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Cyclic Stratum of Frobenius Manifolds, Borel–Laplace (α, β)-Multitransforms, and Integral Representations of Solutions of Quantum Differential Equations

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Abstract

In the first part of this paper, we introduce the notion of *cyclic stratum* of a Frobenius manifold M. This is the set of points of the extended manifold $\mathbb{C}^* \times M$ at which the unit vector field is a cyclic vector for the isomonodromic system defined by the flatness condition of the extended deformed connection. The study of the geometry of the complement of the cyclic stratum is addressed. We show that at points of the cyclic stratum, the isomonodromic system attached to M can be reduced to a scalar differential equation, called the *master differential equation* of M. In the case of Frobenius manifolds coming from Gromov–Witten theory, namely quantum cohomologies of smooth projective varieties, such a construction reproduces the notion of quantum differential equation.

In the second part of the paper, we introduce two multilinear transforms, called *Borel–Laplace* $(\boldsymbol{\alpha}, \boldsymbol{\beta})$*-multitransforms*, on spaces of Ribenboim formal power series with exponents and coefficients in an arbitrary finite-dimensional $\mathbb{C}$-algebra A. When A is specialized to the cohomology of smooth projective varieties, the integral forms of the Borel–Laplace $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-multitransforms are used in order to rephrase the Quantum Lefschetz theorem. This leads to explicit Mellin–Barnes integral representations of solutions of the quantum differential equations for a wide class of smooth projective varieties, including Fano complete intersections in projective spaces.

In the third and final part of the paper, as an application, we show how to use the new analytic tools, introduced in the previous parts, in order to study the quantum differential equations of Hirzebruch surfaces. For Hirzebruch surfaces diffeomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, this analysis reduces to the simpler quantum differential equation of $\mathbb{P}^1$. For Hirzebruch surfaces diffeomorphic to the blow-up of $\mathbb{P}^2$ in one point, the quantum differential equation is integrated via Laplace $(1, 2; \frac{1}{2}, \frac{1}{3})$-multitransforms of solutions of the quantum differential equations of $\mathbb{P}^1$ and $\mathbb{P}^2$, respectively. This leads to explicit integral representations for the Stokes bases of solutions of the quantum differential equations, and finally to the proof of the Dubrovin conjecture for all Hirzebruch surfaces.

In memoria di mio padre

Keywords. Frobenius manifolds, quantum cohomology, isomonodromic deformations, integral transforms, derived categories, Dubrovin conjecture

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Contents

Chapter 1

Introduction

1.1 Background

In the last decades, we have been witnessing a growing and fruitful interaction between theoretical physics and various branches of geometry, leading to new developments in both disciplines. *Enumerative geometry* – an old subject and an active field in the 19th century – has been revolutionized by new ideas from the physics of string theory. After the categorical axiomatization of physical theories of quantum fields [4, 5, 72], the emergence of new mathematical objects was noticed. In such an inspiring context, rich structures known as Frobenius manifolds naturally arise, together with the construction of several invariants of symplectic and algebraic varieties.

The notion of *Frobenius manifolds* was introduced by B. Dubrovin, who first recognized its emergence in the study of classification of two-dimensional topological field theories [29–31]. A Frobenius manifold consists[1] of a complex manifold M whose tangent spaces admit an associative, commutative, and unital algebra structure $(T_pM, \circ_p)$, holomorphically depending on the point $p \in M$. The structure is further enriched with a non-degenerate symmetric bilinear form η, whose Levi-Civita connection is flat, and which is compatible with the product, that is,

$$\eta(Y \circ W, Z) = \eta(Y, W \circ Z)$$

for any local vector fields Y, W, Z on M. This condition makes $(T_pM, \circ_p, \eta_p)_{p\in M}$ a family of Frobenius algebras. Pretty soon, it was understood that Frobenius manifolds are a unifying notion in mathematics. These structures play a central role in mirror symmetry, theory of unfolding spaces of singularities, and enumerative geometry [48,61,71]. Remarkably enough, results proved for classes of Frobenius manifolds emerging in a certain mathematical theory turn out to be valid in general. This *universality* of Frobenius manifolds usually leads to unexpected connections between the aforementioned mathematical theories [33].

Quantum cohomology, introduced by E. Witten [79] and C. Vafa [77] in their study of topological non-linear sigma model, is one of the most interesting example of Frobenius manifold, associated with any complex smooth projective variety X, or a more general compact symplectic manifold [30, 58, 61]. From the physical point of view, the space X is the target of two-dimensional fields, and the Frobenius algebras that arise are a highly non-linear deformation of the classical cohomological

[1] Precise definitions will be given in the main body of the paper.

ring $H^\bullet(X,\mathbb{C})$. If the classical cohomology ring of a variety encodes information about the intersections of its subvarieties, the non-functorial construction of quantum cohomology is an instrument to understand how they are related by *rational* (or, in the general symplectic case, *pseudo-holomorphic*) curves. This information is codified in the *Gromov–Witten invariants* [45, 79, 80], used to define the quantum perturbation of the product. Gromov–Witten invariants count curves on X: for each $\beta \in H_2(X,\mathbb{Z})/\text{torsion}$, and cycles $Z_1,\dots,Z_n \subseteq X$ in general position, the Gromov–Witten invariant[2]

$$\langle \mathrm{PD}(Z_1),\dots,\mathrm{PD}(Z_n)\rangle^X_{g,n,\beta} \in \mathbb{Q}$$

heuristically equals the number of curves $C \subseteq X$, of genus g, with homology class $[C] = \beta$, and intersecting all the cycles Z_i. Consider the generating function

$$F_0^X(\gamma) = \sum_{n=0}^{\infty}\sum_{\beta}\frac{1}{n!}\langle\underbrace{\gamma,\dots,\gamma}_{n \text{ times}}\rangle^X_{0,n,\beta}, \qquad \gamma \in H^\bullet(X,\mathbb{C}),$$

of genus 0 Gromov–Witten invariants of X, and assume that this sum is convergent on a non-empty domain $\Omega \subseteq H^\bullet(X,\mathbb{C})$. The quantum cohomology $QH^\bullet(X)$ is the Frobenius manifold structure on Ω, the flat metric η being given by the Poincaré pairing

$$\eta(Y,W) := \int_X Y \cup W$$

for any local vector fields[3] Y, W on Ω, and the product $Y \circ W$ of vector fields being defined by the identity

$$\eta(Y \circ W, Z) = (YWZ)\, F_0^X$$

for arbitrary flat local vector fields Y, W, Z on Ω.

1.2 The main problem

At the core of the analytic theory of Frobenius manifolds, there is the local identification of semisimple[4] points $p \in M$ with the parameters of isomonodromic deformations of ordinary differential equations with rational coefficients. Such an identification – one of the main points of the theory of Dubrovin – was originally established in [30–32], and subsequently extended in [22–25].

[2] Here $\mathrm{PD}(\alpha)$ denotes the Poincaré dual class of α.

[3] The tangent space $T_p\Omega$ is canonically identified with $H^\bullet(X,\mathbb{C})$ for any $p \in \Omega$. Thus the $\cup$-product $Y \cup W$ of local vector fields is well defined.

[4] A point $p \in M$ is semisimple if the Frobenius algebra $(T_pM, \circ_p, \eta_p)$ is with no nilpotents.

In this paper, we mainly consider the example of analytic Frobenius manifolds given by the quantum cohomology $QH^\bullet(X)$ of a complex smooth projective variety X, see [30,58,61]. In such a case, points $p \in QH^\bullet(X)$ are parameters of isomonodromic deformations of a linear system of differential equations of the form

$$\frac{\partial}{\partial z}\zeta(z,p) = \left(\boldsymbol{\mathcal{U}}(p) + \frac{1}{z}\boldsymbol{\mu}(p)\right)\zeta(z,p). \tag{1.2.1}$$

Here ζ is a z-dependent vector field of $QH^\bullet(X)$, whereas $\boldsymbol{\mathcal{U}}$ and $\boldsymbol{\mu}$ are $(1,1)$-tensors on $QH^\bullet(X)$: the first[5] is the operator of quantum multiplication by the Euler vector field –a distinguished vector field on $QH^\bullet(X)$ which equals the first Chern class $c_1(X)$ along the locus of small quantum cohomology – the second, called *grading operator*, keeps track of the non-vanishing degrees of $H^\bullet(X,\mathbb{C})$.

Equation (1.2.1) is a rich object associated with the variety X: it encapsulates information not only about its *Gromov–Witten theory*, but also (conjecturally) about its *topology*, its *algebraic geometry*, and their mutual relations. The study of the monodromy of solutions of (1.2.1) is the way to disclose such an amount of information, see [21, 31, 36]. In this paper we address the following:

Main Problem. *Find integral representations of solutions of* (1.2.1) *for Fano complete intersections in Fano varieties.*

We split the main problem into two parts:

(1) reduce the system of differential equations (1.2.1) to a distinguished scalar linear differential equation, the *master differential equation*,

(2) find integral representations of solutions of master differential equations.

The study of these questions leads us to introduce some relevant notions, both in the analytic theory of Frobenius manifolds and in the theory of integral transforms. The first three ingredients are the notions of *cyclic stratum*, *master differential equations* and *master functions* of a Frobenius manifold. The second new analytical tool is a pair of integral multilinear transforms of functions, that we call *Borel–Laplace* $(\boldsymbol{\alpha},\boldsymbol{\beta})$*-multitransforms*. We are going to briefly outline these objects.

1.3 Master functions and master differential equations

The rich geometry of a Frobenius manifold M is (almost) completely encoded in integrability conditions of the *extended deformed connection* or *first structural connection* of M (see [30,32,61]). This is a flat meromorphic connection $\widehat{\nabla}$ defined on the pullback π^*TM of the tangent bundle of M on the extended manifold $\widehat{M} := \mathbb{C}^* \times M$,

[5] Precise definitions will be given in the main body of the paper.

by the natural projection $\pi\colon \widehat{M} \to M$. Equation (1.2.1) is equivalent to the equation

$$\widehat{\nabla}_{\frac{\partial}{\partial z}}\xi = 0, \quad \xi \in \Gamma(\pi^* T^* M), \tag{1.3.1}$$

the one-form ξ and the vector field ζ being identified via a flat metric η on M. We call *master function* at $p \in M$ any function[6] $\Phi_\xi \in \mathcal{O}(\widetilde{\mathbb{C}^*})$ of the form

$$\Phi_\xi(z) = z^{-\frac{d}{2}}\langle \xi(z,p), e(p)\rangle,$$

where ξ is as in (1.3.1), and d is the *charge* of the Frobenius manifold M.

In the first part of the paper, we address the problem of reducing the system of differential equations (1.3.1) to a scalar differential equation, whose coefficients depend on the point $p \in M$. This is a well-known problem in the theory of ordinary differential equations, equivalent to the choice of a *cyclic vector* [28, Lemma II.1.3]. On Frobenius manifold, however, we have a *natural* candidate, namely the unit vector field $e \in \Gamma(TM)$.

In Chapter 2 we introduce the *cyclic stratum* $\widehat{M}^{\rm cyc} \subseteq \widehat{M}$ defined as the set of points (z, p) at which the iterated covariant derivatives

$$e,\ \widehat{\nabla}_{\frac{\partial}{\partial z}}e,\ \widehat{\nabla}^2_{\frac{\partial}{\partial z}}e,\ \dots,\ \widehat{\nabla}^{n-1}_{\frac{\partial}{\partial z}}e, \quad n := \dim_{\mathbb{C}} M, \tag{1.3.2}$$

define a basis of the fiber $\pi^*TM|_{(z,p)}$. The complement of $\widehat{M}^{\rm cyc}$ in $\mathbb{P}^1 \times M$ admits a natural stratification, whose study is addressed in Section 2.6. A particular role is played by the $\mathcal{A}_\Lambda$-stratum of M, defined as the set of points $p \in M$ such that

$$\mathbb{C}^* \times \{p\} \subseteq \widehat{M} \setminus \widehat{M}^{\rm cyc}.$$

Introducing the cyclic coframe $\omega_0, \dots, \omega_{n-1} \in \Gamma(\pi^*T^*M)$ as the dual frame of the iterated covariant derivatives (1.3.2), the system of differential equations (1.3.1), specialized at points $p \in M \setminus \mathcal{A}_\Lambda$, reduces to a scalar differential equation – the *master differential equation* – in the function $\langle \xi, e\rangle$. Hence, at points $p \in M \setminus \mathcal{A}_\Lambda$, we obtain a one-to-one correspondence

$$\{\text{Solutions } \xi \text{ of system (1.3.1) specialized at } p\} \Longleftrightarrow \{\text{Master functions } \Phi_\xi \text{ at } p\}.$$

See Theorems 2.7.4 and 2.7.6. Thus, if integral representations for a basis of master functions are found, the main problem is *solved* at points in $M \setminus \mathcal{A}_\Lambda$.

Some motivational comments for introducing these new tools are in order. The notions of master functions and master differential equations define analogs, for an arbitrary Frobenius manifold, of well-known objects in Gromov–Witten and quantum cohomology theories. Namely, in the case of quantum cohomology the components of Givental's J-function (with respect to an *arbitrary* cohomology basis) define a gen-

[6] Here $\widetilde{\mathbb{C}^*}$ denotes the universal cover of $\mathbb{C}^*$.

erating set of master functions. Moreover, the master differential equation is (up to re-scaling of the unknown function) a quantum differential equation as defined, e.g., in [27, Section 10.3], see Chapter 5. In our opinion the concepts of cyclic stratum, master functions, and master differential equations may represent relevant notions in the analytic theory of Frobenius manifolds. For example, any contingent relations with the geometry of distinguished subsets of Frobenius manifolds (e.g., bifurcation diagram, Maxwell stratum, caustic) deserve further investigations. In that regard, it would be interesting to study relations with results of [22,23], concerning the isomonodromic description of Frobenius manifolds at semisimple coalescing points. This point will be addressed in a future publication.

1.4 Borel–Laplace multitransforms

In Chapter 6, we introduce a pair of multilinear transforms in both a formal and an analytical setting.

For $h \in \mathbb{N}^*$, and a given h-tuple $\boldsymbol{\kappa} \in (\mathbb{C}^*)^h$, we introduce a ring $\mathscr{F}_{\boldsymbol{\kappa}}(A)$ of Ribenboim generalized power series [68, 69] with both coefficients and exponents in a finite-dimensional, commutative, associative, and unitary $\mathbb{C}$-algebra A. The numbers κ_i play a role of "weights" for the exponents of the power series. In such a formal setting, given $\boldsymbol{\alpha}, \boldsymbol{\beta} \in (\mathbb{C}^*)^h$, we introduce the *Borel–Laplace* $(\boldsymbol{\alpha}, \boldsymbol{\beta})$*-multitransforms* as two A-multilinear maps rescaling the weights

$$\mathscr{B}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\colon \bigotimes_{j=1}^{h} \mathscr{F}_{\kappa_j}(A) \to \mathscr{F}_{\boldsymbol{\alpha}^{-1}\cdot\boldsymbol{\beta}^{-1}\cdot\boldsymbol{\kappa}}(A), \quad \boldsymbol{\alpha}^{-1}\cdot\boldsymbol{\beta}^{-1}\cdot\boldsymbol{\kappa} := \left(\frac{\kappa_1}{\alpha_1\beta_1}, \dots, \frac{\kappa_h}{\alpha_h\beta_h}\right),$$

$$\mathscr{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\colon \bigotimes_{j=1}^{h} \mathscr{F}_{\kappa_j}(A) \to \mathscr{F}_{\boldsymbol{\alpha}\cdot\boldsymbol{\beta}\cdot\boldsymbol{\kappa}}(A), \qquad \boldsymbol{\alpha}\cdot\boldsymbol{\beta}\cdot\boldsymbol{\kappa} := (\alpha_1\beta_1\kappa_1, \dots, \alpha_h\beta_h\kappa_h).$$

See Sections 6.2 and 6.3 for precise definitions.

In the analytical setting, given h functions $\Phi_1, \dots, \Phi_h\colon \widetilde{\mathbb{C}^*} \to A$, we define their Borel–Laplace $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-multitransforms by

$$\mathscr{B}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[\Phi_1, \dots, \Phi_h](z) := \frac{1}{2\pi i}\int_\gamma \prod_{j=1}^{h} \Phi_j\left(z^{\frac{1}{\alpha_j\beta_j}}\lambda^{-\beta_j}\right)e^{\lambda}\,\frac{d\lambda}{\lambda},$$

$$\mathscr{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[\Phi_1, \dots, \Phi_h](z) := \int_0^\infty \prod_{i=1}^{h} \Phi_i(z^{\alpha_i\beta_i}\lambda^{\beta_i})e^{-\lambda}\,d\lambda,$$

provided that the integrals exist. The contour γ is a Hankel-type contour beginning from $-\infty$, circling the origin once in the positive direction, and returning to $-\infty$ (see Figure 6.1).

1.5 Main results

Consider a Fano smooth projective variety X, and let $\iota\colon Y \to X$ be a Fano subvariety defined as the zero locus of a regular section of a vector bundle $E \to X$. The classical cohomology groups $H^k(Y,\mathbb{C})$ can be (partially) recovered by the cohomology groups $H^k(X,\mathbb{C})$ by the Lefschetz hyperplane theorem. The quantum Lefschetz theorem is a quantum improvement of the classical result: it describes how to reconstruct the Gromov–Witten theory of Y starting from the Gromov–Witten theory of X (see [15, 17, 60]).

In this paper, by using the quantum Lefschetz theorem, we give explicit integral representations of master functions of Y in terms of Laplace $(\boldsymbol{\alpha},\boldsymbol{\beta})$-multitransforms of master functions of the ambient space X under the following assumptions on X and E:

Case 1. We assume that E is a direct sum of fractional powers of the determinant bundle $\det TX$ of X.

Case 2. We assume that $X = X_1 \times \cdots \times X_h$ is a product of Fano varieties X_i, and that E is the external tensor product of fractional powers of the determinant bundles $\det TX_i$.

Our first main result concerns Case 1. Our Theorem 7.2.1 asserts that any master function of Y, at points $\iota^*\delta \in H^2(Y,\mathbb{C})$ of its small quantum cohomology, can be expressed in terms of iterated Laplace (α,β)-transforms (simple transforms of a single function) of master functions of X at the point $\delta \in H^2(X,\mathbb{C})$. More precisely, if $E = \bigoplus_{j=1}^r L^{\otimes d_j}$, and $\det TX = L^\ell$ for an ample line bundle L, then any master function of Y at $\iota^*\delta$ is a $\mathbb{C}$-linear combination of integrals of the form

$$\begin{aligned}
&e^{-c_\delta z}\mathscr{L}_{\frac{\ell-\sum_{i=1}^s d_i}{d_s},\frac{d_s}{\ell-\sum_{i=1}^{s-1} d_i}}\circ\cdots\circ\mathscr{L}_{\frac{\ell-d_1-d_2}{d_2},\frac{d_2}{\ell-d_1}}\circ\mathscr{L}_{\frac{\ell-d_1}{d_1},\frac{d_1}{\ell}}[\Phi]\\
&\qquad= e^{-c_\delta z}\int_0^\infty\cdots\int_0^\infty \Phi\left(z^{\frac{\ell-\sum_{j=1}^r d_j}{\ell}}\prod_{i=1}^r \zeta_i^{\frac{d_i}{\ell}}\right)e^{-\sum_{i=1}^r \zeta_i}\,d\zeta_1\ldots d\zeta_r,
\end{aligned}$$

where Φ is a master function of X at δ, and $c_\delta \in \mathbb{C}$ is a complex number depending on δ.

Our second main result concerns Case 2. In particular, Theorem 7.3.1 asserts that any master function of Y, at points $\iota^*\delta \in H^2(Y,\mathbb{C})$ of the small quantum locus, can be expressed in terms of Laplace $(\boldsymbol{\alpha},\boldsymbol{\beta})$-multitransforms of master functions of X_j at the point $\delta_j \in H^2(X,\mathbb{C})$, where

$$\delta = \sum_{j=1}^h 1\otimes\cdots\otimes\delta_j\otimes\cdots\otimes 1.$$

More precisely, if $E = \boxtimes_{j=1}^h L_j^{\otimes d_j}$ and $\det TX_j = L_j^{\ell_j}$ for ample line bundles L_j, any master function of Y at $\iota^*\delta$ is a $\mathbb{C}$-linear combination of integrals of the form

$$e^{-c_\delta z}\mathscr{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[\Phi_1,\ldots,\Phi_h](z) = e^{-c_\delta z}\int_0^\infty \prod_{j=1}^h \Phi_j\Big(z^{\frac{\ell_j-d_j}{\ell_j}}\lambda^{\frac{d_j}{\ell_j}}\Big)e^{-\lambda}\,d\lambda,$$

where $(\boldsymbol{\alpha},\boldsymbol{\beta}) = (\frac{\ell_1-d_1}{d_1},\ldots,\frac{\ell_h-d_h}{d_h};\frac{d_1}{\ell_1},\ldots,\frac{d_h}{\ell_h})$, Φ_j is a master function of X_j at δ_j, and $c_\delta \in \mathbb{C}$ is a complex number depending on δ.

Assumptions of Cases 1 and 2 are clearly satisfied when the varieties X and X_j have Picard rank one. Therefore, Theorems 7.2.1 and 7.3.1 can be applied to all Fano complete intersections in $\mathbb{P}^n$ and Fano hypersurfaces in products of projective spaces, in order to obtain explicit Mellin–Barnes integral representations of master functions. In particular, if $Y \subseteq \mathbb{P}^{n-1}$ is a Fano complete intersection defined by homogeneous polynomials of degrees $d_1,\ldots,d_h$, our Theorem 7.4.1 asserts that any master function of Y at $0 \in H^\bullet(Y,\mathbb{C})$ is a linear combination of *one*-dimensional Mellin–Barnes integrals ($j = 0,\ldots,n-1$)

$$G_j(z) := \frac{e^{-cz}}{2\pi\sqrt{-1}}\int_\gamma \Gamma(s)^n\prod_{k=1}^h\Gamma(1-d_ks)z^{-(n-\sum_{k=1}^h d_k)s}\varphi_j(s)\,ds,$$

where $c \in \mathbb{Q}$, γ is a parabola (of the form $\operatorname{Re}s = -\rho_1(\operatorname{Im}s)^2 + \rho_2$, for suitable $\rho_1,\rho_2 \in \mathbb{R}_+$) encircling the poles of the factor $\Gamma(s)^n$ and separating them from the poles of the factors $\Gamma(1-d_ks)$, and the function $\varphi_j(s)$ are defined by

$$\varphi_j(s) := \begin{cases}\exp(2\pi\sqrt{-1}js), & n \text{ even},\\ \exp(2\pi\sqrt{-1}js + \pi\sqrt{-1}s), & n \text{ odd}.\end{cases}$$

In the case of a Fano hypersurface $Y \subseteq \mathbb{P}^{n_1-1}\times\cdots\times\mathbb{P}^{n_h-1}$ defined by a homogeneous polynomial of multi-degree $(d_1,\ldots,d_h)$, then our Theorem 7.4.2 asserts that any master function of Y at $0 \in H^\bullet(Y,\mathbb{C})$ is a linear combination of the h-dimensional Mellin–Barnes integrals ($j = 0,\ldots,n-1$)

$$H_{\boldsymbol{j}}(z) := \frac{e^{-cz}}{(2\pi\sqrt{-1})^h}\int_{\times\gamma_i}\left[\prod_{i=1}^h\Gamma(s_i)^{n_i}\varphi^i_{j_i}(s_i)\right]\Gamma\left(1-\sum_{i=1}^h d_is_i\right)$$
$$\times z^{-\sum_{i=1}^h(n_i-d_i)s_i}\,ds_1\ldots ds_h,$$

where $c \in \mathbb{Q}$, γ_i are parabolas (of the form $\operatorname{Re}s_i = -\rho_{1,i}(\operatorname{Im}s_i)^2 + \rho_{2,i}$, for suitable $\rho_{1,i},\rho_{2,i} \in \mathbb{R}_+$) encircling the poles of the factors $\Gamma(s_i)^{n_i}$, and the functions $\varphi^i_{j_i}(s_i)$ are defined by

$$\varphi^i_{j_i}(s_i) := \begin{cases}\exp(2\pi\sqrt{-1}j_is_i), & n_i \text{ even},\\ \exp(2\pi\sqrt{-1}j_is_i + \pi\sqrt{-1}s_i), & n_i \text{ odd},\end{cases}$$

for any h-tuple $\boldsymbol{j} = (j_1,\ldots,j_h)$ with $0 \leqslant j_h \leqslant n_i - 1$.

Some comments are in order. Given a Fano variety X, Mirror Symmetry provides other kinds of integral representations of solutions of equation (1.3.1).[7] These are complex oscillating integrals associated with the Landau–Ginzburg models mirror to X, see [35, 39–41, 50, 57]. In these representations the cycles of integration are multi-dimensional.[8] This fact typically makes more difficult the study of the asymptotic expansions of solutions, and of the determination of the corresponding validity sectors in $\widetilde{\mathbb{C}^*}$. Furthermore, let us recall another technical issue which may be faced: Landau–Ginzburg models may not have enough critical points, and suitable compactification procedures have to be applied in order to recover the right number, see [43, 66, 70]. This could represent a delicate point for the computation of the Stokes bases of solutions of equation (1.2.1), whose exponential growth is ruled by the critical values of the Landau–Ginzburg potential.

We believe that one-dimensional Mellin–Barnes integrals of Theorem 7.4.1 represent a more advantageous representation of the solutions to the purpose of asymptotic analysis. Moreover, even for multi-dimensional Mellin–Barnes integrals of Theorem 7.4.2 the study of their asymptotics is tame: it is equivalent to the study of the asymptotics of one-dimensional generalized Faxén integrals

$$I(\lambda; c_1, \dots, c_r) := \int_0^\infty \exp\left[-\lambda\left(x^\mu + \sum_{k=1}^r c_k x^{m_k}\right)\right] dx,$$

with $\mu > m_1 > m_2 > \cdots > m_r > 0$, which have saddle points whose exponential contributions dominate algebraic terms in the asymptotic expansion. See [65, Chapter 7], [53, Section 5] for a detailed asymptotic analysis, and also [7, 13, 81] for some special cases. This will be exemplified in Section 11.6.

1.6 Dubrovin conjecture for Hirzebruch surfaces

Equation (1.2.1) has two singularities: a Fuchsian singularity at $z = 0$ and an irregular singularity at $z = \infty$ of Poincaré rank 1. The monodromy of its solutions is quantified by a finite set of matrices:

- a monodromy matrix M_0, quantifying the monodromy of solutions of (1.2.1) at $z = 0$,

[7] More precisely, for the equations $\widehat{\nabla}_{\frac{\partial}{\partial t^\alpha}} \xi = 0$, where $t^1, \dots, t^n$ are coordinates on $QH^\bullet(X)$, and not with respect to the spectral parameter z.

[8] Notice, for example, that already in the case of $\mathbb{P}^n$ these oscillating integrals are over n-dimensional cycles. On the other hand, one-dimensional Mellin–Barnes integral representations of solutions of equation (1.2.1) associated with $\mathbb{P}^n$ were obtained in [46]. Their asymptotics in sectors of $\widetilde{\mathbb{C}^*}$ is easier to study.

- a Stokes matrix S, describing the Stokes phenomenon at $z = \infty$,
- and a central connection matrix C gluing the monodromy data M_0 and S at the two singularities.

Remarkably, the monodromy data define a sort of "system of coordinates" in the space of solutions of WDVV equations: from the knowledge of their numerical values, the whole Frobenius manifold structure can be reconstructed via a Riemann–Hilbert problem [30, 32, 47].

In [31], B. Dubrovin formulated an intriguing conjecture concerning the geometrical meaning of the numerical values of the monodromy data of quantum cohomologies of Fano varieties. In the *qualitative* part of the conjecture, for a given Fano variety X, the semisimplicity condition of $QH^\bullet(X)$ is claimed to be equivalent to the existence of full exceptional collections in the derived category $\mathcal{D}^b(X)$ of coherent sheaves on X. Moreover, in the refined *quantitative* part of the conjecture, formulated in [21, Conjecture 5.2], the Stokes and central connection matrices (S_p, C_p) computed at any point $p \in QH^\bullet(X)$ are claimed to be determined by characteristic classes of X and of objects of a full exceptional collection $\mathfrak{E}_p$ in $\mathcal{D}^b(X)$.

In particular, the central connection matrix C_p is claimed to equal the matrix associated with the morphism

$$\begin{aligned} \mathcal{A}_X^-\colon K_0(X)_{\mathbb{C}} &\to H^\bullet(X,\mathbb{C}), \\ F &\mapsto \frac{(\sqrt{-1})^{\overline{d}}}{(2\pi)^{\frac{d}{2}}}\widehat{\Gamma}_X^-\exp(-\pi\sqrt{-1}c_1(X))\mathrm{Ch}(F), \end{aligned} \tag{1.6.1}$$

where $d = \dim_{\mathbb{C}} X$, $\overline{d}$ is its residue class modulo 2, $\widehat{\Gamma}_X^-$ is the characteristic class of X defined by

$$\widehat{\Gamma}_X^- := \prod_{j=1}^{\dim_{\mathbb{C}} X}\Gamma(1-\delta_j), \quad \delta_j \text{ Chern roots of } TX,$$

where

$$\Gamma(1-t) = \exp\left(\gamma t + \sum_{n=2}^{\infty}\frac{\zeta(n)}{n}t^n\right),$$

and $\mathrm{Ch}(F)$ is the graded Chern character defined on vector bundles by the formula $\mathrm{Ch}(V) := \sum_{j=1}^{\mathrm{rk}V}\exp(2\pi\sqrt{-1}\varepsilon_j)$, ε_j being the Chern roots of V. The matrix of $\mathcal{A}_X^-$ is computed with respect to the exceptional basis $[\mathfrak{E}_p]$ of $K_0(X)_{\mathbb{C}}$, defined by the K-theoretical classes of objects of $\mathfrak{E}_p$, and an arbitrary[9] basis of $H^\bullet(X,\mathbb{C})$. Furthermore, if the central connection matrix C_p is related to the morphism $\mathcal{A}_X^-$ as explained above, then the Stokes matrix S_p *automatically* equals the inverse of the Gram matrix

[9] The choice of a basis of $H^\bullet(X,\mathbb{C})$ in (1.6.1) corresponds to the choice of a system of flat coordinates on $QH^\bullet(X)$ with respect to which the monodromy data (M_0, S, C) are computed.

of the Grothendieck–Euler–Poincaré χ-pairing on $K_0(X)$ with respect to the exceptional basis $[\mathfrak{E}_p]$, see [21, Corollary 5.8].

It is important to stress that the monodromy data (M_0, S, C) are defined up to several choices: the choice of a system of flat coordinates on the Frobenius manifold $QH^\bullet(X)$, choices of normalizations (at both $z = 0$ and $z = \infty$) of solutions of equation (1.2.1), and the choice of an "admissible ray" in $\mathbb{C}^*$. Remarkably, all these operations have a geometrical counterpart in derived categories, see [21, Theorem 5.9]. Deserving special mention is Γ-conjecture II of [36]: it consists of an equivalent conjectural statement about the central connection matrix, though with respect to a choice of a solution in "Levelt form" at $z = 0$ not *natural* from the point of view of the theory of Frobenius manifolds. See [21, Section 5.6] for details.

The explicit computation of the monodromy data of quantum cohomologies is typically a rather delicate operation. To the best knowledge of the author, the only cases in which the computation of the complete set of monodromy data (S, C) of equation (1.2.1) has been carried out in all the details (including the determination of the corresponding full exceptional collections) are the cases of projective spaces [32, 46] and of complex Grassmannians [21, 36]. We believe that the main results of the current paper, namely the integral representations described in Theorems 7.2.1, 7.3.1, 7.4.1, and 7.4.2, will represent a fundamental tool for the development of this study [20].

As an application, in Chapters 10 and 11, we will show how to use the Laplace $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-multitransform, and the main results described above, in order to prove the quantitative part of the Dubrovin conjecture for Hirzebruch surfaces [49]. These are surfaces $\mathbb{F}_k$, $k \in \mathbb{Z}$, defined as the total space of the projective bundle $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-k))$ on $\mathbb{P}^1$. The interest of this example is highlighted by the fact that

- only two Hirzebruch surfaces are Fano varieties (namely $\mathbb{F}_0$ and $\mathbb{F}_1$),
- all others Hirzebruch surfaces are deformation equivalent to either $\mathbb{F}_0$ or $\mathbb{F}_1$.

Results of A. Bayer already suggested the non-necessity of the Fano assumption for the validity of the qualitative part of the Dubrovin conjecture, see [9]. Moreover, X. Hu proved that, in a smooth family of complete varieties, the existence of full exceptional collection on a fiber preserves for the fibers in a neighborhood, see [51]. See also [11, Corollary B] for an analogue result for arbitrary semiorthogonal decompositions. To the best of our knowledge, the study of the monodromy of the isomonodromic systems (1.2.1) associated with Hirzebruch surfaces, developed in Chapters 10 and 11, represents the first example in literature which addresses also the quantitative part of the Dubrovin conjecture, in both the non-Fano case and the case of deformations of the complex structures.

The case of Hirzebruch surfaces $\mathbb{F}_{2k}$ (resp. $\mathbb{F}_{2k+1}$) can be reduced to the single case of $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ (resp. $\mathbb{F}_1 = \mathrm{Bl}_{\mathrm{pt}}\mathbb{P}^2$). The monodromy data of $QH^\bullet(\mathbb{F}_0)$ can easily be reconstructed from the monodromy data of $QH^\bullet(\mathbb{P}^1)$, see Theorem 10.3.3.

In the case of $QH^\bullet(\mathbb{F}_1)$, the computation is more delicate, and reduces to the study of the quantum differential equation

$$\begin{aligned}(283z-24)\vartheta^4\Phi+(283z^2-590z+24)\vartheta^3\Phi+(-2264z^2+192z+3)\vartheta^2\Phi\\ -4z^2(2547z^2+350z-104)\vartheta\Phi\\ +z^2(-3113z^3-9924z^2+1476z+192)\Phi=0,\end{aligned}$$

where $\vartheta := z\frac{d}{dz}$. In Section 11.4, we show that the solutions of this equation can be expressed as linear combinations of integrals of the form

$$e^{-z}\mathscr{L}_{(1,2;\frac{1}{2},\frac{1}{3})}[\Phi_1,\Phi_2;z]=e^{-z}\int_0^\infty \Phi_1\big(z^{\frac{1}{2}}\lambda^{\frac{1}{2}}\big)\Phi_2\big(z^{\frac{2}{3}}\lambda^{\frac{1}{3}}\big)e^{-\lambda}d\lambda,$$

where Φ_1 and Φ_2 are solutions of quantum differential equations of $\mathbb{P}^1$ and $\mathbb{P}^2$, respectively, that is,

$$\vartheta^2\Phi_1=4z^2\Phi_1,\quad \vartheta^3\Phi_2=27z^3\Phi_2.$$

This allows the study of the asymptotics of solutions in sectors of $\widetilde{\mathbb{C}^*}$, to reconstruct the Stokes bases of solutions of the quantum differential equation of $\mathbb{F}_1$, and finally to the computation of both Stokes and central connection matrices, see Theorem 11.8.2.

From these results, the quantitative part of the Dubrovin conjecture is proved for all Hirzebruch surfaces $\mathbb{F}_k$, by making explicit the exceptional collections in $\mathcal{D}^b(\mathbb{F}_k)$ which arise from the monodromy data, see Theorems 10.3.3 and 11.8.3.

1.7 Plan of the paper

The paper is organized as follows. In Chapter 2, we introduce the notion of *cyclic stratum* in the general context of Frobenius manifolds theory. A first study of the geometry of the cyclic stratum, and its complement in the extended manifold $\mathbb{C}^*\times M$, is addressed.

In Chapter 3, we recall basic definitions in Gromov–Witten theory, including the definition of the Frobenius manifold structure on the quantum cohomology of a smooth projective variety. In Chapter 4, we recall the definitions of topological-enumerative solution of the isomonodromic system (1.2.1), and also of its monodromy data. We also recall the main properties and natural transformations of the complete set of monodromy data.

In Chapter 5, we recall the definition of Givental's J-function, and we explain how it is related to the space of master functions, see Theorem 5.1.2 and Corollary 5.1.3. We recall the formulation of the quantum Lefschetz theorem, and we obtain an upper bound for the dimension of the space of master functions of a Fano hypersurface of a smooth projective variety X, see Theorem 5.4.1.

In Chapter 6, we recall the notion of generalized power series in the sense of P. Ribenboim, and we introduce the ring $\mathscr{F}_{\boldsymbol{\kappa}}(A)$ of generalized power series with coefficients and exponents in a finite-dimensional $\mathbb{C}$-algebra. We introduce the notions of Borel–Laplace $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-multitransforms, in both formal and analytic setting, and we prove the compatibility of the two definitions, see Theorem 6.5.1.

In Chapter 7, we explain how the J-function can be identified (in several ways) with elements of rings of Ribenboim generalized power series. We prove the main results of this paper, Theorems 7.2.1, 7.3.1, 7.4.1 and 7.4.2.

In Chapter 8, we recall the notions of exceptional collections in derived categories of coherent sheaves, exceptional bases in K-theory, their mutations and helices. We then describe the refined statement of the Dubrovin conjecture, as formulated in [21].

In Chapter 9, we describe the classical and quantum cohomology rings of Hirzebruch surfaces.

In Chapter 10, we explicitly compute the monodromy data of the quantum cohomologies $QH^\bullet(\mathbb{F}_{2k})$, and we prove the Dubrovin conjecture for Hirzebruch surfaces $\mathbb{F}_{2k}$.

In Chapter 11, we address the study of the quantum differential equations of Hirzebruch surfaces $\mathbb{F}_{2k+1}$. We show how to use the Laplace $(1, 2; \frac{1}{2}, \frac{1}{3})$-multitransform in order to give integral representations of solutions, how to reconstruct Stokes fundamental solutions, and hence how to compute the monodromy data. This leads to a proof of the Dubrovin conjecture for Hirzebruch surfaces $\mathbb{F}_{2k+1}$.

Chapter 2

Cyclic stratum of Frobenius manifolds

2.1 Frobenius manifolds

Given a complex manifold M, we denote by TM (resp. T^*M) its holomorphic tangent (resp. cotangent) bundle. If E is a holomorphic vector bundle on M, we denote by $\bigodot^k E$ its k-th symmetrized tensor power, and by $\Gamma(E)$ the vector space of global holomorphic sections of E.

Definition 2.1.1. A *Frobenius manifold* structure on a complex manifold M of dimension n is defined by giving

(FM1) a symmetric $\mathcal{O}(M)$-bilinear form $\eta \in \Gamma(\bigodot^2 T^*M)$, called *metric*,[1] whose corresponding Levi-Civita connection ∇ is flat,

(FM2) a $(1,2)$-tensor $c \in \Gamma(TM \otimes \bigodot^2 T^*M)$ such that
   (a) the induced multiplication of vector fields $X \circ Y := c(-, X, Y)$, for $X, Y \in \Gamma(TM)$, is *associative*,
   (b) $c^\flat \in \Gamma(\bigodot^3 T^*M)$,
   (c) $\nabla c^\flat \in \Gamma(\bigodot^4 T^*M)$,

(FM3) a vector field $e \in \Gamma(TM)$, called the *unity vector field*, such that
   (a) the bundle morphism $c(-, e, -)\colon TM \to TM$ is the identity morphism,
   (b) $\nabla e = 0$,

(FM4) a vector field $E \in \Gamma(TM)$, called the *Euler vector field*, such that
   (a) $\mathfrak{L}_E c = c$,
   (b) $\mathfrak{L}_E \eta = (2 - d) \cdot \eta$, where $d \in \mathbb{C}$ is called the *charge* of the Frobenius manifold.

