

$$\frac{F(t, x_1, \dots, x_n, r_1 + 1, \dots, r_n + 1)}{F(t, x_1, \dots, x_n, r_1, \dots, r_n)} = \det Q_r, \quad (2.6)$$

where  $F_r(t, x_1, \dots, x_n, r_1, \dots, r_n) := \text{Prob}_{h_0}(h(t, x_i) \leq r_i, i = 1, \dots, n)$  and for  $r_i > \underline{r}(t, x_i)$ , they satisfy the non-Abelian 2D Toda equation (2.5).

The non-Abelian 2D Toda equation forms a well-known integrable system [20]. Lax and Zakharov–Shabat (zero-curvature) forms of the equation can be found in [31, §3.1] (see also [28]). They arise from compatibility of the equations for vectors  $\Phi_r$  [21]

$$\partial_\eta \Phi_r = V_r \Phi_r + \lambda_r \Phi_{r-1}, \quad \partial_t \Phi_r = U_r \Phi_{r+1}$$

with

$$U_r = Q_r Q_{r-1}^{-1}, \quad V_r = -\partial_\eta Q_r Q_r^{-1}.$$

The non-Abelian 2D Toda equation is sometimes written in terms of  $U_r$  and  $V_r$  as

$$\partial_\eta U_r + V_r U_r - U_r V_{r-1} = 0, \quad \partial_\xi V_r + U_{r+1} - U_r = 0.$$

When  $n = 1$ , there is no need to take the determinant in (2.6), and the previous theorem reduces via  $U_r = \frac{F_{r+1} F_{r-1}}{F_r^2}$  to the following more accessible statement.

**Corollary 2.3.** *Let  $h_0$  be deterministic. Then for  $t > 0$  and  $r > \underline{r} = \sup_{|y-x| \leq t} h_0(y)$ , the one-point distributions of PNG,  $F_r(t, x) := \text{Prob}_{h_0}(h(t, x) \leq r)$  satisfy the scalar 2D Toda equation*

$$\frac{1}{4}(\partial_t^2 - \partial_x^2) \log F_r = \frac{F_{r+1} F_{r-1}}{F_r^2} - 1. \quad (2.7)$$

In the case of flat initial data,  $h_0 \equiv 0$ ,  $F_r$  is independent of  $x$ , the  $x$  derivative drops out of (2.7) and taking  $g_r = \log F_r - \log F_{r-1}$  and  $t \mapsto 2t$  we obtain the classic Toda lattice,

$$\ddot{g}_r = e^{g_{r+1} - g_r} - e^{g_r - g_{r-1}}.$$

### 3. The KPZ fixed point

Since we have a general formula for the transition probabilities of a model in the KPZ class, if we zoom out in the 1:2:3 scaling

$$\varepsilon^{\frac{1}{2}} h(\varepsilon^{-\frac{3}{2}} t, \varepsilon^{-1} x) - C_\varepsilon t,$$

we can expect to obtain the transition probabilities of the Markov process which is a fixed point of this rescaling. It was done first for TASEP [18, 26], but the rescaling is most transparent for the PNG formulas. The *strong KPZ universality conjecture* says that we should get the same answer for any model in the class. Of course, this can alternatively be thought of as the definition of the KPZ universality class. Certainly, we do get the same answer for several models which can be solved explicitly like this, and for which the limit has been computed.

### 3.1. Brownian scattering transform

Let us see how it all works with the PNG kernel (2.3). First of all, note that the underlying space  $\mathbb{Z}_{>0}$  in which the Fredholm determinant is taken has the physical meaning of height. So it should be rescaled to  $\varepsilon^{1/2}\mathbb{Z}_{>0}$  which is approximating  $\mathbb{R}_+$ . The operators  $\nabla$  and  $\Delta$  act on functions on this space and are now of order  $\varepsilon^{1/2}$  and  $\varepsilon$ , respectively. We can expand

$$\nabla = \varepsilon^{\frac{1}{2}}\partial + \varepsilon^{\frac{3}{2}}\frac{1}{6}\partial^3 + o(\varepsilon^{\frac{3}{2}}), \quad \Delta = \varepsilon\partial^2 + o(\varepsilon).$$

