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Abstract

A kink is a stationary solution to a cubic one-dimensional wave equation $(\partial_t^2 - \partial_x^2)\phi = \phi - \phi^3$ that has different limits when x goes to $-\infty$ and $+\infty$, like $H(x) = \tanh(\frac{x}{\sqrt{2}})$. Asymptotic stability of this solution under small odd perturbation in the energy space has been studied in a recent work of Kowalczyk, Martel and Muñoz. They have been able to show that the perturbation may be written as the sum $a(t)Y(x) + \psi(t,x)$, where Y is a function in Schwartz space, $a(t)$ a function of time having some decay properties at infinity, and $\psi(t,x)$ satisfies some *local in space* dispersive estimate. These results are likely to be optimal when the initial data belong to the energy space. On the other hand, for initial data that are smooth and have some decay at infinity, one may ask if precise dispersive time decay rates for the solution in the whole space-time, and not just for x in a compact set, may be obtained. The goal of this work is to attack these questions.

Our main result gives, for small odd perturbations of the kink that are smooth enough and have some space decay, explicit rates of decay for $a(t)$ and for $\psi(t,x)$ in the whole space-time domain intersected by a strip $|t| \leq \epsilon^{-4+c}$, for any $c > 0$, where ϵ is the size of the initial perturbation. This limitation is due to some new phenomena that appear along lines $x = \pm\frac{\sqrt{2}}{3}t$ that cannot be detected by a local in space analysis. Our method of proof relies on construction of approximate solutions to the equation satisfied by ψ, conjugation of the latter in order to eliminate several potential terms, and normal forms to get rid of problematic contributions in the nonlinearity. We use also Fermi's golden rule in order to prove that the $a(t)Y$ component decays when time grows.

Keywords. Kink, nonlinear Klein–Gordon equations, normal forms, Fermi's golden rule

Mathematics Subject Classification (2020). 35L71, 35B35, 35B40, 35P25, 35S50

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Contents

Chapter 1

Introduction

This book is devoted to the study of dispersive estimates for small perturbations of a stationary solution (the "kink") of a cubic wave equation of the form

$$(\partial_t^2 - \partial_x^2)\phi = \phi - \phi^3,$$

in one space dimension. Before discussing that equation and stating our results, we shall give a general presentation of the framework in which this study lies.

1.1 Long time existence for perturbed evolution equations

The question of long time (or global) existence of solutions to nonlinear dispersive equations, like the wave equation, has been a major line of research for at least the last fifty years. Let us start from the following simple model that encompasses several equations

$$\big(D_t - p(D_x)\big)u = N(u), \tag{1.1}$$

where $u : (t, x) \mapsto u(t, x)$ is a function defined on $I \times \mathbb{R}^d$, with I interval of $\mathbb{R}$, with values in $\mathbb{C}$, where $D_t = \frac{1}{i}\frac{\partial}{\partial t}$, $p(D_x) = \mathcal{F}^{-1}(p(\xi)\hat{u}(\xi))$, $\mathcal{F}^{-1}$ denoting inverse Fourier transform, and where $N(u)$ is some nonlinearity. The function $p(\xi)$ may be equal to

- $p(\xi) = |\xi|$, in which case (1.1) is an half-wave equation,
- $p(\xi) = \sqrt{1 + |\xi|^2}$, corresponding to a half-Klein–Gordon equation,
- $p(\xi) = \frac{1}{2}|\xi|^2$ in the case of a Schrödinger equation.

The right-hand side in (1.1) is a nonlinear expression, that we denote by $N(u)$, though it may contain also factors like $\frac{D_x}{|D_x|}u$, $\frac{D_x}{\langle D_x\rangle}u$, or their conjugates, or even first-order derivatives of u in general. For instance, a Klein–Gordon equation of the form

$$\big(\partial_t^2 - \Delta + 1\big)\phi = F(\phi, \partial_t\phi, \nabla_x\phi) \tag{1.2}$$

with real-valued ϕ, will be reduced to (1.1) defining $u = (D_t + \sqrt{1 + |D_x|^2})\phi$, so that

$$\partial_t\phi = \frac{i}{2}(u - \bar{u}), \quad \nabla_x\phi = \frac{1}{2}\nabla_x\big(1 + |D_x|^2\big)^{-\frac{1}{2}}(u + \bar{u}),$$

and setting

$$N(u) = F\left(\frac{1}{2}\big(1+|D_x|^2\big)^{-\frac{1}{2}}(u+\bar{u}), \frac{i(u-\bar{u})}{2}, \frac{1}{2}\nabla_x\big(1+|D_x|^2\big)^{-\frac{1}{2}}(u+\bar{u})\right), \tag{1.3}$$

which is a non-local nonlinearity. One may proceed in the same way for a quasi-linear version of (1.2), i.e. equations where the right-hand side of (1.2) contains second-order derivatives, and is linear in these second-order derivatives. Then $N(u)$ depends also on first-order derivatives of $(u, \bar{u})$.

When one wants to study long time existence for solutions of equations like (1.1) or (1.2), one of the possible ways is to try to perturb initial data corresponding to a stationary solution, and to show that this perturbation gives rise to a global solution that will remain, for long or all times, close to the stationary solution. Of course, the simplest stationary solution that one may consider is the zero one, in which case one is led to study (1.1) with small initial data. Since the right-hand side vanishes at least at order two at zero, one may hope that it might be considered as an higher-order perturbation.

This framework has been considered by many authors since the mid-seventies, starting with problems of the form (1.1) in higher space dimensions. Let us explain why the question is easier in high space dimensions describing some classical results.

1.2 The use of dispersion

A key point in the study of equations of the form (1.1) is the use of dispersion. Consider first the linear equation $(D_t - p(D_x))u = 0$. Assuming that $p(\xi)$ is real valued, $p(D_x)$ is self-adjoint when acting on L^2 or on Sobolev spaces, so that one has preservation of the Sobolev norms of u along the evolution: $\|u(t,\cdot)\|_{H^s} = \|u(0,\cdot)\|_{H^s}$ for any t. If one considers instead equation (1.1), a Sobolev energy estimate gives just that, as long as the solution exists, one has for any $t \geq 0$,

$$\|u(t,\cdot)\|_{H^s} \leq \|u(0,\cdot)\|_{H^s} + \int_0^t \|N(u)(\tau,\cdot)\|_{H^s}\, d\tau, \tag{1.4}$$

so that one needs, in order to control uniformly the left-hand side, to be able to estimate the integral term on the right-hand side. If one considers a simple model where $N(u)$ is given by $N(u) = P(u, \bar{u})$, where P is an homogeneous polynomial of order $r \geq 2$, one has, for $s > \frac{d}{2}$ where d is the space dimension, a bound

$$\|N(u)\|_{H^s} \leq C\|u\|_{L^\infty}^{r-1}\|u\|_{H^s},$$

so that (1.4) implies

$$\|u(t,\cdot)\|_{H^s} \leq \|u(0,\cdot)\|_{H^s} + C\int_0^t \|u(\tau,\cdot)\|_{L^\infty}^{r-1}\|u(t,\cdot)\|_{H^s}\, d\tau. \tag{1.5}$$

As a consequence, by Gronwall's lemma,

$$\|u(t,\cdot)\|_{H^s} \leq \|u(0,\cdot)\|_{H^s} \exp\Big(C\int_0^t \|u(\tau,\cdot)\|_{L^\infty}^{r-1}\, d\tau\Big). \tag{1.6}$$

One thus sees that, if we want to get a control of $\|u(t,\cdot)\|_{H^s}$ for large t, one needs to obtain as well a priori estimates for $\|u(\tau,\cdot)\|_{L^\infty}$. In particular, to get a uniform global bounds in (1.6), one would need the right-hand side of this inequality to be bounded, i.e. $\int_0^{+\infty}\|u(\tau,\cdot)\|_{L^\infty}^{r-1}\,d\tau < +\infty$.

One may try to guess what are the best estimates one may expect for $\|u(\tau,\cdot)\|_{L^\infty}$ from those holding true for solutions to the linear equation $(D_t - p(D_x))u = 0$. As the solution is given by

$$u(t,x) = \frac{1}{(2\pi)^d}\int e^{itp(\xi)+ix\xi}\hat{u}_0(\xi)\,d\xi \tag{1.7}$$

where $u_0 = u(0,\cdot)$, one sees from the stationary phase formula that if u_0 is smooth enough and has enough decay at infinity, $\|u(t,\cdot)\|_{L^\infty} = O(t^{-\frac{\kappa}{2}})$, where κ depends on the rank of the Hessian of $p(\xi)$. In the case of the wave equation $p(\xi) = |\xi|$, one has $\kappa = d-1$, while for Schrödinger or Klein–Gordon equations (i.e. $p(\xi) = \frac{1}{2}|\xi|^2$ or $p(\xi) = \sqrt{1+|\xi|^2}$), $\kappa = d$. Conjecturing that the same decay will hold for solutions of the nonlinear equation, we would get that the integral on the right-hand side of (1.6) will converge if $\frac{\kappa}{2}(r-1) > 1$, so that if $\frac{d-1}{2}(r-1) > 1$ for the wave equation and $\frac{d}{2}(r-1) > 1$ for the Klein–Gordon or Schrödinger ones.

1.3 Vector fields methods and global solutions

The above heuristics turn out to give a correct answer for nonlinear wave equations if one considers general nonlinearities: actually, in this case, smooth enough decaying initial data of small size give rise to global solutions when $d \geq 4$ if the nonlinearity does not depend on u and is at least quadratic (i.e. $r \geq 2$) as it has been proved by Klainerman [50], Shatah [75], including for quasi-linear nonlinearities. In the same way, for Klein–Gordon equations with quadratic nonlinearities, global existence holds if $d \geq 3$ (see Klainerman [49], Shatah [76]). Moreover, the solutions scatter, i.e. have the same long time asymptotics as the solution of a linear equation.

Let us recall the "Klainerman vector fields method" that provides a powerful way of proving that type of properties. We consider an equation of the form

$$\Box u = f(\partial_t u, \nabla_x u), \tag{1.8}$$

where u is a function of (t,x) in $\mathbb{R}\times\mathbb{R}^d$, $\Box = \partial_t^2 - \Delta_x$ and f is a smooth function vanishing at least at order 2 at the origin. Instead of $\Box$ in the linear part of (1.8), one may more generally take the operator $\sum_{j,k} g^{jk}(\partial_t u, \nabla_x u)\partial_j\partial_k$, where $x_0 = t$ and the coefficients g^{jk} are smooth and satisfy $\sum_{j,k} g^{jk}(0,0)\partial_j\partial_k = \Box$, so that the method is not limited to semilinear equations, but works as well for quasi-linear ones, that is one of its main interests. For the sake of simplification, we shall just discuss (1.8), referring to the original paper of Klainerman [51] and to the book of Hörmander [42] for the more general case. The Sobolev energy inequality applied to (1.8) together with nonlinear estimates for the right-hand side imply that, if $s > \frac{d}{2}$, the energy $E_s(t) = \|\partial_t u(t,\cdot)\|^2_{H^s} + \|\nabla_x u(t,\cdot)\|^2_{H^s}$ satisfies, as long as $\|u'(\tau,\cdot)\|_{L^\infty}$ is bounded,

$$E_s(t)^{\frac{1}{2}} \leq E_s(0)^{\frac{1}{2}} + C\int_0^t \|u'(\tau,\cdot)\|_{L^\infty} E_s(\tau)^{\frac{1}{2}}\,d\tau, \tag{1.9}$$

where we set u' for $(\partial_t u, \nabla_x u)$. This is the analogous of (1.5) for the solution of (1.8) and in order to exploit this estimate, one needs to show that $t \mapsto \|u'(t,\cdot)\|_{L^\infty}$ is integrable. The Klainerman vector fields method allows one to deduce such a property from L^2 estimates for the action of convenient vector fields on u. More precisely, one introduces the Lie algebra of vector fields tangent to the wave cone $t^2 = |x|^2$, generated by

$$\begin{aligned} &t\partial_{x_j} + x_j\partial_t, && j = 1,\dots,d,\\ &x_i\partial_{x_j} - x_j\partial_{x_i}, && 1 \leq i < j \leq d,\\ &t\partial_t + \sum_{j=1}^d x_j\partial_{x_j} \end{aligned} \tag{1.10}$$

and if one denotes by $(Z_i)_{i\in\mathcal{I}}$ the family of fields given by (1.10) or by the usual derivatives $\partial_t, \partial_{x_j}$, $j = 1,\dots,d$, we set, for $I = \{i_1,\dots,i_p\} \subset \mathcal{I}^p$, $Z^I = Z_{i_1}\cdots Z_{i_p}$ and $|I| = p$. Then, as Z^I commutes to $\Box$ by construction (up to a multiple of the equation), one gets from (1.8) essentially

$$\Box Z^I u = Z^I f(\partial_t u, \nabla_x u) \tag{1.11}$$

from which it follows that, if $t \geq 0$,

$$\|Z^I u(t,\cdot)\|_{L^2} \leq \|Z^I u(0,\cdot)\|_{L^2} + \int_0^t \|Z^I f(\partial_t u, \nabla_x u)(\tau,\cdot)\|_{L^2}\,d\tau. \tag{1.12}$$

Using that Z^I is a composition of vector fields, one deduces from Leibniz rule that, setting $u'_N = (Z^I u')_{|I|\leq N}$,

$$\begin{aligned} \|u'_N(t,\cdot)\|_{L^2} \leq \|u'_N(0,\cdot)\|_{L^2} + \int_0^t C\big(\|u'_{N/2}(\tau,\cdot)\|_{L^\infty}\big)&\\ \times \|u'_{N/2}(\tau,\cdot)\|_{L^\infty}\|u'_N(\tau,\cdot)\|_{L^2}\,d\tau.& \end{aligned} \tag{1.13}$$

This is thus an inequality of the form (1.9), and in order to deduce from it an a priori bound for the left-hand side of (1.13), one again needs a dispersive estimate for $\|u'_{N/2}(\tau,\cdot)\|_{L^\infty}$ in $O(\tau^{-\frac{d-1}{2}})$. This estimate follows from the Klainerman–Sobolev inequality

$$(1+|t|+|x|)^{d-1}\big(1+\big||t|-|x|\big|\big)|w(t,x)|^2 \leq C\sum_{|I|\leq\frac{d+2}{2}} \|Z^I w(t,\dots)\|_{L^2} \tag{1.14}$$

for the proof of which we refer for instance to [42, Proposition 6.5.1]. This implies in particular that, if we take N large enough so that $\frac{N}{2} + \frac{d+2}{2} \leq N$, one has for $t \geq 0$,

$$\|u'_{N/2}(t,\cdot)\|_{L^\infty} \leq C(1+t)^{-\frac{d-1}{2}}\|u'_N(t,\cdot)\|_{L^2}. \tag{1.15}$$

One deduces from (1.13) and (1.15) a priori bounds of the form

$$\|u'_N(t,\cdot)\|_{L^2} \leq A\varepsilon, \tag{1.16}$$

$$\|u'_{N/2}(t,\cdot)\|_{L^\infty} \leq B\varepsilon(1+t)^{-\frac{d-1}{2}} \tag{1.17}$$

by a bootstrap argument when $d \geq 4$: If one assumes that (1.16) and (1.17) hold for t in some interval $[0, T]$, one shows that if A, B have been taken large enough in function of the initial data, and if ε is small enough, then (1.16) and (1.17) hold on the same interval with (A, B) replaced by $(\frac{A}{2}, \frac{B}{2})$. One has just to plug (1.16) and (1.17) in (1.13), and to use that $(1+t)^{-\frac{d-1}{2}}$ is integrable in order to prove (1.16) with A replaced by $\frac{A}{2}$. Concerning (1.17) with B replaced by $\frac{B}{2}$, it follows from (1.15) and (1.16) if B is taken large enough with respect to A. Combining these a priori bounds with local existence theory for smooth data shows that solutions are global, for ε small enough, and satisfy (1.16) and (1.17) for any time.

The same type of arguments works more generally when f in (1.8) vanishes at order $r \geq 2$ at zero and $\frac{(d-1)}{2}(r-1) > 1$.

Of special interest is the limiting case of long range nonlinearities when

$$\frac{d-1}{2}(r-1) = 1.$$

This happens in particular if $d = 3, r = 2$, i.e. for quadratic nonlinearities in three space dimension. In this case, one gets in general that data of size $\varepsilon > 0$ give rise to solutions existing over a time interval of length at least $e^{\frac{c}{\varepsilon}}$ for some $c > 0$, but finite time blow-up may occur. Nevertheless, if the solution satisfies a special structure, the so-called "null condition", global existence holds true (see Klainerman [51]). We again refer to the book of Hörmander [42] and references therein for more discussion of long time existence for wave equations, in particular in two space dimension, and to Alinhac [2] for the study of blow-up phenomena when solutions are not global. We also refer to Christodoulou and Klainerman [11] and to Lindblad and Rodnianski [62] for applications to general relativity.

In Section 1.4 we discuss the case of long range nonlinearities for Schrödinger and Klein–Gordon equations in one space dimension, which is the relevant framework for the problem we study in this book. To conclude the present section, let us make some comments on another well known way of exploiting the dispersive character of wave (or other linear) equations, namely Strichartz estimates. The vector fields method that we described above has the advantage of providing explicit decay rates for the solution (and, combined with other arguments, may even furnish precise information on asymptotic behavior of solutions). Moreover, it applies to quasi-linear equations, even if we described it just on a simple semilinear case. On the other hand, it is limited to the study of equations with small and *decaying* data.

When one deals with semilinear equations, and wants to study solutions whose data do not have further decay than being in some Sobolev space, one may instead use Strichartz estimates. Recall that they are given, for a solution u to a linear wave equation,

$$\begin{aligned} (\partial_t^2 - \Delta)u &= F, \\ u(0,\cdot) = u_0, \quad & \partial_t u(0,\cdot) = u_1, \end{aligned} \tag{1.18}$$

defined on $I \times \mathbb{R}^d$, where I is an interval containing 0, by

$$\|u\|_{L_t^q L_x^r(I\times\mathbb{R}^d)} \le C\big(\|u_0\|_{L^2} + \|u_1\|_{\dot{H}^{-1}} + \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(I\times\mathbb{R}^d)}\big), \tag{1.19}$$

where the indices satisfy

$$\begin{aligned} &\frac{1}{\tilde{q}} + \frac{1}{\tilde{q}'} = 1, \quad \frac{1}{\tilde{r}} + \frac{1}{\tilde{r}'} = 1, \quad \frac{1}{q} + \frac{d}{r} = \frac{d}{2}, \quad \frac{1}{\tilde{q}'} + \frac{d}{\tilde{r}'} = \frac{d}{2} + 2, \\ &\frac{1}{q} + \frac{d-1}{2r} \le \frac{d-1}{4}, \quad \frac{1}{\tilde{q}} + \frac{d-1}{2\tilde{r}} \le \frac{d-1}{4}, \\ &(q,r,d) \neq (2,\infty,3), \quad q,r \ge 2,\ r < \infty \\ &(\tilde{q},\tilde{r},d) \neq (2,\infty,3), \quad \tilde{q},\tilde{r} \ge 2,\ \tilde{r} < \infty. \end{aligned} \tag{1.20}$$

We refer to the book of Tao [83] and references therein for the proof. These estimates express both a smoothing and a time decay property of the solution. Because of that, they are useful both in the study of local existence with non-smooth initial data or for global existence and scattering problems in the semilinear case, including for large data. We shall not pursue here on that matter, as this is not the kind of methods we shall use below, since we are more interested in explicit decay rates of solutions. We refer to [83] for some of the many applications of these Strichartz estimates.

1.4 Klainerman–Sobolev estimates in one dimension

The preceding section was devoted to the use of Klainerman vector fields in the framework of wave equations in higher space dimensions. In the present section, we shall focus on the case of (half-)Klein–Gordon or Schrödinger equations in dimension one, as this is the closest framework to our main theorem. As a prerequisite, we shall describe first how (a variant of) the method of Klainerman vector fields allows one to get dispersive decay estimates for solutions when the nonlinearity vanishes at high enough order at initial time. We start with the simplest model of gauge invariant nonlinearities, to which more general equations may be in any case reduces by the normal forms methods we shall discuss later. Denote thus for ξ in $\mathbb{R}$, $p(\xi) = \sqrt{1+\xi^2}$ or $p(\xi) = \frac{\xi^2}{2}$ and consider equation (1.1) with $N(u) = |u|^{2p}u$ with $p \in \mathbb{N}^*$, i.e.

$$\begin{aligned} \big(D_t - p(D_x)\big)u &= \alpha|u|^{2p}u, \\ u|_{t=1} &= u_0, \end{aligned} \tag{1.21}$$

where for convenience of notation we take the initial data at time $t = 1$, α is a complex number and u_0 will be given in a convenient space. One has the following statement.

Theorem 1.4.1. *Let p be larger than or equal to 2 in (1.21). There are s_0, ρ_0 in $\mathbb{N}$ such that, for any $s \geq s_0$, there are $\varepsilon_0 > 0, C > 0$ and for any $\varepsilon \in]0, \varepsilon_0]$, any $u \in H^s(\mathbb{R})$ satisfying*

$$\|u_0\|_{H^s} + \|xu_0\|_{L^2} \leq \varepsilon, \tag{1.22}$$

the solution to (1.21) is global and satisfies for any $t \geq 1$,

$$\|u(t,\cdot)\|_{H^s} \leq C\varepsilon, \quad \|u(t,\cdot)\|_{W^{\rho_0,\infty}} \leq C\frac{\varepsilon}{\sqrt{t}}, \tag{1.23}$$

where $\|w\|_{W^{\rho_0,\infty}} = \|\langle D_x\rangle^{\rho_0} w\|_{L^\infty}$.

We shall present the proof following arguments due to Hayashi and Tsutsumi [40] in the case of Schrödinger equations. For Klein–Gordon equations, the first proof of such a result is due to Klainerman and Ponce [52] and Shatah [75], using a different method. We shall describe here a unified approach for both equations. Notice also that for Klein–Gordon equations, global existence result hold for much more general nonlinearities. We shall give references to that in the forthcoming sections.

Idea of proof of Theorem 1.4.1. We apply the Klainerman vector fields idea, except that instead of using true vector fields, we make use of the operator

$$L_+ = x + tp'(D_x). \tag{1.24}$$

This operator commutes to the linear part of the equation, $[L_+, D_t - p(D_x)] = 0$. Moreover, because the nonlinearity is gauge invariant, a Leibniz rule holds. Actually, in the case of Schrödinger equations, one has a bound

$$\|L_+(|u|^{2p}u)\|_{L^2} \leq C\|u\|_{L^\infty}^{2p}\|L_+u\|_{L^2} \tag{1.25}$$

that follows using that if $p(\xi) = \frac{\xi^2}{2}$, then $L_+ = x + tD_x$ and then

$$\begin{aligned} L_+(|u|^{2p}u) &= L_+(u^{p+1}\bar{u}^p) \\ &= (p+1)(L_+u)|u|^{2p} - pu^{p+1}\bar{u}^{p-1}\overline{L_+u}. \end{aligned}$$

When $p(\xi) = \sqrt{1+\xi^2}$, one has an estimate similar to (1.25) up to replacing the L^∞ norm by a $W^{\rho_0,\infty}$ one, for some large enough ρ_0, and up to some remainders that do not affect the argument below (see [20]). We shall pursue here the argument in the Schrödinger case. Applying L_+ to (1.21) and using the commutation property seen above and (1.25), we obtain

$$\big(D_t - p(D_x)\big)(L_+u) = O_{L^2}\big(\|u\|_{L^\infty}^{2p}\|L_+u\|_{L^2}\big) \tag{1.26}$$

so that one has by L^2 energy inequality

$$\|L_+u(t,\cdot)\|_{L^2} \leq \|L_+u(1,\cdot)\|_{L^2} + C\int_1^t \|u(\tau,\cdot)\|_{L^\infty}^{2p}\|L_+u(\tau,\cdot)\|_{L^2}\,d\tau. \tag{1.27}$$

The proof of the theorem now proceeds with a bootstrap argument: One wants to find constants $A > 0$, $B > 0$ such that

$$\begin{aligned} \|u(t,\cdot)\|_{H^s} &\leq A\varepsilon, \\ \|L_+ u(t,\cdot)\|_{L^2} &\leq A\varepsilon, \\ \|u(t,\cdot)\|_{L^\infty} &\leq B\frac{\varepsilon}{\sqrt{t}} \end{aligned} \tag{1.28}$$

for any $t \geq 1$, as long as $\varepsilon > 0$ is small enough. Assume that these inequalities hold true for t in some interval $[1, T]$. Then, it is enough to show, using equation (1.21), that for t in the same interval $[1, T]$, one has in fact the better estimates

$$\begin{aligned} \|u(t,\cdot)\|_{H^s} &\leq \frac{A}{2}\varepsilon, \\ \|L_+ u(t,\cdot)\|_{L^2} &\leq \frac{A}{2}\varepsilon, \\ \|u(t,\cdot)\|_{L^\infty} &\leq \frac{B}{2}\frac{\varepsilon}{\sqrt{t}}. \end{aligned} \tag{1.29}$$

Actually, estimates (1.28) hold on some interval $[1, T]$ if one has taken A, B large enough, because of assumptions (1.22) made on the initial data, and of Sobolev embedding in order to get the L^∞ bound.

To show that (1.28) implies the first two estimates (1.29), one uses (1.5) (with r replaced by $2p + 1$) and (1.27). Plugging there the a priori bounds (1.28), one gets for any t in $[1, T]$,

$$\begin{aligned} \|u(t,\cdot)\|_{H^s} &\leq \|u_0\|_{H^s} + CB^{2p}A\varepsilon^{2p+1}\int_1^t \tau^{-p}\,d\tau, \\ \|L_+ u(t,\cdot)\|_{L^2} &\leq \|L_+ u(1,\cdot)\|_{H^s} + CB^{2p}A\varepsilon^{2p+1}\int_1^t \tau^{-p}\,d\tau \end{aligned} \tag{1.30}$$

with $p > 1$. Consequently, using assumption (1.22), taking A large enough and ε small enough, one gets the first two inequalities (1.29). To obtain the last one, one uses Klainerman–Sobolev estimates, that allow one to recover an L^∞ bound *with the right time decay* from an L^2 one for $L_+ u$. In the case we are treating $p(\xi) = \frac{\xi^2}{2}$, this is very easy: one writes, by the usual Sobolev embedding

$$\|w\|_{L^\infty} \leq C\|w\|_{L^2}^{\frac{1}{2}}\|D_x w\|_{L^2}^{\frac{1}{2}}.$$

Applying this with $w = e^{i\frac{x^2}{2t}}u(t,\cdot)$, one gets

$$\|u(t,\cdot)\|_{L^\infty} \leq \frac{C}{\sqrt{t}}\|u(t,\cdot)\|_{L^2}^{\frac{1}{2}}\|L_+ u(t,\cdot)\|_{L^2}^{\frac{1}{2}}. \tag{1.31}$$

Plugging the first two inequalities (1.28) inside the right-hand side, one gets

$$\|u(t,\cdot)\|_{L^\infty} \leq \frac{\varepsilon}{\sqrt{t}}CA,$$

which gives the last bound (1.29) if B is chosen large enough relatively to A and concludes the proof. ∎

1.5 The case of long range nonlinearities

In equation (1.21) we limited ourselves to the case $p > 1$, which may be considered as a short range case: actually, if we consider $|u|^{2p}$ as a potential, the time decay of $\|u(t,\cdot)\|_{L^\infty}$ in $t^{-\frac{1}{2}}$ shows that $\||u(t,\cdot)|^{2p}\|_{L^\infty}$ is time integrable at infinity. This played an essential role in order to bound the integrals on the right-hand side of (1.30). Thought, a variant of Theorem 1.4.1 holds as well when $p = 1$:

Theorem 1.5.1. *Let* $p(\xi) = \sqrt{1+\xi^2}$ *or* $p(\xi) = \frac{\xi^2}{2}$ *in one space dimension,* α *a* real *constant. There are* s_0, ρ_0 *in* $\mathbb{N}$, $\delta > 0$ *such that for any* $s \geq s_0$, *there are* $\varepsilon_0 > 0$, $C > 0$ *so that, for any* $\varepsilon \in]0, \varepsilon_0]$, *any* u_0 *in* $H^s(\mathbb{R})$ *satisfying* (1.22)*, the solution of*

$$\begin{aligned} \big(D_t - p(D_x)\big)u &= \alpha|u|^2 u, \\ u|_{t=1} &= u_0 \end{aligned} \tag{1.32}$$

is defined for any $t \geq 1$ *and satisfies there*

$$\|u(t,\cdot)\|_{H^s} \leq C\varepsilon t^\delta, \quad \|u(t,\cdot)\|_{W^{\rho_0,\infty}} \leq C\frac{\varepsilon}{\sqrt{t}}. \tag{1.33}$$

Remarks. We make the following observations.

- A difference between the conclusion of Theorem 1.4.1 and the above statement is that the Sobolev estimate is not uniform: a slight growth in t^δ is possible. Actually, δ may be taken of the form $C\varepsilon^2$ for some constant C.
- The form of the nonlinearity is important, at the difference with the short range case of the preceding section. For instance, one cannot take on the right-hand side of (1.32) for α an arbitrary complex number. The fact that α should be real is an example of a null condition that has to be imposed in order to get global solutions.
- The proof of the theorem provides also modified scattering for u as t goes to infinity.

Let us give some references. For the Schrödinger case, a first proof of Theorem 1.5.1 and of modified scattering of solutions is due to Hayashi and Naumkin [38]. See also Katayama and Tsutsumi [46] and, more recently, Lindblad and Soffer [65], Kato and Pusateri [47] and Ifrim and Tataru [45]. In the case of Klein–Gordon equations, including in the case of quasi-linear nonlinearities satisfying a null condition, we refer to Moriyama, Tonegawa and Tsutsumi [71], Moriyama [70], Delort [18–20], Lindblad and Soffer [63], Lindblad [64] and Stingo [82]. See also Hani, Pausader, Tzvetkov and Visciglia [37] for some further applications.

Before explaining the general strategy of proof of Theorem 1.5.1, let us describe informally how the dispersive estimate in (1.33) will be proved, using an auxiliary

ODE deduced from (1.32). We make this derivation in the case $p(\xi) = \frac{1}{2}\xi^2$, deferring to next paragraph the case of general p. Denote by $\varphi(x) = -\frac{1}{2}x^2$ and look for a solution to (1.32) under the form

$$u(t,x) = \frac{e^{it\varphi(\frac{x}{t})}}{\sqrt{t}} A\Big(t, \frac{x}{t}\Big), \tag{1.34}$$

where $A(t, y)$ is a smooth function. Plugging this Ansatz inside equation (1.32) with $p(D_x) = \frac{1}{2}D_x^2$, one gets

$$D_t A(t,y) = \frac{\alpha}{t}|A(t,y)|^2 A(t,y) + \frac{1}{2t^2} D_y^2 A(t,y). \tag{1.35}$$

If one ignores the last term (that will be proved a posteriori to be a time integrable remainder), one gets that A solves the ODE

$$D_t A(t,y) = \frac{\alpha}{t}|A(t,y)|^2 A(t,y) \tag{1.36}$$

from which follows, as α is real, that $|A(t, y)| = |A(1, y)|$ for all $t \geq 1$, whence

$$A(t,y) = A(1,y) \exp\big(i\alpha|A(1,y)|^2 \log t\big).$$

One thus gets a uniform bound for A, and also discovers that the phase of oscillation of (1.34) involves a logarithmic modification that reflects modified scattering, i.e. one gets when time goes to infinity

$$u(t,x) \sim \frac{1}{\sqrt{t}} A_0\Big(\frac{x}{t}\Big) \exp\bigg(-i\frac{x^2}{2t} + i\alpha\Big|A_0\Big(\frac{x}{t}\Big)\Big|^2 \log t\bigg)$$

for some function A_0. Of course, to establish this rigorously, one has to show that the last term in (1.35) is really a remainder whose addition to the right-hand side of (1.36) does not modify the analysis of asymptotic behavior of solutions.

One may perform such a derivation in a rigorous way using a wave-packets analysis as in Ifrim and Tataru [45] or using a semiclassical approach as we do here. The idea is the following: because of formula (1.34), u appears naturally as a function of t and $\frac{x}{t}$, so that it is natural to write it in terms of a new unknown v by

$$u(t,x) = \frac{1}{\sqrt{t}} v\Big(t, \frac{x}{t}\Big), \tag{1.37}$$

where v will satisfy an equation

$$D_t v - \frac{1}{2t}(x \cdot D_x + D_x \cdot x)v - p\Big(\frac{D_x}{t}\Big)v = \frac{\alpha}{t}|v|^2 v. \tag{1.38}$$

By (1.34), we expect $v(t, x)$ to oscillate like $e^{it\varphi(x)}$. We compute for any smooth function $a(t, x)$,

$$p\Big(\frac{D_x}{t}\Big)\big(e^{it\varphi(x)}a(t,x)\big) = \big(p(\partial_x\varphi(x))a(t,x) + O(t^{-1})\big)e^{it\varphi(x)}.$$

One expects thus that the main contribution to the left-hand side of (1.38) will be obtained replacing $\frac{D_x}{t}$ by $\partial_x \varphi$. This gives an ODE which is nothing but (1.35) if we replace v by $e^{it\varphi(x)} A(t,x)$. In other words, we obtain an ODE allowing us to describe the asymptotics of the solution starting from the quantum problem given by the PDE (1.36) and reducing it to the classical problem obtained making in (1.38) the substitution $\frac{D_x}{t} \mapsto \partial_x \varphi$. We explain below, in the strategy of proof of Theorem 1.5.1, the rigorous way of doing so controlling the errors.

Strategy of proof of Theorem 1.5.1. The starting point of the proof is the same as for Theorem 1.4.1, except that the inequalities to be bootstrapped read now as

$$\begin{aligned} \|u(t,\cdot)\|_{H^s} &\leq A\varepsilon t^{\delta}, \\ \|L_+ u(t,\cdot)\|_{L^2} &\leq A\varepsilon t^{\delta}, \\ \|u(t,\cdot)\|_{W^{\rho_0,\infty}} &\leq B\frac{\varepsilon}{\sqrt{t}} \end{aligned} \tag{1.39}$$

instead of (1.28), with $\delta > 0$ a small number. Again, one has (1.30) with $p = 1$ and the integral term replaced by $\int_1^t \tau^{-1+\delta}\, d\tau \leq \delta^{-1} t^{\delta}$. If $\varepsilon^2 \delta^{-1}$ is small enough, one deduces from (1.30) that the first two inequalities in (1.39) actually hold with A replaced by $\frac{A}{2}$. On the other hand, one cannot deduce the L^∞ estimate in (1.39) from the Sobolev and L^2 ones using (1.31), as the lack of uniformity in the estimate of $\|L_+ u(t,\cdot)\|_{L^2}$ would just provide a bound in $O(t^{-\frac{1}{2}+0})$ instead of $O(t^{-\frac{1}{2}})$. On thus needs an extra argument to obtain the L^∞ estimates (since the L^2 ones cannot be expected to be improved). There have been several approaches to do so, that all rely on the derivation from the PDE (1.32) of an ODE, that may be used in order to get the optimal L^∞ decay (and the asymptotics of the solution). That ODE may be written either on the solution itself or on its Fourier transform (actually on the profile $e^{itp(\xi)}\hat{u}(t,\xi)$ of the Fourier transform). As indicated in the preceding paragraph, the method we shall use in this book, inspired in part from the approach of Ifrim and Tataru [45] based on wave packets, relies on a semiclassical version of the equation satisfied by a rescaled unknown.

We introduce as a semiclassical parameter $h = \frac{1}{t} \in]0,1]$ and define from the unknown u the new unknown v through (1.37). If we denote $\|v\|_{H_h^s} = \|\langle hD_x\rangle^s v\|_{L^2}$, then $\|u(t,\cdot)\|_{H^s} = \|v(t,\cdot)\|_{H_h^s}$. The last estimate in (1.39) is equivalent to getting an $O(\varepsilon)$ bound for $\|\langle hD_x\rangle^{\rho_0} v(t,\cdot)\|_{L^\infty}$. Plugging (1.37) inside (1.32), one gets

$$\big(D_t - \mathrm{Op}_h^{\mathrm{W}}(x\xi + p(\xi))\big)v = h\alpha|v|^2 v, \tag{1.40}$$

where the semiclassical Weyl quantization $\mathrm{Op}_h^{\mathrm{W}}$ associates to a "symbol" $a(x,\xi)$ the operator

$$v \mapsto \mathrm{Op}_h^{\mathrm{W}}(a)v = \frac{1}{2\pi h}\int e^{i(x-y)\frac{\xi}{h}} a\Big(\frac{x+y}{2},\xi\Big) v(y)\, dy\, d\xi. \tag{1.41}$$

The above formula makes sense for more general functions a than the one

$$a(x,\xi) = x\xi + p(\xi)$$

appearing in (1.40). We do not give here these precise assumptions, referring to Appendix D below. Let us just remark that one may translate the action of operator L_+ on u by

$$L_+u(t,x) = \frac{1}{\sqrt{t}}(\mathcal{L}_+v)\Big(t,\frac{x}{t}\Big) \tag{1.42}$$

with

$$\mathcal{L}_+ = \frac{1}{h}\mathrm{Op}_h^{\mathrm{W}}(x + p'(\xi)) \tag{1.43}$$

so that the second a priori assumption (1.39) may be translated as

$$\|\mathcal{L}_+v\|_{L^2} = O(\varepsilon h^{-\delta}). \tag{1.44}$$

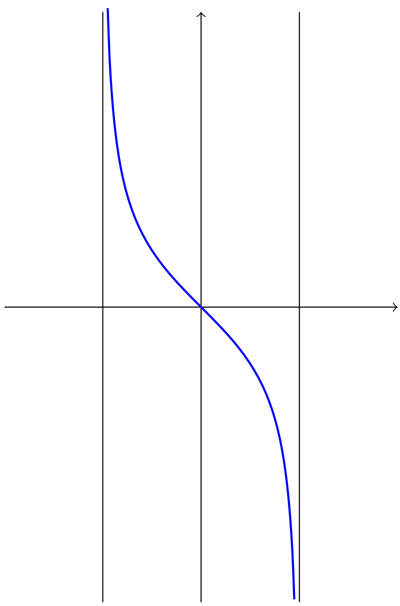
This brings us to introduce the submanifold

$$\Lambda = \{(x,\xi) \in \mathbb{R}\times\mathbb{R} : x + p'(\xi) = 0\} \tag{1.45}$$

that is actually the graph

$$\Lambda = \{(x, d\varphi(x)) : x \in]-1,1[\} \quad \text{with } \varphi(x) = \sqrt{1-x^2} \tag{1.46}$$

given by the following picture.



The idea is to deduce from (1.40) an ODE restricting the symbol $x\xi + p(\xi)$ to Λ. By (1.46) and a direct computation, $(x\xi + p(\xi))|_\Lambda = \varphi(x)$, so that we would want to deduce from (1.40) an ODE of the form

$$\big(D_t - \varphi(x)\big)w = h\alpha|w|^2w + R, \tag{1.47}$$

where w should be conveniently related to v and R being a remainder such that

$$\int_1^{+\infty} \|R(t,\cdot)\|_{W_h^{\rho_0,\infty}}\, dt = O(\varepsilon).$$

We notice first that the a priori bound (1.44) provides a uniform estimate for v cut-off outside a $\sqrt{h}$-neighborhood of Λ. The idea is as follows:

First, contributions to v cut-off for high frequencies have nice bounds if we assume the first a priori estimate (1.39): actually, it implies

$$\|\langle hD_x\rangle^s v(t,\cdot)\|_{L^2} = O(\varepsilon h^{-\delta}),$$

so that if $\chi \in C_0^\infty(\mathbb{R})$ is equal to one close to zero, $\beta > 0$ is small and $s_0 > \frac{1}{2}$, one gets by semiclassical Sobolev estimate

$$\begin{aligned} \|\mathrm{Op}_h^{\mathrm{W}}\big(\chi(h^\beta \xi)\big)v\|_{L^\infty} &\le Ch^{-\frac{1}{2}}\|\langle hD_x\rangle^{s_0}\mathrm{Op}_h^{\mathrm{W}}\big(\chi(h^\beta\xi)\big)v\|_{L^2} \\ &\le Ch^{-\frac{1}{2}+\beta(s-s_0)}\|\langle hD_x\rangle^s v\|_{L^2} \\ &\le C\varepsilon h^{-\frac{1}{2}-\delta+\beta(s-s_0)}. \end{aligned} \tag{1.48}$$

Consequently, for any fixed N in $\mathbb{N}$, if $s\beta$ is large enough, we get an $O(\varepsilon h^N)$ bound for estimate (1.48). This shows that one may assume essentially that $\hat{v}$ is supported for $h^\beta|\xi| \le C$ for some constant, some small $\beta > 0$. In the rest of this section, in order to avoid technicalities, we shall argue as if we had actually $|\xi| \le C$. The case $h^\beta|\xi| \le C$ may be treated similarly, up to an extra loss $h^{-\beta'}$ in the estimates of the remainders, $\beta' > 0$ being as small as we want. This extra loss does not affect the general pattern of the reasoning.

Take γ in $C_0^\infty(\mathbb{R})$, equal to one close to zero, with small enough support, and decompose

$$v = \underline{v}_\Lambda + \underline{v}_{\Lambda^c}, \tag{1.49}$$

where

$$\underline{v}_\Lambda = \mathrm{Op}_h^{\mathrm{W}}\Big(\gamma\Big(\frac{x+p'(\xi)}{\sqrt{h}}\Big)\Big)v,\ \underline{v}_{\Lambda^c} = \mathrm{Op}_h^{\mathrm{W}}\Big((1-\gamma)\Big(\frac{x+p'(\xi)}{\sqrt{h}}\Big)\Big)v, \tag{1.50}$$

i.e. $\underline{v}_\Lambda$ (resp. $\underline{v}_{\Lambda^c}$) is the contribution to v that is microlocally located inside (resp. outside) a $\sqrt{h}$-neighborhood of Λ. Then $\underline{v}_{\Lambda^c}$ satisfies, as a consequence of the L^2 estimate (1.44), a uniform L^∞ bound: define $\gamma_1(z) = \frac{(1-\gamma)(z)}{z}$ and write

$$\begin{aligned} \underline{v}_{\Lambda^c} &= \mathrm{Op}_h^{\mathrm{W}}\Big(\gamma_1\Big(\frac{x+p'(\xi)}{\sqrt{h}}\Big)\Big(\frac{x+p'(\xi)}{\sqrt{h}}\Big)\Big)v \\ &= h^{\frac{1}{2}}\mathrm{Op}_h^{\mathrm{W}}\Big(\gamma_1\Big(\frac{x+p'(\xi)}{\sqrt{h}}\Big)\Big)(\mathcal{L}_+v) + \text{remainder}. \end{aligned} \tag{1.51}$$

Since, at fixed x, $\xi \mapsto \gamma_1((x+p'(\xi))/\sqrt{h})$ is supported inside an interval of length $O(\sqrt{h})$, one may show that the L^∞ norm of the first term on the right-hand side of (1.51) is essentially bounded from above by $h^{-\frac{1}{4}}$ times its L^2 norm, i.e.

$$\|\underline{v}_{\Lambda^c}\|_{L^\infty} \leq C h^{\frac{1}{4}} \|\mathcal{L}_+ v\|_{L^2}. \tag{1.52}$$

(Actually, if one takes into account the fact that on the support of $\hat{v}$ one has $|\xi| \leq ch^{-\beta}$ instead of $|\xi| \leq C$, one would get a power $h^{\frac{1}{4}-\beta'}$ instead of $h^{\frac{1}{4}}$, for some $0 < \beta' \ll 1$ in (1.52), that would not change the estimates below). In any case, combining with (1.44), we get an estimate

$$\|\underline{v}_{\Lambda^c}\|_{L^\infty} = O(\varepsilon h^{\frac{1}{4}-\delta'}), \quad \delta' > 0 \text{ small}. \tag{1.53}$$

If we assume a uniform a priori bound for v (that follows from the third inequality (1.39) and from (1.37)), we see that (1.53) implies that the difference $|v|^2 v - |\underline{v}_\Lambda|^2 \underline{v}_\Lambda$ will be $O(\varepsilon^3 h^{\frac{1}{4}-\delta'})$, so that replacing on the right-hand side of equation (1.40) $h|v|^2 v$ by $h|\underline{v}_\Lambda|^2 \underline{v}_\Lambda$ induces an error of the form of R in (1.47), i.e. we have

$$\big(D_t - \mathrm{Op}_h^{\mathrm{W}}(x\xi + p(\xi))\big)v = h\alpha|\underline{v}_\Lambda|^2 \underline{v}_\Lambda + R. \tag{1.54}$$

We make act next $\mathrm{Op}_h^{\mathrm{W}}(\gamma((x + p'(\xi))/\sqrt{h}))$ on that equality. We get at the left-hand side $(D_t - \mathrm{Op}_h^{\mathrm{W}}(x\xi + p(\xi)))\underline{v}_\Lambda$ and a commutator whose principal contribution may be written as

$$-\frac{h^{\frac{3}{2}}}{i}\mathrm{Op}_h^{\mathrm{W}}\Big(\gamma'\Big(\frac{x + p'(\xi)}{\sqrt{h}}\Big)\Big)(\mathcal{L}_+ v). \tag{1.55}$$

This is of the same form as (1.51), up to an extra h factor, so that, arguing as in (1.52) and (1.53), we bound the L^∞ norm of (1.55) by $C\varepsilon h^{\frac{5}{4}-\delta'} = C\varepsilon t^{-\frac{5}{4}+\delta'}$. As $\delta' > 0$ is small, this is an integrable quantity that may enter in the remainders on the right-hand side of (1.47). As the action of $\mathrm{Op}_h^{\mathrm{W}}(\gamma((x + p'(\xi))/\sqrt{h}))$ on the right-hand side of (1.54) may be written under the same form, up to a modification of the remainder, we get

$$\big(D_t - \mathrm{Op}_h^{\mathrm{W}}(x\xi + p(\xi))\big)\underline{v}_\Lambda = h\alpha|\underline{v}_\Lambda|^2 \underline{v}_\Lambda + R. \tag{1.56}$$

We make now a Taylor expansion of $x\xi + p(\xi)$ on Λ given by (1.45) and (1.46). As $\frac{d}{d\xi}(x\xi + p(\xi))|_\Lambda = 0$, we get

$$x\xi + p(\xi) = \varphi(x) + O\big((x + p'(\xi))^2\big). \tag{1.57}$$

The action of $\mathrm{Op}_h^{\mathrm{W}}((x + p'(\xi))^2)$ on $\underline{v}_\Lambda$ may be written essentially as (1.55), so provides again a contribution to R in (1.56). Finally, plugging (1.57) inside (1.56), we see that we get an equation of the form (1.47) for $w = \underline{v}_\Lambda$. This implies in particular that $\frac{\partial}{\partial t}|\underline{v}_\Lambda(t,\cdot)|^2$ is time integrable (since the coefficient α in (1.56) is real) and thus that $\|\underline{v}_\Lambda(t,\cdot)\|_{L^\infty}$ is bounded. Coming back to the expression (1.37) of u in terms of $v = \underline{v}_\Lambda + \underline{v}_{\Lambda^c}$, remembering (1.53) and adjusting constants, one gets that the a priori assumptions (1.39) imply that the last inequality in these formulas holds true with B replaced by $\frac{B}{2}$ (the reasoning for $W^{\rho_0,\infty}$ norms instead of L^∞ ones being similar). This shows that the bootstrap argument holds. Moreover, the ODE (1.47) may be used also in order to get asymptotics for u when times goes to infinity. ∎

1.6 More general nonlinearities and normal forms

In model (1.32), we considered only a special case of nonlinearity namely $\alpha|u|^2u$. We used this special structure in order to get a Leibniz type rule (see (1.25)). However, we know that we should be able to obtain global solutions even for (some) cubic or quadratic nonlinearities that have a more general form. This is done in [18, 19] for quasi-linear Klein–Gordon equations with a nonlinearity satisfying a null condition (see also Stingo [82]). One makes use of "real" Klainerman vector fields instead of the operator L_+ above. On the other hand, for other equations like Schrödinger ones, the natural operator to be used in order to exploit dispersion is an operator like L_+, that is not a vector field. It is possible to reconcile both points of view using normal forms. Moreover, the use of the latter allows also one to treat quadratic nonlinearities. Consider as a model

$$\begin{aligned}\big(D_t - p(D_x)\big)u &= \alpha_0 u^2 + \alpha|u|^2u,\\ u|_{t=1} &= u_0,\end{aligned} \tag{1.58}$$

where $p(\xi) = \sqrt{1+\xi^2}$, α_0 is a complex number and α a real one. We would like to prove the analogous of Theorem 1.5.1, namely:

Theorem 1.6.1. *There are s_0, ρ_0 in $\mathbb{N}$, $\delta > 0$ such that, for any $s \geq s_0$, there are $\varepsilon_0 > 0$, $C > 0$ so that, for any $\varepsilon \in]0, \varepsilon_0]$, any u_0 in $H^s(\mathbb{R})$ satisfying (1.22), the solution of (1.58) is global and satisfies for any $t \geq 1$,*

$$\|u(t,\cdot)\|_{H^s} \leq C\varepsilon t^\delta, \quad \|u(t,\cdot)\|_{W^{\rho_0,\infty}} \leq C\frac{\varepsilon}{\sqrt{t}}. \tag{1.59}$$

Remarks. We make the following observations.

- Again, one can obtain also the asymptotics of the solution when t goes to infinity, and in particular show modified scattering, and not just the dispersive estimate (1.59).
- One may consider more general quadratic and cubic nonlinearities than on the right-hand side of the first equation in model (1.58), as soon as they satisfy the null condition (see [18, 19, 82]).

The key idea of the proof is essentially to reduce (1.58) to (1.32) by normal forms. One cannot expect to get directly energy estimates on (1.58): for instance, the quadratic part of the nonlinearity has Sobolev norm bounded from above by $C\|u(t,\cdot)\|_{L^\infty}\|u(t,\cdot)\|_{H^s}$, so taking into account the a priori L^∞ estimate in (1.39), by $(C\varepsilon/\sqrt{t})\|u(t,\cdot)\|_{H^s}$. One thus would get an inequality of the form (1.6) with $r = 2$, which would give only exponential time control. Though, as shown first by Shatah [76] and Simon and Taflin [77], one may easily reduce the quadratic nonlinearity in (1.58) to a cubic one.

Lemma 1.6.2. *Define*

$$m(\xi_1, \xi_2) = \Big(\sqrt{1+\xi_1^2} + \sqrt{1+\xi_2^2} - \sqrt{1+(\xi_1+\xi_2)^2}\Big)^{-1}.$$

Then $m(\xi_1, \xi_2)$ is well defined,

$$|m(\xi_1, \xi_2)| \leq C\big(1 + \min(|\xi_1|, |\xi_2|)\big) \tag{1.60}$$

and if one sets

$$\mathrm{Op}(m)(u_1, u_2) = \frac{1}{(2\pi)^2} \int e^{ix(\xi_1+\xi_2)} m(\xi_1, \xi_2) \hat{u}_1(\xi_1) \hat{u}_2(\xi_2)\, d\xi_1\, d\xi_2, \tag{1.61}$$

one has for a fixed ρ_0 and any large enough s,

$$\|\mathrm{Op}(m)(u_1, u_2)\|_{H^s} \leq C\big(\|u_1\|_{W^{\rho_0,\infty}} \|u_2\|_{H^s} + \|u_1\|_{H^s} \|u_2\|_{W^{\rho_0,\infty}}\big). \tag{1.62}$$

Moreover, the map given by $u \mapsto u - \mathrm{Op}(m)(u, u)$ is a diffeomorphism from the open set $H^s \cap \{u \in W^{\rho_0,\infty} : \|u\|_{W^{\rho_0,\infty}} < r\}$ to its image, for small enough r, and if u is in that set, and solves equation (1.58), then $w = u - \mathrm{Op}(m)(u, u)$ solves

$$\big(D_t - p(D_x)\big)w = \alpha|w|^2 w - 2\alpha_0 \mathrm{Op}(m)(w^2, w) + R(w), \tag{1.63}$$

where R is a sum of contributions of degree of homogeneity larger than or equal to 4.

Proof. Estimate (1.60) follows by an immediate computation. It implies that one does not lose derivatives when applying $\mathrm{Op}(m)$ to a couple (u_1, u_2), i.e. that (1.62) holds without losing on s on the right-hand side. This allows one to construct the local diffeomorphism $u \mapsto w$. When one makes act $D_t - p(D_x)$ on $\mathrm{Op}(m)(u, u)$, one gets using equation (1.58), on the one hand

$$\mathrm{Op}(m)\big(p(D_x)u, u\big) + \mathrm{Op}(m)\big(u, p(D_x)u\big) - p(D_x)\mathrm{Op}(m)(u, u) \tag{1.64}$$

which, because of the definition of m is equal to u^2, and on the other hand contributions of the form

$$\mathrm{Op}(m)\big(\alpha_0 u^2 + \alpha|u|^2 u, u\big),\ \mathrm{Op}(m)\big(u, \alpha_0 u^2 + \alpha|u|^2 u\big). \tag{1.65}$$

If we compute the left-hand side of (1.63), we thus see that (1.64) compensates the quadratic term, and that we are left on the right-hand side with the $|u|^2 u$ term and expressions of the form (1.65). If we express u in terms of $w = u - \mathrm{Op}(m)(u, u)$, we shall get the cubic terms on the right-hand side of equation (1.63), and higher-order terms $R(w)$. These higher-order contributions are essentially of the form

$$R_k = \mathrm{Op}(m_k)(w, \dots, w, \bar{w}, \dots, \bar{w})$$

with $k \geq 4$, $m_k = m_k(\xi_1, \dots, \xi_k)$ a smooth function satisfying convenient estimates, and R_k defined as in (1.61) from

$$\begin{aligned}\mathrm{Op}(m_k)(u_1, \dots, u_k) = \frac{1}{(2\pi)^k} \int e^{ix(\xi_1+\dots+\xi_k)} m_k(\xi_1, \dots, \xi_k) \\ \times \hat{u}_1(\xi_1) \dots \hat{u}_k(\xi_k)\, d\xi_1 \cdots d\xi_k.\end{aligned} \tag{1.66}$$

Then $R(w)$ satisfies estimates of the form

$$\|R(w)\|_{H^s} \leq C\|w\|^3_{W^{\rho_0,\infty}}\|w\|_{H^s} \tag{1.67}$$

if w stays in some ball of $W^{\rho_0,\infty}$, i.e. plays the role of a perturbation that is at least quartic. ∎

The preceding lemma thus reduces the case of a quadratic nonlinearity to a cubic one. Of course, the cubic term on the right-hand side of (1.63) is non-local, but this is not a real extra difficulty. Because of that, in order not no be disturbed by unessential technicalities, we shall pursue the reasoning considering a simple variant of (1.63), namely

$$\big(D_t - p(D_x)\big)u = \alpha|u|^2u + \alpha_1 u^3 + \alpha_2 u^2\bar{u}^2 \tag{1.68}$$

with α real, α_1, α_2 complex, forgetting any contribution homogeneous of order larger than or equal to 5 that is in any case easier to treat. Moreover, the special structure of the nonlinear terms on the right-hand side does not matter *except* the fact that α is real.

We have already noticed that a term like u^3 is not compatible with the action of L_+ on the right-hand side. The same holds for $u^2\bar{u}^2$. In order to get around that difficulty, one may try to perform a normal form in order to get rid of cubic or quartic terms. Nevertheless, unlike the quadratic case, one my not eliminate all these terms. Actually, to get rid of $u^2\bar{u}^2$ for instance, one would have to introduce a new unknown of the form $u - \mathrm{Op}(m_4)(u, u, \bar{u}, \bar{u})$, where m_4 would be the inverse of

$$-\sqrt{1+\xi_1^2} - \sqrt{1+\xi_2^2} + \sqrt{1+\xi_3^2} + \sqrt{1+\xi_4^2} - \sqrt{1+(\xi_1+\cdots+\xi_4)^2}. \tag{1.69}$$

But such a quantity vanishes for some values of $(\xi_1, \dots, \xi_4)$. The idea to overcome that difficulty is to use "space-time normal forms" introduced by Germain, Masmoudi and Shatah in [29–32], and Germain and Masmoudi [28] (see also the review paper of Lannes [58] and the works of Hu and Masmoudi [44], Deng, Ionescu, Pausader and Pusateri [21], Wang [84] and Deng and Pusateri [22] for further applications and extensions of the method). These authors define and use space-time normal forms on the profiles of the solutions, namely the functions $e^{-itp(D_x)}u$. Here, we present an equivalent approach based on u itself and on microlocal cut-offs similar to those introduced in (1.50), following [20]. Actually, introducing again from u the unknown v given by (1.37), we rewrite (1.68) as

$$\big(D_t - \mathrm{Op}_h^{\mathrm{W}}\big(x\xi + p(\xi)\big)\big)v = h\alpha|v|^2v + h\alpha_1 v^3 + h^{\frac{3}{2}}\alpha_2 v^2\bar{v}^2 \tag{1.70}$$

using notation (1.41). The idea of space-time normal forms may be described in a geometrical way as follows. As we have seen above, a term like $v^2\bar{v}^2$ in (1.70) may not be fully eliminated by a usual (time) normal form since (1.69) may vanish for some values of the arguments. Though, we have seen in (1.34) that v defined by (1.37) is expected to be a function oscillating as $e^{i\frac{\varphi(x)}{h}}$, which means that we expect that v is "concentrated" on the manifold Λ defined in (1.45), (1.46). This means that, up to remainders having better time decay, we should hope to be able to design a normal form eliminating the term $v^2\bar{v}^2$ of (1.70) as soon as (1.69) does not vanish when the frequencies ξ_1, ξ_2 (corresponding to v) are set equal to $d\varphi(x)$ (by characterization (1.46) of Λ) and ξ_3, ξ_4 (corresponding to $\bar{v}$) are set equal to $-d\varphi(x)$. Notice that restricted to this subset, (1.69) is just equal to -1, so does not vanish. Of course, in order to justify that, we need to explain how we may reduce ourselves to the fact that v may be replaced by a function that is frequency localized on Λ, up to convenient remainders, and show how this allows one to prove energy estimates for the solution of (1.70). Our goal will thus be to prove the following:

Proposition 1.6.3. *The solution v of (1.70) satisfies estimates of the form*

$$\begin{aligned}\|v(t,\cdot)\|_{H^s_h} \leq \|v(1,\cdot)\|_{H^s} + C\int_1^t \|v(\tau,\cdot)\|^2_{W^{\rho_0,\infty}_{h(\tau)}}\big(1+\|v(\tau,\cdot)\|_{W^{\rho_0,\infty}_{h(\tau)}}\big)&\\ \times\|v(\tau,\cdot)\|_{H^s_{h(\tau)}}\frac{d\tau}{\tau}&\end{aligned} \tag{1.71}$$

and

$$\begin{aligned}\|\mathcal{L}_+v(t,\cdot)\|_{L^2} \leq \|\mathcal{L}_+v(1,\cdot)\|_{H^s}&\\ + C\int_1^t \|v(\tau,\cdot)\|^2_{W^{\rho_0,\infty}_{h(\tau)}}\big(1+\|v(\tau,\cdot)\|_{W^{\rho_0,\infty}_{h(\tau)}}\big)&\\ \times\|\mathcal{L}_+v(\tau,\cdot)\|_{L^2}\frac{d\tau}{\tau},&\end{aligned} \tag{1.72}$$

where $h = \frac{1}{t}$, $h(\tau) = \frac{1}{\tau}$, $\|v\|_{H^s_h} = \|\langle hD_x\rangle^s v\|_{L^2}$, $\|v\|_{W^{\rho_0,\infty}_h} = \|\langle hD_x\rangle^{\rho_0} v\|_{L^\infty}$ and $\mathcal{L}_+$ is defined in (1.43).

Remark. These estimates are the translation on v of bounds of the form (1.5) and (1.27) on u according to (1.37). Consequently, if we prove them, we shall get, as in the proof of Theorem 1.5.1, that an a priori set of inequalities of the form (1.39) will imply that the first two of these bounds hold with A replaced by $\frac{A}{2}$.

Proof of the proposition. As indicated before the statement, in order to get (1.71) and (1.72), we have to perform a "space-time" normal form. More precisely, we shall decompose in the $v^3, v^2\bar{v}^2$ terms of (1.70) each factor v as

$$v = \underline{v}_\Lambda + \underline{v}_{\Lambda^c}, \tag{1.73}$$

where $\underline{v}_{\Lambda^c}$ will have better bounds than v, so that cubic or quartic terms involving at least one factor $\underline{v}_{\Lambda^c}$ will provide remainders. In a second step, we shall get rid of the remaining nonlinearities $\alpha_1\underline{v}_\Lambda^3$, $\alpha_2\underline{v}_\Lambda^2\bar{\underline{v}}_\Lambda^2$ by a normal form process. The function $\underline{v}_\Lambda$ in (1.73) will be defined as in (1.49), except that we cut-off around an $O(1)$-neighborhood of Λ instead of an $O(\sqrt{h})$ one, i.e. we define now

$$\underline{v}_\Lambda = \mathrm{Op}_h^{\mathrm{W}}\big(\gamma(x+p'(\xi))\big)v,\ \underline{v}_{\Lambda^c} = \mathrm{Op}_h^{\mathrm{W}}\big((1-\gamma)(x+p'(\xi))\big)v. \tag{1.74}$$

(Actually, the above definition is the correct one when the frequency ξ stays in a compact set. It should be adapted for large ξ, but we forget this technical detail in this introduction.) Then $\underline{v}_{\Lambda^c}$ will satisfy estimates with an $O(h)$ gain, as we may write it essentially under the form

$$\underline{v}_{\Lambda^c} = h\mathrm{Op}_h^{\mathrm{W}}\big(\gamma_1(x + p'(\xi))\big)(\mathcal{L}_+ v), \tag{1.75}$$

where $\gamma_1(z) = \frac{(1-\gamma)(z)}{z}$, so that

$$\|\underline{v}_{\Lambda^c}\|_{L^2} \leq Ch\|\mathcal{L}_+ v\|_{L^2}.$$

Decomposing on the right-hand side of (1.70) $v = \underline{v}_\Lambda + \underline{v}_{\Lambda^c}$, one has thus

$$\big(D_t - \mathrm{Op}_h^{\mathrm{W}}(x\xi + p(\xi))\big)v = h\alpha|v|^2 v + h\alpha_1(\underline{v}_\Lambda)^3 + h^{\frac{3}{2}}\alpha_2 \underline{v}_\Lambda^2 \bar{\underline{v}}_\Lambda^2 + h^2 S(v), \tag{1.76}$$

where $S(v)$, coming from monomials involving at least one factor $\underline{v}_{\Lambda^c}$, satisfies an estimate of the form

$$\|S(v)\|_{L^2} \leq C\|v\|_{L^\infty}^2 \|\mathcal{L}v\|_{L^2} \tag{1.77}$$

as long as $\|v\|_{L^\infty}$ stays bounded. Actually, one has to be more careful when making the above estimates, since Λ has a degeneracy when ξ goes to infinity. The preceding reasoning works for $|\xi|$ staying in a compact set , or equivalently for x staying in a compact subset of $]-1, 1[$. The general case is a little bit more involved, and in particular estimate (1.77) holds with $\|v\|_{L^\infty}$ replaced by $\|v\|_{W_h^{\rho_0,\infty}}$ for some ρ_0.

Since making act the operator $\mathcal{L}_+$ on S makes lose a factor h^{-1} (see the definition (1.43) of $\mathcal{L}_+$), we get that

$$h^2\|\mathcal{L}_+ S(v)\|_{L^2} \leq Ch\|v\|_{L^\infty}^2 \|\mathcal{L}_+ v\|_{L^2}, \tag{1.78}$$

which will be the kind of estimate we want for remainders. By (1.25) with $p = 1$, rewritten in terms of the unknown v, we have also

$$h\|\mathcal{L}_+(|v|^2 v)\|_{L^2} \leq Ch\|v\|_{L^\infty}^2 \|\mathcal{L}_+ v\|_{L^2}. \tag{1.79}$$

On the other hand, the remaining contributions on the right-hand side of (1.77) would not satisfy such estimates, but may now be eliminated by normal forms. Actually, take χ in $C_0^\infty(\mathbb{R})$, equal to one close to zero, and define

$$\begin{aligned} m_4(x, \xi_1, \dots, \xi_4) = \prod_{j=1}^{2} \chi(x + p'(\xi_j)) \prod_{j=3}^{4} \chi(x - p'(\xi_j)) \\ \times \Big[-\sqrt{1 + \xi_1^2} - \sqrt{1 + \xi_2^2} + \sqrt{1 + \xi_3^2} \\ + \sqrt{1 + \xi_4^2} - \sqrt{1 + (\xi_1 + \dots + \xi_4)^2}\Big]^{-1}. \end{aligned} \tag{1.80}$$

This function is well defined, as the term inside the bracket does not vanish on the support of the cut-off: actually (again forgetting what happens for large frequencies), on the support of the cut-off, $\xi_j = d\varphi(x) + o(1)$, $j = 1, 2$, $\xi_j = -d\varphi(x) + o(1)$, $j = 3, 4$, so that the term inside the bracket is equal to $-1 + o(1)$, and thus does not vanish. Consequently, if we define

$$\begin{aligned}
&\mathrm{Op}_h(m_4)\big(\underline{v}_\Lambda, \underline{v}_\Lambda, \bar{\underline{v}}_\Lambda, \bar{\underline{v}}_\Lambda\big) \\
&\quad = \frac{1}{(2\pi)^4} \int e^{ix(\xi_1+\cdots+\xi_4)} m_4(\xi_1, \dots, \xi_4) \\
&\qquad\qquad \times \hat{\underline{v}}_\Lambda(\xi_1)\hat{\underline{v}}_\Lambda(\xi_2)\widehat{\bar{\underline{v}}}_\Lambda(\xi_3)\widehat{\bar{\underline{v}}}_\Lambda(\xi_4)\, d\xi_1 \cdots d\xi_4,
\end{aligned} \tag{1.81}$$

one obtains that

$$\big(D_t - \mathrm{Op}_h^{\mathrm{W}}(x\xi + \sqrt{1+\xi^2})\big)\big(\mathrm{Op}_h(m_4)(\underline{v}_\Lambda, \dots, \bar{\underline{v}}_\Lambda)\big) = \underline{v}_\Lambda^2 \bar{\underline{v}}_\Lambda^2 + \text{remainder},$$

where the remainder, coming from the nonlinearities of the equation, contains at least one h factor. Defining in the same way some cubic symbol m_3, in order to eliminate the $\underline{v}_\Lambda^3$ term in (1.76), one gets that

$$\begin{aligned}
&\big(D_t - \mathrm{Op}_h^{\mathrm{W}}(x\xi + \sqrt{1+\xi^2})\big)\big(v - h\mathrm{Op}_h(m_3)(\underline{v}_\Lambda, \underline{v}_\Lambda, \underline{v}_\Lambda) \\
&\qquad - h^{\frac{3}{2}}\mathrm{Op}_h(m_4)(\underline{v}_\Lambda, \dots, \bar{\underline{v}}_\Lambda)\big) = h^2 S(v) + h\alpha|v|^2 v
\end{aligned} \tag{1.82}$$

for a new $S(v)$ satisfying (1.77).

In other words, we have reduced ourselves to an equation where the right-hand side has the same structure as in (1.7) (up to changing the unknown u to v by (1.37)), modulo a remainder $h^2 S(v)$ that has better time decay. Using estimates of the form (1.78)–(1.79), one thus gets, applying L^2 energy inequalities to (1.82) and denoting

$$w = v - h\mathrm{Op}_h(m_3)(\underline{v}_\Lambda, \underline{v}_\Lambda, \underline{v}_\Lambda) - h^{\frac{3}{2}}\mathrm{Op}_h(m_4)(\underline{v}_\Lambda, \dots, \bar{\underline{v}}_\Lambda),$$

that

$$\|\mathcal{L}_+ w(t,\cdot)\|_{L^2} \le \|\mathcal{L}_+ w(1,\cdot)\|_{L^2} + \int_1^t \|v(\tau,\cdot)\|_{L^\infty}^2 \|\mathcal{L}_+ v(\tau,\cdot)\|_{L^2} \frac{d\tau}{\tau}. \tag{1.83}$$

As one may show that $\|\mathcal{L}_+ w(t,\cdot)\|_{L^2}$ is equivalent to $\|\mathcal{L}_+ v(t,\cdot)\|_{L^2}$, one does get an estimate of the form (1.72). ∎

Remark. As already mentioned, in the proof of Proposition 1.6.3, we argued as if the frequencies were staying in a compact set. When one makes the reasoning taking into account what happens also for large frequencies, one gets a lower bound of the bracket in (1.80) computed for ξ_j in a convenient neighborhood of $\pm d\varphi(x)$ by a negative power of $\langle d\varphi(x)\rangle$. Since for all j, $\langle d\varphi(x)\rangle \sim \langle \xi_j \rangle$ if $(\xi_1, \dots, \xi_4)$ is in the support of (1.80), one may write $\langle d\varphi(x)\rangle \sim 1 + \max_2(|\xi_1|, \dots, |\xi_4|)$, and the bounds one gets in general for a symbol of the form m_4 is

$$|m_4(x, \xi_1, \dots, \xi_4)| \le C\big(1 + \max_2(|\xi_1|, \dots, |\xi_4|)\big)^{N_0} \tag{1.84}$$

for some N_0. Because of that, one gets bounds of type

$$\|\text{Op}_h(m_4)(v, \dots, \bar{v})\|_{H_h^s} \leq C\|v\|^3_{W_h^{\rho_0,\infty}} \|v\|_{H_h^s} \tag{1.85}$$

for any s and with ρ_0 depending only on N_0. In other words, coming back to the unknown u, one obtains an estimate similar to (1.62). These inequalities (1.84) and (1.85) explain why one gets in Proposition 1.6.3 upper bounds involving $W_h^{\rho_0,\infty}$ norms instead of L^∞ ones.

End of proof of Theorem 1.6.1. As for the proof of Theorem 1.5.1, one has just to bootstrap estimates (1.39), showing that if they hold on some time interval and A, B have been taken large enough and ε small enough, then they still hold with A, B replaced by $\frac{A}{2}, \frac{B}{2}$. We have seen after the statement of Proposition 1.6.3 that this holds for the first two inequalities (1.39). To show that the last one holds, with B replaced by $\frac{B}{2}$, one argues as in the proof of Theorem 1.5.1. Actually, in that proof, we did not really use the special form of the nonlinearity in (1.40) (except the fact that α is real), and the same arguments hold for an equation like (1.68). ■

1.7 Perturbations of non-zero stationary solution

Our main goal in this book is to study the perturbation of a non-zero stationary solution of a cubic wave equation in dimension one. In this section, we mention some results and references on that kind of problems. The first set of questions one may ask is the *orbital* stability of stationary solutions.

Let us mention first the result of Henry, Perez and Wreszinski [41] that will be very relevant for us. Consider U a C^2 function on an interval $[a_-, a_+]$ satisfying $U \geq 0$, $U(a_-) = U(a_+) = 0$, $U''(a_\pm) > 0$. Assume moreover that there is a smooth strictly increasing function $x \mapsto H(x)$ solving the equation

$$H''(x) = U'(H(x))$$

such that

$$\lim_{x\to\pm\infty} H(x) = a_\pm$$

and that

$$E_0 = \int_{\mathbb{R}} \Big(\frac{H'(x)^2}{2} + U(H(x))\Big)\, dx < +\infty.$$

Define for any function ϕ and any $q > 0$,

$$d_q(\phi) = \inf_{c\in\mathbb{R}} \int_{\mathbb{R}} \big((\phi'(x) - H'(x+c))^2 + q(\phi(x) - H(x+c))^2\big)\, dx.$$

One may state the main result of [41] as follows.

Theorem 1.7.1. *There are positive constants r, q, k such that if $(t,x) \mapsto \phi(t,x)$ is the solution of*

$$\left(\partial_t^2 - \partial_x^2\right)\phi + U'(\phi) = 0 \tag{1.86}$$

satisfying $\phi(0,\cdot) \in H^1_{\text{loc}}(\mathbb{R})$, $\partial_x\phi(0,\cdot), \partial_t\phi(0,\cdot) \in L^2(\mathbb{R})$, and

$$\begin{gathered} d_q(\phi(0,\cdot)) < r, \\ \int_{\mathbb{R}} \left(\frac{\partial_t\phi(0,x)^2}{2} + \frac{\partial_x\phi(0,x)^2}{2} + U(\phi(0,x)) \right) dx < E_0 + kr^2, \end{gathered} \tag{1.87}$$

then ϕ is globally defined and for any t

$$d_q(\phi(t,\cdot)) \leq r. \tag{1.88}$$

This theorem means that H is orbitally stable, in that sense that an initial data that is close enough to H gives rise to a solution that remains at any time close to a translation of H. It applies in particular to $U(\phi) = \frac{1}{4}(\phi^2 - 1)^2$, $H(x) = \tanh(\frac{x}{\sqrt{2}})$ and $a_\pm = \pm 1$, i.e. it shows the orbital stability of the “kink”, that is the stationary solution $H(x) = \tanh(\frac{x}{\sqrt{2}})$ of the Φ^4 model given by the equation

$$\left(\partial_t^2 - \partial_x^2\right)\phi = \phi - \phi^3. \tag{1.89}$$

The question of orbital stability has been then widely studied for other equations. In particular, we refer to Weinstein [86] for orbital stability of Schrödinger or generalized KdV equations. References to earlier works on that topic may be found in the reference list of that paper.

Once orbital stability is established for a given equation, the next step is to study asymptotic stability. For Schrödinger equations, the first results are due to Buslaev and Perelman [5–7] in dimension one and to Soffer and Weinstein [78] in higher dimension. Buslaev and Perelman consider a one-dimensional Schrödinger equation, of the form

$$i\partial_t\psi = -\partial_x^2\psi + F(|\psi|^2)\psi. \tag{1.90}$$

Under convenient assumptions on F, one may construct soliton solutions of the equation, that have the form

$$e^{-i\beta_0 - it\omega_0 + \frac{i}{2}xv_0}\phi(x - b_0 - tv_0) \tag{1.91}$$

for constants $\beta_0, \omega_0, b_0, v_0$ and where ϕ is a smooth exponentially decaying function. The main result of the above references is that if one solves the initial value problem for (1.90), with initial condition close to the preceding soliton solution, then the solution may be written when time goes to infinity as a sum of a modified soliton, i.e. a function of the form (1.91) (with different values of the parameters $\beta_0, \dots, v_0$), of a solution to a linear Schrödinger equation and of a remainder that converges to zero in L^2.

In the work of Soffer and Weinstein, one introduces a potential in the linear part of the operator, i.e. one considers an equation of the form

$$i\partial_t \phi = -\Delta\phi + (V(x) + \lambda|\phi|^{m-1})\phi \tag{1.92}$$

in $d = 2$ or 3 space dimension, and for $1 < m < \frac{d+2}{d-2}$. They assume, among other things, that the operator $-\Delta + V(x)$ has exactly one eigenvalue, that is moreover strictly negative. They show that for E close to that eigenvalue, there is a solution of (1.92) of the form $e^{-iEt}\psi_E(x)$, with ψ_E smooth and decaying. Then, under some further assumption, they prove that, if one solves the Cauchy problem starting from an initial data that is close to $e^{i\gamma_0}\psi_{E_0}$, for given E_0 close to the eigenvalue, γ_0 real, then the solution may be written at any time t as $e(t)\psi_{E(t)} + R(t)$ where $E(t)$ is real, $e(t)$ is in the unit circle of $\mathbb{C}$ and $R(t)$ goes to zero in a weighted Sobolev space. We refer to [78] for a precise description of the asymptotics of $t \mapsto E(t), e(t)$ when time goes to infinity.

Following the above references, a lot of results concerning asymptotic stability for solutions to nonlinear Schrödinger equations or Gross–Pitaevsky ones have been obtained. Limiting ourselves to one-dimensional problems, and without trying to give an exhaustive list of references, one may cite Buslaev and Sulem [8], Bethuel, Gravejat and Smets [4], Gravejat and Smets [36], Germain, Pusateri and Rousset [35], Cuccagna and Pelinovski [16], Cuccagna and Jenkins [15], Gang and Sigal [25–27], Cuccagna, Georgiev and Visciglia [14]. Still in one space dimension, analogous results have been obtained for (generalized) KdV equations, by Pego and Weinstein [73], Germain, Pusateri and Rousset [34], Martel and Merle [67–69] and for Benjamin–Ono equation by Kenig and Martel [48]. Let us point out that for Schrödinger or gKdV equations, the perturbation of the initial data induces a non-zero translation speed on the stationary solution, so that the perturbed solution is the sum of a *progressive wave* and of a dispersive part. This will be in contrast with the results we shall obtain in this book, where the bound state that is perturbed will remain stationary.

Let us discuss now some results more closely related to our work, concerning nonlinear wave equations. A main breakthrough has been made by Soffer and Weinstein who in [79] consider an equation similar to (1.92), but where the Schrödinger operator is replaced by the wave (or Klein–Gordon) one in three space dimension, namely

$$\partial_t^2 \phi = (\Delta - V(x) - m^2)\phi + \lambda\phi^3, \tag{1.93}$$

where λ is some real constant, $m > 0$ and V is a smooth decaying potential. One assumes among other things that $-\Delta + V + m^2$ has $[m^2, +\infty[$ as continuous spectrum and that there is a unique positive eigenvalue $0 < \Omega^2 < m^2$. One denotes by φ a normalized eigenfunction associated to that eigenvalue, so that for any R, θ in $\mathbb{R}$, $(t, x) \mapsto R\cos(\Omega t + \theta)\varphi(x)$ is a solution to equation (1.93) when $\lambda = 0$. The main result of [79] asserts that if one solves (1.93) with small initial data in weighted Sobolev spaces of smooth enough and decaying enough functions, the solution at time t may be written under the form

$$\phi(t,x) = R(t)\cos(\Omega t + \theta(t))\varphi(x) + \eta(t,x), \tag{1.94}$$

where $R(t) = O(|t|^{-\frac{1}{4}})$ and $\|\eta(t,\cdot)\|_{L^8} = O(|t|^{-\frac{3}{4}})$ when t goes to $\pm\infty$. This result holds under a special non-resonance condition, Fermi's golden rule, that we shall further discuss below in the framework of our problem.

The above breakthrough has been at the origin of many other works. Let us mention in particular Bambusi and Cuccagna [3] who generalized the result of [80] to a wider framework, namely the case when the operator $-\Delta + V(x) + m^2$ has several eigenvalues instead of just one. Closer to our main result in this book, let us mention the work where Cuccagna [13] studies asymptotic stability of a kink solution in three space dimension. More precisely, one considers the solution H of (1.89) as a solution independent of two of the three space variables of the equation $(\partial_t^2 - \Delta)\phi = \phi - \phi^3$ on $\mathbb{R}^3$. The main result of [13] asserts that if one starts from initial data that are a small perturbation of $(H, 0)$ by a smooth compactly supported function on $\mathbb{R}^3$, then the solution of the evolution equation may be written as $H + \phi(t,\cdot)$, where $\phi(t,\cdot)$ is $O(|t|^{-\frac{1}{2}})$ in L^∞. The proof uses the fact that in three space dimension, one has much better dispersive decay than on the real line.

1.8 The kink problem. I

The main goal of this book is to study long time dispersion for small perturbations of the "kink" $H(x) = \tanh(\frac{x}{\sqrt{2}})$ that is a stationary solution of equation (1.89) that we recall below

$$(\partial_t^2 - \partial_x^2)\phi = \phi - \phi^3.$$

We have seen in the preceding section (see Theorem 1.7.1) that H is orbitally stable, and one wants to study its asymptotic stability. In order to do so, one writes ϕ under the form

$$\phi(t,x) = H(x) + \varphi(t\sqrt{2}, x\sqrt{2}) \tag{1.95}$$

and we aim at describing the asymptotics of φ, in particular its dispersive properties, when at initial time φ is small in a convenient weighted Sobolev space. By Theorem 1.7.1, we know that φ is globally defined. It satisfies by direct computation the equation

$$\big(D_t^2 - (D_x^2 + 1 + 2V(x))\big)\varphi = \kappa(x)\varphi^2 + \frac{1}{2}\varphi^3, \tag{1.96}$$

where

$$V(x) = -\frac{3}{4}\cosh^{-2}\Big(\frac{x}{2}\Big), \quad \kappa(x) = \frac{3}{2}\tanh\frac{x}{2}. \tag{1.97}$$

The fact that the linear part of equation (1.96) contains a non-zero potential has two consequences: first, as seen in the preceding section, the operator $D_x^2 + 1 + 2V(x)$ may have bound states (and it has for the potential given by (1.97)). Second, even in the absence of bound states, that operator does not have nice commutation properties with the operator L_+ that we used in order to get dispersion in Sections 1.5 and 1.6.

Let us first discuss some results that are known concerning equations of the form (1.96) either in the case of potentials without bound states, or for equations of that form with $V = 0$ but where the nonlinearities have coefficients that are non- constant functions of x, as on the right-hand side of (1.97). Such results have been proved by Kopylova [53] for linear Klein–Gordon equations in a moving frame and, in the nonlinear case, by Lindblad and Soffer [66], Lindblad, Lührmann and Soffer [60,61], Lindblad, Lührmann, Schlag and Soffer [59], Sterbenz [81]. Very recently, Germain and Pusateri [33] obtained the most general result in that framework. They consider a model version of (1.96) of the form

$$\big(\partial_t^2 - \partial_x^2 + V(x) + m^2\big)\varphi = a(x)\varphi^2, \tag{1.98}$$

where $a(x)$ is a function similar to κ on the right-hand side of (1.96), i.e. a smooth function that has finite limits at $\pm\infty$ and whose derivative is rapidly decaying. The potential V is assumed to be Schwartz and such that $-\partial_x^2 + V$ has no bound state. One of the results of [33] may be stated as follows:

Theorem 1.8.1. *Let V be a* generic *potential without bound state, $m > 0$. There is $\varepsilon_0 > 0$ such that for any $\varepsilon \in]0, \varepsilon_0]$, equation (1.97) has for any (φ_0, φ_1) satisfying*

$$\big\|\big(\sqrt{-\partial_x^2 + V + 1}\varphi_0, \varphi_1\big)\big\|_{H^4} + \big\|\langle x\rangle\big(\sqrt{-\partial_x^2 + V + 1}\varphi_0, \varphi_1\big)\big\|_{H^1} \le \varepsilon$$

a unique global solution corresponding to the initial data $\varphi|_{t=0} = \varphi_0$, $\partial_t\varphi|_{t=0} = \varphi_1$. Moreover, the dispersive estimate

$$\big\|\big(\sqrt{-\partial_x^2 + V + 1}\varphi_0, \varphi_1\big)\big\|_{L^\infty} \le C\varepsilon(1 + |t|)^{-\frac{1}{2}} \tag{1.99}$$

holds and for some small $\delta > 0$

$$\|\varphi(t,\cdot)\|_{H^5} + \|\partial_t\varphi(t,\cdot)\|_{H^4} \le C\varepsilon(1 + |t|)^{\delta}. \tag{1.100}$$

Finally, let us mention that for nonlinearities with coefficients that are *rapidly enough decaying* in x, Lindblad, Lührmann and Soffer [60] (in the case $V \equiv 0$) and Lindblad, Lührmann, Schlag and Soffer [59] (for generic potentials) could show that a dispersive bound like (1.99) *does not* hold in general, and has to be replaced by the product of the right-hand side with a logarithmic loss.

Remark. The assumption that V is generic is explained in Chapter 2 below. The result of [33] is actually more general than Theorem 1.8.1 above. It also applies to non-generic potentials if one makes in addition evenness/oddness assumptions. Let us also mention that the question of asymptotic stability estimates on a compact domain in space, when the linearized equation on the stationary solution has no bound state, has been addressed by Kowalczyk, Martel, Muñoz and Van Den Bosch [57] for some models of semilinear wave equations.

Let us explain the new difficulties one has to take into account to prove a result of the form above in comparison with the case $V \equiv 0$. Clearly, if one wanted to apply the operator

$$L_{+,m} = x + t\frac{D_x}{(m^2 + D_x^2)^{\frac{1}{2}}}$$

(or a "true" Klainerman vector field like $t\partial_x + x\partial_t$) to equation (1.97), its commutator with the potential V would generate a new term with coefficients growing like t, which makes the method inapplicable. In order to circumvent such a difficulty, two approaches are possible. The one implemented by Germain and Pusateri relies on the use of the "modified Fourier transform", which is a version of the Fourier transform well adapted to $-\Delta + V$ instead of being tailored to $-\Delta$. They introduce then the profile g of the solution by

$$g(t,x) = e^{it\sqrt{-\partial_x^2+V+m^2}}\Big(\partial_t - i\sqrt{-\partial_x^2 + V + m^2}\Big)\phi \tag{1.101}$$

and its modified Fourier transform $\tilde{g}(t,\xi)$. The analogue of what does work in the case $V \equiv 0$ would be to get estimates of $\|\partial_\xi \tilde{g}(t,\xi)\|_{L^2}$ (which is related to $\|L_{+,m}\phi\|_{L^2}$ when $V \equiv 0$). It turns out that, in order to get the most general statement of their paper, Germain and Pusateri have to introduce a bigger space than L^2 in which $\partial_\xi \tilde{g}$ has to be estimated, allowing for some degeneracy close to a special frequency. They have then to combine estimates in that space with normal forms constructed from the modified Fourier transform.

The approach we use in this book is the one of wave operators. Let us just say here that, when V is a potential in $\mathcal{S}(\mathbb{R})$, without bound states, one may construct a bounded operator W_+ on L^2 such that

$$W_+^* W_+ = \mathrm{Id}, \quad W_+ W_+^* = \mathrm{Id} \quad \text{and} \quad W_+^*(-\Delta + V)W_+ = -\Delta.$$

Applying W_+^* to (1.98), one thus gets

$$\big(\partial_t^2 - \partial_x^2 + m^2\big)W_+^*\varphi = W_+^*(a(x)\varphi^2).$$

If $w = W_+^*\varphi$, one is thus reduced to an equation of the form

$$\big(\partial_t^2 - \partial_x^2 + m^2\big)w = W_+^*(a(x)(W_+ w)^2), \tag{1.102}$$

i.e. to an equation for which the linear part has again constant coefficients, and thus has nice commutation properties relatively to $t\partial_x + x\partial_t$ or to $L_{+,m}$. Of course, the drawback is that the right-hand side of (1.102) is no longer a local nonlinearity, but involves the operators W_+, W_+^*. In the framework we shall be interested in, namely odd initial conditions and odd coefficient $a(x)$, it turns out that W_+, W_+^* may be expressed from pseudo-differential operators $b(x, D_x)$, with a symbol $b(x,\xi)$ such that $\frac{\partial b}{\partial x}(x,\xi)$ is rapidly decaying when $|x|$ tends to infinity. We shall explain in more detail in Chapter 2 how we treat an equation of the form (1.102). Let us just say now that if we had a cubic nonlinearity on the right-hand side, one could use directly vector fields methods on w. For a quadratic nonlinearity, one has to make use first of normal forms in order to reduce quadratic nonlinearities to cubic ones. The difference with Lemma 1.6.2 is that, because of the presence of W_+, W_+^*, $a(x)$ on the right-hand side of (1.102), one has to consider quadratic corrections of the form (1.61), but with a symbol $m(x, \xi_1, \xi_2)$ that depends also on x. This introduces new commutators, involving quadratic operators associated to the symbol $\frac{\partial m}{\partial x}(x, \xi_1, \xi_2)$. Though, as the latter is rapidly decaying in x, and since we limit ourselves to odd solutions, such terms form remainders that are not fully negligible, but that may be treated more easily than in the more general case considered by Germain and Pusateri [33] or Lindblad, Lührmann and Soffer [60].

1.9 The kink problem II. Coupling with the bound state

In the preceding section, we discussed an equation of the form (1.98) with a potential V that has no bound state. In this section, we go back to the kink problem (1.96), where the potential V given by (1.97) does have bound states, so that the preceding discussion does not apply.

Our starting point has been the paper [56] of Kowalczyk, Martel and Muñoz, where the authors study the asymptotics of solutions of (1.89) when one takes as an initial condition an odd perturbation of $(H, 0)$ that is small enough in the energy norm. They prove that the perturbation of the solution $(\varphi, \partial_t\varphi)$ may be decomposed under the form

$$(\varphi(t,x), \partial_t\varphi(t,x)) = (u_1(t,x), u_2(t,x)) + (z_1(t), z_2(t))Y(x), \tag{1.103}$$

where Y is in $\mathcal{S}(\mathbb{R})$ and is a normalized *odd* eigenfunction of $-\frac{1}{2}\partial_x^2 + V(x)$, $z_j(t)$ are scalar functions of time and $(u_1(t,x), u_2(t,x))$ is the dispersive part of the solution. The main result of [56] states that the functions $t \mapsto z_j(t)$ decay in time in the sense that

$$\int_{-\infty}^{+\infty} \big(|z_1(t)|^4 + |z_2(t)|^4\big)\, dt < +\infty$$

and that the *local* energy of (u_1, u_2) satisfies

$$\int_{-\infty}^{+\infty} \int_{\mathbb{R}} \big((\partial_x u_1)^2 + u_1^2 + u_2^2\big)(t,x) e^{-c_0|x|}\, dt\, dx < +\infty.$$

At the light of the discussion previously given in the case of small perturbations of the zero solution of nonlinear Klein–Gordon equations, or for (1.98) with a potential that has no bound state, the above inequalities raise the following questions: making eventually stronger assumptions on the smoothness/decay of the initial perturbation, could one get an explicit decay rate for the preceding quantities, instead of just integral inequalities? Moreover, could one obtain decay estimates for $\|u_j(t,\cdot)\|_{L^\infty}$ instead of just local in space decay?

A more long term objective might be to obtain for odd perturbations of the kink solution of (1.89) a description as precise as the one that holds when $V \equiv 0$ or when V is a potential without bound state. We are far from being able to achieve that in this paper, where as a first step we aim at describing the perturbed solution up to time ε^{-4} if ε is the small size of the smooth decaying perturbation of the kink at initial time. Recall that if we look for solutions of (1.89) under the form (1.95), we get that the perturbation φ satisfies (1.96), with notation (1.97). We already mentioned that the Schrödinger operator $-\partial_x^2 + 2V(x)$ has discrete spectrum: it has two negative eigenvalues -1 and $-\frac{1}{4}$ and absolutely continuous spectrum $[0, +\infty[$. Eigenvalue -1 will not be of interest to us as it is associated to an even eigenfunction, while we solve (1.96) for odd initial data. Consequently, restricting ourselves to odd solutions, one may decompose the solution of (1.96) as $\varphi = P_{\mathrm{ac}}\varphi + \langle \varphi, Y\rangle Y$, where P_{ac} is the projector on the absolutely continuous spectrum $[0, +\infty[$ and Y is an (odd) normalized eigenfunction associated to eigenvalue $-\frac{1}{4}$. Setting $a(t) = \langle Y, \varphi\rangle$, one may deduce from (1.96) that $(a, P_{\mathrm{ac}}\varphi)$ satisfies a coupled system of ODE/PDE (see (2.9) in Chapter 2).

Our main result asserts the following: Let $c > 0$ be given and consider (1.96) with initial data $\varphi|_{t=1} = \varepsilon\varphi_0$, $\partial_t\varphi|_{t=1} = \varepsilon\varphi_1$ with (φ_0, φ_1) satisfying for some large enough s,

$$\|\varphi_0\|^2_{H^{s+1}} + \|\varphi_1\|^2_{H^s} + \|x\varphi_0\|^2_{H^1} + \|\varphi_1\|^2_{L^2} \le 1. \tag{1.104}$$

Then, if $\varepsilon < \varepsilon_0$ is small enough, the decomposition $\varphi(t,\cdot) = P_{\mathrm{ac}}\varphi(t,\cdot) + a(t)Y$ of the solution of (1.96) satisfies

$$\begin{aligned} |a(t)| + |a'(t)| &= O(\varepsilon(1 + t\varepsilon^2)^{-\frac{1}{2}}),\\ \|P_{\mathrm{ac}}\varphi(t,\cdot)\|_{L^\infty} &= O(t^{-\frac{1}{2}}(\varepsilon^2\sqrt{t})^{\theta'}), \end{aligned} \tag{1.105}$$

where $\theta' \in]0, \frac{1}{2}[$, *as long as* $t \le \varepsilon^{-4+c}$. Let us mention that we limit our study to positive times (that does not reduce generality) and that, in order to simplify some notation, we take the Cauchy data at $t = 1$ instead of $t = 0$. Moreover, the statements we get in Theorem 2.1.1 below give more precise information that (1.105). We just stress here the fact that (1.105) provides the information we are looking for, namely an explicit decay rate for a and $P_{\mathrm{ac}}\varphi$, up to time ε^{-4+c}.

We notice that the dispersive estimate obtained for $\|P_{\mathrm{ac}}\varphi\|_{L^\infty}$ is pretty similar to the bound in $\varepsilon t^{-\frac{1}{2}}$ that holds for small solutions of equations $(\partial_t^2 - \partial_x^2 + 1)u = N(u)$. Here, when $t \le \varepsilon^{-4+c}$, we get that

$$\|P_{\mathrm{ac}}\varphi\|_{L^\infty} = O(\varepsilon^{\frac{c}{2}\theta'}t^{-\frac{1}{2}}),$$

i.e. an estimate in $c(\varepsilon)t^{-\frac{1}{2}}$, with $c(\varepsilon)$ going to zero with zero. Of course, if t goes close to ε^{-4}, the small factor in front of $t^{-\frac{1}{2}}$ in the second estimate (1.105) gets closer and closer to one, and this explains why our result is limited to times that are $O(\varepsilon^{-4+c})$. We shall comment more on that below.

Let us remark also that for dispersive estimates of the form (1.105), there is a "trivial" regime, corresponding to $t \le c\varepsilon^{-2}$. For such times, the ODE satisfied by $a(t)$, from which we shall deduce the first bound (1.105), is in a small time regime, before any singularity could form. On the other hand, to reach a time of size ε^{-4+0}, one has to use the structure of that ODE, namely exploit Fermi's golden rule that we shall discuss in Chapter 2 below, in order to exclude blowing up in finite time, and prove the decay estimate (1.105).

Let us comment more on the limitation to times $t = O(\varepsilon^{-4+0})$ which contrasts with the fact that, when the potential has no bound state, one may obtain dispersive estimates up to infinity. The new difficulty, when bound states are present, comes from the fact that in (1.105), $a(t)$ and $a'(t)$ have a decay in $\frac{\varepsilon}{(1+t\varepsilon^2)^{1/2}}$, which is larger than the rate in $\frac{\varepsilon}{\sqrt{t}}$ that holds for dispersive bounds in the absence of eigenvalues. This has consequences on the estimates satisfied by the dispersive part of the solution $P_{\mathrm{ac}}\varphi(t,\cdot)$. Actually, applying P_{ac} to equation (1.96), one will get an equation that, at first glance, might seem pretty similar to (1.98), since on the range of P_{ac}, $-\partial_x^2 + 2V$ will have no bound state. Though, a major difference appears on the right-hand side: if, for instance, one plugs in the quadratic term of (1.96) the decomposition $\varphi(t,\cdot) = P_{\mathrm{ac}}\varphi(t,\cdot) + a(t)Y$, one might get a source term

$$a(t)^2 P_{\mathrm{ac}}(\kappa(x)Y^2), \tag{1.106}$$

where $a(t)$ has only an $O(\frac{1}{\sqrt{t}})$ decay for $t \gg \varepsilon^{-2}$ (ant not a $\frac{\varepsilon}{\sqrt{t}}$ bound). This has dramatic consequences on the solution to the equation itself. Actually, the solution $P_{\mathrm{ac}}\varphi$ will have to encompass the solution of the linear equation

$$\big(D_t^2 - (D_x^2 + 1 + 2V(x))\big)w = a(t)^2 P_{\mathrm{ac}}(\kappa(x)Y^2)$$

with zero initial data. We shall solve this equation, but will be able to obtain for its solution only a bound in $t^{-\frac{1}{2}}(\varepsilon^2\sqrt{t})^{\theta'}$ for $t \le \varepsilon^{-4+0}$ and some $\theta' > 0$. When doing so, we are not able to obtain $O(t^{-\frac{1}{2}})$ bounds for w along two lines

$$\frac{x}{t} = \pm\sqrt{\frac{2}{3}}$$

when $t \gg \varepsilon^{-4}$. Actually, one might expect a logarithmic loss along these two lines, similar to the ones in the work of Lindblad, Lührmann and Soffer [60] and Lindblad, Lührmann, Schlag and Soffer [59].

Let us also stress on the fact that, besides (1.106), other new terms appear in comparison to the case of potentials without bound states. For instance, a contribution like $P_{\mathrm{ac}}(\kappa(x)(P_{\mathrm{ac}}\varphi)a(t)Y)$ needs also a specific treatment, as it is not amenable to standard normal forms treatment. We describe that in more detail in Section 2.7 of Chapter 2.

To conclude this introduction, let us point out the results of Kopylova and Komech in [54, 55] concerning asymptotic stability of a (moving) kink for a modified version of (1.89). In their model, the Hamiltonian of the equation is tuned in such a way that the projection of equation (1.96) on the absolutely continuous spectrum has coefficients in the nonlinearity that decay when x goes to infinity (instead of converging to some constant) This allows the authors to obtain a description of the dispersive behavior of the corresponding solution for *any* time.

Finally, let us refer to the recent paper of Chen, Liu and Lu [10] concerning asymptotic stability of kinks for sine-Gordon equations. Using the integrability of that equation, they may prove soliton resolution for generic data and show the full asymptotic stability of kinks under space decaying perturbations (see Corollary 1.5 of their paper). In particular, the difference between the solution and the moving kink is shown to decompose, when time goes to infinity, as the sum of an $O(t^{-\frac{1}{2}})$ contribution that involves a logarithmic phase correction and of a more decaying remainder.

Chapter 2

The kink problem

2.1 Statement of the main result

Consider $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a global solution to the nonlinear wave equation

$$(\partial_t^2 - \partial_x^2)\phi = \phi - \phi^3. \tag{2.1}$$

The function

$$H(x) = \tanh\Big(\frac{x}{\sqrt{2}}\Big) \tag{2.2}$$

is a stationary solution of (2.1), and we are interested in describing the dispersive behaviour in large time of solutions to (2.1) corresponding to initial data that are small, smooth, odd and decaying perturbations of the state H. It is known that this state is orbitally stable in the energy space by Henry, Perez and Wreszinski [41], and for odd perturbations in that space, asymptotic stability with space exponential weight is proved by Kowalczyk, Martel and Muñoz [56]. This result describes the dispersive behaviour of the perturbation on compact space domains, but does not give insight into its behaviour in the whole space time. Our goal is to obtain information when (t, x) describes $I_\varepsilon \times \mathbb{R}$, where I_ε is a time interval of length $O(\varepsilon^{-4+0})$, ε being the size of the initial data in a convenient space of smooth decaying functions.

We shall look for solutions to (2.1) under the form

$$\phi(t, x) = H(x) + \varphi\big(t\sqrt{2}, x\sqrt{2}\big). \tag{2.3}$$

We get for φ the equation

$$\big(D_t^2 - (D_x^2 + 1 + 2V(x))\big)\varphi = \kappa(x)\varphi^2 + \frac{1}{2}\varphi^3, \tag{2.4}$$

where $D_t = \frac{1}{i}\frac{\partial}{\partial t}$, $D_x = \frac{1}{i}\frac{\partial}{\partial x}$ and

$$V(x) = -\frac{3}{4}\cosh^{-2}\Big(\frac{x}{2}\Big), \quad \kappa(x) = \frac{3}{2}\tanh\Big(\frac{x}{2}\Big). \tag{2.5}$$

The operator $-\partial_x^2 + 2V$ has $[0, +\infty[$ as its continuous spectrum and has two eigenvalues -1 and $-\frac{1}{4}$. The first one is associated to an even eigenfunction, and the second one to the odd normalized eigenfunction

$$Y(x) = \frac{\sqrt{3}}{2}\tanh\Big(\frac{x}{2}\Big)\cosh^{-1}\Big(\frac{x}{2}\Big) \tag{2.6}$$

(see Nikiforov and Uvarov [72] and Kowalczyk, Martel and Muñoz [56]).

We denote by P_{ac} the spectral projector on the continuous spectrum, restricted to odd functions. The spectral projector on the eigenspace associated to the eigenvalue $-\frac{1}{4}$ is $\varphi \mapsto \langle \varphi, Y\rangle Y$ so that

$$P_{\rm ac}\varphi = \varphi - \langle \varphi, Y\rangle Y, \tag{2.7}$$

where $\langle\,\cdot\,,\cdot\,\rangle$ denotes the L^2 scalar product. If φ solves (2.4), we set

$$a(t) = \langle \varphi, Y\rangle \tag{2.8}$$

so that (2.4) may be written

$$\begin{aligned}\Big(D_t^2 - \frac{3}{4}\Big)a(t) &= \Big\langle Y, \kappa(x)\big(a(t)Y + P_{\rm ac}\varphi\big)^2 + \frac{1}{2}\big(a(t)Y + P_{\rm ac}\varphi\big)^3\Big\rangle,\\ \big(D_t^2 - (D_x^2 + 1 + 2V(x))\big)P_{\rm ac}\varphi &\\ = P_{\rm ac}\Big(\kappa(x)&\big(a(t)Y + P_{\rm ac}\varphi\big)^2 + \frac{1}{2}\big(a(t)Y + P_{\rm ac}\varphi\big)^3\Big).\end{aligned} \tag{2.9}$$

Our main result asserts that, up to a time of order ε^{-4}, the dispersive part $P_{\rm ac}\varphi$ of (2.9) has a time decay in uniform norm of magnitude $t^{-\frac{1}{2}}$, and that the function $a(t)$ in (2.8) has some oscillatory behavior, with decay in $t^{-\frac{1}{2}}$. More precisely, we have:

Theorem 2.1.1. *There is $\rho_0 \in \mathbb{N}$ and for any $\rho \geq \rho_0$, any $c > 0$, any $\theta' \in]0, \frac{1}{2}[$, any large enough N in $\mathbb{N}$, any large enough s in $\mathbb{N}$, there are $\varepsilon_0 \in]0, 1[$, $C > 0$ such that for any couple (φ_0, φ_1) of real-valued* odd *functions in $H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R})$ satisfying*

$$\|\varphi_0\|^2_{H^{s+1}} + \|\varphi_1\|^2_{H^s} + \|x\varphi_0\|^2_{H^1} + \|x\varphi_1\|^2_{L^2} \leq 1, \tag{2.10}$$

the global solution φ of

$$\begin{aligned}\big(D_t^2 - (D_x^2 + 1 + 2V(x))\big)\varphi &= \kappa(x)\varphi^2 + \frac{1}{2}\varphi^3,\\ \varphi|_{t=1} &= \varepsilon\varphi_0,\\ \partial_t\varphi|_{t=1} &= \varepsilon\varphi_1\end{aligned} \tag{2.11}$$

satisfies when $\varepsilon \in]0, \varepsilon_0[$ the following bounds for any $t \in [1, \varepsilon^{-4+c}]$: The oscillatory part a of φ given by (2.8) *may be written*

$$a(t) = e^{it\frac{\sqrt{3}}{2}}g_+(t) - e^{-it\frac{\sqrt{3}}{2}}g_-(t), \tag{2.12}$$

where

$$|g_\pm(t)| \leq C\varepsilon(1 + t\varepsilon^2)^{-\frac{1}{2}},\ |\partial_t g_\pm(t)| \leq C\varepsilon t^{-\frac{1}{2}}(1 + t\varepsilon^2)^{-\frac{1}{2}}. \tag{2.13}$$

The dispersive part $P_{\rm ac}\varphi(t,\cdot)$ satisfies

$$\begin{aligned}\|P_{\rm ac}\varphi(t,\cdot)\|_{W^{\rho,\infty}} &\leq Ct^{-\frac{1}{2}}(\varepsilon^2\sqrt{t})^{\theta'},\\ \|\langle x\rangle^{-2N}P_{\rm ac}\varphi(t,\cdot)\|_{W^{\rho,\infty}} &\leq Ct^{-\frac{3}{4}}(\varepsilon^2\sqrt{t})^{\theta'},\\ \|\langle x\rangle^{-2N}P_{\rm ac}D_t\varphi(t,\cdot)\|_{W^{\rho-1,\infty}} &\leq Ct^{-\frac{3}{4}}(\varepsilon^2\sqrt{t})^{\theta'},\end{aligned} \tag{2.14}$$

where $\|\psi\|_{W^{\rho,\infty}} = \|\langle D_x\rangle^\rho\psi\|_{L^\infty}$.

Remarks. We make the following observations.

- The first estimate (2.14) shows that, up to time essentially equal to ε^{-4}, the dispersive part of the solution decays like $t^{-\frac{1}{2}}$, which is the behavior of small global solutions to nonlinear Klein–Gordon equations (see [18,19,64,82]). Nevertheless, in that case, the upper bound is in $O(\varepsilon t^{-\frac{1}{2}})$, while in (2.14), we have a degeneracy of the factor multiplying $t^{-\frac{1}{2}}$ when t goes to ε^{-4}.
- We construct in the proof some approximate solutions that are $o(t^{-\frac{1}{2}})$ for times $t \leq \varepsilon^{-4+c}$ and ε small. To go past that time seems to require extra arguments – like devising more accurate approximate solutions – in order to get a useful pointwise control of $P_{\mathrm{ac}}\varphi$ for $t > \varepsilon^{-4}$.
- Our estimates are consistent with the ones of Kowalczyk, Martel and Muñoz [56] in time $O(\varepsilon^{-4})$. Actually, it follows from (2.12), (2.13) that if $p > 2$,

$$\int_1^{\varepsilon^{-4+c}} |a(t)|^p \, dt \leq C\varepsilon^{p-2}$$

and

$$\int_1^{\varepsilon^{-4+c}} \Big(\|\langle x\rangle^{-2N-1} P_{\mathrm{ac}}\varphi(t,\cdot)\|_{H^1}^2 + \|\langle x\rangle^{-2N-1} D_t P_{\mathrm{ac}}\varphi(t,\cdot)\|_{L^2}^2 \Big)\, dt \leq C\varepsilon^{4\theta'}$$

for large enough N. These estimates are in accordance with those proved in [56] (when $p = 4$ for the first one) (see Theorem 1.2 in that reference).

2.2 Reduced system

We shall conjugate the second equation (2.9) by the wave operator W_+ associated to $-\frac{1}{2}\partial_x^2 + V(x)$. We discuss in Appendix A.1 below the properties of such an operator. According to Proposition A.1.1 of that Appendix, it may be written, when acting on odd functions, under the form

$$W_+ = b(x, D_x) \circ c(D_x), \tag{2.15}$$

where $b(x,\xi)$ is a symbol of order zero satisfying estimates (A.8) and

$$c(\xi) = e^{i\theta(\xi)}\mathbb{1}_{\xi>0} + e^{-i\theta(\xi)}\mathbb{1}_{\xi<0}$$

for some odd smooth real-valued function θ. Moreover, if we set $A = -\frac{1}{2}\partial_x^2 + V(x)$, $A_0 = -\frac{1}{2}\partial_x^2$, one has by (A.6) and (A.7), for any Borel function m on $\mathbb{R}$,

$$\begin{aligned} m(A)P_{\mathrm{ac}} &= W_+ m(A_0) W_+^*, & m(A_0) &= W_+^* m(A) W_+ \\ W_+ W_+^* &= P_{\mathrm{ac}}, & W_+^* W_+ &= \mathrm{Id}_{L^2} \end{aligned} \tag{2.16}$$

so that applying W_+^* on the second equation (2.9), we get

$$\begin{aligned} \big(D_t^2 - (D_x^2+1)\big)(W_+^* P_{\mathrm{ac}}\varphi) = {} & W_+^*\big(\kappa(x)(a(t)Y + P_{\mathrm{ac}}\varphi)^2\big) \\ & + W_+^*\Big(\frac{1}{2}(a(t)Y + P_{\mathrm{ac}}\varphi)^3\Big). \end{aligned} \tag{2.17}$$

Let us define

$$w = b(x, D_x)^* P_{\mathrm{ac}}\varphi. \tag{2.18}$$

Since $P_{\mathrm{ac}}\varphi$ is real valued, and since because of the symmetry properties (A.9) of $b(x, \xi)$, $b(x, D_x)$ and $b(x, D_x)^*$ preserve the space of real (resp. even, resp. odd) functions, w is still a real-valued odd function. As $c(D_x) \circ c(D_x)^* = \mathrm{Id}$,

$$\begin{aligned} P_{\mathrm{ac}}\varphi &= W_+ W_+^* P_{\mathrm{ac}}\varphi = b(x, D_x)w \\ c(D_x)W_+^* P_{\mathrm{ac}}\varphi &= w, \end{aligned} \tag{2.19}$$

so that making act $c(D_x)$ on (2.17) we see that w solves

$$\begin{aligned} \big(D_t^2 - (D_x^2 + 1)\big)w = b(x, D_x)^*\big(\kappa(x)\big(a(t)Y + b(x, D_x)w\big)^2\big) \\ + \frac{1}{2} b(x, D_x)^* \big(a(t)Y + b(x, D_x)w\big)^3. \end{aligned} \tag{2.20}$$

We shall study from now on the system given by the first equation (2.9) and (2.20). We define

$$\begin{aligned} w_0 &= b(x, D_x)^* P_{\mathrm{ac}}\varphi_0, \\ w_1 &= b(x, D_x)^* P_{\mathrm{ac}}\varphi_1. \end{aligned} \tag{2.21}$$

Since by (2.15) and (2.16), $P_{\mathrm{ac}} = b(x, D_x) \circ b(x, D_x)^*$, and since $b(x, D_x)$ and $[x, b(x, D_x)]$ are bounded on Sobolev spaces, we get from (2.10) that

$$\|w_0\|^2_{H^{s+1}} + \|w_1\|^2_{H^s} + \|xw_0\|^2_{H^1} + \|xw_1\|^2_{L^2} \le C_0 \tag{2.22}$$

for some constant C_0. Denote by $p(D_x)$ the operator

$$p(D_x) = \sqrt{1 + D_x^2} \tag{2.23}$$

and introduce complex-valued odd unknowns

$$\begin{aligned} u_+ &= \big(D_t + p(D_x)\big)w, \\ u_- &= \big(D_t - p(D_x)\big)w = -\bar{u}_+. \end{aligned} \tag{2.24}$$

If $I = (i_1, \dots, i_p)$ is an element of $\{-, +\}^p$, we shall set

$$u_I = (u_{i_1}, \dots, u_{i_p}) \tag{2.25}$$

and we denote also $u_{I,j} = u_{i_j}$, so that equivalently

$$u_I = (u_{I,1}, \dots, u_{I,p}). \tag{2.26}$$

Let us write (2.20) under the equivalent form

$$\big(D_t - p(D_x)\big)u_+ = \sum_{j=0}^{2} F_j^2[a; u_+, u_-] + \sum_{j=0}^{3} F_j^3[a; u_+, u_-], \tag{2.27}$$

where F_j^2 (resp. F_j^3) will be made of terms that are $O(t^{-1})$ (resp. $O(t^{-\frac{3}{2}})$) in L^∞ if the bounds (2.12)–(2.14) hold true, and are given by the following:

- Contribution depending only on a and not on $u_\pm$ are

$$\begin{aligned} F_0^2[a;u_+,u_-] &= F_0^2[a] = a(t)^2 b(x,D_x)^*(\kappa(x)Y^2),\\ F_0^3[a;u_+,u_-] &= F_0^3[a] = \frac{1}{2}a(t)^3 b(x,D_x)^*(Y^3). \end{aligned} \tag{2.28}$$

- Contributions that are homogeneous of degree $j>0$ in (u_+,u_-) are given by the following quantities, where if $|I|=(i_1,\dots,i_p)$, we set $|I|=p$ and $\varepsilon_I=i_1\cdots i_p$:

$$\begin{aligned} F_j^2[a;u_+,u_-] &= a(t)^{2-j}\sum_{|I|=j} F_{j,I}^2[u_I], \quad j=1,2,\\ F_j^3[a;u_+,u_-] &= a(t)^{3-j}\sum_{|I|=j} F_{j,I}^3[u_I], \quad j=1,2,3, \end{aligned} \tag{2.29}$$

with linear terms in (u_+,u_-)

$$\begin{aligned} F_{1,I}^2[u_I] &= \varepsilon_I b(x,D_x)^*\big(Y(x)\kappa(x)b(x,D_x)p(D_x)^{-1}u_I\big),\\ F_{1,I}^3[u_I] &= \frac{3}{4}\varepsilon_I b(x,D_x)^*\big(Y(x)^2 b(x,D_x)p(D_x)^{-1}u_I\big), \end{aligned} \tag{2.30}$$

quadratic terms in (u_+,u_-)

$$\begin{aligned} F_{2,I}^2[u_I] &= \frac{1}{4}\varepsilon_I b(x,D_x)^*\left(\kappa(x)\prod_{\ell=1}^{2} b(x,D_x)p(D_x)^{-1}u_{I,\ell}\right),\\ F_{2,I}^3[u_I] &= \frac{3}{8}\varepsilon_I b(x,D_x)^*\left(Y(x)\prod_{\ell=1}^{2} b(x,D_x)p(D_x)^{-1}u_{I,\ell}\right), \end{aligned} \tag{2.31}$$

and a cubic term in (u_+,u_-)

$$F_{3,I}^3[u_I] = \frac{1}{16}\varepsilon_I b(x,D_x)^*\left(\prod_{\ell=1}^{3} b(x,D_x)p(D_x)^{-1}u_{I,\ell}\right). \tag{2.32}$$

Notice that since κ and Y are odd, as well as $u_\pm$, and $b(x,D_x)$ preserves odd functions, F_j^2, F_j^3 are odd functions.

Let us write now the first equation in (2.9) in terms of a,u_+,u_-. We define

$$a_+(t) = \Big(D_t + \frac{\sqrt{3}}{2}\Big)a, \quad a_-(t) = \Big(D_t - \frac{\sqrt{3}}{2}\Big)a = -\bar{a}_+ \tag{2.33}$$

so that $a=\frac{\sqrt{3}}{3}(a_+-a_-)$ and we rewrite the first equation (2.9) as

$$\begin{aligned} \Big(D_t - \frac{\sqrt{3}}{2}\Big)a_+ &= \sum_{j=0}^{2}(a_+-a_-)^{2-j}\Phi_j[u_+,u_-]\\ &\quad + \sum_{j=0}^{3}(a_+-a_-)^{3-j}\Gamma_j[u_+,u_-], \end{aligned} \tag{2.34}$$

where the terms independent of $u_\pm$ are

$$\begin{aligned}\Phi_0 &= \frac{1}{3}\langle Y, \kappa Y^2\rangle,\\ \Gamma_0 &= \frac{\sqrt{3}}{18}\langle Y, Y^3\rangle\end{aligned} \tag{2.35}$$

and for $j \geq 1$,

$$\begin{aligned}\Phi_j[u_+, u_-] &= \sum_{|I|=j} \Phi_{j,I}[u_I],\\ \Gamma_j[u_+, u_-] &= \sum_{|I|=j} \Gamma_{j,I}[u_I]\end{aligned} \tag{2.36}$$

with linear expressions

$$\begin{aligned}\Phi_{1,I}[u_I] &= \frac{\sqrt{3}}{3}\varepsilon_I\big\langle Y, Y\kappa b(x, D_x)p(D_x)^{-1}u_I\big\rangle,\\ \Gamma_{1,I}[u_I] &= \frac{1}{4}\varepsilon_I\big\langle Y, Y^2 b(x, D_x)p(D_x)^{-1}u_I\big\rangle,\end{aligned} \tag{2.37}$$

quadratic expressions

$$\begin{aligned}\Phi_{2,I}[u_I] &= \frac{1}{4}\varepsilon_I\Big\langle Y, \kappa \prod_{\ell=1}^{2} b(x, D_x)p(D_x)^{-1}u_{I,\ell}\Big\rangle,\\ \Gamma_{2,I}[u_I] &= \frac{\sqrt{3}}{8}\varepsilon_I\Big\langle Y, Y \prod_{\ell=1}^{2} b(x, D_x)p(D_x)^{-1}u_{I,\ell}\Big\rangle,\end{aligned} \tag{2.38}$$

and cubic quantities

$$\Gamma_{3,I}[u_I] = \frac{1}{16}\varepsilon_I\Big\langle Y, \prod_{\ell=1}^{3} b(x, D_x)p(D_x)^{-1}u_{I,\ell}\Big\rangle. \tag{2.39}$$

We shall study from now on system (2.27), (2.34) with initial data at $t = 1$. According to (2.24), (2.21), (2.22), (2.33) and the fact that by (2.8), $a(1) = \langle \varepsilon\varphi_0, Y\rangle$ and $\partial_t a(1) = \langle \varepsilon\varphi_1, Y\rangle$, with φ_0, φ_1 satisfying (2.10), we may assume

$$u_+|_{t=1} = \varepsilon u_{+,0}, \quad a_+|_{t=1} = \varepsilon a_{+,0}, \tag{2.40}$$

where $u_{+,0}$ is a complex-valued odd function in $H^s(\mathbb{R}, \mathbb{C})$ satisfying

$$\begin{aligned}\|u_{+,0}\|_{H^s}^2 + \|xu_{+,0}\|_{L^2}^2 &\leq C_0^2,\\ |a_{+,0}| &\leq C_0^2\end{aligned} \tag{2.41}$$

for some fixed constant C_0.

In the following sections, we shall describe the main steps of the method of proof of our main result.

2.3 Step 1: Writing of the system from multilinear operators

In Section 2.2, we have reduced (2.9) to the system made of equations (2.27) and (2.34). One may rewrite (2.27) on a more synthetic way as

$$\begin{aligned}\big(D_t - p(D_x)\big)u_+ = F_0^2[a] + F_0^3[a] + &\sum_{2\le|I|\le 3} \operatorname{Op}(m_{0,I})[u_I]\\ &+ a(t)\sum_{1\le|I|\le 2}\operatorname{Op}(m'_{1,I})[u_I]\\ &+ a(t)^2\sum_{|I|=1}\operatorname{Op}(m'_{2,I})[u_I]\end{aligned}\tag{2.42}$$

with the following notation: The term $F_0^2[a]$ (resp. $F_0^3[a]$) is the quadratic (resp. cubic) contribution in a obtained setting $w = 0$ on the right-hand side of (2.27). It has structure $a(t)^2 Z_2$ (resp. $a(t)^3 Z_3$) for some $\mathcal{S}(\mathbb{R})$-function Z_2 (resp. Z_3). The other terms on the right-hand side of (2.42) are expressed in terms of multilinear operators $\operatorname{Op}(m)(u_1,\dots,u_p)$, defined if $m(x,\xi_1,\dots,\xi_p)$ is a smooth function satisfying convenient estimates, as

$$\begin{aligned}\operatorname{Op}(m)(u_1,\dots,u_p) = \frac{1}{(2\pi)^p}\int e^{ix(\xi_1+\dots+\xi_p)}&m(x,\xi_1,\dots,\xi_p)\\ &\times\prod_{j=1}^{p}\hat{u}_j(\xi_j)\,d\xi_1\cdots d\xi_p.\end{aligned}\tag{2.43}$$

On the right-hand side of (2.42), we denote by I p-tuples $I = (i_1,\dots,i_p)$ where $i_\ell = \pm$ and set $|I| = p$. Then u_I stands for a p-tuple $u_I = (u_{i_1},\dots,u_{i_p})$ whose components are equal to u_+ or u_- defined in (2.24). The symbols $m_{0,I}$, $m'_{1,I}$, $m'_{2,I}$ are functions of $(x,\xi_1,\dots,\xi_p)$ with $p = |I|$. We do not write explicitly in this presentation of the proof the estimates that are assumed on these functions and their derivatives: we refer to Definition 3.1.1 below and to Appendix B for the precise description of the classes of symbols we consider. Let us just say that symbols $m_{0,I}$ are bounded in x, while their ∂_x-derivatives are rapidly decaying in x. This comes from the fact that the symbol $b(x,\xi)$ and the functions κ, Y in (2.20) satisfy such properties. On the other hand, symbols $m'_{1,I}, m'_{2,I}$ (and more generally any symbol that we shall denote as m' in what follows) decay rapidly in x even without taking derivatives. It turns out that operators with decaying symbol in x acting on functions we shall introduce below will give quantities with a better time decay than operators associated to non-decaying symbols.

2.4 Step 2: First quadratic normal form

The goal of the whole paper is to obtain energy estimates for the solution u_+ to (2.27) and a_+ to (2.34).

As we have seen in Section 1.6 of the Introduction, the first thing to do in order to get Sobolev estimates for an equation like (2.27) is to eliminate the quadratic contributions $\sum_{|I|=2} \mathrm{Op}(m_{0,I})[u_I]$. We do that through a "time normal form" à la Shatah [76] and Simon and Taflin [77] (see also for one-dimensional Klein–Gordon equations Moriyama, Tonegawa and Tsutsumi [71], Moriyama [70], Hayashi and Naumkin [39] and the very recent works of Germain and Pusateri [33], of Lindblad, Lührmann and Soffer [60] and of Lindblad, Lührmann, Schlag and Soffer [59]). Actually, we construct new symbols $(\tilde{m}_{0,I})_{|I|=2}$ such that

$$\begin{aligned}
\big(D_t - p(D_x)\big)\Big(u_+ - \sum_{|I|=2} \mathrm{Op}(\tilde{m}_{0,I})[u_I]\Big) \\
= F_0^2[a] + F_0^3[a] + \sum_{3\le|I|\le4} \mathrm{Op}(m_{0,I})[u_I] + \sum_{|I|=2} \mathrm{Op}(m'_{0,I})[u_I] \\
+ \sum_{j=1}^{3} a(t)^j \sum_{1\le|I|\le4-j} \mathrm{Op}(m'_{j,I})[u_I],
\end{aligned} \tag{2.44}$$

where on the right-hand side, we eliminated the quadratic contributions $\mathrm{Op}(m_{0,I})[u_I]$, but made appear new quadratic terms $\mathrm{Op}(m'_{0,I})[u_I]$ given in terms of new symbols $m'_{0,I}$ that *decay rapidly* when x goes to infinity. These corrections come from the fact that, at the difference with a usual normal form method where one eliminates quadratic expressions like (2.43) with $p = 2$ and a symbol $m(\xi_1, \xi_2)$ independent of x, we have here to cope with symbols $m(x, \xi_1, \xi_2)$. This x dependence makes appear some commutator, given essentially in terms of $\mathrm{Op}(\frac{\partial m}{\partial x}(x, \xi_1, \xi_2))$, with a symbol *rapidly decaying* in x. These commutators are the new quadratic terms $\mathrm{Op}(m'_{0,I})[u_I]$ on the right-hand side of (2.44). As already mentioned, such expressions will have better time decay estimates than the quadratic expressions given by non-space decaying symbols that we have eliminated, and are actually better than most remaining terms on the right-hand side of (2.44). They are not completely negligible, but will be treated only at the end of the reasoning.

