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★**Lectures on dynamical systems.**
Hamiltonian vector fields and symplectic capacities.
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This book is an introduction to dynamical systems. It is based on lectures given by the author at ETH Zurich in the academic year 2004/2005. It can be read with a good knowledge of undergraduate mathematics as the only background.

The book first introduces basic notions of dynamical systems. Then it concentrates on several specific aspects: hyperbolic dynamics, gradient-like flows and Hamiltonian dynamics. This last subject is studied much more deeply than the others. Indeed, this part occupies almost half of the book, in which the author reaches recent deep results on periodic orbits of Hamiltonian dynamics. He introduces in detail the so-called Hofer-Zehnder capacity and gives an accessible proof of the celebrated non-squeezing theorem of Gromov. Let us review in more detail the contents of the book.

The first chapter introduces basic concepts of dynamical systems with many examples. It starts with the concrete example of N-body motion. Then, the notion of orbit is defined, with several standard examples. Translations on the circle or the torus are studied in detail. In particular, equidistribution theorems are proved. A section is devoted to introducing transitive and minimal systems, with a proof of Birkhoff's transitivity theorem. The notion of structural stability is illustrated on the example of the expanding map $z \mapsto z^2$ on the circle. The last part of the chapter deals with measure preserving maps and ergodic theory. In particular, Poincaré's recurrence theorem and Birkhoff's ergodic theorem are proved, as well as the transitivity of ergodic maps for "good" measures.

The second chapter is devoted to hyperbolic fixed points and to their invariant submanifolds. In the first part, the author introduces the notion of hyperbolic fixed point. He then studies the local behavior of a diffeomorphism at such a point. The main result is the Hartman-Grobman theorem, according to which it is conjugate (via a homeomorphism) to its linearization at the fixed point. After that, the author constructs the stable and unstable invariant manifolds of a hyperbolic fixed point. Finally, he defines homoclinic and heteroclinic points and gives several examples.

In the third chapter, the author pushes further the study of hyperbolic dynamics initiated in the previous chapter. The notion of hyperbolic set, which generalises that of hyperbolic fixed point, is introduced. As an example the orbit of a transversal homoclinic point is a hyperbolic set. Then, two important lemmas of Anosov, the so-called "shadowing lemma" and "closing lemma", are given. As an application, the author highlights the chaotic behavior of orbits in the neighbourhood of the orbit of a transversal homoclinic point. Birkhoff's theorem on the existence of infinitely many periodic points and Poincaré's theorem on the existence of infinitely many homoclinic points are proved. Bernoulli systems are introduced and then the main result of the chapter is proved: Smale's theorem according to which Bernoulli systems can be embedded in the neighbourhood of a transversal homoclinic orbit. Corollaries of this result are given. This chaotic behavior is then illustrated on the example of a perturbed mathematical pendulum. The chapter continues with the classical results on the dynamics of torus automorphisms, which also illustrates hyperbolicity. The chapter ends with two smaller sections on the invariant submanifolds of a hyperbolic set and on the structural stability of hyperbolic diffeomorphisms (diffeomorphisms that are C^1-close to a hyperbolic diffeomorphism are conjugated to it by a homeomorphism).

The fourth chapter deals with a subject different from the two preceding ones. It focuses on gradient-like flows. To start, the author recalls the basic facts on vector fields and flows. Then he introduces the notions of limit sets of orbits, attractors and Lyapunov functions. Consequences of the existence of a Lyapunov function on the stability of the system are given, as well as several examples. Then starts the study of gradient-like flows. The author shows that the orbits converge to critical points. He also proves results that underlie Morse theory, in particular the deformation lemma. The study continues with Lyusternik-Shnirel′man theory on the number of rest points of a gradient-like flow. The last section of the chapter is devoted to Morse theory. It is explained how one can understand the topology of a smooth manifold from the critical points of a Morse function. Morse inequalities are proved.

