MR3585514 16-01 13E10 15A69 16Dxx 16E30 16Gxx 16K20 16T05

Skowroński, Andrzej (PL-TORNM)**; Yamagata, Kunio** (J-TAGT2)

★**Frobenius algebras. II.**

Tilted and Hochschild extension algebras.

EMS Textbooks in Mathematics.

European Mathematical Society (*EMS*), *Zürich*, 2017. *x*+619 *pp.*
ISBN 978-3-03719-174-3

The book under review is the second of three volumes devoted to the modern representation theory of finite-dimensional associative algebras, and, in particular, to the representation theory of Frobenius algebras. It is a natural continuation of the first volume [A. Skowroński and K. Yamagata, *Frobenius algebras. I*, EMS Textbk. Math., Eur. Math. Soc., Zürich, 2011; MR2894798]. The aim of the second volume is to provide a comprehensive introduction to the representation theory of hereditary algebras, tilted algebras, Hochschild extension algebras, and especially to the shape of their Auslander-Reiten quivers. The third volume will be more advanced and will discuss Frobenius algebras as orbit algebras of repetitive categories of finite-dimensional associative algebras with respect to actions of certain admissible automorphism groups. In particular, in the third volume covering techniques will be explained which can be used to reduce the representation theory of Frobenius algebras to the representation theory of algebras of small global dimension and then to apply tilting theory. Similarly to the first volume, the second volume is primarily addressed to graduate students as well as to non-specialists. It is again self-contained—the only prerequisites are a basic knowledge of linear algebra and some results of the first volume. The authors present detailed proofs for most of the results. Furthermore, many examples interspersed throughout the book and about 30 or more exercises at the end of each of the chapters will make it easier to understand the theory.

The book can be divided into three parts, which we are going to describe now in more detail. The first part consists of Chapters VII and VIII and makes up about two thirds of the book. In this part hereditary algebras, tilting theory, and tilted algebras are presented. In Chapter I of the first volume the authors already introduced hereditary algebras. In particular, the close relationship between hereditary algebras and path algebras of acyclic quivers was established.

Chapter VII begins with three preliminary sections devoted to the Gabriel quiver of a finite-dimensional associative algebra, a certain generalization of path algebras of finite acyclic quivers, and some consequences of the Snake Lemma that will be needed later to determine the shape of Auslander-Reiten quivers. The chapter ends with a section discussing matrix algebras obtained from two finite-dimensional algebras B, C, and a (B,C)-bimodule as well as some properties of their module categories. The main new concepts in this chapter are the Euler forms and the Coxeter transformation on the Grothendieck group of a finite-dimensional hereditary algebra. The former are a non-symmetric bilinear form and its associated quadratic form, which were introduced by Ringel in 1976; the latter is compatible with the Euler bilinear form and is closely related to the Auslander-Reiten translation of the hereditary algebra. The kind of definiteness of the Euler forms determines the representation type of a finite-dimensional hereditary algebra; namely, the algebra is of finite, tame, or wild representation type if, and only if, its Euler forms are positive definite, positive semidefinite with non-zero radical, or indefinite, respectively. In particular, Gabriel's classical result on finite-dimensional hereditary algebras of finite representation type over an algebraically closed field is proved. It says that a finite-dimensional hereditary algebra A over an algebraically closed field is of finite representation type if, and only if, its Gabriel quiver is a simply laced Dynkin quiver, and the isomorphism classes of finite-dimensional indecomposable

