

## CHAPTER 5

### UNIFORM A PRIORI ESTIMATES FOR THE BBGKY AND BOLTZMANN HIERARCHIES

*This is a revised version of Chapter 5: in the original version, there were inconsistencies in the way the function spaces were introduced, and the present Paragraph 5.1 has been added to this Chapter in order to settle the functional framework.*

*The first two authors wish to thank Thierry Bodineau for his help in the writing of this new version.*

This chapter is devoted to the statement and proof of uniform a priori estimates for mild solutions to the BBGKY hierarchy, defined formally in (4.3.8), which we reproduce here:

$$(5.0.1) \quad F_N(t) = \mathbf{T}(t)F_N(0) + \int_0^t \mathbf{T}(t-\tau)\mathbf{C}_N F_N(\tau) d\tau, \quad F_N = (f_N^{(s)})_{1 \leq s \leq N},$$

as well as for the limit Boltzmann hierarchy defined in (4.4.6)

$$(5.0.2) \quad F(t) = \mathbf{S}(t)F(0) + \int_0^t \mathbf{S}(t-\tau)\mathbf{C}^0 F(\tau) d\tau, \quad F = (f^{(s)})_{s \geq 1}.$$

Those results are obtained in Paragraphs 5.3 and 5.4 by use of a Cauchy-Kowalevskaya type argument. Before that we need to make sense of the formulation (5.0.1), which is not an obvious fact since characteristics of the transport are defined only almost everywhere (see Chapter 4) while the collision operators are defined by integrals on manifolds of codimension 1<sup>(1)</sup>. In Paragraph 5.1 we show that the collision integrals make sense in  $L^\infty$  outside some measure zero sets, provided that they are combined with the transport operator. Then Paragraph 5.2 is devoted to the definition of adequate function spaces in which the equations will be shown to be wellposed, and to the statements of the wellposedness results.

#### 5.1. Rigorous formulation of the BBGKY hierarchy

In this paragraph we show how to make sense of the collision operators in (5.0.1). To this end, we define a new hierarchy by filtering of the transport operator:

$$(5.1.1) \quad G_N(t) = F_N(0) + \int_0^t \mathbf{T}(-\tau)\mathbf{C}_N \mathbf{T}(\tau)G_N(\tau) d\tau.$$

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1. The question of correctly defining the hierarchy is also addressed in the work by S. Simonella, *Evolution of correlation functions in the hard sphere dynamics*, J. Stat. Phys. 155 (2014), no. 6, 1191-1221.

Notice that although  $G_N$  and  $F_N$  are related by the simple fact that

$$G_N(t) = (\mathbf{T}_s(-t)f_N^{(s)}(t))_{1 \leq s \leq N},$$

the hierarchy  $G_N$  has much better regularity properties. In particular one can see (see the discussion in Remark 5.4.4 at the end of this chapter) that writing  $G_N = (g_{n,s})_{1 \leq s \leq N}$  then  $g_{n,s}$  is a continuous function of time, with values in  $L^\infty(\mathcal{D}_s)$ , which is not the case of  $f_N^{(s)}$ . Moreover the idea of combining the collision integral  $\mathcal{C}_{s,s+1}$  with the transport operator  $\mathbf{T}_s(\tau)$  comes from the fact that time can be viewed as the missing coordinate on  $\partial\mathcal{D}_{s+1}$  in the direction orthogonal to the boundary. We then expect to define the collision integral in  $L^\infty$  by using Fubini's theorem.

**5.1.1. A local system of coordinates near the boundary.** — From now on we fix two integers  $1 \leq i \leq s$  and we note that for all  $\delta > 0$ , the change of variables

$$(5.1.2) \quad \begin{aligned} \iota_s &:= \mathcal{D}_s \times [0, \delta] \times \mathbf{S}_1^{d-1} \times \mathbf{R}^d \rightarrow \mathbf{R}^{2d(s+1)} \\ (Z_s, t, \omega, v_{s+1}) &\mapsto Z_{s+1} = (X_s - tV_s, V_s, x_i + \varepsilon\omega - tv_{s+1}, v_{s+1}) \end{aligned}$$

maps the measure  $d\mu_i^- := \varepsilon^{d-1}((v_{s+1} - v_i) \cdot \omega)_- dZ_s dt d\omega dv_{s+1}$  on the Lebesgue measure  $dZ_{s+1}$ . Of course  $Z_{s+1}$  defined in (5.1.2) is simply the mapping of  $\tilde{Z}_{s+1} := (Z_s, x_i + \varepsilon\omega, v_{s+1})$  by the free transport operator. Similarly one can consider a post-collisional situation and notice that as the scattering preserves the measure, we have that for any  $i \leq s$ , with notation (4.4.1),

$$(5.1.3) \quad \iota_s^* := (Z_s, t, \omega, v_{s+1}) \in \mathcal{D}_s \times [0, \delta] \times \mathbf{S}_1^{d-1} \times \mathbf{R}^d \mapsto Z_{s+1} = (X_s - tV_s^*, V_s^*, x_i + \varepsilon\omega - tv_{s+1}^*, v_{s+1}^*)$$

maps the measure  $d\mu_i^+ := \varepsilon^{d-1}((v_{s+1} - v_i) \cdot \omega)_+ dZ_s dt d\omega dv_{s+1}$  on the Lebesgue measure  $dZ_{s+1}$ . In the following we write  $\iota_s^-$  and  $\iota_s^{*-}$  the above mappings where  $t$  is replaced by  $-t$ .

Our aim is to extend this to the case when the free transport in the mappings  $\iota_s, \iota_s^*$  is replaced by the transport  $\Psi_{s+1}$  with exclusion

$$Z_{s+1} = \Psi_{s+1}(-t)\tilde{Z}_{s+1}, \quad \tilde{Z}_{s+1} := (Z_s, x_i + \varepsilon\omega, v_{s+1})$$

so that the image belongs to  $\mathcal{D}_{s+1}$ .

To do so, we are going to consider trajectories away from pathological configurations. From now on we fix  $R_1, R > 0$  (which will go to infinity at the very end), as well as the set

$$B_{R_1, R}^{2(s+1)} := \left\{ Z_{s+1} \in \mathbf{R}^{2d(s+1)} / |X_{s+1}| \leq R_1 \quad \text{and} \quad |V_{s+1}| \leq R \right\}$$

and we define for all  $\delta > 0$ , the sets

$$\begin{aligned} \partial\mathcal{D}_\delta^{i,s+1,\pm} &:= \left\{ Z_{s+1} \in B_{R_1, R}^{2(s+1)} / |x_i - x_{s+1}| = \varepsilon, \quad \pm(v_i - v_{s+1}) \cdot (x_i - x_{s+1}) > 0 \right. \\ &\quad \left. \text{and} \quad \forall (k, \ell) \in [1, s+1]^2 \setminus \{(i, s+1)\}, \quad |x_k - x_\ell| > \varepsilon + R\delta \right\}, \end{aligned}$$

and  $\partial\mathcal{D}_\delta^{i,s+1} := \partial\mathcal{D}_\delta^{i,s+1,+} \cup \partial\mathcal{D}_\delta^{i,s+1,-}$ . When  $\delta = 0$  we write  $\partial\mathcal{D}_0^{i,s+1,\pm} := \partial\mathcal{D}_0^{i,s+1,\pm}$ .