2.5 Step 3: Approximate solution

Our general strategy is to define from the solution u_+ of (2.44) a new unknown $\tilde{u}_+$ that would satisfy similar estimates as those of the bootstrap (1.39) of the introduction. More precisely, we aim at constructing a new unknown $\tilde{u}_+$ for which we could get, for $t \in [1, \varepsilon^{-4+c}]$ with $c > 0$ given, bounds of the following form:

$$\|\tilde{u}_+(t,\cdot)\|_{H^s} = O(\varepsilon t^{\delta}), \tag{2.45}$$

$$\|L_+\tilde{u}_+(t,\cdot)\|_{L^2} = O\big((\varepsilon^2\sqrt{t})^{\theta} t^{\frac{1}{4}}\big), \tag{2.46}$$

$$\|\tilde{u}_+(t,\cdot)\|_{W^{\rho,\infty}} = O\Big(\frac{(\varepsilon^2\sqrt{t})^{\theta'}}{\sqrt{t}}\Big), \tag{2.47}$$

where $\delta > 0$ is small, $\theta' < \theta < \frac{1}{2}$ with θ' close to $\frac{1}{2}$, $s \gg \rho \gg 1$, and where we denoted $\|w\|_{W^{\rho,\infty}} = \|\langle D_x\rangle^\rho w\|_{L^\infty}$. The first estimate (2.45) is the one that would follow by energy inequality for the solution of (1.32), assuming that (2.47) holds (since, for $t \le \varepsilon^{-4+c}$, (2.47) implies a bound in $c(\varepsilon)t^{-\frac{1}{2}}$, with $c(\varepsilon)$ going to zero when ε goes to zero). In the same way, assuming (2.47) and assuming that $\tilde{u}_+$ solves an equation of the form (1.26) with $p = 1$, one could bootstrap a bound of the form (2.46). Finally, an estimate of the form (2.47) will have to be deduced from (2.46) constructing from the PDE solved by $\tilde{u}_+$ an ODE with remainder term controlled from (2.46).

Of course, the right-hand side of (2.44) is far from having the nice structure of the one of (1.32), and this is why we shall have to modify the unknown u_+ in order to eliminate all bad terms on the right-hand side of (2.44). In Chapter 4 of the paper we shall get rid of the contributions $F_0^2[a]$, $F_0^3[a]$. These functions are bounded as well as their space derivatives by $t^{-1}\langle x\rangle^{-N}$ for any N. Clearly, if we make act L_+ on them and compute the L^2 norm, we shall get an $O(1)$ quantity. If we were integrating such a bound, we would deduce that $\|L_+u_+(t,\cdot)\|_{L^2} = O(t)$, a much worse estimate than the one (2.46) we want. We shall thus remove from u_+ the solution of the linear equation with force terms $F_0^2[a] + F_0^3[a]$, i.e. we shall solve

$$\begin{aligned} \big(D_t - p(D_x)\big)U &= F_0^2[a] + F_0^3[a], \\ U|_{t=1} &= 0 \end{aligned} \tag{2.48}$$

and then make the difference between (2.44) and (2.48) in order to eliminate $F_0^2[a]$ and $F_0^3[a]$ from the right-hand side of the new equation obtained in that way. Actually, one needs to take also into account at this stage bilinear terms in (a,u) in (2.44). We thus construct in Proposition 4.1.2 an approximate solution u_+^{app} of

$$\begin{aligned} \big(D_t - p(D_x)\big)u_+^{\text{app}} &= F_0^2(a^{\text{app}}) + F_0^3(a^{\text{app}}) \\ &\quad + a^{\text{app}} \sum_{|I|=1} \mathrm{Op}(m'_{1,I})(u_I^{\text{app}}) + \text{remainder}, \\ u_+^{\text{app}}|_{t=1} &= 0, \end{aligned} \tag{2.49}$$

where a^{app} is some approximation of the function $a(t)$ solving the first equation (2.9).

Let us explain what are the bounds satisfied by the approximate solution u_+^{app} of equation (2.49) that we obtain in Proposition 4.1.2 using the results of Appendix C. We decompose $u_+^{\text{app}} = u'^{\text{app}}_+ + u''^{\text{app}}_+$. The term u'^{app}_+ satisfies the kind of estimates we aim at proving, namely (2.45)–(2.47) (and actually slightly better ones) for times $t = O(\varepsilon^{-4+c})$. On the other hand, inequalities (2.45) and (2.47) hold for u''^{app}_+ (and even actually slightly better ones), but $L_+u''^{\text{app}}_+$ does not verify (2.46). On the other hand, $L_+u''^{\text{app}}_+$ obeys good estimates in L^∞ norms, of the form

$$\|L_+u''^{\text{app}}_+\|_{W^{r,\infty}} = O\big(\log(1+t)\log(1+\varepsilon^2 t)\big) \tag{2.50}$$

that will allow us to estimate conveniently nonlinear terms containing u''^{app}_+. Let us stress that the limitation of our main result to times $O(\varepsilon^{-4})$ comes from the degeneracy of bound (2.46) for $L_+ u'^{\text{app}}_+$ when t becomes larger than ε^{-4}. We do not claim that, in such a regime, an estimate of the form (2.46) would be optimal. But we remark that in the construction of u'^{app}_+ made from the results of Appendix C, the main contribution comes from quantities that have pretty explicit bounds: see Proposition C.1.4 and in particular bound (C.40) with $\omega = 1$ (that gives the main contribution to u'^{app}_+) and (C.42) with $\omega = 1$ (that gives the main contribution to $L_+ u'^{\text{app}}_+$). If we extrapolate estimate (C.40) for $t \gg \varepsilon^{-4}$ (which is of course not legitimate, as we prove it only for times $O(\varepsilon^{-4})$), we see that outside a conical neighborhood of the two lines $x = \pm t\sqrt{2/3}$, an estimate of $|u'^{\text{app}}_+(t,x)|$ in $O(\varepsilon^2 t^{-\frac{1}{2}})$ would hold. On the other hand, along these two lines, a degeneracy happens, and we do not expect to be able to prove that, for $t \gg \varepsilon^{-4}$, $|u'^{\text{app}}_+(t, \pm t\sqrt{2/3})|\sqrt{t}$ remains small (or even bounded). Because of that, we do not hope to push estimates of the form (2.45)–(2.47) for such times, without taking into account first some extra corrections. In particular, going back to (1.105), we do not expect an $O(t^{-\frac{1}{2}})$ bound for $|P_{\text{ac}}\varphi(t,x)|$ along these lines.

Notice that such a phenomenon cannot be detected using weighted space estimates an in [56]: actually, along the lines $x = \pm t\sqrt{2/3}$, a space decaying weight is also time decaying and kills bad bounds of u'^{app}_+ along these lines. We shall comment more extensively on that issue in Section 2.10 below.

In addition to the proof of estimates of the form (2.45)–(2.47), we need, in order to obtain (1.105), to study the solution of the first equation (2.9). We do that in Section 4.2 of Chapter 4. Setting

$$a_+(t) = \Big(D_t + \frac{\sqrt{3}}{2}\Big)a, \quad a_-(t) = \Big(D_t - \frac{\sqrt{3}}{2}\Big)a = -\bar{a}_+,$$

the first equation (2.9) may be rewritten as

$$\begin{aligned}\Big(D_t - \frac{\sqrt{3}}{2}\Big)a_+ = \sum_{j=0}^{2}(a_+ - a_-)^{2-j}\,\Phi_j[u_+, u_-]& \\ + \sum_{j=0}^{3}(a_+ - a_-)^{3-j}\,\Gamma_j[u_+, u_-],&\end{aligned} \tag{2.51}$$

where Φ_j, Γ_j are expressions in the solution u_+ to (2.42) or (2.44). The goal of Section 4.2 is to uncover the structure of a_+. We write

$$a_+(t) = a_+^{\text{app}}(t) + O\big(\varepsilon^3(1 + t\varepsilon^2)^{-\frac{3}{2}}\big),$$

where $a_+^{\text{app}}(t)$ has structure (4.97), that implies in particular

$$a_+^{\text{app}}(t) = e^{it\frac{\sqrt{3}}{2}}g(t) + \text{more decaying terms}. \tag{2.52}$$

The main goal of Section 4.2 is to prove by bootstrap that $g(t)$ satisfies bounds

$$|g(t)| = O(\varepsilon(1 + t\varepsilon^2)^{-\frac{1}{2}}), \quad |\partial_t g(t)| = O(t^{-\frac{3}{2}}). \tag{2.53}$$

(Actually, we get more precise bounds for $\partial_t g$: see (4.99)). These bounds are obtained showing that (2.51) implies that g satisfies an ODE

$$D_t g(t) = \Big(\alpha - i\frac{\sqrt{6}}{18}\hat{Y}_2(\sqrt{2})^2\Big)|g(t)|^2 g(t) + \text{remainder}, \tag{2.54}$$

where Y_2 is some explicit function in $\mathcal{S}(\mathbb{R})$ and α is real. The coefficient of the cubic term on the right-hand side comes from some of the terms on the right-hand side of (2.51) where we replace $u_\pm$ by the approximate solution u_+^{app} determined in Section 4.1. The main contribution to u_+^{app}, integrated against an $\mathcal{S}(\mathbb{R})$ function, may be computed explicitly in terms of g (see Proposition 4.1.3), and brings the right-hand side of (2.54). The key point in that equation is that $\hat{Y}_2(\sqrt{2})^2 < 0$. This implies that g satisfies bounds (2.53) for $t \geq 1$ if $g(1) = O(\varepsilon)$. The inequality $\hat{Y}_2(\sqrt{2})^2 < 0$ is nothing but Fermi's golden rule. Actually, $\hat{Y}_2(\sqrt{2})^2 \leq 0$ holds trivially and the key point is to check that $\hat{Y}_2(\sqrt{2}) \neq 0$. This reduces to showing that some explicit integral is non-zero. Kowalczyk, Martel and Muñoz checked that numerically in [56]. In Appendix G, we compute explicitly this integral by residues.

2.6 Step 4: Reduced form of dispersive equation

The goal of this step is to rewrite equation (2.44) in terms of a new unknown $\tilde{u}_+$ that will satisfy estimates (2.45)–(2.47). We define

$$\tilde{u}_+ = u_+ - \sum_{|I|=2} \mathrm{Op}(\tilde{m}_{0,I})(u_I) - u'^{\text{app}}_+ - u''^{\text{app}}_+, \tag{2.55}$$

and set $\tilde{u}_- = -\overline{\tilde{u}_+}$. Making the difference between (2.44) and (2.49), we show in Section 5.2 (see Proposition 5.2.1) that $\tilde{u}_+$ satisfies

$$\begin{aligned}\big(D_t - p(D_x)\big)\tilde{u}_+ = &\sum_{3\leq|I|\leq4,\, I=(I',I'')} \mathrm{Op}(\tilde{m}_I)(\tilde{u}_{I'}, u_{I''}^{\text{app}})\\ &+ \sum_{|I|=2,\, I=(I',I'')} \mathrm{Op}(m'_{0,I})(\tilde{u}_{I'}, u_{I''}^{\text{app}})\\ &+ \underline{a}^{\text{app}}(t)\sum_{|I|=1}\mathrm{Op}(m'_{1,I})(\tilde{u}_I)\\ &+ \frac{1}{3}\Big(e^{it\frac{\sqrt{3}}{2}}g(t) + e^{-it\frac{\sqrt{3}}{2}}\overline{g(t)}\Big)^2 \sum_{|I|=1}\mathrm{Op}(m'_{0,I})(\tilde{u}_I)\\ &+ \text{remainder},\end{aligned} \tag{2.56}$$

where:

- For $3 \leq |I| \leq 4$, $\tilde{m}_I$ are symbols $\tilde{m}_I(x, \xi_1, \dots, \xi_p)$, $p = |I| = |I'| + |I''|$ which are $O(1)$ as functions of x, but $O(\langle x\rangle^{-\infty})$ if one takes at least one ∂_x-derivative.
- For $1 \leq |I| \leq 2$, $m'_{0,I}, m'_{1,I}$ are symbols that are $O(\langle x\rangle^{-\infty})$, even without taking any derivative.

- Function of time g has been introduced in (2.52) and gives the principal term in the expansion of $a_+^{\mathrm{app}}(t)$ or $a_+(t)$.
- Function $\underline{a}^{\mathrm{app}}(t) = \frac{\sqrt{3}}{3}(\underline{a}_+^{\mathrm{app}}(t) - \underline{a}_-^{\mathrm{app}}(t))$, where

$$\underline{a}_+^{\mathrm{app}}(t) = e^{it\frac{\sqrt{3}}{2}} g(t) + \omega_2 e^{it\sqrt{3}} g(t)^2 + \omega_0 |g(t)|^2 + \omega_{-2} e^{-it\sqrt{3}} \overline{g(t)}^2 \tag{2.57}$$

with convenient constants $\omega_2, \omega_0, \omega_{-2}$ and $\underline{a}_-^{\mathrm{app}}(t) = -\overline{\underline{a}_+^{\mathrm{app}}(t)}$.

We cannot derive directly from equation (2.56) estimate (2.46) for $\tilde{u}_+$, as the right-hand side of (2.56) has not the nice structure (1.32). Before applying an energy method, we shall have to use several normal forms in order to reduce ourselves to such a nice nonlinearity. As a preparation to that step, we show in Corollary 5.2.3 that (2.56) may be rewritten under the following equivalent form:

$$\begin{aligned}
\big(D_t - p(D_x)\big)\tilde{u}_+ &- \sum_{j=-2}^{2} e^{itj\frac{\sqrt{3}}{2}} \mathrm{Op}(b'_{j,+})\tilde{u}_+ - \sum_{j=-2}^{2} e^{itj\frac{\sqrt{3}}{2}} \mathrm{Op}(b'_{j,-})\tilde{u}_- \\
&= \sum_{3\le |I|\le 4,\, I=(I',I'')} \mathrm{Op}(\tilde{m}_I)(\tilde{u}_{I'}, u_{I''}^{\mathrm{app}}) + \sum_{|I|=2} \mathrm{Op}(m'_{0,I})(\tilde{u}_I) \\
&\quad + \sum_{I=(I',I''),\, |I'|=|I''|=1} \mathrm{Op}(m'_{0,I})(\tilde{u}_{I'}, u'^{\mathrm{app},1}_{I''}) \\
&\quad + \sum_{|I|=2} \mathrm{Op}(m'_{0,I})(u'^{\mathrm{app},1}_{I}) + \text{remainder},
\end{aligned} \tag{2.58}$$

where, in comparison with (2.56), all linear terms in $\tilde{u}_+, \tilde{u}_-$ have been sent to the left-hand side, and are expressed from symbols $b'_{j,\pm}(t,x,\xi)$ that are rapidly decaying in x at infinity. Moreover, on the right-hand side, we still use the convention of denoting by $m'_{0,I}$ symbols rapidly decaying in x, while $\tilde{m}_I$ are $O(1)$ in x, with ∂_x-derivatives rapidly decaying in x. Furthermore, in the last two sums in (2.58), we replaced u'^{app} by $u'^{\mathrm{app},1}$, which is actually the main contribution (in terms of time decay) to u'^{app}. If we set $\tilde{u} = [\begin{smallmatrix}\tilde{u}_+\\ \tilde{u}_-\end{smallmatrix}]$, we may rewrite (2.58) as a system of the form

$$\begin{aligned}
\big(D_t - P_0 - \mathcal{V}\big)\tilde{u} = \mathcal{M}_3(\tilde{u}, u^{\mathrm{app}}) &+ \mathcal{M}_4(\tilde{u}, u^{\mathrm{app}}) \\
&+ \mathcal{M}'_2(\tilde{u}, u'^{\mathrm{app},1}) + \text{remainder},
\end{aligned} \tag{2.59}$$

where

$$P_0 = \begin{bmatrix} p(D_x) & 0 \\ 0 & -p(D_x) \end{bmatrix},$$

$\mathcal{V}$ is a 2×2 matrix of operators of the form

$$\mathcal{V} = \sum_{j=-2}^{2} e^{ijt\frac{\sqrt{3}}{2}} \mathrm{Op}(M'_j(t,x,\xi)) \tag{2.60}$$

with M'_j 2×2 matrix of symbols whose entries are given in terms of the $b'_{j,\pm}$ in (2.58), and where the 2-vectors $\mathcal{M}_3$ (resp. $\mathcal{M}_4$, resp. $\mathcal{M}'_2$) come from the cubic (resp. quartic, resp. quadratic) terms on the right-hand side of (2.58).

To obtain the wanted estimates (2.45) and (2.46) for $\tilde{u}_+$, we have next to reduce (2.59) to an equation essentially of the form (1.32). This is the object of Step 5 of the proof.

2.7 Step 5: Normal forms

Equation (2.59) has not structure of the form (1.32), in that sense that if we make act

$$L = \begin{bmatrix} L_+ & 0 \\ 0 & L_- \end{bmatrix},$$

with $L_- = x - tp'(D_x)$, first L does not commute to the potential term $\mathcal{V}$, and second the action of L on the nonlinearities on the right-hand side does not give quantities whose L^2 norm is $O(\|\tilde{u}\|_{L^\infty}^2 \|L\tilde{u}\|_{L^2})$ (which is essentially necessary if we want to get (2.46) by energy estimates). To cope with the lack of commutation of L with $\mathcal{V}$, we shall construct a wave operator and use it to eliminate $\mathcal{V}$ by conjugation of the equation. This is similar to what has been done to pass from the second equation (2.9), that was involving the potential $2V(x)$ to equation (2.17), where there was no longer any potential. The difference here is that $\mathcal{V}$ given by (2.60) is time dependent (with $O(t^{-\frac{1}{2}})$ decay). We thus cannot rely on existing references, and have to construct by hand operators $B(t), C(t)$ (depending on time) such that

$$C(t)\big(D_t - P_0 - \mathcal{V}\big) = (D_t - P_0)C(t). \tag{2.61}$$

In that way, if $\tilde{u}$ solves (2.59), then $C(t)\tilde{u}$ solves the new equation without potential

$$\begin{aligned}(D_t - P_0)C(t)\tilde{u} &= C(t)\mathcal{M}_3(\tilde{u}, u^{\mathrm{app}}) + C(t)\mathcal{M}_4(\tilde{u}, u^{\mathrm{app}}) \\ &\quad + C(t)\mathcal{M}_2'(\tilde{u}, u'^{\mathrm{app},1}) + \text{remainder}\end{aligned} \tag{2.62}$$

(see Proposition 6.1.2). Moreover, since we want to pass from an L^2 bound on $L\tilde{u}$ to an L^2 bound on $LC(t)\tilde{u}$ and conversely, we need to relate $L \circ C(t)$ and L, proving that

$$L \circ C(t) = \tilde{C}(t) \circ L + \tilde{C}_1(t), \tag{2.63}$$

where $\tilde{C}(t)$ is bounded on L^2 uniformly in t and $\tilde{C}_1(t)$ is bounded with a small time growth when t goes to infinity. The construction of operator $C(t)$ is made in Appendix E by a pretty standard series expansion. We notice however that we need to use in that construction the fact that we are dealing with *odd* functions $\tilde{u}$.

Once reduced to (2.62), we still have to handle those nonlinear terms on the right-hand side that do not have a structure of the form (1.32), i.e. we have to cope with nonlinearities that have the same structure as in the model (1.68) of Section 1.6 of the introduction. We have seen there that this problem may be solved using "space-time normal forms". We shall follow essentially the approach of [20], already described in Section 1.6 of the introduction, that we have to adapt to the more general operators $\mathcal{M}_3, \mathcal{M}_4$ on the right-hand side of (2.62). Remark that the components of the vectors

$\mathcal{M}_3, \mathcal{M}_4$ are, according to (2.58), given by expressions $\mathrm{Op}(\tilde{m})(\tilde{u}_\pm, \dots, u_\pm^{\mathrm{app}})$, where $\tilde{m}(x, \xi_1, \dots, \xi_p)$ is a symbol that is $O(1)$ when $|x|$ goes to infinity, but $O(\langle x\rangle^{-\infty})$ if one takes at least one ∂_x-derivative. We have to distinguish between to type of terms, the characteristic and the non-characteristic ones. The former correspond to the case when, among the p arguments of $\mathrm{Op}(\tilde{m})(\tilde{u}_\pm, \dots, u_\pm^{\mathrm{app}})$, $\frac{p+1}{2}$ are equal to $\tilde{u}_+$ or u_+^{app} and $\frac{p-1}{2}$ are equal to $\tilde{u}_-$ or u_-^{app}.

In the case of simple monomial nonlinearities, example of characteristic terms are given by the right-hand side $|u_+|^2 u_+$ of (1.32), which, when making act L_+ on it, may be estimated in L^2 by $\|u_+(t,\cdot)\|_{L^\infty}^2 \|L_+ u_+(t,\cdot)\|_{L^2}$. If $\tilde{m}$ were independent of x, the same would hold for the action of the operator L_+ on any characteristic term like $\mathrm{Op}(\tilde{m})(\tilde{u}_\pm, \dots, \tilde{u}_\pm)$, as $L_+\mathrm{Op}(\tilde{m})(\tilde{u}_\pm, \dots, \tilde{u}_\pm)$ could be expressed from $\mathrm{Op}(\tilde{m})(L_\pm \tilde{u}_\pm, \dots, \tilde{u}_\pm), \dots, \mathrm{Op}(\tilde{m})(\tilde{u}_\pm, \dots, L_\pm \tilde{u}_\pm)$. Using the boundedness properties of $\mathrm{Op}(\tilde{m})$, one would then estimate the L^2 norm of these quantities by $\|\tilde{u}\|_{L^\infty}^{p-1} \|L\tilde{u}\|_{L^2}$. As $p \geq 3$, one could then obtain estimate (2.46) by energy inequality, as in (1.26). Since here $\tilde{m}$ does depend on x, there is no exact commutation relation in the characteristic case between $\mathrm{Op}(\tilde{m})$ and L_+, as some commutators of the form $t\mathrm{Op}(\partial_x \tilde{m})$ have to be taken into account. It turns out that, because $\partial_x \tilde{m}$ is rapidly decaying in x, and because $\tilde{u}_\pm$ is odd, $\|t\mathrm{Op}(\tilde{m})(\tilde{u}_\pm, \dots, \tilde{u}_\pm)\|_{L^2}$ may be also estimate by the right-hand side of (1.26). Actually, the kind of expressions one has to cope with is morally of the form

$$tZ(x)\big(\langle D_x\rangle^{-1}\tilde{u}_\pm\big)^3, \tag{2.64}$$

where Z is in $\mathcal{S}(\mathbb{R})$ (This reflects the fact that $\partial_x \tilde{m}$ is rapidly decaying in x). Since $\tilde{u}_+$ is odd, we may write using the definition of $L_+ = x + t\frac{D_x}{\langle D_x\rangle}$

$$\begin{aligned}\langle D_x\rangle^{-1}\tilde{u}_+ &= ix\int_{-1}^{1}\Big(\frac{D_x}{\langle D_x\rangle}\tilde{u}_+\Big)(\mu x)\,d\mu\\ &= i\frac{x}{t}\int_{-1}^{1}\big((L_+\tilde{u}_+)(\mu x) - \mu x\tilde{u}_+(\mu x)\big)\,d\mu.\end{aligned} \tag{2.65}$$

The rapid decay of $Z(x)$ allows one to absorb the powers of x on the right-hand side of (2.65), and to estimate the L^2 norm of (2.64) by

$$C\big(\|L_+\tilde{u}_+\|_{L^2} + \|\tilde{u}_+\|_{L^2}\big)\|\tilde{u}_+\|_{L^\infty}^2,$$

i.e. by the right-hand side of (1.26) with $p = 1$. Similar arguments apply when the factors $\tilde{u}_\pm$ are replaced by $u_\pm^{\mathrm{app}}$.

The above reasoning disposes of the characteristic components in $\mathcal{M}_j(\tilde{u}, u^{\mathrm{app}})$ in (2.62). The non-characteristic ones are for instance of the form $\mathrm{Op}(\tilde{m})(\tilde{u}_+, \dots, \tilde{u}_+)$ and we no longer have an approximate commutation property of L_+ with such operators. These terms have thus to be eliminated by a space-time normal form. We construct in Proposition 6.2.1, using the results of Appendix F, operators $\hat{\mathcal{M}}_j$, $j = 3, 4$, such that

$$(D_t - P_0)\hat{\mathcal{M}}_j(\tilde{u}, u^{\mathrm{app}}) = \mathcal{M}_j(\tilde{u}, u^{\mathrm{app}})_{\mathrm{nch}} + \text{remainder}, \tag{2.66}$$

where $\mathcal{M}_j(\tilde{u}, u^{\mathrm{app}})_{\mathrm{nch}}$ denotes the non-characteristic contributions to $\mathcal{M}_j(\tilde{u}, u^{\mathrm{app}})$ on the right-hand side of (2.62). Actually, $\mathcal{M}_4(\tilde{u}, u^{\mathrm{app}})_{\mathrm{nch}} = \mathcal{M}_4(\tilde{u}, u^{\mathrm{app}})$ as only $\mathcal{M}_3$ contains characteristic components. In that way, we deduce from (2.62) that

$$\begin{aligned}(D_t - P_0)\big(C(t)\big(\tilde{u} - \hat{\mathcal{M}}_3(\tilde{u}, u^{\mathrm{app}}) - \hat{\mathcal{M}}_4(\tilde{u}, u^{\mathrm{app}})\big)\big)\\ = C(t)\mathcal{M}_2'(\tilde{u}, u'^{\mathrm{app},1}) + \mathcal{R},\end{aligned} \tag{2.67}$$

where the remainder $\mathcal{R}$ satisfies bounds of the form

$$\|L_+\mathcal{R}\|_{L^2} = O\big(\|\tilde{u}_+\|_{L^\infty}^2 \|L_+\tilde{u}_+\|_{L^2}\big)$$

as on the right-hand side of (1.26) with $p = 1$. Notice that to deduce (2.67) from (2.66), we have to compare $(D_t - P_0)C(t)\hat{\mathcal{M}}_j$ and $C(t)(D_t - P_0)\hat{\mathcal{M}}_j$ which by (2.61) makes appear a term $C(t)\mathcal{V}\hat{\mathcal{M}}_j$, but the time and space decay of operator $\mathcal{V}$ allows one to show that such errors form part of the remainder $\mathcal{R}$ in (2.67).

One has still on the right-hand side of (2.67) term $C(t)\mathcal{M}_2'(\tilde{u}, u'^{\mathrm{app},1})$. Again $\mathcal{M}_2'$ may be expressed in terms of quantities $\mathrm{Op}(m')(\tilde{u}_\pm, \tilde{u}_\pm)$ (and similar ones with $\tilde{u}_\pm$ replaced by $u'^{\mathrm{app},1}_\pm$), so that one may gain some time decay using expressions of the form (2.65), but as this term is just quadratic, this gain is not sufficient to include $C(t)\mathcal{M}_2'$ into $\mathcal{R}$ in (2.67). As $C(t) - \mathrm{Id}$ has some time decay, one may prove though that $(C(t) - \mathrm{Id})\mathcal{M}_2'$ is a remainder, but the expression $\mathcal{M}_2'(\tilde{u}, u'^{\mathrm{app},1})$ still needs to be eliminated from the right-hand side of (2.67). We do that in Proposition 6.2.4 of Chapter 6, using results of Appendix F. Actually, a quantity like $\mathrm{Op}(m')(\tilde{u}_\pm, \tilde{u}_\pm)$ may be expressed, using the x-rapid decay of m' and the oddness of $\tilde{u}_\pm$, as a sum of expressions of the form

$$t^{-2}K\big(L_\pm^{\ell_1}\tilde{u}_\pm, L_\pm^{\ell_2}\tilde{u}_\pm\big), \quad 0 \leq \ell_1, \ell_2 \leq 1, \tag{2.68}$$

where K is an operator of form

$$\widehat{K(f_1, f_2)}(\xi_0) = \int k(\xi_0, \xi_1, \xi_2)\hat{f}_1(\xi_1)\hat{f}(\xi_2)\, d\xi_1\, d\xi_2, \tag{2.69}$$

where the kernel k has rapid decay in $\langle \xi_0 - \xi_1 - \xi_2 \rangle$. An operator of form (2.68) slightly misses bounds in $O(t^{-1}\|L_+\tilde{u}_+\|_{L^2})$ when we make act on it $L_\pm$ and take the L^2 norm. But it does satisfy such estimates if we cut-off k in (2.69) on a domain $|\pm\langle\xi_0\rangle \pm \langle\xi_1\rangle \pm \langle\xi_2\rangle| \leq ct^{-\frac{1}{2}}$. Consequently, one may assume that in (2.69), k is supported for $|\pm\langle\xi_0\rangle \pm \langle\xi_1\rangle \pm \langle\xi_2\rangle| \geq ct^{-\frac{1}{2}}$. This extra cut-off allows to construct by normal forms a quadratic term $\hat{\mathcal{M}}_2'(\tilde{u}, u'^{\mathrm{app},1})$ such that

$$(D_t - P_0)\hat{\mathcal{M}}_2'(\tilde{u}, u'^{\mathrm{app},1}) = \mathcal{M}_2'(\tilde{u}, u'^{\mathrm{app},1}) + \text{remainder}.$$

Subtracting this equation from (2.67), one gets finally

$$(D_t - P_0)\mathring{u} = \hat{\mathcal{R}} \tag{2.70}$$

where

$$\mathring{u} = C(t)\Big(\tilde{u} - \sum_{j=3}^{4} \hat{\mathcal{M}}_j(\tilde{u}, u^{\mathrm{app}})\Big) - \hat{\mathcal{M}}'_2(\tilde{u}, u'^{\mathrm{app},1}). \tag{2.71}$$

and where $\hat{\mathcal{R}}$ will satisfy among other things essentially

$$\|L\hat{\mathcal{R}}(t,\cdot)\|_{L^2} = O(t^{-1}\|L_+\tilde{u}_+\|_{L^2}). \tag{2.72}$$

2.8 Step 6: Bootstrap of L^2 estimates

As seen above, the conclusion of the main theorem follows from the bootstrap of estimates (2.45)–(2.47). In Chapter 7, we perform the bootstrap of (2.45) and (2.46), assuming that (2.45)–(2.47) hold on some interval $[1,T]$ with $T \le \varepsilon^{-4+c}$ and showing that (2.45)–(2.46) then actually hold with the implicit constant on the right-hand side divided by 2 for instance. As we have seen, estimate (2.46) cannot be obtained making act L directly on (2.59), as the action of L on the right-hand side of this equation has bad upper bounds in L^2. On the other hand, making act L on (2.70), commuting it to $D_t - P_0$ and using (2.72), one may obtain a bound of the form (2.46) for $\|L_+\mathring{u}_+(t,\cdot)\|_{L^2}$. Actually, to do so with an improved implicit constant, one has to show that the right-hand side of (2.72) is $o(t^{-1}\|L_+\tilde{u}_+\|_{L^2})$ instead of just $O(t^{-1}\|L_+\tilde{u}_+\|_{L^2})$, but this follows from the estimates we get if $t \le \varepsilon^{-4+c}$ and $\varepsilon \ll 1$. The remaining thing to do is then to relate estimates for $L_+\mathring{u}_+$ in L^2 and estimates for $L_+\tilde{u}_+$, i.e. to show that the action of L_+ on the $\hat{\mathcal{M}}_j$, $\hat{\mathcal{M}}'_2$ terms in (2.71) do not perturb significantly the a priori bound of the left-hand side. We do that in Section 7.1 for $\hat{\mathcal{M}}_j$, $j = 3, 4$ and in Section 7.2 for $\hat{\mathcal{M}}'_2$. In this Chapter 7, we also check that the remainder $\hat{\mathcal{R}}$ in (2.70) satisfies (2.72). These estimates heavily rely on the boundedness properties of the different multilinear operators we use, that are discussed in Appendix D. Putting all of that together, we conclude the bootstrap for estimates (2.45)–(2.46) in Proposition 7.3.7.

2.9 Step 7: Bootstrap of L^∞ estimates and end of proof

The only remaining step in order to conclude the proof of the main theorem is to bootstrap bound (2.47). We do that in Chapter 8. We deduce from equation (2.56) satisfied by $\tilde{u}_+$ an ordinary differential equation. We proceed as in [1] for water waves, with simplifications inspired by Ifrim and Tataru [45] (see also [20, 82]). If we write equation (2.56) as $(D_t - p(D_x))\tilde{u}_+ = f_+$ and if we define $\underline{\tilde{u}}_+$, $\underline{f}_+$ by

$$\tilde{u}_+(t,x) = \frac{1}{\sqrt{t}}\underline{\tilde{u}}_+\Big(t, \frac{x}{t}\Big), \quad f_+(t,x) = \frac{1}{\sqrt{t}}\underline{f}_+\Big(t, \frac{x}{t}\Big), \tag{2.73}$$

we obtain

$$\Big(D_t - \mathrm{Op}_h^{\mathrm{W}}\big(x\xi + \sqrt{1+\xi^2}\big)\Big)\underline{\tilde{u}}_+ = \underline{f}_+, \tag{2.74}$$

where we used a Weyl semiclassical quantization, depending on the parameter $h = \frac{1}{t}$, defined in general by

$$\mathrm{Op}_h^{\mathrm{W}}(a(x,\xi)) = \frac{1}{2\pi h}\int e^{i(x-y)\frac{\xi}{h}} a\Big(\frac{x+y}{2},\xi\Big)u(y)\,dy\,d\xi. \tag{2.75}$$

We decompose then $\underline{\tilde{u}}_+ = \underline{\tilde{u}}_\Lambda + \underline{\tilde{u}}_{\Lambda^c}$, where

$$\underline{\tilde{u}}_\Lambda = \mathrm{Op}_h^{\mathrm{W}}\Big(\gamma\Big(\frac{x+p'(\xi)}{\sqrt{h}}\Big)\Big)\underline{\tilde{u}}_+ \tag{2.76}$$

with γ in $C_0^\infty(\mathbb{R})$, equal to one close to zero and with small enough support. Then $\underline{\tilde{u}}_\Lambda$ is localized close to the set $\Lambda = \{(x,\xi) : x = -p'(\xi)\}$, i.e. close to $\{\xi = d\varphi(x)\}$ if $\varphi(x) = \sqrt{1-x^2}$ is the phase of oscillations of solutions to linear Klein–Gordon equations (after rescaling (2.73)). One sees that the L^2 estimates (2.45)–(2.46) allow one to get wanted bounds for the component $\underline{\tilde{u}}_{\Lambda^c}$ (see Proposition 8.1.1). On the other hand, since $\underline{\tilde{u}}_\Lambda$ is microlocalized close to Λ, in the term $\mathrm{Op}_h^{\mathrm{W}}(x\xi + \sqrt{1+\xi^2})\underline{\tilde{u}}_\Lambda$ one may replace the symbol by its restriction to Λ, up to remainders that are well controlled thanks to the L^2 estimates (2.45)–(2.46). This brings an ODE for $\underline{\tilde{u}}_\Lambda$ that implies by integration the wanted bound (2.47). The end of Chapter 8 (Section 8.2) puts together these estimates and those obtained in Section 4.2 for $a(t)$ in order to close the bootstrap argument and prove the main conclusions (2.13) and (2.14).

2.10 Further comments

In the last section of the present chapter, we shall explain what is the difficulty in order to go beyond the time limit ε^{-4}. Since this is much related to a phenomenon extensively discussed in the two papers of Lindblad, Lührmann and Soffer [60] and Lindblad, Lührmann, Schlag and Soffer [59], as well as in the work of Germain and Pusateri [33], let us first recall some of the results of [60].

The authors of that paper consider an equation of the form

$$\big(D_t - \sqrt{1+D_x^2}\big)u = -\frac{1}{2}\langle D_x\rangle^{-1}\big(\alpha(\cdot)(u+\bar{u})^2\big) \tag{2.77}$$

on $\mathbb{R}\times\mathbb{R}$, where α is a smooth decaying function (say $\alpha \in \mathcal{S}(\mathbb{R})$, even if their assumptions are weaker), satisfying $\hat{\alpha}(\sqrt{3}) \neq 0$ or $\hat{\alpha}(-\sqrt{3}) \neq 0$. They prove that if (2.77) is supplemented by an initial data u_0 satisfying $\varepsilon = \|\langle x\rangle^2 u_0\|_{H^4} \ll 1$, then the solution to (2.77) may be decomposed as a sum

$$u(t,\cdot) = u_{\mathrm{free}}(t,x) + u_{\mathrm{mod}}(t,x), \tag{2.78}$$

where u_{free} satisfies the same dispersive estimates as a solution a linear Klein–Gordon equation, namely $\|u_{\mathrm{free}}(t,\cdot)\|_{L^\infty} = O(\varepsilon t^{-\frac{1}{2}})$ when t goes to $+\infty$, and where u_{mod} obeys only the weaker dispersive estimate

$$\|u_{\mathrm{mod}}(t,\cdot)\|_{L^\infty} = O\Big(\varepsilon^2\frac{\log t}{\sqrt{t}}\Big) \tag{2.79}$$

(see [60, Theorem 1.1] and in particular formulas (1.12) and (1.15)). Moreover, the logarithmic loss that appears on the right-hand side of (2.79), in comparison with the decay of linear solution, in unavoidable. Actually, Lindblad, Lührmann and Soffer show that along the rays $x = \pm\sqrt{3}t/2$, $u_{\mathrm{mod}}(t, \pm\sqrt{3}t/2)$ behaves when t goes to $+\infty$ as

$$\frac{a_0^2}{\sqrt{8}} e^{i\frac{\pi}{4}} e^{i\frac{t}{2}} \hat{\alpha}(\mp\sqrt{3}) \frac{\log t}{\sqrt{t}} \tag{2.80}$$

for some complex coefficient $a_0 = O(\varepsilon)$. (See [60, (1.15)] and (1.16) of the same paper for an explicit expression of a_0 in terms of the solution u to (2.77)). On the other hand, outside a conical neighborhood of these two rays, u_{mod} has an $\varepsilon^2 t^{-\frac{1}{2}}$ bound, without any logarithmic loss. In order to relate this with the obstacle that prevents us from going above time ε^{-4} in our own result, let us recall the argument of the introduction of [60] that explains heuristically the appearance of the logarithmic factor in (2.80). The idea is that, since $\alpha(x)$ on the right-hand side of (2.77) is decaying when x goes to infinity, one may replace there $u(t, x)$ by $u(t, 0)$, up to terms that are expected to have a stronger time decay. In that way, an approximation of (2.77) is

$$\big(D_t - \sqrt{1 + D_x^2}\big)u = -\frac{1}{2}\langle D_x\rangle^{-1}\big(\alpha(x)\big(u(t, 0) + \bar{u}(t, 0)\big)^2\big). \tag{2.81}$$

A second approximation (that is justified a posteriori) is to assume that $u(t, 0)$ will have the same asymptotic behavior as a solution to a linear Klein–Gordon equation restricted to $x = 0$ when t goes to infinity. This allows one to replace in (2.81) $u(t, 0)$ by $\varepsilon\frac{e^{it}}{\sqrt{t}}$, so that u_{mod} will be essentially the solution to

$$\big(D_t - \sqrt{1 + D_x^2}\big)u_{\mathrm{mod}} = -\frac{\varepsilon^2}{2t}\big(\langle D_x\rangle^{-1}\alpha\big)\big(e^{2it} + 2 + e^{-2it}\big). \tag{2.82}$$

If more generally one considers an equation of the form

$$\big(D_t - \sqrt{1 + D_x^2}\big)u = \frac{1}{t}Y(x)e^{i\lambda t} \tag{2.83}$$

with Y in $\mathcal{S}(\mathbb{R})$ (or at least smooth enough and decaying enough at infinity), one may rewrite (2.83) as an equation for $u_\lambda(t, x) = e^{-i\lambda t}u(t, x)$ of the form

$$\big(D_t + \lambda - \sqrt{1 + D_x^2}\big)u_\lambda = \frac{1}{t}Y(x). \tag{2.84}$$

If $\lambda < 1$, the operator $\sqrt{1 + D_x^2} - \lambda$ is elliptic and the solution to (2.84) will be $O(t^{-\frac{1}{2}})$ in L^∞ when t goes to infinity: This may be seen using Duhamel formula and integrating by parts, or equivalently defining

$$w_\lambda = u_\lambda + \big(\sqrt{1 + D_x^2} - \lambda\big)^{-1}(t^{-1}Y(x)) \tag{2.85}$$

that satisfies a new equation

$$\big(D_t + \lambda - \sqrt{1 + D_x^2}\big)w_\lambda = \frac{1}{t^2}\tilde{Y}(x), \tag{2.86}$$

where $\tilde{Y}$ is some new $\mathcal{S}(\mathbb{R})$ function and the new right-hand side is time integrable. Because of that, the solution to (2.86) will have the same dispersive time decay rate as a solution to a linear Klein–Gordon equation, i.e. will be $O(t^{-\frac{1}{2}})$ in L^∞. This is what happens for the last two terms on the right-hand side of (2.82). On the other hand, for the first one, one gets an equation of the form (2.83), (2.84) with $\lambda = 2$, so that the symbol $\sqrt{1+\xi^2}-2$ vanishes at $\xi = \pm\sqrt{3}$. In this case, the analysis of the solution to (2.86) expressed from Duhamel formula and Fourier transform shows that an asymptotic behavior of the form (2.80) holds along the two rays $x = \pm t\frac{\sqrt{3}}{2}$.

The logarithmic loss displayed in (2.80) seems incompatible with the known methods used to study global existence and asymptotic behavior for Klein–Gordon equations of the form (1.21) or (2.77) if we no longer assume that $\alpha(\cdot)$ is decaying at infinity. Actually, [60, Theorem 1.1] as well as [59, Theorem 1.1], uses in an essential way the fact that the space decay of this coefficient will provide, along the rays over which (2.80) holds, a time decay that will compensate the logarithmic loss.

Another situation when asymptotic behavior may be obtained for the solution of a problem of the form (2.77), including with nonlinearities involving terms like $(u+\bar{u})^2$, $(u+\bar{u})^3$ (without space decaying pre-factors), appears if the bad term (2.80) vanishes. This happens for the non-resonant case $\hat{\alpha}(\sqrt{3}) = \hat{\alpha}(-\sqrt{3}) = 0$ treated in [60, Theorem 1.6] and [59, Theorem 1.1], when one recovers the same asymptotics as those holding true for equations of the form (2.77) with the function α replaced by a constant.

The second case when (2.80) vanishes is when $a_0 = 0$. This happens for instance when α is an odd function and the initial condition in (2.77) is also odd (see (2.81) where the right-hand side vanishes for odd functions u, so that the contributions coming from (2.82) that were responsible of the bad term (2.80) disappear). Such a situation is studied by Germain and Pusateri [33], in a more general framework. They consider equations of the form

$$\left(\partial_t^2 - \partial_x^2 + V(x) + m^2\right)u = a(x)u^2, \tag{2.87}$$

where $a(x)$ is a smooth function that has different limits at $+\infty$ and $-\infty$ and $V(x)$ an $\mathcal{S}(\mathbb{R})$ potential that has no bound state. They prove a decay estimate for the solution in $O(t^{-\frac{1}{2}})$ when time goes to infinity, under some orthogonality assumption on the solution. This assumption always holds for generic potentials, and in the case of exceptional ones (like the zero potential), it holds under evenness or oddness conditions on V, a and the initial data. One of the key ingredients in the proof of [33, Theorem 1.1] is again related to the fact that a bad frequency $\pm\sqrt{3}$ appears. Actually, it shows up when one tries to perform a variable coefficients normal form. In order to overcome this difficulty, the authors introduce functional spaces, involving dyadic Fourier cut-offs close to the bad frequencies, and measuring the (distorted) Fourier transform of the solution in such spaces.

Let us go back to the problem we study in this book, and in particular to the limitation of our result to times $O(\varepsilon^{-4})$. We already discussed this issue in Section 2.5 after the introduction of the approximate solution in (2.49). Here, we want to explain how the problem we encounter to go beyond time ε^{-4} might be related to some of the works we just described, namely the possible appearance of some extra logarithm in pointwise estimates of the solution along two rays, as in (2.80). Remark first that we are dealing only with odd solutions. As already noticed, this implies that the coefficient a_0 in (2.80) vanishes, so that a solution of a problem of the form (2.77) has $O(t^{-\frac{1}{2}})$ L^∞ estimates. The point is that, in our problem, we do not study an equation of the form (2.77) or (2.87), but a *coupling* between a PDE and an ODE, namely system (2.11) or equivalently, a coupling between the PDE (2.27) and the ODE (2.34). Because of that, our PDE contains a source term given by (2.28), involving expressions of the form

$$a(t)^2 Y_2(x), a(t)^3 Y_3(x), \tag{2.88}$$

where Y_2, Y_3 are $\mathcal{S}(\mathbb{R})$ functions and $a(t)$, solution of the ODE, has an oscillatory behavior of the form

$$\frac{\varepsilon}{\sqrt{1+t\varepsilon^2}} e^{\pm it\frac{\sqrt{3}}{2}}. \tag{2.89}$$

When plugged in (2.88), this shows that our PDE will contain a source term that has a similar structure as the right-hand side of (2.82), with oscillating terms $e^{\pm it\sqrt{3}}$ instead of $e^{\pm 2it}$ and pre-factor $\frac{\varepsilon^2}{1+t\varepsilon^2}$ instead of $\frac{\varepsilon^2}{t}$ (for the quadratic contribution coming from (2.88)). Because of that, and by analogy with the study of [60], we may expect that the solution to our PDE contains contributions that might grow as $\frac{\log t}{\sqrt{t}}$ when t goes to infinity.

In this book, we prove that such a possible growth does not happen before at least time ε^{-4+0}. Let us return to the discussion on that issue that we started in Section 2.5. We introduced in (2.49) a solution u_+^{app} of a linear equation with source terms that are essentially of the form (2.88) (forgetting the second line of the first equation in (2.49)). If we retain only the quadratic term $a(t)^2 Y_2$ in (2.88), and use (2.89), this means that we have to solve essentially an equation of the form

$$\big(D_t - \sqrt{1+D_x^2}\big)U = \frac{\varepsilon^2}{1+t\varepsilon^2} e^{\pm it\sqrt{3}} M(x) \tag{2.90}$$

for some function M in $\mathcal{S}(\mathbb{R})$ and zero initial data at $t=1$. This is an equation of the form (2.83), and as we have seen after (2.84), the delicate case is the one corresponding to the phase $t\sqrt{3}$ in the exponential, so that in the sequel we discuss only (2.90) with sign $+$. Then U is one of the contribution to the approximate solution u_+^{app} of (2.49), and we decompose it as $U = U' + U''$ with essentially

$$U'(t,x) = i\int_1^{\sqrt{t}} e^{i(t-\tau)\sqrt{1+D_x^2}+it\sqrt{3}} M(\cdot)\, \frac{\varepsilon^2\, d\tau}{1+\tau\varepsilon^2}, \tag{2.91}$$

$$U''(t,x) = i\int_{\sqrt{t}}^{t} e^{i(t-\tau)\sqrt{1+D_x^2}+it\sqrt{3}} M(\cdot)\, \frac{\varepsilon^2\, d\tau}{1+\tau\varepsilon^2}. \tag{2.92}$$

This decomposition corresponds to $u_+^{\text{app}} = u_+'^{\text{app}} + u_+''^{\text{app}}$ introduced before (2.50) in Section 2.5, and we may prove some good L^∞ estimate for L_+U'' (see (2.50)) and some good L^2 estimate for L_+U' (of the form (2.46)) for times $t = O(\varepsilon^{-4+0})$. This last L^2 bound degenerates when t goes to ε^{-4}, and actually so does the pointwise estimate of U' that is obtained in Appendix C (see (C.40) with $\omega = 1$). We obtain there for U' a pointwise bound in

$$\frac{(\varepsilon^2\sqrt{t})}{\sqrt{t}}\left\langle t^{\frac{1}{2}}\left(\frac{x}{t} \pm \sqrt{\frac{2}{3}}\right)\right\rangle^{-1}. \tag{2.93}$$

Outside a conical neighborhood of the rays $x = \mp t\sqrt{2/3}$, (2.93) reduces to an $\varepsilon^2 t^{-\frac{1}{2}}$ decay (whatever the value of t). On the other hand, along the lines $x = \mp t\sqrt{2/3}$, we just get a bound in $(\varepsilon^2\sqrt{t})/\sqrt{t}$, that provides an $O(t^{-\frac{1}{2}})$ decay only for $t = O(\varepsilon^{-4})$. Past such a time, estimate (2.93) will no longer remain valid and, at the light of the results of [60] concerning (2.77) and [59], one may not exclude that some $\log t/\sqrt{t}$ behavior might hold along the two preceding rays. Since, unlike in (2.77), we no not have just nonlinearities involving rapidly space decaying coefficients, we do not know how such contributions might be handled in the nonlinear problem.

Chapter 3

First quadratic normal form

In Section 2.2 of the preceding chapter, we have introduced an evolution equation (2.27) for a function u_+. This equation is of the type of (1.58) in the introduction, except that its nonlinearity is non-local (see (2.31) and (2.32)). In this chapter, we shall express these nonlinearities in terms of multilinear operators, that are a special case of classes introduced in Appendix B. This will give us a general framework that will be stable under the reductions we shall have to perform.

The nonlinearity in our equation contains quadratic terms. We have already explained in Section 1.6 of the introduction that such terms have to be eliminated by normal form. This is the goal of Section 3.2 of this chapter, following the guidelines explained in Section 2.4 of Chapter 2.

3.1 Expression of the equation from multilinear operators

Let us define the classes of multilinear operators we shall use. They are special cases of the operators introduced in Appendix B, that will be useful in the rest of the paper. We introduce in this section only the subclasses we need in Chapter 3.

In this chapter, an order function on $\mathbb{R}^p$ is a function from $\mathbb{R}^p$ to $\mathbb{R}_+$ such that there is some $N_0 \in \mathbb{N}$ so that, for any $(\xi_1, \dots, \xi_p), (\xi'_1, \dots, \xi'_p) \in \mathbb{R}^p$,

$$M(\xi'_1, \dots, \xi'_p) \leq C \prod_{j=1}^{p} \langle \xi_j - \xi'_j \rangle^{N_0} M(\xi_1, \dots, \xi_p). \tag{3.1}$$

(In Appendix B, we shall allow order functions depending also on a space variable x.)

Definition 3.1.1. Let M be an order function on $\mathbb{R}^p$, with $p \in \mathbb{N}^*$, $\kappa \in \mathbb{N}$. We denote by $\tilde{S}_{\kappa,0}(M, p)$ the space of smooth functions

$$\begin{gathered}(y, \xi_1, \dots, \xi_p) \mapsto a(y, \xi_1, \dots, \xi_p), \\ \mathbb{R} \times \mathbb{R}^p \to \mathbb{C}\end{gathered} \tag{3.2}$$

satisfying for any $\alpha \in \mathbb{N}^p$,

$$|\partial_\xi^\alpha a(y, \xi)| \leq C M(\xi) M_0(\xi)^{\kappa|\alpha|} \tag{3.3}$$

and for any $\alpha \in \mathbb{N}^p$, any $\alpha'_0 \in \mathbb{N}^*$, any $N \in \mathbb{N}$,

$$|\partial_\xi^\alpha \partial_y^{\alpha'_0} a(y, \xi)| \leq C M(\xi) M_0(\xi)^{\kappa|\alpha|} \big(1 + M_0(\xi)^{-\kappa} |y|\big)^{-N}, \tag{3.4}$$

where $M_0(\xi)$ denotes

$$M_0(\xi_1, \dots, \xi_p) = \Bigg(\sum_{1 \leq i < j \leq p} \langle \xi_i \rangle^2 \langle \xi_j \rangle^2 \Bigg) \Bigg(\sum_{i=1}^{p} \langle \xi_i \rangle^2 \Bigg)^{-\frac{1}{2}} \tag{3.5}$$

and is equivalent to $1 + \max_2(|\xi_1|, \dots, |\xi_p|)$, $\max_2$ standing for the second largest of the arguments.

We denote by $\tilde{S}'_{\kappa,0}(M, p)$ the subspace of $\tilde{S}_{\kappa,0}(M, p)$ of those a for which (3.4) holds including for $\alpha'_0 = 0$.

The symbols of Definition 3.1.1 are the special case of those defined in Definition B.1.2 of Appendix B when there is no x dependence in (B.11). We associate to them operators through the quantization rule

$$\begin{aligned}\mathrm{Op}(a)(v_1, \dots, v_p) = \frac{1}{(2\pi)^p} \int e^{ix(\xi_1 + \dots + \xi_p)} a(x, \xi_1, \dots, \xi_p) \\ \times \prod_{j=1}^{p} \hat{v}_j(\xi_j)\, d\xi_1 \cdots d\xi_p\end{aligned} \tag{3.6}$$

for any $a \in \tilde{S}_{\kappa,0}(M, p)$, any test functions $v_1, \dots, v_p$. This is the rule defined in (B.17) of the appendix in the case of general symbols $a(y, x, \xi)$, specialized to the subclass of symbols that do not depend on x, as in Definition 3.1.1. We shall also impose on our symbols the extra condition

$$a(-y, -\xi_1, \dots, -\xi_p) = (-1)^{p-1} a(y, \xi_1, \dots, \xi_p). \tag{3.7}$$

Under this condition, the operator $\mathrm{Op}(a)$ sends a p-tuple of odd functions to an odd function.

Let us state the symbolic calculus result that is proved in Appendix B (see Corollary B.2.6, (B.42), (B.43)) and that we shall use below.

Proposition 3.1.2. *The following statements hold.*

(i) *Let $n', n'' \in \mathbb{N}^*$, $n = n' + n'' - 1$, let $M'(\xi_1, \dots, \xi_{n'})$, $M''(\xi_{n'}, \dots, \xi_n)$ be two order functions. Let a (resp. b) be in $\tilde{S}_{\kappa,0}(M', n')$ (resp. $\tilde{S}_{\kappa,0}(M'', n'')$). Define*

$$M(\xi_1, \dots, \xi_n) = M'(\xi_1, \dots, \xi_{n'-1}, \xi_{n'} + \dots + \xi_n) M''(\xi_{n'}, \dots, \xi_n). \tag{3.8}$$

There are $\nu \in \mathbb{N}$, depending only on the order functions M' and M'', and a symbol c'_1 in $\tilde{S}'_{\kappa,0}(M M_0^{\nu\kappa}, n)$ such that if

$$\begin{aligned}c(y, \xi_1, \dots, \xi_n) = a(y, \xi_1, \dots, \xi_{n'-1}, \xi_{n'} + \dots + \xi_n) b(y, \xi_{n'}, \dots, \xi_n) \\ + c'_1(y, \xi_1, \dots, \xi_n),\end{aligned} \tag{3.9}$$

then for all test functions $v_1, \dots, v_n$,

$$\mathrm{Op}(a)[v_1, \dots, v_{n'-1}, \mathrm{Op}(b)(v_{n'}, \dots, v_n)] = \mathrm{Op}(c)[v_1, \dots, v_n]. \tag{3.10}$$

Moreover, if a and b satisfy (3.7), so do c and c'_1.

(ii) *If a is in $\tilde{S}_{0,0}(M, 1)$, there is a symbol a^* in $\tilde{S}_{0,0}(M, 1)$ such that* $\mathrm{Op}(a^*) = \mathrm{Op}(a)^*$. *Moreover, if a satisfies (3.7), so does a^*.*

We shall use the above class of symbols to re-express equation (2.27).

Proposition 3.1.3. *For any multiindex* $I = (i_1, \dots, i_p) \in \{-,+\}^p$ *with* $2 \le |I| = p \le 3$, *one may find symbols* $m_{0,I}$ *in* $\tilde{S}_{0,0}(\prod_{j=1}^{p} \langle \xi_j \rangle^{-1}, p)$ *satisfying condition* (3.7), *and for any multiindex* $I = (i_1, \dots, i_p) \in \{-,+\}^p$ *with* $1 \le |I| = p \le 2$, *one may find symbols* $m'_{1,I}$ *in* $\tilde{S}'_{0,0}(\prod_{j=1}^{p} \langle \xi_j \rangle^{-1}, p)$ *satisfying condition* (3.7), *such that equation* (2.27) *may be written*

$$\begin{aligned}\big(D_t - p(D_x)\big)u_+ = F_0^2[a] + F_0^3[a] &+ \sum_{2\le |I| \le 3} \mathrm{Op}(m_{0,I})[u_I] \\ &+ a(t) \sum_{1 \le |I| \le 2} \mathrm{Op}(m'_{1,I})[u_I] \\ &+ a(t)^2 \sum_{|I|=1} \mathrm{Op}(m'_{2,I})[u_I],\end{aligned} \tag{3.11}$$

where u_I *is defined in* (2.25) *and* (2.26).

Proof. Consider first the terms on the right-hand side of equation (2.27) that do not depend on a, i.e. with notation (2.29) $\sum_{|I|=2} F_{2,I}^2[u_I]$ and $\sum_{|I|=3} F_{3,I}^3[u_I]$. These terms are given by the first equality in (2.31) and (2.32). A symbol of the form $\kappa(y) \prod_{\ell=1}^{2} b(y, \xi_j) p(\xi_j)^{-1}$ or $\prod_{\ell=1}^{3} b(y, \xi_j) p(\xi_j)^{-1}$ belongs respectively to $\tilde{S}_{0,0}(\prod_{\ell=1}^{2} \langle \xi_j \rangle^{-1}, 2)$ and $\tilde{S}_{0,0}(\prod_{\ell=1}^{3} \langle \xi_j \rangle^{-1}, 3)$ and because of property (A.9) satisfied by b and the oddness of κ, condition (3.7) holds. If we apply the results of Proposition 3.1.2, we conclude that the contributions to (2.27) that do not depend on a have the structure of the first sum on the right-hand side of (3.11).

Consider next terms of the form $a(t) F_{1,I}^2[u_I]$, $|I| = 1$ or $a(t) F_{2,I}^3[u_I]$, $|I| = 2$ in equation (2.29). They may be expressed from the first line in (2.30) and the second line in (2.31). Since Y is rapidly decaying, the symbols $Y(y)\kappa(y)b(y,\xi)p(\xi)^{-1}$ and $Y(y) \prod_{\ell=1}^{2} b(y, \xi_j) p(\xi_j)^{-1}$ are in $\tilde{S}'_{0,0}(\langle \xi \rangle^{-1}, 1)$ and $\tilde{S}'_{0,0}(\prod_{j=1}^{2} \langle \xi_j \rangle^{-1}, 2)$. Because of the oddness of Y, κ and (A.9), they satisfy (3.7). Using again the composition result of Proposition 3.1.2, and noticing that as soon as at least one of the symbols a and b in (3.9) is in the $\tilde{S}'$ class, so is the composed symbol c, we conclude that the linear term in $a(t)$ on the right-hand side of (2.27) is given by the second sum in (3.11).

In the same way, the contributions $a(t)^2 F_{1,I}^3[u_I]$ coming from the second line (2.29) with $j = 1$, with $F_{1,I}^3$ given by (2.30), provide the last sum in (3.11). This concludes the proof. ■

On the right-hand side of equation (3.11), terms with higher degree of homogeneity in $(a, u_\pm)$ will have better decay estimates. Moreover, an expression of the form $\mathrm{Op}(m')[u_I]$ with $|I| = p$ and a symbol m' in $\tilde{S}'_{0,0}(M, p)$, i.e. with rapid decay in y, will have better time decay than a term $\mathrm{Op}(m)[u_I]$ with $|I| = p$ and a symbol m in $\tilde{S}_{0,0}(M, p)$. Consequently, we expect that the terms in $\sum_{|I|=2} \mathrm{Op}(m_{0,I})[u_I]$ will be, among all $u_\pm$-dependent terms on the right-hand side of (3.11), those having the worst time decay. In next section, we shall get rid of these terms by normal form.

3.2 First quadratic normal form

Proposition 3.2.1. *Define from the symbols $m_{0,I}$, $|I| = 2$ of Proposition 3.1.3 new functions*

$$\tilde{m}_{0,I}(y, \xi_1, \xi_2) = m_{0,I}(y, \xi_1, \xi_2)\big(-p(\xi_1 + \xi_2) + i_1 p(\xi_1) + i_2 p(\xi_2)\big)^{-1} \tag{3.12}$$

if $I = (i_1, i_2)$. Then $\tilde{m}_{0,I}$ belongs to $\tilde{S}_{1,0}(\prod_{j=1}^{2}\langle\xi_j\rangle^{-1}M_0(\xi_1,\xi_2), 2)$. Moreover, there are new symbols

- $(m'_{0,I})_{|I|=2}$ *belonging to* $\tilde{S}'_{1,0}(\prod_{j=1}^{2}\langle\xi_j\rangle^{-1}M_0(\xi), 2)$,
- $(m'_{j,I})_{1\le|I|\le 4-j}$, $1 \le j \le 3$, *in* $\tilde{S}'_{1,0}(\prod_{j=1}^{|I|}\langle\xi_j\rangle^{-1}M_0(\xi)^{\nu}, |I|)$ *for some* ν,
- $(m_{0,I})_{3\le|I|\le 4}$ *belonging to* $\tilde{S}_{1,0}(\prod_{j=1}^{|I|}\langle\xi_j\rangle^{-1}M_0(\xi), |I|)$

such that

$$\begin{aligned}\big(D_t - p(D_x)\big)\Big(u_+ - \sum_{|I|=2}\mathrm{Op}(\tilde{m}_{0,I})[u_I]\Big)\\ = F_0^2[a] + F_0^3[a] + \sum_{3\le|I|\le 4}\mathrm{Op}(m_{0,I})[u_I] + \sum_{|I|=2}\mathrm{Op}(m'_{0,I})[u_I]\\ + \sum_{j=1}^{3} a(t)^j \sum_{1\le|I|\le 4-j}\mathrm{Op}(m'_{j,I})[u_I].\end{aligned} \tag{3.13}$$

Finally, all above symbols satisfy (3.7).

Proof. We notice first that

$$\begin{aligned}\langle\xi_1\rangle + \langle\xi_2\rangle - \langle\xi_1 + \xi_2\rangle &= \frac{1 + 2(\langle\xi_1\rangle\langle\xi_2\rangle - \xi_1\xi_2)}{\langle\xi_1\rangle + \langle\xi_2\rangle + \langle\xi_1 + \xi_2\rangle}\\ &\ge c\big(1 + \max{}_2(|\xi_1|, |\xi_2|)\big)^{-1}\\ &\ge c M_0(\xi_1, \xi_2)^{-1}.\end{aligned} \tag{3.14}$$

This implies that

$$\langle\xi_1 + \xi_2\rangle + \langle\xi_2\rangle - \langle\xi_1\rangle \ge c\big(1 + \max{}_2(|\xi_1 + \xi_2|, |\xi_2|)\big)^{-1}$$

which is larger than the right-hand side of (3.14), except when $|\xi_2| \gg |\xi_1|$. But then the left-hand side is larger than one. Consequently, we deduce from these inequalities that, for any sign i_1, i_2, we have for any $\alpha \in \mathbb{N}^2$,

$$\Big|\partial_\xi^\alpha\big(\langle\xi_1 + \xi_2\rangle + i_1\langle\xi_1\rangle + i_2\langle\xi_2\rangle\big)^{-1}\Big| \le C_\alpha M_0(\xi_1, \xi_2)^{1+|\alpha|}. \tag{3.15}$$

This implies that $\tilde{m}_{0,I}$ belongs to the wanted class of symbols. It obeys trivially (3.7) since $m_{0,I}$ does.

Denoting for $|I| = 2$, $u_I = (u_{i_1}, u_{i_2})$ as in (2.25), we compute

$$\begin{aligned}
&\big(D_t - p(D_x)\big)\big[\mathrm{Op}(\tilde{m}_{0,I})[u_I]\big] \\
&\quad = -\mathrm{Op}(p(\xi)) \circ \mathrm{Op}(\tilde{m}_{0,I})[u_I] + \mathrm{Op}(\tilde{m}_{0,I})[i_1 \mathrm{Op}(p(\xi))u_{i_1}, u_{i_2}] \\
&\qquad + \mathrm{Op}(\tilde{m}_{0,I})[u_{i_1}, i_2 \mathrm{Op}(p(\xi))u_{i_2}] \\
&\qquad + \mathrm{Op}(\tilde{m}_{0,I})[(D_t - i_1 p(D_x))u_{i_1}, u_{i_2}] \\
&\qquad + \mathrm{Op}(\tilde{m}_{0,I})[u_{i_1}, (D_t - i_2 p(D_x))u_{i_2}].
\end{aligned} \tag{3.16}$$

By Corollary B.2.7, the sum of the first three terms on the right-hand side may be written as a contribution to $\sum_{|I|=2} \mathrm{Op}(m'_{0,I})[u_I]$ in (3.13) plus the expression

$$\mathrm{Op}\big((-p(\xi_1 + \xi_2) + i_1 p(\xi_1) + i_2 p(\xi_2))\tilde{m}_{0,I}\big)[u_I]. \tag{3.17}$$

By (3.12), (3.17) will cancel the term $\sum_{|I|=2} \mathrm{Op}(m_{0,I})[u_I]$ in (3.11). Since the other terms on the right-hand side of (3.11) are still present in (3.13), we see that to conclude the proof, we just need to show that the last two terms in (3.16) provide as well contributions to the three sums on the right-hand side of (3.13). We express $(D_t \mp p(D_x))u_\pm$ from (3.11) (or its conjugate). To fix ideas, consider for instance

$$\mathrm{Op}(\tilde{m}_{0,(+,i_2)})[(D_t - p(D_x))u_+, u_{i_2}]. \tag{3.18}$$

If we replace $(D_t - p(D_x))u_+$ by the contribution $F_0^2[a] + F_0^3[a]$, which by (2.28) may be written $a(t)^2 Y_2 + a(t)^3 Y_3$, with odd functions Y_2, Y_3 in $\mathcal{S}(\mathbb{R})$, we see applying Corollary B.2.8 of Appendix B that expression (3.18) will provide contributions to the $\sum_{j=2}^{3} a(t)^j \sum_{|I|=1} \mathrm{Op}(m'_{j,I})[u_I]$ term in (3.13).

We replace next $(D_t - p(D_x))u_+$ in (3.18) by the $a(t)$ or $a(t)^2$ terms in (3.11). We use (i) of Proposition 3.1.2, noticing that if in (3.9), either a is in $\tilde{S}'_{\kappa,0}(M', n')$ or b is in $\tilde{S}'_{\kappa,0}(M'', n'')$, then c is in $\tilde{S}'_{\kappa,0}(M, n)$. Consequently, we get contributions to $a(t) \sum_{2 \le |I| \le 3} \mathrm{Op}(m'_{1,I})[u_I]$ and $a(t)^2 \sum_{|I|=2} \mathrm{Op}(m'_{1,I})[u_I]$ in (3.13). Finally, if we replace in (3.18) $(D_t - p(D_x))u_+$ by the first sum on the right-hand side of (3.11), we obtain contributions to $\sum_{3 \le |I| \le 4} \mathrm{Op}(m_{0,I}[u_I])$ in (3.13) using again (i) of Proposition 3.1.2. This concludes the proof as property (3.7) of the symbols is preserved under composition. ∎

Chapter 4

Construction of approximate solutions

In the preceding chapter, we have performed a quadratic normal form in order to reduce ourselves to an equation of the form (3.13). The right-hand side of this equation contains a source term and in Section 4.1 below, we construct an approximate solution solving the linear equation whose right-hand side is essentially this source term. We explained this part of the proof in Section 2.5, see equations (2.48)–(2.49). The construction of the approximate solution relies on Appendix C below.

On the other hand, because of the coupling between a dispersive equation and the evolution equation for the bound state, we have seen in Section 2.2 that we have also to study an ordinary differential equation (2.34), which is equivalent to the first equation in (2.9). We have explained at the end of Section 2.5 what is the form of that ODE, and how we can show that its solutions are global and decaying using Fermi's golden rule. Section 4.2 below is devoted to the asymptotic analysis of this ODE. Of course, the study is more technical than in the presentation in Chapter 2 since we have to fully take into account those terms on the right-hand side that come from the interaction between the bound state and the dispersive part of our problem.

4.1 Approximate solution to the dispersive equation

The proof of our main theorem being done by bootstrap, we shall assume that we know, on some interval $[1,T]$, an approximation of the function $t \mapsto a(t)$ that is present on the right-hand side of (3.13).

Let $\varepsilon_0 \in]0,1]$, $A, A' > 1$, $\theta' \in]0, \frac{1}{2}[$ (close to $\frac{1}{2}$) be given. Let $T \in [1, \varepsilon^{-4}]$. We shall denote for $t \geq 1$, $\varepsilon \in]0, \varepsilon_0[$,

$$t_\varepsilon = \varepsilon^{-2}\langle t\varepsilon^2\rangle \tag{4.1}$$

and assume given functions

$$\begin{aligned} g : [1,T] &\to \mathbb{C}, & \tilde{u}_\pm : [1,T]\times\mathbb{R} &\to \mathbb{C}, \\ t &\mapsto g(t), & (t,x) &\mapsto \tilde{u}_\pm(t,x) \end{aligned} \tag{4.2}$$

and $x \mapsto Z(x)$ in $\mathcal{S}(\mathbb{R})$, real valued, satisfying the following conditions:

$$|g(t)| \leq At_\varepsilon^{-\frac{1}{2}}, \quad |\partial_t g(t)| \leq A'\big(t_\varepsilon^{-\frac{3}{2}} + (\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'}t^{-\frac{3}{2}}\big), \quad t \in [1,T], \tag{4.3}$$

$$|\langle Z, \tilde{u}_\pm(t,\cdot)\rangle| \leq (\varepsilon^2\sqrt{t})^{\theta'}t^{-\frac{3}{4}}, \quad t \in [1,T]. \tag{4.4}$$

Moreover, we assume given $\widetilde{W}$ a neighborhood of $\{-1,1\}$ in $\mathbb{R}$ and for any λ in $\mathbb{R} - \widetilde{W}$, two functions

$$t \mapsto \varphi_\pm(\lambda,t), \quad t \mapsto \psi_\pm(\lambda,t) \tag{4.5}$$

satisfying for any $t \in [1,T]$, any $\lambda \in \mathbb{R} - \widetilde{\mathcal{W}}$,

$$|\varphi_\pm(\lambda,t)| \le (\varepsilon^2\sqrt{t})^{\theta'} t^{-\frac{1}{2}}, \quad |\psi_\pm(\lambda,t)| \le (\varepsilon^2\sqrt{t})^{\theta'} t^{-1} \tag{4.6}$$

and solving the equation

$$(D_t - \lambda)\varphi_\pm(\lambda,t) = \langle Z, \tilde{u}_\pm\rangle + \psi_\pm(\lambda,t). \tag{4.7}$$

We define from the above data

$$\begin{aligned} a_+^{\mathrm{app}}(t) = {} & e^{it\frac{\sqrt{3}}{2}} g(t) + \omega_2 g(t)^2 e^{it\sqrt{3}} + \omega_0 |g(t)|^2 + \omega_{-2}\overline{g(t)}^2 e^{-it\sqrt{3}} \\ & + e^{it\frac{\sqrt{3}}{2}}\big(g(t)\varphi_+(0,t) - g(t)\varphi_-(0,t)\big) \\ & + e^{-it\frac{\sqrt{3}}{2}}\big(\overline{g(t)}\varphi_+(\sqrt{3},t) - \overline{g(t)}\varphi_-(\sqrt{3},t)\big), \end{aligned} \tag{4.8}$$

where $\omega_0, \omega_2, \omega_{-2}$ are given complex constants. We set

$$a_-^{\mathrm{app}} = -\overline{a_+^{\mathrm{app}}}, \quad a^{\mathrm{app}}(t) = \frac{\sqrt{3}}{3}\big(a_+^{\mathrm{app}}(t) - a_-^{\mathrm{app}}(t)\big). \tag{4.9}$$

We assume given, as in the statement of Proposition 3.2.1, symbols $m'_{1,I}$ for $|I| = 1$ (i.e. $I = +$ or $-$) belonging to the class $\tilde{S}'_{1,0}(\langle\xi\rangle^{-1}, 1)$ satisfying (3.7). We want to construct an approximate solution u_+^{app} to the equation

$$\big(D_t - p(D_x)\big)u_+^{\mathrm{app}} = F_0^2[a^{\mathrm{app}}] + F_0^3[a^{\mathrm{app}}] + a^{\mathrm{app}}(t)\sum_{|I|=1} \mathrm{Op}(m'_{1,I})[u_I^{\mathrm{app}}] \tag{4.10}$$

that is deduced from (3.13) computing the source terms F_0^2, F_0^3 at a^{app}, and retaining from the other terms on the right-hand side only those that are linear both in a and $u_\pm$.

Before stating the main proposition, let us re-express the source term in (4.10).

Lemma 4.1.1. *Under the preceding assumptions on a^{app}, one may rewrite*

$$F_0^2[a^{\mathrm{app}}] + F_0^3[a^{\mathrm{app}}] = I_1 + I_2 + I_3 + R(t,x), \tag{4.11}$$

where

$$I_1(t,x) = \sum_{j\in\{-2,0,2\}} e^{ijt\frac{\sqrt{3}}{2}} M_j(t,x) \tag{4.12}$$

for smooth odd functions of x, $M_j(t,x)$, satisfying for any $\alpha, N \in \mathbb{N}$,

$$\begin{aligned} &|\partial_\xi^\alpha \hat{M}_j(t,\xi)| \le C_{\alpha,N} t_\varepsilon^{-1}\langle\xi\rangle^{-N}, \\ &|\partial_\xi^\alpha \partial_t \hat{M}_j(t,\xi)| \le C_{\alpha,N}\langle\xi\rangle^{-N} t_\varepsilon^{-\frac{1}{2}}\big(t_\varepsilon^{-\frac{3}{2}} + t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'}\big) \end{aligned} \tag{4.13}$$

with constants $C_{\alpha,N}$ depending on A, A' in (4.3)–(4.4), where

$$I_2(t,x) = \sum_{j\in\{-3,-1,1,3\}} e^{ijt\frac{\sqrt{3}}{2}} M_j(t,x) \tag{4.14}$$

for smooth odd functions of x satisfying

$$\begin{aligned} |\partial_\xi^\alpha \hat{M}_j(t,\xi)| &\leq C_{\alpha,N} t_\varepsilon^{-\frac{3}{2}} \langle \xi \rangle^{-N}, \\ |\partial_\xi^\alpha \partial_t \hat{M}_j(t,\xi)| &\leq C_{\alpha,N} \langle \xi \rangle^{-N} t_\varepsilon^{-1} \big(t_\varepsilon^{-\frac{3}{2}} + t^{-\frac{3}{2}} (\varepsilon^2 \sqrt{t})^{\frac{3}{2}\theta'} \big), \end{aligned} \tag{4.15}$$

and where I_3 is a sum of terms

$$I_3(t,x) = \sum_{j=-1}^{1} e^{ijt\sqrt{3}} M_j^3(t,x), \tag{4.16}$$

where M_j^3 are odd and satisfy the following conditions: First, for any j with $|j| \leq 1$, any α, N,

$$\begin{aligned} |\partial_\xi^\alpha \hat{M}_j^3(t,\xi)| &\leq C_{\alpha,N} t_\varepsilon^{-1} t^{-\frac{1}{2}} \langle \xi \rangle^{-N} \\ |\partial_\xi^\alpha \partial_t \hat{M}_j^3(t,\xi)| &\leq C_{\alpha,N} t_\varepsilon^{-1} t^{-\frac{3}{4}} \langle \xi \rangle^{-N}. \end{aligned} \tag{4.17}$$

Moreover, for $j = 1$, and when ξ is a point in a small neighborhood $\mathcal{W}$ of the set $\{\xi : \sqrt{1+\xi^2} = \sqrt{3}\}$, one may find functions $\tilde{\Phi}_1(t,\xi)$, $\tilde{\Psi}_1(t,\xi)$, satisfying

$$|\tilde{\Phi}_1(t,\xi)| \leq C t_\varepsilon^{-1} t^{-\frac{1}{2}}, \quad |\tilde{\Psi}_1(t,\xi)| \leq C t_\varepsilon^{-1} t^{-1} \tag{4.18}$$

such that for $\xi \in \mathcal{W}$,

$$D_t \hat{M}_1^3(t,\xi) = \big(D_t + (\sqrt{3} - \sqrt{1+\xi^2}) \big) \tilde{\Phi}_1(t,\xi) + \tilde{\Psi}_1(t,\xi). \tag{4.19}$$

A similar decomposition holds for $x M_1^3$ instead of M_1^3.

Finally, the remainder R in (4.11) satisfies for any $\alpha, N \in \mathbb{N}$,

$$|\partial_x^\alpha R(t,x)| \leq C_{\alpha,N} t^{-1} t_\varepsilon^{-1} \langle x \rangle^{-N} \tag{4.20}$$

and we have for $M_j(t,x)$ in (4.12) the following explicit expressions:

$$\begin{aligned} M_2(t,x) &= \frac{1}{3} g(t)^2 Y_2(x), \\ M_0(t,x) &= \frac{2}{3} |g(t)|^2 Y_2(x), \\ M_{-2}(t,x) &= \frac{1}{3} \overline{g(t)}^2 Y_2(x), \end{aligned} \tag{4.21}$$

where Y_2 is given by

$$Y_2(x) = b(x, D_x)^* (\kappa(x) Y(x)^2) \in \mathcal{S}(\mathbb{R}). \tag{4.22}$$

Moreover, the constants in all above inequalities depend only on A, A' in (4.3)–(4.4).

Proof. Consider first the contribution $F_0^2[a^{\mathrm{app}}]$ that is given according to (2.28), (4.9) and (4.22) by

$$\frac{1}{3} \big(a_+^{\mathrm{app}} + \overline{a_+^{\mathrm{app}}} \big)^2 Y_2(x).$$

We replace a_+^{app} by its expansion (4.8). We get terms of the following form (up to irrelevant multiplicative constants):

$$e^{it\sqrt{3}}g(t)^2Y_2,\quad |g(t)|^2Y_2,\quad e^{-it\sqrt{3}}\overline{g(t)}^2Y_2, \tag{4.23}$$
$$e^{i(2\ell-3)t\frac{\sqrt{3}}{2}}g(t)^\ell\overline{g(t)}^{3-\ell}Y_2,\quad 0\le\ell\le 3, \tag{4.24}$$

and

$$\begin{aligned}
&e^{it\sqrt{3}}g_2(t)\big(\varphi_+(0,t)-\varphi_-(0,t)+\overline{\varphi_+(\sqrt{3},t)}-\overline{\varphi_-(\sqrt{3},t)}\big)Y_2,\\
&g_0(t)\,\mathrm{Re}\big(\varphi_+(0,t)-\varphi_-(0,t)+\varphi_+(\sqrt{3},t)-\varphi_-(\sqrt{3},t)\big)Y_2,\\
&e^{-it\sqrt{3}}g_{-2}(t)\big(\overline{\varphi_+(0,t)}-\overline{\varphi_-(0,t)}+\varphi_+(\sqrt{3},t)-\varphi_-(\sqrt{3},t)\big)Y_2
\end{aligned} \tag{4.25}$$

with g_{2j}, $j=-1,0,1$ satisfying, according to (4.3), the bounds

$$|g_{2j}(t)|\le C(A)t_\varepsilon^{-1},\quad |\partial_t g_{2j}(t)|\le C(A,A')t_\varepsilon^{-\frac{1}{2}}\big(t_\varepsilon^{-\frac{3}{2}}+t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'}\big), \tag{4.26}$$

and expressions that are, according to conditions (4.3) and (4.6), $O(t_\varepsilon^{-\frac{3}{2}}t^{-\frac{1}{2}}\langle x\rangle^{-N})$ or $O(t_\varepsilon^{-1}t^{-1}\langle x\rangle^{-N})$ for any N, as well as their ∂_x derivatives, so that they will satisfy (4.20). Terms (4.23) give I_1 with actually the explicit expression (4.21) for M_2, M_0, M_{-2}. Terms (4.24) provide contributions to I_2 in (4.14).

To study terms in (4.25) that will provide I_3, let us define

$$\tilde\varphi_\pm(\lambda,t)=e^{-i\lambda t}\varphi_\pm(\lambda,t). \tag{4.27}$$

By (4.7), we have

$$D_t\tilde\varphi_\pm(\lambda,t)=\langle Z,\tilde u_\pm\rangle e^{-i\lambda t}+\psi_\pm(\lambda,t)e^{-i\lambda t}. \tag{4.28}$$

Then all contributions in (4.25) may be written under the form $e^{ijt\sqrt{3}}M_j^\pm(t,x)$, $j=-1,0,1$, with $M_j^\pm$ given by linear combinations of expressions

$$\begin{aligned}
&e^{it\sqrt{3}}g_{2\ell}(t)\tilde\varphi_\pm(\delta\sqrt{3},t)Y_2,\ \ell+\delta=1, 0\le\delta,\ell\le 1,\ \text{if } j=1\\
&g_{-2\ell}(t)\tilde\varphi_\pm(\ell\sqrt{3},t)Y_2,\ g_{2\ell}(t)\overline{\tilde\varphi_\pm(\ell\sqrt{3},t)}Y_2,\ \ell=0,1,\ \text{if } j=0\\
&e^{-it\sqrt{3}}g_{-2\ell}(t)\overline{\tilde\varphi_\pm(\delta\sqrt{3},t)}Y_2,\ \ell+\delta=1, 0\le\delta,\ell\le 1,\ \text{if } j=-1.
\end{aligned} \tag{4.29}$$

Since by (4.28), (4.6), (4.7), (4.4),

$$|D_t\tilde\varphi_\pm(\delta\sqrt{3},t)|\le Ct^{-\frac{3}{4}}(\varepsilon^2\sqrt{t})^{\theta'}$$

we deduce from (4.3) and (4.6) that (4.17) holds for M_j^3 which is a combination of M_j^+ and M_j^-, $-1\le j\le 1$. In the case $j=1$, we have to obtain (4.19), i.e. to find functions $\tilde\Phi_{1,\ell}^\pm$, $\tilde\Psi_{1,\ell}^\pm$, $\ell=0,1$ satisfying (4.18), such that if we define according to the first line in (4.29)

$$M_{1,\ell}^\pm(t,x)=g_{2\ell}(t)\tilde\varphi_\pm\big((1-\ell)\sqrt{3},t\big)Y_2(x), \tag{4.30}$$

for ξ in the neighborhood $\mathcal{W}$ of $\{-\sqrt{2}, \sqrt{2}\}$, we have

$$D_t \hat{M}^{\pm}_{1,\ell}(t,\xi) = \Big(D_t + \big(\sqrt{3} - \sqrt{1+\xi^2}\big)\Big)\tilde{\Phi}^{\pm}_{1,\ell}(t,\xi) + \tilde{\Psi}^{\pm}_{1,\ell}(t,\xi). \tag{4.31}$$

Let us apply (4.7) with λ replaced by $\lambda(\xi) = \sqrt{1+\xi^2} - \ell\sqrt{3}$ and $\xi \in \mathcal{W}$, so that $\lambda(\xi)$ remains close to $\mathbb{Z}\sqrt{3}$, and thus outside a neighborhood of $\{-1, 1\}$. We may then find functions $\varphi_{\pm}(\lambda(\xi), t)$, $\psi_{\pm}(\lambda(\xi), t)$ such that

$$\big(D_t - \sqrt{1+\xi^2} + \ell\sqrt{3}\big)\varphi_{\pm}(\lambda(\xi), t) = \langle Z, \tilde{u}_{\pm}\rangle + \psi_{\pm}(\lambda(\xi), t) \tag{4.32}$$

with estimates of the form

$$|\varphi_{\pm}(\lambda(\xi), t)| \leq (\varepsilon^2\sqrt{t})^{\theta'} t^{-\frac{1}{2}}, \quad |\psi_{\pm}(\lambda(\xi), t)| \leq (\varepsilon^2\sqrt{t})^{\theta'} t^{-1} \tag{4.33}$$

uniformly for ξ in $\mathcal{W}$. Define

$$\tilde{\Phi}^{\pm}_{1,\ell}(t,\xi) = \varphi_{\pm}(\lambda(\xi), t) e^{-it(1-\ell)\sqrt{3}} g_{2\ell}(t)\hat{Y}_2(\xi).$$

Then (4.33) implies that

$$\begin{aligned}\Big(D_t - \big(\sqrt{1+\xi^2} - \sqrt{3}\big)\Big)&\tilde{\Phi}^{\pm}_{1,\ell}(t,\xi) \\ &= \langle Z, \tilde{u}_{\pm}\rangle e^{-it(1-\ell)\sqrt{3}} g_{2\ell}(t)\hat{Y}_2(\xi) \\ &\quad + \psi_{\pm}(\lambda(\xi), t) e^{-it(1-\ell)\sqrt{3}} g_{2\ell}(t)\hat{Y}_2(\xi) \\ &\quad + \varphi_{\pm}(\lambda(\xi), t) e^{-it(1-\ell)\sqrt{3}} D_t g_{2\ell}(t)\hat{Y}_2(\xi).\end{aligned} \tag{4.34}$$

On the other hand, (4.30), (4.28), (4.6) and (4.26) imply that

$$D_t \hat{M}^{\pm}_{1,\ell}(t,\xi) = \langle Z, \tilde{u}_{\pm}\rangle e^{-it(1-\ell)\sqrt{3}} g_{2\ell}(t)\hat{Y}_2(\xi) + R^{\pm}_{1,\ell}(t,\xi) \tag{4.35}$$

with

$$|\partial_\xi^\alpha R^{\pm}_{1,\ell}(t,\xi)| \leq C t^{-1} t_\varepsilon^{-1} (\varepsilon^2\sqrt{t})^{\theta'} \langle \xi \rangle^{-N} \tag{4.36}$$

for any N. Making the difference between (4.34) and (4.35), and using (4.3) and (4.6), we obtain that (4.31) holds, with functions $\Phi^{\pm}_{1,\ell}$, $\Psi^{\pm}_{1,\ell}$ satisfying (4.18) since the last two terms in (4.34) and (4.36) are

$$O\big(t^{-1} t_\varepsilon^{-1} + t_\varepsilon^{-\frac{1}{2}} t^{-1} (\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'}\big) = O(t_\varepsilon^{-1} t^{-1})$$

for $t \leq \varepsilon^{-4}$.

As $xM^{\pm}_{1,\ell}(t,x)$ is also of the form (4.30), with Y_2 replaced by xY_2, the same reasoning applies to that function and shows that (4.19) holds as well for xM_1^3 (with different functions $\tilde{\Phi}_1$, $\tilde{\Psi}_1$ on the right-hand side).

We have thus obtained that the first term $F_0^2[a^{\text{app}}]$ in (4.11) has the wanted structure.

To study $F_0^3[a^{\mathrm{app}}]$, we notice that by (2.28), (4.9), (4.8), it may be written as a linear combination of expressions of the form (4.24) (with Y_2 replaced by another function in $\mathcal{S}(\mathbb{R})$), that have been already treated, and of products of an $\mathcal{S}(\mathbb{R})$ function by expressions that are, by (4.3) and (4.6), $O(t_\varepsilon^{-1}t^{-1})$, so that form part of the remainder term (4.20). ■

We may now state the main proposition of this section.

Proposition 4.1.2. *Assume that properties (4.3)–(4.7) hold. One may construct a function* $u_+^{\mathrm{app}} : [1,T] \times \mathbb{R} \to \mathbb{C}$ *(where* $T < \varepsilon^{-4}$ *is the length of the interval on which* a_+^{app} *is defined by (4.8)), solving the equation*

$$\begin{aligned}\big(D_t - p(D_x)\big)u_+^{\mathrm{app}} &= F_0^2(a^{\mathrm{app}}) + F_0^3(a^{\mathrm{app}}) \\ &\qquad + a^{\mathrm{app}} \sum_{|I|=1} \mathrm{Op}(m'_{1,I})(u_I^{\mathrm{app}}) + R(t,x), \\ u_+^{\mathrm{app}}|_{t=1} &= 0,\end{aligned} \tag{4.37}$$

where $m'_{1,I}$ *is the symbol in the last sum of (3.13), where the remainder* R *satisfies bounds*

$$|\partial_x^\alpha R(t,x)| \le C_{\alpha,N} t_\varepsilon^{-1} t^{-1} \log(1+t)\langle x\rangle^{-N} \tag{4.38}$$

for any α, N *in* $\mathbb{N}$*, with constants* $C_{\alpha,N}(A,A')$ *depending on the constants* A, A' *in (4.3), and where* u_+^{app} *has the following structure: One may decompose*

$$u_+^{\mathrm{app}} = u_+'^{\mathrm{app}} + u_+''^{\mathrm{app}},$$

where $u_+'^{\mathrm{app}}$ *satisfies for any* $r \in \mathbb{N}$*,*

$$\|u_+'^{\mathrm{app}}(t,\cdot)\|_{H^r} \le C(A,A')\varepsilon^2 t^{\frac14}, \tag{4.39}$$

$$\|u_+'^{\mathrm{app}}(t,\cdot)\|_{W^{r,\infty}} \le C(A,A')\varepsilon^2, \tag{4.40}$$

$$\|L_+ u_+'^{\mathrm{app}}(t,\cdot)\|_{H^r} \le C(A,A')t^{\frac14}\big((\varepsilon^2\sqrt{t}) + (\varepsilon^2\sqrt{t})^{\frac78}\varepsilon^{\frac18}\big), \tag{4.41}$$

where

$$L_+ = x + tp'(D_x), \tag{4.42}$$

and where $u_+''^{\mathrm{app}}$ *satisfies for any* r*,*

$$\|u_+''^{\mathrm{app}}(t,\cdot)\|_{H^r} \le C(A,A')\varepsilon\Big(\frac{t\varepsilon^2}{\langle t\varepsilon^2\rangle}\Big)^{\frac12}, \tag{4.43}$$

$$\|u_+''^{\mathrm{app}}(t,\cdot)\|_{W^{r,\infty}} \le C(A,A')\varepsilon^2 \log(1+t)^2, \tag{4.44}$$

$$\|L_+ u_+''^{\mathrm{app}}(t,\cdot)\|_{W^{r,\infty}} \le C(A,A')\log(1+t)\log(1+\varepsilon^2 t). \tag{4.45}$$

For the action of the half-Klein–Gordon operator on $u_+'^{\mathrm{app}}$*, we have estimates*

$$\|(D_t - p(D_x))u_+'^{\mathrm{app}}(t,\cdot)\|_{H^r} \le C(A,A')\varepsilon^2 t^{-\frac34} \tag{4.46}$$

and

$$\|L_+(D_t - p(D_x))u_+'^{\mathrm{app}}(t,\cdot)\|_{H^r} \le C(A,A')t^{-\frac34}\big((\varepsilon^2\sqrt{t}) + (\varepsilon^2\sqrt{t})^{\frac78}\varepsilon^{\frac18}\big). \tag{4.47}$$

Moreover, we may write also another decomposition of u_+^{app}*, of the form*

$$u_+^{\mathrm{app}}(t,x) = u_+^{\mathrm{app},1}(t,x) + \Sigma_+(t,x), \tag{4.48}$$

where $u_+^{\mathrm{app},1}$ *is a sum*

$$u_+^{\mathrm{app},1}(t,x) = \sum_{j\in\{-2,0,2\}} U_{j,+}(t,x), \tag{4.49}$$

where $U_{j,+}$ *solves the equation*

$$\begin{aligned} \big(D_t - p(D_x)\big)U_{j,+} &= e^{itj\frac{\sqrt{3}}{2}} M_j(t,x),\\ U_{j,+}|_{t=1} &= 0, \end{aligned} \tag{4.50}$$

with source term M_j *given by* (4.21)*. The second contribution* Σ_+ *on the right-hand side of* (4.48) *may be also written as a sum*

$$\sum_{j=-3}^{3} \underline{U}_j(t,x),$$

with $\underline{U}_j$ *solving an equation of the form* (4.50)*, with source terms* $e^{ijt\frac{\sqrt{3}}{2}}\underline{M}_j(t,x)$*, where* $\underline{M}_j$ *satisfies for any* α, N,

$$|\partial_\xi^\alpha \hat{\underline{M}}_j(t,\xi)| \le C_{\alpha,N}(A,A')t_\varepsilon^{-1}t^{-\frac{1}{2}}\langle\xi\rangle^{-N} \tag{4.51}$$

and for any symbol m' *in the class* $\tilde{S}'_{0,0}(\langle\xi\rangle^{-1},1)$ *of Definition* 3.1.1*, one has for any* $\alpha, N \in \mathbb{N}$ *estimates*

$$|x^N\partial_x^\alpha \mathrm{Op}(m')(\Sigma_+(t,x))| \le C(A,A')\big(t_\varepsilon^{-\frac{3}{2}} + t^{-1}t_\varepsilon^{-\frac{1}{2}} + t^{-1}\varepsilon^2\big)\log(1+t). \tag{4.52}$$

In addition, all constants $C(A,A')$ *in the above inequality depend only on* A *and* A' *in* (4.3) *and* (4.4)*.*

Moreover, $u_+^{\mathrm{app},1}$ *may be decomposed as* $u_+^{\mathrm{app},1} = u'^{\mathrm{app},1}_+ + u''^{\mathrm{app},1}_+$*, with* $u'^{\mathrm{app},1}_+$ *(resp.* $u''^{\mathrm{app},1}_+$*) satisfying* (4.39)–(4.41) *and* (4.46)*,* (4.47) *(resp.* (4.43)–(4.45)*).*

Finally, all functions above are odd.

Proof. The proof of the proposition will be divided in several steps, and use the results of Appendix C below.

First step. We have decomposed in equation (4.11) the source term of (4.37), i.e. $F_0^2[a^{\mathrm{app}}] + F_0^3[a^{\mathrm{app}}]$. In this first step, we construct a first contribution $u_+^{\mathrm{app},1}$ to the solution of (4.37) taking as forcing term the contribution I_1 given by (4.12) to (4.11), i.e. we solve, with the notation (4.12)

$$\begin{aligned} \big(D_t - p(D_x)\big)u_+^{\mathrm{app},1} &= \sum_{j\in\{-2,0,2\}} e^{itj\frac{\sqrt{3}}{2}} M_j(t,x),\\ u_+^{\mathrm{app},1}|_{t=1} &= 0. \end{aligned} \tag{4.53}$$

The functions M_j on the right-hand side are given by (4.21), satisfy (4.13), and one may thus write $u_+^{\text{app},1}$ under the form (4.49), with $U_{j,+}$ given as the solution of (4.50). We apply Appendix C. The solution of (4.50) is given by (C.3) with $\lambda = j\sqrt{3}/2$ and may be decomposed according to (C.4) in $U'_{j,+} + U''_{j,+}$. We define

$$u'^{\text{app},1}_+ = \sum_{j\in\{-2,0,2\}} U'_{j,+}, \quad u''^{\text{app},1}_+ = \sum_{j\in\{-2,0,2\}} U''_{j,+} \tag{4.54}$$

and check that they give contributions to $u'^{\text{app}}_+, u''^{\text{app}}_+$ that satisfy (4.39)–(4.41) and (4.43)–(4.45). By (4.13), the functions M_j on the right-hand side of (4.53) satisfy (C.7) with $\omega = 1$, i.e. Assumption $(\mathrm{H1})_1$ holds. By (i) of Proposition C.1.1, we thus get bounds of the form (4.39)–(4.41), and by (i) of Proposition C.1.2, we have (4.43)–(4.45). We shall define the contribution $u_+^{\text{app},1}$ in (4.48) by

$$u_+^{\text{app},1} = u'^{\text{app},1}_+ + u''^{\text{app},1}_+, \tag{4.55}$$

i.e. by the right-hand side of (4.49). Moreover, as M_j is odd in x, so are $U_{j,+}$, $U'_{j,+}$ and $U''_{j,+}$.

Second step. We consider now the term involving $\mathrm{Op}(m'_{1,I})$ on the right-hand side of (4.37), where we replace $u_\pm^{\text{app}}$ by $u_\pm^{\text{app},1}$ given by (4.49) (with $u_-^{\text{app},1} = -\overline{u_+^{\text{app},1}}$), i.e.

$$a^{\text{app}}(t) \sum_{|I|=1} \sum_{j\in\{-2,0,2\}} \mathrm{Op}(m'_{I,I})(U_{j,I}) \tag{4.56}$$

with $U_{j,-} = -\overline{U}_{j,+}$. Recall that we decomposed $U_{j,+} = U'_{j,+} + U''_{j,+}$ according to (C.4). Let us examine first the contribution coming from $\mathrm{Op}(m'_{1,I})(U''_{j,I})$ to (4.56). The symbol $m'_{1,I}$ lies in $\tilde{S}'_{1,0}(\langle\xi\rangle^{-1}M_0^\nu, 1)$, which is contained in $\tilde{S}'_{0,0}(1,1)$ (recall that $M_0 \equiv 1$ when there is only one ξ variable), and it satisfies (3.7). Since $U''_{j,+}$ is defined by (C.4) with $\lambda = j\sqrt{3}/2$ from some odd M_j, we may apply Proposition C.2.1, with M_j satisfying Assumption $(\mathrm{H1})_1$, i.e. (C.7) with $\omega = 1$ according to (4.13). We shall thus get from (C.89)

$$\mathrm{Op}(m'_{1,+})(U''_{j,+}) = e^{ijt\frac{\sqrt{3}}{2}} M^{(1)}_{j,+}(t,x) + r_+(t,x) \tag{4.57}$$

with for any α, N, by (C.91),

$$|\partial_x^\alpha r(t,x)| \le C_{\alpha,N}\varepsilon^2 t^{-1}\log(1+t)\langle x\rangle^{-N} \tag{4.58}$$

and where $M^{(1)}_{j,+}$ satisfies by (C.90)

$$\begin{aligned} |\partial_x^\alpha M^{(1)}_{j,+}(t,x)| &\le C_{\alpha,N} t_\varepsilon^{-1}\langle x\rangle^{-N}, \\ |\partial_x^\alpha \partial_t M^{(1)}_{j,+}(t,x)| &\le C_{\alpha,N} t_\varepsilon^{-\frac{1}{2}}\big(t_\varepsilon^{-\frac{3}{2}} + t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'}\big)\langle x\rangle^{-N}. \end{aligned} \tag{4.59}$$

By conjugation, we shall have also

$$\mathrm{Op}(m'_{1,+})(U''_{j,-}) = e^{-ijt\frac{\sqrt{3}}{2}} M^{(1)}_{j,-}(t,x) + r_-(t,x) \tag{4.60}$$

with $M_{j,-}^{(1)}$ (resp. r_-) satisfying also (4.59) (resp. (4.58)). We plug (4.57) and (4.60) in (4.56) and use the expression (4.8)–(4.9) of a^{app}. We get that (4.56) is a sum of quantities of the following form:

- Terms of the form

$$e^{ij't\frac{\sqrt{3}}{2}} M_{j'}^{(1)}(t,x), \quad j' = -3,-1,1,3, \tag{4.61}$$

  coming from the product of the first term in (4.8) (or its conjugate) and of the $M_{j,\pm}^{(1)}$ terms in (4.57) and (4.60). One gets thus smooth odd functions of x, that satisfy by (4.59) and (4.3) estimates

$$\begin{aligned} |\partial_x^\alpha M_{j'}^{(1)}(t,x)| &\le C_{\alpha,N} t_\varepsilon^{-\frac{3}{2}} \langle x\rangle^{-N}, \\ |\partial_x^\alpha \partial_t M_{j'}^{(1)}(t,x)| &\le C_{\alpha,N} t_\varepsilon^{-1} \big(t_\varepsilon^{-\frac{3}{2}} + t^{-\frac{3}{2}} (\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'}\big)\langle x\rangle^{-N}. \end{aligned} \tag{4.62}$$

- Terms satisfying (4.38) and thus contributing to R in (4.37). These terms come from the product of (4.57) or (4.60) with all terms on the right-hand side of (4.8), except $e^{it\sqrt{3}/2}g(t)$ (and its conjugate), and from the product of a^{app} with $r_\pm$ in (4.57) and (4.60). As

$$\varepsilon^2 t^{-1} t_\varepsilon^{-\frac{1}{2}} \le C t^{-1} t_\varepsilon^{-1}$$

  if $t \le \varepsilon^{-4}$, we do get that these terms satisfy (4.38).

- Terms of the form

$$a^{\text{app}}(t) \sum_{|I|=1} \sum_{j\in\{-2,0,2\}} \text{Op}(m'_{1,I})(U'_{j,I}), \tag{4.63}$$

  where $U'_{j,I}$ is given by (C.4) in terms of M_j satisfying Assumption $(\text{H1})_\omega$ with $\omega = 1$. We shall see in fifth step below that (4.63) satisfies also (4.38) and thus contributes to R.

It follows thus from (4.53) and the fact that (4.56) is given by (4.61) up to remainders, that

$$\big(D_t - p(D_x)\big)u_+^{\text{app},1} - a^{\text{app}}(t) \sum_{|I|=1} \text{Op}(m'_{1,I})(u_I^{\text{app},1}) = I_1 - I_2^{(1)} + R(t,x), \tag{4.64}$$

where I_1 is given by (4.12), $I_2^{(1)}$ is the sum of terms (4.61) and R satisfies (4.38). Making the difference between (4.37) and (4.64), we get, taking (4.11) into account

$$\begin{aligned} &\big(D_t - p(D_x)\big)(u_+^{\text{app}} - u_+^{\text{app},1}) \\ &\quad = I_2 + I_3 + I_2^{(1)} + a^{\text{app}}(t) \sum_{|I|=1} \text{Op}(m'_{1,I})(u_I^{\text{app}} - u_I^{\text{app},1}) + R(t,x), \end{aligned} \tag{4.65}$$

with R satisfying (4.38). Notice that by (4.62), $I_2^{(1)}$ has the same form as I_2 given by (4.14) and (4.15) so that we shall be able to treat both terms altogether.

Third step. We now construct an approximate solution in order to eliminate $I_2+I_2^{(1)}$ on the right-hand side of (4.65). Define $u_+^{\mathrm{app},2}$ as the solution to the linear equation

$$\begin{aligned}\big(D_t - p(D_x)\big)u_+^{\mathrm{app},2} &= I_2 + I_2^{(1)},\\ u_+^{\mathrm{app},2}|_{t=1} &= 0.\end{aligned} \tag{4.66}$$

As the right-hand side has structure (4.14) with M_j satisfying (4.15), we may express the solution as a sum $\sum_{j\in\{-3,-1,1,3\}} U_{j,+}(t,x)$, where $U_{j,+}$ is obtained from the j-th term in (4.14) and expressed under form (C.3) with $\lambda = j\sqrt{3}/2$. By (C.4),

$$U_{j,+} = U'_{j,+} + U''_{j,+}$$

and since (4.15) shows that (C.7) holds with $\omega = 3/2$, Assumption $(\mathrm{H1})_{3/2}$ holds. By Proposition C.1.1, bounds (C.18)–(C.20) with $\omega = 3/2$ hold for $U'_{j,+}$, and by Proposition C.1.2, (C.24), (C.25) and (C.27) are true. If we set

$$u'^{\,\mathrm{app},2}_+ = \sum_{j\in\{-3,-1,1,3\}} U'_{j,+},\ u''^{\,\mathrm{app},2}_+ = \sum_{j\in\{-3,-1,1,3\}} U''_{j,+}, \tag{4.67}$$

this shows that these functions provide to $u'^{\mathrm{app}}_+, u''^{\mathrm{app}}_+$ contributions satisfying estimates (4.39)–(4.41) and (4.43)–(4.45).

Let us study

$$a^{\mathrm{app}}(t)\sum_{|I|=1}\mathrm{Op}(m'_{1,I})(u_I^{\mathrm{app},2}). \tag{4.68}$$

If we apply Proposition C.2.1, using that Assumption $(\mathrm{H1})_{3/2}$ holds, we get from (C.89), (C.90), (C.91) and the fact that $a^{\mathrm{app}}(t)$ is $O(t_\varepsilon^{-1/2})$, that the contribution of $u''^{\,\mathrm{app},2}_+$ to (4.68) is $O(t_\varepsilon^{-1}t^{-1}\langle x\rangle^{-N})$, i.e. may be included in R satisfying (4.38). On the other hand, if we replace in (4.68) $u_+^{\mathrm{app},2}$ by $u'^{\,\mathrm{app},2}_+$, we shall get terms of the form (4.63), with $U'_{j,I}$ given by (C.4) in terms of M_j satisfying Assumption $(\mathrm{H1})_\omega$ with $\omega = \frac{3}{2}$. These terms are thus better than those in (4.63) and the fact that they fulfill remainder estimates (4.38) will be seen in Step 5 below.

Consequently, we have shown that

$$\big(D_t - p(D_x)\big)u_+^{\mathrm{app},2} - a^{\mathrm{app}}(t)\sum_{|I|=1}\mathrm{Op}(m'_{1,I})(u_I^{\mathrm{app},2}) = I_2 + I_2^{(1)} + R(t,x) \tag{4.69}$$

with R satisfying (4.38). Making the difference between (4.65) and (4.69), we get

$$\begin{aligned}&\big(D_t - p(D_x)\big)\big(u_+^{\mathrm{app}} - u_+^{\mathrm{app},1} - u_+^{\mathrm{app},2}\big)\\ &\quad = I_3 + a^{\mathrm{app}}(t)\Big(\sum_{|I|=1}\mathrm{Op}(m'_{1,I})(u_I^{\mathrm{app}} - u_I^{\mathrm{app},1} - u_I^{\mathrm{app},2})\Big) + R(t,x).\end{aligned} \tag{4.70}$$

Fourth step. We construct an approximate solution in order to eliminate I_3 in (4.70), i.e. we solve

$$\begin{aligned}\big(D_t - p(D_x)\big)u_+^{\mathrm{app},3} &= I_3,\\ u_+^{\mathrm{app},3}|_{t=1} &= 0\end{aligned} \tag{4.71}$$

with I_3 given by equation (4.16). For each contribution $e^{ijt\sqrt{3}}M_j^3(t,x)$ to (4.16), with $-1 \le j \le 1$, we get an equation of the form (C.2) with $\lambda = j\sqrt{3}$. Moreover, by (4.17)–(4.19) assumptions (C.8)–(C.10) hold (the last two ones being empty if $\lambda = j\sqrt{3}$ with $j = 0$ or -1), i.e. Assumption (H2) of section (C.2) holds. We may thus apply (ii) of Proposition C.1.1 and Proposition C.1.2 that allow to write $u_+^{\mathrm{app},3}$ as a sum

$$u_+^{\mathrm{app},3} = \sum_{j=-1}^{1} U_{j,+}(t,x), \quad U_{j,+} = U'_{j,+} + U''_{j,+} \tag{4.72}$$

with $U'_{j,+}$ satisfying (C.21)–(C.23) and $U''_{j,+}$ satisfying (C.28)–(C.30). If we now set $u_+^{\mathrm{app},3} = u'_+{}^{\mathrm{app},3} + u''_+{}^{\mathrm{app},3}$ with

$$u'_+{}^{\mathrm{app},3} = \sum_{j=-1}^{1} U'_{j,+}(t,x), \quad u''_+{}^{\mathrm{app},3} = \sum_{j=-1}^{1} U''_{j,+}(t,x), \tag{4.73}$$

it follows that (4.39)–(4.41) and (4.43)–(4.45) hold true. Let us check that

$$a^{\mathrm{app}}(t) \sum_{|I|=1} \mathrm{Op}(m'_{1,I})(u_+^{\mathrm{app},3}) \tag{4.74}$$

is a remainder satisfying (4.38). Since we are here under Assumption (H2), we shall apply Proposition C.2.4 splitting each $U_{j,+}$ in (4.72) as

$$U_{j,+} = U'_{j,+,1} + U''_{j,+,1} \tag{4.75}$$

according to (C.110). Then by (C.111), and the fact that $a^{\mathrm{app}} = O(t_\varepsilon^{-\frac{1}{2}})$, the contribution coming from $U''_{j,+,1}$ obeys remainder estimates (4.38), so that (4.74) may be written as a contribution to R in (4.37) and as

$$a^{\mathrm{app}}(t) \sum_{|I|=1} \mathrm{Op}(m'_{1,I})(u'^{\mathrm{app},3}_{+,1}) \tag{4.76}$$

with

$$u'^{\mathrm{app},3}_{+,1} = \sum_{j=-1}^{1} U'_{j,+,1}(t,x). \tag{4.77}$$

We shall see in Step 5 below that (4.76) provides also a contribution to R. Consequently, we have obtained that

$$\big(D_t - p(D_x)\big)u_+^{\mathrm{app},3} - a^{\mathrm{app}}(t) \sum_{|I|=1} \mathrm{Op}(m'_{1,I})(u_I^{\mathrm{app},3}) = I_3 + R(t,x).$$

Making the difference with (4.70), we conclude that u_+^{app} will solve (4.37) if and only if

$$\begin{aligned}\big(D_t - p(D_x)\big)&\Big(u_+^{\mathrm{app}} - \sum_{\ell=1}^{3} u_+^{\mathrm{app},\ell}\Big)\\ &- a^{\mathrm{app}}(t) \sum_{|I|=1} \mathrm{Op}(m'_{1,I})\Big(u_I^{\mathrm{app}} - \sum_{\ell=1}^{3} u_I^{\mathrm{app},\ell}\Big) = R(t,x).\end{aligned}$$

Consequently, we just have to take $u_+^{\mathrm{app}} = u_+^{\mathrm{app},1} + u_+^{\mathrm{app},2} + u_+^{\mathrm{app},3}$. We have checked that then estimates (4.39)–(4.41) and (4.43)–(4.45) hold. It remains to check that terms of the form (4.63) and (4.76) provide remainders, and that estimates (4.46)–(4.47) hold true, as well as the properties of the decomposition (4.48). This will be done in the following steps.

Fifth step. Let us show that (4.63) and (4.76) are remainders. Let us use the same notation $U'_{j,+}$ for either $U'_{j,+}$ in (4.63) or $U'_{j,+,1}$ in (4.77). Notice that since the functions M_j in (4.12), (4.14), (4.16) are odd in x, so are the $U'_{j,+}$ defined from them. Moreover, as $m'_{1,I}$ is in $\tilde{S}'_{1,0}(\langle\xi\rangle^{-1}, 1)$, we may write

$$\mathrm{Op}(m'_{1,\pm})(U'_{j,\pm}) = \mathrm{Op}(\tilde{m}_{1,\pm})(\langle D_x\rangle^{-1} U'_{j,\pm}) \tag{4.78}$$

with $\tilde{m}'_{1,I}$ in $\tilde{S}'_{1,0}(1,1)$. By oddness of $U'_{j,+}$

$$\begin{aligned}\langle D_x\rangle^{-1} U'_{j,+} &= \frac{ix}{2}\int_{-1}^{1}\Big(\frac{D_x}{\langle D_x\rangle} U'_{j,+}\Big)(t,\mu x)\,d\mu \\ &= \frac{ix}{2t}\int_{-1}^{1}\big((L_+ U'_{j,+})(t,\mu x) - \mu x U'_{j,+}(t,\mu x)\big)\,d\mu.\end{aligned} \tag{4.79}$$

As $\tilde{m}_{1,I}$ has rapidly decaying coefficients in x, we rewrite (4.78) as a linear combination of expressions

$$\frac{1}{t}\mathrm{Op}(\hat{m}'_{1,I})\Big(\int_{-1}^{1} (L_\pm^k U'_{j,\pm})(t,\mu x)\mu^{1-k}\,d\mu\Big), \quad k = 0,1, \tag{4.80}$$

for new symbols $\hat{m}'_{1,I}$ in the class $\tilde{S}'_{1,0}(1,1)$. Using (C.92) with $\omega = 1$ or (C.112), we bound any L^∞ norm of $x^\beta \partial_x^\alpha$ acting on (4.80) by $C\varepsilon^2 t^{-1}$. Taking into account that $a^{\mathrm{app}}(t)$ is $O(t_\varepsilon^{-1/2})$, we see that (4.63) and (4.76) satisfy (4.38) (using again $t \le \varepsilon^{-4}$).

Sixth step. We shall prove estimates (4.46) and (4.47). Recall that by definition

$$u'^{\mathrm{app}}_+ = u'^{\ \mathrm{app},1}_+ + u'^{\ \mathrm{app},2}_+ + u'^{\ \mathrm{app},3}_+$$

with $u'^{\ \mathrm{app},1}_+$ given by (4.54), $u'^{\ \mathrm{app},2}_+$ given by (4.67) and $u'^{\ \mathrm{app},3}_+$ given by (4.73). Consequently, the term $(D_t - p(D_x))u'^{\mathrm{app}}_+$ is a sum of expressions $(D_t - p(D_x))U'_{j,+}$, where $U'_{j,+}$ is given by an integral of the form (C.4) (resp. (C.110)) with M replaced by an M_j satisfying either (4.13) (for those coming from (4.54)) or (4.15) (for those coming from (4.67)) (resp. satisfying (4.17) for those coming from (4.73)). Consequently, for contributions of the form (C.4),

$$\big(D_t - p(D_x)\big)U'_{j,+} = -\frac{1}{2t}\int_1^{+\infty} e^{i(t-\tau)p(D_x)+i\lambda_j\tau}\tilde{\chi}\Big(\frac{\tau}{\sqrt{t}}\Big)M_j(\tau,\cdot)\,d\tau, \tag{4.81}$$

where $\tilde{\chi}(\tau) = \tau\chi'(\tau)$ and λ_j is some integer multiple of $\frac{\sqrt{3}}{2}$. In other words, we obtain still an expression of the form of the first line in (C.4), but with a gain of a factor t^{-1}. Estimates (4.39) and (4.41) that we have already obtained for u'^{app}_+ furnish thus (4.46) and (4.47) multiplying them by t^{-1} (the change of cut-off $\tilde{\chi}$ does not matter, as it has support contained in the one of χ). This shows also that (4.46) and (4.47) hold for $u'^{\mathrm{app},1} + u'^{\mathrm{app},2}$. The case of $u'^{\mathrm{app},3}$ is similar, using (C.110) to get an expression of the form (4.81), but with $\tilde{\chi}(\frac{\tau}{\sqrt{t}})$ replaced by $\tilde{\chi}(\frac{\tau}{t})$, i.e. again an integral of form (C.110) with the gain of a pre-factor t^{-1}.

Seventh step. We have to establish still (4.48). The contribution $u_+^{\mathrm{app},1}$ on the right-hand side is the one that has been defined in the first step by (4.53), with right-hand side given in terms of M_j defined in (4.21). The term Σ_+ in (4.48) is thus given by $u_+^{\mathrm{app},2} + u_+^{\mathrm{app},3}$ introduced in (4.67) and (4.72). These functions are constructed as sums of contributions $\underline{U}_j$ that satisfy equations of the form (4.50), where the source term satisfies (4.15) or (4.17) and thus (4.51). It remains to show (4.52). As m' has rapidly decaying coefficients in x, we may forget the x^N factor in (4.52), and are thus reduced to the study of $\partial_x^\alpha \mathrm{Op}(m')(u_+^{\mathrm{app},2})$ and $\partial_x^\alpha \mathrm{Op}(m')(u_+^{\mathrm{app},3})$.

Consider first $\partial_x^\alpha \mathrm{Op}(m')(u_+^{\mathrm{app},2})$. By (4.67), we express that from

$$\partial_x^\alpha \mathrm{Op}(m')(U'_{j,+}),\ \partial_x^\alpha \mathrm{Op}(m')(U''_{j,+}). \tag{4.82}$$

As Assumption (H1)$_\omega$ holds with $\omega = \frac{3}{2}$, according to (4.15), the second term above is given by (C.89) of Proposition C.2.1. It follows from (C.90) and (C.91) that its modulus is smaller than

$$t_\varepsilon^{-\frac{3}{2}} + \varepsilon^3 t^{-1} \log(1+t),$$

so than the right-hand side of (4.52). On the other hand, $\mathrm{Op}(m')(U'_{j,+})$ has been expressed in fifth step under the form (4.80). If we plug there estimates (C.92), we see that the modulus of the first term in (4.82) is $O(\varepsilon^3 t^{-1})$, so better than the right-hand side of (4.52).

Consider next $\partial_x^\alpha \mathrm{Op}(m')(u_+^{\mathrm{app},3})$. Solving (4.71), we have written $u_+^{\mathrm{app},3}$ under the form $\sum_{j=-1}^{1}(U'_{j,+,1} + U''_{j,+,1})$ according to (4.75). If we plug this decomposition in $\partial_x^\alpha \mathrm{Op}(m')(\cdot)$, we get on the one hand expressions of the form (C.111), that are bounded by the right-hand side of (4.52). For the contribution $\partial_x^\alpha \mathrm{Op}(m')(U'_{j,+,1})$, we use again that we can write an expression of the form (4.80) and bounds (C.112). We get an estimate in $O(\varepsilon^2 t^{-1})$ that is better than the right-hand side of (4.52). This concludes the proof. ∎

To conclude this section, let us compute some integrals that will be useful in the sequel.

Proposition 4.1.3. *Let Y_2 be the function defined in* (4.22)*. The functions $U_{j,+}$, $j = -2, 0, 2$, on the right-hand side of* (4.49) *satisfy the following:*

$$\int U_{2,+}(t,x) p(D_x)^{-1} Y_2 \, dx = (\alpha_2 + i\beta_2) e^{it\sqrt{3}} g(t)^2 + r(t), \tag{4.83}$$

where α_2 is real,

$$\beta_2 = -\frac{\sqrt{2}}{6} \hat{Y}_2(\sqrt{2})^2 \tag{4.84}$$

for the function Y_2 defined in (2.6), and where $r(t)$ satisfies

$$|r(t)| \leq C(A, A')\Big(\varepsilon^2 t^{-\frac{3}{2}} + t_\varepsilon^{-2} + \varepsilon t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'}\Big) \leq C(A, A')t_\varepsilon^{-1}. \tag{4.85}$$

Moreover,

$$\int U_{0,+}(t,x)p(D_x)^{-1}Y_2\,dx = \alpha_0|g(t)|^2 + r(t), \tag{4.86}$$

$$\int U_{2,-}(t,x)p(D_x)^{-1}Y_2\,dx = \alpha_{-2}\overline{g(t)}^2 e^{-it\sqrt{3}} + r(t), \tag{4.87}$$

where α_0, α_{-2} are real constants, and where r satisfies (4.85). Finally, the function Σ_+ in (4.48) satisfies

$$\begin{aligned}\Big|\int \Sigma_+(t,x)p(D_x)^{-1}Y_2\,dx\Big|\\ \leq C(A, A')\big(t_\varepsilon^{-\frac{3}{2}} + \varepsilon^2 t_\varepsilon^{-1} + t^{-1}t_\varepsilon^{-\frac{1}{2}}\big)\log(1+t).\end{aligned} \tag{4.88}$$

Proof. Let us establish (4.83). The function $U_{2,+}$ is defined as the solution of (4.50) with $j = 2$ and M_2 on the right-hand side given by (4.21). We write (4.83) as

$$\frac{1}{2\pi}\int \hat{U}_{2,+}(t,\xi)p(\xi)^{-1}\hat{Y}_2(-\xi)\,d\xi.$$

Since Y_2 is odd, we get from equation (C.124) applied with $\hat{Z}(\xi) = -p(\xi)^{-1}\hat{Y}_2(\xi)$, $\hat{M}(t,\xi) = \hat{M}_2(t,\xi)$, $\lambda = \sqrt{3}$, a contribution to r and two integral terms. By (4.21), the second one is

$$-\frac{e^{it\sqrt{3}}}{6\pi}\int \frac{(1-\chi_{\sqrt{3}})(\xi)}{\sqrt{3}-\sqrt{1+\xi^2}}\frac{\hat{Y}_2(\xi)^2}{\sqrt{1+\xi^2}}\,d\xi g(t)^2 \tag{4.89}$$

which may be written since Y_2 is real and odd, under the form $\alpha_2' e^{it\sqrt{3}}g(t)^2$ for some real α_2'.

Using the definition (C.123) of χ_λ, and the fact that $\hat{Y}_2(\xi)^2$ is even, the first term on the right-hand side of (C.124) brings the contribution

$$\begin{aligned}-\frac{i}{3\pi}e^{it\sqrt{3}}g(t)^2\lim_{\sigma\to 0+}\int_0^{+\infty}\int e^{i\tau(\sqrt{1+\xi^2}-\sqrt{3})-\sigma\tau}\chi(\xi-\sqrt{2})\\ \times\frac{\hat{Y}_2(\xi)^2}{\sqrt{1+\xi^2}}\,d\xi\,d\tau.\end{aligned} \tag{4.90}$$

Denote by $\xi(\zeta)$ the reciprocal of the change of variables $\xi \mapsto \zeta = \sqrt{3} - \sqrt{1+\xi^2}$ defined from a neighborhood of $\xi = \sqrt{2}$ to a neighborhood of $\zeta = 0$. We rewrite (4.90) as

$$-\frac{i}{3\pi}e^{it\sqrt{3}}g(t)^2\lim_{\sigma\to 0+}\int_0^{+\infty}\int e^{-i\tau\zeta-\sigma\tau}\chi(\xi(\zeta)-\sqrt{2})\hat{Y}_2(\xi(\zeta))^2\frac{d\zeta}{|\xi(\zeta)|}\,d\tau. \tag{4.91}$$

Notice that

$$\lim_{\sigma\to 0+}\int_0^{+\infty} e^{-i\tau\zeta-\sigma\tau}\,d\tau = -i(\zeta - i0)^{-1} = \pi\delta_0 - i\,\text{p.v.}\,\frac{1}{\zeta}.$$

Plugging in (4.91), we obtain an expression $\alpha_2' + i\beta_2$ with α_2' real and β_2 given by (4.84).

To obtain (4.86) and (4.87), we apply again Proposition C.3.1 but with $\lambda = 0$ or $\lambda = -\sqrt{3}$ so that $\chi_\lambda = 0$ and in (C.124) the first term on the right-hand side disappears. Only the second one and r remain, so that one gets no imaginary contribution to (4.86) and (4.87).

Finally, let us prove (4.88). As Y_2 is in $\mathcal{S}(\mathbb{R})$, the integral may be expressed as an integral of $\text{Op}(m')(\Sigma_+)$ for the symbol $m' = Y_2(x)p(\xi)^{-1}$, so that (4.52) brings the conclusion. ■

4.2 Asymptotic analysis of the ODE

In this section, we shall prove that solutions of the ordinary differential equation (2.34) have a certain asymptotic expansion by a bootstrap argument.

We make some a priori assumptions on the functions Φ_j and Γ_j on the right-hand side of (2.34).

Assumption (H$'_1$). Assume that u_+ is a solution to equation (2.27) defined on the set $[1,T]\times\mathbb{R}$ for some $T \le \varepsilon^{-4}$ such that the functions Φ_2 and Γ_j, $j = 1,2,3$, defined on (2.36) satisfy the inequality

$$\begin{aligned}&|\Phi_2(u_+(t,\cdot),u_-(t,\cdot))| + \sum_{j=1}^{3} t_\varepsilon^{-\frac{3}{2}+\frac{j}{2}}|\Gamma_j(u_+(t,\cdot),u_-(t,\cdot))|\\ &\le B't^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{2\theta'}\end{aligned} \tag{4.92}$$

for some constant B', some $\theta' \in \,]0,\frac{1}{2}[$ (close to $\frac{1}{2}$), all $t\in[1,T]$, and assume that the function Φ_1 given by (2.36) satisfies for any $t\in[1,T]$,

$$\begin{aligned}\Big|&\Phi_1(u_+(t,\cdot),u_-(t,\cdot)) - \frac{\sqrt{3}}{3}\langle Y, Y\kappa(x)b(x,D_x)p(D_x)^{-1}\big(u_+^{\text{app}} - u_-^{\text{app}}\big)\rangle\\ &- \big(\langle Z,\tilde{u}_+\rangle - \langle Z,\tilde{u}_-\rangle\big)\Big| \le B't^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{2\theta'},\end{aligned} \tag{4.93}$$

where u_+^{app} is the approximate solution constructed in Section 4.1, Z is a function in $\mathcal{S}(\mathbb{R})$, $\tilde{u}_\pm$ are functions verifying inequality (4.4) such that for any λ in $\mathbb{R}-\{-1,1\}$, one may find functions $\varphi_\pm(\lambda,t)$ and $\psi_\pm(\lambda,t)$ as in (4.5), solving equation (4.7) and such that estimates (4.6) hold true, for λ outside a given neighborhood $\widetilde{W}$ of $\{-1,1\}$ in $\mathbb{R}$.