The operator  $\partial$  acts on functions on  $\mathbb{R}$ . Now our formula for PNG rescales as

$$\begin{aligned} P_{\varepsilon^{1/2}h_0(\varepsilon^{-1}x)}(\varepsilon^{\frac{1}{2}}h(\varepsilon^{-\frac{3}{2}}x_i, \varepsilon^{-1}t) - C_\varepsilon t \leq r_i) \\ = \det(I - \varepsilon^{-1}K(\varepsilon^{-\frac{3}{2}}t, \varepsilon^{-1}\vec{x}, \varepsilon^{-\frac{1}{2}}(\vec{r} + C_\varepsilon t), \varepsilon^{\frac{1}{2}}h_0(\varepsilon^{-1}x)))_{\ell^2(\varepsilon^{1/2}\mathbb{Z}_{>0})}. \end{aligned}$$

The  $\varepsilon^{-1}$  in front of the  $K$  is because we want the measure on  $\varepsilon^{-1/2}\mathbb{Z}_{>0}$  to give mass  $\varepsilon^{1/2}$  to each site so that it approximates Lebesgue measure on  $[0, \infty)$ . We are assuming  $h_0(x) = \lim_{\varepsilon \searrow 0} \varepsilon^{1/2}h_0(\varepsilon^{-1}x)$  and look at the three terms in the second part of the kernel (2.4) and their approximations up to terms vanishing in  $\varepsilon$ :

$$\begin{aligned} & \underbrace{\tau_{\varepsilon^{-1/2}(r_i + C_\varepsilon t)} e^{-2\varepsilon^{-\frac{3}{2}}t \nabla - \varepsilon^{-1}x_i \Delta}}_{\tau_{r_i + C_\varepsilon t} e^{-2\varepsilon^{-1}t \partial - (t/3)\partial^3 - x_i \partial^2}} \underbrace{T_{\varepsilon^{-1}x - \varepsilon^{-3/2}t, \varepsilon^{-1}x + \varepsilon^{-3/2}t}^{\varepsilon^{\frac{1}{2}}h_0(\varepsilon^{-1}\cdot)}}_{T^{h_0}} \\ & \cdot \underbrace{e^{2\varepsilon^{-\frac{3}{2}}t \nabla + \varepsilon^{-1}x_j \Delta} \tau_{-\varepsilon^{-1/2}(r_j + C_\varepsilon t)}}_{e^{2\varepsilon^{-1}t \partial + (t/3)\partial^3 + x_j \partial^2} \tau_{-r_j - C_\varepsilon t}}. \end{aligned}$$

In the first and third terms, there is a huge shift  $\tau_{r_i + C_\varepsilon t}$  followed by  $e^{-2\varepsilon^{-1}t\partial} = e^{-2\varepsilon^{-1}t\partial} = \tau_{-2\varepsilon^{-1}t}$  since we are on  $\mathbb{R}$ . So we take  $C_\varepsilon = 2\varepsilon^{-1}$ . In the middle one, the operator converges to what we call the *Brownian scattering transform*  $T^{h_0}$  of  $h_0$ . To summarize, we get in the 1:2:3 limit

$$K = e^{-\frac{t}{3}\partial^3 - x_i \partial^2 + r_i \partial} T^{h_0} e^{\frac{t}{3}\partial^3 + x_j \partial^2 - r_j \partial}. \quad (3.1)$$

The presentation is a bit informal, and  $T^{\mathfrak{h}_0}$  is not really defined by itself. For  $t \neq 0$ , the operators on the outside are convolution operators with a completely explicit kernel in terms of the Airy function  $\text{Ai}$ :

$$e^{\frac{t}{3}\partial^3+x\partial^2}(u_1, u_2) = t^{-\frac{1}{3}} e^{\frac{2x^3}{3t^2} - \frac{(u_1-u_2)x}{t}} \text{Ai}(-t^{-\frac{1}{3}}(u_1 - u_2) + t^{-\frac{4}{3}}x^2). \quad (3.2)$$

Although it is not immediate, at  $t = 0$  it is not too hard to check that it converges to the usual heat kernel

$$e^{x\partial^2}(u_1, u_2) = \frac{1}{\sqrt{4\pi x}} e^{-\frac{(u_2-u_1)^2}{4x}}.$$

As for  $T^{\mathfrak{h}_0}$ , we are asymptotically computing the probability for a Brownian motion (of variance 2) to hit  $\mathfrak{h}$  (see Figure 5) but surrounded by backward (!) heat equations

$$T^{\mathfrak{h}} = \lim_{L \rightarrow \infty} e^{-L\partial^2} P_{-L,L}^{\text{hit}(\mathfrak{h})} e^{-L\partial^2}. \quad (3.3)$$