---

[1] In what follows, we will denote by $(-)^\flat$ and $(-)^\sharp$ the *musical isomorphisms* induced by the metric η. These are the isomorphisms between vector spaces of mixed tensors. If $v \in \Gamma(TM)$, the one-form $v^\flat \in \Gamma(T^*M)$ is defined by $v^\flat(w) = \eta(w, v)$, where $w \in \Gamma(TM)$. Conversely, if $\xi \in \Gamma(T^*M)$, the vector field $\xi^\sharp \in \Gamma(TM)$ is uniquely defined by the identity

$$\xi(w) = \eta(w, \xi^\sharp),$$

where $w \in \Gamma(TM)$. Thus, $(-)^\flat\colon \Gamma(TM) \to \Gamma(T^*M)$ and $(-)^\sharp\colon \Gamma(T^*M) \to \Gamma(TM)$ are mutually inverse. In components, these operations are also known as "lowering" and "raising" of indices, respectively. These operations naturally extend to mixed tensors. For example, given a $(1,2)$-tensor $c \in \Gamma(TM \otimes T^*M \otimes T^*M)$, the tensor $c^\flat$ is the $(0,3)$-tensor defined by

$$c^\flat(v_1, v_2, v_3) = \eta(v_1, c(v_2, v_3)),$$

where $v_1, v_2, v_3 \in \Gamma(TM)$.

At any point $p \in M$ the triple $(T_pM, \eta_p, \circ_p)$ is a complex *Frobenius algebra*, namely an associative commutative algebra with unity whose product is compatible with the metric, in the sense that

$$\eta_p(a \circ_p b, c) = \eta_p(a, b \circ_p c) \quad \text{for all } a, b, c \in T_pM,$$

by axioms (FM2-a), (FM2-b), (FM3-a). Moreover, there exist an open neighborhood $\Omega \subseteq M$ of p and a function $F\colon \Omega \to \mathbb{C}$ such that

$$c^\flat = \nabla^3 F,$$
$$\eta = \nabla_e \nabla^2 F.$$

This follows from axiom (FM2-b). Any such a function F will be called *potential* of M.

Remark 2.1.2. The Euler vector field E is an affine vector field, i.e.

$$\nabla^2 E = 0.$$

This follows[2] from axioms (FM1) and (FM4-b).

Convention. In this paper, we assume that the flat endomorphism $X \mapsto \nabla_X E$ of TM is *diagonalizable*. By introducing ∇-flat coordinates $\boldsymbol{t} = (t^\alpha)_{\alpha=1}^n$ on M, with respect to which the metric η is constant and the connection ∇ coincides with partial derivatives, we have that

$$E = \sum_{\alpha=1}^{n} ((1 - q_\alpha)t^\alpha + r_\alpha) \frac{\partial}{\partial t^\alpha}, \quad q_\alpha, r_\alpha \in \mathbb{C}.$$

Following [30–32], we choose flat coordinates $\boldsymbol{t}$ so that $\frac{\partial}{\partial t^1} \equiv e$ and $r_\alpha \neq 0$ only if $q_\alpha = 1$. This can always be done, up to an affine change of coordinates.

[2]For a generic vector field X on a pseudo-Riemannian manifold (M, g), a simple computation (invoking the first Bianchi identities) shows that

$$\nabla_\beta \nabla_\alpha X_\lambda = \sum_\mu R_{\lambda\alpha\beta\mu} X^\mu + \frac{1}{2}(\nabla_\beta K_{\alpha\lambda} + \nabla_\alpha K_{\beta\lambda} - \nabla_\lambda K_{\alpha\beta}),$$

where

$$K_{\alpha\beta} = (\mathfrak{L}_X g)_{\alpha\beta} = \nabla_\alpha X_\beta + \nabla_\beta X_\alpha.$$

If X is Killing conformal, and $\mathfrak{L}_X g = \omega g$ for a function ω, then

$$\nabla_\beta \nabla_\alpha X_\lambda = \sum_\mu R_{\lambda\alpha\beta\mu} X^\mu + \frac{1}{2}(g_{\alpha\lambda}\partial_\beta\omega + g_{\beta\lambda}\partial_\alpha\omega - g_{\alpha\beta}\partial_\lambda\omega).$$

In our case $R = 0$ and ω is a constant function.

Remark 2.1.3. The associativity of the algebra is equivalent to the following conditions for F, called WDVV-equations:

$$\sum_{\gamma,\delta=1}^{n} \partial_\alpha \partial_\beta \partial_\gamma F \, \eta^{\gamma\delta} \partial_\delta \partial_\epsilon \partial_\nu F = \sum_{\gamma,\delta=1}^{n} \partial_\nu \partial_\beta \partial_\gamma F \, \eta^{\gamma\delta} \partial_\delta \partial_\epsilon \partial_\alpha F,$$

while axiom (FM4) is equivalent to

$$\eta_{\alpha\beta} = \partial_1 \partial_\alpha \partial_\beta F, \quad \mathfrak{L}_E F = (3-d)F + Q(\boldsymbol{t}),$$

with $Q(\boldsymbol{t})$ a quadratic expression in parameters t_α. Conversely, given a solution of the WDVV equations, satisfying the quasi-homogeneity conditions above, a structure of Frobenius manifold is naturally defined on an open subset of the space of parameters t^α.

Definition 2.1.4. Define the *grading operator* of M to be the tensor $\boldsymbol{\mu} \in \Gamma(TM \otimes T^*M)$ defined by

$$\boldsymbol{\mu}(Y) := \frac{2-d}{2} Y - \nabla_Y E, \quad Y \in \Gamma(TM).$$

In what follows we will also denote by $\boldsymbol{\mathcal{U}}$ the $(1,1)$-tensor defined by $\circ$-multiplication by the Euler vector field, i.e.

$$\boldsymbol{\mathcal{U}}(Y) := E \circ Y, \quad Y \in \Gamma(TM).$$

We denote by μ and $\mathcal{U}$ the matrices of components of the tensors $\boldsymbol{\mu}$, and $\boldsymbol{\mathcal{U}}$, respectively, with respect to the system $\boldsymbol{t}$ of ∇-flat coordinates.

2.2 Semisimple points and bifurcation set

Definition 2.2.1. A point $p \in M$ is *semisimple* if and only if the corresponding Frobenius algebra $(T_pM, *_p, \eta_p, \frac{\partial}{\partial t^1}|_p)$ is without nilpotents. Denote by $M_{\rm ss}$ the open dense subset of M of semisimple points.

In this paper, only generically semisimple Frobenius manifolds are considered. In other words, we will always assume $M_{\rm ss} \neq \emptyset$.

On $M_{\rm ss}$ there are n well-defined idempotent vector fields $\pi_1, \dots, \pi_n \in \Gamma(TM_{\rm ss})$, satisfying

$$\pi_i * \pi_j = \delta_{ij}\pi_i, \quad \eta(\pi_i, \pi_j) = \delta_{ij}\eta(\pi_i, \pi_i), \quad i, j = 1, \dots, n.$$

Theorem 2.2.2 ([29, 30, 32]). *The idempotent vector fields pairwise commute, that is,* $[\pi_i, \pi_j] = 0$ *for* $i, j = 1, \dots, n$. *Hence, there exist holomorphic local coordinates* $(u_1, \dots, u_n)$ *on* $M_{\rm ss}$ *such that* $\frac{\partial}{\partial u_i} = \pi_i$ *for* $i = 1, \dots, n$.

Definition 2.2.3. The coordinates $(u_1, \dots, u_n)$ of Theorem 2.2.2 are called *canonical coordinates*.

Proposition 2.2.4 ([30,32]). *Canonical coordinates are uniquely defined up to ordering and shifts by constants. The eigenvalues of the tensor* $\boldsymbol{\mathcal{U}}$ *define a system of canonical coordinates in a neighborhood of any semisimple point of* $M_{\rm ss}$.

Definition 2.2.5. Given a Frobenius manifold M, we call *bifurcation set* of M the set $\mathcal{B}_M$ of points $p \in M$ at which the spectrum of the operator $\mathcal{U}(p)$ is not simple, i.e. $u_i(p) = u_j(p)$ for some $i \neq j$.

Following the terminology of [21,23,25], the points of $\mathcal{B}_M$ which are semisimple are called *semisimple coalescing points*. We define the[3] *Maxwell stratum* of M to be the closure of the set of semisimple coalescing points, i.e. $\mathcal{M}_M := \overline{M_{\rm ss} \cap \mathcal{B}_M}$.

The *caustic* of M is the set-theoretic difference $\mathcal{K}_M := \mathcal{B}_M \setminus M_{\rm ss}$.

Lemma 2.2.6. *We have* $\mathcal{B}_M = \mathcal{M}_M \cup \mathcal{K}_M$. ■

Definition 2.2.7. We call *orthonormalized idempotent frame* a frame $(f_i)_{i=1}^n$ of $TM_{\rm ss}$ defined by

$$f_i := \eta(\pi_i, \pi_i)^{-\frac{1}{2}} \pi_i, \quad i = 1, \dots, n, \tag{2.2.1}$$

for arbitrary choices of signs of the square roots. The Ψ-matrix is the matrix of change of tangent frames $(\Psi_{i\alpha})_{i,\alpha=1}^n$, defined by

$$\frac{\partial}{\partial t^\alpha} = \sum_{i=1}^n \Psi_{i\alpha} f_i, \quad \alpha = 1, \dots, n.$$

Remark 2.2.8. In the orthonormalized idempotent frame, the operator $\boldsymbol{\mathcal{U}}$ is represented by a diagonal matrix, and the operator $\boldsymbol{\mu}$ by an antisymmetric matrix:

$$\begin{aligned} U &:= \mathrm{diag}(u_1, \dots, u_n), & \Psi \mathcal{U} \Psi^{-1} &= U, \\ V &:= \Psi \mu \Psi^{-1}, & V^T + V &= 0. \end{aligned}$$

2.3 Extended deformed connection

Given a Frobenius manifold M, we introduce the extended manifold $\widehat{M} := \mathbb{C}^* \times M$, and consider the pullback $\pi^* TM$ of the tangent bundle of M along the obvious projection $\pi \colon \widehat{M} \to M$. We will denote the natural lifts on $\widehat{M}$ of the tensors $\eta, c, e, E, \boldsymbol{\mu}$, $\boldsymbol{\mathcal{U}}$ by the same symbols. Moreover, we also denote by ∇ the pull-backed Levi-Civita connection: it is the connection on the vector bundle $\pi^* TM$, uniquely defined by the

[3] The name is taken from singularity theory: for Frobenius structures defined on the universal space of unfoldings of singularities the two notions coincide, see [1–3].

further requirement that

$$\nabla_{\frac{\partial}{\partial z}} Y = 0 \quad \text{for all } Y \in \pi^{-1}\mathscr{T}_M,$$

where z denotes the natural coordinate on $\mathbb{C}^*$, and $\mathscr{T}_M$ denotes the tangent sheaf of M. We are going now to define a second connection $\widehat{\nabla}$ on π^*TM which is a deformation of ∇.

Definition 2.3.1. We define the *extended deformed connection* $\widehat{\nabla}$ as the connection on π^*TM given by

$$\widehat{\nabla}_X Y = \nabla_X Y + zX \circ Y, \quad \widehat{\nabla}_{\frac{\partial}{\partial z}} Y = \nabla_{\frac{\partial}{\partial z}} Y + \boldsymbol{\mathcal{U}}(Y) - \frac{1}{z}\boldsymbol{\mu}(Y)$$

for all $X, Y \in \Gamma(\pi^*TM)$.

Theorem 2.3.2 ([32]). *The extended deformed connection $\widehat{\nabla}$ is flat. More precisely, its flatness is equivalent to the totality of the following conditions:*

(1) $\nabla c^\flat \in \Gamma(\odot^4 T^*M)$,

(2) *the product on each tangent space of M is associative,*

(3) $\nabla^2 E = 0$,

(4) $\mathfrak{L}_E c = c$. ■

The connection $\widehat{\nabla}$ induces a flat connection on π^*T^*M, denoted by the same symbol.

2.4 Cyclic stratum, and cyclic (co)frame

Definition 2.4.1. Given a Frobenius manifold M, we define infinitely many sections $e_j \in \Gamma(\pi^*TM)$ as

$$e_k := \widehat{\nabla}^k_{\frac{\partial}{\partial z}} e, \quad k \in \mathbb{N}.$$

We will call the *cyclic stratum* $\widehat{M}^{\text{cyc}}$ to be the maximal open subset U of $\widehat{M}$ such that the bundle $\pi^*TM|_U$ is trivial and the collection of sections $(e_k|_U)_{k=0}^{n-1}$ defines a basis of each fiber. On $\widehat{M}^{\text{cyc}}$ we will also introduce the dual coframe $(\omega_j)_{j=0}^{n-1}$, by imposing

$$\langle \omega_j, e_k \rangle = \delta_{jk}. \tag{2.4.1}$$

The frame $(e_k)_{k=0}^{n-1}$ will be called *cyclic frame*, and its dual $(\omega_j)_{j=0}^{n-1}$ *cyclic coframe*.

Definition 2.4.2. Define the matrix-valued function $\Lambda = (\Lambda_{i\alpha}(z, p))$, holomorphic on $\widehat{M}^{\text{cyc}}$, by the equation

$$\frac{\partial}{\partial t^\alpha} = \sum_{i=0}^{n-1} \Lambda_{i\alpha} e_i, \quad \alpha = 1, \ldots, n. \tag{2.4.2}$$

Remark 2.4.3. The Λ-matrix should be thought as an analogue of the Ψ-matrix. The former matrix relates the flat coordinate frame $(\frac{\partial}{\partial t^\alpha})_{\alpha=1}^n$ to the cyclic frame $(e_i)_{i=0}^{n-1}$, and the latter matrix relates the flat coordinate frame $(\frac{\partial}{\partial t^\alpha})_{\alpha=1}^n$ to the normalized idempotent frame $(f_i)_{i=1}^n$.

Lemma 2.4.4. *For $j = 1, \dots, n-1$, we have*

$$\widehat{\nabla}_{\frac{\partial}{\partial z}} \omega_j = -\omega_{j-1}.$$

Proof. From (2.4.1), for any $k = 0, \dots, n-2$, we have

$$\begin{aligned} \langle \widehat{\nabla}_{\frac{\partial}{\partial z}} \omega_j, e_k \rangle + \langle \omega_j, e_{k+1} \rangle = 0 \implies \widehat{\nabla}_{\frac{\partial}{\partial z}} \omega_j, e_k \rangle &= -\delta_{j,k+1} \\ \implies \widehat{\nabla}_{\frac{\partial}{\partial z}} \omega_j &= -\omega_{j-1}. \end{aligned}$$ ∎

Proposition 2.4.5. *The vector fields e_k, with $k \in \mathbb{N}$, have the following form:*

$$e_k = \sum_{j=0}^{k} \frac{1}{z^j} p_j^k(E),$$

where the vector fields $p_j^k(E)$ do not depend on z and satisfy the difference equations

$$\begin{aligned} &p_0^{k+1}(E) = E \circ p_0^k(E), \\ &p_j^{k+1}(E) = E \circ p_j^k(E) - \boldsymbol{\mu}(p_{j-1}^k(E)) + (1-j) p_{j-1}^k(E), \quad j = 1, \dots, k, \\ &p_{k+1}^{k+1}(E) = -\mu(p_k^k(E)) - k p_k^k(E), \end{aligned}$$

with the only initial datum $p_j^0(E) = \delta_{0j} \cdot e$. ∎

2.5 Properties of the function $\det \Lambda$

The holomorphic function $\det \Lambda \colon \widehat{M}^{\text{cyc}} \to \mathbb{C}^*$ extends meromorphically to a function on $\mathbb{P}^1 \times M$.

Theorem 2.5.1. *The function* $\det \Lambda$ *is a meromorphic function on* $\mathbb{P}^1 \times M$ *of the form*

$$\det \Lambda(z, p) = \frac{z^{\binom{n-1}{2}}}{z^{\binom{n-1}{2}} A_0(p) + \dots + A_{\binom{n-1}{2}}(p)},$$

where $A_0, \dots, A_{\binom{n-1}{2}}$ are holomorphic functions on M. Moreover, if $n > 2$ and if the eigenvalues of the grading operator $\boldsymbol{\mu}$ are not pairwise distinct, then the function $A_{\binom{n-1}{2}}$ is identically zero.

We need a preliminary result.

Lemma 2.5.2. *For $k \in \{0, \dots, n-1\}$, the polyvector field*

$$e_0 \wedge \cdots \wedge e_k \in \Gamma\Big(\bigwedge^{k+1} \pi^* TM\Big)$$

admits a pole at $\{0\} \times M$ of order at most $\binom{k}{2}$. More precisely, we have

$$e_0 \wedge \cdots \wedge e_k = w_0 + \frac{1}{z} w_1 + \cdots + \frac{1}{z^{\binom{k}{2}}} w_{\binom{k}{2}}, \quad w_j \in \Gamma\Big(\bigwedge^{k+1} \pi^* TM\Big),$$

with

$$w_{\binom{k}{2}} = (-1)^{\binom{k}{2}}\, e \wedge E \wedge \boldsymbol{\mu}(E) \wedge \boldsymbol{\mu}^2(E) \wedge \cdots \wedge \boldsymbol{\mu}^{k-1}(E).$$

Proof. By induction on k. For the base cases $k = 0$ and $k = 1$, we have $e_0 = e$ and $e_0 \wedge e_1 = e \wedge E$, respectively. So, for $k = 0, 1$ the claim holds true.

Assume that $e_0 \wedge \cdots \wedge e_{k-1}$ is of the form

$$e_0 \wedge \cdots \wedge e_{k-1} = w_0 + \frac{1}{z} w_1 + \cdots + \frac{1}{z^{\binom{k-1}{2}}} w_{\binom{k-1}{2}}$$

with

$$w_{\binom{k-1}{2}} = (-1)^{\binom{k-1}{2}}\, e \wedge E \wedge \boldsymbol{\mu}(E) \wedge \boldsymbol{\mu}^2(E) \wedge \cdots \wedge \boldsymbol{\mu}^{k-2}(E).$$

We have

$$e_0 \wedge \cdots \wedge e_k = \left(\sum_{j=0}^{\binom{k-1}{2}} z^{-j} w_j \right) \wedge \left(\sum_{\ell=0}^{k} z^{-\ell} p_\ell^k(E) \right).$$

We claim that the coefficient $w_{\binom{k-1}{2}} \wedge p_k^k(E)$ of $z^{-\binom{k-1}{2}-k}$ vanishes. Indeed, $p_k^k(E)$ is proportional to e: we have

$$p_k^k(E) = \frac{d}{2}\left(\frac{d}{2} - 1\right) \cdots \left(\frac{d}{2} - k + 1\right) e, \quad k \geqslant 0,$$

as it can easily be seen by induction (the key property is $\mu(e) = -\frac{d}{2} e$, together with the last difference equation of Proposition 2.4.5). Consequently, we have

$$w_{\binom{k-1}{2}} \wedge p_k^k(E) = c \cdot (e \wedge \cdots \wedge e) = 0.$$

Hence, the (possibly non-vanishing) most polar term of $e_0 \wedge \cdots \wedge e_k$ equals

$$\begin{aligned} z^{-\binom{k-1}{2}-k+1} \cdot w_{\binom{k-1}{2}} \wedge p_{k-1}^k(E) &= z^{-\binom{k}{2}} \cdot w_{\binom{k-1}{2}} \wedge ((-1)^{k-1} \mu^{k-1}(E)) \\ &= z^{-\binom{k}{2}} (-1)^{\binom{k}{2}}\, e \wedge E \wedge \boldsymbol{\mu}(E) \wedge \cdots \wedge \boldsymbol{\mu}^{k-1}(E). \end{aligned}$$

For the first equality we have used the difference equation for $p_{k-1}^k(E)$ of Proposition 2.4.5. ■

Proof of Theorem 2.5.1. The polyvector field $e_0 \wedge \cdots \wedge e_{n-1}$ has the form

$$e_0 \wedge \cdots \wedge e_{n-1} = w_0(p) + \frac{1}{z} w_1(p) + \cdots + \frac{1}{z^{\binom{n-1}{2}}} w_{\binom{n-1}{2}}(p), \tag{2.5.1}$$

where $w_0, w_1, \dots, w_{\binom{n-1}{2}}$ are holomorphic n-vector fields on M, by Lemma 2.5.2. Introduce holomorphic functions $A_0(p), \dots, A_{\binom{n-1}{2}}(p)$ such that

$$w_j(p) = A_j(p) \cdot \frac{\partial}{\partial t^1} \wedge \dots \wedge \frac{\partial}{\partial t^n}.$$

From the identity

$$\frac{\partial}{\partial t^1} \wedge \dots \wedge \frac{\partial}{\partial t^n} = \det \Lambda \cdot e_0 \wedge \dots \wedge e_{n-1},$$

we deduce

$$1 = \det \Lambda(z,p)\left(A_0(p) + \frac{1}{z} A_1(p) + \dots \frac{1}{z^{\binom{n-1}{2}}} A_{\binom{n-1}{2}}(p)\right).$$

The last statement on $A_{\binom{n-1}{2}}$ follows from the explicit formula for $w_{\binom{n-1}{2}}$ given in Lemma 2.5.2. ∎

Theorem 2.5.3. *We have*

$$A_0(p) = \frac{\prod_{i<j}(u_j(p) - u_i(p))}{\mathrm{Jac}(p)}, \quad \mathrm{Jac}(p) := \det\left(\frac{\partial u_i}{\partial t^\alpha}\right)\Big|_p.$$

Proof. The polyvector field w_0 in equation (2.5.1) is

$$w_0 = \bigwedge_{j=0}^{n-1} p_0^j(E).$$

By Proposition 2.4.5, we have

$$p_0^j(E) = E^{\circ j}, \quad j \in \mathbb{N},$$

and using the idempotent vielbein $(\frac{\partial}{\partial u_i})_{i=1}^n$, we can write w_0 as follows:

$$\begin{aligned} w_0 &= \begin{vmatrix} 1 & \dots & 1 \\ u_1 & \dots & u_n \\ u_1^2 & \dots & u_n^2 \\ & \vdots & \\ u_1^{n-1} & \dots & u_n^{n-1} \end{vmatrix} \frac{\partial}{\partial u_1} \wedge \dots \wedge \frac{\partial}{\partial u_n} \\ &= \left(\prod_{i<j}(u_j - u_i)\right) \frac{\partial}{\partial u_1} \wedge \dots \wedge \frac{\partial}{\partial u_n} \\ &= \left(\prod_{i<j}(u_j - u_i)\right) \cdot \frac{1}{\mathrm{Jac}} \cdot \frac{\partial}{\partial t^1} \wedge \dots \wedge \frac{\partial}{\partial t^n}. \end{aligned}$$ ∎

Remark 2.5.4. We also have

$$\frac{\partial}{\partial t^1} \wedge \dots \wedge \frac{\partial}{\partial t^n} = \det \Psi f_1 \wedge \dots \wedge f_n = \frac{\det \Psi}{\prod_{i=1}^n \eta(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_i})^{\frac{1}{2}}} \frac{\partial}{\partial u_1} \wedge \dots \wedge \frac{\partial}{\partial u_n},$$

so that

$$\mathrm{Jac}(p) = \left.\frac{\det \Psi}{\prod_{i=1}^n \eta(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_i})^{\frac{1}{2}}}\right|_p = \left.\frac{(\det \eta)^{\frac{1}{2}}}{\prod_{i=1}^n \eta(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_i})^{\frac{1}{2}}}\right|_p.$$

The last equality follows from $\Psi^T \Psi = \eta$.

2.6 Geometry of the complement of the cyclic stratum in $\mathbb{P}^1 \times M$

Let us consider the tuple of functions $(A_0, \dots, A_{\binom{n-1}{2}})$, and extend it to the sequence $(A_k)_{k\in\mathbb{N}}$ by setting $A_k = 0$ for $k > \binom{n-1}{2}$. Set

$$\overline{n} := \min\{j \in \mathbb{N} : A_h(p) = 0 \text{ for all } p \in M \text{ and all } h > j\}.$$

We necessarily have $0 \leqslant \overline{n} \leqslant \binom{n-1}{2}$. By Theorem 2.5.1, we have $\overline{n} < \binom{n-1}{2}$ if $\boldsymbol{\mu}$ has not simple spectrum. The function $\det \Lambda$ takes the form

$$\det \Lambda = \frac{z^{\overline{n}}}{z^{\overline{n}} A_0(p) + z^{\overline{n}-1} A_1(p) \cdots + A_{\overline{n}}(p)}.$$

Define the subsets $\mathcal{P}_\Lambda, M_0, M_\infty \subseteq \mathbb{P}^1 \times M$ and $\mathcal{A}_\Lambda, \mathcal{I}_\Lambda^\infty, \mathcal{I}_\Lambda^0 \subseteq M$ by

$$\begin{aligned}
\mathcal{P}_\Lambda &:= \{(z,p) \in \widehat{M} : z^{\overline{n}} A_0(p) + \cdots + A_{\overline{n}}(p) = 0\},\\
M_0 &:= \{0\} \times M,\\
M_\infty &:= \{\infty\} \times M,\\
\mathcal{A}_\Lambda &:= \{p \in M : A_0(p) = \cdots = A_{\overline{n}}(p) = 0\},\\
\mathcal{I}_\Lambda^\infty &:= \{p \in M : A_0(p) = 0\},\\
\mathcal{I}_\Lambda^0 &:= \{p \in M : A_{\overline{n}}(p) = 0\}.
\end{aligned}$$

Lemma 2.6.1. *We have the obvious inclusions*

$$\mathbb{C}^* \times \mathcal{A}_\Lambda \subseteq \mathcal{P}_\Lambda, \quad \mathcal{A}_\Lambda \subseteq \mathcal{I}_\Lambda^0 \cap \mathcal{I}_\Lambda^\infty. \qquad \blacksquare$$

The set $\mathcal{P}_\Lambda$ is an analytic subspace of $\mathbb{P}^1 \times M$ of codimension 1 along which the function $\det \Lambda$ admits a pole. The function $\det \Lambda$ admits poles along a further analytic subspace, namely $\{\infty\} \times \mathcal{I}_\Lambda^\infty$. See Table 2.1 and Figure 2.1.

The set $\mathcal{P}_\Lambda$ is the complement $\widehat{M} \setminus \widehat{M}^{\mathrm{cyc}}$ of the cyclic stratum. The complement of $\widehat{M}^{\mathrm{cyc}}$ in $\mathbb{P}^1 \times M$ is the disjoint union

$$\mathcal{P}_\Lambda \uplus M_0 \uplus M_\infty.$$

The geometry of $\mathcal{P}_\Lambda$ is rather complicated: in general it admits several irreducible components. For example, $\mathcal{A}_\Lambda$ itself does, and consequently also $\mathbb{C}^* \times \mathcal{A}_\Lambda$. The projection $\pi\colon \widehat{M} \to M$, if restricted to $\mathcal{P}_\Lambda \setminus (\mathbb{C}^* \times \mathcal{A}_\Lambda)$, defines a ramified covering of degree $\overline{n}$.

Poles of $\det\Lambda$	$\mathcal{P}_\Lambda \uplus (\{\infty\}\times\mathcal{I}^\infty_\Lambda)$
Zeros of $\det\Lambda$	$M_0 \setminus (\{0\}\times\mathcal{I}^0_\Lambda)$
Indeterminacy locus of $\det\Lambda$	$\{0\}\times\mathcal{I}^0_\Lambda$

Table 2.1. Location of poles, zeros and indeterminacy locus for the meromorphic function $\det\Lambda$ on $\mathbb{P}^1\times M$.

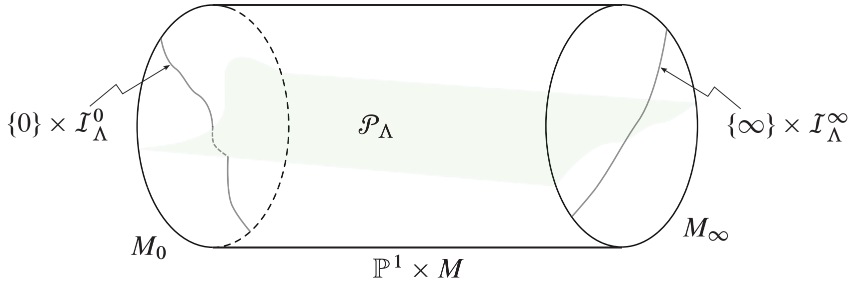

Figure 2.1. Configuration of the sets $\mathcal{P}_\Lambda$, $\{\infty\}\times\mathcal{I}^\infty_\Lambda$, and $\{0\}\times\mathcal{I}^0_\Lambda$ in $\mathbb{P}^1\times M$.

The set $\{0\}\times\mathcal{I}^0_\Lambda$ is an analytic subspace of $\mathbb{P}^1\times M$ of codimension 2 and it is the *indeterminacy locus* of the function $\det\Lambda$.

Each of the sets $\mathcal{I}^\infty_\Lambda, \mathcal{I}^0_\Lambda, \mathcal{A}_\Lambda$ seems to be strictly related to other distinguished subsets of the Frobenius manifold M, namely its bifurcation set $\mathcal{B}_M$, and its two components, the Maxwell stratum $\mathcal{M}_M$ and the caustic $\mathcal{K}_M$. We limit to the following observation.

Theorem 2.6.2. *We have $\mathcal{I}^\infty_\Lambda \subseteq \mathcal{B}_M$.*

Proof. Let $p\notin\mathcal{B}_M$. On the complement of $\mathcal{B}_M$, the eigenvalues $(u_1,\dots,u_n)$ define a holomorphic system of coordinates. Hence, $\operatorname{Jac}(p)\neq 0$. Moreover, by definition we have $\prod_{i<j}(u_j(p)-u_i(p))\neq 0$. Hence, $p\notin\mathcal{I}^\infty_\Lambda$ by Theorem 2.5.3. ∎

In order to obtain more precise results on contingent relations between the sets $\mathcal{I}^\infty_\Lambda$, $\mathcal{I}^0_\Lambda$, $\mathcal{A}_\Lambda$ and $\mathcal{B}_M$, $\mathcal{M}_M$, $\mathcal{K}_M$ a more detailed study of the polyvector fields $p^k_j(E)$ of Proposition 2.4.5 is needed. We plan to address this problem in a future project. We conclude this section with three low-dimensional examples.

Example. For two-dimensional Frobenius manifolds, we have $\mathcal{I}^\infty_\Lambda=\mathcal{B}_M$. In this case, indeed, we have

$$e_0=e,\ e_1=E+\frac{d}{2z}e \implies e_0\wedge e_1=e\wedge E.$$

The bivector $e\wedge E$ vanishes if and only if $u_1=u_2$.

Example. Consider the A_3-Frobenius manifold, that is, the space $M \cong \mathbb{C}^3$ of polynomials $f(x, \boldsymbol{a}) = x^4 + a_2 x^2 + a_1 x + a_0$, where $\boldsymbol{a} = (a_0, a_1, a_2) \in \mathbb{C}^3$ are natural coordinate. Fix $\boldsymbol{a}_o \in M$, and define the Kodaira–Spencer isomorphism

$$\kappa \colon T_{\boldsymbol{a}_o} M \to \mathbb{C}[x]/\langle \partial_x f(x, \boldsymbol{a}_o) \rangle$$

by identifying ∂_{a_i} with the class of the partial derivative $\partial_{a_i} f(x, \boldsymbol{a}_o)$. This allows to pull back the product of the Jacobi–Milnor algebra $\mathbb{C}[x]/\langle \partial_x f(x, \boldsymbol{a}_o) \rangle$ on $T_{\boldsymbol{a}_o} M$. Consider the Grothendieck residue metric

$$\eta_{\boldsymbol{a}} \left(\frac{\partial}{\partial a_i}, \frac{\partial}{\partial a_j} \right) := \frac{1}{2\pi i} \int_{\Gamma_{\boldsymbol{a}}} \left. \frac{\frac{\partial f}{\partial a_i} \frac{\partial f}{\partial a_j}}{\frac{\partial f}{\partial x}} \right|_{(u, \boldsymbol{a})} du,$$

where $\Gamma_{\boldsymbol{a}}$ is a circle, positively oriented, bounding a disc containing all the roots of $\frac{\partial f}{\partial x}(u, \boldsymbol{a})$. One can show that the coordinates $\boldsymbol{t} = (t_1, t_2, t_3)$ given by

$$t_1 = a_0 - \frac{1}{8} a_2^2, \quad t_2 = a_1, \quad t_3 = a_2,$$

are flat for the metric η. In $\boldsymbol{t}$-coordinates, the Euler vector field is given by

$$E = t_1 \frac{\partial}{\partial t_1} + \frac{3t_2}{4} \frac{\partial}{\partial t_2} + \frac{t_3}{2} \frac{\partial}{\partial t_3}.$$

The Maxwell stratum is the set $\{t_2 = 0\}$, and the caustic is the set $\{8t_3^3 + 27t_2^2 = 0\}$. We have the following formulas for the Λ-matrix and for $\det \Lambda$: Setting

$$a := z^2 t_3^5 - 21 z^2 t_2^2 t_3^2 - 64 z^2 t_1^2 t_3 - 12 t_3 - 18 z t_2^2 - 72 z^2 t_1 t_2^2$$

and

$$b := -3 z^2 t_3^4 - 16 z t_3^2 - 64 z^2 t_1 t_3^2 + 63 z^2 t_2^2 t_3 + 192 z^2 t_1^2 + 48 z t_1 + 48,$$

we get

$$\Lambda(z, \boldsymbol{t}) = \begin{pmatrix} 1 & \frac{a}{2 z t_2 (8 z t_3^3 - 6 t_3 + 27 z t_2^2)} & \frac{b}{4 z (8 z t_3^3 - 6 t_3 + 27 z t_2^2)} \\ 0 & \frac{4 (9 z t_2^2 + 16 z t_1 t_3)}{t_2 (8 z t_3^3 - 6 t_3 + 27 z t_2^2)} & -\frac{4(-4 z t_3^2 + 24 z t_1 + 3)}{8 z t_3^3 - 6 t_3 + 27 z t_2^2} \\ 0 & -\frac{32 z t_3}{t_2 (8 z t_3^3 - 6 t_3 + 27 z t_2^2)} & \frac{48 z}{8 z t_3^3 - 6 t_3 + 27 z t_2^2} \end{pmatrix}$$

and

$$\det \Lambda(z, \boldsymbol{t}) = \frac{64 z}{(8 t_2 t_3^3 + 27 t_2^3) z - 6 t_2 t_3}.$$

We have

$$\mathcal{I}_\Lambda^\infty = \mathcal{B}_M, \quad \mathcal{I}_\Lambda^0 = \mathcal{M}_M \cup \{t_3 = 0\}, \quad \mathcal{A}_\Lambda = \mathcal{M}_M.$$

Example. The $A_2 \times A_2$-Frobenius manifold is the Frobenius structure M on $\mathbb{C}^4$, with flat coordinates $(\boldsymbol{t}, \boldsymbol{s}) = (t_0, t_1, s_0, s_1)$, defined by the WDVV–potential

$$F(\boldsymbol{t}, \boldsymbol{s}) = \frac{1}{2}(t_0^2 t_1 + s_0^2 s_1) + \frac{1}{24}(t_1^4 + s_1^4).$$

In these coordinates, the unit vector field is $e = \frac{\partial}{\partial t_0} + \frac{\partial}{\partial s_0}$, and the flat metric η has components

$$\eta = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The Euler field equals

$$E = t_0 \frac{\partial}{\partial t_0} + \frac{2}{3} t_1 \frac{\partial}{\partial t_1} + s_0 \frac{\partial}{\partial s_0} + \frac{2}{3} s_1 \frac{\partial}{\partial s_1}.$$

The bifurcation diagram $\mathcal{B}_M$ equals $\mathcal{B}_M = \mathcal{M}_M \cup \mathcal{K}_M$, where the Maxwell stratum is

$$\mathcal{M}_M = \left\{ -8t_1^3\left(9(s_0 - t_0)^2 + 4s_1^3\right) + \left(4s_1^3 - 9(s_0 - t_0)^2\right)^2 + 16t_1^6 = 0 \right\},$$

and the caustic is

$$\mathcal{K}_M = \{t_1 = 0\} \cup \{s_1 = 0\}.$$

After some computations, one finds that

$$\det \Lambda(z, \boldsymbol{t}, \boldsymbol{s}) = 729z^2 \cdot \Big(4s_1 t_1 (z^2(-8t_1^3(9(s_0 - t_0)^2 + 4s_1^3) + (4s_1^3 - 9(s_0 - t_0)^2)^2 + 16t_1^6) + 45(s_0 - t_0)^2)\Big)^{-1}.$$

We have

$$\mathcal{I}_\Lambda^\infty = \mathcal{B}_M, \quad \mathcal{I}_\Lambda^0 = \mathcal{K}_M \cup \{s_0 = t_0\}, \quad \mathcal{A}_\Lambda = \mathcal{K}_M \cup \{s_0 = t_0, s_1^3 = t_1^3\}.$$

2.7 Master differential equation and master functions

Let $\xi \in \Gamma(\pi^* T^* M)$ be a $\widehat{\nabla}$-flat section. Consider the corresponding vector field $\zeta \in \Gamma(\pi^* TM)$ via musical isomorphism, i.e. such that

$$\xi(v) = \eta(\zeta, v)$$

for all $v \in \Gamma(\pi^* TM)$.

The vector field ζ satisfies the following system[4] of equations:

$$\frac{\partial}{\partial t^\alpha} \zeta = z \mathcal{C}_\alpha \zeta, \quad \alpha = 1, \dots, n, \tag{2.7.1}$$

$$\frac{\partial}{\partial z} \zeta = \left(\mathcal{U} + \frac{1}{z} \mu \right) \zeta. \tag{2.7.2}$$

Here $\mathcal{C}_\alpha$ is the $(1, 1)$-tensor defined by $(\mathcal{C}_\alpha)_\gamma^\beta := c_{\alpha\gamma}^\beta$.

[4] We consider the joint system (2.7.1)–(2.7.2) in matrix notations (ζ is a column vector whose entries are the components $\zeta^\alpha(z, \boldsymbol{t})$ with respect to $\frac{\partial}{\partial t^\alpha}$). Bases of solutions are arranged in invertible $n \times n$-matrices, called *fundamental systems of solutions*.

Multiply by η (on the left) the left-hand and right-hand sides of (2.7.1)–(2.7.2): we obtain the equivalent system of differential equations

$$\begin{cases} \dfrac{\partial}{\partial t^\alpha}\xi = z\mathcal{C}_\alpha^T \xi, \quad \alpha = 1, \dots, n, \\ \dfrac{\partial}{\partial z}\xi = \left(\mathcal{U}^T - \dfrac{1}{z}\mu\right)\xi, \end{cases} \tag{2.7.3}$$

where ξ is a column vector whose entries are the components $\xi_\alpha(z, \boldsymbol{t})$ with respect to dt^α. At points $(z, p) \in \widehat{M}^{\text{cyc}}$, let us introduce the column vector $\overline{\xi}$ by

$$\overline{\xi} = (\Lambda^{-1})^T \xi, \tag{2.7.4}$$

where Λ is defined as in (2.4.2). The entries of $\overline{\xi}$ are the components $\overline{\xi}_j$ with respect to the cyclic coframe ω_j. The vector $\overline{\xi}$ satisfies the system

$$\begin{cases} \dfrac{\partial\overline{\xi}}{\partial t^\alpha} = \left(z(\Lambda^{-1})^T \mathcal{C}_\alpha \Lambda^T + \dfrac{\partial(\Lambda^{-1})^T}{\partial t^\alpha}\Lambda^T\right)\overline{\xi}, \\ \dfrac{\partial\overline{\xi}}{\partial z} = \left((\Lambda^{-1})^T \mathcal{U}^T \Lambda^T - \dfrac{1}{z}(\Lambda^{-1})^T \mu \Lambda^T + \dfrac{\partial(\Lambda^{-1})^T}{\partial z}\Lambda^T\right)\overline{\xi}. \end{cases} \tag{2.7.5}$$

Proposition 2.7.1. *Let $\xi \in \Gamma(\pi^* T^* M)$ be a $\widehat{\nabla}$-flat section, and let $(\overline{\xi}_j(z, p))_{j=0}^{n-1}$ be its components with respect to the cyclic co-frame, i.e. $\xi = \sum_j \overline{\xi}_j \omega_j$. We have*

$$\frac{\partial\overline{\xi}_j}{\partial z} = \overline{\xi}_{j+1}, \quad j = 0, \dots, n-2.$$

Proof. We have

$$\begin{aligned} 0 = \widehat{\nabla}_{\frac{\partial}{\partial z}} \xi &= \sum_j \frac{\partial\overline{\xi}_j}{\partial z}\omega_j + \sum_j \overline{\xi}_j \widehat{\nabla}_{\frac{\partial}{\partial z}}\omega_j \\ &= \sum_j \frac{\partial\overline{\xi}_j}{\partial z}\omega_j - \sum_j \overline{\xi}_j \omega_{j-1}, \end{aligned}$$

by Lemma 2.4.4. The claim follows. ■

Corollary 2.7.2. *The system of differential equations* (2.7.5) *is the companion system of a* scalar *differential equation in $\overline{\xi}_0$.* ■

Remark 2.7.3. Note that $\xi_1 = \overline{\xi}_0$. Indeed, we have $e_0 = e = \frac{\partial}{\partial t^1}$, so that $\Lambda_{i1} = \delta_{i1}$. The claim then follows from (2.7.4).