We consider on the interval $[1, T]$ the solution a_+ of equation (2.34), namely

$$\Big(D_t - \frac{\sqrt{3}}{2}\Big)a_+ = \sum_{j=0}^{2}(a_+ - a_-)^{2-j}\Phi_j[u_+, u_-] + \sum_{j=0}^{3}(a_+ - a_-)^{3-j}\Gamma_j[u_+, u_-] \tag{4.94}$$

with an initial condition at $t = 1$ satisfying

$$|a_+(1)| \le A_0\varepsilon \tag{4.95}$$

for some constant A_0. We introduce as a second assumption an estimate on a_+, that we give in terms of upper bounds (4.99) below:

Assumption ($\mathbf{H}_2'$). The solution of equation (4.94) with initial condition (4.95) exists on some interval $[1, T]$ with $T \le \varepsilon^{-4}$ and satisfies on that interval the following requirements: One may write

$$a_+(t) = a_+^{\mathrm{app}}(t) + S(t), \tag{4.96}$$

where $a_+^{\mathrm{app}}(t)$ has the structure

$$\begin{aligned} a_+^{\mathrm{app}}(t) = {} & e^{it\frac{\sqrt{3}}{2}}g(t) + \omega_2 g(t)^2 e^{it\sqrt{3}} + \omega_0|g(t)|^2 + \omega_{-2}\overline{g(t)}^2 e^{-it\sqrt{3}} \\ & + e^{it\frac{\sqrt{3}}{2}}g(t)\big(\varphi_+(0,t) - \varphi_-(0,t)\big) \\ & + e^{-it\frac{\sqrt{3}}{2}}\overline{g(t)}\big(\varphi_+(\sqrt{3},t) - \varphi_-(\sqrt{3},t)\big) \end{aligned} \tag{4.97}$$

and where

$$S(t) = \omega_3 g(t)^3 e^{3it\frac{\sqrt{3}}{2}} + \omega_{-1}|g(t)|^2\overline{g(t)}e^{-it\frac{\sqrt{3}}{2}} + \omega_{-3}\overline{g(t)}^3 e^{-3it\frac{\sqrt{3}}{2}} \tag{4.98}$$

with the following notation:

- The coefficients ω_j in (4.97) (resp. (4.98)) are real (resp. complex) constants that will be chosen below.
- The function g satisfies, for some constants A, A' and $t \in [1, T]$,

  $$|g(t)| \le At_\varepsilon^{-\frac{1}{2}},\ |\partial_t g(t)| \le A'(t_\varepsilon^{-\frac{3}{2}} + t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'}), \tag{4.99}$$

  where $\theta' \in]0, \frac{1}{2}[$ is close to $\frac{1}{2}$ and has been introduced in (H_1').
- The functions $\varphi_\pm(0,t)$, $\varphi_\pm(\sqrt{3},t)$ satisfy conditions (4.5)–(4.7) with Z and $\tilde{u}_\pm$ introduced in (4.93), i.e. one has estimates

  $$\begin{aligned} |\varphi_\pm(\lambda,t)| &\le (\varepsilon^2\sqrt{t})^{\theta'}t^{-\frac{1}{2}}, \quad |\psi_\pm(\lambda,t)| \le (\varepsilon^2\sqrt{t})^{\theta'}t^{-1} \\ |\langle Z, \tilde{u}_\pm(t,\cdot)\rangle| &\le (\varepsilon^2\sqrt{t})^{\theta'}t^{-\frac{3}{4}} \end{aligned} \tag{4.100}$$

  (when ε is small enough) and one has the equation

  $$(D_t - \lambda)\varphi_\pm(\lambda,t) = \langle Z, \tilde{u}_\pm(t,\cdot)\rangle + \psi_\pm(\lambda,t) \tag{4.101}$$

  for $\lambda = 0$ or $\sqrt{3}$.

We shall bootstrap Assumption (H$'_2$), i.e. estimates (4.99) assuming that (H$'_1$) holds:

Proposition 4.2.1. *Let $c \in]0,1[$ and $\theta' \in]0, \frac{1}{2}[$, θ' close to $\frac{1}{2}$. There are constants $A, A', \varepsilon_0 > 0$ such that if Assumption (H$'_1$) holds and if the solution a_+ of (4.94) exists on $[1,T]$ and has structure (4.96) with g satisfying (4.99) on $[1,T]$, then if $\varepsilon \in]0, \varepsilon_0[$, $T \leq \varepsilon^{-4+c}$, one has actually, for any $t \in [1,T]$,*

$$|g(t)| \leq \frac{1}{2} A t_\varepsilon^{-\frac{1}{2}}, \quad |\partial_t g(t)| \leq \frac{1}{2} A' \Big(t_\varepsilon^{-\frac{3}{2}} + t^{-\frac{3}{2}} (\varepsilon^2 \sqrt{t})^{\frac{3}{2}\theta'} \Big). \tag{4.102}$$

As a first step towards the proof of the proposition, let us rewrite equation (4.94).

Lemma 4.2.2. *There are a real constant γ_1 and complex constants $\gamma_3, \gamma_{-1}, \gamma_{-3}$ such that, under the assumptions of the proposition,*

$$\begin{aligned}\Big(D_t - \frac{\sqrt{3}}{2}\Big) a_+ = e^{it\frac{\sqrt{3}}{2}} & |g(t)|^2 g(t) \Big(\gamma_1 - i \frac{\sqrt{6}}{18} \hat{Y}_2 (\sqrt{2})^2 \Big) \\ & + e^{3it\frac{\sqrt{3}}{2}} g(t)^3 \gamma_3 + e^{-it\frac{\sqrt{3}}{2}} |g(t)|^2 \overline{g(t)} \gamma_{-1} \\ & + e^{-3it\frac{\sqrt{3}}{2}} \overline{g(t)}^3 \gamma_{-3} \\ & + (a_+ - a_-)^2 \Phi_0 + (a_+ - a_-)^3 \Gamma_0 \\ & + (a_+ - a_-)(\langle Z, \tilde{u}_+ \rangle - \langle Z, \tilde{u}_- \rangle) + r(t),\end{aligned} \tag{4.103}$$

where $r(t)$ satisfies

$$|r(t)| \leq C(A, A', B') t^{-\frac{3}{2}} (\varepsilon^2 \sqrt{t})^{2\theta'} \tag{4.104}$$

for a constant depending only on the constants A, A', B' of (4.99), (4.92), (4.93).

Proof. Consider the right-hand side of equation (4.94). By (4.92), the Φ_2 contribution is bounded by $B' t^{-\frac{3}{2}} (\varepsilon^2 \sqrt{t})^{2\theta'}$, so satisfies (4.104). By (4.96), (4.97), (4.99), (4.100)

$$|a_+(t)| + |a_-(t)| \leq C(A) t_\varepsilon^{-\frac{1}{2}} \tag{4.105}$$

so that (4.92) implies that the contributions $(a_+ - a_-)^{3-j} \Gamma_j$, $j = 1,2,3$, to (4.94) satisfy (4.104). We are thus left with studying

$$\Phi_0 (a_+ - a_-)^2 + \Phi_1 [u_+, u_-](a_+ - a_-) + \Gamma_0 (a_+ - a_-)^3. \tag{4.106}$$

The first and last terms in (4.106) are present on the right-hand side of (4.103). Consider $(a_+ - a_-)\Phi_1$. By (4.93), up to another contribution to r, we get on the one hand the last but one term on the right-hand side of (4.103) and the quantity

$$\frac{\sqrt{3}}{3} (a_+ - a_-) \langle Y, Y \kappa(x) b(x, D_x) p(D_x)^{-1} (u_+^{\text{app}} - u_-^{\text{app}}) \rangle$$

that, according to the definition (4.22) of Y_2, may be written

$$\frac{\sqrt{3}}{3} (a_+ - a_-) \langle Y_2, p(D_x)^{-1} (u_+^{\text{app}} - u_-^{\text{app}}) \rangle. \tag{4.107}$$

We replace above u_+^{app} by expansion (4.48). According to (4.88),

$$|\langle Y_2, p(D_x)^{-1}\Sigma_+\rangle| \le C(A,A')\big(t_\varepsilon^{-\frac{3}{2}} + t^{-1}\varepsilon^2 + t^{-1}t_\varepsilon^{-\frac{1}{2}}\big)\log(1+t).$$

If we use also (4.105) and (4.1), we conclude, since

$$t_\varepsilon^{-2} \le Ct^{-\frac{3}{2}}(\varepsilon^2\sqrt{t}), \quad t_\varepsilon^{-\frac{1}{2}}t^{-1}\varepsilon^2 \le Ct^{-\frac{3}{2}}(\varepsilon^2\sqrt{t}), \quad t^{-1}t_\varepsilon^{-1} \le Ct^{-\frac{3}{2}}(\varepsilon^2\sqrt{t}),$$

that (4.107) satisfies inequality (4.104) (if we absorb the logarithm using that we assume $\varepsilon^2\sqrt{t} \le \varepsilon^{\frac{c}{2}}$, $\theta' < \frac{1}{2}$, and that we take ε small). We are thus left with the contribution to (4.107) of

$$\frac{\sqrt{3}}{3}(a_+ - a_-)\langle Y_2, p(D_x)^{-1}(u_+^{\text{app},1} - u_-^{\text{app},1})\rangle \tag{4.108}$$

with $u_+^{\text{app},1}$ given by (4.49). The bracket above has been computed in (4.83), (4.86) and (4.87). It is in particular $O(C(A,A')t_\varepsilon^{-1})$. By equations (4.96)–(4.100) the difference $a_+ - e^{it\sqrt{3}/2}g$ is bounded by $C(A)(t_\varepsilon^{-1} + t_\varepsilon^{-1/2}t^{-1/2}(\varepsilon^2\sqrt{t})^{\theta'})$, so that if we replace in (4.108) a_+ by $e^{it\sqrt{3}/2}g$, we get an error bounded by

$$C(A,A')\Big(t_\varepsilon^{-2} + t_\varepsilon^{-\frac{3}{2}}t^{-\frac{1}{2}}(\varepsilon^2\sqrt{t})^{\theta'}\Big) \le C(A,A')t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{2\theta'}, \tag{4.109}$$

so that we get a remainder. Consequently, using again (4.49), we have reduced (4.108) to

$$\frac{\sqrt{3}}{3}\Big(g(t)e^{it\frac{\sqrt{3}}{2}} + \overline{g(t)}e^{-it\frac{\sqrt{3}}{2}}\Big)\Big[\sum_{j\in\{-2,0,2\}}\langle Y_2, p(D_x)^{-1}(U_{j,+} + \overline{U}_{j,+})\rangle\Big] \tag{4.110}$$

up to remainders. We have computed the bracket above in (4.83), (4.86) and (4.87). Up to terms bounded by the product of (4.85) with $t_\varepsilon^{-1/2}$, which still provides remainders satisfying (4.104), we get that (4.110) is given by

$$e^{3it\frac{\sqrt{3}}{2}}\gamma_3 g(t)^3 + e^{it\frac{\sqrt{3}}{2}}\tilde{\gamma}_1|g(t)|^2 g(t) + e^{-it\frac{\sqrt{3}}{2}}\gamma_{-1}|g(t)|^2\overline{g(t)} + e^{-3it\frac{\sqrt{3}}{2}}\gamma_{-3}\overline{g(t)}^3,$$

where γ_j are complex constants, with $\tilde{\gamma}_1 = \frac{\sqrt{3}}{3}(2\alpha_0 + \alpha_2 + \alpha_{-2} + i\beta_2)$, where α_0, α_2, α_{-2} are real and β_2 is given by (4.84). We obtain thus the first four terms on the right-hand side of (4.103). This concludes the proof. ■

We shall next compute from expression (4.96) of a_+ and from (4.103) an equation satisfied by g.

Lemma 4.2.3. *One may choose the coefficients ω_j, $-3 \le j \le 3$, $j \ne 1$, in (4.97) and (4.98) such that if a_+ is given by (4.96) and satisfies (4.103), then g solves*

$$D_t g(t) = \Big(\alpha - i\frac{\sqrt{6}}{18}\hat{Y}_2(\sqrt{2})^2\Big)|g(t)|^2 g(t) + r_1(t), \tag{4.111}$$

where α is real, $\hat{Y}_2(\sqrt{2})^2$ is negative and $r_1(t)$ satisfies

$$\begin{aligned}|r_1(t)| \le C(A)t_\varepsilon^{-\frac{1}{2}}t^{-1}(\varepsilon^2\sqrt{t})^{\theta'} + C(A, A', B')\Big(t_\varepsilon^{-2} + t_\varepsilon^{-1}t^{-1}(\varepsilon^2\sqrt{t})^{\theta'} \\ + t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{2\theta'} + t_\varepsilon^{-\frac{1}{2}}t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'} + t^{-2}(\varepsilon^2\sqrt{t})^{\frac{5}{2}\theta'}\Big),\end{aligned} \tag{4.112}$$

where $C(\cdot)$ are constants depending only on the indicated quantities.

Proof. Let us express in a more explicit way the right-hand side of (4.103). By equations (4.96)–(4.100),

$$\begin{aligned}\Big|a_+(t) - \Big(e^{it\frac{\sqrt{3}}{2}}g(t) + \omega_2 g(t)^2 e^{it\sqrt{3}} + \omega_0|g(t)|^2 + \omega_{-2}\overline{g(t)}^2 e^{-it\sqrt{3}}\Big)\Big| \\ \le C(A)t_\varepsilon^{-\frac{1}{2}}t^{-\frac{1}{2}}(\varepsilon^2\sqrt{t})^{\theta'} + C(A)t_\varepsilon^{-\frac{3}{2}}\end{aligned} \tag{4.113}$$

for constants $C(A)$ depending only on A.

It follows that

$$\begin{aligned}(a_+(t) - a_-(t))^2 = e^{it\sqrt{3}}g(t)^2 + 2|g(t)|^2 + e^{-it\sqrt{3}}\overline{g(t)}^2 \\ + 2e^{3it\frac{\sqrt{3}}{2}}g(t)^3(\omega_2 + \omega_{-2}) \\ + 2e^{it\frac{\sqrt{3}}{2}}|g(t)|^2 g(t)(2\omega_0 + \omega_2 + \omega_{-2}) \\ + 2e^{-it\frac{\sqrt{3}}{2}}|g(t)|^2\overline{g(t)}(2\omega_0 + \omega_2 + \omega_{-2}) \\ + 2e^{-3it\frac{\sqrt{3}}{2}}\overline{g(t)}^3(\omega_2 + \omega_{-2}) + r(t),\end{aligned} \tag{4.114}$$

where r satisfies (4.112).

In the same way

$$\begin{aligned}(a_+(t) - a_-(t))^3 = e^{3it\frac{\sqrt{3}}{2}}g(t)^3 + 3e^{it\frac{\sqrt{3}}{2}}|g(t)|^2 g(t) \\ + 3e^{-it\frac{\sqrt{3}}{2}}|g(t)|^2\overline{g(t)} + e^{-3it\frac{\sqrt{3}}{2}}\overline{g(t)}^3 + r(t)\end{aligned} \tag{4.115}$$

where r satisfies (4.112). We plug (4.114)–(4.115) in the right-hand side of (4.103). We get, as Φ_0, Γ_0 given by (2.35) are real constants, the expression

$$\begin{aligned}& e^{it\sqrt{3}}\Phi_0 g(t)^2 + 2|g(t)|^2\Phi_0 + e^{-it\sqrt{3}}\Phi_0\overline{g(t)}^2 \\ & + e^{it\frac{\sqrt{3}}{2}}|g(t)|^2 g(t)\Big(\underline{\gamma}_1 - i\frac{\sqrt{6}}{18}\hat{Y}_2(\sqrt{2})^2\Big) \\ & + e^{3it\frac{\sqrt{3}}{2}}g(t)^3\underline{\gamma}_3 + e^{-it\frac{\sqrt{3}}{2}}|g(t)|^2\overline{g(t)}\underline{\gamma}_{-1} + e^{-3it\frac{\sqrt{3}}{2}}\overline{g(t)}^3\underline{\gamma}_{-3} \\ & + e^{it\frac{\sqrt{3}}{2}}g(t)(\langle Z, \tilde{u}_+\rangle - \langle Z, \tilde{u}_-\rangle) \\ & + e^{-it\frac{\sqrt{3}}{2}}\overline{g(t)}(\langle Z, \tilde{u}_+\rangle - \langle Z, \tilde{u}_-\rangle) + r(t),\end{aligned} \tag{4.116}$$

where $\underline{\gamma}_j$, $j = -3, -1, 1, 3$, are new constants with $\underline{\gamma}_1$ real, $\underline{\gamma}_{-3}, \underline{\gamma}_{-1}, \underline{\gamma}_3$ depending on $\omega_{-2}, \omega_0, \omega_2$ but not on $\omega_{-3}, \omega_{-1}, \omega_3$, and where $r(t)$ satisfies (4.112), and contains in particular the product of $\langle Z, \tilde{u}_\pm\rangle$ with $a_+(t) - e^{it\sqrt{3}/2}g(t)$, $a_-(t) + e^{it\sqrt{3}/2}\overline{g(t)}$, according to estimates (4.113) and (4.100).

On the other hand, we may compute the left-hand side of (4.103) replacing a_+ by its expression (4.96). We get, using (4.101) with $\lambda = 0$ or $\sqrt{3}$,

$$\begin{aligned}\Big(D_t - \frac{\sqrt{3}}{2}\Big)a_+ = e^{it\frac{\sqrt{3}}{2}} D_t g &+ \frac{\sqrt{3}}{2} e^{it\sqrt{3}}\omega_2 g(t)^2 - \frac{\sqrt{3}}{2}\omega_0|g(t)|^2\\ &- 3\frac{\sqrt{3}}{2}\omega_{-2}e^{-it\sqrt{3}}\overline{g(t)}^2 + \sqrt{3}\omega_3 e^{3it\frac{\sqrt{3}}{2}}g(t)^3\\ &- \sqrt{3}\omega_{-1}e^{-it\frac{\sqrt{3}}{2}}|g(t)|^2\overline{g(t)}\\ &- 2\sqrt{3}\omega_{-3}e^{-3it\frac{\sqrt{3}}{2}}\overline{g(t)}^3\\ &+ e^{it\frac{\sqrt{3}}{2}}g(t)(\langle Z, \tilde{u}_+\rangle - \langle Z, \tilde{u}_-\rangle)\\ &+ e^{-it\frac{\sqrt{3}}{2}}\overline{g(t)}(\langle Z, \tilde{u}_+\rangle - \langle Z, \tilde{u}_-\rangle) + r_1(t),\end{aligned} \tag{4.117}$$

where $r_1(t)$ is made of terms of the form

$$\begin{aligned}&O(|gD_t g|), && O(|D_t g\varphi_\pm(0,t)|), && O(|D_t g\varphi_\pm(\sqrt{3},t)|),\\ &O(|g\psi_\pm(0,t)|), && O(|g\psi_\pm(\sqrt{3},t)|), && O(|g^2 D_t g|).\end{aligned} \tag{4.118}$$

By a priori estimate (4.99) and (4.100), these terms are bounded by

$$\begin{aligned}C(A,A')\Big(t_\varepsilon^{-2} + t_\varepsilon^{-\frac{1}{2}}t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'} + t^{-\frac{1}{2}}t_\varepsilon^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{\theta'} + t^{-2}(\varepsilon^2\sqrt{t})^{\frac{5}{2}\theta'}\Big)\\ + C(A)t_\varepsilon^{-\frac{1}{2}}t^{-1}(\varepsilon^2\sqrt{t})^{\theta'},\end{aligned} \tag{4.119}$$

the last contribution coming from the first two terms in the second line of (4.118). We choose now the free parameters ω_j, $j \in \{-3, \dots, 3\} - \{1\}$ setting

$$\begin{aligned}\omega_3 &= \frac{\sqrt{3}}{3}\underline{\gamma}_3, & \omega_2 &= \frac{2\sqrt{3}}{3}\Phi_0, & \omega_0 &= -\frac{4\sqrt{3}}{3}\Phi_0,\\ \omega_{-1} &= -\frac{\sqrt{3}}{3}\underline{\gamma}_{-1}, & \omega_{-2} &= -\frac{2\sqrt{3}}{9}\Phi_0, & \omega_{-3} &= -\frac{\sqrt{3}}{6}\underline{\gamma}_{-3}\end{aligned}$$

(which is possible as $\underline{\gamma}_{-3}, \underline{\gamma}_{-1}, \underline{\gamma}_3$ do not depend on $\omega_{-3}, \omega_{-1}, \omega_3$). In that way, when we make the difference between the two expressions (4.116) and (4.117) of $(D_t - \frac{\sqrt{3}}{2})$, we obtain equation (4.111) with a remainder satisfying (4.119). This concludes the proof, as $\hat{Y}_2(\sqrt{2})$ being purely imaginary (since Y_2 is real and odd), $\hat{Y}_2(\sqrt{2})^2 \le 0$ and moreover, by Proposition G.1.2, $\hat{Y}_2(\sqrt{2}) \neq 0$. ■

Proof of Proposition 4.2.1. Let us show first that under the assumptions of the proposition, the first inequality of (4.102) holds if A has been chosen large enough, ε small enough and $t \leq \varepsilon^{-4+c}$. In a first step, consider the case when $\varepsilon^2 t$ is small, i.e. let us show that there is $\tau_0 \in]0,1]$ such that if $1 \leq t \leq \frac{\tau_0}{\varepsilon^2}$, and ε is small enough,

$$|g(t)| \leq \frac{A}{4} t_\varepsilon^{-\frac{1}{2}}. \tag{4.120}$$

Since for these t one has $\frac{\varepsilon^2}{2} \leq t_\varepsilon^{-1} \leq \varepsilon^2$, the a priori bound (4.99), equation (4.111) and estimates (4.112) imply that, for any such t,

$$|g(t)| \leq |g(1)| + KA^3\varepsilon^3 t + C(A, A', B')(\varepsilon^{1+\theta'} + \varepsilon^{4\theta'}),$$

where $K = |\alpha - i\frac{\sqrt{6}}{18}\hat{Y}_2(\sqrt{2})^2|$ and $C(\cdot)$ is a new constant depending on A, A', B' (and τ_0). If A is taken such that

$$|g(1)| \leq \frac{A}{8}\frac{\varepsilon}{\sqrt{2}},$$

and τ_0 small enough so that

$$KA^2\tau_0 < \frac{1}{16\sqrt{2}},$$

and if we take ε small enough, we get, using that θ' is close to $\frac{1}{2}$, that

$$|g(t)| \leq \frac{A}{4\sqrt{2}}\varepsilon \leq \frac{A}{4} t_\varepsilon^{-\frac{1}{2}},$$

i.e. (4.120).

We shall thus study from now on equation (4.111) for $t \geq \frac{\tau_0}{\varepsilon^2}$ and initial condition at $\frac{\tau_0}{\varepsilon^2}$ bounded by $\frac{A}{4\sqrt{2}}\varepsilon$. In this regime, for some new constant $C(A, A', B')$, (4.112) implies

$$|r_1(t)| \leq C(A, A', B')\big(t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{\theta'} + t^{-2}\big), \tag{4.121}$$

remembering that t stays in $[\tau_0\varepsilon^{-2}, \varepsilon^{-4+c}]$. For t in $[\tau_0, \varepsilon^{-2+c}]$, set

$$e(t) = \varepsilon^{-1}(1+t)^{\frac{1}{2}} g\Big(\frac{t}{\varepsilon^2}\Big). \tag{4.122}$$

We deduce from (4.111) and (4.121) that if $\beta = -\frac{\sqrt{6}}{18}\hat{Y}_2(\sqrt{2})^2 > 0$,

$$\partial_t e(t) = \frac{1}{2}\frac{e(t)}{1+t} + \frac{-\beta + i\alpha}{1+t}|e(t)|^2 e(t) + R(t), \tag{4.123}$$

where

$$\begin{aligned} |R(t)| &\leq C(A, A', B')\Big(\frac{(1+t)^{\frac{1}{2}}}{t^{\frac{3}{2}}}(\varepsilon\sqrt{t})^{\theta'} + \varepsilon\frac{(1+t)^{\frac{1}{2}}}{t^2}\Big) \\ &\leq \frac{C(A, A', B')}{1+t}(1+\tau_0^{-1})^{\frac{3}{2}}\big(\varepsilon^{\frac{\theta'}{2}c} + \varepsilon\tau_0^{-\frac{1}{2}}\big). \end{aligned} \tag{4.124}$$

Denote $w(t) = |e(t)|^2$. Then

$$\partial_t w(t) = \frac{1}{1+t}(w(t) - 2\beta w(t)^2 + Q(t)), \tag{4.125}$$

where according to (4.124), for $t \in [\tau_0, \varepsilon^{-2+c}]$,

$$|Q(t)| \leq C\left(\varepsilon^{\frac{\theta'}{2}c} + \varepsilon\tau_0^{-\frac{1}{2}}\right)|w(t)|^{\frac{1}{2}} \tag{4.126}$$

for some constant depending on A, A', B', τ_0. Moreover, we have

$$w(\tau_0) \leq \left(\frac{A}{4}\right)^2. \tag{4.127}$$

We fix A large enough so that $(\frac{A}{2})^2 - 2\beta(\frac{A}{2})^4 \leq -\frac{A}{2}$ and then take $\varepsilon < \varepsilon_0$ small enough (in function of A, A', B', τ_0) such that (4.126) implies $|Q(t)| \leq \frac{1}{2}|w(t)|^{1/2}$. Then it follows that if, at some time t_*, $w(t_*)$ reaches $(\frac{A}{2})^2$, the right-hand side of (4.125) is strictly negative. Consequently, taking (4.127) into account, we get $w(t) \leq (\frac{A}{2})^2$ for any t in $[\tau_0, \varepsilon^{-2+c}]$. Using (4.122), we conclude that

$$|g(t)| \leq \frac{A}{2} t_\varepsilon^{-\frac{1}{2}}$$

for t in $[\frac{\tau_0}{\varepsilon^2}, \varepsilon^{-4+c}]$. This gives the first inequality of (4.102).

To get the second one, we notice that we may bound the right-hand side of (4.112) by

$$C(A)\left(t_\varepsilon^{-\frac{3}{2}} + t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'}\right)$$
$$+ C(A, A', B')\left(\varepsilon + (\varepsilon^2\sqrt{t})^{\frac{\theta'}{2}}\right)\left(t_\varepsilon^{-\frac{3}{2}} + t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'}\right)$$

for new constants $C(A), C(A, A', B')$, depending only on the indicated arguments. Plugging this in (4.111), we get

$$|\partial_t g(t)| \leq K|g(t)|^3 + (C(A) + C(A, A', B')e(t, \varepsilon))\left(t_\varepsilon^{-\frac{3}{2}} + t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'}\right)$$

with

$$\lim_{\varepsilon \to 0+} \sup_{t \in [1, \varepsilon^{-4+c}]} e(t, \varepsilon) = 0.$$

If we plug there the first inequality of (4.102), choose A' large enough relatively to A, so that

$$K\left(\frac{A}{2}\right)^3 + C(A) \leq \frac{A'}{4}$$

and then take ε small enough relatively to A, A', B', we get the second inequality of (4.102). This concludes the proof. ■

Chapter 5

Reduced form of dispersive equation

In Section 3.2, we performed a quadratic normal form on equation (3.11) satisfied by u_+ in order to get equation (3.13). On the other hand, in Section 4.1, we constructed some approximate solution solving equation (4.37). Making the difference between (3.13) and (4.37), we shall get an equation for the action of $D_t - p(D_x)$ on

$$\tilde{u}_+ = u_+ - \sum_{|I|=2} \mathrm{Op}(\tilde{m}_{0,I})(u_I) - u_+^{\mathrm{app}}.$$

The goal of this chapter is to invert in convenient spaces the map $u_+ \mapsto \tilde{u}_+$, to obtain an expression for u_+ in terms of $\tilde{u}_+$ and to write down the equation satisfied by $\tilde{u}_+$ in closed form.

5.1 A fixed point theorem

We establish first some abstract theorem. We consider E, F two Banach spaces with norms $\|\cdot\|_E, \|\cdot\|_F$. We consider also two other normed spaces $\tilde{E}, \tilde{F}$ such that $E \cap \tilde{E}$ (resp. $F \cap \tilde{F}$) is also a Banach space. We set $B_F(r), B_E(r)$ for the closed ball of center zero, radius r in F, E. We assume given a function

$$\begin{aligned} \Phi : (E \cap F) \times (E \cap F) &\to E \cap F, \\ (u'', f) &\mapsto \Phi(u'', f) \end{aligned} \tag{5.1}$$

satisfying the following estimates: There are $C > 0, \sigma > 0$ such that for any parameter $\lambda \geq 1$, any u'', f, f_1, f_2 in $E \cap F$, one has

$$\|\Phi(u'', f)\|_E \leq C\big(\|u''\|_F + \|f\|_F\big)\big(\|u''\|_E + \|f\|_E\big), \tag{5.2}$$

$$\begin{aligned} \|\Phi(u'', f)\|_F \leq{}& C\lambda^\sigma\big(\|u''\|_F + \|f\|_F\big)^2 \\ &+ C\lambda^{-1}\big(\|u''\|_F + \|f\|_F\big)\big(\|u''\|_E + \|f\|_E\big), \end{aligned} \tag{5.3}$$

$$\begin{aligned} &\|\Phi(u'', f_1) - \Phi(u'', f_2)\|_E \\ &\quad \leq C\big(\|u''\|_F + \|f_1\|_F + \|f_2\|_F\big)\|f_1 - f_2\|_E \\ &\qquad + C\big(\|u''\|_E + \|f_1\|_E + \|f_2\|_E\big)\|f_1 - f_2\|_F, \end{aligned} \tag{5.4}$$

$$\begin{aligned} &\|\Phi(u'', f_1) - \Phi(u'', f_2)\|_F \\ &\quad \leq C\big(\lambda^\sigma\big(\|u''\|_F + \|f_1\|_F + \|f_2\|_F\big) \\ &\qquad + \lambda^{-1}\big(\|u''\|_E + \|f_1\|_E + \|f_2\|_E\big)\big)\|f_1 - f_2\|_F \\ &\qquad + C\lambda^{-1}\big(\|u''\|_F + \|f_1\|_F + \|f_2\|_F\big)\big)\|f_1 - f_2\|_E. \end{aligned} \tag{5.5}$$

We assume also that if, in addition to preceding assumptions, u'' is in $\tilde{F}$ and f is in $\tilde{E}$, then $\Phi(u'', f)$ is in $\tilde{E}$, with estimate

$$\|\Phi(u'', f)\|_{\tilde{E}} \leq C(\|u''\|_{\tilde{F}}\|u''\|_E + \big(\|u''\|_F + \|f\|_F\big)\|f\|_{\tilde{E}}\big) \tag{5.6}$$

and if f_1, f_2 are in $\tilde{E}$,

$$\|\Phi(u'', f_1) - \Phi(u'', f_2)\|_{\tilde{E}} \leq C\big(\|u''\|_F + \|f_1\|_F + \|f_2\|_F\big)\|f_1 - f_2\|_{\tilde{E}}. \tag{5.7}$$

Lemma 5.1.1. *There is $r_0 > 0$ such that for any r in $]0, r_0[$, any $\lambda \geq 1$, any $u', u'', \tilde{u}$ in $B_E(r\lambda) \cap B_F(r\lambda^{-\sigma})$, the fixed point problem*

$$f = u' + \tilde{u} + \Phi(u'', f) \tag{5.8}$$

has a unique solution f in $B_E(3r\lambda) \cap B_F(3r\lambda^{-\sigma})$. Moreover, if one defines inductively

$$\begin{aligned} \Phi^1(u'', a, g) &= a + \Phi(u'', g), \\ \Phi^{n+1}(u'', a, g) &= \Phi^n(u'', a, \Phi^1(u'', a, g)) = \Phi^1(u'', a, \Phi^n(u'', a, g)), \end{aligned} \tag{5.9}$$

and if one sets

$$\mathcal{E}_\lambda = \lambda^\sigma\big(\|u''\|_F + \|u'\|_F + \|\tilde{u}\|_F\big) + \lambda^{-1}\big(\|u''\|_E + \|u'\|_E + \|\tilde{u}\|_E\big),$$

one has for any $N \geq 1$ and a new constant $C > 0$,

$$\begin{aligned} &\|f - \Phi^N(u'', u' + \tilde{u}, u')\|_E \\ &\quad\leq C^{N+1}\mathcal{E}_\lambda^N\|f - u'\|_E \\ &\qquad\qquad + C^{N+1}\mathcal{E}_\lambda^{N-1}\big(\|u''\|_E + \|u'\|_E + \|\tilde{u}\|_E\big)\|f - u'\|_F, \\ &\|f - \Phi^N(u'', u' + \tilde{u}, u')\|_F \\ &\quad\leq C^{N+1}\mathcal{E}_\lambda^N\|f - u'\|_F + C^{N+1}\mathcal{E}_\lambda^N\lambda^{-1}\|f - u'\|_E. \end{aligned} \tag{5.10}$$

Furthermore, if one assumes that $u', \tilde{u}$ are also in $\tilde{E}$ and u'' is also in $\tilde{F}$, then f is in $\tilde{E}$ and one has for any $N \geq 1$,

$$\|f - \Phi^N(u'', u' + \tilde{u}, u')\|_{\tilde{E}} \leq C^N\big(\|u'\|_F + \|\tilde{u}\|_F + \|u''\|_F\big)^N\|f - u'\|_{\tilde{E}}. \tag{5.11}$$

Proof. We define the usual sequence of approximations

$$\begin{aligned} f_{N+1} &= \Phi^{N+1}(u'', u' + \tilde{u}, u') = u' + \tilde{u} + \Phi(u'', f_N), \\ f_0 &= 0 \end{aligned}$$

using notation (5.9). By (5.2) and (5.3), we have

$$\|f_{N+1}\|_E \leq \|u'\|_E + \|\tilde{u}\|_E + C\big(\|u''\|_F + \|f_N\|_F\big)\big(\|u''\|_E + \|f_N\|_E\big)$$

and

$$\|f_{N+1}\|_F \le \|u'\|_F + \|\tilde{u}\|_F + C\big(\lambda^\sigma\big(\|u''\|_F + \|f_N\|_F\big) + \lambda^{-1}\big(\|u''\|_E + \|f_N\|_E\big)\big)\big(\|u''\|_F + \|f_N\|_F\big).$$

It follows that if $u', u'', \tilde{u}$ are in $B_F(r\lambda^{-\sigma}) \cap B_E(\lambda r)$ with r small enough, one has for any N,

$$\|f_{N+1}\|_E \le \frac{4}{3}\big(\|u'\|_E + \|\tilde{u}\|_E\big) + \frac{1}{3}\|u''\|_E,$$
$$\|f_{N+1}\|_F \le \frac{4}{3}\big(\|u'\|_F + \|\tilde{u}\|_F\big) + \frac{1}{3}\|u''\|_F.$$

In particular, $(f_N)_N$ remains bounded in $B_F(3r\lambda^{-\sigma}) \cap B_E(3\lambda r)$. Moreover, by (5.4) and (5.5) and the above bounds, for r small enough, $(f_N)_N$ converges in $E \cap F$ to a limit f satisfying

$$f = u' + \tilde{u} + \Phi(u'', f) = \Phi^1(u'', u' + \tilde{u}, f).$$

Then (5.10) with $N = 1$ follows from (5.4) and (5.5). One obtains the general case by induction, using (5.4) and (5.5). In the same way, (5.11) follows from (5.7). ∎

We shall apply the preceding lemma with $E = H^s(\mathbb{R})$, $F = W^{\rho,\infty}(\mathbb{R})$, $s > 0$, $\lambda = t \ge 1$, $\rho \in \mathbb{N}$. We define the spaces $\tilde{E}, \tilde{F}$ by

$$\tilde{E} = \{f \in L^2(\mathbb{R}) : xf \in L^2(\mathbb{R})\}, \quad \tilde{F} = \{f \in W^{\rho,\infty}(\mathbb{R}) : xf \in W^{\rho,\infty}(\mathbb{R})\} \tag{5.12}$$

and we endow them with norms depending on the parameter t:

$$\|f\|_{\tilde{E}} = t\|f\|_{L^2} + \|xf\|_{L^2}, \quad \|f\|_{\tilde{F}} = t\|f\|_{W^{\rho,\infty}} + \|xf\|_{W^{\rho,\infty}}.$$

The functions u', u'' of (5.8) will be the functions $u'^{\text{app}}_+, u''^{\text{app}}_+$ of Proposition 4.1.2. By (4.39)–(4.41) applied with a large enough r, and using (4.42), we get

$$\begin{aligned} &\|u'^{\text{app}}_+(t,\cdot)\|_E \le C(A,A')\varepsilon^2 t^{\frac{1}{4}},\\ &\|u'^{\text{app}}_+(t,\cdot)\|_F \le C(A,A')\varepsilon^2,\\ &\|u'^{\text{app}}_+(t,\cdot)\|_{\tilde{E}} \le C(A,A')\big(\varepsilon^2 t^{\frac{5}{4}} + t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^{\frac{7}{8}}\varepsilon^{\frac{1}{8}}\big). \end{aligned} \tag{5.13}$$

In particular, for ε small, $t^\sigma\|u'^{\text{app}}_+(t,\cdot)\|_F + t^{-1}\|u'^{\text{app}}_+(t,\cdot)\|_E$ may be made as small as we want (uniformly in $t \le \varepsilon^{-4}$) if $\varepsilon > 0$ is small enough. In the same way, by (4.43)–(4.45)

$$\begin{aligned} &\|u''^{\text{app}}_+(t,\cdot)\|_E \le C(A,A')\varepsilon,\\ &\|u''^{\text{app}}_+(t,\cdot)\|_F \le C(A,A')\varepsilon^2(\log(1+t))^2,\\ &\|u''^{\text{app}}_+(t,\cdot)\|_{\tilde{F}} \le C(A,A')t\varepsilon^2(\log(1+t))^2. \end{aligned} \tag{5.14}$$

Again, for $t \le \varepsilon^{-4}$, we see that $t^\sigma\|u''^{\text{app}}_+(t,\cdot)\|_F + t^{-1}\|u''^{\text{app}}_+(t,\cdot)\|_E$ may be made as small as we want for $\varepsilon > 0$ small.

We shall take some function $\tilde{u}_+$ in $B_E(\lambda r) \cap B_F(\lambda^{-\sigma} r) \cap \tilde{E}$, and shall solve in u_+ the equation

$$\tilde{u}_+ = u_+ - \sum_{|I|=2} \mathrm{Op}(\tilde{m}_{0,I})(u_I) - u'^{\mathrm{app}}_+ - u''^{\mathrm{app}}_+, \tag{5.15}$$

where $\tilde{m}_{0,I}$ are symbols in $\tilde{S}_{1,0}(\prod_{j=1}^2 \langle \xi_j \rangle^{-1} M_0, 2)$ defined in Proposition 3.2.1. Setting $f_+ = u_+ - u''^{\mathrm{app}}_+$, we rewrite (5.15) as

$$f_+ = u'^{\mathrm{app}}_+ + \tilde{u}_+ + \Phi(u''^{\mathrm{app}}_+, f_+), \tag{5.16}$$

where

$$\Phi(u''^{\mathrm{app}}_+, f_+) = \sum_{|I|=2} \mathrm{Op}(\tilde{m}_{0,I})\big((u''^{\mathrm{app}} + f)_I\big). \tag{5.17}$$

Let us check that the assumptions of Lemma 5.1.1 are satisfied by the preceding map.

Lemma 5.1.2. *If we take $E = H^s(\mathbb{R})$, $F = W^{\rho,\infty}(\mathbb{R})$, with s, ρ large enough and $\tilde{E}, \tilde{F}$ defined by (5.12), then inequalities (5.2) to (5.7) are satisfied by the function Φ defined by (5.17).*

Proof. To prove (5.2) we have to check that, for any I with $|I| = 2$,

$$\|\mathrm{Op}(\tilde{m}_{0,I})\big((u'' + f)_I\big)\|_{H^s} \le C\big(\|u''\|_{W^{\rho,\infty}} + \|f\|_{W^{\rho,\infty}}\big)\big(\|u''\|_{H^s} + \|f\|_{H^s}\big)$$

which follows from (D.32) if ρ is large enough, since Proposition D.1.6 applies in particular to symbols that are independent of x, which is the case of elements of $\tilde{S}_{1,0}(\prod_{j=1}^2 \langle \xi_j \rangle^{-1} M_0, 2)$ according to Definition 3.1.1. In the same way, (5.3) may be written

$$\begin{aligned}\|\mathrm{Op}(\tilde{m}_{0,I})&\big((u'' + f)_I\big)\|_{W^{\rho,\infty}} \\ &\le C\Big(t^\sigma\big(\|u''\|_{W^{\rho,\infty}} + \|f\|_{W^{\rho,\infty}}\big) \\ &\qquad + t^{-1}\big(\|u''\|_{H^s} + \|f\|_{H^s}\big)\Big)\big(\|u''\|_{W^{\rho,\infty}} + \|f\|_{W^{\rho,\infty}}\big)\end{aligned}$$

which follows from (D.39) with $r = 1$ if $(s - \rho)\sigma$ is large enough. Inequalities (5.4) and (5.5) are proved in the same way using the bilinearity of $\mathrm{Op}(\tilde{m}_{0,I})$.

Let us prove (5.6) and (5.7). To simplify notation, consider for instance the case $I = (2, 0)$. It is enough to prove the estimates

$$\|\mathrm{Op}(\tilde{m}_{0,I})(f_1, f_2)\|_{L^2} \le C\|f_1\|_{W^{\rho,\infty}}\|f_2\|_{L^2}, \tag{5.18}$$

$$\|x\mathrm{Op}(\tilde{m}_{0,I})(f_1, f_2)\|_{L^2} \le C\big(t\|f_1\|_{W^{\rho,\infty}} + \|xf_1\|_{W^{\rho,\infty}}\big)\|f_2\|_{L^2}, \tag{5.19}$$

$$\|x\mathrm{Op}(\tilde{m}_{0,I})(f_1, f_2)\|_{L^2} \le C\|f_1\|_{W^{\rho,\infty}}\big(t\|f_2\|_{L^2} + \|xf_2\|_{L^2}\big) \tag{5.20}$$

(and the symmetric ones) in order to get (5.6) and (5.7). But (5.18) (resp. (5.19)) follows from (D.33) (resp. (D.37)) if on the right-hand side of the latter inequality we estimate

$$\|L_\pm v_j\|_{W^{\rho_0,\infty}} \le C\big(\|xv_j\|_{W^{\rho_0,\infty}} + t\|v_j\|_{W^{\rho_0+1,\infty}}\big).$$

To get (5.20), one applies instead (D.33) after commuting x to $\mathrm{Op}(\tilde{m}_{0,I})$ in order to put it against the f_2 argument.

This concludes the proof of the lemma. ∎

We may now state the main result of this section, that will show that the implicit equation (5.16) may be solved in f_+, and that we get an expansion for f_+ in terms of $u'^{\mathrm{app}}_+, u''^{\mathrm{app}}_+$ and $\tilde{u}_+$.

Proposition 5.1.3. *Let $u'^{\mathrm{app}}_+, u''^{\mathrm{app}}_+$ be function satisfying (5.13)–(5.14). Let also $\tilde{u}_+$ be a function of $(t,x) \in [1,T] \times \mathbb{R}$, with $T \leq \varepsilon^{-4+c}$ satisfying for some $0 < \theta' < \theta < \frac{1}{2}$ (θ' and θ being close to $\frac{1}{2}$), some $\delta > 0$, some constant D the following estimates*

$$\begin{aligned} \|\tilde{u}_+(t,\cdot)\|_E &\leq D\varepsilon t^\delta, \\ \|\tilde{u}_+(t,\cdot)\|_F &\leq D\frac{(\varepsilon^2\sqrt{t})^{\theta'}}{\sqrt{t}}, \\ \|\tilde{u}_+(t,\cdot)\|_{\tilde{E}} &\leq Dt^{\frac{5}{4}}(\varepsilon^2\sqrt{t})^{\theta}. \end{aligned} \tag{5.21}$$

Then, if ε is small enough, there is a unique function f_+ in $E \cap F$ with

$$\begin{aligned} \|f_+\|_F &\leq 3\max(C(A,A'),D)\max\left(\varepsilon^2(\log(1+t))^2, \frac{(\varepsilon^2\sqrt{t})^{\theta'}}{\sqrt{t}}\right), \\ \|f_+\|_E &\leq 3\max(C(A,A'),D)\varepsilon t^\delta \end{aligned} \tag{5.22}$$

such that, setting $f_- = -\bar{f}_+$,

$$f_+ = u'^{\mathrm{app}}_+ + \tilde{u}_+ + \sum_{|I|=2} \mathrm{Op}(\tilde{m}_{0,I})\big((u''^{\mathrm{app}} + f)_I\big). \tag{5.23}$$

Moreover, one may find symbols $(m_I)_{2\leq|I|\leq 4}$ in the class $\tilde{S}_{1,0}(\prod_{j=1}^{|I|}\langle\xi_j\rangle^{-1}M_0^\nu, |I|)$ for some ν, such that one may write the solution f_+ to (5.23) under the form

$$f_+ = u'^{\mathrm{app}}_+ + \tilde{u}_+ + \sum_{\substack{2\leq|I|\leq 4\\ I=(I',I'')}} \mathrm{Op}(m_I)\big(\tilde{u}_{I'}, u^{\mathrm{app}}_{I''}\big) + R, \tag{5.24}$$

where R satisfies

$$\|R(t,\cdot)\|_{H^s} \leq C'(A,A',D)\left(\frac{(\varepsilon^2\sqrt{t})^{\theta'}t^\sigma}{\sqrt{t}}\right)^4 \varepsilon t^\delta, \tag{5.25}$$

$$\|xR(t,\cdot)\|_{L^2} \leq C'(A,A',D)\left(\frac{(\varepsilon^2\sqrt{t})^{\theta'}t^\sigma}{\sqrt{t}}\right)^4 t^{\frac{5}{4}}(\varepsilon^2\sqrt{t})^\theta \tag{5.26}$$

for some new constants $C'(A,A',D)$, $\sigma > 0$ as small as we want.

Proof. Equation (5.23) may be written under the form (5.16) with Φ given by (5.17). We have seen in Lemma 5.1.2 that inequalities (5.2) to (5.7) hold true, with the spaces $E, F, \tilde{E}, \tilde{F}$ defined in that lemma. By (5.13), (5.14) and (5.21), if $t \leq \varepsilon^{-4}$ and ε is small enough, we can make $t^\sigma\|u'^{\text{app}}_+(t,\cdot)\|_F$, $t^\sigma\|u''^{\text{app}}_+(t,\cdot)\|_F$, $t^\sigma\|\tilde{u}'_+(t,\cdot)\|_F$ and $t^{-1}\|u'^{\text{app}}_+(t,\cdot)\|_E$, $t^{-1}\|u''^{\text{app}}_+(t,\cdot)\|_E$, $t^{-1}\|\tilde{u}'_+(t,\cdot)\|_E$ as small as we want. We may thus apply Lemma 5.1.1, that gives the solution f_+ to (5.23) and its uniqueness. This lemma gives as well the first inequality of (5.22). To get the second one, we deduce from (5.8) and (5.2) that

$$\|f_+\|_E \leq \|u'^{\text{app}}_+\|_E + \|\tilde{u}_+\|_E + \sigma(\varepsilon)\big(\|f_+\|_E + \|u''^{\text{app}}_+\|_E\big), \tag{5.27}$$

where $\sigma(\varepsilon)$ is controlled by $\|f_+\|_F$ and $\|u''^{\text{app}}_+\|_F$, so goes to zero if ε goes to zero by the first inequality of (5.22) and (5.14). Using (5.13), (5.14), (5.21), it follows that, for ε small enough,

$$\|f_+\|_E \leq 3\max(C(A,A'),D)\varepsilon t^\delta. \tag{5.28}$$

In the same way, we get from (5.8) and (5.6),

$$\|f_+\|_{\tilde{E}} \leq \|u'^{\text{app}}_+\|_{\tilde{E}} + \|\tilde{u}_+\|_{\tilde{E}} + C\|u''^{\text{app}}_+\|_{\tilde{F}}\|u''^{\text{app}}_+\|_E + \sigma(\varepsilon)\|f_+\|_{\tilde{E}},$$

where $\sigma(\varepsilon)$ is controlled by $\|u''^{\text{app}}_+\|_F + \|f_+\|_F$, so goes to zero with ε. Plugging (5.13), (5.14), (5.21) in this inequality, we get for ε small enough, and some new constant $\tilde{C}(A,A',D)$,

$$\|f_+\|_{\tilde{E}} \leq \tilde{C}(A,A',D)t^{\frac{5}{4}}(\varepsilon^2\sqrt{t})^\theta. \tag{5.29}$$

We apply next (5.10) with $N=4$. We obtain, using (5.13), (5.14), (5.21), (5.22) that

$$\big\|f_+ - \Phi^4(u''^{\text{app}}_+, u'^{\text{app}}_+ + \tilde{u}_+, u'^{\text{app}}_+)\big\|_E \leq C'(A,A',D)\bigg(\frac{(\varepsilon^2\sqrt{t})^{\theta'}t^\sigma}{\sqrt{t}}\bigg)^4 \varepsilon t^\delta \tag{5.30}$$

since we assume $t \leq \varepsilon^{-4+c}$ with some $c>0$. In the same way, by (5.11)

$$\begin{aligned}\big\|f_+ - \Phi^4(u''^{\text{app}}_+, u'^{\text{app}}_+ + \tilde{u}_+, u'^{\text{app}}_+)\big\|_{\tilde{E}} \\ \leq C'(A,A',D)\bigg(\frac{(\varepsilon^2\sqrt{t})^{\theta'}t^\sigma}{\sqrt{t}}\bigg)^4 t^{\frac{5}{4}}(\varepsilon^2\sqrt{t})^\theta.\end{aligned} \tag{5.31}$$

The right-hand side of (5.30) (resp. (5.31)) is controlled by (5.25) (resp. (5.26)).

To finish the proof, we have to rewrite $\Phi^4(u''^{\text{app}}_+, u'^{\text{app}}_+ + \tilde{u}_+, u'^{\text{app}}_+)$ as the main term on the right-hand side of (5.24), up to remainders. Let us show by induction that one may write

$$\Phi^N(u''^{\text{app}}_+, u'^{\text{app}}_+ + \tilde{u}_+, u'^{\text{app}}_+) = u'^{\text{app}}_+ + \tilde{u}_+ + \sum_{\substack{2\leq|I|\leq N+1\\ I=(I',I'')}} \operatorname{Op}(m_I^N)(\tilde{u}_{I'}, u^{\text{app}}_{I''}) \tag{5.32}$$

for some new symbols m_I^N in $\tilde{S}_{1,0}(\prod_{j=1}^{|I|}\langle\xi_j\rangle^{-1}M_0^\nu, |I|)$ for some ν. For $N=1$ this follows from the definition (5.9) of Φ^1 and of (5.17). The general case follows using (5.9) and Corollary B.2.6, i.e. the stability of operators of the form $\operatorname{Op}(m_I^N)$ by composition.

We apply (5.32) with $N = 4$, and according to (5.30) and (5.31), equality (5.24) will be proved if we show that the contribution to the right-hand side of (5.32) given by I with $|I| = 5$ forms part of R in (5.24). Using (D.33), we estimate the H^s norm of such a term by

$$C\big(\|\tilde{u}_+\|_{W^{\rho_0,\infty}} + \|u'^{\mathrm{app}}_+\|_{W^{\rho_0,\infty}} + \|u''^{\mathrm{app}}_+\|_{W^{\rho_0,\infty}}\big)^4 \\ \times \big(\|\tilde{u}_+\|_{H^s} + \|u'^{\mathrm{app}}_+\|_{H^s} + \|u''^{\mathrm{app}}_+\|_{H^s}\big),$$

so by the right-hand side of (5.25), using (5.13), (5.14), (5.21).

To study the L^2 norm of the product of x and of the terms in the sum (5.32) with $|I| = 5$, we rewrite the latter, decomposing $u^{\mathrm{app}} = u'^{\mathrm{app}} + u''^{\mathrm{app}}$ under the form

$$\sum_{\substack{|I|=5\\ I=(I',I'',I''')}} \mathrm{Op}(\tilde{m}_I^5)(\tilde{u}_{I'}, u'^{\mathrm{app}}_{I''}, u''^{\mathrm{app}}_{I'''}) \tag{5.33}$$

with symbols $\tilde{m}_I^5$ in $\tilde{S}_{1,0}(\prod_{j=1}^5 \langle \xi_j \rangle^{-1} M_0^{\nu}, 5)$.

In (5.33), we distinguish the cases $|I'''| < 5$ and $|I'''| = 5$. In the first one, we use (D.36), making play the special role to one argument different from $u''^{\mathrm{app}}_\pm$. We obtain a bound in

$$\big(\|\tilde{u}_+\|_{W^{\rho_0,\infty}} + \|u'^{\mathrm{app}}_+\|_{W^{\rho_0,\infty}} + \|u''^{\mathrm{app}}_+\|_{W^{\rho_0,\infty}}\big)^4 \big(\|u'^{\mathrm{app}}_+\|_{\tilde{E}} + \|\tilde{u}_+\|_{\tilde{E}}\big)$$

which is controlled by the right-hand side of (5.26). When $|I'''| = 5$, we use (D.37), to obtain a bound in

$$\|u''^{\mathrm{app}}_+\|^3_{W^{\rho_0,\infty}} \|u''^{\mathrm{app}}_+\|_{L^2} \|u''^{\mathrm{app}}_+\|_{\tilde{F}} \le C(A, A')t\big(\log(1+t)\big)^8 \varepsilon^9$$

by (5.14). Since $t \le \varepsilon^{-4+c}$, the last bound is smaller, for ε small enough, than

$$C'(A, A', D)\bigg(\frac{(\varepsilon^2\sqrt{t})^{\theta'}}{\sqrt{t}}\bigg)^4 t^{\frac{5}{4}} (\varepsilon^2\sqrt{t})^{\theta},$$

so than the right-hand side of (5.26). This concludes the proof. ■

5.2 Reduction of the dispersive equation

The goal of this section is to deduce from equation (3.13) satisfied by u_+ an equation satisfied by the function $\tilde{u}_+$ defined in (5.15). More precisely, we shall prove:

Proposition 5.2.1. *We fix $c > 0$, $0 < \theta' < \theta < \frac{1}{2}$, with θ' close to $\frac{1}{2}$ and $\delta > 0$ small. We take numbers satisfying $s \gg \rho \gg 1$ (that may depend on the preceding parameters c, θ, θ'). Let $\varepsilon \in]0, 1]$ and $T \in [1, \varepsilon^{-4+c}]$. Assume we are given on interval $[1, T]$ a solution $u^{\mathrm{app}}_+ = u'^{\mathrm{app}}_+ + u''^{\mathrm{app}}_+$ of (4.37) satisfying bounds (4.39)–(4.41) and (4.43)–(4.45). Assume also given a function u_+ in $C([1, T], H^s(\mathbb{R}))$, odd, solution* *of* (3.13) *and such that, if we define* $\tilde{u}_+$ *by* (5.15)*, i.e.*

$$\tilde{u}_+ = u_+ - \sum_{|I|=2} \mathrm{Op}(\tilde{m}_{0,I})(u_I) - u'^{\mathrm{app}}_+ - u''^{\mathrm{app}}_+, \tag{5.34}$$

then $\tilde{u}_+$ *satisfies for* $t \in [1, T]$ *the bounds*

$$\begin{aligned} \|\tilde{u}_+(t,\cdot)\|_{H^s} &\leq D\varepsilon t^{\delta}, \\ \|\tilde{u}_+(t,\cdot)\|_{W^{\rho,\infty}} &\leq D\frac{(\varepsilon^2\sqrt{t})^{\theta'}}{\sqrt{t}}, \\ \|L_+\tilde{u}_+(t,\cdot)\|_{L^2} &\leq Dt^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^{\theta} \end{aligned} \tag{5.35}$$

for some constant D*. Then* $\tilde{u}_+$ *solves the equation*

$$\begin{aligned} &\big(D_t - p(D_x)\big)\tilde{u}_+ \\ &\quad= \sum_{\substack{3\leq |I|\leq 4\\ I=(I',I'')}} \mathrm{Op}(\tilde{m}_I)(\tilde{u}_{I'}, u^{\mathrm{app}}_{I''}) + \sum_{\substack{|I|=2\\ I=(I',I'')}} \mathrm{Op}(m'_{0,I})(\tilde{u}_{I'}, u^{\mathrm{app}}_{I''}) \\ &\qquad + \underline{a}^{\mathrm{app}}(t) \sum_{|I|=1} \mathrm{Op}(m'_{1,I})(\tilde{u}_I) \\ &\qquad + \frac{1}{3}\Big(e^{it\frac{\sqrt{3}}{2}}g(t) + e^{-it\frac{\sqrt{3}}{2}}\overline{g(t)}\Big)^2 \sum_{|I|=1} \mathrm{Op}(m'_{0,I})(\tilde{u}_I) + R(t,x), \end{aligned} \tag{5.36}$$

where for some $\nu \in \mathbb{N}$*,* $\tilde{m}_I$ *are symbols in* $\tilde{S}_{1,0}(\prod_{j=1}^{|I|}\langle\xi_j\rangle^{-1}M_0(\xi)^{\nu}, |I|)$*,* $3 \leq |I| \leq 4$*, where* $m'_{0,I}, \tilde{m}'_{1,I}$ *are in* $\tilde{S}'_{1,0}(\prod_{j=1}^{|I|}\langle\xi_j\rangle^{-1}M_0(\xi)^{\nu}, |I|)$*, all these symbols satisfying* (3.7)*, and where*

$$\underline{a}^{\mathrm{app}}(t) = \frac{\sqrt{3}}{3}\big(\underline{a}^{\mathrm{app}}_+(t) - \underline{a}^{\mathrm{app}}_-(t)\big) \tag{5.37}$$

with $\underline{a}^{\mathrm{app}}_+(t)$ *being given by the first four terms on the right-hand side of* (4.8)*, namely*

$$\underline{a}^{\mathrm{app}}_+(t) = e^{it\frac{\sqrt{3}}{2}}g(t) + \omega_2 g(t)^2 e^{it\sqrt{3}} + \omega_0|g(t)|^2 + \omega_{-2}\overline{g(t)}^2 e^{-it\sqrt{3}} \tag{5.38}$$

and

$$\underline{a}^{\mathrm{app}}_-(t) = -\overline{\underline{a}^{\mathrm{app}}_+(t)},$$

and where $R(t,x)$ *satisfies the following bounds for* $t \in [1, T]$:

$$\|R(t,\cdot)\|_{H^s} \leq \varepsilon t^{\delta-1}e(t,\varepsilon), \tag{5.39}$$

$$\|L_{\pm}R(t,\cdot)\|_{L^2} \leq t^{-\frac{3}{4}}(\varepsilon^2\sqrt{t})^{\theta}e(t,\varepsilon), \tag{5.40}$$

where

$$\lim_{\varepsilon\to 0+}\ \sup_{1\leq t\leq \varepsilon^{-4+c}} e(t,\varepsilon) = 0. \tag{5.41}$$

As a preparation for the proof, let us rewrite equation (3.13) replacing in its left-hand side u_+ by the expression of that function that follows from (5.34), namely

$$\begin{aligned}\big(D_t - p(D_x)\big)\big(\tilde u_+ + u'^{\mathrm{app}}_+ + u''^{\mathrm{app}}_+\big)\\ = F_0^2[a] + F_0^3[a] + \sum_{3\le |I|\le 4} \mathrm{Op}(m_{0,I})[u_I] + \sum_{|I|=2}\mathrm{Op}(m'_{0,I})[u_I]\\ + \sum_{j=1}^{3} a(t)^j \sum_{1\le |I|\le 4-j}\mathrm{Op}(m'_{1,I})[u_I].\end{aligned} \tag{5.42}$$

Recall that we have written in (4.37) an expression for $(D_t - p(D_x))u_+^{\mathrm{app}}$. Making the difference between (5.42) and (4.37), we get that $(D_t - p(D_x))\tilde u_+$ is equal to the sum of the following expressions:

$$F_0^2[a] - F_0^2[a^{\mathrm{app}}] + F_0^3[a] - F_0^3[a^{\mathrm{app}}], \tag{5.43}$$

$$\sum_{3\le |I|\le 4}\mathrm{Op}(m_{0,I})[u_I], \tag{5.44}$$

$$\sum_{|I|=2}\mathrm{Op}(m'_{0,I})[u_I], \tag{5.45}$$

$$a(t)\sum_{|I|=1}\mathrm{Op}(m'_{1,I})[u_I] - a^{\mathrm{app}}(t)\sum_{|I|=1}\mathrm{Op}(m'_{1,I})[u_I^{\mathrm{app}}], \tag{5.46}$$

$$a(t)\sum_{2\le |I|\le 3}\mathrm{Op}(m'_{0,I})[u_I], \tag{5.47}$$

$$a(t)^j \sum_{1\le |I|\le 4-j}\mathrm{Op}(m'_{0,I})[u_I], \quad j = 2,3, \tag{5.48}$$

$$-R(t,x), \tag{5.49}$$

where R satisfies (4.38).

We shall analyze successively the expressions (5.43) to (5.49), using (5.34), in order to rewrite their sum as the right-hand side of (5.36) with a new remainder R.

We first write in a lemma some elementary inequalities that we shall refer to in the sequel.

Lemma 5.2.2. *We denote by $e(t,x)$ any real-valued function defined on the interval $[1,\varepsilon^{-4+c}]$, satisfying (5.41). We have then the following inequalities:*

$$t_\varepsilon^{-1}t^{-\gamma} = O(\varepsilon t^{-1} e(t,\varepsilon)) \quad \text{if } \gamma > \frac{1}{2}, \tag{5.50}$$

$$|\log\varepsilon| t_\varepsilon^{-\gamma} t^{-\frac12} = O\big(t^{-\frac34}(\varepsilon^2\sqrt{t})^\theta e(t,\varepsilon)\big) \quad \text{if } \gamma \ge \frac12,\ \theta < \frac12, \tag{5.51}$$

$$\big(\varepsilon^\gamma + (\varepsilon^2\sqrt{t})^{\gamma'} t^{-1}\big)\varepsilon t^\delta = O\big(\varepsilon t^{\delta-1} e(t,\varepsilon)\big) \quad \text{if } \delta > 0,\ \gamma\ge 4,\ \gamma' > 0, \tag{5.52}$$

$$\begin{aligned}(\varepsilon^2\sqrt{t})^\gamma |\log\varepsilon|^4 t^{-\frac34}(\varepsilon^2\sqrt{t})^\theta = O\big(t^{-\frac34}(\varepsilon^2\sqrt{t})^\theta e(t,\varepsilon)\big)\\ \text{if } \gamma > 0,\ 0<\theta<\frac12,\end{aligned} \tag{5.53}$$

$$(\varepsilon^2\sqrt{t})^{\gamma}|\log\varepsilon|t^{-\frac{3}{2}-\alpha}\big(t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^{\theta}\big) = O\big(\varepsilon t^{\delta-1}e(t,\varepsilon)\big)$$
$$\text{if } \frac{1}{2}-\theta<\gamma\leq\frac{1}{2}-\theta+2\delta,\ \alpha\geq 0, \tag{5.54}$$

$$|\log\varepsilon|^2\varepsilon t^{-\frac{1}{2}} = O\big(t^{-\frac{3}{4}}(\varepsilon^2\sqrt{t})^{\theta}e(t,\varepsilon)\big) \quad \text{if } 0<\theta<\frac{1}{2}, \tag{5.55}$$

$$|\log\varepsilon|^2\varepsilon t_\varepsilon^{-\frac{1}{2}}t^{-\gamma} = O(\varepsilon t^{-1}e(t,\varepsilon)) \quad \text{if } \frac{1}{2}<\gamma<1, \tag{5.56}$$

$$\varepsilon^2 t_\varepsilon^{-1}t^{\frac{1}{4}} = O(\varepsilon t^{-1}e(t,\varepsilon)). \tag{5.57}$$

Proof of Proposition 5.2.1. Since $(D_t - p(D_x))\tilde{u}_+$ is given by (5.43) to (5.49), we have to write each of these terms as contributions to the right-hand side of (5.36). We study them successively.

Terms of the form (5.43). Recall that $a = \frac{\sqrt{3}}{3}(a_+ - a_-)$ with $a_- = -\bar{a}_+$ (see (2.33)) and that $a_+(t)$ is given by (4.96). Since by (4.99), $g(t)$ is $O(t_\varepsilon^{-1/2})$, it follows from (4.96), (4.98) that $a_+(t) - a_+^{\mathrm{app}}(t) = O(t_\varepsilon^{-3/2})$. The definition (2.28) of $F_0^2[a]$, $F_0^3[a]$ implies that for any α, N integers

$$\big|\partial_x^{\alpha}\big(F_0^j[a] - F_0^j[a^{\mathrm{app}}]\big)(t,x)\big| \leq C_{\alpha,N}t_\varepsilon^{-2}\langle x\rangle^{-N},\ j=2,3. \tag{5.58}$$

Thus (5.50) implies that (5.39) holds (even with $\delta=0$) and (5.51) implies that (5.40) is true for any $\theta<\frac{1}{2}$. So these terms contribute to R in (5.36).

Terms of the form (5.44). Notice that if $\tilde{u}_+$ satisfies estimates (5.35), then it satisfies bounds (5.21) (with a new constant D) in view of the definition of $E = H^s$, $F = W^{\rho,\infty}$ and (5.12) of $\tilde{E}$. Moreover, if we set $f_+ = u_+ - u''^{\mathrm{app}}_+$, equation (5.34) may be written as (5.23). Then Proposition 5.1.3 implies that for ε small enough, there is a unique solution f_+ solving equation (5.23), and we have an expansion (5.24) for f_+ in terms of $\tilde{u}, u^{\mathrm{app}}$. We may rewrite this as

$$u_+ = u_+^{\mathrm{app}} + \tilde{u}_+ + \sum_{\substack{2\leq|I|\leq 4\\ I=(I',I'')}}\mathrm{Op}(m_I)(\tilde{u}_{I'}, u^{\mathrm{app}}_{I''}) + R \tag{5.59}$$

with symbols m_I in $\tilde{S}_{1,0}(\prod_{j=1}^{|I|}\langle\xi_j\rangle^{-1}M_0^{\nu}, |I|)$ and R satisfying (5.25) and (5.26). We plug expansion (5.59) inside (5.44). Recall that by Proposition 3.2.1, the symbols $m_{0,I}$ in (5.44) belong to $\tilde{S}_{1,0}(\prod_{j=1}^{|I|}\langle\xi_j\rangle^{-1}M_0, |I|)$. By Corollary B.2.6, we shall get terms of the following form:

$$\mathrm{Op}(\tilde{m}_I)(\tilde{u}_{I'}, u^{\mathrm{app}}_{I''}), \quad 3\leq|I|\leq 4,\ I=(I',I''), \tag{5.60}$$

where $\tilde{m}_I$ is some new symbol in $\tilde{S}_{1,0}(\prod_{j=1}^{|I|}\langle\xi_j\rangle^{-1}M_0^{\nu}, |I|)$ for some new ν;

$$\mathrm{Op}(\tilde{m}_I)(U_1, U_2, \dots, U_k), \quad k=|I| \tag{5.61}$$

with $\tilde{m}_I$ as above and either

$$k\geq 5, \quad U_\ell\in\{\tilde{u}_\pm, u'^{\mathrm{app}}_\pm, u''^{\mathrm{app}}_\pm\} \tag{5.62}$$

or

$$k \geq 3, \quad U_\ell \in \{\tilde{u}_\pm, u'^{\text{app}}_\pm, u''^{\text{app}}_\pm, R\} \tag{5.63}$$

with R satisfying (5.25), (5.26), one of the U_ℓ at least being equal to R.

Terms of the form (5.60) are present on the right-hand side of (5.36). We have to show that (5.61) contributes to the remainder in that formula. By (D.32), under (5.62), the H^s norm of (5.61) is bounded from above by

$$C\left(\|\tilde{u}_+\|_{W^{\rho,\infty}} + \|u'^{\text{app}}_+\|_{W^{\rho,\infty}} + \|u''^{\text{app}}_+\|_{W^{\rho,\infty}}\right)^{k-1} \\ \times \left(\|\tilde{u}_+\|_{H^s} + \|u'^{\text{app}}_+\|_{H^s} + \|u''^{\text{app}}_+\|_{H^s}\right).$$

By (5.35), (5.13), (5.14), and since $k \geq 5$, we obtain a bound in

$$C\left(\varepsilon^2|\log\varepsilon|^2 + \frac{(\varepsilon^2\sqrt{t})^{\theta'}}{\sqrt{t}}\right)^4 \varepsilon t^\delta \tag{5.64}$$

so that (5.52) implies that (5.39) holds. On the other hand, consider the action of $L_\pm$ on (5.61) and let us estimate the L^2 norm of the resulting expression by the right-hand side of (5.40). If we multiply (5.61) by x, we have to study

$$x\mathrm{Op}(\tilde{m}_I)(U_1, \ldots, U_{k-1}, U_k). \tag{5.65}$$

Consider first the case when among the U_ℓ's in (5.61), at least one of them is equal to $\tilde{u}_\pm$ or $u'^{\text{app}}_\pm$, say U_k. We apply (D.36) (with $j = k$) and obtain thus for the L^2 norm of the relevant quantity at time τ a bound in

$$C\left(\|\tilde{u}_+\|_{W^{\rho,\infty}} + \|u'^{\text{app}}_+\|_{W^{\rho,\infty}} + \|u''^{\text{app}}_+\|_{W^{\rho,\infty}}\right)^{k-1} \\ \times \left(\tau\|\tilde{u}_+\|_{L^2} + \|L_+\tilde{u}_+\|_{L^2} + \tau\|u'^{\text{app}}_+\|_{L^2} + \|L_+u'^{\text{app}}_+\|_{L^2}\right). \tag{5.66}$$

By (5.35), (4.40), (4.44), (4.39), (4.41), and the fact that $k \geq 5$, we obtain a bound at time τ in

$$C\left(\varepsilon^2|\log\varepsilon|^2 + \frac{(\varepsilon^2\sqrt{\tau})^{\theta'}}{\sqrt{\tau}}\right)^4 \tau^{\frac{5}{4}}(\varepsilon^2\sqrt{\tau})^\theta. \tag{5.67}$$

By (5.53) we get a bound of the form (5.40) for (5.66).

Consider next the case when in (5.61), all the U_ℓ are equal to $u''^{\text{app}}_\pm$. In this case, we use (D.37) (with $\rho > \rho_0$) to estimate the L^2 norm of (5.65) at time τ. We get a bound by

$$C\|u''^{\text{app}}_+\|^{k-2}_{W^{\rho,\infty}}\left(\tau\|u''^{\text{app}}_+\|_{W^{\rho,\infty}} + \|L_+u''^{\text{app}}_+\|_{W^{\rho,\infty}}\right)\|u''^{\text{app}}_+\|_{L^2}. \tag{5.68}$$

By (4.43)–(4.45) we get an estimate by

$$C\varepsilon(\varepsilon^2\sqrt{\tau})^4|\log\varepsilon|^8\tau^{-1} + \varepsilon(\varepsilon^2\sqrt{\tau})^3|\log\varepsilon|^8\tau^{-\frac{3}{2}}$$

to which (5.53) largely applies.

On the other hand, the L^2 norm of the product of (5.61) by τ is estimated using (D.33) by (5.66) or (5.68) as well. We thus have obtained that, under condition (5.62), (5.61) forms part of the remainder in (5.36).

Let us study now case (5.63). If we compute the H^s norm of (5.61) applying (D.32), we obtain a bound in

$$\begin{aligned} &C\big(\|\tilde{u}_+\|_{W^{\rho,\infty}} + \|u'^{\text{app}}_+\|_{W^{\rho,\infty}} + \|u''^{\text{app}}_+\|_{W^{\rho,\infty}} + \|R\|_{W^{\rho,\infty}}\big)^{k-1}\|R\|_{H^s} \\ &\quad + C\big(\|\tilde{u}_+\|_{W^{\rho,\infty}} + \|u'^{\text{app}}_+\|_{W^{\rho,\infty}} + \|u''^{\text{app}}_+\|_{W^{\rho,\infty}} + \|R\|_{W^{\rho,\infty}}\big)^{k-2} \\ &\qquad \times \big(\|\tilde{u}_+\|_{H^s} + \|u'^{\text{app}}_+\|_{H^s} + \|u''^{\text{app}}_+\|_{H^s}\big)\|R\|_{W^{\rho,\infty}}. \end{aligned} \tag{5.69}$$

By (5.25), that allows to bound $\|R\|_{W^{\rho,\infty}}$ by Sobolev injection, (4.40), (4.44), (5.35), the first line is bounded by (5.25), so it satisfies (5.39). The second line of (5.69) is also estimated in that way. Notice that the assumption $k \geq 3$ is not used here, and that $k \geq 2$ suffices.

If we compute instead the L^2 norm of the product of (5.61) by x from an expression of the form (5.65) with U_k replaced by R and apply (D.36), we obtain an estimate at time τ in

$$\begin{aligned} &C\big(\|\tilde{u}_+\|_{W^{\rho,\infty}} + \|u'^{\text{app}}_+\|_{W^{\rho,\infty}} + \|u''^{\text{app}}_+\|_{W^{\rho,\infty}} + \|R\|_{W^{\rho,\infty}}\big)^{k-1} \\ &\quad \times \big(\tau\|R\|_{L^2} + \|xR\|_{L^2}\big). \end{aligned} \tag{5.70}$$

The first factor is $O(\varepsilon^{2\theta'})$ by (4.40), (4.44), (5.35) and (5.25) (coupled with Sobolev injection). The last one is bounded from above using (5.25) and (5.26), so that it satisfies (5.40) using (5.53). The L^2 norm of the product of (5.61) by τ is also estimated by (5.70). Again, only $k \geq 2$ is used.

Terms of the form (5.45). We plug in (5.45) expansion (5.59). By Corollary B.2.6, we get terms of the form

$$\mathrm{Op}(m'_{0,I})(\tilde{u}_{I'}, u^{\text{app}}_{I''}), \quad |I| = 2,\ I = (I', I'') \tag{5.71}$$

and terms of higher degree of homogeneity. We may thus write these terms as

$$\mathrm{Op}(\tilde{m}'_I)(U_1, \dots, U_k), \quad |I| = k, \tag{5.72}$$

where $\tilde{m}'_I$ is in $\tilde{S}'_{1,0}(\prod_{j=1}^{|I|}\langle\xi_j\rangle^{-1}M_0^{\nu}, |I|)$ for some ν and where either

$$k \geq 3, \quad U_\ell \in \{\tilde{u}_\pm, u'^{\text{app}}_\pm, u''^{\text{app}}_\pm\} \tag{5.73}$$

or

$$k \geq 2, \quad U_\ell \in \{\tilde{u}_\pm, u'^{\text{app}}_\pm, u''^{\text{app}}_\pm, R\} \tag{5.74}$$

with at least one factor equal to R. Terms (5.72) under condition (5.74) provide remainders satisfying (5.39) and (5.40), as it has been seen in (5.69) and (5.70). (The fact that $k \geq 3$ there has not been used.)

Terms (5.71) are present on the right-hand side of (5.36). Let us show that terms (5.72) under condition (5.73), provide contributions to R in (5.36). To estimate the H^s norm of (5.72), we may first split the symbols in new ones satisfying the support condition of Corollary D.2.12, i.e. for instance $|\xi_1| + \cdots + |\xi_{k-1}| \leq K(1 + |\xi_k|)$. We shall apply estimate (D.78) with $n = k$, $\ell = k - 1$. Let ℓ' be the number of indices j between 1 and $k - 1$ such that in (5.72), U_j is equal to $\tilde{u}_{\pm}$ or $u'^{\text{app}}_{\pm}$. Then by (D.78)

$$\begin{aligned}
&\|\mathrm{Op}(\tilde{m}'_I)(U_1, \dots, U_k)\|_{H^s} \\
&\quad \leq C t^{-(k-1)+\sigma} \big(\|L_+ \tilde{u}_+\|_{L^2} + \|L_+ u'^{\text{app}}_+\|_{L^2} + \|\tilde{u}_+\|_{H^s} + \|u'^{\text{app}}_+\|_{H^s}\big)^{\ell'} \\
&\qquad \times \big(\|L_+ u''^{\text{app}}_+\|_{W^{\rho_0,\infty}} + \|u''^{\text{app}}_+\|_{W^{\rho_0,\infty}} + t^{-\frac{1}{2}}\|u''^{\text{app}}_+\|_{H^s}\big)^{k-1-\ell'} \\
&\qquad \times \big(\|\tilde{u}_+\|_{H^s} + \|u'^{\text{app}}_+\|_{H^s} + \|u''^{\text{app}}_+\|_{H^s}\big).
\end{aligned} \tag{5.75}$$

Since $k \geq 3$, we obtain from (4.39)–(4.41), (4.43)–(4.45) and (5.35) a bound in

$$C t^{\sigma-2}\big(t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^{\theta}|\log\varepsilon|^2\big)^2 \varepsilon t^{\delta} \leq C t^{-1} e(t,\varepsilon)\varepsilon t^{\delta}$$

if σ is taken small enough, so that (5.39) holds.

We consider next the L^2 norm of (5.72) multiplied by x or t. The rapid decay of symbols in the $S'_{\kappa,0}$ class relatively to $M_0(\xi)^{-\kappa}|y|$ given by (B.13) implies that the product of $\tilde{m}'_I$ by x is still a symbol of the form $\tilde{m}'_I$ (with a new value of ν). We thus have to estimate just

$$t\|\mathrm{Op}(\tilde{m}'_I)(U_1, \dots, U_k)\|_{L^2} \tag{5.76}$$

with U_ℓ satisfying (5.73). If at least one U_j is equal to $\tilde{u}_{\pm}$ or $u'^{\text{app}}_{\pm}$, we use (D.71) with that value of j. We get a bound of (5.76) in

$$\begin{aligned}
&C\big(\|\tilde{u}_+\|_{W^{\rho_0,\infty}} + \|u'^{\text{app}}_+\|_{W^{\rho_0,\infty}} + \|u''^{\text{app}}_+\|_{W^{\rho_0,\infty}}\big)^{k-1} \\
&\qquad \times \big(\|L_+\tilde{u}_+\|_{L^2} + \|L_+ u'^{\text{app}}_+\|_{L^2} + \|\tilde{u}_+\|_{L^2} + \|u'^{\text{app}}_+\|_{L^2}\big).
\end{aligned} \tag{5.77}$$

If all U_j are equal to $u''^{\text{app}}_{\pm}$, we use (D.72) in order to obtain a bound in

$$C\|u''^{\text{app}}_+\|^{k-2}_{W^{\rho_0,\infty}}\big(\|L_+ u''^{\text{app}}_+\|_{W^{\rho_0,\infty}} + \|u''^{\text{app}}_+\|_{W^{\rho_0,\infty}}\big)\|u''^{\text{app}}_+\|_{L^2}. \tag{5.78}$$

By (4.39)–(4.41), (4.43)–(4.45) and (5.35), the sum of (5.77) and (5.78) is estimated at time τ (since $k \geq 3$) by

$$C\Big(\frac{(\varepsilon^2\sqrt{\tau})^{\theta'}}{\sqrt{\tau}} + \varepsilon^2|\log\varepsilon|^2\Big)^2 \tau^{\frac{1}{4}}(\varepsilon^2\sqrt{\tau})^{\theta} + \varepsilon^3|\log\varepsilon|^4. \tag{5.79}$$

By (5.53), the first term is smaller than the right-hand side of (5.40). The same holds true trivially for the last term in (5.79). This finishes the proof that terms (5.45) contributes to the remainder in (5.36).

Terms of the form (5.46). We need to prove that (5.46) contributes to the remainder and to the $\underline{a}^{\text{app}}\sum_{|I|=1}\mathrm{Op}(\tilde{m}'_{0,I})(u_I)$ terms on the right-hand side of (5.36). Substitute (5.59) in (5.46). We get the following terms:

$$\big(a(t)-a^{\text{app}}(t)\big)\sum_{|I|=1}\mathrm{Op}(m'_{1,I})(u_I^{\text{app}})+\big(a(t)-\underline{a}^{\text{app}}(t)\big)\sum_{|I|=1}\mathrm{Op}(m'_{1,I})(\tilde{u}_I),\tag{5.80}$$

$$\underline{a}^{\text{app}}(t)\sum_{|I|=1}\mathrm{Op}(m'_{1,I})(\tilde{u}_I),\tag{5.81}$$

$$a(t)\sum_{|I|=1}\sum_{\substack{2\le|\tilde{I}|\le 4\\ \tilde{I}=(\tilde{I}',\tilde{I}'')}}\mathrm{Op}(m'_{1,I})\mathrm{Op}(m_{\tilde{I}})(\tilde{u}_{\tilde{I}'},u_{\tilde{I}''}^{\text{app}}),\tag{5.82}$$

$$a(t)\sum_{|I|=1}\mathrm{Op}(m'_{1,I})(R),\tag{5.83}$$

where R satisfies (5.25), (5.26).

By (5.38), (4.8), (4.6), (4.3) and (4.96), (4.98),

$$a^{\text{app}}(t)-\underline{a}^{\text{app}}(t)=O\big(t_\varepsilon^{-\frac{1}{2}}t^{-\frac{1}{2}}(\varepsilon^2\sqrt{t})^{\theta'}\big)$$

and

$$a(t)-a^{\text{app}}(t)=O\big(t_\varepsilon^{-\frac{3}{2}}\big)=O\big(t_\varepsilon^{-\frac{1}{2}}t^{-\frac{1}{2}}(\varepsilon^2\sqrt{t})^{\theta'}\big).$$

By (D.31), the H^s norm of (5.80) is thus bounded from above at time τ by

$$C\tau_\varepsilon^{-\frac{1}{2}}\tau^{-\frac{1}{2}}(\varepsilon^2\sqrt{\tau})^{\theta'}\big(\|u'^{\text{app}}_{+}\|_{H^s}+\|u''^{\text{app}}_{+}\|_{H^s}+\|\tilde{u}_+\|_{H^s}\big)\le C\tau^{-1}(\varepsilon^2\sqrt{\tau})^{\theta'}\varepsilon\tau^{\delta}$$

using (4.39), (4.43), (5.35). This quantity satisfies (5.39). If we make act $L_\pm$ on (5.80) and use (D.71) to estimate the L^2 norm, we obtain a bound in

$$C\tau_\varepsilon^{-\frac{1}{2}}\tau^{-\frac{1}{2}}(\varepsilon^2\sqrt{\tau})^{\theta'}\big(\|L_+u'^{\text{app}}_{+}\|_{L^2}+\|L_+\tilde{u}_+\|_{L^2}+\|u'^{\text{app}}_{+}\|_{L^2}+\|\tilde{u}_+\|_{L^2}\big)$$

for the contribution of $u'^{\text{app}}_{\pm}$ and $\tilde{u}_\pm$ to (5.80). Using (5.35) and (4.39), (4.41), we get by (5.53) the wanted estimate of the form (5.40). On the other hand, if we consider the contribution $(a(t)-a^{\text{app}}(t))\mathrm{Op}(m'_{I,1})u''^{\text{app}}_{\pm}$ to (5.80) on which acts $L_\pm$, we may estimate the L^2 norm from the L^∞ one, as $m'_{1,I}(x,\xi)$ is rapidly decaying in x. Then, by (D.77) with $\ell=n=1$, we obtain a bound in

$$\begin{aligned}Ct|a-a^{\text{app}}|\Big(t^{-r}\big(\|u''^{\text{app}}_{+}\|_{W^{\rho_0,\infty}}+t^{-\frac{1}{2}}\|u''^{\text{app}}_{+}\|_{H^s}\big)\\ +t^{-1+\sigma}\big(\|u''^{\text{app}}_{+}\|_{W^{\rho_0,\infty}}+\|L_+u''^{\text{app}}_{+}\|_{W^{\rho_0,\infty}}\big)\Big).\end{aligned}\tag{5.84}$$

As $a-a^{\text{app}}=O(t_\varepsilon^{-\frac{3}{2}})$, it follows, taking for instance $r=1$, and using (4.43), (4.44), (4.45) that (5.84) at time τ may be estimated, if σ is small enough, from

$$C\tau_\varepsilon^{-\frac{3}{2}}\tau^{\sigma}|\log\varepsilon|^2\le C\tau_\varepsilon^{-\frac{1}{2}}\tau^{-\frac{1}{2}}\varepsilon^{1-2\sigma}|\log\varepsilon|^2.$$

By (5.51), (5.40) will hold largely. We have thus obtained that (5.80) is a remainder.

Term (5.81) is present on the right-hand side of (5.36).

Consider next (5.82). By Corollary B.2.6, the composition $\mathrm{Op}(m'_{1,I}) \circ \mathrm{Op}(m_{\tilde{I}})$ may be written under the form $\mathrm{Op}(m'_{1,\tilde{I}})$ for new symbols $m'_{1,\tilde{I}}$ in

$$\tilde{S}'_{1,0}\Bigg(\prod_{j=1}^{|\tilde{I}|}\langle\xi_j\rangle^{-1}M_0^{\nu}, |\tilde{I}|\Bigg)$$

for some ν and $2 \le |\tilde{I}| \le 4$. Consequently, we write (5.82) under the form

$$a(t)\sum_{\substack{2\le|\tilde{I}|\le 4\\ \tilde{I}=(\tilde{I}',\tilde{I}'')}} \mathrm{Op}(m'_{1,\tilde{I}})(\tilde{u}_{\tilde{I}'}, u^{\mathrm{app}}_{\tilde{I}''}). \tag{5.85}$$

Since such expressions will appear also in the study of terms of the form (5.47), we postpone their study.

Finally, let us study (5.83). As $\mathrm{Op}(m'_{1,I})$ is bounded on H^s, the Sobolev norm of (5.83) is $O(t_\varepsilon^{-1/2}\|R(t,\cdot)\|_{H^s})$. Using (5.25), it satisfies (5.39). If we make act $L_\pm$ on (5.83), the rapid decay of $m'_{1,I}$ and (5.25), show that we obtain at time τ an expression whose L^2 norm is bounded from above by

$$C\tau_\varepsilon^{-\frac{1}{2}}(\varepsilon^2\sqrt{\tau})^{4\theta'}\tau^{-1+4\sigma}(\varepsilon\tau^{\delta})$$

that trivially satisfies (5.40).

This concludes the study of terms of the form (5.46).

Terms of the form (5.47) (and (5.85)). We study now expressions of the form (5.47) and the related ones introduced in (5.85).

We plug expansion (5.59) in (5.47). By Corollary B.2.6, we get again terms of the form (5.85), with $2 \le |\tilde{I}| \le 6$ instead of $2 \le |\tilde{I}| \le 4$, and terms of the form

$$a(t)\mathrm{Op}(\tilde{m}'_{1,I})(U_1,\dots,U_k), \quad |I| = k \ge 2 \tag{5.86}$$

with again $\tilde{m}'_{1,I}$ in $\tilde{S}'_{1,0}(\prod_{j=1}^{|I|}\langle\xi_j\rangle^{-1}M_0^{\nu}, |I|)$, U_ℓ belonging to

$$\{\tilde{u}_\pm, u'^{\mathrm{app}}_{\pm}, u''^{\mathrm{app}}_{\pm}, R\},$$

one of the arguments at least being equal to R satisfying (5.25) and (5.26). We have already checked that terms of this last form provide remainders (even without the pre-factor $a(t)$) (see (5.69) and (5.70), where the assumption $k \ge 3$ was not used). We are thus reduced to the study of terms of the form (5.85), with $|\tilde{I}| \ge 2$ in the sum. If $|\tilde{I}| \ge 3$, we get terms of the form (5.72) with conditions (5.73), that have been seen to be remainders. We must thus just study

$$a(t)\mathrm{Op}(\tilde{m}'_{1,I})(U_1, U_2) \tag{5.87}$$

with $|I| = 2$, $U_1, U_2 \in \{\tilde{u}_\pm, u'^{\mathrm{app}}_{\pm}, u''^{\mathrm{app}}_{\pm}\}$. Moreover, we may assume, in order to bound the Sobolev norm, that $\tilde{m}'_{1,I}$ is supported for $|\xi_1| \le K(1 + |\xi_2|)$ for instance.

Applying (D.78) with $\ell' = \ell = 1$ if $U_1 = \tilde{u}_\pm$ or $u'^{\text{app}}_\pm$ and $\ell = 1, \ell' = 0$ if $U_1 = u''^{\text{app}}_\pm$, we bound the H^s norm of (5.87) by

$$\begin{aligned}|a(t)|t^{-1+\sigma}\big(&\|L_+\tilde{u}_+\|_{L^2} + \|\tilde{u}_+\|_{H^s} + \|L_+u'^{\text{app}}_+\|_{L^2} + \|u'^{\text{app}}_+\|_{H^s}\\ &+ \|L_+u''^{\text{app}}_+\|_{W^{\rho_0,\infty}} + \|u''^{\text{app}}_+\|_{W^{\rho_0,\infty}} + t^{-\frac{1}{2}}\|u''^{\text{app}}_+\|_{H^s}\big)\\ &\times\big(\|\tilde{u}_+\|_{H^s} + \|u'^{\text{app}}_+\|_{H^s} + \|u''^{\text{app}}_+\|_{H^s}\big).\end{aligned}$$

As $a(t) = O(t_\varepsilon^{-\frac{1}{2}})$, one gets at time τ a bound in $\varepsilon\tau^{\delta-1}e(\tau,\varepsilon)$ using (4.39)–(4.41), (4.43)–(4.45) and (5.35). It follows that (5.39) will hold. On the other hand, if we make act $L_\pm$ on (5.87) and compute the L^2 norm, we get a bound given by

$$|a(t)| = O(t_\varepsilon^{-\frac{1}{2}})$$

multiplied by (5.77) or (5.78) with $k = 2$. Using again (4.39)–(4.41), (4.43)–(4.45) and (5.35), we obtain at time τ an upper bound in

$$\begin{aligned}C\tau_\varepsilon^{-\frac{1}{2}}\Big(\Big(&\frac{(\varepsilon^2\sqrt{\tau})^{\theta'}}{\sqrt{\tau}} + \varepsilon^2|\log\varepsilon|^2\Big)\tau^{\frac{1}{4}}(\varepsilon^2\sqrt{\tau})^{\theta}\\ &+ \log(1+\tau)\log(1+\tau\varepsilon^2)\varepsilon\Big(\frac{\tau\varepsilon^2}{\langle\tau\varepsilon^2\rangle}\Big)^{\frac{1}{2}}\Big).\end{aligned}$$

By (5.53), (5.55), (5.40) will hold true. This concludes the estimate of these terms.

Terms of the form (5.48). Terms (5.48) with $|I| \geq 2$ are of the same form as (5.47), with a smaller pre-factor $a(t)^j$, so they are remainders. We have thus to study

$$a(t)^j \sum_{|I|=1} \text{Op}(m'_{0,I})(u_I), \quad j = 2, 3. \tag{5.88}$$

By (4.96), (4.97), (4.98), (4.100) and the definition of $a(t) = \frac{\sqrt{3}}{3}(a_+ - a_-)$, one may write (5.88) from the term

$$\frac{1}{3}\sum_{|I|=1}\Big(e^{it\frac{\sqrt{3}}{2}}g(t) + e^{-it\frac{\sqrt{3}}{2}}\overline{g(t)}\Big)^2\text{Op}(m'_{0,I})(u_I) \tag{5.89}$$

and from terms like

$$\tilde{a}(t)\sum_{|I|=1}\text{Op}(m'_{0,I})(u_I), \tag{5.90}$$

where

$$|\tilde{a}(t)| \leq Ct_\varepsilon^{-1}(t^{-\frac{1}{2}}(\varepsilon^2\sqrt{t})^{\theta'} + t_\varepsilon^{-\frac{1}{2}}). \tag{5.91}$$

Terms (5.89) are present on the right-hand side of (5.36). We have to show that (5.90) provides remainders. The H^s norm of these terms in bounded from above, using the Sobolev boundedness of $\text{Op}(m'_{0,I})$ and estimates (4.39), (4.43) and (5.35) by $C\varepsilon t^{\delta-1}\varepsilon^{2\theta'}$ so that (5.39) will hold.

On the other hand, if we make act $L_\pm$ on (5.90) and compute the L^2 norm, we have to estimate by (5.91) expressions of the form

$$t t_\varepsilon^{-1}\big(t^{-\frac{1}{2}}(\varepsilon^2\sqrt{t})^{\theta'} + t_\varepsilon^{-\frac{1}{2}}\big)\|\mathrm{Op}(\tilde{m}'_{0,I})U\|_{L^2}, \tag{5.92}$$

where $\tilde{m}'_{0,I}$ is of the same form as $m'_{0,I}$ and $U = \tilde{u}_\pm$ or $u'^{\mathrm{app}}_\pm$ or $u''^{\mathrm{app}}_\pm$.

When $U = \tilde{u}_\pm$ or $u'^{\mathrm{app}}_\pm$, we use (D.71) to bound (5.92) by

$$C t_\varepsilon^{-1}\big(t^{-\frac{1}{2}}(\varepsilon^2\sqrt{t})^{\theta'} + t_\varepsilon^{-\frac{1}{2}}\big)\big(\|L_+\tilde{u}_+\|_{L^2} + \|L_+u'^{\mathrm{app}}_+\|_{L^2} + \|\tilde{u}_+\|_{L^2} + \|u'^{\mathrm{app}}_+\|_{L^2}\big).$$

Using (4.39), (4.41) and (5.35), we see from (5.53) that (5.40) will hold. On the other hand, if $U = u''^{\mathrm{app}}_+$, we estimate the L^2 norm in (5.92) from an L^∞ one, using the rapid decay of $\tilde{m}'_{0,I}$, and we use (D.77) with $\ell = n = 1$, $r = 1$, in order to obtain a bound in

$$t^\sigma t_\varepsilon^{-1}\big(t^{-\frac{1}{2}}(\varepsilon^2\sqrt{t})^{\theta'} + t_\varepsilon^{-\frac{1}{2}}\big)\big(\|u''^{\mathrm{app}}_+\|_{W^{\rho_0,\infty}} + \|L_+u''^{\mathrm{app}}_+\|_{W^{\rho_0,\infty}} + t^{-\frac{1}{2}}\|u''^{\mathrm{app}}_+\|_{H^s}\big).$$

By (4.43)–(4.45), we bound this by

$$C|\log\varepsilon|^2\varepsilon t^{-\frac{1}{2}}(t^\sigma\varepsilon)$$

so that, since $t \le \varepsilon^{-4}$ and σ may be taken as small as we want, (5.55) implies that (5.40) holds. This concludes the study of terms (5.48).

Terms of the form (5.49). These terms satisfy (4.38). It follows immediately from (5.50) that (5.39) holds. Using (5.51), we get as well (5.40).

This concludes the proof of Proposition 5.2.1. ∎

The reduced equation (5.36) obtained in Proposition 5.2.1 still needs one more reduction before we are able to deal with it. Recall that in Proposition 4.1.2, we have decomposed u^{app}_+ under the form (4.48) $u^{\mathrm{app}}_+ = u^{\mathrm{app},1}_+ + \Sigma_+$, where $u^{\mathrm{app},1}_+$ was given by (4.49). We refined this decomposition in (4.54) as

$$\begin{aligned} u^{\mathrm{app},1}_+ &= u'^{\mathrm{app},1}_+ + u''^{\mathrm{app},1}_+,\\ u'^{\mathrm{app},1}_+ &= \sum_{j\in\{-2,0,2\}} U'_{j,+}(t,x),\\ u''^{\mathrm{app},1}_+ &= \sum_{j\in\{-2,0,2\}} U''_{j,+}(t,x), \end{aligned} \tag{5.93}$$

where $U'_{j,+}, U''_{j,+}$ are defined in (C.4) from the right-hand side of (4.50), namely

$$\begin{aligned} U'_{j,+}(t,x) &= i\int_1^{+\infty} e^{i(t-\tau)p(D_x)+ij\frac{\sqrt{3}}{2}}\chi\Big(\frac{\tau}{\sqrt{t}}\Big)M_j(\tau,\cdot)\,d\tau,\\ U''_{j,+}(t,x) &= i\int_{-\infty}^{t} e^{i(t-\tau)p(D_x)+ij\frac{\sqrt{3}}{2}}(1-\chi)\Big(\frac{\tau}{\sqrt{t}}\Big)M_j(\tau,\cdot)\,d\tau \end{aligned} \tag{5.94}$$

with M_j given by (4.21). Let us prove the following corollary of Proposition 5.2.1.

Corollary 5.2.3. *Under the assumptions of Proposition 5.2.1, $\tilde{u}_+$ solves an equation of the form*

$$
\begin{aligned}
&(D_t - p(D_x))\tilde{u}_+ - \sum_{j=-2}^{2} e^{itj\frac{\sqrt{3}}{2}}\mathrm{Op}(b'_{j,+})\tilde{u}_+ - \sum_{j=-2}^{2} e^{itj\frac{\sqrt{3}}{2}}\mathrm{Op}(b'_{j,-})\tilde{u}_- \\
&\quad = \sum_{\substack{3\le|I|\le 4\\ I=(I',I'')}} \mathrm{Op}(\tilde{m}_I)(\tilde{u}_{I'}, u^{\mathrm{app}}_{I''}) + \sum_{|I|=2}\mathrm{Op}(m'_{0,I})(\tilde{u}_I) \\
&\qquad + \sum_{\substack{I=(I',I'')\\ |I'|=|I''|=1}} \mathrm{Op}(m'_{0,I})(\tilde{u}_{I'}, u'^{\mathrm{app},1}_{I''}) \\
&\qquad + \sum_{|I|=2}\mathrm{Op}(m'_{0,I})(u'^{\mathrm{app},1}_{I}) + R_+(t,x),
\end{aligned}
\tag{5.95}
$$

where $(\tilde{m}_I)_{3\le|I|\le 4}$ is as in the statement of Proposition 5.2.1, where $(m'_{0,I})_{|I|=2}$ are symbols in the class $\tilde{S}'_{1,0}(\prod_{j=1}^{2}\langle\xi_j\rangle^{-1}M_0(\xi),2)$, where R_+ satisfies (5.39) and (5.40), and where the symbols $b'_{j,\pm}$ satisfy (3.7) and the following estimates for α, β, N in $\mathbb{N}$: If $j=-1$ or $j=1$,

$$
\begin{aligned}
|\partial_x^\alpha\partial_\xi^\beta b'_{j,\pm}(t,x,\xi)| &\le C_{\alpha,\beta,N} t_\varepsilon^{-\frac{1}{2}}\langle x\rangle^{-N}\langle\xi\rangle^{-1}, \\
|\partial_t\partial_x^\alpha\partial_\xi^\beta b'_{j,\pm}(t,x,\xi)| &\le C_{\alpha,\beta,N}\big(t_\varepsilon^{-\frac{3}{2}} + (\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'}t^{-\frac{3}{2}}\big)\langle x\rangle^{-N}\langle\xi\rangle^{-1},
\end{aligned}
\tag{5.96}
$$

and if $j=-2,0,2$,

$$
\begin{aligned}
|\partial_x^\alpha\partial_\xi^\beta b'_{j,\pm}(t,x,\xi)| &\le C_{\alpha,\beta,N} t_\varepsilon^{-1}\langle x\rangle^{-N}\langle\xi\rangle^{-1}, \\
|\partial_t\partial_x^\alpha\partial_\xi^\beta b'_{j,\pm}(t,x,\xi)| &\le C_{\alpha,\beta,N} t_\varepsilon^{-\frac{1}{2}}\big(t_\varepsilon^{-\frac{3}{2}} + (\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'}t^{-\frac{3}{2}}\big)\langle x\rangle^{-N}\langle\xi\rangle^{-1}.
\end{aligned}
\tag{5.97}
$$

Proof. Let us analyze the different terms on the right-hand side of (5.36). The first sum appears unchanged in (5.95).

By the definition (5.38) of $\underline{a}^{\mathrm{app}}_+$, the fact that $\underline{a}^{\mathrm{app}} = \frac{\sqrt{3}}{3}(\underline{a}^{\mathrm{app}}_+ + \overline{\underline{a}^{\mathrm{app}}_+})$ and (4.3), the $\underline{a}^{\mathrm{app}}(t)\sum_{|I|=1}\mathrm{Op}(m'_{1,I})(\tilde{u}_I)$ term in (5.36) contributes to the terms involving $b'_{j,\pm}$ on the left-hand side of (5.95). The same holds true for the last but one term in (5.36). We are thus left with studying

$$
\sum_{\substack{|I|=2\\ I=(I',I'')}} \mathrm{Op}(m'_{0,I})(\tilde{u}_{I'}, u^{\mathrm{app}}_{I''}). \tag{5.98}
$$

First step. If $|I''|=0$, we get the $\sum_{|I|=2}\mathrm{Op}(m'_{0,I})(\tilde{u}_I)$ contribution in (5.95).

Second step. We consider next the contributions to (5.98) with $|I'|=1$, $|I''|=1$. As one may decompose

$$
u^{\mathrm{app}}_+ = u'^{\mathrm{app},1}_+ + u''^{\mathrm{app},1}_+ + \Sigma_+
$$

by (4.48) and (4.55), we shall get three type of terms:

$$\sum_{\substack{I=(I',I'')\\|I'|=|I''|=1}} \mathrm{Op}(m'_{0,I})(\tilde{u}_{I'}, u'^{\mathrm{app},1}_{I''}), \tag{5.99}$$

$$\sum_{\substack{I=(I',I'')\\|I'|=|I''|=1}} \mathrm{Op}(m'_{0,I})(\tilde{u}_{I'}, u''^{\mathrm{app},1}_{I''}), \tag{5.100}$$

$$\sum_{\substack{I=(I',I'')\\|I'|=|I''|=1}} \mathrm{Op}(m'_{0,I})(\tilde{u}_{I'}, \Sigma_{I''}). \tag{5.101}$$

Term (5.99) appears on the right-hand side of (5.95). From (5.93), we may rewrite (5.100) as a sum of expressions

$$\mathrm{Op}(m'_{0,I})(\tilde{u}_{I'}, U''_{j,I''}), \quad j = -2, 0, 2. \tag{5.102}$$

We shall apply Proposition C.2.2 with $\kappa = 1, \omega = 1$. Since $U''_{j,+}$ is defined by (5.94) from a M_j given by (4.21), thus satisfying by (4.3) inequalities (C.7) with $\omega = 1$, Assumption $(\mathrm{H1})_1$ of Proposition C.2.1 is satisfied, and so Proposition C.2.2 applies. It follows from (C.106), applied with $\lambda = j\frac{\sqrt{3}}{2}$, $j = -2, 0, 2$, that (5.102) may be written as

$$e^{ijt\frac{\sqrt{3}}{2}} \mathrm{Op}(b_1^j)\tilde{u}_{I'} + \mathrm{Op}(b_2^j)\tilde{u}_{I'} \tag{5.103}$$

where b_1^j (resp. b_2^j) satisfies (3.7) and the first two lines (resp. the last line) in (C.107) with $\omega = 1$. The first term in (5.103) brings thus contributions to the last two sums on the left-hand side of (5.95), for $j = -2, 0, 2$, with symbols satisfying (5.97) and (3.7).

We have to check next that the last term in (5.103) contributes to the remainders. By the last line in (C.107) and (D.32), (5.35)

$$\|\mathrm{Op}(b_2^j)\tilde{u}_{I'}\|_{H^s} \le C\varepsilon^2 t^{-1}\log(1+t)\varepsilon t^{\delta}$$

from which a remainder estimate of the form (5.39) follows. If we make act $L_\pm$ on $\mathrm{Op}(b_2^j)\tilde{u}_{I'}$ and use (D.71) with $n = 1$ and the bounds (C.107) for the semi-norms of b_2^j (with $\omega = 1$), we obtain from (5.35)

$$\|L_\pm \mathrm{Op}(b_2^j)\tilde{u}_{I'}\|_{L^2} \le C\varepsilon^2 t^{-1}\log(1+t)t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^{\theta} \tag{5.104}$$

so that a bound of form (5.40) holds.

It remains to study (5.101). Recall the definition of Σ_+ given after (4.50): this function is a sum

$$\sum_{j=-3}^{3} \underline{U}_j(t,x),$$

where $\underline{U}_j$ solves (4.50) with source term $e^{ijt\frac{\sqrt{3}}{2}}\underline{M}_j$, where $\underline{M}_j$ satisfies (4.51), i.e. the first inequality in (C.8). We may then decompose each $\underline{U}_j$ as $\underline{U}'_{j,1} + \underline{U}''_{j,1}$, according to (C.110) with $\lambda = j\frac{\sqrt{3}}{2}$ and rewrite the terms in (5.101) from

$$\mathrm{Op}(m'_{0,I})(\tilde{u}_{I'}, \underline{U}'_{j,1,I''}), \quad \mathrm{Op}(m'_{0,I})(\tilde{u}_{I'}, \underline{U}''_{j,1,I''}) \tag{5.105}$$

to which Proposition C.2.5 applies. This allows us to rewrite these terms in the form $\mathrm{Op}(b)(\tilde{u}_\pm)$, where b satisfies estimates (C.117), namely

$$|\partial_y^{\alpha'_0}\partial_\xi b(t,y,\xi)| \leq C t_\varepsilon^{-\frac{1}{2}} t^{-1}\log(1+t)\langle y\rangle^{-N}\langle\xi\rangle^{-1}. \tag{5.106}$$

By (D.32) and (5.35), we thus get

$$\begin{aligned}\|\mathrm{Op}(b)(\tilde{u}_\pm)\|_{H^s} &\leq C t_\varepsilon^{-\frac{1}{2}} t^{-1}\log(1+t)\|\tilde{u}_+\|_{H^s}\\ &\leq C t_\varepsilon^{-\frac{1}{2}} t^{-1}\log(1+t)\varepsilon t^{\delta}.\end{aligned}$$

An estimate of the form (5.39) follows at once. If we make act $L_\pm$ on $\mathrm{Op}(b)(\tilde{u}_\pm)$, use the rapid decay in y of (5.106) and (D.71), we obtain an estimate of the L^2 norm by the right-hand side of (5.104), with ε^2 replaced by $t_\varepsilon^{-1/2} \leq \varepsilon$. This suffices to imply that (5.40) holds, and thus shows that (5.101) is a remainder.

Third step. We study finally contributions to (5.98) where $|I'| = 0$. Again, we use (4.48) and (4.55) to write

$$u_+^{\mathrm{app}} = u'^{\mathrm{app},1}_+ + u''^{\mathrm{app},1}_+ + \Sigma_+.$$

Plugging this expression inside the terms (5.98) with $|I'| = 0$, we shall get expressions given by

$$\begin{aligned}
&\mathrm{Op}(m'_{0,I})\big(u'^{\mathrm{app},1}_I\big), && |I| = 2, && (5.107)\\
&\mathrm{Op}(m'_{0,I})\big(\Sigma_{I'}, u'^{\mathrm{app},1}_{I''}\big), && |I'| = |I''| = 1,\ I = (I', I''), && (5.108)\\
&\mathrm{Op}(m'_{0,I})(\Sigma_I), && |I| = 2, && (5.109)\\
&\mathrm{Op}(m'_{0,I})\big(u''^{\mathrm{app},1}_I\big), && |I| = 2, && (5.110)\\
&\mathrm{Op}(m'_{0,I})\big(\Sigma_{I'}, u''^{\mathrm{app},1}_{I''}\big), && |I'| = |I''| = 1,\ I = (I', I''), && (5.111)\\
&\mathrm{Op}(m'_{0,I})\big(u'^{\mathrm{app},1}_{I'}, u''^{\mathrm{app},1}_{I''}\big), && |I'| = |I''| = 1,\ I = (I', I''), && (5.112)
\end{aligned}$$

where $m'_{0,I}$ are still elements of $\tilde{S}'_{1,0}(\prod_{j=1}^{|I|}\langle\xi_j\rangle^{-1}M_0^{\nu}, |I|)$.

Term (5.107) appears on the right-hand side of (5.95).

Term (5.108) is treated as (5.101): actually, $u'^{\mathrm{app},1}_+$ satisfies (4.39)–(4.41) as has been established after (4.54), and these bounds are better than inequalities (5.35) for $\tilde{u}_+$.

Term (5.109) may be treated in the same way: we have seen in the study of (5.101) that $\mathrm{Op}(m'_{0,I})(\cdot, \Sigma_{I''})$ may be written as $\mathrm{Op}(b)\cdot$ for b satisfying (5.106) (see (5.105)). By (4.52), we shall get for any N,

$$\begin{aligned}\|x^N\mathrm{Op}(m'_{0,I})(\Sigma_I)\|_{H^s} &\leq C\|x^N\mathrm{Op}(b)(\Sigma_\pm)\|_{H^s}\\ &\leq C t_\varepsilon^{-\frac{1}{2}} t^{-1}(\log(1+t))^2\big(t_\varepsilon^{-\frac{3}{2}} + t^{-1}t_\varepsilon^{-\frac{1}{2}} + t^{-1}\varepsilon^2\big).\end{aligned} \tag{5.113}$$

By (5.56), we see that (5.39) will hold. Estimating the action of $L_\pm$ on $\mathrm{Op}(m'_{0,I})(\Sigma_I)$ in L^2, we get an upper bound by the right-hand side of (5.113) multiplied by t. Then (5.55) shows that (5.40) holds.

To study (5.110), we recall that $u''^{\mathrm{app},1}_{+}$ is given by (4.54), where $U''_{j,+}$ is given by the second formula (C.4) in terms of an M that satisfies (4.13), i.e. such that (C.7) with $\omega = 1$ (Assumption $(\mathrm{H1})_1$) holds. We may thus apply Corollary C.2.3 with $\omega = 1$. It follows that the H^s norm of (5.110) is bounded from above by

$$C\big(t_\varepsilon^{-2} + \varepsilon^4 t^{-2}(\log(1+t))^2\big).$$

This largely implies (5.39). On the other hand, the L^2 norm of the action of $L_\pm$ on (5.110) is bounded by

$$C\big(tt_\varepsilon^{-2} + \varepsilon^4 t^{-1}(\log(1+t))^2\big).$$

Then (5.55) implies that (5.40) largely holds.

Terms (5.111) may be treated in a similar way as (5.109): we have seen that $\mathrm{Op}(\tilde{m}'_I)(\Sigma_{I'}, u''^{\mathrm{app},1}_{I''})$ may be written as $\mathrm{Op}(b)u''^{\mathrm{app},1}_{\pm}$ with b satisfying (5.106). By the expression (4.54) of

$$u''^{\mathrm{app},1}_{+} = \sum_{j\in\{-2,0,2\}} U''_{j,+},$$

where $U''_{j,+}$ is defined by the second formula (C.4) with $\lambda = j\frac{\sqrt{3}}{2}$ and $M = M_j$ given by (4.21), we see that we may apply Proposition C.2.1 with $\omega = 1$. Taking into account the time decaying factor on the right-hand side of (5.106), it follows from (C.89)–(C.91) that

$$\begin{aligned} &|\partial_x^\alpha \mathrm{Op}(m'_{0,I})(\Sigma_{I'}, u''^{\mathrm{app},1}_{I''})| \\ &\quad \le Ct_\varepsilon^{-\frac{1}{2}} t^{-1}(\log(1+t))\big(t_\varepsilon^{-1} + \varepsilon^2 t^{-1}\log(1+t)\big)\langle x\rangle^{-N}. \end{aligned} \tag{5.114}$$

Thus the H^s norm of (5.111) is bounded from above by the t-depending factor in (5.114). By (5.56), we get that (5.39) largely holds. If we make act $L_\pm$ on (5.111) and estimate the L^2 norm, we get a bound in

$$Ct_\varepsilon^{-\frac{1}{2}}\log(1+t)\big(t_\varepsilon^{-1} + \varepsilon^2 t^{-1}\log(1+t)\big).$$

Thus (5.55) implies (5.40).

It just remains to treat (5.112). Notice that (5.112) is of the same form as (5.100) with $\tilde{u}_{I'}$ replaced by $u'^{\mathrm{app},1}_{I'}$, so that may be written under a similar form as (5.103), namely

$$e^{ijt\frac{\sqrt{3}}{2}}\mathrm{Op}(b_1^j)u'^{\mathrm{app},1}_{I'} + \mathrm{Op}(b_2^j)u'^{\mathrm{app},1}_{I'}, \tag{5.115}$$

where b_1^j (resp. b_2^j) satisfies the first two lines (resp. the last line) in (C.107) with $\omega = 1$. We have checked after (5.103) that the second term in that formula is a remainder. Since as seen above, $u'^{\mathrm{app},1}_{+}$ satisfies (4.39)–(4.41), which are better estimates than those verified by $\tilde{u}_+$, it follows that the last term in (5.115) is also a remainder. Let us prove that, because of the better bounds satisfied by $u'^{\text{app},1}_+$ versus $\tilde{u}_+$, the first term in (5.115) is a remainder as well. By the estimates of b_1 in (C.107) and (D.32),

$$\|\mathrm{Op}(b_1^j)u'^{\text{app},1}_+\|_{H^s} \leq C t_\varepsilon^{-1}\|u'^{\text{app},1}_+\|_{H^s} \leq C t_\varepsilon^{-1}\varepsilon^2 t^{\frac{1}{4}}$$

according to (4.39) written for $u'^{\text{app},1}_+$. By (5.57), we conclude that (5.39) holds. To estimate $\|L_\pm \mathrm{Op}(b_1^j)u'^{\text{app},1}_+\|_{L^2}$, we are reduced, by the fact that b_1^j is rapidly decaying in x, to bounding $t\|\mathrm{Op}(b_1^j)u'^{\text{app},1}_+\|_{L^2}$. According to (D.71) and the bounds (C.107) of b_1^j, we thus get an estimate in

$$t_\varepsilon^{-1}\big(\|u'^{\text{app},1}_+\|_{L^2} + \|L_+ u'^{\text{app},1}_+\|_{L^2}\big) \leq C t_\varepsilon^{-1} t^{\frac{1}{4}}\big((\varepsilon^2\sqrt{t}) + (\varepsilon^2\sqrt{t})^{\frac{7}{8}}\varepsilon^{\frac{1}{8}}\big)$$

by (4.41). As in (5.40) $\theta < \frac{1}{2}$, (5.53) shows that (5.40) holds.

This ends the study of term (5.112) and thus the proof of Corollary 5.2.3. ∎

Chapter 6

Normal forms

This chapter is devoted to the completion of Step 5 of the proof of our main theorem, that is described in Section 2.7 of Chapter 2. We recall here some elements of the strategy. The preceding steps of the proof allowed us to reduce ourselves to an equation (5.95) for a new unknown $\tilde{u}_+$. In this chapter, we first write a system made of that equation and of the one obtained by conjugation. In that way, if we set $\tilde{u}_- = -\overline{\tilde{u}_+}$ and $\tilde{u} = [\begin{smallmatrix}\tilde{u}_+\\ \tilde{u}_-\end{smallmatrix}]$, the system we get on $\tilde{u}$ may be written (see equation (6.17) below)

$$\big(D_t - P_0 - \mathcal{V}\big)\tilde{u} = \mathcal{M}_3(\tilde{u}, u^{\mathrm{app}}) + \mathcal{M}_4(\tilde{u}, u^{\mathrm{app}}) + \mathcal{M}_2'(\tilde{u}, u'^{\mathrm{app},1}) + \mathcal{R}, \qquad (6.1)$$

where $\mathcal{R}$ is a remainder and the other terms in the equation have the following structure:

- Operator P_0 is just
$$P_0 = \begin{bmatrix} p(D_x) & 0 \\ 0 & -p(D_x) \end{bmatrix}. \qquad (6.2)$$
- Operator $\mathcal{V}$ is a 2×2 matrix of linear operators acting on $\tilde{u}$.

Each of these operators is a pseudo-differential operator of order -1, whose coefficients depend on the approximate solution u^{app} constructed in Chapter 4. The main contribution to $\mathcal{V}$ has thus entries of the following simplified form:

$$e^{\pm it\frac{\sqrt{3}}{2}} t_\varepsilon^{-\frac{1}{2}} c(x)\langle D_x\rangle^{-1}, \qquad (6.3)$$

where $c(x)$ is in $\mathcal{S}(\mathbb{R})$ and again $t_\varepsilon^{-\frac{1}{2}} = \frac{\varepsilon}{(1+t\varepsilon^2)^{1/2}}$. The left-hand side of (6.1) is thus a vectorial version of the scalar operator

$$D_t - p(D_x) - t_\varepsilon^{-\frac{1}{2}} \operatorname{Re}\big(c(x)\langle D_x\rangle^{-1} e^{it\frac{\sqrt{3}}{2}}\big). \qquad (6.4)$$

We get thus a perturbation of the constant coefficients operator $p(D_x) = \sqrt{1 + D_x^2}$ by a potential term, rapidly decaying in x. We already encountered such a perturbation in Chapter 2, except that there the potential was *autonomous*. Here, it is time dependent and has some decay when t goes to infinity. Because of that, we cannot apply the results of Chapter 2 or of Appendix A to eliminate term $\mathcal{V}$ in (6.1) through conjugation. Nevertheless, one may construct by hand some wave operators for a time depending perturbation of $D_t - p(D_x)$ like the one in (6.4). That construction is made on the Fourier transform side: we introduce in Lemma 6.1.1 below a class of operators, obtained composing at the left and the right the last term in (6.4) by (inverse) Fourier transform. In Appendix E below we design "by hand" wave operators for such perturbations of $p(D_x)$, so that, conjugating (6.1) through them, we may eliminate $\mathcal{V}$ from that equation, exactly as we got rid of potential $2V$ in the second equation of (2.9) in Section 2.1 of Chapter 2 (see equation (2.17)).

The second part of this chapter is devoted to a normal form procedure allowing one to eliminate non-characteristic contributions to the quadratic, cubic and quartic terms $\mathcal{M}_2', \mathcal{M}_3, \mathcal{M}_4$ in (6.1). Characteristic contributions are terms like $|\tilde{u}_+|^2\tilde{u}_+$ that obey a Leibniz type rule of the form

$$\|L_+(|\tilde{u}_+|^2\tilde{u}_+)\|_{L^2} \leq C\|\tilde{u}_+\|^2_{W^{\rho_0,\infty}}\|L_+\tilde{u}_+\|_{L^2}$$

up to remainders. These contributions may be safely kept on the right-hand side of (6.1). The non-characteristic terms are those that do not satisfy such a Leibniz rule, and that have to be eliminated by normal form. We explained this idea on a simple model in Section 1.6 of the introduction, and gave more details in Section 2.7. In the present chapter, we apply this method to $\mathcal{M}_3, \mathcal{M}_4$ that have essentially the same structure as the models discussed there.

We have also to eliminate the quadratic term $\mathcal{M}_2'(\tilde{u}, u'^{\mathrm{app},1})$ on the right-hand side of (6.1). Since the arguments $\tilde{u}, u'^{\mathrm{app},1}$ are odd, and $\mathcal{M}_2'$ is morally of the form $a(x)\tilde{u}_\pm\tilde{u}_\pm$, with $a(x)$ rapidly decaying, one may express each factor $\tilde{u}_\pm$ using (2.65) in terms of $L_\pm\tilde{u}_\pm$ gaining a t^{-1} decay for each factor. Nevertheless, this gain is not sufficient to be able to consider $\mathcal{M}_2'$ as a remainder. One get operators of the form (2.68)–(2.69), and we explained at the end of Section 2.7 how to eliminate these expressions performing again some elementary normal form.

6.1 Expression of the equation as a system

Let us first fix some notation. From $\tilde{u}_+$, $\tilde{u}_- = -\overline{\tilde{u}_+}$, u_+^{app}, $u_-^{\mathrm{app}} = -\overline{u_+^{\mathrm{app}}}$, u'^{app}_+ and $u'^{\mathrm{app}}_- = -\overline{u'^{\mathrm{app}}_+}$, we introduce the vector-valued functions

$$\tilde{u} = \begin{bmatrix}\tilde{u}_+\\ \tilde{u}_-\end{bmatrix}, \quad u^{\mathrm{app}} = \begin{bmatrix}u_+^{\mathrm{app}}\\ u_-^{\mathrm{app}}\end{bmatrix}, \quad u'^{\mathrm{app}} = \begin{bmatrix}u'^{\mathrm{app}}_+\\ u'^{\mathrm{app}}_-\end{bmatrix}. \tag{6.5}$$

In order to write (5.95) as a system on $\tilde{u}$, let us define, when $I = \pm$,

$$b_I'(t,x,\xi) = \sum_{j=-2}^{2} e^{itj\frac{\sqrt{3}}{2}} b_{j,I}'(t,x,\xi), \tag{6.6}$$

where $b_{j,\pm}'$ satisfies (5.96), (5.97). Denoting $\bar{b}'^{\vee}_{\pm}(t,x,\xi) = \overline{b_\pm'(t,x,-\xi)}$, we define the matrix of symbols

$$M'(t,x,\xi) = \begin{bmatrix} b_+'(t,x,\xi) & b_-'(t,x,\xi)\\ -\bar{b}'^{\vee}_-(t,x,\xi) & -\bar{b}'^{\vee}_+(t,x,\xi)\end{bmatrix}. \tag{6.7}$$

As $\overline{\mathrm{Op}(b_\pm')w} = \mathrm{Op}(\bar{b}'^{\vee}_\pm)\bar{w}$, if we denote by $\mathrm{Op}(M')$ the quantization of M' defined entry by entry, and define $\overline{\mathrm{Op}(M')}$ by

$$\overline{\mathrm{Op}(M')}\tilde{u} = \overline{\mathrm{Op}(M')\bar{\tilde{u}}},$$

the form of M' shows that

$$\mathrm{Op}(M') = \begin{bmatrix} \mathrm{Op}(b'_+) & \mathrm{Op}(b'_-) \\ -\overline{\mathrm{Op}(b'_-)} & -\overline{\mathrm{Op}(b'_+)} \end{bmatrix} \tag{6.8}$$

or equivalently, if $N_0 = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$,

$$\overline{\mathrm{Op}(M')}N_0 + N_0\mathrm{Op}(M') = 0. \tag{6.9}$$

If we define for $j = -2, \dots, 2$,

$$M'_j(t,x,\xi) = \begin{bmatrix} b'_{j,+}(t,x,\xi) & b'_{j,-}(t,x,\xi) \\ -\bar{b}'^{\vee}_{-j,-}(t,x,\xi) & -\bar{b}'^{\vee}_{-j,+}(t,x,\xi) \end{bmatrix},$$

we have

$$\begin{gathered} M'(t,x,\xi) = \sum_{j=-2}^{2} e^{ijt\frac{\sqrt{3}}{2}} M'_j(t,x,\xi), \\ \overline{\mathrm{Op}(M'_j)}N_0 + N_0\mathrm{Op}(M'_{-j}) = 0. \end{gathered} \tag{6.10}$$

We shall set also, if $m(x,\xi_1,\dots,\xi_n)$ is a multilinear symbol,

$$\overline{m}^{\vee}(x,\xi_1,\dots,\xi_n) = \overline{m(x,-\xi_1,\dots,-\xi_n)} \tag{6.11}$$

so that $\overline{\mathrm{Op}(m)} = \mathrm{Op}(\overline{m}^{\vee})$ if we set again

$$\overline{\mathrm{Op}(m)}(w_1,\dots,w_n) = \overline{\mathrm{Op}(m)(\overline{w}_1,\dots,\overline{w}_n)}.$$

If $I = (i_1,\dots,i_n) \in \{-,+\}^n$ and $u_I = (u_{i_1},\dots,u_{i_n})$, we denote $\bar{I} = (-i_1,\dots,-i_n)$

$$u_{\bar{I}} = (u_{-i_1},\dots,u_{-i_n}) = -(\bar{u}_{i_1},\dots,\bar{u}_{i_n}) = -\overline{u_I} \tag{6.12}$$

according to our definition $u_- = -\bar{u}_+$. Then if m_I is in $S_{\kappa,0}(M,|I|)$, we shall get that

$$\overline{\mathrm{Op}(m_I)(u_I)} = \overline{\mathrm{Op}(m_I)}(\overline{u_I}) = (-1)^{|I|}\overline{\mathrm{Op}(m_I)}(u_{\bar{I}}) = (-1)^{|I|}\mathrm{Op}(\bar{m}_I^{\vee})(u_{\bar{I}}). \tag{6.13}$$

Let us use this notation to express nonlinear quantities constructed from (5.95). We define first the quadratic terms, that will come from the right-hand side of (5.95), namely

$$\begin{aligned} \mathcal{M}'_2(\tilde{u}, u'^{\mathrm{app},1}) = & \sum_{\substack{I=(I',I'') \\ |I'|=0,\,|I''|=2}} \begin{bmatrix} \mathrm{Op}(m'_{0,I})(u'^{\mathrm{app},1}_{I''}) \\ \mathrm{Op}(\bar{m}'^{\vee}_{0,I})(u'^{\mathrm{app},1}_{\bar{I}''}) \end{bmatrix} \\ & + \sum_{\substack{I=(I',I'') \\ |I'|=|I''|=1}} \begin{bmatrix} \mathrm{Op}(m'_{0,I})(\tilde{u}_{I'}, u'^{\mathrm{app},1}_{I''}) \\ \mathrm{Op}(\bar{m}'^{\vee}_{0,I})(\tilde{u}_{\bar{I}'}, u'^{\mathrm{app},1}_{\bar{I}''}) \end{bmatrix} \\ & + \sum_{\substack{I=(I',I'') \\ |I'|=2,\,|I''|=0}} \begin{bmatrix} \mathrm{Op}(m'_{0,I})(\tilde{u}_{I'}) \\ \mathrm{Op}(\bar{m}'^{\vee}_{0,I})(\tilde{u}_{\bar{I}'}) \end{bmatrix} \end{aligned} \tag{6.14}$$

and the cubic and quartic expressions, given for $j = 3, 4$ by

$$\mathcal{M}_j(\tilde{u}, u^{\mathrm{app}}) = \begin{bmatrix} \sum_{I=(I',I''),|I|=j} \mathrm{Op}(\tilde{m}_I)(\tilde{u}_{I'}, u^{\mathrm{app}}_{I''}) \\ (-1)^j \sum_{I=(I',I''),|I|=j} \mathrm{Op}(\overline{\tilde{m}}_I^{\vee})(\tilde{u}_{\bar{I}'}, u^{\mathrm{app}}_{\bar{I}''}) \end{bmatrix}. \tag{6.15}$$

We also set

$$\mathcal{R}(t,x) = \begin{bmatrix} R_+(t,x) \\ \overline{R_+(t,x)} \end{bmatrix} \tag{6.16}$$

where R_+ is the last term in (5.95).