We do not really mean to make sense of (3.3) directly. What we really mean is that for any  $t \neq 0$ ,

$$e^{-\frac{t}{3}\partial^3} T^{\mathfrak{h}} e^{\frac{t}{3}\partial^3} = \lim_{L \rightarrow \infty} e^{-\frac{t}{3}\partial^3 - L\partial^2} P_{-L,L}^{\text{hit}(\mathfrak{h})} e^{\frac{t}{3}\partial^3 - L\partial^2}$$

makes sense (as a trace class operator on  $L^2([r, \infty))$ ) and (3.1) is not really a product of operators but an informal way to write the family in  $t, \vec{x}, \vec{r}$  of integral kernels  $K(t, \vec{x}, \vec{r}, u_1, u_2)$  in  $u_1, u_2$  which solve

$$\begin{aligned} \partial_t K &= -\frac{1}{3}(\partial_{u_1}^3 + \partial_{u_2}^3) K, & \partial_{x_i} K &= -\partial_{u_1}^2 K, & \partial_{x_j} K &= \partial_{u_2}^2 K, \\ \partial_{r_i} K &= \partial_{u_1} K, & \partial_{r_j} K &= \partial_{u_2} K. \end{aligned} \quad (3.4)$$

When  $\mathfrak{h}$  has compact support, which in this world means that it is equal to  $-\infty$  outside a box  $[-\ell, \ell]$ , suppose  $L \gg \ell$ , and we want to compute

$$\lim_{L \rightarrow \infty} e^{-\frac{t}{3}\partial^3 - x_i\partial^2 + r_i\partial} e^{-L\partial^2} P_{-L,L}^{\text{hit}(\mathfrak{h})} e^{-L\partial^2} e^{\frac{t}{3}\partial^3 + x_j\partial^2 - r_j\partial}. \quad (3.5)$$

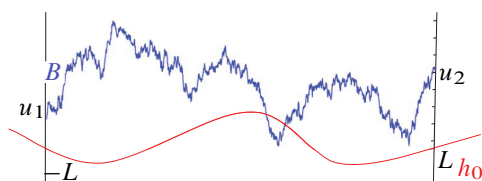


**Figure 5.** The quantity  $P_{-L,L}^{\text{hit}(\mathfrak{h})}(u_1, u_2)$  is the probability density that a Brownian motion  $B(x)$  is at  $u_2$  at time  $L$  given that it starts at  $u_1$  at time  $-L$  and that it hits the hypograph of  $\mathfrak{h}$  along the way.

But the Brownian semigroup is  $e^{t\partial^2}$ , so since the Brownian particle is free on the intervals  $[-L, -\ell)$  and  $(\ell, L]$ , we can write

$$P_{-L,L}^{\text{hit}(\mathfrak{h})} = e^{(L-\ell)\partial^2} P_{-\ell,\ell}^{\text{hit}(\mathfrak{h})} e^{(L-\ell)\partial^2},$$

and (3.5) becomes, even without taking the limit, completely well defined:

$$e^{-\frac{t}{3}\partial^3 + (-x_i - \ell)\partial^2 + r_i\partial} P_{-\ell,\ell}^{\text{hit}(\mathfrak{h})} e^{\frac{t}{3}\partial^3 + (x_j - \ell)\partial^2 - r_j\partial}. \quad (3.6)$$

For example, in the case of one narrow wedge at 0,  $P_{-\ell,\ell}^{\text{hit}(\mathfrak{h})}$  is just  $e^{\ell\partial^2} \mathbf{1}_{(-\infty, 0]} e^{\ell\partial^2}$ , and after adding in the term  $-e^{(x_j - x_i)\partial^2 + (r_i - r_j)\partial} \mathbf{1}_{x_i < x_j}$ , the limit of the first term in (2.4), one obtains from (3.2) the *extended Airy kernel*.

One could also, say, start with two narrow wedges at  $\pm 1$ , in which case  $P_{-\ell,\ell}^{\text{hit}(\mathfrak{h})}$  is given by

$$e^{(\ell-1)\partial^2} \mathbf{1}_{(-\infty, 0]} e^{2\partial^2} \mathbf{1}_{(-\infty, 0]} e^{(\ell-1)\partial^2}.$$

This approach leads to a practical formula for any compactly supported data.