A-modules are in bijection with the positive roots of the Euler quadratic form of A (or equivalently, the positive roots of the root system corresponding to the underlying Dynkin diagram of the Gabriel quiver of A). It should be remarked that in the case of infinite representation type the authors only prove that the Euler quadratic form of A is positive semidefinite with non-zero radical (resp. indefinite) if, and only if, the Gabriel quiver of A is Euclidean (resp. neither Dynkin nor Euclidean). The equivalence of this fact to A being of tame or wild representation type is outside the scope of the book. The Auslander-Reiten quiver Γ_A of the hereditary algebra A has a unique connected component $\mathcal{P}(A)$ that contains all projective indecomposable modules and their successors and a unique connected component $\mathcal{Q}(A)$ that contains all injective indecomposable modules and their predecessors. The connected component $\mathcal{P}(A)$ is called the postprojective component of Γ_A and the connected component $\mathcal{Q}(A)$ is called the preinjective component Γ_A. An indecomposable finite-dimensional hereditary algebra A has finite representation type if, and only if, $\Gamma_A = \mathcal{P}(A) = \mathcal{Q}(A)$. If A is of infinite representation type, then any connected component of $\Gamma_A \setminus [\mathcal{P}(A) \cup \mathcal{Q}(A)]$ is called regular and its union is denoted by $\mathcal{R}(A)$. The Coxeter transformation is used to determine the structure of the postprojective and preinjective components of Γ_A. In the case that the Gabriel quiver of A is Euclidean, the authors prove that the additive category of $\mathcal{R}(A)$ is abelian and closed under extensions. Moreover, the structure of $\mathcal{R}(A)$ and its relationship to $\mathcal{P}(A)$ and $\mathcal{Q}(A)$ is determined. In the remaining wild case the additive category of $\mathcal{R}(A)$ is not abelian. In this case it is shown that a regular component of the Auslander-Reiten quiver is isomorphic to the translation quiver $\mathbb{Z}\mathbb{A}_\infty$ and that indecomposable modules in regular components are uniquely determined by their composition factors. All this and much more is explained by the authors in sufficient detail.

In Chapter VIII the reader is introduced to tilting theory and tilted algebras. Tilting theory is one of the main tools of modern representation theory. It goes back to the use of reflection functors by Bernstein, Gelfand, and Ponomarev to give an elegant non-computational proof of Gabriel's theorem on hereditary algebras of finite representation type and to the study of Coxeter functors by Auslander, Platzeck, and Reiten. The first definition of a tilting module is due to Brenner and Butler and was later reformulated in its current form by Happel and Ringel with an essential addition by Bongartz. Tilting theory is a generalization of Morita equivalence. Namely, the tilting theorem of Brenner and Butler relates a torsion pair in the category of finite-dimensional modules over a finite-dimensional associative algebra A to a torsion pair in the category of finite-dimensional modules over the endomorphism algebra $\mathrm{End}_A(T)$ of a tilting module T for A via naturally defined homological functors. As a consequence, the Grothendieck groups of A and $\mathrm{End}_A(T)$ are isomorphic. The representation theory of tilted algebras (i.e., endomorphism algebras of tilting modules over finite-dimensional hereditary algebras) is very well understood. The authors prove that tilted algebras have global dimension at most two. Moreover, their Auslander-Reiten quivers admit an acyclic component containing a canonical section connecting the torsion-free part and the torsion part of the torsion pair defined by the tilting module. A criterion for deciding whether a given algebra is tilted that independently is due to Liu and the first author is established. The chapter concludes by proving a theorem of Ringel asserting that a hereditary algebra has a regular tilting module if, and only if, its Gabriel quiver is neither Dynkin nor Euclidean, and its Grothendieck group has rank at least three.

The second part is Chapter IX (the shortest chapter in the book taking up about 60 pages), in which the authors discuss the shape and the structure of connected components of Auslander-Reiten quivers of finite-dimensional associative algebras. This is done by using the functorial approach proposed by Auslander and Reiten. The chapter begins with characterizations of almost split sequences in terms of minimal projective resolutions of simple functors. As an application, a theorem of Igusa and Todorov is proved. Then Liu's theory of degrees of irreducible homomorphisms between indecomposable modules is presented. This is used to find the shape of the connected components of stable Auslander-Reiten quivers of finite-dimensional associative algebras. In particular, the structure of regular components is described completely. Moreover, the authors prove a result due to the first author asserting that the generalized standard acyclic regular components of Auslander-Reiten quivers of finite-dimensional associative algebras are the connected components of tilted algebras determined by the regular tilting modules over hereditary algebras of wild type. Chapter IX concludes with some results on stable equivalences between module categories of finite-dimensional associative algebras which will be needed later.

The third and last part of the book is Chapter X, which consists of about 130 pages. In this chapter the authors introduce finite-dimensional self-injective algebras that are obtained as Hochschild extension algebras of finite-dimensional associative algebras by self-duality bimodules. Many of the results in this chapter are due to the second author alone or in collaboration with the first author. In the last section results from Chapter VII are used to determine the structure of the Auslander-Reiten quiver of an indecomposable Hochschild extension algebra of a finite-dimensional hereditary algebra by a self-duality bimodule.

The book under review is very well written and will be especially useful for graduate students and non-experts because of the detailed exposition, the wealth of examples and exercises, as well as the carefully chosen mixture of basic and more advanced results. As I mentioned in my review of the first volume, an index of symbols would have been very helpful for the prospective reader. *Jörg Feldvoss*