Note that  $(\partial\mathcal{D}_\delta^{i,s+1,+})_{\delta>0}$  are decreasing families.

### 5.1.2. Definition of the truncated collision integral. —

The collision operator is obtained by integration on each component of the boundary  $\partial\mathcal{D}^{i,s+1,\pm}$  with respect to a partial set of variables, namely  $\omega, v_{s+1}$ , with the measure  $d\mu_i^\pm$ . For functions in  $L^\infty$  (which are defined almost everywhere), such integrals are defined by Fubini's theorem.

More precisely, let us define truncated collision operators as follows: for any  $\delta > 0$  and any continuous function  $\varphi_{s+1}$  defined on  $\mathcal{D}_{s+1}$ ,

$$\begin{aligned} (\mathcal{C}_{s,s+1}^{\pm,\delta}\varphi_{s+1})(Z_s) &:= \sum_{i=1}^s (\mathcal{C}_{s,s+1}^{\pm,i,\delta}\varphi_{s+1})(Z_s) \\ &:= (N-s)\varepsilon^{d-1} \sum_{i=1}^s \int_{\mathbf{S}_1^{d-1} \times \mathbf{R}^d} (\omega \cdot (v_{s+1} - v_i))_{\pm} \\ &\quad \times \varphi_{s+1}(Z_s, x_i + \varepsilon\omega, v_{s+1}) \left( \prod_{(k,\ell) \in [1,s+1]^2 \setminus \{(i,s+1)\}} \mathbb{1}_{|x_k - x_\ell| > \varepsilon + \delta R} \right) d\omega dv_{s+1}. \end{aligned}$$

In the above integral to simplify notation we have written  $x_{s+1} = x_i + \varepsilon\omega$  in the exclusion function  $\prod_{(k,\ell) \in [1,s+1]^2 \setminus \{(i,s+1)\}} \mathbb{1}_{|x_k - x_\ell| > \varepsilon + \delta R}$ .

Now let us fix  $T > 0$  and let us make sense of the functions  $\mathcal{C}_{s,s+1}^{\pm,\delta} \mathbf{T}_{s+1}(t) \varphi_{s+1}$  in  $L^\infty$ , for  $\varphi_{s+1}$  belonging to  $L^\infty(\mathcal{D}_{s+1})$  and  $t \in [0, T]$ .

- We start by proving that those functions are locally integrable on  $\mathcal{D}_s \times [0, T]$  (equipped with the Lebesgue measure  $dZ_s dt$ ).

In the case when  $t \in [0, \delta]$  then writing

$$\mathcal{C}_{s,s+1}^{\pm,\delta}(\mathbf{T}_{s+1}(t)\varphi_{s+1}) = \mathcal{C}_{s,s+1}^{\pm,\delta}(\varphi_{s+1}(\tilde{Z}_{s+1}))$$

then by definition there is no recollision since  $\tilde{Z}_{s+1}$  belongs to  $\partial\mathcal{D}_\delta^{i,s+1,\pm}$ . Using the change of variables (5.1.2) in the pre-collisional case, and (5.1.3) in the post-collisional one, one finds that for any function  $\varphi_{s+1}$  belonging to  $L^\infty(\mathcal{D}_{s+1}) \subset L_{loc}^1(\mathcal{D}_{s+1})$ , the volumic integral is well defined: the domain of integration is indeed included in  $\iota_s^-(B_{R_1}^s \times [0, \delta] \times \mathbf{S}_1^{d-1} \times B_R^1) \cup \iota_s^{-*}(B_{R_1}^s \times [0, \delta] \times \mathbf{S}_1^{d-1} \times B_R^1)$ , or in other words in

$$\begin{aligned} &\{Z_{s+1} \in B_{R_1,R}^{2(s+1)} / \exists t \in [0, \delta], \quad |x_i - x_{s+1} + t(v_i - v_{s+1})| = \varepsilon\} \\ &\cup \{Z_{s+1} \in B_{R_1,R}^{2(s+1)} / \exists t \in [0, \delta], \quad |x_i - x_{s+1} + t(v_i^* - v_{s+1}^*)| = \varepsilon\} \end{aligned}$$

the volume of which is  $O(R\delta\varepsilon^{d-1}R^{d(s+1)}R_1^{ds})$ . Then,

$$\left| \int_0^\delta \int_{\mathcal{D}_s} (\mathcal{C}_{s,s+1}^{\pm,i,\delta} \mathbf{T}_{s+1}(t)\varphi_{s+1}) dZ_s dt \right| \leq C_d \delta \varepsilon^{d-1} R_1^{ds} R^{d(s+1)+1} \|\varphi_{s+1}\|_{L^\infty(\mathcal{D}_{s+1})}.$$

Next we cover  $[0, T]$  by  $T/\delta$  intervals  $[n\delta, (n+1)\delta]$

$$\int_{n\delta}^{(n+1)\delta} \int_{\mathcal{D}_s} (\mathcal{C}_{s,s+1}^{\pm,i,\delta} \mathbf{T}_{s+1}(t)\varphi_{s+1}) dZ_s dt = \int_0^\delta \int_{\mathcal{D}_s} (\mathcal{C}_{s,s+1}^{\pm,i,\delta} \mathbf{T}_{s+1}(\tau) \mathbf{T}_{s+1}(n\delta)\varphi_{s+1}) dZ_s d\tau$$

and we know that thanks to Alexander [2] (see also Paragraph 4.1),

$$|(\mathbf{T}_{s+1}(n\delta)\varphi_{s+1})(Z_{s+1})| \leq \|\varphi_{s+1}\|_{L^\infty(\mathcal{D}_{s+1})}.$$

As above one infers after changing variables that

$$\int_{n\delta}^{(n+1)\delta} \int_{\mathcal{D}_s} \left| \left( \mathcal{C}_{s,s+1}^{\pm,i,\delta} \mathbf{T}_{s+1}(t) \varphi_{s+1} \right) \right| dZ_s dt \leq C_d \delta \varepsilon^{d-1} R_1^{ds} R^{d(s+1)+1} \|\varphi_{s+1}\|_{L^\infty(\mathcal{D}_{s+1})}$$

and therefore finally

$$\int_0^T \int_{\mathcal{D}_s} \left| \left( \mathcal{C}_{s,s+1}^{\pm,i,\delta} \mathbf{T}_{s+1}(t) \varphi_{s+1} \right) \right| dZ_s dt \leq C_d T \varepsilon^{d-1} R_1^{ds} R^{d(s+1)+1} \|\varphi_{s+1}\|_{L^\infty(\mathcal{D}_{s+1})}.$$

Then, by Fubini's theorem, we conclude that  $\mathcal{C}_{s,s+1}^{\pm,i,\delta} \mathbf{T}_{s+1}(t) \varphi_{s+1} \in L^1([0, T] \times \mathcal{D}_s)$ , in particular they are measurable functions.

