Contents

1 Introduction to Spectral Geometry 1

2 Detailed introduction to abstract spectral theory 7
2.1 Linear operators . . . 7
2.1.1 Bounded operators . . . 7
2.1.2 Lax–Milgram theorem . . . 8
2.1.3 Unbounded operators . . . 10
2.2 Closed operators . . . 12
2.3 Adjoint operators . . . 14
2.3.1 Introduction . . . 14
2.3.2 Symmetry and self-adjointness . . . 16
2.3.3 An important result concerning self-adjoint operators . . . 17
2.4 Spectrums and resolvent set . . . 19
2.4.1 Spectrum versus point spectrum . . . 19
2.4.2 The self-adjoint case . . . 25
2.5 Spectral theory of compact operators . . . 27
2.5.1 The notion of compact operators . . . 27
2.5.2 Algebraic properties of compact operators . . . 28
2.5.3 Main spectral theorem for compact self-adjoint operators . . . 30
2.5.4 A proof using the Riesz–Schauder theory . . . 30
2.6 The spectral theorem multiplication operator form for unbounded operators on a Hilbert space . . . 35
2.6.1 Spectral theorem multiplication operator form . . . 35
2.6.2 Functional calculus . . . 38
2.6.3 Functional calculus in Quantum Mechanics . . . 39
2.7 Some complements on operators theory . . . 40
2.7.1 Closable operators . . . 40
2.7.2 Unbounded operators with compact resolvents . . . 41
2.7.3 Numerical range and applications . . . 44
2.8 Exercises . . . 49

3 The Laplacian on a compact Riemannian manifold 51
3.1 Basic Riemannian Geometry . . . 51
3.1.1 Differential Geometry: conventions and notations . . . 51
3.1.2 Riemannian manifolds and examples . . . 57
3.1.3 Metric structure and geodesics on a Riemannian manifold . . . 59
3.1.4 Curvatures on a Riemannian manifold . . . 61
3.1.5 Integration on a Riemannian manifold . . . 64

3.2 Analysis on manifolds 66
3.2.1 Distributions on a Riemannian manifold 66
3.2.2 Sobolev spaces on a Riemannian manifold 67
3.2.3 The Laplacian operator and the Green formula 68
3.3 Exercises 69

4 Spectrum of the Laplacian on a compact manifold 71
4.1 Physical examples 71
4.1.1 The wave equation on a string 71
4.1.2 The heat equation 72
4.1.3 The Schrödinger equation 73
4.2 A class of spectral problems 75
4.2.1 The closed eigenvalue problem 75
4.2.2 The Dirichlet eigenvalue problem 75
4.2.3 The Neumann eigenvalue problem 76
4.2.4 Other problems 76
4.3 Spectral theorem for the Laplacian 76
4.4 A detailed proof by a variational approach 78
4.4.1 Variational generic abstract eigenvalue problem 78
4.4.2 The closed eigenvalue problem 81
4.4.3 The Dirichlet eigenvalue problem 84
4.4.4 The Neumann eigenvalue problem 87
4.4.5 A remark on the variational formulation 89
4.5 The minimax principle and applications 90
4.5.1 A physical example 90
4.5.2 The minimax theorem 91
4.5.3 Properties of the first eigenvalue 95
4.5.4 Monotonicity domain principle 97
4.5.5 A perturbation of metric result 98
4.6 Complements: The Schrödinger operator and the Hodge–de Rham Laplacian 100
4.6.1 The Schrödinger operator 100
4.6.2 The Hodge–de Rham Laplacian 102

5 Direct problems in spectral geometry 107
5.1 Explicit calculation of the spectrum 107
5.1.1 Flat tori 107
5.1.2 Rectangular domains with boundary conditions 109
5.1.3 Spheres 109
5.1.4 Harmonic oscillator 110
5.2 Qualitative properties of the spectrum 113
5.2.1 From the bottom of the spectrum 113
5.2.2 ... to the large eigenvalues: the Weyl formula 117

5.2.3 Weyl formula for a flat torus . . . 117
5.2.4 The semi-classical paradigm . . . 118
5.3 The spectral partition function Z_M . . . 119
5.4 Eigenvalues and eigenfunctions of surfaces . . . 121
5.4.1 Spectrum of surfaces . . . 121
5.4.2 Eigenfunctions of surfaces and Courant's nodal theorem . . . 122
5.5 Exercises . . . 122

6 Intermezzo: "Can one hear the holes of a drum?" 125
6.1 The main result . . . 125
6.2 Some useful spaces . . . 126
6.3 Electrostatic capacity and the Poincaré inequality . . . 129
6.4 A detailed proof of the main Theorem 6.1.1 . . . 131

7 Inverse problems in spectral geometry 139
7.1 Can one hear the shape of a drum? . . . 139
7.2 Length spectrum and trace formulas . . . 139
7.3 Milnor's counterexample . . . 142
7.4 Prescribing the spectrum on a manifold . . . 146
7.5 Heat kernel and spectral geometry . . . 147
7.5.1 The heat equation . . . 147
7.5.2 The heat kernel . . . 148
7.5.3 The spectral partition function Z_M as a trace . . . 158
7.6 The Minakshisundaram–Pleijel expansion and the Weyl formula . 159
7.7 Two planar isospectral nonisometric domains . . . 161
7.7.1 Construction of the domains D_1 and D_2 . . . 161
7.7.2 Isospectrality of the domains D_1 and D_2 . . . 163
7.8 Few words about Laplacian and conformal geometry . . . 166
7.8.1 Conformal geometry on surfaces . . . 166
7.8.2 Spectral zeta function and uniformization theorem on surfaces . . . 167
7.8.3 Ricci flow on surfaces . . . 172
7.8.4 What about three manifolds? . . . 173

Bibliography 177

Index 185