Theorem 2.7.4. *Consider the system of differential equations* (2.7.3)*, specialized at a point $p \in M \setminus \mathcal{A}_\Lambda$. The system can be reduced to a single scalar ordinary differential equation of order n in the unknown function ξ_1. The scalar differential equation admits at most $\binom{n-1}{2}$ apparent singularities.*

Proof. If $p \in M \setminus \mathcal{A}_\Lambda$, then there exist $\overline{n}$ complex numbers $z_1, \dots, z_{\overline{n}}$, not necessarily distinct, such that $(z_i, p) \notin \widehat{M}^{\mathrm{cyc}}$. The numbers z_i are the zeros of the denominator of the function $\det \Lambda(z, p)$. ■

The scalar differential equation to which system (2.7.3) can be reduced will be called the *master differential equation* of M.

Definition 2.7.5. Fix a point $p \in M$. Consider the system of differential equations (2.7.3) specialized at p, and set $\mathcal{X}_p$ be the $\mathbb{C}$-vector space of its solutions. Then let $\nu_p \colon \mathcal{X}_p \to \mathcal{O}(\widetilde{\mathbb{C}^*})$ be the morphism defined by

$$\xi \mapsto \Phi_\xi(z), \quad \Phi_\xi(z) := z^{-\frac{d}{2}} \langle \xi(z, p), e(p) \rangle,$$

where d is the charge of the Frobenius manifold. Set $\mathcal{S}_p(M) := \operatorname{im}(\nu_p)$. Elements of $\mathcal{S}_p(M)$ will be called *master functions* at p.

Theorem 2.7.6. *At points $p \in M \setminus \mathcal{A}_\Lambda$ the morphism ν_p is injective.*

Proof. Given $\Phi_\xi \in \mathcal{S}_p(M)$, the function $\xi_1(z) = z^{\frac{d}{2}} \Phi_\xi(z)$ is a solution of the master differential equation at p. By Theorem 2.7.4, the solution $\xi(z)$ can be reconstructed from the component $\xi_1(z)$ only. ■

Chapter 3

Gromov–Witten theory

3.1 Notations and conventions

Let X be a smooth projective variety over $\mathbb{C}$. In order not to introduce superstructures, in what follows we assume that $H^{\text{odd}}(X, \mathbb{C}) = 0$. Denote by $b_k(X)$ the k-th Betti number of X.

Attached to X there is an infinite-dimensional $\mathbb{C}$-vector space $\mathcal{P}_X$, called the *big phase space*, defined as the infinite product of countable many copies of the classical cohomology space of X, that is,

$$\mathcal{P}_X := \prod_{n\in\mathbb{N}} H^\bullet(X, \mathbb{C}).$$

Let us fix a homogeneous basis $(T_0, \dots, T_N)$ of $H^\bullet(X, \mathbb{C})$ such that

- $T_0 = 1$,
- $T_1, \dots, T_r$ is a nef integral basis of $H^2(X, \mathbb{Z})$.

In particular, $b_2(X) = r$. Set $\boldsymbol{t} = (t^0, \dots, t^N)$, the dual coordinates of $H^\bullet(X, \mathbb{C})$.

We denote by $(\tau_p T_0, \dots, \tau_p T_N)$ the corresponding basis of the p-th copy of $H^\bullet(X, \mathbb{C})$ in $\mathcal{P}_X$. The element $\tau_p T_\alpha$ will be called a *descendant* of T_α with level p. The coordinate of a point $\boldsymbol{\gamma} \in \mathcal{P}_X$ with respect to the basis $(\tau_p T_\alpha)_{\alpha,p}$ will be denoted by $\boldsymbol{t}^\bullet = (t^{\alpha,p})_{\alpha,p}$. Instead of denoting by $\boldsymbol{\gamma} = (t^{\alpha,p}\tau_p T_\alpha)_{\alpha,p}$ a generic element of $\mathcal{P}_X$ we will usually write this as a formal series

$$\boldsymbol{\gamma} = \sum_{\alpha=1}^{m}\sum_{p=0}^{\infty} t^{\alpha,p}\tau_p T_\alpha.$$

We identify $H^\bullet(X, \mathbb{C})$ with the 0-th factor of $\mathcal{P}_X$, called the *small phase space*. This allow us to identify $t^\alpha \equiv t^{\alpha,0}$ for $\alpha = 0, \dots, N$.

Denote by η: $H^\bullet(X, \mathbb{C}) \times H^\bullet(X, \mathbb{C}) \to H^\bullet(X, \mathbb{C})$ the Poincaré pairing defined by

$$\eta(u, v) := \int_X u \cup v,$$

and we set $\eta_{\alpha\beta} := \eta(T_\alpha, T_\beta)$ for $\alpha, \beta = 0, \dots, N$. The numbers $\eta_{\alpha\beta}$ will be collected in the Gram[1] matrix $\eta = (\eta_{\alpha\beta})_{\alpha,\beta=0}^N$, with inverse matrix $\eta^{-1} = (\eta^{\alpha\beta})_{\alpha,\beta=0}^N$. We also

[1] We denote the metric tensor and its Gram matrix by the same symbol η. This is a standard abuse of notation.

introduce the dual basis $(T^0, \dots, T^N)$ of $H^\bullet(X, \mathbb{C})$, by setting

$$T^\alpha := \sum_{\lambda=0}^{N} T_\lambda \eta^{\lambda\alpha}, \quad \alpha = 0, \dots, N.$$

Define the *Novikov ring* Λ_X as the ring of formal sums

$$\sum_{\beta \in H_2(X,\mathbb{Z})} a_\beta \mathbf{Q}^\beta, \quad a_\beta \in \mathbb{Q},$$

such that

$$\operatorname{card}\left\{\beta : a_\beta \neq 0 \text{ and } \int_\beta \omega < C\right\} < \infty \quad \text{for any } C \in \mathbb{R},$$

where ω is the Kähler form of X.

3.2 Descendant Gromov–Witten invariants

For any given $g, n \in \mathbb{N}$ and $\beta \in H_2(X, \mathbb{Z})$, denote by $\overline{\mathcal{M}}_{g,n}(X, \beta)$ the Kontsevich–Manin moduli stack of genus g, n-pointed stable maps of degree β with target X: it parametrizes isomorphism classes of pairs $((C, \boldsymbol{x}), f)$, where

- C is a genus g nodal connected projective curve,
- $\boldsymbol{x} = (x_1, \dots, x_n)$ is an n-tuple of pairwise distinct points of the smooth locus of C,
- $f\colon C \to X$ is a morphism with $f_*[C] = \beta$,
- a morphism between two pairs $((C, \boldsymbol{x}), f)$ and $((C', \boldsymbol{x}'), f')$ is a morphism $\sigma\colon C \to C'$ such that $\sigma(x_i) = x_i'$ for all i, and making commutative the diagram

$$\begin{tikzcd} C \arrow[rr, "\sigma"] \arrow[dr, "f"'] & & C' \arrow[dl, "f'"] \\ & X, & \end{tikzcd}$$

- the group of automorphisms of $((C, \boldsymbol{x}), f)$ is finite.

The moduli space $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is a proper Deligne–Mumford stack of virtual dimension

$$\operatorname{vir}\dim_{\mathbb{C}} \overline{\mathcal{M}}_{g,n}(X, \beta) := (1-g)(\dim_{\mathbb{C}} X - 3) + \int_\beta c_1(X) + n.$$

We denote by $\mathcal{L}_i$, with $i = 1, \dots, n$, the i-th *tautological line bundle* on $\overline{\mathcal{M}}_{g,n}(X, \beta)$ whose fiber at the point $[((C, \boldsymbol{x}), f)] \in \overline{\mathcal{M}}_{g,n}(X, \beta)$ is the cotangent space $T^*_{x_i} C$. Set $\psi_j := c_1(\mathcal{L}_j)$ for $j = 1, \dots, n$.

We have naturally defined *evaluation morphisms*

$$\mathrm{ev}_i\colon \overline{\mathcal{M}}_{g,n}(X,\beta) \to X, \quad [((C,\boldsymbol{x}), f)] \mapsto f(x_i)$$

for $i = 1, \dots, n$.

Definition 3.2.1. Let $d_1, \dots, d_n$ be non-negative integers. The *genus g descendant Gromov–Witten invariants* (or *genus g gravitational correlators*) are the rational numbers defined by the integrals

$$\langle \tau_{d_1}\alpha_1, \dots, \tau_{d_n}\alpha_n \rangle^X_{g,n,\beta} := \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{virt}}} \prod_{j=1}^n \psi_j^{d_j} \cup \mathrm{ev}_j^*(\alpha_j),$$

where $\alpha_1, \dots, \alpha_n \in H^\bullet(X, \mathbb{C})$, and the class

$$[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{virt}} \in \mathrm{CH}_D(\overline{\mathcal{M}}_{g,n}(X,\beta)), \quad D = \mathrm{vir}\dim_{\mathbb{C}} \overline{\mathcal{M}}_{g,n}(X,\beta),$$

denotes the virtual fundamental class of $\overline{\mathcal{M}}_{g,n}(X,\beta)$.

Definition 3.2.2. The *genus g total descendant potential* of X is the generating function $\mathcal{F}^X_g \in \Lambda_X[\![\boldsymbol{t}^\bullet]\!]$ of descendant GW-invariants of X defined by

$$\begin{aligned}\mathcal{F}^X_g(\boldsymbol{t}^\bullet, \mathbf{Q}) &:= \sum_{n=0}^\infty \sum_{\beta \in \mathrm{Eff}(X)} \frac{\mathbf{Q}^\beta}{n!} \langle \boldsymbol{\gamma}, \dots, \boldsymbol{\gamma} \rangle^X_{g,n,\beta} \\ &= \sum_{n=0}^\infty \sum_\beta \sum_{\alpha_1,\dots,\alpha_n=0}^N \sum_{p_1,\dots,p_n=0}^\infty \frac{t^{\alpha_1,p_1} \dots t^{\alpha_n,p_n}}{n!} \\ &\qquad \cdot \langle \tau_{p_1} T_{\alpha_1}, \dots, \tau_{p_n} T_{\alpha_n} \rangle^X_{g,n,\beta} \mathbf{Q}^\beta.\end{aligned}$$

Setting $t^{\alpha,0} = t^\alpha$ and $t^{\alpha,p} = 0$ for $p > 0$, we obtain the *genus g Gromov–Witten potential* of X

$$F^X_g(\boldsymbol{t}, \mathbf{Q}) := \sum_{n=0}^\infty \sum_\beta \sum_{\alpha_1,\dots,\alpha_n=0}^N \frac{t^{\alpha_1} \dots t^{\alpha_n}}{n!} \langle T_{\alpha_1}, \dots, T_{\alpha_n} \rangle^X_{g,n,\beta} \mathbf{Q}^\beta.$$

It will also be convenient to introduce the *genus g correlation functions* defined by the derivatives

$$\langle\!\langle \tau_{d_1} T_{\alpha_1}, \dots, \tau_{d_n} T_{\alpha_n} \rangle\!\rangle_g := \frac{\partial}{\partial t^{\alpha_1,d_1}} \cdots \frac{\partial}{\partial t^{\alpha_n,d_n}} \mathcal{F}^X_g(\boldsymbol{t}^\bullet, \mathbf{Q}) \Big|_{\substack{t^{\alpha,p}=0 \text{ for } p>0 \\ t^{\alpha,0}=t^\alpha}}.$$

3.3 Quantum cohomology

Let $\beta_1, \dots, \beta_r \in H_2(X, \mathbb{Z})$ be the homology classes dual to $T_1, \dots, T_r$. By the divisor axiom, the genus 0 Gromov–Witten potential $F^X_0(\boldsymbol{t}, \mathbf{Q})$ can be seen as an element of the ring $\mathbb{C}[\![t^0, \mathbf{Q}^{\beta_1} e^{t^1}, \dots, \mathbf{Q}^{\beta_r} e^{t^r}, t^{r+1}, \dots, t^N]\!]$. In what follows we will be interested in the cases when F_0^X is a convergent series expansion

$$F_0^X \in \mathbb{C}\{t^0, \mathbf{Q}^{\beta_1} e^{t^1}, \dots, \mathbf{Q}^{\beta_r} e^{t^r}, t^{r+1}, \dots, t^N\}. \tag{3.3.1}$$

Without loss of generality we can put $\mathbf{Q} = 1$. Under assumption (3.3.1), $F_0^X(\boldsymbol{t})$ defines an analytic function in an open neighborhood $\Omega \subseteq H^\bullet(X, \mathbb{C})$ of the point

$$t^i = 0, \quad i = 0, r+1, \dots, N, \qquad \operatorname{Re} t^i \to -\infty, \quad i = 1, 3, \dots, r.$$

The function F_0^X is a solution of WDVV equations [58,61,76,78], and thus it defines an analytic Frobenius manifold structure on Ω. Using the canonical identifications of tangent spaces $T_p\Omega \cong H^\bullet(X;\mathbb{C})$: $\partial_{t^\alpha} \mapsto T_\alpha$, the unit vector field is $e = \partial_{t^0} \equiv 1$, and the Euler vector field is

$$E := c_1(X) + \sum_{\alpha=0}^{N} \left(1 - \frac{1}{2} \deg T_\alpha\right) t^\alpha T_\alpha,$$

which satisfies

$$\mathfrak{L}_E F_0^X = (3 - \dim_{\mathbb{C}} X) F_0^X.$$

The Frobenius manifold structure on Ω can be extended by analytic continuation. The resulting maximal Frobenius structure is called *quantum cohomology of X*, denoted $QH^\bullet(X)$.

In the recent paper [18], a useful convergence criterion for formal power series solutions of WDVV equations is given. In the case of quantum cohomologies of Fano varieties, we have the following result.

Assume that X is Fano, and let us consider the finite-dimensional $\mathbb{C}$-algebra $(H^\bullet(X, \mathbb{C}), \circ_0)$, where the product $\circ_0$ is defined by

$$T_\alpha \circ_0 T_\gamma = \sum_{\lambda=0}^{N} c_{\alpha\gamma}^{\lambda} T_\lambda, \quad \alpha, \gamma = 0, \dots, N,$$

where

$$c_{\alpha\gamma}^{\lambda} := \sum_{\varepsilon=0}^{N} \sum_{\beta \in \mathrm{Eff}(X)} \langle T_\alpha, T_\gamma, T_\varepsilon \rangle_{0,3,\beta}^X \eta^{\varepsilon\lambda}, \quad \alpha, \gamma = 0, \dots, N.$$

Notice that the sums defining the structure constants $c_{\alpha\beta}^{\lambda}$ are finite, due to the Fano assumption.

Theorem 3.3.1 ([18]). *If $(H^\bullet(X, \mathbb{C}), \circ_0)$ is semisimple, then the Gromov–Witten potential $F_0^X(\boldsymbol{t}, \mathbf{Q})$ is convergent. That is, condition (3.3.1) holds.* ∎

For a further convergence result, beyond the Fano case, see [18, Sec. 6]. See also [19], where the convergence criteria of [18] have been generalized to solutions of the more general "oriented associativity equations" [62].

Chapter 4

Monodromy data of quantum cohomology

4.1 Topological-enumerative solution

For $\beta = 0, \dots, N$ and $k \in \mathbb{N}$, introduce the functions

$$\theta_{\beta,k}(\boldsymbol{t}) := \langle\!\langle \tau_k T_\beta, 1 \rangle\!\rangle_0 \big|_{\mathbf{Q}=1}, \quad \theta_\beta(z, \boldsymbol{t}) := \sum_{k=0}^{\infty} \theta_{\beta,k}(\boldsymbol{t}) z^k.$$

Define the matrix $\Theta(z, \boldsymbol{t})$ by

$$\Theta(z, \boldsymbol{t})^\alpha_\beta := \sum_{\lambda=0}^{N} \eta^{\alpha\lambda} \frac{\partial \theta_\beta(z, \boldsymbol{t})}{\partial t^\lambda}, \quad \alpha, \beta = 0, \dots, N.$$

Denote by R the matrix associated with the morphism

$$c_1(X) \cup \colon H^\bullet(X, \mathbb{C}) \to H^\bullet(X, \mathbb{C}), \quad x \mapsto c_1(X) \cup x,$$

with respect to the basis $(T_0, \dots, T_N)$.

Let us consider the joint system (2.7.1)–(2.7.2) attached to the Frobenius manifold $QH^\bullet(X)$.

Theorem 4.1.1 ([23, 32]). *The matrix $Z_{\text{top}}(z, \boldsymbol{t}) := \Theta(z, \boldsymbol{t}) z^\mu z^R$ is a fundamental system of solutions of the joint system* (2.7.1)–(2.7.2). ∎

The fundamental system of solutions $Z_{\text{top}}(z, \boldsymbol{t})$ is called *topological-enumerative solution* of the joint system (2.7.1)–(2.7.2).

Let $M_0(\boldsymbol{t})$ be the monodromy matrix defined by

$$Z_{\text{top}}(e^{2\pi\sqrt{-1}} z, \boldsymbol{t}) = Z_{\text{top}}(z, \boldsymbol{t}) M_0(\boldsymbol{t}), \quad z \in \widetilde{\mathbb{C}^*}.$$

Lemma 4.1.2. *We have*

$$M_0(\boldsymbol{t}) = \exp(2\pi\sqrt{-1}\mu) \exp(2\pi\sqrt{-1} R).$$

In particular, M_0 does not depend on $\boldsymbol{t}$. ∎

4.2 Stokes rays and ℓ-chamber decomposition

Definition 4.2.1. We call *Stokes rays* at a point $p \in \Omega$ the oriented rays $R_{ij}(p)$ in $\mathbb{C}$ defined by

$$R_{ij}(p) := \left\{ -\sqrt{-1}\,(\overline{u_i(p)} - \overline{u_j(p)})\rho : \rho \in \mathbb{R}_+ \right\},$$

where $(u_1(p), \dots, u_n(p))$ is the spectrum of the operator $\mathcal{U}(p)$ (with a fixed arbitrary order).

Fix an oriented ray ℓ in the universal cover $\widetilde{\mathbb{C}^*}$.

Definition 4.2.2. We say that ℓ is *admissible* at $p \in \Omega$ if the projection of the ray ℓ on $\mathbb{C}^*$ does not coincide with any Stokes ray $R_{ij}(p)$.

Definition 4.2.3. Define the open subset O_ℓ of points $p \in \Omega$ by the following conditions:

(1) the eigenvalues $u_i(p)$ are pairwise distinct,

(2) ℓ is admissible at p.

We call *ℓ-chamber* of Ω any connected component of O_ℓ.

4.3 Stokes fundamental solutions at $z = \infty$

Fix an oriented ray $\ell \equiv \{\arg z = \varphi\}$ in $\widetilde{\mathbb{C}^*}$. For $m \in \mathbb{Z}$, define the following sectors in $\widetilde{\mathbb{C}^*}$:

$$\Pi_{L,m}(\varphi) := \{z \in \widetilde{\mathbb{C}^*} : \varphi + 2\pi m < \arg z < \varphi + \pi + 2\pi m\},$$
$$\Pi_{R,m}(\varphi) := \{z \in \widetilde{\mathbb{C}^*} : \varphi - \pi + 2\pi m < \arg z < \varphi + 2\pi m\}.$$

Denote by $\mathcal{B}_X$ the bifurcation diagram of the quantum cohomology of X.

Theorem 4.3.1 ([30, 32]). *There exists a unique formal solution $Z_{\text{form}}(z, \boldsymbol{t})$ of the joint system* (2.7.1)–(2.7.2) *of the form*

$$Z_{\text{form}}(z, \boldsymbol{t}) = \Psi(\boldsymbol{t})^{-1} G(z, \boldsymbol{t}) \exp(zU(\boldsymbol{t})),$$
$$G(z, \boldsymbol{t}) = I + \sum_{k=1}^{\infty} \frac{1}{z^k} G_k(\boldsymbol{t}),$$

where the matrices $G_k(\boldsymbol{t})$ are holomorphic on $\Omega \setminus \mathcal{B}_X$.

Theorem 4.3.2 ([30, 32]). *Let $m \in \mathbb{Z}$. There exist unique fundamental systems of solutions $Z_{L,m}(z, \boldsymbol{t})$ and $Z_{R,m}(z, \boldsymbol{t})$ of the joint system* (2.7.1)–(2.7.2) *with respective asymptotic expansion*

$$Z_{L,m}(z, \boldsymbol{t}) \sim Z_{\text{form}}(z, \boldsymbol{t}), \quad |z| \to \infty,\ z \in \Pi_{L,m}(\varphi),$$
$$Z_{R,m}(z, \boldsymbol{t}) \sim Z_{\text{form}}(z, \boldsymbol{t}), \quad |z| \to \infty,\ z \in \Pi_{R,m}(\varphi).$$

Definition 4.3.3. The solutions $Z_{L,m}(z, \boldsymbol{t})$ and $Z_{R,m}(z, \boldsymbol{t})$ are called *Stokes fundamental solutions* of the joint system (2.7.1), (2.7.2) on the sectors $\Pi_{L,m}(\varphi)$ and $\Pi_{R,m}(\varphi)$, respectively.

4.4 Monodromy data

Let $\ell \equiv \{\arg z = \varphi\}$ be an oriented ray in $\widetilde{\mathbb{C}^*}$ and consider the corresponding Stokes fundamental systems of solutions $Z_{L,m}(z,\boldsymbol{t})$, $Z_{R,m}(z,\boldsymbol{t})$ for $m \in \mathbb{Z}$.

Definition 4.4.1. We define the *Stokes* and *central connection* matrices $S^{(m)}(p)$, $C^{(m)}(p)$, with $m \in \mathbb{Z}$, at the point $p \in O_\ell$ by the identities

$$Z_{L,m}(z,\boldsymbol{t}(p)) = Z_{R,m}(z,\boldsymbol{t}(p))S^{(m)}(p), \quad z \in \widetilde{\mathbb{C}^*},$$
$$Z_{R,m}(z,\boldsymbol{t}(p)) = Z_{\text{top}}(z,\boldsymbol{t}(p))C^{(m)}(p), \quad z \in \widetilde{\mathbb{C}^*}.$$

Set $S(p) := S^{(0)}(p)$ and $C(p) := C^{(0)}(p)$.

Definition 4.4.2. The *monodromy data* at the point $p \in O_\ell$ are defined as the 4-tuple of matrices $(\mu, R, S(p), C(p))$, where

- μ is the matrix associated to the grading operator,
- R is the matrix associated to the operator $c_1(X)\cup : H^\bullet(X,\mathbb{C}) \to H^\bullet(X,\mathbb{C})$,
- $S(p), C(p)$ are the Stokes and central connection matrices at p, respectively.

Definition 4.4.3. Fix a point $p \in O_\ell$ with canonical coordinates $(u_i(p))_{i=1}^n$. Define the oriented rays $L_j(p,\varphi)$, $j = 1,\dots,n$, in the complex plane by the equations

$$L_j(p,\varphi) := \left\{u_j(p) + \rho e^{\sqrt{-1}(\frac{\pi}{2}-\varphi)} : \rho \in \mathbb{R}_+\right\}.$$

The ray $L_j(p,\varphi)$ is oriented from $u_j(p)$ to ∞. We say that $(u_i(p))_{i=1}^n$ are in *ℓ-lexicographical order* if $L_j(p,\varphi)$ is on the left of $L_k(p,\varphi)$ for $1 \leqslant j < k \leqslant n$.

In what follows, it is assumed that the ℓ-lexicographical order of canonical coordinates is fixed at all points of ℓ-chambers.

Lemma 4.4.4 ([21,32]). *If the canonical coordinates $(u_i(p))_{i=1}^n$ are in ℓ-lexicographical order at $p \in O_\ell$, then the Stokes matrices $S^{(m)}(p)$, $m \in \mathbb{Z}$, are upper triangular with ones along the diagonal.*

By Lemma 4.1.2, the matrices μ and R determine the monodromy of solutions of the qDE,

$$M_0 := \exp(2\pi\sqrt{-1}\mu)\exp(2\pi\sqrt{-1}R).$$

Moreover, μ and R do not depend on the point p. The following theorem furnishes a refinement of this property.

Theorem 4.4.5 ([21,30,32]). *The monodromy data (μ, R, S, C) are constant in each ℓ-chamber. Moreover, they satisfy the following identities:*

$$CS^TS^{-1}C^{-1} = M_0, \tag{4.4.1}$$
$$S = C^{-1}\exp(-\pi\sqrt{-1}R)\exp(-\pi\sqrt{-1}\mu)\eta^{-1}(C^T)^{-1}, \tag{4.4.2}$$
$$S^T = C^{-1}\exp(\pi\sqrt{-1}R)\exp(\pi\sqrt{-1}\mu)\eta^{-1}(C^T)^{-1}. \tag{4.4.3}$$

Theorem 4.4.6 ([21]). *The Stokes and central connection matrices S_m, C_m, with $m \in \mathbb{Z}$, can be reconstructed from the monodromy data (μ, R, S, C):*

$$S^{(m)} = S, \quad C^{(m)} = M_0^{-m} C, \quad m \in \mathbb{Z}. \tag{4.4.4}$$

Remark 4.4.7. Points of O_ℓ are semisimple. The results of [21, 22, 24, 25] imply that the monodromy data (μ, R, S, C) are well defined also at points $p \in \Omega_{\rm ss} \cap \mathcal{B}_\Omega$, and that Theorem 4.4.5 still holds true.

Remark 4.4.8. Note that from the knowledge of the monodromy data (μ, R, S, C) the Gromov–Witten potential $F_0^X(\boldsymbol{t})$ can be reconstructed via a Riemann–Hilbert boundary value problem, see [21, 23, 32, 47]. Hence, the monodromy data may be interpreted as a *system of coordinates* in the space of solutions of WDVV equations.

4.5 Natural transformations of monodromy data

The definition of the Stokes and central connection matrices is subordinate to several non-canonical choices:

(1) the choice of an oriented ray ℓ in $\widetilde{\mathbb{C}^*}$,

(2) the choice of an ordering of canonical coordinates $u_1, \dots, u_n$ on each ℓ-chamber,

(3) the choice of signs in (2.2.1), and hence of the branch of the Ψ-matrix on each ℓ-chamber.

Different choices affect the numerical values of the data (S, C).

For different choices of the oriented ray ℓ, the transformation of S and C can be described in terms of an action of the braid group $\mathcal{B}_n$, described in Section 4.6.
For different choices of ordering of canonical coordinates, the Stokes and central connection matrices transform as follows:

$$S \mapsto \Pi S \Pi^{-1}, \quad C \mapsto C \Pi^{-1}, \quad \Pi \text{ permutation matrix.}$$

For different choices of the branch of the Ψ-matrix, we have a transformation of the following type:

$$S \mapsto ISI, \quad C \mapsto CI, \quad I = \operatorname{diag}(\pm 1, \dots, \pm 1).$$

See [21, 23] for more details.

Moreover, let us also add that the value of all the monodromy data is affected by different choices of the system of flat coordinates $\boldsymbol{t}$.

Proposition 4.5.1. *Let $(\tilde{t}^0, \dots, \tilde{t}^N)$ be a system of flat coordinates on Ω related to $(t^0, \dots, t^N)$ by the transformations*

$$\tilde{t}^\alpha = A^\alpha_\beta t^\beta + c^\alpha, \quad A^\alpha_\beta, c^\alpha \in \mathbb{C}, \quad \alpha, \beta = 0, \dots, N.$$

The monodromy data $(\widetilde{\mu}, \widetilde{R}, \widetilde{S}, \widetilde{C})$*, computed with respect to the coordinates* $\widetilde{\boldsymbol{t}}$*, are related to the data* (μ, R, S, C)*, computed with respect to* $\boldsymbol{t}$*, as follows:*

$$\widetilde{\mu} = A\mu A^{-1}, \quad \widetilde{R} = ARA^{-1}, \quad \widetilde{S} = S, \quad \widetilde{C} = AC.$$

Proof. The transformation of μ, R is due to their tensorial nature: they are (1,1)-tensors on Ω. Notice that $\widetilde{\Psi} = \Psi A^{-1}$, $\widetilde{Z}_{R,0} = AZ_{R,0}$ and $\widetilde{Z}_{\text{top}} = AZ_{\text{top}}A^{-1}$ so that

$$\widetilde{C} = \widetilde{Z}_{\text{top}}^{-1}\widetilde{Z}_{R,0} = AZ_{\text{top}}^{-1}A^{-1}AZ_{R,0} = AC.$$

Equation (4.4.2), together with $\widetilde{\eta} = (A^{-1})^T\eta A^{-1}$, shows that $\widetilde{S} = S$. ∎

Remark 4.5.2. In particular, Proposition 4.5.1 applies in the case of deformations of the complex structures of X. Consider a smooth proper map $f: \mathcal{F} \to B$ with a connected base space B, and set $X_b := f^{-1}(b)$ with $b \in B$. Given $b_1, b_2 \in B$, there exists a diffeomorphism $\varphi: X_{b_1} \to X_{b_2}$, which allows to identify (co)homology groups:

$$\varphi_*: H_\bullet(X_{b_1}, \mathbb{Z}) \to H_\bullet(X_{b_2}, \mathbb{Z})$$

and

$$\varphi^*: H^\bullet(X_{b_2}, \mathbb{Z}) \to H^\bullet(X_{b_1}, \mathbb{Z}).$$

By using the isomorphisms φ_*, φ^*, and by invoking the deformation axiom of Gromov–Witten invariants (see e.g. [27, Section 7.3]), we can identify the quantum cohomologies $QH^\bullet(X_{b_1})$ and $QH^\bullet(X_{b_2})$: the deformation of the complex structure just represents a change of flat coordinates on the same Frobenius manifold.

4.6 Action of the braid group $\mathcal{B}_n$

Consider the braid group $\mathcal{B}_n$ with generators $\beta_1, \dots, \beta_{n-1}$ satisfying the relations

$$\begin{aligned} \beta_i\beta_j &= \beta_j\beta_i, \quad |i-j| > 1, \\ \beta_i\beta_{i+1}\beta_i &= \beta_{i+1}\beta_i\beta_{i+1}. \end{aligned}$$

Let $\mathcal{U}_n$ be the set of upper triangular $(n \times n)$-matrices with ones along the diagonal.

Definition 4.6.1. Given $U \in \mathcal{U}_n$, define the matrices $A^{\beta_i}(U)$, with $i = 1, \dots, n-1$, as follows:

$$\begin{aligned} \left(A^{\beta_i}(U)\right)_{hh} &:= 1, \quad h = 1, \dots, n, \ h \neq i, i+1, \\ \left(A^{\beta_i}(U)\right)_{i+1,i+1} &:= -U_{i,i+1}, \\ \left(A^{\beta_i}(U)\right)_{i,i+1} &:= \left(A^{\beta_i}(U)\right)_{i+1,i} = 1, \end{aligned}$$

and all other entries of $A^{\beta_i}(U)$ are equal to zero.

Lemma 4.6.2 ([30,32]). *The braid group $\mathcal{B}_n$ acts on $\mathcal{U}_n \times \mathrm{GL}(n,\mathbb{C})$ as follows:*

$$\begin{aligned}\mathcal{B}_n \times \mathcal{U}_n \times \mathrm{GL}(n,\mathbb{C}) &\to \mathcal{U}_n \times \mathrm{GL}(n,\mathbb{C}),\\ (\beta_i, U, C) &\mapsto (A^{\beta_i}(U)\cdot U \cdot A^{\beta_i}(U),\ C\cdot A^{\beta_i}(U)^{-1}).\end{aligned}$$

We denote by $(U,C)^{\beta_i}$ the action of β_i on (U,C). ■

Fix an oriented ray $\ell_o \equiv \{\arg z = \varphi_o\}$ in $\widetilde{\mathbb{C}^*}$, and denote by $\overline{\ell_o}$ its projection on $\widetilde{\mathbb{C}^*}$. Let $p_o \in O_{\ell_o}$, and let (S_0, C_0) be the Stokes and central connection matrices computed at p_o with respect to ℓ_o, the ℓ_o-lexicographical order of canonical coordinates $u_i(p_o)$, and a suitable determination of the Ψ-matrix at p_o. If we let the oriented ray rotate, so that it crosses some Stokes rays $R_{ij}(p_o)$, the values of (S_0, C_0) will change. We can describe this difference of values in terms of the braid group action of Lemma 4.6.2.

Theorem 4.6.3 ([21,30,32]). *Consider a continuous map $\varphi\colon [0,1] \to \mathbb{R}$, with $\varphi(0) = \varphi_o$, and set $\ell(t) := \{\arg z = \varphi(t)\}$ for any $t \in [0,1]$. Assume that*

- *the rays $\ell(0)$ and $\ell(1)$ are admissible at p_o,*
- *there exists a unique $t_o \in]0,1[$ such that $\ell(t_o)$ is not admissible at p_o,*
- *there exist $i_1,\dots,i_k \in \{1,\dots,n\}$, with $|i_a - i_b| > 1$ for $a \neq b$, such that the projected ray $\overline{\ell}(t) \subseteq \mathbb{C}$ crosses the rays $(R_{i_j,i_j+1})_{j=1}^{k}$ in the counterclockwise (resp. clockwise) direction, as $t \to t_o^-$.*

Denote by (S_i, C_i), with $i = 0,1$, the Stokes and central connection matrices at p_o with respect to the oriented ray $\ell(i)$, with $i = 0,1$. We have

$$(S_1, C_1) = (S_0, C_0)^{\beta}, \quad \beta = \prod_{j=1}^{k} \beta_{i_j} \quad \left(\text{resp. } \beta = \left(\prod_{j=1}^{k} \beta_{i_j}\right)^{-1}\right). \qquad \blacksquare$$

Remark 4.6.4. An arbitrary rotation of ℓ can be decomposed into the composition of elementary rotations satisfying the assumptions of Theorem 4.6.3.

Furthermore, the braid group action also describes how the values of Stokes and central connection matrices in different ℓ-chambers (for a fixed oriented rays ℓ) are related to each other.

Fix an oriented ray $\ell \equiv \{\arg z = \varphi\}$ in $\widetilde{\mathbb{C}^*}$, and denote by $\overline{\ell}$ its projection on $\mathbb{C}^*$. Let $\Omega_{\ell,1}, \Omega_{\ell,2}$ be two ℓ-chambers and let $p_i \in \Omega_{\ell,i}$ for $i = 1,2$. The difference of values of the Stokes and central connection matrices (S_1, C_1) and (S_2, C_2), at p_1 and p_2, respectively, can be described by the action of the braid group $\mathcal{B}_n$ of Lemma 4.6.2.

Theorem 4.6.5 ([21,30,32]). *Consider a continuous path $\gamma\colon [0,1] \to \Omega$ such that*

- *$\gamma(0) = p_1$ and $\gamma(1) = p_2$,*
- *there exists a unique $t_o \in [0,1]$ such that ℓ is not admissible at $\gamma(t_o)$,*

- *there exist* $i_1, \dots, i_k \in \{1, \dots, n\}$, *with* $|i_a - i_b| > 1$ *for* $a \neq b$, *such that the rays*[1] $(R_{i_j, i_j+1}(t))_{j=1}^{r}$ *(resp.* $(R_{i_j, i_j+1}(t))_{j=r+1}^{k}$*) cross the ray* $\overline{\ell}$ *in the clockwise (resp. counterclockwise) direction, as* $t \to t_o^-$.

Then we have

$$(S_2, C_2) = (S_1, C_1)^{\beta}, \quad \beta := \left(\prod_{j=1}^{r} \beta_{i_j} \right) \cdot \left(\prod_{h=r+1}^{k} \beta_{i_h} \right)^{-1}. \qquad \blacksquare$$

Remark 4.6.6. In the general case, the points p_1 and p_2 can be connected by concatenations of paths γ satisfying the assumptions of Theorem 4.6.5.

Remark 4.6.7. The action of $\mathcal{B}_n$ on (S, C) also describes the analytic continuation of the Frobenius manifold structure on Ω, see [32, Lecture 4].

[1]Here the labeling of Stokes rays is the one prolonged from the initial point $t = 0$.

Chapter 5

J-function and quantum Lefschetz theorem

5.1 J-function and master functions

Definition 5.1.1. The J-function of X is the $H^\bullet(X, \Lambda_X)[\![\hbar^{-1}]\!]$-valued function of $\boldsymbol{\tau} \in H^\bullet(X, \mathbb{C})$ defined by

$$J_X(\boldsymbol{\tau}) := 1 + \sum_{\alpha,\lambda=0}^{N} \sum_{n=0}^{\infty} \hbar^{-(n+1)} \langle\!\langle \tau_n T_\alpha, 1 \rangle\!\rangle_0 \eta^{\alpha\lambda} T_\lambda.$$

The following result will be crucial for us. For its proof see Appendix A.

Theorem 5.1.2. *Let $\alpha = 0, \dots, N$ and $\delta \in H^2(X, \mathbb{C})$. The $(1, \alpha)$-entry of the matrix $\eta Z_{\mathrm{top}}(z, \delta)$ equals*

$$z^{\frac{\dim X}{2}} \int_X T_\alpha \cup J_X(\delta + \log z \cdot c_1(X))\Big|_{\substack{\mathbf{Q}=1 \\ \hbar=1}}.$$

Corollary 5.1.3. *Let $\delta \in H^2(X, \mathbb{C})$. The components of the function*

$$J(\delta + \log z \cdot c_1(X))\Big|_{\substack{\mathbf{Q}=1 \\ \hbar=1}},$$

with respect to any basis of $H^\bullet(X, \mathbb{C})$, span the space of master functions $\mathcal{S}_\delta(X)$.

Proof. The functions $z^{-\frac{\dim X}{2}} [\eta Z_{\mathrm{top}}(z, \delta)]^1_\alpha$ define a generating set of the space of master functions $\mathcal{S}_\delta(X)$. The claim follows by Theorem 5.1.2. ∎

In the notations of Section 3.1, set

$$\delta = \sum_{i=1}^{r} t^i T_i.$$

Any formal differential operator $P \in \mathbb{C}[\![\hbar \frac{\partial}{\partial t^1}, \dots, \hbar \frac{\partial}{\partial t^r}, e^{t^1}, \dots, e^{t^r}, \hbar]\!]$ such that

$$P J_X(\delta) = 0$$

is called a *quantum differential operator*. The equation $PY = 0$ is called a *quantum differential equation*, see e.g. [27, Section 10.3]. By Corollary 5.1.3, the master differential equation, defined as in Section 2.7 at a point δ of the complement of the $\mathcal{A}_\Lambda$-stratum of $QH^\bullet(X)$, is equivalent to a differential equation for master functions

$$\widetilde{P}_\delta(\vartheta, z)\Phi = 0, \quad \vartheta := z\frac{d}{dz},$$

for a suitable differentiable operator $\widetilde{P}_\delta$.

5.2 Twisted Gromov–Witten invariants

Given a holomorphic vector bundle $E \to X$ and an invertible multiplicative[1] characteristic class $\boldsymbol{c}$, one can introduce a $(E, \boldsymbol{c})$-*twisted* version of the Gromov–Witten theory of X.