For noncompactly supported data, it looks like one has to take a limit. For example, for flat initial data  $\mathfrak{h}_0 = 0$ , one might take approximations which are 0 on  $[-N, N]$  and  $-\infty$  on  $[-N, N]^C$ , compute the scattering operator by the reflection principle and send  $N \rightarrow \infty$ . One finds, much to one's satisfaction, that the limiting operator  $T^0$  is exactly the reflection operator  $\varrho f(u) = f(-u)$ . Then one can check by complex analysis that  $e^{-(1/3)\partial^3} \varrho e^{(1/3)\partial^3}(u, v) = 2^{-1/3} \text{Ai}(2^{-1/3}(u + v))$ , the kernel for  $F_{\text{GOE}}(2^{-1/3}r)$ ; the rescaling by  $2^{-1/3}$  represents the slight difference between the one-point distribution, at time 1 of the KPZ fixed point starting from  $\mathfrak{h}_0 \equiv 0$ , and the limiting fluctuations of the top eigenvalue of a standard  $N \times N$  matrix from the Gaussian orthogonal ensemble.

Writing the operator as a product such as in (3.1) can be misleading, especially as these operator products are on  $L^2(\mathbb{R})$  while the Fredholm determinants are being taken on  $L^2([0, \infty))$ . Otherwise, the pre- and postfactors  $e^{-(t/3)\partial^3} T e^{(t/3)\partial^3}$  would just be a conjugation and the determinant would not describe any non-trivial dynamics.

For general noncompactly supported data, one might wonder if a limiting operation always works. Actually, there is a trick to avoid taking any limit at all, which we call *splitting*. Recall that the hitting is being done from above, i.e., hitting  $\mathfrak{h}$  means entering its hypograph. We can write, with the obvious notation

$$P_{-\ell,\ell}^{\text{hit}(\mathfrak{h})} = e^{2\ell\partial^2} - P_{-\ell,\ell}^{\text{no hit}(\mathfrak{h})}$$

and slip the two terms into (3.6). By the Markov property of Brownian motion,

$$P_{-\ell,\ell}^{\text{no hit}(\mathfrak{h})}(u, v) = \int_{\mathfrak{h}(0)}^{\infty} dz P_{-\ell,0}^{\text{no hit}(\mathfrak{h})}(u, z) P_{0,\ell}^{\text{no hit}(\mathfrak{h})}(z, v).$$

Now one can go back and write  $P_{-\ell,0}^{\text{no hit}(\mathfrak{h})} = e^{\ell\partial^2} - P_{-\ell,0}^{\text{hit}(\mathfrak{h})}$  and  $P_{0,\ell}^{\text{no hit}(\mathfrak{h})} = e^{\ell\partial^2} - P_{0,\ell}^{\text{hit}(\mathfrak{h})}$ . One of these sees  $\mathfrak{h}^+$  which is just the function  $\mathfrak{h}$  on  $[0, \infty)$ , and the other sees  $\mathfrak{h}^-$  which is  $\mathfrak{h}(-x)$ ,  $x \in [0, \infty)$ . Let  $\tau_{\mathfrak{h}^+}$  and  $\tau_{\mathfrak{h}^-}$  be the hitting times of  $\mathfrak{h}^\pm$  of a Brownian motion starting at  $z$ . Since there has to be a first time it hits, after which we do not care, we can write

$$P_{0,\ell}^{\text{hit}(\mathfrak{h})}(z, v) = \int_0^\ell P_z(\tau_{\mathfrak{h}^+} \in dx) e^{(\ell-x)\partial^2} (B(x), v),$$

where  $B$  is the hitting Brownian motion. But now when we put it back into (3.6), all the  $\ell$ s cancel, and we can let  $\ell \rightarrow \infty$  free of charge. Putting it all together, we get the extended kernel

$$\begin{aligned} \mathfrak{K}_{ij}^{\text{ext}}(t, \vec{x}, \vec{r}, \mathfrak{h}) &= -e^{(x_j-x_i)\partial^2 + (r_i-r_j)\partial} 1_{x_i < x_j} \\ &\quad + (S_{t,x_i,r_i} - S_{t,x_i,r_i}^{\mathfrak{h}^-})^* (S_{t,-x_j,r_j} - S_{t,-x_j,r_j}^{\mathfrak{h}^+}), \end{aligned} \quad (3.7)$$

where

$$S_{t,x,r}^{\mathfrak{h}}(z, v) = E_z[S_{t,x-\tau_{\mathfrak{h}},r}(B(\tau), v)], \quad S_{t,x,r} = e^{\frac{t}{3}\partial^3 + x\partial^2 - r\partial}.$$

One needs an upper bound on  $\mathfrak{h}_0$  for it to make sense. For example, if  $\mathfrak{h}_0(x) = ax^2$ , it is not hard to see (for example, from the variational formula (1.8)) that the fixed point explodes to  $+\infty$  at time  $t = a^{-1}$ . Height functions should naturally be upper semicontinuous, and a nice class of such functions is given by those that are bounded by  $C_1 + C_2|x|$  for some  $C_1, C_2 < \infty$ . No lower bound is needed, and  $\mathfrak{K}^{\text{ext}}$  given by (3.7) is a nice continuous map from this space equipped with the local Hausdorff topology into trace class operators [18].

