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The Monge-Ampère equation and its applications. (English)

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In this very interesting book, the author presents some old and new results involving the Monge-Ampère equation. This equation in its full generality reads as

(MA)
$$det(D^2u) = f(x, u, \nabla u) \text{ in } \Omega$$

where $\Omega \subset \mathbb{R}^n$ is an open set, $u : \Omega \to \mathbb{R}$ is a convex function and $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, +\infty)$ is a given function. The unknown is the function u, while f represents the data of the problem. In most of the text, f is supposed to depend only on the variable x. Without any doubt, (MA) is the most known fully nonlinear second-order PDE, and its importance is motivated by the fact that it arises in many fields and applications such as antenna design, differential geometry, image processing, nonlinear elasticity, fluid dynamics, and optimal mass transportation to name a few.

In the introduction, the author explains why the equation (MA) is fully nonlinear degenerate elliptic and why one can restrict the search to convex functions u when looking for the candidates for the solution. The introduction is ended with a short historic note on the developments of the theory, going back to the classical works of Minkowski, Alexandrov and Pogorelov on the one hand and to more recent advancements by Caffarelli, De Philippis, Evans, the author, Krylov, Trudinger, Wang and others.

The text has four subsequent chapters and an appendix collecting some useful classical results from linear algebra, Hausdorff measures, convex geometry and functions and some tools from measure theory, nonlinear analysis and PDEs. The appendix definitely makes the book self-contained. In what follows, we briefly present the content of each of the chapters.

Chapter 2 is devoted to the concept of weak solutions à la Alexandrov to (MA). The author presents here the existence and uniqueness of weak solutions of the Dirichlet problem associated to (MA) and a first regularity result and its applications: the C^1 regularity of solutions in dimension two.

The idea behind the notion of Alexandrov solution is simple, yet profound: for a C^2 function u, the area formula implies that for any $E \subset \Omega$ Borel set we have

$$\int_E \det(D^2 u) dx = |\nabla u(E)|.$$

Therefore, if $f: \Omega \to [0, +\infty)$ is given and u is only C^1 , (MA) can be given the sense of

$$\int_{E} f(x)dx = |\nabla u(E)|$$

for any Borel set $E \subset \Omega$. This further motivates the definition of the *Monge-Ampère measure*, defined for any convex function $u: \Omega \to \mathbb{R}$ (and $\Omega \subset \mathbb{R}^n$ convex) as

$$\mu_u(E) := \bigg| \bigcup_{x \in E} \partial u(x) \bigg|,$$

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where $\partial u(x) \subset \mathbb{R}^n$ stands for the subdifferential of u in x. Thus, using this object one can simply define weak solutions (which need to be convex) to (MA) for merely Borel measure right hand sides ν by requiring

$$\mu_u(E) = \nu(E)$$

for any $E \subset \Omega$ Borel set. Besides the existence and uniqueness of weak Alexandrov solutions of the Dirichlet problem associated to (MA), the author discusses in this chapter in particular the Alexandrov maximum principle, a comparison principle, stability of weak solutions and gives an application involving the Minkowski problem for curvature measures.

In Chapter 3, the author studies the question of existence of smooth solutions to (MA). First, the existence of smooth solutions is obtained via the a classical *continuity method*. The main theorem here can be summarized as follows: if Ω is uniformly convex with $\partial\Omega$ of class $C^{k+2,\alpha}$ (for some $k \geq 2$, $\alpha \in (0,1)$) and if $f: \Omega \to [0, +\infty)$ is of class $C^{k,\alpha}$, $f \geq c_0 > 0$, then for any boundary data g of class $C^{k+2,\alpha}(\partial\Omega)$, there exists a unique $u \in C^{k+2,\alpha}(\overline{\Omega})$ solution to the Dirichlet problem (with boundary data g) to (MA). This theorem is very much in the same spirit as the classical Schauder estimates for linear uniformly elliptic equations.

Next, this chapter deals with Pogorelov's counterexample to interior regularity (in dimensions $n \ge 3$), provided the boundary data are not smooth. Then, Progorelov's interior estimates and regularity of weak solutions are presented.

Chapter 4 presents various $(C^{1,\alpha}, W^{2,p} \text{ and } C^{2,\alpha})$ interior regularity estimates for weak solutions to (MA). Here the main assumption is that the right hand side f is bounded away from zero and infinity. This one represents the longest chapter of the textbook. In order to obtain the desired estimates, a fine (geometric) analysis of the so-called *sections* is necessary, which is presented in a very clear way. In addition to the main theorems on the regularity estimates, the author gives interesting applications to Petty's theorem, to the optimal transport problem with quadratic cost and to the semigeostrophic equations. A Liouville type theorem is also presented.

The main part of the textbook ends with Chapter 5, where the author collects further results and extensions related to Monge-Ampère type equations. The presented results here are stated without proofs. These include: the existence of smooth solutions with general right hand sides (depending also on u and ∇u); the study of linearized Monge-Ampère equations; more general classes of Monge-Ampère type equations (motivated by optimal transport problems with general costs; the authors describes here in particular the famous Ma-Trudinger-Wang condition) and some general prescribed Jacobian equations.

The text is very well written and it presents the many times involved and sophisticated arguments in a very clear and elegant manner. By this, the techniques are approachable for non-experts as well. Therefore, it is an excellent resource for graduate students and researchers who would like to deepen their knowledge on Monge-Ampère type problems, with a particular emphasis on the regularity theory.

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