

Abstract

In 2022, F. Da Lio, M. Gianocca, and T. Rivière developed a new general method to establish Morse stability results in geometric analysis, and they successfully applied it to conformally invariant Lagrangians in dimension 2 (whose critical points include harmonic maps with values into closed Riemannian manifolds). By Morse stability, we mean that the Morse index of a sequence of critical points of bounded energy is lower semicontinuous, while the sum of the Morse index and the nullity is upper semicontinuous.

In this memoir, we apply this theory to Willmore immersions and show their Morse stability, provided that the limiting map and the (eventual) bubbles are also immersions, i.e., free of branch points. Our result covers the case of degenerating Riemann surfaces for which the ratio of the second residue (considered by P. Laurain and T. Rivière, 2018) by the length of the minimal shrinking geodesic of the underlying sequence of Riemann surfaces is smaller than a universal constant (that only depends on the ambient dimension). Along the way, we prove Rellich and Hardy–Rellich inequalities in annuli for the bilaplacian in arbitrary dimensions and for a natural family of fourth-order elliptic operators with regular singularities in dimension 2, provided that the conformal class of the annuli is larger than a universal constant depending only on the dimension and the parameter $m \geq 1$ of the second family of operators. Those results had subsequent applications to the Morse stability of biharmonic maps in dimension 4 and the Morse stability of Willmore immersions that develop branch points in the limit.

Keywords. Willmore surfaces, conformally invariant problems, Morse theory, Lorentz spaces, energy quantisation, Rellich inequalities, Hardy–Rellich inequalities

Mathematics Subject Classification (2020): 49Q10 (primary); 35J35, 34L15, 35P15, 53A05 (secondary)

Acknowledgements. The authors would like to thank the anonymous referee for their extremely detailed comments that helped to improve the presentation of this memoir.