The KPZ fixed point is the 1:2:3 scaling invariant Markov process of height functions which sits in the middle of the class, much as Brownian motion sits in the middle of the universality class of stochastic processes without long-range correlations which scale to it in the diffusive 1:2 scaling (see Figure 6). It is defined through its transition probabilities, whose finite-dimensional distributions are given by

$$P_{\mathfrak{h}_0}(\mathfrak{h}(t, x_1) \leq r_1, \dots, \mathfrak{h}(t, x_n) \leq r_n) = \det(I - \mathfrak{K}^{\text{ext}}(t, \vec{x}, \vec{r}, \mathfrak{h}_0))_{L^2([0, \infty))^n}.$$

It is proved in [18] that these are indeed transition probabilities and satisfy the Chapman–Kolmogorov equations. So the stochastic integrability is  $\mathfrak{h}_0 \mapsto$  Brownian scattering transform  $\mapsto$  linear evolution in height/space/time  $\xrightarrow{\text{det}}$  transition probs, analogous to scattering/linear evolution/inverse scattering for classical integrable PDE.

The following basic properties all follow from the formula; they can also be proved by approximation from appropriate models:

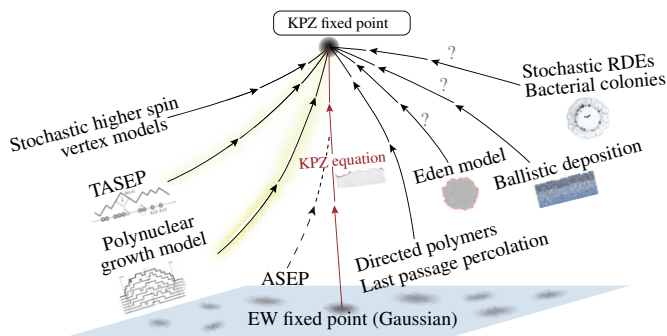


**Figure 6.** Schematic of the space of models with EW and KPZ fixed points under *zooming out*.

- (1) (1:2:3 invariance)  $\varepsilon^{1/2} \mathfrak{h}(\varepsilon^{-3/2}t, \varepsilon^{-1}x; \varepsilon^{-1/2}\mathfrak{h}_0(\varepsilon x)) \stackrel{\text{dist}}{=} \mathfrak{h}(t, x; \mathfrak{h}_0)$ .
- (2) (Invariance of Brownian motion) If  $B(x)$  is a two-sided Brownian motion with diffusion coefficient 2, then for each  $t > 0$ ,  $\mathfrak{h}(t, x; B) - \mathfrak{h}(t, 0; B)$  is also two-sided Brownian motion in  $x$  with diffusion coefficient 2.
- (3) (Stationarity in space)  $\mathfrak{h}(t, x + u; \mathfrak{h}_0(x - u)) \stackrel{\text{dist}}{=} \mathfrak{h}(t, x; \mathfrak{h}_0)$ .
- (4) (Reflection invariance)  $\mathfrak{h}(t, -x; \mathfrak{h}_0(-x)) \stackrel{\text{dist}}{=} \mathfrak{h}(t, x; \mathfrak{h}_0)$ .
- (5) (Affine invariance)  $\mathfrak{h}(t, x; \mathfrak{h}_0(x) + a + cx) \stackrel{\text{dist}}{=} \mathfrak{h}(t, x + \frac{1}{2}ct; \mathfrak{h}_0(x)) + a + cx + \frac{1}{4}c^2t$ .
- (6) (Skew time reversibility)  $P(\mathfrak{h}(t, x; \mathfrak{g}) \leq -\mathfrak{f}(x), x \in \mathbb{R}) = P(\mathfrak{h}(t, x; \mathfrak{f}) \leq -\mathfrak{g}(x), x \in \mathbb{R})$ .