The system obtained taking equation (5.95) and the conjugated equation may be written as follows, denoting $\mathcal{V}$ the operator $\mathrm{Op}(M')$ given by (6.8) and P_0 the matrix of operators given by (6.2):

$$\big(D_t - P_0 - \mathcal{V}\big)\tilde{u} = \mathcal{M}_3(\tilde{u}, u^{\mathrm{app}}) + \mathcal{M}_4(\tilde{u}, u^{\mathrm{app}}) + \mathcal{M}_2'(\tilde{u}, u'^{\mathrm{app},1}) + \mathcal{R}. \tag{6.17}$$

In order to apply the results of Appendix E below, we need to re-express operator $\mathcal{V}$ on the Fourier transform side.

Lemma 6.1.1. *For $j = -2, \dots, 2$, there are two by two matrices*

$$Q_j(t,\xi,\eta) = \Big[\frac{\xi}{\langle\xi\rangle}\frac{\eta}{\langle\eta\rangle} q_{j,(k,\ell)}(t,\xi,\eta)\Big]_{1\le k,\ell\le 2}$$

whose entries satisfy estimates

$$\begin{aligned} |\partial_\xi^\alpha \partial_\eta^\beta q_{j,(k,\ell)}| &\le C_N t_\varepsilon^{-\frac{1}{2}} \langle |\xi| - |\eta|\rangle^{-N} \langle\eta\rangle^{-1}, \\ |\partial_\xi^\alpha \partial_\eta^\beta \partial_t q_{j,(k,\ell)}| &\le C_N \big(t_\varepsilon^{-\frac{3}{2}} + (\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'} t^{-\frac{3}{2}}\big) \langle |\xi| - |\eta|\rangle^{-N} \langle\eta\rangle^{-1} \end{aligned} \tag{6.18}$$

for any α, β, N if $j = -1, 1$, and

$$\begin{aligned} |\partial_\xi^\alpha \partial_\eta^\beta q_{j,(k,\ell)}| &\le C_N t_\varepsilon^{-1} \langle |\xi| - |\eta|\rangle^{-N} \langle\eta\rangle^{-1}, \\ |\partial_\xi^\alpha \partial_\eta^\beta \partial_t q_{j,(k,\ell)}| &\le C_N t_\varepsilon^{-\frac{1}{2}} \big(t_\varepsilon^{-\frac{3}{2}} + (\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'} t^{-\frac{3}{2}}\big) \langle |\xi| - |\eta|\rangle^{-N} \langle\eta\rangle^{-1} \end{aligned} \tag{6.19}$$

for any α, β, N if $j = -2, 0, 2$, such that, if we define the operator K_{Q_j} by

$$\widehat{K_{Q_j} f}(\xi) = \int Q_j(t,\xi,\eta) \hat{f}(\eta)\, d\eta \tag{6.20}$$

for f a $\mathbb{C}^2$-valued function, the operator $\mathcal{V}$ acting on odd *functions may be written as*

$$\mathcal{V} = \sum_{j=-2}^{2} e^{itj\frac{\sqrt{3}}{2}} K_{Q_j}. \tag{6.21}$$

Moreover, one has $\overline{\mathcal{V}} N_0 = -N_0 \mathcal{V}$.

Proof. If $f = [\begin{smallmatrix} f_+ \\ f_- \end{smallmatrix}]$, we have according to the definition (6.8) of $\mathcal{V} = \mathrm{Op}(M')$ and (6.10)

$$\mathrm{Op}(M')f = \sum_{j=-2}^{2} e^{itj\frac{\sqrt{3}}{2}} \mathrm{Op}(M'_j)f, \tag{6.22}$$

$$\mathrm{Op}(M'_j)f = \begin{bmatrix} \mathrm{Op}(b'_{j,+})f_+ + \mathrm{Op}(b'_{j,-})f_- \\ -\mathrm{Op}(\bar{b}'^{\vee}_{-j,-})f_+ - \mathrm{Op}(\bar{b}'^{\vee}_{-j,+})f_- \end{bmatrix}. \tag{6.23}$$

The Fourier transform of the first line of (6.23) may be written

$$\int \hat{b}'_{j,+}(t,\xi-\eta,\eta)\hat{f}_+(\eta)\,d\eta + \int \hat{b}'_{j,-}(t,\xi-\eta,\eta)\hat{f}_-(\eta)\,d\eta, \tag{6.24}$$

where $\hat{b}'_{j,\pm}$ is the Fourier transform relatively to the first variable. Since $b'_{j,\pm}$ satisfies (3.7), if we set

$$\tilde{q}_{j,(1,1)}(t,\xi,\eta) = \hat{b}'_{j,+}(t,\xi-\eta,\eta),\ \tilde{q}_{j,(1,2)}(t,\xi,\eta) = \hat{b}'_{j,-}(t,\xi-\eta,\eta),$$

we see that $\tilde{q}_{j,(k,\ell)}(t,-\xi,-\eta) = \tilde{q}_{j,(k,\ell)}(t,\xi,\eta)$. If we make act (6.24) on odd functions f_+, f_-, we may rewrite this expression as the sum for $(k,\ell) = (1,1)$ or $(1,2)$ of

$$\frac{1}{2}\int \big(\tilde{q}_{j,(k,\ell)}(t,\xi,\eta) - \tilde{q}_{j,(k,\ell)}(t,\xi,-\eta)\big)\hat{f}_\pm(\eta)\,d\eta$$

(with f_+ if $(k,\ell) = (1,1)$ and f_- if $(k,\ell) = (1,2)$). In other words, we may assume that $\tilde{q}_{j,(1,1)}(t,\xi,\eta)$ is odd in η. Since that function is even in (ξ,η), it has also to be odd in ξ. By (5.96)–(5.97), $x \mapsto b'_j(t,x,\eta)$ is in $\mathcal{S}(\mathbb{R})$, and the function is C^∞ in η. It follows that the Fourier transform in x of these functions satisfies

$$|\partial_\xi^\alpha \partial_\eta^\beta \partial_t^{\ell-1} \hat{b}'_{j,I}(t,\xi-\eta,\eta)| \le C_{\alpha,\beta,N}\mathcal{T}_j^\ell(t,\varepsilon)\langle|\xi|-|\eta|\rangle^{-N}\langle\eta\rangle^{-1}$$

for any α,β,N, $\ell = 1,2$, where $\mathcal{T}_j^\ell(t,\varepsilon)$ is the time dependent pre-factor in the ℓ-th equation in (5.96) (resp. (5.97)). After the preceding reductions, it follows that $\tilde{q}_{j,(k,\ell)}$ satisfies for all $\alpha,\beta,N \in \mathbb{N}$, $\ell = 1,2$,

$$|\partial_\xi^\alpha \partial_\eta^\beta \partial_t^{\ell-1} \tilde{q}_{j,(k,\ell)}(t,\xi,\eta)| \le C_{\alpha,\beta,N}\mathcal{T}_j^\ell(t,\varepsilon)\langle|\xi|-|\eta|\rangle^{-N}\langle\eta\rangle^{-1}.$$

Since we have seen that this function is odd in ξ and odd in η, we may write it as $\frac{\xi}{\langle\xi\rangle}\frac{\eta}{\langle\eta\rangle}q_{j,(k,\ell)}(t,\xi,\eta)$, where $q_{j,(k,\ell)}$ satisfies (6.18)–(6.19). It follows that we have written the first component of the Fourier transform $\widehat{\mathcal{V}f}$ of (6.22) as the first component of $\sum_{j=-2}^{2} e^{itj\sqrt{3}/2}\widehat{K_{Q_j}f}(\xi)$. Since the reasoning is the same for the second component, we get (6.21).

The last statement of the lemma follows from (6.9). ■

We may now eliminate the operator $\mathcal{V}$ on the left-hand side of (6.17), using the results of Appendix E.

Proposition 6.1.2. *Fix m in $]0, \frac{1}{2}[$ close to $\frac{1}{2}$, and set as in the example following Definition E.1.1, $\iota = \min(1-2m, \frac{3}{4}c\theta') > 0$. There is $\varepsilon_0 > 0$ such that, for any $\mathcal{V}$ of the form (6.21), defined in terms of matrices Q_j whose coefficients satisfy (6.18) and (6.19), with $\varepsilon \in]0, \varepsilon_0[$, there are operators $B(t)$, $C(t)$, defined for $t \in [1, T]$ ($T \leq \varepsilon^{-4+c}$), bounded on $H^s(\mathbb{R})$, satisfying the properties of Propositions E.1.1 and E.1.3 of Appendix E, such that, if $\tilde{u}$ solves (6.17) and satisfies estimates (5.35), then $C(t)\tilde{u}$ solves*

$$
\begin{aligned}
(D_t - P_0)C(t)\tilde{u} = C(t)\mathcal{M}_3(\tilde{u}, u^{\mathrm{app}}) + C(t)\mathcal{M}_4(\tilde{u}, u^{\mathrm{app}})\\
+ C(t)\mathcal{M}_2'(\tilde{u}, u'^{\mathrm{app},1}) + C(t)\mathcal{R}
\end{aligned}
\tag{6.25}
$$

with $\mathcal{R}$ satisfying for any $t \in [1, T]$,

$$
\|\mathcal{R}(t,\cdot)\|_{H^s} \leq \varepsilon t^{\delta-1} e(t,\varepsilon), \tag{6.26}
$$

$$
\|L\mathcal{R}(t,\cdot)\|_{H^s} \leq t^{-\frac{3}{4}} (\varepsilon^2\sqrt{t})^{\theta} e(t,\varepsilon), \tag{6.27}
$$

where e satisfies (5.41). Moreover, $C(t)\tilde{u}$ is odd if $\tilde{u}$ is odd and $N_0 C(t)\tilde{u} = -\overline{C(t)\tilde{u}}$.

Proof. By (E.9), it holds $(D_t - P_0 - \mathcal{V})B(t) = B(t)(D_t - P_0)$ and by (E.14), we have $\tilde{u} = B(t)C(t)\tilde{u}$. Replacing $\tilde{u}$ by this value on the left-hand side of (6.17), composing at the left with $C(t)$ and using again (E.14), we obtain (6.25). Since $\mathcal{V}(t)$ preserves odd functions and satisfies $\overline{\mathcal{V}(t)}N_0 = -N_0\mathcal{V}(t)$, the last statement of the proposition follows from (E.23) and the fact that $N_0\tilde{u} = -\overline{\tilde{u}}$. This concludes the proof, as estimates (6.26) and (6.27) are just rewriting of (5.39) and (5.40). ∎

6.2 Normal forms

Our next objective will be to eliminate by normal forms most of the contributions on the right-hand side of (6.25). We shall construct first the relevant operators in order to do so.

Let us fix some notation. Let n be in $\mathbb{N}^*$. Consider $\mathbb{C}^2$-valued test functions v_j, defined on $[1, T] \times \mathbb{R}$ for some T, of the form

$$
(t,x) \mapsto v_j(t,x) = \begin{bmatrix} v_{j,+}(t,x) \\ v_{j,-}(t,x) \end{bmatrix} \tag{6.28}
$$

with $v_{j,\pm}$ odd in x and satisfying $v_{j,-} = -\overline{v_{j,+}}$. If $n \geq 3$, we shall consider n-linear maps

$$
(v_1, \dots, v_n) \mapsto \tilde{\mathcal{M}}_j(v_1, \dots, v_n) \tag{6.29}
$$

sending $\mathbb{C}^2$-valued functions to $\mathbb{C}^2$-valued functions and having the following structure (using notation (B.17)):

$$
\tilde{\mathcal{M}}_n(v_1, \dots, v_n) = \begin{bmatrix} \sum_{|I|=n} \mathrm{Op}^t(\tilde{m}_I)(v_{1,i_1}, \dots, v_{n,i_n}) \\ (-1)^n \sum_{|I|=n} \mathrm{Op}^t(\overline{\tilde{m}_I^\vee})(v_{1,-i_1}, \dots, v_{n,-i_n}) \end{bmatrix}, \tag{6.30}
$$

where $I = (i_1, \dots, i_n) \in \{-, +\}^n$, $\tilde{m}_I$ is in $S_{1,\beta}(M_0^\nu \prod_{j=1}^n \langle \xi_j \rangle^{-1}, n)$ for some $\beta > 0$ small, $\nu \in \mathbb{N}$, where $\overline{\tilde{m}}_I^\vee$ is defined by (6.11), and where the form of the second line of (6.30) versus the first one just reflects the fact that $\mathcal{M}_n(v_1, \dots, v_n)$ will have a structure with respect to conjugation similar to the one in (6.14) or (6.15) (see (6.13)). Moreover, we assume that $\tilde{m}_I$ satisfies

$$\tilde{m}(y, x, \xi_1, \dots, \xi_n) = (-1)^{n-1} \tilde{m}(-y, -x, -\xi_1, \dots, -\xi_n) \tag{6.31}$$

so that the associated operator preserves odd functions (see (3.7)).

Proposition 6.2.1. *Let $n \geq 3$. For any I with $|I| = n$ one may find symbols $\hat{m}_I$ in $S_{4,\beta}(M_0^\nu \prod_{j=1}^n \langle \xi_j \rangle^{-1} \langle x \rangle^{-\infty}, n)$ such that, if one sets*

$$\hat{\mathcal{M}}_n(v_1, \dots, v_n) = \begin{bmatrix} \sum_{|I|=n} \mathrm{Op}^t(\hat{m}_I)(v_{1,i_1}, \dots, v_{n,i_n}) \\ (-1)^n \sum_{|I|=n} \mathrm{Op}^t(\overline{\hat{m}}_I^\vee)(v_{1,-i_1}, \dots, v_{n,-i_n}) \end{bmatrix} \tag{6.32}$$

one may write

$$\begin{aligned} R_n(v_1, \dots, v_n) &\stackrel{\text{def}}{=} (D_t - P_0)\hat{\mathcal{M}}_n(v_1, \dots, v_n) - \tilde{\mathcal{M}}(v_1, \dots, v_n) \\ &\quad - \sum_{j=1}^n \hat{\mathcal{M}}_n(v_1, \dots, (D_t - P_0)v_j, \dots, v_n) \end{aligned} \tag{6.33}$$

under the following form:

$$R_n(v_1, \dots, v_n) = \begin{bmatrix} R_{n,+}(v_1, \dots, v_n) \\ R_{n,-}(v_1, \dots, v_n) \end{bmatrix} \tag{6.34}$$

with $R_{n,-} = \overline{R_{n,+}}$, and $R_{n,+}$ satisfies the following: One may write $R_{n,+}(v_1, \dots, v_n)$ as a sum

$$R_{n,+}(v_1, \dots, v_n) = \sum_{|I|=n} \mathrm{Op}^t(r_I)(v_{1,i_1}, \dots, v_{n,i_n}) \tag{6.35}$$

with symbols r_I in the class $S_{4,\beta}(M_0^\nu \prod_{j=1}^n \langle \xi_j \rangle^{-1}, n)$ for some $\nu \in \mathbb{N}$. Moreover, $L_+ R_{n,+}(v_1, \dots, v_n)$ may be written as a sum of terms of the following form:

$$\sum_{|I|=n} \sum_{j=1}^n \mathrm{Op}^t(r_{I,j})(v_{1,i_1}, \dots, L_{i_j} v_{j,i_j}, \dots, v_{n,i_n}) \tag{6.36}$$

with $r_{I,j}$ in $S_{4,\beta}(M_0^\nu \prod_{j=1}^n \langle \xi_j \rangle^{-1}, n)$,

$$\sum_{|I|=n} \mathrm{Op}^t(r_I)(v_{1,i_1}, \dots, v_{n,i_n}) \tag{6.37}$$

for symbols r_I in $S_{4,\beta}(M_0^\nu \prod_{j=1}^n \langle \xi_j \rangle^{-1}, n)$, and

$$t \sum_{|I|=n} \mathrm{Op}^t(r'_I)(v_{1,i_1}, \dots, v_{n,i_n}) \tag{6.38}$$

for symbols r'_I *in* $S'_{4,\beta}(M_0^\nu \prod_{j=1}^n \langle \xi_j \rangle^{-1}, n)$. *Moreover,* $\hat{m}_I$ *satisfies*

$$\hat{m}_I(-y,-x,-\xi_1,\dots,-\xi_n) = (-1)^{n-1}\hat{m}_I(y,x,\xi_1,\dots,\xi_n) \tag{6.39}$$

if $\tilde{m}_I$ *does so in* (6.30).

We shall prove the proposition expressing (6.33) in terms of the semiclassical quantization of symbols introduced in (B.14) in Appendix B. If $h = \frac{1}{t}$, we introduce for any function v_j, $j = 1,\dots,n$, the function $\underline{v}_j$ defined by

$$v_j(t,x) = \frac{1}{\sqrt{t}}\underline{v}_j\Big(t,\frac{x}{t}\Big) = \Theta_t \underline{v}_j(t,x) \tag{6.40}$$

according to (B.15). By (B.16), each term on the first line of (6.30) may be written

$$\mathrm{Op}^t(\tilde{m}_I)(v_{1,i_1},\dots,v_{n,i_n})(t,x) = h^{\frac{n}{2}}\mathrm{Op}_h(\tilde{m}_I)(\underline{v}_{1,i_1},\dots,\underline{v}_{n,i_n})\Big(t,\frac{x}{t}\Big) \tag{6.41}$$

and similarly for the first line of (6.32). The first line on the right-hand side of (6.33) may be written as the sum in I of

$$\begin{aligned}(D_t - p(D_x))&\mathrm{Op}^t(\hat{m}_I)(v_{1,i_1},\dots,v_{n,i_n}) - \mathrm{Op}^t(\tilde{m}_I)(v_{1,i_1},\dots,v_{n,i_n})\\ &- \sum_{j=1}^n \mathrm{Op}^t(\hat{m}_I)(v_{1,i_1},\dots,(D_t - i_j p(D_x))v_{j,i_j},\dots,v_{n,i_n}).\end{aligned} \tag{6.42}$$

It follows from (6.41) that the first term in (6.42) may be written as

$$h^{\frac{n}{2}}\Big(D_t - \mathrm{Op}_h\Big(x\xi + p(\xi) - i\frac{n}{2}h\Big)\Big)(\mathrm{Op}_h(\hat{m}_I)(\underline{v}_{1,i_1},\dots,\underline{v}_{n,i_n}))\Big(t,\frac{x}{t}\Big).$$

The other terms in (6.42) admit analogous expressions, so that (6.42) may be rewritten as $h^{\frac{n}{2}}\underline{R}^I_{n,+}(\underline{v}_{1,i_1},\dots,\underline{v}_{n,i_n})(t,\frac{x}{t})$ with

$$\begin{aligned}&\underline{R}^I_{n,+}(\underline{v}_{1,i_1},\dots,\underline{v}_{n,i_n})(t,x)\\ &\quad= \Big(D_t - \mathrm{Op}_h\Big(x\xi + p(\xi) - i\frac{n}{2}h\Big)\Big)(\mathrm{Op}_h(\hat{m}_I)(\underline{v}_{1,i_1},\dots,\underline{v}_{n,i_n}))\\ &\qquad - \mathrm{Op}_h(\tilde{m}_I)(\underline{v}_{1,i_1},\dots,\underline{v}_{n,i_n})\\ &\qquad - \sum_{j=1}^n \mathrm{Op}_h(\hat{m}_I)\Big[\underline{v}_{1,i_1},\dots,\Big(D_t - \mathrm{Op}_h\Big(x\xi + i_j p(\xi) - i\frac{h}{2}\Big)\Big)\underline{v}_{i,i_j},\\ &\qquad\qquad \dots,\underline{v}_{n,i_n}\Big].\end{aligned} \tag{6.43}$$

We shall study (6.43) both when I is characteristic and I is non-characteristic, according to the terminology introduced in Definition F.1.1, that we recall in the statements of the following two lemmas.

Lemma 6.2.2. *Let* $I = (i_1,\dots,i_n)$ *be characteristic, i.e.* $i_1 + \dots + i_n = 1$, *and take* $\hat{m}_I = 0$ *in* (6.43). *Then if* $\mathcal{L}_\pm = \frac{1}{h}\mathrm{Op}_h(x \pm p'(\xi))$, *the term* $\mathcal{L}_\pm \underline{R}^I_{n,+}(\underline{v}_{1,i_1},\dots,\underline{v}_{n,i_n})$

may be written as a sum of the following expressions:

$$\begin{aligned}\mathrm{Op}_h(r_{I,j})(\underline{v}_{1,i_1},\dots,\mathcal{L}_{i_j}\underline{v}_{j,i_j},\dots,\underline{v}_{n,i_n}),\\ \mathrm{Op}_h(r_I)(\underline{v}_{1,i_1},\dots,\underline{v}_{n,i_n}),\\ \frac{1}{h}\mathrm{Op}_h(r_I')(\underline{v}_{1,i_1},\dots,\underline{v}_{n,i_n})\end{aligned} \tag{6.44}$$

with $r_{I,j}, r_I$ *in* $S_{4,\beta}(M_0^\nu\prod_{j=1}^n\langle\xi_j\rangle^{-1},n)$ *and* r_I' *in* $S'_{4,\beta}(M_0^\nu\prod_{j=1}^n\langle\xi_j\rangle^{-1},n)$ *for some* ν.

Proof. We just have to apply Proposition F.2.1 of Appendix F. ■

We shall consider next the case of non-characteristic indices.

Lemma 6.2.3. *Let* $I=(i_1,\dots,i_n)$ *be non-characteristic, i.e.* $i_1+\dots+i_n\neq 1$. *Then one may find a symbol* $\hat{m}_I$ *in* $S_{4,\beta}(M_0^\nu\prod_{j=1}^n\langle\xi_j\rangle^{-1}\langle x\rangle^{-\infty},n)$, *for some* ν, *such that* $\underline{R}^I_{n,+}(\underline{v}_{1,i_1},\dots,\underline{v}_{n,i_n})$ *given by* (6.43) *may be written as a sum of terms*

$$\begin{aligned}\mathrm{Op}_h(r_I^1)(\underline{v}_{1,i_1},\dots,\underline{v}_{n,i_n}),\\ h\mathrm{Op}_h(r_I)(\underline{v}_{1,i_1},\dots,\underline{v}_{n,i_n}),\\ \mathrm{Op}_h(r_I')(\underline{v}_{1,i_1},\dots,\underline{v}_{n,i_n})\end{aligned} \tag{6.45}$$

with symbols r_I^1 *in* $S_{4,\beta}(M_0^\nu\prod_{j=1}^n\langle\xi_j\rangle^{-1},n)$, r_I *in* $S_{4,\beta}(M_0^\nu\prod_{j=1}^n\langle\xi_j\rangle^{-1}\langle x\rangle^{-1},n)$, *and* r_I' *in* $S'_{4,\beta}(M_0^\nu\prod_{j=1}^n\langle\xi_j\rangle^{-1},n)$. *Moreover,* $\mathcal{L}_+\underline{R}^I_{n,+}(\underline{v}_{1,i_1},\dots,\underline{v}_{n,i_n})$ *may be written under the form* (6.44) *and* $\hat{m}_I$ *satisfies* (6.39) *if* $\tilde{m}_I$ *does so.*

Proof. We apply Proposition F.3.1 and define $\hat{m}_I$ to be the symbol a_I of that statement, that satisfies (F.7). According to (F.20) (with m_I replaced by $\tilde{m}_I$ in its right-hand side), (6.43) may be written as the sum of (F.22) and of the last two lines in (F.21). This gives (6.45).

To get the last statement of the lemma, we use that $\underline{R}^I_{n,+}$ is also given by (F.21). We have thus to show that the action of $\mathcal{L}_+=\frac{1}{h}\mathrm{Op}_h(x+p'(\xi))$ on the three terms in (F.21) may be rewritten under the form (6.44). For $\frac{1}{h}\mathrm{Op}_h(p'(\xi))$ this follows from the composition result of Proposition B.2.1. For the product of $\frac{x}{h}$ by (F.21), this is a consequence of the fact that in these formulas $m_{I,j}$ and r_I are in classes $S_{4,\beta}(M_0^\nu\prod_{j=1}^n\langle\xi_j\rangle^{-1}\langle x\rangle^{-1},n)$. In the case of r_I', the fact that the symbol belongs to the class $S'_{4,\beta}(M_0^\nu\prod_{j=1}^n\langle\xi_j\rangle^{-1},n)$ means that it is rapidly decaying in $M_0(\xi)^{-4}|y|$, so may be multiplied by x (and even by x/h), up to a loss on the exponent ν. This concludes the proof since the definition (F.9) of a_I (with m_I replaced by $\tilde{m}_I$) shows that it satisfies (6.39) if $\tilde{m}_I$ does (taking the cut-off γ even). ■

Proof of Proposition 6.2.1. We just have to translate the above two lemmas going back to functions $v_1,\dots,v_n$ from $\underline{v}_1,\dots,\underline{v}_n$ through (6.40). The first component $R_{n,+}$ of (6.33) is then $h^{\frac{n}{2}}\underline{R}^I_{n,+}(\underline{v}_{1,i_1},\dots,\underline{v}_{n,i_n})$ with $\underline{R}^I_{n,+}$ given by (6.43). In the characteristic case, (6.43) with $\hat{m}_I=0$ and (6.41) show that equation (6.35) holds, and Lemma 6.2.2 implies that $L_+ R_{n,+}$ is of the form (6.36). In the non-characteristic case, these properties follow from Lemma 6.2.3. ■

Proposition 6.2.1 will allow us to treat by normal form the contributions $\mathcal{M}_3, \mathcal{M}_4$ on the right-hand side of (6.25). We need also a result that will allow us to treat $\mathcal{M}'_2$.

We consider a bilinear map $(v_1, v_2) \mapsto \tilde{\mathcal{M}}'_2(v_1, v_2)$ of the form

$$\tilde{\mathcal{M}}'_2(v_1, v_2) = \begin{bmatrix} \sum_{|I|=2} \mathrm{Op}(m'_{0,I})(v_{1,i_1}, v_{2,i_2}) \\ \sum_{|I|=2} \mathrm{Op}(\bar{m}'^{\vee}_{0,I})(v_{1,-i_1}, v_{2,-i_2}) \end{bmatrix}, \tag{6.46}$$

where $m'_{0,I}$ is in $\tilde{S}'_{1,0}(\prod_{j=1}^2 \langle \xi_j \rangle^{-1} M_0(\xi), 2)$ and satisfies (3.7). Our goal is to prove:

Proposition 6.2.4. *One may find an operator $(v_1, v_2) \mapsto \hat{\tilde{\mathcal{M}}}'_2(v_1, v_2)$, that may be written*

$$\hat{\tilde{\mathcal{M}}}'_2(v_1, v_2) = \begin{bmatrix} \sum \sum_{(i_1,i_2)\in\{-,+\}^2} Q_{i_1,i_2}(v_{1,i_1}, v_{2,i_2}) \\ \sum \sum_{(i_1,i_2)\in\{-,+\}^2} \overline{Q_{i_1,i_2}(v_{1,i_1}, v_{2,i_2})} \end{bmatrix} \tag{6.47}$$

with operators $Q_{i_1,i_2}(v_{1,i_1}, v_{2,i_2})$ of the form (F.35), preserving the space of odd functions, such that, if we set

$$\begin{aligned} R_2(v_1, v_2) = (D_t - P_0)\hat{\tilde{\mathcal{M}}}'_2(v_1, v_2) - \tilde{\mathcal{M}}'_2(v_1, v_2) - \hat{\tilde{\mathcal{M}}}'_2\big((D_t - P_0)v_1, v_2\big) \\ - \hat{\tilde{\mathcal{M}}}'_2\big(v_1, (D_t - P_0)v_2\big) \end{aligned} \tag{6.48}$$

and if v_1, v_2 are odd functions, then $R_2 = \left[\begin{smallmatrix} R_{2,+} \\ R_{2,-} \end{smallmatrix}\right]$ with $R_{2,-} = \overline{R_{2,+}}$ and $R_{2,+}$ being a sum

$$R_{2,+}(v_1, v_2) = t^{-2} \sum_{(i_1,i_2)\in\{-,+\}^2} \sum_{\ell_1=0}^{1} \sum_{\ell_2=0}^{1} K^{\ell_1,\ell_2}_{L,i_1,i_2}\big(L^{\ell_1}_{i_1} v_{1,i_1}, L^{\ell_2}_{i_2} v_{2,i_2}\big) \tag{6.49}$$

with $K^{\ell_1,\ell_2}_{L,i_1,i_2}$ in the class $\mathcal{K}'_{1,\frac{1}{2}}(1, i_1, i_2)$ of Definition F.4.1.

Proof. We just have to apply Corollary F.4.4 to the first component of equality (6.48) changing the definition of the notation $K^{\ell_1,\ell_2}_{L,i_1,i_2}$ on the right-hand side of (6.49). ■

We shall use the results established so far in that section in order to rewrite equation (6.25). Recall first that by (E.8), (E.9), (E.14), where $\mathcal{V}$ is the operator (6.21), we have

$$(D_t - P_0)C(t) = C(t)(D_t - P_0 - \mathcal{V}) \tag{6.50}$$

when both sides of these equalities act on odd functions.

Recall the form of operators $\mathcal{M}_j$ in (6.15): these operators may be written as

$$\mathcal{M}_j(\tilde{u}, u^{\mathrm{app}}) = \sum_{\ell=0}^{j} \mathcal{M}^\ell_j(\underbrace{\tilde{u}, \dots, \tilde{u}}_{\ell}, \underbrace{u^{\mathrm{app}}, \dots, u^{\mathrm{app}}}_{j-\ell}), \quad j = 3, 4, \tag{6.51}$$

where

$$\mathcal{M}_j^\ell(v_1,\dots,v_j) = \begin{bmatrix} \sum_{\substack{I'=(i_1,\dots,i_\ell)\\ I''=(i_{\ell+1},\dots,i_j)}} \mathrm{Op}(\tilde{m}_{I',I''})(v_{1,i_1},\dots,v_{j,i_j}) \\ \sum_{\substack{I'=(i_1,\dots,i_\ell)\\ I''=(i_{\ell+1},\dots,i_j)}} (-1)^j \mathrm{Op}(\overline{\tilde{m}}^\vee_{I',I''})(v_{1,-i_1},\dots,v_{j,-i_j}) \end{bmatrix} \tag{6.52}$$

and the symbols $\tilde{m}_{I',I''}$ are in $\tilde{S}_{1,0}(\prod_{j=1}^{|I|}\langle\xi_j\rangle^{-1}M_0(\xi)^\nu,|I|)$, with $3\le |I| = j\le 4$, according to Proposition 5.2.1. According to Corollary D.1.7, each of these symbols may be replaced by a symbol in $S_{1,\beta}(\prod_{j=1}^{|I|}\langle\xi_j\rangle^{-1}M_0(\xi)^\nu,|I|)$, for $\beta>0$ small, up to adding to (6.51) some remainder satisfying (D.35) for an arbitrary r. In other words, we may rewrite (6.51) under the form

$$\mathcal{M}_j(\tilde{u},u^{\mathrm{app}}) = \sum_{\ell=0}^{j}\mathcal{M}_j^\ell(\tilde{u},\dots,\tilde{u},u^{\mathrm{app}},\dots,u^{\mathrm{app}}) + \tilde{\mathcal{R}}_j(\tilde{u},u^{\mathrm{app}}), \tag{6.53}$$

where $\mathcal{M}_j^\ell$ is of the form (6.52) with symbols $\tilde{m}_{I',I''}$ in

$$S_{1,\beta}\Bigg(\prod_{j=1}^{|I|}\langle\xi_j\rangle^{-1}M_0(\xi)^\nu,|I|\Bigg),$$

with $\beta>0$ and where $\tilde{\mathcal{R}}_j$ satisfies

$$\|\tilde{\mathcal{R}}_j(\tilde{u},u^{\mathrm{app}})\|_{H^s}\le Ct^{-2}\big(\|\tilde{u}\|_{H^s}+\|u^{\mathrm{app}}\|_{H^s}\big)^j \tag{6.54}$$

and setting $L=\left[\begin{smallmatrix} L_+ & 0\\ 0 & L_-\end{smallmatrix}\right]$,

$$\begin{aligned}\|L\tilde{\mathcal{R}}_j(\tilde{u},u^{\mathrm{app}})\|_{L^2}\le Ct^{-2}&\big(\|\tilde{u}\|_{H^s}+\|u^{\mathrm{app}}\|_{H^s}\big)^{j-1}\\ &\times\big(\|\tilde{u}\|_{H^s}+\|u^{\mathrm{app}}\|_{H^s}+\|L\tilde{u}\|_{L^2}\\ &\qquad+\|Lu'^{\mathrm{app}}\|_{L^2}+\|Lu''^{\mathrm{app}}\|_{W^{\rho_0,\infty}}\big),\end{aligned} \tag{6.55}$$

where in (6.55), we decomposed the factor u^{app} that eventually replaces v_n in (D.35) as $u^{\mathrm{app}} = u'^{\mathrm{app}} + u''^{\mathrm{app}}$, and used the second (resp. third) of these estimates if v_n is substituted by u'^{app} (resp. u''^{app}).

In the same way, operators $\mathcal{M}_2'$ in (6.14) may be written as

$$\mathcal{M}_2'(\tilde{u},u'^{\mathrm{app},1}) = \mathcal{M}'^0_2(u'^{\mathrm{app},1},u'^{\mathrm{app},1}) + \mathcal{M}'^1_2(\tilde{u},u'^{\mathrm{app},1}) + \mathcal{M}'^2_2(\tilde{u},\tilde{u}), \tag{6.56}$$

where $\mathcal{M}'^\ell_2$ is given by the $(\ell+1)$-st contribution in (6.14). Applying again Corollary D.1.7, we may assume that

$$\mathcal{M}'^\ell_2(v_1,v_2) = \begin{bmatrix} \sum_{\substack{I'=(i_1,\dots,i_\ell)\\ I''=(i_{\ell+1},\dots,i_j)}} \mathrm{Op}(m'_{0,I',I''})(v_{1,i_1},v_{2,i_2}) \\ \sum_{\substack{I'=(i_1,\dots,i_\ell)\\ I''=(i_{\ell+1},\dots,i_j)}} \mathrm{Op}(\bar{m}'^\vee_{0,I',I''})(v_{1,-i_1},v_{2,-i_2}) \end{bmatrix} \tag{6.57}$$

up to replacing (6.56) by

$$\begin{aligned}\mathcal{M}_2'(\tilde u, u'^{\mathrm{app},1}) = \mathcal{M}'^0_2(u'^{\mathrm{app},1}, u'^{\mathrm{app},1}) + \mathcal{M}'^1_2(\tilde u, u'^{\mathrm{app},1}) + \mathcal{M}'^2_2(\tilde u, \tilde u)\\ + \tilde{\mathcal{R}}_2(\tilde u, u'^{\mathrm{app},1}),\end{aligned} \tag{6.58}$$

where $\tilde{\mathcal{R}}_2$ satisfies

$$\begin{aligned}\|\tilde{\mathcal{R}}_2(\tilde u, u'^{\mathrm{app},1})\|_{H^s} &\le Ct^{-2}\big(\|\tilde u\|_{H^s} + \|u'^{\mathrm{app},1}\|_{H^s}\big)^2,\\ \|L\tilde{\mathcal{R}}_2(\tilde u, u'^{\mathrm{app},1})\|_{L^2} &\le Ct^{-2}\big(\|\tilde u\|_{H^s} + \|u'^{\mathrm{app},1}\|_{H^s}\big)\\ &\quad\times\big(\|\tilde u\|_{H^s} + \|u'^{\mathrm{app},1}\|_{H^s}\\ &\qquad + \|L\tilde u\|_{L^2} + \|Lu'^{\mathrm{app},1}\|_{L^2}\big)\end{aligned} \tag{6.59}$$

and where the symbols $m'_{0,I',I''}$ in (6.57) are now in $S'_{1,\beta}\Big(\prod_{j=1}^2\langle\xi_j\rangle^{-1}M_0(\xi),2\Big)$ for some $\beta > 0$.

Let us apply to each $\mathcal{M}_j^\ell$ on the right-hand side of (6.53) Proposition 6.2.1 setting $\tilde{\mathcal{M}}_j = \mathcal{M}_j^\ell$ in order to define by (6.32) an operator $\tilde{\tilde{\mathcal{M}}}_j$ that we denote just by $\hat{\mathcal{M}}_j^\ell$, $0 \le \ell \le j$, $j = 3, 4$. In the same way, apply to each $\mathcal{M}'^\ell_2$, $\ell = 0, 1, 2$ Proposition 6.2.4 in order to define operators $\hat{\mathcal{M}}'^\ell_2$, $\ell = 0, 1, 2$. Denote

$$\begin{aligned}\hat{\mathcal{M}}_j(\tilde u, u^{\mathrm{app}}) &= \sum_{\ell=0}^{j}\hat{\mathcal{M}}_j^\ell(\underbrace{\tilde u,\dots,\tilde u}_{\ell},\underbrace{u^{\mathrm{app}},\dots,u^{\mathrm{app}}}_{j-\ell}), \quad j = 3, 4,\\ \hat{\mathcal{M}}_2'(\tilde u, u'^{\mathrm{app},1}) &= \sum_{\ell=0}^{2}\hat{\mathcal{M}}'^\ell_2(\underbrace{\tilde u,\dots,\tilde u}_{\ell},\underbrace{u'^{\mathrm{app},1},\dots,u'^{\mathrm{app},1}}_{2-\ell}).\end{aligned} \tag{6.60}$$

Let us prove:

Corollary 6.2.5. *Let $\tilde u$ satisfy the assumptions of Proposition 6.1.2, so that equation (6.25) holds. Then, with the above notation,*

$$(D_t - P_0)\bigg(C(t)\Big(\tilde u - \sum_{j=3}^4\hat{\mathcal{M}}_j(\tilde u, u^{\mathrm{app}})\Big) - \hat{\mathcal{M}}_2'(\tilde u, u'^{\mathrm{app},1})\bigg) = \hat{\mathcal{R}}, \tag{6.61}$$

where $\hat{\mathcal{R}}$ is the sum of contributions of the following form:

$$C(t)\mathcal{V}(t)\hat{\mathcal{M}}_j^\ell(\underbrace{\tilde u,\dots,\tilde u}_{\ell},\underbrace{u^{\mathrm{app}},\dots,u^{\mathrm{app}}}_{j-\ell}), \qquad j = 3, 4,\ 0 \le \ell \le j \tag{6.62}$$

$$\big(C(t) - \mathrm{Id}\big)\mathcal{M}'^\ell_2(\underbrace{\tilde u,\dots,\tilde u}_{\ell},\underbrace{u'^{\mathrm{app},1},\dots,u'^{\mathrm{app},1}}_{2-\ell}), \quad 0 \le \ell \le 2, \tag{6.63}$$

$$\begin{aligned}&-C(t)\hat{\mathcal{M}}_j^\ell(\underbrace{\tilde u,\dots,\tilde u,(D_t - P_0)\tilde u,\dots,\tilde u}_{\ell},u^{\mathrm{app}},\dots,u^{\mathrm{app}}),\\ &-C(t)\hat{\mathcal{M}}_j^\ell(\underbrace{\tilde u,\dots,\tilde u}_{\ell},u^{\mathrm{app}},\dots,u^{\mathrm{app}},(D_t - P_0)u^{\mathrm{app}},\dots,u^{\mathrm{app}})\end{aligned} \tag{6.64}$$

for $j = 3, 4$, $0 \leq \ell \leq j$,

$$\begin{aligned}
&-C(t)\hat{\mathcal{M}}'^{\ell}_2(\underbrace{\tilde{u}, \dots, (D_t - P_0)\tilde{u}, \dots, \tilde{u}}_{\ell}, u'^{\mathrm{app},1}, \dots, u'^{\mathrm{app},1}),\\
&-C(t)\hat{\mathcal{M}}'^{\ell}_2(\underbrace{\tilde{u}, \dots, \tilde{u}}_{\ell}, u'^{\mathrm{app},1}, \dots, (D_t - P_0)u'^{\mathrm{app},1}, \dots, u'^{\mathrm{app},1})
\end{aligned} \tag{6.65}$$

for $0 \leq \ell \leq 2$, *of remainders of type*

$$C(t)R_j(\underbrace{\tilde{u}, \dots, \tilde{u}}_{\ell}, \underbrace{u^{\mathrm{app}}, \dots, u^{\mathrm{app}}}_{j-\ell}), \quad j = 3, 4,\ 0 \leq \ell \leq j, \tag{6.66}$$

where R_j *is of the form* (6.34) *and*

$$R_2(\underbrace{\tilde{u}, \dots, \tilde{u}}_{\ell}, \underbrace{u'^{\mathrm{app},1}, \dots, u'^{\mathrm{app},1}}_{2-\ell}), \quad 0 \leq \ell \leq 2, \tag{6.67}$$

where $R_2 = \begin{bmatrix} R_{2,+} \\ R_{2,-} \end{bmatrix}$ *with* $R_{2,-} = \overline{R_{2,+}}$, *and* $R_{2,+}$ *given by* (6.49), *and of contributions*

$$C(t)\big(\mathcal{R}(t,x) + \tilde{\mathcal{R}}_3 + \tilde{\mathcal{R}}_4\big) + \tilde{\mathcal{R}}_2, \tag{6.68}$$

where $\mathcal{R}$ *is given by equation* (6.16) *and satisfies* (6.26)–(6.27) *and with* $\tilde{\mathcal{R}}_2$ *(resp.* $\tilde{\mathcal{R}}_3$, *resp.* $\tilde{\mathcal{R}}_4$*) satisfying* (6.59) *(resp.* (6.54), *resp.* (6.55)*).*

Proof. We write, using (6.50), for $j = 3, 4$,

$$\begin{aligned}
(D_t - P_0)C(t)\hat{\mathcal{M}}_j(\tilde{u}, u^{\mathrm{app}}) = &-C(t)\mathcal{V}(t)\hat{\mathcal{M}}_j(\tilde{u}, u^{\mathrm{app}})\\
&+ C(t)(D_t - P_0)\hat{\mathcal{M}}_j(\tilde{u}, u^{\mathrm{app}}).
\end{aligned} \tag{6.69}$$

We plug in the right-hand side of this equality (6.33) with $\tilde{\mathcal{M}}$ (resp. $\hat{\tilde{\mathcal{M}}}_n$) replaced by $\mathcal{M}^{\ell}_j$ (resp. $\hat{\mathcal{M}}^{\ell}_j$) according to the notation defined before (6.60). In the same way, we express

$$(D_t - P_0)\hat{\mathcal{M}}'_2(\tilde{u}, u'^{\mathrm{app},1})$$

from (6.48) with $\tilde{\mathcal{M}}'_2$ (resp. $\hat{\tilde{\mathcal{M}}}'_2$) replaced by $\mathcal{M}'^{\ell}_2$ (resp. $\hat{\tilde{\mathcal{M}}}'^{\ell}_2$). Making the difference between (6.25) (where we substitute (6.53) and (6.58)) and these expressions, we obtain the contributions (6.62) to (6.68). This concludes the proof. ■

Chapter 7

Bootstrap: L^2 estimates

The proof of the main theorem relies on a bootstrap argument of the type described in Sections 1.4 and 1.5 of the introduction (see estimates (1.28), (1.29) and (1.39)). In our setting, the bounds to be bootstrapped will be actually (2.45), (2.46), (2.47) of Section 2.5 in Chapter 2 (see (7.3) below). In the present chapter our objective is to bootstrap the first and last estimates (7.3) (see Proposition 7.3.7 below). We have thus to bound the Sobolev norm of the solution $\tilde{u}$ of (6.61), and the L^2 norm of $L\tilde{u}$. This is done by energy inequality, and the main task is to estimate the right-hand side of (6.61) in Sobolev spaces or the action of L on that right-hand side in L^2. We do that first for cubic and quartic terms, then for quadratic ones, and finally for terms of higher order.

7.1 Estimates for cubic and quartic terms

We consider $\mathbb{C}$-valued functions $u'^{\text{app}}_{+}, u''^{\text{app}}_{+}$, defined on some interval $[1,T]$, with $T \le \varepsilon^{-4+c}$ for some given $c>0$, and that satisfy on that interval, for a given large r in $\mathbb{N}$ and some constant $C(A,A')$ bounds (4.39)–(4.41) and (4.43)–(4.45) that we recall below:

$$\begin{aligned} \|u'^{\text{app}}_{+}(t,\cdot)\|_{H^r} &\le C(A,A')\varepsilon^2 t^{\frac{1}{4}},\\ \|u'^{\text{app}}_{+}(t,\cdot)\|_{W^{r,\infty}} &\le C(A,A')\varepsilon^2,\\ \|L_+u'^{\text{app}}_{+}(t,\cdot)\|_{H^r} &\le C(A,A')t^{\frac{1}{4}}\big((\varepsilon^2\sqrt{t}) + (\varepsilon^2\sqrt{t})^{\frac{7}{8}}\varepsilon^{\frac{1}{8}}\big) \end{aligned} \tag{7.1}$$

and

$$\begin{aligned} \|u''^{\text{app}}_{+}(t,\cdot)\|_{H^r} &\le C(A,A')\varepsilon\bigg(\frac{t\varepsilon^2}{\langle t\varepsilon^2\rangle}\bigg)^{\frac{1}{2}},\\ \|u''^{\text{app}}_{+}(t,\cdot)\|_{W^{r,\infty}} &\le C(A,A')\varepsilon^2\log(1+t)^2,\\ \|L_+u''^{\text{app}}_{+}(t,\cdot)\|_{W^{r,\infty}} &\le C(A,A')\log(1+t)\log(1+\varepsilon^2 t). \end{aligned} \tag{7.2}$$

Moreover, we shall assume that the solution $\tilde{u} = [\begin{smallmatrix}\tilde{u}_+\\ \tilde{u}_-\end{smallmatrix}]$ (with $\tilde{u}_- = -\overline{\tilde{u}_+}$) of (6.61) satisfies a priori estimates (5.35), i.e. having fixed $c>0$, $\theta'<\theta<\frac{1}{2}$ with θ' close to $\frac{1}{2}$, and $\delta>0$ small, for some $1 \ll \rho \ll s$, we have

$$\begin{aligned} \|\tilde{u}_+(t,\cdot)\|_{H^s} &\le D\varepsilon t^{\delta},\\ \|\tilde{u}_+(t,\cdot)\|_{W^{\rho,\infty}} &\le D\frac{(\varepsilon^2\sqrt{t})^{\theta'}}{\sqrt{t}},\\ \|L_+\tilde{u}_+(t,\cdot)\|_{L^2} &\le Dt^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^{\theta}. \end{aligned} \tag{7.3}$$

We recall also that we have defined from u_+^{app} the function $u_+^{\text{app},1}$ in (4.48), that we decomposed in (4.55) as $u'^{\text{app},1}_+ + u''^{\text{app},1}_+$ and we have seen after (4.54) that $u'^{\text{app},1}_+$ satisfies the same estimates as u'^{app}_+, so that we shall have

$$\begin{aligned} \|u'^{\text{app},1}_+(t,\cdot)\|_{H^r} &\le C(A,A')\varepsilon^2 t^{\frac14},\\ \|u'^{\text{app},1}_+(t,\cdot)\|_{W^{r,\infty}} &\le C(A,A')\varepsilon^2,\\ \|L_+u'^{\text{app},1}_+(t,\cdot)\|_{H^r} &\le C(A,A')t^{\frac14}\big((\varepsilon^2\sqrt{t}) + (\varepsilon^2\sqrt{t})^{\frac78}\varepsilon^{\frac18}\big). \end{aligned} \tag{7.4}$$

We may assume that r in (7.1) and (7.4) is as large as we want since the smoothness of the approximate solution u^{app} is independent of s: these functions are actually C^∞, since their x dependence comes only from stationary solution to our initial problem.

Our goal in that section is to deduce from (7.1) to (7.4) bounds for the cubic and quartic terms on the left-hand side of (6.61) and in (6.62) and (6.64).

Proposition 7.1.1. *Let $\hat{\mathcal{M}}_j(\tilde u, u^{\text{app}})$, $j = 3, 4$, be given by the first line in (6.60). There is a function $(t,\varepsilon)\mapsto e(t,\varepsilon)$, depending on the constants A, A', D in (7.1)–(7.3), satisfying $\lim_{\varepsilon\to 0+}\sup_{1\le t\le \varepsilon^{-4+c}} e(t,\varepsilon) = 0$, such that the following bounds hold:*

$$\|C(t)\hat{\mathcal{M}}_j(\tilde u, u^{\text{app}})\|_{H^s} \le C\varepsilon t^\delta\big((\varepsilon^2\sqrt{t})^{2\theta'}t^{-1} + \varepsilon^4 t^\sigma\big) \le \varepsilon t^\delta e(t,\varepsilon), \tag{7.5}$$

$$\|LC(t)\hat{\mathcal{M}}_j(\tilde u, u^{\text{app}})\|_{L^2} \le t^{\frac14}(\varepsilon^2\sqrt{t})^\theta e(t,\varepsilon) \tag{7.6}$$

for any $t \in [1, \varepsilon^{-4+c}]$, any $\sigma > 0$.

Proof. We prove first (7.5). By (E.19), $C(t)$ is bounded on H^s, uniformly in t staying in the wanted interval. By (6.60) we have thus to bound

$$\|\hat{\mathcal{M}}_j^\ell(\underbrace{\tilde u,\dots,\tilde u}_{\ell}, \underbrace{u^{\text{app}},\dots,u^{\text{app}}}_{j-\ell})\|_{H^s}, \quad 0\le \ell\le j,\ j = 3,4 \tag{7.7}$$

(where each $\hat{\mathcal{M}}_j^\ell$ has form (6.32)) by the right-hand side of (7.5). By (D.32), (7.7) is bounded from above by

$$C\big[\|\tilde u\|_{H^s}\|\tilde u\|^{\ell-1}_{W^{\rho_0,\infty}}\|u^{\text{app}}\|^{j-\ell}_{W^{\rho_0,\infty}} + \|u^{\text{app}}\|_{H^s}\|u^{\text{app}}\|^{j-\ell-1}_{W^{\rho_0,\infty}}\|\tilde u\|^{\ell}_{W^{\rho_0,\infty}}\big] \tag{7.8}$$

with the convention that the first (resp. second) term in the bracket should be replaced by zero if $\ell = 0$ (resp. $\ell = j$). As

$$u_\pm^{\text{app}} = u'^{\text{app}}_\pm + u''^{\text{app}}_\pm, \quad u^{\text{app}} = \begin{bmatrix} u_+^{\text{app}}\\ u_-^{\text{app}}\end{bmatrix},$$

it follows from (7.1) and (7.2) that

$$\begin{aligned} \|u^{\text{app}}\|_{H^s} &\le \tilde C(A,A')\varepsilon\Big(\frac{t\varepsilon^2}{\langle t\varepsilon^2\rangle}\Big)^{\frac12},\\ \|u^{\text{app}}\|_{W^{\rho_0,\infty}} &\le \tilde C(A,A')\varepsilon^2(\log(1+t))^2 \end{aligned} \tag{7.9}$$

for $t \le \varepsilon^{-4}$. Using also (7.3), we bound (7.8) by

$$C\varepsilon t^{\delta}\Big(\big(\varepsilon^2(\log(1+t))^2\big)^{j-1} + \Big(\frac{(\varepsilon^2\sqrt{t})^{\theta'}}{\sqrt{t}}\Big)^{j-1}\Big). \tag{7.10}$$

Since $j \ge 3$, we have obtained a bound by the right-hand side of (7.5).

Let us prove (7.6). By (E.20)–(E.22), it suffices to bound by the right-hand side of (7.6) the quantities

$$\|L\hat{\mathcal{M}}_j(\tilde{u}, u^{\mathrm{app}})\|_{L^2},\ \|\hat{\mathcal{M}}_j(\tilde{u}, u^{\mathrm{app}})\|_{L^2} t^{\frac{1}{2}-m}\varepsilon^{\iota},$$

where m is close to $\frac{1}{2}$. The estimate of the second term is a consequence of (7.5). To study the first one, we recall that $L = \left[\begin{smallmatrix} L_+ & 0 \\ 0 & L_- \end{smallmatrix}\right]$ with $L_\pm = x \pm tp'(D_x)$, so that we have to estimate

$$t\|\hat{\mathcal{M}}_j(\tilde{u}, u^{\mathrm{app}})\|_{L^2},\ \|x\hat{\mathcal{M}}_j(\tilde{u}, u^{\mathrm{app}})\|_{L^2}. \tag{7.11}$$

By (7.10), the first term is estimated by (as $j \ge 3$)

$$t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^{\theta} e(t,\varepsilon) \tag{7.12}$$

with

$$e(t,\varepsilon) = O\big(\varepsilon^2 t^{\delta}(\log(1+t))^4(\varepsilon^2\sqrt{t})^{\frac{3}{2}-\theta} + \varepsilon t^{-\frac{1}{4}+\delta}(\varepsilon^2\sqrt{t})^{2\theta'-\theta}\big).$$

If $t \le \varepsilon^{-4}$, $\theta' < \theta < \frac{1}{2}$ is close enough to $\frac{1}{2}$, so that $2\theta' - \theta \ge 0$, and if δ is small enough, one gets that e satisfies the condition in the statement. This concludes the proof of (7.6) for the first term in (7.11). To study the second one, we have to bound by $t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^{\theta} e$ the norm $\|x\hat{\mathcal{M}}_j^{\ell}(\tilde{u}, \dots, \tilde{u}, u^{\mathrm{app}}, \dots, u^{\mathrm{app}})\|_{L^2}$, $\ell = 0, \dots, j$. Consider first the case $\ell > 0$, so that at least one of the arguments is equal to $\tilde{u}$. By the form (6.32) of $\hat{\mathcal{M}}_j^{\ell}$, we may apply (D.36), putting the L^2 norm on that argument equal to $\tilde{u}$, i.e. we obtain a bound in

$$C\big[\|\tilde{u}\|^{j-1}_{W^{\rho_0,\infty}} + \|u^{\mathrm{app}}\|^{j-1}_{W^{\rho_0,\infty}}\big]\big[t\|\tilde{u}\|_{L^2} + \|L\tilde{u}\|_{L^2}\big]. \tag{7.13}$$

The contribution of the first term in the last bracket has already been estimates by (7.12) in the study of the first term (7.11). The second term gives rise, according to (7.9) and (7.3), to a quantity bounded by

$$Ct^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^{\theta}\Big(\frac{(\varepsilon^2\sqrt{t})^{\theta'}}{\sqrt{t}} + \varepsilon^2(\log(1+t))^2\Big)^2$$

which is also of the form (7.12). It just remains to study the term

$$\|x\hat{\mathcal{M}}_j^{\ell}(u^{\mathrm{app}}, \dots, u^{\mathrm{app}})\|_{L^2}.$$

We decompose one of the arguments u^{app}, say the last one, as $u^{\mathrm{app}} = u'^{\mathrm{app}} + u''^{\mathrm{app}}$. We estimate then the L^2 norm of the function $x\hat{\mathcal{M}}_j^{\ell}(u^{\mathrm{app}}, \dots, u^{\mathrm{app}}, u'^{\mathrm{app}})$ (resp. of $x\hat{\mathcal{M}}_j^{\ell}(u^{\mathrm{app}}, \dots, u^{\mathrm{app}}, u''^{\mathrm{app}})$) using (D.36) with $n = j$ (resp. (D.37) with $n = j$). We obtain a bound in

$$\begin{aligned}C\|u^{\mathrm{app}}\|_{W^{\rho_0,\infty}}^{j-1}&\big(t\|u'^{\mathrm{app}}\|_{L^2}+\|Lu'^{\mathrm{app}}\|_{L^2}\big)\\&+C\|u^{\mathrm{app}}\|_{W^{\rho_0,\infty}}^{j-2}\|u^{\mathrm{app}}\|_{L^2}\big(t\|u''^{\mathrm{app}}\|_{W^{\rho_0,\infty}}+\|Lu''^{\mathrm{app}}\|_{W^{\rho_0,\infty}}\big).\end{aligned}\tag{7.14}$$

Using (7.9), (7.1), (7.2), we obtain a bound in

$$\begin{aligned}C\varepsilon^4(\log(1+t))^4&\big(\varepsilon^2t^{\frac54}+t^{\frac14}\big(\varepsilon^2\sqrt{t}+(\varepsilon^2\sqrt{t})^{\frac78}\varepsilon^{\frac18}\big)\big)\\&+C\varepsilon^2(\log(1+t))^2\varepsilon\big(\varepsilon^2t(\log(1+t))^2+\log(1+t)\log(1+\varepsilon^2t)\big)\end{aligned}\tag{7.15}$$

which is largely of form (7.12). This concludes the proof. ■

We shall study next term (6.62).

Proposition 7.1.2. *With notation (5.41) for $e(t,\varepsilon)$, one has the following bounds for $0\le\ell\le j$, $j=3,4$:*

$$\|C(t)\mathcal{V}(t)\hat{\mathcal{M}}_j^\ell(\underbrace{\tilde u,\dots,\tilde u}_{\ell},u^{\mathrm{app}},\dots,u^{\mathrm{app}})\|_{H^s}\le t^{-1}\varepsilon t^{\delta}e(t,\varepsilon),\tag{7.16}$$

$$\|LC(t)\mathcal{V}(t)\hat{\mathcal{M}}_j^\ell(\underbrace{\tilde u,\dots,\tilde u}_{\ell},u^{\mathrm{app}},\dots,u^{\mathrm{app}})\|_{H^s}\le t^{-1}\big(t^{\frac14}(\varepsilon^2\sqrt{t})^{\theta}\big)e(t,\varepsilon).\tag{7.17}$$

Proof. Recall that $\hat{\mathcal{M}}_j$ is given by (6.60) in terms of operators $\hat{\mathcal{M}}_j^\ell$ defined in (6.32). Moreover, recall that $\mathcal{V}(t)$ in (6.17) is by definition the operator $\mathrm{Op}(M')$ given by (6.8), in function of symbols $b'_\pm$ satisfying (5.96)–(5.97). This means that in particular $t_\varepsilon^{1/2}b'_\pm$ are elements of the class $\tilde S'_{\kappa,\beta}(\langle\xi\rangle^{-1},1)$ (for any κ,β as these symbols depend only on one frequency variable). Moreover, the symbols $\hat m_I$ in (6.32) belong to $S_{4,\beta}(M_0^\nu\prod_{\ell=1}^j\langle\xi_\ell\rangle^{-1},j)$. It follows from the composition result of Corollary B.2.6 that the components of $\mathcal{V}(t)\hat{\mathcal{M}}_j^\ell(\tilde u,\dots,u^{\mathrm{app}})$ may be written under the form

$$t_\varepsilon^{-\frac12}\mathrm{Op}^t(m')(\tilde u_\pm,\dots,\tilde u_\pm,u_\pm^{\mathrm{app}},\dots,u_\pm^{\mathrm{app}})\tag{7.18}$$

for some symbol m' in the class $S'_{4,\beta}(M_0^\nu\prod_{\ell=1}^j\langle\xi_\ell\rangle^{-1},j)$ (for some new ν), and any choice of the signs $\pm$. We use (D.32) together with the boundedness of $C(t)$ on H^s, to estimate the left-hand side of (7.16) by

$$Ct_\varepsilon^{-\frac12}\big(\|u^{\mathrm{app}}\|_{W^{\rho,\infty}}+\|\tilde u\|_{W^{\rho,\infty}}\big)^{j-1}\big(\|u^{\mathrm{app}}\|_{H^s}+\|\tilde u\|_{H^s}\big).\tag{7.19}$$

Using estimates (7.9), (7.3) and $j\ge3$, we bound this largely by the right-hand side of (7.16).

Let us prove (7.17). By (E.20)–(E.22) it is enough to estimate

$$\varepsilon^{\iota}t^{\frac12-m}\|\mathcal{V}(t)\hat{\mathcal{M}}_j^\ell(\tilde u,\dots,u^{\mathrm{app}})\|_{L^2},\ \|L\mathcal{V}(t)\hat{\mathcal{M}}_j^\ell(\tilde u,\dots,u^{\mathrm{app}})\|_{L^2}$$

by the right-hand side of (7.17). The first term satisfies the wanted bound as a consequence of (7.19), since the exponent $\frac12-m$ is close to zero. By (7.18), the study of the second one is reduced to

$$t_\varepsilon^{-\frac{1}{2}} \|L_\pm \mathrm{Op}^t(m')(\tilde{u}_\pm, \dots, \tilde{u}_\pm, u_\pm^{\mathrm{app}}, \dots, u_\pm^{\mathrm{app}})\|_{L^2} \tag{7.20}$$

for m' in the class $S'_{4,\beta}(M_0^\nu \prod_{\ell=1}^j \langle \xi_\ell \rangle^{-1}, j)$. As $L_\pm = x \pm tp'(\xi)$, and the symbol $m'(y, x, \xi_1, \dots, \xi_j)$ is decaying like $\langle M_0(\xi)^{-\kappa} y \rangle^{-N}$ for any N, we are reduced to bounding by the right-hand side of (7.17) the quantity

$$t t_\varepsilon^{-\frac{1}{2}} \|\mathrm{Op}^t(m')(\tilde{u}_\pm, \dots, \tilde{u}_\pm, u_\pm^{\mathrm{app}}, \dots, u_\pm^{\mathrm{app}})\|_{L^2} \tag{7.21}$$

for a new m'. If there is at least one argument equal to $\tilde{u}_\pm$ in (7.21), we use estimate (D.71), making play the special role devoted to v_j there to such an $\tilde{u}_\pm$ argument. We obtain a bound of (7.21) in

$$C t_\varepsilon^{-\frac{1}{2}} \big(\|\tilde{u}\|_{W^{\rho,\infty}} + \|u^{\mathrm{app}}\|_{W^{\rho,\infty}}\big)^{j-1} \big(\|\tilde{u}\|_{L^2} + \|L\tilde{u}\|_{L^2}\big). \tag{7.22}$$

By (7.9) and (7.3), this is bounded by

$$C t_\varepsilon^{-\frac{1}{2}} \bigg(\frac{(\varepsilon^2\sqrt{t})^{\theta'}}{\sqrt{t}} + \varepsilon^2(\log(1+t))^2\bigg)^2 \big(t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^{\theta}\big) \tag{7.23}$$

since $j \geq 3$. Again this is largely bounded by the right-hand side of (7.17).

Consider next the case when all arguments in (7.21) are equal to u^{app}. Decompose one of these arguments, say the last one, as $u^{\mathrm{app}} = u'^{\mathrm{app}} + u''^{\mathrm{app}}$. By linearity, we get a contribution in $\mathrm{Op}^t(m')(u_\pm^{\mathrm{app}}, \dots, u_\pm^{\mathrm{app}}, u'^{\mathrm{app}}_\pm)$ for which (7.21) may be estimated by (7.22) with $\tilde{u}$ replaced by u'^{app} in the last factor. As by (7.1) the L^2 bounds of u'^{app} and Lu'^{app} are better than the corresponding ones for $\tilde{u}$, $L\tilde{u}$ in (7.3), we get that (7.23) holds again. We are thus left with

$$t t_\varepsilon^{-\frac{1}{2}} \|\mathrm{Op}^t(m')(u''^{\mathrm{app}}_\pm, \dots, u''^{\mathrm{app}}_\pm)\|_{L^2}.$$

We use then (D.72) to estimate this by

$$C t_\varepsilon^{-\frac{1}{2}} \|u''^{\mathrm{app}}\|_{W^{\rho_0,\infty}}^{j-2} \|u''^{\mathrm{app}}\|_{L^2} \big(\|u''^{\mathrm{app}}\|_{W^{\rho_0,\infty}} + \|Lu''^{\mathrm{app}}\|_{W^{\rho_0,\infty}}\big). \tag{7.24}$$

By (7.2), we thus get a bound in

$$t_\varepsilon^{-\frac{1}{2}} \varepsilon^2 (\log(1+t))^2 \varepsilon \bigg(\frac{t\varepsilon^2}{\langle t\varepsilon^2 \rangle}\bigg)^{\frac{1}{2}} \log(1+t) \log(1+t\varepsilon^2).$$

Distinguishing the cases $t\varepsilon^2 \leq 1$, $t\varepsilon^2 \geq 1$, one checks that this is smaller than

$$t^{-\frac{3}{4}} (\varepsilon^2\sqrt{t})^{\frac{1}{2}} e(t, \varepsilon),$$

so than the right-hand side of (7.17). This concludes the proof. ∎

7.2 Estimates for quadratic terms

We shall study in this section the quadratic term in (6.61) and (6.63).

Proposition 7.2.1. *Let $\hat{\mathcal{M}}_2'$ be given by the second line in (6.60). One has the following bounds:*

$$\|\hat{\mathcal{M}}_2'(\tilde{u}, u^{\mathrm{app},1})\|_{H^s} \leq \varepsilon t^\delta e(t,\varepsilon), \tag{7.25}$$

$$\|L\hat{\mathcal{M}}_2'(\tilde{u}, u^{\mathrm{app},1})\|_{L^2} \leq t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^\theta e(t,\varepsilon) \tag{7.26}$$

for any $t \in [1, \varepsilon^{-4+c}]$, where $e(t,\varepsilon)$ satisfies (5.41).

To prove the proposition, we shall study the three terms in the definition of $\hat{\mathcal{M}}_2'$.

Lemma 7.2.2. *One has the following estimates:*

$$\|\hat{\mathcal{M}}'^2_2(\tilde{u},\tilde{u})\|_{H^s} \leq C\varepsilon t^\delta \big(t^{-\frac{1}{2}+\sigma}(\varepsilon^2\sqrt{t})^\theta\big), \tag{7.27}$$

$$\|L\hat{\mathcal{M}}'^2_2(\tilde{u},\tilde{u})\|_{L^2} \leq t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^\theta e(t,\varepsilon) \tag{7.28}$$

for any t in $[1, \varepsilon^{-1+c}]$, any $\sigma > 0$, if s is large enough relatively to $\frac{1}{\sigma}$.

Proof. By definition, $\hat{\mathcal{M}}'^2_2$ is obtained applying Proposition 6.2.4 to $\mathcal{M}'^2_2$ given by the first term on the right-hand side of the second line in (6.60). It has structure (6.47). We thus have to study

$$\|Q'_{i_1,i_2}(\tilde{u}_{i_1},\tilde{u}_{i_2})\|_{H^s}, \tag{7.29}$$

$$\|L_\pm Q'_{i_1,i_2}(\tilde{u}_{i_1},\tilde{u}_{i_2})\|_{L^2} \tag{7.30}$$

to obtain respectively (7.27) and (7.28), where Q'_{i_1,i_2} are operators of the form (F.35), preserving the space of odd functions. To bound (7.29), we thus have to study

$$t^{-\frac{3}{2}}\|K^{\ell_1,\ell_2}_{H,i_1,i_2}(L^{\ell_1}_{i_1}\tilde{u}_{i_1}, L^{\ell_2}_{i_2}\tilde{u}_{i_2})\|_{H^s}, \tag{7.31}$$

where $0 \leq \ell_1, \ell_2 \leq 1$.

If $\ell_1 = \ell_2 = 0$, we apply inequality (F.46) of Corollary F.5.2, with $\omega = \frac{1}{2}$. We obtain a bound of (7.31) in

$$Ct^{-\frac{7}{4}}\|\tilde{u}_+\|^2_{H^s}. \tag{7.32}$$

If $\ell_1 = 0, \ell_2 = 1$ (or the symmetric case), we apply (F.58), which gives for (7.31) an estimate in

$$Ct^{-\frac{3}{4}}\|\tilde{u}_+\|^2_{H^s}. \tag{7.33}$$

If $\ell_1 = \ell_2 = 1$, we use (F.57) in order to bound (7.31) by

$$Ct^{-\frac{3}{4}+\sigma}\big(\|L_+\tilde{u}_+\|_{L^2} + \|\tilde{u}_+\|_{H^s}\big)\|\tilde{u}_+\|_{H^s} \tag{7.34}$$

where $\sigma > 0$ is as small as we want (if s is large enough). Plugging in these estimates (7.3), we obtain a bound in

$$C\varepsilon t^{-\frac{3}{4}+\sigma+\delta}t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^\theta, \tag{7.35}$$

which gives (7.27).

Consider next (7.30) and decompose $L_\pm = x \pm tp'(D_x)$. The action of $tp'(D_x)$ on $Q'_{i_1,i_2}(\tilde{u}_{i_1}, \tilde{u}_{i_2})$ has L^2 norm bounded from above, according to (F.35), by

$$t^{-\frac{1}{2}} \| K^{\ell_1,\ell_2}_{H,i_1,i_2}(L^{\ell_1}_{i_1}\tilde{u}_{i_1}, L^{\ell_2}_{i_2}\tilde{u}_{i_2})\|_{L^2}. \tag{7.36}$$

When $\ell_1 = \ell_2 = 0$ (resp. $(\ell_1, \ell_2) = (1,0)$ or $(0,1)$), we apply (F.46) with $s = 0$ (resp. (F.50) and (F.51)) to bound this by

$$Ct^{-\frac{3}{4}+\sigma}\big(\|\tilde{u}_+\|_{H^s} + \|L_+\tilde{u}_+\|_{L^2}\big)\|\tilde{u}_+\|_{H^s}$$

for any $\sigma > 0$, so by (7.35), which is better that what we want.

On the other hand, if $\ell_1 = \ell_2 = 1$ in (7.36), we apply (F.50) or (F.51) with f_2 or f_1 replaced by $L_+\tilde{u}_+$. We obtain for (7.36) an estimate in

$$Ct^{-\frac{3}{4}+\sigma}\big(\|L_+\tilde{u}_+\|_{L^2} + \|\tilde{u}_+\|_{H^s}\big)^2. \tag{7.37}$$

Using (7.3), we obtain a better bound than (7.28). We are left with studying

$$t^{-\frac{3}{2}} \| x K^{\ell_1,\ell_2}_{H,i_1,i_2}(L^{\ell_1}_{i_1}\tilde{u}_{i_1}, L^{\ell_2}_{i_2}\tilde{u}_{i_2})\|_{L^2}. \tag{7.38}$$

We noticed at the end of the proof of Proposition F.5.1 that an operator xK may be written as an operator K_1 of the same type as K, up to the loss of a factor t^ω (here $t^{\frac{1}{2}}$). It follows that (7.38) will be bounded by $t^{-\frac{1}{2}}$ times (7.36), which is better than the estimate already obtained for the other contribution to (7.30). This concludes the proof. ∎

Proof of Proposition 7.2.1. We remark first that the conclusion of Lemma 7.2.2 holds for the three terms on the right-hand side of the second formula in (6.60) that defines $\hat{\mathcal{M}}'_2$. We have seen it for the last one. It holds for the other two terms as, by the end of the statement in Proposition 4.1.2, $u'^{\text{app},1}_+$ satisfies the same estimates (7.1) as u'^{app}. Since these bounds are better than inequalities (7.3) satisfied by $\tilde{u}$ (for $t \le \varepsilon^{-4}$), the proof of Lemma 7.2.2 thus applies as well to $\hat{\mathcal{M}}'^0_2$, $\hat{\mathcal{M}}'^1_2$ in (6.60). Consequently, (7.25) and (7.26) hold. ∎

We want next to study quadratic terms on the right-hand side of (6.61), i.e. terms of the form (6.63).

Proposition 7.2.3. *Let $\mathcal{M}'_2$ be given by* (6.14) *and denote by $e(t,\varepsilon)$ a function satisfying* (5.41). *We have bounds*

$$\|(C(t) - \text{Id})\mathcal{M}'_2(\tilde{u}, u'^{\text{app},1})\|_{H^s} \le t^{-1}\varepsilon t^\delta e(t,\varepsilon), \tag{7.39}$$

$$\|L(C(t) - \text{Id})\mathcal{M}'_2(\tilde{u}, u'^{\text{app},1})\|_{L^2} \le t^{-1}t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^\theta e(t,\varepsilon). \tag{7.40}$$

Proof. We write the proof for the component of $\mathcal{M}'_2$ that is quadratic in $\tilde{u}$. This implies the general case, as $u'^{\text{app},1}$ satisfies better estimates than those holding true for $\tilde{u}$.

Recall that by (6.14), the components of $\mathcal{M}_2'$ are of the form $\mathrm{Op}(m'_{0,I})(\tilde{u}_I)$ with $m'_{0,I}$ in $\tilde{S}'_{1,0}(\prod_{j=1}^{2}\langle\xi_j\rangle^{-1}M_0, 2)$. If we apply estimate (D.78) with $\ell' = \ell = 1$ and $n = 2$, we obtain

$$\|\mathcal{M}_2'(\tilde{u},\tilde{u})\|_{H^s} \leq Ct^{-1+\sigma}\big(\|L\tilde{u}\|_{L^2} + \|\tilde{u}\|_{H^s}\big)\|\tilde{u}\|_{H^s}.$$

Plugging there (7.3), we get a bound in

$$C(\varepsilon t^{\delta})t^{-\frac{3}{4}+\sigma}(\varepsilon^2\sqrt{t})^{\theta}. \tag{7.41}$$

Since $\|C(t) - \mathrm{Id}\|_{\mathcal{L}(L^2)} = O(\varepsilon^{\iota}t^{-m+\delta'+\frac{1}{4}})$ by (E.19), we obtain an estimate in

$$C\varepsilon t^{\delta-1}\big[\varepsilon^{\iota}t^{\frac{1}{2}-m+\delta'+\sigma}(\varepsilon^2\sqrt{t})^{\theta}\big].$$

Since m may be taken as close to $\frac{1}{2}$ as we want (see the example following Definition E.1.1 where m is introduced), and since δ', σ may also be taken as small as wanted (in function of the fixed parameters c, θ, θ'), for $t \leq \varepsilon^{-4+c}$, the factor between brackets is of the form $e(t,\varepsilon)$ in (7.39).

To prove (7.40), we write by (E.20)

$$L(C(t) - \mathrm{Id})\mathcal{M}_2' = (\tilde{C}(t) - \mathrm{Id})L\mathcal{M}_2' + \tilde{C}_1(t)\mathcal{M}_2'. \tag{7.42}$$

Since $\|\mathcal{M}_2'(\tilde{u},\tilde{u})\|_{L^2}$ is estimated by (7.41), and since $\|\tilde{C}_1(t)\|_{\mathcal{L}(L^2)}$ is bounded by (E.22) with m close to $\frac{1}{2}$, we see that the L^2 norm of the last term in (7.42) is smaller than the right-hand side of (7.40) (for $t \leq \varepsilon^{-4}$).

On the other hand, by the definition of L, $\|L\mathcal{M}_2'(\tilde{u},\tilde{u})\|_{L^2}$ is bounded from above by $t\|\mathrm{Op}(m'_{0,I})(\tilde{u}_I)\|_{L^2}$, with $m'_{0,I}$ in $\tilde{S}'_{1,0}(\prod_{j=1}^{2}\langle\xi_j\rangle^{-1}, 2)$. Using (D.76), we estimate this by

$$Ct^{-1+\sigma}\big(\|L_+\tilde{u}_+\|_{L^2} + \|\tilde{u}_+\|_{H^s}\big)^2 \leq Ct^{-1+\sigma}\big(t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^{\theta}\big)^2.$$

Since $\|\tilde{C}(t) - \mathrm{Id}\|_{\mathcal{L}(L^2)} = O(\varepsilon^{\iota}t^{-m+\delta'+\frac{1}{4}})$ with m close to $\frac{1}{2}$ by (E.21), we see that the L^2 norm of the first term on the right-hand side of (7.42) is bounded from above by

$$Ct^{-1}t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^{\theta}\big[(\varepsilon^2\sqrt{t})^{\theta}t^{\frac{1}{2}-m+\delta'+\sigma}\varepsilon^{\iota}\big]$$

and again, if $\frac{1}{2} - m, \delta', \sigma$ have been taken small enough, the bracket is of the form $e(t,\varepsilon)$, whence a bound by the right-hand side of (7.40). This concludes the proof. ■

7.3 Higher-order terms

In this section, we shall bound expressions of the form (6.64)–(6.65) that appear as contributions of higher order of homogeneity if one replaces $(D_t - P_0)\tilde{u}$ by its expression coming from (6.17). We study first the first line in (6.64).

Proposition 7.3.1. *Denote for* $1 \le \ell \le j$, $j = 3, 4$,

$$F(t) = C(t)\hat{\mathcal{M}}_j^\ell\big(\tilde{u}, \dots, (D_t - P_0)\tilde{u}, \dots, \tilde{u}, u^{\mathrm{app}}, \dots, u^{\mathrm{app}}\big). \tag{7.43}$$

Then under a priori assumptions (7.1) *and* (7.3)*, one has the following bounds:*

$$\|F(t)\|_{H^s} \le t^{-1}\varepsilon t^\delta e(t,\varepsilon), \tag{7.44}$$

$$\|LF(t)\|_{L^2} \le t^{-1}\big(t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^\theta\big)e(t,\varepsilon) \tag{7.45}$$

with e *satisfying* (5.41).

To prove the proposition, we first re-express $F(t)$ replacing on the right-hand side $(D_t - P_0)\tilde{u}$ by its value.

Lemma 7.3.2. *The components of*

$$\hat{\mathcal{M}}_j^\ell\big(\tilde{u}, \dots, (D_t - P_0)\tilde{u}, \dots, \tilde{u}, u^{\mathrm{app}}, \dots, u^{\mathrm{app}}\big)$$

may be written as sums of terms of the following form:

$$t_\varepsilon^{-\frac{1}{2}}\mathrm{Op}^t(m')(\tilde{u}_{I'}, u_{I''}^{\mathrm{app}}), \quad j = |I'| + |I''| \ge 3, \tag{7.46}$$

where m' *is in* $S'_{4,\beta}(M_0^\nu \prod_{\ell=1}^j \langle\xi_\ell\rangle^{-1}, j)$,

$$\mathrm{Op}^t(m)(\tilde{u}_{I'}, u_{I''}^{\mathrm{app}}), \quad j = |I'| + |I''| \ge 5, \tag{7.47}$$

where m *is in* $S_{4,\beta}(M_0^\nu \prod_{\ell=1}^j \langle\xi_\ell\rangle^{-1}, j)$,

$$\mathrm{Op}^t(m)\big(\mathcal{R}_{j'}(\tilde{u}, u^{\mathrm{app}}), \tilde{u}_{I'}, u_{I''}^{\mathrm{app}}\big), \quad j = |I'| + |I''|, \tag{7.48}$$

where $j' \ge 3$*,* $j \ge 2$*,* m *is in* $S_{4,\beta}(M_0^\nu \prod_{\ell=1}^{j+1} \langle\xi_\ell\rangle^{-1}, j+1)$ *and* $\mathcal{R}_{j'}$ *satisfies* (6.54) *and* (6.55)*,*

$$\mathrm{Op}^t(m')\big(\tilde{u}_{I'}, u_{I''}^{\prime\mathrm{app},1}, u_{I'''}^{\mathrm{app}}\big), \quad j = |I'| + |I''| + |I'''| \ge 4, \tag{7.49}$$

where m' *is in* $S'_{4,\beta}(M_0^\nu \prod_{\ell=1}^j \langle\xi_\ell\rangle^{-1}, j)$,

$$\mathrm{Op}^t(m)\big(\tilde{\mathcal{R}}_2(\tilde{u}, u^{\prime\mathrm{app},1}), \tilde{u}_{I'}, u_{I''}^{\mathrm{app}}\big), \quad j = |I'| + |I''|, \tag{7.50}$$

with $j \ge 2$*,* m *is in* $S_{4,\beta}(M_0^\nu \prod_{\ell=1}^{j+1} \langle\xi_\ell\rangle^{-1}, j+1)$*,* $\tilde{\mathcal{R}}_2$ *satisfying* (6.59)*,*

$$\mathrm{Op}^t(m)(\mathcal{R}, \tilde{u}_{I'}, u_{I''}^{\mathrm{app}}), \quad j = |I'| + |I''| \ge 2, \tag{7.51}$$

where $\mathcal{R}$ *satisfies estimates* (5.39) *and* (5.40) *and where* m *is a symbol in the class* $S_{4,\beta}(M_0^\nu \prod_{\ell=1}^{j+1} \langle\xi_\ell\rangle^{-1}, j+1)$.

Proof. Recall that by (6.17)

$$(D_t - P_0)\tilde{u} = \mathcal{V}(t)\tilde{u} + \mathcal{M}_3(\tilde{u}, u^{\mathrm{app}}) + \mathcal{M}_4(\tilde{u}, u^{\mathrm{app}}) + \mathcal{M}'_2(\tilde{u}, u^{\prime\mathrm{app},1}) + \mathcal{R}. \tag{7.52}$$

Recall that $\hat{\mathcal{M}}_j^\ell$ is an operator of the form (6.32), so that its components computed at $(\tilde{u}, \dots, \tilde{u}, u^{\text{app}}, \dots, u^{\text{app}})$ may be written

$$\text{Op}^t(m)(\tilde{u}_{i_1}, \dots, \tilde{u}_{i_\ell}, u^{\text{app}}_{i_{\ell+1}}, \dots, u^{\text{app}}_{i_j}) \tag{7.53}$$

with $i_j = \pm$ and m element of $S_{4,\beta}(M_0^\nu \prod_{\ell=1}^j \langle \xi_\ell \rangle^{-1}, j)$ for some $\beta > 0$. We have to compute (7.53) when one of its $\tilde{u}$ arguments, say the first one, is replaced by $(D_t - P_0)\tilde{u}$, so by the right-hand side of (7.52). If we replace $(D_t - P_0)\tilde{u}$ by $\mathcal{V}(t)\tilde{u}$ and use that $\mathcal{V}(t)$ is constructed from operators $\text{Op}(b'_\pm)$ in (6.8) that satisfy (5.96) and (5.97), i.e. are such that $t_\varepsilon^{1/2} b'_\pm = c'_\pm$ is in $S'_{\kappa,\beta}(\langle \xi \rangle^{-1}, 1)$ (for any κ, β), we get a contribution

$$t_\varepsilon^{-\frac{1}{2}} \text{Op}^t(m)\big(\text{Op}(c'_{i_1})\tilde{u}_{i_1}, \tilde{u}_{i_2}, \dots, \tilde{u}_{i_\ell}, u^{\text{app}}_{i_{\ell+1}}, \dots, u^{\text{app}}_{i_j}\big).$$

By the composition result of Corollary B.2.6, we get a term of the form (7.46).

Let us study next (7.53) with the first argument replaced by

$$\mathcal{M}_3(\tilde{u}, u^{\text{app}}) + \mathcal{M}_4(\tilde{u}, u^{\text{app}})$$

coming from (7.52). According to definition (6.15) of $\mathcal{M}_j$ and to (6.53), we shall get contributions

$$\text{Op}^t(m)\big(\text{Op}(\tilde{m}_I)(\tilde{u}_{I'}, u^{\text{app}}_{I''}), \tilde{u}_{i_2}, \dots, \tilde{u}_{i_\ell}, u^{\text{app}}_{i_{\ell+1}}, \dots, u^{\text{app}}_{i_j}\big) \tag{7.54}$$

with $|I| = 3$ or 4 and $\tilde{m}$ in $\tilde{S}_{1,\beta}(M_0(\xi)^\nu \prod_{j=1}^{|I|} \langle \xi_j \rangle^{-1}, |I|)$, with $\beta > 0$ and

$$\text{Op}^t(m)\big(\tilde{\mathcal{R}}_{j',\pm}(\tilde{u}, u^{\text{app}}), \tilde{u}_{i_2}, \dots, u^{\text{app}}_{i_j}\big) \tag{7.55}$$

for

$$\tilde{R}_{j'} = \begin{bmatrix} \tilde{\mathcal{R}}_{j',+} \\ \tilde{\mathcal{R}}_{j',-} \end{bmatrix}$$

satisfying (6.54) and (6.55) with $j' = 3$ or 4. By Corollary (B.19), (7.54) may be written as a term homogeneous of degree larger than or equal to 5 that has the structure (7.47). Moreover, (7.55) provides terms of the form (7.48).

We have to study then the term (7.53) where the first argument is replaced by the $\mathcal{M}'_2(\tilde{u}, u'^{\text{app},1})$ term in (7.52). By (6.58) and (6.57), we get contributions of the form

$$\text{Op}^t(m)\big[\text{Op}(m'_{0,I',I''})(\tilde{u}_{I'}, u'^{\text{app},1}_{I''}), \tilde{u}_{i_2}, \dots, \tilde{u}_{i_\ell}, u^{\text{app}}_{i_{\ell+1}}, \dots, u^{\text{app}}_{i_j}\big] \tag{7.56}$$

with $|I'| + |I''| = 2$, $j \geq 3$, and

$$\text{Op}^t(m)\big[\tilde{\mathcal{R}}_{2,\pm}(\tilde{u}, u'^{\text{app},1}), \tilde{u}_{i_2}, \dots, u^{\text{app}}_{i_j}\big]. \tag{7.57}$$

Again by Corollary B.2.6, (7.56) brings a contribution of the form (7.49) and (7.57) an expression of type (7.50).

Finally, we have to replace one argument of (7.53) by the last term $\mathcal{R}$ in (7.52). This brings (7.51). This concludes the proof of the lemma. ∎

Proof of Proposition 7.3.1. Let us prove (7.44) and (7.45). We have to estimate all contributions from (7.46) to (7.51). As already seen, (E.19) to (E.22) allow us to ignore the action of operator $C(t)$ on the definition (7.43) of $F(t)$, so that we need to study only the Sobolev norm of (7.46) to (7.51), and the L^2 norm of the action of L on these two quantities.

Term (7.46). This term is of the form (7.18) and has already been estimated by the wanted quantities.

Term (7.47). The Sobolev norm of this term may be bounded from above, according to (D.32), by

$$C\big(\|\tilde{u}\|_{W^{\rho_0,\infty}} + \|u^{\mathrm{app}}\|_{W^{\rho_0,\infty}}\big)^4\big(\|\tilde{u}\|_{H^s} + \|u^{\mathrm{app}}\|_{H^s}\big).$$

Using (7.1) and (7.3), we bound this by

$$Ct^{-2}(\varepsilon^2\sqrt{t})^{4\theta'}\varepsilon t^{\delta} \tag{7.58}$$

which is better than the right-hand side of (7.44). If we make act $L_{\pm}$ on (7.47) and compute the L^2 norm, we get on the one hand the product of (7.58) by t, which is smaller than the right-hand side of (7.45) and $\|x\mathrm{Op}^t(m)(\tilde{u}_{I'}, u^{\mathrm{app}}_{I''})\|_{L^2}$. This is a quantity of the same form as the second term in (7.11), except that $j \geq 5$. We thus obtain a bound by (7.13), when at least one of the arguments in (7.47) is equal to $\tilde{u}$. By (7.1)–(7.3) and $j \geq 5$, this is controlled by the right-hand side of (7.45). If all the arguments are equal to u^{app}, we get instead a bound by (7.14) with $j \geq 5$, so by (7.15) multiplied by $\|u^{\mathrm{app}}\|^2_{W^{\rho_0,\infty}} \leq Ct^{-1}$ when $t \leq \varepsilon^{-4+c}$ by (7.1) and (7.2). Since (7.15) was controlled by (7.12), we get again a bound of the form (7.45).

Term (7.48). By (D.32), the H^s norm of (7.48) is bounded by

$$\begin{aligned} C\|\tilde{\mathcal{R}}_{j'}(\tilde{u}, u^{\mathrm{app}})\|_{H^s}&\big(\|\tilde{u}\|_{W^{\rho_0,\infty}} + \|u^{\mathrm{app}}\|_{W^{\rho_0,\infty}}\big)^2 \\ &+ \|\tilde{\mathcal{R}}_{j'}(\tilde{u}, u^{\mathrm{app}})\|_{W^{\rho_0,\infty}}\big(\|\tilde{u}\|_{W^{\rho_0,\infty}} + \|u^{\mathrm{app}}\|_{W^{\rho_0,\infty}}\big) \\ &\qquad \times \big(\|\tilde{u}\|_{H^s} + \|u^{\mathrm{app}}\|_{H^s}\big) \end{aligned} \tag{7.59}$$

since $j \geq 2$ in (7.48). Using Sobolev injection, we may bound $\|\tilde{\mathcal{R}}_{j'}\|_{W^{\rho_0,\infty}}$ from $\|\tilde{\mathcal{R}}_{j'}\|_{H^s}$. By (6.54) and (7.1)–(7.3), we largely get an estimate of the form (7.44).

If we make act $L_{\pm}$ on (7.48), and use that

$$x\mathrm{Op}^t(m)(v_1, \ldots, v_n) - \mathrm{Op}^t(m)(xv_1, \ldots, v_n)$$

is of the form $\mathrm{Op}^t(m_1)(v_1, \ldots, v_n)$ for a new symbol m_1 of the same form as m, we reduce the estimate of the L^2 norm of the action of $L_{\pm}$ on (7.48) to bounding

$$\begin{aligned} &t\|\mathrm{Op}^t(m)\big(\tilde{\mathcal{R}}_{j',\pm}(\tilde{u}, u^{\mathrm{app}}), \tilde{u}_{I'}, u^{\mathrm{app}}_{I''}\big)\|_{L^2}, \\ &\|\mathrm{Op}^t(m)\big(L\tilde{\mathcal{R}}_{j',\pm}(\tilde{u}, u^{\mathrm{app}}), \tilde{u}_{I'}, u^{\mathrm{app}}_{I''}\big)\|_{L^2}. \end{aligned}$$

By (D.33), we get an estimate in

$$\big(t\|\tilde{\mathcal{R}}_{j'}(\tilde{u},u^{\mathrm{app}})\|_{L^2}+\|L_\pm\tilde{\mathcal{R}}_{j'}(\tilde{u},u^{\mathrm{app}})\|_{L^2}\big)\big(\|\tilde{u}\|_{W^{\rho_0,\infty}}+\|u^{\mathrm{app}}\|_{W^{\rho_0,\infty}}\big)^2. \quad (7.60)$$

By (6.54), (6.55), (7.1)–(7.3), this is largely estimated by the right-hand side of (7.45).

Term (7.49). This term is of the form (7.18), except that there is no $t_\varepsilon^{-1/2}$ factor, that we may have an argument $u'^{\mathrm{app},1}$ instead of u^{app}, and that the number of arguments is larger than or equal to 4. By (7.19), the H^s norm of (7.49) is bounded from above by

$$C\big(\|u'^{\mathrm{app},1}\|_{W^{\rho_0,\infty}}+\|u^{\mathrm{app}}\|_{W^{\rho_0,\infty}}+\|\tilde{u}\|_{W^{\rho_0,\infty}}\big)^3 \times\big(\|u^{\mathrm{app}}\|_{H^s}+\|\tilde{u}\|_{H^s}+\|u'^{\mathrm{app},1}\|_{H^s}\big).$$

Using (7.1)–(7.4) we get a better estimate than (7.44). If we make act $L_\pm$ on (7.49) and compute the L^2 norm, we obtain a quantity of the form (7.20), without the pre-factor $t_\varepsilon^{-1/2}$. We obtain thus an upper bound given by (7.22) or (7.24) without the $t_\varepsilon^{-1/2}$ factor, but with $j\geq 4$ and an argument $u'^{\mathrm{app},1}$ replacing eventually an u^{app}. By (7.1)–(7.4),

$$\big(\|u'^{\mathrm{app},1}\|_{W^{\rho_0,\infty}}+\|u^{\mathrm{app}}\|_{W^{\rho_0,\infty}}+\|\tilde{u}\|_{W^{\rho_0,\infty}}\big)^3\big(\|\tilde{u}\|_{L^2}+\|L\tilde{u}\|_{L^2}\big)$$

is smaller than the right-hand side of (7.44). On the other hand, the contribution of the form (7.24) is bounded from above by

$$C\|u''^{\mathrm{app}}\|^2_{W^{\rho_0,\infty}}\|u''^{\mathrm{app}}\|_{L^2}\big(\|u''^{\mathrm{app}}\|_{W^{\rho_0,\infty}}+\|Lu''^{\mathrm{app}}\|_{W^{\rho_0,\infty}}\big)\leq C\varepsilon^5(\log(1+t))^6$$

by (7.2). As $t\leq\varepsilon^{-4+c}$, we estimate this by $\frac{1}{t}\varepsilon e(t,\varepsilon)$, so by the right-hand side of (7.45).

Term (7.50). This is a term of form (7.48). The H^s norm may be bounded by (7.59), with $\tilde{\mathcal{R}}_{j'}$ replaced by $\tilde{\mathcal{R}}_2$. It follows from (6.59), Sobolev injection and (7.1)–(7.4) that we largely get a bound of the form (7.44). If we make act $L_\pm$ and estimate the L^2 norm, we get a bound of the form (7.60), with $\tilde{\mathcal{R}}_{j'}$ replaced by $\tilde{\mathcal{R}}_2$. Again, by (6.59), (7.1)–(7.4), we obtain the conclusion.

Term (7.51). This is a term of the form (7.48), with $\tilde{\mathcal{R}}_{j'}$ replaced by $\mathcal{R}$. Again, we may apply (7.59) to bound the H^s norm. According to (5.39), we obtain a bound by the right-hand side of (7.44). To study the L^2 norm of the action of $L_\pm$ on (7.51), we use that we have again a bound of the form (7.60) with $\tilde{\mathcal{R}}_{j'}$ replaced by $\mathcal{R}$. As the last factor in (7.60) is $O(t^{-1})$ by (7.1)–(7.3), we conclude that we get an upper bound by (7.45) using (5.39), (5.40). This concludes the proof of Proposition 7.3.1 ∎

Our next task is to study the second line in (6.64).

Proposition 7.3.3. *Denote now*

$$F(t)=C(t)\hat{\mathcal{M}}^\ell_j(\tilde{u},\dots,\tilde{u},u^{\mathrm{app}},\dots,(D_t-P_0)u^{\mathrm{app}},\dots,u^{\mathrm{app}}). \quad (7.61)$$

Then under assumptions (7.1)–(7.4)

$$\|F(t)\|_{H^s} \le t^{-1}\varepsilon t^{\delta} e(t,\varepsilon), \tag{7.62}$$

$$\|LF(t)\|_{H^s} \le t^{-1} t^{\frac{1}{4}} (\varepsilon^2 \sqrt{t})^{\theta} e(t,\varepsilon). \tag{7.63}$$

Proof. Recall that $(D_t - p(D_x))u_+^{\mathrm{app}}$ is given by (4.37). Together with the definition (2.28) of F_0^2, F_0^3, with the fact that by (4.3), (4.6), (4.8), a^{app} is $O(t_\varepsilon^{-1/2})$, and with estimates (4.38), this implies that

$$(D_t - p(D_x))u_+^{\mathrm{app}} = Z(t,x) + a^{\mathrm{app}}(t) \sum_{|I|=1} \mathrm{Op}(m'_{1,I})(u_I^{\mathrm{app}}), \tag{7.64}$$

where $m'_{1,I}$ is in $\tilde{S}'_{1,0}(\langle\xi\rangle^{-1}, 1)$ and $Z(t,x)$ satisfies for any α, N,

$$|\partial_x^\alpha Z(t,x)| \le C_{\alpha,N} t_\varepsilon^{-1} \langle x\rangle^{-N}. \tag{7.65}$$

Notice that we may consider as well $m'_{1,I}$ as an element of $S'_{1,\beta}(\langle\xi\rangle^{-1}, 1)$ for $\beta > 0$, since for symbols depending only on one frequency variable, this does not make any difference. We plug (7.64) inside (7.61). Using the form (6.32) of $\hat{\mathcal{M}}_j^\ell$ and the composition result of Corollary B.2.6, we write (7.61), where we forget factor $C(t)$ that does not affect the estimates, as a sum of terms (up to permutations of the arguments)

$$t_\varepsilon^{-\frac{1}{2}} \mathrm{Op}^t(m')(\tilde{u}_\pm, \dots, u_\pm^{\mathrm{app}}), \tag{7.66}$$

$$\mathrm{Op}^t(m)(Z, \tilde{u}_\pm, \dots, u_\pm^{\mathrm{app}}), \tag{7.67}$$

where the number of arguments $(\tilde{u}_\pm, \dots, u_\pm^{\mathrm{app}})$ in term (7.66) (resp. term (7.67)) is j (resp. $j-1$) with $j \ge 3$, and m' belongs to $S'_{4,\beta}(M_0^\nu \prod_{\ell=1}^j \langle\xi_\ell\rangle^{-1}, j)$, m to $S_{4,\beta}(M_0^\nu \prod_{\ell=1}^j \langle\xi_\ell\rangle^{-1}, j)$ for some ν. Expression (7.66) is of the form (7.46), so satisfies the wanted bounds (7.62)–(7.63) by the first point in the proof of Proposition 7.3.1. The H^s norm of (7.67) is bounded by (D.32) by

$$\begin{aligned} &C\big(\|\tilde{u}\|_{H^s} + \|u^{\mathrm{app}}\|_{H^s}\big)\big(\|\tilde{u}\|_{W^{\rho_0,\infty}} + \|u^{\mathrm{app}}\|_{W^{\rho_0,\infty}}\big)\|Z\|_{W^{\rho_0,\infty}} \\ &\quad + C\big(\|\tilde{u}\|_{W^{\rho_0,\infty}} + \|u^{\mathrm{app}}\|_{W^{\rho_0,\infty}}\big)^2 \|Z\|_{H^s} \end{aligned}$$

so by the right-hand side of (7.62), by (7.1)–(7.3) and (7.65).

Let us bound next the L^2 norm of the action of $L_\pm$ on (7.67). We decompose each factor $u_\pm^{\mathrm{app}} = u'^{\mathrm{app}}_\pm + u''^{\mathrm{app}}_\pm$. Consider first the case of the resulting expression where at least one of the last $j-1$ arguments in (7.67) is equal to $\tilde{u}_\pm$ or $u'^{\mathrm{app}}_\pm$, say the last one. We have to estimate

$$\begin{aligned} &t\|\mathrm{Op}^t(m)(Z, \tilde{u}_\pm, \dots, u_\pm^{\mathrm{app}}, w)\|_{L^2}, \\ &\|x\mathrm{Op}^t(m)(Z, \tilde{u}_\pm, \dots, u_\pm^{\mathrm{app}}, w)\|_{L^2} \end{aligned} \tag{7.68}$$

with $w = \tilde{u}_\pm$ or $u'^{\mathrm{app}}_\pm$. Up to commuting x to $\mathrm{Op}^t(m)$ in order to put it against Z, it is enough to bound the first expression. We use (D.73) with the special index j equal to the last one. Recalling the t_ε^{-1} factor in (7.65), we get a bound in

$$\begin{aligned} &C t_\varepsilon^{-1}\big(\|\tilde{u}\|_{W^{\rho_0,\infty}} + \|u^{\mathrm{app}}\|_{W^{\rho_0,\infty}}\big)^{j-2} \\ &\qquad \times \big(\|\tilde{u}\|_{L^2} + \|L\tilde{u}\|_{L^2} + \|u'^{\mathrm{app}}\|_{L^2} + \|L_\pm u'^{\mathrm{app}}\|_{L^2}\big) \end{aligned} \tag{7.69}$$

which by (7.1)–(7.3) is smaller than the right-hand side of (7.63) (as $j-2 \geq 1$). On the other hand, if we consider (7.68) with all arguments $(\tilde{u}_\pm, \dots, u_\pm^{\mathrm{app}}, w)$ replaced by $u''^{\mathrm{app}}_\pm$, we use (D.74) and get instead of (7.69), by (7.2)

$$\begin{aligned} &C t_\varepsilon^{-1}\|u''^{\mathrm{app}}\|^{j-3}_{W^{\rho_0,\infty}}\big(\|Lu''^{\mathrm{app}}\|_{W^{\rho_0,\infty}} + \|u''^{\mathrm{app}}\|_{W^{\rho_0,\infty}}\big)\|u''^{\mathrm{app}}\|_{L^2} \\ &\leq C t_\varepsilon^{-1}\varepsilon \log(1+t)\log(1+t\varepsilon^2). \end{aligned}$$

This is much better than (7.63). This concludes the proof. ■

Let us move now to the study of (6.65).

Proposition 7.3.4. *Denote*

$$\begin{aligned} F(t) = C(t)\hat{\mathcal{M}}'^0_2&\big((D_t - P_0)u'^{\mathrm{app},1}, u'^{\mathrm{app},1}\big) \\ &+ C(t)\hat{\mathcal{M}}'^0_2\big(u'^{\mathrm{app},1}, (D_t - P_0)u'^{\mathrm{app},1}\big) \\ &+ C(t)\hat{\mathcal{M}}'^1_2\big((D_t - P_0)\tilde{u}, u'^{\mathrm{app},1}\big) \\ &+ C(t)\hat{\mathcal{M}}'^1_2\big(\tilde{u}, (D_t - P_0)u'^{\mathrm{app},1}\big) \\ &+ C(t)\hat{\mathcal{M}}'^2_2\big((D_t - P_0)\tilde{u}, \tilde{u}\big) + C(t)\hat{\mathcal{M}}'^2_2\big(\tilde{u}, (D_t - P_0)\tilde{u}\big). \end{aligned} \tag{7.70}$$

Then

$$\|F(t)\|_{H^s} \leq t^{-1}\varepsilon t^\delta e(t,\varepsilon), \tag{7.71}$$

$$\|L_\pm F(t)\|_{L^2} \leq t^{-1}\big(t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^\theta\big)e(t,\varepsilon). \tag{7.72}$$

Before starting the proof, we recall some estimates for $(D_t - P_0)\tilde{u}$.

Lemma 7.3.5. *Under a priori assumptions* (7.43)–(7.45) *we have the following estimates:*

$$\|(D_t - P_0)\tilde{u}\|_{H^s} \leq C\varepsilon t^{\delta - \frac{1}{2}}, \tag{7.73}$$

$$L(D_t - P_0)\tilde{u} = f_1 + x f_2 \tag{7.74}$$

with

$$\|f_1\|_{L^2} \leq C t^{-\frac{1}{2}}\big(t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^\theta\big), \tag{7.75}$$

$$\|f_2\|_{L^2} \leq C t^{-1}(\varepsilon^2\sqrt{t})^{2\theta'}\varepsilon t^\delta. \tag{7.76}$$

Proof. Recall that $(D_t - P_0)\tilde{u}$ is given by (7.52) and that $\mathcal{V}(t)$ may be expressed, according to (6.8), from operators $t_\varepsilon^{-1/2}\mathrm{Op}^t(c'_\pm)$ with $c'_\pm$ in the class $S'_{\kappa,\beta}(\langle\xi\rangle^{-1}, 1)$. By boundedness of these operators on H^s and (7.3), we get for $\|\mathcal{V}(t)\tilde{u}\|_{H^s}$ a bound by the right-hand side of (7.73).

The action of L on $\mathcal{V}(t)\tilde{u}$ will have L^2 norm bounded from above by

$$t_\varepsilon^{-\frac{1}{2}}\|x\mathrm{Op}^t(c'_\pm)\tilde{u}\|_{L^2} + t t_\varepsilon^{-\frac{1}{2}}\|\mathrm{Op}^t(c'_\pm)\tilde{u}\|_{L^2}.$$

By (D.71) with $n = 1$ and (7.3), we get a bound by the right-hand side of (7.75).

Consider next the $\mathcal{M}_j(\tilde{u}, u^{\mathrm{app}})$ terms, $j = 3, 4$, on the right-hand side of (7.52). By (6.53), these terms are given on the one hand by the contributions $\tilde{\mathcal{R}}_j$, which by (6.54) are largely bounded in H^s by the right-hand side of (7.73), and which by (6.55) contribute to f_1 in (7.74) if we apply L on them. On the other hand, the main terms in (6.53) are of the form $\mathrm{Op}^t(\tilde{m}_{I',I''})(\tilde{u}_{I'}, u^{\mathrm{app}}_{I''})$. By (D.32) and (7.1)–(7.3), they satisfy (7.73). Let us study $L_\pm\mathrm{Op}^t(\tilde{m}_{I',I''})(\tilde{u}_{I'}, u^{\mathrm{app}}_{I''})$. We apply Proposition F.2.1 and Corollary F.2.2 (translated in the non-semiclassical framework). This allows us to re-express this quantity from

$$\mathrm{Op}^t(\tilde{m})\big(L_\pm v_1, v_2, \dots, v_j\big), \tag{7.77}$$

$$\mathrm{Op}^t(\tilde{r})(v_1, \dots, v_j), \tag{7.78}$$

$$t\mathrm{Op}^t(\tilde{r}')(v_1, \dots, v_j), \tag{7.79}$$

$$x\mathrm{Op}^t(\tilde{r})(v_1, \dots, v_j) \tag{7.80}$$

where $v_\ell = \tilde{u}_\pm$ or $v_\ell = u'^{\mathrm{app}} + u''^{\mathrm{app}}$, where $\tilde{m}, \tilde{r}$ are in $S_{4,\beta}(M_0^\nu \prod_{\ell=1}^j \langle\xi_\ell\rangle^{-1}, j)$ and $\tilde{r}'$ is in $S'_{4,\beta}(M_0^\nu \prod_{\ell=1}^j \langle\xi_\ell\rangle^{-1}, j)$.

We estimate the L^2 norm of (7.77) using (D.33) with the special index equal to the first one, when v_1 is replaced either by $\tilde{u}_\pm$ or $u'^{\mathrm{app}}_\pm$. We largely get a bound by (7.75) as $j \geq 3$ using (7.1)–(7.3). If v_1 is replaced by $u''^{\mathrm{app}}_\pm$, we still use (D.33), but make play the special role to the second argument. We obtain a bound in

$$\|L_+ u''^{\mathrm{app}}_+\|_{W^{\rho_0,\infty}}\big(\|u^{\mathrm{app}}_+\|_{W^{\rho_0,\infty}} + \|\tilde{u}\|_{W^{\rho_0,\infty}}\big)\big(\|u^{\mathrm{app}}_+\|_{L^2} + \|\tilde{u}_+\|_{L^2}\big) \tag{7.81}$$

which is largely controlled by (7.75) by (7.1)–(7.3).

The L^2 norm of (7.78) (or of the coefficient of x in (7.80)) is bounded from above by the right-hand side of (7.75) (or (7.76)) again by (D.33), (7.1)–(7.3) and the fact that $j \geq 3$.

Consider (7.79). If at least one v_ℓ is replaced by $\tilde{u}_\pm$ or $u'^{\mathrm{app}}_\pm$, we use (D.71), with the special index equal to this ℓ. By (7.1)–(7.3) we largely get an estimate (7.75). If all v_ℓ are equal to $u''^{\mathrm{app}}_\pm$, we use instead (D.72), from which (7.75) largely follows.

To finish the proof of the lemma, we still have to study the last two terms on the right-hand side of (7.52). Contribution $\mathcal{M}'(\tilde{u}, u'^{\mathrm{app}})$ has structure (6.58). The remainders $\mathcal{R}_2$ largely satisfy bounds (7.73), (7.75). The other terms are, by (6.57), of the form $\mathrm{Op}^t(\tilde{m}')(v_1, v_2)$ with $\tilde{m}'$ in $S'_{1,\beta}(M_0(\xi)\prod_{j=1}^2\langle\xi_j\rangle^{-1}, 2)$ and v_1, v_2 equal to $\tilde{u}_\pm$ or $u'^{\mathrm{app},1}_\pm$. By (D.32) and (7.3)–(7.4), the Sobolev estimate (7.73) holds. On the other hand, by (D.76) (and the rapid decay in x of symbols in $S'_{1,\beta}(M_0(\xi)\prod_{j=1}^2\langle\xi_j\rangle^{-1}, 2)$), we have

$$\begin{aligned}\|L_\pm\mathrm{Op}^t(\tilde{m}')(v_1, v_2)\|_{L^2} \leq C t^{-1+\sigma}\big(&\|L_+\tilde{u}_\pm\|_{L^2} + \|L_+ u'^{\mathrm{app},1}_+\|_{L^2}\\ &+ \|\tilde{u}_+\|_{H^s} + \|u'^{\mathrm{app},1}_+\|_{H^s}\big)^2\end{aligned}$$

if $s\sigma$ is large enough. Using (7.3)–(7.4) and taking $\sigma < \frac{1}{4}$, we estimate this by the right-hand side of (7.75).

Finally, the last term $\mathcal{R}$ in (7.52) satisfies (5.39)–(5.40), so also (7.73) and (7.75) for the action of L on it. This concludes the proof of the lemma. ∎

Proof of Proposition 7.3.4. We shall prove successively (7.71) and (7.72).

Step 1: Proof of (7.71). Since $C(t)$ is bounded on H^s, we may ignore it. We thus need to study $\|\hat{\mathcal{M}}_2'(v_1, v_2)\|_{H_h^s}$, where (up to symmetries)

$$v_1 = (D_t - P_0)\tilde{u} \text{ or } (D_t - P_0)u'^{\text{app},1}, \quad v_2 = \tilde{u} \text{ or } u'^{\text{app},1}. \tag{7.82}$$

Recall that $\hat{\mathcal{M}}_2'$ is given by (6.47) in term of operators Q_{i_1,i_2} of the form (F.35). We have thus to bound

$$t^{-\frac{3}{2}}\|K_{H,i_1,i_2}^{\ell_1,\ell_2}(L_{i_1}^{\ell_1}v_{1,i_1}, L_{i_2}^{\ell_2}v_{2,i_2})\|_{H^s} \tag{7.83}$$

with operators $K_{H,i_1,i_2}^{\ell_1,\ell_2}$ in the class $\mathcal{K}'_{1,\frac{1}{2}}(1, i_1, i_2)$ introduced in Definition F.4.1.

Consider first the case $v_1 = (D_t - P_0)u'^{\text{app},1}$. We apply Corollary F.5.4 when ℓ_1 or ℓ_2 is non-zero and (F.46) if $\ell_1 = \ell_2 = 0$. We obtain for $\sigma > 0$ small and $s\sigma$ large enough a bound of (7.83) by

$$\begin{aligned}Ct^{-\frac{3}{4}}\Big(&t^{\sigma}\|L(D_t - P_0)u'^{\text{app},1}\|_{L^2}\big(\|\tilde{u}\|_{H^s} + \|u'^{\text{app},1}\|_{H^s}\big)\\ &+ t^{\sigma}\big(\|L\tilde{u}\|_{L^2} + \|Lu'^{\text{app},1}\|_{L^2}\big)\|(D_t - P_0)u'^{\text{app},1}\|_{H^s}\\ &+ \|(D_t - P_0)u'^{\text{app},1}\|_{H^s}\big(\|\tilde{u}\|_{H^s} + \|u'^{\text{app},1}\|_{H^s}\big)\Big).\end{aligned} \tag{7.84}$$

By the end of the statement of Proposition 4.1.2, $u'^{\text{app},1}_+$ satisfies estimates of the form (4.46)–(4.47) and also (4.39)–(4.41). Moreover, $\tilde{u}$ satisfies (7.3). Plugging these estimates in (7.84), we get a better upper bound than (7.71).

Consider next the case $v_1 = (D_t - P_0)\tilde{u}$, $\ell_1 = 1$ in (7.83). Decompose

$$K_{H,i_1,i_2}^{\ell_1,\ell_2} = K_< + K_>,$$

where $K_<$ (resp. $K_>$) is defined by the same formula (F.25) as $K_{H,i_1,i_2}^{\ell_1,\ell_2}$, but with the function k cut-off for $|\xi_1| \leq 2\langle\xi_2\rangle$ (resp. $|\xi_2| \leq 2\langle\xi_1\rangle$). We need to bound

$$t^{-\frac{3}{2}}\|K_<(L_{i_1}(D_t - i_1p(D_x))\tilde{u}_{i_1}, L_{i_2}^{\ell_2}v_{2,i_2})\|_{H^s}, \tag{7.85}$$

$$t^{-\frac{3}{2}}\|K_>(L_{i_1}(D_t - i_1p(D_x))\tilde{u}_{i_1}, L_{i_2}^{\ell_2}v_{2,i_2})\|_{H^s}, \tag{7.86}$$

where $\ell_2 = 0$ or 1 and $v_2 = \tilde{u}$ or $u'^{\text{app},1}$. Consider first expression (7.85). We decompose the first argument in $K_<$ under the form $g_1 + g_2$, where, for $\chi \in C_0^\infty(\mathbb{R})$, equal to one close to zero,

$$g_1 = (1 - \chi)(t^{-\beta}D_x)\big(L_{i_1}(D_t - i_1p(D_x))\tilde{u}_{i_1}\big), \tag{7.87}$$

$$g_2 = \chi(t^{-\beta}D_x)(f_{1,i_1} + xf_{2,i_1}), \tag{7.88}$$

where we used decomposition (7.74). Using the definition of L_{i_1} and (7.73), we may rewrite g_1 as a sum $g_1 = tg_1' + xg_1''$ with according to (7.73), for any $\sigma_0 \le s$,

$$\|g_1'\|_{H^{\sigma_0}} + \|g_1''\|_{H^{\sigma_0}} \le t^{-\beta(s-\sigma_0)}\varepsilon t^{\delta-\frac{1}{2}}. \tag{7.89}$$

Applying (F.38)–(F.40) (with the roles of f_1, f_2 interchanged), we see that (7.85) with the first argument of $K_<$ replaced by g_1 has Sobolev norm bounded from above by

$$Ct^{\frac{1}{4}-\beta(s-\sigma_0)}\varepsilon t^{\delta-\frac{1}{2}}\big(\|\tilde{u}\|_{H^s} + \|u'^{\mathrm{app},1}\|_{H^s}\big).$$

If $s\beta$ is large enough, we get an estimate by the right-hand side of (7.71). On the other hand, if we replace the first argument of $K_<$ in (7.85) by g_2, we reduce ourselves to

$$t^{-\frac{3}{2}}\|K_<(\tilde{\chi}(t^{-\beta}D_x)\tilde{f}_{1,i_1}, L_{i_2}^{\ell_2}v_2)\|_{H^s}, \tag{7.90}$$

$$t^{-\frac{3}{2}}\|K_<(x\tilde{\chi}(t^{-\beta}D_x)\tilde{f}_{2,i_1}, L_{i_2}^{\ell_2}v_2)\|_{H^s} \tag{7.91}$$

for new functions $\tilde{f}_1, \tilde{f}_2$ satisfying the same estimates (7.75)–(7.76) as f_1, f_2 and $\tilde{\chi}$ in $C_0^\infty(\mathbb{R})$. Decomposing $L_{i_2} = x + i_2tp'(D_x)$ and using (F.38)–(F.39) with the roles of f_1, f_2 interchanged, we bound (7.90) by

$$t^{-\frac{3}{4}}\|\tilde{\chi}(t^{-\beta}D_x)\tilde{f}_{1,i_1}\|_{H^{\sigma_0}}\|v_2\|_{H^s}.$$

By (7.75) and (7.3)–(7.4), this is smaller than

$$t^{-\frac{3}{4}+\beta\sigma_0}t^{-\frac{1}{2}}\big(t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^\theta\big)\varepsilon t^\delta$$

so than the right-hand side of (7.71) if $t \le \varepsilon^{-4+c}$ and β is small enough. To study (7.91), we decompose again L_{i_2} as above and use (F.39) and (F.40), to obtain a bound in

$$t^{-\frac{1}{4}}\|\tilde{\chi}(t^{-\beta}D_x)\tilde{f}_2\|_{H^{\sigma_0}}\|v_2\|_{H^s}.$$

By (7.76) for $\tilde{f}_2$ and (7.3), (7.4), we obtain a bound by the right-hand side of (7.71).

Let us study next (7.86). If $\ell_2 = 1$, we use (F.52) (with f_1 and f_2 interchanged) and if $\ell_2 = 0$ we use (F.58). We bound thus (7.86) by

$$Ct^{-\frac{3}{4}}\|(D_t - P_0)\tilde{u}\|_{H^s}\big(t^{\beta\sigma_0}\big(\|L\tilde{u}\|_{L^2} + \|Lu'^{\mathrm{app},1}\|_{L^2}\big) + \|\tilde{u}\|_{H^s} + \|u'^{\mathrm{app},1}\|_{H^s}\big).$$

If we use (7.73), (7.3), (7.4), we bound this by the right-hand side of (7.71), using again $t \le \varepsilon^{-4+c}$, and taking β small enough.

To conclude Step 1, we still have to consider (7.83) with $v_1 = (D_t - P_0)\tilde{u}$ and $\ell_1 = 0$, i.e. to bound

$$t^{-\frac{3}{2}}\|K_{H,i_1,i_2}^{0,\ell_2}(D_t - i_1p(D_x))\tilde{u}_{i_1}, L_{i_2}^{\ell_2}v_{2,i_2})\|_{H^s}.$$

Expressing L_{i_2} and using (F.54) and (F.46), we obtain a bound in

$$t^{-\frac{3}{4}}\|(D_t - P_0)\tilde{u}\|_{H^s}\big(\|\tilde{u}\|_{H^s} + \|u'^{\mathrm{app},1}\|_{H^s}\big).$$

Using (7.73), (7.3), (7.4), we obtain a bound of the form (7.71). This concludes the proof of Step 1.

Step 2: Proof of (7.72). Again, properties (E.20)–(E.22) of operator $C(t)$ allow us to ignore it in the proof of the estimates. We shall have thus to bound $\|L\hat{\mathcal{M}}_2'(v_1, v_2)\|_{L^2}$ where $\hat{\mathcal{M}}_2'$ has structure (6.47) and v_1, v_2 are given by equation (7.82). If we express $L_\pm = x \pm tp'(D_x)$, we are reduced to studying

$$t^{-\frac{1}{2}}\|K^{\ell_1,\ell_2}_{H,i_1,i_2}\big(L^{\ell_1}_{i_1}v_{1,i_1}, L^{\ell_2}_{i_2}v_{2,i_2}\big)\|_{L^2}, \tag{7.92}$$

$$t^{-\frac{3}{2}}\|xK^{\ell_1,\ell_2}_{H,i_1,i_2}\big(L^{\ell_1}_{i_1}v_{1,i_1}, L^{\ell_2}_{i_2}v_{2,i_2}\big)\|_{L^2}. \tag{7.93}$$

By Definition F.4.1 of the class $\mathcal{K}'_{1,1/2}(i)$, $xK^{\ell_1,\ell_2}_{H,i_1,i_2}$ may be written as $t^{\frac{1}{2}}\tilde{K}^{\ell_1,\ell_2}_{H,i_1,i_2}$ for another operator in $\mathcal{K}'_{1,1/2}(i)$. It is thus enough to bound (7.92).

We consider first the case $v_1 = (D_t - P_0)u'^{\text{app},1}$. By (F.50), (F.47), we bound (7.92) by

$$\begin{aligned}Ct^{-\frac{3}{4}}&\big(\|(D_t - P_0)u'^{\text{app},1}\|_{H^s} + t^\sigma\|L(D_t - P_0)u'^{\text{app},1}\|_{L^2}\big)\\ &\times\big(\|Lu'^{\text{app},1}\|_{L^2} + \|L\tilde{u}\|_{L^2} + \|u'^{\text{app},1}\|_{L^2} + \|\tilde{u}\|_{L^2}\big)\end{aligned}$$

for any $\sigma > 0$ (if $s\sigma$ is large enough). Since by Proposition 4.1.2, $u'^{\text{app},1}$ satisfies (4.46)–(4.47), we deduce from (7.3)–(7.4) an estimate better than (7.72).

Consider next the case $v_1 = (D_t - P_0)\tilde{u}$, $\ell_1 = 1$ in (7.92). We replace $L(D_t - P_0)\tilde{u}$ by the right-hand side of (7.74). By (F.47) and (F.51), the f_1 contribution to (7.92) is bounded from above by

$$Ct^{-\frac{3}{4}}\|f_1\|_{L^2}\big(t^\sigma\big(\|Lu'^{\text{app},1}\|_{L^2} + \|L\tilde{u}\|_{L^2}\big) + \|u'^{\text{app}}\|_{H^s} + \|\tilde{u}\|_{H^s}\big).$$

Using (7.75), (7.3), (7.4), we get an estimate in

$$Ct^{-1}\big(t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^\theta\big)\big((\varepsilon^2\sqrt{t})^\theta t^\sigma + \varepsilon t^{\delta-\frac{1}{4}}\big).$$

If σ is small enough, and since $t \le \varepsilon^{-4+c}$, we get a bound of the form (7.72).

On the other hand, if we replace $(D_t - P_0)\tilde{u}$ by xf_2, (7.92) is reduced to

$$t^{-\frac{1}{2}}\|K^{\ell_1,\ell_2}_{H,i_1,i_2}(xf_{2,i_1}, L^{\ell_2}_{i_2}v_{2,i_2})\|_{L^2}. \tag{7.94}$$

A ∂_{ξ_1}-integration by parts in (F.25) using (F.27) shows that (7.94) is reduced to

$$\|\tilde{K}^{\ell_1,\ell_2}_{H,i_1,i_2}(f_{2,i_1}, L^{\ell_2}_{i_2}v_{2,i_2})\|_{L^2}$$

for a new operator in the same class. Using (F.47) and (F.51), we get a bound in

$$Ct^{-\frac{1}{4}}\|f_2\|_{L^2}\big(\big(\|Lu'^{\text{app},1}\|_{L^2} + \|L\tilde{u}\|_{L^2}\big)t^\sigma + \|u'^{\text{app},1}\|_{H^s} + \|\tilde{u}\|_{H^s}\big).$$

Using (7.76), (7.3), (7.4), we obtain a bound of the form (7.72).