• Returning to the control of the  $L^\infty$  norm, we find from the above analysis that for any subset  $A$  of  $[0, \delta] \times \mathcal{D}_s$ ,

$$\int_A \left| \left( \mathcal{C}_{s,s+1}^{\pm,i,\delta} \mathbf{T}_{s+1}(t) \varphi_{s+1} \right) \right| dZ_s dt \leq C_d |A| R^{d+1} \varepsilon^{d-1} \|\varphi_{s+1}\|_{L^\infty(\mathcal{D}_{s+1})},$$

since the domain of integration is included in  $\iota_s^-(A \times \mathbf{S}_1^{d-1} \times B_R^1) \cup \iota_s^{-*}(A \times \mathbf{S}_1^{d-1} \times B_R^1)$ . It is then easy to conclude that

$$\left| \left( \mathcal{C}_{s,s+1}^{\pm,i,\delta} \mathbf{T}_{s+1}(t) \varphi_{s+1} \right) (Z_s) \right| \leq C_d R^{d+1} \varepsilon^{d-1} \|\varphi_{s+1}\|_{L^\infty(\mathcal{D}_{s+1})}$$

almost everywhere in  $[0, \delta] \times \mathcal{D}_s$  (since the set where these inequalities are not satisfied is of measure 0). We then extend the reasoning to any set of the type  $[n\delta, (n+1)\delta] \times \mathcal{D}_s$  as in the previous paragraph: for any subset  $A_n$  of  $[n\delta, (n+1)\delta] \times \mathcal{D}_s$ , we have

$$\begin{aligned} \int_{A_n} \left( \mathcal{C}_{s,s+1}^{\pm,i,\delta} \mathbf{T}_{s+1}(t) \varphi_{s+1} \right) (Z_s) dZ_s dt &= \int_{A_n} \left( \mathcal{C}_{s,s+1}^{\pm,i,\delta} \mathbf{T}_{s+1}(t - n\delta) \mathbf{T}_{s+1}(n\delta) \varphi_{s+1} \right) (Z_s) dZ_s dt \\ &= \int_{A_n^\delta} \left( \mathcal{C}_{s,s+1}^{\pm,i,\delta} \mathbf{T}_{s+1}(\tau) \mathbf{T}_{s+1}(n\delta) \varphi_{s+1} \right) (Z_s) dZ_s d\tau \end{aligned}$$

where  $A_n^\delta := \{(\tau, Z_s) / (\tau + n\delta, Z_s) \in A_n\}$ . Since  $|A_n^\delta| = |A_n|$  we find that

$$\int_{A_n} \left| \left( \mathcal{C}_{s,s+1}^{\pm,i,\delta} \mathbf{T}_{s+1}(t) \varphi_{s+1} \right) (Z_s) \right| dZ_s dt \leq C_d |A_n| R^{d+1} \varepsilon^{d-1} \|\varphi_{s+1}\|_{L^\infty(\mathcal{D}_{s+1})},$$

so

$$\left| \left( \mathcal{C}_{s,s+1}^{\pm,i,\delta} \mathbf{T}_{s+1}(t) \varphi_{s+1} \right) (Z_s) \right| \leq C_d R^{d+1} \varepsilon^{d-1} \|\varphi_{s+1}\|_{L^\infty(\mathcal{D}_{s+1})}$$

almost everywhere in  $[n\delta, (n+1)\delta] \times \mathcal{D}_s$ . Finally this implies that

$$\left| \left( \mathcal{C}_{s,s+1}^{\pm,i,\delta} \mathbf{T}_{s+1}(t) \varphi_{s+1} \right) (Z_s) \right| \leq C_d R^{d+1} \varepsilon^{d-1} \|\varphi_{s+1}\|_{L^\infty(\mathcal{D}_{s+1})}$$

almost everywhere in  $[0, T] \times \mathcal{D}_s$ .

We have thus defined truncated collision integrals far from the singular points of the boundary of  $\mathcal{D}_{s+1}$ . It remains then to check that the sequence of operators thus constructed is a Cauchy sequence with respect to the truncation parameter in  $L^\infty$ , outside a set of measure going to zero with the truncation parameter.

### 5.1.3. Removing the truncation. —

Let  $0 < \delta' < \delta$  be given and consider the truncated operators

$$\mathcal{C}_{s,s+1}^{\pm,i,\delta',\delta} := \mathcal{C}_{s,s+1}^{\pm,i,\delta'} - \mathcal{C}_{s,s+1}^{\pm,i,\delta}.$$

We shall prove that the partial integral  $\mathcal{C}_{s,s+1}^{\pm,i,\delta',\delta} \mathbf{T}_{s+1}(t) \varphi_{s+1}$  is small (of the order  $\sqrt{\delta}$ ) outside a small subset of  $\mathcal{D}_s \times [0, T]$ , of measure going to zero with  $\delta$ . Indeed we have

$$\int_0^{\delta'} \int_{\mathcal{D}_s} \left| \left( \mathcal{C}_{s,s+1}^{\pm,i,\delta',\delta} \mathbf{T}_{s+1}(t) \varphi_{s+1} \right) \right| dZ_s dt = \int_{V_{\delta,\delta'}} |\varphi_{s+1}(Z_{s+1})| dZ_{s+1},$$

where  $V_{\delta,\delta'}$  is a subdomain of

$$\begin{aligned} & \left\{ Z_{s+1} \in B_{R_1,R}^{2(s+1)} / \exists t \in [0, \delta'], (j, j') \neq (i, s+1), \quad |x_i - x_{s+1} + t(v_i - v_{s+1})| = \varepsilon \right. \\ & \quad \left. \text{and } \varepsilon \leq |x_j - x_{j'} + t(v_j - v_{j'})| \leq \varepsilon + R\delta \right\} \\ & \cup \left\{ Z_{s+1} \in B_{R_1,R}^{2(s+1)} / \exists t \in [0, \delta'], (j, j') \neq (i, s+1), \ell \neq i, s+1, \quad |x_i - x_{s+1} + t(v_i^* - v_{s+1}^*)| = \varepsilon \right. \\ & \quad \left. \text{and } \begin{cases} \text{either } \varepsilon \leq |x_i - x_\ell + t(v_i^* - v_\ell)| \leq \varepsilon + R\delta \\ \text{or } \varepsilon \leq |x_{s+1} - x_\ell + t(v_{s+1}^* - v_\ell)| \leq \varepsilon + R\delta \quad \text{or } \varepsilon \leq |x_j - x_{j'} + t(v_j - v_{j'})| \leq \varepsilon + R\delta \end{cases} \right\}. \end{aligned}$$