If we start from a narrow wedge at 0, the spatial process is called the *parabolic Airy process*

$$\mathcal{A}(x) = \mathfrak{h}(1, x; \mathfrak{d}_0).$$

If we add a parabola  $\hat{\mathcal{A}}(x) = \mathcal{A}(x) + x^2$ , it turns out to be stationary in  $x$ ; for example, for each  $x$ ,  $\hat{\mathcal{A}}(x) \stackrel{\text{dist}}{=} F_{\text{GUE}}(x)$ . You get the same thing if you take the large  $N$  limit of the top line of Dyson Brownian motion version of GUE, i.e., the eigenvalues of a dynamical  $N \times N$  Hermitian matrix whose upper diagonal entries are independent stationary Ornstein–Uhlenbeck processes. From last passage models like PNG, the KPZ fixed point inherits the *preservation of max* property: For any  $\mathfrak{f}_1, \mathfrak{f}_2$ ,

$$\mathfrak{h}(t, x; \mathfrak{f}_1 \vee \mathfrak{f}_2) \stackrel{\text{dist}}{=} \mathfrak{h}(t, x; \mathfrak{f}_1) \vee \mathfrak{h}(t, x; \mathfrak{f}_2).$$

The *parabolic Airy sheet* is the two-parameter process

$$\mathcal{A}(x, y) = \mathfrak{h}(1, y; \mathfrak{d}_x), \quad (3.8)$$

and its stationary version is  $\hat{\mathcal{A}}(x, y) = \mathcal{A}(x, y) - (x - y)^2$ . Using stationarity in space,  $\mathfrak{h}(1, y; \mathfrak{d}_x) \stackrel{\text{dist}}{=} \mathfrak{h}(1, y - x; \mathfrak{d}_0)$ , and by reflection invariance,  $\mathfrak{h}(1, y - x; \mathfrak{d}_0) \stackrel{\text{dist}}{=} \mathfrak{h}(1, x - y; \mathfrak{d}_0)$ . Therefore,

$$\mathcal{A}(x, y) \stackrel{\text{dist}}{=} \mathcal{A}(y, x),$$

and the Airy sheet is an Airy process in each of its coordinates.

From the max property, the KPZ fixed point satisfies the variational formula (1.8). This is the closest thing we have to a stochastic equation for the KPZ fixed point, and it is incomplete because we do not exactly know what the joint distributions of  $\mathcal{A}(x, y)$  are. Of course, many things about it can be understood by approximation from various models.

#### 4. The KPZ fixed point and integrable PDE

Let  $F(t, x, r)$  be the one-point distribution of PNG. We now perform the same scaling on it as a solution of the 2D Toda lattice. So we are interested in the limit  $\mathfrak{F} = \lim_{\varepsilon \searrow 0} \mathfrak{F}_\varepsilon$ , where the latter is defined by

$$F(t, x, r) = \mathfrak{F}_\varepsilon(\varepsilon^{\frac{3}{2}}t, \varepsilon x, \varepsilon^{\frac{1}{2}}(r - 2t)).$$

We expand  $\mathfrak{F}_\varepsilon$  in Taylor series even though it really lives on  $\varepsilon^{1/2}\mathbb{Z}$ . Under this scaling, (2.7) becomes

$$\frac{1}{4}(\varepsilon^3 \partial_t^2 - \varepsilon^2 4 \partial_{\mathfrak{t}}^2 + \varepsilon 4 \partial_r^2 - \varepsilon^2 \partial_x^2) \log \mathfrak{F} = \frac{\mathfrak{F}(\cdot + \varepsilon^{\frac{1}{2}}) \mathfrak{F}(\cdot - \varepsilon^{\frac{1}{2}})}{\mathfrak{F}^2} - 1, \quad (4.1)$$

where for notational simplicity we removed the  $\varepsilon$  from the subscript in  $\mathfrak{F}_\varepsilon$ . Expanding (4.1) out and multiplying by  $\mathfrak{F}^2$ , we get

$$\begin{aligned} \varepsilon^2 \left[ \mathfrak{F} \partial_{\mathfrak{t}}^2 \mathfrak{F} - \partial_t \mathfrak{F} \partial_r \mathfrak{F} + \frac{1}{4} \mathfrak{F} \partial_x^2 \mathfrak{F} - \frac{1}{4} (\partial_x \mathfrak{F})^2 \right. \\ \left. + \frac{1}{12} \mathfrak{F} \partial_r^4 \mathfrak{F} - \frac{1}{3} \partial_r \mathfrak{F} \partial_r^3 \mathfrak{F} + \frac{1}{4} (\partial_r^2 \mathfrak{F})^2 \right] + o(\varepsilon^2) = 0. \end{aligned}$$

The vanishing of the square brackets constitutes the Hirota form of the Kadomtsev–Petviashvili equation KP-II.