Given E, there exists a complex

$$0 \to E^0_{g,n,\beta} \to E^1_{g,n,\beta} \to 0$$

of locally free orbi-sheaves on $\overline{\mathcal{M}}_{g,n}(X, \beta)$ whose cohomology sheaves are

$$R^0\mathrm{ft}_{n+1,*}(\mathrm{ev}^*_{n+1}E) \quad \text{and} \quad R^1\mathrm{ft}_{n+1,*}(\mathrm{ev}^*_{n+1}E),$$

respectively. Here, the forgetful and evaluation morphisms ft_{n+1}, ev_{n+1} at the last marked point fit in the diagram

$$\begin{tikzcd} & \overline{\mathcal{M}}_{g,n+1}(X,\beta) \arrow[dl, "\mathrm{ft}_{n+1}"'] \arrow[dr, "\mathrm{ev}_{n+1}"] & \\ \overline{\mathcal{M}}_{g,n}(X,\beta) & & X. \end{tikzcd}$$

Let us introduce an *obstruction K-class*

$$E_{g,n,\beta} \in K^0(\overline{\mathcal{M}}_{g,n}(X, \beta)),$$

defined as the K-theoretic difference

$$E_{g,n,\beta} := [E^0_{g,n,\beta}] - [E^1_{g,n,\beta}].$$

It is possible to show that such a difference does not depend on the choice of the complex.

Definition 5.2.1. The $(E, \boldsymbol{c})$-*twisted Gromov–Witten invariants (with descendants)* of X are the intersection numbers

$$\langle \tau_1^{d_1}\alpha_1 \otimes \cdots \otimes \tau_n^{d_n}\alpha_n\rangle^{X,E,\boldsymbol{c}}_{g,n,\beta} := \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{virt}}} \boldsymbol{c}(E_{g,n,\beta}) \cup \prod_{j=1}^{n} \psi_j^{d_j} \cup \mathrm{ev}_j^*(\alpha_j),$$

where $\alpha_1, \dots, \alpha_n \in H^\bullet(X, \mathbb{C})$.

Remark 5.2.2. If $\boldsymbol{c}$ is the trivial characteristic class, then we recover the untwisted Gromov–Witten invariants of X.

[1]A characteristic class $\boldsymbol{c}$ is said to be *multiplicative* if $\boldsymbol{c}(E_1 \oplus E_2) = \boldsymbol{c}(E_1)\boldsymbol{c}(E_2)$. It is *invertible* if $\boldsymbol{c}(E)$ is invertible in $H^\bullet(Y, \mathbb{C})$ for any vector bundle E on a manifold Y.

5.3 Quantum Lefschetz theorem

Introduce a $\mathbb{C}^*$-action on the total space E defined by fiberwise multiplication. Note that the $\mathbb{C}^*$-equivariant Euler class $\boldsymbol{e}$ is invertible over the field of fractions $\mathbb{Q}(\lambda)$ of $H^\bullet_{\mathbb{C}^*}(\mathrm{pt}) \cong \mathbb{Q}[\lambda]$. Taking $\boldsymbol{c} = \boldsymbol{e}$ we refer to the twisted Gromov–Witten invariants as *Euler-twisted Gromov–Witten invariants*.

Exactly as in the untwisted case, $(E, \boldsymbol{c})$-twisted Gromov–Witten invariants can be collected in generating functions. In particular, we can introduce the *Euler-twisted J-function* as the $H^\bullet(X, \Lambda_X[\lambda])[\![\hbar^{-1}]\!]$-valued function on $H^\bullet(X, \mathbb{C})$ by

$$J_{E,\boldsymbol{e}}(\boldsymbol{\tau}) = 1 + \sum_{\alpha,k,n,\beta} \hbar^{-n-1} \frac{\mathbf{Q}^\beta}{k!} \langle \tau_n T_\alpha, 1, \boldsymbol{\tau}, \dots, \boldsymbol{\tau} \rangle^{X,E,\boldsymbol{e}}_{0,k+2,\beta} T^\alpha.$$

Assume now that the vector bundle E is convex,[2] i.e. $H^1(C, f^*E) = 0$ for all stable maps $f\colon C \to X$ with C of genus zero. Let Y be a smooth subvariety of X defined by the zero locus of a regular section of E.

Theorem 5.3.1 ([15, 17]). *The non-equivariant limit $J_{E,\boldsymbol{e}}|_{\lambda=0}$ exists. Moreover, it is related to the function J_Y by the equation*

$$\iota^* J_{E,\boldsymbol{e}}|_{\lambda=0}(\boldsymbol{\tau}) \overset{\iota_*}{=} J_Y(\iota^* \boldsymbol{\tau}), \quad \tau \in H^\bullet(X, \mathbb{C}), \tag{5.3.1}$$

where $\iota\colon Y \hookrightarrow X$ is the inclusion.

Remark 5.3.2. The symbol $\overset{\iota_*}{=}$ means that identity (5.3.1) holds true after application of the morphism $\iota_*\colon \Lambda_X \to \Lambda_Y$ defined by $\mathbf{Q}^\beta \mapsto \mathbf{Q}^{\iota_*\beta}$.

Remark 5.3.3. If $\dim_{\mathbb{C}} X > 3$, then ι^* is an isomorphism, by the hyperplane Lefschetz theorem.

Assume that

$$E = \bigoplus_{i=1}^{s} L_i,$$

where L_i are nef line bundles on X such that $c_1(E) \leqslant c_1(X)$. In such a case, the quantum Lefschetz theorem prescribes how to compute the non-equivariant limit $J_{E,\boldsymbol{e}}(\delta)|_{\lambda=0}$ at points of the small quantum locus $\delta \in H^2(X, \mathbb{C})$.

Introduce the *hypergeometric modification* $I_{X,Y}$ of the function J_X as follows: write $J_X = \sum_\beta J_\beta \mathbf{Q}^\beta$, and for $\delta \in H^2(X, \mathbb{C})$ define

$$I_{X,Y}(\delta) := \sum_\beta J_\beta(\delta) \mathbf{Q}^\beta \prod_{i=1}^{s} \prod_{m=1}^{\langle c_1(L_i), \beta \rangle} (c_1(L_i) + m\hbar). \tag{5.3.2}$$

[2] Globally generated vector bundles and direct sums of nef line bundles are automatically convex.

Theorem 5.3.4 ([17]). *The function $I_{X,Y}$ admits an expansion of the form*

$$I_{X,Y}(\delta) = F(\delta) + \frac{1}{\hbar}G(\delta) + O\left(\frac{1}{\hbar^2}\right), \quad \delta \in H^2(X,\mathbb{C}),$$

where F is $H^0(X,\Lambda_X)$-valued and G takes values in $H^0(X,\Lambda_X) \oplus H^2(X,\Lambda_X)$. Moreover, we have

$$J_{E,\boldsymbol{e}}(\varphi(\delta))|_{\lambda=0} = \frac{I_{X,Y}(\delta)}{F(\delta)}, \quad \varphi(\delta) := \frac{G(\delta)}{F(\delta)}.$$

Proposition 5.3.5 ([16, 17]). *Moreover, if $c_1(X) > c_1(E)$, then we have*

$$\begin{aligned} F(\delta) &\equiv 1, \\ G(\delta) &= \delta + H(\delta)\cdot 1, \\ H(\delta) &= \sum_\beta \left(w_\beta \mathbf{Q}^\beta e^{\int_\beta \delta}\right)\cdot \delta_{1,\langle \beta, c_1(X)-c_1(E)\rangle} \end{aligned}$$

for suitable rational coefficients $w_\beta \in \mathbb{Q}$.

Proof. The function $I_{X,Y}(\delta)$ is homogeneous of degree 0 with respect to the gradings

$$\begin{aligned} \deg \mathbf{Q}^\beta &= \int_\beta c_1(X) - \int_\beta c_1(E), \\ \deg \hbar &= 1, \\ \deg T_\alpha &= k \quad \text{if } T_\alpha \in H^{2k}(X,\mathbb{C}). \end{aligned}$$

This is easily seen from the expansion of J_X given in Lemma A.2. Hence, $F(\delta)$ is given from the only contribution of the term $J_0(\delta) = 1 + \frac{\delta}{\hbar} + \cdots$ and $H(\delta)$ from the terms for which $\deg \mathbf{Q}^\beta = 1$. ∎

5.4 Inequality for dimensions of spaces of master functions

Let $Y \subseteq X$ be the zero locus of a regular section of a vector bundle $E \to X$, sum of nef line bundles, with $c_1(E) < c_1(X)$. Denote by $\iota\colon Y \to X$ the inclusion. We always assume that both X and Y have vanishing odd cohomology.

For a point $\boldsymbol{\tau} \in QH^\bullet(X)$, denote by $\mathcal{S}_{\boldsymbol{\tau}}(X) := \mathcal{S}_{\boldsymbol{\tau}}(QH^\bullet(X))$ the space of master functions as $\boldsymbol{\tau}$.

Theorem 5.4.1. *Let $\delta \in H^2(X,\mathbb{C})$. We have*

$$\dim_{\mathbb{C}} \mathcal{S}_{\iota^*\delta}(Y) \leqslant \dim_{\mathbb{C}} \mathcal{S}_{\delta+c}(X), \tag{5.4.1}$$

where $c := c_1(X) - c_1(E)$.

Proof. By the adjunction formula, we have

$$\iota^* c = c_1(Y).$$

The components of the function $J_X(\delta + c \log z)|_{\mathbf{Q}=1,\hbar=1}$, with respect to any basis of $H^\bullet(X, \mathbb{C})$, span the space $\mathcal{S}_{\delta+c}(X)$. Analogously, the components of the function $J_Y(\iota^*\delta + c_1(Y) \log z)|_{\mathbf{Q}=1,\hbar=1}$, with respect to any basis of $H^\bullet(Y, \mathbb{C})$, span the space $\mathcal{S}_{\iota^*\delta}(Y)$.

By Theorems 5.3.1, 5.3.4 and Proposition 5.3.5, we have

$$J_Y(\iota^*\delta + c_1(Y) \log z)|_{\substack{\mathbf{Q}=1\\ \hbar=1}} = e^{-zH(\delta)} \cdot \iota^* I_{X,Y}(\delta + c \log z)|_{\substack{\mathbf{Q}=1\\ \hbar=1}}.$$

The components of the right side are obtained by linear combinations and rescaling of the components of $J_X(\delta + c \log z)|_{\mathbf{Q}=1,\hbar=1}$: such a linear combination is due to the hypergeometric modification (5.3.2), namely the $\cup$-multiplication by an invertible class. The claim follows. ∎

Theorem 5.4.2. *Let Y be a hyperplane section of X. Assume that $d := \dim_{\mathbb{C}} X$ is odd, and that the following inequalities of Betti numbers hold true:*

$$b_{d-1}(X) < \frac{1}{2} b_{d-1}(Y). \tag{5.4.2}$$

Then $\iota^*(H^2(X, \mathbb{C}))$ is contained in the $\mathcal{A}_\Lambda$-stratum of the Frobenius manifold $QH^\bullet(Y)$. In particular, along $\iota^*(H^2(X, \mathbb{C}))$ the canonical coordinates of $QH^\bullet(Y)$ coalesce.

Proof. From the hyperplane Lefschetz theorem we deduce that (5.4.2) holds true if and only if $\dim_{\mathbb{C}} H^\bullet(X, \mathbb{C}) < \dim_{\mathbb{C}} H^\bullet(Y, \mathbb{C})$. Then for any $\delta \in H^2(X, \mathbb{C})$ we have $\dim_{\mathbb{C}} \mathcal{S}_{\iota^*\delta}(Y) < \dim_{\mathbb{C}} H^\bullet(Y, \mathbb{C})$, by (5.4.1). Hence, the master differential equation of $QH^\bullet(Y)$ at $\iota^*\delta$ is not of order $\dim_{\mathbb{C}} H^\bullet(Y, \mathbb{C})$. This implies that the denominator of $\det \Lambda$ is identically zero at $\iota^*\delta$. The last statement follows from Lemma 2.6.1 and Theorem 2.6.2. ∎

Chapter 6

Borel–Laplace (α, β)-multitransforms

6.1 Algebras of Ribenboim's generalized power series

Let $(M, +, 0)$ be a *monoid*, i.e. a commutative semigroup with neutral element. We say that a partial order relation $\leqslant$ on M defines a *strictly ordered monoid* $(M, +, 0, \leqslant)$ if the following compatibility condition holds true:

$$\text{if } a < b, \text{ then } a + s < b + s \text{ for all } s \in M.$$

Let R be a commutative ring with unit. The set

$$R[\![M]\!] := R^M$$

of all functions $f: M \to R$ is equipped with a natural R-module structure, with respect to pointwise addition and multiplication by scalars. An element $f \in R[\![M]\!]$ will usually be denoted by

$$f = \sum_{a \in M} f(a) Z^a,$$

where Z is an indeterminate. Given two functions $f, g \in R[\![M]\!]$, we could be tempted to define their product as

$$f \cdot g := \sum_{s \in M} \Bigg(\sum_{(p,q) \in X_s(f,g)} f(p) \cdot g(q) \Bigg) Z^s, \tag{6.1.1}$$

where we set

$$X_s(f, g) := \{(p, q) \in M \times M \,:\, p + q = s,\ f(p) \neq 0,\ g(q) \neq 0\}.$$

In general the set $X_s(f, g)$ is *not* finite, and consequently the product $f \cdot g$ could be not defined.

Definition 6.1.1. Let $(M, +, 0, \leqslant)$ be a strictly ordered monoid. The R-submodule of $R[\![M]\!]$ which consists of all functions $f: M \to R$ whose support

$$\operatorname{supp}(f) := \{s \in M \,:\, f(s) \neq 0\}$$

is

(1) *Artinian*, i.e. every subset of $\operatorname{supp}(f)$ admits a minimal element,

(2) *narrow*, i.e. every subset of $\operatorname{supp}(f)$ of pairwise incomparable elements is finite,

is called the set of *generalized power series* with coefficients in R and exponents in M. It is denoted by $R[\![M, \leqslant]\!]$.

Proposition 6.1.2 ([68, 69]). *Given $f, g \in R[\![M, \leqslant]\!]$, the set $X_s(f, g)$ is finite, and the product* (6.1.1) *is well defined. The set $R[\![M, \leqslant]\!]$ inherits the structure of an associative R-algebra.*

Remark 6.1.3. If $(M, \leqslant)$ is itself Artinian and narrow, then all its subsets are Artinian and narrow. Consequently, $R[\![M, \leqslant]\!] = R[\![M]\!]$.

6.2 The algebra $\mathscr{F}_{\boldsymbol{\kappa}}(A)$

Let $\boldsymbol{\kappa} := (\kappa_1, \ldots, \kappa_h) \in (\mathbb{C}^*)^h$. Consider an associative, commutative, unitary and finite-dimensional $\mathbb{C}$-algebra $(A, +, \cdot, 1_A)$. Denote by $\mathrm{Nil}(A)$ the nilradical of A, that is,

$$\mathrm{Nil}(A) := \{a \in A : \text{there exists an } n \in \mathbb{N} \text{ such that } a^n = 0\}.$$

Set $\mathbb{N}_A := \{n \cdot 1_A : n \in \mathbb{N}\}$. Define the monoid $M_{A,\boldsymbol{\kappa}}$ as the (external) direct sum of monoids

$$M_{A,\boldsymbol{\kappa}} := \left(\bigoplus_{j=1}^{h} \kappa_j \mathbb{N}_A \right) \oplus \mathrm{Nil}(A).$$

We have two maps $\nu_{\boldsymbol{\kappa}} : M_{A,\boldsymbol{\kappa}} \to \mathbb{N}^h$ and $\iota_{\boldsymbol{\kappa}} : M_{A,\boldsymbol{\kappa}} \to A$ defined by

$$\nu_{\boldsymbol{\kappa}}((\kappa_i n_i 1_A)_{i=1}^h, r) := (n_i)_{i=1}^h$$

and

$$\iota_{\boldsymbol{\kappa}}((\kappa_i n_i 1_A)_{i=1}^h, r) := \sum_{i=1}^{h} \kappa_i n_i 1_A + r.$$

On $M_{A,\boldsymbol{\kappa}}$ we can define the partial order

$$x \leqslant y \iff \nu_{\boldsymbol{\kappa}}(x) \leqslant \nu_{\boldsymbol{\kappa}}(y),$$

the order on $\mathbb{N}^h$ being the lexicographical one. This order makes $(M_{A,\boldsymbol{\kappa}}, \leqslant)$ a strictly ordered monoid.

We denote by $\mathscr{F}_{\boldsymbol{\kappa}}(A)$ the ring $A[\![M_{A,\boldsymbol{\kappa}}, \leqslant]\!]$.

By the universal property of the direct sums of monoids, the natural inclusions $M_{A,\kappa_i} \to M_{A,\boldsymbol{\kappa}}$ induce a unique morphism

$$\rho_{\boldsymbol{\kappa}} : \bigoplus_{i=1}^{h} M_{A,\kappa_i} \to M_{A,\boldsymbol{\kappa}}.$$

Definition 6.2.1. Let $r_o \in \mathrm{Nil}(A)$. We say that an element $f \in \mathscr{F}_{\boldsymbol{\kappa}}(A)$ is *concentrated at r_o* if

$$\mathrm{supp}(f) \subseteq \left(\bigoplus_{i=1}^{h} \kappa_i \mathbb{N}_A \right) \times \{r_o\}.$$

6.3 Formal Borel–Laplace $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-multitransforms

Given two h-tuples $\boldsymbol{\alpha}, \boldsymbol{\beta} \in (\mathbb{C}^*)^h$, we set $\boldsymbol{\alpha} \cdot \boldsymbol{\beta} := (a_i \beta_i)_{i=1}^h$, and $\boldsymbol{\alpha}^{-1} := (\frac{1}{\alpha_i})_{i=1}^h$.

Definition 6.3.1. Let $F \in \mathbb{C}[\![x]\!]$ be a formal power series $F(x) = \sum_{k=0}^{\infty} a_k x^k$. For $\alpha \in \mathrm{Nil}(A)$ define $F(\alpha) \in A$ by the finite sum

$$F(\alpha) = \sum_{k=0}^{\infty} a_k \alpha^k.$$

If F is invertible, i.e. $a_0 \neq 0$, then $F(\alpha)$ is invertible in A.

In what follows we will usually take $F(x) = \Gamma(\lambda + x)$ with $\lambda \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}$, where Γ denotes the Euler Gamma function.

Definition 6.3.2. Let $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\kappa} \in (\mathbb{C}^*)^h$. We define the *Borel* $(\boldsymbol{\alpha}, \boldsymbol{\beta})$*-multitransform* as the A-linear morphism

$$\mathscr{B}_{\boldsymbol{\alpha},\boldsymbol{\beta}} \colon \bigotimes_{j=1}^{h} \mathscr{F}_{\kappa_j}(A) \to \mathscr{F}_{\boldsymbol{\alpha}^{-1} \cdot \boldsymbol{\beta}^{-1} \cdot \boldsymbol{\kappa}}(A),$$

which is defined, on decomposable elements, by

$$\mathscr{B}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\left(\bigotimes_{j=1}^{h}\left(\sum_{s_j \in M_{A,\kappa_j}} f_{s_j}^{j} Z^{s_j}\right)\right)$$
$$:= \sum_{\substack{s_j \in M_{A,\kappa_j}\\ j=1,\dots,h}} \frac{\prod_{i=1}^{h} f_{s_i}^{i}}{\Gamma\big(1 + \sum_{\ell=1}^{h} \iota_{\kappa_\ell}(s_\ell)\beta_\ell\big)} Z^{\rho_{\boldsymbol{\kappa}}\left(\bigoplus_{\ell=1}^{h} \frac{s_\ell}{\alpha_\ell \beta_\ell}\right)}.$$

Definition 6.3.3. Let $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\kappa} \in (\mathbb{C}^*)^h$. We define the *Laplace* $(\boldsymbol{\alpha}, \boldsymbol{\beta})$*-multitransform* as the A-linear morphism

$$\mathscr{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}} \colon \bigotimes_{j=1}^{h} \mathscr{F}_{\kappa_j}(A) \to \mathscr{F}_{\boldsymbol{\alpha} \cdot \boldsymbol{\beta} \cdot \boldsymbol{\kappa}}(A),$$

which is defined, on decomposable elements, by

$$\mathscr{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\left(\bigotimes_{j=1}^{h}\left(\sum_{s_j \in M_{A,\kappa_j}} f_{s_j}^{j} Z^{s_j}\right)\right)$$
$$:= \sum_{\substack{s_j \in M_{A,\kappa_j}\\ j=1,\dots,h}} \left(\prod_{i=1}^{h} f_{s_i}^{i}\right) \Gamma\left(1 + \sum_{\ell=1}^{h} \iota_{\kappa_\ell}(s_\ell)\beta_\ell\right) Z^{\rho_{\boldsymbol{\kappa}}(\bigoplus_{\ell=1}^{h} \alpha_\ell \beta_\ell s_\ell)}.$$

In the case $h = 1$, the Borel–Laplace $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-multitransform simplify as follows.

Definition 6.3.4. Given $\alpha, \beta \in \mathbb{C}^*$, we define two A-linear maps

$$\mathscr{B}_{\alpha,\beta} \colon \mathscr{F}_\kappa(A) \to \mathscr{F}_{\frac{\kappa}{\alpha\beta}}(A), \quad \mathscr{L}_{\alpha,\beta} \colon \mathscr{F}_\kappa(A) \to \mathscr{F}_{\alpha\beta\kappa}(A), \quad \kappa \in \mathbb{C}^*$$

called respectively (α, β)-*Borel and Laplace transforms*, through the formulas

$$\mathscr{B}_{\alpha,\beta}\Bigg[\sum_{s \in M_{A,\kappa}} f_s Z^s\Bigg] := \sum_{s \in M_{A,\kappa}} \frac{f_s}{\Gamma(1+\beta s)} Z^{\frac{s}{\alpha\beta}},$$

$$\mathscr{L}_{\alpha,\beta}\Bigg[\sum_{s \in M_{A,\kappa}} f_s Z^s\Bigg] := \sum_{s \in M_{A,\kappa}} f_s \Gamma(1+\beta s) Z^{\alpha\beta s}.$$

Theorem 6.3.5. *The Borel–Laplace (α, β)-transform are inverses of each other, i.e.*

$$\mathscr{B}_{\alpha,\beta} \circ \mathscr{L}_{\alpha,\beta} = \mathrm{Id}, \quad \mathscr{L}_{\alpha,\beta} \circ \mathscr{B}_{\alpha,\beta} = \mathrm{Id}. \qquad \blacksquare$$

6.4 Analytic Borel–Laplace $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-multitransforms

Definition 6.4.1. Let $\boldsymbol{\alpha}, \boldsymbol{\beta} \in (\mathbb{C}^*)^h$. The Borel $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-multitransform of an h-tuple of A-valued functions $(\Phi_1, \dots, \Phi_h)$ is defined, when the integral exists, by

$$\mathscr{B}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[\Phi_1, \dots, \Phi_h](z) := \frac{1}{2\pi i} \int_\gamma \prod_{j=1}^h \Phi_j\Big(z^{\frac{1}{\alpha_j \beta_j}} \lambda^{-\beta_j}\Big) e^\lambda \frac{d\lambda}{\lambda},$$

where γ is a Hankel-type contour of integration, see Figure 6.1.

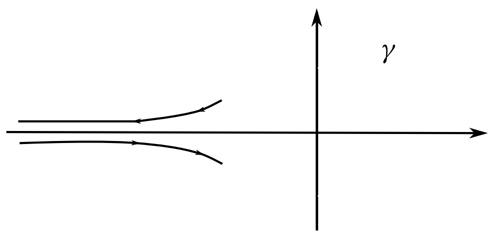

Figure 6.1. Hankel-type contour of integration defining Borel $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-multitransform.

Definition 6.4.2. Let $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_h)$ and $\boldsymbol{\beta} := (\beta_1, \dots, \beta_h)$ be h-tuples in $(\mathbb{C}^*)^h$. The $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-Laplace transform of an h-tuple of functions $(\Phi_1, \dots, \Phi_h)$ is defined, when the integral exists, by

$$\mathscr{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[\Phi_1, \dots, \Phi_h](z) := \int_0^\infty \prod_{i=1}^h \Phi_i(z^{\alpha_i \beta_i} \lambda^{\beta_i}) \exp(-\lambda)\, d\lambda.$$

Proposition 6.4.3. *Let $(e_1, \dots, e_n)$ be a basis of A and let $\Phi_1, \dots, \Phi_h$ be A-valued functions. Write $\Phi_i = \sum_j \Phi_i^j e_j$ for $\mathbb{C}$-valued component functions Φ_i^j. The components of $\mathscr{B}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[\Phi_1, \dots, \Phi_h]$ (resp. $\mathscr{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[\Phi_1, \dots, \Phi_h]$) are $\mathbb{C}$-linear combinations of the $h \cdot n$ $\mathbb{C}$-valued functions $\mathscr{B}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[\Phi_1^{i_1}, \dots, \Phi_h^{i_h}]$ (resp. $\mathscr{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[\Phi_1^{i_1}, \dots, \Phi_h^{i_h}]$), where $(i_1, \dots, i_h) \in \{1, \dots, n\}^{\times h}$.*

Proof. Let $c_{jk}^i \in \mathbb{C}$ be the structure constants of the algebra A, so that

$$e_j e_k = \sum_i c_{jk}^i e_i.$$

We have

$$\mathscr{B}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[\Phi_1, \dots, \Phi_h] = \sum_{\boldsymbol{a},\boldsymbol{i}} c_{i_1 i_2}^{a_1} c_{a_1 i_3}^{a_2} \cdots c_{a_{h-2} i_h}^{a_{h-1}} e_{a_{h-1}} \mathscr{B}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[\Phi_1^{i_1}, \dots, \Phi_h^{i_h}].$$

Similarly for the Laplace multitransform. ∎

6.5 Analytification of elements of $\mathscr{F}_{\boldsymbol{\kappa}}(A)$

Let $s = ((\kappa_i n_i 1_A)_{i=1}^h, r) \in M_{A,\boldsymbol{\kappa}}$. We define the *analytification* $\widehat{Z^s}$ of the monomial $Z^s \in \mathscr{F}_{\boldsymbol{\kappa}}(A)$ to be the A-valued holomorphic function

$$\widehat{Z^s} \colon \widetilde{\mathbb{C}^*} \to A, \quad \widehat{Z^s}(z) := z^{\sum_{i=1}^h \kappa_i n_i} \sum_{j=1}^{\infty} \frac{r^j}{j!} \log^j z.$$

Notice that the sum is finite, since $r \in \mathrm{Nil}(A)$.

Let $f \in \mathscr{F}_{\boldsymbol{\kappa}}(A)$ be a series

$$f(Z) = \sum_{s \in M_{A,\boldsymbol{\kappa}}} f_a Z^s$$

such that

$$\operatorname{card}\operatorname{supp}(f) \leqslant \aleph_0.$$

The *analytification* $\widehat{f}$ of f is the A-valued holomorphic function defined if the series absolutely converges, by

$$\widehat{f} \colon W \subseteq \widetilde{\mathbb{C}^*} \to A, \quad \widehat{f}(z) := \sum_{s \in M_{A,\boldsymbol{\kappa}}} f_a \widehat{Z^s}(z).$$

Theorem 6.5.1. *Let $f_i \in \mathscr{F}_{\kappa_i}(A)$ such that*

- card supp$(f_i) \leqslant \aleph_0$ *for $i = 1, \dots, h$,*
- *the functions $\widehat{f_i}$ are well defined on $\mathbb{R}_+$.*

We have

$$\widehat{\mathscr{B}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\left[\bigotimes_{j=1}^{h} f_j\right]} = \mathscr{B}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[\widehat{f}_1,\dots,\widehat{f}_h],$$

$$\widehat{\mathscr{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\left[\bigotimes_{j=1}^{h} f_j\right]} = \mathscr{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[\widehat{f}_1,\dots,\widehat{f}_h],$$

provided that both sides are well defined.

Proof. It is sufficient to prove the statement on monomials $Z^{s_1},\dots,Z^{s_h}$. To this end, let $s_j=(\kappa_j n_j 1_A, r_j)$ for $j=1,\dots,h$. We have

$$\begin{aligned}
&\mathscr{B}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\left[\bigotimes_{j=1}^{h} Z^{s_j}\right]\\
&\quad=\frac{1}{\Gamma\big(1+\sum_{\ell=1}^{h}\iota_{\kappa_\ell}(s_\ell)\beta_\ell\big)} Z^{\rho_\kappa\left(\bigoplus_{\ell=1}^{h}\frac{s_\ell}{\alpha_\ell\beta_\ell}\right)}\\
&\quad=\frac{1}{\Gamma\big(1+\sum_{\ell=1}^{h}(\kappa_\ell n_\ell 1_A+r_\ell)\beta_\ell\big)} Z^{\left(\left(\frac{\kappa_j}{\alpha_j\beta_j}n_j 1_A\right)_{j=1}^{h},\ \frac{r_1}{\alpha_1\beta_1}+\cdots+\frac{r_h}{\alpha_h\beta_h}\right)}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\widehat{\mathscr{B}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\left[\bigoplus_{j=1}^{h} Z^{s_j}\right]}(z)\\
&\quad=\frac{z^{\sum_{i=1}^{h}\frac{\kappa_i n_i}{\alpha_i\beta_i}}}{\Gamma\big(1+\sum_{\ell=1}^{h}(\kappa_\ell n_\ell 1_A+r_\ell)\beta_\ell\big)}\sum_{j=1}^{\infty}\frac{\left(\frac{r_1}{\alpha_1\beta_1}+\cdots+\frac{r_h}{\alpha_h\beta_h}\right)^j}{j!}\log^j z.
\end{aligned}$$

On the other hand, we have

$$\widehat{Z^{s_j}}(z)=z^{\kappa_j n_j}\sum_{\ell}\frac{r_j^{\ell}}{\ell!}\log^{\ell} z,$$

so that

$$\begin{aligned}
&\mathscr{B}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[\widehat{Z^{s_1}},\dots,\widehat{Z^{s_h}}](z)\\
&\quad=\frac{1}{2\pi i}\int_{\gamma}\prod_{j=1}^{h}\widehat{Z^{s_j}}\Big(z^{\frac{1}{\alpha_j\beta_j}}\lambda^{-\beta_j}\Big)e^{\lambda}\frac{d\lambda}{\lambda}\\
&\quad=\frac{1}{2\pi i}\int_{\gamma}e^{\lambda}\frac{d\lambda}{\lambda}\prod_{j=1}^{h}\Big(z^{\frac{1}{\alpha_j\beta_j}}\lambda^{-\beta_j}\Big)^{\kappa_j n_j}\sum_{\ell}\frac{r_j^{\ell}}{\ell!}\log^{\ell}\Big(z^{\frac{1}{\alpha_j\beta_j}}\lambda^{-\beta_j}\Big)
\end{aligned}$$

$$= \frac{z^{\sum_{i=1}^{h} \frac{\kappa_i n_i}{\alpha_i \beta_i}}}{2\pi i} \int_\gamma e^\lambda \frac{d\lambda}{\lambda^{1+\sum_{\ell=1}^{h} \kappa_\ell n_\ell \beta_\ell}} \prod_{j=1}^{h} \sum_{\ell} \frac{r_j^\ell}{\ell!} \log^\ell \left(z^{\frac{1}{\alpha_j \beta_j}} \lambda^{-\beta_j} \right)$$

$$= \frac{z^{\sum_{i=1}^{h} \frac{\kappa_i n_i}{\alpha_i \beta_i}}}{2\pi i} \int_\gamma e^\lambda \frac{d\lambda}{\lambda^{1+\sum_{\ell=1}^{h} \kappa_\ell n_\ell \beta_\ell}} \sum_{\ell_1,\dots,\ell_h} \prod_{j=1}^{h} \frac{r_j^{\ell_j}}{\ell_j!} \log^{\ell_j} \left(z^{\frac{1}{\alpha_j \beta_j}} \lambda^{-\beta_j} \right).$$

We have

$$\prod_{j=1}^{h} \frac{r_j^{\ell_j}}{\ell_j!} \log^{\ell_j} (z^{\frac{1}{\alpha_j \beta_j}} \lambda^{-\beta_j})$$

$$= \prod_{j=1}^{h} \sum_{w,u=0}^{\infty} \frac{r_j^{\ell_j}}{w!u!} \left(\frac{\log z}{\alpha_j \beta_j} \right)^w (-\beta_j \log \lambda)^u \delta_{w+u,\ell_j}$$

$$= \sum_{\substack{w_1,\dots,w_h \\ u_1,\dots,u_h}} \prod_{j=1}^{h} \frac{r_j^{\ell_j}}{w_j! u_j!} \left(\frac{\log z}{\alpha_j \beta_j} \right)^{w_j} (-\beta_j \log \lambda)^{u_j} \delta_{w_j+u_j,\ell_j},$$

and

$$\frac{1}{2\pi i} \int_\gamma e^\lambda \frac{d\lambda}{\lambda^{1+\sum_{\ell=1}^{h} \kappa_\ell n_\ell \beta_\ell}} (-\log \lambda)^{u_j} = \left(\frac{1}{\Gamma} \right)^{(u_j)} \left(1 + \sum_{\ell=1}^{h} \kappa_\ell n_\ell \beta_\ell \right),$$

because of the Hankel formula (see e.g. [64])

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_\gamma e^\lambda \frac{d\lambda}{\lambda^z}.$$

Thus, we have

$$\mathscr{B}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[\widehat{Z^{s_1}}, \dots, \widehat{Z^{s_h}}](z) = z^{\sum_{i=1}^{h} \frac{\kappa_i n_i}{\alpha_i \beta_i}} \sum_{\substack{\ell_1,\dots,\ell_h \\ w_1,\dots,w_h \\ u_1,\dots,u_h}} \prod_{j=1}^{h} \frac{r_j^{\ell_j} \beta_j^{u_j}}{w_j! u_j!} \left(\frac{\log z}{\alpha_j \beta_j} \right)^{w_j}$$

$$\cdot \left(\frac{1}{\Gamma} \right)^{(u_j)} \left(1 + \sum_{\ell=1}^{h} \kappa_\ell n_\ell \beta_\ell \right) \delta_{w_j+u_j,\ell_j}.$$

This coincides with the formula of $widehat\mathscr{B}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[\bigotimes_{j=1}^{h} Z^{s_j}](z)$. The proof for the Laplace multitransform is similar, based on the identity

$$\Gamma(z) = \int_0^\infty \lambda^{z-1} e^{-\lambda} \, d\lambda. \qquad \blacksquare$$

Chapter 7

Integral representations of solutions of qDEs

7.1 J_X-function as element of $\mathscr{F}_{\boldsymbol{\kappa}}(X)$

Let X be a variety with nef anticanonical bundle.[1] Introduce the basis $(\beta_1, \dots, \beta_r)$ of $H_2(X, \mathbb{Z})$ Poincaré dual to $(T^1, \dots, T^n)$, so that

$$\int_{\beta_i} T_j = \int_X T^i \cup T_j = \delta_{i,j}.$$

Set

$$c_1(X) = \sum_{j=1}^{\mathrm{r}} c^{\alpha_{i_j}} T_{\alpha_{i_j}}, \quad c^{\alpha_{i_j}} \in \mathbb{N}^*.$$

Consider the $\mathbb{C}$-algebra $H^\bullet(X, \mathbb{C})$. For brevity, we set

$$\mathscr{F}_{\boldsymbol{\kappa}}(X) := \mathscr{F}_{\boldsymbol{\kappa}}(H^\bullet(X, \mathbb{C}))$$

for any $\boldsymbol{\kappa} \in (\mathbb{C}^*)^h$.

The J_X-function restricted to the small quantum locus of $QH^\bullet(X)$ admits the following expansion:

$$\begin{aligned} &J_X(\delta + \log z \cdot c_1(X))|_{\substack{\mathbf{Q}=1\\ \hbar=1}} \\ &\quad = e^\delta z^{c_1(X)} + \sum_\alpha \sum_{\beta \neq 0} \sum_{k=0}^{\infty} e^\delta z^{\int_\beta c_1(X)} z^{c_1(X)} \langle \tau_k T_\alpha, 1 \rangle^X_{0,2,\beta} T^\alpha. \end{aligned}$$

Such a series can be seen as an element of $\mathscr{F}_{\boldsymbol{\kappa}}(X)$ for different choices of $\boldsymbol{\kappa}$. We describe two possible choices. In both cases, we have a series in $\mathscr{F}_{\boldsymbol{\kappa}}(X)$ concentrated at $c_1(X)$.

Choice 1. Set $h = 1$ and $\kappa = c$, where c is a common divisor of the numbers

$$c^{\alpha_{i_1}}, \dots, c^{\alpha_{i_{\mathrm{r}}}}.$$

The series can be rearranged as follows:

$$J_X(\delta + \log z \cdot c_1(X))|_{\substack{\mathbf{Q}=1\\ \hbar=1}} = \sum_{d \in \mathbb{N}} J_d(\delta) z^{dc + c_1(X)},$$

[1] We recall that this means $\int_C c_1(X) \geqslant 0$ for all curves C in X. If the strict inequality holds true for any C, then X is Fano by the Nakai–Moishezon theorem. Varieties with nef anticanonical bundle can be thought as an interpolation between Fano and Calabi–Yau varieties.

where

$$J_d(\delta) = e^{\delta} \sum_{\alpha,k} \langle \tau_k T_\alpha, 1\rangle_{0,2,d\cdot \mathrm{PD}(T)} T^\alpha, \quad d \in \mathbb{N},\ T \in H^2(X,\mathbb{Z}),\ c_1(X) = cT.$$

In particular, $J_0(\delta) = e^{\delta}$.

Choice 2. Set $h = \mathrm{r}$ and $\boldsymbol{\kappa} = (c^{\alpha_{i_1}}, \dots, c^{\alpha_{i_\mathrm{r}}})$. By expanding the sum over β over the basis $(\beta_1, \dots, \beta_r)$, the sum above becomes

$$J_X(\delta + \log z \cdot c_1(X))|_{\substack{\mathbf{Q}=1\\ \hbar=1}} = \sum_{d\in\mathbb{N}^\mathrm{r}} J_d(\delta) z^{d_1 c^{\alpha_{i_1}} + \dots + d_\mathrm{r} c^{\alpha_{i_\mathrm{r}}} + c_1(X)},$$

where

$$J_d(\delta) = e^{\delta} \sum_{\alpha,k} \langle \tau_k T_\alpha, 1\rangle_{0,2,d_1\beta_{\alpha_{i_1}} + \dots + d_\mathrm{r}\beta_{\alpha_{i_\mathrm{r}}}} T^\alpha, \quad d \in \mathbb{N}^\mathrm{r}.$$

In particular, $J_0(\delta) = e^{\delta}$.

7.2 Integral representations of the first kind

Let X be a Fano smooth projective variety. Assume that $\det T_X = L^{\otimes \ell}$ with L ample line bundle. Let $\iota\colon Y \subseteq X$ be a smooth subvariety defined as the zero locus of a regular section of the vector bundle $E = \bigoplus_{j=1}^s L^{\otimes d_j}$, where the numbers $d_j \in \mathbb{N}^*$ are such that $\sum_{j=1}^s d_j < \ell$.

Theorem 7.2.1. *Let $\delta \in H^2(X,\mathbb{C})$, and let $\mathcal{S}_\delta(X)$ be the corresponding space of master functions of $QH^\bullet(X)$. There exists a complex number $c_\delta \in \mathbb{C}$ such that the space of master functions $\mathcal{S}_{\iota^*\delta}(Y)$ is contained in image of the $\mathbb{C}$-linear map*

$$\mathscr{S}_{(\ell,\boldsymbol{d})}\colon \mathcal{S}_\delta(X) \to \mathcal{O}(\widetilde{\mathbb{C}^*})$$

defined by

$$\mathscr{S}_{(\ell,\boldsymbol{d})}[\Phi](z) := e^{-c_\delta z} \mathscr{L}_{\frac{\ell - \sum_{i=1}^s d_i}{d_s}, \frac{d_s}{\ell - \sum_{i=1}^{s-1} d_i}} \circ \dots \circ \mathscr{L}_{\frac{\ell-d_1-d_2}{d_2}, \frac{d_2}{\ell-d_1}} \circ \mathscr{L}_{\frac{\ell-d_1}{d_1}, \frac{d_1}{\ell}}[\Phi](z).$$

In other words, any element of $\mathcal{S}_{\iota^*\delta}(Y)$ is of the form

$$e^{-c_\delta z} \int_0^\infty \dots \int_0^\infty \Phi\left(z^{\frac{\ell - \sum_{j=1}^s d_j}{\ell}} \prod_{i=1}^s \zeta_i^{\frac{d_i}{\ell}} \right) e^{-\sum_{i=1}^s \zeta_i} d\zeta_1 \dots d\zeta_s \tag{7.2.1}$$

for some $\Phi \in \mathcal{S}_\delta(X)$. Moreover, $c_\delta \neq 0$ only if $\sum_j d_j = \ell - 1$.