Consider finally the case $v_1 = (D_t - P_0)\tilde{u}$, $\ell_1 = 0$ in (7.92). By (F.47), we get a bound of (7.92) by

$$Ct^{-\frac{3}{4}}\|(D_t - P_0)\tilde{u}\|_{H^s}\big(\|L\tilde{u}\|_{L^2} + \|Lu'^{\mathrm{app},1}\|_{L^2} + \|\tilde{u}\|_{L^2} + \|u'^{\mathrm{app},1}\|_{L^2}\big).$$

If we plug there (7.73) and (7.3)–(7.4), we get an estimate of the form (7.72). This concludes the proof. ■

This concludes the study of terms of the form (6.65). It remains to study (6.66), (6.67) and (6.68).

Proposition 7.3.6. *The following statements hold.*

(i) *Denote*

$$F(t) = C(t)R_j(\underbrace{\tilde{u},\dots,\tilde{u}}_{\ell}, u^{\mathrm{app}},\dots,u^{\mathrm{app}}), \quad j = 3,4,\ 0 \le \ell \le j, \tag{7.95}$$

with R_j of the form (6.34)–(6.35). Then there is a function e satisfying (5.41) such that

$$\|F(t)\|_{H^s} \le t^{-1}\varepsilon t^{\delta} e(t,\varepsilon), \tag{7.96}$$

$$\|L_\pm F(t)\|_{L^2} \le t^{-1}\big(t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^{\theta}\big)e(t,\varepsilon). \tag{7.97}$$

(ii) *Denote*

$$F(t) = C(t)R_2(\underbrace{\tilde{u},\dots,\tilde{u}}_{\ell}, u'^{\mathrm{app},1},\dots,u'^{\mathrm{app},1})$$

with $0 \le \ell \le 2$ and $R_2 = \left[\begin{smallmatrix} R_{2,+} \\ R_{2,-}\end{smallmatrix}\right]$ given by (6.49). Then (7.96) and (7.97) hold.

(iii) *Let $F(t) = C(t)(\mathcal{R}(t,\cdot) + \tilde{\mathcal{R}}_3(t,\cdot) + \tilde{\mathcal{R}}_4(t,\cdot)) + \tilde{\mathcal{R}}_2(t,\cdot)$ with $\mathcal{R}, \tilde{\mathcal{R}}_j$ as in (6.68). Then (7.96) and (7.97) hold.*

Proof. (i) By (6.35) and (D.32) (and the boundedness of $C(t)$ on H^s), we bound $\|F(t)\|_{H^s}$ by

$$C\big(\|\tilde{u}\|_{W^{\rho_0,\infty}} + \|u^{\mathrm{app}}\|_{W^{\rho_0,\infty}}\big)^{j-1}\big(\|\tilde{u}\|_{H^s} + \|u^{\mathrm{app}}\|_{H^s}\big).$$

As $j \ge 3$, (7.1) and (7.3) imply (7.96).

To prove (7.97), we use once again that by (E.20)–(E.22), we may ignore the factor $C(t)$, and have to estimate LR_j in L^2. This expression is a sum of quantities of the form (6.36)–(6.38), so of the form (7.77)–(7.79) with $v_\ell = \tilde{u}_\pm$ or $v_\ell = u'^{\mathrm{app}}_\pm + u''^{\mathrm{app}}_\pm$.

When v_1 in (7.77) is replaced by $\tilde{u}_\pm$ or $u'^{\mathrm{app}}_\pm$, we use (D.33) to estimate the L^2 norm of these terms by

$$C\big(\|\tilde{u}\|_{W^{\rho_0,\infty}} + \|u^{\mathrm{app}}\|_{W^{\rho_0,\infty}}\big)^{j-1}\big(\|L\tilde{u}\|_{L^2} + \|Lu'^{\mathrm{app}}\|_{L^2}\big)$$

so by the right-hand side of (7.97) by (7.1)–(7.3), since $j \geq 3$. If $v_1 = u''^{\text{app}}$, we have a bound by (7.81) so by

$$\frac{1}{t} t^{\frac{1}{4}} (\varepsilon^2 \sqrt{t})^{\theta} \big((\varepsilon^2 \sqrt{t})^{\frac{1}{2}+\theta'-\theta} t^{\delta} \log(1+t) \log(1+t\varepsilon^2) \big) \tag{7.98}$$

which is bounded by the right-hand side of (7.97) for $\delta > 0$ small, θ, θ' close to $\frac{1}{2}$ if $t \leq \varepsilon^{-4+c}$.

Expression (7.78) is controlled as (7.77). For (7.79), we use (D.71) if at least one of the functions v_j is equal to $\tilde{u}_{\pm}$ or $u'^{\text{app}}_{\pm}$, which brings the wanted estimate (7.97) by (7.1)–(7.3). If all arguments v_j are equal to $u''^{\text{app}}_{\pm}$, we use (D.72), that brings again an estimate of the form (7.98). This concludes the proof of (i).

(ii) Again, we may forget operator $C(t)$. We have to study

$$t^{-2} \| K^{\ell_1,\ell_2}_{L,i_1,i_2} (L^{\ell_1}_{i_1} v_{1,i_1}, L^{\ell_2}_{i_2} v_{2,i_2}) \|_{H^s}, \tag{7.99}$$

$$t^{-2} \| L_{\pm} K^{\ell_1,\ell_2}_{L,i_1,i_2} (L^{\ell_1}_{i_1} v_{1,i_1}, L^{\ell_2}_{i_2} v_{2,i_2}) \|_{L^2} \tag{7.100}$$

with $K^{\ell_1,\ell_2}_{L,i_1,i_2}$ in $\mathcal{K}'_{1/2,1}(i)$, and v_1, v_2 equal to $\tilde{u}$ or $u'^{\text{app},1}$. Since estimates (7.4) are better than (7.3), we may argue just in the case $v_1 = v_2 = \tilde{u}$. Then (7.99) is just (7.31) multiplied by $t^{-\frac{1}{2}}$. It is then estimated by (7.32)–(7.34) multiplied by $t^{-\frac{1}{2}}$ and thus by (7.35) multiplied by $t^{-\frac{1}{2}}$, so by $\varepsilon t^{\delta-1} t^{\sigma} (\varepsilon^2 \sqrt{t})^{\theta}$. For $t \leq \varepsilon^{-4+c}$, this is of the form of the right-hand side of (7.96) if σ is small enough. Let us bound next (7.100). Using the expression $L_{\pm} = x \pm t p'(D_x)$, we have to estimate

$$t^{-1} \| K^{\ell_1,\ell_2}_{L,i_1,i_2} (L^{\ell_1}_{i_1} v_{1,i_1}, L^{\ell_2}_{i_2} v_{2,i_2}) \|_{L^2}, \tag{7.101}$$

$$t^{-2} \| x K^{\ell_1,\ell_2}_{L,i_1,i_2} (L^{\ell_1}_{i_1} v_{1,i_1}, L^{\ell_2}_{i_2} v_{2,i_2}) \|_{L^2}. \tag{7.102}$$

By (F.47), (F.50), (F.51), we bound (7.101) by

$$C t^{-\frac{5}{4}} \big(\| L\tilde{u} \|_{L^2} t^{\sigma} + \| \tilde{u} \|_{H^s} \big)^2.$$

Using (7.3), we obtain

$$C t^{-1} ((\varepsilon^2 \sqrt{t})^{\theta} t^{\frac{1}{4}}) t^{2\sigma} (\varepsilon^2 \sqrt{t})^{\theta}$$

which is smaller than the right-hand side of (7.97) for $t \leq \varepsilon^{-4+c}$ if σ is small enough.

Finally, to study (7.102), we notice, as after (7.38), that this expression may be bounded by $t^{-\frac{1}{2}}$ times (7.101), so has the wanted bounds.

(iii) The contributions $C(t)\tilde{\mathcal{R}}_3$, $C(t)\tilde{\mathcal{R}}_4$, $\tilde{\mathcal{R}}_2$ are estimated by (6.59), (6.54), (6.55), so largely by the right-hand side of (7.96)–(7.97), using (7.1)–(7.3). The fact that $C(t)\mathcal{R}$ satisfies these estimates follows from inequalities (5.39)–(5.40) satisfied by $\mathcal{R}$ (or (6.26)–(6.27)). This concludes the proof. ■

We conclude this chapter summarizing the estimates we have obtained.

Proposition 7.3.7. *Let $c > 0$ (small) be given, $0 < \theta' < \theta < \frac{1}{2}$ with θ' close to $\frac{1}{2}$. Let $T \in [1, \varepsilon^{-4+c}]$ and assume that we are given on $[1,T] \times \mathbb{R}$ functions $\tilde{u}_+$, u'^{app}_+, u''^{app}_+, $u'^{\text{app},1}_+$ that satisfy estimates (7.1)–(7.4), for some small $\delta > 0$, some constants $C(A, A')$, D, any ε in an interval $]0, \varepsilon_0]$, and such that $\tilde{u}$ solves (6.61). Then there are $D_0 > 0$, $\varepsilon'_0 \in]0, \varepsilon_0]$ such that if $D \geq D_0$ and $\varepsilon \in]0, \varepsilon'_0]$, for any $t \in [1, T]$, the L^2 estimates in (7.3) may be improved to*

$$\|\tilde{u}_+(t,\cdot)\|_{H^s} \leq \frac{D}{2}\varepsilon t^{\delta}, \tag{7.103}$$

$$\|L_+\tilde{u}_+(t,\cdot)\|_{L^2} \leq \frac{D}{2} t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^{\theta}. \tag{7.104}$$

Proof. By Corollary 6.2.5, we know that

$$(D_t - P_0)\mathring{u} = \hat{\mathcal{R}} \tag{7.105}$$

if we define

$$\mathring{u} = C(t)\Big(\tilde{u} - \sum_{j=3}^{4} \hat{\mathcal{M}}_j(\tilde{u}, u^{\text{app}})\Big) - \hat{\mathcal{M}}'_2(\tilde{u}, u'^{\text{app},1}). \tag{7.106}$$

By Proposition 7.1.1, Proposition 7.2.1 and the boundedness properties (E.19)–(E.22) of $C(t)$, we have

$$\|\mathring{u} - C(t)\tilde{u}\|_{H^s} \leq \varepsilon t^{\delta} e(t,\varepsilon), \tag{7.107}$$

$$\|L(\mathring{u} - C(t)\tilde{u})\|_{L^2} \leq t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^{\theta} e(t,\varepsilon), \tag{7.108}$$

where e satisfies (5.41).

The right-hand side $\hat{\mathcal{R}}$ of (7.105) is the sum of terms (6.62)–(6.68). These terms have been estimated in Proposition 7.1.2, Proposition 7.2.3, Proposition 7.3.1, Proposition 7.3.3, Proposition 7.3.4, Proposition 7.3.6, which imply that

$$\begin{aligned} \|\hat{R}(t,\cdot)\|_{H^s} &\leq \varepsilon t^{\delta-1} e(t,\varepsilon), \\ \|L\hat{R}(t,\cdot)\|_{L^2} &\leq t^{-1} t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^{\theta} e(t,\varepsilon). \end{aligned} \tag{7.109}$$

By the fact that L commutes to $(D_t - P_0)$, it follows from the energy inequality applied to (7.105) that

$$\|\mathring{u}(t,\cdot)\|_{H^s} \leq \|\mathring{u}(1,\cdot)\|_{H^s} + \varepsilon t^{\delta} e(t,\varepsilon), \tag{7.110}$$

$$\|L\mathring{u}(t,\cdot)\|_{L^2} \leq \|L\mathring{u}(1,\cdot)\|_{L^2} + t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^{\theta} e(t,\varepsilon) \tag{7.111}$$

and then, by (7.107)–(7.108) and (E.14), (E.19)–(E.22) that

$$\|\tilde{u}(t,\cdot)\|_{H^s} \leq C\|\tilde{u}(1,\cdot)\|_{H^s} + \varepsilon t^{\delta} e(t,\varepsilon), \tag{7.112}$$

$$\|L\tilde{u}(t,\cdot)\|_{L^2} \leq C\big(\|L\tilde{u}(1,\cdot)\|_{L^2} + \|\tilde{u}(1,\cdot)\|_{L^2}\big) + t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^{\theta} e(t,\varepsilon) \tag{7.113}$$

for some constant C, some new factors $e(t, \varepsilon)$. Recall that $\tilde{u}_+$ has been defined from u_+ in (5.34), and that since this function is $O(\varepsilon)$ at time $t = 1$ in the space $\{f \in H^s : xf \in L^2\}$ by (2.24) and (2.22), we may take D so large that the first term on the right-hand side of (7.112)–(7.113) is smaller than $\frac{D}{4}\varepsilon$. If ε is small enough, we thus get (7.103)–(7.104) using (5.41). ∎

Chapter 8

L^∞ estimates and end of bootstrap

The goal of this chapter is to conclude the bootstrap argument that gives our main theorem. At the end of the preceding chapter, we have seen that assuming a priori estimates (7.3), we could prove that the first and last ones hold with a better constant. Here, we shall bootstrap the $W^{\rho,\infty}$ bound in (7.3). Once this is done, we still have to go back to the original unknowns of the statement of our main Theorem 2.1.1 and to deduce from estimates of $\tilde{u}$ and from the study made in Section 4.2 the bounds of the quantities that appear in that theorem.

8.1 L^∞ estimates

One cannot deduce an L^∞ estimate of the form of the second inequality in (7.3) from the Sobolev estimates satisfied by $\tilde{u}_+, L_+\tilde{u}_+$ through Klainerman–Sobolev inequalities: the fact that $\|L_+\tilde{u}_+\|_{L^2}$ admits only an $O(t^{\frac{1}{4}})$ bound would be too rough in order to do so. Instead, we deduce from the equation satisfied by $\tilde{u}$ an ODE, that will allow us to get the wanted L^∞ bound.

We shall reduce ourselves to the semiclassical framework, defining from the solution $\tilde{u} = \begin{bmatrix} \tilde{u}_+ \\ \tilde{u}_- \end{bmatrix}$ of (6.61) a function $\underline{\tilde{u}} = \begin{bmatrix} \underline{\tilde{u}}_+ \\ \underline{\tilde{u}}_- \end{bmatrix}$ by

$$\tilde{u}_\pm = \frac{1}{\sqrt{t}}\underline{\tilde{u}}_\pm\Big(t, \frac{x}{t}\Big) = (\Theta_t \underline{\tilde{u}})(t, x) \tag{8.1}$$

using notation (B.15). We set $h = t^{-1}$ and decompose for a given $\rho \geq 0$,

$$\langle hD_x\rangle^\rho \underline{\tilde{u}}_\pm = \underline{\tilde{u}}^\rho_{\pm,\Lambda} + \underline{\tilde{u}}^\rho_{\pm,\Lambda^c} \tag{8.2}$$

with according to notation (D.91)

$$\underline{\tilde{u}}^\rho_{\pm,\Lambda} = \mathrm{Op}_h^{\mathrm{W}}\Big(\gamma\Big(\frac{x \pm p'(\xi)}{\sqrt{h}}\Big)\Big)\mathrm{Op}_h^{\mathrm{W}}(\langle\xi\rangle^\rho)\underline{\tilde{u}}_\pm, \tag{8.3}$$

where $\gamma \in C_0^\infty(\mathbb{R})$ has small enough support and is equal to 1 close to zero. We denote by $\tilde{u}^\rho_{\pm,\Lambda}, \tilde{u}^\rho_{\pm,\Lambda^c}$ the functions corresponding to $\underline{\tilde{u}}^\rho_{\pm,\Lambda}, \underline{\tilde{u}}^\rho_{\pm,\Lambda^c}$ by a change of variables of the form (8.1).

The contribution $\underline{\tilde{u}}^\rho_{\pm,\Lambda^c}$ has nice L^∞ bounds by Klainerman–Sobolev estimates:

Proposition 8.1.1. *For any $\sigma > 0$, any s with $s\sigma$ large enough, one has the following estimate:*

$$\|\tilde{u}^\rho_{\pm,\Lambda^c}\|_{L^\infty} \leq Ct^{-\frac{3}{4}+\sigma}\big(\|L_\pm\tilde{u}_\pm\|_{L^2} + \|\tilde{u}_\pm\|_{H^s}\big). \tag{8.4}$$

Proof. Translating that on $\underline{\tilde{u}}^{\rho}_{\pm,\Lambda^c}$, this means

$$\|\underline{\tilde{u}}^{\rho}_{\pm,\Lambda^c}\|_{L^\infty} \leq Ch^{\frac{1}{4}-\sigma}\big(\|\mathcal{L}_\pm \underline{\tilde{u}}_\pm\|_{L^2} + \|\underline{\tilde{u}}_\pm\|_{H^s_h}\big).$$

This is just statement (D.87) in Proposition D.3.4. ∎

We study from now on the function $\underline{\tilde{u}}^{\rho}_{\pm,\Lambda}$. We first prove some bounds for expressions (5.43)–(5.49), whose sum is equal to $(D_t - p(D_x))\tilde{u}_+$. If $W(t,x)$ is some function and $\underline{W}$ is defined from W by (8.1), i.e. $W(t,\cdot) = \Theta_t \underline{W}(t,\cdot)$, we denote by $\underline{W}^{\rho}_{\Lambda}$ the function defined by (8.3) with sign $+$ and $\underline{\tilde{u}}_\pm$ replaced by $\underline{W}$, and we shall call W^{ρ}_{Λ} the function $W^{\rho}_{\Lambda} = \Theta_t \underline{W}^{\rho}_{\Lambda}$.

Lemma 8.1.2. *Let*

$$a(t) = \frac{\sqrt{3}}{3}(a_+(t) - a_-(t)), \quad a^{\mathrm{app}}(t) = \frac{\sqrt{3}}{3}(a^{\mathrm{app}}_+(t) - a^{\mathrm{app}}_-(t)),$$

where $a_- = -\bar{a}_+$, $a^{\mathrm{app}}_- = -\overline{a^{\mathrm{app}}_+}$, and where a_+, a^{app}_+ satisfy by (4.96)–(4.100)

$$|a^{\mathrm{app}}_+(t)| \leq Ct_\varepsilon^{-\frac{1}{2}}, \quad |a_+(t) - a^{\mathrm{app}}_+(t)| \leq Ct_\varepsilon^{-\frac{3}{2}} \tag{8.5}$$

for t in the interval $[1,T]$, $T \leq \varepsilon^{-4+c}$, where these functions are defined. Assume moreover that on that interval, the functions $\tilde{u}_+, u'^{\mathrm{app}}_+, u''^{\mathrm{app}}_+$ satisfy (7.1)–(7.3). Then the quantities (5.43)–(5.49) satisfy the following estimates, with a constant C depending on the constants A, A', D in (7.1)–(7.3):

$$\|(5.43)\|_{W^{\rho,\infty}} \leq Ct^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^\theta, \tag{8.6}$$
$$\|(5.44)\|_{W^{\rho,\infty}} \leq Ct^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^\theta, \tag{8.7}$$
$$\|(5.45)\|_{W^{\rho,\infty}} \leq Ct^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^\theta, \tag{8.8}$$
$$\|(5.46)^{\rho}_{\Lambda}\|_{L^\infty} \leq Ct^{-\frac{3}{2}+\sigma}(\varepsilon^2\sqrt{t})^\theta, \tag{8.9}$$
$$\|(5.47)\|_{W^{\rho,\infty}} \leq Ct^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^\theta, \tag{8.10}$$
$$\|(5.48)\|_{W^{\rho,\infty}} \leq Ct^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^\theta, \tag{8.11}$$
$$\|(5.49)\|_{W^{\rho,\infty}} \leq Ct^{-\frac{3}{2}+\sigma}(\varepsilon^2\sqrt{t})^\theta, \tag{8.12}$$

where $\sigma > 0$ may be taken as small as one wants if $s\sigma$ is large enough (s being the index of Sobolev estimates (7.1)–(7.3)) relatively to ρ, and where in (8.9) one uses the notation W^{ρ}_{Λ} defined before the statement of the lemma.

Proof. We prove the inequalities separately.

Inequality (8.6). This inequality follows from (5.58) and the fact that $t_\varepsilon^{-\frac{1}{2}} \leq \varepsilon$.

Inequality (8.7). We have seen in the proof of Proposition 5.2.1 that (5.44) is a sum of terms of the form (5.60) or (5.61), with conditions (5.62) or (5.63), i.e. may be written from

$$\mathrm{Op}(m)(v_1,\dots,v_n), \tag{8.13}$$

where m is in $\tilde{S}_{1,0}(\prod_{j=1}^n \langle\xi_j\rangle^{-1} M_0^\nu, n)$, with $n \geq 3$ and v_j equal to $\tilde{u}_\pm$ or $u'^{\mathrm{app}}\pm$ or $u''^{\mathrm{app}}_\pm$ or R (with R satisfying (5.25)–(5.26)). In particular, by Sobolev estimates, one has

$$\|R(t,\cdot)\|_{W^{\rho,\infty}} \leq C\left(\frac{(\varepsilon^2\sqrt{t})^{\theta'} t^\sigma}{\sqrt{t}}\right)^4 \varepsilon t^\delta. \tag{8.14}$$

If we apply (D.39), we obtain for the $W^{\rho,\infty}$ norm of (8.13) a bound in

$$\begin{aligned}&\big(\|\tilde{u}_+\|_{W^{\rho,\infty}} + \|u'^{\mathrm{app}}_+\|_{W^{\rho,\infty}} + \|u''^{\mathrm{app}}_+\|_{W^{\rho,\infty}} + \|R\|_{W^{\rho,\infty}}\big)^2\\ &\quad\times\Big(t^\sigma\big(\|\tilde{u}_+\|_{W^{\rho,\infty}} + \|u'^{\mathrm{app}}_+\|_{W^{\rho,\infty}} + \|u''^{\mathrm{app}}_+\|_{W^{\rho,\infty}} + \|R\|_{W^{\rho,\infty}}\big)\\ &\qquad + t^{-1}\big(\|\tilde{u}_+\|_{H^s} + \|u'^{\mathrm{app}}_+\|_{H^s} + \|u''^{\mathrm{app}}_+\|_{H^s} + \|R\|_{H^s}\big)\Big).\end{aligned}$$

By (7.1)–(7.3) and (5.25), (8.14), this is smaller than the right-hand side of (8.7) (if we use that $(\varepsilon^2\sqrt{t})^{3\theta'-\theta} t^\sigma \leq C$ for $t \leq \varepsilon^{-4+c}$).

Inequality (8.8). Expression (5.45) to estimate has been seen to be of the form (5.71) or (5.72), with either (5.73) or (5.74). Terms corresponding to (5.73) are of the form (8.13) and, as we have just seen, satisfy the wanted bound. We have just to consider expressions (5.71) or (5.72) under (5.74), i.e. quantities of the form

$$\mathrm{Op}(m')(v_1,v_2), \tag{8.15}$$

where m' is in $\tilde{S}'_{1,0}(\prod_{j=1}^2 \langle\xi_j\rangle^{-1} M_0^\nu, 2)$, and v_1, v_2 taken among $\tilde{u}_\pm$, $u'^{\mathrm{app}}_\pm$, $u''^{\mathrm{app}}_\pm$, R. If both v_1, v_2 are different from $u''^{\mathrm{app}}_\pm$, we use (D.77) with $r = 2$, $n = 2$, $\ell = 0$. We get a bound in

$$\begin{aligned}&t^{-2+\sigma}\big(\|u'^{\mathrm{app}}_+\|_{H^s} + \|\tilde{u}_+\|_{H^s} + \|R\|_{H^s}\\ &\qquad + \|L_+ u'^{\mathrm{app}}_+\|_{L^2} + \|L_+\tilde{u}_+\|_{L^2} + \|L_+ R\|_{L^2}\big)^2\end{aligned} \tag{8.16}$$

(estimating the $W^{\rho_0,\infty}$ norm from the H^s one). It follows from (5.25) and (5.26) that $\|L_+R\|_{L^2} \leq C(t^{\frac14}(\varepsilon^2\sqrt{t})^\theta)$. Using also (7.1) and (7.3), we estimate (8.16) by the right-hand side of (8.8), when $t \leq \varepsilon^{-4+c}$ if σ is small enough. Consider next the case when v_1 or v_2 is equal to $u''^{\mathrm{app}}_\pm$. If for instance $v_1 = u''^{\mathrm{app}}_\pm$ and $v_2 = \tilde{u}_\pm$ or $u'^{\mathrm{app}}_\pm$ or R, we apply (D.77) with $n = 2$, $\ell = 1$. The first term on the right-hand side of this expression is largely estimated by (8.8) if r is taken large enough. The second one is smaller than

$$\begin{aligned}&Ct^{-2+\sigma}\big(\|u''^{\mathrm{app}}_+\|_{W^{\rho,\infty}} + \|L_+u''^{\mathrm{app}}_+\|_{W^{\rho,\infty}}\big)\\ &\quad\times\big(\|u'^{\mathrm{app}}_+\|_{H^s} + \|\tilde{u}_+\|_{H^s} + \|R\|_{H^s}\\ &\qquad + \|L_+u'^{\mathrm{app}}_+\|_{L^2} + \|L_+\tilde{u}_+\|_{L^2} + \|L_+R\|_{L^2}\big).\end{aligned}$$

By (7.1)–(7.3) and (5.25)–(5.26), this is largely bounded by the right-hand side of inequality (8.8).

If v_1 and v_2 are both equal to $u''^{\text{app}}_{\pm}$, we use (D.77) with $\ell = n = 2$. We obtain a bound in $t^{-2+\sigma}(\log(1+t))^2(\log(1+t\varepsilon^2))^2$ for the second contribution to the right-hand side of (D.77). If σ is small enough, this is better than (8.8) since $\theta \leq \frac{1}{2}$.

Inequality (8.9). It follows from (D.82) (with a large enough r) translated in the non-semiclassical framework, that for any function W

$$\|W^{\rho}_{\Lambda}\|_{L^\infty} \leq C\big(t^{-\frac{1}{4}+\sigma}\|W\|_{L^2} + t^{-2}\|W\|_{H^s}\big). \tag{8.17}$$

To estimate (8.9), we decompose expression (5.46) as the sum of (5.80)–(5.83). Consider first the nonlinear quantity (5.82), that may be written as (5.85). By (D.88) and the fact that $a(t) = O(t_\varepsilon^{-1/2})$, its contribution to (8.9) is bounded from above by

$$t^{\sigma}t_\varepsilon^{-\frac{1}{2}}\big(\|\mathrm{Op}(m')(v_1,\dots,v_n)\|_{W^{\rho,\infty}} + t^{-r}\|\mathrm{Op}(m')(v_1,\dots,v_n)\|_{H^s}\big) \tag{8.18}$$

for any r, if $\sigma > 0$ and $s\sigma$ is large enough, m' being in $\tilde{S}'_{1,0}(\prod_{j=1}^{n}\langle\xi_j\rangle^{-1}M_0^{\nu}, n)$, $2 \leq n \leq 4$, v_j being equal to $\tilde{u}_{\pm}$ or $u'^{\text{app}}_{\pm}$ or $u''^{\text{app}}_{\pm}$. Since (8.18) involves expressions of the form (8.13) or (8.15), we already know that the first term is estimated by the right-hand side of (8.9). The second term is easily bounded, as r is arbitrary.

We have thus just to consider the linear expressions (5.80), (5.81), (5.83). As $a(t) = O(t_\varepsilon^{-1/2})$, $a(t) - a^{\text{app}}(t) = O(t_\varepsilon^{-3/2})$ by (8.5), the expressions to study are of the form

$$\begin{aligned} &t_\varepsilon^{-\frac{1}{2}}\mathrm{Op}(m')\tilde{u}_{\pm},\\ &t_\varepsilon^{-\frac{1}{2}}\mathrm{Op}(m')R, \end{aligned} \tag{8.19}$$

$$\begin{aligned} &t_\varepsilon^{-\frac{3}{2}}\mathrm{Op}(m')u'^{\text{app}}_{\pm},\\ &t_\varepsilon^{-\frac{3}{2}}\mathrm{Op}(m')u''^{\text{app}}_{\pm}, \end{aligned} \tag{8.20}$$

where m' is in $\tilde{S}'_{1,0}(\langle\xi\rangle^{-1}, 1)$. We replace in (8.17) W by (8.19) or (8.20). It follows from (D.71) and (D.32) with $n = 1$ that the contribution of (8.19) to the right-hand side of (8.17) is bounded from above by

$$t^{-\frac{5}{4}+\sigma}t_\varepsilon^{-\frac{1}{2}}\big(\|\tilde{u}_{\pm}\|_{H^s} + \|R\|_{H^s} + \|L_{\pm}\tilde{u}_{\pm}\|_{L^2} + \|L_{\pm}R\|_{L^2}\big).$$

Combined with (7.1), (7.3) and (5.25)–(5.26), this gives an estimate in $t^{-\frac{3}{2}+\sigma}(\varepsilon^2\sqrt{t})^{\theta}$ as wanted.

To study the contribution of (8.20) to the right-hand side of (8.17), we just apply the Sobolev boundedness of $\mathrm{Op}(m')$ to get

$$t_\varepsilon^{-\frac{3}{2}}t^{-\frac{1}{4}+\sigma}\big(\|u'^{\text{app}}_{+}\|_{H^s} + \|u''^{\text{app}}_{+}\|_{H^s}\big).$$

Combining with (7.1) and (7.2), we get again the wanted bound. This concludes the study of (8.9).

Inequality (8.10). Expression (5.47) is made of terms of the form (5.45) or (5.44) multiplied by the decaying factor $a(t)$. It is thus estimated by better quantities than the right-hand side of (8.7)–(8.8).

Inequality (8.11). To estimate (5.48), we notice first that terms in that expression corresponding to $|I| \geq 2$ have already been treated in the proof of (8.7) and (8.8). It remains thus to study the linear terms, that are of the form

$$a(t)^j \operatorname{Op}(m')u_\pm, \quad j \geq 2,$$

with m' in $\tilde{S}'_{1,0}(\langle\xi\rangle^{-1}, 1)$. By expression (5.59) of u_+, we shall get terms of the form (5.82) with $a(t)$ replaced by $a(t)^2$. These terms have already been considered in the study of (8.7) and (8.8) (see (8.13) and (8.15)). We obtain also linear terms in

$$\begin{aligned} &a(t)^j \operatorname{Op}(m')\tilde{u}_\pm, && a(t)^j \operatorname{Op}(m')u'^{\text{app}}_\pm, \\ &a(t)^j \operatorname{Op}(m')u''^{\text{app}}_\pm, && a(t)^j \operatorname{Op}(m')R \end{aligned} \tag{8.21}$$

with $j \geq 2$. To study those terms in (8.21) of the form $a(t)^j \operatorname{Op}(m')w$ with $w = \tilde{u}_\pm$ or $u'^{\text{app}}_\pm$ or R, we use (D.77) with $n = 1$, $\ell = 0$. We obtain an estimate of the $W^{\rho,\infty}$ norm in

$$\begin{aligned} Ct_\varepsilon^{-1}t^{-1+\sigma}\big(\|u'^{\text{app}}_+\|_{H^s} + \|\tilde{u}_+\|_{H^s} + \|R_+\|_{H^s} \\ + \|L_+\tilde{u}_+\|_{L^2} + \|L_+u'^{\text{app}}_+\|_{L^2} + \|L_+R\|_{L^2}\big). \end{aligned}$$

Combined with (7.1)–(7.2) and (5.25)–(5.26), this largely implies a bound by the right-hand side of (8.11). Finally, the $W^{\rho,\infty}$ norm of the terms in (8.21) involving $u''^{\text{app}}_\pm$ is estimated using (D.77) when $n = 1, \ell = 1$. One obtains

$$Ct_\varepsilon^{-1}t^{-1+\sigma}\big(\|u''^{\text{app}}_+\|_{H^s} + \|u''^{\text{app}}_+\|_{W^{\rho,\infty}} + \|L_+u''^{\text{app}}_+\|_{W^{\rho,\infty}}\big)$$

which by (7.2) is also largely estimated by (8.11).

Inequality (8.12). Finally, (8.12) follows from the fact that (5.49) satisfies bounds (4.38), that largely imply (8.12). ∎

We may deduce from the above lemma an L^∞ bound for $(D_t - p(D_x))\tilde{u}_+$.

Proposition 8.1.3. *Denote $f_+ = (D_t - p(D_x))\tilde{u}_+$ and define $\underline{f}_+$ by*

$$f_+(t,x) = \frac{1}{\sqrt{t}}\underline{f}_+\Big(t, \frac{x}{t}\Big) = \Theta_t \underline{f}_+(t,x) \tag{8.22}$$

using notation (B.15)*. According to* (D.91)*, define*

$$\underline{f}^\rho_{+,\Lambda} = \operatorname{Op}_h^{\mathrm{W}}\Big(\gamma\Big(\frac{x + p'(\xi)}{\sqrt{h}}\Big)\Big)\operatorname{Op}_h^{\mathrm{W}}(\langle\xi\rangle^\rho)\underline{f}_+. \tag{8.23}$$

Then, under a priori assumption (7.3) *on $\tilde{u}_+$, for any $\sigma > 0$, any s such that $s\sigma$ is large enough, one has*

$$\|\underline{f}^\rho_{+,\Lambda}(t,\cdot)\|_{L^\infty} \leq Ch^{1-\sigma}(\varepsilon^2\sqrt{t})^\theta. \tag{8.24}$$

Proof. Recall that

$$f_+ = (D_t - p(D_x))\tilde{u}_+$$

is given by the sum of expressions (5.43)–(5.49). Call $f_{+,2}$ contribution (5.46) and $f_{+,1}$ the sum of all other contributions. Define $\underline{f}^\rho_{+,j,\Lambda}$, $j = 1, 2$, from $\underline{f}_{+,j}$ as in (8.23). Then (8.9) shows that $\underline{f}^\rho_{+,2,\Lambda}$ satisfies (8.24). To obtain the same estimates for $\underline{f}^\rho_{+,1,\Lambda}$, we apply (D.88) in order to bound the different contributions to $\underline{f}^\rho_{+,1,\Lambda}$ in L^∞ from (8.6)–(8.8) and (8.10)–(8.12), using moreover (7.73) in order to estimate the H^s norm in (D.88) (taking the power N in the pre-factor h^N large enough). This concludes the proof. ∎

We shall now write an ODE satisfied by function (8.3).

Proposition 8.1.4. *Assume a priori assumptions (7.1)–(7.3). There is a real-valued function θ_h, supported in $]-1,1[$ such that $\underline{\tilde{u}}^\rho_{+,\Lambda}$ defined by (8.3) satisfies*

$$\big(D_t - \theta_h(x)\sqrt{1-x^2}\big)\underline{\tilde{u}}^\rho_{+,\Lambda} = O_{L^\infty}\big(t^{-1+\sigma}(\varepsilon^2\sqrt{t})^\theta\big), \tag{8.25}$$

where $\sigma > 0$ is as small as one wants (if s in estimate (7.3) is large enough relatively to $\frac{1}{\sigma}$).

Proof. Denote as in the preceding proposition $f_+ = (D_t - p(D_x))\tilde{u}_+$, so that

$$(D_t - p(D_x))\big(\langle D_x\rangle^\rho \tilde{u}_+\big) = \langle D_x\rangle^\rho f_+.$$

If $\underline{f}_+$ is given by (8.22) and $\underline{\tilde{u}}_+$ by (8.1), this is equivalent to

$$\Big(D_t - \mathrm{Op}_h^{\mathrm{W}}\big(x\xi + \sqrt{1+\xi^2}\big)\Big)\mathrm{Op}_h^{\mathrm{W}}(\langle\xi\rangle^\rho)\underline{\tilde{u}}_+ = \mathrm{Op}_h^{\mathrm{W}}(\langle\xi\rangle^\rho)\underline{f}_+. \tag{8.26}$$

We make act $\mathrm{Op}_h^{\mathrm{W}}(\gamma(\frac{x+p'(\xi)}{\sqrt{h}}))$ on (8.26). By (D.94) and the definition (8.3) of $\underline{\tilde{u}}^\rho_{+,\Lambda}$, we obtain

$$\Big(D_t - \mathrm{Op}_h^{\mathrm{W}}\big(x\xi + \sqrt{1+|\xi|}\big)\Big)\underline{\tilde{u}}^\rho_{+,\Lambda} = \underline{f}^\rho_{+,\Lambda} + R_1 + R_2 \tag{8.27}$$

with

$$R_1 = h\mathrm{Op}_h^{\mathrm{W}}\Big(\gamma_{-1}\Big(\frac{x+p'(\xi)}{\sqrt{h}}\Big)\Big(\frac{x+p'(\xi)}{\sqrt{h}}\Big)\Big)\mathrm{Op}_h^{\mathrm{W}}(\langle\xi\rangle^\rho)\underline{\tilde{u}}_+, \tag{8.28}$$

$$R_2 = h^{\frac{3}{2}}\mathrm{Op}_h^{\mathrm{W}}(r)\mathrm{Op}_h^{\mathrm{W}}(\langle\xi\rangle^\rho)\underline{\tilde{u}}_+, \tag{8.29}$$

where $|\partial_z^\alpha \gamma_{-1}(z)| \le C_\alpha\langle z\rangle^{-1-\alpha}$ and r satisfies

$$|\partial_x^{\alpha_1}\partial_\xi^{\alpha_2}(h\partial_h)^k r(x,\xi,h)| \le Ch^{-\frac{\alpha_1+\alpha_2}{2}}\Big\langle\frac{x+p'(\xi)}{\sqrt{h}}\Big\rangle^{-1}. \tag{8.30}$$

By [82, Lemma 4.2], R_1 may be replaced by

$$h^{\frac{1}{2}}\mathrm{Op}_h^{\mathrm{W}}\Big(\gamma_{-1}\Big(\frac{x+p'(\xi)}{\sqrt{h}}\Big)(x+p'(\xi))\langle\xi\rangle^\rho\chi(h^\beta\xi)\Big)\underline{\tilde{u}}_+ \tag{8.31}$$

modulo a quantity estimated in L^∞ by

$$Ch^{\frac{5}{4}-\sigma}\big(\|\mathcal{L}_+\underline{\tilde{u}}_+\|_{L^2} + \|\tilde{u}_+\|_{H^s}\big) \tag{8.32}$$

for some $\sigma > 0$, σ going to zero with β. By a priori assumption (7.3) (translated on $\underline{\tilde{u}}_+$) this is estimated by the right-hand side of (8.25). By [82, estimate (4.25) of Lemma 4.3], the L^∞ norm of (8.31) is also controlled by (8.32), so by the right-hand side of (8.25).

Let us check that R_2 given by (8.29) is also bounded by the same quantity. This follows from semiclassical Sobolev injection together with the a priori Sobolev estimate in (7.3). Moreover, by (8.24), the $\underline{f}^{\rho}_{+,\Lambda}$ contribution in (8.27) is also bounded by the right-hand side of (8.25).

It remains to write the left-hand side of (8.27) as the left-hand side of (8.25), up to some new contributions to the right-hand side of the latter. This follows from Proposition D.3.6, where the right-hand side of the second inequality of (D.93) is again estimated using (7.3). This concludes the proof. ∎

8.2 Bootstrap of L^∞ estimates

We have shown in Proposition 7.3.7 that under a priori assumptions (7.1)–(7.4), we could improve the Sobolev estimates in (7.3) to (7.103)–(7.104). Our first goal here will be to improve also the L^∞ estimate.

Proposition 8.2.1. *Assume that* (7.1)–(7.3) *hold true on an interval* $[1, T]$. *Let* $c > 0$ *be given. Then if* D *in* (7.3) *has been taken large enough, there is* $\varepsilon_0 \in]0, 1]$ *such that, for all* $\varepsilon \in]0, \varepsilon_0]$, *all* $1 \le t \le T \le \varepsilon^{-4+c}$, *one has the bound*

$$\|\tilde{u}_+\|_{W^{\rho,\infty}} \le \frac{D}{2}\frac{(\varepsilon^2\sqrt{t})^{\theta'}}{\sqrt{t}}. \tag{8.33}$$

Proof. We have to bound $\langle D_x\rangle^\rho \tilde{u}_+$ in L^∞. By (8.1) and the notation introduced after (8.3) for $\tilde{u}^{\rho}_{+,\Lambda}$, $\tilde{u}^{\rho}_{+,\Lambda^c}$, it suffices to show

$$\|\tilde{u}^{\rho}_{+,\Lambda}\|_{L^\infty} \le \frac{D}{4}t^{-\frac{1}{2}}(\varepsilon^2\sqrt{t})^{\theta'}, \tag{8.34}$$

$$\|\tilde{u}^{\rho}_{+,\Lambda^c}\|_{L^\infty} \le \frac{D}{4}t^{-\frac{1}{2}}(\varepsilon^2\sqrt{t})^{\theta'}. \tag{8.35}$$

By (8.4) and a priori estimate (7.3), one may bound (8.35) by $Ct^{-\frac{1}{2}+\sigma}(\varepsilon^2\sqrt{t})^{\theta}$. Since $\theta' < \theta$ and $t \le \varepsilon^{-4+c}$, we bound this by the quantity $Ct^{-\frac{1}{2}}(\varepsilon^2\sqrt{t})^{\theta'}e(t,\varepsilon)$, where e satisfies (5.41), if σ has been taken small enough relatively to $c(\theta-\theta')$.

We are left with estimating (8.34). It is equivalent to show that

$$\|\underline{\tilde{u}}^{\rho}_{+,\Lambda}\|_{L^\infty} \le \frac{D}{4}(\varepsilon^2\sqrt{t})^{\theta'}$$

if ε is small enough. Computing $\partial_t|\underline{\tilde{u}}^{\rho}_{+,\Lambda}(t,x)|^2$ from (8.25) and integrating in time, we get

$$|\underline{\tilde{u}}^{\rho}_{+,\Lambda}(t,x)| \leq |\underline{\tilde{u}}^{\rho}_{+,\Lambda}(1,x)| + C\int_1^t \tau^{-1+\sigma}(\varepsilon^2\sqrt{\tau})^{\theta}\,d\tau.$$

If D has been taken large enough so that $\|\underline{\tilde{u}}^{\rho}_{+,\Lambda}(1,\cdot)\|_{L^\infty} \leq \frac{D}{8}\varepsilon$, we get the wanted estimate, using again that $t \leq \varepsilon^{-4+c}$ and that σ may be taken small relatively to $c(\theta-\theta')$. This concludes the proof. ■

Propositions 7.3.7 and 8.2.1 allowed us to bootstrap estimates (7.3). To be able to finish the proof of the main theorem, we shall have to bootstrap as well the inequalities satisfied by g. We prove first some technical lemmas.

Proposition 8.2.2. *Let Z be a function in $\mathcal{S}(\mathbb{R})$. Assume that the function $\tilde{u}_+$ satisfies estimate (7.3). For any neighborhood $\mathcal{W}$ of $\{-1,1\}$ in $\mathbb{R}$, there is $\varepsilon_0 > 0$ (depending only on $\mathcal{W}$ and on the constants in (7.3)) such that for any λ in $\mathbb{R}-\mathcal{W}$, there are functions $\varphi_\pm(\lambda,t)$, $\psi_\pm(\lambda,t)$ defined for $t \in [1,\varepsilon^{-4+c}]$, $\varepsilon \in]0,\varepsilon_0]$, satisfying the estimates*

$$|\varphi_\pm(\lambda,t)| \leq t^{-\frac{1}{2}}(\varepsilon^2\sqrt{t})^{\theta'}, \tag{8.36}$$

$$|\psi_\pm(\lambda,t)| \leq t^{-1}(\varepsilon^2\sqrt{t})^{\theta'} \tag{8.37}$$

and solving the equation

$$(D_t-\lambda)\varphi_\pm(\lambda,t) = \langle Z,\tilde{u}_\pm\rangle + \psi_\pm(\lambda,t). \tag{8.38}$$

Moreover, denoting $\langle Z,\tilde{u}\rangle$ for the vector $\left[\begin{smallmatrix}\langle Z,\tilde{u}_+\rangle\\ \langle Z,\tilde{u}_-\rangle\end{smallmatrix}\right]$, one has the bound

$$|\langle Z,\tilde{u}\rangle| \leq t^{-\frac{3}{4}}(\varepsilon^2\sqrt{t})^{\theta'}. \tag{8.39}$$

Proof. We shall use the following notation: we set $f = o(g)$ when we may write $|f| \leq |g|e(t,\varepsilon)$ for some $e(t,\varepsilon)$ satisfying (5.41). In particular, for any given N, taking ε small enough, we may bound $|f|$ by $\frac{1}{N}|g|$.

We prove the proposition in the case of sign $+$. Let us show first that on the right-hand side of (8.38), we may replace $\langle Z,\tilde{u}_+\rangle$ by $\langle Z(C(t)\tilde{u})_+\rangle$, up to a contribution to ψ_+. Since $((\mathrm{Id}-C(t))\tilde{u})_+$ is odd, and Z is in $\mathcal{S}$, we may use (4.79) to write

$$\begin{aligned}\langle Z,((\mathrm{Id}-C(t))\tilde{u})_+\rangle &= \frac{1}{t}\int_{-1}^{1}\langle Z^1,(L(\mathrm{Id}-C(t))\tilde{u})_+(\mu x)\rangle\,d\mu\\ &\quad -\frac{1}{t}\int_{-1}^{1}\langle Z^2,((\mathrm{Id}-C(t))\tilde{u})_+(\mu x)\rangle\mu\,d\mu\end{aligned} \tag{8.40}$$

for new functions Z^1, Z^2 in $\mathcal{S}(\mathbb{R})$. By (7.3) and L^2 boundedness of $C(t)$, the last term is $O(\varepsilon t^{\delta-1}) = o((\varepsilon^2\sqrt{t})^{\theta'}t^{-1})$. It may thus be integrated to $\psi_+(\lambda,t)$. In the first term on the right-hand side of (8.40) we write using (E.20)

$$L(\mathrm{Id}-C(t))\tilde{u} = (\mathrm{Id}-\tilde{C}(t))L\tilde{u} + \tilde{C}_1(t)\tilde{u}.$$

By (E.21), (E.22) and (7.3), we get

$$\|L(\mathrm{Id}-C(t))\tilde{u}\|_{L^2} \leq C(\varepsilon^2\sqrt{t})^{\theta'}\big[\varepsilon^{\iota}t^{-m+\frac{1}{2}+\delta'}(\varepsilon^2\sqrt{t})^{\theta-\theta'} + \varepsilon^{1+\iota-2\theta'}t^{\frac{1}{2}-m+\delta-\frac{\theta'}{2}}\big]. \tag{8.41}$$

As θ, θ' are fixed with $\theta' < \theta < \frac{1}{2}$ and θ' close to $\frac{1}{2}$, and as $\delta', \frac{1}{2}-m$ may be taken as small as we want, the bracket above is $o(1)$ when $t \leq \varepsilon^{-4+c}$ and ε goes to zero. Thus (8.41) plugged in the first term on the right-hand side of (8.40) shows that this term is $o(t^{-1}(\varepsilon^2\sqrt{t})^{\theta'})$, so satisfies (8.37). We are thus reduced to studying equation

$$(D_t-\lambda)\varphi_+(\lambda,t) = \langle Z, (C(t)\tilde{u})_+\rangle + \psi_+(\lambda,t). \tag{8.42}$$

Recall the function $\mathring{u}$ defined in (7.106). We may write

$$\begin{aligned}\langle Z, (C(t)\tilde{u})_+\rangle &= \langle Z, \mathring{u}_+\rangle + \psi_1(t),\\ \psi_1(t) &= \langle Z, (\hat{\mathcal{M}}_2'(\tilde{u}, u'^{\mathrm{app},1}))_+\rangle + \sum_{j=3}^{4}\langle Z, (C(t)\hat{\mathcal{M}}_j(\tilde{u}, u'^{\mathrm{app},1}))_+\rangle.\end{aligned} \tag{8.43}$$

By (7.5), we may bound the last sum by

$$Ct^{-1}(\varepsilon^2\sqrt{t})^{\theta'}\big(t^{\delta}(\varepsilon^2\sqrt{t})^{\theta'}\varepsilon + \varepsilon^{5-2\theta'}t^{1-\frac{\theta'}{2}+\delta+\sigma}\big).$$

As $t \leq \varepsilon^{-4+c}$, this is smaller than the right-hand side of (8.37) (for δ, σ small).

Let us show that the first term on the right-hand side of the expression of ψ_1 satisfies also (8.37). It suffices to show that $\|\hat{\mathcal{M}}_2'(\tilde{u}, u'^{\mathrm{app},1})\|_{L^2} = o(t^{-1}(\varepsilon^2\sqrt{t})^{\theta'})$. Recall that $\hat{\mathcal{M}}_2'(\tilde{u}, u'^{\mathrm{app},1})$ is given by (6.60) in terms of expressions $\hat{\mathcal{M}}_2'^{\ell}$, that have structure (6.47), i.e. that may be written from expressions

$$t^{-\frac{3}{2}}K^{\ell_1,\ell_2}\big(L_{\pm}^{\ell_1}f_{1,\pm}, L_{\pm}^{\ell_2}f_{2,\pm}\big), \tag{8.44}$$

where $0 \leq \ell_1, \ell_2 \leq 1$, K^{ℓ_1,ℓ_2} is in $\mathcal{K}'_{1,1/2}(1,\pm,\pm)$ and f_1, f_2 equal to $\tilde{u}$ or $u'^{\mathrm{app},1}$ (see (F.35)). If we apply (F.47), (F.50), (F.51), we obtain a bound for the L^2 norm of (8.44) in

$$Ct^{-\frac{3}{2}-\frac{1}{4}+\sigma}\big(\|L_+\tilde{u}_+\|_{L^2} + \|L_+u'^{\mathrm{app},1}_+\|_{L^2} + \|\tilde{u}_+\|_{H^s} + \|u'^{\mathrm{app},1}_+\|_{H^s}\big)^2$$

so according to (7.3) and (7.4) by

$$Ct^{-\frac{3}{2}+\sigma}(\varepsilon^2\sqrt{t})^{\theta}t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^{\theta}$$

which is better than (8.37). On the right-hand side of (8.42), up to incorporating ψ_1 to ψ_+, we thus may replace $\langle Z, (C(t)\tilde{u})_+\rangle$ by $\langle Z, \mathring{u}_+\rangle$, i.e. we reduced equation (8.42) to

$$(D_t-\lambda)\varphi_+(\lambda,t) = \langle Z, \mathring{u}_+\rangle + \psi_+ \tag{8.45}$$

for a new ψ_+. Since $\mathring{u}_+$ is odd and Z in $\mathcal{S}(\mathbb{R})$, we may write using (4.79) again

$$\langle Z, \mathring{u}_+\rangle = \frac{1}{t}\int_{-1}^{1}\langle Z^1, (L_+\mathring{u}_+)(\mu\cdot)\rangle\, d\mu - \frac{1}{t}\int_{-1}^{1}\langle Z^2, \mathring{u}_+(\mu\cdot)\rangle \mu\, d\mu \tag{8.46}$$

for new functions Z^1, Z^2 in the space $\mathcal{S}(\mathbb{R})$. By inequality (7.110), the last term is $O(\varepsilon t^{\delta-1}) = o((\varepsilon^2\sqrt{t})^{\theta'} t^{-1})$. It may thus be incorporated to $\psi_+(\lambda, t)$. We decompose the first integral on the right-hand side of (8.46) as $I_1 + I_2$, with

$$\begin{aligned} I_2 &= \int_{-1}^{1}\Big\langle Z^1, \Big(\chi\Big(\sqrt{t}(\lambda - \sqrt{1+D_x^2}\Big)\Big)(L_+\mathring{u}_+)\Big)(\mu\cdot)\Big\rangle\, d\mu \\ &= \int_{-1}^{1}\Big\langle \chi\Big(\sqrt{t}(\lambda - \sqrt{1+D_x^2}\Big)\Big)\Big(Z^1\Big(\frac{\cdot}{\mu}\Big)\Big), L_+\mathring{u}_+\Big\rangle \frac{d\mu}{\mu}, \end{aligned} \tag{8.47}$$

where $\chi \in C_0^\infty(\mathbb{R})$ is real valued, equal to one close to zero. By Cauchy–Schwarz,

$$|I_2| \le \int_{-1}^{1}\Big\|\chi\Big(\sqrt{t}(\lambda - \sqrt{1+D_x^2}\Big)\Big)\Big(Z^1\Big(\frac{\cdot}{\mu}\Big)\Big)\Big\|_{L^2}\frac{d\mu}{\mu}\|L_+\mathring{u}_+\|_{L^2}. \tag{8.48}$$

Since $\lambda \notin \mathcal{W}$, $\|\chi(\sqrt{t}(\lambda - \sqrt{1+\xi^2}))\|_{L^2(d\xi)} = O(t^{-\frac{1}{4}})$, so that the L^2 norm inside the above integral is bounded by

$$Ct^{-\frac{1}{4}}\Big\|Z^1\Big(\frac{\cdot}{\mu}\Big)\Big\|_{L^1} = O(\mu C t^{-\frac{1}{4}}).$$

By (7.111), it follows that the contribution of I_2 to the first term in (8.46) satisfies (8.37), so may be incorporated to ψ_+. We have thus written by (8.40) and (8.46)

$$\langle Z, \mathring{u}_+\rangle = \frac{1}{t}I_1 + \psi_+^1, \tag{8.49}$$

where ψ_+^1 satisfies the same estimates as ψ_+ (with an arbitrary small multiplicative constant on the right-hand side) and

$$I_1 = \int_{-1}^{1}\Big\langle Z^1, \Big((1-\chi)\Big(\sqrt{t}(\lambda - \sqrt{1+D_x^2}\Big)\Big)(L_+\mathring{u}_+)\Big)(\mu\cdot)\Big\rangle\, d\mu. \tag{8.50}$$

We thus reduced (8.45) to

$$(D_t - \lambda)\varphi_+(\lambda, t) = \frac{1}{t}I_1 + \psi_+(\lambda, t) \tag{8.51}$$

for a new ψ_+. We define

$$\begin{aligned} \varphi_+(\lambda, t) = \frac{1}{t}\int_{-1}^{1}\Big\langle Z^1, \Big(\frac{(1-\chi)(\sqrt{t}(\lambda - \sqrt{1+D_x^2}))}{\sqrt{1+D_x^2} - \lambda}L_+\mathring{u}_+\Big)(\mu\cdot)\Big\rangle\, d\mu \\ + = \frac{1}{\sqrt{t}}\int_{-1}^{1}\Big\langle \chi_1\Big(\sqrt{t}(\lambda - \sqrt{1+D_x^2}\Big)\Big)\Big(Z^1\Big(\frac{\cdot}{\mu}\Big)\Big), L_+\mathring{u}_+\Big\rangle \frac{d\mu}{\mu}, \end{aligned} \tag{8.52}$$

where $\chi_1(z) = \frac{\chi(z)-1}{z}$. Arguing as in (8.48) and using inequality (7.111), we obtain that $\varphi_+(\lambda,t)$ satisfies (8.36). If we compute $(D_t-\lambda)\varphi_+(\lambda,t)$, we get the following terms:

$$\frac{i}{t}\varphi_+(\lambda,t), \tag{8.53}$$

$$\frac{1}{t}\int_{-1}^{1}\Big\langle Z^1, \Big(\frac{(1-\chi)\big(\sqrt{t}(\lambda-\sqrt{1+D_x^2})\big)}{\sqrt{1+D_x^2}-\lambda}(D_t-p(D_x))L_+\mathring{u}_+\Big)(\mu\cdot)\Big\rangle\, d\mu, \tag{8.54}$$

$$\frac{1}{t}I_1(t), \tag{8.55}$$

$$-\frac{i}{2t^{\frac{3}{2}}}\int_{-1}^{1}\Big\langle Z^1, \Big(\chi'(\sqrt{t}(\lambda-\sqrt{1+D_x^2}))L_+\mathring{u}_+\Big)(\mu\cdot)\Big\rangle\, d\mu. \tag{8.56}$$

According to (8.51), we shall have proved (8.38) (in the case of sign $+$) if we show that (8.53), (8.54), (8.56) satisfy estimates (8.37), with a small constant in front of the right-hand side of this inequality. For (8.53), this follows from (8.52) and (8.36). We may rewrite (8.54) as

$$\frac{1}{\sqrt{t}}\int_{-1}^{1}\Big\langle \chi_1\Big(\sqrt{t}(\lambda-\sqrt{1+D_x^2})\Big)\Big(Z^1\Big(\frac{\cdot}{\mu}\Big)\Big), (D_t-\sqrt{1+D_x^2})L_+\mathring{u}_+\Big\rangle \frac{d\mu}{\mu}.$$

Arguing as in (8.48), we estimate that by

$$Ct^{-\frac{3}{4}}\|(D_t-\sqrt{1+D_x^2})L_+\mathring{u}_+\|_{L^2}.$$

Since L_+ commutes to $(D_t-\sqrt{1+D_x^2})$, it follows from (7.105) and (7.109) that this is bounded by

$$t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^\theta e(t,\varepsilon) = o(t^{-1}(\varepsilon^2\sqrt{t})^{\theta'})$$

which implies an estimate of the form (8.37). Finally, (8.56) is bounded by

$$Ct^{-\frac{3}{2}}\int_{-1}^{1}\Big\|\chi'\Big(\sqrt{t}(\lambda-\sqrt{1+D_x^2})\Big)\Big(Z^1\Big(\frac{\cdot}{\mu}\Big)\Big)\Big\|_{L^2}\|L_+\mathring{u}_+\|_{L^2}\frac{d\mu}{\mu} \le Ct^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^\theta$$

according to (7.111). This is again better than needed.

Finally, estimate (8.39) follows from (8.40) (that is bounded by (8.37)), (8.43), the fact that ψ_1 is $o(t^{-1}(\varepsilon^2\sqrt{t})^{\theta'})$, (8.46) were we plug (7.110) and (7.111). This concludes the proof. ■

Our next task will be to show that a priori assumptions (7.1)–(7.3) imply that inequalities (4.92)–(4.93) that we assume in Section 4.2 in order to get estimates for the solution of the ODE (4.94), hold.

Lemma 8.2.3. *Assume that estimates* (7.1)–(7.3) *hold. Then inequality* (4.92) *is true, with a constant B' depending only on the constants A, A', D in* (7.1)–(7.3).

Proof. We divide the proof into two steps.

Step 1. Consider first the contribution Φ_2 on the left-hand side of (4.92). Recall that Φ_2 is given by (2.36), (2.38) so may be written as a sum of terms

$$\iint e^{ix(\xi_1+\xi_2)} m'(x,\xi_1,\xi_2)\hat{u}_\pm(\xi_1)\hat{u}_\pm(\xi_2)\,d\xi_1\,d\xi_2\,dx \tag{8.57}$$

with

$$m'(x,\xi_1,\xi_2) = \kappa(x)Y(x)b(x,\xi_1)b(x,\xi_2)p(\xi_1)^{-1}p(\xi_2)^{-1}.$$

By estimates (A.8) satisfied by b, and the fact that Y is in $\mathcal{S}(\mathbb{R})$, we have that m' belongs to $\tilde{S}'_{0,0}(\prod_{j=1}^2 \langle \xi_j \rangle^{-1}, 2)$ and Φ_2 is thus a sum of expressions

$$\int \mathrm{Op}(m')(u_\pm, u_\pm)\,dx.$$

On the other hand, recall that u_+ is related to $\tilde{u}_+$ by (5.59), with a remainder R satisfying (5.25) and (5.26). By Corollary B.2.6, we get that (8.57) may be written as a sum of expressions

$$\int \mathrm{Op}(\tilde{m}')(v_1,\dots,v_n)\,dx \tag{8.58}$$

where $n \geq 2$ and v_j is equal to $u'^{\mathrm{app}}_\pm$ or $u''^{\mathrm{app}}_\pm$, or $\tilde{u}_\pm$ or R, with a symbol $\tilde{m}'$ in $\tilde{S}'_{1,0}(\prod_{j=1}^2 \langle \xi_j \rangle^{-1} M_0^\nu, 2)$ for some ν.

Consider first the case when at least one of the arguments v_j, say the last one, is not equal to $u''^{\mathrm{app}}_\pm$. Since $\tilde{m}'$ is rapidly decaying as $\langle M_0(\xi)^{-1}|y|\rangle^{-N}$, we may estimate (8.58) from the L^2 norm of the integrand. If $n = 2$, we use (D.76) when v_1 is different from $u''^{\mathrm{app}}_\pm$ and (D.75) if $v_1 = u''^{\mathrm{app}}_\pm$. We obtain for (8.58) a bound in

$$\begin{aligned} Ct^{-2+\sigma}\big(&\|L_+\tilde{u}_+\|_{L^2} + \|L_+u'^{\mathrm{app}}_+\|_{L^2} + \|L_+R\|_{L^2} + \|\tilde{u}_+\|_{H^s} + \|u'^{\mathrm{app}}_+\|_{H^s} \\ &+ \|R\|_{H^s} + \|L_+u''^{\mathrm{app}}_+\|_{W^{\rho_0,\infty}} + \|u''^{\mathrm{app}}_+\|_{W^{\rho_0,\infty}}\big) \\ \times \big(&\|L_+\tilde{u}_+\|_{L^2} + \|L_+u'^{\mathrm{app}}_+\|_{L^2} + \|L_+R\|_{L^2} + \|\tilde{u}_+\|_{H^s} \\ &+ \|u'^{\mathrm{app}}_+\|_{H^s} + \|R\|_{H^s}\big). \end{aligned} \tag{8.59}$$

We plug there (7.1)–(7.3) and (5.25)–(5.26). We obtain a bound in $t^{-\frac{3}{2}+\sigma}(\varepsilon^2\sqrt{t})^{2\theta}$. As $\theta > \theta'$ and $t \leq \varepsilon^{-4+c}$, we see that if σ is small enough, this is smaller than the right-hand side of (4.92).

If $n \geq 3$ in (8.58), and again at least one v_j, say the last one, is different from $u''^{\mathrm{app}}_\pm$, we use Corollary D.2.8. By (D.71), we estimate then (8.58) by

$$\begin{aligned} Ct^{-1}\big(&\|u'^{\mathrm{app}}_+\|_{W^{\rho_0,\infty}} + \|u''^{\mathrm{app}}_+\|_{W^{\rho_0,\infty}} + \|\tilde{u}_+\|_{W^{\rho_0,\infty}} + \|R_+\|_{W^{\rho_0,\infty}}\big)^{n-1} \\ &\times \big(\|L_+\tilde{u}_+\|_{L^2} + \|L_+u'^{\mathrm{app}}_+\|_{L^2} + \|L_+R\|_{L^2} + \|\tilde{u}_+\|_{L^2} \\ &\quad + \|u'^{\mathrm{app}}_+\|_{L^2} + \|R\|_{L^2}\big). \end{aligned}$$

Using (7.1)–(7.3) and (5.25) (together with Sobolev injection), (5.26), we get a bound in $t^{-2}(\varepsilon^2\sqrt{t})^{2\theta'}(\varepsilon^2\sqrt{t})^{\theta}t^{\frac{1}{4}}$, which is better than what we want.

It remains to study (8.58) when all arguments v_j are equal to $u''^{\mathrm{app}}_{\pm}$. Again by the rapid decay in x of the symbol $\tilde{m}'$, it is enough to control the L^∞ norm of the integrand (up to changing the definition of $\tilde{m}'$). We may use then (D.77) with $n = \ell \geq 2$. We obtain a bound in

$$t^{-2+\sigma}\big(\|u''^{\mathrm{app}}_{+}\|_{W^{\rho_0,\infty}} + \|L_+ u''^{\mathrm{app}}_{+}\|_{W^{\rho_0,\infty}} + t^{-\frac{1}{2}}\|u''^{\mathrm{app}}_{+}\|_{H^s}\big)^2. \tag{8.60}$$

Using (7.2) and the fact that $\theta' < \frac{1}{2}$, $\sigma \ll 1$, one controls that by $t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{2\theta'}$ for $t \leq \varepsilon^{-4+c}$. This concludes the proof of (4.92) for contribution Φ_2.

Step 2. We study next the term $t_\varepsilon^{-\frac{3}{2}+\frac{j}{2}}\Gamma_j(u_+, u_-)$ in (4.92), for $1 \leq j \leq 3$. Recall that Γ_j is given by (2.36)–(2.39). It has thus again the structure (8.58) with $n = j$, as it follows from the expression (5.59) of u_+ in terms of u^{app}_{+}, $\tilde{u}_+$, R and the composition results of Appendix B. If $j \geq 2$, our preceding reasoning implies the wanted bound. We thus just have to consider

$$t_\varepsilon^{-1}\int \mathrm{Op}(\tilde{m}')(v)\,dv \tag{8.61}$$

with $\tilde{m}'$ in $\tilde{S}'_{1,0}(\langle\xi\rangle^{-1}, 1)$ and $v = u'^{\mathrm{app}}_{\pm}, u''^{\mathrm{app}}_{\pm}, \tilde{u}_\pm, R$. When v is not equal to $u''^{\mathrm{app}}_{\pm}$, we use (D.71) in order to bound (8.61) by

$$\begin{aligned} C t_\varepsilon^{-1} t^{-1}\big(&\|L_+ u'^{\mathrm{app}}_{+}\|_{L^2} + \|L_+\tilde{u}_+\|_{L^2} + \|L_+ R\|_{L^2} \\ &+ \|u'^{\mathrm{app}}_{+}\|_{L^2} + \|\tilde{u}_+\|_{L^2} + \|R\|_{L^2}\big) \end{aligned}$$

which by (7.1)–(7.3) and (5.25)–(5.26) is bounded from above by $t_\varepsilon^{-1}t^{-1}(\varepsilon^2\sqrt{t})^{\theta}t^{\frac{1}{4}}$. One checks that this quantity is $O(t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{2\theta'})$ using $\theta' < \theta < \frac{1}{2}$.

If v in (8.61) is equal to $u''^{\mathrm{app}}_{\pm}$, we bound (8.61) by

$$C t_\varepsilon^{-1}\|\mathrm{Op}(\tilde{m}')v\|_{L^\infty}$$

(for a new symbol $\tilde{m}'$). We use (D.77) to get a bound in

$$t_\varepsilon^{-1} t^{-1+\sigma}\big[\|u''^{\mathrm{app}}_{+}\|_{W^{\rho_0,\infty}} + \|L_+ u''^{\mathrm{app}}_{+}\|_{W^{\rho_0,\infty}} + t^{-\frac{1}{2}}\|u''^{\mathrm{app}}_{+}\|_{H^s}\big]. \tag{8.62}$$

Using (7.2), one bounds the bracket by $t^{\sigma'}t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^{\frac{1}{2}}$ for any $\sigma' > 0$. As $t \leq \varepsilon^{-4+c}$, one concludes that if σ, σ' are small enough, (8.62) is $O(t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{2\theta'})$. This concludes the proof of the lemma. ∎

We show next that a priori assumptions (7.1)–(7.3) imply as well estimates (4.93).

Lemma 8.2.4. *Assume that estimates* (7.1)–(7.3) *hold true. Then inequality* (4.93) *holds true with a constant B' depending only on A, A', D in* (7.1)–(7.3).

Proof. Recall that $\Phi_1(u_+, u_-)$ is given by (2.36), i.e. taking (2.37) into account, by

$$\frac{\sqrt{3}}{3}\langle Y, Y(x)\kappa(x)b(x, D_x)p(D_x)^{-1}(u_+ - u_-)\rangle. \tag{8.63}$$

Expressing u_+ using (5.59), we get that, if we define

$$Z = \frac{\sqrt{3}}{3} p(D_x)^{-1} b(x, D_x)^* (\kappa(x) Y(x)^2),$$

the term inside the modulus on the left-hand side of (4.93) may be written as the sum of an expression $\langle Z, R\rangle$ with R satisfying (5.25) and of expressions of the form (8.58) with $n \geq 2$. We have seen that these last quantities may be bounded by (8.59) or (8.60), and thus by the right-hand side of (4.93). On the other hand, by (5.25) $\langle Z, R\rangle$ is also $O(t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{2\theta'})$. This concludes the proof. ■

Corollary 8.2.5. *Assume that estimates (7.1)–(7.3) hold true. Then Assumption (H_1') of Section 4.2 holds.*

Proof. We have seen that by Lemmas 8.2.3 and 8.2.4, inequalities (4.92) and (4.93) hold. It remains to check that for any $\lambda \in \mathbb{R} - \{-1, 1\}$, there are functions $\varphi_\pm(\lambda, t)$, $\psi_\pm(\lambda, t)$ as at the end of the statement of condition (H_1'). But this is exactly the statement of Proposition 8.2.2. ■

8.3 End of bootstrap argument

We give here the proof of Theorem 2.1.1. We shall have to gather all estimates we proved in the preceding chapters. We first restate the main estimates in Theorem 2.1.1.

Proposition 8.3.1. *There is ρ_0 in $\mathbb{N}$ and for any $\rho \geq \rho_0$, any $c \in]0, 1[$, any $\theta' \in]0, \frac{1}{2}[$ close to $\frac{1}{2}$, any large enough $N \in \mathbb{N}$, there are $\varepsilon_0 > 0$, $C > 0$ such that if $0 < \varepsilon < \varepsilon_0$, the solution φ of equation (2.11) with odd initial conditions with bounds (2.10) satisfies for $t \in [1, \varepsilon^{-4+c}]$ the following estimates (using notation (2.7) and (2.8)):*

$$\begin{aligned} \|P_{\mathrm{ac}}\varphi(t,\cdot)\|_{W^{\rho,\infty}} &\leq C t^{-\frac{1}{2}} (\varepsilon^2\sqrt{t})^{\theta'}, \\ \|\langle x\rangle^{-2N} P_{\mathrm{ac}}\varphi(t,\cdot)\|_{W^{\rho,\infty}} &\leq C t^{-\frac{3}{4}} (\varepsilon^2\sqrt{t})^{\theta'}, \\ \|\langle x\rangle^{-2N} D_t P_{\mathrm{ac}}\varphi(t,\cdot)\|_{W^{\rho-1,\infty}} &\leq C t^{-\frac{3}{4}} (\varepsilon^2\sqrt{t})^{\theta'} \end{aligned} \tag{8.64}$$

and $a(t)$ may be written as $a(t) = e^{it\frac{\sqrt{3}}{2}} g_+(t) - e^{-it\frac{\sqrt{3}}{2}} g_-(t)$ with

$$\begin{aligned} |g_\pm(t)| &\leq C\varepsilon(1 + t\varepsilon^2)^{-\frac{1}{2}}, \\ |\partial_t g_\pm(t)| &\leq C\varepsilon t^{-\frac{1}{2}} (1 + t\varepsilon^2)^{-\frac{1}{2}}. \end{aligned} \tag{8.65}$$

Proof. Recall that we have defined in (2.18) and (2.19)

$$w = b(x, D_x)^* P_{\mathrm{ac}}\varphi, \quad P_{\mathrm{ac}}\varphi = b(x, D_x) w. \tag{8.66}$$

We have introduced in (2.24)

$$u_+ = (D_t + p(D_x)) w. \tag{8.67}$$

We shall prove the following inequalities, where the last two ones are just the restatement of (8.65):

$$\begin{aligned}\|u_+(t,\cdot)\|_{W^{\rho,\infty}} &\le Ct^{-\frac{1}{2}}(\varepsilon^2\sqrt{t})^{\theta'},\\ \|u_+(t,\cdot)\|_{H^s} &\le C\varepsilon t^{\delta}\end{aligned}\tag{8.68}$$

and

$$\begin{aligned}|g_\pm(t)| &\le C\varepsilon(1+t\varepsilon^2)^{-\frac{1}{2}},\\ |\partial_t g_\pm(t)| &\le C\varepsilon t^{-\frac{1}{2}}(1+t\varepsilon^2)^{-\frac{1}{2}}.\end{aligned}\tag{8.69}$$

We shall deduce these estimates from bounds on $\tilde{u}_+$ that we establish by bootstrap of (7.3). Actually, let us show that if (7.3) holds on some interval $[1,T]$ with $T \le \varepsilon^{-4+c}$ with a constant D, then it still holds with D replaced by $\frac{D}{2}$, as soon as D has been fixed large enough, and ε smaller than some ε_0 (depending on D). Proposition 7.3.7 shows that this statement holds for the Sobolev and L^2 estimate as soon as bounds (7.1), (7.2), (7.4) hold true (with constants A, A' that may depend on D). By Proposition 8.2.1, the $W^{\rho,\infty}$ estimate of $\tilde{u}_+$ may also be bootstrapped.

Let us next show that we may bootstrap as well estimate (4.99) on g. According to Proposition 4.2.1, we may do so as soon as Assumption (H_1') holds true. By Corollary 8.2.5, this follows under a priori conditions (7.1)–(7.3). Property (7.3) is the bootstrap assumption. On the other hand, (7.1), (7.2), (7.4) hold, for convenient constants $C(A,A')$ by Proposition 4.1.2 as soon as (4.3)–(4.7) hold. The first of these inequalities is the bootstrap assumption (4.99) on g. The other ones are (8.36)–(8.39), that, according to Proposition 8.2.2, hold under the bootstrap assumption (7.3).

Let us now deduce (8.68) from estimates (7.1)–(7.3) and (4.3), that hold on $[1,\varepsilon^{-4+c}]$ for ε small, according to our bootstrap assumption. Recall that u_+ is given by (5.59) (or (5.24)) by

$$u_+ = u'^{\text{app}}_+ + u''^{\text{app}}_+ + \tilde{u}_+ + \sum_{\substack{2\le|I|\le 4\\ I=(I',I'')}} \mathrm{Op}(\tilde{m}_I)(\tilde{u}_{I'}, u^{\text{app}}_{I''}) + R, \tag{8.70}$$

where R satisfies (5.25). This (and Sobolev injection) shows that R satisfies better bounds than those given by (8.68). By (7.1)–(7.3), the first three terms in (8.70) satisfy also the wanted bounds. Finally, the terms in the sum are also estimated by these bounds using (7.1)–(7.3) and (D.32), (D.39).

Let us check inequalities (8.69). Recall that $a(t) = \frac{\sqrt{3}}{3}(a_+(t) - a_-(t))$, where $a_- = -\overline{a_+}$ and a_+ is given by (4.96). We set then, using notation (4.97) and (4.98),

$$g_+(t) = \frac{\sqrt{3}}{3}e^{-it\frac{\sqrt{3}}{2}}(a^{\text{app}}_+(t) + S(t)) \tag{8.71}$$

and $g_-(t) = -\overline{g_+(t)}$. It follows from the expressions of a^{app}_+, S and (4.97)–(4.101) that

$$g_+(t) = O\big(t_\varepsilon^{-\frac{1}{2}}\big), \quad \partial_t g_+(t) = O\big(t_\varepsilon^{-\frac{1}{2}}t^{-\frac{1}{2}}\big).$$

It remains to prove (8.64). By (2.19) and (2.24),

$$P_{\mathrm{ac}}\varphi = b(x, D_x)w = \frac{1}{2}b(x, D_x)p(D_x)^{-1}(u_+ - u_-). \tag{8.72}$$

By Proposition D.1.5, the operator $b(x, D_x)p(D_x)^{-1}\langle D_x\rangle^{-\alpha}$ is bounded on $W^{\rho',\infty}$ if $\alpha > 0$. It follows that the first estimate (8.64) follows from (8.68) if we modify the value of ρ on the left-hand side of (8.64).

To obtain the weighted estimates in (8.64), let us write from (8.72) and (2.24)

$$\langle x\rangle^{-2N} P_{\mathrm{ac}}\varphi = \frac{1}{2}\langle x\rangle^{-2N} b(x, D_x)p(D_x)^{-1}(u_+ - u_-), \tag{8.73}$$

$$\langle x\rangle^{-2N} D_t P_{\mathrm{ac}}\varphi = \frac{1}{2}\langle x\rangle^{-2N} b(x, D_x)(u_+ + u_-). \tag{8.74}$$

On the right-hand side of (8.73), we replace u_+ by its expression (8.70). We have to bound the following quantities:

$$\begin{aligned}&\|\langle x\rangle^{-2N} b(x, D_x)p(D_x)^{-1}u'^{\mathrm{app}}_+\|_{W^{\rho,\infty}},\\ &\|\langle x\rangle^{-2N} b(x, D_x)p(D_x)^{-1}\tilde{u}_+\|_{W^{\rho,\infty}},\end{aligned} \tag{8.75}$$

$$\|\langle x\rangle^{-2N} b(x, D_x)p(D_x)^{-1}u''^{\mathrm{app}}_+\|_{W^{\rho,\infty}}, \tag{8.76}$$

$$\sum_{\substack{2\le|I|\le4\\ I=(I',I'')}} \|\langle x\rangle^{-2N} b(x, D_x)p(D_x)^{-1}\mathrm{Op}(m_I)(\tilde{u}_{I'}, u^{\mathrm{app}}_{I''})\|_{W^{\rho,\infty}}, \tag{8.77}$$

$$\|\langle x\rangle^{-2N} b(x, D_x)p(D_x)^{-1}R\|_{W^{\rho,\infty}}. \tag{8.78}$$

If $N = 2$, the assumptions of Proposition D.2.5 with $n = 1$ are satisfied. We may thus apply Corollary D.2.11 with $\ell = 0$. Taking into account (7.1) and (7.3), we obtain for (8.75) a bound in

$$t^{-\frac{3}{4}+\sigma}(\varepsilon^2\sqrt{t})^{\theta} + t^{-1}\frac{(\varepsilon^2\sqrt{t})^{\theta'}}{\sqrt{t}}.$$

For (8.76), we apply also Corollary D.2.11, but with $\ell = 1$. We obtain by (7.2) a bound in

$$t^{-1+\sigma}\log(1+t)\log(1+t\varepsilon^2) = O\big(t^{-\frac{3}{4}+2\sigma}(\varepsilon^2\sqrt{t})^{\frac{1}{2}}\big).$$

modulo a bound in $t^{-1}\frac{(\varepsilon^2\sqrt{t})^{\theta'}}{\sqrt{t}}$. To estimate (8.77), we use again Corollary D.2.11, with $n = |I|$ and ℓ equal to the number of arguments equal to $u''^{\mathrm{app}}_{\pm}$, $n - \ell$ equal to the number of arguments equal to $\tilde{u}_{\pm}$ or $u'^{\mathrm{app}}_{\pm}$. If N is taken large enough, we get better estimates than those holding for (8.75) and (8.76). Finally, Sobolev injection and (5.25) provide for (8.78) a better upper bound than the one in (8.64). We thus got estimates of $\|\langle x\rangle^{-N} P_{\mathrm{ac}}\varphi(t,\cdot)\|_{W^{\rho,\infty}}$ in $t^{-\frac{3}{4}}(\varepsilon^2\sqrt{t})^{\theta'}$ since σ is as small as we want, $t \le \varepsilon^{-4+c}$, and $\theta < \frac{1}{2}$. This implies the second inequality of (8.64).

The proof of the last inequality (8.64) is similar, starting from (8.74). ■

Appendix A

Scattering for time independent potential

This appendix is devoted to the construction of wave operators for a Schrödinger operator of the form

$$A = -\frac{1}{2}\frac{d^2}{dx^2} + V(x),$$

where V is a real-valued potential in $\mathcal{S}(\mathbb{R})$. If W_+ stands for the wave operator defined by (A.5) below, one knows that $W_+ W_+^* = P_{\text{ac}}$, $W_+^* W_+ = \text{Id}_{L^2}$, where P_{ac} is the spectral projector associated to the absolutely continuous spectrum of A. Moreover, one has the intertwining property

$$W_+^* A W_+ = -\frac{1}{2}\frac{d^2}{dx^2}.$$

Our main result below is that, under convenient assumptions on V, operator W_+ acting on odd functions may be represented from pseudo-differential operators (see Proposition A.1.1). Let us mention that, even if we give quite complete proofs, our approach here is not original, and that we strongly rely on the classical paper of Deift and Trubowitz [17] and on the work of Weder [85].

A.1 Statement of main proposition

We consider $V : \mathbb{R} \to \mathbb{R}$ a potential belonging to $\mathcal{S}(\mathbb{R})$. Then the operator

$$-\frac{1}{2}\Delta + V = -\frac{1}{2}\frac{d^2}{dx^2} + V$$

is a self-adjoint operator whose spectrum is made of an absolutely continuous part, equal to $[0, +\infty[$, and of finitely many negative eigenvalues (see [17]). For ξ in $\mathbb{R}$, we define the Jost function $f_1(x, \xi)$ (resp. $f_2(x, \xi)$) as the unique solution to

$$-\frac{d^2}{dx^2} f + 2V(x) f = \xi^2 f \tag{A.1}$$

that satisfies $f_1(x, \xi) \sim e^{ix\xi}$ when x goes to $+\infty$ (resp. $f_2(x, \xi) \sim e^{-ix\xi}$ when x goes to $-\infty$). We set

$$\begin{aligned} m_1(x, \xi) &= e^{-ix\xi} f_1(x, \xi), \\ m_2(x, \xi) &= e^{ix\xi} f_2(x, \xi). \end{aligned} \tag{A.2}$$

We shall say that the potential V is generic if

$$\int_{-\infty}^{+\infty} V(x) m_1(x, 0)\, dx \neq 0. \tag{A.3}$$

Notice that the above integral is convergent as $m_1(x,\xi)$ is bounded when x goes to $+\infty$ and has at most polynomial growth as x goes to $-\infty$ (see [17, Lemma 1] and Lemma A.1.1 below). We say that V is very exceptional if

$$\int_{-\infty}^{+\infty} V(x)m_1(x,0)\,dx = 0 \quad \text{and} \quad \int_{-\infty}^{+\infty} V(x)xm_1(x,0)\,dx = 0. \tag{A.4}$$

If one sets $V(x) = -\frac{3}{4}\cosh^{-2}(\frac{x}{2})$, as for the potential of interest in this paper (see equation (2.5)), it is proved in [13, Lemma 2.1] that the transmission coefficient of this potential satisfies $T(0) = 1$ (see [17] or below for the definition of the transmission coefficient). This implies on the one hand that (A.3) does not hold (as (A.3) is equivalent to $T(0) = 0$ – see [17, 85] or (A.32) below) and that moreover

$$\int xV(x)m_1(x,0)\,dx = 0,$$

i.e. that (A.4) holds, as follows from (A.26) and (A.31).

We denote by W_+ the wave operator associated to $A = -\frac{1}{2}\Delta + V$, defined as the strong limit

$$W_+ = s - \lim_{t\to+\infty} e^{itA}e^{-itA_0}, \tag{A.5}$$

where $A_0 = -\frac{1}{2}\Delta$. One knows (see Weder [85] and references therein) that

$$W_+W_+^* = P_{\mathrm{ac}}, \quad W_+^*W_+ = \mathrm{Id}_{L^2}, \tag{A.6}$$

where P_{ac} is the orthogonal projector on the absolutely continuous spectrum and, more generally, that if $\mathfrak{b}$ is any Borel function on $\mathbb{R}$,

$$\mathfrak{b}(A)P_{\mathrm{ac}} = W_+\mathfrak{b}(A_0)W_+^*, \quad \mathfrak{b}(A_0) = W_+^*\mathfrak{b}(A)W_+. \tag{A.7}$$

Notice that since A and A_0 preserve the space of odd functions, so do W_+, W_+^*. For odd w, we shall obtain an expression for W_+w given by the following proposition.

Proposition A.1.1. *Assume that V is an even potential that is either generic or very exceptional. Let $\chi_\pm$ be smooth functions, supported for $\pm x \geq -1$, with values in the interval $[0,1]$, with $\chi_-(x) = \chi_+(-x)$, $\chi_+(x) + \chi_-(x) \equiv 1$. There are an odd smooth real-valued function θ, and a smooth function $(x,\xi) \mapsto b(x,\xi)$ satisfying*

$$\begin{aligned} \left|\partial_\xi^\beta b(x,\xi)\right| &\leq C_\beta && \text{for all } \beta \in \mathbb{N}, \\ \left|\partial_x^\alpha \partial_\xi^\beta b(x,\xi)\right| &\leq C_{\alpha\beta N}\langle x\rangle^{-N} && \text{for all } \alpha \in \mathbb{N}^*,\ \beta \in \mathbb{N},\ N \in \mathbb{N}, \end{aligned} \tag{A.8}$$

and

$$\overline{b(x,-\xi)} = b(x,\xi), b(-x,-\xi) = b(x,\xi) \tag{A.9}$$

such that if we set $c(\xi) = e^{i\theta(\xi)}\mathbb{1}_{\xi>0} + e^{-i\theta(\xi)}\mathbb{1}_{\xi<0}$, then for any odd function w,

$$W_+w = b(x,D_x) \circ c(D_x)w \tag{A.10}$$

with

$$b(x,D)v = \frac{1}{2\pi}\int e^{ix\xi}b(x,\xi)\hat{w}(\xi)\,d\xi.$$

A.2 Proof of main proposition

We shall give here the proof of Proposition A.1.1, relying on the results of Deift and Trubowitz [17] and Weder [85].

If V is a real-valued even potential, the Jost functions satisfy by uniqueness $f_1(-x,\xi) = f_2(x,\xi)$ so that (A.2) implies that

$$m_1(-x,\xi) = m_2(x,\xi). \tag{A.11}$$

By [17, Lemma 1], m_1 solves the Volterra equation

$$m_1(x,\xi) = 1 + \int_x^{+\infty} D_\xi(x'-x)2V(x')m_1(x',\xi)\,dx' \tag{A.12}$$

where

$$D_\xi(x) = \int_0^x e^{2ix'\xi}\,dx' = \frac{e^{2ix\xi}-1}{2i\xi}. \tag{A.13}$$

If V is in $\mathcal{S}(\mathbb{R})$, then [17, Lemma 1 (ii)] shows that

$$\begin{aligned} \left|\partial_x^\alpha \partial_\xi^\beta (m_1(x,\xi)-1)\right| &\le C_{\alpha\beta N}\langle x\rangle^{-N}\langle \xi\rangle^{-1-\beta} \quad \text{for all } x > -M,\ \xi\in\mathbb{R},\\ \left|\partial_x^\alpha \partial_\xi^\beta (m_2(x,\xi)-1)\right| &\le C_{\alpha\beta N}\langle x\rangle^{-N}\langle \xi\rangle^{-1-\beta} \quad \text{for all } x < M,\ \xi\in\mathbb{R}, \end{aligned} \tag{A.14}$$

holds for m_1 (and thus also for m_2) when $\alpha=\beta=0$. To get also estimates for the derivatives, we need to establish the following lemma, whose proof relies on the same ideas as in [17]:

Lemma A.2.1. *Denote for any β, N in $\mathbb{N}$ by $\Omega_N^\beta(x)$ a smooth positive function such that $\Omega_N^\beta(x) = \langle x\rangle^{-N}$ for $x \ge 1$ and $\Omega_N^\beta(x) = \langle x\rangle^\beta$ for $x\le -1$. Then for any N,α,β in $\mathbb{N}$, there is $C>0$ such that for any ξ with $\operatorname{Im}\xi \ge 0$, any x,*

$$\left|\partial_x^\alpha \partial_\xi^\beta (m_1(x,\xi)-1)\right| \le C\,\Omega_N^{\beta+1}(x)\langle\xi\rangle^{-1-\beta}. \tag{A.15}$$

Proof. Following the proof of [17, Lemma 1], we write

$$m_1(x,\xi) = 1 + \sum_{n=1}^{+\infty} g_n(x,\xi) \tag{A.16}$$

with

$$g_n(x,\xi) = \int_{x\le x_1\le\cdots\le x_n} \prod_{j=1}^n D_\xi(x_j - x_{j-1})2V(x_j)\,dx_1\cdots dx_n, \tag{A.17}$$

using the convention $x_0 = x$. Set $\Omega(x) = \Omega_0^1(x)$ and

$$K_\xi(y,y') = D_\xi(y-y')\Omega(y')^{-1}2V(y)\Omega(y).$$

Then we may rewrite g_n as

$$g_n(x,\xi) = \Omega(x) \int_{x \le x_1 \le \cdots \le x_n} \prod_{j=1}^{n} K_\xi(x_j, x_{j-1}) \Omega(x_n)^{-1}\, dx_1 \cdots dx_n,$$

or equivalently

$$g_n(x,\xi) = \Omega(x) \int_{y_1 \ge 0, \dots, y_n \ge 0} \prod_{j=1}^{n} K_\xi(x + y_1 + \cdots + y_j, x + y_1 + \cdots + y_{j-1}) \times \Omega(x + y_1 + \cdots + y_n)^{-1}\, dy_1 \cdots dy_n. \tag{A.18}$$

By (A.13), we have

$$\left|\partial_\xi^\beta D_\xi(y)\right| \le C_\beta \langle \xi \rangle^{-1} \langle y \rangle^{1+\beta}.$$

Fix some integer m. The definition of K_ξ implies that for $\alpha + \beta \le m$

$$\begin{aligned} &\left|\partial_x^\alpha \partial_\xi^\beta K_\xi(x + y_1 + \cdots + y_j, x + y_1 + \cdots + y_{j-1})\right| \\ &\quad \le C \langle \xi \rangle^{-1} \Omega(x + y_1 + \cdots + y_{j-1})^{-1} \langle x + y_1 + \cdots + y_j \rangle^{-1-\beta} \\ &\qquad \times W(x + y_1 + \cdots + y_j) \langle y_j \rangle^{1+\beta}, \end{aligned} \tag{A.19}$$

where W is some smooth rapidly decaying function. When $y_1 \ge 0, \dots, y_j \ge 0$, we may bound

$$\langle y_j \rangle^{1+\beta} \Omega(x + y_1 + \cdots + y_{j-1})^{-1} \langle x + y_1 + \cdots + y_j \rangle^{-1-\beta} \le C\,\Omega(x)^\beta.$$

Consequently, (A.18) implies that

$$|\partial_x^\alpha \partial_\xi^\beta g_n(x,\xi)| \le C\,\Omega(x)^{\beta+1} \langle \xi \rangle^{-n} \times \int_{y_1 \ge 0, \dots, y_n \ge 0} \prod_{j=1}^{n} W(x + y_1 + \cdots + y_j)\, dy_1 \cdots dy_n. \tag{A.20}$$

Define $G(x) = \int_x^{+\infty} W(z)\, dz$, so that the last integral above may be written

$$(-1)^{n-1} \int_{y_1 \ge 0, \dots, y_{n-1} \ge 0} \prod_{j=1}^{n-1} G'(x + y_1 + \cdots + y_j) \times G(x + y_1 + \cdots + y_{n-1})\, dy_1 \cdots dy_{n-1} = \frac{1}{n!} G(x)^n.$$

As $|G(x)| \le C_N \Omega_N^0(x)$ for any N, it follows from (A.20) that, for any N,

$$|\partial_x^\alpha \partial_\xi^\beta g_n(x,\xi)| \le \frac{C_N^{n+1}}{n!} \langle \xi \rangle^{-n} \Omega_N^{\beta+1}(x). \tag{A.21}$$

If we sum for $n \ge \beta + 1$, we get a bound by the right-hand side of (A.15).

We are thus left with studying

$$\sum_{n=1}^{\beta} \partial_x^\alpha \partial_\xi^\beta g_n(x,\xi). \tag{A.22}$$

Notice that (A.21) summed for $n = 1, \ldots, \beta$ gives, when $|\xi| \le 1$, estimate (A.15) for (A.22) as well. Assume from now on that $|\xi| \ge 1$ and let us prove by induction on $n = 1, \ldots, \beta$ that $|\partial_x^\alpha \partial_\xi^\beta g_n(x,\xi)|$ is bounded by the right-hand side of (A.15). We may write from (A.17)

$$\begin{aligned} g_n(x,\xi) &= \int_{x\le x_1} D_\xi(x_1 - x) 2V(x_1) g_{n-1}(x_1,\xi)\, dx_1 \\ &= \int_{y_1 \ge 0} D_\xi(y_1) 2V(y_1 + x) g_{n-1}(y_1 + x, \xi)\, dy_1 \end{aligned} \tag{A.23}$$

with $g_0 \equiv 1$. We use in (A.23) the last expression (A.13) for D_ξ. We have then to consider two kind of terms. The first one is

$$\begin{aligned} &\int_{y_1\ge 0} \frac{e^{2iy_1\xi}}{\xi} 2V(y_1 + x) g_{n-1}(y_1 + x, \xi)\, dy_1 \\ &\quad = -\frac{1}{2i\xi^2} 2V(x) g_{n-1}(x,\xi) \\ &\qquad - \int_{y_1\ge 0} \frac{e^{2iy_1\xi}}{2i\xi^2} \partial_{y_1}\big(2V(y_1 + x) g_{n-1}(y_1 + x, \xi)\big)\, dy_1. \end{aligned}$$

Repeating the integrations by parts, we end up with contributions that, according to the induction hypothesis (and the fact that $g_0 \equiv 1$), satisfy estimates of the form (A.15) (with $\Omega_N^\beta(x)$ replaced by $\langle x\rangle^{-N}$), and an integral term of the form

$$\int_{y_1\ge 0} \frac{e^{2iy_1\xi}}{\xi^{M+1}} \partial_{y_1}^M \big(2V(y_1 + x) g_{n-1}(y_1 + x, \xi)\big)\, dy_1 \tag{A.24}$$

for M as large as we want. If $M = \beta$, we see that (A.24) satisfies (A.15). The second type of terms coming from (A.23) to consider is

$$\frac{1}{\xi} \int_{y_1\ge 0} 2V(y_1 + x) g_{n-1}(y_1 + x, \xi)\, dy_1$$

which trivially satisfies (A.15) by the induction hypothesis applied to g_{n-1}. This concludes the proof. ■

In order to obtain the representation (A.10) for $W_+ w$, when w is odd, we recall first the definition of the transmission and reflection coefficients. The Wronskian of $(f_1(x,\xi), f_1(x,-\xi))$ (resp. $(f_2(x,\xi), f_2(x,-\xi))$) is non-zero for any ξ in $\mathbb{R}^*$ (see [17, p. 144]), so that, for real $\xi \neq 0$, we may find unique coefficients $T_1(\xi), T_2(\xi)$

non-zero, $R_1(\xi)$, $R_2(\xi)$ such that

$$\begin{aligned} f_2(x,\xi) &= \frac{R_1(\xi)}{T_1(\xi)} f_1(x,\xi) + \frac{1}{T_1(\xi)} f_1(x,-\xi) \\ f_1(x,\xi) &= \frac{R_2(\xi)}{T_2(\xi)} f_2(x,\xi) + \frac{1}{T_2(\xi)} f_2(x,-\xi). \end{aligned} \tag{A.25}$$

By [17, Theorem I], these functions extend as smooth functions on $\mathbb{R}$, and they satisfy the following properties:

$$\begin{aligned} T_1(\xi) = T_2(\xi) &\overset{\text{def}}{=} T(\xi), \\ T(\xi)\overline{R_2(\xi)} + R_1(\xi)\overline{T(\xi)} &= 0, \\ |T(\xi)|^2 + |R_j(\xi)|^2 = 1, &\quad j = 1,2, \\ \overline{T(\xi)} = T(-\xi), &\quad \overline{R_j(\xi)} = R_j(-\xi). \end{aligned} \tag{A.26}$$

If the potential V is even, we have seen that

$$f_1(-x,\xi) = f_2(x,\xi),$$

so that, plugging this equality in the first relation of (A.25), comparing to the second one, and using that $T_1 = T_2$, we conclude that

$$R_1(\xi) = R_2(\xi). \tag{A.27}$$

We denote by $R(\xi)$ this common value. The integral representations of the scattering coefficients (see [17, p. 145])

$$\begin{aligned} \frac{R(\xi)}{T(\xi)} &= \frac{1}{2i\xi} \int e^{2ix\xi} 2V(x) m_1(x,\xi)\, dx, \\ \frac{1}{T(\xi)} &= 1 - \frac{1}{2i\xi} \int 2V(x) m_1(x,\xi)\, dx \end{aligned} \tag{A.28}$$

together with (A.15) and the fact that $V \in \mathcal{S}(\mathbb{R})$, show that for any N, β,

$$\partial_\xi^\beta R(\xi) = O(\langle \xi \rangle^{-N}), \quad \partial_\xi^\beta (T(\xi) - 1) = O(\langle \xi \rangle^{-1-\beta}). \tag{A.29}$$

We need the following lemma:

Lemma A.2.2. *The functions T, R satisfy*

$$T(0) = 1 + R(0) \tag{A.30}$$

in the following two cases:

- *The generic case $\int V(x) m_1(x,0)\, dx \neq 0$.*
- *The very exceptional case $\int V(x) m_1(x,0)\, dx = 0$ and $\int V(x) x m_1(x,0)\, dx = 0$.*

Proof. Summing the two equalities (A.28) and making an expansion at $\xi = 0$ using (A.15), we get

$$\begin{aligned} R(\xi) + 1 = T(\xi)\Big(1 - \frac{1}{i\xi}\int_{-\infty}^{+\infty} V(x)m_1(x,\xi)\,dx \\ + \frac{1}{i\xi}\int_{-\infty}^{+\infty} e^{2ix\xi}V(x)m_1(x,\xi)\,dx\Big) \\ = T(\xi)\Big(1 + 2\int_{-\infty}^{+\infty} xV(x)m_1(x,0)\,dx + O(\xi)\Big), \quad \xi \to 0, \end{aligned}$$

so that

$$R(0) + 1 - T(0) = 2T(0)\int_{-\infty}^{+\infty} xV(x)m_1(x,0)\,dx. \tag{A.31}$$

In the generic case, by (A.28),

$$T(\xi) = i\xi\Big(-\int_{-\infty}^{+\infty} V(x)m_1(x,0)\,dx + O(\xi)\Big)^{-1}, \quad \xi \to 0, \tag{A.32}$$

so that $T(0) = 0$. This shows that (A.31) vanishes in the two considered cases. ■

Proof of Proposition A.1.1. We have to prove that W_+ acting on odd functions is given by (A.10). Recall (see for instance Weder [85] formula (2.20), Schechter [74]) that W_+w is given by

$$W_+w = F_+^*\hat{w}, \tag{A.33}$$

where F_+^* is the adjoint of the distorted Fourier transform, given by

$$F_+^*\Phi = \frac{1}{2\pi}\int \psi_+(x,\xi)\Phi(\xi)\,d\xi, \tag{A.34}$$

where

$$\psi_+(x,\xi) = \mathbb{1}_{\xi>0}T(\xi)f_1(x,\xi) + \mathbb{1}_{\xi<0}T(-\xi)f_2(x,-\xi). \tag{A.35}$$

Let $\chi_\pm$ be the functions defined in the statement of Proposition A.1.1 and write

$$\psi_+(x,\xi) = \chi_+(x)\psi_+(x,\xi) + \chi_-(x)\psi_+(x,\xi).$$

Replace in $\chi_+\psi_+$ (resp. $\chi_-\psi_+$) ψ_+ by (A.35), where we express f_2 from f_1 (resp. f_1 for f_2) using the first (resp. second) formula (A.25). We get, using notation (A.2),

$$\begin{aligned} \psi_+(x,\xi) = \chi_+(x)\Big(e^{ix\xi}\big(T(\xi)m_1(x,\xi)\mathbb{1}_{\xi>0} + m_1(x,\xi)\mathbb{1}_{\xi<0}\big) \\ + e^{-ix\xi}R(-\xi)m_1(x,-\xi)\mathbb{1}_{\xi<0}\Big) \\ + \chi_-(x)\Big(e^{ix\xi}\big(m_2(x,-\xi)\mathbb{1}_{\xi>0} + T(-\xi)m_2(x,-\xi)\mathbb{1}_{\xi<0}\big) \\ + e^{-ix\xi}R(\xi)m_2(x,\xi)\mathbb{1}_{\xi>0}\Big). \end{aligned} \tag{A.36}$$

Using (A.11), we deduce from (A.33), (A.34) and (A.36) that

$$W_+ w = \frac{1}{2\pi}\int e^{ix\xi} e_1(x,\xi)\hat{w}(\xi)\,d\xi + \frac{1}{2\pi}\int e^{-ix\xi} e_2(x,\xi)\hat{w}(\xi)\,d\xi \tag{A.37}$$

with

$$\begin{aligned} e_1(x,\xi) &= \chi_+(x)m_1(x,\xi)\big(T(\xi)\mathbb{1}_{\xi>0} + \mathbb{1}_{\xi<0}\big)\\ &\qquad + \chi_-(x)m_1(-x,-\xi)\big(\mathbb{1}_{\xi>0} + T(-\xi)1_{\xi<0}\big),\\ e_2(x,\xi) &= \chi_+(x)R(-\xi)m_1(x,-\xi)\mathbb{1}_{\xi<0} + \chi_-(x)R(\xi)m_1(-x,\xi)\mathbb{1}_{\xi>0}. \end{aligned} \tag{A.38}$$

If w is odd, we may rewrite (A.37) as

$$W_+ w = \frac{1}{2\pi}\int e^{ix\xi} a(x,\xi)\hat{w}(\xi)\,d\xi$$

with

$$\begin{aligned} a(x,\xi) &= e_1(x,\xi) - e_2(x,-\xi)\\ &= \chi_+(x)m_1(x,\xi)\big((T(\xi)-R(\xi))\mathbb{1}_{\xi>0} + \mathbb{1}_{\xi<0}\big)\\ &\qquad + \chi_-(x)m_1(-x,-\xi)\big(\mathbb{1}_{\xi>0} + (T(-\xi)-R(-\xi))\mathbb{1}_{\xi<0}\big). \end{aligned} \tag{A.39}$$

By properties (A.26), $|T(\xi)-R(\xi)|^2 = 1$ and by (A.30), $T(0)-R(0) = 1$. We may thus find a unique smooth real-valued function $\theta(\xi)$, satisfying $\theta(0) = 0$, such that $T(\xi)-R(\xi) = e^{2i\theta(\xi)}$. Moreover, using (A.26), one gets that θ is odd, and by (A.29) it satisfies $\partial^\beta\theta(\xi) = O(\langle\xi\rangle^{-1-\beta})$. We define

$$c(\xi) = e^{i\theta(\xi)}\mathbb{1}_{\xi>0} + e^{-i\theta(\xi)}\mathbb{1}_{\xi<0} \tag{A.40}$$

so that in (A.39)

$$\begin{aligned} (T(\xi)-R(\xi))\mathbb{1}_{\xi>0} + \mathbb{1}_{\xi<0} &= e^{i\theta(\xi)}c(\xi),\\ \mathbb{1}_{\xi>0} + (T(-\xi)-R(-\xi))\mathbb{1}_{\xi<0} &= e^{-i\theta(\xi)}c(\xi) \end{aligned}$$

and $a(x,\xi) = b(x,\xi)c(\xi)$, where b is a smooth function satisfying (A.8) given by

$$b(x,\xi) = \chi_+(x)m_1(x,\xi)e^{i\theta(\xi)} + \chi_-(x)m_1(-x,-\xi)e^{-i\theta(\xi)}.$$

We thus got $W_+ w = b(x,D_x)\circ c(D_x)w$ for odd w. Moreover, the definition of f_1 and m_1 shows that $\overline{f_1(x,\xi)} = f_1(x,-\xi), \overline{m_1(x,\xi)} = m_1(x,-\xi)$, so that it follows from the expression of b that equalities (A.9) hold. ∎

Remarks. We make the following observations.

- The proof of the last result shows that b satisfies better estimates than those written in (A.8): Actually, on the right-hand side of these inequalities, one could insert a factor $\langle\xi\rangle^{-\beta}$. We wrote the estimates without this factor because we shall have in any case to consider also more general classes of symbols, for which only (A.8) holds.

- The difference between generic or very exceptional potentials versus exceptional ones appears, as is well known, when considering the action of the Fourier multiplier $c(\xi)$ on L^∞ based spaces. Since $\partial^\beta\theta(\xi) = O(\langle\xi\rangle^{-1-\beta})$ when $|\xi| \to +\infty$, $c(\xi) - 1$ coincides with a symbol of order -1 outside a neighborhood of zero. Consequently, if $\chi_0 \in C_0^\infty(\mathbb{R})$ is equal to one close to zero, $(1 - \chi_0)(D_x)c(D_x)$ is bounded on L^∞. On the other hand, $\chi_0(\xi)c(\xi)$ is Lipschitz at zero if the potential is generic or very exceptional, since $\theta(0) = 0$, so that $\chi_0(D_x)c(D_x)$ is also bounded on L^∞. In the exceptional potential case, $c(\xi)$ has a jump at $\xi = 0$, and L^∞ bounds for $c(D_x)$ do not hold.

Appendix B

(Semiclassical) pseudo-differential operators

This appendix is devoted to the definition and main properties of classes of multilinear pseudo-differential operators and their semiclassical counterparts. Recall that the symbol of a pseudo-differential operator of order $m \in \mathbb{R}$ is in general a smooth function $(x, \xi) \mapsto a(x, \xi)$ defined on $\mathbb{R}^d \times \mathbb{R}^d$, satisfying for any multi-indices α, β estimates of the form

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \le C_{\alpha,\beta} \langle \xi \rangle^{m-\rho|\beta|+\delta|\alpha|}, \tag{B.1}$$

where $0 \le \delta \le \rho \le 1$ (see Hörmander [42, 43]). One associates to such a symbol an operator acting on test functions in $\mathcal{S}(\mathbb{R})$ by a quantization rule, that may be given for instance by the usual quantization

$$\mathrm{Op}(a)u = \frac{1}{(2\pi)^d} \int e^{ix\cdot\xi} a(x, \xi) \hat{u}(\xi)\, d\xi = \frac{1}{(2\pi)^d} \int e^{i(x-y)\cdot\xi} a(x, \xi) u(y)\, dy\, d\xi$$

or by the Weyl quantization

$$\mathrm{Op}^{\mathrm{W}}(a)u = \frac{1}{(2\pi)^d} \int e^{i(x-y)\cdot\xi} a\Big(\frac{x+y}{2}, \xi\Big) u(y)\, dy\, d\xi.$$

We shall be here more interested in the semiclassical version of this calculus, namely smooth symbols $(x, \xi, h) \mapsto a(x, \xi, h)$ that depend on a parameter $h \in]0, 1]$, and that satisfy bounds of the form

$$|\partial_x^\alpha \partial_\xi^\beta (h\partial_h)^k a(x, \xi, h)| \le C_{\alpha,\beta,k} M(x, \xi) \tag{B.2}$$

with a fixed "weight function" $M(x, \xi)$ (see Dimassi and Sjöstrand [24]). For instance, a function satisfying (B.1) with $\rho = \delta = 0$ obeys these inequalities with $M \equiv 1$. One defines then the semiclassical quantization of a by the formulas

$$\begin{aligned} \mathrm{Op}_h(a)u = a(x, hD_x, h)u &= \frac{1}{(2\pi)^d} \int e^{ix\cdot\xi} a(x, h\xi, h) \hat{u}(\xi)\, d\xi \\ &= \frac{1}{(2\pi h)^d} \int e^{i(x-y)\cdot\frac{\xi}{h}} a(x, \xi, h) u(y)\, dy\, d\xi \end{aligned} \tag{B.3}$$

or for the Weyl quantization by

$$\mathrm{Op}_h^{\mathrm{W}}(a)u = \frac{1}{(2\pi h)^d} \int e^{i(x-y)\cdot\frac{\xi}{h}} a\Big(\frac{x+y}{2}, \xi, h\Big) u(y)\, dy d\xi. \tag{B.4}$$

One has then a symbolic calculus. Assume for instance that we are given two symbols a, b satisfying (B.2) with $M \equiv 1$. Then there is a symbol c in the same class such that

$\mathrm{Op}_h(a) \circ \mathrm{Op}_h(b) = \mathrm{Op}_h(c)$. Moreover, one may get an asymptotic expansion of c in terms of powers of the semiclassical parameter h, whose first terms are given by

$$c(x,\xi,h) = a(x,\xi,h)b(x,\xi,h) + \frac{h}{i}\sum_{j=1}^{d} \partial_{\xi_j} a(x,\xi,h)\partial_{x_j} b(x,\xi,h) + \cdots. \tag{B.5}$$

It turns out that we shall be interested only in the case of one variable $d = 1$, but with more general classes of symbols. In Appendix A, we have used symbols $b(x,\xi)$ satisfying inequalities (A.8). It turns out that, if one translates in the semiclassical framework the operators $b(x, D_x)$ (see (B.15) and (B.16) below), one is led to consider instead of (B.3) the more general operator

$$b\Big(\frac{x}{h}, hD_x\Big)u = \frac{1}{2\pi}\int e^{ix\xi} b\Big(\frac{x}{h}, h\xi\Big)\hat{u}(\xi)\,d\xi. \tag{B.6}$$

Of course, the function $(x,\xi) \mapsto b(\frac{x}{h},\xi)$ does not satisfy the estimates in (B.2), since ∂_x-derivatives make lose powers of h^{-1}. On the other hand, because of (A.8), taking a ∂_x-derivatives makes gain a weight in $\langle\frac{x}{h}\rangle^{-N}$ for any N. We shall thus consider symbols depending on two space variables, $(y,x,\xi) \mapsto a(y,x,\xi,h)$, such that at fixed y, $(x,\xi,h) \mapsto a(y,x,\xi,h)$ satisfies the estimates in (B.2), and that for any $\ell > 0$, $(x,\xi,h) \mapsto \partial_y^\ell a(y,x,\xi,h)$ satisfies (B.2) with on the right-hand side of these inequalities an arbitrarily decaying factor in $\langle\frac{x}{h}\rangle^{-N}$. We shall quantify such symbols as

$$\mathrm{Op}_h(a)u = a\Big(\frac{x}{h}, x, hD_x, h\Big)u = \frac{1}{2\pi}\int e^{ix\xi} a\Big(\frac{x}{h}, x, h\xi, h\Big)\hat{u}(\xi)\,d\xi. \tag{B.7}$$

In that way, instead of getting for the composition of two such symbols an expansion of the form (B.5), we shall obtain

$$c(y,x,\xi,h) = a(y,x,\xi,h)b(y,x,\xi,h) + hr_1 + r_1', \tag{B.8}$$

where r_1 is in the same class as a, b and where r_1' is rapidly decaying in $\frac{x}{h}$, i.e. satisfies (B.2) with on the right-hand side an extra arbitrary factor in $\langle\frac{x}{h}\rangle^{-N}$.

It turns out that we shall not just need linear, but also multilinear operators, defined instead of (B.7) by formula (B.14) below. The goal of this chapter is thus to define such operators and study their composition properties, establishing the generalization of formulas of the form (B.8) to this multilinear framework.

B.1 Classes of symbols and their quantization

We shall use classes of semiclassical multilinear pseudo-differential operators, analogous to those introduced in [20]. We shall use also the non-semiclassical counterparts of these operators that are deduced from the former by conjugation through dilations. We refer to Dimassi and Sjöstrand [24] for a reference text on semiclassical calculus. Recall first the following definition.

Definition B.1.1. An order function on $\mathbb{R} \times \mathbb{R}^p$ is a function M from $\mathbb{R} \times \mathbb{R}^p$ to $\mathbb{R}_+$, $(x, \xi_1, \dots, \xi_p) \mapsto M(x, \xi_1, \dots, \xi_p)$, such that there is N_0 in $\mathbb{N}$, $C > 0$ and for any $(x, \xi_1, \dots, \xi_p)$, $(x', \xi_1', \dots, \xi_p')$ in $\mathbb{R} \times \mathbb{R}^p$,

$$M(x', \xi_1', \dots, \xi_p') \le C \langle x - x' \rangle^{N_0} \prod_{j=1}^{p} \langle \xi_j - \xi_{j'} \rangle^{N_0} M(x, \xi_1, \dots, \xi_p). \tag{B.9}$$

An example of an order function that we use several times is

$$M_0(\xi_1, \dots, \xi_p) = \Big(\sum_{1 \le i < j \le p} \langle \xi_i \rangle^2 \langle \xi_j \rangle^2 \Big)^{\frac{1}{2}} \Big(\sum_{i=1}^{p} \langle \xi_i \rangle^2 \Big)^{-\frac{1}{2}}. \tag{B.10}$$

Actually, this function is smooth and is equivalent to $1 + \max_2(|\xi_1|, \dots, |\xi_p|)$, where $\max_2(|\xi_1|, \dots, |\xi_p|)$ is the second largest among $|\xi_1|, \dots, |\xi_p|$.

We shall introduce several classes of semiclassical symbols, depending on a semiclassical parameter $h \in]0, 1]$:

Definition B.1.2. Let p be in $\mathbb{N}^*$, M an order function on $\mathbb{R} \times \mathbb{R}^p$, M_0 the function defined in (B.10). Let (β, κ) be in $[0, +\infty[\times \mathbb{N}$. We denote by $S_{\kappa,\beta}(M, p)$ the space of smooth functions

$$\begin{gathered} (y, x, \xi_1, \dots, \xi_p, h) \mapsto a(y, x, \xi_1, \dots, \xi_p, h), \\ \mathbb{R} \times \mathbb{R} \times \mathbb{R}^p \times]0, 1] \to \mathbb{C} \end{gathered} \tag{B.11}$$

satisfying for any $\alpha_0 \in \mathbb{N}$, $\alpha \in \mathbb{N}^p$, $k \in \mathbb{N}$, $N \in \mathbb{N}$, $\alpha_0' \in \mathbb{N}^*$ the bounds

$$\big|\partial_x^{\alpha_0} \partial_\xi^{\alpha} (h\partial_h)^k a(y, x, \xi, h)\big| \le C M(x, \xi) M_0(\xi)^{\kappa(\alpha_0 + |\alpha|)} \big(1 + \beta h^\beta M_0(\xi)\big)^{-N} \tag{B.12}$$

and

$$\begin{aligned} &\big|\partial_y^{\alpha_0'} \partial_x^{\alpha_0} \partial_\xi^{\alpha} (h\partial_h)^k a(y, x, \xi, h)\big| \\ &\quad \le C M(x, \xi) M_0(\xi)^{\kappa(\alpha_0 + |\alpha|)} \big(1 + \beta h^\beta M_0(\xi)\big)^{-N} \big(1 + M_0(\xi)^{-\kappa} |y|\big)^{-N}, \end{aligned} \tag{B.13}$$

where ξ stands for $(\xi_1, \dots, \xi_p)$.

We denote by $S'_{\kappa,\beta}(M, p)$ the subspace of $S_{\kappa,\beta}(M, p)$ of those symbols that satisfy (B.13) including for $\alpha_0' = 0$.

We shall set $S'^{N'}_{\kappa,\beta}(M, p)$ for the space of functions satisfying the bound in (B.13) including for the case $\alpha_0' = 0$, but with the last factor $(1+M_0(\xi)^{-\kappa}|y|)^{-N}$ replaced by $(1 + M_0(\xi)^{-\kappa}|y|)^{-N'}$, for a fixed power N' instead of for all N.

Remarks. We make the following observations.

- If $p = 1$, then $M_0(\xi) = 1$ and symbols of the class $S_{\kappa,\beta}(M, 1)$ that do not depend on y are just usual symbols of pseudo-differential operators as defined in [24] for instance. For symbols depending on y, we impose that if we take at least one

∂_y-derivative, we get a rapid decay in $|y|$ in the case of the class $S_{\kappa,\beta}(M,1)$. For elements of $S'_{\kappa,\beta}(M,1)$, this rapid decay has to hold including without taking any ∂_y-derivative. Notice also that when $p=1$, the classes we define do not depend on the parameters κ,β.

- The parameter κ in the definition of the classes of symbols measures the power of $M_0(\xi)$ that we lose when taking ∂_x- or ∂_ξ-derivatives. As these losses involve only "small frequencies", they will be affordable.
- When $\beta>0$, we have an extra gain in $\langle h^\beta M_0(\xi)\rangle^{-N}$ for any N, that allows to trade off the loss $M_0(\xi)^\kappa$ for $h^{-\beta\kappa}$. If β is small, this reduces these losses to those ones used usually in definitions of semiclassical symbols as in [24]. Moreover, an element of $S_{\kappa,0}(M,p)$ may be always reduced to an element of $S_{\kappa,\beta}(M,p)$ multiplying it by $\chi(h^\beta M_0(\xi))$ for some χ in $C_0^\infty(\mathbb{R})$.

We shall quantize symbols in $S_{\kappa,\beta}(M,p)$ as p-linear operators acting a p-tuple of functions by

$$\begin{aligned}\mathrm{Op}_h(a)(\underline{v}_1,\dots,\underline{v}_p)&\\ &=\frac{1}{(2\pi)^p}\int e^{ix(\xi_1+\cdots+\xi_p)}a\Big(\frac{x}{h},x,h\xi_1,\dots,h\xi_p\Big)\prod_{j=1}^p\hat{\underline{v}}_j(\xi_j)\,d\xi_1\cdots d\xi_p\\ &=\frac{1}{(2\pi h)^p}\int e^{i\sum_{j=1}^p(x-x'_j)\frac{\xi_j}{h}}a\Big(\frac{x}{h},x,\xi_1,\dots,\xi_p\Big)\prod_{j=1}^p\underline{v}_j(x'_j)\,dx'd\xi.\end{aligned}\tag{B.14}$$

We shall call (B.14) the semiclassical quantization of a. We shall also use a classical quantization, depending on the parameter $t=\frac{1}{h}\geq 1$, related to (B.14) through conjugation by dilations: If $t\geq 1$, and $\underline{v}$ is a test function on $\mathbb{R}$, define the L^2 isometry Θ_t by

$$\Theta_t\underline{v}(x)=\frac{1}{\sqrt{t}}\underline{v}\Big(\frac{x}{t}\Big).\tag{B.15}$$

We shall set for a an element of $S_{\kappa,\beta}(M,p)$,

$$\mathrm{Op}^t(a)(v_1,\dots,v_p)=h^{\frac{p-1}{2}}\Theta_t\circ\mathrm{Op}_h(a)\big(\Theta_{t^{-1}}v_1,\dots,\Theta_{t^{-1}}v_p\big)\tag{B.16}$$

with $h=t^{-1}$. Explicitly, we get from (B.14)

$$\begin{aligned}\mathrm{Op}^t(a)(v_1,\dots,v_p)&\\ &=\frac{1}{(2\pi)^p}\int e^{ix(\xi_1+\cdots+\xi_p)}a\Big(x,\frac{x}{t},\xi_1,\dots,\xi_p\Big)\prod_{j=1}^p\hat{v}_j(\xi_j)\,d\xi_1\cdots d\xi_p.\end{aligned}\tag{B.17}$$

Remark that if $a(y,x,\xi)$ is independent of x, then $\mathrm{Op}^t(a)$ is independent of t, and if $p=1$, $\mathrm{Op}^t(a)$ is just the usual pseudo-differential operator of symbol $a(y,\xi)$. In this case, we shall just write $\mathrm{Op}(a)$ for $\mathrm{Op}^t(a)$.

B.2 Symbolic calculus

We prove first a proposition generalizing [20, Proposition 1.5].

Proposition B.2.1. *Let* n', n'' *be in* $\mathbb{N}^*$, $n = n' + n'' - 1$. *Let* $M'(x, \xi_1, \dots, \xi_{n'})$ *and* $M''(x, \xi_{n'}, \dots, \xi_n)$ *be two order functions on* $\mathbb{R} \times \mathbb{R}^{n'}$ *and* $\mathbb{R} \times \mathbb{R}^{n''}$, *respectively. In particular, they satisfy* (B.9) *and we shall denote by* N_0'' *an integer such that*

$$M''(x', \xi_{n'}, \dots, \xi_n) \le C\langle x - x'\rangle^{N_0''} M''(x, \xi_{n'}, \dots, \xi_n). \tag{B.18}$$

Let $(\kappa, \beta) \in \mathbb{N} \times [0, 1]$, a *in* $S_{\kappa,\beta}(M', n')$, b *in* $S_{\kappa,\beta}(M'', n'')$. *Assume either that* $(\kappa, \beta) = (0, 0)$ *or* $0 < \beta\kappa \le 1$ *or that symbol* b *is independent of* x. *Define*

$$M(x, \xi_1, \dots, \xi_n) = M'(x, \xi_1, \dots, \xi_{n'-1}, \xi_{n'} + \cdots + \xi_n) M''(x, \xi_{n'}, \dots, \xi_n). \tag{B.19}$$

Then there is ν *in* $\mathbb{N}$, *that depends only on* N_0'' *in* (B.18), *and symbols*

$$c_1 \in S_{\kappa,\beta}(MM_0^{\nu\kappa}, n), c_1' \in S'_{\kappa,\beta}(MM_0^{\nu\kappa}, n) \tag{B.20}$$

such that one may write

$$\mathrm{Op}_h(a)[v_1, \dots, v_{n'-1}, \mathrm{Op}_h(b)(v_{n'}, \dots, v_n)] = \mathrm{Op}_h(c)[v_1, \dots, v_n], \tag{B.21}$$

where

$$\begin{aligned} c(y, x, \xi_1, \dots, \xi_n) = a(y, x, \xi_1, &\dots, \xi_{n'-1}, \xi_{n'} + \cdots + \xi_n) \\ &\times b(y, x, \xi_{n'}, \dots, \xi_n) \\ &+ hc_1(y, x, \xi_1, \dots, \xi_n) + c_1'(y, x, \xi_1, \dots, \xi_n). \end{aligned} \tag{B.22}$$

Moreover, if b *is independent of* y, c_1' *in* (B.22) *vanishes and if* b *is independent of* x, *then* c_1 *vanishes. In addition, if* a *is in* $S'_{\kappa,\beta}(M', n')$ *or* b *is in* $S'_{\kappa,\beta}(M'', n'')$, *then* c *and* c_1 *are in* $S'_{\kappa,\beta}(MM_0^{\nu\kappa}, n)$.

Let us prove first a lemma:

Lemma B.2.2. *Let* $\xi' = (\xi_1, \dots, \xi_{n'-1})$ *and* $\xi'' = (\xi_{n'}, \dots, \xi_n)$, $\xi = (\xi', \xi'')$. *Then*

$$M_0(\xi', \xi_{n'} + \cdots + \xi_n) \le CM_0(\xi), \quad M_0(\xi'') \le CM_0(\xi). \tag{B.23}$$

Moreover, if ζ *is a real number and* $|\zeta|/M_0(\xi)$ *is small enough,*

$$\max\big(M_0(\xi', \xi_{n'} + \cdots + \xi_n - \zeta), M_0(\xi'')\big) \ge cM_0(\xi) \tag{B.24}$$

for some $c > 0$.

Proof. Estimate (B.23) follows from the fact that $M_0(\xi_1, \dots, \xi_n)$ is equivalent to $1 + \max_2(|\xi_1|, \dots, |\xi_n|)$.

To prove estimate (B.24), we may assume that $|\xi_n| \ge |\xi_{n-1}| \ge \cdots \ge |\xi_{n'}|$ and $|\xi_1| \ge |\xi_2| \ge \cdots \ge |\xi_{n'-1}|$. Moreover, if $n = n'$, then (B.24) is trivial, so that we may assume $n' < n$.

Case 1. Assume $|\xi_n| \geq |\xi_1|$. If $|\xi_n| \sim |\xi_{n-1}|$, then both $M_0(\xi'')$ and $M_0(\xi)$ are of the magnitude of $\langle \xi_{n-1}\rangle$, so (B.24) is trivial.

Let us assume that $|\xi_{n-1}| \ll |\xi_n|$.

- If in addition $|\xi_n| \sim |\xi_1|$, then $M_0(\xi) \sim \langle \xi_n\rangle \sim \langle \xi_1\rangle$ and
$$\langle \xi_{n'} + \cdots + \xi_n - \zeta\rangle \sim \langle \xi_n\rangle,$$
so that
$$M_0(\xi', \xi_{n'} + \cdots + \xi_n - \zeta) \sim M_0(\xi', \xi_n) \sim \langle \xi_n\rangle \sim \langle \xi_1\rangle$$
and (B.24) holds.
- If $|\xi_1| \ll |\xi_n|$, then $M_0(\xi) \sim \max(\langle \xi_1\rangle, \langle \xi_{n-1}\rangle)$ and $M_0(\xi'') \sim \langle \xi_{n-1}\rangle$, so that $M_0(\xi', \xi_{n'} + \cdots + \xi_n - \zeta) \sim M_0(\xi', \xi_n) \sim \langle \xi_1\rangle$ and (B.24) holds again.

Case 2. Assume $|\xi_1| \geq |\xi_n|$. Then $M_0(\xi) \sim \max(\langle \xi_2\rangle, \langle \xi_n\rangle)$.