Let  $\mathfrak{K} = I - \mathfrak{K}^{\text{ext}}$ , the fixed point kernel from (3.7), and  $Q = \mathfrak{K} \mathfrak{K}^{\text{ext}}(0, 0)$  which is an  $n \times n$  matrix-valued function of  $t, x_1, \dots, x_n, r_1, \dots, r_n$ , and the initial height profile  $\mathfrak{h}_0$ . Let  $\mathcal{D}_r = \partial_{r_1} + \dots + \partial_{r_n}$ ,  $\mathcal{D}_x = \partial_{x_1} + \dots + \partial_{x_n}$ . One can check that for  $\mathfrak{F} := P_{\mathfrak{h}_0}(\mathfrak{h}(t, x_1) \leq r_1, \dots, \mathfrak{h}(t, x_n) \leq r_n)$ ,

$$\mathcal{D}_r \log \mathfrak{F} = \text{tr } Q,$$

in other words, the  $n$ -dimensional distributions can be reconstructed from  $Q$ . One can prove [24] the following as a general consequence for Fredholm determinants of kernels satisfying evolution equations (3.4).

**Theorem 4.1.** *The function  $Q$  and its derivative  $q = \mathcal{D}_r Q$  solve the matrix Kadomtsev–Petviashvili (KP) equation*

$$\partial_t q + \frac{1}{2} \mathcal{D}_r q^2 + \frac{1}{12} \mathcal{D}_r^3 q + \frac{1}{4} \mathcal{D}_x^2 Q + \frac{1}{2} [q, \mathcal{D}_x Q] = 0,$$

where  $[A, B] = AB - BA$ .

In particular, the one-point marginals of the KPZ fixed point (i.e., in the case  $n = 1$ ),  $\phi = \partial_r^2 \log \mathfrak{F}$ , satisfy the scalar KP-II equation

$$\partial_t \phi + \frac{1}{2} \partial_r \phi^2 + \frac{1}{12} \partial_r^3 \phi + \frac{1}{4} \partial_r^{-1} \partial_x^2 \phi = 0. \quad (4.2)$$

Classical examples include the following.

**Example 4.2.** For narrow wedge initial data  $\mathfrak{h}_0 = \mathfrak{b}_0$ , the distribution of  $\mathfrak{h}(t, x) + \frac{x^2}{t}$  does not depend on  $x$  and should be the Tracy–Widom GUE scaled by  $t^{1/3}$ . So we look for a self-similar solution to KP (4.2) of the form

$$\phi^{\text{nw}}(t, x, r) = t^{-\frac{2}{3}} \psi^{\text{nw}}(t^{-\frac{1}{3}} r + t^{-\frac{4}{3}} x^2),$$

which gives the ODE

$$(\psi^{\text{nw}})''' + 12\psi^{\text{nw}}(\psi^{\text{nw}})' - 4r(\psi^{\text{nw}})' - 2\psi^{\text{nw}} = 0. \quad (4.3)$$

The transformation  $\psi^{\text{nw}} = -q^2$  takes (4.3) into Painlevé II:

$$q'' = rq + 2q^3. \quad (4.4)$$

As  $r \rightarrow -\infty$ , the solution is approximately  $\phi^{\text{nw}}(t, x, r) \sim -(\frac{r}{2t} + \frac{x^2}{2t^2})$ , picking out the Hastings–McLeod solution  $q(r) \sim -\text{Ai}(r)$  as  $r \rightarrow \infty$ . Thus, we recover  $F(t, x, r) = F_{\text{GUE}}(t^{-1/3}r + t^{-4/3}x^2)$ ,

$$F_{\text{GUE}}(s) = e^{-\partial^{-2}q^2} = \exp\left\{-\int_s^\infty du (u-s)q^2(u)\right\}.$$