Proof. Set $\rho := c_1(L)$, and let $\rho^* \in H_2(X,\mathbb{Z})$ be its Poincaré dual homology class. In particular, we have $c_1(X) = \ell\rho$ and $c_1(E) = (\sum_{i=1}^s d_i)\rho$. By the adjunction formula, we have $c_1(Y) = \iota^*(c_1(X) - c_1(E))$. From Lemma A.2, we have

$$\begin{aligned} J_X(\delta + \log z \cdot c_1(X))|_{\substack{\mathbf{Q}=1\\ \hbar=1}} &= \sum_{d\in\mathbb{N}} J_{d\rho^*}(\delta) z^{d\ell + c_1(X)} \\ &= \sum_{d\in\mathbb{N}} J_{d\rho^*}(\delta) z^{d\ell+\ell\rho}, \end{aligned} \tag{7.2.2}$$

where $J_{d\rho^*}(\delta) = e^{\delta} \sum_{\alpha,k} \langle \tau_k T_\alpha, 1\rangle^X_{0,2,d\rho^*} T^\alpha$. Analogously, from (5.3.2) we have

$$\begin{aligned} &I_{X,Y}(\delta + (c_1(X) - c_1(E))\log z)|_{\substack{\mathbf{Q}=1\\ \hbar=1}} \\ &= \sum_{d\in\mathbb{N}} J_{d\rho^*}(\delta + (c_1(X) - c_1(E))\log z) \prod_{i=1}^{s} \prod_{m=1}^{\langle d_i\rho, d\rho^*\rangle} (d_i\rho + m) \\ &= \sum_{d\in\mathbb{N}} J_{d\rho^*}(\delta) z^{d(\ell - \sum d_i) + c_1(X) - c_1(E)} \prod_{i=1}^{s} \prod_{m=1}^{d\cdot d_i} (d_i\rho + m) \\ &= \sum_{d\in\mathbb{N}} J_{d\rho^*}(\delta) z^{d(\ell-\sum d_i) + (\ell - \sum d_i)\rho} \prod_{i=1}^{s} \frac{\Gamma(1 + d_i\rho + dd_i)}{\Gamma(1+d_i\rho)}. \end{aligned} \tag{7.2.3}$$

On the one hand, from (7.2.2), one can see that $J_X(\delta + \log z \cdot c_1(X)))|_{\mathbf{Q}=1,\hbar=1}$ is the analytification $\widehat{\mathrm{J}}_X$ of the series $\mathrm{J}_X \in \mathscr{F}_\ell(X)$, concentrated at $c_1(X) = \ell\rho$, defined by

$$\mathrm{J}_X(Z) = \sum_{d\in\mathbb{N}} J_{d\rho^*}(\delta) Z^{d\ell \oplus c_1(X)}.$$

On the other hand, one recognizes in equation (7.2.3) the analytification of the iteration of Laplace transforms

$$\begin{aligned} \mathrm{I}_{X,Y} := \prod_{i=1}^{s} \frac{1}{\Gamma(1+d_i\rho)} \cdot \Big(&\mathscr{L}_{\frac{\ell - \sum_{i=1}^{s} d_i}{d_s}, \frac{d_s}{\ell - \sum_{i=1}^{s-1} d_i}} \\ &\circ \cdots \circ \mathscr{L}_{\frac{\ell - d_1 - d_2}{d_2}, \frac{d_2}{\ell - d_1}} \circ \mathscr{L}_{\frac{\ell - d_1}{d_1}, \frac{d_1}{\ell}} [\mathrm{J}_X] \Big), \end{aligned} \tag{7.2.4}$$

which is an element of $\mathscr{F}_{\frac{\ell - \sum_{i=1}^{s} d_i}{\ell}}(X)$. By Theorems 5.3.1, 5.3.4, 6.5.1, and Proposition 5.3.5, we have

$$J_Y(\iota^*\delta + c_1(Y)\log z)|_{\substack{\mathbf{Q}=1\\ \hbar=1}} = \iota^* \widehat{\mathrm{I}}_{X,Y}(\delta + (c_1(X) - c_1(E))) \exp(-zH(\delta)|_{\mathbf{Q}=1}),$$

where $H(\delta)$ is defined in Proposition 5.3.5. Thus, the components of the right-hand side, with respect to any basis of $H^\bullet(Y, \mathbb{C})$, span the space of master functions $\mathcal{S}_{\iota^*\delta}(Y)$, by Corollary 5.1.3. The factor $\iota^* \prod_{i=1}^{s} \Gamma(1 + d_i\rho)^{-1}$ coming from (7.2.4) can be eliminated by a change of basis of $H^\bullet(Y, \mathbb{C})$. By $H^\bullet(X, \mathbb{C})$-linearity of the Laplace (α, β)-transforms, the claim follows by setting $c_\delta := H(\delta)|_{\mathbf{Q}=1}$. ■

Remark 7.2.2. Integral (7.2.1) is convergent for any $z \in \widetilde{\mathbb{C}^*}$. This follows from the exponential asymptotics of Theorem 4.3.2 for $z \to \infty$, the Fano assumption on Y (i.e. $\sum_{j=1}^{s} d_j < \ell$), and the asymptotics $|\Phi(z)| < C|\log z|^{\dim_{\mathbb{C}} X}$ for $z \to 0^+$ (see Theorem 5.1.2 and Corollary 5.1.3).

Remark 7.2.3. Formula (7.2.4) generalizes [37, Lemma 8.1].

7.3 Integral representations of the second kind

Let $X_1, \dots, X_h$ be Fano smooth projective varieties. Assume that $\det T_{X_j} = L_j^{\otimes \ell_j}$ for ample line bundles L_j. Let Y be a smooth subvariety of $X := \prod_{j=1}^{h} X_j$ defined as the zero locus of a regular section of the line bundle

$$E = \boxtimes_{j=1}^{h} L_j^{\otimes d_j},$$

where the numbers $d_j \in \mathbb{N}^*$ are such that $d_j < \ell_j$ for any $j = 1, \dots, h$.

By Künneth isomorphism, any element of $H^2(X, \mathbb{C})$ is of the form

$$\boldsymbol{\delta} = \sum_{i=1}^{h} 1 \otimes \cdots \otimes \delta_i \otimes \cdots \otimes 1 \quad \text{with } \delta_i \in H^2(X_i, \mathbb{C}).$$

Denote by $\iota \colon Y \to X$ the inclusion.

Theorem 7.3.1. *Let $\boldsymbol{\delta} \in H^2(X, \mathbb{C})$, $\delta_i \in H^2(X_i, \mathbb{C})$ be as above, and let $\mathcal{S}_{\delta_i}(X_i)$ be the corresponding space of master functions of $QH^\bullet(X_i)$. There exists a rational number $c_{\boldsymbol{\delta}} \in \mathbb{Q}$ such that the space of master functions $\mathcal{S}_{\iota^*\delta}(Y)$ is contained in image of the $\mathbb{C}$-linear map $\mathscr{P}_{(\boldsymbol{\ell},\boldsymbol{d})} \colon \bigotimes_{j=1}^{h} \mathcal{S}_{\delta_j}(X_j) \to \mathcal{O}(\widetilde{\mathbb{C}^*})$ defined by*

$$\mathscr{P}_{(\ell,\boldsymbol{d})}[\Phi_1, \dots, \Phi_h](z) := e^{-c_\delta z} \mathscr{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[\Phi_1, \dots, \Phi_h](z),$$

where

$$(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \left(\frac{\ell_1 - d_1}{d_1}, \dots, \frac{\ell_h - d_h}{d_h}; \frac{d_1}{\ell_1}, \dots, \frac{d_h}{\ell_h}\right).$$

In other words, any element of $\mathcal{S}_{\iota^*\delta}(Y)$ is of the form

$$e^{-c_\delta z} \int_0^\infty \prod_{j=1}^{h} \Phi_j\left(z^{\frac{\ell_j - d_j}{\ell_j}} \lambda^{\frac{d_j}{\ell_j}}\right) e^{-\lambda}\, d\lambda \tag{7.3.1}$$

for some $\Phi_j \in \mathcal{S}_{\delta_j}(X)$ with $j = 1, \dots, h$. Moreover, $c_\delta \neq 0$ only if $d_j = \ell_j - 1$ for some j.

Proof. Set $\rho_i := c_1(L_i)$ and let $\rho_i^* \in H_2(X_i, \mathbb{Z})$ be its Poincaré dual homology class, for any $i = 1, \dots, h$. By the Künneth isomorphism, and by the universal property of coproduct of algebras (i.e. tensor product), we have injective[2] maps

$$H^\bullet(X_i,\mathbb{C}) \to H^\bullet(X,\mathbb{C}).$$

In order to ease the computations, in the next formulas we will not distinguish an element of $H^\bullet(X_i,\mathbb{C})$ with its image in $H^\bullet(X,\mathbb{C})$. So, for example we will write

$$c_1(E) = \sum_{p=1}^{h} d_p\rho_p.$$

The same will be applied for elements in $H_2(X,\mathbb{Z})$.

We have

$$\begin{aligned} J_X(\boldsymbol{\delta} + c_1(X)\log z)|_{\substack{\mathbf{Q}=1\\ \hbar=1}} &= \bigotimes_{i=1}^{h} J_{X_i}(\delta_i + c_1(X_i)\log z)|_{\substack{\mathbf{Q}=1\\ \hbar=1}} \\ &= \bigotimes_{i=1}^{h} \sum_{k_i\in\mathbb{N}} J_{i,k_i\rho_i^*}(\delta_i) z^{k_i\ell_i+\ell_i\rho_i}, \end{aligned} \tag{7.3.2}$$

where

$$J_{i,k_i\rho_i^*}(\delta_i) = e^{\delta_i}\sum_{\alpha,j}\langle\tau_j T_{\alpha,i},1\rangle_{0,2,k_i\rho_i^*}^{X_i} T_i^\alpha.$$

Analogously, from (5.3.2), we deduce the formula

$$\begin{aligned} &I_{X,Y}(\boldsymbol{\delta} + (c_1(X)-c_1(E))\log z)|_{\substack{\mathbf{Q}=1\\ \hbar=1}} \\ &= \sum_{k_1,\dots,k_h\in\mathbb{N}} \bigotimes_{i=1}^{h} J_{i,k_i\rho_i^*}(\delta_i) z^{k_i(\ell_i-d_i)+(\ell_i-d_i)\rho_i} \\ &\quad\cdot \prod_{m=1}^{\langle\sum_p d_p\rho_p,\sum_p k_p\rho_p^*\rangle}\left(\sum_p d_p\rho_p + m\right) \\ &= \sum_{k_1,\dots,k_h\in\mathbb{N}} \bigotimes_{i=1}^{h} J_{i,k_i\rho_i^*}(\delta_i) z^{k_i(\ell_i-d_i)+(\ell_i-d_i)\rho_i} \\ &\quad\cdot \prod_{m=1}^{\sum_p d_pk_p}\left(\sum_p d_p\rho_p + m\right) \\ &= \sum_{k_1,\dots,k_h\in\mathbb{N}} \bigotimes_{i=1}^{h} J_{i,k_i\rho_i^*}(\delta_i) z^{k_i(\ell_i-d_i)+(\ell_i-d_i)\rho_i} \\ &\quad\cdot \frac{\Gamma(1+\sum_p d_pk_p+\sum_p d_p\rho_p)}{\Gamma(1+\sum_p d_p\rho_p)}. \end{aligned} \tag{7.3.3}$$

[2]In particular, we have inclusions $\mathscr{F}_{\boldsymbol{k}}(X_j) \to \mathscr{F}_{\boldsymbol{k}}(X)$.

Each element in the tensor product (7.3.2) can easily be recognized as the analytification $\widehat{\mathrm{J}}_{X_i}$ of a series $\mathrm{J}_{X_i} \in \mathscr{F}_{\ell_i}(X)$, for each $i = 1, \ldots, h$. The function in equation (7.3.3) can be identified with the analytification of the Laplace $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-multitransform

$$\mathrm{I}_{X,Y} = \left(\bigotimes_{i=1}^{h} \frac{1}{\Gamma(1 + \sum_p d_p \rho_p)} \right) \cup_X \mathscr{L}_{\alpha,\beta} \left[\bigotimes_{i=1}^{h} \mathrm{J}_{X_i} \right], \tag{7.3.4}$$

where

$$(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \left(\frac{\ell_1 - d_1}{d_1}, \ldots, \frac{\ell_h - d_h}{d_h}; \frac{d_1}{\ell_1}, \ldots, \frac{d_h}{\ell_h} \right).$$

The series $\mathrm{I}_{X,Y}$ can be seen as an element of $\mathscr{F}_{\boldsymbol{\kappa}}(X)$, with $\boldsymbol{\kappa} = (\ell_i - d_i)_{i=1}^{h}$, via the Künneth isomorphism. By Theorems 5.3.1, 5.3.4, 6.5.1, and Proposition 5.3.5, we have

$$J_Y(\iota^* \boldsymbol{\delta} + c_1(Y) \log z)|_{\substack{\mathbf{Q}=1 \\ \hbar=1}} = \iota^* \widehat{\mathrm{I}}_{X,Y}(\boldsymbol{\delta} + (c_1(X) - c_1(E))) \exp(-zH(\boldsymbol{\delta})|_{\mathbf{Q}=1}).$$

Thus, the components of the right-hand side, with respect to any basis of $H^\bullet(Y, \mathbb{C})$, span the space of master functions $\mathcal{S}_{\iota^*\delta}(Y)$, by Corollary 5.1.3. Notice that the factor $\iota^* \bigotimes_{i=1}^{s} \Gamma(1 + \sum_p d_p \rho_p)^{-1}$ coming from (7.3.4) can be eliminated by a change of basis of $H^\bullet(Y, \mathbb{C})$. By $H^\bullet(X, \mathbb{C})$-linearity of the Laplace $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-multitransform, the claim follows by setting $c_{\boldsymbol{\delta}} := H(\boldsymbol{\delta})|_{\mathbf{Q}=1}$. ∎

Remark 7.3.2. Integral (7.3.1) is convergent for any $z \in \widetilde{\mathbb{C}^*}$. This follows from the exponential asymptotics of Theorem 4.3.2 for $z \to \infty$, the assumption $d_j < \ell_j$ for any $j = 1, \ldots, h$, and the asymptotics $|\Phi_j(z)| < C |\log z|^{\dim_{\mathbb{C}} X_j}$ for $z \to 0^+$ (see Theorem 5.1.2 and Corollary 5.1.3).

Remark 7.3.3. Formula (7.3.4) generalizes [37, Lemma 8.1].

7.4 Master functions as Mellin–Barnes integrals

When applied to the case of Fano complete intersections in products of projective spaces, Theorems 7.2.1 and 7.3.1 give explicit Mellin–Barnes integral representations of solutions of the qDE.

Theorem 7.4.1. *Let Y be a Fano complete intersection in $\mathbb{P}^{n-1}$ defined by h homogeneous polynomials of degrees $d_1, \ldots, d_h$. There exists a unique $c \in \mathbb{Q}$ such that any master functions in $\mathcal{S}_0(Y)$ is a linear combination of the Mellin–Barnes integrals*

$$G_j(z) = \frac{e^{-cz}}{2\pi\sqrt{-1}} \int_\gamma \Gamma(s)^n \prod_{k=1}^{h} \Gamma(1 - d_k s) z^{-(n - \sum_{k=1}^{h} d_k)s} \varphi_j(s)\, ds$$

for $j = 0, \dots, n-1$. *The path of integration* γ *is a parabola of the form*

$$\mathrm{Re}\, s = -\rho_1 (\mathrm{Im}\, s)^2 + \rho_2,$$

for suitable $\rho_1, \rho_2 \in \mathbb{R}_+$, *such that* γ *encircles the poles of* $\Gamma(s)^n$, *and separates them from the poles of the factors* $\Gamma(1 - d_k s)$. *The functions* φ_j *are given by*

- *for* n *even:*

$$\varphi_j(s) := \exp\big(2\pi\sqrt{-1}\, js\big), \quad j = 0, \dots, n-1,$$

- *for* n *odd:*

$$\varphi_j(s) := \exp\big(2\pi\sqrt{-1}\, js + \pi\sqrt{-1}s\big), \quad j = 0, \dots, n-1.$$

Moreover, $c \neq 0$ *only if* $\sum_k d_k = n - 1$.

Proof. The functions

$$g_j(z) := \frac{1}{2\pi\sqrt{-1}} \int_\gamma \Gamma(s)^n z^{-ns} \varphi_j(s)\, ds, \quad j = 0, \dots, n-1,$$

are a basis of the space of master functions $\mathcal{S}_0(\mathbb{P}^{n-1})$, see [46, Lemma 5]. The result follows by applying Theorem 7.2.1 to the case $X = \mathbb{P}^{n-1}$, $\ell = n$. ■

Theorem 7.4.2. *Let* Y *be a Fano hypersurface of* $\mathbb{P}^{n_1 - 1} \times \dots \times \mathbb{P}^{n_h - 1}$ *defined by a homogeneous polynomial of multi-degree* $(d_1, \dots, d_h)$. *There exists a unique* $c \in \mathbb{Q}$ *such that any master function in* $\mathcal{S}_0(Y)$ *is a linear combination of the multi-dimensional Mellin–Barnes integrals*

$$H_{\boldsymbol{j}}(z) := \frac{e^{-cz}}{(2\pi\sqrt{-1})^h} \int_{\times\gamma_i} \left[\prod_{i=1}^h \Gamma(s_i)^{n_i} \varphi^i_{j_i}(s_i)\right] \cdot \Gamma\left(1 - \sum_{i=1}^h d_i s_i\right) z^{-\sum_{i=1}^h (n_i - d_i) s_i}\, ds_1 \dots ds_h$$

for $\boldsymbol{j} = (j_1, \dots, j_h) \in \prod_{i=1}^h \{0, \dots, n_i - 1\}$. *The paths* γ_i *are parabolas of the form*

$$\mathrm{Re}\, s_i = -\rho_{1,i} (\mathrm{Im}\, s_i)^2 + \rho_{2,i},$$

for suitable $\rho_{1,i}, \rho_{2,i} \in \mathbb{R}_+$, *so that they encircle the poles of the factors* $\Gamma(s_i)^{n_i}$. *The function* $\varphi^i_{j_i}$ *is defined as follows:*

- *for* n_i *even:*

$$\varphi^i_{j_i}(s_i) := \exp\big(2\pi\sqrt{-1}\, j_i s_i\big), \quad j_i = 0, \dots, n_i - 1,$$

- *for* n_i *odd:*

$$\varphi^i_{j_i}(s_i) := \exp\big(2\pi\sqrt{-1}\, j_i s_i + \pi\sqrt{-1}s_i\big), \quad j_i = 0, \dots, n_i - 1.$$

Moreover, $c \neq 0$ *only if* $d_i = n_i - 1$ *for some* $i = 1, \dots, h$.

Proof. The result follows by application of Theorem 7.3.1 to the case $X_i = \mathbb{P}^{n_i-1}$, $\ell_i = n_i$. For each factor $\mathbb{P}^{n_i-1}$ a basis of the space $\mathcal{S}_0(\mathbb{P}^{n_i-1})$ is given by the integrals

$$g^i_{j_i}(z) := \frac{1}{2\pi\sqrt{-1}}\int_{\gamma_i}\Gamma(s)^{n_i}z^{-n_is}\varphi^i_{j_i}(s)\,ds, \quad j_i = 0,\dots,n_i-1. \qquad \blacksquare$$

Example. Consider the complex Grassmannian $\mathbb{G} := \mathbb{G}(2,4)$: it can be realized as a quadric in $\mathbb{P}^5$, by Plücker embedding. It can be shown that the space $\mathcal{S}_0(\mathbb{G})$ is the space of solutions Φ of the qDE given by

$$\vartheta^5\Phi - 1024z^4\vartheta\Phi - 2048z^4\Phi = 0, \quad \vartheta := z\frac{d}{dz}. \tag{7.4.1}$$

By Theorem 7.4.1, any solution of (7.4.1) is a linear combination of the functions

$$G_j(z) = \frac{1}{2\pi\sqrt{-1}}\int_\gamma \Gamma(s)^6\Gamma(1-2s)z^{-4s}\exp\big(2\pi\sqrt{-1}js\big)\,ds, \quad j = 0,\dots,5.$$

Recalling the reflection and duplication formulas for Γ-function (see e.g. [64]),

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \quad \Gamma(2z) = \pi^{-\frac{1}{2}}2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right),$$

it is easy to see that the function

$$G_0(z) = \frac{2\pi^{\frac{3}{2}}}{2\pi\sqrt{-1}}\int_\gamma \frac{\Gamma(s)^5}{\Gamma\big(s+\frac{1}{2}\big)}\frac{4^{-s}}{\sin(2\pi s)}z^{-4s}\,ds$$

is a solution of (7.4.1). In [23, Section 6] the solutions

$$\Phi_1(z) = \frac{1}{2\pi\sqrt{-1}}\int_\gamma \frac{\Gamma(s)^5}{\Gamma\big(s+\frac{1}{2}\big)}4^{-s}z^{-4s}\,ds$$

and

$$\Phi_2(z) = \frac{1}{2\pi\sqrt{-1}}\int_\gamma \Gamma(s)^5\Gamma\left(\frac{1}{2}-s\right)e^{i\pi s}4^{-s}z^{-4s}\,ds$$

of equation (7.4.1) were found and studied. It is not difficult to see that Φ_1 and Φ_2 are linear combinations of the functions G_j.

Remark 7.4.3. This example can be extended to Grassmannians $\mathbb{G}(k,n)$ and other families of partial flag varieties. In the case of Grassmannians it gives different integral representations of solutions with respect to those obtained from the quantum Satake identification [42, 55]. More in general, it would be interesting to do a comparison with the integral representations of solutions obtained from the Abelian–Nonabelian correspondence [14].

Chapter 8

Dubrovin conjecture

8.1 Exceptional collections and exceptional bases

Let X be a smooth complex projective variety, and denote by $\mathcal{D}^b(X)$ the bounded derived category of coherent sheaves on X, see [38,52]. Given $E, F \in \mathrm{Ob}(\mathcal{D}^b(X))$, define $\mathrm{Hom}^\bullet(E, F)$ as the $\mathbb{C}$-vector space[1]

$$\mathrm{Hom}^\bullet(E, F) := \bigoplus_{k\in\mathbb{Z}} \mathrm{Hom}(E, F[k]).$$

An object $E \in \mathrm{Ob}(\mathcal{D}^b(X))$ is said to be *exceptional* if $\mathrm{Hom}^\bullet(E, E)$ is a one-dimensional $\mathbb{C}$-algebra, generated by the identity morphism.

A collection $\mathfrak{E} = (E_1, \dots, E_n)$ of objects of $\mathcal{D}^b(X)$ is said to be an *exceptional collection* if

(1) each object E_i is exceptional,

(2) we have $\mathrm{Hom}^\bullet(E_j, E_i) = 0$ for $j > i$.

Moreover, an exceptional collection $\mathfrak{E}$ is *full* if it generates $\mathcal{D}^b(X)$, i.e. any triangular subcategory containing all objects of $\mathfrak{E}$ is equivalent to $\mathcal{D}^b(X)$ via the inclusion functor.

Consider the Grothendieck group $K_0(X) \equiv K_0(\mathcal{D}^b(X))$, and let χ to be the Grothendieck–Euler–Poincaré bilinear form

$$\chi([V], [F]) := \sum_k (-1)^k \dim_{\mathbb{C}} \mathrm{Hom}(V, F[k]), \quad V, F \in \mathcal{D}^b(X).$$

Definition 8.1.1. A basis $(e_i)_{i=1}^n$ of $K_0(X)_{\mathbb{C}}$ is called *exceptional* if $\chi(e_i, e_i) = 1$ for $i = 1, \dots, n$, and $\chi(e_j, e_i) = 0$ for $1 \leqslant i < j \leqslant n$.

Lemma 8.1.2. *Let $(E_i)_{i=1}^n$ be a full exceptional collection in $\mathcal{D}^b(X)$. The K-classes $([E_i])_{i=1}^n$ form an exceptional basis of $K_0(X)_{\mathbb{C}}$.* ∎

8.2 Mutations and helices

Let $\mathfrak{E} = (E_1, \dots, E_n)$ be an exceptional collection in $\mathcal{D}^b(X)$. For $i = 1, \dots, n-1$ define the collections

$$\mathbb{L}_i\mathfrak{E} := (E_1, \dots, E_{i-1}, E'_{i+1}, E_i, E_{i+2}, \dots, E_n),$$
$$\mathbb{R}_i\mathfrak{E} := (E_1, \dots, E_{i-1}, E_{i+1}, E''_i, E_{i+2}, \dots, E_n),$$

[1] Notice that the category $\mathcal{D}^b(X)$ is a $\mathbb{C}$-linear category.

where the objects E'_{i+1}, E''_i sit in the distinguished triangles

$$E'_{i+1}[-1] \to \mathrm{Hom}^\bullet(E_i, E_{i+1}) \otimes E_i \to E_{i+1} \to E'_{i+1},$$
$$E''_i \to E_i \to \mathrm{Hom}^\bullet(E_i, E_{i+1})^* \otimes E_{i+1} \to E''_i[1].$$

Remark 8.2.1. The object E'_{i+1} (resp. E''_i) is uniquely defined *up to unique isomorphism*, because of the exceptionality of E_i (resp. E_{i+1}), see [21, Section 3.3].

Proposition 8.2.2 ([12, 44]). *For any i, with $0 < i < n$, the collections $\mathbb{L}_i\mathfrak{E}, \mathbb{R}_i\mathfrak{E}$ are exceptional. The mutation operators $\mathbb{L}_i, \mathbb{R}_i$ satisfy the following identities:*

$$\mathbb{L}_i\mathbb{R}_i = \mathbb{R}_i\mathbb{L}_i = \mathrm{Id},$$
$$\mathbb{R}_i\mathbb{R}_j = \mathbb{R}_j\mathbb{R}_i \quad \text{if } |i-j| > 1, \qquad \mathbb{R}_{i+1}\mathbb{R}_i\mathbb{R}_{i+1} = \mathbb{R}_i\mathbb{R}_{i+1}\mathbb{R}_i.$$

Moreover, if $\mathfrak{E}$ is full, then also $\mathbb{L}_i\mathfrak{E}$ and $\mathbb{R}_i\mathfrak{E}$ are full. ■

Denote by $\beta_1, \dots, \beta_{n-1}$ the generators of the braid group $\mathcal{B}_n$, satisfying the relations

$$\beta_i\beta_{i+1}\beta_i = \beta_{i+1}\beta_i\beta_{i+1}, \qquad \beta_i\beta_j = \beta_j\beta_i \quad \text{if } |i-j| > 1.$$

We define the left action of $\mathcal{B}_n$ on the set of exceptional collections of length n by identifying the action of β_i with $\mathbb{L}_i$.

Definition 8.2.3. Let $\mathfrak{E} = (E_1, \dots, E_n)$ be a full exceptional collection. We define the *helix* generated by $\mathfrak{E}$ to be the infinite family $(E_i)_{i\in\mathbb{Z}}$ of exceptional objects such that

$$(E_{1-kn}, E_{2-kn}, \dots, E_{n-kn}) = \mathfrak{E}^\beta, \quad \beta = (\beta_{n-1} \dots \beta_1)^{kn}, \quad k \in \mathbb{Z}.$$

Any family of n consecutive exceptional objects $(E_{i+k})_{k=1}^n$ is called a *foundation* of the helix.

Lemma 8.2.4 ([44]). *The following statements hold:*

(1) *Any foundation is a full exceptional collection.*

(2) *For $i, j \in \mathbb{Z}$, we have* $\mathrm{Hom}^\bullet(E_i, E_j) \cong \mathrm{Hom}^\bullet(E_{i-n}, E_{j-n})$. ■

The action of the braid group on the set of exceptional collections in $\mathcal{D}^b(X)$ admits a K-theoretical analogue on the set of exceptional bases of $K_0(X)_\mathbb{C}$, see [21, 44].

8.3 Γ-classes and graded Chern character

Let V be a complex vector bundle on X of rank r, and $\delta_1, \dots, \delta_r$ its Chern roots, so that $c_j(V) = s_j(\delta_1, \dots, \delta_r)$, where s_j is the j-th elementary symmetric polynomial.

Definition 8.3.1. Let Q be an indeterminate, and $F \in \mathbb{C}[\![Q]\!]$ be of the form

$$F(Q) = 1 + \sum_{n \geqslant 1} \alpha_n Q^n.$$

The F-*class* of V is the characteristic class $\widehat{F}_V \in H^\bullet(X)$ defined by

$$\widehat{F}_V := \prod_{j=1}^{r} F(\delta_j).$$

Definition 8.3.2. The $\Gamma^{\pm}$-*classes* of V are the characteristic classes associated with the Taylor expansions

$$\Gamma(1 \pm Q) = \exp\left(\mp \gamma Q + \sum_{m=2}^{\infty} (\mp 1)^m \frac{\zeta(m)}{m} Q^m \right) \in \mathbb{C}[\![Q]\!],$$

where γ is the Euler–Mascheroni constant and ζ is the Riemann zeta function.

If $V = TX$, then we denote $\widehat{\Gamma}^{\pm}_X$ its Γ-classes.

Definition 8.3.3. The *graded Chern character* of the complex vector bundle V is the characteristic class $\mathrm{Ch}(V) \in H^\bullet(X)$ defined by $\mathrm{Ch}(V) := \sum_{j=1}^{r} \exp(2\pi\sqrt{-1}\delta_j)$.

8.4 Statement of the conjecture

Let X be a Fano variety. In [31] Dubrovin conjectured that many properties of the qDE of X, in particular its monodromy, Stokes and central connection matrices, are encoded in the geometry of exceptional collections in $\mathcal{D}^b(X)$. The following conjecture is a refinement of the original version in [31].

Conjecture 8.4.1 ([21]). *Let X be a smooth Fano variety of Hodge–Tate type.*

(1) *The quantum cohomology $QH^\bullet(X)$ has semisimple points if and only if there exists a full exceptional collection in $\mathcal{D}^b(X)$.*

(2) *If $QH^\bullet(X)$ is generically semisimple, then for any oriented ray ℓ of slope $\varphi \in [0, 2\pi[$ there is a map from the set of ℓ-chambers to the set of helices with a marked foundation.*

(3) *Let Ω_ℓ be an ℓ-chamber and $\mathfrak{E}_\ell = (E_1, \dots, E_n)$ the corresponding exceptional collection (the marked foundation). Denote by S and C Stokes and central connection matrices computed in Ω_ℓ with respect to a basis $(T_\alpha)_{\alpha=1}^{n}$ of $H^\bullet(X, \mathbb{C})$.*

   (a) *The matrix S is the inverse of the Gram matrix of the χ-pairing in $K_0(X)_{\mathbb{C}}$ with respect to the exceptional basis $[\mathfrak{E}_\ell]$,*

$$(S^{-1})_{ij} = \chi(E_i, E_j).$$

(b) *The matrix C coincides with the matrix associated with the $\mathbb{C}$-linear morphism*

$$\mathit{\Pi}_X^-: K_0(X)_{\mathbb{C}} \to H^\bullet(X),$$
$$F \mapsto \frac{(\sqrt{-1})^{\overline{d}}}{(2\pi)^{\frac{d}{2}}} \widehat{\Gamma}_X^- \exp(-\pi\sqrt{-1}c_1(X))\mathrm{Ch}(F),$$

where $d := \dim_{\mathbb{C}} X$, and $\overline{d}$ is the residue class $d \pmod 2$. The matrix is computed with respect to the exceptional basis $[\mathfrak{E}_\ell]$ and the pre-fixed basis $(T_\alpha)_{\alpha=1}^n$.

Remark 8.4.2. If point (3.b) holds true, then automatically also point (3.a) holds true. This follows from identity (4.4.2) and the Hirzebruch–Riemann–Roch theorem, see [21, Corollary 5.8].

Remark 8.4.3. In [9], A. Bayer suggested dropping any reference to X being Fano in the formulation of the Dubrovin conjecture. He proved indeed that the semisimplicity of the quantum cohomology preserves under blow-ups at any number of points. It follows that point (1) of Conjecture 8.4.1 (the *qualitative* part) still holds true after blowing up X at an arbitrary number of points, which may yield a non-Fano variety. To the best of our knowledge, however, there is no non-Fano example for which both the Stokes and central connection matrices have been explicitly computed. In Chapters 10 and 11 we will provide the first example, in the case of Hirzebruch surfaces.

Remark 8.4.4. Assume the validity of points (3.a) and (3.b) of Conjecture 8.4.1. The action of the braid group $\mathcal{B}_n$ on the Stokes and central connection matrices (cf. Lemma 4.6.2) is compatible with the action of $\mathcal{B}_n$ on the marked foundations attached at each ℓ-chambers. Different choices of the branch of the Ψ-matrix correspond to shifts of objects of the marked foundation. The matrix M_0^{-1} is identified with the canonical operator $\kappa: K_0(X)_{\mathbb{C}} \to K_0(X)_{\mathbb{C}}$, $[F] \mapsto (-1)^d[F \otimes \omega_X]$. Equations (4.4.4) imply that the connection matrices $C^{(m)}$, with $m \in \mathbb{Z}$, correspond to the matrices of the morphism $\mathit{\Pi}_X^-$ with respect to the foundations $(\mathfrak{E}_\ell \otimes \omega_X^{\otimes m})[md]$. The statement $S^{(m)} = S$ coincides with the Hom-periodicity described in point (2) of Lemma 8.2.4, see [21, Theorem 5.9].

Remark 8.4.5. Conjecture 8.4.1 relates two different aspects of the geometry of X, namely its *symplectic structure* (Gromov–Witten theory) and its *complex structure* (the derived category $\mathcal{D}^b(X)$). Heuristically, Conjecture 8.4.1 follows from the homological mirror symmetry conjecture of M. Kontsevich, see [21, Section 5.5].

Remark 8.4.6. In the paper [54] it was underlined the role of Γ-classes for refining the original version of Dubrovin's conjecture [31]. Subsequently, in [34] and [36, Γ-conjecture II] two equivalent versions of point (3.b) above were given. However, in both these versions, different choices of solutions in Levelt form of the qDE at $z = 0$

are chosen with respect to the natural ones in the theory of Frobenius manifolds, see [21, Section 5.6].

Remark 8.4.7. Point (3.b) of Conjecture 8.4.1 allows to identify K-classes with solutions of the joint system of equations (2.7.1)–(2.7.2). Under this identification, Stokes fundamental solutions correspond to exceptional bases of K-theory. In the approach of [26,75], where the equivariant case is addressed, such an identification is more fundamental and *a priori*: it is defined via explicit integral representations of solutions of the joint system of qDE and qKZ equations.

Remark 8.4.8. Note that the existence of a map between ℓ-chambers and helices with a marked foundation, discussed in point (2) of Conjecture 8.4.1, is an important aspect of the Dubrovin conjecture. A careful study of such a correspondence may hide several delicate open problems. Consider, for instance, the study of injectivity and surjectivity of such a map. This study is closely related (possibly equivalent) to the study of the freeness and transitivity of the braid group action on the set of exceptional collections. These are well-known open problems, whose answer is known in a few special cases only, see [44]. In the remaining sections of this paper, we will address the study of point (3) of Conjecture 8.4.1, but not of point (2).

Chapter 9

Quantum cohomology of Hirzebruch surfaces

9.1 Preliminaries on Hirzebruch surfaces

Hirzebruch surfaces $\mathbb{F}_k$, with $k \in \mathbb{Z}$, are defined as the total space of $\mathbb{P}^1$-projective bundles on $\mathbb{P}^1$, namely

$$\mathbb{F}_k := \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-k)), \quad k \in \mathbb{Z},$$

where $\mathcal{O}(n)$ are line bundles on $\mathbb{P}^1$. More explicitly, they can be defined as hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^1$ by

$$\mathbb{F}_k := \{([a_0 : a_1 : a_2], [b_1 : b_2]) \in \mathbb{P}^2 \times \mathbb{P}^1 : a_1 b_1^k = a_2 b_2^k\}, \quad k \in \mathbb{N}. \tag{9.1.1}$$

Hirzebruch surfaces have the following properties:

- the surfaces $(\mathbb{F}_{2k})_{k\in\mathbb{N}}$ are all diffeomorphic,
- the surfaces $(\mathbb{F}_{2k+1})_{k\in\mathbb{N}}$ are all diffeomorphic,
- the surfaces $\mathbb{F}_n$ and $\mathbb{F}_m$ with $n \neq m$ are not biholomorphic,
- the only Fano Hirzebruch surfaces are $\mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_1 \cong \mathrm{Bl}_{\mathrm{pt}}\mathbb{P}^2$,
- the surfaces $\mathbb{F}_n$ and $\mathbb{F}_m$ are deformation equivalent if and only if n and m have the same parity.

See [10, 49].

Remark 9.1.1. Let $0 \leqslant m \leqslant \frac{1}{2}n$. Consider the family $\mathcal{F}$ defined by the equation

$$\mathcal{F} := \{([a_0 : a_1 : a_2], [b_1 : b_2], t) \in \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{C} : a_1 b_1^n - a_2 b_2^n + t a_0 b_1^{n-m} b_2^m = 0\}.$$

The central fiber over $t = 0$ is $\mathbb{F}_n$. Any non-central fiber over $t \neq 0$ is isomorphic to $\mathbb{F}_{n-2m}$. See [56, Example 2.16]. See also [74] and [63, Example 0.1.10].

Remark 9.1.2. The only possible complex structures on $\mathbb{S}^2 \times \mathbb{S}^2$ are the even Hirzebruch surfaces $\mathbb{F}_{2k}$, with $k \in \mathbb{N}$, and the only possible complex structures on the connected sum $\mathbb{P}^2 \# \overline{\mathbb{P}^2}$ are the odd Hirzebruch surfaces $\mathbb{F}_{2k+1}$, with $k \in \mathbb{N}$, see [67].

9.2 Classical cohomology of Hirzebruch surfaces

Using the explicit polynomial description (9.1.1) of the Hirzebruch surfaces, let us define the following subvarieties of $\mathbb{F}_k$:

$$\Sigma_1^k := \{a_1 = a_2 = 0\}, \quad \Sigma_2^k := \{a_2 = b_1 = 0\},$$
$$\Sigma_3^k := \{a_1 = b_2 = 0\}, \quad \Sigma_4^k := \{a_0 = 0\}.$$

Each of these subvarieties naturally define a cycle in $H_2(\mathbb{F}_k,\mathbb{Z})$. Notice that, under the identification

$$\mathbb{F}_k \equiv \mathcal{O}(-k) \cup \infty\text{section},$$

we can

(1) identify Σ_1^k with the 0-section of $\mathcal{O}(-k)$,
(2) identify Σ_4^k with the ∞-section,
(3) identify both Σ_2^k and Σ_3^k with (the compactification of) two fibers of $\mathcal{O}(-k)$.

Using the original notations of Hirzebruch, we denote by

- $\tau_k \in H_2(\mathbb{F}_k,\mathbb{C})$ the homology class defined by Σ_1^k,
- $\varepsilon_k \in H_2(\mathbb{F}_k,\mathbb{C})$ the homology class defined by Σ_4^k,
- $\nu_k \in H_2(\mathbb{F}_k,\mathbb{C})$ the homology class defined by both Σ_2^k and Σ_3^k.