- If $|\xi_n| \geq |\xi_2|$ and $|\xi_n| \sim |\xi_{n-1}|$, then $M_0(\xi'') \sim \langle \xi_n\rangle$, so that (B.24) holds.
- If $|\xi_n| \geq |\xi_2|$ and $|\xi_n| \gg |\xi_{n-1}|$, then we have $|\xi_{n'} + \cdots + \xi_n - \zeta| \sim |\xi_n|$, so that $M_0(\xi', \xi_{n'} + \cdots + \xi_n - \zeta) \sim \langle \xi_n\rangle$ and (B.24) holds.
- If $|\xi_2| \geq |\xi_n|$, then $M_0(\xi', \xi_{n'} + \cdots + \xi_n - \zeta) \sim \langle \xi_2\rangle$, so that (B.24) holds as well.

This concludes the proof. ■

Proof of Proposition B.2.1. Going back to the definition (B.14) of quantization, we may write the composition (B.21) as the right-hand side of this expression, with a symbol c given by the oscillatory integral

$$c(y,x,\xi) = \frac{1}{2\pi}\int e^{-iz\zeta} a(y,x,\xi',\xi_{n'} + \cdots + \xi_n - \zeta) \times b(y-z, x-hz, \xi'')\,dz\,d\zeta. \tag{B.25}$$

We decompose

$$a(y,x,\xi',\xi_{n'} + \cdots + \xi_n - \zeta) = a(y,x,\xi',\xi_{n'} + \cdots + \xi_n) - \zeta\tilde{a}(y,x,\xi',\xi_{n'} + \cdots + \xi_n,\zeta) \tag{B.26}$$

with

$$\tilde{a}(y,x,\xi',\tilde{\xi},\zeta) = \int_0^1 \Big(\frac{\partial a}{\partial \tilde{\xi}}\Big)(y,x,\xi',\tilde{\xi} - \lambda\zeta)\,d\lambda. \tag{B.27}$$

It follows from (B.23) that

$$M_0(\xi', \xi_{n'} + \cdots + \xi_n - \lambda\zeta) \leq C(M_0(\xi) + \langle\zeta\rangle). \tag{B.28}$$

Using (B.12) and the definition of order functions, we get that $\tilde{a}$ satisfies

$$\begin{aligned} &|\partial_x^{\alpha_0}\partial_\xi^{\alpha}\partial_\zeta^{\gamma}(h\partial_h)^k \tilde{a}(y,x,\xi',\xi_{n'} + \cdots + \xi_n,\zeta)| \\ &\quad\leq C(M_0(\xi) + \langle\zeta\rangle)^{\kappa(1+|\alpha|+|\gamma|+\alpha_0)}\langle\zeta\rangle^{N_0} M'(x,\xi',\xi_{n'} + \cdots + \xi_n) \\ &\qquad\times \int_0^1 \big(1 + \beta h^{\beta} M_0(\xi', \xi_{n'} + \cdots + \xi_n - \lambda\zeta)\big)^{-N}\,d\lambda \end{aligned} \tag{B.29}$$

for any $\alpha, \alpha_0, \gamma, k, N$. If one takes at least one ∂_y-derivative, the same estimate holds, with an extra factor

$$\big(1 + (M_0(\xi) + \langle\zeta\rangle)^{-\kappa}|y|\big)^{-N} \tag{B.30}$$

using (B.13) and (B.28). If we plug (B.26) in (B.25), we get the first term on the right-hand side of (B.22) and, by integration by parts, the following two contributions:

$$-\frac{i}{2\pi}\int e^{-iz\zeta}\tilde{a}(y,x,\xi',\xi_{n'}+\cdots+\xi_n,\zeta)\frac{\partial b}{\partial y}(y-z,x-hz,\xi'')\,dz\,d\zeta, \tag{B.31}$$

$$-\frac{ih}{2\pi}\int e^{-iz\zeta}\tilde{a}(y,x,\xi',\xi_{n'}+\cdots+\xi_n,\zeta)\frac{\partial b}{\partial x}(y-z,x-hz,\xi'')\,dz\,d\zeta. \tag{B.32}$$

Let us show that (B.31) (resp. (B.32)) provides the contribution c_1' (resp. hc_1) in equation (B.22).

Study of (B.31). If we insert under integral (B.31) a cut-off $(1-\chi_0)(\zeta)$ for some C_0^∞ function χ_0 equal to one close to zero and make N_1 integrations by parts in z, we gain a factor ζ^{-N_1}, up to making act on $\frac{\partial b}{\partial y}(y-z,x-hz,\xi'')$ at most N_1 ∂_z-derivatives. By (B.12) and (B.13), each of these ∂_z-derivatives makes lose $\langle hM_0(\xi'')^\kappa\rangle$ if it falls on the x argument of $\frac{\partial b}{\partial y}$, and does not make lose anything if it falls on the y argument. Consequently, if $\beta=\kappa=0$, or if b is independent of x, we get no loss, while if $\kappa\beta>0$, we get a loss that may be compensated since, in this case, we get by (B.12) and (B.13) a factor $\langle h^\beta M_0(\xi'')\rangle^{-N}$ in the estimates, with an arbitrary N. Since we assume $\beta\kappa\le 1$, $\langle h^\beta M_0(\xi'')\rangle^{-N}\langle hM_0(\xi'')^\kappa\rangle^{N_1} = O(\langle h^\beta M_0(\xi'')\rangle^{-N/2})$ if N is large enough relatively to N_1. In other words, up to changing the definition of b, we may insert under (B.31) an extra factor decaying like $\langle\zeta\rangle^{-N_1}$ as well as its derivatives, for a given N_1.

We perform next N_2 integrations by parts using the operator

$$\big\langle z(\langle\zeta\rangle + M_0(\xi))^{-\kappa}\big\rangle^{-2}\big(1 - (\langle\zeta\rangle + M_0(\xi))^{-2\kappa} zD_\zeta\big). \tag{B.33}$$

By estimates (B.28) and (B.29), each of these integrations by parts makes gain a factor $\langle z(\langle\zeta\rangle + M_0(\xi))^{-\kappa}\rangle^{-1}$. Using (B.29), (B.13), the definition (B.19) of M and (B.18), we bound the modulus of (B.31) by

$$\begin{aligned} CM(x,\xi)\int &\langle\zeta\rangle^{-N_1+N_0}\big\langle z(\langle\zeta\rangle + M_0(\xi))^{-\kappa}\big\rangle^{-N_2}(\langle\zeta\rangle + M_0(\xi))^\kappa \\ &\times \langle hz\rangle^{N_0''}\big(1 + M_0(\xi)^{-\kappa}|y-z|\big)^{-N} \\ &\times \int_0^1 \big(1 + \beta h^\beta M_0(\xi',\xi_{n'}+\cdots+\xi_n-\lambda\zeta)\big)^{-N}\,d\lambda \\ &\times (1+\beta h^\beta M_0(\xi''))^{-N}\,dz\,d\zeta \end{aligned} \tag{B.34}$$

for arbitrary N_1, N_2, N and given N_0, N_0'' (coming from (B.9) and (B.18)), the factor in $(1+M_0(\xi)^{-\kappa}|y-z|)^{-N}$ coming from the last factor in (B.13) of $\frac{\partial b}{\partial y}$. If $N_1 - N_0$ is large enough, and if we integrate for $|\zeta|\ge cM_0(\xi)$, then the factor $\langle\zeta\rangle^{-N_1+N_0}$

provides a decay in $M_0(\xi)^{-N'}$ for any given N'. On the other hand, if we integrate for $|\zeta| \le cM_0(\xi)$, we may use (B.24) that shows that the product of the last two factors in (B.34) is smaller than $C(1+\beta h^\beta M_0(\xi))^{-N}$. We thus get a bound in

$$\begin{aligned}
&CM(x,\xi)(1+\beta h^\beta M_0(\xi))^{-N}\\
&\qquad\times\int \langle\zeta\rangle^{-N_1+N_0+N}\big\langle z(\langle\zeta\rangle+M_0(\xi))^{-\kappa}\big\rangle^{-N_2}(\langle\zeta\rangle+M_0(\xi))^{\kappa}\\
&\qquad\times\langle hz\rangle^{N_0''}\big(1+M_0(\xi)^{-\kappa}|y-z|\big)^{-N}\,dz\,d\zeta\\
&\quad\le CM(x,\xi)\big(1+\beta h^\beta M_0(\xi)\big)^{-N}M_0(\xi)^{(2+N_0'')\kappa}\big(1+M_0(\xi)^{-\kappa}|y|\big)^{-N}
\end{aligned}\tag{B.35}$$

if $N_1 \gg N_2 \gg N+N_0+N_0''$. We thus get an estimate of the form (B.13), with $\alpha_0=0$, $\alpha=0$, and the order function M replaced by $M(x,\xi)M_0(\xi)^{\kappa(2+N_0'')}$.

If we make the same computation after taking a $\partial_x^{\alpha_0}$ and a ∂_ξ^α-derivative of (B.31), we replace, according to estimate (B.29), the factor $(M_0(\xi)+\langle\zeta\rangle)^\kappa$ in (B.34) by $(M_0(\xi)+\langle\zeta\rangle)^{\kappa(1+\alpha_0+|\alpha|)}$, so that we obtain again a bound of the form (B.13), with still M replaced by $M(x,\xi)M_0(\xi)^{\nu\kappa}$ with $\nu=2+N_0''$.

Study of (B.32). The difference with the preceding case is that the ∂_x-derivative acting on b makes lose an extra factor $M_0(\xi)^\kappa$, and that we do not have in (B.34) the factor in $(1+M_0(\xi)^{-\kappa}|y-z|)^{-N}$. Instead of (B.35), we thus get a bound in

$$CM(x,\xi)M_0(\xi)^{\nu\kappa}\big(1+\beta h^\beta M_0(\xi)\big)^{-N}$$

for some ν depending only on N_0''. On the other hand, if one takes a ∂_y-derivative of (B.32), either it falls on b, which reduces one to an expression of the form (B.31), or on $\tilde a$, so that one gains a factor (B.30) in the estimates. In both cases, it shows that a bound of form (B.13) holds. One studies in the same way the derivatives, and shows that (B.32) provides the hc_1 contribution in (B.22).

If b does not depend on y, then (B.31) vanishes identically so that there is no c_1' contribution in (B.33). If it is independent of x, the term hc_1 given by (B.32) vanishes.

Finally, if one assumes that b is in $S'_{\kappa,\beta}(M'',n'')$, then estimates of the form (B.35), i.e. with the factor $(1+M_0(\xi)^{-\kappa}|y-z|)^{-N}$, hold also for the study of term (B.32), so that we get that c_1 in (B.22) is also in $S'_{\kappa,\beta}(MM_0^\nu,n)$. In the same way, if a is in $S'_{\kappa,\beta}(M',n')$, one gets in (B.29) an extra factor of the form (B.30) on the right-hand side, so that (B.32) is again in $S'_{\kappa,\beta}(M,n)$. This concludes the proof. ∎

Let us write a special case of Proposition B.2.1.

Corollary B.2.3. *Let $p(\xi)=\langle\xi\rangle$ and let $b(y,\xi_1,\dots,\xi_n)$ be a function satisfying estimates*

$$\begin{aligned}
|\partial_\xi^\alpha b(y,\xi)| &\le C\prod_{j=1}^n\langle\xi_j\rangle^{-1}M_0(\xi)^{1+|\alpha|},\\
|\partial_y^{\alpha_0'}\partial_\xi^\alpha b(y,\xi)| &\le C_N\prod_{j=1}^n\langle\xi_j\rangle^{-1}M_0(\xi)^{1+|\alpha|}\langle y\rangle^{-N}
\end{aligned}\tag{B.36}$$

for all $\alpha_0' \in \mathbb{N}^*$, $\alpha \in \mathbb{N}^n$, $N \in \mathbb{N}$. *Then*

$$\begin{aligned}&\mathrm{Op}_h(p(\xi))\big[\mathrm{Op}_h(b)(v_1,\dots,v_n)\big]\\&\quad= \mathrm{Op}_h\big(p(\xi)b(y,\xi)\big)(v_1,\dots,v_n) + \mathrm{Op}_h(c_1')(v_1,\dots,v_n),\end{aligned} \tag{B.37}$$

where c_1' *satisfies*

$$|\partial_y^{\alpha_0'}\partial_\xi^\alpha c_1'(y,\xi)| \le C_N \prod_{j=1}^n \langle \xi_j\rangle^{-1} M_0(\xi)^{1+|\alpha|}\langle y\rangle^{-N} \tag{B.38}$$

for all α_0', α, N.

Proof. We may not directly apply the proposition, as the order function it would provide on the right-hand side of (B.38) would not be the right one. Though, we may apply its proof that shows that the composed operator (B.37) is given by (B.31) with $\tilde{a}$ given by (B.27), i.e.

$$-\frac{i}{2\pi}\int_0^1\int e^{-iz\zeta}p'(\xi_1+\cdots+\xi_n-\lambda\zeta)\frac{\partial b}{\partial y}(y-z,\xi_1,\dots,\xi_n)\,dz\,d\zeta\,d\lambda. \tag{B.39}$$

Performing integrations by parts in z,ζ, we may bound the modulus of (B.39) by

$$C\int \langle z\rangle^{-N}\langle \zeta\rangle^{-N}\langle y-z\rangle^{-N}\,dz\,d\zeta \prod_{j=1}^n\langle\xi_j\rangle^{-1}M_0(\xi)$$

which gives (B.38) performing the same computations for the derivatives. ■

We shall use also the following corollary.

Corollary B.2.4. *Let* b *be a symbol in* $S_{\kappa,\beta}(M,n)$ *for some order function* M, *some* n *in* $\mathbb{N}^*$, *with* (κ,β) *satisfying the assumptions of Proposition* B.2.1. *Assume moreover that* $b(y,x,\xi_1,\dots,\xi_n)$ *is supported inside* $|\xi_1|+\cdots+|\xi_{n-1}| \le C\langle\xi_n\rangle$. *There is* $\nu \ge 0$ *such that for any* $s \ge 0$, *one may write*

$$\langle hD\rangle^s \mathrm{Op}_h\big(b\langle\xi_n\rangle^{-s}\big) = \mathrm{Op}_h(c) \tag{B.40}$$

with a symbol c *in* $S_{\kappa,\beta}(MM_0^\nu,n)$. *The result holds also if* b *(and then* c*) satisfy* (B.13) *with the last exponent* N *replaced by* 2, *i.e. if* b *is in* $S'^2_{\kappa,\beta}(M,n)$, *then* c *lies in* $S'^2_{\kappa,\beta}(MM_0^\nu,n)$.

Proof. We apply Proposition B.2.1 with $a(\xi) = \langle\xi\rangle^s \in S_{\kappa,\beta}(\langle\xi\rangle^s,1)$ (for any (κ,β)) and for second symbol $b(y,x,\xi_1,\dots,\xi_n)\langle\xi_n\rangle^{-s}$. Notice that, because of the support assumption on b, this symbol belongs to the class $S_{\kappa,\beta}(M(x,\xi)(\sum_{j=1}^n\langle\xi_j\rangle)^{-s},n)$. Then by (B.20), c in (B.40) belongs to $S_{\kappa,\beta}(\tilde{M}(x,\xi)M_0^{\nu\kappa},n)$, where ν depends only on the exponent N_0'' in (B.18), which is independent of s, and where $\tilde{M}$ is given, according to (B.19), by

$$\tilde{M}(x,\xi_1,\dots,\xi_n) = \langle\xi_1+\cdots+\xi_n\rangle^s M(x,\xi)\Big(\sum_{j=1}^n\langle\xi_j\rangle\Big)^{-s} \le CM(x,\xi).$$

The conclusion follows, as the last statement of the corollary comes from the fact that when taking a ∂_y-derivative of c given by (B.25), it falls on the b factor as $a(\xi) = \langle \xi \rangle^s$ and makes appear a gain $(1 + M_0(\xi)^{-\kappa}|y - z|)^{-2}$ if we assume that (B.13) holds with last exponent equal to 2. ∎

Let us state a result on the adjoint. Since we shall need it only for linear operators, we limit ourselves to that case.

Proposition B.2.5. *Let $M(x, \xi)$ be an order function on $\mathbb{R} \times \mathbb{R}$ and let a be an element of $S_{0,0}(M, 1)$. Define*

$$a^*(y, x, \xi) = \frac{1}{2\pi} \int e^{-iz\zeta} \bar{a}(y - z, x - hz, \xi - \zeta)\, dz\, d\zeta. \tag{B.41}$$

Then a^* belongs to $S_{0,0}(M, 1)$ and $(\mathrm{Op}_h(a))^* = \mathrm{Op}_h(a^*)$.

Proof. By a direct computation $(\mathrm{Op}_h(a))^*$ is given by $\mathrm{Op}_h(a^*)$ if a^* is defined by (B.41). Making ∂_z and ∂_ζ integrations by parts, one checks that a^* belongs to the wanted class. ∎

Remark. It follows from (B.25), (B.31), (B.32), that if a, b in the statement of Proposition B.2.1 satisfy

$$\begin{aligned} a(-y, -x, -\xi_1, \dots, -\xi_{n'}) &= (-1)^{n'-1} a(y, x, \xi_1, \dots, \xi_{n'}), \\ b(-y, -x, -\xi_1, \dots, -\xi_{n''}) &= (-1)^{n''-1} b(y, x, \xi_1, \dots, \xi_{n''}), \end{aligned} \tag{B.42}$$

then symbol c in (B.22) satisfies

$$c(-y, -x, -\xi_1, \dots, -\xi_n) = (-1)^{n-1} a(y, x, \xi_1, \dots, \xi_n) \tag{B.43}$$

and a similar statement for c_1, c_1'. One has an analogous property for a^*.

To conclude this appendix, let us translate Propositions B.2.1 and B.2.5 in the framework of the non-semiclassical quantization introduced in (B.16) and (B.17).

Corollary B.2.6. *The following statements hold.*

(i) *Let n', n'' be in $\mathbb{N}^*$, $n = n' + n'' - 1$, M', M'' two order functions on $\mathbb{R} \times \mathbb{R}^{n'}$ and $\mathbb{R} \times \mathbb{R}^{n''}$, respectively. Let (κ, β) be in $\mathbb{N} \times [0, 1]$, a in $S_{\kappa,\beta}(M', n')$, b in $S_{\kappa,\beta}(M'', n'')$. Assume that either $(\kappa, \beta) = (0, 0)$ or $0 < \kappa\beta \leq 1$ or that b is independent of x. Then if M is defined in (B.19), there are ν in $\mathbb{N}$, symbols c_1 in $S_{\kappa,\beta}(MM_0^{\nu\kappa}, n)$, c_1' in $S'_{\kappa,\beta}(MM_0^{\nu\kappa}, n)$ such that if*

$$\begin{aligned} c(y, x, \xi_1, \dots, \xi_n) = a(y, x, \xi_1, \dots, \xi_{n'-1}, \xi_{n'} + \dots + \xi_n) \\ \times b(y, x, \xi_{n'}, \dots, \xi_n) \\ + t^{-1} c_1(y, x, \xi_1, \dots, \xi_n) \\ + c_1'(y, x, \xi_1, \dots, \xi_n), \end{aligned} \tag{B.44}$$

then for any functions $v_1, \dots, v_n$,

$$\mathrm{Op}^t(a)[v_1, \dots, v_{n'-1}, \mathrm{Op}^t(b)(v_{n'}, \dots, v_n)] = \mathrm{Op}^t(c)[v_1, \dots, v_n]. \quad \text{(B.45)}$$

Moreover, if b is independent of x, then c_1 vanishes in (B.44). Finally, if a is in $S'_{\kappa,\beta}(M', n')$ or b is in $S'_{\kappa,\beta}(M'', n'')$, then c is in $S'_{\kappa,\beta}(MM_0^{\nu\kappa}, n)$.

(ii) *In the same way, if a is in $S_{0,0}(M, 1)$, then $\mathrm{Op}^t(a)^* = \mathrm{Op}^t(a^*)$, for a symbol a^* in the same class. Moreover, if a satisfies (B.42), so does a^*.*

Proof. Statement (i) is just the translation of Proposition B.2.1. Statement (ii) follows from Proposition B.2.5. ∎

We get also translating Corollary B.2.3:

Corollary B.2.7. *Under the assumptions and notation of Corollary B.2.3, one has*

$$\begin{aligned}&\mathrm{Op}\big(p(\xi)\big)\mathrm{Op}(b)(v_1, \dots, v_n)\\ &\quad = \mathrm{Op}(p(\xi_1 + \dots + \xi_n)b)(v_1, \dots, v_n) + \mathrm{Op}(c'_1)(v_1, \dots, v_n)\end{aligned}$$

with c'_1 in the class $\tilde{S}'_{1,0}(\prod_{j=1}^n \langle \xi_j \rangle^{-1} M_0(\xi), n)$ of Definition 3.1.1.

We shall use also:

Corollary B.2.8. *Let $n \geq 2$. Let $M(\xi_1, \dots, \xi_n)$ be an order function on $\mathbb{R}^n$ (independent of x) and let $a(y, \xi_1, \dots, \xi_n)$ be a symbol in $S_{\kappa,0}(M, n)$, independent of x, for some κ in $\mathbb{N}$. Let Z be a function in $\mathcal{S}(\mathbb{R})$. Denote*

$$\tilde{M}(\xi_1, \dots, \xi_{n-1}) = M(\xi_1, \dots, \xi_{n-1}, 0).$$

There is a symbol a' in $S'_{\kappa,0}(\tilde{M}, n-1)$, independent of x, such that for any test functions $v_1, \dots, v_{n-1}$,

$$\mathrm{Op}(a)[v_1, \dots, v_{n-1}, Z] = \mathrm{Op}(a')[v_1, \dots, v_{n-1}]. \quad \text{(B.46)}$$

Moreover, if Z is odd and $a(-y, -\xi_1, \dots, -\xi_n) = (-1)^{n-1} a(y, \xi_1, \dots, \xi_n)$, then

$$a'(-y, -\xi_1, \dots, -\xi_{n-1}) = (-1)^{n-2} a(y, \xi_1, \dots, \xi_{n-1}).$$

Proof. By (B.17), we have that (B.46) holds if we define

$$a'(y, \xi_1, \dots, \xi_{n-1}) = \frac{1}{2\pi} \int e^{iy\xi_n} a(y, \xi_1, \dots, \xi_{n-1}, \xi_n) \hat{Z}(\xi_n)\, d\xi_n. \quad \text{(B.47)}$$

If $\alpha' = (\alpha_1, \dots, \alpha_{n-1}) \in \mathbb{N}^{n-1}$ and $\xi' = (\xi_1, \dots, \xi_{n-1})$, we deduce from (B.12) with $\beta = 0$ that

$$|\partial_{\xi'}^{\alpha'} a'(y, \xi_1, \dots, \xi_{n-1})| \leq C \int M(\xi', \xi_n) M_0(\xi', \xi_n)^{\kappa|\alpha'|} |\hat{Z}(\xi_n)|\, d\xi_n.$$

Using (B.9) both for M and M_0, we obtain a bound in $\tilde{M}(\xi')M_0(\xi')^{\kappa|\alpha'|}$. To check that actually our symbol a' is in $S'_{\kappa,0}(\tilde{M}, n-1)$, i.e. that it is rapidly decaying in $(1 + M_0(\xi')^{-\kappa}|y|)^{-N}$, we just make in (B.47) ∂_{ξ_n}-integrations by parts, and perform the same estimate. One bounds ∂_y-derivatives in the same way. Finally, the last statement of the corollary follows from (B.47) and the oddness of $\hat{Z}$. ∎

Appendix C

Bounds for forced linear Klein–Gordon equations

The goal of this appendix is to obtain some Sobolev or L^∞ estimates of solutions of half-Klein–Gordon equations with zero initial data and force term that is time oscillating. The kind of equations we want to study is of the form

$$\begin{aligned}\big(D_t - \sqrt{1+D_x^2}\big)U &= e^{i\lambda t} t_\varepsilon^{-1} M(x),\\ U|_{t=1} &= 0,\end{aligned} \tag{C.1}$$

where M is in $\mathcal{S}(\mathbb{R})$, $t_\varepsilon^{-1} = \frac{\varepsilon^2}{1+t\varepsilon^2}$ and λ is a real number different from one. This restriction means that the right-hand side of the equation oscillates at a frequency which is non-characteristic when one restricts the symbol $\sqrt{1+\xi^2}$ of the operator on the left-hand side to frequency zero. Our goal is to prove estimates for U or $L_+U = (x + t\frac{D_x}{\langle D_x\rangle})U$ for large times. Actually, we shall split the solution as $U = U' + U''$, where U' is obtained writing the Duhamel formula to express U and restricting the time integral to times that are $O(\sqrt{t})$. It turns out that, when time t stays smaller than ε^{-4+0}, $L_+U'(t,\cdot)$ has L^2 estimates that are $o(t^{\frac{1}{4}})$, which is acceptable for our applications. On the other hand L_+U'' would not enjoy such bounds, but it has good estimates in L^∞-like spaces.

Equation (C.1) is actually just a simplified model of the problem we study in this Appendix. For the applications to our main problem, i.e. the description of some approximate solutions (see Section 2.5 of Chapter 2), we need more general right-hand sides than in (C.1). Though, the method of proof of our estimates is quite the same as for the model above. It relies on the explicit writing of the solution using Duhamel formula and the stationary phase formula.

We shall close this appendix with explicit computations that are used in the main part of this text to check Fermi's golden rule.

C.1 Linear solutions to half-Klein–Gordon equations

We consider a function $(t,x) \mapsto M(t,x)$ that is C^1 in time, with values in $\mathcal{S}(\mathbb{R})$. If λ is in $\mathbb{R}$, $\lambda \neq 1$, we denote by $U(t,x)$ the solution to

$$\begin{aligned}(D_t - p(D_x))U &= e^{i\lambda t} M(t,x),\\ U|_{t=1} &= 0,\end{aligned} \tag{C.2}$$

where $p(D_x) = \sqrt{1+D_x^2}$, and where we study the solution for t in an interval $[1,T]$. We write the solution by Duhamel formula as

$$U(t,x) = i\int_1^t e^{i(t-\tau)p(D_x)+i\lambda\tau} M(\tau,\cdot)\,d\tau. \tag{C.3}$$

We fix some function χ in $C^\infty(\mathbb{R})$, equal to one close to $]-\infty, \frac{1}{4}]$, supported in $]-\infty, \frac{1}{2}]$. Then for t larger than some constant (say $t \geq 16$), we may write (C.3) as $U = U' + U''$ where

$$\begin{aligned} U'(t,x) &= i \int_1^{+\infty} e^{i(t-\tau)p(D_x)+i\lambda\tau} \chi\Big(\frac{\tau}{\sqrt{t}}\Big) M(\tau,\cdot)\, d\tau \\ U''(t,x) &= i \int_{-\infty}^{t} e^{i(t-\tau)p(D_x)+i\lambda\tau} (1-\chi)\Big(\frac{\tau}{\sqrt{t}}\Big) M(\tau,\cdot)\, d\tau. \end{aligned} \tag{C.4}$$

Our goal is to obtain Sobolev and L^∞ estimates for U', U'' and for the result of the action on U', U'' of the operator

$$L_\pm = x \pm tp'(D_x) = x \pm t\frac{D_x}{\langle D_x\rangle}, \tag{C.5}$$

under two sets of assumptions on M, that we describe now. We shall take ε in $]0,1]$ and for $t \geq 1$, we recall that we defined in (4.1)

$$t_\varepsilon = \varepsilon^{-2}\langle t\varepsilon^2\rangle = (\varepsilon^{-4} + t^2)^{\frac{1}{2}}. \tag{C.6}$$

For ω in $[1, +\infty[$, $\theta' \in]0, \frac{1}{2}[$, close to $\frac{1}{2}$, we introduce the following:

Assumption (H1)$_\omega$. For any α, N in $\mathbb{N}$, any t in $[1,T]$, x in $\mathbb{R}$, ε in $]0,1]$, one has bounds

$$\begin{aligned} |\partial_x^\alpha M(t,x)| &\leq C_{\alpha,N} t_\varepsilon^{-\omega}\langle x\rangle^{-N}, \\ |\partial_x^\alpha \partial_t M(t,x)| &\leq C_{\alpha,N} t_\varepsilon^{-\omega+\frac{1}{2}}(t_\varepsilon^{-\frac{3}{2}} + t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'})\langle x\rangle^{-N}. \end{aligned} \tag{C.7}$$

The second type of assumption we shall make on M is more technical. If $\lambda > 1$, we denote by $\pm\xi_\lambda$ the two roots of $\sqrt{1+\xi^2} = \lambda$ (with $\xi_\lambda > 0$) and set $\mathcal{W}_\lambda$ for a small open neighborhood of the set $\{\xi_\lambda, -\xi_\lambda\}$. We introduce:

Assumption (H2). For any α, N, the x-Fourier transform of $M(t,x)$ satisfies bounds

$$\begin{aligned} |\partial_\xi^\alpha \hat{M}(t,\xi)| &\leq C_{\alpha,N} t^{-\frac{1}{2}} t_\varepsilon^{-1}\langle \xi\rangle^{-N}, \\ |\partial_t\partial_\xi^\alpha \hat{M}(t,\xi)| &\leq C_{\alpha,N} t^{-\frac{3}{4}} t_\varepsilon^{-1}\langle \xi\rangle^{-N}. \end{aligned} \tag{C.8}$$

Moreover, for ξ in $\mathcal{W}_\lambda$, one may decompose

$$D_t\hat{M}(t,\xi) = (D_t + \lambda - \sqrt{1+\xi^2})\Phi(t,\xi) + \Psi(t,\xi), \tag{C.9}$$

where Φ, Ψ satisfy the following bounds:

$$\begin{aligned} |\Phi(t,\xi)| &\leq Ct^{-\frac{1}{2}} t_\varepsilon^{-1}, \\ |\Psi(t,\xi)| &\leq Ct^{-1} t_\varepsilon^{-1} \end{aligned} \tag{C.10}$$

and a similar decomposition holds for xM instead of M. Of course, conditions (C.9) and (C.10) are void if $\lambda < 1$.

For future reference, let us state some elementary inequalities that hold if $\theta' < \frac{1}{2}$ is close enough to $\frac{1}{2}$, $\varepsilon^2\sqrt{t} \leq 1$ and $\omega \geq 1$:

$$\begin{aligned}\int_1^{\sqrt{t}} \tau_\varepsilon^{-\omega+\frac{1}{2}}\big(\tau_\varepsilon^{-\frac{3}{2}} + \tau^{-\frac{3}{2}}(\varepsilon^2\sqrt{\tau})^{\frac{3}{2}\theta'}\big)\,d\tau &\leq C\varepsilon^{2\omega}(\varepsilon^2\sqrt{t} + \varepsilon^{3\theta'-1}) \\ &\leq C\varepsilon^{2\omega},\end{aligned} \tag{C.11}$$

$$\begin{aligned}&\int_{\sqrt{t}}^{t} \tau_\varepsilon^{-\omega+\frac{1}{2}}\big(\tau_\varepsilon^{-\frac{3}{2}} + \tau^{-\frac{3}{2}}(\varepsilon^2\sqrt{\tau})^{\frac{3}{2}\theta'}\big)\,d\tau \\ &\quad\leq C\varepsilon^{2\omega}\Big(\frac{\varepsilon^2 t}{\langle\varepsilon^2 t\rangle} + \varepsilon^{\frac{3}{2}\theta'}(\varepsilon^2\sqrt{t})^{-\frac{1}{2}+\frac{3}{4}\theta'}\Big) \\ &\quad\leq C\min\Big(\varepsilon^{2\omega-1}\Big(\frac{\varepsilon^2 t}{\langle\varepsilon^2 t\rangle}\Big)^{\frac{1}{2}}, \varepsilon^{2\omega}\Big),\end{aligned} \tag{C.12}$$

$$\int_1^{\sqrt{t}} \tau^a\tau_\varepsilon^{-\omega}\,d\tau \leq C\varepsilon^{2\omega}t^{\frac{1}{2}+\frac{a}{2}}, \quad a > -1, \tag{C.13}$$

$$\int_{\sqrt{t}}^{t} \tau^{-a}\tau_\varepsilon^{-1}\,d\tau \leq C\varepsilon^{2a}\Big(\frac{\varepsilon^2 t}{\langle\varepsilon^2 t\rangle}\Big)^{1-a} \leq C\varepsilon\Big(\frac{\varepsilon^2 t}{\langle\varepsilon^2 t\rangle}\Big)^{\frac{1}{2}}, \quad \frac{1}{2} \leq a < 1, \tag{C.14}$$

$$\begin{aligned}&\int_{\sqrt{t}}^{t} \tau_\varepsilon^{-\omega+\frac{1}{2}}\big(\tau_\varepsilon^{-\frac{3}{2}} + \tau^{-\frac{3}{2}}(\varepsilon^2\sqrt{\tau})^{\frac{3}{2}\theta'}\big)\sqrt{\tau}\,d\tau \\ &\quad\leq C\varepsilon^{2\omega-1}\Big(\Big(\frac{\varepsilon^2 t}{\langle\varepsilon^2 t\rangle}\Big)^{\frac{3}{2}} + \varepsilon^{\frac{3}{2}\theta'}\Big(\frac{\varepsilon^2 t}{\langle\varepsilon^2 t\rangle}\Big)^{\frac{3}{4}\theta'}\Big) \\ &\quad\leq C\varepsilon^{2\omega-1}\Big(\frac{\varepsilon^2 t}{\langle\varepsilon^2 t\rangle}\Big)^{\frac{1}{2}},\end{aligned} \tag{C.15}$$

$$\int_{\sqrt{t}}^{t} \tau^{\frac{1}{2}}\tau_\varepsilon^{-1}\,d\tau \leq C\sqrt{t}\frac{\varepsilon^2 t}{\langle\varepsilon^2 t\rangle}. \tag{C.16}$$

Let us state two propositions giving the bounds we shall get for U', U'' under either Assumption (H1)$_\omega$ or Assumption (H2). We denote below

$$\|v\|_{W^{\rho,\infty}} = \|\langle D_x\rangle^\rho v\|_{L^\infty} \tag{C.17}$$

for any $\rho \geq 0$.

Proposition C.1.1. *The following statements hold.*

(i) *Assume that* (H1)$_\omega$ *holds for some* $\omega \geq 1$*. Then for any* $r \geq 0$*, there is* $C_r > 0$ *such that* U' *given by* (C.4) *satisfies for any* $\varepsilon \in]0,1]$, $t \in [1, \varepsilon^{-4}]$,

$$\|U'(t,\cdot)\|_{H^r} \leq C_r\varepsilon(\varepsilon^{2(\omega-1)}(\varepsilon^2\sqrt{t})^{\frac{1}{2}}), \tag{C.18}$$

$$\|U'(t,\cdot)\|_{W^{r,\infty}} \leq C_r\varepsilon^{2\omega}, \tag{C.19}$$

$$\|L_+U'(t,\cdot)\|_{H^r} \leq C_r t^{\frac{1}{4}}(\varepsilon^{2(\omega-1)}(\varepsilon^2\sqrt{t})). \tag{C.20}$$

(ii) *Under Assumption (H2), there is, for any $r \geq 1$, a constant $C_r > 0$ such that U' satisfies for any $\varepsilon \in]0, 1]$, $t \in [1, \varepsilon^{-4}]$,*

$$\|U'(t,\cdot)\|_{H^r} \leq C_r \varepsilon (\varepsilon^2 \sqrt{t})^{\frac{1}{2}}, \tag{C.21}$$

$$\|U'(t,\cdot)\|_{W^{r,\infty}} \leq C_r \varepsilon^2 t^{-\frac{1}{4}}, \tag{C.22}$$

$$\|L_+ U'(t,\cdot)\|_{H^r} \leq C_r t^{\frac{1}{4}} (\varepsilon^{\frac{1}{8}} (\varepsilon^2 \sqrt{t})^{\frac{7}{8}}). \tag{C.23}$$

Let us state now the bounds we shall prove for U''.

Proposition C.1.2. *The following statements hold.*

(i) *Under Assumption $(H1)_\omega$ with $\omega \geq 1$, one has for any $r \geq 0$, the following bounds:*

$$\|U''(t,\cdot)\|_{H^r} \leq C_r \varepsilon^{2\omega-1} \Big(\frac{\varepsilon^2 t}{\langle \varepsilon^2 t\rangle}\Big)^{\frac{1}{2}}, \tag{C.24}$$

$$\|U''(t,\cdot)\|_{W^{r,\infty}} \leq C_r \varepsilon^{2\omega} \log(1+t), \tag{C.25}$$

$$\|L_+ U''(t,\cdot)\|_{W^{r,\infty}} \leq C_r \log(1+t) \log(1+\varepsilon^2 t) \qquad \text{if } \omega = 1, \tag{C.26}$$

$$\|L_+ U''(t,\cdot)\|_{W^{r,\infty}} \leq C_r \varepsilon^{2(\omega-1)} \log(1+t) \Big(\frac{\varepsilon^2 t}{\langle \varepsilon^2 t\rangle}\Big) \qquad \text{if } \omega > 1. \tag{C.27}$$

(ii) *Under Assumption (H2), one has for any $r \geq 0$, the following bounds:*

$$\|U''(t,\cdot)\|_{H^r} \leq C_r \varepsilon \Big(\frac{\varepsilon^2 t}{\langle \varepsilon^2 t\rangle}\Big)^{\frac{1}{2}}, \tag{C.28}$$

$$\|U''(t,\cdot)\|_{W^{r,\infty}} \leq C_r \varepsilon^2 (\log(1+t))^2, \tag{C.29}$$

$$\|L_+ U''(t,\cdot)\|_{W^{r,\infty}} \leq C_r \log(1+t) \log(1+\varepsilon^2 t). \tag{C.30}$$

Remark. Notice that we obtain Sobolev estimates for $L_+ U'(t,\cdot)$ in (C.20), (C.23), while we bound $L_+ U''(t,\cdot)$ in $W^{r,\infty}$ spaces in (C.26), (C.27), (C.30). Actually, we could not obtain for the $L_+ U''$ contribution to $L_+ U$ as good Sobolev estimates as those that hold for $L_+ U'$, and this is the reason for our splitting $U = U' + U''$.

Study of the U' contribution. We shall prove Proposition C.1.1. By (C.4) and (C.5)

$$U'(t,x) = \frac{i}{2\pi} \int_1^{+\infty} \int e^{i((t-\tau)\sqrt{1+\xi^2} + \lambda\tau + x\xi]} \chi\Big(\frac{\tau}{\sqrt{t}}\Big) \hat{M}(\tau,\xi)\, d\xi\, d\tau \tag{C.31}$$

and

$$\begin{aligned} L_+ U'(t,x) = \frac{i}{2\pi} \int_1^{+\infty} \int & e^{i((t-\tau)\sqrt{1+\xi^2} + \lambda\tau + x\xi)} \chi\Big(\frac{\tau}{\sqrt{t}}\Big) \\ & \times \Big(\tau \frac{\xi}{\langle \xi\rangle} \hat{M}(\tau,\xi) + \widehat{xM}(\tau,\xi)\Big)\, d\xi\, d\tau. \end{aligned} \tag{C.32}$$

We shall estimate first the above integrals when either $\lambda < 1$, so that the coefficient of τ in the phase $\lambda - \sqrt{1+\xi^2}$ never vanishes, or when $\lambda > 1$ but $\hat{M}(\tau,\xi)$ is supported outside a neighborhood of the two roots $\pm\xi_\lambda$ of that expression.

Lemma C.1.3. *Assume that either $\lambda < 1$ or $\lambda > 1$ and there is a neighborhood $\mathcal{W}_\lambda$ of $\{-\xi_\lambda, \xi_\lambda\}$ such that $\hat{M}(\cdot, \xi)$ vanishes for ξ in $\mathcal{W}_\lambda$. Assume also $t \le \varepsilon^{-4}$.*

(i) *Under Assumption (H1)$_\omega$, estimates (C.18)–(C.20) hold true.*

(ii) *Under Assumption (H2), estimates (C.21)–(C.23) hold true.*

Proof. Let us prove first the Sobolev bounds (C.18), (C.20), (C.21) and (C.23). By (C.31), $\hat{U}'(t,\xi)$ may be written as

$$e^{it\sqrt{1+\xi^2}} \int_1^{+\infty} e^{i(\lambda-\sqrt{1+\xi^2})\tau} \chi\Big(\frac{\tau}{\sqrt{t}}\Big) N(\tau,\xi)\, d\tau, \tag{C.33}$$

where $N(\tau,\xi)$ satisfies for any N, any α, according to (C.7) and (C.8),

$$|\partial_\xi^\alpha \partial_\tau^j N(\tau,\xi)| \le C_{\alpha,N} \tau_\varepsilon^{-\omega+\frac{j}{2}} \big(\tau_\varepsilon^{-\frac{3}{2}} + \tau^{-\frac{3}{2}}(\varepsilon^2\sqrt{\tau})^{\frac{3}{2}\theta'}\big)^j \langle\xi\rangle^{-N}, \quad j=0,1, \tag{C.34}$$

under Assumption (H1)$_\omega$ and

$$|\partial_\xi^\alpha \partial_\tau^j N(\tau,\xi)| \le C_{\alpha,N} \tau^{-\frac{1}{2}} \tau_\varepsilon^{-1} \tau^{-\frac{j}{4}} \langle\xi\rangle^{-N}, \quad j=0,1, \tag{C.35}$$

under Assumption (H2). In the same way, by (C.32), $\widehat{L_+U'}(t,\xi)$ may be written under the form (C.33), where N satisfies, according to (C.7) and (C.8),

$$|\partial_\xi^\alpha \partial_\tau^j N(\tau,\xi)| \le C_{\alpha,N} \tau^{1-j} \tau_\varepsilon^{-\omega} \langle\xi\rangle^{-N}, \quad j=0,1, \tag{C.36}$$

under Assumption (H1)$_\omega$ and

$$|\partial_\xi^\alpha \partial_\tau^j N(\tau,\xi)| \le C_{\alpha,N} \tau^{\frac{1}{2}-\frac{j}{4}} \tau_\varepsilon^{-1} \langle\xi\rangle^{-N}, \quad j=0,1, \tag{C.37}$$

under Assumption (H2).

Since $N(\tau,\xi)$ is supported outside a neighborhood of the zeros of $\sqrt{1+\xi^2}-\lambda$, we may perform in integral (C.33) one ∂_τ-integration by parts. Taking moreover an $L^2(\langle\xi\rangle^r d\xi)$ norm, we obtain quantities bounded in the following way:

- If N satisfies (C.34), we obtain a control of (C.33) in terms of $C\varepsilon^{2\omega}$ and of (C.11). This gives an $\varepsilon^{2\omega}$ estimate, better than the right-hand side (C.18).
- If N satisfies (C.35), we obtain an upper bound by the right-hand side of (C.13), which is better than (C.21).
- If N satisfies (C.36), the $L^2(\langle\xi\rangle^r d\xi)$ norm of (C.33) is bounded by (C.13) with $a=0$, so by (C.20).
- If N satisfies (C.37), that same norm is bounded by (C.13), thus by the right-hand side of (C.23).

We have thus proved Lemma C.1.3 for Sobolev estimates. It remains to establish (C.19) and (C.22). Since $\hat{M}$ is rapidly decaying in ξ, it is sufficient to estimate the L^∞ norm of U'. Notice that the $d\xi$-integral in (C.31) may be written as

$$\int e^{it\left(\left(1-\frac{\tau}{t}\right)\sqrt{1+\xi^2}+\frac{x}{t}\xi\right)} \hat{M}(\tau,\xi)\, d\xi \tag{C.38}$$

and that on the support of $\chi(\tau/\sqrt{t})$, $|\tau/t| \ll 1$, so that the stationary phase formula implies that (C.38) is smaller in modulus than $Ct^{-\frac{1}{2}}\tau_\varepsilon^{-\omega}\mathbb{1}_{\tau<\sqrt{t}}$ under conditions (C.7) and $Ct^{-\frac{1}{2}}\tau^{-\frac{1}{2}}\tau_\varepsilon^{-1}\mathbb{1}_{\tau<\sqrt{t}}$ under condition (C.8). Integrating in τ, we get bounds in $O(\varepsilon^{2\omega})$ and $O(\varepsilon^2 t^{-\frac{1}{4}})$, respectively, as in (C.19) and (C.22). This concludes the proof. ∎

Lemma C.1.3 provides Proposition C.1.1 when either $\lambda < 1$ or $\lambda > 1$ and $\hat{M}$ in (C.31) and (C.32) is cut-off outside a neighborhood of $\sqrt{1+\xi^2} = \lambda$. We have thus to study now the case when $\lambda > 1$ and $\hat{M}$ is supported in a small neighborhood of one of the roots $\pm\xi_\lambda$ of that equation. More precisely, we have to study, in order to estimate the contribution to U', the expressions

$$\tilde{U}'_\pm(t,x) = \int_1^{+\infty}\int e^{it\left(\left(1-\frac{\tau}{t}\right)\sqrt{1+\xi^2}+\lambda\frac{\tau}{t}+\frac{x}{t}\xi\right)}\chi\Big(\frac{\tau}{\sqrt{t}}\Big)N_\pm(\tau,\xi)\,d\tau\,d\xi, \tag{C.39}$$

where $N_\pm$ is supported close to $\pm\xi_\lambda$ and satisfies (C.34) or (C.35), and, in order to estimate the contribution to L_+U', an expression of the form (C.39) with $N_\pm$ satisfying (C.36) or (C.37). We shall show actually the more precise result:

Proposition C.1.4. *For any α in $\mathbb{N}$, we have the following bounds:*

$$|\partial_x^\alpha \tilde{U}'_\pm(t,x)| \le C_\alpha \varepsilon^{2\omega}\langle t^{-\frac{1}{2}}(\lambda x \pm t\xi_\lambda)\rangle^{-1} \tag{C.40}$$

if $N_\pm$ satisfies (C.34),

$$|\partial_x^\alpha \tilde{U}'_\pm(t,x)| \le C_\alpha \varepsilon^{2} t^{-\frac{1}{4}}\langle t^{-\frac{7}{8}}(\lambda x \pm t\xi_\lambda)\rangle^{-1} \tag{C.41}$$

if $N_\pm$ satisfies (C.35),

$$|\partial_x^\alpha \tilde{U}'_\pm(t,x)| \le C_\alpha \varepsilon^{2\omega} t^{\frac{1}{2}}\langle t^{-\frac{1}{2}}(\lambda x \pm t\xi_\lambda)\rangle^{-1} \tag{C.42}$$

if $N_\pm$ satisfies (C.36), and

$$|\partial_x^\alpha \tilde{U}'_\pm(t,x)| \le C_\alpha \varepsilon^{2} t^{\frac{1}{4}}\langle t^{-\frac{7}{8}}(\lambda x \pm t\xi_\lambda)\rangle^{-1} \tag{C.43}$$

if $N_\pm$ satisfies (C.37).

It follows immediately from (C.40) (resp. (C.41)) that (C.18) and (C.19) (resp. (C.21) and (C.22)) hold true. In the same way, computing the L^2 norms of (C.42) (resp. (C.43)) we obtain upper bounds by (C.20) (resp. (C.23)). Consequently, Proposition C.1.1 will be proved if we establish Proposition C.1.4.

Lemma C.1.5. *One may write the derivatives of $\tilde{U}'_\pm$ given by (C.39) under the form*

$$\partial_x^\alpha \tilde{U}'_\pm(t,x) = \int_1^{+\infty} e^{i\psi_\pm(\tau,t,z_\pm)}\tilde{\chi}_\pm(t,\tau,z_\pm)J_\alpha(\tau,t,z_\pm)\,d\tau + R_\alpha^\pm, \tag{C.44}$$

where $\tilde{\chi}_\pm$ is supported for $\tau \le \sqrt{t}$ and for $|z_\pm| \le c$, and where

$$z_\pm = \frac{x}{t} \pm \frac{\xi_\lambda}{\lambda}, \quad \tilde{\chi}_\pm = O(1), \quad \partial_\tau\tilde{\chi}_\pm = O(t^{-\frac{1}{2}}), \tag{C.45}$$

where $\psi_\pm(\tau, t, z_\pm)$ *satisfies*

$$|\partial_\tau \psi_\pm(\tau, t, z_\pm)| \sim |z_\pm|, \quad \partial_\tau^2 \psi_\pm = 0 \tag{C.46}$$

on the support of the integrand, if t is large enough, and where J_α *satisfies the bounds*

$$\begin{aligned} |J_\alpha(\tau, t, z_\pm)| &\leq C_\alpha t^{-\frac{1}{2}} \tau_\varepsilon^{-\omega}, \\ |\partial_\tau J_\alpha(\tau, t, z_\pm)| &\leq C_\alpha t^{-\frac{1}{2}} \tau_\varepsilon^{-\omega+\frac{1}{2}} \big(\tau_\varepsilon^{-\frac{3}{2}} + t^{-1}\tau_\varepsilon^{-\frac{1}{2}} + \tau^{-\frac{3}{2}}(\varepsilon^2\sqrt{\tau})^{\frac{3}{2}\theta'}\big) \end{aligned} \tag{C.47}$$

if $N_\pm$ *satisfies* (C.34)*, and*

$$\begin{aligned} |J_\alpha(\tau, t, z_\pm)| &\leq C_\alpha t^{-\frac{1}{2}} \tau_\varepsilon^{-1} \tau^{-\frac{1}{2}}, \\ |\partial_\tau J_\alpha(\tau, t, z_\pm)| &\leq C_\alpha t^{-\frac{1}{2}} \tau_\varepsilon^{-1} \tau^{-\frac{3}{4}} \end{aligned} \tag{C.48}$$

if $N_\pm$ *satisfies* (C.35).

In the same way, $\partial_x^\alpha \tilde{U}'_\pm$ *is given by an integral of the form* (C.44) *with* J_α *satisfying*

$$\begin{aligned} |J_\alpha(\tau, t, z_\pm)| &\leq C_\alpha t^{-\frac{1}{2}} \tau_\varepsilon^{-\omega} \tau, \\ |\partial_\tau J_\alpha(\tau, t, z_\pm)| &\leq C_\alpha t^{-\frac{1}{2}} \tau_\varepsilon^{-\omega} \end{aligned} \tag{C.49}$$

if $N_\pm$ *satisfies* (C.36)*, and*

$$\begin{aligned} |J_\alpha(\tau, t, z_\pm)| &\leq C_\alpha t^{-\frac{1}{2}} \tau_\varepsilon^{-1} \tau^{\frac{1}{2}}, \\ |\partial_\tau J_\alpha(\tau, t, z_\pm)| &\leq C_\alpha t^{-\frac{1}{2}} \tau_\varepsilon^{-1} \tau^{\frac{1}{4}} \end{aligned} \tag{C.50}$$

if $N_\pm$ *satisfies* (C.37)*. Finally, the remainder* $R_\alpha^\pm$ *in* (C.44) *satisfies*

$$\begin{aligned} |R_\alpha^\pm| &\leq C_{\alpha,N} \varepsilon^{2\omega} t^{-N} \langle \lambda x \pm t\xi_\lambda \rangle^{-N} \quad \text{under (H1)}_\omega, \\ |R_\alpha^\pm| &\leq C_{\alpha,N} \varepsilon^{2} t^{-N} \langle \lambda x \pm t\xi_\lambda \rangle^{-N} \quad \text{under (H2)}, \end{aligned} \tag{C.51}$$

for any N *in* $\mathbb{N}$.

Proof. For t bounded, estimates of the form (C.51) follow from (C.34), (C.36) and ∂_ξ-integration by parts. Assume $t \gg 1$. We treat the case of sign $+$ and set z for z_+ in (C.45). We consider the $d\xi$ integral in (C.39), expressed in terms of z instead of x. The oscillatory phase may be written as $t\phi(t, \tau, z, \xi)$ with

$$\frac{\partial \phi}{\partial \xi}(t, \tau, z, \xi) = \Big(\frac{\xi}{\sqrt{1+\xi^2}} - \frac{\xi_\lambda}{\lambda}\Big) - \frac{\tau}{t}\frac{\xi}{\sqrt{1+\xi^2}} + z. \tag{C.52}$$

Since we assume $t \gg 1$, $\frac{\tau}{t} \leq \frac{1}{\sqrt{t}} \ll 1$ in (C.52). If $|z| \geq c > 0$, under this condition on t, and for $|\xi - \xi_\lambda| \ll 1$, we see from (C.52) that $\big|\frac{\partial\phi}{\partial\xi}(t, \tau, z, \xi)\big| \sim |z|$, so that, performing ∂_ξ-integration by parts, we get again estimates of the form (C.51).

We may thus assume from now on that $t \gg 1$, $|z| \ll 1$. For $z = 0$, $\frac{\tau}{t} = 0$, (C.52) vanishes at $\xi = \xi_\lambda$, and since the ∂_ξ-derivative at this point is $\lambda^{-3} \neq 0$, we have for $t \gg 1$, $|z| \ll 1$, a unique critical point $\xi(t, \tau, z)$ close to ξ_λ. Moreover, it follows from (C.52) that

$$\frac{\partial \xi}{\partial \tau}(t,\tau,z) = O\Big(\frac{1}{t}\Big), \quad \frac{\partial^2 \xi}{\partial \tau^2}(t,\tau,z) = O\Big(\frac{1}{t^2}\Big). \tag{C.53}$$

We rewrite the phase ϕ as

$$\phi(t,\tau,z,\xi) = \phi^c(t,\tau,z) + \frac{1}{2}A(t,\tau,z,\xi)^2(\xi - \xi(t,\tau,z))^2, \tag{C.54}$$

where the critical value $\phi^c(t,\tau,z)$ satisfies

$$|\partial_\tau \phi^c(t,\tau,z)| = O(t^{-1}), \quad |\partial_\tau^2 \phi^c(t,\tau,z)| = O(t^{-2}) \tag{C.55}$$

and where A is strictly positive for $\frac{\tau}{t} \ll 1$, $|z| \ll 1$, $|\xi - \xi_\lambda| \ll 1$ and satisfies for any γ,

$$|\partial_\tau \partial_\xi^\gamma A(t,\tau,z,\xi)| = O(t^{-1}). \tag{C.56}$$

We introduce the change of variables $\zeta = A(t,\tau,z,\xi)(\xi - \xi(t,\tau,z))$ for ξ close to ξ_λ and its inverse $\xi = \Xi(t,\tau,z,\zeta)$. By (C.53) and (C.56), we have

$$\frac{\partial \zeta}{\partial \tau} = O(t^{-1}), \quad \frac{\partial^{\gamma+1}\Xi}{\partial \zeta^\gamma \partial \tau} = O(t^{-1}) \tag{C.57}$$

for any γ. Then the expression of $\partial_x^\alpha \tilde{U}'_+$ may be written from (C.39)

$$\partial_x^\alpha \tilde{U}'_+(t,x) = \int_1^{+\infty} e^{it\phi^c(t,\tau,z)} \chi\Big(\frac{\tau}{\sqrt{t}}\Big) J_\alpha(t,\tau,z)\, d\tau, \tag{C.58}$$

where

$$J_\alpha(t,\tau,z) = \int e^{it\frac{\zeta^2}{2}} \tilde{N}_\alpha(t,\tau,z,\zeta)\, d\zeta, \tag{C.59}$$

where $\tilde{N}_\alpha$ is supported close to $\zeta = 0$ and satisfies when $\tau \le \sqrt{t}$, by (C.57), the following estimates for any γ in $\mathbb{N}$:

$$\begin{aligned} &|\partial_\zeta^\gamma \tilde{N}_\alpha(t,\tau,z,\zeta)| \le C\tau_\varepsilon^{-\omega}, \\ &|\partial_\tau \partial_\zeta^\gamma \tilde{N}_\alpha(t,\tau,z,\zeta)| \le C\tau_\varepsilon^{-\omega+\frac{1}{2}}\Big(\tau_\varepsilon^{-\frac{3}{2}} + \tau^{-\frac{3}{2}}(\varepsilon^2\sqrt{\tau})^{\frac{3}{2}\theta'} + \tau_\varepsilon^{-\frac{1}{2}}t^{-1}\Big) \end{aligned} \tag{C.60}$$

if $N_\pm$ in (C.39) satisfies (C.34),

$$|\partial_\zeta^\gamma \tilde{N}_\alpha(t,\tau,z,\zeta)| \le C\tau^{-\frac{1}{2}}\tau_\varepsilon^{-1}, \quad |\partial_\tau \partial_\zeta^\gamma \tilde{N}_\alpha(t,\tau,z,\zeta)| \le C\tau^{-\frac{3}{4}}\tau_\varepsilon^{-1} \tag{C.61}$$

if $N_\pm$ satisfies (C.35),

$$|\partial_\zeta^\gamma \tilde{N}_\alpha(t,\tau,z,\zeta)| \le C\tau\tau_\varepsilon^{-\omega}, \quad |\partial_\tau \partial_\zeta^\gamma \tilde{N}_\alpha(t,\tau,z,\zeta)| \le C\tau_\varepsilon^{-\omega} \tag{C.62}$$

if $N_\pm$ satisfies (C.36), and

$$|\partial_\zeta^\gamma \tilde{N}_\alpha(t,\tau,z,\zeta)| \le C\tau^{\frac{1}{2}}\tau_\varepsilon^{-1}, \quad |\partial_\tau \partial_\zeta^\gamma \tilde{N}_\alpha(t,\tau,z,\zeta)| \le C\tau^{\frac{1}{4}}\tau_\varepsilon^{-1} \tag{C.63}$$

if $N_\pm$ satisfies (C.37). If we apply the stationary phase formula to equation (C.59), we gain a factor $t^{-\frac{1}{2}}$, which, according to (C.60)–(C.63) provides bounds of the form (C.47)–(C.50). To get expressions of the form (C.44), we still have to replace the phase $t\phi^c$ of (C.58) by ψ_+. By the Taylor–Lagrange formula relatively to τ and (C.55),

$$\phi^c(t,\tau,z) = \phi^c(t,0,z) + \tau(\partial_\tau\phi^c)(t,0,z) + O\Big(\frac{\tau^2}{t^2}\Big).$$

Moreover, by the definition of the phase ϕ of (C.39),

$$\big(\partial_\tau\phi^c\big)(t,0,z) = \frac{1}{t}\Big(\lambda - \sqrt{1+\xi(t,0,z)^2}\Big)$$

and by (C.52), the critical point $\xi(t,0,z)$ satisfies

$$\frac{\xi(t,0,z)}{\langle\xi(t,0,z)\rangle} = \frac{\xi_\lambda}{\lambda} - z = \frac{\xi_\lambda}{\langle\xi_\lambda\rangle} - z$$

so that

$$\sqrt{1+\xi(t,0,z)^2} = \lambda - \lambda^2\xi_\lambda z + O(z^2), \quad z\to 0.$$

We thus get

$$\begin{aligned} \phi^c(t,\tau,z) &= \phi^c(t,0,z) + \frac{\tau}{t}\big(\lambda^2\xi_\lambda z + O(z^2)\big) + r(t,\tau,z),\\ r(t,\tau,z) &= O\Big(\frac{\tau^2}{t^2}\Big), \quad \partial_\tau r(t,\tau,z) = O\Big(\frac{\tau}{t^2}\Big). \end{aligned} \tag{C.64}$$

We define

$$\begin{aligned} \psi_+(t,\tau,z) &= t\big(\phi^c(t,\tau,z) - r(t,\tau,z)\big),\\ \tilde\chi_+(t,\tau,z) &= \chi\Big(\frac{\tau}{\sqrt{t}}\Big)e^{itr(t,\tau,z)}. \end{aligned} \tag{C.65}$$

Plugging (C.64) in (C.58), we deduce from (C.65) that for $|z|\ll 1$, the properties of $\tilde\chi_+, \psi_+$ in (C.45), (C.46) do hold. This concludes the proof of the lemma. ■

Proof of Proposition C.1.4. Since $R_\alpha^\pm$ in (C.44) satisfy better estimates than those we want, by (C.51), we just consider the integral in the expansion of $\partial_x^\alpha\tilde U'_\pm$.

Under condition (C.34), J_α satisfies (C.47). It follows from (C.13) that the modulus of the integral in (C.44) is $O(\varepsilon^{2\omega})$. On the other hand, if we multiply (C.44) by $z_\pm$, use (C.46), integrate by parts in τ in (C.44) and use (C.45), we deduce from (C.11) and (C.13) a bound in $t^{-\frac{1}{2}}\varepsilon^{2\omega}$ for the resulting expression. Together with the definition (C.45) of $z_\pm$, this brings (C.40).

To prove (C.41), we proceed in the same way. Under estimates (C.35), (C.48) holds for J_α. By (C.13), this provides for (C.44) an estimate in $\varepsilon^2 t^{-\frac{1}{4}}$. On the other hand, if we multiply equation (C.44) by $z_\pm$ and integrate by parts, we get using (C.48) and (C.13) an estimate in $\varepsilon^2 t^{-\frac{3}{8}}$. Together with the first one, this implies (C.41).

One obtains (C.42) (resp. (C.43)) in the same way from (C.49) (resp. (C.50)) and (C.13). ■

Study of the U'' contribution. According to (C.4) and (C.5) we have

$$U''(t,x) = \frac{i}{2\pi}\int_{-\infty}^{t}\int e^{i((t-\tau)\sqrt{1+\xi^2}+\lambda\tau+x\xi)}(1-\chi)\Big(\frac{\tau}{\sqrt{t}}\Big)\hat{M}(\tau,\xi)\,d\xi\,d\tau \quad \text{(C.66)}$$

and

$$\begin{aligned} L_+U''(t,x) = \frac{i}{2\pi}\int_{-\infty}^{t}\int e^{i((t-\tau)\sqrt{1+\xi^2}+\lambda\tau+x\xi)}(1-\chi)\Big(\frac{\tau}{\sqrt{t}}\Big) \\ \times\Big(\tau\frac{\xi}{\langle\xi\rangle}\hat{M}(\tau,\xi)+\widehat{xM}(\tau,\xi)\Big)\,d\xi\,d\tau. \end{aligned} \quad \text{(C.67)}$$

We treat first the case when $\lambda < 1$ or $\lambda > 1$ and $\hat{M}$ is supported for ξ outside a neighborhood of $\pm\xi_\lambda$.

Lemma C.1.6. *Assume $\lambda < 1$ or $\lambda > 1$ and $\hat{M}$ supported outside a neighborhood of $\{-\xi_\lambda, \xi_\lambda\}$.*

(i) *Under Assumption (H1)$_\omega$, estimates (C.24)–(C.27) hold true.*

(ii) *Under Assumption (H2), estimates (C.28)–(C.30) hold true.*

Proof. We write $\hat{U}''(t,\xi)$ as

$$\int_{-\infty}^{t} e^{i(\lambda-\sqrt{1+\xi^2})\tau}(1-\chi)\Big(\frac{\tau}{\sqrt{t}}\Big)N(\tau,\xi)\,d\tau\, e^{it\sqrt{1+\xi^2}} \quad \text{(C.68)}$$

with N satisfying condition (C.34) under Assumption (H1)$_\omega$ and condition (C.35) under Assumption (H2). In the same way, $\widehat{L_+U''}$ is given by (C.68) with N satisfying (C.36) when Assumption (H1)$_\omega$ holds and (C.37) under Assumption (H2).

We perform one ∂_τ-integration by parts in (C.68) and compute the $L^2(\langle\xi\rangle^r)$ norm. When N satisfies (C.34), we obtain from (C.12) (and from (C.13) if ∂_τ falls on $(1-\chi)(\tau/\sqrt{t})$) a bound of the form (C.24). If instead of computing the $L^2(\langle\xi\rangle^r d\xi)$ norm, we estimate the $L^1(\langle\xi\rangle^r d\xi)$ one, we get (C.25) from (C.12) and (C.13).

Under condition (C.35) we get an estimate of the $L^2(\langle\xi\rangle^r d\xi)$ norm of (C.68) by

$$C\int_{\sqrt{t}}^{t}\tau_\varepsilon^{-1}\tau^{-\frac{3}{4}}\,d\tau + C\varepsilon^2 t^{-\frac{1}{2}}$$

which is smaller than the right-hand side of (C.28) by (C.14).

We are left with proving (C.26), (C.27), (C.29) and (C.30). Integrating by parts in τ in (C.66) and (C.67), we have thus to bound the integrals

$$\int e^{i(\lambda t+x\xi)}N(t,\xi)\,d\xi, \quad \text{(C.69)}$$

$$\int_{-\infty}^{t}\int e^{i((t-\tau)\sqrt{1+\xi^2}+\lambda\tau+x\xi)}\partial_\tau\Big(N(\tau,\xi)(1-\chi)\Big(\frac{\tau}{\sqrt{t}}\Big)\Big)\,d\xi\,d\tau, \quad \text{(C.70)}$$

where N satisfies (C.35) (to get (C.29)) or (C.36) (to obtain (C.26)–(C.27)) or (C.37) (to get (C.30)). The $W^{r,\infty}$ norm of (C.69) is bounded from above by the L^1 norm of

$\langle\xi\rangle^r N(\tau,\xi)$, that has immediately the wanted estimates. Let us study (C.70). Since the integrand is in $\mathcal{S}(\mathbb{R})$ relatively to ξ, stationary phase shows that the $d\xi$-integral is $O(\langle t-\tau\rangle^{-\frac{1}{2}})$, with bounds given by the right-hand side of (C.35)–(C.37). Consequently, the contribution of (C.70) to (C.29) will be estimated by

$$C\int_{\sqrt{t}}^{t}\langle t-\tau\rangle^{-\frac{1}{2}}\frac{\varepsilon^2}{1+\tau\varepsilon^2}\tau^{-\frac{3}{4}}\,d\tau, \tag{C.71}$$

its contribution to (C.26)–(C.27) will be bounded by

$$C\int_{\sqrt{t}}^{t}\langle t-\tau\rangle^{-\frac{1}{2}}\frac{\varepsilon^{2\omega}}{(1+\tau\varepsilon^2)^{\omega}}\,d\tau, \tag{C.72}$$

and its contribution to (C.30) will be controlled by

$$C\int_{\sqrt{t}}^{t}\langle t-\tau\rangle^{-\frac{1}{2}}\frac{\varepsilon^2}{1+\tau\varepsilon^2}\tau^{\frac{1}{4}}\,d\tau. \tag{C.73}$$

One checks that (C.71) (resp. (C.72), resp. (C.73)) is bounded from above by the right-hand side of (C.29) (resp. (C.26)–(C.27), resp. (C.30)). This concludes the proof of the lemma. ■

We have obtained estimates (C.24)–(C.30) when $\hat{M}$ in (C.66)–(C.67) is supported away from the zeros of $\lambda-\sqrt{1+\xi^2}$. We shall next obtain these bounds for $\hat{M}$ supported in a small neighborhood of this set. We prove first these estimates under Assumption $(\mathrm{H1})_\omega$, i.e. those of (i) in the statement of Proposition C.1.2. We have to study again the integral

$$\int_{-\infty}^{t}\int e^{i((t-\tau)\sqrt{1+\xi^2}+\lambda\tau+x\xi)}(1-\chi)\Big(\frac{\tau}{\sqrt{t}}\Big)N(\tau,\xi)\,d\xi\,d\tau, \tag{C.74}$$

where N will satisfy (C.34) or (C.36) and is supported close to $\pm\xi_\lambda$.

Lemma C.1.7. *Assume $\lambda>1$ and N supported in a small enough neighborhood of $\{\xi_\lambda,-\xi_\lambda\}$. Then if N satisfies* (C.34) *(resp.* (C.36)*), estimates* (C.24) *and* (C.25) *(resp.* (C.26)–(C.27)*) hold true.*

Proof. Introduce $\Omega(\tau,\zeta)=\frac{e^{i\tau\zeta}-1}{i\zeta}$ and write (C.74), after making a ∂_τ-integration by parts, as the sum of the following quantities:

$$\int e^{i(t\sqrt{1+\xi^2}+x\xi)}\Omega\Big(t,\lambda-\sqrt{1+\xi^2}\Big)N(t,\xi)\,d\xi, \tag{C.75}$$

$$\begin{aligned}-\int_{-\infty}^{t}\int e^{i(t\sqrt{1+\xi^2}+x\xi)}\Omega\Big(\tau,\lambda-\sqrt{1+\xi^2}\Big)&\\ \times\,\partial_\tau\Big((1-\chi)\Big(\frac{\tau}{\sqrt{t}}\Big)N(\tau,\xi)\Big)\,d\xi\,d\tau.&\end{aligned} \tag{C.76}$$

Assume for instance that ξ stays in a small neighborhood of ξ_λ on the support of N, and make the change of variables $\zeta = \lambda - \sqrt{1+\xi^2}$ in the integrals, with ζ staying close to zero.

Consider first the case when N satisfies (C.34) and let us prove (C.25). We estimate the modulus of (C.75) by

$$\int_{|\zeta|\ll 1} |\Omega(t,\zeta)| \frac{\varepsilon^{2\omega}}{(1+t\varepsilon^2)^\omega}\, d\zeta \leq \frac{C\varepsilon^{2\omega}}{(1+t\varepsilon^2)^\omega} \log t$$

which is controlled by the right-hand side of (C.25). In the same way, we bound the modulus of (C.76) by

$$C \int_{\sqrt{t}}^t \Big(\tau_\varepsilon^{-\omega+\frac{1}{2}} \Big(\tau_\varepsilon^{-\frac{3}{2}} + \tau^{-\frac{3}{2}} (\varepsilon^2\sqrt{\tau})^{\frac{3}{2}\theta'} \Big) + \frac{1}{\sqrt{t}} \tau_\varepsilon^{-\omega} \mathbb{1}_{\tau\sim\sqrt{t}} \Big) \int_{|\zeta|\ll 1} |\Omega(\tau,\zeta)|\, d\zeta\, d\tau.$$

As

$$\int_{|\zeta|\ll 1} |\Omega(\tau,\zeta)|\, d\zeta = O(\log\tau) = O(\log t),$$

we obtain using (C.12) and (C.13) a bound in $\varepsilon^{2\omega}\log(1+t)$ as wanted. Assume next that N satisfies (C.36), and let us show (C.26)–(C.27). We estimate then (C.75) by

$$\frac{C\varepsilon^{2\omega} t}{(1+t\varepsilon^2)^\omega} \int_{|\zeta|\ll 1} |\Omega(t,\zeta)|\, d\zeta$$

that is bounded by (C.26)–(C.27). On the other hand, (C.76) may be controlled by

$$\int_{\sqrt{t}}^t \log\tau \frac{\varepsilon^{2\omega}}{(1+\tau\varepsilon^2)^\omega}\, d\tau,$$

that is bounded by (C.26) if $\omega = 1$, (C.27) if $\omega > 1$.

To finish the proof of the lemma, we still need to get (C.24). The H^r norm of (C.75) and (C.76) is bounded from above respectively by

$$\big\| \Omega\big(t, \lambda - \sqrt{1+\xi^2}\big) N(t,\xi) \big\|_{L^2(\langle\xi\rangle^r d\xi)} \tag{C.77}$$

and by

$$\int_{\sqrt{t}}^t \Big\| \Omega\big(\tau, \lambda - \sqrt{1+\xi^2}\big) \partial_\tau \Big((1-\chi)\Big(\frac{\tau}{\sqrt{t}}\Big) N(\tau,\xi) \Big) \Big\|_{L^2(\langle\xi\rangle^r d\xi)} d\tau. \tag{C.78}$$

We consider again the case when N is supported in a small neighborhood of ξ_λ and use $\zeta = \lambda - \sqrt{1+\xi^2}$ as the variable of integration. Since

$$\|\Omega(\tau,\zeta)\mathbb{1}_{|\zeta|\ll 1}\|_{L^2(d\zeta)} = O(\sqrt{\tau}),$$

we estimate, in view of (C.34), (C.77) and (C.78) by (C.24) again using (C.15) and (C.13). This concludes the proof. ■

Lemma C.1.7 concludes the proof of (i) of Proposition C.1.2. In order to finish the proof of (ii), we need to show the following.

Lemma C.1.8. *Consider equation* (C.66) *(resp.* (C.67)*) when* $\hat{M}$ *is supported close to* $\{-\xi_\lambda, \xi_\lambda\}$ *and when Assumption* (H2) *holds i.e. under conditions* (C.8)–(C.10)*. Then estimates* (C.28) *and* (C.29) *(resp.* (C.30)*) hold true.*

Proof. Notice first that the term $\widehat{xM}$ under the integral (C.67) satisfies the same hypothesis as $\hat{M}$ under integral (C.66) (see the lines below (C.10)). Since the right-hand side of (C.30) is larger than the one in (C.29), it suffices to show (C.28) and (C.29) for expression (C.66), and (C.30) for (C.67) where one forgets the $\widehat{xM}$ term. We thus have to study an expression

$$\int_{-\infty}^{t}\int e^{i((t-\tau)\sqrt{1+\xi^2}+\lambda\tau+x\xi)}(1-\chi)\Big(\frac{\tau}{\sqrt{t}}\Big)\tau^j N(\tau,\xi)\,d\xi\,d\tau, \tag{C.79}$$

where, according to conditions (C.8)–(C.10), N is supported in a small neighborhood of $\{-\xi_\lambda, \xi_\lambda\}$ and there are functions ϕ, ψ such that the following estimates hold:

$$\begin{aligned} |N(t,\xi)| + |\phi(t,\xi)| &\le Ct^{-\frac{1}{2}}t_\varepsilon^{-1},\\ |\partial_t N(t,\xi)| &\le Ct^{-\frac{3}{4}}t_\varepsilon^{-1},\\ |\psi(t,\xi)| &\le Ct^{-1}t_\varepsilon^{-1},\\ D_t N(t,\xi) &= (D_t + \lambda - \sqrt{1+\xi^2})\phi(t,\xi) + \psi(t,\xi), \end{aligned} \tag{C.80}$$

and where $j = 0$ in the case of bounds (C.28)–(C.29) and $j = 1$ for (C.30).

Let χ_0 be in $C_0^\infty(\mathbb{R})$, equal to one close to zero, and write the integral in (C.79) as $I_L^j + I_R^j$, where

$$\begin{aligned} I_L^j = \int_{-\infty}^{t}\int & e^{i((t-\tau)\sqrt{1+\xi^2}+\lambda\tau+x\xi)}\chi_0\Big(\big(\lambda-\sqrt{1+\xi^2}\big)\sqrt{t}\Big)\\ &\times(1-\chi)\Big(\frac{\tau}{\sqrt{t}}\Big)\tau^j N(\tau,\xi)\,d\tau\,d\xi. \end{aligned} \tag{C.81}$$

Since $\lambda > 1$, the $d\xi$ integral is $O(t^{-\frac{1}{2}})$, and using the estimate of N in (C.80), we get by (C.14) and (C.16)

$$|I_L^0| \le C\frac{\varepsilon}{\sqrt{t}}\Big(\frac{t\varepsilon^2}{\langle t\varepsilon^2\rangle}\Big)^{\frac{1}{2}}, \quad |I_L^1| \le C\frac{t\varepsilon^2}{\langle t\varepsilon^2\rangle}$$

which are better than the right-hand side of (C.29), (C.30), respectively. To study I_R^j, we make a ∂_τ-integration by parts and write this term as a sum of

$$-i\sqrt{t}\int e^{i(\lambda t+x\xi)}\chi_1\Big(\sqrt{t}\big(\lambda-\sqrt{1+\xi^2}\big)\Big)t^j N(t,\xi)\,d\xi, \tag{C.82}$$

where $\chi_1(z) = \frac{1-\chi_0(z)}{z}$, of

$$\begin{aligned} i\sqrt{t}\int_{-\infty}^{t}\int & e^{i((t-\tau)\sqrt{1+\xi^2}+\lambda\tau+x\xi)}\\ &\times\chi_1\Big(\big(\lambda-\sqrt{1+\xi^2}\big)\sqrt{t}\Big)\partial_\tau\Big((1-\chi)\Big(\frac{\tau}{\sqrt{t}}\Big)\tau^j\Big)N(\tau,\xi)\,d\xi\,d\tau \end{aligned} \tag{C.83}$$

and of

$$-\sqrt{t}\int_{-\infty}^{t}\int e^{i((t-\tau)\sqrt{1+\xi^2}+\lambda\tau+x\xi)} \times \chi_1\Big(\big(\lambda-\sqrt{1+\xi^2}\big)\sqrt{t}\Big)(1-\chi)\Big(\frac{\tau}{\sqrt{t}}\Big)\tau^j D_\tau N(\tau,\xi)\,d\xi\,d\tau. \tag{C.84}$$

We plug the last equality (C.80) in (C.84). We get on the one hand

$$-\sqrt{t}\int_{-\infty}^{t}\int e^{i((t-\tau)\sqrt{1+\xi^2}+\lambda\tau+x\xi)} \times \chi_1\Big(\big(\lambda-\sqrt{1+\xi^2}\big)\sqrt{t}\Big)(1-\chi)\Big(\frac{\tau}{\sqrt{t}}\Big)\tau^j \psi(\tau,\xi)\,d\xi\,d\tau \tag{C.85}$$

and, after another integration by parts, the terms

$$i\sqrt{t}\int e^{i(\lambda t+x\xi)}\chi_1\Big(\sqrt{t}\big(\lambda-\sqrt{1+\xi^2}\big)\Big)t^j\phi(t,\xi)\,d\xi \tag{C.86}$$

and

$$-i\sqrt{t}\int_{\sqrt{t}}^{t}\int e^{i((t-\tau)\sqrt{1+\xi^2}+\lambda\tau+x\xi)} \times \chi_1\Big(\big(\lambda-\sqrt{1+\xi^2}\big)\sqrt{t}\Big)\partial_\tau\Big((1-\chi)\Big(\frac{\tau}{\sqrt{t}}\Big)\tau^j\Big)\phi(\tau,\xi)\,d\xi\,d\tau. \tag{C.87}$$

Notice that since N and ϕ satisfy the same bound (C.80), a bound for (C.82) will also provide a bound for (C.86). In the same way, an estimate for (C.83) will bring one for (C.87). We are just reduced, in order to get (C.29) and (C.30), to estimate the L^∞ norms of (C.82), (C.83) and (C.85).

We estimate the modulus of (C.82) by

$$C\frac{\varepsilon^2 t^j}{\langle t\varepsilon^2\rangle}\int_{|\zeta|<c}\frac{d\zeta}{\langle\sqrt{t}\zeta\rangle}\le C\frac{\varepsilon^2 t^{j-\frac12}}{\langle t\varepsilon^2\rangle}\log(1+t)$$

which is better than the right-hand side of (C.29) (resp. (C.30)) if $j=0$ (resp. $j=1$). We bound (C.83) by

$$C\sqrt{t}\int_{|\zeta|<c}\frac{d\zeta}{\langle\sqrt{t}\zeta\rangle}\int_{\sqrt{t}}^{t}\tau^{-\frac12}\frac{\varepsilon^2}{1+\tau\varepsilon^2}\Big|\partial_\tau\Big(\tau^j(1-\chi)\Big(\frac{\tau}{\sqrt{t}}\Big)\Big)\Big|\,d\tau.$$

If $j=0$, we get a bound in $\log(1+t)\varepsilon^2 t^{-\frac14}$, better than (C.29), and if $j=1$, we obtain using (C.13), a bound in

$$\varepsilon^2 t^{\frac14}\log(1+t)$$

which is better than (C.30) since $t\le\varepsilon^{-4}$.

Finally, we estimate (C.85) by, using (C.80),

$$\log(1+t)\int_{\sqrt{t}}^{t}\tau^{j-1}\frac{\varepsilon^2}{1+\tau\varepsilon^2}\,d\tau$$

which is bounded by (C.29) if $j=0$ and by (C.30) if $j=1$. We have thus established these two estimates. To get the remaining bound (C.28), we just plug inside (C.66) bound (C.8) of $\hat{M}$ and use (C.14). This concludes the proof. ∎

C.2 Action of linear and bilinear operators

The goal of this section is to study the action of some operators on a function of the form (C.3), and on its decomposition $U=U'+U''$ given by (C.4). These operators will be of the form $\mathrm{Op}(m')$, given by the non-semiclassical quantization (B.17), for symbols $m'(y,\xi)$ that do not depend on x and belong to the class $\tilde{S}'_{\kappa,0}(1,j)$, $j=1,2$, defined in Definition 3.1.1.

We study first linear operators.

Proposition C.2.1. *Let $(t,x)\mapsto M(t,x)$ be a function satisfying Assumption (H1)$_\omega$, i.e. inequalities (C.7). Assume moreover that M is an odd function of x. Let m' be a symbol in the class $\tilde{S}'_{0,0}(1,1)$ of Definition 3.1.1, i.e. a function $m'(y,\xi)$ on $\mathbb{R}\times\mathbb{R}$ such that*

$$|\partial_y^{\alpha_0'}\partial_\xi^{\alpha}m'(y,\xi)|\le C(1+|y|)^{-N} \tag{C.88}$$

for any N,α_0',α, and that m' satisfies $m'(-y,-\xi)=m'(y,\xi)$, so that $\mathrm{Op}(m')$ will preserve odd functions. Then, for U'' defined from M by (C.4), we have

$$\mathrm{Op}(m')U''=e^{i\lambda t}M_1(t,x)+r(t,x), \tag{C.89}$$

where $M_1(t,x)$ is an odd function of x, satisfying for any $\alpha,N\in\mathbb{N}$,

$$\begin{aligned}|\partial_x^{\alpha}M_1(t,x)|&\le C_{\alpha,N}t_\varepsilon^{-\omega}\langle x\rangle^{-N},\\ |\partial_x^{\alpha}\partial_t M_1(t,x)|&\le C_{\alpha,N}t_\varepsilon^{-\omega+\frac12}\Big(t_\varepsilon^{-\frac32}+t^{-\frac32}(\varepsilon^2\sqrt{t})^{\frac32\theta'}\Big)\langle x\rangle^{-N}\end{aligned} \tag{C.90}$$

and where $r(t,x)$ is such that for any α,N,

$$|\partial_x^{\alpha}r(t,x)|\le C_{\alpha,N}\big(\varepsilon^{2\omega}t^{-1}\log(1+t)\big)\langle x\rangle^{-N}. \tag{C.91}$$

Moreover, if L_+ is the operator (C.5), for any $\alpha\in\mathbb{N}$, $k=0,1$,

$$\begin{aligned}\int_{-1}^{1}\|\partial_x^{\alpha}\mathrm{Op}(m')((L_+^kU')(t,\mu\cdot))\|_{L^\infty}\,d\mu&\le C_\alpha\varepsilon^{2\omega},\\ \int_{-1}^{1}\|\partial_x^{\alpha}\mathrm{Op}(m')((L_+^kU')(t,\mu\cdot))\|_{L^2}\,d\mu&\le C_\alpha\varepsilon^{2\omega}.\end{aligned} \tag{C.92}$$

Proof. The definition (B.17) of $\mathrm{Op}(m')$ and the expression (C.4) of U'' imply that

$$\begin{aligned}\mathrm{Op}(m')U'' = \frac{i}{2\pi}\int_{-\infty}^{t}\int e^{i(x\xi+(t-\tau)\sqrt{1+\xi^2}+\lambda\tau)}m'(x,\xi)\\ \times(1-\chi)\Big(\frac{\tau}{\sqrt{t}}\Big)\hat{M}(\tau,\xi)\,d\xi\,d\tau.\end{aligned}\tag{C.93}$$

We decompose $\hat{M}(\tau,\xi) = \hat{M}'(\tau,\xi) + \hat{M}''(\tau,\xi)$, where $\hat{M}'$ is supported for ξ in a small neighborhood of the two roots $\pm\xi_\lambda$ of $\sqrt{1+\xi^2} = \lambda$ and $\hat{M}''$ vanishes close to that set when $\lambda > 1$, and $\hat{M}' = 0$ if $\lambda < 1$. Moreover, $\hat{M}'(\tau,\xi)$, $\hat{M}''(\tau,\xi)$ are odd in ξ, because M is odd in x. We define then

$$\begin{aligned}B'(x,\tau,\xi) &= e^{ix\xi}m'(x,\xi)\hat{M}'(\tau,\xi),\\ B''(x,\tau,\xi) &= e^{ix\xi}m'(x,\xi)\hat{M}''(\tau,\xi).\end{aligned}\tag{C.94}$$

By the evenness of m', we have

$$B'(-x,\tau,-\xi) = -B'(x,\tau,\xi),\quad B''(-x,\tau,-\xi) = -B''(x,\tau,\xi).\tag{C.95}$$

Let us study first the contribution of $\hat{M}''$ to (C.93), given by

$$\frac{i}{2\pi}\int_{-\infty}^{t}\int e^{i((t-\tau)\sqrt{1+\xi^2}+\lambda\tau)}B''(x,\tau,\xi)(1-\chi)\Big(\frac{\tau}{\sqrt{t}}\Big)\,d\xi\,d\tau.\tag{C.96}$$

We perform one ∂_τ-integration by parts, that provides on the one hand $e^{i\lambda t}M_1(t,x)$, where

$$M_1(t,x) = \frac{1}{2\pi}\int\Big(\lambda-\sqrt{1+\xi^2}\Big)^{-1}B''(x,t,\xi)\,d\xi$$

satisfies (C.90) by (C.94), (C.88) and (C.7), and is odd in x by (C.95), and on the other hand a contribution

$$\frac{1}{2\pi}\int_{-\infty}^{t}\int e^{i((t-\tau)\sqrt{1+\xi^2}+\lambda\tau)}N(x,\tau,\xi)\,d\xi\,d\tau,\tag{C.97}$$

where

$$N(x,\tau,\xi) = -\partial_\tau\Big(B''(x,\tau,\xi)(1-\chi)\Big(\frac{\tau}{\sqrt{t}}\Big)\Big)\Big(\lambda-\sqrt{1+\xi^2}\Big)^{-1}$$

satisfies by (C.88) and (C.7)

$$\begin{aligned}|\partial_x^\alpha\partial_\xi^\beta N(x,\tau,\xi)| \le C\langle x\rangle^{-N}\langle\xi\rangle^{-N}\tau_\varepsilon^{-\omega}\\ \times\big(\tau_\varepsilon^{-1}+\tau^{-1}\mathbb{1}_{\tau\sim\sqrt{t}}+\tau_\varepsilon^{\frac{1}{2}}\tau^{-\frac{3}{2}}(\varepsilon^2\sqrt{\tau})^{\frac{3}{2}\theta'}\big).\end{aligned}\tag{C.98}$$

By the oddness of $\hat{M}$ in ξ, $N(x,\tau,0)\equiv 0$. Consequently, if we apply the stationary phase formula to the ∂_ξ-integral in (C.97) at the unique (non-degenerate) critical point

$\xi = 0$, we gain a decaying factor in $\langle t-\tau\rangle^{-1}$ instead of $\langle t-\tau\rangle^{-\frac{1}{2}}$. Taking (C.98) into account, and using (C.12), we obtain for (C.97) and its ∂_x-derivatives a bound in

$$C_N\langle x\rangle^{-N}\int_{\sqrt{t}}^{t}\langle t-\tau\rangle^{-1}\tau_\varepsilon^{-\omega}\big(\tau_\varepsilon^{-1}+\tau^{-1}\mathbb{1}_{\tau\sim\sqrt{t}}+\tau_\varepsilon^{\frac{1}{2}}\tau^{-\frac{3}{2}}(\varepsilon^2\sqrt{\tau})^{\frac{3}{2}\theta'}\big)\,d\tau$$
$$\le C_N\langle x\rangle^{-N}\varepsilon^{2\omega}t^{-1}\log(1+t)$$

which is bounded by (C.91).

Let us study next the contribution of $\hat{M}'$ to (C.93). We get

$$\int_1^t\int e^{i((t-\tau)\sqrt{1+\xi^2}+\lambda\tau)}B'(x,\tau,\xi)(1-\chi)\Big(\frac{\tau}{\sqrt{t}}\Big)\,d\xi\,d\tau. \tag{C.99}$$

Write for $1\le\tau\le t$

$$B'(x,\tau,\xi)=B'(x,t,\xi)+(\tau-t)\tilde{B}'(x,\tau,t,\xi), \tag{C.100}$$

where $\tilde{B}'$ satisfies by (C.7) and (C.88)

$$|\partial_x^\alpha\partial_\xi^\beta\tilde{B}'(x,\tau,t,\xi)|\le C\tau_\varepsilon^{-\omega}\big(\tau_\varepsilon^{-1}+\tau_\varepsilon^{\frac{1}{2}}\tau^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'}\big)\langle x\rangle^{-N}$$

and is supported for ξ close to $\{-\xi_\lambda,\xi_\lambda\}$. If we substitute in the integral (C.99) expression $(\tau-t)\tilde{B}'$ to B', and use that, since $\xi_\lambda\neq 0$, $\tilde{B}'$ is supported far away the critical point $\xi=0$ of the phase, we may gain a factor $\langle t-\tau\rangle^{-N}$ for any N by ∂_ξ-integration by parts. We thus get a contribution to (C.99) and to its ∂_x-derivatives bounded by

$$C_N\langle x\rangle^{-N}\int_{\sqrt{t}}^{t}\langle t-\tau\rangle^{-N}\tau_\varepsilon^{-\omega}\big(\tau_\varepsilon^{-1}+\tau_\varepsilon^{\frac{1}{2}}\tau^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'}\big)\,d\tau.$$

This again provides a contribution to (C.91). We are left with studying (C.99) with $B'(x,\tau,\xi)$ replaced by $B'(x,t,\xi)$ according to (C.100), i.e.

$$\begin{aligned}&\int_1^t\int e^{i((t-\tau)\sqrt{1+\xi^2}+\lambda\tau)}(1-\chi)\Big(\frac{\tau}{\sqrt{t}}\Big)B'(x,t,\xi)\,d\xi\,d\tau\\&\quad=e^{i\lambda t}\int T\big(t,\sqrt{1+\xi^2}-\lambda\big)B'(x,t,\xi)\,d\xi\end{aligned} \tag{C.101}$$

with

$$T(t,\zeta)=T_1(t,\zeta)+T_2(t,\zeta)$$

and

$$T_1(t,\zeta)=\int_0^{t-1}e^{i\tau\zeta}\,d\tau,$$
$$T_2(t,\zeta)=-\int_0^{t-1}e^{i\tau\zeta}\chi\Big(\frac{t-\tau}{\sqrt{t}}\Big)\,d\tau.$$

Note that if $\varphi \in \mathcal{S}(\mathbb{R})$,

$$\begin{aligned}\int T_1(t,\zeta)\varphi(\zeta)\,d\zeta &= \int_0^{t-1} \hat{\varphi}(-\tau)\,d\tau = \int_0^{+\infty} \hat{\varphi}(-\tau)\,d\tau + O(t^{-\infty}),\\ \int T_2(t,\zeta)\varphi(\zeta)\,d\zeta &= O(t^{-\infty}).\end{aligned} \tag{C.102}$$

Using that B' is supported close to $\xi = \pm\xi_\lambda$, and that $\xi_\lambda \neq 0$, we may use in the last integral in (C.101) $\zeta = \sqrt{1+\xi^2} - \lambda$ as a variable of integration close to this point. We express thus (C.101) from integrals of the form (C.102), with φ expressed from B'. The definition (C.94) of B' and (C.88), (C.7) imply that the principal term on the first line (C.102) brings to (C.101) a contribution in $e^{i\lambda t} M_1(t,x)$ with M_1 satisfying estimates (C.90). The other contributions, as well as their ∂_x-derivatives, are $O(t_\varepsilon^{-\omega} t^{-N} \langle x\rangle^{-N})$ for any N, so satisfy (C.91).

It remains to prove (C.92). We express L_+U' from (C.32), which allows us to write $\mathrm{Op}(m')((L_+U')(\mu\cdot))$ as the sum of two expressions

$$\frac{i}{2\pi}\int_1^{+\infty}\int e^{i((t-\tau)\sqrt{1+\xi^2}+\lambda\tau)}\chi\Big(\frac{\tau}{\sqrt{t}}\Big) B_j^\mu(x,\tau,\xi)\,d\tau\,d\xi, \quad j = 1,2, \tag{C.103}$$

with

$$\begin{aligned} B_1^\mu(x,\tau,\xi) &= e^{ix\xi\mu} m'(x,\mu\xi)\widehat{xM}(\tau,\xi),\\ B_2^\mu(x,\tau,\xi) &= e^{ix\xi\mu} m'(x,\mu\xi)\tau\frac{\xi}{\langle\xi\rangle}\hat{M}(\tau,\xi).\end{aligned} \tag{C.104}$$

When $j = 1$, we use the stationary phase formula in ξ to make appear a $\langle t-\tau\rangle^{-\frac{1}{2}}$ factor. Using also (C.7) and (C.88), we get for any ∂_x-derivative of (C.103) with $j = 1$ a bound in

$$C\int_1^{\sqrt{t}} \langle t-\tau\rangle^{-\frac{1}{2}} \tau_\varepsilon^{-\omega}\,d\tau \langle x\rangle^{-N} \leq C\varepsilon^{2\omega}\langle x\rangle^{-N}. \tag{C.105}$$

When $j = 2$, we notice that because $\hat{M}$ is odd in ξ, $B_2^\mu(x,\tau,\xi)$ vanishes at second order at $\xi = 0$. Consequently, stationary phase formula in (C.103) makes gain a factor in $\langle t-\tau\rangle^{-\frac{3}{2}}$, so that (C.103) is controlled, using again (C.13), by

$$C\int_1^{\sqrt{t}} \langle t-\tau\rangle^{-\frac{3}{2}} \tau\tau_\varepsilon^{-\omega}\,d\tau \langle x\rangle^{-N} \leq C\varepsilon^{2\omega}\langle x\rangle^{-N}.$$

Bounds (C.92) follow from this inequality and (C.105). This concludes the proof of (C.92) when $k = 1$. If $k = 0$, the estimate is similar to the one with B_1^μ above. ■

Let us prove a similar result to Proposition C.2.1 for some bilinear operators.

Proposition C.2.2. *Let M and U'' be as in the statement of Proposition C.2.1. Let m' be a symbol in $\tilde{S}'_{\kappa,0}(\prod_{j=1}^2 \langle\xi_j\rangle^{-1}, 2)$ for some $\kappa \geq 0$, satisfying*

$$m'(-y,-\xi_1,-\xi_1) = -m'(y,\xi_1,\xi_2).$$

Then for any function v,

$$\mathrm{Op}(m')(U'', v) = e^{i\lambda t}\mathrm{Op}(b_1)v + \mathrm{Op}(b_2)v, \tag{C.106}$$

where b_1, b_2 *satisfy for any* α_0', α, N *the following estimates:*

$$\begin{aligned} |\partial_y^{\alpha_0'}\partial_\xi^\alpha b_1(t,y,\xi)| &\leq C t_\varepsilon^{-\omega}\langle y\rangle^{-N}\langle\xi\rangle^{-1}, \\ |\partial_y^{\alpha_0'}\partial_\xi^\alpha \partial_t b_1(t,y,\xi)| &\leq C t_\varepsilon^{-\omega+\frac{1}{2}}\big(t_\varepsilon^{-\frac{3}{2}} + t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'}\big)\langle y\rangle^{-N}\langle\xi\rangle^{-1}, \\ |\partial_y^{\alpha_0'}\partial_\xi^\alpha b_2(t,y,\xi)| &\leq C\varepsilon^{2\omega}t^{-1}\log(1+t)\langle y\rangle^{-N}\langle\xi\rangle^{-1}. \end{aligned} \tag{C.107}$$

Moreover, $b_j(t,-y,-\xi) = b_j(t,y,\xi)$.

Proof. By expression (C.4) of U'', we have

$$\begin{aligned} \mathrm{Op}(m')(U'', v) &= \frac{i}{(2\pi)^2}\int_{-\infty}^t \iint e^{i(x(\xi_1+\xi)+(t-\tau)\sqrt{1+\xi_1^2}+\lambda\tau)} \\ &\qquad \times m'(x,\xi_1,\xi)(1-\chi)\Big(\frac{\tau}{\sqrt{t}}\Big)\hat{M}(\tau,\xi_1)\hat{v}(\xi)\,d\xi\,d\xi_1\,d\tau \\ &= \mathrm{Op}(b)v \end{aligned}$$

if

$$\begin{aligned} b(t,x,\xi) = \frac{i}{2\pi}\int_{-\infty}^t \iint & e^{i(x\xi_1+(t-\tau)\sqrt{1+\xi_1^2}+\lambda\tau)} \\ & \times m'(x,\xi_1,\xi)(1-\chi)\Big(\frac{\tau}{\sqrt{t}}\Big)\hat{M}(\tau,\xi_1)\,d\xi_1\,d\tau. \end{aligned} \tag{C.108}$$

We notice that if we consider ξ as a parameter, the function

$$(y,\xi_1) \mapsto m'(y,\xi_1,\xi)\hat{M}(\tau,\xi_1)$$

satisfies estimates of the form (C.88) for every τ, as the losses in

$$M_0(\xi_1,\xi)^\kappa = O(\langle\xi_1\rangle^\kappa)$$

appearing when one takes derivatives in the definition of symbol classes in (B.13) are compensated by the rapid decay of $\hat{M}(\tau,\xi_1)$. We obtain thus an integral of the form (C.93) (with ξ replaced by ξ_1), depending on an extra parameter ξ. By (the proof of) Proposition C.2.1, we obtain thus that (C.108) has an expression of the form (C.89), i.e. $e^{i\lambda t}b_1 + b_2$, with b_1, (resp. b_2) satisfying bounds of the form (C.90) (resp. (C.91)), which gives (C.107), using also that $m'(x,\xi_1,\xi)$ in equation (C.108) is $O(\langle\xi\rangle^{-1})$. The evenness of b_j in (y,ξ) comes from the oddness of m' and $\hat{M}$. This concludes the proof. ■

Corollary C.2.3. *Under the assumptions of Proposition C.2.2, one has the following estimates for any* α, N:

$$|\partial_x^\alpha \mathrm{Op}(m')(U'', U'')| \leq C\langle x\rangle^{-N}\big(t_\varepsilon^{-2\omega} + \varepsilon^{4\omega}t^{-2}(\log(1+t))^2\big). \tag{C.109}$$

Proof. By (C.106), we may write

$$\mathrm{Op}(m')(U'', U'') = e^{i\lambda t}\mathrm{Op}(b_1)U'' + \mathrm{Op}(b_2)U''$$

with b_1, b_2 satisfying (C.107). We may apply (C.89) to each term above, using that b_1, b_2 satisfy estimates of the form (C.88), with an extra pre-factor given by the first and last estimates (C.107). Using the first bound (C.90) and (C.91), we reach the conclusion. ∎

We have obtained in the preceding results estimates under assumptions of the form (C.7) for the function M in (C.4), i.e. under Assumption $(\mathrm{H1})_\omega$. We shall need also variants of the preceding results when Assumption (H2), i.e. (C.8) holds instead. In this case, we shall split the function U defined in (C.3) in a different way than in (C.4), cutting at time of order $\tau \sim ct$ instead of $\tau \sim \sqrt{t}$. More precisely, we set

$$\begin{aligned} U &= U_1' + U_1'', \\ U_1'(t,x) &= i\int_1^{+\infty} e^{i(t-\tau)p(D_x)+i\lambda\tau}\chi\Big(\frac{\tau}{t}\Big)M(\tau,\cdot)\,d\tau, \\ U_1''(t,x) &= i\int_{-\infty}^{t} e^{i(t-\tau)p(D_x)+i\lambda\tau}(1-\chi)\Big(\frac{\tau}{t}\Big)M(\tau,\cdot)\,d\tau \end{aligned} \tag{C.110}$$

Proposition C.2.4. *Let us assume that M is odd in x, satisfies the first inequality of (C.8) and that m' satisfies (C.88). We have then the following estimates for any $\alpha, N \in \mathbb{N}$:*

$$|\partial_x^\alpha \mathrm{Op}(m')U_1''| \le C_{\alpha N}\langle x\rangle^{-N} t_\varepsilon^{-\frac{1}{2}} t^{-1}\log(1+t) \tag{C.111}$$

and for $\ell = 0, 1$,

$$\begin{aligned} \int_{-1}^{1}\Big(&\|\partial_x^\alpha \mathrm{Op}(m')\big((L_+^\ell\, U_1')(t,\mu\cdot)\big)\|_{L^2} \\ &+ \|\partial_x^\alpha \mathrm{Op}(m')\big((L_+^\ell U_1')(t,\mu\cdot)\big)\|_{L^\infty}\Big)\,d\mu \le C_\alpha \varepsilon^2. \end{aligned} \tag{C.112}$$

Estimate (C.112) holds as soon as (C.88) is true for some large enough N.

Proof. We denote

$$B(x,\tau,\xi_1) = e^{ix\xi_1}m'(x,\xi_1)\hat{M}(\tau,\xi_1),$$

that satisfies by the first inequality of (C.8) and (C.88)

$$|\partial_x^{\alpha_0}\partial_{\xi_1}^{\alpha} B(x,\tau,\xi_1)| \le C_{\alpha_0,\alpha}\langle x\rangle^{-N}\langle \xi_1\rangle^{-N}\tau^{-\frac{1}{2}}\tau_\varepsilon^{-1}$$

and that vanishes at $\xi_1 = 0$ as M is odd. Then as in (C.93), (C.96)

$$\begin{aligned} \mathrm{Op}(m')U_1'' = \frac{i}{2\pi}\int_{-\infty}^{t}\int & e^{i((t-\tau)\sqrt{1+\xi_1^2}+\lambda\tau)} \\ &\times (1-\chi)\Big(\frac{\tau}{t}\Big)B(x,\tau,\xi_1)\,d\xi_1\,d\tau. \end{aligned} \tag{C.113}$$

Using stationary phase in ξ_1 and the fact that B vanishes at $\xi_1 = 0$, we get for some $a \in]0,1[$,

$$|\partial_x^\alpha \mathrm{Op}(m')U_1''(t,x)| \le C \int_{at}^{t} \langle t-\tau\rangle^{-1}\tau_\varepsilon^{-1}\tau^{-\frac{1}{2}}\,d\tau\langle x\rangle^{-N}$$

which is bounded by the right-hand side of (C.111).

To prove estimate (C.112) with $\ell = 1$, we express $\mathrm{Op}(m')((L_+U')(\mu\cdot))$ under form (C.103), except that the cut-off $\chi(\tau/\sqrt{t})$ has to be replaced by $\chi(\tau/t)$, i.e. we have to study

$$\frac{i}{2\pi}\int_1^{+\infty}\int e^{i((t-\tau)\sqrt{1+\xi_1^2}+\lambda\tau)}\chi\Big(\frac{\tau}{t}\Big)B_j^\mu(x,\tau,\xi_1)\,d\xi_1\,d\tau, \tag{C.114}$$

where B_j^μ, $j = 1,2$, is given by (C.104). If $j = 1$, we get from the first inequality of (C.8), (C.88) and stationary phase in ξ_1 a bound of ∂_x-derivatives of (C.114) by

$$C\langle x\rangle^{-N}\int_1^{at}\langle t-\tau\rangle^{-\frac{1}{2}}\tau^{-\frac{1}{2}}\tau_\varepsilon^{-1}\,d\tau \tag{C.115}$$

for some $a \in]0,1[$, whence the $O(\varepsilon^2)$ wanted bound for the L^2 and L^∞ norms. If $j = 2$, using stationary phase and the fact that B_2^μ vanishes at order 2 at $\xi = 0$, we get an estimate in

$$C\langle x\rangle^{-N}\int_1^{at}\langle t-\tau\rangle^{-\frac{3}{2}}\tau^{\frac{1}{2}}\tau_\varepsilon^{-1}\,d\tau \tag{C.116}$$

which is also $O(\varepsilon^2)$. This concludes the proof of (C.112) when $\ell = 1$. If $\ell = 0$, we may use directly (C.115) to get the estimate. Notice that to get (C.112), we do not need that (C.115) and (C.116) hold for any N, but just for a large enough N (actually $N = 1$ suffices), so that (C.88) has to be assumed only for some large enough N. ∎

Let us write a version of Proposition C.2.2 under Assumption (H2) as well.

Proposition C.2.5. *Let M be as in Proposition C.2.4 and m' in $\tilde{S}'_{\kappa,0}(\prod_{j=1}^2\langle\xi_j\rangle^{-1},2)$. Then $\mathrm{Op}(m')(U_1',v)$ and $\mathrm{Op}(m')(U_1'',v)$ may be written as $\mathrm{Op}(b)v$ for all symbols $b(t,y,\xi)$ satisfying the estimates*

$$|\partial_y^{\alpha'_0}\partial_\xi^\alpha b(t,y,\xi)| \le Ct_\varepsilon^{-\frac{1}{2}}t^{-1}\log(1+t)\langle y\rangle^{-N}\langle\xi\rangle^{-1}. \tag{C.117}$$

Proof. Consider first $\mathrm{Op}(m')(U_1'',v)$ that may be written using expression (C.110) of U_1'' as

$$\mathrm{Op}(m')(U_1'',v) = \frac{1}{2\pi}\int e^{ix\xi}b(t,x,\xi)\hat{v}(\xi)\,d\xi \tag{C.118}$$

with

$$b(t,x,\xi) = \frac{i}{2\pi}\int_{-\infty}^{t}\int e^{ix\xi_1+i((t-\tau)\sqrt{1+\xi_1^2}+\lambda\tau)} \times m'(x,\xi_1,\xi)\hat{M}(\tau,\xi_1)(1-\chi)\Big(\frac{\tau}{t}\Big)\,d\xi_1\,d\tau.$$

Using again stationary phase with respect to ξ_1 and the fact that $\hat{M}(\tau,0) = 0$ to gain a decaying factor in $\langle t-\tau\rangle^{-1}$, we obtain for the $\partial_x^{\alpha'_0}\partial_\xi^{\alpha}$-derivatives of b an upper bound in

$$C\int_{at}^{t}\langle t-\tau\rangle^{-1}\tau^{-\frac{1}{2}}\tau_\varepsilon^{-1}\,d\tau\langle x\rangle^{-N}\langle\xi\rangle^{-1}\quad (a\in]0,1[) \tag{C.119}$$

since, as seen at the beginning of the proof of Proposition C.2.2,

$$(y,\xi_1)\mapsto m'(y,\xi_1,\xi)\hat{M}(\tau,\xi_1)$$

and its derivatives have bounds in

$$C\langle y\rangle^{-N}\tau^{-\frac{1}{2}}\tau_\varepsilon^{-1}\langle\xi_1\rangle^{-N}\langle\xi\rangle^{-1}$$

according to (C.8). As (C.119) is bounded by the right-hand side of (C.117), we get the wanted conclusion for $\mathrm{Op}(m')(U_1'',v)$.

Consider now the case of $\mathrm{Op}(m')(U_1',v)$, i.e.

$$\frac{1}{(2\pi)^2}\int e^{ix(\xi_1+\xi)}m'(x,\xi_1,\xi)\hat{U}_1'(\xi_1)\hat{v}(\xi)\,d\xi_1\,d\xi.$$

We may rewrite it as

$$\frac{1}{2\pi}\int e^{ix\xi}b(t,x,\xi)\hat{v}(\xi)\,d\xi$$

with, for any N,

$$b(t,x,\xi)=\int K_N(t,x-y,x,\xi)\langle D_y\rangle^{2N-1}U_1'(y)\,dy, \tag{C.120}$$

where

$$K_N(t,z,x,\xi)=\frac{1}{2\pi}\int e^{iz\xi_1}\langle\xi_1\rangle^{-2N+1}m'(x,\xi_1,\xi)\,d\xi_1.$$

By the assumption on m', estimates of the form (B.13) hold (with y on the right-hand side of this inequality replaced by x) whence

$$|\partial_x^{\alpha'_0}\partial_\xi^{\alpha}\partial_{\xi_1}^{\alpha_1}m'(x,\xi_1,\xi)|\le C(1+|x|\langle\xi_1\rangle^{-\kappa})^{-N'}\langle\xi\rangle^{-1}\langle\xi_1\rangle^{-1+\kappa(|\alpha|+|\alpha_1|)}$$

for any N'. We conclude that for any α,β,N',N'', one has estimates

$$|\partial_x^{\alpha}\partial_\xi^{\beta}K_N(t,z,x,\xi)|\le C\langle x\rangle^{-N'}\langle z\rangle^{-N''}\langle\xi\rangle^{-1}$$

if N is taken large enough relatively to N',N'',α,β. Plugging this in (C.120), we conclude that for any N',N'',α,β, there is N such that

$$|\partial_x^{\alpha}\partial_\xi^{\beta}b(t,x,\xi)|\le C\langle x\rangle^{-N'}\sup_y|\langle y\rangle^{-N''}\langle D_y\rangle^{2N-1}U_1'(y)|\langle\xi\rangle^{-1}. \tag{C.121}$$

Since U_1' is odd, we may write

$$\begin{aligned}\langle D_y\rangle^{2N-1}U_1'(y)&=i\frac{y}{2}\int_{-1}^{1}(D_x\langle D_x\rangle^{2N-1}U_1')(\mu y)\,d\mu\\&=i\frac{y}{2t}\int_{-1}^{1}\big((L_+\langle D_x\rangle^{2N}U_1')(\mu y)-\mu y(\langle D_x\rangle^{2N}U_1')(\mu y)\big)\,d\mu\end{aligned}$$

using the definition (C.5) of L_+. We get finally

$$\begin{aligned}&|\langle y\rangle^{-N''}\langle D_y\rangle^{2N-1}U_1'(y)|\\&\quad\le \frac{C}{t}\big(\|\langle y\rangle^{-N''+1}L_+\langle D_x\rangle^{2N}U_1'\|_{L^\infty}+\|\langle y\rangle^{-N''+2}\langle D_x\rangle^{2N}U_1'\|_{L^\infty}\big).\end{aligned}\tag{C.122}$$

We may apply estimate (C.112) with U_1' replaced by $\langle D_x\rangle^{2N}U_1'$ (as $\langle D_x\rangle^{2N}M(\tau,\cdot)$ in (C.110) satisfies the same assumption as $M(\tau,\cdot)$), and the pre-factor $\langle y\rangle^{-N''+1}$, $\langle y\rangle^{-N''+2}$ on the right-hand side of (C.122) satisfies estimates of the form (C.88) with some large fixed N (instead of for any N). By the last statement in Proposition C.2.4, this is enough to apply (C.112). Plugging this in (C.121), we get for that expression a bound in $\varepsilon^2 t^{-1}\langle x\rangle^{-N'}\langle\xi\rangle^{-1}$, which is controlled by the right-hand side of (C.117) since $t\le\varepsilon^{-4}$. This concludes the proof. ∎

C.3 An explicit computation

In this last section of this chapter, we make an explicit computation that will be used in relation with Fermi's golden rule.

Let χ be in $C_0^\infty(\mathbb{R})$, even, equal to one close to zero. If $\lambda>1$ and if $\pm\xi_\lambda$ are still the two roots of $\sqrt{1+\xi^2}-\lambda=0$, set

$$\chi_\lambda(\xi)=\chi(\xi-\xi_\lambda)+\chi(\xi+\xi_\lambda).\tag{C.123}$$

If $\lambda<1$, set $\chi_\lambda\equiv 0$.

Proposition C.3.1. *Let M be a function satisfying (C.7) with $\omega=1$, that is odd in x. Let U be defined from M by (C.3) ant let Z be an odd function in $\mathcal{S}(\mathbb{R})$. Then*

$$\begin{aligned}&\int \hat{U}(t,\xi)\hat{Z}(\xi)\,d\xi\\&\quad=\lim_{\sigma\to 0+} ie^{i\lambda t}\int_0^{+\infty}\int e^{i\tau(\sqrt{1+\xi^2}-\lambda+i\sigma)}\chi_\lambda(\xi)\hat{M}(t,\xi)\hat{Z}(\xi)\,d\xi\,d\tau\\&\qquad+e^{i\lambda t}\int\frac{(1-\chi_\lambda)(\xi)}{\lambda-\sqrt{1+\xi^2}}\hat{M}(t,\xi)\hat{Z}(\xi)\,d\xi+r(t),\end{aligned}\tag{C.124}$$

where r satisfies

$$|r(t)|\le C\big(\varepsilon^2 t^{-\frac{3}{2}}+t_\varepsilon^{-2}+\varepsilon t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'}\big).\tag{C.125}$$

Remark. It is clear that the limit on the right-hand side of (C.124) exists and may be computed from $(\sqrt{1+\xi^2}-\lambda+i0)^{-1}$. We keep it nevertheless under the form (C.124) as this will be more convenient for us when using the proposition.

To prove the proposition, we shall write the left-hand side of (C.124), according to (C.3), under the form

$$i\int_1^t\int e^{i(t-\tau)\sqrt{1+\xi^2}+i\lambda\tau}\hat{M}(\tau,\xi)\hat{Z}(\xi)\,d\xi\,d\tau.\tag{C.126}$$

We decompose

$$\hat{M}(\tau,\xi) = \hat{M}'(\tau,\xi) + \hat{M}''(\tau,\xi),$$
$$\hat{M}'(\tau,\xi) = \hat{M}(\tau,\xi)\chi_\lambda(\xi), \tag{C.127}$$
$$\hat{M}''(\tau,\xi) = \hat{M}(\tau,\xi)(1-\chi_\lambda)(\xi).$$

We notice that $\hat{M}''$ vanishes at order one at $\xi = 0$ by the oddness assumption on M.

Lemma C.3.2. *Expression* (C.126) *with* $\hat{M}$ *replaced by* $\hat{M}''$ *may be written as*

$$e^{i\lambda t}\int \frac{(1-\chi_\lambda)(\xi)}{\lambda - \sqrt{1+\xi^2}}\hat{M}(t,\xi)\hat{Z}(\xi)\,d\xi \tag{C.128}$$

modulo a remainder satisfying (C.125).

Proof. The expression under study is the sum of (C.128) and of

$$-\int e^{i(t-1)\sqrt{1+\xi^2}+i\lambda}\hat{M}(1,\xi)\frac{(1-\chi_\lambda)(\xi)}{\lambda-\sqrt{1+\xi^2}}\hat{Z}(\xi)\,d\xi \tag{C.129}$$

and

$$-\int_1^t\int e^{i(t-\tau)\sqrt{1+\xi^2}+i\lambda\tau}\partial_\tau\hat{M}(\tau,\xi)\frac{(1-\chi_\lambda)(\xi)}{\lambda-\sqrt{1+\xi^2}}\hat{Z}(\xi)\,d\xi\,d\tau. \tag{C.130}$$

In (C.129) and (C.130), the integrand vanishes at order 2 at $\xi = 0$ by the oddness of M and Z. The stationary phase formula in ξ allows thus to gain a factor $t^{-\frac{3}{2}}$ or $\langle t-\tau\rangle^{-\frac{3}{2}}$. Taking into account (C.7) with $\omega = 1$, we thus bound (C.129) by $C\varepsilon^2 t^{-\frac{3}{2}}$ and (C.130) from

$$\int_1^t \langle t-\tau\rangle^{-\frac{3}{2}}\Big(\frac{\varepsilon^4}{(1+\tau\varepsilon^2)^2} + \frac{\varepsilon^{1+3\theta'}}{(1+\tau\varepsilon^2)^{\frac{1}{2}}}\tau^{-\frac{3}{2}(1-\frac{\theta'}{2})}\Big)\,d\tau$$
$$\leq C\big(t_\varepsilon^{-2} + \varepsilon t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'}\big)$$

(using $t \leq \varepsilon^{-4}$). We thus get quantities controlled as in (C.125). ∎

The lemma implies the proposition when $\lambda < 1$. We shall assume from now on that $\lambda > 1$ and study (C.126) with $\hat{M}$ replaced by $\hat{M}'$.

End of the proof of Proposition C.3.1. By the Taylor formula, we write for $1 \leq \tau \leq t$,

$$\hat{M}'(\tau,\xi) = \hat{M}'(t,\xi) + (\tau - t)H(t,\tau,\xi),$$

where according to (C.7) with $\omega = 1$, H satisfies for any α,

$$|\partial_\xi^\alpha H(t,\tau,\xi)| \leq C_\alpha \tau_\varepsilon^{-\frac{1}{2}}\big(\tau_\varepsilon^{-\frac{3}{2}} + \tau^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'}\big).$$

Integral (C.126) with $\hat{M}$ replaced by $\hat{M}'$ may be written as the sum $J_1 + J_2$, where

$$\begin{aligned} J_1 &= i \int_1^t \int e^{i(t-\tau)\sqrt{1+\xi^2}+i\lambda\tau} \hat{M}'(t,\xi)\hat{Z}(\xi)\, d\xi\, d\tau, \\ J_2 &= i \int_1^t \int e^{i(t-\tau)\sqrt{1+\xi^2}+i\lambda\tau} (\tau - t) H(t,\tau,\xi)\hat{Z}(\xi)\, d\xi\, d\tau. \end{aligned} \tag{C.131}$$

Since H is supported close to $\pm\xi_\lambda$, so far away from zero, we can make in J_2 any number of integrations by parts in ξ in order to gain a decaying factor in $\langle t-\tau\rangle^{-N}$ for any N, so that

$$|J_2| \le C \int_1^t \langle t-\tau\rangle^{-N} \big(\tau_\varepsilon^{-2} + \tau_\varepsilon^{-\frac{1}{2}} \tau^{-\frac{3}{2}} (\varepsilon^2 \sqrt{t})^{\frac{3}{2}\theta'}\big)\, d\tau$$

which is better than the right-hand side of (C.125). On the other hand, we may write

$$\begin{aligned} J_1 &= i e^{i\lambda t} \int_0^{t-1} \int e^{i\tau(\sqrt{1+\xi^2}-\lambda)} \hat{M}'(t,\xi)\hat{Z}(\xi)\, d\xi\, d\tau \\ &= \lim_{\sigma\to 0+} i e^{i\lambda t} \int_0^{+\infty} \int e^{i\tau(\sqrt{1+\xi^2}-\lambda+i\sigma)} \hat{M}'(t,\xi)\hat{Z}(\xi)\, d\xi\, d\tau + J_1', \end{aligned} \tag{C.132}$$

where

$$J_1' = -i e^{i\lambda t} \lim_{\sigma\to 0+} \int_{t-1}^{+\infty} \int e^{i\tau(\sqrt{1+\xi^2}-\lambda+i\sigma)} \hat{M}'(t,\xi)\hat{Z}(\xi)\, d\xi\, d\tau.$$

The first term on the right-hand side of (C.132) provides the first term on the right-hand side of (C.124). Moreover, in the expression of J_1', we can make as many integrations by parts in ξ as we want to get a decaying factor in $\langle\tau\rangle^{-N}$ for any N. This shows that J_1' is $O(\varepsilon^2 t^{-N})$, so may be incorporated to r in (C.124). This concludes the proof. ∎

Appendix D

Action of multilinear operators on Sobolev and Hölder spaces

In Appendix B, we have introduced multilinear operators that generalize the linear operators (B.3). In this appendix, we want to discussed Sobolev boundedness properties of such operators. For linear ones like (B.3), given in terms of symbols satisfying (B.1) with $M(x,\xi) \equiv 1$, such bounds are well known: see for instance Dimassi and Sjöstrand [24]. We generalize these bounds to multilinear operators, under the form

$$\|\mathrm{Op}_h(a)(\underline{v}_1,\dots,\underline{v}_n)\|_{H_h^s} \leq C \sum_{j=1}^{n} \prod_{\ell \neq j} \|\underline{v}_\ell\|_{W_h^{\rho_0,\infty}} \|\underline{v}_j\|_{H_h^s}, \tag{D.1}$$

where $\|\underline{v}\|_{W_h^{\rho_0,\infty}} = \|\langle hD_x\rangle^{\rho_0}\underline{v}\|_{L^\infty}$ and $\|\underline{v}\|_{H_h^s} = \|\langle hD_x\rangle^s\underline{v}\|_{L^2}$ with $s \geq 0$ and ρ_0 a large enough number independent of s. Notice that such an estimate is the natural generalization of the standard bound $\|uv\|_{H^s} \leq \|u\|_{L^\infty}\|v\|_{H^s} + \|u\|_{H^s}\|v\|_{L^\infty}$, that holds for any $s \geq 0$, to a framework of multilinear operators more general than the product.

We give also, in the case when the symbol $a(\frac{x}{h}, x, \xi_1, \dots, \xi_n)$ in (D.1) is rapidly decaying in $\frac{x}{h}$, other estimates of the form

$$\|\mathrm{Op}_h(a)(\underline{v}_1,\dots,\underline{v}_n)\|_{L^2} \leq Ch \prod_{j=1}^{n-1} \|\underline{v}_j\|_{W_h^{\rho_0,\infty}} (\|\mathcal{L}_\pm \underline{v}_n\|_{L^2} + \|\underline{v}_n\|_{L^2}) \tag{D.2}$$

for any *odd* functions $\underline{v}_1, \dots, \underline{v}_n$, where

$$\mathcal{L}_\pm = x \pm \frac{D_x}{\langle D_x\rangle}.$$

The important point here is that the rapid decay in $\frac{x}{h}$ of the symbol a allows one to gain on the right-hand side a small factor h. We have already explained in Chapter 2 where this gain comes from: The quantity inside the norm on the left-hand side of (D.2) is $h = t^{-1}$ times a generalization of expression (2.64). We have seen that thanks to (2.65), one may express any of the functions $\underline{v}_j$, say $\underline{v}_n$, from $\mathcal{L}_\pm \underline{v}_n$, up to a loss of $\frac{x}{h}$ that is compensated by the rapid decay of a relatively to that variable. Such properties explain why terms like r_1' in (B.8) may be considered somewhat as remainders: they do not involve a factor h in their estimate, but the fact that they decay rapidly in $\frac{x}{h}$ allows one to use (D.2) and thus to recover in that way an $O(h)$ bound.

Let us indicate more precisely what are the Sobolev bounds we shall get with respect to the symbols defined in Appendix B. Recall that we introduced classes of symbols $\tilde{S}_{\kappa,0}(M,p)$, $\tilde{S}'_{\kappa,0}(M,p)$ in Definition 3.1.1 and their (generalized) semiclassical counterparts $S_{\kappa,\beta}(M,p)$, $S'_{\kappa,\beta}(M,p)$ in Definition B.1.2. We shall study first the action of operators associated to the $\tilde{S}_{\kappa,0}(M,p)$, $S_{\kappa,\beta}(M,p)$ classes and then, in the second section of this appendix, the case of operators associated to classes of decaying symbols $\tilde{S}'_{\kappa,0}(M,p)$, $S'_{\kappa,\beta}(M,p)$.

D.1 Action of quantization of non-space-decaying symbols

We introduce the following notation. If $\underline{v}$ is a function depending on the semiclassical parameter $h \in]0,1]$, we set

$$\|\underline{v}\|_{H^s_h} = \|\langle hD_x\rangle^s \underline{v}\|_{L^2} \tag{D.3}$$

for any $s \in \mathbb{R}$. For ρ in $\mathbb{N}$, we define

$$\|\underline{v}\|_{W^{\rho,\infty}_h} = \|\langle hD_x\rangle^\rho \underline{v}\|_{L^\infty}. \tag{D.4}$$

Proposition D.1.1. *Let n be in $\mathbb{N}^*$, κ in $\mathbb{N}$, $\nu \geq 0$. There is ρ_0 in $\mathbb{N}$ such that, for any $\beta \geq 0$, any symbol a in the class $S_{\kappa,\beta}(M_0^\nu, n)$ of Definition B.1.2 (with M_0 given by (B.10)), the following holds true, under the restriction that, for* (i) *and* (ii)*, either $(\kappa,\beta) = (0,0)$ or $0 < \kappa\beta \leq 1$ or $a(y,x,\xi_1,\dots,\xi_n)$ is independent of x:*

(i) *Assume moreover that $a(y,x,\xi_1,\dots,\xi_n)$ is supported in the domain*

$$|\xi_1| + \cdots + |\xi_{n-1}| \leq K(1+|\xi_n|)$$

for some constant K. Then, for any $s \geq 0$, there is $C > 0$ such that, for any test functions $\underline{v}_1,\dots,\underline{v}_n$,

$$\|\mathrm{Op}_h(a)(\underline{v}_1,\dots,\underline{v}_n)\|_{H^s_h} \leq C \prod_{j=1}^{n-1} \|\underline{v}_j\|_{W^{\rho_0,\infty}_h} \|\underline{v}_n\|_{H^s_h} \tag{D.5}$$

uniformly in $h \in]0,1]$.

