**Addendum to Proposition III.5, p. 84**
**E. Zehnder, Lectures on Dynamical Systems**

In the proof of Proposition III.5 we have studied the homoclinic points $p \in \Lambda \setminus Q$, and it remains to examine the homoclinic points $p \in \Lambda \cap Q$ near the hyperbolic fixed point. For this purpose we consider $p \in \Lambda \cap W^{-}_{\text{loc}}(Q, \varphi)$ in a sufficiently small neighborhood $Q$ of the origin and prove the following linearized version of the Inclination Lemma due to Janko Latschev. Recalling the notation of the proof we work with the $(\xi, \eta)$ coordinates and denote by $Q'$ the image of $Q$ under the coordinate transformation.

**Linearized Inclination Lemma.** *We consider the points* $(0, \eta) \in W^{-}_{\text{loc}}(Q', \hat{\varphi})$ *and let* $L \subset T_{(0,\eta)}\mathbb{R}^n$ *be a subspace of the tangent space, which is transversal to the space* $E_- = T_{(0,\eta)} W^{-}_{\text{loc}}(Q', \hat{\varphi})$, *so that* $\dim L + \dim E_- = n$. *Then*

$$d\hat{\varphi}^{-j}(0, \eta)L \to E_+$$

*as* $j \to \infty$.

*Proof.* Abbreviating the linear spaces $L_n := d\hat{\varphi}^{-n}(0, \eta)L$ we shall show that for every $\varepsilon > 0$ there exists a positive integer $N$ such that for $n \geq N$ and for non-vanishing vectors $v = (v_+, v_-) \in L_n$ we have the estimates

$$\frac{|v_-|}{|v_+|} \leq \varepsilon.$$

As a consequence the vector space $L_n$ is the graph of a linear map $E_+ \to E_-$ whose operator norm is smaller than or equal to $\varepsilon$, so that the lemma follows.

Since $0 < \alpha < 1$, we can choose $\varepsilon > 0$ so small that

$$\vartheta := \alpha + \frac{\varepsilon}{2} < 1 \quad \text{and} \quad \frac{1}{\alpha} - \frac{\varepsilon}{2} > 1.$$

If $|\eta| > 0$ is sufficiently small we find, in view of our choice of the coordinates, a constant $K > 0$ such that

$$d\hat{\varphi}^{-1}(0, \eta) = \begin{pmatrix} A_+^{-1} + \rho_+(\eta) & 0 \\ \rho(\eta) & A_-^{-1} + \rho_-(\eta) \end{pmatrix},$$

where $|\rho_+(\eta)| \leq K \cdot \eta$, $|\rho_-(\eta)| \leq K \cdot \eta$ and $|\rho(\eta)| \leq K \cdot \eta$. We now choose the integer $N_0$ so large that $\hat{\varphi}^{-N_0}(0, \eta) = (0, \eta_0)$ satisfies

$$K \cdot \eta_0 \leq \frac{\varepsilon}{2}(1 - \vartheta).$$

For $n \geq N_0$ we have $L_n = d\hat{\varphi}^{-(n-N_0)}(0,\eta_0)L_{N_0}$. If $v \in L_{N_0}$ we can estimate, using $\frac{1}{\alpha} - K\eta_0 \geq \frac{1}{\alpha} - \frac{\varepsilon}{2} > 1$,

$$\frac{|(d\hat{\varphi}^{-1}(0,\eta_0)v)_-|}{|(d\hat{\varphi}^{-1}(0,\eta_0)v)_+|} \leq \frac{K\eta_0|v_+| + (\alpha + K\eta_0)|v_-|}{(\frac{1}{\alpha} - K\eta_0)|v_+|} \leq K\eta_0 + \vartheta\frac{|v_-|}{|v_+|}.$$

Defining the sequence of numbers $\lambda_j$ by

$$\lambda_j := \frac{|(d\hat{\varphi}^{-j}(0,\eta_0)v)_-|}{|(d\hat{\varphi}^{-j}(0,\eta_0)v)_+|}, \quad j \geq 0,$$

we have just proved that $\lambda_1 \leq K \cdot \eta_0 + \vartheta \cdot \lambda_0$. Analogously, $\lambda_{j+1} \leq K \cdot \eta_0 + \vartheta \cdot \lambda_j$ and consequently, if $j \geq N_1$ for a sufficiently large number $N_1$,

$$\begin{aligned} \lambda_{j+1} &\leq \vartheta^{j+1}\lambda_0 + K\eta_0 \sum_{s=0}^{j} \vartheta^s \\ &\leq \vartheta^{j+1}\lambda_0 + K \cdot \eta_0 \frac{1}{1-\vartheta} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

We have proved, for $n \geq N_0 + N_1$, that indeed $\frac{|v_-|}{|v_+|} \leq \varepsilon$ for the non-vanishing vectors $v \in L_n$, and the lemma follows. $\square$

From the lemma we easily deduce the desired estimate near the origin. Indeed, recalling the estimate $|d\hat{\varphi}(0)v_+| \leq \alpha|v_+|$ for $v_+ \in E_+$ we choose $\alpha < \beta < 1$. In view of the continuity of the derivative $d\hat{\varphi}$ and in view of the convergence $L_n \to E_+$ and $\hat{\varphi}^{-n}(0,\eta) \to 0$ as $n \to \infty$, we find an integer $N'$ such that, for all $n \geq N'$, we have the estimate

$$|d\hat{\varphi}(\hat{\varphi}^{-n}(0,\eta))v| \leq \beta|v|$$

for all $v \in L_n$.

The case $p \in \Lambda \cap W^+_{\text{loc}}(Q,\varphi)$ is treated the same way, and Proposition III.5 follows. For the Inclination Lemma (the $\lambda$-Lemma) we refer to the book [81] by Jacob Palis, Jr., and Welington de Melo.