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Holden, Helge (N-NUST);
Karlsen, Kenneth H. [Karlsen, Kenneth Hvistendahl] (N-OSLO-CMA);
Lie, Knut-Andreas (N-SINT-A); **Risebro, Nils Henrik** (N-OSLO-CMA)
★**Splitting methods for partial differential equations with rough solutions.**
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This book considers operator splitting methods applied to nonlinear mixed hyperbolic-parabolic partial differential equations (PDEs) with the property that the solutions are rough, with limited regularity and may even contain jump discontinuities (shocks). It focuses on degenerate convection-diffusion equations and includes hyperbolic conservation laws, the heat equation, porous medium equations, two-phase reservoir flow equations and applications to sedimentation. The main interest is in problems where convection dominates diffusion; getting accurate solutions in such cases is often a very complicated task.

Many realistic models are often formulated by PDEs which contain terms (operators) that are mathematically very different. The whole problem of obtaining a computational solution can be very complicated. However, in general, these problems are the sum of evolution operators which have been extensively studied in the literature and for which practical algorithms are available.

A strategy to deal with complicated problems is to “divide and conquer”, and it is in this spirit that the operator splitting methods are designed: they approximate each sub-equation by an appropriately tailored method and then the numerical solution is obtained by composition. Since the operators in each sub-equation may be very different, they may also require very different numerical and analytical techniques. This technique allows also for great flexibility, since one can replace a given scheme designed for an elementary operator by another scheme for the same operator. The use of operator splitting techniques may also reduce memory requirements, increase the stability range, and even provide methods that are unconditionally stable. Since the book focuses on rough solutions, only first- and eventually (symmetric) second-order operator splitting methods are considered here.

The use of splitting methods for PDEs of the kind analyzed in the book under review presents some peculiarities worth emphasizing. Thus, for instance, the solution of the whole problem may be smooth, whereas the solutions corresponding to the sub-problems considered with splitting can develop singularities in finite time. In consequence, weak solutions and the question of uniqueness have to be analyzed. The equations have to be interpreted in the sense of distributions (usually weak solutions in the sense of distributions). The authors present a theoretical framework for operator splitting methods based on recent hyperbolic techniques (developed for scalar problems and weakly coupled systems) which allows them to treat a range of problems, from purely hyperbolic equations possessing shock wave solutions to uniformly parabolic convection diffusion equations possessing smooth solutions. Hyperbolic problems are solved throughout the book using a variety of different schemes: monotone schemes (upwind, Godunov), quasi-monotone schemes, front tracking, large-time-steps, Godunov or Glim methods, characteristic Galerkin methods, second-order MUSCL schemes and higher-order nonoscillatory central schemes.

The book is organized into six chapters. It mainly focuses on theoretical aspects when the exact solutions for the sub-problems are replaced by approximate solvers, and a deep theoretical framework is provided. A full chapter is dedicated to presenting a rigorous convergence theory for operator splitting methods in the context of weakly coupled systems of strongly degenerate convection-diffusion equations. In fact, a central issue of the book is to analyze under which precise circumstances the numerical solution obtained by splitting converges, and, if it does, whether the limit is the correct solution of the original equation or not. Next, the authors consider a variety of semidiscrete and fully discrete product formulas (dimensional splitting, viscous splitting, and flux splitting) and verify that the previously developed conditions needed to apply the abstract convergence theory hold. This part is very technical and collects results from the recent literature, to which the authors have made a significant contribution. Several numerical examples illustrate the results.

The last chapter is dedicated to operator splitting for examples not covered by the theory presented in previous chapters (scalar equations or weakly coupled systems). It illustrates how operator splitting methods are a viable numerical strategy for a wide variety of problems. The codes for most of the numerical examples can be uploaded from the book's Web site: www.math.ntnu.no/operatorsplitting. *Sergio Blanes*