Basic Noncommutative Geometry

by Masoud Khalkhali

ZURICH: EUROPEAN MATHEMATICAL SOCIETY, 2013, 257 PP., SECOND EDITION, SOFTCOVER, 38.00 EUR, ISBN 978-3-03719-128-6

REVIEWED BY JOHN ROE

Noncommutative geometry, in the sense in which this book uses the term, is to a very large extent the creation of a single mathematician, Alain Connes. Chapter 3, indeed, begins with an extended quotation from Connes's summary of his talk at a memorable Oberwolfach conference in 1981: the first public presentation of *cyclic cohomology*. At that time there were hardly any published references, and photocopies of Connes's elegantly handwritten notebooks were valuable treasures among us graduate students finding our way in the new field. A glance through the bibliography will confirm that, more than 30 years later, Connes's work is still the primary source both for foundations and for new developments in noncommutative geometry.

What, then, *is* noncommutative geometry, in Connes's sense? It is the product of a vision of mathematics informed by a particular history. One component of that history is as old as Fermat and Descartes: the *algebraization of geometry*, or the realization that the geometry of a "space" can be studied by way of the algebraic properties of functions on that space. By choosing different classes of functions, one studies different kinds of geometry (such as algebraic geometry, differential geometry, topology, and so on).

The other component is more recent: it is the theory of algebras of operators on Hilbert space. In the 1920s, mathematicians and physicists realized that quantum theory required a radical restructuring of the foundations of physics, whereby the observables were no longer modeled by *functions* on the phase space of a system, but instead by (self-adjoint) *operators* on a Hilbert space: operators that need not commute. This led Murray and von Neumann to develop a theory of (not necessarily commutative) algebras of operators on Hilbert space, a theory that generated qualitatively new phenomena such as "real-valued dimensions."

It is the operator-theoretic and analytic elements arising from Hilbert space that distinguish Connes's noncommutative geometry from other similar proposals. From a physical perspective, the use of Hilbert space seems to be forced on one by the idea of *positivity*: physical theory has to calculate probabilities, which have to be positive real numbers, so that the idea that a *state* of a quantum system is a *positive* linear functional on the algebra of observables seems to be built into the theory at a fundamental level. Via a classical construction (named after Gelfand, Naĭmark, and Segal) this gives rise to Hilbert space representations of the algebra of observables.

In modern terminology, an algebra of operators on a Hilbert space H, which is norm-closed and closed under the adjoint operation, is called a *C^*-algebra*. If X is a compact Hausdorff space, then $C(X)$, the algebra of continuous complex-valued functions on X, is a C^*-algebra (with unit), and another theorem of Gelfand and Naimark says that *every* commutative C^*-algebra with unit is of this sort. In other words, all of topology—at least of compact Hausdorff spaces—is subsumed as a special (commutative) case of the theory of C^*-algebras. This is a fundamental motivation in Connes's program insofar as it suggests that we frame the theory of *noncommutative* C^*-algebras as some kind of "noncommutative topology."

To justify this language, one needs two things:

(a) A rich family of *examples* of noncommutative C^*-algebras (and related operator algebras) that have significant geometric content.
(b) A variety of *techniques* for studying noncommutative operator algebras, which "extend to the noncommutative world" familiar tools of topology and geometry: cohomology, *K*-theory, differential forms, curvature, Riemannian metrics, and so on.

Khalkhali's book introduces the student to many of these examples and techniques. The first chapter is an extensive survey of examples of "algebra-geometry correspondence," including noncommutative spaces such as crossed products and noncommutative tori, vector bundles and projective modules, algebraic function fields (Riemann surfaces), various approaches to noncommutative algebraic geometry, Hopf algebras, and quantum groups. Continuing with the theme of examples, the second chapter focuses on the *noncommutative quotient*, Connes's generalization of the group measure space construction of Murray and von Neumann and a primary motivation for noncommutative geometry. Within this chapter one finds an excellent discussion of groupoids (which provide a general framework for several kinds of noncommutative quotient construction) and of Morita equivalence.

In the third chapter of the book, the focus shifts to (b) as mentioned previously, with a detailed presentation of cyclic (co)homology theory, including the recent development of *Hopf-cyclic* cohomology. In noncommutative geometry, the cyclic cohomology of an algebra A serves as a model for the "de Rham homology" of the "underlying space" of A. Thus, if elements $a \in A$ are thought of as some kinds of "functions" on this underlying space, cyclic (co)homology for A should be obtained by manipulating symbols a and da, for $a \in A$, subject to suitable rules. Problems both algebraic and analytic arise in developing this theory. *Algebraically*, it is not at all clear exactly what the right properties are for the "noncommutative differential" suggested by the symbol "d" above; *analytically*, the example of manifolds already suggests that to be effective the theory will need to be applied not to a C^*-algebra (like the algebra of all *continuous* functions) but to a suitable subalgebra of "smooth" elements. Questions related to the choice of such subalgebras play an important role in noncommutative geometry, but they are not emphasized here. On the other hand, the algebraic aspects of cyclic theory are developed with great clarity from two or three different perspectives.

The theme of the fourth chapter is *K*-theory and its relationship to cyclic cohomology. It turns out that, of all the tools of algebraic topology, *K*-theory is the one that is most immediately amenable to noncommutative generalization. In ordinary topology, the relationship between *K*-theory and (rational) cohomology is expressed by the *Chern character*. This naturally leads one to ask whether there is a noncommutative Chern character relating *K*-theory and cyclic (co)homology. The answer (as explained in this chapter) is yes: indeed, Connes's development of cyclic theory was expressly guided by the expectation that a "good" Chern character must exist.

A fruitful source of *K*-theory classes, especially on noncommutative spaces arising from the "noncommutative quotient" construction mentioned earlier, is the *index theory* of elliptic operators. By asking whether *all K*-classes arise from index theory, one arrives at the *Baum-Connes* conjecture, which relates the noncommutative quotient to other "desingularized" quotients such as the Borel construction from homotopy theory. The fourth chapter of the book will leave the reader well prepared to engage this material. Earlier chapters will similarly prepare the reader well to study current work on noncommutative geometry in relation to Hopf algebras, quantum groups, or spectral triples.

This book will be very valuable to students and others seeking an orientation to noncommutative geometry.

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