In particular,  $|V_{\delta,\delta'}| \leq C(R, \varepsilon) \delta \delta'$ . Arguing as in the previous section we deduce the estimate on  $[0, T]$

$$(5.1.4) \quad \int_0^T \int_{\mathcal{D}_s} \left| \left( \mathcal{C}_{s,s+1}^{\pm,i,\delta',\delta} \mathbf{T}_{s+1}(t) \varphi_{s+1} \right) \right| dZ_s dt \leq C(R, T) \delta \|\varphi_{s+1}\|_{L^\infty(\mathcal{D}_{s+1})},$$

uniformly in  $\delta'$ . Finally we introduce the set

$$I_{\delta,\delta',i,\pm} = \left\{ (t, Z_s) \in [0, T] \times \mathcal{D}_s \mid \left| \left( \mathcal{C}_{s,s+1}^{\pm,i,\delta',\delta} \mathbf{T}_{s+1}(t) \varphi_{s+1} \right) (Z_s) \right| \geq \sqrt{\delta} \right\}.$$

Thanks to the Bienaymé-Tchebichev inequality and to (5.1.4), we have uniformly in  $\delta'$

$$|I_{\delta,\delta',i,\pm}| = O(\sqrt{\delta}).$$

Note furthermore that  $I_{\delta,\delta',i,\pm}$  is a decreasing function of  $\delta$ . On the complement of  $I_{\delta,\delta',i,\pm}$ , for any function  $\varphi_{s+1} \in L^\infty(\mathcal{D}_{s+1})$

$$\|\mathcal{C}_{s,s+1}^{\pm,i,\delta',\delta} \mathbf{T}_{s+1}(t) \varphi_{s+1}\|_{L^\infty} \leq C(R) \|\varphi_{s+1}\|_{L^\infty} \sqrt{\delta}.$$

This tells us exactly that the sequence  $\mathcal{C}_{s,s+1}^{\pm,i,\delta} \mathbf{T}_{s+1}(t) \varphi_{s+1}$  is a Cauchy sequence and converges weakly-\* in  $L^\infty([0, T] \times \mathcal{D}_s)$  as  $\delta \rightarrow 0$ .

### 5.1.4. Dependence with respect to time and conclusion. —

Finally to define  $\mathcal{C}_{s,s+1} \mathbf{T}_{s+1}(t)$  on time-dependent functions belonging to  $C([0, T]; L^\infty(\mathcal{D}_{s+1}))$  supported in  $[0, T] \times B_{R_1,R}^{2(s+1)}$ , we notice that the above arguments are very easily adapted to the case of piecewise constant functions in time, denoted  $PC([0, T]; L^\infty(\mathcal{D}_{s+1}))$ . Then we conclude by density of  $PC([0, T]; L^\infty(\mathcal{D}_{s+1}))$  in  $C([0, T]; L^\infty(\mathcal{D}_{s+1}))$ . Indeed if  $\varphi_{s+1}$  is a function in  $C([0, T]; L^\infty(\mathcal{D}_{s+1}))$  supported in  $[0, T] \times B_{R_1,R}^{2(s+1)}$  and if  $(\varphi_{s+1}^n)_{n \in \mathbb{N}}$  is a sequence of approximations of  $\varphi_{s+1}$ , we have the following estimate

$$\|\mathcal{C}_{s,s+1}^{\pm} \mathbf{T}_{s+1}(t) (\varphi_{s+1}^n(t) - \varphi_{s+1}^m(t))\|_{L^\infty} \leq C(R) \|\varphi_{s+1}^n(t) - \varphi_{s+1}^m(t)\|_{L^\infty},$$

which tends to 0 as  $n, m \rightarrow \infty$ , uniformly in  $t \in [0, T]$ .

Letting  $R_1$  and  $R$  go to infinity, we conclude that the operator  $\mathcal{C}_{s,s+1} \mathbf{T}_{s+1}(t)$  is well defined on functions of  $C([0, T]; L^\infty(\mathcal{D}_{s+1}))$  with bounded support in  $V_{s+1}$  (or decaying sufficiently fast at infinity). A quantitative estimate of this decay will be given by introducing appropriate weighted spaces in the next section.

Notice that for the Boltzmann hierarchy (5.0.2), the collision operators are defined by integrals on manifolds of codimension  $d$  but since free transport preserves continuity one can require that all functions under study are continuous.

## 5.2. Functional spaces and statement of the results

In order to obtain uniform a priori bounds for mild solutions to the (filtered) BBGKY hierarchy, we need to introduce some norms on the space of sequences  $(g_s)_{s \geq 1}$ . Given  $\varepsilon > 0$ ,  $\beta > 0$ , an integer  $s \geq 1$ , and a measurable function  $g_s : \mathcal{D}_s \rightarrow \mathbf{R}$ , we let

$$(5.2.1) \quad |g_s|_{\varepsilon, s, \beta} := \sup_{Z_s \in \mathcal{D}_s} (|g_s(Z_s)| \exp(\beta E_0(Z_s)))$$

where  $E_0$  is the free Hamiltonian:

$$(5.2.2) \quad E_0(Z_s) := \sum_{1 \leq i \leq s} \frac{|v_i|^2}{2}.$$

Note that the dependence on  $\varepsilon$  of the norm is through the constraint  $Z_s \in \mathcal{D}_s$ .

We also define, for a continuous function  $g_s : \mathbf{R}^{2ds} \rightarrow \mathbf{R}$ ,

$$(5.2.3) \quad |g_s|_{0, s, \beta} := \sup_{Z_s \in \mathbf{R}^{2ds}} (|g_s(Z_s)| \exp(\beta E_0(Z_s))).$$

**Definition 5.2.1.** — For  $\varepsilon > 0$  and  $\beta > 0$ , we denote  $X_{\varepsilon, s, \beta}$  the Banach space of measurable functions  $\mathcal{D}_s \rightarrow \mathbf{R}$  with finite  $|\cdot|_{\varepsilon, s, \beta}$  norm, and similarly  $X_{0, s, \beta}$  is the Banach space of continuous functions  $\mathbf{R}^{2ds} \rightarrow \mathbf{R}$  with finite  $|\cdot|_{0, s, \beta}$  norm.