(ii) *Without any support condition on the symbol, we have instead*

$$\|\mathrm{Op}_h(a)(\underline{v}_1,\dots,\underline{v}_n)\|_{H^s_h} \leq C \sum_{j=1}^{n} \prod_{\ell \neq j} \|\underline{v}_\ell\|_{W^{\rho_0,\infty}_h} \|\underline{v}_j\|_{H^s_h}. \tag{D.6}$$

(iii) *For any $j = 1,\dots,n$, we have also the estimate (without any restriction on (κ,β) or a)*

$$\|\mathrm{Op}_h(a)(\underline{v}_1,\dots,\underline{v}_n)\|_{L^2} \leq C \prod_{\ell \neq j} \|\underline{v}_\ell\|_{W^{\rho_0,\infty}_h} \|\underline{v}_j\|_{L^2}. \tag{D.7}$$

Moreover, the above estimates hold true under a weaker assumption than in Definition B.1.2 of the symbols: namely it is enough to assume that bounds (B.13) hold with $N = 2$ (instead of for all N) for the last exponent in this formula.

Before giving the proof, we establish a lemma.

Lemma D.1.2. *Let a be in the class $S'_{\kappa,0}(M_0^\nu, n)$ of Definition B.1.2 (or more generally a symbol satisfying (B.13) for any $\alpha'_0, \alpha_0, k \in \mathbb{N}, \alpha \in \mathbb{N}^p$, with the last factor replaced by $(1 + M_0^{-\kappa}|y|)^{-2}$). There are ρ_0 in $\mathbb{N}$ depending only on ν, and a family of functions $a_{k_1,\dots,k_{n-1}}(\underline{v}_1, \dots, \underline{v}_{n-1}, y, x, \xi)$ indexed by $(k_1, \dots, k_{n-1}) \in \mathbb{N}^{n-1}$ satisfying bounds*

$$|\partial_x^\alpha \partial_\xi^{\alpha'} a_{k_1,\dots,k_{n-1}}(\underline{v}_1, \dots, \underline{v}_{n-1}, y, x, \xi)| \\ \le C 2^{-\max(k_1,\dots,k_{n-1})} \langle y \rangle^{-2} \prod_{j=1}^{n-1} \|\underline{v}_j\|_{W_h^{\rho_0,\infty}} \tag{D.8}$$

for $0 \le \alpha, \alpha' \le 2$, such that if we set for any y

$$a(y, x, hD_1, \dots, hD_n)(\underline{v}_1, \dots, \underline{v}_{n-1}, \underline{v}_n) \\ = \frac{1}{(2\pi)^n} \int e^{ix(\xi_1+\dots+\xi_n)} a(y, x, h\xi_1, \dots, h\xi_n) \prod_{j=1}^n \hat{\underline{v}}_j(\xi_j)\, d\xi_1 \cdots d\xi_n \tag{D.9}$$

and use a similar notation for $a_{k_1,\dots,k_{n-1}}(\underline{v}_1, \dots, \underline{v}_{n-1}, y, x, hD_x)\underline{v}_n$, then

$$a(y, x, hD_1, \dots, hD_n)(\underline{v}_1, \dots, \underline{v}_{n-1}, \underline{v}_n) \\ = \sum_{k_1=0}^{+\infty} \cdots \sum_{k_{n-1}=0}^{+\infty} a_{k_1,\dots,k_{n-1}}(\underline{v}_1, \dots, \underline{v}_{n-1}, y, x, hD_x)\underline{v}_n. \tag{D.10}$$

Proof. We take a Littlewood–Paley decomposition of the identity, $\mathrm{Id} = \sum_{k=0}^{+\infty} \Delta_k^h$, where $\Delta_0^h = \mathrm{Op}_h(\psi(\xi))$, $\Delta_k^h = \mathrm{Op}_h(\varphi(2^{-k}\xi))$ for $k > 0$, with convenient functions $\psi \in C_0^\infty(\mathbb{R})$, $\varphi \in C_0^\infty(\mathbb{R} - \{0\})$. We also take $\tilde{\psi}$ in $C_0^\infty(\mathbb{R})$, $\tilde{\varphi}$ in $C_0^\infty(\mathbb{R} - \{0\})$ with $\tilde{\psi}\psi = \psi$, $\tilde{\varphi}\varphi = \varphi$. We set $\tilde{\varphi}_k(\xi) = \tilde{\varphi}(2^{-k}\xi)$ for $k > 0$, $\tilde{\varphi}_0(\xi) = \tilde{\psi}(\xi)$. Plugging this decomposition on each factor $\underline{v}_j$, $j = 1, \dots, n-1$ in (D.9), we obtain an expression of the form (D.10) if we define

$$a_{k_1,\dots,k_{n-1}}(\underline{v}_1, \dots, \underline{v}_{n-1}, y, x, \xi) \\ = \frac{1}{(2\pi)^{n-1}} \int e^{ix(\xi_1+\dots+\xi_{n-1})} a(y, x, h\xi_1, \dots, h\xi_{n-1}, \xi) \\ \times \prod_{j=1}^{n-1} \tilde{\varphi}_{k_j}(h\xi_j) \widehat{\Delta_{k_j}^h \underline{v}_j}(\xi_j)\, d\xi_1 \cdots d\xi_{n-1}. \tag{D.11}$$

We may rewrite this as

$$a_{k_1,\dots,k_{n-1}}(\underline{v}_1, \dots, \underline{v}_{n-1}, y, x, \xi) \\ = h^{-(n-1)} \int K_{k_1,\dots,k_{n-1}}\Big(y, x, \frac{x - x'_1}{h}, \dots, \frac{x - x'_{n-1}}{h}, \xi\Big) \\ \times \prod_{j=1}^{n-1} \Delta_{k_j}^h \underline{v}_j(x'_j)\, dx'_1 \cdots dx'_{n-1} \tag{D.12}$$

with

$$\begin{aligned} &K_{k_1,\dots,k_{n-1}}(y,x,z_1,\dots,z_{n-1},\xi)\\ &\quad= \frac{1}{(2\pi)^{n-1}}\int e^{i(z_1\xi_1+\dots+z_{n-1}\xi_{n-1})}a(y,x,\xi_1,\dots,\xi_{n-1},\xi)\\ &\qquad\times\prod_{j=1}^{n-1}\tilde{\varphi}_{k_j}(\xi_j)\,d\xi_1\cdots d\xi_{n-1}. \end{aligned}\tag{D.13}$$

By the definition of $M_0(\xi_1,\dots,\xi_{n-1},\xi_n)$, on the support of $\prod_{j=1}^{n-1}\tilde{\varphi}_{k_j}(\xi_j)$, one has

$$M_0(\xi_1,\dots,\xi_{n-1},\xi_n) = O(2^{\hat{k}}) \quad \text{if } \hat{k} = \max(k_1,\dots,k_{n-1}).$$

As a is in the class $S'_{\kappa,0}(M_0^\nu,n)$, this implies that a in (D.13) is $O(2^{\nu\hat{k}})$. Moreover, if we perform two ∂_{ξ_j}-integrations by parts in (D.13), we gain a factor in $\langle 2^{-\hat{k}\kappa}z_j\rangle^{-2}$ under the integral, for $j = 1,\dots,n-1$, according to (B.13). In addition, we have also a decaying factor in $\langle 2^{-\hat{k}\kappa}|y|\rangle^{-2}$. It follows that for $\alpha,\alpha' \le 1$,

$$\begin{aligned} &|\partial_x^\alpha\partial_\xi^{\alpha'}K_{k_1,\dots,k_{n-1}}(y,x,z_1,\dots,z_{n-1},\xi)|\\ &\qquad\le C2^{(\kappa(\alpha+\alpha'+2)+\nu+n-1)\hat{k}}\prod_{j=1}^{n-1}\langle 2^{-\kappa\hat{k}}z_j\rangle^{-2}\langle y\rangle^{-2}. \end{aligned}\tag{D.14}$$

Plugging this estimate in (D.12) and using

$$|\Delta_{k_j}^h \underline{v}_j(x'_j)| \le C2^{-k_j\rho_0}\|\langle hD_x\rangle^{\rho_0}\underline{v}_j\|_{L^\infty}$$

we see that if ρ_0 has been taken large enough relatively to ν,κ, we get bounds of the form (D.8). This concludes the proof. ∎

Proof of Proposition D.1.1. (i) We reduce first to the case $s = 0$. Actually, by Corollary B.2.4, that applies under the restrictions in the statement on (κ,β) or a, the operator

$$(\underline{v}_1,\dots,\underline{v}_n) \mapsto \langle hD_x\rangle^s \mathrm{Op}_h(a)(\underline{v}_1,\dots,\underline{v}_{n-1},\langle hD_x\rangle^{-s}\underline{v}_n)$$

may be written as $\mathrm{Op}_h(\tilde{a})(\underline{v}_1,\dots,\underline{v}_n)$ for some symbol $\tilde{a}$ in $S_{\kappa,\beta}(M_0^{\nu'},n)$ for some ν' that does not depend on s. It is thus sufficient to show that

$$\|\mathrm{Op}_h(\tilde{a})(\underline{v}_1,\dots,\underline{v}_n)\|_{L^2} \le C\prod_{j=1}^{n-1}\|\underline{v}_j\|_{W_h^{\rho_0,\infty}}\|\underline{v}_n\|_{L^2}. \tag{D.15}$$

By expression (B.14), we have

$$\begin{aligned} \mathrm{Op}_h(\tilde{a})(\underline{v}_1,\dots,\underline{v}_n) &= \tilde{a}\Big(\frac{x}{h},x,hD_1,\dots,hD_n\Big)(\underline{v}_1,\dots,\underline{v}_n)\\ &= \tilde{a}(-\infty,x,hD_1,\dots,hD_n)(\underline{v}_1,\dots,\underline{v}_n)\\ &\quad+\int_{-\infty}^{\frac{x}{h}}(\partial_y\tilde{a})(y,x,hD_1,\dots,hD_n)(\underline{v}_1,\dots,\underline{v}_n)\,dy. \end{aligned}\tag{D.16}$$

As $\partial_y \tilde{a}$ is in $S'_{\kappa,0}(M_0^\nu, n)$ (for some ν), we may apply at any fixed y expansion (D.10) to $\partial_y \tilde{a}$. The symbols $a_{k_1,\dots,k_{n-1}}$ on the right-hand side satisfy (D.8), so that we may apply to them the Calderón–Vaillancourt theorem [9] in the version of Cordes [12], considering $y, \underline{v}_1, \dots, \underline{v}_{n-1}$ as parameters. One gets in that way for any $y, \underline{v}_1, \dots, \underline{v}_n$,

$$\begin{aligned} &\|\partial_y \tilde{a}(y, x, hD_1, \dots, hD_n)(\underline{v}_1, \dots, \underline{v}_n)\|_{L^2} \\ &\quad \le C \sum_{k_1} \cdots \sum_{k_{n-1}} 2^{-\max(k_1,\dots,k_{n-1})} \langle y\rangle^{-2} \prod_{j=1}^{n-1} \|\underline{v}_j\|_{W_h^{\rho_0,\infty}} \|\underline{v}_n\|_{L^2}. \end{aligned} \tag{D.17}$$

The fact that the L^2 norm of the last term in (D.16) is bounded from above by the right-hand side of (D.5) (with $s = 0$) follows from that inequality. If we apply the version of Lemma D.1.2 without parameter y to $\tilde{a}(-\infty, x, \xi_1, \dots, \xi_n)$, we obtain also an inequality of the form (D.17) (without factor $\langle y\rangle^{-2}$ on the right-hand side), which implies for the first term on the right-hand side of (D.16) the wanted estimate. This concludes the proof.

(ii) We just split a as a sum of symbols for which

$$\sum_{\ell \ne j} |\xi_\ell| \le K(1 + |\xi_j|), \quad j = 1, \dots, n,$$

and apply (i) to each of them.

(iii) It is enough to prove (D.7) with $j = n$ for instance. Remember that in the proof of (i), we use that the support condition on a and the restrictions on (κ, β) or a only to reduce the case of H_h^s to L^2 estimates. Once this has been done, inequality (D.15) has been proved without any support condition on $\tilde{a}$, nor on (κ, β), so that it implies (D.7). This concludes the proof, the last statement of the Proposition coming from the fact that Lemma D.1.2 has been proved for symbols satisfying the indicated property and that Corollary B.2.4 used at the beginning of the proof holds also under such a condition. ∎

It will be useful to be able to decompose a symbol belonging to $S_{\kappa,0}(M_0^\nu, n)$ as a sum of a symbol in $S_{\kappa,\beta}(M_0^\nu, n)$ for some small $\beta > 0$ and a symbol whose quantization satisfies better estimates than (D.6) and (D.7). Define

$$\mathcal{L}_\pm = \frac{1}{h}\mathrm{Op}_h(x \pm p'(\xi)). \tag{D.18}$$

Corollary D.1.3. *Let $a(y, x, \xi_1, \dots, \xi_n)$ be in $S_{\kappa,0}(M_0^\nu, n)$ for some $\kappa \ge 0$, some $\nu \ge 0$, some $n \ge 2$. Let $\beta > 0$ (small), $r \in \mathbb{R}_+$. One may decompose $a = a_1 + a_2$, where a_1 is in $S_{\kappa,\beta}(M_0^\nu, n)$ and a_2 is such that if s satisfies $(s - \rho_0 - 1)\beta \ge r + \frac{n+1}{2}$,*

$$\|\mathrm{Op}_h(a_2)(\underline{v}_1, \dots, \underline{v}_n)\|_{H_h^s} \le Ch^r \prod_{j=1}^{n} \|\underline{v}_j\|_{H_h^s}, \tag{D.19}$$

$$\|\mathcal{L}_\pm \mathrm{Op}_h(a_2)(\underline{v}_1, \dots, \underline{v}_n)\|_{L^2} \le Ch^r \prod_{j=1}^{n-1} \|\underline{v}_j\|_{H_h^s} (\|\underline{v}_n\|_{L^2} + \|\mathcal{L}_\pm \underline{v}_n\|_{L^2}) \tag{D.20}$$

and

$$\|\mathcal{L}_\pm \mathrm{Op}_h(a_2)(\underline{v}_1,\dots,\underline{v}_n)\|_{L^2} \le C h^r \prod_{j=1}^{n-1} \|\underline{v}_j\|_{H_h^s}(\|\underline{v}_n\|_{L^2} + \|\mathcal{L}_\pm \underline{v}_n\|_{W_h^{\rho_0,\infty}}). \quad \text{(D.21)}$$

(In the last two estimates, we could make play the special role devoted to n to any other index).

A similar statement holds replacing classes $S_{\kappa,0}$ (resp. $S_{\kappa,\beta}$) by $S'_{\kappa,0}$ (resp. $S'_{\kappa,\beta}$).

Proof. Take χ in $C_0^\infty(\mathbb{R})$ equal to one close to zero and define $a_1 = a\chi(h^\beta M_0(\xi))$, $a_2 = a(1-\chi)(h^\beta M_0(\xi))$. Then a_1 is in $S_{\kappa,\beta}(M_0^\nu, n)$ as it satisfies (B.12)–(B.13). Let us show that a_2 obeys (D.19)–(D.20). Decomposing a_2 in a sum of several symbols, we may assume for instance that it is supported for $|\xi_1| + \cdots + |\xi_{n-1}| \le K\langle \xi_n \rangle$. Then, by the definition of a_2, there is at least one index j, $1 \le j \le n-1$, such that $|\xi_j| \ge ch^{-\beta}$ on the support of a_2, for instance $j = n-1$. Applying (D.5), we get

$$\begin{aligned} &\|\mathrm{Op}_h(a_2)(\underline{v}_1,\dots,\underline{v}_n)\|_{H_h^s} \\ &\quad \le C \prod_{j=1}^{n-1} \|\underline{v}_j\|_{W_h^{\rho_0,\infty}} \|\mathrm{Op}_h((1-\tilde\chi)(h^{-\beta}\xi))\underline{v}_{n-1}\|_{W_h^{\rho_0,\infty}} \|\underline{v}_n\|_{H_h^s} \end{aligned} \quad \text{(D.22)}$$

for some new function $\tilde\chi$ equal to one close to zero. By semiclassical Sobolev injection,

$$\|\underline{v}_j\|_{W_h^{\rho_0,\infty}} \le C h^{-\frac{1}{2}} \|\underline{v}_j\|_{H_h^s}$$

if $s > \rho_0 + \frac{1}{2}$, and

$$\begin{aligned} &\|\mathrm{Op}_h((1-\tilde\chi)(h^\beta\xi))\underline{v}_{n-1}\|_{W_h^{\rho_0,\infty}} \\ &\quad \le C h^{-\frac{1}{2}} \|\mathrm{Op}_h((1-\tilde\chi)(h^{-\beta}\xi))\underline{v}_{n-1}\|_{H_h^{\rho_0+1}} \\ &\quad \le C h^{-\frac{1}{2}+(s-\rho_0-1)\beta} \|\underline{v}_{n-1}\|_{H_h^s}. \end{aligned} \quad \text{(D.23)}$$

If s is as in the statement, we get (D.19).

To obtain (D.20), we notice that

$$\begin{aligned} \mathcal{L}_\pm \mathrm{Op}_h(a_2)(\underline{v}_1,\dots,\underline{v}_n) = &\pm \frac{1}{h}\mathrm{Op}_h(p'(\xi))\mathrm{Op}_h(a_2)(\underline{v}_1,\dots,\underline{v}_n) \\ &+ i\,\mathrm{Op}_h\Big(\frac{\partial a_2}{\partial \xi_n}\Big)(\underline{v}_1,\dots,\underline{v}_n) \\ &+ \mathrm{Op}_h(a_2)\Big(\underline{v}_1,\dots,\underline{v}_{n-1},\frac{x}{h}\underline{v}_n\Big). \end{aligned} \quad \text{(D.24)}$$

The L^2 norm of the first two terms on the right-hand side is bounded from above by $Ch^r \prod_{j=1}^{n-1} \|\underline{v}_j\|_{H_h^s}\|\underline{v}_n\|_{L^2}$ if we use (D.7) and (D.23), for s as in the statement. On the other hand, in the third term, the last argument of $\mathrm{Op}_h(a_2)$ in (D.24) may be written $\mathcal{L}_\pm \underline{v}_n \mp \frac{1}{h}\mathrm{Op}_h(p'(\xi))$, so that we get an upper bound by the right-hand side of (D.20) using again (D.7) and (D.23).

We may also estimate the last term in (D.24) using (D.7), but putting the L^2 norm on $\underline{v}_{n-1}$, i.e. writing

$$\begin{aligned}&\|\mathrm{Op}_h(a_2)(\underline{v}_1,\dots,\underline{v}_{n-1},\mathcal{L}_\pm\underline{v}_n)\|_{L^2}\\&\quad\le C\prod_{j=1}^{n-2}\|\underline{v}_j\|_{W_h^{\rho_0,\infty}}\|\mathrm{Op}_h((1-\tilde{\chi})(h^\beta\xi))\underline{v}_{n-1}\|_{L^2}\|\mathcal{L}_\pm\underline{v}_n\|_{W_h^{\rho_0,\infty}}.\end{aligned}$$

Bounding the last but one factor by $h^{\beta s}\|\underline{v}_{n-1}\|_{H_h^s}$, we get as well (D.21). The last statement of the corollary concerning classes $S'_{\kappa,0}$, $S'_{\kappa,\beta}$ holds in the same way. ∎

Let us state next a corollary of Proposition D.1.1.

Corollary D.1.4. *Let $\nu\ge 0, n\in\mathbb{N}^*$. There is $\rho_0\in\mathbb{N}$ such that for any $\kappa\ge 0$, any $\beta\ge 0$, for any $j=1,\dots,n$, any a in $S_{\kappa,\beta}(M_0^\nu,n)$, there is $C>0$ such that for any $\underline{v}_1,\dots,\underline{v}_n$,*

$$\Big\|\frac{x}{h}\mathrm{Op}_h(a)(\underline{v}_1,\dots,\underline{v}_n)\Big\|_{L^2}\le C\prod_{\ell\ne j}\|\underline{v}_\ell\|_{W_h^{\rho_0,\infty}}(h^{-1}\|\underline{v}_j\|_{L^2}+\|\mathcal{L}_\pm\underline{v}_j\|_{L^2})\quad\text{(D.25)}$$

and for any $j\ne j'$, $1\le j,j'\le n$,

$$\begin{aligned}\Big\|\frac{x}{h}\mathrm{Op}_h(a)(\underline{v}_1,\dots,\underline{v}_n)\Big\|_{L^2}&\le C\Big(\prod_{\ell\ne j,j'}\|\underline{v}_\ell\|_{W_h^{\rho_0,\infty}}\Big)\|\underline{v}_{j'}\|_{L^2}\\&\quad\times\big(h^{-1}\|\underline{v}_j\|_{W_h^{\rho_0,\infty}}+\|\mathcal{L}_\pm\underline{v}_j\|_{W_h^{\rho_0,\infty}}\big).\end{aligned}\quad\text{(D.26)}$$

Proof. Let us prove (D.25) with $j=n$ for instance. By the definition of the quantization

$$\frac{x}{h}\mathrm{Op}_h(a)(\underline{v}_1,\dots,\underline{v}_n)=\mathrm{Op}_h(a)\Big(\underline{v}_1,\dots,\underline{v}_{n-1},\frac{x}{h}\underline{v}_n\Big)+i\,\mathrm{Op}_h\Big(\frac{\partial a}{\partial\xi_n}\Big)(\underline{v}_1,\dots,\underline{v}_n).$$

If we write $\frac{x}{h}=\mathcal{L}_\pm\mp h^{-1}p'(D_x)$, and apply (D.7) with $j=n$, we obtain (D.25). One obtains (D.26) in the same way, applying estimate (D.7) with j replaced by j', and using that $p'(hD_x)$ is bounded from $W_h^{\rho_0',\infty}$ to $W_h^{\rho_0,\infty}$ if $\rho_0'>\rho_0$. This concludes the proof. ∎

We shall also use some L^∞ estimates.

Proposition D.1.5. *Let $\nu\in[0,+\infty[$, $\kappa\ge 0$, $n\in\mathbb{N}^*$, $\beta\ge 0$. Let $q>1$ and let a be a symbol in $S_{\kappa,\beta}(M_0^\nu\prod_{j=1}^n\langle\xi_j\rangle^{-q},n)$. (It is actually enough to assume that in estimates (B.13), the last exponent N is equal to 2). Assume that $(\kappa,\beta)=(0,0)$ or $0<\kappa\beta\le 1$, or that $a(y,x,\xi)$ is independent of x. Then there are ρ_0 in $\mathbb{N}$ and, for any integer $\rho\ge\rho_0$, a constant $C>0$ such that for any $\underline{v}_1,\dots,\underline{v}_n$,*

$$\|\mathrm{Op}_h(a)(\underline{v}_1,\dots,\underline{v}_n)\|_{W_h^{\rho,\infty}}\le C\prod_{j=1}^n\|\underline{v}_j\|_{W_h^{\rho,\infty}}.\quad\text{(D.27)}$$

If we have just $a \in S_{\kappa\beta}(M_0^\nu \prod_{j=1}^n \langle \xi_j \rangle^{-1}, n)$, *we get for any* r *in* $\mathbb{N}$, *any* $\sigma > 0$, *any* s, ρ *with* $(s - \rho - 1)\sigma \geq r + \frac{1}{2}$ *and* $\rho \geq \rho_0$, *the bound*

$$\begin{aligned} &\|\mathrm{Op}_h(a)(\underline{v}_1, \dots, \underline{v}_n)\|_{W_h^{\rho,\infty}} \\ &\quad \leq C h^{-\sigma} \prod_{j=1}^n \|\underline{v}_j\|_{W_h^{\rho,\infty}} + C h^r \sum_{j=1}^n \prod_{\ell \neq j} \|\underline{v}_\ell\|_{W_h^{\rho,\infty}} \|\underline{v}_j\|_{H_h^s}. \end{aligned} \tag{D.28}$$

Proof. One may assume that a is supported for $|\xi_1| + \cdots + |\xi_{n-1}| \leq K(1 + |\xi_n|)$. One may use Corollary B.2.4, whose assumptions are satisfied, in order to reduce (D.27) to estimate

$$\|\mathrm{Op}_h(a)(\underline{v}_1, \dots, \underline{v}_n)\|_{L^\infty} \leq C \prod_{j=1}^{n-1} \|\underline{v}_j\|_{W_h^{\rho_0,\infty}} \|\underline{v}_n\|_{L^\infty}. \tag{D.29}$$

We apply (D.16) to reduce (D.29) to bounds of the form

$$\begin{aligned} &\|a(-\infty, x, hD_1, \dots, hD_n)(\underline{v}_1, \dots, \underline{v}_n)\|_{L^\infty} \\ &\quad \leq C \prod_{j=1}^{n-1} \|\underline{v}_j\|_{W_h^{\rho_0,\infty}} \|\underline{v}_n\|_{L^\infty}, \\ &\int_{-\infty}^{+\infty} \|\partial_y a(y, x, hD_1, \dots, hD_n)(\underline{v}_1, \dots, \underline{v}_n)\|_{L^\infty} \\ &\quad \leq C \prod_{j=1}^{n-1} \|\underline{v}_j\|_{W_h^{\rho_0,\infty}} \|\underline{v}_n\|_{L^\infty}. \end{aligned} \tag{D.30}$$

We may decompose $\partial_y a(y, x, hD_1, \dots, hD_n)$ using equality (D.10). Each contribution in the sum is given by a symbol satisfying estimate (D.8), with an extra factor $\langle \xi_n \rangle^{-q}$ on the right-hand side, coming from the fact that our symbol a was in $S_{\kappa,\beta}(M_0^\nu \prod_{j=1}^n \langle \xi_j \rangle^{-q}, n)$. The kernel of the corresponding operator will then be bounded in modulus by

$$C h^{-1} G\Big(\frac{x - x'}{h}\Big) 2^{-\max(k_1, \dots, k_{n-1})} \langle y \rangle^{-2} \prod_{j=1}^{n-1} \|\underline{v}_j\|_{W_h^{\rho_0,\infty}}$$

with some L^1 function G. The second estimate (D.30) follows from that. The first one is proved in the same way.

Finally, to get (D.28), we assume again a supported as above and decompose it as $a = a_1 + a_2$, with $a_1 = a\chi(h^\sigma \xi_n)$ for some $\sigma > 0$ and χ in $C_0^\infty(\mathbb{R})$ equal to one close to zero. Then a_1 is in $h^{-\sigma} S_{\kappa\beta}(M_0^\nu \prod_{j=1}^n \langle \xi_j \rangle^{-2}, n)$ (for a new value of ν), so that (D.27) applies, with a loss $h^{-\sigma}$, which provides the first term on the right-hand side of (D.28). On the other hand, we estimate $\|\mathrm{Op}_h(a_2)(\underline{v}_1, \dots, \underline{v}_n)\|_{W_h^{\rho,\infty}}$ from $C h^{-\frac{1}{2}} \|\mathrm{Op}_h(a_2)(\underline{v}_1, \dots, \underline{v}_n)\|_{H_h^{\rho+1}}$ by semiclassical Sobolev injection, and then this quantity by the last term on the right-hand side of (D.28) with $r = \sigma(s - \rho - 1) - \frac{1}{2}$. This concludes the proof. ∎

Let us translate the preceding results in the non-semiclassical case using the transformation Θ_t defined in (B.15) and (B.16)–(B.17). We translate first Proposition D.1.1.

Proposition D.1.6. *Let a be a symbol satisfying the assumptions of Proposition D.1.1 and (κ, β) satisfying also the assumptions of that proposition in the case of statements* (i) *and* (ii) *below (in particular, if a is independent of x, these statements hold for any (κ, β) with $\kappa \geq 0, \beta \geq 0$).*

(i) *If moreover a is supported for $|\xi_1| + \cdots + |\xi_{n-1}| \leq K(1 + |\xi_n|)$, one has for any $s \geq 0$ the bound*

$$\|\mathrm{Op}^t(a)(v_1, \dots, v_n)\|_{H^s} \leq C \prod_{j=1}^{n-1} \|v_j\|_{W^{\rho_0,\infty}} \|v_n\|_{H^s} \tag{D.31}$$

with some ρ_0 independent of s, Op^t being defined in (B.16)*.*

(ii) *Without any support assumption on the symbol of a, one has*

$$\|\mathrm{Op}^t(a)(v_1, \dots, v_n)\|_{H^s} \leq C \sum_{j=1}^{n} \prod_{\ell \neq j} \|v_\ell\|_{W^{\rho_0,\infty}} \|v_j\|_{H^s}. \tag{D.32}$$

(iii) *For any $j = 1, \dots, n$, one has also*

$$\|\mathrm{Op}^t(a)(v_1, \dots, v_n)\|_{L^2} \leq C \prod_{\ell \neq j} \|v_\ell\|_{W^{\rho_0,\infty}} \|v_j\|_{L^2}. \tag{D.33}$$

Proof. One combines Proposition D.1.1, (B.16) and the fact that by (B.15),

$$\|\Theta_t \underline{v}\|_{H^s} = \|\underline{v}\|_{H^s_h}$$

and

$$\|\Theta_t \underline{v}\|_{W^{\rho,\infty}} = h^{\frac{1}{2}} \|\underline{v}\|_{W^{\rho,\infty}_h}$$

if $h = t^{-1}$. ∎

To get non-semiclassical versions of Corollaries D.1.3 and D.1.4, let us notice that by (B.15)

$$L_\pm \Theta_t \underline{v} = \frac{1}{\sqrt{t}} (\mathcal{L}_\pm \underline{v}) \Big(\frac{x}{t}\Big)$$

is $\mathcal{L}_\pm$ is defined by (D.18) and

$$L_\pm = x \pm t p'(D_x). \tag{D.34}$$

We have then:

Corollary D.1.7. *Let $a(y, x, \xi_1, \dots, \xi_n)$ be a symbol in $S_{\kappa,0}(M_0^\nu, n)$ for some $\kappa \geq 0$, some $\nu \geq 0$, some $n \geq 2$. Let $\beta > 0$ be small and r in $\mathbb{R}_+$. One may decompose $a = a_1 + a_2$, where a_1 is in $S_{\kappa,\beta}(M_0^\nu, n)$ and a_2 satisfies, if $(s - \rho_0)\beta$ is large* *enough relatively to* r, n*,*

$$\|\mathrm{Op}^t(a_2)(v_1,\dots,v_n)\|_{H^s} \leq C t^{-r} \prod_{j=1}^{n} \|v_j\|_{H^s},$$

$$\|L_\pm \mathrm{Op}^t(a_2)(v_1,\dots,v_n)\|_{L^2} \leq C t^{-r} \prod_{j=1}^{n-1} \|v_j\|_{H^s}\big(\|v_n\|_{L^2} + \|L_\pm v_n\|_{L^2}\big), \tag{D.35}$$

$$\|L_\pm \mathrm{Op}^t(a_2)(v_1,\dots,v_n)\|_{L^2} \leq C t^{-r} \bigg(\prod_{j=1}^{n-1} \|v_j\|_{H^s}\bigg)\big(\|v_n\|_{L^2} + \|L_\pm v_n\|_{W^{\rho,\infty}}\big).$$

Moreover, in the last two estimates, one may make play the special role devoted to n *to any other index.*

Proof. Again, we combine (B.15)–(B.16) and the estimates in (D.19)–(D.21) (up to a change of notation for r). ■

In the same way, we get from Corollary D.1.4:

Corollary D.1.8. *With the notation of Corollary D.1.4, we have*

$$\|x\mathrm{Op}^t(a)(v_1,\dots,v_n)\|_{L^2} \leq C \prod_{\ell\neq j} \|v_\ell\|_{W^{\rho_0,\infty}}(t\|v_j\|_{L^2} + \|L_\pm v_j\|_{L^2}) \tag{D.36}$$

for any $1 \leq j \leq n$*. Moreover, for any* $j \neq j'$*,* $1 \leq j, j' \leq n$*,*

$$\begin{aligned}&\|x\mathrm{Op}^t(a)(v_1,\dots,v_n)\|_{L^2}\\ &\quad\leq C \prod_{\ell\neq j,j'} \|v_\ell\|_{W^{\rho_0,\infty}} \|v_{j'}\|_{L^2}(t\|v_j\|_{W^{\rho_0,\infty}} + \|L_\pm v_j\|_{W^{\rho_0,\infty}}).\end{aligned} \tag{D.37}$$

Finally, it follows from Proposition D.1.5:

Proposition D.1.9. *Under the assumptions and with notation of Proposition D.1.5, one has for* $\rho \geq \rho_0$*,*

$$\|\mathrm{Op}^t(a)(v_1,\dots,v_n)\|_{W^{\rho,\infty}} \leq C \prod_{j=1}^{n} \|v_j\|_{W^{\rho,\infty}} \tag{D.38}$$

if a *is in* $S_{\kappa,\beta}(M_0^\nu \prod_{j=1}^n \langle\xi_j\rangle^{-q}, n)$ *for some* $q > 1$ *and*

$$\begin{aligned}&\|\mathrm{Op}^t(a)(v_1,\dots,v_n)\|_{W^{\rho,\infty}}\\ &\quad\leq C t^{\sigma} \prod_{j=1}^{n} \|v_j\|_{W^{\rho,\infty}} + C t^{-r} \sum_{j=1}^{n} \prod_{\ell\neq j} \|v_\ell\|_{W^{\rho,\infty}} \|v_j\|_{H^s}\end{aligned} \tag{D.39}$$

if $q = 1$*,* $\sigma > 0$ *and* $(s - \rho)\sigma$ *is large enough relatively to* r*.*

D.2 Action of quantization of space decaying symbols

In this section we study the action of operators associated to symbols belonging to the classes $S'_{\kappa,\beta}(M_0^\nu, n)$ on Sobolev or Hölder spaces of *odd* functions. The oddness of the functions, together with the fact that elements in the S' class are symbols $a(y, x, \xi)$ rapidly decaying in y, will allow us to re-express the functions $\underline{v}$ on which acts the operator from $h\mathcal{L}_\pm \underline{v}$ (using notation (D.18)), thus gaining a power of h. Actually, it is not necessary that a be rapidly decaying in y, and we shall give statements with less stringent decay assumptions.

Proposition D.2.1. *Let n be in $\mathbb{N}^*$, κ in $\mathbb{N}$, $\nu \geq 0$. There is ρ_0 in $\mathbb{N}$ such that, for any $\beta \geq 0$, any symbol $a(y, x, \xi_1, \dots, \xi_n)$, supported in the domain*

$$|\xi_1| + \cdots + |\xi_{n-1}| \leq K(1 + |\xi_n|)$$

for some constant K, and such that for some ℓ, $1 \leq \ell \leq n-1$, a belongs to the class $S'^{2\ell+2}_{\kappa,\beta}(M_0^\nu, n)$ introduced at the end of Definition B.1.2, with $\kappa \geq 0$ and either $(\kappa, \beta) = (0, 0)$ or $0 < \kappa\beta \leq 1$ or a is independent of x, the following holds true:

(i) *For any $s \geq 0$, any* odd *test functions $\underline{v}_1, \dots, \underline{v}_n$, and any choice of signs $\varepsilon_j \in \{-, +\}$, $j = 1, \dots, \ell$,*

$$\begin{aligned} &\|\mathrm{Op}_h(a)(\underline{v}_1, \dots, \underline{v}_n)\|_{H_h^s} \\ &\quad \leq C h^\ell \prod_{j=1}^{\ell} \left(\|\mathcal{L}_{\varepsilon_j} \underline{v}_j\|_{W_h^{\rho_0,\infty}} + \|\underline{v}_j\|_{W_h^{\rho_0,\infty}}\right) \\ &\quad\quad \times \prod_{j=\ell+1}^{n-1} \|\underline{v}_j\|_{W_h^{\rho_0,\infty}} \|\underline{v}_n\|_{H_h^s}. \end{aligned} \tag{D.40}$$

(ii) *Assume in addition to the preceding assumptions that $\beta > 0$. Then, for any $0 \leq \ell' \leq \ell$, one has*

$$\begin{aligned} &\|\mathrm{Op}_h(a)(\underline{v}_1, \dots, \underline{v}_n)\|_{H_h^s} \\ &\quad \leq C h^{\ell - \frac{1}{2}\ell' - \sigma(\beta)} \prod_{j=1}^{\ell'} \left(\|\mathcal{L}_{\varepsilon_j} \underline{v}_j\|_{L^2} + \|\underline{v}_j\|_{L^2}\right) \\ &\quad\quad \times \prod_{j=\ell'+1}^{\ell} \left(\|\mathcal{L}_{\varepsilon_j} \underline{v}_j\|_{W_h^{\rho_0,\infty}} + \|\underline{v}_j\|_{W_h^{\rho_0,\infty}}\right) \\ &\quad\quad \times \prod_{j=\ell+1}^{n-1} \|\underline{v}_j\|_{W_h^{\rho_0,\infty}} \|\underline{v}_n\|_{H_h^s}, \end{aligned} \tag{D.41}$$

where $\sigma(\beta) > 0$ goes to zero when β goes to zero ($\sigma(\beta) = \ell'(\rho_0 + \frac{1}{2})\beta$ holds).

Proof. We shall prove (i) and (ii) simultaneously. We notice first that, by our support condition on $(\xi_1, \dots, \xi_n)$, $M_0(\xi) \sim 1 + |\xi_1| + \dots + |\xi_{n-1}|$, so that, up to changing ν, we may study the H_h^s norm of

$$\mathrm{Op}_h(\tilde{a})\big(\mathrm{Op}_h(\langle\xi\rangle^{-1})\underline{v}_1, \dots, \mathrm{Op}_h(\langle\xi\rangle^{-1})\underline{v}_\ell, \underline{v}_{\ell+1}, \dots, \underline{v}_n\big) \tag{D.42}$$

for a new symbol $\tilde{a}$ satisfying the same assumptions as a. Moreover, when $\beta > 0$, this symbol is rapidly decaying in $h^\beta M_0(\xi)$ according to (B.12)–(B.13), so that, modifying again $\tilde{a}$, we rewrite (D.42) as

$$\begin{aligned}\mathrm{Op}_h(\tilde{a})\big(\mathrm{Op}_h(\langle\xi\rangle^{-1}\langle\beta h^\beta\xi\rangle^{-\gamma})\underline{v}_1, \dots, \mathrm{Op}_h(\langle\xi\rangle^{-1}\langle\beta h^\beta\xi\rangle^{-\gamma})\underline{v}_\ell,&\\ \underline{v}_{\ell+1}, \dots, \underline{v}_n\big)&\end{aligned} \tag{D.43}$$

with $\gamma > 0$ to be chosen. We use now that if f is an odd function, we may write

$$f(x) = \frac{x}{2}\int_{-1}^{1}(\partial f)(\mu x)\,d\mu.$$

Consequently, for $j = 1, \dots, \ell$,

$$\mathrm{Op}_h\big(\langle\xi\rangle^{-1}\langle\beta h^\beta\xi\rangle^{-\gamma}\big)\underline{v}_j = \frac{ix}{2h}\int_{-1}^{1}\Big(\mathrm{Op}_h\Big(\langle\beta h^\beta\xi\rangle^{-\gamma}\frac{\xi}{\langle\xi\rangle}\Big)\underline{v}_j\Big)(\mu_j x)\,d\mu_j, \tag{D.44}$$

that we rewrite using (D.18)

$$\begin{aligned}&\mathrm{Op}_h\big(\langle\xi\rangle^{-1}\langle\beta h^\beta\xi\rangle^{-\gamma}\big)\underline{v}_j\\ &\quad= ih\frac{\varepsilon_j}{2}\frac{x}{h}\int_{-1}^{1}\big(\mathrm{Op}_h\big(\langle\beta h^\beta\xi\rangle^{-\gamma}\big)\mathcal{L}_{\varepsilon_j}\underline{v}_j\big)(\mu_j x)\,d\mu_j\\ &\qquad\quad - ih\frac{\varepsilon_j}{2}\frac{x}{h}\int_{-1}^{1}\Big(\mathrm{Op}_h\big(\langle\beta h^\beta\xi\rangle^{-\gamma}\big)\frac{x}{h}\underline{v}_j\Big)(\mu_j x)\,d\mu_j.\end{aligned} \tag{D.45}$$

We may thus write (D.45) as a linear combination of expressions of the form

$$h\Big(\frac{x}{h}\Big)^q\int_{-1}^{1}\mu_j^{q'}V_j(\mu_j x)\,d\mu_j, \tag{D.46}$$

where $q = 0, 1, 2$, $q' \in \mathbb{N}$ and $V_j(x)$ is of the form

$$V_j(x) = \mathrm{Op}_h\big(b_j(\beta h^\beta\xi)\big)\mathcal{L}_{\varepsilon_j}\underline{v}_j \quad \text{or} \quad V_j(x) = \mathrm{Op}_h\big(b_j(\beta h^\beta\xi)\big)\underline{v}_j \tag{D.47}$$

with $|\partial^k b_j(\xi)| = O(\langle\xi\rangle^{-\gamma-k})$. We plug these expressions inside (D.43). We remark that when we commute each factor $\frac{x}{h}$ with $\tilde{a}$, we get again an operator given by a symbol similar to $\tilde{a}$, up to changing ν. Moreover, the $\langle M_0^{-\kappa}y\rangle^{-2\ell-2}$ decay of $\tilde{a}(y, x, \xi)$ that we assume shows that for $q \le 2\ell$, $(\frac{x}{h})^q\tilde{a}(\frac{x}{h}, x, \xi)$ may be written $\tilde{a}_1(\frac{x}{h}, x, \xi)$ with $\tilde{a}_1(y, x, \xi)$ in $S'^{2}_{\kappa,\beta}(M_0^\nu, n)$ (for a new ν). Consequently, we may write (D.43) as a combination of quantities of the form

$$h^{\ell}\int_{-1}^{1}\cdots\int_{-1}^{1}\mathrm{Op}_h(\tilde{a}_1)\big(V_1(\mu_1\cdot),\dots,V_\ell(\mu_\ell\cdot),\underline{v}_{\ell+1},\dots,\underline{v}_n\big)\\ \times P(\mu_1,\dots,\mu_\ell)\,d\mu_1\cdots d\mu_\ell, \tag{D.48}$$

where V_j are given by (D.47) and P is some polynomial.

If we apply (D.5) (together with the remark at the end of the statement of Proposition D.1.1) and use that $\mathrm{Op}_h(b_j(\beta h^\beta\xi))$ is bounded from $W_h^{\rho_0,\infty}$ to itself, uniformly in h, we obtain (D.40). To prove (D.41), we apply again (D.5) and use that, for factors indexed by $j=1,\dots,\ell'$, we may write if $\gamma\geq\rho_0+1$ and $\beta>0$

$$\begin{aligned}\|\mathrm{Op}_h\big(b_j(\beta h^\beta\xi)\big)w\|_{W_h^{\rho_0,\infty}} &= \|\mathrm{Op}_h\big(\langle\xi\rangle^{\rho_0}b_j(\beta h^\beta\xi)\big)w\|_{L^\infty}\\ &\leq Ch^{-\frac12}\|\mathrm{Op}_h\big(\langle\xi\rangle^{\rho_0}\langle\beta h^\beta\xi\rangle^{-\gamma}\big)w\|_{L^2}^{\frac12}\\ &\quad\times\|\mathrm{Op}_h\big(\langle\xi\rangle^{\rho_0}\xi\langle\beta h^\beta\xi\rangle^{-\gamma}\big)w\|_{L^2}^{\frac12}\\ &\leq Ch^{-\frac12-\beta(\rho_0+\frac12)}\|w\|_{L^2}\end{aligned}$$

if $\gamma\geq\rho_0$. This brings (D.41) with $\sigma(\beta)=\ell'(\rho_0+\frac12)\beta$. ∎

When we want to estimate only the L^2 norms, instead of the H^s ones, we have the following statement:

Proposition D.2.2. *Let n be in $\mathbb{N}^*$, $\kappa\in\mathbb{N}$, $\beta\geq0$, $\nu\geq0$. There is $\rho_0\in\mathbb{N}$ such that, for any symbol a in $S'_{\kappa,\beta}(M_0^\nu\prod_{j=1}^n\langle\xi_j\rangle^{-1},n)$ and for any odd functions $\underline{v}_1,\dots,\underline{v}_n$, one has the following estimate:*

$$\|\mathrm{Op}_h(a)(\underline{v}_1,\dots,\underline{v}_n)\|_{L^2}\leq Ch\prod_{j=1}^{n-1}\|\underline{v}_j\|_{W_h^{\rho_0,\infty}}[\|\mathcal{L}_\pm\underline{v}_n\|_{L^2}+\|\underline{v}_n\|_{L^2}]. \tag{D.49}$$

Moreover, when $n\geq2$, we have also the bound

$$\begin{aligned}&\|\mathrm{Op}_h(a)(\underline{v}_1,\dots,\underline{v}_n)\|_{L^2}\\ &\quad\leq Ch\prod_{j=1}^{n-2}\|\underline{v}_j\|_{W_h^{\rho_0,\infty}}\big[\|\mathcal{L}_\pm\underline{v}_{n-1}\|_{W_h^{\rho_0,\infty}}+\|\underline{v}_n\|_{W_h^{\rho_0,\infty}}\big]\|\underline{v}_n\|_{L^2}.\end{aligned} \tag{D.50}$$

Estimate (D.49) (resp. (D.50)) holds as well for n (resp. $(n-1,n)$) replaced by any $j\in\{1,\dots,n\}$ (resp. $j,j'\in\{1,\dots,n\}$, $j\neq j'$). Moreover, it suffices to assume that a is in $S'^4_{\kappa,\beta}(M_0^\nu\prod_{j=1}^n\langle\xi_j\rangle^{-1},n)$ instead of $a\in S'_{\kappa,\beta}(M_0^\nu\prod_{j=1}^n\langle\xi_j\rangle^{-1},n)$.

Proof. Because of the assumption on a, we may write

$$\mathrm{Op}_h(a)(\underline{v}_1,\dots,\underline{v}_n)=\mathrm{Op}_h(\tilde{a})(\underline{v}_1,\dots,\underline{v}_{n-1},\mathrm{Op}_h(\langle\xi\rangle^{-1})\underline{v}_n)) \tag{D.51}$$

with $\tilde{a}$ in $S'_{\kappa,\beta}(M_0^\nu \prod_{j=1}^{n-1}\langle\xi_j\rangle^{-1}, n)$ (or $\tilde{a}$ in $S'^4_{\kappa,\beta}(M_0^\nu \prod_{j=1}^{n-1}\langle\xi_j\rangle^{-1}, n)$). We use next equation (D.45) (with $\gamma = 0$) in order to express $\mathrm{Op}_h(\langle\xi\rangle^{-1})\underline{v}_n$ as a combination of terms of the form (D.46) with $j = n$ and V_n given by (D.47). We obtain thus for (D.51) an expression in terms of integrals

$$h\int_{-1}^{1} \mathrm{Op}_h(\tilde{a}_1)[\underline{v}_1, \dots, \underline{v}_{n-1}, V_n(\mu_n\cdot)]P(\mu_n)\, d\mu_n \tag{D.52}$$

for some polynomial P, some $\tilde{a}_1 \in S'^2_{\kappa,\beta}(M_0^\nu \prod_{j=1}^{n-1}\langle\xi_j\rangle^{-1}, n)$. Applying (D.7), we get (D.49).

To obtain (D.50), we make appear the $\mathrm{Op}_h(\langle\xi\rangle^{-1})$ operator on argument $\underline{v}_{n-1}$ instead of $\underline{v}_n$ in (D.51), use (D.45) with $j = n-1$, obtain an expression of the form (D.52) with the roles of n and $n-1$ interchanged, and apply again (D.7). ∎

Let us also establish some corollaries and variants of the above results.

Corollary D.2.3. *Let n, κ, β, ν be as in Proposition D.2.2. Let a be a symbol in the class $S_{\kappa,\beta}(M_0^\nu \prod_{j=1}^{n+1}\langle\xi_j\rangle^{-1}, n+1)$. Let Z be in $\mathcal{S}(\mathbb{R})$. Then for any odd functions $\underline{v}_1, \dots, \underline{v}_n$,*

$$\begin{aligned}&\Big\|\mathrm{Op}_h(a)\Big[Z\Big(\frac{x}{h}\Big), \underline{v}_1, \dots, \underline{v}_n\Big]\Big\|_{L^2}\\ &\quad\le Ch\prod_{j=1}^{n-1}\|\underline{v}_j\|_{W_h^{\rho_0,\infty}}\big(\|\mathcal{L}_\pm \underline{v}_n\|_{L^2} + \|\underline{v}_n\|_{L^2}\big).\end{aligned} \tag{D.53}$$

If $n \ge 2$, we have also

$$\begin{aligned}&\Big\|\mathrm{Op}_h(a)\Big[Z\Big(\frac{x}{h}\Big), \underline{v}_1, \dots, \underline{v}_n\Big]\Big\|_{L^2}\\ &\quad\le Ch\prod_{j=1}^{n-2}\|\underline{v}_j\|_{W_h^{\rho_0,\infty}}\big(\|\mathcal{L}_\pm \underline{v}_{n-1}\|_{W_h^{\rho_0,\infty}} + \|\underline{v}_{n-1}\|_{W_h^{\rho_0,\infty}}\big)\|v_n\|_{L^2}.\end{aligned} \tag{D.54}$$

Proof. We write

$$a(y,x,\xi) = \langle y\rangle^4 \tilde{a}(y,x,\xi).$$

Then, according to the last remark in the statement, Proposition D.2.2 applies to $\tilde{a}$. Moreover, we may write $\mathrm{Op}_h(a)[Z(\frac{x}{h}), \underline{v}_1, \dots, \underline{v}_n]$ as a sum of expressions

$$\Big(\frac{x}{h}\Big)^q \mathrm{Op}_h(\tilde{a})\Big[Z\Big(\frac{x}{h}\Big), \underline{v}_1, \dots, \underline{v}_n\Big], \quad 0 \le q \le 4. \tag{D.55}$$

The commutator

$$\frac{x}{h}\mathrm{Op}_h(\tilde{a})\Big[Z\Big(\frac{x}{h}\Big), \underline{v}_1, \dots, \underline{v}_n\Big] - \mathrm{Op}_h(\tilde{a})\Big[\frac{x}{h}Z\Big(\frac{x}{h}\Big), \underline{v}_1, \dots, \underline{v}_n\Big]$$

is again of the form $\mathrm{Op}_h(\tilde{a}_1)[Z(\frac{x}{h}), \underline{v}_1, \dots, \underline{v}_n]$, for a new symbol satisfying the same assumptions as a, eventually with a different ν. Finally, we express (D.55) as a sum of expressions $\mathrm{Op}_h(\tilde{a}_1)[Z_1(\frac{x}{h}), \underline{v}_1, \dots, \underline{v}_n]$, for new symbols $\tilde{a}_1$ and a new $\mathcal{S}(\mathbb{R})$ function Z_1. If we apply (D.49) (resp. (D.50)), we get (D.53) (resp. (D.54)). ∎

We have also the following variant of Proposition D.2.2, that we state only for bilinear operators.

Proposition D.2.4. *Let $\nu, \kappa \geq 0$. There is $\rho_0 \in \mathbb{N}$ such that, for any a in the class $S'_{\kappa,0}(M_0^\nu \prod_{j=1}^2 \langle \xi_j \rangle^{-1}, 2)$, any odd functions $\underline{v}_1, \underline{v}_2$, one has the following estimates:*

$$\begin{aligned} &\|\mathrm{Op}_h(a)(\underline{v}_1, \underline{v}_2)\|_{L^2} \\ &\quad \leq C h^2 \big(\|\mathcal{L}_\pm \underline{v}_1\|_{W_h^{\rho_0,\infty}} + \|\underline{v}_1\|_{W_h^{\rho_0,\infty}}\big)\big(\|\mathcal{L}_\pm \underline{v}_2\|_{L^2} + \|\underline{v}_2\|_{L^2}\big) \end{aligned} \tag{D.56}$$

for any choice of the signs $\pm$ on the right-hand side. The symmetric inequality holds as well.

If moreover s, σ are positive with $s\sigma \geq 2(\rho_0 + 1)$, we get

$$\|\mathrm{Op}_h(a)(\underline{v}_1, \underline{v}_2)\|_{L^2} \leq C h^{\frac{3}{2}-\sigma} \prod_{j=1}^{2} \big(\|\mathcal{L}_\pm \underline{v}_j\|_{L^2} + \|\underline{v}_j\|_{H_h^s}\big). \tag{D.57}$$

Proof. To get (D.56), we write

$$\mathrm{Op}_h(a)(\underline{v}_1, \underline{v}_2) = \mathrm{Op}_h(\tilde{a})\big(\mathrm{Op}_h(\langle \xi \rangle^{-1})\underline{v}_1, \mathrm{Op}_h(\langle \xi \rangle^{-1})\underline{v}_2\big)$$

with some $\tilde{a}$ in $S_{\kappa,0}(M_0^\nu, 2)$. We use next (D.45) (with $\gamma = 0$) for $j = 1, 2$ in order to reduce ourselves to expressions of the form (D.48) with $\ell = 2$. Applying (D.7), we get the conclusion.

To obtain (D.57), we may assume that a is supported for $|\xi_1| \leq 2(1 + |\xi_2|)$ for instance. Let $\beta > 0$, $\chi \in C_0^\infty(\mathbb{R})$, equal to one close to zero and decompose

$$a(y, x, \xi_1, \xi_2) = a(y, x, \xi_1, \xi_2)\chi(h^{-\beta}\xi_1) + a(y, x, \xi_1, \xi_2)(1 - \chi)(h^{-\beta}\xi_1).$$

If we apply (D.7) to the second symbol, we obtain an estimate to the corresponding contribution to (D.57) by

$$C\|\mathrm{Op}_h((1 - \chi)(h^\beta \xi))\underline{v}_1\|_{W_h^{\rho_0,\infty}} \|\underline{v}_2\|_{L^2}.$$

By semiclassical Sobolev injection, this is bounded from above by

$$C h^{-\frac{1}{2} + \beta(s - \rho_0 - 1)} \|\underline{v}_1\|_{H_h^s} \|\underline{v}_2\|_{L^2},$$

so by the right-hand side of (D.57) if $\beta(s - (\rho_0 + 1)) \geq 2 - \sigma$.

Consider next $\mathrm{Op}_h(a_1)(\underline{v}_1, \underline{v}_2)$ with $a_1 = a\chi(h^{-\beta}\xi_1)$, so that a_1 is in the class $S'_{\kappa,\beta}(M_0^\nu \prod_{j=1}^2 \langle \xi_j \rangle^{-1}, 2)$. Since $\beta > 0$, we may rewrite as in (D.43), $\mathrm{Op}_h(a_1)(\underline{v}_1, \underline{v}_2)$ as

$$\mathrm{Op}_h(\tilde{a}_1)\big[\mathrm{Op}_h\big(\langle \xi \rangle^{-1} \langle h^\beta \xi \rangle^{-\gamma}\big)\underline{v}_1, \mathrm{Op}_h\big(\langle \xi \rangle^{-1}\big)\underline{v}_2\big]$$

with $\tilde{a}_1$ in $S'^2_{\kappa,\beta}(M_0^\nu, 2)$, hence under form (D.48) with $\ell = 2$, V_1 (resp. V_2) being given by (D.47) with $b_j = O(\langle \xi \rangle^{-\gamma})$ (resp. $O(1)$). Applying (D.7), we get, in view of the definition of the V_j a bound in

$$Ch^2\Big(\|\mathrm{Op}_h(b_1(h^\beta\xi))\mathcal{L}_\pm\underline{v}_1\|_{W_h^{\rho_0,\infty}} + \|\mathrm{Op}_h(b_1(h^\beta\xi))\underline{v}_1\|_{W_h^{\rho_0,\infty}}\Big) \\ \times\big(\|\mathcal{L}_\pm\underline{v}_2\|_{L^2} + \|\underline{v}_2\|_{L^2}\big).$$

Using the semiclassical Sobolev injection, the first factor is bounded from above by

$$Ch^{-\frac{1}{2}-\beta(\rho_0+1)}\big(\|\mathcal{L}_\pm\underline{v}_1\|_{L^2} + \|\underline{v}_1\|_{L^2}\big).$$

We set $\sigma = \beta(\rho_0+1)$ and get the conclusion under the condition $s\sigma \geq 2(\rho_0+1)$. ■

We prove now an L^∞ estimate that is a counterpart of (D.40).

Proposition D.2.5. *Let $\kappa \in \mathbb{N}$, $\nu \geq 0$, $n \in \mathbb{N}$. There is $\rho_0 \in \mathbb{N}$ such that, for any $\rho \geq \rho_0$, any a in $S'^{2n+2}_{\kappa,0}(M_0^\nu, n)$, any $\ell \leq n$, one has for any odd functions $\underline{v}_1, \dots, \underline{v}_n$, any $r \geq 0$, the estimate*

$$\begin{aligned}\|\mathrm{Op}_h(a)(\underline{v}_1,\dots,\underline{v}_n)\|_{W_h^{\rho,\infty}} &\leq Ch^r\prod_{j=1}^n\big(\|\underline{v}_j\|_{W_h^{\rho_0,\infty}} + \|\underline{v}_j\|_{H_h^s}\big)\\ &\quad + Ch^{\frac{n}{2}+\frac{\ell}{2}-\sigma}\prod_{j=1}^{\ell}\big(\|\underline{v}_j\|_{W_h^{\rho,\infty}} + \|\mathcal{L}_\pm\underline{v}_j\|_{W_h^{\rho,\infty}}\big)\\ &\quad\times\prod_{j=\ell+1}^{n}\big(\|\underline{v}_j\|_{L^2} + \|\mathcal{L}_\pm\underline{v}_j\|_{L^2}\big)\end{aligned} \tag{D.58}$$

for any $\sigma > 0$, any s such that

$$s \geq s_0(\rho,\kappa)\Big(1 + \frac{r+1}{\sigma}\Big) \tag{D.59}$$

(where $s_0(\rho,\kappa)$ is some explicit function of (ρ,κ)).

Proof. Set $|\xi|^2 = \xi_1^2 + \dots + \xi_n^2$. Take $\chi \in C_0^\infty(\mathbb{R})$ equal to one close to zero and let $\beta > 0$ to be chosen. Decompose $a = a_1 + a_2$ with

$$\begin{aligned}a_1(y,x,\xi_1,\dots,\xi_n) &= a(y,x,\xi_1,\dots,\xi_n)\chi(h^{2\beta}|\xi|^2),\\ a_2(y,x,\xi_1,\dots,\xi_n) &= a(y,x,\xi_1,\dots,\xi_n)(1-\chi)(h^{2\beta}|\xi|^2).\end{aligned} \tag{D.60}$$

Let us assume in addition that a_2 is supported for instance for

$$|\xi_1| + \dots + |\xi_{n-1}| \leq K(1 + |\xi_n|).$$

By semiclassical Sobolev injection, we have

$$\|\mathrm{Op}_h(a_2)(\underline{v}_1,\dots,\underline{v}_n)\|_{W_h^{\rho,\infty}} \leq Ch^{-\frac{1}{2}}\|\mathrm{Op}_h(a_2)(\underline{v}_1,\dots,\underline{v}_n)\|_{H_h^{\rho+1}}. \tag{D.61}$$

If we use estimates (B.12) and (B.13), we see that the action of an hD_x-derivative on $\mathrm{Op}_h(a_2)(\underline{v}_1, \dots, \underline{v}_n)$ makes lose at most one power of $\langle \xi_n \rangle^{\max(1,\kappa)}$ (since ξ_n is the largest frequency). Consequently, (D.61) is bounded from above by

$$Ch^{-\frac{1}{2}} \|\mathrm{Op}_h(\tilde{a}_2)(\underline{v}_1, \dots, \underline{v}_{n-1}, \langle hD_x \rangle^{(\rho+1)\max(1,\kappa)} \underline{v}_n)\|_{L^2}$$

for a symbol $\tilde{a}_2$ that has the same support properties as a_2. We apply next (D.7) with $j = n$, and remember that, by definition of a_2, $\tilde{a}_2$ is supported for $|\xi_{j_0}| \geq ch^{-\beta}$ for some j_0. We thus get a bound either by

$$Ch^{-\frac{1}{2}} \prod_{j=1}^{n-1} \|\underline{v}_j\|_{W_h^{\rho_0,\infty}} \|\mathrm{Op}_h\big(\langle \xi \rangle^{(\rho+1)\max(1,\kappa)} \chi_1(h^\beta \xi)\big)\underline{v}_n\|_{L^2} \tag{D.62}$$

if $j_0 = n$, or

$$\begin{aligned} Ch^{-\frac{1}{2}} & \prod_{1 \leq j \leq n-1,\, j \neq j_0} \|\underline{v}_j\|_{W_h^{\rho_0,\infty}} \|\mathrm{Op}_h(\chi_1(h^\beta \xi))\underline{v}_{j_0}\|_{W_h^{\rho_0,\infty}} \\ & \times \|\mathrm{Op}_h\Big(\langle \xi \rangle^{(\rho+1)\max(1,\kappa)} \chi_1(h^\beta \xi)\Big)\underline{v}_n\|_{L^2} \end{aligned} \tag{D.63}$$

if $j_0 < n$, where $\chi_1 \in C^\infty(\mathbb{R})$ is equal to one close to infinity and to zero close to zero. Writing (using semiclassical embedding)

$$\begin{aligned} \|\mathrm{Op}_h\big(\langle \xi \rangle^m \chi_1(h^\beta \xi)\big)\underline{v}_n\|_{L^2} &\leq Ch^{\beta(s-m)} \|\underline{v}_n\|_{H_h^s}, \\ \|\mathrm{Op}_h\big(\chi_1(h^\beta \xi)\big)\underline{v}_{j_0}\|_{W_h^{\rho_0,\infty}} &\leq Ch^{-\frac{1}{2}+\beta(s-(\rho_0+1))} \|\underline{v}_{j_0}\|_{H_h^s}, \end{aligned}$$

we obtain for (D.62) and (D.63) an estimate in

$$Ch^r \prod_{j=1}^{n} \big(\|\underline{v}_j\|_{W_h^{\rho_0,\infty}} + \|\underline{v}_j\|_{H_h^s}\big) \tag{D.64}$$

if

$$\begin{aligned} \beta\big(s - (\rho+1)\max(1,\kappa)\big) &\geq r + \frac{1}{2}, \\ \beta(s - (\rho_0 + 1)) &\geq r + 1. \end{aligned} \tag{D.65}$$

Consider next a_1, which satisfies

$$h^{3\beta n} a_1 \in S'^{2n+2}_{\kappa,\beta}\bigg(M_0^\nu \prod_{j=1}^{n} \langle \xi_j \rangle^{-3}, n\bigg).$$

We may write $\mathrm{Op}_h(a_1)(\underline{v}_1, \dots, \underline{v}_n)$ under form (D.48) with $\ell = n$ and a new symbol $\tilde{a}_1$, such that

$$h^{3\beta n} \tilde{a}_1 \in S'^{2}_{\kappa,\beta}\bigg(M_0^\nu \prod_{j=1}^{n} \langle \xi_j \rangle^{-2}, n\bigg)$$

(for a new ν). We apply (D.27) that implies

$$\begin{aligned}\|\mathrm{Op}_h(\tilde{a}_1)(\underline{v}_1,\dots,\underline{v}_n)\|_{W_h^{\rho,\infty}}\\ \leq C h^{n(1-3\beta)}\int_{-1}^{1}\cdots\int_{-1}^{1}\prod_{j=1}^{n}\|V_j(\mu_j\cdot)\|_{W_h^{\rho,\infty}}\,d\mu_1\cdots d\mu_n,\end{aligned} \tag{D.66}$$

where V_j is given by (D.47) with $\gamma>\rho+1$. For $j=\ell+1,\dots,n$, we use semiclassical Sobolev injection to estimate

$$\int_{-1}^{1}\|V_j(\mu_j\cdot)\|_{W_h^{\rho,\infty}}\,d\mu_j\leq C h^{-\frac{1}{2}-\beta(\rho+1)}\big(\|\mathcal{L}_{\pm}\underline{v}_j\|_{L^2}+\|\underline{v}_j\|_{L^2}\big)$$

whence finally a bound of (D.66) in

$$\begin{aligned}C h^{n(1-3\beta)-\frac{n-\ell}{2}-\beta(\rho+1)(n-\ell)}\prod_{j=1}^{\ell}\big(\|\underline{v}_j\|_{W_h^{\rho,\infty}}+\|\mathcal{L}_{\pm}\underline{v}_j\|_{W_h^{\rho,\infty}}\big)\\ \times\prod_{j=\ell+1}^{n}\big(\|\underline{v}_j\|_{L^2}+\|\mathcal{L}_{\pm}\underline{v}_j\|_{L^2}\big).\end{aligned}$$

Combining this with (D.64) and taking $\beta=\frac{\sigma}{3n+(n-\ell)(\rho+1)}$, we get the conclusion if s satisfies the inequality in the statement. ∎

The same type of reasoning as above may be used to remove the assumption $\beta>0$ in (ii) of Proposition D.2.1.

Proposition D.2.6. *Let a be a symbol in $S_{\kappa,0}(M_0^{\nu},n)$ independent of x, satisfying the assumptions of Proposition D.2.1. Then, for any $\beta>0$ with $\kappa\beta\leq 1$, one may decompose $a=a_1+a_2$ with a_1 in $S'^{2\ell+2}_{\kappa,\beta}(M_0^{\nu},n)$ and a_2 is such that*

$$\|\mathrm{Op}_h(a_2)(\underline{v}_1,\dots,\underline{v}_n)\|_{H_h^s}\leq C h^r\sum_{j=1}^{n-1}\Big(\prod_{\ell\neq i}\|\underline{v}_j\|_{W_h^{\rho_0,\infty}}\Big)\|\underline{v}_j\|_{H_h^s}\|\underline{v}_n\|_{H_h^s} \tag{D.67}$$

as soon as $\beta(s-\rho_0-1)\geq r+\frac{1}{2}$.

As a consequence, one has the estimate, for $1\leq\ell\leq n-1$, $0\leq\ell'\leq\ell$,

$$\begin{aligned}&\|\mathrm{Op}_h(a)(\underline{v}_1,\dots,\underline{v}_n)\|_{H_h^s}\\ &\leq C h^{\ell-\frac{\ell'}{2}-\sigma}\prod_{j=1}^{\ell'}\big(\|\mathcal{L}_{\varepsilon_j}\underline{v}_j\|_{L^2}+\|\underline{v}_j\|_{H_h^s}\big)\\ &\times\prod_{j=\ell'+1}^{\ell}\big(\|\mathcal{L}_{\varepsilon_j}\underline{v}_j\|_{W_h^{\rho_0,\infty}}+\|\underline{v}_j\|_{W_h^{\rho_0,\infty}}+\|\underline{v}_j\|_{H_h^s}\big)\\ &\times\prod_{j=\ell+1}^{n-1}\big(\|\underline{v}_j\|_{W_h^{\rho_0,\infty}}+\|\underline{v}_j\|_{H_h^s}\big)\|\underline{v}_n\|_{H_h^s},\end{aligned} \tag{D.68}$$

where $\sigma>0$ is any small number and s is such that $(s-\rho_0-1)\sigma$ is large enough.

Proof. We shall decompose $a = a_1 + a_2$ as at the beginning of the proof of Corollary D.1.3. By (D.22)–(D.23), estimate (D.67) holds if $(s - \rho_0 - 1)\beta \geq r + \frac{1}{2}$. On the other hand, applying (D.41) to $\mathrm{Op}_h(a_1)$, and expressing $\sigma(\beta)$ from β, one gets a bound of $\|\mathrm{Op}_h(a_1)(\underline{v}_1, \dots, \underline{v}_n)\|_{H_h^s}$ by the right-hand side of (D.68). Since, for r large enough, the right-hand side of (D.67) may be estimated by (D.68) (using semiclassical Sobolev injection to bound some $W_h^{\rho_0,\infty}$ norm by $h^{-\frac{1}{2}}$ times an H_h^s one), we get the conclusion. ∎

Let us translate the inequalities proved in this section in the non-semiclassical framework, using (B.15)–(B.17).

Corollary D.2.7. *Under the assumptions of Proposition D.2.1, one has the following estimates:*

(i) *For any $s \geq 0$, any odd test functions $v_1, \dots, v_n$, and any choice of signs $\varepsilon_j \in \{-, +\}$, $j = 1, \dots, \ell$,*

$$\|\mathrm{Op}^t(a)(v_1, \dots, v_n)\|_{H^s} \leq C t^{-\ell} \prod_{j=1}^{\ell} \big(\|L_{\varepsilon_j} v_j\|_{W^{\rho_0,\infty}} + \|v_j\|_{W^{\rho_0,\infty}}\big) \times \prod_{j=\ell+1}^{n-1} \|v_j\|_{W^{\rho_0,\infty}} \|v_n\|_{H^s} \tag{D.69}$$

with $L_\pm$ defined in (D.34).

(ii) *If moreover $\beta > 0$, one has for any $0 \leq \ell' \leq \ell$,*

$$\begin{aligned} \|\mathrm{Op}^t(a)(v_1, \dots, v_n)\|_{H^s} &\leq C t^{-\ell+\sigma(\beta)} \prod_{j=1}^{\ell'} \big(\|L_{\varepsilon_j} v_j\|_{L^2} + \|v_j\|_{L^2}\big) \\ &\quad \times \prod_{j=\ell'+1}^{\ell} \big(\|L_{\varepsilon_j} v_j\|_{W^{\rho_0,\infty}} + \|v_j\|_{W^{\rho_0,\infty}}\big) \\ &\quad \times \prod_{j=\ell+1}^{n-1} \|v_j\|_{W^{\rho_0,\infty}} \|v_n\|_{H^s} \end{aligned} \tag{D.70}$$

with $\sigma(\beta) > 0$ going to zero when β goes to zero.

This is just a restatement of Proposition D.2.1. Proposition D.2.2 gives:

Corollary D.2.8. *Under the assumptions and with the notation of Proposition D.2.2, one has the following estimates for any j, $1 \leq j \leq n$:*

$$\begin{aligned} \|\mathrm{Op}^t(a)(v_1, \dots, v_n)\|_{L^2} \\ \leq C t^{-1} \prod_{\ell \neq j, 1 \leq \ell \leq n} \|v_\ell\|_{W^{\rho_0,\infty}} \big(\|L_\pm v_j\|_{L^2} + \|v_j\|_{L^2}\big), \end{aligned} \tag{D.71}$$

and if $n \geq 2$, for any $j \neq j'$, $1 \leq j, j' \leq n$,

$$\|\mathrm{Op}^t(a)(v_1, \dots, v_n)\|_{L^2} \leq Ct^{-1} \prod_{\ell \neq j, j', 1 \leq \ell \leq n} \|v_j\|_{W^{\rho_0,\infty}} \times \big(\|L_\pm v_{j'}\|_{W^{\rho_0,\infty}} + \|v_{j'}\|_{W^{\rho_0,\infty}}\big)\|v_j\|_{L^2}. \tag{D.72}$$

Moreover, these estimates hold as soon as $a \in S'^4_{\kappa,\beta}(M_0^\nu \prod_{j=1}^n \langle \xi_j \rangle^{-1}, n)$.

In the same way, we have the bounds of Corollary D.2.3:

Corollary D.2.9. *With the notation of Corollary D.2.3, one has for any j,*

$$\|\mathrm{Op}^t(a)(Z, v_1, \dots, v_n)\|_{L^2} \leq Ct^{-1} \prod_{1 \leq \ell \leq n, \ell \neq j} \|v_j\|_{W^{\rho_0,\infty}} \big(\|L_\pm v_j\|_{L^2} + \|v_j\|_{L^2}\big) \tag{D.73}$$

and if $n \geq 2$, $j \neq j'$ are in $\{1, \dots, n\}$,

$$\|\mathrm{Op}^t(a)(Z, v_1, \dots, v_n)\|_{L^2} \leq Ct^{-1} \prod_{\ell \neq j, j', 1 \leq \ell \leq n} \|v_j\|_{W^{\rho_0,\infty}} \big(\|L_\pm v_{j'}\|_{W^{\rho_0,\infty}} + \|v_{j'}\|_{W^{\rho_0,\infty}}\big)\|v_j\|_{L^2}. \tag{D.74}$$

Next we restate Proposition D.2.4.

Corollary D.2.10. *With the notation and under the assumptions of Proposition D.2.4, one has for any odd functions v_1, v_2,*

$$\|\mathrm{Op}^t(a)(v_1, v_2)\|_{L^2} \leq Ct^{-2}\big(\|L_\pm v_1\|_{W^{\rho_0,\infty}} + \|v_1\|_{W^{\rho_0,\infty}}\big) \times \big(\|L_\pm v_2\|_{L^2} + \|v_2\|_{L^2}\big), \tag{D.75}$$

$$\|\mathrm{Op}^t(a)(v_1, v_2)\|_{L^2} \leq Ct^{-2+\sigma} \prod_{j=1}^2 \big(\|L_\pm v_j\|_{L^2} + \|v_j\|_{H^s}\big) \tag{D.76}$$

if $s, \sigma > 0$ are such that $s\sigma \geq 2(\rho_0 + 1)$.

Finally, we translate the estimates of Propositions D.2.5 and D.2.6.

Corollary D.2.11. *With the notation and under the assumptions of Proposition D.2.5, one has, for any odd functions $v_1, \dots, v_n$, any $0 \leq \ell \leq n$, any $r \geq 0$,*

$$\begin{aligned}\|\mathrm{Op}^t(a)(v_1, \dots, v_n)\|_{W^{\rho,\infty}} &\leq Ct^{-r} \prod_{j=1}^n \big(\|v_j\|_{W^{\rho_0,\infty}} + t^{-\frac{1}{2}}\|v_j\|_{H^s}\big) \\ &\quad + Ct^{-n+\sigma} \prod_{j=1}^{\ell} \big(\|v_j\|_{W^{\rho,\infty}} + \|L_\pm v_j\|_{W^{\rho,\infty}}\big) \\ &\quad \times \prod_{j=\ell+1}^n \big(\|v_j\|_{L^2} + \|L_\pm v_j\|_{L^2}\big)\end{aligned} \tag{D.77}$$

if $s \geq s_0(\rho, \kappa)(1 + \frac{r+1}{\sigma})$ for some function $s_0(\rho, \kappa)$.

Corollary D.2.12. *With the notation and under the assumption of Proposition D.2.6, one has for any odd functions* $v_1, \dots, v_n$*, any* ℓ*,* $1 \le \ell \le n-1$*, any* $0 \le \ell' \le \ell$*,*

$$\begin{aligned} &\|\mathrm{Op}^t(a)(v_1,\dots,v_n)\|_{H^s} \\ &\quad \le C t^{-\ell+\sigma} \prod_{j=1}^{\ell'} \big(\|L_{\varepsilon_j} v_j\|_{L^2} + \|v_j\|_{H^s}\big) \\ &\qquad \times \prod_{j=\ell'+1}^{\ell} \big(\|L_{\varepsilon_j} v_j\|_{W^{\rho_0,\infty}} + \|v_j\|_{W^{\rho_0,\infty}} + t^{-\frac{1}{2}}\|v_j\|_{H^s}\big) \\ &\qquad \times \prod_{j=\ell+1}^{n-1} \big(\|v_j\|_{W^{\rho_0,\infty}} + t^{-\frac{1}{2}}\|v_j\|_{H^s}\big)\|v_n\|_{H^s} \end{aligned} \tag{D.78}$$

for any small $\sigma > 0$*, as soon as* $(s-\rho_0-1)\sigma$ *is large enough. The same estimate holds true if we apply on the right-hand side any permutation on the indices* $\{1,\dots,n-1\}$*.*

D.3 Weyl calculus

In Chapter 8, we use a different quantization of symbols $a(x,\xi)$ on $\mathbb{R}\times\mathbb{R}$. We give its definition and properties here. Our classes of symbols will be variants of those introduced in Definition B.1.2.

Definition D.3.1. Let $\delta' \in [0, \frac{1}{2}]$, $\beta \ge 0$, and let $(x,\xi) \mapsto M(x,\xi)$ be a weight function on $\mathbb{R}\times\mathbb{R}$. One denotes by $S^{\mathrm{W}}_{\delta',\beta}(M)$ the space of smooth functions

$$(h,x,\xi) \mapsto a(x,\xi,h)$$

defined on $]0,1] \times \mathbb{R}\times\mathbb{R}$ satisfying estimates

$$|\partial_x^{\alpha_1}\partial_\xi^{\alpha_2}(h\partial_h)^k a(x,\xi,h)| \le C M(x,\xi) h^{-\delta'(\alpha_1+\alpha_2)}(1+\beta h^\beta|\xi|)^{-N} \tag{D.79}$$

for any α_1, α_2, k, N in $\mathbb{N}$.

Remark. Notice that for $\beta > 0$, we assume a rapid decay of the symbol in $\langle h^\beta \xi\rangle^{-N}$. This is not the same condition as in (B.12) and (B.13), where the rapid decay was in $\langle h^\beta M_0(\xi)\rangle^{-N}$, which, when there is only one ξ variable, is just $O(1)$. Notice also that instead of having a loss in $M_0(\xi)^\kappa$ for each derivative acting on the symbol, we allow a $h^{-\delta'}$ loss. Finally, at the difference of (B.11), we consider symbols that do not depend on the y variable.

For a in $S^{\mathrm{W}}_{\delta',\beta}(M)$, we define the Weyl quantization by

$$\mathrm{Op}_h^{\mathrm{W}}(a)\underline{v} = \frac{1}{2\pi h}\iint e^{\frac{i}{h}(x-y)\xi} a\Big(\frac{x+y}{2},\xi,h\Big)\underline{v}(y)\,dy\,d\xi \tag{D.80}$$

for any test function $\underline{v}$. We recall some results of [82] that we use in Chapter 8.

Proposition D.3.2. *Let ρ be in $\mathbb{R}_+$, $\Gamma(x,\xi,h)$ a function satisfying*

$$|\partial_x^{\alpha_1}\partial_\xi^{\alpha_2}(h\partial_h)^k\Gamma(x,\xi,h)| \leq Ch^{-\frac{\alpha_1+\alpha_2}{2}}\Big\langle\frac{x\pm p'(\xi)}{\sqrt{h}}\Big\rangle^{-1} \tag{D.81}$$

for any α_1,α_2,k in $\mathbb{N}$. Then, for any $\sigma>0$, any $r\geq 0$, any s such that $s\sigma$ is large enough, we have

$$\|\mathrm{Op}_h^{\mathrm{W}}(\Gamma)\mathrm{Op}_h^{\mathrm{W}}(\langle\xi\rangle^\rho)\underline{v}\|_{L^\infty} \leq C\big(h^{-\frac{1}{4}-\sigma}\|\underline{v}\|_{L^2} + h^r\|\underline{v}\|_{H_h^s}\big). \tag{D.82}$$

Proof. Fix $\beta>0$ small. Decompose $\Gamma=\Gamma\chi(h^\beta\xi)+\Gamma(1-\chi)(h^\beta)$ for χ in $C_0^\infty(\mathbb{R})$ equal to one close to zero. By [82, Lemma 3.9], we may write

$$\mathrm{Op}_h^{\mathrm{W}}(\Gamma\chi(h^\beta\xi)) = \mathrm{Op}_h^{\mathrm{W}}(r_1)\mathrm{Op}_h^{\mathrm{W}}(\tilde{\chi}(h^\beta\xi)) + h^N\mathrm{Op}_h^{\mathrm{W}}(r_2) \tag{D.83}$$

and

$$\mathrm{Op}_h^{\mathrm{W}}\big(\Gamma(1-\chi)(h^\beta\xi)\big) = \mathrm{Op}_h^{\mathrm{W}}(r_3)\mathrm{Op}_h^{\mathrm{W}}\big((1-\tilde{\chi}_1)(h^\beta\xi)\big) + h^N\mathrm{Op}_h^{\mathrm{W}}(r_4), \tag{D.84}$$

where r_j are in $S^{\mathrm{W}}_{1/2,\beta}(1)$, N is arbitrary, $\tilde{\chi},\tilde{\chi}_1$ are in $C_0^\infty(\mathbb{R})$ equal to one close to zero. By semiclassical Sobolev injection and Proposition D.3.3 below, the last term in (D.83)–(D.84) acting on $\mathrm{Op}_h^{\mathrm{W}}(\langle\xi\rangle^\rho)\underline{v}$ has L^∞ norm estimated by the last term in (D.82). Moreover, r_1 satisfies estimates of the form (D.81), so that we may apply [82, Proposition 3.11] to estimate

$$\|\mathrm{Op}_h^{\mathrm{W}}(r_1)\mathrm{Op}_h^{\mathrm{W}}(\tilde{\chi}(h^\beta\xi)\langle\xi\rangle^\rho)\underline{v}\|_{L^\infty}$$

by the first term on the right-hand side of (D.82) with σ linear in β. Finally, by semiclassical Sobolev injection and Proposition D.3.3, the L^∞ norm of the first term on the right-hand side of (D.84) is bounded from above by

$$Ch^{-\frac{1}{2}}\|\mathrm{Op}_h^{\mathrm{W}}\big(\langle\xi\rangle^\rho(1-\tilde{\chi}_1)(h^\beta\xi)\big)\|_{H_h^1}$$

which is estimated by $h^r\|\underline{v}\|_{H_h^s}$ is $s\beta$ is large enough. This concludes the proof. ∎

One has also Sobolev estimates (see [24] or [82, Proposition 3.10]):

Proposition D.3.3. *Let $\beta\geq 0$, $\delta'\in[0,\frac{1}{2}]$, $r\in\mathbb{R}$, a in $S^{\mathrm{W}}_{\delta',\beta}(\langle\xi\rangle^r)$. Then $\mathrm{Op}_h^{\mathrm{W}}(a)$ is bounded from H_h^s to H_h^{s-r} for any s in $\mathbb{R}$, with operator norm bounded uniformly in h.*

We state next [82, Proposition 4.4].

Proposition D.3.4. *Let γ be in $C_0^\infty(\mathbb{R})$, equal to one close to zero. Let $\mathcal{L}_+$ be the operator (D.18) that may be written as well*

$$\mathcal{L}_+ = \frac{1}{h}\mathrm{Op}_h^{\mathrm{W}}(x+p'(\xi)).$$

For ρ in $\mathbb{N}$, $\underline{v}$ a function, define

$$\underline{v}^{\rho}_{\Lambda^c} = \mathrm{Op}_h^{\mathrm{W}}\Big((1-\gamma)\Big(\frac{x+p'(\xi)}{\sqrt{h}}\Big)\Big)\mathrm{Op}_h^{\mathrm{W}}(\langle\xi\rangle^{\rho})\underline{v}. \tag{D.85}$$

Then for any $\sigma > 0$, any s such that $s\sigma$ is large enough, one has estimates

$$\|\underline{v}^{\rho}_{\Lambda^c}\|_{L^2} \le Ch^{\frac{1}{2}-\sigma}\big(\|\mathcal{L}_+\underline{v}\|_{L^2} + \|\underline{v}\|_{H^s_h}\big), \tag{D.86}$$

$$\|\underline{v}^{\rho}_{\Lambda^c}\|_{L^\infty} \le Ch^{\frac{1}{4}-\sigma}\big(\|\mathcal{L}_+\underline{v}\|_{L^2} + \|\underline{v}\|_{H^s_h}\big). \tag{D.87}$$

Let us prove next an L^∞ estimate for $\mathrm{Op}_h^{\mathrm{W}}(\gamma(\frac{x+p'(\xi)}{\sqrt{h}}))$.

Proposition D.3.5. *Let γ be in $C_0^\infty(\mathbb{R})$, with small enough support. Then for any $\sigma > 0$, $N > 0$, we have as soon as $s\sigma$ is large enough relatively to N,*

$$\Big\|\mathrm{Op}_h^{\mathrm{W}}\Big(\gamma\Big(\frac{x+p'(\xi)}{\sqrt{h}}\Big)\Big)\underline{v}\Big\|_{L^\infty} \le Ch^{-\sigma}\big(\|\underline{v}\|_{L^\infty} + h^N\|\underline{v}\|_{H^s_h}\big). \tag{D.88}$$

Proof. Let $\beta > 0$, χ in $C_0^\infty(\mathbb{R})$ equal to one close to zero. Decompose

$$\underline{v} = \mathrm{Op}_h^{\mathrm{W}}\big(\chi(h^\beta\xi)\big)\underline{v} + \mathrm{Op}_h^{\mathrm{W}}\big((1-\chi)(h^\beta\xi)\big)\underline{v}.$$

By semiclassical Sobolev injection, Proposition D.3.3 and the fact that

$$\|\mathrm{Op}_h^{\mathrm{W}}\big((1-\chi)(h^\beta\xi)\big)\|_{\mathcal{L}(H^s_h,H^{s'}_h)} = O(h^{\beta(s-s')})$$

if $s > s'$, we have

$$\begin{aligned}
&\Big\|\mathrm{Op}_h^{\mathrm{W}}\Big(\gamma\Big(\frac{x+p'(\xi)}{\sqrt{h}}\Big)\Big)\mathrm{Op}_h^{\mathrm{W}}\big((1-\chi)(h^\beta\xi)\big)\underline{v}\Big\|_{L^\infty}\\
&\le Ch^{-\frac{1}{2}}\Big\|\mathrm{Op}_h^{\mathrm{W}}\Big(\gamma\Big(\frac{x+p'(\xi)}{\sqrt{h}}\Big)\Big)\mathrm{Op}_h^{\mathrm{W}}\big((1-\chi)(h^\beta\xi)\big)\underline{v}\Big\|_{H^1_h}\\
&\le Ch^{-\frac{1}{2}+\beta(s-1)}\|\underline{v}\|_{H^s_h}
\end{aligned}$$

which is estimated by the right-hand side of (D.88) if $s\beta$ is large enough. On the other hand, by [82, Lemma 3.9], we may write for any N,

$$\mathrm{Op}_h^{\mathrm{W}}\Big(\gamma\Big(\frac{x+p'(\xi)}{\sqrt{h}}\Big)\Big)\mathrm{Op}_h^{\mathrm{W}}\big(\chi(h^\beta\xi)\big) = \mathrm{Op}_h^{\mathrm{W}}\big(\Gamma(x,\xi,h)\big) + h^N\mathrm{Op}_h^{\mathrm{W}}(r)$$

for some r in $S^{\mathrm{W}}_{1/2,\beta}(1)$ and a symbol Γ in $S^{\mathrm{W}}_{1/2,\beta}(1)$ supported inside

$$|\xi| \le h^{-\beta} \quad \text{and} \quad |x+p'(\xi)| \le c\sqrt{h}$$

for some small c. According to [20, Lemma 1.2.6], we know that setting

$$\varphi(x) = \sqrt{1-x^2}$$

for $|x| < 1$, if $|x + p'(\xi)| < c\langle\xi\rangle^{-2}$ for some small enough c, then

$$|\xi - d\varphi(x)| \le C\langle\xi\rangle^3 |x + p'(\xi)|.$$

It follows that

$$\Gamma(x,\xi,h) = \Gamma(x,\xi,h)\mathbb{1}_{|\xi - d\varphi(x)| < ch^{\frac{1}{2}-3\beta}}.$$

The kernel of $\mathrm{Op}_h^{\mathrm{W}}(\Gamma)$ is

$$\frac{1}{2\pi h}\int e^{\frac{i}{h}(x-y)\xi}\Gamma\Big(\frac{x+y}{2},\xi,h\Big)\,d\xi \tag{D.89}$$

that may be written

$$\begin{aligned}\frac{1}{2\pi\sqrt{h}}&e^{\frac{i}{h}(x-y)d\varphi\left(\frac{x+y}{2}\right)}\\ &\times\int e^{i(x-y)\frac{\zeta}{\sqrt{h}}}\Gamma\Big(\frac{x+y}{2}, d\varphi\Big(\frac{x+y}{2}\Big)+\sqrt{h}\zeta, h\Big)\,d\zeta.\end{aligned} \tag{D.90}$$

The integral is of the form

$$\int e^{i(x-y)\frac{\zeta}{\sqrt{h}}}A(x,y,\zeta)\,d\zeta,$$

with A supported for $|\zeta| \le Ch^{-3\beta}$ and satisfying $\partial_\zeta^\alpha A = O(1)$. It follows that (D.89) is

$$O\Big(h^{-\frac{1}{2}-3\beta}\Big\langle\frac{x-y}{\sqrt{h}}\Big\rangle^{-2}\Big),$$

which implies that operator (D.89) has $\mathcal{L}(L^\infty)$ norm that is $O(h^{-3\beta})$.

On the other hand, $\|h^N \mathrm{Op}_h^{\mathrm{W}}(r)\underline{v}\|_{L^\infty}$ is bounded by the last term on the right-hand side of (D.88) using again semiclassical Sobolev injection. ∎

We shall use also [82, Proposition 4.11] that we reproduce below.

Proposition D.3.6. *Define*

$$\underline{v}_\Lambda^\rho = \mathrm{Op}_h^{\mathrm{W}}\Big(\gamma\Big(\frac{x+p'(\xi)}{\sqrt{h}}\Big)\Big)\mathrm{Op}_h^{\mathrm{W}}(\langle\xi\rangle^\rho)\underline{v}, \tag{D.91}$$

where $\gamma \in C_0^\infty(\mathbb{R})$ has small enough support. There is $(\theta_h)_{h\in]0,1]}$ a family of smooth functions, real valued, supported in an interval $[-1+ch^{2\beta}, 1-ch^{2\beta}]$ for some small constant $c > 0$, with $\partial_h^\alpha\theta_h = O(h^{-2\beta\alpha})$ for some small $\beta > 0$, such that, still denoting $\varphi(x) = \sqrt{1-x^2}$ for $|x| < 1$,

$$\mathrm{Op}_h^{\mathrm{W}}(x\xi + p(\xi))\underline{v}_\Lambda^\rho = \varphi(x)\theta_h(x)\underline{v}_\Lambda^\rho + hR \tag{D.92}$$

where

$$\begin{aligned}\|R\|_{L^2} &\le Ch^{\frac{1}{2}-\sigma}\big(\|\mathcal{L}_+\underline{v}\|_{L^2} + \|\underline{v}\|_{H_h^s}\big),\\ \|R\|_{L^\infty} &\le Ch^{\frac{1}{4}-\sigma}\big(\|\mathcal{L}_+\underline{v}\|_{L^2} + \|\underline{v}\|_{H_h^s}\big)\end{aligned} \tag{D.93}$$

for any $\sigma > 0$, any s such that $s\sigma$ is large enough.

Finally, let us reproduce [82, Lemma 4.5].

Lemma D.3.7. *Let* γ *be as in Proposition* D.3.6. *One may write*

$$\begin{aligned}&\Big[D_t - \mathrm{Op}_h^{\mathrm{W}}\big(x\xi + p(\xi)\big), \mathrm{Op}_h^{\mathrm{W}}\Big(\gamma\Big(\frac{x+p'(\xi)}{\sqrt{h}}\Big)\Big)\Big]\\&\quad = h\mathrm{Op}_h^{\mathrm{W}}\Big(\gamma_{-1}\Big(\frac{x+p'(\xi)}{\sqrt{h}}\Big)\frac{x+p'(\xi)}{\sqrt{h}}\Big) + h^{\frac{3}{2}}\mathrm{Op}_h^{\mathrm{W}}(r),\end{aligned} \tag{D.94}$$

where $\gamma_{-1}(z)$ *satisfies for any* α, $|\partial_z^\alpha \gamma_{-1}(z)| \leq C_\alpha \langle z\rangle^{-1-\alpha}$ *and where* r *satisfies estimates* (D.81).

Appendix E

Wave operators for time dependent potentials

The goal of this chapter is to construct wave operators for some time dependent perturbations of a constant coefficients operator. We consider a reference operator P_0 independent of time, and a perturbation of P_0 of the form $P(t) = P_0 + \mathcal{V}(t)$, given in terms of a time depending potential $\mathcal{V}(t)$. Our goal is to construct a "wave operator" $B(t)$ such that

$$(D_t - P(t))B(t) = B(t)(D_t - P_0). \tag{E.1}$$

We did something similar in Appendix A in the autonomous case, when $\mathcal{V}(t)$ does not depend on time, and is given by a potential smooth and decaying in space. Here, we shall have to consider a potential $\mathcal{V}(t)$ that depends on time. As mentioned in the introduction of Chapter 6, a scalar model for the kind of operators $P(t)$ we want to consider is given by

$$D_t - p(D_x) - t_\varepsilon^{-\frac{1}{2}} \operatorname{Re}\big(c(x)\langle D_x\rangle^{-1} e^{it\frac{\sqrt{3}}{2}}\big), \tag{E.2}$$

where $p(\xi) = \sqrt{1+\xi^2}$ and c is in $\mathcal{S}(\mathbb{R})$. The potential perturbing the autonomous problem is given here in terms of

$$t_\varepsilon^{-\frac{1}{2}} c(x)\langle D_x\rangle^{-1} e^{\pm it\frac{\sqrt{3}}{2}}.$$

As a function of x, this is still a smooth rapidly decaying function, but we have now also t dependence. On the one hand, this time dependence might be considered as an advantage, since it makes the potential smaller and smaller as time growth. On the other side, it makes impossible to use stationary arguments in order to construct wave operators. Of course, there are well known results concerning scattering by time dependent potentials. We refer for instance to the book of Dereziński and Gérard [23], in particular Sections 3.3 and 3.4. Though, these results would not apply to our problem, as they demand better time decay of the potential and of its space derivatives as the one we have in (E.2). We thus have to construct $B(t)$ by hand, composing (E.1) at the left with Fourier transform, at the right with inverse Fourier transform and defining a wave operator through iterated integrals.

E.1 Statement of the result

In order to state the result, we have to introduce some notation.

Definition E.1.1. Let a, b be in $\mathbb{N}$, $m \geq 0$, $\iota \geq 0$. We denote by $\Sigma_{0,0}^{\iota,m}$ the space of functions $(t,\xi,\eta) \mapsto q(t,\xi,\eta)$ defined on $[1,+\infty[\times\mathbb{R}\times\mathbb{R}$, with values in $\mathbb{C}$, that are Lipschitz in time, smooth in (ξ,η), and satisfy for any N in $\mathbb{N}$, any $j = 0, 1$, any

$t \geq 1$, any $(\xi, \eta) \in \mathbb{R}^2$, any $(\alpha, \alpha') \in \mathbb{N}^2$,

$$|\partial_t^j \partial_\xi^\alpha \partial_\eta^{\alpha'} q(t, \eta, \xi)| \leq C_{\alpha\alpha' N} \varepsilon^\iota t^{-m-j} \langle |\xi| - |\eta| \rangle^{-N}. \tag{E.3}$$

We denote by $\Sigma_{a,b}^{\iota,m}$ the space of functions q of the form $q = (\frac{\xi}{\langle\xi\rangle})^a (\frac{\eta}{\langle\eta\rangle})^b q_1$ with q_1 in $\Sigma_{0,0}^{\iota,m}$.

Example. Let us give an example of functions in the preceding class. Let $q = q_{j,(k,\ell)}$, where $q_{j,(k,\ell)}$ is one of the functions defined in Lemma 6.1.1. Assume that these functions are defined and satisfy (6.18) or (6.19) for t in some interval $[1, T]$ with $4 \leq T \leq \varepsilon^{-4+c}$. Extend this function to $[1, +\infty[$ by

$$q(t, \xi, \eta) \mathbb{1}_{t<T} + q(2T - t, \xi, \eta) \mathbb{1}_{t>T} \chi_0\Big(\frac{t}{T}\Big), \tag{E.4}$$

where $\chi_0 \in C^\infty(\mathbb{R})$ is equal to one on $]-\infty, \frac{5}{4}]$ and to zero on $[\frac{7}{4}, +\infty[$. If we denote this extension still by q, we get a Lipschitz function of time on $[1, +\infty[$ that satisfies (6.18) or (6.19) for any $t \geq 1$. Notice that these inequalities imply estimates of the form (E.3) when we take T in (E.4) smaller than ε^{-4+c} for some $c > 0$, so that (E.4) is supported for $t \leq C\varepsilon^{-4+c}$. Actually, writing for any $m \in]0, \frac{1}{2}[$, $t_\varepsilon^{-1/2} \leq t^{-m} \varepsilon^{1-2m}$, it follows from (6.18) that q belongs to $\Sigma_{0,0}^{\iota,m}$ if $\iota = \min(1 - 2m, 3c\theta'/4) > 0$. In the same way, under condition (6.19), we obtain an element of $\Sigma_{0,0}^{\iota,m+1/2}$. The matrix Q_j of Lemma 6.1.1 has thus entries in $\Sigma_{1,1}^{\iota,m}$.

We consider in this section an operator $\mathcal{V}$ defined in the following way. Assume that we are given matrices Q_j with entries in $\Sigma_{0,0}^{\iota,m}$ for $m > 0$, $\iota > 0$ and $-2 \leq j \leq 2$. Let $\lambda_j = j\frac{\sqrt{3}}{2}$ and define

$$\mathcal{V}(t) = \sum_{j=-2}^{2} e^{i\lambda_j t} K_{Q_j}, \tag{E.5}$$

where, when q is in $\Sigma_{0,0}^{\iota,m}$, and f is a scalar-valued function, $K_q f$ is defined by

$$\widehat{K_q f}(\xi) = \int q(t, \xi, \eta) \hat{f}(\eta)\, d\eta, \tag{E.6}$$

and when Q_j is a 2×2 matrix, and f is $\mathbb{C}^2$-valued, $K_{Q_j} f$ is defined in the natural way. We shall assume also that operator $\mathcal{V}$ satisfies

$$\overline{\mathcal{V}(t)} N_0 = -N_0 \mathcal{V}(t) \tag{E.7}$$

with $N_0 = [\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}]$ (see (6.9)) and that $\mathcal{V}(t)$ preserves the space of odd functions. If

$$P_0 = \begin{bmatrix} p(D_x) & 0 \\ 0 & -p(D_x) \end{bmatrix},$$

we define

$$P(t) = P_0 + \mathcal{V}(t). \tag{E.8}$$

We want to construct a family of operators $B(t)$ so that, for any f in $L^2(\mathbb{R})$ such that $(D_t - P_0)f$ is in $L^2(\mathbb{R})$ for any t,

$$\big(D_t - P(t)\big)B(t)f = B(t)(D_t - P_0)f. \tag{E.9}$$

We shall prove:

Proposition E.1.2. *For any $t \geq 1$, let $\mathcal{V}(t)$ be a bounded operator on $L^2(\mathbb{R})$. Assume that $t \mapsto \mathcal{V}(t)$ is compactly supported and define for any $t \geq 1$, $n \in \mathbb{N}^*$,*

$$B_n(t) = (-i)^n \int \prod_{j=1}^{n} e^{-i\tau_j P_0}\mathcal{V}(t+\tau_j)e^{i\tau_j P_0}\mathbb{1}_{0<\tau_1<\cdots<\tau_n}\, d\tau_1 \cdots d\tau_n, \tag{E.10}$$

where, for non-commuting variables $A_1, \dots, A_n$, $\prod_{j=1}^n A_j$ denotes $A_1A_2\cdots A_n$. Set also $B_0(t) = \mathrm{Id}$. Assume that for any f in $L^2(\mathbb{R})$, one may find a sequence $(\alpha_n)_n$ in ℓ^1 such that one has

$$\sup_{t\geq 1}\|B_n(t)f\|_{L^2} \leq \alpha_n. \tag{E.11}$$

Define

$$B(t)f = \sum_{n=0}^{+\infty} B_n(t)f, \tag{E.12}$$

that exists because of our assumptions. Then $B(t)$ solves equation (E.9). Moreover, define $C_0(t) = \mathrm{Id}$ and for n in $\mathbb{N}^*$,

$$C_n(t) = i^n \int \prod_{j=1}^{n} e^{-i\tau_j P_0}\mathcal{V}(t+\tau_j)e^{i\tau_j P_0}\mathbb{1}_{0<\tau_n<\cdots<\tau_1}\, d\tau_1 \cdots d\tau_n. \tag{E.13}$$

If we assume that the analogous of (E.11) holds for C_n, and define then $C(t)$ as in (E.12), one has

$$B(t)C(t) = C(t)B(t) = \mathrm{Id}. \tag{E.14}$$

Proof. Let us denote $A(t,s) = -ie^{-isP_0}\mathcal{V}(t+s)e^{isP_0}$. Then

$$[D_t - D_s, A(t,s)] = [P_0, A(t,s)]$$

and by (E.10)

$$B_n(t) = \int \prod_{j=1}^{n} A(t,\tau_j)\mathbb{1}_{0<\tau_1<\cdots<\tau_n}\, d\tau_1 \cdots d\tau_n \tag{E.15}$$

so that

$$\begin{aligned}
[D_t - P_0, B_n] &= \int (D_{\tau_1} + \cdots + D_{\tau_n})\Big(\prod_{j=1}^{n} A(t,\tau_j)\Big)\mathbb{1}_{0<\tau_1<\cdots<\tau_n}\, d\tau_1 \cdots d\tau_n \\
&= -\int \prod_{j=1}^{n} A(t,\tau_j)(D_{\tau_1} + \cdots + D_{\tau_n})\mathbb{1}_{0<\tau_1<\cdots<\tau_n}\, d\tau_1 \cdots d\tau_n \\
&= iA(t,0)B_{n-1}(t).
\end{aligned}$$

Using (E.8), and making the convention $B_{-1}(t) = 0$, we rewrite this as

$$(D_t - P(t))B_n(t) = B_n(t)(D_t - P_0) - \mathcal{V}(t)(B_n(t) - B_{n-1}(t)).$$

If we denote by $S_n(t) = \sum_{n'=0}^{n} B_{n'}(t)$ the partial sum, we get

$$(D_t - P(t))S_n(t) = S_n(t)(D_t - P_0) - \mathcal{V}(t)B_n(t). \tag{E.16}$$

If we make act this on a function f in $L^2(\mathbb{R})$ such that $(D_t - P_0)f$ is in L^2, we get when n goes to infinity, in view of (E.11) and (E.12), the conclusion (E.9).

We still have to show that $C(t)$ is the inverse of $B(t)$. To this end, let us denote for $j = 0, \dots, n-1$, $\varphi_j(\tau_j, \tau_{j+1}) = \mathbb{1}_{\tau_{j+1} > \tau_j}$ and rewrite the definition of $B_n(t)$ given in (E.15) as

$$B_n(t) = \int \prod_{j=1}^{n} A(t, \tau_j)\chi(\tau_1, \dots, \tau_n) \prod_{j'=1}^{n-1} \varphi_{j'}(\tau_{j'}, \tau_{j'+1})\, d\tau_1 \cdots d\tau_n,$$

where $\chi(\tau_1, \dots, \tau_n) = \prod_{\ell=1}^{n} \mathbb{1}_{0<\tau_\ell}$. In the same way, (E.13) may be written as

$$C_n(t) = (-1)^n \int \prod_{j=1}^{n} A(t, \tau_j)\chi(\tau_1, \dots, \tau_n) \prod_{j'=1}^{n-1} (1 - \varphi_{j'})(\tau_{j'}, \tau_{j'+1})\, d\tau_1 \cdots d\tau_n.$$

We thus get for $1 \le \ell \le n$,

$$C_\ell(t) \circ B_{n-\ell}(t) = (-1)^\ell \int \prod_{j=1}^{n} A(t, \tau_j)\chi(\tau_1, \dots, \tau_n) \prod_{j'=1}^{\ell-1} (1 - \varphi_{j'})(\tau_{j'}, \tau_{j'+1}) \times \prod_{j'=\ell+1}^{n-1} \varphi_{j'}(\tau_{j'}, \tau_{j'+1})\, d\tau_1 \cdots d\tau_n$$

using the convention $\prod_{j=1}^{0} = \prod_{j=n}^{n-1} = 1$. This may be rewritten for $\ell = 1, \dots, n-1$,

$$\begin{aligned} C_\ell(t) \circ B_{n-\ell}(t) = (-1)^\ell \int \prod_{j=1}^{n} A(t, \tau_j)\chi(\tau_1, \dots, \tau_n) \prod_{j'=1}^{\ell} (1 - \varphi_{j'})(\tau_{j'}, \tau_{j'+1}) \\ \times \prod_{j'=\ell+1}^{n-1} \varphi_{j'}(\tau_{j'}, \tau_{j'+1})\, d\tau_1 \cdots d\tau_n \\ - (-1)^{\ell-1} \int \prod_{j=1}^{n} A(t, \tau_j)\chi(\tau_1, \dots, \tau_n) \prod_{j'=1}^{\ell-1} (1-\varphi_{j'})(\tau_{j'}, \tau_{j'+1}) \\ \times \prod_{j'=\ell}^{n-1} \varphi_{j'}(\tau_{j'}, \tau_{j'+1})\, d\tau_1 \cdots d\tau_n. \end{aligned}$$

It follows that $\sum_{\ell=0}^{n} C_\ell(t)B_{n-\ell}(t) = 0$ when $n \geq 1$, which implies $C(t) \circ B(t) = \mathrm{Id}$. In the same way $B(t) \circ C(t) = \mathrm{Id}$. ∎

In the rest of this chapter, we shall show that the preceding proposition may be applied to an operator of the form (E.5), if one makes convenient assumptions on the Q_j. Moreover, we shall obtain for the operators $B(t)$ and $C(t)$ estimates in other spaces than L^2. More precisely, we shall prove the proposition below, where we use the following notation. Set, according to (D.34),

$$L_\pm = x \pm tp'(D_x), \quad L = \begin{bmatrix} L_+ & 0 \\ 0 & L_- \end{bmatrix} \tag{E.17}$$

so that

$$[D_t - P_0, L] = 0. \tag{E.18}$$

In the following sections, we shall prove:

Proposition E.1.3. *Let $B_n(t)$ and $C_n(t)$ be defined respectively by* (E.10) *and* (E.13), *in terms of $\mathcal{V}$ given by* (E.5) *with Q_j a 2×2 matrix of elements of $\Sigma_{1,1}^{\iota,m}$, for some $\iota > 0$ small, some $m \in \,]0, \frac{1}{2}[$, close to $\frac{1}{2}$. Then for ε small enough,* (E.11) *and the corresponding inequality for $C_n(t)$ holds, so that*

$$\sum_{n=0}^{+\infty} B_n(t) = B(t) \quad \textit{and} \quad \sum_{n=0}^{+\infty} C_n(t) = C(t)$$

are well defined as operators acting on $L^2(\mathbb{R})$. Moreover, the operators $B(t)$, $C(t)$ are bounded on $H^s(\mathbb{R})$ for any $s \geq 0$ and satisfy for small $\delta' > 0$,

$$\begin{aligned} \|B(t) - \mathrm{Id}\|_{\mathcal{L}(H^s)} &\leq C\varepsilon^\iota t^{-m+\delta'+\frac{1}{4}}, \\ \|C(t) - \mathrm{Id}\|_{\mathcal{L}(H^s)} &\leq C\varepsilon^\iota t^{-m+\delta'+\frac{1}{4}}. \end{aligned} \tag{E.19}$$

One may also write for any f in $L^2(\mathbb{R};\mathbb{C}^2)$ such that $Lf \in L^2(\mathbb{R};\mathbb{C}^2)$,

$$L \circ C(t)f = \tilde{C}(t)Lf + \tilde{C}_1(t)f, \tag{E.20}$$

where

$$\|\tilde{C}(t) - \mathrm{Id}\|_{\mathcal{L}(L^2)} \leq C\varepsilon^\iota t^{-m+\delta'+\frac{1}{4}}, \tag{E.21}$$

$$\|\tilde{C}_1(t)\|_{\mathcal{L}(L^2)} \leq C\varepsilon^\iota t^{\frac{1}{2}-m}. \tag{E.22}$$

Moreover, under condition (E.7), *one has*

$$\overline{B(t)}N_0 = N_0B(t), \quad \overline{C(t)}N_0 = N_0C(t) \tag{E.23}$$

and if $\mathcal{V}(t)$ preserves the space of odd functions, so do $B(t)$ and $C(t)$.

E.2 Technical lemmas

In this section, we prove some technical lemmas that will be used to obtain Proposition E.1.3.

Lemma E.2.1. *For ξ, η, λ real, denote*

$$\phi_{\pm}(\xi, \eta, \lambda) = \langle \xi \rangle \pm \langle \eta \rangle + \lambda. \tag{E.24}$$

There is $C > 0$ such that for any λ in $\mathbb{R}$, any $t \geq 1$,

$$\int_{|\phi_{\pm}(\xi,\eta,\lambda)|<1} \langle t\phi_{\pm}(\xi, \eta, \lambda)\rangle^{-1}\, d\eta \leq Ct^{-\frac{1}{2}}, \tag{E.25}$$

$$\int_{|\phi_{\pm}(\xi,\eta,\lambda)|<1} \langle t\phi_{\pm}(\xi, \eta, \lambda)\rangle^{-1} \frac{|\eta|}{\langle \eta \rangle}\, d\eta \leq Ct^{-1}\log(1+t). \tag{E.26}$$

Proof. We compute first the integrals over the domain $\eta \geq c$ or $\eta \leq -c$ for some constant $c > 0$. On these domains, $\eta \mapsto \zeta = \phi_{\pm}(\xi, \eta, \lambda)$ is a change of variables, whose Jacobian has uniform lower and upper bounds. The corresponding integrals are thus bounded by

$$C\int_{|\zeta|<1} \langle t\zeta \rangle^{-1}\, d\zeta \leq Ct^{-1}\log(t+1).$$

We compute next the integrals for $|\eta| < c$. If c is small enough, we may write on this domain

$$\phi_{\pm}(\xi, \eta, \lambda) = \phi_{\pm}(\xi, 0, \lambda) + g(\eta)^2,$$

where $g(0) = 0$, $g'(0) \neq 0$, so that we may bound the two integrals (E.25) and (E.26), respectively, by

$$C\int_{|\zeta|<c'} \langle \rho + t\zeta^2 \rangle^{-1}\, d\zeta, \quad C\int_{|\zeta|<c'} \langle \rho + t\zeta^2 \rangle^{-1} |\zeta|\, d\zeta,$$

where $c' > 0$ is some constant, and ρ is some real number depending on ξ, λ, t. These two integrals are smaller than the right-hand side of (E.25) and (E.26), respectively, uniformly in ρ. ∎

We study now composition of operators defined by (E.6) from symbols in the classes of Definition E.1.1, and we prove also Sobolev estimates for such operators.

Lemma E.2.2. *The following statements hold.*

(i) *If ℓ is in $\mathbb{N}$, set $\mu(\ell) = \frac{1}{2}$ if $\ell = 0$ and let $\mu(\ell)$ be strictly smaller than 1 if $\ell \geq 1$. Let $N \geq 2$. There is a constant $C > 0$ such that if two functions q_1, q_2 satisfy estimates*

$$\begin{aligned} |q_1(\xi, \eta)| &\leq K_1 \langle |\xi| - |\eta| \rangle^{-N} \left(\frac{|\eta|}{\langle \eta \rangle}\right)^b, \\ |q_2(\xi, \eta)| &\leq K_2 \langle |\xi| - |\eta| \rangle^{-N} \left(\frac{|\xi|}{\langle \xi \rangle}\right)^a, \end{aligned} \tag{E.27}$$

where a, b are in $\{0, 1\}$, then the function given by

$$q_3(\xi,\eta) = \int q_1(\xi,\zeta)q_2(\zeta,\eta)\langle t\phi_\pm(\xi,\zeta,\lambda)\rangle^{-1}\,d\zeta \tag{E.28}$$

satisfies

$$|q_3(\xi,\eta)| \le CK_1K_2t^{-\mu(b+a)}\langle|\xi|-|\eta|\rangle^{-N}. \tag{E.29}$$

(ii) *Let s be in $\mathbb{R}_+$, $\delta' > 0$, $N \ge s+2$. There is $C > 0$ such that if a function $(\xi,\eta) \mapsto q(\xi,\eta)$ satisfies*

$$|q(\xi,\eta)| \le K\langle|\xi|-|\eta|\rangle^{-N}\Big(\frac{|\xi|}{\langle\xi\rangle} + \frac{|\eta|}{\langle\eta\rangle}\Big), \tag{E.30}$$

then the operator K_q defined by (E.6) *satisfies*

$$\|K_q\|_{\mathcal{L}(H^s)} \le CKt^{-\frac{3}{4}+\delta'}. \tag{E.31}$$

(iii) *If instead of* (E.30)*, q satisfies*

$$|q(\xi,\eta)| \le K\langle|\xi|-|\eta|\rangle^{-N}\frac{|\xi|}{\langle\xi\rangle}\frac{|\eta|}{\langle\eta\rangle}, \tag{E.32}$$

one gets instead of (E.31)

$$\|K_q\|_{\mathcal{L}(H^s)} \le CKt^{-1+\delta'}. \tag{E.33}$$

Proof. (i) If in (E.28) we integrate for $\phi_\pm(\xi,\zeta,\lambda) \ge 1$, then (E.29) holds trivially, as a consequence of (E.27), with factor t^{-1} instead of $t^{-\mu(b+a)}$. If we integrate for $|\phi_\pm(\xi,\zeta,\lambda)| < 1$, the contribution to q_3 is bounded from above by

$$CK_1K_2\langle|\xi|-|\eta|\rangle^{-N}\int_{|\phi_\pm(\xi,\zeta,\lambda)|<1}\langle t\phi_\pm(\xi,\zeta,\lambda)\rangle^{-1}\Big(\frac{|\zeta|}{\langle\zeta\rangle}\Big)^{a+b}\,d\zeta.$$

Applying Lemma E.2.1, we get (E.29).

(ii) Since $N \ge s+2$, the $\mathcal{L}(H^s)$ estimate is reduced to an $\mathcal{L}(L^2)$ one for $N \ge 2$ using the decay in $\langle|\xi|-|\eta|\rangle$ in (E.30). If the kernel of the operator K_q is cut-off for $|\phi_\pm(\xi,\eta,\lambda)| \ge 1$, then Schur's lemma shows that estimate (E.31) holds with t^{-1} instead of $t^{-\frac{3}{4}+\delta'}$. We have thus to study

$$f \mapsto \int q(\xi,\eta)\langle t\phi_\pm(\xi,\eta,\lambda)\rangle^{-1}\mathbb{1}_{|\phi_\pm(\xi,\eta,\lambda)|<1}f(\eta)\,d\eta.$$

By Schur's lemma and (E.30), the $\mathcal{L}(L^2)$ norm of this operator is bounded from above by

$$\begin{aligned} CK\Big(\sup_\xi\int\langle|\xi|-|\eta|\rangle^{-N}\langle t\phi_\pm(\xi,\eta,\lambda)\rangle^{-1}\frac{|\eta|}{\langle\eta\rangle}\,d\eta\Big)^{\frac{1}{2}} \\ \times\Big(\sup_\eta\int\langle|\xi|-|\eta|\rangle^{-N}\langle t\phi_\pm(\xi,\eta,\lambda)\rangle^{-1}\,d\xi\Big)^{\frac{1}{2}} \end{aligned} \tag{E.34}$$

and by the symmetric quantity. Using (E.25) and (E.26), we get (E.31).

(iii) We make the same reasoning as above, except that (E.34) is now replaced by

$$CK\Big(\sup_{\xi}\int\langle|\xi|-|\eta|\rangle^{-N}\langle t\phi_{\pm}(\xi,\eta,\lambda)\rangle^{-1}\frac{|\eta|}{\langle\eta\rangle}\,d\eta\Big)^{\frac{1}{2}}$$
$$\times\Big(\sup_{\eta}\int\langle|\xi|-|\eta|\rangle^{-N}\langle t\phi_{\pm}(\xi,\eta,\lambda)\rangle^{-1}\frac{|\xi|}{\langle\xi\rangle}\,d\xi\Big)^{\frac{1}{2}}.$$

We conclude by (E.26). ∎

Let us define a class that will contain functions obtained from those of Definition E.1.1 by introduction of an extra variable.

Definition E.2.3. We denote by $\widetilde{\Sigma}_{0,0}^{\iota,m,m_0}$ the space of functions

$$(t,v,\xi,\eta)\mapsto q(t,v,\xi,\eta),$$

defined for $t\geq 1$, $v\geq 0$, ξ,η in $\mathbb{R}$, that are Lipschitz and compactly supported in v and satisfy for any N and $j=0,1$,

$$|\partial_v^j q(t,v,\xi,\eta)|\leq C_N\varepsilon^{\iota}t^{1-m}(1+v)^{-m_0-j}\langle|\xi|-|\eta|\rangle^{-N}. \tag{E.35}$$

For a,b in $\mathbb{N}$, we denote by $\widetilde{\Sigma}_{a,b}^{\iota,m,m_O}$ the space of functions that may be written

$$\Big(\frac{\xi}{\langle\xi\rangle}\Big)^{a}\Big(\frac{\eta}{\langle\eta\rangle}\Big)^{b}q_1$$

with q_1 in $\widetilde{\Sigma}_{0,0}^{\iota,m,m_0}$.

We shall also allow q to depend on extra parameters, estimates (E.35) being uniform in these parameters.

Notice that if q belongs to the class $\Sigma_{a,b}^{\iota,m}$ of Definition E.1.1 and is compactly supported in time, then $\tilde{q}(t,v,\xi,\eta)=tq(t(1+v),\xi,\eta)$ is in $\widetilde{\Sigma}_{a,b}^{\iota,m,m_0}$ if $m\geq m_0$.

We shall discuss some operators constructed from functions in $\widetilde{\Sigma}_{a,b}^{\iota,m,m_0}$. In the following discussion, we shall identify operators and their kernels.

Let Q be in $\widetilde{\Sigma}_{a,b}^{\iota,m,m_0}\otimes\mathcal{M}_2(\mathbb{R})$ (i.e. a 2×2 matrix of elements of $\widetilde{\Sigma}_{a,b}^{\iota,m,m_0}$). If λ is in $\mathbb{R}$, we consider the operator from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$ given at fixed t,v by the kernel in (ξ,η)

$$S(t,v,Q,\lambda)=e^{-itvP_0(\xi)}Q(t,v,\xi,\eta)e^{itv(P_0(\eta)+\lambda)}. \tag{E.36}$$

If we decompose

$$Q(t,v,\xi,\eta)=\sum_{j=1}^{2}\sum_{k=1}^{2}q_{jk}(t,v,\xi,\eta)E_{jk},$$

where

$$E_{jk}=(\delta_j^{j'}\delta_k^{k'})_{1\leq j',k'\leq 2}, \tag{E.37}$$

we may write

$$S(t, v, Q, \lambda) = \sum_{j=1}^{2} \sum_{k=1}^{2} S_{jk}(t, v, Q, \lambda) \tag{E.38}$$

with

$$S_{jk}(t, v, Q, \lambda) = q_{jk}(t, v, \xi, \eta) e^{itv\phi_{jk}(\xi,\eta,\lambda)} E_{jk}, \tag{E.39}$$

where

$$\phi_{jk}(\xi, \eta, \lambda) = (-1)^j p(\xi) - (-1)^k p(\eta) + \lambda. \tag{E.40}$$

We assume given functions Q^ℓ in $\widetilde{\Sigma}^{\iota^\ell, m^\ell, m_0^\ell}_{a^\ell, b^\ell} \otimes \mathcal{M}_2(\mathbb{R})$ and real numbers λ_ℓ for ℓ in $\mathbb{N}^*$. We set

$$\underline{Q}_n = (Q^n, \dots, Q^1), \quad \underline{\lambda} = (\lambda^n, \dots, \lambda^1). \tag{E.41}$$

We define inductively a sequence of operators by their kernels, starting with

$$M_1(t, u, \underline{Q}_1, \underline{\lambda}_1) = \int_u^{+\infty} S(t, v, Q^1, \lambda^1)\, dv \tag{E.42}$$

and for $n \geq 1$,

$$\begin{aligned} M_{n+1}&(t, u, \underline{Q}_{n+1}, \underline{\lambda}_{n+1}) \\ &= \int_u^{+\infty} S(t, v, Q^{n+1}, \lambda^{n+1}) \circ M_n(t, v, \underline{Q}_n, \underline{\lambda}_n)\, dv. \end{aligned} \tag{E.43}$$

Notice that the above integrals converge since S is compactly supported in v. According to our convention of identification between kernels and operators, we shall set for a function f

$$M_n(t, v, \underline{Q}_n, \underline{\lambda}_n) f(\xi) = \int M_n(t, v, \underline{Q}_n, \underline{\lambda}_n)(\xi, \eta) f(\eta)\, d\eta. \tag{E.44}$$

We shall prove the following estimates:

Lemma E.2.4. *Let $m, m_0^n, m_0', \iota, a, b$ satisfy*

$$m_0^n, m_0' > \frac{1}{4}, \quad a, b \in \mathbb{N},\ a + b \geq 1, \quad \iota > 0, \quad m > 0. \tag{E.45}$$

Let Q be in $\widetilde{\Sigma}^{\iota, m, m_0'}_{a,b} \otimes \mathcal{M}_2(\mathbb{R})$, λ in $\mathbb{R}$, and let K_N be the best constant C_N in (E.35) for the entries of Q. In the same way, denote by $K_{N,\ell}$ the best constant in (E.35) for the entries of Q_ℓ, $\ell = 1, \dots, n$. There are for any $N \geq 2$, any $\delta' > 0$, a constant C_N that does not depend on K_N, $K_{N,\ell}$ and a symbol $\tilde{Q}$ in

$$\widetilde{\Sigma}^{\iota + \iota_n, m + m^n - \frac{1}{2}, m_0^n + m_0' - \frac{1}{2}}_{a, b^n} \otimes \mathcal{M}_2(\mathbb{R})$$

if $a^n + b = 0$, *and in*

$$\widetilde{\Sigma}^{\iota+\iota_n, m+m^n-\delta', m_0^n+m_0'-\delta'}_{a,b^n} \otimes \mathcal{M}_2(\mathbb{R})$$

if $a^n + b \geq 1$, *whose* N*-th semi-norm is bounded from above by* $C_N K_N K_{N,n}$, *such that if* $n \geq 1$,

$$\begin{aligned}\int_u^{+\infty} & S(t, v, Q, \lambda) \circ M_n(t, v, \underline{Q}_n, \underline{\lambda}_n)\, dv \\ &= \int_u^{+\infty} S(t, v, \tilde{Q}, \tilde{\lambda}) \circ M_{n-1}(t, v, \underline{Q}_{n-1}, \underline{\lambda}_{n-1})\, dv + R_n(t, u),\end{aligned} \tag{E.46}$$

where $\tilde{\lambda} = \lambda + \lambda_n$ *and* R_n *satisfies for any* f *in* $L^2(\mathbb{R})$ *and any* $\delta' > 0$,

$$\|\sup_u |R_n(t, u) f|\|_{L^2} \leq C K_2 \varepsilon^{\iota} t^{-m+\frac{1}{4}+\delta'} \|\sup_u |M_n(t, u, \underline{Q}_n, \underline{\lambda}_n) f|\|_{L^2}. \tag{E.47}$$

If $n = 0$, *then* (E.46) *holds as well without the integral term on the right-hand side.*

Proof. In the left-hand side of (E.46) we plug (E.38). Then the kernel of that operator is the sum in j, k, $1 \leq j, k \leq 2$, of

$$\int_u^{+\infty} \int S_{jk}(t, v, Q, \lambda)(\xi, \zeta) M_n(t, v, \underline{Q}_n, \underline{\lambda}_n)(\zeta, \eta)\, d\zeta\, dv. \tag{E.48}$$

Let us define for $1 \leq j, k \leq 2$ the operator

$$\begin{aligned}L_{jk\lambda}(\xi, \zeta) = {} & \langle t(1 + v)\phi_{jk}(\xi, \zeta, \lambda)\rangle^{-2} \\ & \times \big(1 + t(1 + v)\phi_{jk}(\xi, \zeta, \lambda)(1 + v) D_v\big),\end{aligned} \tag{E.49}$$

where we used notation (E.40). Then, by (E.39),

$$\begin{aligned}L_{jk\lambda} S_{jk}(\xi, \zeta) = S_{jk}(\xi, \zeta) + {} & \frac{t(1 + v)\phi_{jk}(\xi, \zeta, \lambda)}{\langle t(1 + v)\phi_{jk}(\xi, \zeta, \lambda)\rangle^2}(1 + v) \\ & \times D_v q_{jk}(t, v, \xi, \zeta, \lambda) e^{itv\phi_{jk}(\xi, \zeta, \lambda)} E_{jk}.\end{aligned} \tag{E.50}$$

We plug the expression of S_{jk} deduced from (E.50) inside (E.48). We obtain on the one hand

$$\begin{aligned}-\int_u^{+\infty} \int & \frac{t(1 + v)\phi_{jk}(\xi, \zeta, \lambda)}{\langle t(1 + v)\phi_{jk}(\xi, \zeta, \lambda)\rangle^2}(1 + v) D_v q_{jk}(t, v, \xi, \zeta, \lambda) \\ & \times e^{itv\phi_{jk}(\xi, \zeta, \lambda)} E_{jk} M_n(t, v, \underline{Q}_n, \underline{\lambda}_n)(\zeta, \eta)\, d\zeta\, dv\end{aligned} \tag{E.51}$$

and on the other hand

$$\int_u^{+\infty} \int L_{jk\lambda} S_{jk}(t, v, Q, \lambda)(\xi, \zeta) M_n(t, v, \underline{Q}_n, \underline{\lambda}_n)(\zeta, \eta)\, d\zeta\, dv. \tag{E.52}$$

Using the expression (E.49) of $L_{jk\lambda}$, we perform in (E.52) one integration by parts in v. We get the following contributions:

$$\int_u^{+\infty}\int\Big(\big\langle t(1+v)\phi_{jk}(\xi,\zeta,\lambda)\big\rangle^{-2} - D_v\Big((1+v)\frac{t(1+v)\phi_{jk}(\xi,\zeta,\lambda)}{\langle t(1+v)\phi_{jk}(\xi,\zeta,\lambda)\rangle^2}\Big)\Big) \times S_{jk}(t,v,Q,\lambda)(\xi,\zeta)M_n(t,v,\underline{Q}_n,\underline{\lambda}_n)(\zeta,\eta)\,d\zeta\,dv, \tag{E.53}$$

$$-\int_u^{+\infty}\int\frac{t(1+v)\phi_{jk}(\xi,\zeta,\lambda)}{\langle t(1+v)\phi_{jk}(\xi,\zeta,\lambda)\rangle^2}S_{jk}(t,v,Q,\lambda)(\xi,\zeta) \times (1+v)D_vM_n(t,v,\underline{Q}_n,\underline{\lambda}_n)(\zeta,\eta)\,d\zeta\,dv, \tag{E.54}$$

$$-\frac{1}{i}\int\frac{t(1+u)^2\phi_{jk}(\xi,\zeta,\lambda)}{\langle t(1+u)\phi_{jk}(\xi,\zeta,\lambda)\rangle^2}S_{jk}(t,u,Q,\lambda)(\xi,\zeta) \times M_n(t,u,\underline{Q}_n,\underline{\lambda}_n)(\zeta,\eta)\,d\zeta. \tag{E.55}$$

Let us show that (E.51), (E.53), (E.54), (E.55) may be written as contributions to the right-hand side of (E.46).