As it is easily seen, the three classes $\tau_k, \varepsilon_k, \nu_k$ are not $\mathbb{Z}$-linearly independent. They are indeed related by the equation

$$\varepsilon_k = \tau_k + k\nu_k. \tag{9.2.1}$$

Finally, let us also introduce a homogeneous basis $(T_{0,k}, T_{1,k}, T_{2,k}, T_{3,k})$ of the classical cohomology $H^\bullet(\mathbb{F}_k,\mathbb{Z})$, where

$$T_{0,k} := 1, \quad T_{1,k} := \mathrm{PD}(\varepsilon_k), \quad T_{2,k} := \mathrm{PD}(\nu_k), \quad T_{3,k} := \mathrm{PD}(\mathrm{pt}),$$

where $\mathrm{PD}(\alpha)$ denotes the Poincaré dual class of $\alpha \in H_\bullet(\mathbb{F}_k,\mathbb{Z})$. We denote the corresponding dual coordinates by $(t^{0,k}, t^{1,k}, t^{2,k}, t^{3,k})$.

By the Leray–Hirsch theorem, the classical cohomology algebra is generated by the classes $(T_{1,k}, T_{2,k})$. More precisely, we have the following result.

Theorem 9.2.1. *In the classical cohomology ring $H^\bullet(\mathbb{F}_k,\mathbb{Z})$, the following identities hold true:*

(1) $T_{1,k}^2 = k \cdot T_{3,k}$,
(2) $T_{2,k}^2 = 0$,
(3) $T_{1,k}T_{2,k} = T_{3,k}$.

Hence, the following presentation of algebras holds:

$$H^\bullet(\mathbb{F}_k,\mathbb{C}) \cong \frac{\mathbb{C}[T_{1,k}, T_{2,k}]}{\langle T_{2,k}^2, T_{1,k}^2 - k\cdot T_{1,k}T_{2,k}\rangle}.$$

The Poincaré metric in the basis $(T_{i,k})_{i=0}^3$ is given by

$$\eta_k = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & k & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{9.2.2}$$

Proposition 9.2.2 ([59]). *Let $k \in \mathbb{N}$. The collection*

$$(\mathcal{O}, \mathcal{O}(\Sigma_2^k), \mathcal{O}(\Sigma_4^k), \mathcal{O}(\Sigma_2^k + \Sigma_4^k))$$

is a full exceptional collection in $\mathcal{D}^b(\mathbb{F}_k)$. The corresponding Gram matrix of the χ-pairing is

$$\begin{pmatrix} 1 & 2 & 2+k & 4+k \\ 0 & 1 & k & 2+k \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Proof. The Gram matrix can easily be computed by the Hirzebruch–Riemann–Roch theorem. ∎

9.3 Quantum cohomology of Hirzebruch surfaces

There exist only two classes of deformation equivalence of Hirzebruch surfaces, namely $(\mathbb{F}_{2k})_{k\in\mathbb{N}}$ and $(\mathbb{F}_{2k+1})_{k\in\mathbb{N}}$. Hence, by the deformation axiom of Gromov–Witten invariants [27], the quantum cohomology algebra of $\mathbb{F}_{2k}$ (resp. $\mathbb{F}_{2k+1}$) can be identified with the one of $\mathbb{F}_0$ (resp. $\mathbb{F}_1$), as explained in Remark 4.5.2. Notice that the quantum cohomology algebras of $\mathbb{F}_0$ and $\mathbb{F}_1$ coincide with the corresponding Batyrev rings [8]. This does not hold true for other Hirzebruch surfaces $\mathbb{F}_k$ with $k \neq 0, 1$, being not Fano [73]. See also [6] for a presentation of the quantum cohomology algebra of $\mathbb{F}_1$.

9.3.1 Case of $\mathbb{F}_{2k}$

The diffeomorphism $\varphi_{2k}\colon \mathbb{F}_{2k} \to \mathbb{F}_0$ induces isomorphisms in homology and cohomology. We have $(\varphi_{2k})_*(\tau_{2k}) = \tau_0$ and $(\varphi_{2k})_*(\nu_{2k}) = \nu_0$, so that from equations (9.2.1) and (9.2.2) we deduce

$$\varphi_{2k}^*(T_{0,0}) = T_{0,2k}, \tag{9.3.1}$$
$$\varphi_{2k}^*(T_{1,0}) = T_{1,2k} - kT_{2,2k}, \tag{9.3.2}$$
$$\varphi_{2k}^*(T_{2,0}) = T_{2,2k}, \tag{9.3.3}$$
$$\varphi_{2k}^*(T_{3,0}) = T_{3,2k}. \tag{9.3.4}$$

Thus, we can identify the quantum cohomologies $QH^\bullet(\mathbb{F}_0)$ and $QH^\bullet(\mathbb{F}_{2k})$ via the change of coordinates

$$\begin{aligned} t^{0,2k} &= t^{0,0}, & t^{1,2k} &= t^{1,0}, \\ t^{2,2k} &= t^{2,0} - kt^{1,0}, & t^{3,2k} &= t^{3,0}. \end{aligned} \tag{9.3.5}$$

Theorem 9.3.1. *For any $k \geqslant 0$, the following isomorphism of algebras holds true:*

$$QH^\bullet(\mathbb{F}_{2k}) \cong \frac{\mathbb{C}[T_{1,2k}, T_{2,2k}, q_1, q_2]}{\langle T_{2,2k}^{\circ 2} - q_1^k q_2, (T_{1,2k} - k \cdot T_{2,2k})^{\circ 2} - q_1 \rangle},$$

where $q_1 = \exp(t^{1,2k})$ and $q_2 = \exp(t^{2,2k})$.

Proof. The assertion follows from the presentation of the quantum cohomology algebra of $QH^\bullet(\mathbb{F}_0) \cong QH^\bullet(\mathbb{P}^1) \otimes QH^\bullet(\mathbb{P}^1)$, and formulas (9.3.1)–(9.3.5). ■

Lemma 9.3.2. *For all $k \geqslant 0$ we have*

$$T_{1,2k} \circ T_{2,2k} = T_{3,2k} + k q_1^k q_2. \tag{9.3.6}$$

Proof. By homogeneity, let $\lambda_{0,2k}, \lambda_{1,2k}, \lambda_{2,2k}, \lambda_{3,2k}$ be the dual basis of $H_\bullet(\mathbb{F}_{2k}, \mathbb{C})$ of the basis $(T_{i,2k})_{i=0}^3$. By the deformation axiom of Gromov–Witten invariants, for any $r, s \in \mathbb{N}$, we have

$$\begin{aligned}
&\langle T_{1,2k}, T_{2,2k}, T_{3,2k}\rangle_{0,3,r\lambda_{1,2k}+s\lambda_{1,2k}}^{\mathbb{F}_{2k}} \\
&\quad = \langle T_{1,0} + kT_{2,0}, T_{2,0}, T_{3,0}\rangle_{0,3,r\lambda_{0,0}+s(\lambda_{1,0}-k\lambda_{0,0})}^{\mathbb{F}_0} \\
&\quad = \langle T_{1,0}, T_{2,0}, T_{3,0}\rangle_{0,3,(r-sk)\lambda_{0,0}+s\lambda_{1,0}}^{\mathbb{F}_0} + k\langle T_{2,0}, T_{2,0}, T_{3,0}\rangle_{0,3,(r-sk)\lambda_{0,0}+s\lambda_{1,0}}^{\mathbb{F}_0} \\
&\quad = \langle \sigma, 1, \sigma\rangle_{0,3,(r-sk)H}^{\mathbb{P}^1} \langle 1, \sigma, \sigma\rangle_{0,3,(r-sk)H}^{\mathbb{P}^1} \\
&\qquad\quad + k\langle 1, 1, \sigma\rangle_{0,3,(r-sk)H}^{\mathbb{P}^1} \langle \sigma, \sigma, \sigma\rangle_{0,3,(r-sk)H}^{\mathbb{P}^1} \\
&\quad = k \cdot \delta_{1,2(r-ks)+1}\delta_{3,2s+1}.
\end{aligned}$$

Here we used the class $H \in H_2(\mathbb{P}^1, \mathbb{Z})$ to be the hyperplane class and $\sigma \in H^2(\mathbb{P}^2, \mathbb{Z})$ to be its dual. This gives the quantum correction in (9.3.6). ■

9.3.2 Case of $\mathbb{F}_{2k+1}$

The diffeomorphism $\varphi_{2k+1}\colon \mathbb{F}_{2k+1} \to \mathbb{F}_1$ induces an isomorphism φ_{2k+1}^* in cohomology given by

$$\varphi_{2k+1}^*(T_{0,1}) = T_{0,2k+1}, \tag{9.3.7}$$
$$\varphi_{2k+1}^*(T_{1,1}) = T_{1,2k+1} - kT_{2,2k+1}, \tag{9.3.8}$$
$$\varphi_{2k+1}^*(T_{2,1}) = T_{2,2k+1}, \tag{9.3.9}$$
$$\varphi_{2k+1}^*(T_{3,1}) = T_{3,2k+1}. \tag{9.3.10}$$

We can identify the quantum cohomologies $QH^\bullet(\mathbb{F}_1)$ and $QH^\bullet(\mathbb{F}_{2k+1})$ via the change of coordinates

$$\begin{aligned}
t^{0,2k+1} &= t^{0,1}, & t^{1,2k+1} &= t^{1,1}, \\
t^{2,2k+1} &= t^{2,1} - kt^{1,1}, & t^{3,2k+1} &= t^{3,1}.
\end{aligned} \tag{9.3.11}$$

Theorem 9.3.3. *For any $k \geqslant 0$, the following isomorphism of algebras holds true:*

$$QH^\bullet(\mathbb{F}_{2k+1}) \cong \frac{\mathbb{C}[T_{1,2k+1}, T_{2,2k+1}, q_1, q_2]}{A_k},$$

where

$$\begin{aligned} A_k := \langle & T_{2,2k+1}^{\circ 2} - (T_{1,2k+1} - (k+1)T_{2,2k+1})q_1^k q_2, \\ & (T_{1,2k+1} - kT_{2,2k+1}) \circ (T_{1,2k+1} - (k+1)T_{2,2k+1}) - q_1 \rangle \end{aligned}$$

and $q_1 := \exp(t^{1,2k+1})$ *and* $q_2 := \exp(t^{2,2k+1})$.

Proof. The following presentation for $QH^\bullet(\mathbb{F}_1)$ holds true:

$$QH^\bullet(\mathbb{F}_1) \cong \frac{\mathbb{C}[T_{1,1}, T_{2,1}, q_1, q_2]}{\langle T_{2,1}^{\circ 2} - (T_{1,1} - T_{2,1})q_2,\ T_{1,1}^{\circ 2} - T_{1,1} \circ T_{2,1} - q_1 \rangle}.$$

The result follows by formulas (9.3.7)-(9.3.10) and (9.3.11). ∎

Chapter 10

Dubrovin conjecture for Hirzebruch surfaces $\mathbb{F}_{2k}$

10.1 $\mathcal{A}_\Lambda$-stratum and Maxwell stratum of $QH^\bullet(\mathbb{F}_{2k})$

Fix a point $p = t^{1,2k}T_{1,2k} + t^{2,2k}T_{2,2k}$ of the small quantum cohomology of $\mathbb{F}_{2k}$. The matrix form of the tensor $\mathcal{U}$ is given by

$$\mathcal{U}(p) = \begin{pmatrix} 0 & 2q_1 + 2kq_1^k q_2 & 2q_1^k q_2 & 0 \\ 2 & 0 & 0 & 2q_1^k q_2 \\ 2-2k & 0 & 0 & 2q_1 - 2kq_1^k q_2 \\ 0 & 2+2k & 2 & 0 \end{pmatrix}.$$

The canonical coordinates are given by

$$\begin{aligned} u_1(p) &= -2\big(q_1^{\frac12} - q_1^{\frac k2} q_2^{\frac12}\big), \quad u_2(p) = 2\big(q_1^{\frac12} - q_1^{\frac k2} q_2^{\frac12}\big), \\ u_3(p) &= -2\big(q_1^{\frac12} + q_1^{\frac k2} q_2^{\frac12}\big), \quad u_4(p) = 2\big(q_1^{\frac12} + q_1^{\frac k2} q_2^{\frac12}\big). \end{aligned}$$

The Ψ-matrix at the point p is given by

$$\Psi(p) = \begin{pmatrix} -\frac{iq_1^{\frac12(-\frac k2-\frac12)}}{2\sqrt[4]{q_2}} & \frac{iq_1^{\frac12(-\frac k2-\frac12)}(\sqrt{q_1}-kq_1^{\frac k2}\sqrt{q_2})}{2\sqrt[4]{q_2}} & -\frac12 iq_1^{\frac{k-1}{4}}\sqrt[4]{q_2} & \frac12 iq_1^{\frac{k+1}{4}}\sqrt[4]{q_2} \\ -\frac{iq_1^{\frac12(-\frac k2-\frac12)}}{2\sqrt[4]{q_2}} & -\frac{iq_1^{\frac12(-\frac k2-\frac12)}(\sqrt{q_1}-kq_1^{\frac k2}\sqrt{q_2})}{2\sqrt[4]{q_2}} & \frac12 iq_1^{\frac{k-1}{4}}\sqrt[4]{q_2} & \frac12 iq_1^{\frac{k+1}{4}}\sqrt[4]{q_2} \\ \frac{q_1^{\frac12(-\frac k2-\frac12)}}{2\sqrt[4]{q_2}} & -\frac{q_1^{\frac12(-\frac k2-\frac12)}(k\sqrt{q_2}q_1^{\frac k2}+\sqrt{q_1})}{2\sqrt[4]{q_2}} & -\frac12 q_1^{\frac{k-1}{4}}\sqrt[4]{q_2} & \frac12 q_1^{\frac{k+1}{4}}\sqrt[4]{q_2} \\ \frac{q_1^{\frac12(-\frac k2-\frac12)}}{2\sqrt[4]{q_2}} & \frac{q_1^{\frac12(-\frac k2-\frac12)}(k\sqrt{q_2}q_1^{\frac k2}+\sqrt{q_1})}{2\sqrt[4]{q_2}} & \frac12 q_1^{\frac{k-1}{4}}\sqrt[4]{q_2} & \frac12 q_1^{\frac{k+1}{4}}\sqrt[4]{q_2} \end{pmatrix}.$$

Proposition 10.1.1. *The small quantum cohomology of Hirzebruch surfaces $\mathbb{F}_{2k}$ is contained in the $\mathcal{I}^0_\Lambda$-stratum of $QH^\bullet(\mathbb{F}_{2k})$. Moreover, the point p is in the $\mathcal{A}_\Lambda$-stratum of $QH^\bullet(\mathbb{F}_{2k})$ if and only if $q_1 = q_1^k q_2$.*

Proof. By Theorem 2.5.1, the function $\det\Lambda$ takes the form

$$\det\Lambda(z,p) = \frac{z^2}{z^2A_0(p) + zA_1(p) + A_2(p)},$$

where A_0, A_1, A_2 are holomorphic functions on $QH^\bullet(\mathbb{F}_{2k})$. If p is a point of the small quantum locus, an explicit computation shows that

$$\det\Lambda(z,p) = -\frac{1}{256}(q_1 - q_2q_1^k)^{-1},$$

so that $A_1(p) = A_2(p) = 0$. The claim follows. ∎

Corollary 10.1.2. *Along the small quantum locus of $QH^\bullet(\mathbb{F}_{2k})$ the $\mathcal{A}_\Lambda$-stratum coincides with the Maxwell stratum $\mathcal{M}_{\mathbb{F}_{2k}}$.*

Proof. If $q_1 = q_1^k q_2$, we have coalescences of canonical coordinates u_1, u_2, u_3, u_4. Any point of the small quantum locus, however, is semisimple. ∎

10.2 Small qDE of $\mathbb{F}_{2k}$

In the coordinates $(t^{\alpha,2k})_{\alpha=0}^3$, the grading tensor μ has matrix $\mu = \operatorname{diag}(-1, 0, 0, 1)$. The isomonodromic system (2.7.3) is

$$\mathcal{H}_k^{\mathrm{ev}}\colon \quad \begin{cases} \dfrac{\partial \xi_1}{\partial z} = (2-2k)\xi_3 + 2\xi_2 + \dfrac{1}{z}\xi_1, \\ \dfrac{\partial \xi_2}{\partial z} = (2k+2)\xi_4 + \xi_1(2kq_1^k q_2 + 2q_1), \\ \dfrac{\partial \xi_3}{\partial z} = 2\xi_1 q_1^k q_2 + 2\xi_4, \\ \dfrac{\partial \xi_4}{\partial z} = 2\xi_2 q_1^k q_2 + \xi_3(2q_1 - 2kq_1^k q_2) - \dfrac{1}{z}\xi_4. \end{cases}$$

In the complement of the $\mathcal{A}_\Lambda$-stratum, it can be reduced to the single equation in ξ_1, the *master differential equation*

$$\begin{aligned} z^4 \frac{\partial^4 \xi_1}{\partial z^4} &- z^2\left[z^2(8q_1^k q_2 + 8q_1) - 1\right]\frac{\partial^2 \xi_1}{\partial z^2} \\ &- 3z\frac{\partial \xi_1}{\partial z} - \left(-16z^4(q_1 - q_1^k q_2)^2 - 3\right)\xi_1 = 0. \end{aligned} \tag{10.2.1}$$

Given a solution $\xi_1(z,t)$ of equation (10.2.1), we can reconstruct a solution of the system $\mathcal{H}_k^{\mathrm{ev}}$ through the formulas

$$\begin{aligned} \xi_2 = &-\frac{(-4(k+1)q_2 z^2 q_1^k + 4(k+1)q_1 z^2 + k - 1)}{16z^3(q_1 - q_2 q_1^k)}\xi_1 \\ &- \frac{(4(3k-1)q_2 z^2 q_1^k + 4(k-3)q_1 z^2 - k + 1)}{16z^2(q_1 - q_2 q_1^k)}\frac{\partial \xi_1}{\partial z} + \frac{(k-1)}{16(q_1 - q_2 q_1^k)}\frac{\partial^3 \xi_1}{\partial z^3}, \end{aligned}$$

$$\begin{aligned} \xi_3 = &-\frac{(-4q_2 z^2 q_1^k + 4q_1 z^2 + 1)}{16z^3(q_1 - q_2 q_1^k)}\xi_1 - \frac{(12q_2 z^2 q_1^k + 4q_1 z^2 - 1)}{16z^2(q_1 - q_2 q_1^k)}\frac{\partial \xi_1}{\partial z} \\ &+ \frac{1}{16(q_1 - q_2 q_1^k)}\frac{\partial^3 \xi_1}{\partial z^3}, \end{aligned}$$

$$\xi_4 = -\frac{(4q_2 z^2 q_1^k + 4q_1 z^2 - 1)}{8z^2}\xi_1 - \frac{1}{8z}\frac{\partial \xi_1}{\partial z} + \frac{1}{8}\frac{\partial^2 \xi_1}{\partial z^2}.$$

By looking for solution of the form

$$\xi_1(z,t) = z \cdot \Phi(z,t),$$

equation (10.2.1) can be rewritten as the *(small) quantum differential equation*

$$z(\vartheta^4\Phi - 2\vartheta^3\Phi) - 8z^3(q_1 + q_1^k q_2)[\vartheta^2\Phi + \vartheta\Phi] + 16z^5(q_1 - q_1^k q_2)^2\Phi = 0,$$

where $\vartheta := z\frac{\partial}{\partial z}$.

10.3 Proof for $QH^\bullet(\mathbb{F}_{2k})$

Let us specialize the system $\mathcal{H}_k^{\mathrm{ev}}$ at the point $0 \in QH^\bullet(\mathbb{F}_{2k})$, for which $q_1 = q_2 = 1$:

$$\mathcal{H}_k'\colon\quad \begin{cases} \dfrac{\partial \xi_1}{\partial z} = (2-2k)\xi_3 + 2\xi_2 + \dfrac{1}{z}\xi_1, \\ \dfrac{\partial \xi_2}{\partial z} = (2k+2)\xi_4 + \xi_1(2k+2), \\ \dfrac{\partial \xi_3}{\partial z} = 2\xi_1 + 2\xi_4, \\ \dfrac{\partial \xi_4}{\partial z} = 2\xi_2 + \xi_3(2-2k) - \dfrac{1}{z}\xi_4. \end{cases}$$

The point $p = 0$ is in the $\mathcal{A}_\Lambda$-stratum of $QH^\bullet(\mathbb{F}_{2k})$, and so in the Maxwell stratum. Hence, the study of monodromy data of the system of differential equations $\mathcal{H}_k'$ fits in the analysis developed in [22, 23]. In particular, the isomonodromy property is justified by [23, Theorem 4.5]. As explained in Remark 4.5.2, we can reduce the computation of the monodromy data of the system $\mathcal{H}_k'$ to the single case of $\mathcal{H}_0'$. The system $\mathcal{H}_0'$ can in turn be integrated using solutions of the isomonodromic system of $QH^\bullet(\mathbb{P}^1)$ (see [32, Lemma 4.10]).

Proposition 10.3.1. *Let $(\varphi_1^{(i)}, \varphi_2^{(i)})$ with $i = 1, 2$ be two solutions of system (2.7.3) for the quantum cohomology of $\mathbb{P}^1$, specialized at $0 \in H^2(\mathbb{P}^1, \mathbb{C})$, i.e.*

$$\begin{cases} \dfrac{\partial \varphi_1}{\partial z} = 2\varphi_2 + \dfrac{1}{2z}\varphi_1, \\ \dfrac{\partial \varphi_2}{\partial z} = 2\varphi_1 - \dfrac{1}{2z}\varphi_2. \end{cases}$$

Then the tensor product

$$\begin{pmatrix} \varphi_1^{(1)} \\ \varphi_2^{(1)} \end{pmatrix} \otimes \begin{pmatrix} \varphi_1^{(2)} \\ \varphi_2^{(2)} \end{pmatrix} = \begin{pmatrix} \varphi_1^{(1)} \cdot \varphi_1^{(2)} \\ \varphi_1^{(1)} \cdot \varphi_2^{(2)} \\ \varphi_2^{(1)} \cdot \varphi_1^{(2)} \\ \varphi_2^{(1)} \cdot \varphi_2^{(2)} \end{pmatrix}$$

is a solution of the system $\mathcal{H}_0'$. ∎

Remark 10.3.2. In order to explicitly compute the monodromy data of $\mathscr{H}'_{\mathrm{ev}}$ one could still develop the study of solutions of the small quantum differential equation, and then reconstruct the Stokes solutions of $\mathscr{H}'_k$ doing a similar argument to the one developed in [23, Section 6] for the quantum cohomology of $\mathbb{G}(2,4)$.

Theorem 10.3.3. *The central connection matrix of $QH^\bullet(\mathbb{F}_{2k})$, computed at the point $0 \in QH^\bullet(\mathbb{F}_{2k})$, with respect to an oriented admissible line ℓ of slope $\varphi \in \,]\frac{\pi}{2}, \frac{3\pi}{2}[$ and for a suitable choice of the determination of the Ψ-matrix, is equal to*

$$C_k = \begin{pmatrix} \frac{1}{2\pi} & \frac{1}{2\pi} & \frac{1}{2\pi} & \frac{1}{2\pi} \\ -i + \frac{\gamma}{\pi} & -i + \frac{\gamma}{\pi} & \frac{\gamma}{\pi} & \frac{\gamma}{\pi} \\ -\frac{(k-1)(\gamma - i\pi)}{\pi} & \frac{i\pi k - \gamma k + \gamma}{\pi} & -i + \frac{\gamma - \gamma k}{\pi} & \frac{\gamma - \gamma k}{\pi} \\ \frac{2(\gamma - i\pi)^2}{\pi} & \frac{2\gamma(\gamma - i\pi)}{\pi} & \frac{2\gamma(\gamma - i\pi)}{\pi} & \frac{2\gamma^2}{\pi} \end{pmatrix},$$

and the corresponding Stokes matrix is equal to

$$S = \begin{pmatrix} 1 & -2 & -2 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The matrix C_k is the matrix associated with the morphism

$$\begin{aligned} Д^-_{\mathbb{F}_{2k}} \colon K_0(\mathbb{F}_{2k})_{\mathbb{C}} &\to H^\bullet(\mathbb{F}_{2k}, \mathbb{C}), \\ [\mathscr{F}] &\mapsto \frac{1}{2\pi} \widehat{\Gamma}^-_{\mathbb{F}_{2k}} \cup e^{-\pi i c_1(\mathbb{F}_{2k})} \cup \mathrm{Ch}(\mathscr{F}), \end{aligned}$$

with respect to

- *an exceptional basis $\mathfrak{E} := (E_i)_{i=1}^4$ of $K_0(\mathbb{F}_{2k})_{\mathbb{C}}$,*
- *the basis $(T_{i,2k})_{i=0}^3$ of $H^\bullet(\mathbb{F}_{2k}, \mathbb{C})$.*

The exceptional basis $\mathfrak{E}$ is the one obtained by acting on the exceptional basis

$$\big([\mathcal{O}], [\mathcal{O}(\Sigma_2^{2k})], [\mathcal{O}(\Sigma_4^{2k})], [\mathcal{O}(\Sigma_2^{2k} + \Sigma_4^{2k})]\big)$$

with the element $(J_k^{-1}, b_k) \in (\mathbb{Z}/2\mathbb{Z})^4 \rtimes \mathcal{B}_4$, where

$$J_k := \begin{cases} (1, 1, (-1)^{p+1}, (-1)^p) & \text{if } k = 2p+1, \\ (1, 1, (-1)^p, (-1)^p) & \text{if } k = 2p, \end{cases}$$
$$b_k := \beta_3^k.$$

Proof. We divide the proof into three steps.

Step 1. Let us first show that for suitable choices of the oriented line ℓ and Ψ-matrix, the central connection matrix computed at the point $0 \in QH^\bullet(\mathbb{F}_0)$ is given in the following form:

$$C_0 := \begin{pmatrix} \frac{1}{2\pi} & \frac{1}{2\pi} & \frac{1}{2\pi} & \frac{1}{2\pi} \\ -i+\frac{\gamma}{\pi} & -i+\frac{\gamma}{\pi} & \frac{\gamma}{\pi} & \frac{\gamma}{\pi} \\ -i+\frac{\gamma}{\pi} & \frac{\gamma}{\pi} & -i+\frac{\gamma}{\pi} & \frac{\gamma}{\pi} \\ \frac{2(\gamma-i\pi)^2}{\pi} & \frac{2\gamma(\gamma-i\pi)}{\pi} & \frac{2\gamma(\gamma-i\pi)}{\pi} & \frac{2\gamma^2}{\pi} \end{pmatrix}. \tag{10.3.1}$$

According to [21, Corollary 6.11], the central connection matrix C of $QH^\bullet(\mathbb{P}^1)$ computed at the point 0, with respect to an oriented line ℓ of slope $\varphi \in \left]\frac{\pi}{2}, \frac{3\pi}{2}\right[$ and with respect to the following choice of Ψ-matrix

$$\Psi_0 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix},$$

equals

$$C := \frac{i}{\sqrt{2\pi}} \begin{pmatrix} 1 & 1 \\ 2(\gamma - \pi i) & 2\gamma \end{pmatrix}.$$

This is the matrix associated with the morphism

$$\begin{aligned} Д^-_{\mathbb{P}^1} \colon K_0(\mathbb{P}^1)_{\mathbb{C}} &\to H^\bullet(\mathbb{P}^1, \mathbb{C}), \\ [\mathscr{F}] &\mapsto \frac{i}{(2\pi)^{\frac{1}{2}}} \widehat{\Gamma}^-_{\mathbb{P}^1} \cup e^{-\pi i c_1(\mathbb{P}^1)} \cup \mathrm{Ch}(\mathscr{F}), \end{aligned}$$

with respect to the bases

- $([\mathscr{O}], [\mathscr{O}(1)])$ of $K_0(\mathbb{P}^1)_{\mathbb{C}}$ (the Beilinson basis),
- $(1, \sigma)$ of $H^\bullet(\mathbb{P}^1, \mathbb{C})$.

By taking the Kronecker tensor square $C^{\otimes 2}$, we obtain the central connection matrix of $QH^\bullet(\mathbb{P}^1 \times \mathbb{P}^1)$ computed at the point 0, with respect to the same line ℓ (which is still admissible) and with respect to the choice of the Ψ-matrix given by the Kronecker tensor square $\Psi_0^{\otimes 2}$:

$$C^{\otimes 2} = \begin{pmatrix} -\frac{1}{2\pi} & -\frac{1}{2\pi} & -\frac{1}{2\pi} & -\frac{1}{2\pi} \\ -\frac{\gamma-i\pi}{\pi} & -\frac{\gamma}{\pi} & -\frac{\gamma-i\pi}{\pi} & -\frac{\gamma}{\pi} \\ -\frac{\gamma-i\pi}{\pi} & -\frac{\gamma-i\pi}{\pi} & -\frac{\gamma}{\pi} & -\frac{\gamma}{\pi} \\ -\frac{2(\gamma-i\pi)^2}{\pi} & -\frac{2\gamma(\gamma-i\pi)}{\pi} & -\frac{2\gamma(\gamma-i\pi)}{\pi} & -\frac{2\gamma^2}{\pi} \end{pmatrix}.$$

By changing all the signs of the rows of the Kronecker tensor square $\Psi_0^{\otimes 2}$, i.e. acting with $(-1,-1,-1,-1) \in (\mathbb{Z}/2\mathbb{Z})^4$ on $C^{\otimes 2}$, we obtain the matrix $-C^{\otimes 2}$ associated with the morphism

$$\begin{aligned} Д^-_{\mathbb{P}^1\times\mathbb{P}^1} \colon K_0(\mathbb{P}^1 \times \mathbb{P}^1)_{\mathbb{C}} &\to H^\bullet(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{C}), \\ [\mathscr{F}] &\mapsto \frac{1}{2\pi} \widehat{\Gamma}^-_{\mathbb{P}^1\times\mathbb{P}^1} \cup e^{-\pi i c_1(\mathbb{P}^1\times\mathbb{P}^1)} \cup \mathrm{Ch}(\mathscr{F}), \end{aligned}$$

written with respect to the bases

- $([\mathcal{O}], [\mathcal{O}(1,0)], [\mathcal{O}(0,1)], [\mathcal{O}(1,1)])$ of $K_0(\mathbb{P}^1 \times \mathbb{P}^1)_{\mathbb{C}}$,
- $(1, \sigma \otimes 1, 1 \otimes \sigma, \sigma \otimes \sigma)$ of $H^\bullet(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{C}) \cong H^\bullet(\mathbb{P}^1, \mathbb{C})^{\otimes 2}$.

See [21, Proposition 5.11]. In the notations introduced before for Hirzebruch surfaces, this exceptional collection is

$$\left(\mathcal{O}, \mathcal{O}(\Sigma_4^0), \mathcal{O}(\Sigma_2^0), \mathcal{O}(\Sigma_2^0 + \Sigma_4^0)\right).$$

It is a 3-block exceptional collection,[1] coherently with the fact that $0 \in QH^\bullet(\mathbb{F}_0)$ is a semisimple coalescing point, see [23, Section 6] and [21, Remark 5.4]. In particular, the braids $\beta_{2,3}$ and $\beta_{2,3}^{-1}$ act as a mere permutation of the central objects, and of the two central columns of the matrix $-C^{\otimes 2}$. Such a permuted matrix is exactly the matrix C_0 in (10.3.1), and it corresponds to the matrix associated with the morphism $\mathit{\Pi}_{\mathbb{F}_0}^{-}$ with respect to the collection

$$\left(\mathcal{O}, \mathcal{O}(\Sigma_2^0), \mathcal{O}(\Sigma_4^0), \mathcal{O}(\Sigma_2^0 + \Sigma_4^0)\right).$$

In conclusion, we have proved that, for suitable choices of ℓ and Ψ, the central connection matrix computed at $0 \in QH^\bullet(\mathbb{F}_0)$ is

$$C_0 = \begin{pmatrix} \frac{1}{2\pi} & \frac{1}{2\pi} & \frac{1}{2\pi} & \frac{1}{2\pi} \\ -i + \frac{\gamma}{\pi} & -i + \frac{\gamma}{\pi} & \frac{\gamma}{\pi} & \frac{\gamma}{\pi} \\ -i + \frac{\gamma}{\pi} & \frac{\gamma}{\pi} & -i + \frac{\gamma}{\pi} & \frac{\gamma}{\pi} \\ \frac{2(\gamma - i\pi)^2}{\pi} & \frac{2\gamma(\gamma - i\pi)}{\pi} & \frac{2\gamma(\gamma - i\pi)}{\pi} & \frac{2\gamma^2}{\pi} \end{pmatrix},$$

which coincides with the matrix associated with the collection

$$\left(\mathcal{O}, \mathcal{O}(\Sigma_2^0), \mathcal{O}(\Sigma_4^0), \mathcal{O}(\Sigma_2^0 + \Sigma_4^0)\right).$$

Step 2. Equations (9.3.5) and Proposition 4.5.1 imply that the central connection matrix computed at $0 \in QH^\bullet(\mathbb{F}_{2k})$, with respect to the same choices of ℓ and Ψ, is

$$C_k = \begin{pmatrix} \frac{1}{2\pi} & \frac{1}{2\pi} & \frac{1}{2\pi} & \frac{1}{2\pi} \\ -i + \frac{\gamma}{\pi} & -i + \frac{\gamma}{\pi} & \frac{\gamma}{\pi} & \frac{\gamma}{\pi} \\ -\frac{(k-1)(\gamma - i\pi)}{\pi} & \frac{i\pi k - \gamma k + \gamma}{\pi} & -i + \frac{\gamma - \gamma k}{\pi} & \frac{\gamma - \gamma k}{\pi} \\ \frac{2(\gamma - i\pi)^2}{\pi} & \frac{2\gamma(\gamma - i\pi)}{\pi} & \frac{2\gamma(\gamma - i\pi)}{\pi} & \frac{2\gamma^2}{\pi} \end{pmatrix}.$$

---

[1] An exceptional collection $(E_1, \dots, E_n)$ is a *k-block exceptional collection* if it is possible to decompose it into k exceptional sub-collections $\mathfrak{B}_1, \dots, \mathfrak{B}_k$, called *blocks*, such that

- they are consecutive, i.e. of the form $\mathfrak{B}_1 = (E_1, \dots, E_{j_1})$, $\mathfrak{B}_2 = (E_{j_1+1}, \dots, E_{j_2})$, ..., $\mathfrak{B}_k = (E_{j_{k-1}+1}, \dots, E_{j_k})$, with $1 \leqslant j_1 < j_2 < \dots < j_k \leqslant n$,
- we have $\mathrm{Hom}^\bullet(E_j, E_i) = 0$ if E_i and E_j belong to a same block $\mathfrak{B}_h$.

In particular, inside each block $\mathfrak{B}_h$, mutations act as permutations of exceptional objects. See [21, Section 3.6.4], and references therein.

The corresponding Stokes matrix is independent of k, and it is equal to

$$S = \begin{pmatrix} 1 & -2 & -2 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{10.3.2}$$

Step 3. Let us define the matrix $J_k \in (\mathbb{Z}/2\mathbb{Z})^4$ as follows:

$$J_k := \begin{cases} (1, 1, (-1)^{p+1}, (-1)^p) & \text{if } k = 2p+1, \\ (1, 1, (-1)^p, (-1)^p) & \text{if } k = 2p. \end{cases}$$

We claim that by acting on $C_k J_k$ with the braid β_3^{-k} we obtain the matrix associated with $\varPi^-_{\mathbb{F}_{2k}}$ and with respect to the exceptional collection

$$\left(\mathcal{O}, \mathcal{O}(\Sigma_2^{2k}), \mathcal{O}(\Sigma_4^{2k}), \mathcal{O}(\Sigma_2^{2k} + \Sigma_4^{2k})\right),$$

namely the matrix

$$E_k := \begin{pmatrix} \frac{1}{2\pi} & \frac{1}{2\pi} & \frac{1}{2\pi} & \frac{1}{2\pi} \\ -i + \frac{\gamma}{\pi} & -i + \frac{\gamma}{\pi} & \frac{\gamma}{\pi} & \frac{\gamma}{\pi} \\ -\frac{(k-1)(\gamma - i\pi)}{\pi} & \frac{i\pi k - \gamma k + \gamma}{\pi} & -\frac{(k-1)(\gamma - i\pi)}{\pi} & \frac{i\pi k - \gamma k + \gamma}{\pi} \\ \frac{2(\gamma - i\pi)^2}{\pi} & \frac{2\gamma(\gamma - i\pi)}{\pi} & \frac{2\gamma(i\pi(k-1)+\gamma)}{\pi} & \frac{2\gamma(i\pi k + \gamma)}{\pi} \end{pmatrix}.$$

Note that the claim is equivalent to the following statement: the matrix $A^{\beta}(J_k \cdot S \cdot J_k)$, with $\beta = \beta_3^{-k}$ and S as in (10.3.2), is equal to

$$E_k^{-1} C_k J_k = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & k+1 & k \\ 0 & 0 & -k & 1-k \end{pmatrix} \cdot J_k. \tag{10.3.3}$$

Given a generic 4×4 unipotent upper triangular matrix X, the action of subsequent powers of the braid β_3, or of its inverse β_3^{-1}, simply changes the sign of the entry in position $(3, 4)$: more precisely, we have

$$[X^{\beta}]_{3,4} = (-1)^n [X]_{3,4} \quad \text{if } \beta = \beta_3^{\pm n}.$$

For example, by acting twice with the braid β_3 we have

$$\begin{pmatrix} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & a & c & b - cf \\ 0 & 1 & e & d - ef \\ 0 & 0 & 1 & -f \\ 0 & 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & a & b - cf & c + f(b - cf) \\ 0 & 1 & d - ef & e + f(d - ef) \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In particular, the matrix $A^{\beta}(X)$, with $\beta = \beta_3^{-k}$, is equal to

$$\prod_{j=1}^{k}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&(-1)^j x&1\\0&0&1&0\end{pmatrix},\qquad x = X_{3,4}.$$

In the case $X = J_k \cdot S \cdot J_k$, we have

$$x = (-1)^{k+1}2.$$

So, in conclusion, we have to prove that the following identity holds for all $k \geqslant 0$:

$$\prod_{j=1}^{k}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&(-1)^{j+k+1}2&1\\0&0&1&0\end{pmatrix} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&k+1&k\\0&0&-k&1-k\end{pmatrix}\cdot J_k.$$

We prove the claim by induction on k. The base case $k = 0$ is evidently true. Let us assume that the statement holds true for $k-1$, and let us prove it for k. We have

$$\begin{aligned}&\prod_{j=1}^{k}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&(-1)^{j+k+1}2&1\\0&0&1&0\end{pmatrix}\\ &=\left[\prod_{j=1}^{k-1}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&(-1)^{j+k+1}2&1\\0&0&1&0\end{pmatrix}\right]\cdot\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-2&1\\0&0&1&0\end{pmatrix}\\ &=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&k&k-1\\0&0&1-k&2-k\end{pmatrix}\cdot J_{k-1}\cdot\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-2&1\\0&0&1&0\end{pmatrix},\end{aligned}$$

and in both cases k even/odd, the last term is easily seen to be equal to (10.3.3). ∎

Chapter 11

Dubrovin conjecture for Hirzebruch surfaces $\mathbb{F}_{2k+1}$

11.1 $\mathcal{A}_\Lambda$-stratum and Maxwell stratum of $QH^\bullet(\mathbb{F}_{2k+1})$

Fix a point

$$p = t^{1,2k+1}T_{1,2k+1} + t^{2,2k+1}T_{2,2k+1}$$

of the small quantum cohomology of $\mathbb{F}_{2k+1}$. The matrix associated to the $\mathcal{U}$-tensor at p is

$$\mathcal{U}(p) = \begin{pmatrix} 0 & 2q_1 & 0 & 3q_1^{k+1}q_2 \\ 2 & kq_1^k q_2 & q_1^k q_2 & 0 \\ 1-2k & k(-kq_2q_1^k - q_1^k q_2) & -kq_2q_1^k - q_1^k q_2 & 2q_1 \\ 0 & 2k+3 & 2 & 0 \end{pmatrix}.$$

The canonical coordinates are the roots $u_1(p), u_2(p), u_3(p), u_4(p)$ of the polynomial

$$j(u) := u^4 + u^3 q_1^k q_2 - 8q_1u^2 - 36uq_1^{k+1}q_2 - 27q_2^2 q_1^{2k+1} + 16q_1^2.$$

Hence the bifurcation set $\mathcal{B}_{\mathbb{F}_{2k+1}}$, along the small quantum cohomology, is defined by the zero locus of the discriminant of $j(u)$, i.e.

$$\mathcal{B}_{\mathbb{F}_{2k+1}} = \{p : q_1^{2k+2}q_2^2(27q_2^2q_1^{2k} + 256q_1)^3 = 0\}.$$

Since any point of the small quantum cohomology of $\mathbb{F}_{2k+1}$ is semisimple, the set above actually coincides with the Maxwell stratum $\mathcal{M}_{\mathbb{F}_{2k+1}}$. The determinant of the Λ-matrix is given by

$$\det \Lambda(z, p) = -\frac{z}{(27q_1^{2k}q_2^2 + 256q_1)z - 24q_2q_1^k}.$$

Hence, the $\mathcal{A}_\Lambda$-stratum is given by

$$\mathcal{A}_\Lambda := \{p : 27q_1^{2k}q_2^2 + 256q_1 = 0\}. \tag{11.1.1}$$

Also in this case, the Maxwell stratum and the $\mathcal{A}_\Lambda$-stratum coincide along the small quantum cohomology of $\mathbb{F}_{2k+1}$.