**Example 4.3.** For flat initial data  $\mathfrak{h}_0(x) \equiv 0$ , there is no  $x$  dependence and KP-II (4.2) reduces to KdV. We look for a self-similar solution of KdV of the form

$$\phi^{\text{fl}}(t, r) = \left(\frac{t}{4}\right)^{-\frac{2}{3}} \psi^{\text{fl}}\left(\left(\frac{t}{4}\right)^{-\frac{1}{3}}\right)$$

which reduces it to the ODE

$$(\psi^{\natural})''' + 12(\psi^{\natural})'\psi^{\natural} - r(\psi^{\natural})' - 2\psi^{\natural} = 0.$$

Miura's transform

$$\psi^{\natural} = \frac{1}{2}(q' - q^2)$$

brings this to Painlevé II (4.4), with the same behaviour as  $r \rightarrow -\infty$  as in the previous example. So we recover  $F(t, x, r) = F_{\text{GOE}}(4^{1/3}t^{-1/3}r)$ ,

$$F_{\text{GOE}}(r) = e^{-\partial^{-2} \frac{1}{2}(q' + q^2)} = \exp\left\{-\frac{1}{2} \int_r^\infty du q(u)\right\} F_{\text{GUE}}(r)^{\frac{1}{2}}.$$

These distributions arose earlier as the limits of rescaled top eigenvalues from, respectively, the Gaussian unitary and Gaussian orthogonal ensembles. But unlike other limit distributions in probability, they have no scaling invariance. KP is invariant under

$$\phi(t, x, r) \mapsto \alpha^{-2}\phi(\alpha^{-3}t, \alpha^{-2}x, \alpha^{-1}r), \quad \mathfrak{h}_0(x) \mapsto \alpha^{-1}\mathfrak{h}_0(\alpha^2x).$$

This is not surprising; we essentially derived it by taking this scaling. So now we have embedded the Tracy–Widom distributions into a system which does have scaling invariance, i.e., we have enlarged our perspective enough to see the underlying invariance principle.

This begs the question: Where do the KP equations come from, and what do they mean? One of the key characteristics of KP is solitons. But our solutions have no solitons at all. Universal non-linear partial differential equations like KP and KdV, or the non-linear Schrödinger equation, are generally obtained from more complicated equations by rescalings, asymptotic expansions and/or ansatz of special forms. Unlike other basic PDEs like the heat and wave equations, Burgers or the Euler equations, they are not derived from first-principle physical mechanisms, but only as model equations governing the non-linear effects in certain asymptotic regimes. It is all “this is what you get in the limit” as opposed to “the stuff is doing this”. For example, KP is usually derived from water wave equations in the limit of long waves with small amplitude and slow dependence on the transverse coordinate. KdV is derived from one-way travelling waves in Fermi–Pasta–Ulam, etc. Since these model equations appear as limits of so many equations, they have an opportunity to inherit integrability in the scaling limit from even just one integrable example within their domain of attraction. So it is not surprising that many of them are integrable.

Our case is unusual. There is no “more complicated equation” reducing to KP in the scaling limit. And the KP equations are discovered not through the scaling limit, but through the exact formulas themselves, which are then recognized as examples of inverse scattering solutions of KP. None of this was very easy to predict.

One could perhaps think of the full evolution equation, for, say, PNG, as the more complicated equation. But this just leads to a bigger mystery. It is certainly not true that one had any reason to expect that one-point functions for PNG satisfy a closed equation. In general, in a many-body problem like this, one expects to write down an infinite hierarchy of equations, for example, for  $n$  particle density functions, the so-called BBGKY hierarchy. So one thing to keep in mind is that the biggest surprise is not that one-point distributions of the KPZ fixed point are governed by KP, but that they are governed by any closed equation at all.

The KPZ equation itself is not stochastically integrable, though it seems to have partial solvability. It can be obtained in the limit of other models as a weakly asymmetric or weak noise limit in which the asymmetry or noise is scaled down as we scale the solution with diffusive scaling. Some of the approximating models are themselves partially solvable. For example, for the asymmetric exclusion process, formulas for narrow wedge and step Bernoulli initial data were obtained in the breakthrough works [29, 30]. From the formulas, one has convergence to the KPZ fixed point for these special initial data. For general initial data, it is a very recent result [2]. An earlier attempt [25] contains a mistake in a shift argument used to identify the limiting fluctuations (however, it does prove tightness and even works for non-nearest neighbour exclusion height functions in the 1:2:3 scaling, a model that is not even partially solvable). That the KPZ equation converges to the KPZ fixed point, i.e., it is a member of the universality class bearing its name, is proved in [33, 34]. The key challenge for the future is to devise methods to prove the convergence to the KPZ fixed point for the larger class of non-solvable KPZ models.

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