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Metric measure geometry. Gromov's theory of convergence and concentration of metrics and measures.

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This book is dedicated to the study, completion and generalization of the theory of metric geometry on the space of metric measure spaces, originally developed by *M. Gromov* in his book [Metric structures for Riemannian and non-Riemannian spaces. Transl. from the French by Sean Michael Bates. With appendices by M. Katz, P. Pansu, and S. Semmes. Edited by J. LaFontaine and P. Pansu. 3rd printing. Modern Birkhäuser Classics. Basel: Birkhäuser (2007; Zbl 1113.53001)] which in turn is based on the idea of the concentration of measure phenomena due to *P. Lévy* [Problèmes concrets d'analyse fonctionnelle. - Seconde éd. des Leçons d'analyse fonctionnelle. - Avec un complément par F. Pellegrino. Paris: Gauthier-Villars, (1951; Zbl 43.-32302)] and *V. D. Mil'man* [Funct. Anal. Appl. 5, 288–295 (1972); translation from Funkts. Anal. Prilozh. 5, No. 4, 28–37 (1971; Zbl 239.46018)] and [in: Les processus stochastiques, Coll. Paul Lévy, Palaiseau/Fr. 1987, Astérisque 157–158, 273–301 (1988; Zbl 681.46021)].

Contents. Introduction, 1. Prelimimaries from measure theory, 2. The Lévy-Milman concentration phenomenon, 3. Gromov-Hausdorff distance and distance matrix, 4. Box distance, 5. Observable distance and measurement, 6. The space of pyramids, 7. Asymptotic concentration, 8. Dissipation, 9. Curvature and concentration, Bibliography, Index.

The author's new theory is based of the notion of observable distance.

A *metric measure space*, short *mm-space*, is a triple (X, d_X, μ_X), where (X, d_X) is a complete separable metric space and μ_X is a Borel probability measure of X. A sequence of mm-spaces X_n, $n = 1, 2, \dots$, is called a *Lévy family* if $\lim_{n\to\infty} \inf_{c\in\mathbb{R}} \mu_{X_n}(|f_n - c| > \varepsilon) = 0$, for any sequence of 1-Lipschitz continuous functions $f_n : X_n \to \mathbb{R}$, $n = 1, 2, \dots$, and for any $\varepsilon > 0$. Example: the sequence of unit spheres $S^n(1)$, $n = 1, 2, \dots$, is a Lévy family, where the measure on $S^n(1)$ is taken to be the Riemannian volume measure normalized as the total measure to be 1.

Let $I := [0, 1)$ and X be a topological space with a Borel probability measure μ_X. A map $\varphi : I \to X$ is called a *parameter* of X if φ is a Borel-measurable map such that $\varphi_*\mathcal{L}^1 = \mu_X$, where $\mathcal{L}^1$ denotes the one-dimensional Lebesgue measure on I.

For a pseudo-metric ρ on I, let $\mathcal{L}ip_1(\rho)$ denotes the set of 1-Lipschitz function on I with respect to ρ. Denote by $\mathbf{D}$ the set of pseudo-metrics ρ on I such that every element of $\mathcal{L}ip_1(\rho)$ is a Borel-measure function. For two pseudo-metrics ρ_1, $\rho_2 \in \mathbf{D}$, denote $d_{\mathrm{conc}}(\rho_1, \rho_2) := d_H(\mathcal{L}ip_1(\rho_1), \mathcal{L}ip_1(\rho_2))$, where d_H is a Hausdorff distance. If X, Y are two mm-spaces, then the *observable distance* $d_{\mathrm{conc}}(X, Y)$ is defined by $d_{\mathrm{conc}}(X, Y) := \inf_{\varphi,\psi} d_{\mathrm{conc}}(\varphi^* d_X, \psi^* d_Y)$, where $\varphi : I \to X$ and $\psi : I \to Y$ run over all parameters of X and Y, respectively. It is said that a sequence of mm-spaces X_n, $n = 1, 2, \dots$, *concentrates* to an mm-space X if $\lim_{n\to\infty} d_{\mathrm{conc}}(X_n, X) = 0$. It is proved (Theorem 5.13) that d_{conc} is a metric on the set χ of mm-isomorphism classes of mm-spaces. The d_{conc}-convergence of mm-spaces is a generalization of the Lévy property for spheres. The d_{conc}-convergence of mm-spaces is called *concentration* of mm-spaces. A typical example $X_n \to Y$ of concentration is obtained by a sequence of fibrations $F_n \to X_n \to Y$ such that $\{F_n\}_{n=1}^{\infty}$ is a Lévy family. This example shows that the concentration of mm-spaces is the analogous of collapsing of Riemannian manifolds. An example proves that the concentration is strictly weaker than measured-Gromov-Hausdorff convergence and that this is more suitable for the study of a sequence of manifolds whose dimensions are unbounded. However, because the d_{conc} is not easily investigated, the author considers the so-called *box distance* between mm-spaces, which is closely related to the measured-Gromov-Hausdorff convergence of mm-spaces. For two pseudo-metrics ρ_1, ρ_2 on I, $\square(\rho_1, \rho_2)$ is the minimum of $\varepsilon \geq 0$ for which there exists a Borel subset $I_0 \subset I$ such that: (1) $|\rho_1(s, t) - \rho_2(s, t)| \leq \varepsilon$, for all s, $t \in I_0$, (2) $L^1(I_0) \geq 1 - \varepsilon$. The *box distance* $\square(X, Y)$ between two mm-spaces X and Y is the infimum $\square(\varphi^* d_X, \psi^* d_Y)$, where $\varphi : I \to X$, $\psi : I \to Y$ run over all parameters of X and Y respectively, where $\varphi^* d_X(s, t) := d_X(\varphi(s), \varphi(t)$, for s, $t \in I$.