For sequences of measurable functions  $G = (g_s)_{s \geq 1}$ , with  $g_s : \mathcal{D}_s \rightarrow \mathbf{R}$ , we let for  $\varepsilon > 0$ ,  $\beta > 0$ , and  $\mu \in \mathbf{R}$ ,

$$\|G\|_{\varepsilon, \beta, \mu} := \sup_{s \geq 1} (|g_s|_{\varepsilon, s, \beta} \exp(\mu s)).$$

We define similarly for  $G = (g_s)_{s \geq 1}$ , with  $g_s : \mathbf{R}^{2ds} \rightarrow \mathbf{R}$  continuous,

$$\|G\|_{0, \beta, \mu} := \sup_{s \geq 1} (|g_s|_{0, s, \beta} \exp(\mu s)).$$

**Definition 5.2.2.** — For  $\varepsilon \geq 0$ ,  $\beta > 0$ , and  $\mu \in \mathbf{R}$ , we denote  $\mathbf{X}_{\varepsilon, \beta, \mu}$  the Banach space of sequences of functions  $G = (g_s)_{1 \leq s \leq N}$ , with  $g_s \in X_{\varepsilon, s, \beta}$  and  $\|G\|_{\varepsilon, \beta, \mu} < \infty$ , and similarly  $\mathbf{X}_{0, \beta, \mu}$  the Banach space of sequences of continuous functions  $G = (g_s)_{s \geq 1}$ , with  $g_s \in X_{0, s, \beta}$  and  $\|G\|_{0, \beta, \mu} < \infty$ .

The following inclusions hold:

$$(5.2.4) \quad \text{if } \beta' \leq \beta \text{ and } \mu' \leq \mu, \text{ then } X_{\varepsilon, s, \beta'} \subset X_{\varepsilon, s, \beta}, \quad \mathbf{X}_{\varepsilon, \beta', \mu'} \subset \mathbf{X}_{\varepsilon, \beta, \mu}.$$

**Remark 5.2.3.** — *These norms are rather classical in statistical physics (up to replacing the  $L^\infty$  norm by an  $L^1$  norm), where probability measures are called “ensembles”.*

*At the canonical level, the ensemble  $\mathbb{1}_{Z_s \in \mathcal{D}_s} e^{-\beta E_0(Z_s)} dZ_s$  is a normalization of the Lebesgue measure, where  $\beta \sim \theta^{-1}$  (and  $\theta$  is the absolute temperature) specifies fluctuations of energy. The Boltzmann-Gibbs principle states that the average value of any quantity in the canonical ensemble is its equilibrium value at temperature  $\theta$ .*

*The micro-canonical level consists in restrictions of the ensemble to energy surfaces.*

*At the grand-canonical level the number of particles may vary, with variations indexed by chemical potential  $\mu \in \mathbf{R}$ .*

Existence and uniqueness for (5.0.1) comes from the theory of linear transport equations which provides a unique, global solution to the Liouville equation (4.2.1) by the method of characteristics. Nevertheless, in order to obtain a similar result for the limiting hierarchy (5.0.2), we need to obtain uniform a priori estimates with respect to  $N$ , on the marginals  $f_N^{(s)}$  for any fixed  $s$ . We shall thus deal with both systems (5.0.1) and (5.0.2) simultaneously, using analytical-type techniques which will provide short-time existence (with uniform bounds) in the spaces of  $\mathbf{X}_{\varepsilon, \beta, \mu}$ -valued functions of time (resp.  $\mathbf{X}_{0, \beta, \mu}$ ). Actually the parameters  $\beta$  and  $\mu$  will themselves depend on time: in the sequel we choose for simplicity a linear dependence in time, though other, decreasing functions of time could be chosen just as well. Such a time dependence on the parameters of the function spaces is a situation which occurs whenever continuity estimates involve a loss, which is the case here since the continuity estimates on the collision operators lead to a deterioration in the parameters  $\beta$  and  $\mu$ .

**Definition 5.2.4.** — *Given  $T > 0$ , a positive function  $\beta$  and a real valued function  $\mu$  both defined on  $[0, T]$ , we denote by  $\mathbf{X}_{\varepsilon, \beta, \mu}$  the space of time continuous functions*

$$G : t \in [0, T] \mapsto G(t) = (g_s(t))_{s \geq 1} \in \mathbf{X}_{\varepsilon, \beta(t), \mu(t)},$$

*such that*

$$(5.2.5) \quad \begin{aligned} \|G\|_{\varepsilon, \beta, \mu} &:= \sup_{0 \leq t \leq T} \|G(t)\|_{\varepsilon, \beta(t), \mu(t)} < \infty, \\ \lim_{s \rightarrow t^-} \|G(t) - G(s)\|_{\mathbf{X}_{\varepsilon, \beta(t), \mu(t)}} &= 0. \end{aligned}$$

*We define similarly*

$$\|G\|_{0, \beta, \mu} := \sup_{0 \leq t \leq T} \|G(t)\|_{0, \beta(t), \mu(t)}.$$

We shall prove the following uniform bounds for the BBGKY hierarchy.

**Theorem 6 (Uniform estimates for the BBGKY hierarchy).** — *Let  $\beta_0 > 0$  and  $\mu_0 \in \mathbf{R}$  be given. There is a time  $T > 0$  as well as two nonincreasing functions  $\beta > 0$  and  $\mu$  defined on  $[0, T]$ , satisfying  $\beta(0) = \beta_0$  and  $\mu(0) = \mu_0$ , such that in the Boltzmann-Grad scaling  $N\varepsilon^{d-1} \equiv 1$ , any family of initial marginals  $F_N(0) = (f_N^{(s)}(0))_{1 \leq s \leq N}$  in  $\mathbf{X}_{\varepsilon, \beta_0, \mu_0}$  gives rise to a unique solution  $G_N(t) = (\mathbf{T}_s(-t)f_N^{(s)}(t))_{1 \leq s \leq N}$  in  $\mathbf{X}_{\varepsilon, \beta, \mu}$  to the BBGKY hierarchy (5.0.1) satisfying the following bound:*

$$\|G_N\|_{\varepsilon, \beta, \mu} \leq 2 \|F_N(0)\|_{\varepsilon, \beta_0, \mu_0}.$$

**Remark 5.2.5.** — *The proof of Theorem 6 provides a lower bound for the time  $T$  on which one has a uniform bound, in terms of the initial parameters  $\beta_0$ ,  $\mu_0$  and the dimension  $d$ : one finds*

$$(5.2.6) \quad T \geq C_d e^{\mu_0} (1 + \beta_0^{\frac{1}{2}})^{-1} \max_{\beta \in [0, \beta_0]} \beta e^{-\beta} (\beta_0 - \beta)^{\frac{d+1}{2}},$$

where  $C_d$  is a constant depending only on  $d$ .

In particular if  $d \ll \beta_0$ , there holds  $\max_{\beta \in [0, \beta_0]} \beta e^{-\beta} (\beta_0 - \beta)^{\frac{d+1}{2}} = \beta_0^{\frac{d+1}{2}} (1 + o(1))$ , hence an existence time of the order of  $e^{\mu_0} \beta_0^{d/2}$ .

The proof of Theorem 6 uses neither the fact that the BBGKY hierarchy is closed by the transport equation satisfied by  $f_N$ , nor possible cancellations of the collision operators. It only relies on crude estimates and in particular the limiting hierarchy satisfies the same result, proved similarly. Note that the functional setting is simpler in the case of the Boltzmann hierarchy as all functions are continuous with respect to all parameters.