11.2 Small qDE of $\mathbb{F}_1$

At the point p, the grading operator μ has matrix

$$\mu = \operatorname{diag}(-1, 0, 0, 1).$$

Hence the isomonodromic system of differential equations (2.7.3) for $QH^\bullet(\mathbb{F}_{2k+1})$ is given by

$$\mathcal{H}_k^{\mathrm{od}}\colon \quad \begin{cases} \dfrac{\partial \xi_1}{\partial z} = (1-2k)\xi_3 + 2\xi_2 + \dfrac{\xi_1}{z}, \\ \dfrac{\partial \xi_2}{\partial z} = (2k+3)\xi_4 + k\xi_2 q_2 q_1^k + k\xi_3(-kq_2q_1^k - q_2q_1^k) + 2\xi_1 q_1, \\ \dfrac{\partial \xi_3}{\partial z} = \xi_2 q_2 q_1^k + \xi_3(-kq_2q_1^k - q_2q_1^k) + 2\xi_4, \\ \dfrac{\partial \xi_4}{\partial z} = 3\xi_1 q_2 q_1^{k+1} + 2\xi_3 q_1 - \dfrac{\xi_4}{z}. \end{cases}$$

As explained in Remark 4.5.2, the computation of the monodromy data of $\mathcal{H}_k^{\mathrm{od}}$ can be reduced to the single case $\mathcal{H}_0^{\mathrm{od}}$.

The point $0 \in QH^\bullet(\mathbb{F}_1)$ is not in the $\mathcal{A}_\Lambda$-stratum, as it follows from (11.1.1). At the point $0 \in QH^\bullet(\mathbb{F}_1)$, indeed, the system $\mathcal{H}_0^{\mathrm{od}}$ can be reduced to the *small quantum differential equation*

$$\begin{aligned} &(283z-24)\vartheta^4\Phi + (283z^2 - 590z + 24)\vartheta^3\Phi \\ &\quad + (-2264z^2 + 192z + 3)\vartheta^2\Phi \\ &\quad - 4z^2(2547z^2 + 350z - 104)\vartheta\Phi \\ &\quad + z^2(-3113z^3 - 9924z^2 + 1476z + 192)\Phi = 0. \end{aligned} \tag{11.2.1}$$

Given a solution $\Phi(z)$ of (11.2.1), the corresponding solution of the system $\mathcal{H}_0^{\mathrm{od}}$ can be reconstructed by the formulas

$$\xi_1(z) = z \cdot \Phi(z), \tag{11.2.2}$$

$$\begin{aligned} \xi_2(z) = \frac{1}{z^2(283z-24)}\big(&169z^3\xi_1'(z) + z^3\xi_1''(z) + 204z^3\xi_1(z) \\ &- 8z^3{\xi_1}^{(3)}(z) - 9z^2\xi_1'(z) - 105z^2\xi_1(z) - 8z\xi_1'(z) \\ &+ 9z\xi_1(z) + 8\xi_1(z)\big), \end{aligned} \tag{11.2.3}$$

$$\begin{aligned} \xi_3(z) = \frac{1}{z^2(283z-24)}\big(&-55z^3\xi_1'(z) - 2z^3\xi_1''(z) - 408z^3\xi_1(z) \\ &+ 16z^3{\xi_1}^{(3)}(z) - 6z^2\xi_1'(z) - 73z^2\xi_1(z) + 16z\xi_1'(z) \\ &+ 6z\xi_1(z) - 16\xi_1(z)\big), \end{aligned} \tag{11.2.4}$$

$$\begin{aligned} \xi_4(z) = \frac{1}{z^2(283z-24)}\big(&-28z^3\xi_1'(z) + 35z^3\xi_1''(z) - 218z^3\xi_1(z) \\ &+ 3z^3{\xi_1}^{(3)}(z) - 35z^2\xi_1'(z) - 3z^2\xi_1''(z) + 16z^2\xi_1(z) \\ &+ 6z\xi_1'(z) + 35z\xi_1(z) - 6\xi_1(z)\big). \end{aligned} \tag{11.2.5}$$

These formulas are obtained by the identity

$$\xi = \Lambda^T \begin{pmatrix} \xi_1 \\ \xi_1' \\ \xi_1'' \\ \xi_1^{(3)} \end{pmatrix},$$

where the Λ-matrix at $0 \in QH^\bullet(\mathbb{F}_1)$ is

$$\Lambda(z,0) = \begin{pmatrix} 1 & \frac{204z^3-105z^2+9z+8}{z^2(283z-24)} & \frac{-408z^3-73z^2+6z-16}{z^2(283z-24)} & \frac{-218z^3+16z^2+35z-6}{z^2(283z-24)} \\ 0 & \frac{169z^2-9z-8}{z(283z-24)} & \frac{-55z^2-6z+16}{z(283z-24)} & \frac{-28z^2-35z+6}{z(283z-24)} \\ 0 & \frac{z}{283z-24} & -\frac{2z}{283z-24} & \frac{35z-3}{283z-24} \\ 0 & -\frac{8z}{283z-24} & \frac{16z}{283z-24} & \frac{3z}{283z-24} \end{pmatrix}.$$

Remark 11.2.1. The quantum differential equation (11.2.1) has one apparent singularity at $z = \frac{24}{283}$. This coincides with the zero of the denominator of the determinant of the Λ-matrix:

$$\det \Lambda(z,0) = \frac{z}{24 - 283z}.$$

The Ψ-matrix at the point $0 \in QH^\bullet(\mathbb{F}_1)$ is given by

$$\Psi = \begin{pmatrix} \alpha_1^{\frac{1}{2}}\varepsilon_1 & \alpha_1^{\frac{1}{2}}\delta_1 & \alpha_1^{\frac{1}{2}}\sigma_1 & \alpha_1^{\frac{1}{2}}\upsilon_1 \\ \alpha_2^{\frac{1}{2}}\varepsilon_2 & \alpha_2^{\frac{1}{2}}\delta_2 & \alpha_2^{\frac{1}{2}}\sigma_2 & \alpha_2^{\frac{1}{2}}\upsilon_2 \\ \alpha_3^{\frac{1}{2}}\varepsilon_3 & \alpha_3^{\frac{1}{2}}\delta_3 & \alpha_3^{\frac{1}{2}}\sigma_3 & \alpha_3^{\frac{1}{2}}\upsilon_3 \\ \alpha_4^{\frac{1}{2}}\varepsilon_4 & \alpha_4^{\frac{1}{2}}\delta_4 & \alpha_4^{\frac{1}{2}}\sigma_4 & \alpha_4^{\frac{1}{2}}\upsilon_4 \end{pmatrix},$$

where the numbers $\alpha_i, \varepsilon_i, \delta_i, \sigma_i, \upsilon_i$ satisfy the algebraic equations

$$\begin{aligned} \alpha_i^4 + \alpha_i^3 - 6\alpha_i^2 - 283 &= 0, \\ 283\varepsilon_i^4 + 6\varepsilon_i^2 - \varepsilon_i - 1 &= 0, \\ 283\delta_i^4 - 2\delta_i^2 - 9\delta_i - 1 &= 0, \\ 283\sigma_i^4 - 32\sigma_i^2 - \sigma_i + 1 &= 0, \\ 283\upsilon_i^4 - 283\upsilon_i^3 + 105\upsilon_i^2 - 17\upsilon_i + 1 &= 0. \end{aligned}$$

Their numerical approximations are

$$\begin{aligned} \alpha_1 &\approx 4.21193, & \varepsilon_1 &\approx 0.237421, \\ \alpha_2 &\approx -0.204399 - 3.73457i, & \varepsilon_2 &\approx -0.0146116 + 0.266969i, \\ \alpha_3 &\approx -0.204399 + 3.73457i, & \varepsilon_3 &\approx -0.0146116 - 0.266969i, \\ \alpha_4 &\approx -4.80313, & \varepsilon_4 &\approx -0.208197, \end{aligned}$$

$$
\begin{aligned}
&\delta_1 \approx 0.353808, && \sigma_1 \approx 0.194489,\\
&\delta_2 \approx -0.122264 - 0.276482i, && \sigma_2 \approx -0.240929 - 0.0719476i,\\
&\delta_3 \approx -0.122264 + 0.276482i, && \sigma_3 \approx -0.240929 + 0.0719476i,\\
&\delta_4 \approx -0.10928, && \sigma_4 \approx 0.28737,\\
&\upsilon_1 \approx 0.28983, &&\\
&\upsilon_2 \approx 0.279666 - 0.0511337i, &&\\
&\upsilon_3 \approx 0.279666 + 0.0511337i, &&\\
&\upsilon_4 \approx 0.150837. &&
\end{aligned}
$$

The reader can check that $\Psi^T\Psi = \eta$, and that

$$\Psi \mathcal{U} \Psi^{-1} = \mathrm{diag}(x_1, x_2, x_3, x_4),$$

where the canonical coordinates x_i are the roots of the polynomial

$$x^4 + x^3 - 8x^2 - 36x - 11 = 0.$$

Their numerical approximations are

$$
\begin{aligned}
x_1 &\approx 3.7996,\\
x_2 &\approx -2.23455 + 1.94071i,\\
x_3 &\approx -2.23455 - 1.94071i,\\
x_4 &\approx -0.3305.
\end{aligned}
$$

11.3 Coordinates on $\mathcal{S}(\mathbb{P}^1) \otimes \mathcal{S}(\mathbb{P}^2)$

Consider the spaces $\mathcal{S}(\mathbb{P}^1)$ and $\mathcal{S}(\mathbb{P}^2)$ of solutions of the qDEs of $\mathbb{P}^1$ and $\mathbb{P}^2$ specialized at the origins of $H^2(\mathbb{P}^1, \mathbb{C})$ and $H^2(\mathbb{P}^2, \mathbb{C})$, respectively: these equations are

$$\vartheta^2 \Phi_1 = 4z^2 \Phi_1, \tag{11.3.1}$$

$$\vartheta^3 \Phi_2 = 27z^3 \Phi_2. \tag{11.3.2}$$

Solutions $\Phi_1(z)$ of equation (11.3.1) have the following expansion at $z = 0$:

$$\Phi_1(z) = \sum_{m=0}^{\infty} (A_{m,1} + A_{m,0} \log z) \frac{z^{2m}}{(m!)^2}, \tag{11.3.3}$$

where $A_{0,0}$ and $A_{0,1}$ are arbitrary complex numbers, and the other coefficients are uniquely determined by the difference equations

$$A_{m-1,0} = A_{m,0}, \tag{11.3.4}$$

$$A_{m-1,1} = \frac{A_{m,0}}{m} + A_{m,1}. \tag{11.3.5}$$

In particular, notice that from equation (11.3.5) we deduce that

$$A_{m,1} = A_{0,1} - A_{0,0} H_m, \quad m \geqslant 0,$$

where $H_m := \sum_{i=1}^{m} \frac{1}{i}$ denotes the m-th harmonic number.

Analogously, solutions $\Phi_2(z)$ of equation (11.3.2) have the following expansion at $z = 0$:

$$\Phi_2(z) = \sum_{n=0}^{\infty} (B_{n,2} + B_{n,1} \log z + B_{n,0} \log^2 z) \frac{z^{3n}}{(n!)^3}, \tag{11.3.6}$$

where $B_{0,0}, B_{0,1}, B_{0,2}$ are arbitrary complex numbers, and the other coefficients are uniquely determined by the difference equations

$$B_{n-1,0} = B_{n,0}, \tag{11.3.7}$$

$$B_{n-1,1} = \frac{2}{n} B_{n,0} + B_{n,1}, \tag{11.3.8}$$

$$B_{n-1,2} = \frac{2}{3n^2} B_{n,0} + \frac{1}{n} B_{n,1} + B_{n,2}. \tag{11.3.9}$$

From the difference equation (11.3.8) we deduce that

$$B_{n,1} = B_{0,1} - 2 B_{0,0} H_n.$$

The products $A_{0,i} B_{0,j}$, with $i = 0, 1$ and $j = 0, 1, 2$, define a natural system of coordinates on the tensor product $\mathcal{S}(\mathbb{P}^1) \otimes_{\mathbb{C}} \mathcal{S}(\mathbb{P}^2)$.

11.4 Solutions of qDE of $\mathbb{F}_1$ as Laplace $(1, 2; \frac{1}{2}, \frac{1}{3})$-multitransforms

According to Theorem 7.3.1, the space of solutions of the quantum differential equation (11.2.1) can be reconstructed from the spaces of solutions of the qDEs (11.3.1) and (11.3.2). From the polynomial equation (9.1.1), indeed, it follows that Theorem 7.3.1 applies with the specialization of the parameters $h = 2$, $\boldsymbol{\ell} = (2, 3)$ and $\boldsymbol{d} = (1, 1)$.

Hence, we expect to reconstruct the solutions of the differential equation (11.2.1) via a $\mathbb{C}$-bilinear operator

$$\mathscr{P} \colon \mathcal{S}(\mathbb{P}^1) \otimes \mathcal{S}(\mathbb{P}^2) \to \mathcal{O}(\widetilde{\mathbb{C}^*})$$

involving the Laplace $(1, 2; \frac{1}{2}, \frac{1}{3})$-multitransform:

$$\begin{aligned} \mathscr{P}[\Phi_1, \Phi_2](z) &:= e^{-cz} \mathscr{L}_{(1,2;\frac{1}{2},\frac{1}{3})}[\Phi_1, \Phi_2] \\ &= e^{-cz} \int_0^{\infty} \Phi_1\big(z^{\frac{1}{2}} \lambda^{\frac{1}{2}}\big) \Phi_2\big(z^{\frac{2}{3}} \lambda^{\frac{1}{3}}\big) e^{-\lambda}\, d\lambda, \end{aligned}$$

for a suitable number $c \in \mathbb{Q}$ to be determined.

Lemma 11.4.1. *We have $c = 1$.*

Proof. Along the locus of small quantum cohomology, the J-function of $\mathbb{P}^{n-1}$ is

$$J_{\mathbb{P}^{n-1}}(\delta) = e^{\frac{\delta}{\hbar}} \sum_{d=0}^{\infty} \mathbf{Q}^d e^{dt} \frac{1}{\left(\prod_{k=1}^{d}(H + k\hbar)\right)^n}, \quad \delta = tH,$$

where $H \in H^2(\mathbb{P}^{n-1}, \mathbb{C})$ denotes the hyperplane class. It follows that the I-function $I_{\mathbb{P}^1\times\mathbb{P}^2,\mathbb{F}_1}$ equals

$$\begin{aligned} I_{\mathbb{P}^1\times\mathbb{P}^2,\mathbb{F}_1}&(\delta_1 \otimes 1 + 1 \otimes \delta_2) \\ &= e^{\frac{\delta_1}{\hbar}} \otimes e^{\frac{\delta_2}{\hbar}} \cdot \sum_{d_1,d_2\geqslant 0} \mathbf{Q}_1^{d_1}\mathbf{Q}_2^{d_2} \frac{e^{t^1 d_1}}{\left(\prod_{k=1}^{d_1}(H_1 + k\hbar)\right)^2} \otimes \frac{e^{t^2 d_2}}{\left(\prod_{k=1}^{d_2}(H_2 + k\hbar)\right)^3} \\ &\qquad \cdot \prod_{j=1}^{d_1+d_2} (H_1 \otimes 1 + 1 \otimes H_2 + j\hbar) \\ &= 1 + \frac{1}{\hbar}\left(\mathbf{Q}_1^{d_1} e^{t^1} + \delta_1 \otimes 1 + 1 \otimes \delta_2\right) + O\left(\frac{1}{\hbar^2}\right), \end{aligned}$$

where we set:

- $H_1 \in H^2(\mathbb{P}^1, \mathbb{C})$ and $H_2 \in H^2(\mathbb{P}^2, \mathbb{C})$ are the hyperplane classes,
- $\delta_1 = t^1 H_1$ and $\delta_2 = t^2 H_2$ with $t^1, t^2 \in \mathbb{C}$,
- $\mathbf{Q}_i = \mathbf{Q}^{\beta_i}$, β_i being the dual homology class of H_i, for $i = 1, 2$.

In the notations of Proposition 5.3.5, we have $H(\delta_1 \otimes 1 + 1 \otimes \delta_2) = \mathbf{Q}_1^{d_1} e^{t^1}$. The number c equals

$$c = H(0)|_{\mathbf{Q}=1} = 1. \qquad \blacksquare$$

For brevity, in all the remaining part of this section, we will simply write $\mathscr{L}$ to denote the Laplace $(1, 2; \frac{1}{2}, \frac{1}{3})$-multitransform.

11.4.1 The subspace $\mathscr{H}$

The space $\mathcal{S}(\mathbb{P}^1) \otimes \mathcal{S}(\mathbb{P}^2)$ has dimension 6. We are going to identify a subspace $\mathscr{H}$ of dimension 4 which is isomorphically mapped to the space $\mathcal{S}(\mathbb{F}_1)$ via the operator $\mathscr{P}$.

Theorem 11.4.2. *Let $\Phi_1(z)$ and $\Phi_2(z)$ be two solutions of the quantum differential equations of $\mathbb{P}^1$ and $\mathbb{P}^2$, respectively, namely*

$$\vartheta^2\Phi_1(z) = 4z^2\Phi_1(z), \quad \vartheta^3\Phi_2(z) = 27z^3\Phi_2(z).$$

The function

$$\Phi(z) := e^{-z}\mathscr{L}[\Phi_1, \Phi_2; z]$$

is a solution of the quantum differential equation of $\mathbb{F}_1$ *if the following vanishing conditions are satisfied:*

$$\mathscr{D}_1[\Phi_1, \Phi_2; z] = 0, \quad \mathscr{D}_2[\Phi_1, \Phi_2; z] = 0,$$

where

$$\begin{aligned}\mathscr{D}_1[\Phi_1, \Phi_2; z] &:= 2z^2 \mathscr{L}[\vartheta\Phi_1, \Phi_2; z] - \frac{2}{9}\mathscr{L}[\vartheta\Phi_1, \vartheta^2\Phi_2; z] \\ &\quad + \frac{4}{9} z \mathscr{L}[\Phi_1, \vartheta^2\Phi_2; z], \\ \mathscr{D}_2[\Phi_1, \Phi_2; z] &:= z^3 \mathscr{L}[\Phi_1, \Phi_2; z] - \frac{z^2}{3}\mathscr{L}[\Phi_1, \vartheta\Phi_2; z] \\ &\quad - \frac{z}{9}\mathscr{L}[\Phi_1, \vartheta^2\Phi_2; z] + \frac{z}{6}\mathscr{L}[\vartheta\Phi_1, \vartheta\Phi_2; z].\end{aligned}$$

Proof. Let us look for solutions of equation (11.2.1) in the form

$$\Phi(z) = e^{-z}\mathscr{L}_{(1,2;\frac{1}{2},\frac{1}{3})}[\Phi_1, \Phi_2; z],$$

where Φ_1 and Φ_2 are solutions of the quantum differential equation for $\mathbb{P}^1$ and $\mathbb{P}^2$, respectively, that is,

$$\vartheta^2\Phi_1 = 4z^2\Phi_1, \tag{11.4.1}$$

$$\vartheta^3\Phi_2 = 27z^3\Phi_2. \tag{11.4.2}$$

Given arbitrary functions f and g, we have

$$\begin{aligned}\mathscr{L}[s^2 f(s), g(s); z] = z\Big\{&\mathscr{L}[f(s), g(s); z] + \frac{1}{2}\mathscr{L}[\vartheta_s f(s), g(s); z] \\ &+ \frac{1}{3}\mathscr{L}[f(s), \vartheta_s g(s); z] - \mathcal{I}(f, g)\Big\},\end{aligned}$$

with

$$\mathcal{I}(f, g) := \lambda \cdot f(z^{\frac{1}{2}}\lambda^{\frac{1}{2}}) g(z^{\frac{2}{3}}\lambda^{\frac{1}{3}}) e^{-\lambda}\Big|_{\lambda=0}^{\lambda=\infty}. \tag{11.4.3}$$

Applying the previous identity to Φ_1 and Φ_2, and using equations (11.4.1)–(11.4.2), we deduce the following identities:

$$\begin{aligned}\mathscr{L}[\vartheta^2\Phi_1, \Phi_2; z] &= 4z\Big\{\mathscr{L}[\Phi_1, \Phi_2; z] + \frac{1}{2}\mathscr{L}[\vartheta\Phi_1, \Phi_2; z] \\ &\quad + \frac{1}{3}\mathscr{L}[\Phi_1, \vartheta\Phi_2; z]\Big\} + \mathscr{R}_1, \\ \mathscr{L}[\vartheta^3\Phi_1, \Phi_2; z] &= 8(z + z^2)\mathscr{L}[\Phi_1, \Phi_2; z] + (8z + 4z^2)\mathscr{L}[\vartheta\Phi_1, \Phi_2; z] \\ &\quad + \frac{8}{3}(z + z^2)\mathscr{L}[\Phi_1, \vartheta\Phi_2; z] \\ &\quad + \frac{4}{3} z \mathscr{L}[\vartheta\Phi_1, \vartheta\Phi_2; z] + \mathscr{R}_2,\end{aligned}$$

$$\begin{aligned}
\mathscr{L}[\vartheta^4\Phi_1,\Phi_2;z] &= 16(z+4z^2+z^3)\mathscr{L}[\Phi_1,\Phi_2;z]\\
&\quad + 8(3z+5z^2+z^3)\mathscr{L}[\vartheta\Phi_1,\Phi_2;z]\\
&\quad + \frac{16}{3}(z+5z^2+z^3)\mathscr{L}[\Phi_1,\vartheta\Phi_2;z]\\
&\quad + \frac{16}{3}(z+z^2)\mathscr{L}[\vartheta\Phi_1,\vartheta\Phi_2;z]\\
&\quad + \frac{16}{9}z^2\mathscr{L}[\Phi_1,\vartheta^2\Phi_2;z]+\mathscr{R}_3,\\
\mathscr{L}[\Phi_1,\vartheta^3\Phi_2;z] &= 27z^2\Big\{\mathscr{L}[\Phi_1,\Phi_2;z]+\frac{1}{2}\mathscr{L}[\vartheta\Phi_1,\Phi_2;z]\\
&\quad + \frac{1}{3}\mathscr{L}[\Phi_1,\vartheta\Phi_2;z]\Big\}+\mathscr{R}_4,\\
\mathscr{L}[\Phi_1,\vartheta^4\Phi_2;z] &= \frac{9}{2}z^2\Big\{18\mathscr{L}[\Phi_1,\Phi_2;z]+12\mathscr{L}[\Phi_1,\vartheta\Phi_2;z]\\
&\quad + 2\mathscr{L}[\Phi_1,\vartheta^2\Phi_2;z]+9\mathscr{L}[\vartheta\Phi_1,\Phi_2;z]\\
&\quad + 3\mathscr{L}[\vartheta\Phi_1,\vartheta\Phi_2;z]\Big\}+\mathscr{R}_5,\\
\mathscr{L}[\vartheta\Phi_1,\vartheta^3\Phi_2;z] &= 54z^3\mathscr{L}[\Phi_1,\Phi_2;z]+27(z^2+z^3)\mathscr{L}[\vartheta\Phi_1,\Phi_2;z]\\
&\quad + 18z^3\mathscr{L}[\Phi_1,\vartheta\Phi_2;z]\\
&\quad + 9z^2\mathscr{L}[\vartheta\Phi_1,\vartheta\Phi_2;z]+\mathscr{R}_6,\\
\mathscr{L}[\vartheta^2\Phi_1,\vartheta^2\Phi_2;z] &= 36z^3\mathscr{L}[\Phi_1,\Phi_2;z]+18z^3\mathscr{L}[\vartheta\Phi_1,\Phi_2;z]\\
&\quad + 12z^3\mathscr{L}[\Phi_1,\vartheta\Phi_2;z]+4z\mathscr{L}[\Phi_1,\vartheta^2\Phi_2;z]\\
&\quad + 2z\mathscr{L}[\vartheta\Phi_1,\vartheta^2\Phi_2;z]+\mathscr{R}_7,\\
\mathscr{L}[\vartheta^3\Phi_1,\vartheta\Phi_2;z] &= 8(z+z^2)\mathscr{L}[\Phi_1,\vartheta\Phi_2;z]\\
&\quad + (8z+4z^2)\mathscr{L}[\vartheta\Phi_1,\vartheta\Phi_2;z]\\
&\quad + \frac{8}{3}(z+z^2)\mathscr{L}[\Phi_1,\vartheta^2\Phi_2;z]\\
&\quad + \frac{4}{3}z\mathscr{L}[\vartheta\Phi_1,\vartheta^2\Phi_2;z]+\mathscr{R}_8,\\
\mathscr{L}[\vartheta^2\Phi_1,\vartheta\Phi_2;z] &= 4z\Big\{\mathscr{L}[\Phi_1,\vartheta\Phi_2;z]+\frac{1}{2}\mathscr{L}[\vartheta\Phi_1,\vartheta\Phi_2;z]\\
&\quad + \frac{1}{3}\mathscr{L}[\Phi_1,\vartheta^2\Phi_2;z]\Big\}+\mathscr{R}_9,
\end{aligned}$$

where $\mathscr{R}_j$ with $j=1,\dots,9$ denote some negligible boundary terms due to the cumulations of terms like (11.4.3). Using these identities, after some computations, we can rewrite the quantum differential equation (11.2.1) as follows:

$$(-72+1674z+283z^2)\mathscr{D}_1[\Phi_1,\Phi_2]+(36+724z+4811z^2)\mathscr{D}_2[\Phi_1,\Phi_2]=0. \blacksquare$$

An explicit computation shows that $\mathscr{D}_1[\Phi_1,\Phi_2;z]$ and $\mathscr{D}_2[\Phi_1,\Phi_2;z]$ have the following expansions:

$$\mathscr{D}_1[\Phi_1,\Phi_2;z] = \Theta_1(z)\log^3 z + \Theta_2(z)\log^2 z + \Theta_3(z)\log z + \Theta_4(z),$$
$$\mathscr{D}_2[\Phi_1,\Phi_2;z] = \Lambda_1(z)\log^3 z + \Lambda_2(z)\log^2 z + \Lambda_3(z)\log z + \Lambda_4(z),$$

where the functions $\Theta_i(z)$ and $\Lambda_i(z)$ are of the form

$$\Theta_i(z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(m+n)!}{(m!)^2(n!)^3}\big(\mathcal{A}_1^{(i)}(m,n) + \mathcal{A}_2^{(i)}(m,n)z + \mathcal{A}_3^{(i)}(m,n)z^2\big)z^{m+2n}, \tag{11.4.4}$$

$$\Lambda_i(z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(m+n)!}{(m!)^2(n!)^3}\big(\mathcal{B}_1^{(i)}(m,n) + \mathcal{B}_2^{(i)}(m,n)z + \mathcal{B}_3^{(i)}(m,n)z^2\big)z^{m+2n+1} \tag{11.4.5}$$

for $i=1,2,3,4$. See Appendix B for the explicit expressions of the coefficients $\mathcal{A}_j^{(i)}$ and $\mathcal{B}_j^{(i)}$.

Lemma 11.4.3. *For all $m,n\geqslant 1$ and $i=1,2,3,4$, the following identities hold true:*

$$(m+n)\mathcal{A}_1^{(i)}(m,n) + m^2\mathcal{A}_1^{(i)}(m-1,n) + n^3\mathcal{A}_1^{(i)}(m,n-1) = 0, \tag{11.4.6}$$
$$(m+n)\mathcal{B}_1^{(i)}(m,n) + m^2\mathcal{B}_1^{(i)}(m-1,n) + n^3\mathcal{B}_1^{(i)}(m,n-1) = 0, \tag{11.4.7}$$
$$\mathcal{A}_1^{(i)}(m,0) + m\mathcal{A}_2^{(i)}(m-1,0) = 0, \tag{11.4.8}$$
$$\mathcal{B}_1^{(i)}(m,0) + m\mathcal{B}_2^{(i)}(m-1,0) = 0, \tag{11.4.9}$$
$$\mathcal{A}_1^{(i)}(0,n) + n^2\mathcal{A}_3^{(i)}(0,n-1) = 0, \tag{11.4.10}$$
$$\mathcal{B}_1^{(i)}(0,n) + n^2\mathcal{B}_3^{(i)}(0,n-1) = 0. \tag{11.4.11}$$

Proof. The reader can check the validity of these identities using the explicit expressions in Appendix B, equations (11.3.4), (11.3.5), (11.3.7), (11.3.8), (11.3.9), and the following identities (see e.g. [64]):

$$\psi^{(k)}(z+1) = \psi^{(k)}(z) + \frac{(-1)^k k!}{z^{k+1}}, \quad k\geqslant 0,$$
$$\psi^{(0)}(n) = H_{n-1} - \gamma, \quad n\geqslant 1, \quad \psi(z) := \frac{\Gamma'(z)}{\Gamma(z)}. \qquad \blacksquare$$

Theorem 11.4.4. *Let $\Phi_1(z)\in\mathcal{S}(\mathbb{P}^1)$, $\Phi_2(z)\in\mathcal{S}(\mathbb{P}^2)$ be as in equations (11.3.3) and (11.3.6), respectively. Then the function $\Phi(z) := e^{-z}\mathscr{L}[\Phi_1,\Phi_2;z]$ is a solution of the qDE of $\mathbb{F}_1$ if*

$$A_{0,0}B_{0,0} = 0, \quad 4A_{0,1}B_{0,0} = 3A_{0,0}B_{0,1}. \tag{11.4.12}$$

Proof. Let us rearrange the double series (11.4.4) as follows:

$$\begin{aligned}
\Theta_i(z) = \Bigg\{ & \mathcal{A}_1^{(i)}(0,0) + \underbrace{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{(m+n)!}{(m!)^2(n!)^3}\mathcal{A}_1^{(i)}(m,n)z^{m+2n}}_{(\star)} \\
& + \underbrace{\sum_{m=1}^{\infty} \frac{1}{m!}\mathcal{A}_1^{(i)}(m,0)z^m}_{(\star\star)} + \underbrace{\sum_{n=1}^{\infty} \frac{1}{(n!)^2}\mathcal{A}_1^{(i)}(0,n)z^{2n}}_{(\star\star\star)} \\
& + \underbrace{\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{(m+n)!}{(m!)^2(n!)^3}\mathcal{A}_2^{(i)}(m,n)z^{1+m+2n}}_{(\star)} \\
& + \underbrace{\sum_{m=0}^{\infty} \frac{1}{m!}\mathcal{A}_2^{(i)}(m,0)z^{1+m}}_{(\star\star)} \\
& + \underbrace{\sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \frac{(m+n)!}{(m!)^2(n!)^3}\mathcal{A}_3^{(i)}(m,n)z^{2+m+2n}}_{(\star)} \\
& + \underbrace{\sum_{n=0}^{\infty} \frac{1}{(n!)^2}\mathcal{A}_3^{(i)}(0,n)z^{2+2n}\Bigg\}}_{(\star\star\star)},
\end{aligned}$$

where

(1) the $(\star)$-labelled summands cancel by equation (11.4.6),

(2) the $(\star\star)$-labelled summands cancel by equation (11.4.8),

(3) the $(\star\star\star)$-labelled summands cancel by equation (11.4.10).

The proof for $\Lambda_i(z)$ is identical. ∎

Definition 11.4.5. Let $\mathcal{H}$ denote the four-dimensional subspace of $\mathcal{S}(\mathbb{P}^1) \otimes \mathcal{S}(\mathbb{P}^2)$ defined by the linear equations (11.4.12).

Corollary 11.4.6. *The space $\mathcal{H}$ is isomorphic to the space of solutions $\mathcal{S}(\mathbb{F}_1)$ via the operator $\mathcal{P}$.* ∎

11.4.2 Bases of $\mathcal{S}(\mathbb{P}^1)$

Define

$$g(z) := \frac{1}{2\pi i}\int_{\mathcal{L}_1} \Gamma\left(\frac{s}{2}\right)^2 z^{-s}\, ds, \tag{11.4.13}$$

where $\mathcal{L}_1$ is a (positively oriented) parabola $\mathrm{Re}\, s = -c \cdot (\mathrm{Im}\, s)^2 + c'$, for suitable $c, c' \in \mathbb{R}_+$ so that it encircles all the poles of the integrand at $s \in 2\mathbb{Z}_{\leqslant 0}$. It is easy to see that the integral in (11.4.13) converges for all $z \in \widetilde{\mathbb{C}^*}$ and that its value does not depend on the particular choice of c, c'.

Proposition 11.4.7. *The functions $g(e^{-i\pi} z)$ and $g(z)$ define a basis of solutions of the qDE of $\mathbb{P}^1$.* ∎

Define the bases $(g_1(z), g_2(z))$ and $(s_1(z), s_2(z))$ of $\mathcal{S}(\mathbb{P}^1)$ by

$$\begin{pmatrix} g_1(z) \\ g_2(z) \end{pmatrix} = M_1 \begin{pmatrix} g(e^{-\pi i} z) \\ g(z) \end{pmatrix}, \quad \begin{pmatrix} s_1(z) \\ s_2(z) \end{pmatrix} = M_2 \begin{pmatrix} g(e^{-\pi i} z) \\ g(z) \end{pmatrix},$$

where

$$M_1 := \begin{pmatrix} -\frac{i\gamma}{4\pi} & \frac{i(\gamma + i\pi)}{4\pi} \\ \frac{i}{4\pi} & -\frac{i}{4\pi} \end{pmatrix}, \quad M_2 := \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}.$$

Lemma 11.4.8. *For $z \to 0$, the following asymptotic expansions hold true:*

$$g_1(z) = \log z + O(z^2 \log z),$$
$$g_2(z) = 1 + O(z^2 \log z).$$

Proof. The proof is a simple computation of residues: by modifying the paths of integration $\mathcal{L}_1$, one obtains the asymptotic expansions of g as a sum of residues of the integrand. ∎

Lemma 11.4.9. *We have*

$$g(z) \sim \frac{2\pi^{\frac{1}{2}}}{z^{\frac{1}{2}}} e^{-2z}, \quad z \to \infty,$$

in the sector $|\arg z| < \frac{3}{2}\pi$.

Proof. The estimate follows from application of steepest descent method. ∎

11.4.3 Bases of $\mathcal{S}(\mathbb{P}^2)$

Define

$$h(z) := \frac{1}{2\pi i} \int_{\mathcal{L}_2} \Gamma\left(\frac{s}{3}\right)^3 e^{\frac{\pi i s}{3}} z^{-s}\, ds, \tag{11.4.14}$$

where $\mathcal{L}_2$ is a (positively oriented) parabola $\mathrm{Re}\, s = -c \cdot (\mathrm{Im}\, s)^2 + c'$, for suitable $c, c' \in \mathbb{R}_+$ so that it encircles all the poles of the integrand at $s \in 3\mathbb{Z}_{\leqslant 0}$. It is easy to see that the integral in (11.4.14) converges for all $z \in \widetilde{\mathbb{C}^*}$ and that its value does not depend on the particular choice of c, c'.

Proposition 11.4.10. *The functions $h(e^{-\frac{2i\pi}{3}} z), h(z), h(e^{\frac{2i\pi}{3}} z)$ define a basis of solutions of the qDE of $\mathbb{P}^2$.* ∎

Define the bases $(h_1(z), h_2(z), h_3(z))$ and $(p_1(z), p_2(z), p_3(z))$ of $\mathcal{S}(\mathbb{P}^2)$ by

$$\begin{pmatrix} h_1(z) \\ h_2(z) \\ h_3(z) \end{pmatrix} = N_1 \begin{pmatrix} h(e^{-\frac{2i\pi}{3}} z) \\ h(z) \\ h(e^{\frac{2i\pi}{3}} z) \end{pmatrix}, \quad \begin{pmatrix} p_1(z) \\ p_2(z) \\ p_3(z) \end{pmatrix} = N_2 \begin{pmatrix} h(e^{-\frac{2i\pi}{3}} z) \\ h(z) \\ h(e^{\frac{2i\pi}{3}} z) \end{pmatrix}, \tag{11.4.15}$$

where

$$N_1 := \begin{pmatrix} \frac{-18\gamma^2-\pi^2}{216\pi^2} & \frac{-18\gamma^2-24i\gamma\pi+7\pi^2}{216\pi^2} & \frac{18\gamma^2+12i\gamma\pi+5\pi^2}{108\pi^2} \\ \frac{\gamma}{12\pi^2} & \frac{3\gamma+2i\pi}{36\pi^2} & \frac{-3\gamma-i\pi}{18\pi^2} \\ -\frac{1}{12\pi^2} & -\frac{1}{12\pi^2} & \frac{1}{6\pi^2} \end{pmatrix},$$

$$N_2 := \begin{pmatrix} -1 & 3 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The basis (p_1, p_2, p_3) will be studied later, in Section 11.7, where it will be used to construct Stokes bases of solutions. We now focus on the properties of the basis (h_1, h_2, h_3).

Lemma 11.4.11. *For $z \to 0$, the following asymptotic expansions hold true:*

$$\begin{aligned} h_1(z) &= \log^2 z + O(z^3 \log^2 z), \\ h_2(z) &= \log z + O(z^3 \log^2 z), \\ h_3(z) &= 1 + O(z^3 \log^2 z). \end{aligned}$$

Proof. The proof is a simple computations of residues: by modifying the paths of integration $\mathcal{L}_2$, one obtains the asymptotic expansions of h as a sum of residues of the integrand. ■

Lemma 11.4.12. *We have*

$$h(z) \sim e^{-\frac{5}{3}\pi i} \frac{\sqrt{3}}{z} \exp\left(3e^{\frac{2\pi i}{3}} z\right), \quad z \to \infty,$$

in the sector $-\pi < \arg z < \frac{5}{3}\pi$.

Proof. The estimate follows from the steepest descent method. ■

11.5 Basis of solutions Υ of $\mathcal{S}(\mathbb{F}_1)$

Theorem 11.5.1. *The tensors*

$$\frac{1}{3} g_1 \otimes h_2 + \frac{1}{4} g_2 \otimes h_1, \quad g_1 \otimes h_3, \quad g_2 \otimes h_2, \quad g_2 \otimes h_3 \tag{11.5.1}$$

define a basis of the subspace $\mathcal{H}$.

Proof. Each of the vectors given in (11.5.1) satisfy the constraints (11.4.12), by Lemmata 11.4.8 and 11.4.11. ∎

Corollary 11.5.2. *The functions*

$$\begin{aligned}
\Upsilon_1 &:= \mathscr{P}\left(\frac{1}{3}g_1 \otimes h_2 + \frac{1}{4}g_2 \otimes h_1\right),\\
\Upsilon_2 &:= \mathscr{P}(g_1 \otimes h_3),\\
\Upsilon_3 &:= \mathscr{P}(g_2 \otimes h_2),\\
\Upsilon_4 &:= \mathscr{P}(g_2 \otimes h_3)
\end{aligned}$$

define a basis of solutions of the qDE of $\mathbb{F}_1$. ∎

Remark 11.5.3. Explicit double Mellin–Barnes integral representations of solutions $\Upsilon_1, \dots, \Upsilon_4$ can be obtained: for any j, k we have

$$\mathscr{P}\big(g(e^{\pi k i}z) \otimes h(e^{\frac{2\pi j i}{3}}z)\big) = \frac{e^{-z}}{(2\pi i)^2}\int_{\mathscr{L}_1\times\mathscr{L}_2} \Gamma\left(\frac{s}{2}\right)^2 \Gamma\left(\frac{t}{3}\right)^3 \Gamma\left(1-\frac{s}{2}-\frac{t}{3}\right) \cdot e^{-\pi i k s + \frac{\pi i}{3}t(1-2j)} z^{-\frac{s}{2}-\frac{2t}{3}}\, dt\, ds.$$

The functions Υ_i are linear combinations of the integrals above, in accordance with Theorem 7.4.2.