The final four chapters are devoted to Hamiltonian dynamics. The fifth chapter introduces basic definitions as well as the geometric framework: symplectic manifolds. It starts with a section on symplectic vector spaces and on the group of symplectic matrices. Then several sections are devoted to recalling the basics of differential calculus on smooth manifolds (differential forms, Lie derivative, vector fields, exterior derivative, etc.). After this preparation, the author is able to define symplectic manifolds and to prove the fundamental Darboux theorem that symplectic manifolds of the same dimension are all locally equivalent. The author then focuses on Hamiltonian diffeomorphisms. He introduces them and gives their first properties: invariance of the Hamiltonian function, invariance of the symplectic form, invariance of the Liouville measure. Along the way he also introduces Poisson brackets. Then, a section deals with the possibility of representing Hamiltonian diffeomorphisms by single functions (the so-called generating function) via a change of coordinates. The last section is devoted to the special case of integrable systems and to the proof of the Arnold-Jost theorem (sometimes called the Arnold-Liouville theorem).

The sixth chapter introduces the questions that will be studied in the two last chapters. It deals with quite recent results in symplectic topology. The author starts with Dacorogna-Moser's theorem that the total volume is basically the only invariant of volume preserving diffeomorphisms. He then raises the question of what happens in the case of symplectic diffeomorphisms. He states the famous Gromov "non-squeezing theorem" that one cannot symplectically embed a large ball in a thin symplectic cylinder and concludes that there should exist some other invariant than the total volume. In the same spirit of symplectic rigidity, the author states the also famous theorem that a uniform limit of symplectic maps is symplectic. He then moves on to questions on the existence of closed orbits on prescribed regular level hypersurfaces of autonomous Hamiltonians ("energy surfaces"): they exist on almost every energy surface. The chapter ends with examples of such closed orbits on the surface of a ball or a symplectic cylinder, where the notion of symplectic capacity emerges. It serves as an introduction to the next chapter where a capacity is defined.

The seventh chapter is devoted to symplectic capacities and their applications to the questions considered in chapter VI. The author first defines symplectic capacities. He then shows that the existence of such an invariant gives an obstruction to symplectic embeddings, in particular Gromov's non-squeezing theorem. Conversely, the non-squeezing theorem also implies the existence of a symplectic capacity, the so-called Gromov width. In the second section, the Hofer-Zehnder capacity is introduced. By definition, the Hofer-Zehnder capacity of an open set U is the supremum of all maxima of functions in a certain class. The fundamental properties satisfied by the functions in this class are that their Hamiltonian flow is supported in U and has no non-constant periodic orbit of period less than 1. The proof that this actually gives a symplectic capacity occupies the rest of the chapter. The hard part is the proof of the normalisation axioms for the symplectic cylinder, which reduces to the search for periodic orbits on its boundary. To look for these periodic orbits, one has to see them as the critical points of a certain "action functional" and therefore to apply variational methods to this functional. To prepare this program, a section is devoted to recalling "minimax" principles that allow one to find critical points of functions on Hilbert spaces. To apply these principles to the action functional, one needs to work on well-chosen Sobolev spaces. All the material needed to work with Sobolev spaces is recalled in detail with proofs. The end of the proof comes in the last section, and it uses these minimax principles as well as a linking lemma based on the Leray-Schauder degree.

The last chapter gives applications of the Hofer-Zehnder capacity to Hamiltonian dynamics. The theorem stated in chapter VI that almost every energy surface contains a closed characteristic is proved. Hypersurfaces of contact types are defined and it is proved that they always carry a closed characteristic (a conjecture of Weinstein, proved by Viterbo) as a consequence of the fact that they admit a neighbourhood of finite capacity. This applies to proving that energy surfaces of Hamiltonian functions arising from classical mechanics carry closed characteristics. The last two sections deal with neighbourhoods of periodic orbits and Floquet theory. For example, according to Poincaré's theorem, periodic orbits generally come in 1-parameter families.

The book is very well written. The proofs are presented in detail and are easy to follow. All the material is carefully introduced with a lot of recollection. Moreover, each chapter ends with a quick and helpful overview of the literature on the subject. The first five chapters are a good introduction to topological dynamics, while the last three are more advanced. The book might seem a bit heterogeneous since it starts at the very beginning of dynamical systems and ends with advanced recent symplectic topology theorems. Nevertheless, the author manages to maintain the same level of difficulty throughout the whole book and makes very accessible several results in symplectic topology which are reputed to be very hard. *Vincent Humilière*