Contributions of (E.51) *and* (E.53). We make act (E.51) and (E.53) on a function f. We shall get an expression

$$\int_u^{+\infty}\int K(v,\xi,\zeta)\big(M_n(t,v,\underline{Q}_n,\underline{\lambda}_n)f\big)(\zeta)\,d\zeta\,dv, \tag{E.56}$$

where, by the fact that q_{jk} in (E.39) is in $\widetilde{\Sigma}^{\iota,m,m_0'}_{a,b}$ and (E.35), the kernel K satisfies the bound

$$|K(v,\xi,\zeta)| \le CK_2\langle t(1+v)\phi_{jk}(\xi,\zeta,\lambda)\rangle^{-1}\Big(\frac{|\xi|}{\langle\xi\rangle}\Big)^a\Big(\frac{|\zeta|}{\langle\zeta\rangle}\Big)^b \times \varepsilon^\iota t^{1-m}(1+v)^{-m_0'}\langle|\xi|-|\eta|\rangle^{-2}. \tag{E.57}$$

We bound the modulus of (E.56) by

$$\int_0^{+\infty}\int|K(v,\xi,\zeta)|\big(\sup_w|M_n(t,w,\underline{Q}_n,\underline{\lambda}_n)f(\zeta)|\big)\,d\zeta\,dv.$$

Then the L^2 norm in ξ of the supremum in u of (E.56) is bounded from above by

$$\int_0^{+\infty}\Big\|\int|K(v,\xi,\zeta)|\big(\sup_w|M_n(t,w,\underline{Q}_n,\underline{\lambda}_n)f(\zeta)|\big)\,d\zeta\Big\|_{L^2(d\xi)}\,dv. \tag{E.58}$$

As $a+b\ge 1$, (E.57) shows that we may apply to the $d\zeta$-integral, which is of the form of the right-hand side of (E.30), estimate (E.31), with t replaced by $t(1+v)$. We obtain that (E.58) is smaller than

$$CK_2\int_0^{+\infty}\varepsilon^\iota t^{\frac14-m+\delta'}(1+v)^{-m_0'-\frac34+\delta'}\,dv\,\big\|\sup_w|M_n(t,w,\underline{Q}_n,\underline{\lambda}_n)f|\big\|_{L^2}$$

with $\delta'>0$ as small as we want. Since by assumption $m_0'>\frac14$, we obtain a bound of the form (E.47), that shows that (E.51) and (E.53) contribute to R_n in (E.46).

Contribution of (E.55). This is an expression similar to (E.53), except that we no not have a dv integral and have a factor $(1+u)^2$ instead of $(1+v)$. Consequently, for the L^2 norm of that operator acting on f, we get a bound of the form (E.58) but without dv-integration and an extra factor $(1+u)$, and with K estimated at u instead of v. This implies again that we obtain a contribution to R_n.

Contribution of (E.54). By (E.43) at order $n-1$,

$$D_v M_n(t,v,\underline{Q}_n,\underline{\lambda}_n) = iS(t,v,Q^n,\lambda^n)\circ M_{n-1}(t,v,\underline{Q}_{n-1},\underline{\lambda}_{n-1}).$$

Plugging this in (E.54), we get the expression

$$\begin{aligned} -i\int_u^{+\infty}\iint \frac{t(1+v)\phi_{jk}(\xi,\zeta,\lambda)}{\langle t(1+v)\phi_{jk}(\xi,\zeta,\lambda)\rangle^2} S_{jk}(t,v,Q,\lambda)(\xi,\zeta) \\ \times(1+v)S(t,v,Q^n,\lambda^n)(\zeta,\eta')M_{n-1}(t,v,\underline{Q}_{n-1},\underline{\lambda}_{n-1})(\eta',\eta)\,d\zeta\,d\eta'\,dv. \end{aligned} \tag{E.59}$$

We write by (E.38)

$$S(t,v,Q^n,\lambda^n) = \sum_{k'=1}^{2}\sum_{\ell=1}^{2} S_{k'\ell}(t,v,Q^n,\lambda^n).$$

By (E.39) and the fact that $E_{jk}E_{k'\ell} = \delta_k^{k'}E_{j\ell}$, we have

$$\begin{aligned} \sum_{k'=1}^{2} S_{jk}(t,v,Q,\lambda)(\xi,\zeta)S_{k',\ell}(t,v,Q^n,\lambda^n)(\zeta,\eta') \\ = q_{jk}(t,v,\xi,\zeta)q^n_{k\ell}(t,v,\zeta,\eta')e^{itv\phi_{jk}(\xi,\zeta,\lambda)+itv\phi_{k\ell}(\zeta,\eta',\lambda_n)}E_{j\ell}, \end{aligned} \tag{E.60}$$

where $q^n_{k\ell}$ denote the entries of matrix Q^n. By (E.40), the phase in the exponential is $\phi_{j\ell}(\xi,\eta',\lambda+\lambda^n)$. Define

$$\begin{aligned} \tilde{q}_{j\ell}(t,v,\xi,\eta',\lambda) = -i(1+v)\int\sum_{k=1}^{2} q_{jk}(t,v,\xi,\zeta)q^n_{k\ell}(t,v,\zeta,\eta') \\ \times t(1+v)\phi_{jk}(\xi,\zeta,\lambda)\langle t(1+v)\phi_{jk}(\xi,\eta,\lambda)\rangle^{-2}\,d\zeta. \end{aligned} \tag{E.61}$$

Since q_{jk} is in $\widetilde{\Sigma}^{\iota,m,m_0'}_{a,b}$, estimate (E.35) shows that we may write this function as $(\frac{\xi}{\langle\xi\rangle})^a$ multiplied by a function that will satisfy the first estimate (E.27), with K_1 bounded by $\varepsilon^\iota t^{1-m}(1+v)^{-m_0'}$. In the same way, since $q^n_{k\ell}$ is in $\widetilde{\Sigma}^{\iota^n,m^n,m_0^n}_{a^n,b^n}$, it may be written as $(\frac{\eta'}{\langle\eta'\rangle})^{b^n}$ times a function satisfying the second estimate (E.27), with a replaced by a^n and K_2 bounded by $\varepsilon^{\iota^n}t^{1-m_n}(1+v)^{-m_0^n}$. By (i) of Lemma E.2.2, applied with t replaced by $t(1+v)$, we see that (E.61) may be written as a product of $(\frac{\xi}{\langle\xi\rangle})^a(\frac{\eta'}{\langle\eta'\rangle})^{b^n}$ times a quantity bounded from above by

$$CK_NK_{N,n}\varepsilon^{\iota+\iota^n}t^{\frac{3}{2}-m-m^n}(1+v)^{\frac{1}{2}-m_0^n-m_0'}\langle|\xi|-|\eta'|\rangle^{-N}$$

if $b + a^n = 0$ and by

$$CK_N K_{N,n} \varepsilon^{\iota+\iota^n} t^{1-m-m^n+\delta'} (1+v)^{-m_0^n - m_0' + \delta'} \langle |\xi| - |\eta'| \rangle^{-N}$$

for any $\delta' > 0$ if $b + a^n \geq 1$, according to (E.29).

If one takes a ∂_v-derivative of (E.61), one gains an extra decay factor in $(1+v)^{-1}$. Consequently, equation (E.61) defines a symbol in the class $\widetilde{\Sigma}_{a,b^n}^{\iota+\iota^n, m+m^n-\frac{1}{2}, m_0^n+m_0'-\frac{1}{2}}$ (resp. in the class $\widetilde{\Sigma}_{a,b^n}^{\iota+\iota^n, m+m^n-\delta', m_0^n+m_0'-\delta'}$) if $b + a^n = 0$ (resp. $b + a^n \geq 1$). Since the phases in equation (E.60) satisfy

$$\phi_{jk}(\xi, \zeta, \lambda) + \phi_{k\ell}(\zeta, \eta', \lambda^n) = \phi_{j\ell}(\xi, \eta', \lambda + \lambda^n),$$

this shows that (E.59) may be written under the form of the first integral on the right-hand side of (E.46), with a matrix function $\tilde{Q}$, depending on λ, but with estimates uniform in λ, whose entries are respectively in the classes of the statement of the lemma. This concludes the proof as, in the case $n = 0$, one has just to estimate terms of the form (E.51), (E.53), (E.55). ■

Our next goal will be to obtain bounds for (E.43) iterating (E.46). We introduce some notation.

Let p, n be in $\mathbb{N}^*$. Assume given for each (n, p) a sequence $(X_{(n,p)}^j)_{1 \leq j \leq n}$, where $X_{(n,p)}^j$ is an element

$$X_{(n,p)}^j = \left(\iota_{(n,p)}^j, m_{(n,p)}^j, m_{(n,p),0}^j, a_{(n,p)}^j, b_{(n,p)}^j\right) \tag{E.62}$$

of $]0, +\infty[\times]\frac{1}{4}, +\infty[\times]\frac{1}{4}, +\infty[\times \mathbb{N} \times \mathbb{N}$ satisfying the following conditions:

$$\begin{aligned} &\text{If } p \leq n, \text{ then } m_{(n,p),0}^j > \frac{3}{8},\ j = 1, \dots, n. \\ &\text{If } p \geq n+1, \text{ then } m_{(n,p),0}^j > \frac{3}{8},\ j = 1, \dots, n-1, \text{ and } m_{(n,p),0}^n > \frac{1}{4}. \end{aligned} \tag{E.63}$$

$$\begin{aligned} &\text{For } 1 \leq j', j'' \leq n, a_{(n,p)}^{j'} + b_{(n,p)}^{j''} \geq 1 \text{ except eventually if} \\ &j' < j'' = p \text{ (this exception being void if } p > n \text{ or } p = 1). \end{aligned} \tag{E.64}$$

For any $X_{(n,p)}^j$ of the form (E.62), we denote for short by $\widetilde{\Sigma}(X_{(n,p)}^j)$ the class

$$\widetilde{\Sigma}(X_{(n,p)}^j) = \widetilde{\Sigma}_{a_{(n,p)}^j, b_{(n,p)}^j}^{\iota_{(n,p)}^j, m_{(n,p)}^j, m_{(n,p),0}^j}$$

of Definition E.2.3.

If $(X_{(n+1,p)}^j)_{1 \leq j \leq n+1}$ is a sequence of the form (E.62), we define from it the concatenated sequence $(X_{(n,p)}^{j,\mathrm{C}})_{1 \leq j \leq n}$ and the truncated sequence $(X_{(n,p)}^{j,\mathrm{T}})_{1 \leq j \leq n}$ in the following way: We just set

$$X_{(n,p)}^{j,\mathrm{T}} = X_{(n+1,p)}^j, \quad j = 1, \dots, n, \tag{E.65}$$

while we denote

$$X_{(n,p)}^{j,\mathrm{C}} = \left(\iota_{(n,p)}^{j,\mathrm{C}}, m_{(n,p)}^{j,\mathrm{C}}, m_{(n,p),0}^{j,\mathrm{C}}, a_{(n,p)}^{j,\mathrm{C}}, b_{(n,p)}^{j,\mathrm{C}}\right),$$

where the components of the preceding vector are defined in the following way:

$$\iota^{n,\mathrm{C}}_{(n,p)} = \iota^{n+1}_{(n+1,p)} + \iota^{n}_{(n+1,p)}, \quad \iota^{j,\mathrm{C}}_{(n,p)} = \iota^{j}_{(n+1,p)}, \quad j = 1,\dots,n-1. \tag{E.66}$$

If $n \neq p-1$, we set for $j = 1,\dots,n-1$,

$$\begin{aligned} m^{n,\mathrm{C}}_{(n,p)} &= m^{n+1}_{(n+1,p)} + m^{n}_{(n+1,p)} - \delta', & m^{j,\mathrm{C}}_{(n,p)} &= m^{j}_{(n+1,p)}, \\ m^{n,\mathrm{C}}_{(n,p),0} &= m^{n+1}_{(n+1,p),0} + m^{n}_{(n+1,p),0} - \delta', & m^{j,\mathrm{C}}_{(n,p),0} &= m^{j}_{(n+1,p),0}, \end{aligned} \tag{E.67}$$

where $\delta' > 0$ is as small as wanted (in particular, δ' will be small enough so that the lower bound (E.63) still holds with $m^{j}_{(n,p),0}$ replaced by $m^{j}_{(n,p),0} - \delta'$).

If $n = p-1$, we define instead of (E.67), for $j = 1,\dots,p-2$,

$$\begin{aligned} m^{p-1,\mathrm{C}}_{(p-1,p)} &= m^{p}_{(p,p)} + m^{p-1}_{(p,p)} - \frac{1}{2}, & m^{j,\mathrm{C}}_{(p-1,p)} &= m^{j}_{(p,p)}, \\ m^{p-1,\mathrm{C}}_{(p-1,p),0} &= m^{p}_{(p,p),0} + m^{p-1}_{(p,p),0} - \frac{1}{2}, & m^{j,\mathrm{C}}_{(p-1,p),0} &= m^{j}_{(p,p),0}. \end{aligned} \tag{E.68}$$

Finally, we set for all (n, p),

$$\begin{aligned} a^{n,\mathrm{C}}_{(n,p)} &= a^{n+1}_{(n+1,p)}, & b^{n,\mathrm{C}}_{(n,p)} &= b^{n}_{(n+1,p)}, \\ a^{j,\mathrm{C}}_{(n,p)} &= a^{j}_{(n+1,p)}, & b^{j,\mathrm{C}}_{(n,p)} &= b^{j}_{(n+1,p)}, \quad j = 1,\dots,n-1. \end{aligned} \tag{E.69}$$

Let us check that if the sequence $(X^{j}_{(n+1,p)})_{1\le j\le n+1}$ satisfies (E.63)–(E.64) (with n replaced by $n+1$), then $(X^{j,\mathrm{C}}_{(n,p)})_{1\le j\le n}$ satisfies also (E.63)–(E.64).

Verification of condition (E.63).

Case $p \le n$. As $n \neq p-1$, (E.67) applies and shows that

$$m^{j,\mathrm{C}}_{(n,p),0} = m^{j}_{(n+1,p),0}$$

for $j = 1,\dots,n-1$. On the other hand, by (E.63) with n replaced by $n+1$,

$$m^{j}_{(n+1,p),0} > \frac{3}{8},$$

so that the first condition (E.63) holds for $m^{j,\mathrm{C}}_{(n,p),0}$ if $j = 1,\dots,n-1$. To get it for $m^{n,\mathrm{C}}_{(n,p),0}$, we write by (E.67) that

$$m^{n,\mathrm{C}}_{(n,p),0} = m^{n+1}_{(n+1,p),0} + m^{n}_{(n+1,p),0} - \delta' > \frac{3}{8} + \frac{3}{8} - \delta' > \frac{3}{8}$$

using the first line in (E.63) with n replaced by $n+1$.

Case $p = n+1$. By (E.68), we have

$$m^{j,\mathrm{C}}_{(p-1,p),0} = m^{j}_{(p,p),0}$$

for $j = 1, \dots, p-2$, and by the first line in (E.63) (with n replaced by $n+1 = p$), this is strictly larger than $\frac{3}{8}$, so that the second line of (E.63) holds for $m^{j,C}_{(p-1,p),0}$, $j = 1, \dots, p-2$. On the other hand, still by (E.68)

$$m^{p-1,C}_{(p-1,p),0} = m^{p}_{(p,p),0} + m^{p-1}_{(p,p),0} - \frac{1}{2} > \frac{3}{8} + \frac{3}{8} - \frac{1}{2} = \frac{1}{4}$$

so that the last condition (E.63) holds for $m^{p-1,C}_{(p-1,p),0}$. We thus got (E.63) for $m^{j,C}_{(n,p),0}$ when $n = p-1$.

Case $p \geq n+2$. Again, we may apply (E.67) to write for $j = 1, \dots, n-1$,

$$m^{j,C}_{(n,p),0} = m^{j}_{(n+1,p),0} > \frac{3}{8}$$

by the second condition of (E.63) with n replaced by $n+1$. On the other hand, still by (E.67)

$$m^{n,C}_{(n,p),0} = m^{n+1}_{(n+1,p),0} + m^{n}_{(n+1,p),0} - \delta' > \frac{1}{4} + \frac{3}{8} - \delta' > \frac{3}{8}$$

using (E.63) with n replaced by $n+1$. This is better than what we need to ensure the last condition (E.63) for $m^{n,C}_{(n,p),0}$. This concludes the verification.

Verification of (E.64). We assume that (E.64) holds at rank $n+1$, i.e.

> For $1 \leq j', j'' \leq n+1$, $a^{j'}_{(n+1,p)} + b^{j''}_{(n+1,p)} \geq 1$ except eventually if $j' < j'' = p$.

Let us check (E.64) for $a^{j',C}_{(n,p)}, b^{j'',C}_{(n,p)}$. If both j' and j'' are strictly smaller than n, then (E.69) shows that the wanted property holds. On the other hand, if $j'' \leq n$, $j' < n$, then

$$a^{j',C}_{(n,p)} + b^{j'',C}_{(n,p)} = a^{j'}_{(n+1,p)} + b^{j''}_{(n+1,p)}$$

by (E.69), and this expression is larger than or equal to one, except eventually if $j' < j'' = p$, whence again (E.64). It remains to study the case $j' = n$. We have then

$$a^{n,C}_{(n,p)} + b^{j'',C}_{(n,p)} = a^{n+1}_{(n+1,p)} + b^{j''}_{(n+1,p)}.$$

The inequality $n+1 < j'' = p$ cannot hold, so that the above quantity is always larger than or equal to one. This shows that (E.64) is satisfied by $(X^{j,C}_{(n,p)})_{1 \leq j \leq n}$.

We may state our main proposition.

Proposition E.2.5. *Let n be in $\mathbb{N}$, p in $\mathbb{N}^*$. Assume a sequence $(X^{j}_{(n+1,p)})_{1 \leq j \leq n+1}$ of the form (E.62) is given, satisfying (E.63) and (E.64), with n replaced by $n+1$. For $j = 1, \dots, n+1$, let $Q^{j}_{(n+1,p)}$ be an element of $\widetilde{\Sigma}(X^{j}_{(n+1,p)}) \otimes \mathcal{M}_2(\mathbb{R})$. Denote by $K^{j}_{(n+1,p)}$ the semi-norm provided by the best constant in estimate (E.35), in the special case $N = 2$. Set as in (E.41),*

$$\underline{Q}_{n+1} = (Q^{n+1}_{(n+1,p)}, \dots, Q^{1}_{(n+1,p)}).$$

Then there exists a universal constant C_0 *such that, for any function* f *in* L^2*, any* $\underline{\lambda}_{n+1} = (\lambda^{n+1}, \dots, \lambda^1)$ *in* $\mathbb{R}^{n+1}$*, one has when* $p > n+1$ *or* $p = 1$ *the bounds*

$$\begin{aligned}&\big\|\sup_{u>0}|M_{n+1}(t,u,\underline{Q}_{n+1},\underline{\lambda}_{n+1})f|\big\|_{L^2}\\&\quad\le C_0^{n+1}\underline{K}_{(n+1,p)}\varepsilon^{\underline{\iota}(n+1,p)}t^{-\underline{m}(n+1,p)}\|f\|_{L^2},\end{aligned}\tag{E.70}$$

where

$$\begin{aligned}\underline{\iota}_{(n+1,p)} &= \sum_{j=1}^{n+1}\iota^j_{(n+1,p)},\\ \underline{m}_{(n+1,p)} &= \sum_{j=1}^{n+1}m^j_{(n+1,p)} - (n+1)\Big(\delta' + \frac{1}{4}\Big),\\ \underline{K}_{(n+1,p)} &= K^1_{(n+1,p)}\cdots K^{n+1}_{(n+1,p)},\end{aligned}\tag{E.71}$$

while if $2 \le p \le n+1$*, one gets instead*

$$\begin{aligned}&\big\|\sup_{u>0}|M_{n+1}(t,u,\underline{Q}_{n+1},\underline{\lambda}_{n+1})f|\big\|_{L^2}\\&\quad\le C_0^{n+1}\underline{K}_{(n+1,p)}\varepsilon^{\underline{\iota}(n+1,p)}t^{-\underline{m}(n+1,p)+\frac{1}{2}-(\delta'+\frac{1}{4})}\|f\|_{L^2}.\end{aligned}\tag{E.72}$$

The proposition will be deduced from the following lemma.

Lemma E.2.6. *Let* $\underline{Q}_{n+1}$ *be as in the statement of Proposition E.2.5. There are* $C > 0$*, a sequence*

$$\underline{Q}_n^{\mathrm{T}} = (Q^{j,\mathrm{T}}_{(n,p)})_{1\le j\le n},$$

with $Q^{j,\mathrm{T}}_{(n,p)}$ *in* $\widetilde{\Sigma}(X^{j,\mathrm{T}}_{(n,p)}) \otimes \mathcal{M}_2(\mathbb{R})$ *with semi-norms* $K^{j,\mathrm{T}}_{(n,p)}$ *satisfying*

$$K^{j,\mathrm{T}}_{(n,p)} \le K^j_{(n+1,p)},\tag{E.73}$$

a sequence

$$\underline{Q}_n^{\mathrm{C}} = (Q^{j,\mathrm{C}}_{(n,p)})_{1\le j\le n},,$$

with $Q^{j,\mathrm{C}}_{(n,p)}$ *in* $\widetilde{\Sigma}(X^{j,\mathrm{C}}_{(n,p)}) \otimes \mathcal{M}_2(\mathbb{R})$ *and semi-norms* $K^{j,\mathrm{C}}_{(n,p)}$ *satisfying*

$$\begin{aligned}K^{j,\mathrm{C}}_{(n,p)} &\le K^j_{(n+1,p)}, \quad j = 1,\dots,n-1,\\ K^{n,\mathrm{C}}_{(n,p)} &\le CK^n_{(n+1,p)}K^{n+1}_{(n+1,p)},\end{aligned}\tag{E.74}$$

such that

$$\begin{aligned}&\big\|\sup_{u>0}|M_{n+1}(t,u,\underline{Q}_{n+1},\underline{\lambda}_{n+1})f|\big\|_{L^2}\\&\quad\le \big\|\sup_{u>0}|M_n(t,u,\underline{Q}_n^{\mathrm{C}},\underline{\lambda}_n^{\mathrm{C}})f|\big\|_{L^2}\\&\qquad + Ct^{-m^{n+1}_{(n+1,p)}+\frac{1}{4}+\delta'}\varepsilon^{\iota^{n+1}_{(n+1,p)}}K^{n+1}_{(n+1,p)}\\&\qquad\quad\times\big\|\sup_{u>0}|M_n(t,u,\underline{Q}_n^{\mathrm{T}},\underline{\lambda}_n^{\mathrm{T}})f|\big\|_{L^2}\end{aligned}\tag{E.75}$$

for other sequences of real numbers $\underline{\lambda}_n^{\mathrm{C}}, \underline{\lambda}_n^{\mathrm{T}}$.

Proof. We apply Lemma E.2.4 with

$$\underline{Q}_n = (Q^n_{(n+1,p)}, \dots, Q^1_{(n+1,p)}),$$
$$Q = Q^{n+1}_{(n+1,p)},$$
$$\underline{Q}_{n-1} = (Q^{n-1}_{(n+1,p)}, \dots, Q^1_{(n+1,p)}).$$

The left-hand side of equation (E.46) is then, according to equation (E.43), equal to $M_{n+1}(t, u, \underline{Q}_{n+1}, \underline{\lambda}_{n+1})$. Let us check that condition (E.45) holds. By (E.63) with n replaced by $n+1$, we have

$$m^{n+1}_{(n+1,p),0} > \frac{1}{4}, \quad m^{n}_{(n+1,p),0} > \frac{1}{4}.$$

We have to check that

$$a^{n+1}_{(n+1,p)} + b^{n+1}_{(n+1,p)} \geq 1,$$

that follows from (E.64) at order $n+1$. Let us check that the first term on the right-hand side of (E.46) may be written as $M_n(t, u, \underline{Q}^{\mathrm{C}}_n, \underline{\lambda}^{\mathrm{C}}_n)$, so that it will provide the first term on the right-hand side of (E.75). We shall define the sequence $\underline{Q}^{\mathrm{C}}_n$ by

$$Q^{n,\mathrm{C}}_{(n,p)} = \tilde{Q}, \; Q^{j,\mathrm{C}}_{(n,p)} = Q^{j}_{(n+1,p)}, \quad j = 1, \dots, n-1, \tag{E.76}$$

where $\tilde{Q}$ is introduced in the statement of Lemma E.2.4. Let us check that we get for the elements of the sequence $(X^{j,\mathrm{C}}_{(n,p)})_{1 \leq j \leq n}$ the expressions in (E.66)–(E.69). For $j = 1, \dots, n-1$, this follows from the definition of $Q^{j,\mathrm{C}}_{(n,p)}$ in (E.76). Consider now $\tilde{Q}$. The class to which it belongs depends on the fact that

$$b^{n+1}_{(n+1,p)} + a^{n}_{(n+1,p)} \geq 1 \tag{E.77}$$

or not. By (E.64) at order $n+1$, (E.77) holds except if $n+1 = p > n$. Consequently, when $n \neq p-1$, we have according to Lemma E.2.4 that $\iota^{n,\mathrm{C}}_{(n,p)}, m^{n,\mathrm{C}}_{(n,p)}, m^{n,\mathrm{C}}_{(n,p),0}$ are given by (E.66)–(E.67) and $a^{n,\mathrm{C}}_{(n,p)}, b^{n,\mathrm{C}}_{(n,p)}$ by (E.69). If $n = p-1$, then we know only that

$$b^{n+1}_{(n+1,p)} + a^{n}_{(n+1,p)} \geq 0,$$

and in this case, the lemma shows that $m^{n,\mathrm{C}}_{(n,p)}$ and $m^{n,\mathrm{C}}_{(n,p),0}$ are given by the expressions in equation (E.68). We thus obtain that the first term on the right-hand side of equation (E.46) is $M_n(t, u, \underline{Q}^{\mathrm{C}}_n, \underline{\lambda}^{\mathrm{C}}_n)$ for a convenient sequence $\underline{\lambda}^{\mathrm{C}}_n$. Moreover, again by Lemma E.2.4, the semi-norm of $\tilde{Q} = Q^{n,\mathrm{C}}_{(n,p)}$ (corresponding to $N = 2$ in (E.35)) is controlled according to the last inequality in (E.74), the case of the semi-norms of

$$Q^{j,\mathrm{C}}_{(n,p)} = Q^{j}_{(n+1,p)}, \quad j = 1, \dots, n-1,$$

being trivial.

We have next to check that the remainder R_n in (E.46) provides the last contribution to (E.75). This follows from (E.47) and the fact that, by definition, $\underline{Q}^{\mathrm{T}}_n$ is the truncated sequence $(Q^n_{(n+1,p)}, \dots, Q^1_{(n,p)})$. This concludes the proof. ■

Proof of Proposition E.2.5. We proceed by induction on n. If $n = 0$, the last statement in Lemma E.2.4 shows that we get (E.70). We assume from now on that $n \geq 1$. Assume that (E.70) and (E.72) have been proved at order n instead of $n + 1$.

Case $p \geq n + 2$. We apply inequality (E.75). On its right-hand side, we may apply the induction hypothesis to $M_n(t, u, \underline{Q}_n^{\mathrm{C}}, \underline{\lambda}_n^{\mathrm{C}})$ and $M_n(t, u, \underline{Q}_n^{\mathrm{T}}, \underline{\lambda}_n^{\mathrm{T}})$. Since $p > n$, it follows that estimate (E.70) (with $n + 1$ replaced by n) for $M_n(t, u, \underline{Q}_n^{\mathrm{C}}, \underline{\lambda}_n^{\mathrm{C}})$ will hold, with $\underline{\iota}_{(n+1,p)}$, $\underline{m}_{(n+1,p)}$, $\underline{K}_{(n+1,p)}$ replaced by

$$\begin{aligned}
\underline{\iota}^{\mathrm{C}}_{(n,p)} &= \sum_{j=1}^{n} \iota^{j,\mathrm{C}}_{(n,p)}, \\
\underline{m}^{\mathrm{C}}_{(n,p)} &= \sum_{j=1}^{n} m^{j,\mathrm{C}}_{(n,p)} - n\Big(\delta' + \frac{1}{4}\Big), \\
\underline{K}^{\mathrm{C}}_{(n,p)} &= \prod_{j=1}^{n} K^{j,\mathrm{C}}_{(n,p)},
\end{aligned}$$

respectively. Using (E.66), (E.67), (E.74), we get a bound of the first term on the right-hand side of (E.75) by

$$C_0^n C \prod_{j=1}^{n+1} K^j_{(n+1,p)} \varepsilon^{\underline{\iota}_{(n+1,p)}} t^{-\underline{m}_{(n+1,p)}} \|f\|_{L^2}. \tag{E.78}$$

On the other hand, if we apply inequality (E.70) (with $n + 1$ replaced by n) to $M_n(t, u, \underline{Q}_n^{\mathrm{T}}, \underline{\lambda}_n^{\mathrm{T}})$ and use (E.73), we bound the last term in (E.75) by

$$C t^{-m^{n+1}_{(n+1,p)} + \frac{1}{4} + \delta'} \varepsilon^{\iota^{n+1}_{(n+1,p)}} K^{n+1}_{(n+1,p)} C_0^n \underline{K}^{\mathrm{T}}_{(n,p)} \varepsilon^{\underline{\iota}^{\mathrm{T}}_{(n,p)}} t^{-\underline{m}^{\mathrm{T}}_{(n,p)}} \|f\|_{L^2}, \tag{E.79}$$

where we denoted

$$\begin{aligned}
\underline{\iota}^{\mathrm{T}}_{(n,p)} &= \sum_{j=1}^{n} \iota^{j,\mathrm{T}}_{(n,p)} = \sum_{j=1}^{n} \iota^{j}_{(n+1,p)}, \\
\underline{m}^{\mathrm{T}}_{(n,p)} &= \sum_{j=1}^{n} m^{j,\mathrm{T}}_{(n,p)} - n\Big(\frac{1}{4} + \delta'\Big) = \sum_{j=1}^{n} m^{j}_{(n+1,p)} - n\Big(\frac{1}{4} + \delta'\Big), \\
\underline{K}^{\mathrm{T}} &= \prod_{j=1}^{n} K^{j,\mathrm{T}}_{(n,p)} = \prod_{j=1}^{n} K^{j}_{(n+1,p)}
\end{aligned}$$

according to the definition of $X^{j,\mathrm{T}}_{(n,p)}$ in (E.65). Taking (E.71) into account, we bound again (E.79) by (E.78).

Case $p = n + 1$. We apply again (E.75). On the right-hand side, the first term may be estimated again from (E.70) with $n + 1$ replaced by $n = p - 1$, since we have

$p > p - 1$. The exponent $\underline{m}^{\mathrm{C}}_{(n,p)}$ of t on the right-hand side will be here

$$\underline{m}^{\mathrm{C}}_{(p-1,p)} = \sum_{j=1}^{p-1} m^{j,\mathrm{C}}_{(p-1,p)} - (p-1)\Big(\delta' + \frac{1}{4}\Big)$$
$$= \sum_{j=1}^{p} m^{j}_{(p,p)} - (p-1)\Big(\delta' + \frac{1}{4}\Big) - \frac{1}{2}$$

according to (E.68). On the other hand, the last term in (E.75) will be estimated by (E.70) at order n instead of $n+1$, and thus by (E.79). We thus get a bound of the form (E.72).

Case $2 \le p \le n$. We apply again (E.75). The first term on the right-hand side may be estimated from the induction hypothesis (E.72), applied with $n+1$ replaced by n, to $M_n(t, u, \underline{Q}^{\mathrm{C}}_n, \underline{\lambda}^{\mathrm{C}}_n)$. Since $n \neq p-1$, the exponent $m^{j,\mathrm{C}}_{(n,p)}$ are given by (E.67), so that

$$\underline{m}^{\mathrm{C}}_{(n,p)} = \sum_{j=1}^{n} m^{j,\mathrm{C}}_{(n,p)} - n\Big(\delta' + \frac{1}{4}\Big) \ge \underline{m}_{(n+1,p)} + \frac{1}{4}$$

which largely allows to bound the first term by

$$C_0^n C \underline{K}_{(n+1,p)} \varepsilon^{\underline{\iota}(n+1,p)} t^{-\underline{m}_{(n+1,p)} + \frac{1}{2} - (\delta' + \frac{1}{4})} \|f\|_{L^2}. \tag{E.80}$$

The second term on the right-hand side of (E.75) is estimated using the induction assumption for $M_n(t, u, \underline{Q}^{\mathrm{T}}_n, \underline{\lambda}^{\mathrm{T}}_n)$, i.e. writing for this expression (E.72) with $n+1$ replaced by n. One gets again a bound of the form (E.80).

Case $p = 1$. In this case, we proceed as when $p > n+1$: We prove (E.70) by induction, using at each step (E.75), and the fact that the condition $n \neq p - 1 = 0$ holding for all $n \ge 1$, we may use at each step (E.67). This concludes the proof. ∎

E.3 Proof of Proposition E.1.3

We shall prove first Sobolev estimates.

Lemma E.3.1. *Let $B_n(t)$ (resp. $C_n(t)$) be given by (E.10) (resp. (E.13)) with $\mathcal{V}(\cdot)$ of the form (E.5), Q_j being in $\Sigma^{\iota,m}_{1,1}$ for some $\iota > 0$, some $m \in \,]0, \frac{1}{2}[$ close to $\frac{1}{2}$ (as in the example following Definition E.1.1). There are $K > 0$, $\delta' > 0$ small, such that for any n in $\mathbb{N}^*$,*

$$\begin{aligned} \|B_n(t)\|_{\mathcal{L}(H^s)} &\le \big(K\varepsilon^{\iota} t^{-(m-\delta'-\frac{1}{4})}\big)^n, \\ \|C_n(t)\|_{\mathcal{L}(H^s)} &\le \big(K\varepsilon^{\iota} t^{-(m-\delta'-\frac{1}{4})}\big)^n. \end{aligned} \tag{E.81}$$

The same conclusion holds true if Q_j is in $\Sigma^{\iota,m}_{2,0}$ for all j or Q_j is in $\Sigma^{\iota,m}_{0,2}$ for all j.

Proof. We shall estimate $\|\langle D_x\rangle^s B_n(t)\langle D_x\rangle^{-s}\|_{\mathcal{L}(L^2)}$. By (E.10),

$$\langle D_x\rangle^s B_n(t)\langle D_x\rangle^{-s} = \int \prod_{j=1}^{n} e^{-i\tau_j P_0}\langle D_x\rangle^s(-i)\mathcal{V}(t+\tau_j)\langle D_x\rangle^{-s} e^{i\tau_j P_0} \times \mathbb{1}_{0<\tau_1<\cdots<\tau_n}\, d\tau_1\cdots\tau_n. \tag{E.82}$$

By (E.5), this may be written as a sum of 5^n terms

$$\sum_{i_1=-2}^{2}\cdots\sum_{i_n=-2}^{2}\int\prod_{j=1}^{n}(-i)e^{-i\tau_j P_0}\langle D_x\rangle^s K_{Q_{i_{n+1-j}}(t+\tau_j)} \times e^{i\tau_j P_0+i(t+\tau_j)\lambda_{i_{n+1-j}}}\langle D_x\rangle^{-s}\mathbb{1}_{0<\tau_1<\cdots<\tau_n}\, d\tau_1\cdots d\tau_n, \tag{E.83}$$

where by assumption Q_{i_j} is an element of $\Sigma^{\iota,m}_{1,1}$ (resp. $\Sigma^{\iota,m}_{2,0}$, resp. $\Sigma^{\iota,m}_{0,2}$) for all j. We shall set $(a,b)=(1,1)$ (resp. $(2,0)$, resp. $(0,2)$). Composing (E.83) by Fourier transform on the left and inverse Fourier transform on the right, as in (E.6), we reduce ourselves to the $\mathcal{L}(L^2)$ boundedness of an operator that may be written, setting $\tau_j = v_j t$ in the integral, as the sum in $i_1,\dots,i_n$ of

$$\int\prod_{j=1}^{n} S(t,v_j,\tilde{Q}_{i_{n+1-j}},\lambda_{i_{n+1-j}})\mathbb{1}_{0<v_1<\cdots<v_n}\, dv_1\cdots dv_n, \tag{E.84}$$

where $\tilde{Q}_{i_{n+1-j}}$ is defined from $Q_{i_{n+1-j}}$ by

$$\tilde{Q}_{i_{n+1-j}}(t,v_j,\xi,\eta) = e^{it\lambda_{i_{n+1-j}}t}\langle\xi\rangle^s Q_{i_{n+1-j}}(t(1+v_j),\xi,\eta)\langle\eta\rangle^{-s} \tag{E.85}$$

and $S(t,v_j,\tilde{Q}_{i_{n+1-j}},\lambda_{i_{n+1-j}})$ is defined in (E.36). Since $Q_{i_{n+1-j}}$ belongs to the class $\Sigma^{\iota,m}_{a,b}$ of Definition E.1.1, $\tilde{Q}_{i_{n+1-j}}$ is in the class $\widetilde{\Sigma}^{\iota,m,m_0}_{a,b}$ of Definition E.2.3, taking for m_0 any number $m_0\le m$. Since m is taken close to $\frac{1}{2}$, we may assume that $m_0>\frac{3}{8}$. In other words, the integral in (E.84) is of the form $M_n(t,0,\underline{\tilde{Q}}^n,\underline{\lambda}^n)$, with notation (E.43) with $\underline{\tilde{Q}}=(\tilde{Q}_{i_n},\dots,\tilde{Q}_{i_1})$.

We shall apply Proposition E.2.5 with $n+1$ replaced by n and $p=n+1$. This is possible since, if in condition (E.64), $a_j=b_j=1$ for all j, or $a_j=2, b_j=0$ for all j, or $a_j=0, b_j=2$ for all j, inequality $a_{n'}+b_{n''}\ge 1$ is always satisfied. We deduce from (E.70) that the $\mathcal{L}(L^2)$ norm of (E.84) is bounded from above by

$$\left(\tilde{K}\varepsilon^\iota t^{-(m-\delta'-\frac{1}{4})}\right)^n$$

for some $\tilde{K}>0$. Since we have 5^n terms in the sum (E.83), (E.81) follows for $B_n(t)$. Since according to (E.13), $C_n(t)$ may be written as $B_n(t)^*$ for some $B_n(t)$ of the form (E.10), we get also the second estimate of (E.81).

This concludes the proof. ∎

We want next to obtain $\mathcal{L}(L^2)$ bounds for $L\circ C_n(t)$, where L is defined in equation (E.17). We compute first the composition between L and an operator of the form $e^{-i\tau P_0}\mathcal{V}(t+\tau)e^{i\tau P_0}$, where $\mathcal{V}$ is of the form (E.5).

Lemma E.3.2. *Let Q be a 2×2 matrix of functions in the class $\Sigma_{1,1}^{\iota,m}$ of Definition E.1.1. Let λ be in $\mathbb{R}$ and set $\mathcal{V}_Q(t) = e^{i\lambda t} K_Q$ according to notation (E.5)–(E.6). Then one may find 2×2 matrices Q' (resp. Q'') with entries in $\Sigma_{2,0}^{\iota,m}$ (resp. $\Sigma_{2,0}^{\iota,m}$ or $\Sigma_{0,1}^{\iota,m}$) such that*

$$\begin{aligned} &L \circ \big(e^{-i\tau P_0} \mathcal{V}_Q(t+\tau) e^{i\tau P_0}\big) \\ &\quad = \big(e^{-i\tau P_0} \mathcal{V}_{Q'}(t+\tau) e^{i\tau P_0}\big) \circ L + \big(e^{-i\tau P_0} \mathcal{V}_{Q''}(t+\tau) e^{i\tau P_0}\big). \end{aligned} \tag{E.86}$$

Proof. Using notation (E.37), we write

$$Q(t,\xi,\eta) = \sum_{j=1}^{2} \sum_{k=1}^{2} q_{jk}(t,\xi,\eta) E_{jk} \frac{\xi}{\langle \xi \rangle} \frac{\eta}{\langle \eta \rangle}$$

with q_{jk} in $\Sigma_{0,0}^{\iota,m}$. We have to compute the action of L on the operator with kernel

$$\begin{aligned} \sum_{1 \le j,k \le 2} \frac{e^{i\lambda(t+\tau)}}{2\pi} \int e^{i(x\xi - y\eta) + i\tau((-1)^j p(\xi) - (-1)^k p(\eta))} E_{jk} \\ \times \frac{\xi}{\langle \xi \rangle} \frac{\eta}{\langle \eta \rangle} q_{jk}(t+\tau,\xi,\eta)\, d\xi\, d\eta. \end{aligned} \tag{E.87}$$

One gets, using expression (E.17) of L,

$$\begin{aligned} \sum_{1 \le j,k \le 2} \frac{e^{i\lambda(t+\tau)}}{2\pi} \int e^{i(x\xi - y\eta) + i\tau((-1)^j p(\xi) - (-1)^k p(\eta))} E_{jk} \\ \times \big(x + (-1)^{j+1} t p'(\xi)\big) \frac{\xi}{\langle \xi \rangle} \frac{\eta}{\langle \eta \rangle} q_{jk}(t+\tau,\xi,\eta)\, d\xi d\eta. \end{aligned} \tag{E.88}$$

As $p'(\xi) = \frac{\xi}{\langle \xi \rangle}$, we have

$$\begin{aligned} \big(x + (-1)^{j+1} t p'(\xi)\big) \frac{\xi}{\langle \xi \rangle} \frac{\eta}{\langle \eta \rangle} &= (-1)^j \frac{\xi}{\langle \xi \rangle} \Big(x \frac{\eta}{\langle \eta \rangle} (-1)^j - y \frac{\xi}{\langle \xi \rangle} (-1)^k\Big) \\ &\quad + (-1)^{j+k} \frac{\xi^2}{\langle \xi \rangle^2} \big(y + (-1)^{k+1} t p'(\eta)\big). \end{aligned} \tag{E.89}$$

We plug (E.89) in (E.88). The last term in (E.89) gives an expression of the form of the first term on the right-hand side of (E.86), where the operator $e^{-i\tau P_0} \mathcal{V}_{Q'}(t+\tau) e^{i\tau P_0}$ is given by an expression of the form (E.87), with $\frac{\xi}{\langle \xi \rangle} \frac{\eta}{\langle \eta \rangle} q_{jk}$ replaced by

$$(-1)^{j+k} \frac{\xi^2}{\langle \xi \rangle^2} q_{jk},$$

i.e. Q' is given by

$$Q'(t,\xi,\eta) = \sum_{j=1}^{2} \sum_{k=1}^{2} q_{jk}(t,\xi,\eta) (-1)^{j+k} E_{jk} \frac{\xi^2}{\langle \xi \rangle^2}.$$

This is an element of $\Sigma_{2,0}^{\iota,m}$ as wanted.

On the other hand, if we plug the first term of the right-hand side of (E.89) in (E.88) and perform one integration by parts, we get

$$(-1)^{j+1}\sum_{j=1}^{2}\sum_{k=1}^{2}\frac{e^{i\lambda(t+\tau)}}{2\pi}\int e^{i(x\xi-y\eta)+i\tau((-1)^j p(\xi)-(-1)^k p(\eta))} \\ \times\Big((-1)^j\frac{\eta}{\langle\eta\rangle}D_\xi+(-1)^k\frac{\xi}{\langle\xi\rangle}D_\eta\Big)\Big(\frac{\xi}{\langle\xi\rangle}q_{jk}(t+\tau,\xi,\eta)\Big)\,d\xi d\eta.$$

We get an operator of the form of the last term in (E.86), with a symbol Q'' that may be written as the sum of an element in $\Sigma^{\iota,m}_{2,0}$ and an element in $\Sigma^{\iota,m}_{0,1}$. This concludes the proof of the lemma. ■

We may prove now the following statement.

Lemma E.3.3. *For any n in $\mathbb{N}^*$, one may find operators $C_n^p(t)$, $0\le p\le n$, such that*

$$L\circ C_n(t)=C_n^0(t)\circ L+\sum_{p=1}^{n}C_n^p(t) \tag{E.90}$$

which have the following structure: Operator $C_n^0(t)$ is of the form

$$\int\prod_{j=1}^{n}e^{-i\tau_j P_0}i\mathcal{V}'(t+\tau_j)e^{i\tau_j P_0}\mathbb{1}_{0<\tau_n<\cdots<\tau_1}\,d\tau_1\cdots d\tau_n, \tag{E.91}$$

where $\mathcal{V}'(t)=\sum_{\ell=-2}^{2}e^{i\lambda_\ell t}K_{Q'_\ell}$, with Q'_ℓ matrices with entries in $\Sigma^{\iota,m}_{2,0}$. Operator $C_n^p(t)$ for $1\le p\le n$ has the structure

$$\int\prod_{j=1}^{p-1}e^{-i\tau_j P_0}i\mathcal{V}'(t+\tau_j)e^{i\tau_j P_0}\times e^{-i\tau_p P_0}i\mathcal{V}''(t+\tau_p)e^{i\tau_p P_0} \\ \times\prod_{j=p+1}^{n}e^{-i\tau_j P_0}i\mathcal{V}(t+\tau_j)e^{i\tau_j P_0}\mathbb{1}_{0<\tau_n<\cdots<\tau_1}\,d\tau_1\cdots d\tau_n, \tag{E.92}$$

where $\mathcal{V}$ is as in (E.5), $\mathcal{V}'$ is as above and $\mathcal{V}''$ is a sum $\mathcal{V}''(t)=\sum_{\ell=-2}^{2}e^{i\lambda_\ell t}K_{Q''_\ell}$, with Q''_ℓ matrices with entries in $\Sigma^{\iota,m}_{2,0}$ or $\Sigma^{\iota,m}_{0,1}$. Moreover, one has the estimates

$$\|C_n^0(t)\|_{\mathcal{L}(L^2)}\le\big(\tilde{K}\varepsilon^\iota t^{\delta'+\frac14-m}\big)^n, \tag{E.93}$$

$$\|C_n^p(t)\|_{\mathcal{L}(L^2)}\le\big(\tilde{K}\varepsilon^\iota t^{\delta'+\frac14-m}\big)^n t^{\frac12-\left(\delta'+\frac14\right)},\ 1\le p\le n. \tag{E.94}$$

Proof. We start from expression (E.13) of $C_n(t)$. If we compose at the left with L and use (E.86), we obtain the sum of an expression of the form (E.92) with $p=1$ and a quantity of the form (E.13), with the product replaced by

$$e^{-i\tau_1 P_0}i\mathcal{V}'(t+\tau_1)e^{i\tau_1 P_0}\circ L\circ\prod_{j=2}^{n}e^{-i\tau_j P_0}i\mathcal{V}(t+\tau_j)e^{i\tau_j P_0}. \tag{E.95}$$

If we iterate, we obtain $C_n^0(t) \circ L$ with $C_n^0(t)$ given by (E.91) and the sum for p going from 1 to n of (E.92).

We have next to obtain (E.93) and (E.94). By duality, we may replace (E.91) by

$$(-1)^n \int \prod_{j=1}^{n} e^{-i\tau_j P_0} i \mathcal{V}'(t+\tau_j)^* e^{i\tau_j P_0} \mathbb{1}_{0<\tau_1<\cdots<\tau_n} \, d\tau_1 \cdots d\tau_n \tag{E.96}$$

and (E.92) by

$$\begin{aligned}(-1)^n \int \prod_{j=1}^{n-p} & e^{-i\tau_j P_0} i \mathcal{V}(t+\tau_j)^* e^{i\tau_j P_0} \\ & \times e^{-i\tau_{n+1-p} P_0} i \mathcal{V}''(t+\tau_{n+1-p})^* e^{i\tau_{n+1-p} P_0} \\ & \times \prod_{j=n+2-p}^{n} e^{-i\tau_j P_0} i \mathcal{V}'(t+\tau_j)^* e^{i\tau_j P_0} \mathbb{1}_{0<\tau_1<\cdots<\tau_n} \, d\tau_1 \cdots d\tau_n\end{aligned} \tag{E.97}$$

for $1 \leq p \leq n$.

Consider first (E.96). We have an operator of the form (E.83) (with $s = 0$) whose $\mathcal{L}(L^2)$ boundedness reduces to the one of an expression of the form (E.84) in terms of symbols $\tilde{Q}_{i_{n+1-j}}$ given by (E.85) from symbols in the class $\Sigma_{0,2}^{\iota,m}$ because of the definition of $\mathcal{V}'(t+\tau_j)$. It follows from the last statement in Lemma E.3.1 that the same estimate as (E.81) holds, which gives a bound of the $\mathcal{L}(L^2)$ norm of (E.96) by the right-hand side of (E.93).

Let us study expression (E.97) and show that its $\mathcal{L}(L^2)$ norm is bounded from above by the right-hand side of (E.94). Operator (E.97) is of the form (E.84), with a sequence of symbols $(\tilde{Q}_{i_n}, \ldots, \tilde{Q}_{i_1})$ with $\tilde{Q}_{i_j}$ belonging to the classes $\widetilde{\Sigma}_{a_j,b_j}^{\iota,m,m_0}$, where $(a_j, b_j)_{1\leq j\leq n}$ has the following form:

$$\begin{aligned}(a_n, b_n) = (1,1),\ \ldots,\ (a_{p+1}, b_{p+1}) = (1,1),\ (a_p, b_p) = (0,2) \text{ or } (1,0),\\ (a_{p-1}, b_{p-1}) = (0,2),\ \ldots,\ (a_1, b_1) = (0,2).\end{aligned} \tag{E.98}$$

The only couples (j', j'') such that $a_{j'} + b_{j''}$ may be eventually equal to zero are those with $j' < j'' = p$, i.e. those for which condition (E.64) is satisfied. We thus obtain that (E.97) is of the form (E.84) and has $\mathcal{L}(L^2)$ norm bounded from above by (E.70) and (E.72), so by the right-hand side of (E.94). This concludes the proof. ∎

Proof of Proposition E.1.3. Since m is taken close to $\frac{1}{2}$ and δ' close to zero, the exponent of t on the right-hand side of (E.81) is negative. As $\iota > 0$, for ε small enough, we have

$$\|B_n(t)\|_{\mathcal{L}(H^s)} \leq \frac{1}{2^n}, \quad \|C_n(t)\|_{\mathcal{L}(H^s)} \leq \frac{1}{2^n}.$$

In particular, (E.11) and its counterpart for $C_n(t)$ holds, so that $B(t)$ and $C(t)$ are well defined, bounded on H^s and satisfy (E.19)

Since by (E.93), $\|C_n^0(t)\|_{\mathcal{L}(L^2)}$ satisfies the same estimate as $\|B_n(t)\|_{\mathcal{L}(H^s)}$ and $\|C_n(t)\|_{\mathcal{L}(H^s)}$, the operator

$$\tilde{C}(t) = \mathrm{Id} + \sum_{n=1}^{+\infty} C_n^0(t)$$

is well defined and satisfies (E.21). We notice next that if we set for $n \geq 1$,

$$\tilde{C}_{1,n}(t) = \sum_{p=1}^{n} C_n^p(t),$$

we have by (E.94)

$$\|\tilde{C}_{1,n}(t)\|_{\mathcal{L}(L^2)} \leq Cn(\tilde{K}\varepsilon^\iota)^n t^{(n-1)(\delta'+\frac{1}{4}-m)} t^{\frac{1}{2}-m}.$$

Since $\delta' + \frac{1}{4} - m < 0$, we get after summation estimate (E.22) for

$$\tilde{C}_1(t) = \sum_{n=1}^{+\infty} \tilde{C}_{1,n}(t).$$

We still have to check the last assertions of the proposition. To prove (E.23), it suffices to check that for any n, $N_0 B_n(t) = \overline{B_n(t)} N_0$ for any n, and the corresponding equality for $C_n(t)$. Because of (E.10) and (E.13), it is enough to show that

$$N_0 e^{-i\tau P_0} \mathcal{V}(t+\tau) e^{i\tau P_0} = -e^{i\tau P_0} \overline{\mathcal{V}(t+\tau)} e^{-i\tau P_0} N_0.$$

But this equality follows from (E.7) and the fact that $N_0 e^{i\tau P_0} = e^{-i\tau P_0} N_0$.

Moreover, if $\mathcal{V}$ preserves the space of odd functions, so do $B_n(t)$ and $C_n(t)$ because of their definition, and of the fact that P_0 preserves such spaces. This concludes the proof. ∎

Appendix F

Division lemmas and normal forms

We have discussed in Section 1.6 normal forms for an equation of the form

$$\big(D_t - p(D_x)\big)u = N(u),$$

where $p(\xi) = \sqrt{1+\xi^2}$ and $N(u)$ is some polynomial in $u, \bar{u}$. We distinguish among the monomials of u the characteristic ones, that are those of the form

$$u^{p+1}\bar{u}^p = |u|^{2p}u$$

and the non-characteristics ones, of the form $u^p\bar{u}^q$ with $p-q \neq 1$. We have seen that if $L_+ = x + tp'(D_x)$, a characteristic monomial will satisfy essentially an equality of the form

$$L_+(|u|^{2p}u) = (p+1)(L_+u)|u|^{2p} - pu^{p+1}\bar{u}^{p-1}\overline{L_+u} + \text{remainder}, \tag{F.1}$$

that allows one to obtain for the L^2 norm of the left side a bound in $\|u\|_{L^\infty}^{2p}\|L_+u\|_{L^2}$.

Our first goal in this appendix is to give a proof of inequalities of that form for more general characteristic nonlinearities, given in terms of the kind of non-local multilinear operators that we have to use in the proof of the main theorem of the book. Section F.2 below is devoted to that, except that we put ourselves in the semiclassical framework that is very convenient for the proofs.

For non-characteristic nonlinearities, (F.1) non-longer works, and as explained in Section 1.6, one has then to eliminate such nonlinearities by space-time normal forms. We perform in section (F.4) these space-time normal forms in the semiclassical framework, for general non-characteristic nonlinearities given by the multilinear pseudo-differential operators introduced in Appendix B. The method is the one outlined in Section 1.6, extended to these general multilinear expressions. We make also normal forms for quadratic contributions given in terms of symbols with space decaying symbols, along the lines of the end of Section 2.7.

F.1 Division lemmas

We establish in this section some division lemmas, which are variants of similar results obtained in [20].

Definition F.1.1. For n in $\mathbb{N}^*$, denote by Γ_n the set of multi-indices $I = (i_1, \dots, i_n)$ with $i_j = \pm 1$ for $j = 1, \dots, n$. Denote by Γ_n^{ch} the subset of Γ_n made by those $I = (i_1, \dots, i_n)$ such that $\sum_{j=1}^n i_j = 1$ and $\Gamma_n^{\text{nch}} = \Gamma_n - \Gamma_n^{\text{ch}}$.

Let us fix some notation. If $I = (i_1, \dots, i_n)$ is in Γ_n and as above

$$p(\xi) = \sqrt{1+\xi^2},$$

we define

$$g_I(\xi_1, \dots, \xi_n) = -p\big(\xi_1 + \dots + \xi_n\big) + \sum_{j=1}^{n} i_j p(\xi_j). \tag{F.2}$$

Set also $\varphi(x) = \sqrt{1-x^2}$ for $|x| < 1$, so that by [20, Lemma 1.8], if $\gamma \in C_0^\infty(\mathbb{R})$ has small enough support

$$\begin{aligned} a_\pm(x,\xi) &= \frac{x \pm p'(\xi)}{\xi \mp d\varphi(x)} \gamma\big(\langle\xi\rangle^2 (x \pm p'(\xi))\big), \\ b_\pm(x,\xi) &= \frac{\xi \mp d\varphi(x)}{x \pm p'(\xi)} \gamma\big(\langle\xi\rangle^2 (x \pm p'(\xi))\big) \end{aligned} \tag{F.3}$$

satisfy estimates

$$\begin{aligned} |\partial_x^\alpha \partial_\xi^\beta a_\pm(x,\xi)| &\le C_{\alpha\beta} \langle\xi\rangle^{-3+2|\alpha|-|\beta|}, \\ |\partial_x^\alpha \partial_\xi^\beta b_\pm(x,\xi)| &\le C_{\alpha\beta} \langle\xi\rangle^{3+2|\alpha|-|\beta|}. \end{aligned} \tag{F.4}$$

Proposition F.1.2. *Recall notation (B.10) for the function $M_0(\xi_1, \dots, \xi_n)$ and the class of symbols introduced in Definition B.1.2 for $\beta \ge 0$, $\kappa \ge 0$. Let $\nu \ge 0$.*

(i) *Let I be a multi-index in $(i_1, \dots, i_\ell)$ be in Γ_n and let m_I be a symbol in $S_{1,\beta}(\prod_{j=1}^n \langle\xi_j\rangle^{-1} M_0(\xi)^\nu, n)$. Then we may find symbols*

$$m_{I,\ell} \in S_{4,\beta}\Bigg(\prod_{j=1}^{n} \langle\xi_j\rangle^{-1} M_0(\xi)^{4+\nu} \langle x\rangle^{-1}, n\Bigg), \quad \ell = 1, \dots, n, \tag{F.5}$$

such that if γ is in $C_0^\infty(\mathbb{R})$ and has small enough support, one may write

$$\begin{aligned} &m_I(y, x, \xi_1, \dots, \xi_n) \\ &\quad = m_I(y, x, \xi_1, \dots, \xi_n) \prod_{\ell=1}^{n} \gamma\big(M_0(\xi)^4 (x + i_\ell p'(\xi_\ell))\big) \\ &\qquad + \sum_{\ell=1}^{n} (x + i_\ell p'(\xi_\ell)) m_{I,\ell}(y, x, \xi_1, \dots, \xi_n). \end{aligned} \tag{F.6}$$

(ii) *Assume that I is in Γ_n^{nch}. Then we may find a symbol*

$$a_I \in S_{4,\beta}\Bigg(\prod_{j=1}^{n} \langle\xi_j\rangle^{-1} M_0(\xi)^\nu \langle x\rangle^{-\infty}, n\Bigg) \tag{F.7}$$

and symbols $m_{I,j}$ as in (F.5) such that

$$\begin{aligned} m_I(y, x, \xi_1, \dots, \xi_n) &= g_I(\xi_1, \dots, \xi_n) a_I(y, x, \xi_1, \dots, \xi_n) \\ &\quad + \sum_{\ell=1}^{n} (x + i_\ell p'(\xi_\ell)) m_{I,\ell}(y, x, \xi_1, \dots, \xi_n). \end{aligned} \tag{F.8}$$

Proof. Define

$$m_{I,1}(y,x,\xi_1,\dots,\xi_n) = m_I(y,x,\xi_1,\dots,\xi_n)\frac{(1-\gamma)(M_0(\xi)^4(x+i_1p'(\xi_1)))}{x+i_1p'(\xi_1)},$$
$$m_I^{(1)}(y,x,\xi_1,\dots,\xi_n) = m_I(y,x,\xi_1,\dots,\xi_n)\gamma\big(M_0(\xi)^4(x+i_1p'(\xi_1))\big)$$

and write

$$m_I(y,x,\xi_1,\dots,\xi_n) = m_1^{(1)}(y,x,\xi_1,\dots,\xi_n) + m_{I,1}(y,x,\xi_1,\dots,\xi_n)(x+i_1p'(\xi_1)).$$

Then $m_{I,1}$ satisfies (F.5), and repeating the process with m_I replaced by $m_{I,1}$, successively with respect to $\xi_2,\dots,\xi_n$, we get (F.6).

(ii) Equality (F.8) is obtained from (F.6) defining

$$a_I = m_I g_I^{-1}\prod_{j=1}^n \gamma\big(M_0(\xi)^4(x+i_\ell p'(\xi_\ell))\big) \tag{F.9}$$

and showing that a_I belongs to $S_{4,\beta}\big(\prod_{j=1}^n\langle\xi_j\rangle^{-1}M_0(\xi)^{\nu+1}\langle x\rangle^{-\infty},n\big)$. This is done in [20, proof of (i) of Proposition 2.2] (with the parameter κ in that reference set to 2). ∎

F.2 Commutation results

We study now the action of the operator $\mathcal{L}_+ = \frac{1}{h}\mathrm{Op}_h(x+p'(\xi))$ introduced in (D.8) on characteristic terms.

Proposition F.2.1. *Let I be in Γ_n^{ch} for some (odd) $n \geq 3$ and let ν be nonnegative. Let m_I be an element of $S_{1,\beta}(\prod_{j=1}^n\langle\xi_j\rangle^{-1}M_0(\xi)^\nu,n)$ with $\beta>0$. Then, for some new value of ν, there are symbols $m_{I,j}$ in $S_{4,\beta}(\prod_{j=1}^n\langle\xi_j\rangle^{-1}M_0^\nu,n)$, $j=1,\dots,n$, r in $S_{4,\beta}(\prod_{j=1}^n\langle\xi_j\rangle^{-1}M_0^\nu,n)$, r' in $S'_{4,\beta}(\prod_{j=1}^n\langle\xi_j\rangle^{-1}M_0^\nu,n)$, such that for any functions $\underline{v}_1,\dots,\underline{v}_n$,*

$$\begin{aligned}\mathcal{L}_+\mathrm{Op}_h(m_I)(\underline{v}_1,\dots,\underline{v}_n) = \sum_{j=1}^n &\mathrm{Op}_h(m_{I,j})(\underline{v}_1,\dots,\mathcal{L}_{i_j}\underline{v}_j,\dots,\underline{v}_n)\\ &+\mathrm{Op}_h(r)(\underline{v}_1,\dots,\underline{v}_n)\\ &+\frac{1}{h}\mathrm{Op}_h(r')(\underline{v}_1,\dots,\underline{v}_n).\end{aligned} \tag{F.10}$$

Proof. We write decomposition (F.6) of m_I, denoting the first term on the right-hand side by $m_I^{(1)}$. This is an element of $S_{4,\beta}(\prod_{j=1}^n\langle\xi_j\rangle^{-1}M_0^\nu,n)$ supported in

$$\bigcap_{\ell=1}^n\{(y,x,\xi_1,\dots,\xi_n) : |x+i_\ell p'(\xi_\ell)| < \alpha M_0(\xi_1,\dots,\xi_n)^{-4}\} \tag{F.11}$$

for some small $\alpha > 0$. It is proved in the proof of [20, Proposition 2.2] that on domain (F.11), one has $|\xi_\ell| \leq CM_0(\xi)$ for any $\ell = 1, \dots, n$ and that $\langle d\varphi(x)\rangle \sim M_0(\xi)$ (see [20, formulas (2.10)–(2.13), and the lines following them as well as Lemma 1.8]). Let us show that

$$\begin{aligned} m_I^{(1)}(y, x, \xi_1, \dots, \xi_n)\Big(p'(\xi_1 + \dots + \xi_n) - \sum_{j=1}^n p'(\xi_j)\Big) \\ = \sum_{j=1}^n m_{I,j}(y, x, \xi_1, \dots, \xi_n)(x + i_j p'(\xi_j)) \end{aligned} \tag{F.12}$$

for symbols $m_{I,j}$ in $S_{4,\beta}(\prod_{j=1}^n \langle \xi_j\rangle^{-1} M_0(\xi)^{3+\nu}\langle x\rangle^{-\infty}, n)$. Actually, expanding the bracket in the left hand side of (F.12) on $\xi_j = i_j d\varphi(x)$, $j = 1, \dots, n$ and using $\sum_{j=1}^n i_j = 1$, one may write the left-hand side of (F.12) as

$$\sum_{j=1}^n m_I^{(1)}(y, x, \xi_1, \dots, \xi_n)(\xi_j - i_j d\varphi(x))\tilde{e}_j(x, \xi) \tag{F.13}$$

with

$$\begin{aligned} \tilde{e}_j(x, \xi) = \int_0^1 \Big(p''\big((1-\mu)d\varphi(x) + \mu(\xi_1 + \dots + \xi_n)\big) \\ - \sum_{j=1}^n p''\big((1-\mu)i_j d\varphi(x) + \mu\xi_j\big)\Big)\, d\mu. \end{aligned} \tag{F.14}$$

Notice that on the set (F.11) containing the support of $m_I^{(1)}$, x stays for any ξ in a compact subset of $]-1, 1[$ and that for any α in $\mathbb{N}^*$,

$$\langle \partial^\alpha d\varphi(x)\rangle = O\big(\langle d\varphi(x)\rangle^{1+2\alpha}\big) = O\big(M_0(\xi)^{1+2\alpha}\big) = O\big(M_0(\xi)^{3\alpha}\big),$$

so that each ∂_x^α-derivative of $\tilde{e}_j(x, \xi)$ is $O(M_0(\xi)^{3\alpha})$ on that support. Moreover, we may write using (F.3)

$$(\xi_j - i_j d\varphi(x))\tilde{e}_j(x, \xi) = (x + i_j p'(\xi_j))b_+(x, \xi_j)\tilde{e}_j(x, \xi)$$

if (x, ξ) stays in (F.11) and the function γ in (F.3) is conveniently chosen. Plugging this in (F.13) and defining

$$m_{I,j}(y, x, \xi_1, \dots, \xi_n) = m_I^{(1)}(y, x, \xi_1, \dots, \xi_n)b_+(x, \xi_j)\tilde{e}_j(x, \xi),$$

we get (F.12), with a symbol $m_{I,j}$ in the wanted class because of (F.4) and of the fact that $|\xi_j| = O(M_0(\xi))$ on (F.11). We use now Proposition B.2.1 to write

$$\begin{aligned} &\mathrm{Op}_h(p'(\xi)) \circ \mathrm{Op}_h\big(m_I^{(1)}(y, x, \xi_1, \dots, \xi_n)\big) \\ &\quad = \mathrm{Op}_h\big(p'(\xi_1 + \dots + \xi_n)m_I^{(1)}(y, x, \xi_1, \dots, \xi_n)\big) \\ &\qquad + h\mathrm{Op}_h\big(r_1(y, x, \xi_1, \dots, \xi_n)\big) + \mathrm{Op}_h\big(r_1'(y, x, \xi_1, \dots, \xi_n)\big) \end{aligned} \tag{F.15}$$

with r_1 in $S_{4,\beta}(\prod_{j=1}^n \langle\xi_j\rangle^{-1} M_0^\nu, n)$, and r_1' in $S_{4,\beta}'(\prod_{j=1}^n \langle\xi_j\rangle^{-1} M_0^\nu, n)$ for some ν. Using (F.12), we may rewrite the first term on the right-hand side as

$$\sum_{j=1}^n \mathrm{Op}_h\big(m_I^{(1)}(y,x,\xi_1,\dots,\xi_n)p'(\xi_j)\big) \\ + \sum_{j=1}^n \mathrm{Op}_h\big(m_{I,j}(y,x,\xi_1,\dots,\xi_n)(x+i_j p'(\xi_j))\big). \tag{F.16}$$

Using that $\sum_{j=1}^n i_j = 1$, and that $\mathcal{L}_+ = \frac{1}{h}\mathrm{Op}_h(x+p'(\xi))$, it follows from (F.6), (F.15), (F.16) and Proposition B.2.1 that $\mathcal{L}_+\mathrm{Op}_h(m_I)$ is the sum of terms of the following form:

$$\begin{aligned} &\frac{i_j}{h}\mathrm{Op}_h\big(m_I^{(1)}(y,x,\xi_1,\dots,\xi_n)(x+i_j p'(\xi_j))\big), \quad j=1,\dots,n,\\ &\frac{1}{h}\mathrm{Op}_h\big(m_{I,j}(y,x,\xi_1,\dots,\xi_n)(x+i_j p'(\xi_j))\big), \quad j=1,\dots,n,\\ &\mathrm{Op}_h\big(r_1(y,x,\xi_1,\dots,\xi_n)\big) + \frac{1}{h}\mathrm{Op}_h\big(r_1'(y,x,\xi_1,\dots,\xi_n)\big) \end{aligned} \tag{F.17}$$

with $m_{I,j}$ in $S_{4,\beta}\big(\prod_{j=1}^n \langle\xi_j\rangle^{-1} M_0(\xi)^\nu \langle x\rangle^{-1}, n\big)$ coming from (F.6) or (F.16). To conclude the proof, we just have to apply again Proposition B.2.1 to the first two lines of (F.17), in order to rewrite them as the sum on the right-hand side of (F.10), up to new contributions to the remainders. ∎

In the non-characteristic case, we cannot expect an equality of the form (F.10). Instead, we shall have:

Corollary F.2.2. *Let I be in Γ_n^{nch}. Then there are symbols $m_{I,j}$, r, r' as in the statement of Proposition F.2.1 and a symbol r_1 in $S_{4,\beta}(\prod_{j=1}^n \langle\xi_j\rangle^{-1} M_0^\nu, n)$ for some ν, such that*

$$\begin{aligned} \mathcal{L}_+\mathrm{Op}_h(m_I)(\underline{v}_1,\dots,\underline{v}_n) = &\sum_{j=1}^n \mathrm{Op}_h(m_{I,j})(\underline{v}_1,\dots,\mathcal{L}_{i_j}\underline{v}_j,\dots,\underline{v}_n)\\ &+ \mathrm{Op}_h(r)(\underline{v}_1,\dots,\underline{v}_n)\\ &+ \frac{1}{h}\mathrm{Op}_h(r')(\underline{v}_1,\dots,\underline{v}_n)\\ &+ \frac{x}{h}\mathrm{Op}_h(r_1)(\underline{v}_1,\dots,\underline{v}_n). \end{aligned} \tag{F.18}$$

Proof. We may reproduce the proof of Proposition F.2.1, except that, when Taylor expanding the bracket on the left-hand side of (F.12) on $\xi_j = i_j d\varphi(x)$, we shall get the right-hand side of this equality and the extra term

$$m_I^{(1)}(y,x,\xi_1,\dots,\xi_n)\Big(p'\Big(\sum_{j=1}^n i_j d\varphi(x)\Big) - \sum_{j=1}^n p'\big(i_j d\varphi(x)\big)\Big) \tag{F.19}$$

which does not vanish if $\sum_{j=1}^{n} i_j \neq 1$. Since

$$p'(\xi) = \frac{\xi}{\langle \xi \rangle} \quad \text{and} \quad d\varphi(x) = -x\langle d\varphi(x)\rangle,$$

with $\langle d\varphi(x)\rangle = O(M_0(\xi))$ on the support of $m_I^{(1)}$, we see that (F.19) may be written as xr_1 for some r_1 as in the statement. This gives the last contribution to (F.18), the preceding ones being those furnished by the proof of Proposition F.2.1. ■

The last term in (F.18) does not enjoy nice estimates. Because of that, non-characteristic terms have to be eliminated by normal forms. We describe such normal forms in next section.

F.3 Normal forms for non-characteristic terms

Proposition F.3.1. *With the notation and under the assumptions of* (ii) *of Proposition* F.1.2, *one may write for any* $\underline{v}_1, \dots, \underline{v}_n$,

$$\begin{aligned} \Big(D_t - \mathrm{Op}_h\Big(x\xi + p(\xi) - in\frac{h}{2}\Big)\Big)&\mathrm{Op}_h(a_I)(\underline{v}_1, \dots, \underline{v}_n) \\ = \mathrm{Op}_h(m_I)&(\underline{v}_1, \dots, \underline{v}_n) \\ &+ \sum_{j=1}^{n} \mathrm{Op}_h(a_I)[\underline{v}_1, \dots, (D_t - \mathrm{Op}_h(\lambda_{i_j}))\underline{v}_j, \dots, \underline{v}_n] \\ &+ \underline{R}(\underline{v}_1, \dots, \underline{v}_n), \end{aligned} \tag{F.20}$$

where $\lambda_{i_j}(x,\xi) = x\xi + i_j p(\xi) - \frac{i}{2}h$, *and where* $\underline{R}$ *is the sum of terms of the following form*

$$\begin{aligned} &h\mathrm{Op}_h(m_{I,j})(\underline{v}_1, \dots, \mathcal{L}_{i_j}\underline{v}_j, \dots, \underline{v}_n), \quad 1 \le j \le n, \\ &\mathrm{Op}_h(r_I')(\underline{v}_1, \dots, \underline{v}_n), \\ &h\mathrm{Op}_h(r_I)(\underline{v}_1, \dots, \underline{v}_n), \end{aligned} \tag{F.21}$$

where $m_{I,j}$ *is a symbol in* $S_{4,\beta}(\prod_{j=1}^{n}\langle \xi_j\rangle^{-1}M_0^{\nu}\langle x\rangle^{-1}, n)$, r_I *(resp.* r_I' *) belongs to the class* $S_{4,\beta}(\prod_{j=1}^{n}\langle \xi_j\rangle^{-1}M_0^{\nu}\langle x\rangle^{-\infty}, n)$ *(resp.* $S'_{4,\beta}(\prod_{j=1}^{n}\langle \xi_j\rangle^{-1}M_0^{\nu}, n)$*) for some* ν. *The first line in* (F.21) *may also be written as*

$$\mathrm{Op}_h(r_I^1)(\underline{v}_1, \dots, \underline{v}_n) \tag{F.22}$$

for a symbol r_I^1 *in* $S_{4,\beta}(\prod_{j=1}^{n}\langle \xi_j\rangle^{-1}M_0^{\nu}, n)$.

Proof. Notice first that by the definition (B.14) of Op_h and the fact that $h = \frac{1}{t}$, one has

$$\begin{aligned} \big(D_t - \mathrm{Op}_h(x\xi)\big)&\mathrm{Op}_h(a_I)(\underline{v}_1, \dots, \underline{v}_n) \\ &= \sum_{j=1}^{n} \mathrm{Op}_h(a_I)\big(\underline{v}_1, \dots, (D_t - \mathrm{Op}_h(x\xi))\underline{v}_j, \dots, \underline{v}_n\big) \\ &\qquad + ih\mathrm{Op}_h\big((x\partial_x a_I)(y, x, \xi)\big)(\underline{v}_1, \dots, \underline{v}_n). \end{aligned} \tag{F.23}$$

Moreover, by Proposition B.2.1 and the definition (F.2) of g_I,

$$
\begin{aligned}
-\mathrm{Op}_h(p(\xi))\mathrm{Op}_h(a_I)(\underline{v}_1,\dots,\underline{v}_n)
&= \mathrm{Op}_h(a_I g_I)(\underline{v}_1,\dots,\underline{v}_n)\\
&\quad - \sum_{j=1}^{n} i_j \mathrm{Op}_h(a_I)\big(\underline{v}_1,\dots,\mathrm{Op}_h(p(\xi))\underline{v}_j,\dots,\underline{v}_n\big)\\
&\quad + h\mathrm{Op}_h(r_I)(\underline{v}_1,\dots,\underline{v}_n) + \mathrm{Op}_h(r_I')(\underline{v}_1,\dots,\underline{v}_n),
\end{aligned}
\tag{F.24}
$$

where r_I is in $S_{4,\beta}(\prod_{j=1}^n\langle\xi_j\rangle^{-1}M_0^\nu\langle x\rangle^{-\infty},n)$ and r_I' in $S'_{4,\beta}(\prod_{j=1}^n\langle\xi_j\rangle^{-1}M_0^\nu,n)$. Notice that $p(\xi)$ is in $S_{\kappa,\beta}(\langle\xi\rangle,1)$ (for any κ,β since, this symbol depending only on one variable ξ, $M_0(\xi)=1$), so that, to get from Proposition B.2.1 symbols r_I, r_I' in the indicated classes, we would need that a_I be in $S_{4,\beta}(M_0^\nu\prod_{j=1}^n\langle\xi_j\rangle^{-2}\langle x\rangle^{-\infty},n)$ instead of (F.7). But by (F.9), a_I is supported in (F.11), and we have seen just after this formula that this implies that $|\xi_\ell|\le CM_0(\xi)$ for any ℓ. Consequently, the above property for a_I does hold, for large enough ν. If we make the sum of (F.23) and (F.24), we get that the left-hand side of (F.20) is given by the sum on the right-hand side of (F.20), contributions to $\underline{R}$ of the form of the last two lines in (F.21) and the term $\mathrm{Op}_h(a_Ig_I)(\underline{v}_1,\dots,\underline{v}_n)$. By (F.8), we thus get the first term on the right-hand side of (F.20) and expressions

$$
-\mathrm{Op}_h\big(m_{I,\ell}(y,x,\xi_1,\dots,\xi_n)(x+i_\ell p'(\xi_\ell))\big)(\underline{v}_1,\dots,\underline{v}_n).
$$

Using again Proposition B.2.1, we write these terms as contributions to $\underline{R}$ given by (F.21). This concludes the proof. ∎

F.4 Quadratic normal forms for space decaying symbols

In Section 3.2 we have performed an easy quadratic normal form, that allowed us to get rid of the quadratic term on the right-hand side of (3.11), given by $\mathrm{Op}_h(m_{0,I})[u_I]$, with $|I|=2$ and $m_{0,I}$ in $\tilde{S}_{0,0}(\prod_{j=1}^2\langle\xi_j\rangle^{-1},2)$. This procedure made appear a new quadratic term $\mathrm{Op}_h(m_{0,I}')[u_I]$ on the right-hand side of equation (3.13), given in terms of a symbol $m_{0,I}'$ in $\tilde{S}_{0,0}'(\prod_{j=1}^2\langle\xi_j\rangle^{-1},2)$. We shall have to perform also a normal form to eliminate such terms. We define a new class of operators.

Definition F.4.1. Let $\omega\in[0,1]$, and $i=(i_1,i_2,i_3)$ in $\{-1,1\}^3$. We denote by $\mathcal{K}_{\kappa,\omega}$, resp. $\mathcal{K}_{\kappa,\omega}'(i)$, the space of operators of the form

$$
\begin{aligned}
(f_1,f_2)\mapsto \frac{1}{2\pi}\int_{-1}^{1}\int_{-1}^{1}\int & e^{ix\xi_0}k(t,\xi_0,\xi_1,\xi_2,\mu_1,\mu_2)\\
&\times \hat{f}(\xi_1)\hat{f}(\xi_2)\,d\xi_0\,d\xi_1\,d\xi_2\,d\mu_1\,d\mu_2,
\end{aligned}
\tag{F.25}
$$

where k is a smooth function of $(t,\xi_0,\xi_1,\xi_2,\mu_1,\mu_2)$ that satisfies for some ν in $\mathbb{N}$, any $N, \gamma_0, \gamma_1, \gamma_2, \mu_1, \mu_2, j$ in $\mathbb{N}$,

$$\begin{aligned}&|\partial_t^j \partial_{\xi_0}^{\gamma_0}\partial_{\xi_1}^{\gamma_1}\partial_{\xi_2}^{\gamma_2} k(t,\xi_0,\xi_1,\xi_2,\mu_1,\mu_2)| \\ &\quad \leq C M_0(\xi_1,\xi_2)^{\nu+(\gamma_0+\gamma_1+\gamma_2)\kappa}\langle \xi_0-\mu_1\xi_1-\mu_2\xi_2\rangle^{-N} t^{\omega(\gamma_0+\gamma_1+\gamma_2)-j},\end{aligned} \tag{F.26}$$

resp. that satisfies

$$\begin{aligned}&|\partial_t^j \partial_{\xi_0}^{\gamma_0}\partial_{\xi_1}^{\gamma_1}\partial_{\xi_2}^{\gamma_2} k(t,\xi_0,\xi_1,\xi_2,\mu_1,\mu_2)| \\ &\quad \leq C M_0(\xi_1,\xi_2)^{\nu+(\gamma_0+\gamma_1+\gamma_2)\kappa}\langle \xi_0-\mu_1\xi_1-\mu_2\xi_2\rangle^{-N} t^{\omega(\gamma_0+\gamma_1+\gamma_2)-j} \\ &\qquad \times \langle t^{\omega}(i_0\langle\xi_0\rangle - i_1\langle\xi_1\rangle - i_2\langle\xi_2\rangle)\rangle^{-1}\end{aligned} \tag{F.27}$$

in the case of $\mathcal{K}'_{\kappa,\omega}(i)$), where $M_0(\xi_1,\xi_2)$ still denoted the second largest among $\langle\xi_1\rangle$ and $\langle\xi_2\rangle$.

If k satisfies

$$k(t,-\xi_0,-\xi_1,-\xi_2) = -k(t,\xi_0,\xi_1,\xi_2), \tag{F.28}$$

then (F.25) sends a couple of two odd functions or two even functions to an odd function. If k satisfies

$$k(t,-\xi_0,-\xi_1,-\xi_2) = k(t,\xi_0,\xi_1,\xi_2), \tag{F.29}$$

then (F.25) sends a couple (f_1, f_2) with f_1 odd, f_2 even or f_1 even, f_2 odd to an odd function.

Let us check first that we may express operators of the form $\mathrm{Op}(m')(v_1,v_2)$ with m' in $\tilde{S}'_{1,0}(M_0(\xi_1,\xi_2)\prod_{j=1}^2\langle\xi_j\rangle^{-1},2)$ in terms of operators $\mathcal{K}_{\kappa,\omega}$.

Lemma F.4.2. *Let m' be in $\tilde{S}'_{1,0}(M_0\prod_{j=1}^2\langle\xi_j\rangle^{-1},2)$. Let $i_1,i_2\in\{-1,1\}^2$ be any choice of signs. Then if $L_\pm$ is defined by (C.5), one may find operators K_{ℓ_1,ℓ_2} in $\mathcal{K}_{1,0}$, $0\leq \ell_1,\ell_2\leq 1$, such that the action of* $\mathrm{Op}(m')$ *on any couple of odd functions (v_1,v_2) (as defined in (3.6)) may be written as*

$$t^{-2}\sum_{\ell_1=0}^{1}\sum_{\ell_2=0}^{1} K_{\ell_1,\ell_2}(L_{i_1}^{\ell_1}v_1, L_{i_2}^{\ell_1}v_2). \tag{F.30}$$

Moreover, if m satisfies (3.7), then K_{ℓ_1,ℓ_2} is given by a symbol k satisfying (F.28) if $\ell_1+\ell_2 = 0$ or 2 and (F.29) if $\ell_1+\ell_1=1$.

Proof. We may rewrite

$$\mathrm{Op}(m')(v_1,v_2) = \mathrm{Op}(m'_1)(\langle D_x\rangle^{-1}v_1, \langle D_x\rangle^{-1}v_2)$$

with m'_1 in $\tilde{S}_{1,0}(M_0,2)$. Using the oddness of v_j, we write

$$\begin{aligned}\langle D_x\rangle^{-1}v_j &= \frac{i}{2}x\int_{-1}^{1}(D_x\langle D_x\rangle^{-1}v_j)(\mu_j x)\,d\mu_j \\ &= \frac{i}{2}\frac{x}{t}i_j\int_{-1}^{1}\big((L_{i_j}v_j)(\mu_j x) - \mu_j x v_j(\mu_j x)\big)\,d\mu_j\end{aligned} \tag{F.31}$$

for any choice of the signs $i_j = \pm$. By definition (3.6) of the quantization and inequalities (3.4) satisfied by elements of the class S', one may rewrite expressions like $\mathrm{Op}(m_1')(xf_1, f_2)$ as sums of expressions of the form $\mathrm{Op}(\tilde{m}_1')(f_1, f_2)$, for new symbols $\tilde{m}_1'$ in $\tilde{S}_{1,0}(M_0^\nu, 2)$ for some ν. Using (F.31), we thus see that $\mathrm{Op}(m')(v_1, v_2)$ may be rewritten as a sum of terms

$$t^{-2}\int_{-1}^{1}\int_{-1}^{1}\mu_1^{1-\ell_1}\mu_2^{1-\ell_2}\mathrm{Op}(\tilde{m}')\big[(L_{i_1}^{\ell_1}v_1)(\mu_1\cdot),(L_{i_2}^{\ell_2}v_2)(\mu_2\cdot)\big]\,d\mu_1\,d\mu_2$$

for some symbols $\tilde{m}'$ in $S'_{1,0}(M_0^\nu, 2)$. By (3.6), we have

$$\begin{aligned}
&\mathrm{Op}(\tilde{m}')[f_1(\mu_1\cdot), f_2(\mu_2\cdot)]\\
&\quad=\frac{1}{(2\pi)^2}\int e^{ix(\mu_1\xi_1+\mu_2\xi_2)}m'(x,\mu_1\xi_1,\mu_2\xi_2)\hat{f}_1(\xi_1)\hat{f}_2(\xi_2)\,d\xi_1\,d\xi_2\\
&\quad=\frac{1}{2\pi}\int e^{ix\xi_0}k(\xi_0,\xi_1,\xi_2,\mu_1,\mu_2)\hat{f}_1(\xi_1)\hat{f}_2(\xi_2)\,d\xi_1\,d\xi_2
\end{aligned}$$

with

$$k(\xi_0,\xi_1,\xi_2,\mu_1,\mu_2)=\frac{1}{(2\pi)^2}\hat{m}'(\xi_0-\mu_1\xi_1-\mu_2\xi_2,\mu_1\xi_1,\mu_2\xi_2).$$

It follows from estimates (3.4) that hold for any α, α_0', that inequalities (F.26) are true for some ν, $\kappa = 1$, $\omega = 0$, which implies the conclusion as the last statement follows from the transfer of property (3.7) to k by inspection. ∎

Proposition F.4.3. *Let K be in $\mathcal{K}_{\kappa,0}$. Let $i = (i_0, i_1, i_2) \in \{-,+\}^3$. One may find operators K_L, K_H in $\mathcal{K}'_{\kappa,\frac{1}{2}}(i)$ such that for any f_1, f_2,*

$$\begin{aligned}
&(D_t - i_0 p(D_x))(\sqrt{t}K_H(f_1,f_2))\\
&\quad= K(f_1,f_2)+\sqrt{t}K_H\big((D_t-i_1p(D_x))f_1,f_2\big)\\
&\qquad+\sqrt{t}K_H\big(f_1,(D_t-i_2p(D_x))f_2\big)+K_L(f_1,f_2).
\end{aligned}\tag{F.32}$$

If K satisfies (F.28) (resp. (F.29)), so do K_H, K_L.

Proof. Take χ in $C_0^\infty(\mathbb{R})$ equal to one close to zero and set $\chi_1(z) = \frac{1-\chi(z)}{z}$. Define from the function k associated to K by (F.25) a new function

$$\begin{aligned}
k_H(t,\xi_0,\xi_1,\xi_2,\mu_1,\mu_2) &= k(\xi_0,\xi_1,\xi_2,\mu_1,\mu_2)\\
&\quad\times\chi_1\big(\sqrt{t}(-i_0\langle\xi_0\rangle+i_1\langle\xi_1\rangle+i_2\langle\xi_2\rangle)\big).
\end{aligned}\tag{F.33}$$

Then k_H satisfies (F.27) with $\omega = \frac{1}{2}$. Call K_H the associated operator. If we make act $D_t - i_0 p(D_x)$ on $\sqrt{t}K_H(f_1, f_2)$, we get the second and third terms on the right-hand side of (F.32), an operator associated to the function

$$k(\xi_0,\xi_1,\xi_2,\mu_1,\mu_2)(1-\chi)\big(\sqrt{t}(-i_0\langle\xi_0\rangle+i_1\langle\xi_1\rangle+i_2\langle\xi_2\rangle)\big)\tag{F.34}$$

and contributions coming from the action of D_t on k_H, that may be written as contributions to K_L in (F.32) (with even an extra factor $t^{-1/2}$). Finally, we see that (F.34) provides K on the right-hand side of (F.32), modulo another contribution to K_L. This concludes the proof as the last statement follows from (F.34). ■

Corollary F.4.4. *Let m' be in $S'_{1,0}(\prod_{j=1}^{2}\langle\xi_j\rangle^{-1}, 2)$. One may find for any i_1, i_2 in $\{-,+\}$, any ℓ_1, ℓ_2 in $\{0,1\}$ operators*

$$K_{H,i_1,i_2}^{\ell_1,\ell_2}, \quad K_{L,i_1,i_2}^{\ell_1,\ell_2}$$

in the class $\mathcal{K}'_{1,\frac{1}{2}}(1, i_1, i_2)$ such that for any odd functions v_1, v_2, if one sets

$$Q_{i_1,i_2}(v_1, v_2) = t^{-\frac{3}{2}} \sum_{\ell_1=0}^{1}\sum_{\ell_2=0}^{1} K_{H,i_1,i_2}^{\ell_1,\ell_2}\big(L_{i_1}^{\ell_1} v_1, L_{i_2}^{\ell_2} v_2\big), \tag{F.35}$$

then

$$\begin{aligned}\big(D_t - p(D_x)\big)&Q_{i_1,i_2}(v_1, v_2)\\ &= \mathrm{Op}(m')(v_1, v_2) + Q_{i_1,i_2}\big((D_t - i_1 p(D_x))v_1, v_2\big)\\ &\qquad + Q_{i_1,i_2}\big(v_1, (D_t - i_2 p(D_x))v_2\big) + R_{i_1,i_2}(v_1, v_2),\end{aligned} \tag{F.36}$$

where

$$\begin{aligned}R_{i_1,i_2}(v_1, v_2) = t^{-2} &\sum_{\ell_1=0}^{1}\sum_{\ell_2=0}^{1} K_{L,i_1,i_2}^{\ell_1,\ell_2}\big(L_{i_1}^{\ell_1} v_1, L_{i_2}^{\ell_2} v_2\big)\\ &+ 2it^{-\frac{5}{2}} \sum_{\ell_1=0}^{1}\sum_{\ell_2=0}^{1} K_{H,i_1,i_2}^{\ell_1,\ell_2}\big(L_{i_1}^{\ell_1} v_1, L_{i_2}^{\ell_2} v_2\big).\end{aligned} \tag{F.37}$$

Moreover, if m' satisfies (3.7), $K_{H,i_1,i_2}^{\ell_1,\ell_2}, K_{L,i_1,i_2}^{\ell_1,\ell_2}$ satisfy (F.28) if $\ell_1 + \ell_2 = 0$ or 2 and (F.29) if $\ell_1 + \ell_2 = 1$. In particular, Q_{i_1,i_2} sends a couple of odd functions to an odd function.

Proof. By Lemma F.4.2, we may write $\mathrm{Op}(m')(v_1, v_2)$ under the form (F.30). We apply to each K_{ℓ_1,ℓ_2} in (F.30) Proposition F.4.3. If we define $K_{H,i_1,i_2}^{\ell_1,\ell_2}$ (resp. $K_{L,i_1,i_2}^{\ell_1,\ell_2}$) from the operator K_H (resp. K_L) in equation (F.32), and use that L_{i_ℓ} commutes to $D_t - i_\ell p'(D_x)$, we obtain (F.36) for the Q_{i_1,i_2} defined in equation (F.35). The last statement of the corollary follows from the last statement in Proposition F.4.3 and Lemma F.4.2. ■

F.5 Sobolev estimates

We shall prove Sobolev estimates for operators introduced in Definition F.4.1.

Proposition F.5.1. *Let $\omega \in [0,1]$, $\kappa \geq 0$, let K be an operator in the class $\mathcal{K}'_{\kappa,\omega}(i)$ (for a triple $i = (i_1, i_2, i_3) \in \{-,+\}^3$). Assume moreover that the function k in (F.25) is supported for $|\xi_2| \leq 2\langle \xi_1 \rangle$. There exists $\sigma_0 \in \mathbb{R}_+$ (depending on the exponent ν in (F.27)) such that the following estimates hold true for any s in $\mathbb{R}_+$, any test functions f_1, f_2:*

$$\|K(f_1, f_2)\|_{H^s} \leq C t^{-\frac{\omega}{2}} \|f_2\|_{H^{\sigma_0}} \|f_1\|_{H^s}, \tag{F.38}$$

$$\begin{aligned} &\|K(f_1, x f_2)\|_{H^s} + \|K(x f_1, f_2)\|_{H^s} + \|x K(f_1, f_2)\|_{H^s} \\ &\quad \leq C t^{\frac{\omega}{2}} \|f_2\|_{H^{\sigma_0}} \|f_1\|_{H^s}, \end{aligned} \tag{F.39}$$

$$\|K(x f_1, x f_2)\|_{H^s} \leq C t^{\frac{3\omega}{2}} \|f_2\|_{H^{\sigma_0}} \|f_1\|_{H^s}. \tag{F.40}$$

Proof. By (F.25), we have to prove, in order to establish (F.38), that the operator

$$\begin{aligned} (g_1, g_2) \mapsto \int_{-1}^{1} \int_{-1}^{1} \int & \langle \xi_0 \rangle^s k(t, \xi_0, \xi_1, \xi_2, \mu_1, \mu_2) \langle \xi_1 \rangle^{-s} \langle \xi_2 \rangle^{-\sigma_0} \\ & \times g_1(\xi_1) g_2(\xi_2)\, d\xi_1\, d\xi_2\, d\mu_1\, d\mu_2 \end{aligned} \tag{F.41}$$

is bounded from $L^2 \times L^2$ to L^2, with operator norm $O(t^{-\frac{\omega}{2}})$. Because of our support assumptions, $M_0(\xi_1, \xi_2) \leq C\langle \xi_2 \rangle$, so that we may control the factor $M_0(\xi_1, \xi_2)$ in (F.27) by $C\langle \xi_2 \rangle$, i.e. M_0^ν will be bounded using $\langle \xi_2 \rangle^{-\sigma_0}$ if σ_0 is taken large enough. Moreover, as $s \geq 0$, $\langle \xi_0 \rangle^s \langle \xi_0 - \mu_1 \xi_1 - \mu_2 \xi_2 \rangle^{-N} \langle \xi_1 \rangle^{-s} = O(1)$ when $|\xi_2| \leq 2\langle \xi_1 \rangle$ if N is large enough relatively to s. The proof of (F.38) is thus reduced to the proof that operators of the form

$$(g_1, g_2) \mapsto \int_{-1}^{1} \int_{-1}^{1} \int \tilde{k}(t, \xi_0, \xi_1, \xi_2, \mu_1, \mu_2) g_1(\xi_1) g_2(\xi_2)\, d\xi_1\, d\xi_2\, d\mu_1\, d\mu_2 \tag{F.42}$$

are bounded from $L^2 \times L^2$ to L^2, with operator norm $O(t^{-\frac{\omega}{2}})$, if $\tilde{k}$ satisfies

$$\begin{aligned} |\tilde{k}(t, \xi_0, \xi_1, \xi_2, \mu_1, \mu_2)| \leq C & \langle \xi_0 - \mu_1 \xi_1 - \mu_2 \xi_2 \rangle^{-1} \langle \xi_2 \rangle^{-2} \\ & \times \langle t^\omega (i_0 \langle \xi_0 \rangle - i_1 \langle \xi_1 \rangle - i_2 \langle \xi_2 \rangle) \rangle^{-1}. \end{aligned} \tag{F.43}$$

The operator norm of (F.42) is bounded from above by

$$\begin{aligned} C \int_{-1}^{1} \int_{-1}^{1} & \left(\sup_{\xi_0} \int |\tilde{k}(t, \xi_0, \xi_1, \xi_2, \mu_1, \mu_2)|\, d\xi_1\, d\xi_2 \right)^{\frac{1}{2}} \\ & \times \left(\sup_{\xi_1, \xi_2} \int |\tilde{k}(t, \xi_0, \xi_1, \xi_2, \mu_1, \mu_2)|\, d\xi_0 \right)^{\frac{1}{2}} d\mu_1\, d\mu_2. \end{aligned} \tag{F.44}$$

Notice that there is $C > 0$ such that for any α, β in $\mathbb{R}$, any $\mu \in [-1,1]$,

$$\int \langle t^\omega (\alpha + \langle \xi \rangle) \rangle^{-1} \langle \beta + \mu \xi \rangle^{-1}\, d\xi \leq C |\mu|^{-\frac{1}{2}} t^{-\frac{\omega}{2}} \tag{F.45}$$

uniformly in α, β. Actually, if we integrate for $|\xi| \geq 1$, we bound (F.45) by

$$C|\mu|^{-\frac{1}{2}}\left(\int_{|\xi|>1} \langle t^{\omega}(\alpha+\langle\xi\rangle)\rangle^{-2}\, d\xi\right)^{\frac{1}{2}}.$$

If one takes in the above integral computed either on domain $\xi > 1$ or $\xi < -1$, $\eta = \langle\xi\rangle$ as a new variable of integration, we get a bound by the right-hand side of (F.45). If one integrates for $|\xi| < 1$ on the left-hand side of (F.45), we bound the corresponding quantity by

$$\int_{|\xi|<1} \left\langle t^{\omega}(\alpha+\sqrt{1+\xi^2})\right\rangle^{-1} d\xi \leq C\int \langle\alpha' + t^{\omega}\zeta^2\rangle^{-1}\, d\zeta \leq Ct^{-\frac{\omega}{2}}$$

which is better than the bound we want. We use (F.43) and (F.45) with $\xi = \xi_0$ to estimate the second factor in (F.44) by $t^{-\frac{\omega}{4}}$ and (F.45) with $\xi = \xi_1$ to estimate the first integral factor by $t^{-\frac{\omega}{2}}|\mu_1|^{-\frac{1}{2}}$. We obtain that (F.44) is $O(t^{-\frac{\omega}{2}})$ from which (F.38) follows.

To get estimate (F.39), we notice that, by (F.25), $K(xf_1, f_2)$ (resp. $K(f_1, xf_2)$, resp. $xK(f_1, f_2)$) may be written as $K_1(f_1, f_2)$ for an operator K_1 of the form (F.25), obtained replacing k by $D_{\xi_1}k$ (resp. $D_{\xi_2}k$, resp. $-D_{\xi_0}k$). Since by (F.27) these D_{ξ_j}-derivatives make lose t^{ω} (and change the value of the exponent ν), we get (F.39) from (F.38) (with a new value of σ_0).

One obtains (F.40) in a same way. ∎

Corollary F.5.2. *Let K be an element of $\mathcal{K}'_{\kappa,\omega}(i)$ for $\omega \in [0, 1]$, $\kappa \geq 0$, $i \in \{-, +\}^3$. The following estimates hold true for any $s \geq 0$ and some σ_0 independent of s:*

$$\|K(f_1, f_2)\|_{H^s} \leq Ct^{-\frac{\omega}{2}}\left(\|f_1\|_{H^{\sigma_0}}\|f_2\|_{H^s} + \|f_1\|_{H^s}\|f_2\|_{H^{\sigma_0}}\right), \tag{F.46}$$

$$\begin{aligned} &\|K(f_1, f_2)\|_{L^2} \leq Ct^{-\frac{\omega}{2}}\|f_1\|_{L^2}\|f_2\|_{H^{\sigma_0}},\\ &\|K(f_1, f_2)\|_{L^2} \leq Ct^{-\frac{\omega}{2}}\|f_1\|_{H^{\sigma_0}}\|f_2\|_{L^2}, \end{aligned} \tag{F.47}$$

$$\begin{aligned} &\|K(xf_1, f_2)\|_{L^2} + \|K(f_1, xf_2)\|_{L^2} + \|xK(f_1, f_2)\|_{L^2}\\ &\quad\leq Ct^{\frac{\omega}{2}}\|f_1\|_{L^2}\|f_2\|_{H^{\sigma_0}},\\ &\|K(xf_1, f_2)\|_{L^2} + \|K(f_1, xf_2)\|_{L^2} + \|xK(f_1, f_2)\|_{L^2}\\ &\quad\leq Ct^{\frac{\omega}{2}}\|f_1\|_{H^{\sigma_0}}\|f_2\|_{L^2}, \end{aligned} \tag{F.48}$$

$$\begin{aligned} &\|K(xf_1, f_2)\|_{H^s} + \|K(f_1, xf_2)\|_{H^s} + \|xK(f_1, f_2)\|_{H^s}\\ &\quad\leq Ct^{\frac{\omega}{2}}\left(\|f_1\|_{H^{\sigma_0}}\|f_2\|_{H^s} + \|f_1\|_{H^s}\|f_2\|_{H^{\sigma_0}}\right). \end{aligned} \tag{F.49}$$

Proof. We may split $K = K_< + K_>$, where $K_>$ (resp. $K_<$) is given by an expression of the form (F.25) with k supported for $|\xi_2| \leq 2\langle\xi_1\rangle$ (resp. $|\xi_1| \leq 2\langle\xi_2\rangle$). If we apply (F.38) to $K_>$ and the symmetric inequality to $K_<$, we obtain (F.46).

Let us prove (F.47). It suffices to show that the two estimates hold for $K_>$ for instance. The first one follows from (F.38) with $s = 0$. To get the second one, we notice that it is enough to establish the $L^2 \times L^2 \to L^2$ boundedness of

$$(g_1, g_2) \mapsto \int_{-1}^{1} \int_{-1}^{1} k(t, \xi_0, \xi_1, \xi_2, \mu_1, \mu_2) \langle \xi_1 \rangle^{-\sigma_0} g_1(\xi_1) g_2(\xi_2)\, d\xi_1\, d\xi_2\, d\mu_1\, d\mu_2$$

with operator norm $O(t^{-\frac{\omega}{2}})$. Since $|\xi_2| \le 2\langle \xi_1 \rangle$ on the support, if σ_0 has been taken large enough, we see that we may rewrite this under the form (F.42), with some $\tilde{k}$ fulfilling (F.43) so that the conclusion follows.

Finally, estimates (F.48) follow from (F.47), noticing that, as in the proof of (F.39), we may reduce ourselves to operator $K_1(f_1, f_2)$ satisfying the same assumptions as K, up to the loss of a factor t^{ω}. This concludes the proof, as (F.49) follows from (F.39) and the above decomposition $K = K_< + K_>$. ∎

Corollary F.5.3. *Let $\beta > 0$, K, σ_0 as in Corollary F.5.2 and take s large enough so that $(s - \sigma_0)\beta \ge 1$. Then*

$$\|K(L_\pm f_1, f_2)\|_{L^2} \le C t^{-\frac{\omega}{2}} \big(t^{\beta\sigma_0} \|L_\pm f_1\|_{L^2} + \|f_1\|_{H^s} \big) \|f_2\|_{L^2}, , \tag{F.50}$$

$$\|K(f_1, L_\pm f_2)\|_{L^2} \le C t^{-\frac{\omega}{2}} \|f_1\|_{L^2} \big(t^{\beta\sigma_0} \|L_\pm f_2\|_{L^2} + \|f_2\|_{H^s} \big). \tag{F.51}$$

Proof. Let χ be in $C_0^\infty(\mathbb{R})$, $\chi \equiv 1$ close to zero. Decompose

$$L_\pm f_1 = \chi(t^{-\beta} D_x)(L_\pm f_1) + (1 - \chi)(t^{-\beta} D_x)(L_\pm f_1).$$

Write

$$\begin{aligned}(1 - \chi)(t^{-\beta} D_x)(L_\pm f_1) &= x(1 - \chi)(t^{-\beta} D_x) f_1 + i t^{-\beta} \chi'(t^{-\beta} D_x) f_1 \\ &\quad \pm t(1 - \chi)(t^{-\beta} D_x) \frac{D_x}{\langle D_x \rangle} f_1.\end{aligned}$$

If one applies the second estimate in (F.47) and (F.48), one gets then

$$\begin{aligned}&\|K\big((1 - \chi)(t^{-\beta} D_x) L_\pm f_1, f_2\big)\|_{L^2} \\ &\quad \le C\Big(t^{\frac{\omega}{2}} \|(1 - \chi)(t^{-\beta} D_x) f_1\|_{H^{\sigma_0}} \\ &\qquad + t^{-\frac{\omega}{2}} \big(\|\chi'(t^{-\beta} D_x) f_1\|_{H^{\sigma_0}} + t\|(1 - \chi)(t^{-\beta} D_x) f_1\|_{H^{\sigma_0}} \big)\Big) \|f_2\|_{L^2}.\end{aligned}$$

Since $(s - \sigma_0)\beta \ge 1$, this is bounded by $C t^{-\frac{\omega}{2}} \|f_1\|_{H^s} \|f_2\|_{L^2}$.

On the other hand, by the second estimate (F.47)

$$\begin{aligned}\|K\big(\chi(t^{-\beta} D_x) L_\pm f_1, f_2\big)\|_{L^2} &\le C t^{-\frac{\omega}{2}} \|\chi(t^{-\beta} D_x) L_\pm f_1\|_{H^{\sigma_0}} \|f_2\|_{L^2} \\ &\le C t^{-\frac{\omega}{2} + \beta\sigma_0} \|L_\pm f_1\|_{L^2} \|f_2\|_{L^2}.\end{aligned}$$

This concludes the proof of (F.50), and thus of the corollary since (F.51) is just the symmetric estimate. ∎

Let us get next some Sobolev estimates for $K(L_\pm f_1, L_\pm f_2)$.

Corollary F.5.4. *Let K be in the class $\mathcal{K}'_{\kappa,\omega}(i)$. Assume moreover that k in (F.25) is supported for $|\xi_1| \leq 2\langle \xi_2 \rangle$. Let s, σ_0, β be as in Corollary F.5.3. Then, if $(s-\sigma_0)\beta \geq 1$,*

$$\|K(L_\pm f_1, L_\pm f_2)\|_{H^s} \leq Ct^{1-\frac{\omega}{2}} \|f_2\|_{H^s} \big(t^{\beta\sigma_0} \|L_\pm f_1\|_{L^2} + \|f_2\|_{H^s}\big), \tag{F.52}$$

$$\|K(L_\pm f_1, f_2)\|_{H^s} + \|K(f_1, L_\pm f_2)\|_{H^s} \leq Ct^{1-\frac{\omega}{2}} \|f_1\|_{H^s} \|f_2\|_{H^s}, \tag{F.53}$$

$$\begin{aligned} \|K(xf_1, f_2)\|_{H^s} + \|K(f_1, xf_2)\|_{H^s} &\leq Ct^{\frac{\omega}{2}} \|f_1\|_{H^s} \|f_2\|_{H^s}, \\ \|K(xf_1, xf_2)\|_{H^s} &\leq Ct^{3\frac{\omega}{2}} \|f_1\|_{H^s} \|f_2\|_{H^s}. \end{aligned} \tag{F.54}$$

Proof. Take χ in $C_0^\infty(\mathbb{R})$, equal to one close to zero and write $K(L_\pm f_1, L_\pm f_2)$ as a linear combination of the four terms

$$\begin{aligned} I &= tK\Big(\chi(t^{-\beta} D_x) L_\pm f_1, \frac{D_x}{\langle D_x \rangle} f_2\Big), \\ II &= tK\Big((1-\chi)(t^{-\beta} D_x) L_\pm f_1, \frac{D_x}{\langle D_x \rangle} f_2\Big), \\ III &= K\big(\chi(t^{-\beta} D_x) L_\pm f_1, xf_2\big), \\ IV &= K\big((1-\chi)(t^{-\beta} D_x) L_\pm f_1, xf_2\big). \end{aligned} \tag{F.55}$$

We apply (F.38) (with f_1 and f_2 exchanged since we assume here $|\xi_1| \leq 2\langle \xi_2 \rangle$ on the support instead of $|\xi_2| \leq 2\langle \xi_1 \rangle$) in order to estimate the H^s norm of I by

$$Ct^{1-\frac{\omega}{2}} \|\chi(t^{-\beta} D_x) L_\pm f_1\|_{H^{\sigma_0}} \|f_2\|_{H^s} \leq Ct^{1-\frac{\omega}{2}+\beta\sigma_0} \|L_\pm f_1\|_{L^2} \|f_2\|_{H^s} \tag{F.56}$$

which is bounded by the right-hand side of (F.52).

To study II, we write it as a combination of terms

$$\begin{aligned} &t^2 K\Big((1-\chi)(t^{-\beta} D_x) \frac{D_x}{\langle D_x \rangle} f_1, \frac{D_x}{\langle D_x \rangle} f_2\Big), \\ &tK\Big(x(1-\chi)(t^{-\beta} D_x) f_1, \frac{D_x}{\langle D_x \rangle} f_2\Big), \\ &it^{1-\beta} K\Big(\chi'(t^{-\beta} D_x) f_1, \frac{D_x}{\langle D_x \rangle} f_2\Big). \end{aligned}$$

We estimate their H^s norm using (F.38) and (F.39) (with f_1 and f_2 interchanged) by

$$\begin{aligned} &Ct^{2-\frac{\omega}{2}} \|f_2\|_{H^s} \big(\|(1-\chi)(t^{-\beta} D_x) f_1\|_{H^{\sigma_0}} + \|\chi'(t^{-\beta} D_x) f_1\|_{H^{\sigma_0}}\big) \\ &\quad \leq Ct^{2-(s-\sigma_0)\beta-\frac{\omega}{2}} \|f_1\|_{H^s} \|f_2\|_{H^s}. \end{aligned}$$

This implies a bound by the right-hand side of (F.52) since $(s-\sigma_0)\beta \geq 1$.

By (F.39) (with f_1 and f_2 exchanged), we estimate the H^s norm of III by

$$Ct^{\frac{\omega}{2}} \|\chi(t^{-\beta} D_x) L_\pm f_1\|_{H^{\sigma_0}} \|f_2\|_{H^s}$$

that we bound by the right-hand side of (F.52) as in (F.56) since $\omega \leq 1$.

We write IV as a combination of terms

$$
\begin{aligned}
&tK\Big((1-\chi)(t^{-\beta}D_x)\frac{D_x}{\langle D_x\rangle}f_1, xf_2\Big),\\
&K\big(x(1-\chi)(t^{-\beta}D_x)f_1, xf_2\big),\\
&it^{-\beta}K\big(\chi'(t^{-\beta}D_x)f_1, xf_2\big).
\end{aligned}
$$

We estimate the H^s norm of these quantities using (F.39) and (F.40) with f_1 and f_2 interchanged. We get

$$
\begin{aligned}
C\big(t^{1+\frac{\omega}{2}}+t^{3\frac{\omega}{2}}\big)\|(1-\chi)(t^{-\beta}D_x)f_1\|_{H^{\sigma_0}}\|f_2\|_{H^s}&\\
+Ct^{-\beta+\frac{\omega}{2}}\|\chi'(t^{-\beta}D_x)f_1\|_{H^{\sigma_0}}\|f_2\|_{H^s}&.
\end{aligned}
$$

As $(s-\sigma_0)\beta\geq\omega$, this implies a bound by the right-hand side of (F.52). This concludes the proof of (F.52)

To prove (F.53), we decompose $K(L_\pm f_1, f_2)$ (resp. $K(f_1, L_\pm f_2)$) as the sum of $\pm tK(\frac{D_x}{\langle D_x\rangle}f_1, f_2)$ (resp. $\pm tK(f_1, \frac{D_x}{\langle D_x\rangle}f_2)$) and of $K(xf_1, f_2)$ (resp. $K(f_1, xf_2)$) and we apply (F.38) and (F.39) to get the conclusion.

Finally, (F.54) is just a consequence of (F.39) and (F.40). ∎

We translate finally the preceding corollary when one does not make any assumption of support on the frequencies.

Corollary F.5.5. *Let K be in the class $\mathcal{K}'_{\kappa,\omega}(i)$. With the notation of Corollary F.5.4, one has the following inequalities:*

$$
\begin{aligned}
\|K(L_\pm f_1, L_\pm f_2)\|_{H^s}\leq Ct^{1-\frac{\omega}{2}}\Big(&t^{\beta\sigma_0}\big(\|L_\pm f_1\|_{L^2}\|f_2\|_{H^s}\\
&+\|f_1\|_{H^s}\|L_\pm f_2\|_{L^2}\big)+\|f_1\|_{H^s}\|f_2\|_{H^s}\Big)
\end{aligned}
\tag{F.57}
$$

and

$$
\|K(f_1, L_\pm f_2)\|_{H^s}+\|K(L_\pm f_1, f_2)\|_{H^s}\leq Ct^{1-\frac{\omega}{2}}\|f_1\|_{H^s}\|f_2\|_{H^s},\tag{F.58}
$$

(with any choice of the signs $\pm$ in the left and right-hand side of these inequalities).

Proof. One decomposes $K=K_<+K_>$ as in the proof of Corollary F.5.2 and applies (F.52) and (F.53). ∎

Appendix G

Verification of Fermi's golden rule

The goal of this Appendix is to check that Fermi's golden rule, used in Chapter 4 (see Lemma 4.2.3 and the proof of Proposition 4.2.1) does hold. We already know that from Kowalcyk, Martel and Muñoz, who gave a numerical verification of the condition. We shall prove here that it may actually be checked analytically.

G.1 Reductions

We want to prove the following:

Proposition G.1.1. *Let Y_2 be the function defined in* (4.22)*. Then* $\hat{Y}_2(\sqrt{2}) \neq 0$.

Let us prove here the following reduction:

Lemma G.1.2. *Define the integral*

$$I = \int_{\mathbb{R}} e^{2ix\sqrt{2}}\Big(\cosh^2 x + \frac{1}{2} + i\sqrt{2}\sinh x\cosh x\Big)\frac{\sinh^3 x}{\cosh^7 x}\,dx. \tag{G.1}$$

If $I \neq 0$, then $\hat{Y}_2(\sqrt{2}) \neq 0$.

Proof. Recall that by (4.22), Y_2 is given by

$$Y_2(x) = b(x, D_x)^*(\kappa(x)Y(x)^2), \tag{G.2}$$

where κ, Y are defined in (2.5)–(2.6) and $b(x, D_x)$ has been introduced in Proposition A.1.1. Since $b(x, D_x)^*$ preserves real-valued functions and odd functions, we see that Y_2 is real valued and odd. By Proposition A.1.1, $W_+^* = c(D_x)^* \circ b(x, D_x)^*$ (when acting on odd functions), where $c(\xi)$ has modulus one. In order to show that $\hat{Y}_2(\sqrt{2}) \neq 0$, it thus suffices, according to (G.2), to prove that

$$\widehat{W_+^*(\kappa(x)Y^2)}(\sqrt{2}) \neq 0.$$

Recall that by (A.33) and (A.34),

$$W_+ w = \frac{1}{2\pi}\int \psi_+(x,\xi)\hat{w}(\xi)\,d\xi \tag{G.3}$$

with, by (A.35),

$$\psi_+(x,\xi) = \mathbb{1}_{\xi>0}T(\xi)f_1(x,\xi) + \mathbb{1}_{\xi<0}T(-\xi)f_2(x,-\xi), \tag{G.4}$$

where f_1, f_2 are the two Jost functions introduced at the beginning of Appendix A and $T(\xi)$ is defined in (A.26). We thus get

$$\begin{aligned}\widehat{W_+^*(\kappa(x)Y^2)}(\sqrt{2}) &= \int \overline{\psi_+(x,\sqrt{2})}\kappa(x)Y(x)^2\,dx \\ &= \overline{T(\sqrt{2})}\int \overline{f_1(x,\sqrt{2})}\kappa(x)Y(x)^2\,dx.\end{aligned} \tag{G.5}$$

Since the transmission coefficient $T(\sqrt{2})$ is non-zero, it remains to prove that if I given by (G.1) is different from zero, the same is true for the last integral in (G.5), or since κY^2 is real valued, that

$$\int f_1(x,\sqrt{2})\kappa(x)Y(x)^2\,dx \neq 0. \tag{G.6}$$

One checks by a direct computation that the function

$$e^{ix\sqrt{2}}\Big(1+\frac{1}{2}\cosh^{-2}\Big(\frac{x}{2}\Big)+i\sqrt{2}\tanh\frac{x}{2}\Big)(1+i\sqrt{2})^{-1}$$

solves (A.1) with $\xi = \sqrt{2}$ and is equivalent to $e^{ix\sqrt{2}}$ when x goes to $+\infty$, so that is the Jost function $f_1(x,\sqrt{2})$. If one plugs that value in (G.6) and uses the definition (2.5)–(2.6) of κ, Y, one obtains that (G.6) is just a non-zero multiple of (G.1). This concludes the proof. ∎

G.2 Proof of the non-vanishing of $\hat{Y}_2(\sqrt{2})$

In order to prove Proposition G.1.1, it remains to show that I given by (G.1) is non-zero. We compute explicitly this integral by residues.

Lemma G.2.1. *One has*

$$I = -\frac{2i\pi}{\sinh(\pi\sqrt{2})}. \tag{G.7}$$

Proof. Denote

$$F(z) = e^{2iz\sqrt{2}}\Big(\cosh^2 z + \frac{1}{2} + i\sqrt{2}\sinh z\cosh z\Big)\frac{\sinh^3 z}{\cosh^7 z}. \tag{G.8}$$

This is a meromorphic function on $\mathbb{C}$ with poles $z_k = i\frac{\pi}{2}(2k+1)$, $k \in \mathbb{Z}$. Let $\mathcal{R}_k$ be the rectangle in the complex plane with vertices at $\pm k\pi, \pm k\pi + ik\pi$ for k in $\mathbb{N}^*$. In order to show that

$$I = 2i\pi\sum_{k=0}^{+\infty}\operatorname{Res}(F, z_k) \tag{G.9}$$

we have to check that

$$\int_0^1 |F(\pm k\pi + itk\pi)|k\,dt \to 0, \qquad \int_{-1}^1 |F(tk\pi + ik\pi)|k\,dt \to 0$$

when k goes to $+\infty$. As $F(-\bar{z}) = -\overline{F(z)}$, we just have to prove

$$k \int_0^1 \big(|F(k\pi + itk\pi)| + |F(tk\pi + ik\pi)|\big)\,dt \to 0 \tag{G.10}$$

when $k \to +\infty$. As $F(z)$ is a sum of expressions of the form $e^{2iz\sqrt{2}} \frac{\sinh^p z}{\cosh^q z}$ with p, q in $\mathbb{N}$, $p < q$, and bounding

$$\left|\frac{\sinh^p z}{\cosh^q z}\right| \le e^{(p-q)\,\mathrm{Re}\,z} \left|\frac{(1-e^{-2z})^p}{(1+e^{-2z})^q}\right|,$$

we obtain when $0 \le t \le 1$, $k \in \mathbb{N}^*$,

$$|F(tk\pi + ik\pi)| \le e^{-2k\pi\sqrt{2} - tk\pi},$$

$$|F(k\pi + itk\pi)| \le e^{-2k\pi\sqrt{2}t - k\pi} \frac{(1+e^{-2k\pi})^p}{(1-e^{-2k\pi})^q}$$

from which (G.10) follows.

Using

$$\cosh(z_k + w) = i(-1)^k \sinh w \quad \text{and} \quad \sinh(z_k + w) = i(-1)^k \cosh w,$$

we may write

$$F(z_k + w) = e^{-\pi\sqrt{2}(2k+1)} G(w),$$

$$G(w) = e^{2i\sqrt{2}w}\Big(-\sinh^2 w + \frac{1}{2} - i\sqrt{2}\sinh w \cosh w\Big)\frac{\cosh^3 w}{\sinh^7 w}$$

so that $\mathrm{Res}(F, z_k) = e^{-\pi\sqrt{2}(2k+1)}\,\mathrm{Res}(G, 0)$. One checks by direct computation that $\mathrm{Res}(G, 0) = -2$. It follows that (G.9) is given by

$$I = -4i\pi e^{-\pi\sqrt{2}} \sum_{k=0}^{+\infty} e^{-2\pi k\sqrt{2}} = -\frac{2i\pi}{\sinh(\pi\sqrt{2})}$$

whence (G.7). ∎

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MEMOIRS OF THE EUROPEAN MATHEMATICAL SOCIETY

Jean-Marc Delort, Nader Masmoudi

Long-Time Dispersive Estimates for Perturbations of a Kink Solution of One-Dimensional Cubic Wave Equations

A kink is a stationary solution to a cubic one-dimensional wave equation $(\partial_t^2 - \partial_x^2)\phi = \phi - \phi^3$ that has different limits when x goes to $-\infty$ and $+\infty$, like $H(x) = \tanh(x/\sqrt{2})$. Asymptotic stability of this solution under small odd perturbation in the energy space has been studied in a recent work of Kowalczyk, Martel and Muñoz. They have been able to show that the perturbation may be written as the sum $a(t)Y(x) + \psi(t,x)$, where Y is a function in Schwartz space, $a(t)$ a function of time having some decay properties at infinity, and $\psi(t,x)$ satisfies some *local in space* dispersive estimate. These results are likely to be optimal when the initial data belong to the energy space. On the other hand, for initial data that are smooth and have some decay at infinity, one may ask if precise dispersive time decay rates for the solution in the whole space-time, and not just for x in a compact set, may be obtained. The goal of this work is to attack these questions.

Our main result gives, for small odd perturbations of the kink that are smooth enough and have some space decay, explicit rates of decay for $a(t)$ and for $\psi(t,x)$ in the whole space-time domain intersected by a strip $|t| \leq \varepsilon^{-4+c}$, for any $c > 0$, where ε is the size of the initial perturbation. This limitation is due to some new phenomena that appear along lines $x = \pm\sqrt{2}/3\,t$ that cannot be detected by a local in space analysis. Our method of proof relies on construction of approximate solutions to the equation satisfied by ψ, conjugation of the latter in order to eliminate several potential terms, and normal forms to get rid of problematic contributions in the nonlinearity. We use also Fermi's golden rule in order to prove that the $a(t)Y$ component decays when time grows.

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