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★**An introduction to noncommutative geometry.**

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This excellent introduction to noncommutative geometry gives a precise 'current state of affairs' of the subject. Ever since the invention of noncommutative geometry by Alain Connes in the 1980s, more and more applications of it have been found in both mathematics and physics. In this book, the author succeeds in giving the reader a broad and accessible overview of these. In the meanwhile, the book remains within the scope of the EMS lecture series by giving a rapid introduction to the subject and providing an extensive guide to the more advanced literature.

In Chapter 1, it is explained how noncommutative geometry generalizes ordinary commutative geometry. In three steps, the spectral triple describing the spin geometry of an ordinary Riemannian Spin manifold is derived. It consists of the algebra of smooth functions on such a manifold together with a Dirac operator, which interact via their representation on the Hilbert space of spinors on M. In order to make this construction as explicit as possible, the author discusses in Chapter 2 the spectral triple describing the spin geometry of the Riemann sphere. A central role is played by a spinor calculus, which allows for an explicit description of the Dirac operator on the sphere.

In Chapter 3, the axiomatic foundation of spectral triples is laid out. One starts with a data set consisting of an algebra A and a selfadjoint operator D, which are both represented on a Hilbert space H. They satisfy the following two conditions: (1) The commutator of any element in A with D is a bounded operator (on H). (2) The resolvent of D is a compact operator. Then, step by step, the following additional conditions are imposed: reality, first-order condition, smoothness of the algebra, orientation, finiteness and Poincaré duality.

In Chapter 4, this abstract data set is exemplified once more, but now with an honest noncommutative geometry. This is the famous noncommutative torus, given as the algebra generated by two unitaries u, v that satisfy the relation $uv = \lambda vu$ for λ a complex number of modulus one. A family of spin geometries on the noncommutative torus is then constructed, and it is shown that it satisfies all the above additional conditions on a spectral triple.

In Chapter 5, a general noncommutative integral is derived for any (suitably summable) spectral triple. This Dixmier trace reduces to the ordinary Riemann integral in the commutative case; this is derived via the Wodzicki residue. Another facet of noncommutative geometry is addressed in Chapter 6 where quantization is described. By now, quantization has a thorough mathematical basis involving groupoids and C^*-algebras on them, which are briefly described in this chapter. Also, the special role played by index theorems and K-theory is mentioned in the last two sections.

As a preparation for what is to come in Chapter 8, a unitary equivalence relation for noncommutative geometries is derived in Chapter 7. It involves a unitary operator on a Hilbert space that intertwines both the algebra and the Dirac operator. There is also a weaker notion of equivalence, which is Morita equivalence. In some sense, this equivalence relation is better adapted to the noncommutative situation since it also affects the distance defined by the Dirac operator. This is in contrast with unitary equivalence, for which the distance is invariant. As an example, there follows a discussion on Morita equivalent spin geometries on the noncommutative torus.

In Chapter 8, the previously mentioned equivalence relations are set to work. Both the spectral action and the fermionic action are shown to be unitarily invariant. The spectral action is given as the number of eigenvalues of D up to some cut-off, whereas the fermionic action is given by an inner product (involving D) in the Hilbert space. On the product of an ordinary manifold with an "almost commutative geometry", the fluctuation of D coming from Morita equivalence gives rise—via the spectral and fermionic action—to the Standard Model of physics coupled to the Einstein-Hilbert action. In the remainder of the chapter, some background is given on the computational techniques used for the derivation of this action.

Finally, in Chapter 10 some new directions in noncommutative geometry are listed. A flavour of noncommutative field theories is given, as well as a special class of noncommutative geometries, known as isospectral deformations. Also, the recently established interactions between noncommutative geometry and quantum group theory are sketched.

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