**Theorem 7 (Existence for the Boltzmann hierarchy).** — *Let  $\beta_0 > 0$  and  $\mu_0 \in \mathbf{R}$  be given. There is a time  $T > 0$  as well as two nonincreasing functions  $\beta > 0$  and  $\boldsymbol{\mu}$  defined on  $[0, T]$ , satisfying  $\beta(0) = \beta_0$  and  $\boldsymbol{\mu}(0) = \mu_0$ , such that any family of initial marginals  $F(0) = (f^{(s)}(0))_{s \geq 1}$  in  $\mathbf{X}_{0, \beta_0, \mu_0}$  gives rise to a unique solution  $G(t) = (\mathbf{S}_s(-t) f^{(s)}(t))_{s \geq 1}$  in  $\mathbf{X}_{0, \beta, \boldsymbol{\mu}}$  to the Boltzmann hierarchy (5.0.2), satisfying the following bound:*

$$\|G\|_{0, \beta, \boldsymbol{\mu}} \leq 2 \|F(0)\|_{0, \beta_0, \mu_0}.$$

### 5.3. Main steps of the proofs

The proofs of Theorems 6 and 7 are typical of analytical-type results, such as the classical Cauchy-Kowalevskaya theorem. We follow here Ukai's approach [45], which turns out to be remarkably short and self-contained.

Let us give the main steps of the proof: we start by noting that the conservation of energy for the  $s$ -particle flow is reflected in identities

$$(5.3.1) \quad |\mathbf{T}_s(t) g_s|_{\varepsilon, s, \beta} = |g_s|_{\varepsilon, s, \beta} \quad \text{and} \quad \|\mathbf{T}(t) G_N\|_{\varepsilon, \beta, \boldsymbol{\mu}} = \|G_N\|_{\varepsilon, \beta, \boldsymbol{\mu}},$$

for all parameters  $\beta > 0$ ,  $\boldsymbol{\mu} \in \mathbf{R}$ , and for all  $g_s \in X_{\varepsilon, s, \beta}$ ,  $G_N = (g_s)_{1 \leq s \leq N} \in \mathbf{X}_{\varepsilon, \beta, \boldsymbol{\mu}}$ , and all  $t \geq 0$ . Similarly,

$$(5.3.2) \quad |\mathbf{S}_s(t) g_s|_{0, s, \beta} = |g_s|_{0, s, \beta} \quad \text{and} \quad \|\mathbf{S}(t) G\|_{0, \beta, \boldsymbol{\mu}} = \|G\|_{0, \beta, \boldsymbol{\mu}},$$

for all parameters  $\beta > 0$ ,  $\boldsymbol{\mu} \in \mathbf{R}$ , and for all  $g_s \in X_{0, s, \beta}$ ,  $G = (g_s)_{s \geq 1} \in \mathbf{X}_{0, \beta, \boldsymbol{\mu}}$ , and all  $t \geq 0$ .

Next assume that in the Boltzmann-Grad scaling  $N\varepsilon^{d-1} \equiv 1$ , there holds the bound

$$(5.3.3) \quad \forall 0 < \varepsilon \leq \varepsilon_0, \quad \left\| \int_0^t \mathbf{T}(-\tau) \mathbf{C}_N \mathbf{T}(\tau) G_N(\tau) d\tau \right\|_{\varepsilon, \beta, \boldsymbol{\mu}} \leq \frac{1}{2} \|G_N\|_{\varepsilon, \beta, \boldsymbol{\mu}},$$

for some functions  $\beta$  and  $\boldsymbol{\mu}$  as in the statement of Theorem 6. Under (5.3.3), the linear operator

$$\mathfrak{L}: \quad G_N \in \mathbf{X}_{\varepsilon, \beta, \boldsymbol{\mu}} \mapsto \left( t \mapsto \int_0^t \mathbf{T}(-\tau) \mathbf{C}_N \mathbf{T}(\tau) G_N(\tau) d\tau \right) \in \mathbf{X}_{\varepsilon, \beta, \boldsymbol{\mu}}$$



is linear continuous from  $\mathbf{X}_{\varepsilon,\beta,\mu}$  to itself with norm strictly smaller than one. In particular, the operator  $\text{Id} - \mathfrak{L}$  is invertible in the Banach algebra  $\mathcal{L}(\mathbf{X}_{\varepsilon,\beta,\mu})$ . Hence, there exists a unique solution  $G_N$  in  $\mathbf{X}_{\varepsilon,\beta,\mu}$  to  $(\text{Id} - \mathfrak{L})G_N = F_N(0)$ , an equation which is equivalent to (5.0.1).

The reasoning is identical for Theorem 7, replacing (5.3.3) by

$$(5.3.4) \quad \left\| \int_0^t \mathbf{S}(-\tau) \mathbf{C}^0 \mathbf{S}(\tau) G(\tau) d\tau \right\|_{0,\beta,\mu} \leq \frac{1}{2} \|G\|_{0,\beta,\mu}.$$

The next section is devoted to the proofs of (5.3.3) and (5.3.4).

#### 5.4. Continuity estimates

In order to prove (5.3.3) and (5.3.4), we first establish bounds, in the above defined functional spaces, for the collision operators defined in (4.3.2) and (4.4.3), and for the total collision operators. In  $\mathcal{C}_{s,s+1}$ , the sum in  $i$  over  $[1, s]$  will imply a loss in  $\mu$ , while the linear velocity factor will imply a loss in  $\beta$ .

The next statement concerns the BBGKY collision operator.

**Proposition 5.4.1.** — *Given  $\beta > 0$  and  $\mu \in \mathbf{R}$ , for  $1 \leq s \leq N - 1$ , the collision operator  $\mathcal{C}_{s,s+1}$  satisfies the bound, for all  $G_N = (g_s)_{1 \leq s \leq N} \in \mathbf{X}_{\varepsilon,\beta,\mu}$  in the Boltzmann-Grad scaling  $N\varepsilon^{d-1} \equiv 1$ , and for almost all  $t$  and  $Z_s$ ,*

$$(5.4.1) \quad |(\mathcal{C}_{s,s+1} \mathbf{T}_{s+1}(t) g_{s+1})(Z_s)| \leq C_d \beta^{-\frac{d}{2}} \left( s\beta^{-\frac{1}{2}} + \sum_{1 \leq i \leq s} |v_i| \right) e^{-\beta E_0(Z_s)} |g_{s+1}|_{\varepsilon,s+1,\beta},$$

for some  $C_d > 0$  depending only on  $d$ .