Concentration of mm-spaces is equivalent to convergence of associated pyramids using the box distance function, where a *pyramid* is a family of mm-spaces that forms a directed set with respect to some natural order relation between mm-spaces, called the *Lipschitz order*. A metric ρ is defined on the set Π of pyramids induced from the box distance function. Each mm-space X is associated with a pyramid $\mathcal{P}_X$ consisting of all descendents of the mm-space with respect to the Lipschitz order. Then the map $\iota : \chi \ni X \to \mathcal{P}_X \in \Pi$ is a 1-Lipschitz continuous topological embedding map with respect to d_{conc}. Also it is proved that Π is a

compactification of χ with respect to d_{conc} (Proposition 7.29).
Then the author studies sequences of mm-spaces such that d_{conc}-diverges but has proper asymptotic behavior. A sequence of mm-spaces X_n, $n = 1, 2, \ldots$, is said to be *asymptotic* if the associated pyramid $\mathcal{P}_{X_n}$ converges in Π, and this sequence *asymptotically concentrates* if it is a d_{conc}-Cauchy sequence. Any asymptotically concentrating sequence of mm-spaces is asymptotic. Some examples (Example 7.36) and counter-examples (Theorem 7.40, Corollary 7.42) are given. Theorem 7.27, one of main theorems in the book, states that the map $\iota : \chi \to \Pi$, $\iota(X) = \mathcal{P}_X$, extends to the d_{conc}-completion of χ, so that the space Π of pyramids is also a compactification of the d_{conc}-completion of the space χ.
If γ^n denotes the standard Gaussian measure on $\mathbb{R}^n$, then both associated pyramids $\mathcal{P}_{S^n(\sqrt{n})}$ and $\mathcal{P}_{(\mathbb{R}^n,\gamma^n)}$ converge to a common pyramid as $n \to \infty$ (Theorem 7.40), a result which is a generalization of the Maxwell-Bolzmann law (or the Poincaré limit theorem).
About the spectral compactness the author proves that any spectrally compact and asymptotic sequence of mm-spaces asymptotically concentrates if the observable diameter is bounded from above (Theorem 7.52). A sequence of mm-spaces *spectrally concentrates* if it is spectrally compact and asymptotically concentrates. An example is given.
There is some notion of *dissipation* (δ and infinitely dissipation) for a sequence of mm-spaces, which is opposite to concentration and means that the mm-spaces disperse into many small pieces far apart of each other. A nondissipation theorem is proved (Theorem 8.8).
In the last chapter (Chapter 9) the author studies the relation between curvature and concentration. The concept of *curvature-dimension condition* for an mm-space in the sense of *J. Lott* and *C. Villani* [Ann. Math. (2) 169, No. 3, 903–991 (2009; Zbl 1178.53038)] and *K.-T. Sturm* [Acta Math. 196, No. 1, 65–131 (2006; Zbl 1105.53035); ibid. 196, No. 1, 133–177 (2006; Zbl 1106.53032)] was used by the author together with *K. Funano* in [Geom. Funct. Anal. 23, No. 3, 888–936 (2013; Zbl 1277.53038)] to prove that if a sequence of mm-spaces satisfies this condition and concentrates to an mm-space, then the limit also satisfies the curvature-dimension condition. This stability result of the curvature-dimension condition by concentration has an application to the eigenvalues of the Laplacian of Riemannian manifolds. Finally the author proves the stability of a lower bound of the so-called *Alexandrov curvature* (Theorem 57).

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