*Proof.* — Recall that as in (4.3.2),

$$\begin{aligned} (\mathcal{C}_{s,s+1} \mathbf{T}_{s+1}(t) g^{(s+1)})(t, Z_s) &:= (N - s) \varepsilon^{d-1} \\ &\times \sum_{i=1}^s \int_{\mathbf{S}_1^{d-1} \times \mathbf{R}^d} \omega \cdot (v_{s+1} - v_i) \mathbf{T}_{s+1}(t) g^{(s+1)}(t, Z_s, x_i + \varepsilon \omega, v_{s+1}) d\omega dv_{s+1}. \end{aligned}$$

Estimating each term in the sum separately, regardless of possible cancellations between “gain” and “loss” terms, it is obvious that

$$|\mathcal{C}_{s,s+1} \mathbf{T}_{s+1}(t) g_{s+1}(Z_s)| \leq \kappa_d \varepsilon^{d-1} (N - s) |g_{s+1}|_{\varepsilon,s+1,\beta} \sum_{1 \leq i \leq s} I_i(V_s),$$

where  $\kappa_d$  is the volume of the unit ball of  $\mathbf{R}^d$ , and where

$$I_i(V_s) := \int_{\mathbf{R}^d} (|v_{s+1}| + |v_i|) \exp\left(-\frac{\beta}{2} \sum_{j=1}^{s+1} |v_j|^2\right) dv_{s+1}.$$

Since a direct calculation gives

$$I_i(V_s) \leq C_d \beta^{-\frac{d}{2}} \left( \beta^{-\frac{1}{2}} + |v_i| \right) \exp\left(-\frac{\beta}{2} \sum_{1 \leq j \leq s} |v_j|^2\right),$$

the result (5.4.1) is deduced directly in the Boltzmann-Grad scaling  $N\varepsilon^{d-1} \equiv 1$ . Proposition 5.4.1 is proved.  $\square$

A similar result holds for the limiting collision operator.

**Proposition 5.4.2.** — Given  $\beta > 0$ ,  $\mu \in \mathbf{R}$ , the collision operator  $\mathcal{C}_{s,s+1}^0$  satisfies the following bound, for all  $g_{s+1} \in X_{0,s+1,\beta}$  :

$$(5.4.2) \quad |(\mathcal{C}_{s,s+1}^0 g_{s+1})(Z_s)| \leq C_d \beta^{-\frac{d}{2}} \left( s \beta^{-\frac{1}{2}} + \sum_{1 \leq i \leq s} |v_i| \right) e^{-\beta E_0(Z_s)} |g_{s+1}|_{0,s+1,\beta},$$

for some  $C_d > 0$  depending only on  $d$ .

*Proof.* — There holds

$$|(\mathcal{C}_{s,s+1}^0 g_{s+1})(Z_s)| \leq \sum_{1 \leq i \leq s} \int_{\mathbf{S}^{d-1} \times \mathbf{R}^d} (|v_{s+1}| + |v_i|) (|g_{s+1}(v_i^*, v_{s+1}^*)| + |g_{s+1}(v_i, v_{s+1})|) d\omega dv_{s+1},$$

omitting most of the arguments of  $g_{s+1}$  in the integrand. By definition of  $|\cdot|_{0,s,\beta}$  norms and conservation of energy (5.3.1), there holds

$$\begin{aligned} |g_{s+1}(v_i^*, v_{s+1}^*)| + |g_{s+1}(v_i, v_{s+1})| &\leq (e^{-\beta E_0(Z_s^*)} + e^{-\beta E_0(Z_s)}) |g_{s+1}|_{0,\beta} \\ &= 2e^{-\beta E_0(Z_s)} |g_{s+1}|_{0,s+1,\beta}, \end{aligned}$$

where  $Z_s^*$  is identical to  $Z_s$  except for  $v_i$  and  $v_{s+1}$  changed to  $v_i^*$  and  $v_{s+1}^*$ . This gives

$$|(\mathcal{C}_{s,s+1}^0 g_{s+1})(Z_s)| \leq C_d |g_{s+1}|_{0,s+1,\beta} e^{-\beta E_0(Z_s)} \sum_{1 \leq i \leq s} I_i(V_s),$$

borrowing notation from the proof of Proposition 5.4.1, and we conclude as above.  $\square$

Propositions 5.4.1 and 5.4.2 are the key to the proof of (5.3.3) and (5.3.4). Let us first prove a continuity estimate based on Proposition 5.4.1, which implies directly (5.3.3).

**Lemma 5.4.3.** — Let  $\beta_0 > 0$  and  $\mu_0 \in \mathbf{R}$  be given. For all  $\lambda > 0$  and  $t > 0$  such that  $\lambda t < \beta_0$ , there holds the bound

$$(5.4.3) \quad e^{s(\mu_0 - \lambda t)} \left| \int_0^t \mathbf{T}_s(-\tau) \mathcal{C}_{s,s+1} \mathbf{T}_{s+1}(\tau) g_{s+1}(\tau) d\tau \right|_{\varepsilon, s, \beta_0 - \lambda t} \leq \bar{c}(\beta_0, \mu_0, \lambda, T) \|G_N\|_{\varepsilon, \beta, \mu},$$

for all  $G_N = (g_s)_{1 \leq s \leq N} \in \mathbf{X}_{\varepsilon, \beta, \mu}$ , with  $\bar{c}(\beta_0, \mu_0, \lambda, T)$  computed explicitly in (5.4.9) below. In particular there is  $T > 0$  depending only on  $\beta_0$  and  $\mu_0$  such that for an appropriate choice of  $\lambda$  in  $(0, \beta_0/T)$ , there holds for all  $t \in [0, T]$

$$(5.4.4) \quad e^{s(\mu_0 - \lambda t)} \left| \int_0^t \mathbf{T}_s(-\tau) \mathcal{C}_{s,s+1} \mathbf{T}_{s+1}(\tau) g_{s+1}(\tau) d\tau \right|_{\varepsilon, s, \beta_0 - \lambda t} \leq \frac{1}{2} \|G_N\|_{\varepsilon, \beta, \mu}.$$

*Proof.* — Let us define, for all  $\lambda > 0$  and  $t > 0$  such that  $\lambda t < \beta_0$ , the functions

$$(5.4.5) \quad \beta_0^\lambda(t) := \beta_0 - \lambda t \quad \text{and} \quad \mu_0^\lambda(t) := \mu_0 - \lambda t.$$

By conservation of energy (5.3.1), there holds the bound

$$\left| \int_0^t \mathbf{T}_s(-\tau) \mathcal{C}_{s,s+1} \mathbf{T}_{s+1}(\tau) g_{s+1}(\tau) d\tau \right|_{\varepsilon, s, \beta_0^\lambda(t)} \leq \sup_{Z_s \in \mathbf{R}^{2ds}} \int_0^t e^{\beta_0^\lambda(t) E_0(Z_s)} |\mathcal{C}_{s,s+1} \mathbf{T}_{s+1}(\tau) g_{s+1}(\tau, Z_s)| d\tau.$$

Estimate (5.4.1) from Proposition 5.4.1 gives

$$\begin{aligned} &e^{\beta_0^\lambda(t) E_0(Z_s)} |\mathcal{C}_{s,s+1} \mathbf{T}_{s+1}(\tau) g_{s+1}(\tau, Z_s)| \\ &\leq C_d (\beta_0^\lambda(\tau))^{-\frac{d}{2}} |g_{s+1}(\tau)|_{\varepsilon, s+1, \beta_0^\lambda(\tau)} \left( s (\beta_0^\lambda(\tau))^{-\frac{1}{2}} + \sum_{1 \leq i \leq s} |v_i| \right) e^{\lambda(\tau-t) E_0(Z_s)}. \end{aligned}$$

By definition of norms  $\|\cdot\|_{\varepsilon,\beta,\mu}$  and  $\|\|\cdot\|\|_{\varepsilon,\beta,\mu}$  we have

$$(5.4.6) \quad \begin{aligned} |g_{s+1}(\tau)|_{\varepsilon,s+1,\beta_0^\lambda(\tau)} &\leq e^{-(s+1)\mu_0^\lambda(\tau)} \|G_N(\tau)\|_{\varepsilon,\beta_0^\lambda(\tau),\mu_0^\lambda(\tau)} \\ &\leq e^{-(s+1)\mu_0^\lambda(\tau)} \|G_N\|_{\varepsilon,\beta,\mu}. \end{aligned}$$

The above bounds yield, since  $\beta_0^\lambda$  and  $\mu_0^\lambda$  are nonincreasing,

$$\begin{aligned} e^{s\mu_0^\lambda(t)} \left| \int_0^t \mathbf{T}_s(-\tau) \mathcal{C}_{s,s+1} \mathbf{T}_{s+1}(\tau) g_{s+1}(\tau) d\tau \right|_{\varepsilon,s,\beta_0^\lambda(t)} \\ \leq C_d \|G_N\|_{\varepsilon,\beta,\mu} e^{-\mu_0^\lambda(T)} (\beta_0^\lambda(T))^{-\frac{d}{2}} \sup_{Z_s \in \mathbf{R}^{2ds}} \int_0^t \bar{C}(\tau, t, Z_s) d\tau, \end{aligned}$$

where, for  $\tau \leq t$ ,

$$(5.4.7) \quad \bar{C}(\tau, t, Z_s) := \left( s(\beta_0^\lambda(\tau))^{-\frac{1}{2}} + \sum_{1 \leq i \leq s} |v_i| \right) e^{\lambda(\tau-t)(s+E_0(Z_s))}.$$

Since

$$(5.4.8) \quad \sup_{Z_s \in \mathbf{R}^{2ds}} \int_0^t \bar{C}(\tau, t, Z_s) d\tau \leq \frac{C_d}{\lambda} \left( 1 + (\beta_0^\lambda(T))^{-\frac{1}{2}} \right),$$

there holds finally

$$e^{s\mu_0^\lambda(t)} \left| \int_0^t \mathbf{T}_s(-\tau) \mathcal{C}_{s,s+1} \mathbf{T}_{s+1}(\tau) g_{s+1}(\tau) d\tau \right|_{\varepsilon,s,\beta_0^\lambda(t)} \leq \bar{c}(\beta_0, \mu_0, \lambda, T) \|G_N\|_{\varepsilon,\beta,\mu},$$

where, with a possible change of the constant  $C_d$ ,

$$(5.4.9) \quad \bar{c}(\beta_0, \mu_0, \lambda, T) := C_d e^{-\mu_0^\lambda(T)} \lambda^{-1} (\beta_0^\lambda(T))^{-\frac{d}{2}} \left( 1 + (\beta_0^\lambda(T))^{-\frac{1}{2}} \right).$$

The result (5.4.3) follows. To deduce (5.4.4) we need to find  $T > 0$  and  $\lambda > 0$  such that  $\lambda T < \beta_0$  and

$$(5.4.10) \quad C_d (1 + (\beta_0 - \lambda T)^{-\frac{1}{2}}) e^{-\mu_0 + \lambda T} (\beta_0 - \lambda T)^{-\frac{d}{2}} = \frac{\lambda}{2}.$$

With  $\beta := \lambda T \in (0, \beta_0)$ , condition (5.4.10) becomes

$$\begin{aligned} T &= C_d e^{\mu_0} \beta e^{-\beta} \frac{(\beta_0 - \beta)^{\frac{d+1}{2}}}{1 + (\beta_0 - \beta)^{\frac{1}{2}}} \\ &\geq C_d e^{\mu_0} (1 + \beta_0^{\frac{1}{2}})^{-1} \beta e^{-\beta} (\beta_0 - \beta)^{\frac{d+1}{2}}, \end{aligned}$$

up to changing the constant  $C_d$  and (5.4.4) follows. Notice that (5.2.6) is a consequence of this computation.  $\square$

The proof of the corresponding result (5.3.4) for the Boltzmann hierarchy is identical, since the estimates for  $\mathcal{C}_{s,s+1}^0$  and  $\mathcal{C}_{s,s+1}$  are essentially identical (compare estimate (5.4.1) from Proposition 5.4.1 with estimate (5.4.2) from Proposition 5.4.2).

**Remark 5.4.4.** — *The above arguments provide the global in time wellposedness of the BBGKY hierarchy for each fixed  $N$  — though with no uniform bound on  $N$ . Indeed the exponential weight  $\exp(-\mu_0 N - \beta_0 E_0(Z_N)) \mathbb{1}_{\mathcal{D}_N}$  is an invariant measure for the flow of the transport equation*

$$\partial_t f_N + V_N \cdot \nabla_{X_N} f_N = 0.$$

The maximum principle then implies that for all  $t \geq 0$

$$0 \leq f_N(t, Z_N) \leq \exp(-\mu_0 N - \beta_0 E_0(Z_N)) \mathbb{1}_{\mathcal{D}_N}.$$

By integration we find

$$0 \leq f_N^{(s)}(t, Z_s) \leq \exp(-\mu_0 N - \beta_0 E_0(Z_s)) \mathbb{1}_{\mathcal{D}_s}$$

As the measure  $\exp(-\mu_0 N - \beta_0 E_0(Z_s)) \mathbb{1}_{\mathcal{D}_s}$  is invariant by the flow  $\mathbf{T}_s$ , we get by filtering that with the notation introduced in Paragraph 5.1,  $G_N = (g_{N,s})_{1 \leq s \leq N}$  satisfies

$$0 \leq g_{N,s}(t, Z_s) \leq \exp(-\mu_0 N - \beta_0 E_0(Z_s)) \mathbb{1}_{\mathcal{D}_s}$$

hence a bound for which no parameters depend on  $t$  (though the bound is very poor in  $N$ ).

Then we can iterate the fixed point method used in the proof of Theorem 6 to prove that the marginals belong for all time to the space  $\mathbf{X}_{\varepsilon, \beta, \mu}$  and not only on a short time interval. However the size of the functions grows with  $N$  so that fact cannot be used to obtain a convergence result.