11.6 Asymptotics of Laplace $(1,2;\frac{1}{2},\frac{1}{3})$-multitransforms

Consider the integral

$$\mathcal{I}(z) := \int_0^\infty \Phi_1(z^{\frac{1}{2}}\lambda^{\frac{1}{2}})\Phi_2(z^{\frac{2}{3}}\lambda^{\frac{1}{3}})e^{-\lambda}\, d\lambda,$$

where

$$\Phi_1(z) = z^{D_1}\exp(zu_1), \quad \Phi_2(z) = z^{D_2}\exp(zu_2),$$

with $D_1, D_2, u_1, u_2 \in \mathbb{C}$. The integral $\mathcal{I}(z)$ is convergent for all $z \in \widetilde{\mathbb{C}^*}$.

Set $z = re^{i\kappa}$ with $r > 0$, and change variable of integration $\lambda = \alpha z$:

$$\mathcal{I}(z) = z^{1+D_1+D_2}\int_0^{e^{-i\kappa}\infty} \alpha^{\frac{D_1}{2}+\frac{D_2}{3}} \exp\big\{z(-\alpha + u_1\alpha^{\frac{1}{2}} + u_2\alpha^{\frac{1}{3}})\big\}\, d\alpha.$$

Change variable $\alpha = \beta^6$, by taking the principal determination of the sixth root:

$$\mathcal{I}(z) = 6z^{1+D_1+D_2}\int_0^{e^{-\frac{i\kappa}{6}}\infty} \beta^{5+3D_1+2D_2} \exp\big\{z(-\beta^6 + u_1\beta^3 + u_2\beta^2)\big\}\, d\beta. \quad (11.6.1)$$

Define

$$f(\beta; u_1, u_2) := -\beta^6 + u_1\beta^3 + u_2\beta^2 \quad \text{for } \beta \in \mathbb{C},$$

and consider the z-dependent downward flow in the β-plane defined by

$$\frac{d\beta}{dt} = -\overline{z}\frac{\partial \overline{f}}{\partial \overline{\beta}}, \quad \frac{d\overline{\beta}}{dt} = -z\frac{\partial f}{\partial \beta}. \tag{11.6.2}$$

The equilibria points β_c are the critical points of f, that is,

$$\left.\frac{\partial f}{\partial \beta}\right|_{\beta=\beta_c} = 0.$$

For a fixed z, we associate to each critical point β_c a curve $\mathcal{L}_c$, a *Lefschetz thimble*, defined as the set-theoretic union of the trajectories of the flow (11.6.2) starting at β_c for $t \to -\infty$. Morse and Picard–Lefschetz theory guarantee that the cycles $\mathcal{L}_c$ are smooth one-dimensional submanifolds of $\mathbb{C}$, piecewise smoothly dependent on the parameter z, and they represent a basis for the inverse limit of relative homology groups

$$\varprojlim_T H_1(\mathbb{C}, \mathbb{C}_{T,z}), \quad \mathbb{C}_{T,z} := \{\beta \in \mathbb{C} : \mathrm{Re}(zf(\beta; u_1, u_2)) < -T\}, \ T \in \mathbb{R}_+.$$

Lemma 11.6.1. *The Lefschetz thimble $\mathcal{L}_c$ is the steepest descent path at β_c: the function $t \mapsto \mathrm{Im}(zf(\beta; u_1, u_2))$ is constant on $\mathcal{L}_c$ and the function $t \mapsto \mathrm{Re}(zf(\beta; u_1, u_2))$ is strictly decreasing along the flow.*

Proof. We have

$$\frac{d}{dt}[\mathrm{Im}(zf)] = \left(\frac{d\beta}{dt}\frac{\partial}{\partial\beta} + \frac{d\overline{\beta}}{dt}\frac{\partial}{\partial\overline{\beta}}\right)\left[\frac{zf - \overline{zf}}{2i}\right] = 0,$$

$$\frac{d}{dt}[\mathrm{Re}(zf)] = \left(\frac{d\beta}{dt}\frac{\partial}{\partial\beta} + \frac{d\overline{\beta}}{dt}\frac{\partial}{\partial\overline{\beta}}\right)\left[\frac{zf + \overline{zf}}{2}\right] = -\left|z\frac{\partial f}{\partial\beta}\right|^2. \qquad \blacksquare$$

We are interested in the following cases, by Lemmata 11.4.9 and 11.4.12:

$$u_1 = \pm 2, \quad u_2 = 3\zeta_3^k, \quad \zeta_3 := \exp\frac{2\pi i}{3}, \quad k = 0, 1, 2. \tag{11.6.3}$$

For any possible pair (u_1, u_2), define β_+ as the critical point of $f(\beta; u_1, u_2)$ with maximal real part (the bold one in Table 11.1).

Lemma 11.6.2. *We have*

$$\mathcal{I}(z) \sim 6z^{\frac{1}{2}+D_1+D_2}\beta_+^{5+2D_1+3D_2}\left(\frac{2\pi}{9u_1\beta_+ + 8u_2}\right)^{\frac{1}{2}} \exp z(-\beta_+^6 + u_1\beta_+^3 + u_2\beta_+^2)$$

for $|z| \to \infty$ in the sector $|\arg z - \arg \overline{f(\beta_+)}| < \pi$.

u_1	u_2	β_c	$f(\beta_c)$	$f(\beta_c)-1$
2	3	-0.724492	0.6695	-0.3305
2	3	0.	0.	$-1.$
2	3	**1.22074**	4.7996	3.7996
2	3	$-0.248126-1.03398i$	$-1.23455+1.94071i$	$-2.23455+1.94071i$
2	3	$-0.248126+1.03398i$	$-1.23455-1.94071i$	$-2.23455-1.94071i$
2	$3e^{\frac{2i\pi}{3}}$	0.	0.	$-1.$
2	$3e^{\frac{2i\pi}{3}}$	$-0.771392-0.731875i$	$-1.23455-1.94071i$	$-2.23455-1.94071i$
2	$3e^{\frac{2i\pi}{3}}$	$-0.610372+1.0572i$	4.7996	3.7996
2	$3e^{\frac{2i\pi}{3}}$	$0.362246-0.627428i$	0.6695	-0.3305
2	$3e^{\frac{2i\pi}{3}}$	$\mathbf{1.01952+0.302108}\boldsymbol{i}$	$-1.23455+1.94071i$	$-2.23455+1.94071i$
2	$3e^{-\frac{1}{3}(2i\pi)}$	0.	0.	$-1.$
2	$3e^{-\frac{1}{3}(2i\pi)}$	$-0.771392+0.731875i$	$-1.23455+1.94071i$	$-2.23455+1.94071i$
2	$3e^{-\frac{1}{3}(2i\pi)}$	$-0.610372-1.0572i$	4.7996	3.7996
2	$3e^{-\frac{1}{3}(2i\pi)}$	$0.362246+0.627428i$	0.6695	-0.3305
2	$3e^{-\frac{1}{3}(2i\pi)}$	$\mathbf{1.01952-0.302108}\boldsymbol{i}$	$-1.23455-1.94071i$	$-2.23455-1.94071i$
-2	3	-1.22074	4.7996	3.7996
-2	3	0.	0.	$-1.$
-2	3	**0.724492**	0.6695	-0.3305
-2	3	$0.248126-1.03398i$	$-1.23455-1.94071i$	$-2.23455-1.94071i$
-2	3	$0.248126+1.03398i$	$-1.23455+1.94071i$	$-2.23455+1.94071i$
-2	$3e^{\frac{2i\pi}{3}}$	0.	0.	$-1.$
-2	$3e^{\frac{2i\pi}{3}}$	$-1.01952-0.302108i$	$-1.23455+1.94071i$	$-2.23455+1.94071i$
-2	$3e^{\frac{2i\pi}{3}}$	$-0.362246+0.627428i$	0.6695	-0.3305
-2	$3e^{\frac{2i\pi}{3}}$	$0.610372-1.0572i$	4.7996	3.7996
-2	$3e^{\frac{2i\pi}{3}}$	$\mathbf{0.771392+0.731875}\boldsymbol{i}$	$-1.23455-1.94071i$	$-2.23455-1.94071i$
-2	$3e^{-\frac{1}{3}(2i\pi)}$	0.	0.	$-1.$
-2	$3e^{-\frac{1}{3}(2i\pi)}$	$-1.01952+0.302108i$	$-1.23455-1.94071i$	$-2.23455-1.94071i$
-2	$3e^{-\frac{1}{3}(2i\pi)}$	$-0.362246-0.627428i$	0.6695	-0.3305
-2	$3e^{-\frac{1}{3}(2i\pi)}$	$0.610372+1.0572i$	4.7996	3.7996
-2	$3e^{-\frac{1}{3}(2i\pi)}$	$\mathbf{0.771392-0.731875}\boldsymbol{i}$	$-1.23455+1.94071i$	$-2.23455+1.94071i$

Table 11.1. For any possible value of the pair (u_1,u_2), we list the corresponding critical points β_c of the function $f(\beta;u_1,u_2)$, and the corresponding critical values $f(\beta_c)$. Notice that the numbers $f(\beta_c)-1$, with $\beta_c\neq 0$, equal all possible values of the canonical coordinates x_1,x_2,x_3,x_4 at the origin of $QH^\bullet(\mathbb{F}_1)$. In bold, we represent the critical point β_+ with maximal real part.

Proof. After choosing an orientation for each Lefschetz thimble, the path of integration $\gamma_z \equiv e^{-i\frac{\kappa}{6}} \cdot \mathbb{R}_+$, defining the function $\mathcal{I}$ in equation (11.6.1), can be expressed as integer combination, $\gamma_z = \sum_{j=1}^{5} n_j(z)\mathcal{L}_j$ with $n_j \in \mathbb{Z}$, of the thimbles $\mathcal{L}_c$ for any value of z not on a Stokes ray $\mathcal{R}_{ij}$, defined by

$$\mathcal{R}_{ij} := \left\{z \in \widetilde{\mathbb{C}^*} : z = r\left(\overline{f(\beta_{c,i})} - \overline{f(\beta_{c,j})}\right),\, r \in \mathbb{R}_+\right\}, \quad i, j = 1, \dots, 5,$$

where $\beta_{c,i}$ are the critical points of (11.6.2). If we let z vary, the Lefschetz thimbles change. When z crosses a Stokes ray $\mathcal{R}_{ij}$, Lefschetz thimbles jump discontinuously: in particular, for z on a Stokes ray there exists a flow line of (11.6.2) connecting two critical points β_c. A detailed analysis of the phase portrait of the flow (11.6.2), for each pair (u_1, u_2) as in (11.6.3), shows that in the sector $|\arg z - \arg \overline{f(\beta_+)}| < \pi$ we have $\gamma_z = \pm\mathcal{L}_{\beta_+} \pm \mathcal{L}_0^1 \pm \mathcal{L}'$, where $\mathcal{L}_0^1$ is only one half of the Lefschetz thimble $\mathcal{L}_0$, and $\mathcal{L}'$ denotes the sum of Lefschetz thimbles attached to other critical points β_c. Hence, we have three contributions in the asymptotics of $\mathcal{I}(z)$: one from the integration along $\mathcal{L}_{\beta_+}$, one from other critical points, the last one from the integration along $\mathcal{L}_0^1$. The last two contributions are easily seen to be negligible with respect to the first one. So, by the steepest descent method, we obtain the estimate

$$\mathcal{I}(z) \sim \pm 6 z^{\frac{1}{2}+D_1+D_2} \beta_+^{5+2D_1+3D_2} \left(-\frac{2\pi}{f''(\beta_+)}\right)^{\frac{1}{2}} \exp z f(\beta_+).$$

See Figure 11.1. ∎

Remark 11.6.3. Note that the arbitrariness of the orientations of the Lefschetz thimbles can be incorporated in the choice of the entries of the Ψ-matrix. Consequently, it will affect the monodromy data by the action of the group $(\mathbb{Z}/2\mathbb{Z})^4$.

Proposition 11.6.4. *Let now Φ_1, Φ_2 be two functions with asymptotic expansions*

$$\Phi_1(z) \sim z^{D_1} \exp(z u_1), \quad \Phi_2(z) \sim z^{D_2} \exp(z u_2) \tag{11.6.4}$$

for $|z| \to \infty$ in the sectors

$$A_1 < \arg z < B_1, \quad A_2 < \arg z < B_2, \tag{11.6.5}$$

respectively. We have

$$\mathscr{L}_{(1,2;\frac{1}{2},\frac{1}{3})}[\Phi_1, \Phi_2; z] \sim C z^{\frac{1}{2}+D_1+D_2} \exp z(-\beta_+^6 + u_1\beta_+^3 + u_2\beta_+^2),$$

where

$$C := 6\beta_+^{5+2D_1+3D_2} \left(\frac{2\pi}{9u_1\beta_+ + 8u_2}\right)^{\frac{1}{2}},$$

for $|z| \to \infty$ in the sector $A' < \arg z < B'$, where

$$A' := \max\left\{A_1 - 3\arg\beta_+,\ A_2 - 2\arg\beta_+,\ \arg\overline{f(\beta_+)} - \pi\right\},$$
$$B' := \min\left\{B_1 - 3\arg\beta_+,\ B_2 - 2\arg\beta_+,\ \arg\overline{f(\beta_+)} + \pi\right\}.$$

(See Table 11.2.)

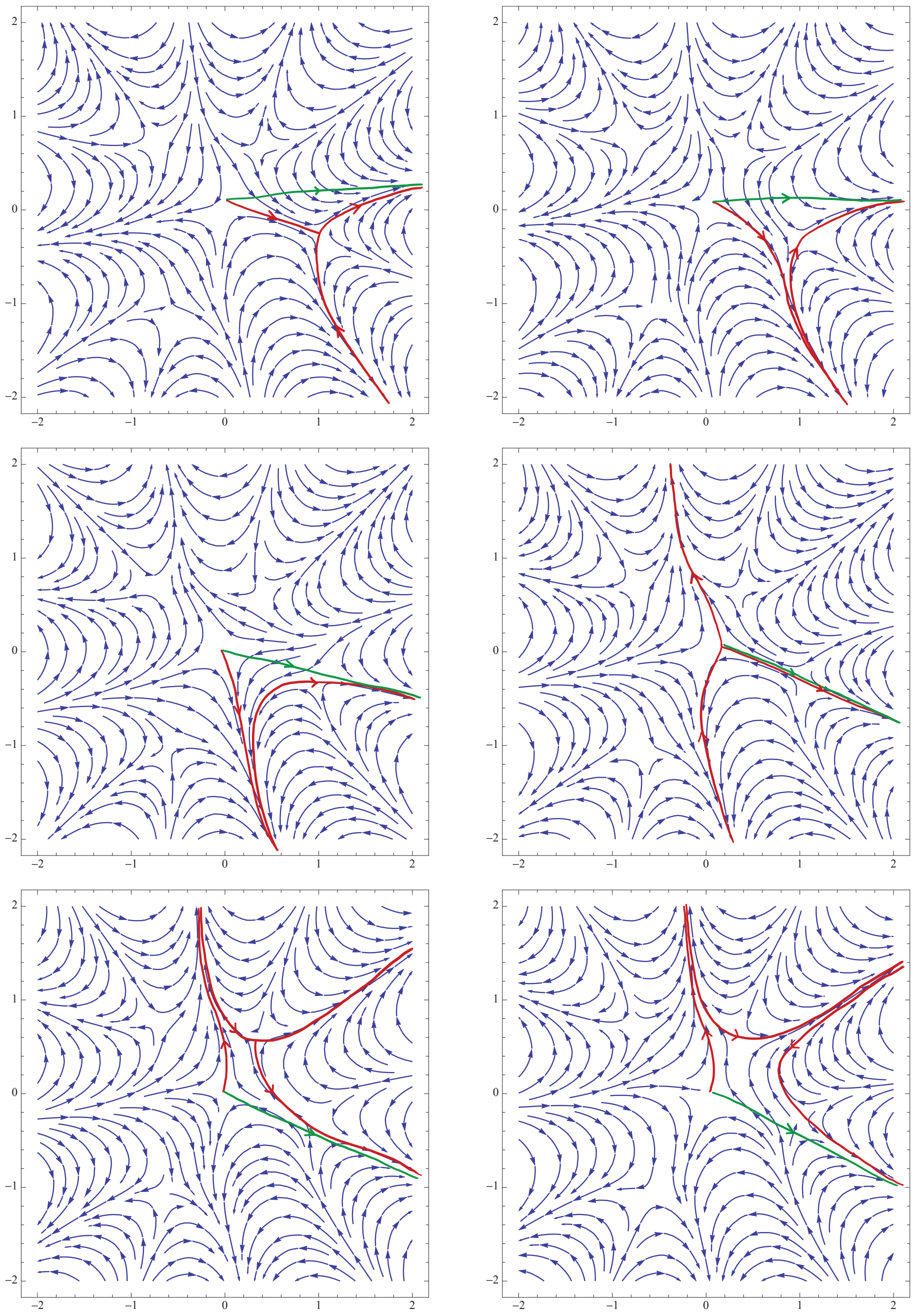

Figure 11.1. In this figure we represent the downward flow (11.6.2) and the mutations of Lefschetz thimbles for $|z| = 10^5$, and $|\arg z - \arg \overline{f(\beta_+)}| < \pi$ for the pair $(u_1, u_2) = (2, 3e^{\frac{4\pi i}{3}})$. Lefschetz thimbles are in red. The path of integration in equation (11.6.1) is drawn in green. *To be continued on the next page.*

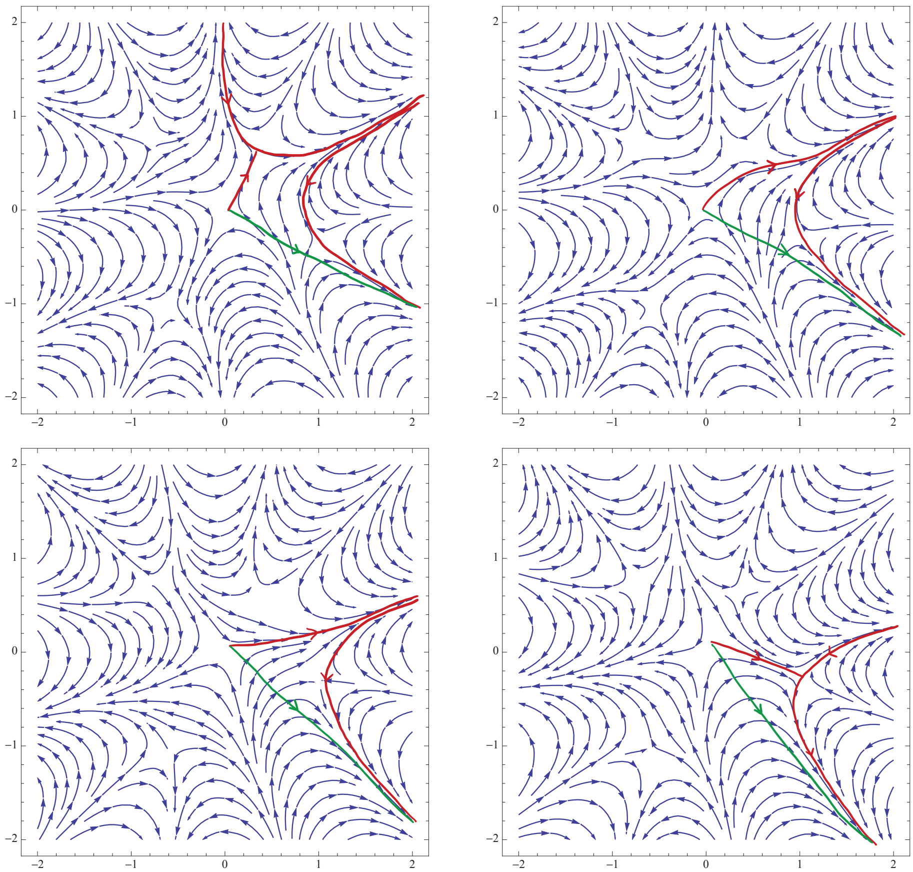

Figure 11.1 (continued). Notice that, for a certain range of values of arg z, there is also a contribution in the asymptotic expansion coming from a third critical point. Such a term is negligible, since it is dominated by the exponential term from the critical point β_+.

u_1	u_2	A'	B'
2	3	$-\pi$	$\frac{\pi}{3}$
2	$3e^{\frac{2\pi i}{3}}$	−3.71775	0.471036
2	$3e^{\frac{4\pi i}{3}}$	−1.00423	1.62336
−2	3	$-\pi$	$\frac{\pi}{3}$
−2	$3e^{\frac{2\pi i}{3}}$	−1.00423	−0.706554
−2	$3e^{\frac{4\pi i}{3}}$	−1.62336	1.00423

Table 11.2. In this table we represent the values A' and B' predicted in Proposition 11.6.4 for all possible values of u_1 and u_2.

Proof. The statement follows by application of the steepest descent path method and Lemma 11.6.2. Notice that the sector $A' < \arg z < B'$ is chosen so that the critical point of the logarithm of the integrand lies in the region (11.6.5) of validity of the asymptotic expansions (11.6.4). ∎

11.7 Stokes basis of the qDE of $\mathbb{F}_1$

Set
$$s_{ij} := s_i \otimes p_j \in \mathcal{S}(\mathbb{P}^1) \otimes \mathcal{S}(\mathbb{P}^2)$$
for $i = 1, 2$ and $j = 1, 2, 3$. See equation (11.4.15).

Theorem 11.7.1. *The following linear combinations of the tensors s_{ij} define a basis of $\mathcal{H}$:*
$$s_{11} - 5s_{22} - 6s_{23}, \quad s_{12} + s_{23}, \quad s_{13} - s_{22} - 2s_{23}, \quad s_{21} - 4s_{22} - 5s_{23}.$$

Proof. Define the column vectors

- $\boldsymbol{g} = (g_1, g_2)^T$ and $\boldsymbol{s} = (s_1, s_2)^T$, bases of $\mathcal{S}(\mathbb{P}^1)$,
- $\boldsymbol{h} = (h_1, h_2, h_3)^T$ and $\boldsymbol{p} = (p_1, p_2, p_3)^T$, bases of $\mathcal{S}(\mathbb{P}^2)$, respectively.

In what follow we denote by $A \otimes B$ the Kronecker tensor product of two matrices A and B. Hence we denote

- by $\boldsymbol{g} \otimes \boldsymbol{h}$ the basis $(g_i \otimes h_j)_{i,j}$ of $\mathcal{S}(\mathbb{P}^1) \otimes \mathcal{S}(\mathbb{P}^2)$,
- by $\boldsymbol{s} \otimes \boldsymbol{p}$ the basis $(s_i \otimes p_j)_{i,j}$ of $\mathcal{S}(\mathbb{P}^1) \otimes \mathcal{S}(\mathbb{P}^2)$.

We have
$$\boldsymbol{g} \otimes \boldsymbol{h} = [(M_1 M_2^{-1}) \otimes (N_1 N_2^{-1})] \boldsymbol{s} \otimes \boldsymbol{p}, \tag{11.7.1}$$
where we represent the basis $\boldsymbol{g} \otimes \boldsymbol{h}$ and $\boldsymbol{s} \otimes \boldsymbol{p}$ as column vectors. Multiply on the left both sides of (11.7.1) by the matrix
$$E_1 := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$
We thus obtain the relation
$$\boldsymbol{s} \otimes \boldsymbol{p} = X \begin{pmatrix} g_1 \otimes h_1 \\ g_1 \otimes h_2 \\ g_1 \otimes h_3 \\ \frac{1}{3} g_1 \otimes h_2 + \frac{1}{4} g_2 \otimes h_1 \\ g_2 \otimes h_2 \\ g_2 \otimes h_3 \end{pmatrix}, \tag{11.7.2}$$

where X is the matrix

$$X = [(M_1 M_2^{-1}) \otimes (N_1 N_2^{-1})]^{-1} E_1^{-1}$$
$$= \begin{pmatrix} 54 & 36(\gamma + 11i\pi) & * & * & * & * \\ -54 & -36(\gamma + i\pi) & * & * & * & * \\ 54 & 36(\gamma + 3i\pi) & * & * & * & * \\ 54 & 36(\gamma + 9i\pi) & * & * & * & * \\ -54 & -36(\gamma - i\pi) & * & * & * & * \\ 54 & 36(\gamma + i\pi) & * & * & * & * \end{pmatrix}.$$

Multiply on the left each sides of (11.7.2) by the matrix

$$E_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & -5 & -6 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 & -4 & -5 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We obtain

$$\begin{pmatrix} s_{11} - 5s_{22} - 6s_{23} \\ s_{12} + s_{23} \\ s_{13} - s_{22} - 2s_{23} \\ s_{21} - 4s_{22} - 5s_{23} \\ s_{22} + s_{23} \\ s_{23} \end{pmatrix} = E_2 X \begin{pmatrix} g_1 \otimes h_1 \\ g_1 \otimes h_2 \\ g_1 \otimes h_3 \\ \frac{1}{3} g_1 \otimes h_2 + \frac{1}{4} g_2 \otimes h_1 \\ g_2 \otimes h_2 \\ g_2 \otimes h_3 \end{pmatrix},$$

and we have

$$E_2 X = \left(\begin{array}{cc|cccc} 0 & 0 & & & & \\ 0 & 0 & & C_1 & & \\ 0 & 0 & & & & \\ 0 & 0 & & & & \\ \hline 0 & 72i\pi & * & * & * & * \\ 54 & 36(\gamma + i\pi) & * & * & * & * \end{array}\right). \tag{11.7.3}$$

This proves the claim. ∎

Remark 11.7.2. The matrix C_1 in equation (11.7.3) is

$$\begin{pmatrix} 24(-3i\gamma - 2\pi)\pi & -216i\pi & 36\pi(-5i\gamma + 9\pi) & 3\pi(-42i\gamma^2 + 92\gamma\pi + 17i\pi^2) \\ 72i\gamma\pi & 216i\pi & 36\pi(5i\gamma + \pi) & 3\pi(42i\gamma^2 + 12\gamma\pi - i\pi^2) \\ -72i\gamma\pi & -216i\pi & 36\pi(-5i\gamma + \pi) & 3\pi(-42i\gamma^2 + 12\gamma\pi + i\pi^2) \\ -48\pi^2 & 0 & 0 & -48\gamma\pi^2 \end{pmatrix}.$$

Corollary 11.7.3. *The functions*

$$\Sigma_1 := \mathscr{P}(s_{11} - 5s_{22} - 6s_{23}), \quad \Sigma_2 := \mathscr{P}(s_{12} + s_{23}),$$
$$\Sigma_3 := \mathscr{P}(s_{13} - s_{22} - 2s_{23}), \quad \Sigma_4 := \mathscr{P}(s_{21} - 4s_{22} - 5s_{23})$$

define a basis of solutions of the qDE of $\mathbb{F}_1$. ∎

Proposition 11.7.4. *The Stokes basis* Ξ_R *of* $\mathscr{H}_0^{\mathrm{od}}$ *on the sector* $\Pi_R(\varepsilon)$ *can be reconstructed, using formulas* (11.2.2)–(11.2.5)*, from a basis* $\boldsymbol{\Sigma}_{\boldsymbol{\lambda}}$ *of solutions of the qDE of* $\mathbb{F}_1$ *of the form*

$$\lambda_1\Sigma_2, \quad \lambda_2\Sigma_3+\lambda_3\Sigma_2, \quad \lambda_4\Sigma_4+\lambda_5\Sigma_3+\lambda_6\Sigma_2, \quad \lambda_7\Sigma_1+\lambda_8\Sigma_4+\lambda_9\Sigma_3+\lambda_{10}\Sigma_2,$$

for a suitable choice of the coefficients $\lambda_j \in \mathbb{C}$*, with* $j = 1, \dots, 10$.

Proof. The canonical coordinates x_1, x_2, x_3, x_4 are in lexicographical order with respect to a line of slope $\varepsilon > 0$ sufficiently small. The functions above have the expected exponential growth $\exp(x_i z)$ in the sector $\Pi_R(\varepsilon)$ defined by an oriented line of slope ε. This follows from the data in Tables 11.1 and 11.2, and from the configuration of the Stokes rays $R_{ij} := \{-r\sqrt{-1}(\overline{x_i} - \overline{x_j}) : r \in \mathbb{R}_+\}$: these are given by

$$R_{12} = \{\arg z = \pi\}, \qquad R_{13} = \{\arg z = 2.36573\},$$
$$R_{14} = \{\arg z = 1.88197\}, \quad R_{23} = \{\arg z = 0.775863\},$$
$$R_{24} = \{\arg z = 1.25962\}, \quad R_{34} = \left\{\arg z = \frac{\pi}{2}\right\},$$

see Figure 11.2. ∎

Remark 11.7.5. Notice that, according to Proposition 11.6.4, the function Σ_3 has the expected exponential growth $\exp(zx_2)$ in the sector in which this is minimal with respect to the dominance relation, i.e. in which it is dominated by any other exponential $\exp(zx_1), \exp(zx_3), \exp(zx_4)$. Hence, we expect that $\lambda_3 = 0$.

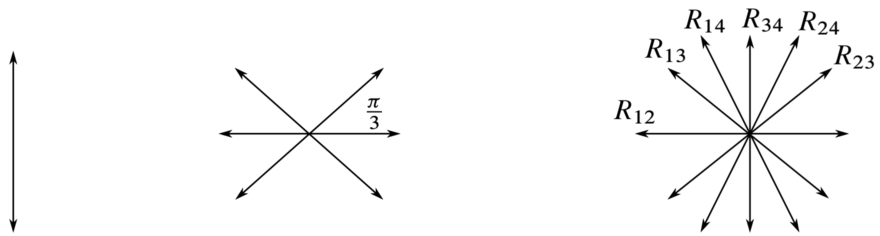

Figure 11.2. From the left to the right: Stokes rays corresponding to the origin of the quantum cohomology of $\mathbb{P}^1$, $\mathbb{P}^2$, and $\mathbb{F}_1$, respectively.

11.8 Computation of the central connection and Stokes matrices

Denote by $\mathcal{H}_0''$ the system of differential equations $\mathcal{H}_0^{\mathrm{od}}$ specialized at $0 \in QH^\bullet(\mathbb{F}_1)$. Consider the fundamental system of solutions of $\mathcal{H}_0''$

$$\Xi_\Upsilon(z) := \begin{pmatrix} z\Upsilon_1(z) & z\Upsilon_2(z) & z\Upsilon_3(z) & z\Upsilon_4(z) \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix},$$

reconstructed from the basis $(\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4)$ of the qDE of $\mathbb{F}_1$ (see Corollary 11.5.2) by formulas (11.2.2)–(11.2.5).

Proposition 11.8.1. *We have*

$$\Xi_\Upsilon(z) = \Xi_{\mathrm{top}}(z) \cdot C_0,$$

where

$$C_0 := \begin{pmatrix} \frac{1}{18} & 0 & 0 & 0 \\ -\frac{\gamma}{18} & \frac{1}{2} & 0 & 0 \\ -\frac{\gamma}{18} & -\frac{1}{2} & \frac{1}{3} & 0 \\ \frac{6\gamma^2+\pi^2}{72} & -\frac{\gamma}{2} & -\frac{\gamma}{3} & 1 \end{pmatrix}. \tag{11.8.1}$$

Proof. From Lemmata 11.4.8 and 11.4.11, we can compute the asymptotic expansions of $\Upsilon_i(z)$ for $z \to 0$. We have

$$\begin{aligned}
\Upsilon_1(z) = {} & \frac{1}{72}\left(16\log^2(z) - 20\gamma\log(z) + 6\gamma^2 + \pi^2\right) + \frac{1}{18}z(\log(z) - \gamma - 2) \\
& + \frac{1}{72}z^2\left(16\log^2(z) - 20\gamma\log(z) - 17\log(z) + 6\gamma^2 + \pi^2 + 13\gamma + 2\right) \\
& + \frac{z^3}{1944}\left(432\log^2(z) - 540\gamma\log(z) - 750\log(z) + 162\gamma^2\right. \\
& \qquad \left. + 27\pi^2 + 426\gamma + 311\right) + \cdots, \\
\Upsilon_2(z) = {} & \frac{1}{2}(\log(z) - \gamma) - \frac{z}{2} + \frac{1}{8}z^2(4\log(z) - 4\gamma + 5) \\
& + \frac{1}{36}z^3(18\log(z) - 18\gamma - 37) \\
& + \frac{1}{192}z^4(24\log(z) - 24\gamma + 13) + \cdots, \\
\Upsilon_3(z) = {} & -\frac{\gamma}{3} + \frac{2\log(z)}{3} + \frac{z}{3} + \frac{1}{12}z^2(8\log(z) - 4\gamma - 9) \\
& + \frac{1}{54}z^3(36\log(z) - 18\gamma - 17) \\
& + \frac{1}{288}z^4(48\log(z) - 24\gamma - 49) + \cdots, \\
\Upsilon_4(z) = {} & 1 + z^2 + z^3 + \frac{z^4}{4} + \cdots.
\end{aligned}$$

After some computations, one finds the first terms of the asymptotic expansion of $\Xi_\Upsilon(z)$ for $z \to 0$:

$$\Xi_\Upsilon(z) = \begin{pmatrix} \frac{z(16\log^2(z)-20\gamma\log(z)+6\gamma^2+\pi^2)}{72} & \frac{z(\log(z)-\gamma)}{2} & \frac{z(2\log(z)-\gamma)}{3} & z \\ \frac{\log(z)}{6} - \frac{\gamma}{9} & 0 & \frac{1}{3} & 0 \\ \frac{\log(z)}{9} + \frac{z(\log(z)-\gamma-1)}{18} - \frac{\gamma}{18} & \frac{1}{2} - \frac{z}{2} & \frac{z}{3} & 0 \\ \frac{z(2\log(z)-\gamma-1)}{18} + \frac{1}{18z} & \frac{z}{2} & 0 & 0 \end{pmatrix} + \text{h.o.t.}$$

The leading term of the asymptotic expansion of $\Xi_{\text{top}}(z)$ for $z \to 0$ is

$$\begin{aligned} \Xi_{\text{top}}(z) &= \eta z^{\mu} z^{R} + \text{h.o.t.} \\ &= \begin{pmatrix} 4z\log^2(z) & 3z\log(z) & 2z\log(z) & z \\ 3\log(z) & 1 & 1 & 0 \\ 2\log(z) & 1 & 0 & 0 \\ \frac{1}{z} & 0 & 0 & 0 \end{pmatrix} + \text{h.o.t.}, \end{aligned}$$

where $\mu = \operatorname{diag}(-1, 0, 0, 1)$ and R is the operator of $\cup$-multiplication by $c_1(\mathbb{F}_1)$ on $H^\bullet(\mathbb{F}_1, \mathbb{C})$, that is,

$$R = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 \end{pmatrix}.$$

By comparison of the leading terms of the asymptotic expansions of Ξ_Υ and Ξ_{top}, one obtains the matrix C_0 in formula (11.8.1). ∎

Theorem 11.8.2. *The central connection and Stokes matrices at* $0 \in QH^\bullet(\mathbb{F}_1)$*, computed with respect to an admissible oriented line of slope* $\varepsilon > 0$ *sufficiently small, equal*

$$C = \begin{pmatrix} \frac{1}{2\pi} & -\frac{1}{2\pi} & \frac{1}{2\pi} & -\frac{1}{2\pi} \\ \frac{\gamma}{\pi} & -\frac{\gamma}{\pi} & i + \frac{\gamma}{\pi} & -i - \frac{\gamma}{\pi} \\ \frac{1}{2}\left(-i + \frac{\gamma}{\pi}\right) & -\frac{\gamma + i\pi}{2\pi} & \frac{1}{2}\left(-i + \frac{\gamma}{\pi}\right) & -\frac{\gamma + i\pi}{2\pi} \\ \gamma\left(-i + \frac{2\gamma}{\pi}\right) & \gamma\left(-i - \frac{2\gamma}{\pi}\right) & \frac{2\gamma(\gamma + i\pi)}{\pi} & -\frac{2(\gamma + i\pi)^2}{\pi} \end{pmatrix}, \tag{11.8.2}$$

$$S = \begin{pmatrix} 1 & 2 & -1 & -3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{11.8.3}$$

Proof. Denote

- by $\Xi_{\boldsymbol{\lambda}}$ the fundamental system of solutions of $\mathcal{H}_0''$ constructed from the basis $\boldsymbol{\Sigma}_{\boldsymbol{\lambda}}$ of Proposition 11.7.4,

- by Ξ_{Σ} the fundamental system of solutions of $\mathscr{H}_0''$ constructed from the basis $\boldsymbol{\Sigma}$ of Corollary 11.7.3.

We have

$$\Xi_{\boldsymbol{\lambda}} = \Xi_{\Sigma} \cdot \begin{pmatrix} 0 & 0 & 0 & \lambda_7 \\ \lambda_1 & \lambda_3 & \lambda_6 & \lambda_{10} \\ 0 & \lambda_2 & \lambda_5 & \lambda_9 \\ 0 & 0 & \lambda_4 & \lambda_8 \end{pmatrix} = \Xi_{\Upsilon} \Pi^T C_1^T \begin{pmatrix} 0 & 0 & 0 & \lambda_7 \\ \lambda_1 & \lambda_3 & \lambda_6 & \lambda_{10} \\ 0 & \lambda_2 & \lambda_5 & \lambda_9 \\ 0 & 0 & \lambda_4 & \lambda_8 \end{pmatrix},$$

where C_1 is as in Remark 11.7.2 and

$$\Pi := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus, we obtain

$$\Xi_{\boldsymbol{\lambda}} = \Xi_{\mathrm{top}} C_{\boldsymbol{\lambda}}, \quad C_{\boldsymbol{\lambda}} := C_0 \Pi^T C_1^T \begin{pmatrix} 0 & 0 & 0 & \lambda_7 \\ \lambda_1 & \lambda_3 & \lambda_6 & \lambda_{10} \\ 0 & \lambda_2 & \lambda_5 & \lambda_9 \\ 0 & 0 & \lambda_4 & \lambda_8 \end{pmatrix},$$

where C_0 is given by (11.8.1). In order to determine the values of $\boldsymbol{\lambda}$ for which $\Xi_{\boldsymbol{\lambda}}$ is the Stokes basis, let us compute the product

$$C_{\boldsymbol{\lambda}}^T \eta e^{\pi i \mu} e^{\pi i R} C_{\boldsymbol{\lambda}}. \tag{11.8.4}$$

If $\Xi_{\boldsymbol{\lambda}}$ is the Stokes basis, then the matrix above is the inverse of the Stokes matrix S, by equation (4.4.2): in particular, it is an upper triangular matrix with ones along the main diagonal. An explicit computation gives the following result: the columns of (11.8.4) are

$$\begin{pmatrix} -576\pi^4\lambda_1^2 \\ -576\pi^4\lambda_1\lambda_3 \\ -576\pi^4\lambda_1\lambda_6 \\ -576\pi^4\lambda_1(3\lambda_7+\lambda_{10}) \end{pmatrix},$$

$$\begin{pmatrix} 576\pi^4\lambda_1(2\lambda_2-\lambda_3) \\ -576\pi^4(\lambda_2-\lambda_3)^2 \\ -576\pi^4(\lambda_3\lambda_6+\lambda_2(\lambda_4+\lambda_5-2\lambda_6)) \\ 576\pi^4(\lambda_2(\lambda_7-\lambda_8-\lambda_9+2\lambda_{10})-\lambda_3(3\lambda_7+\lambda_{10})) \end{pmatrix},$$

$$\begin{pmatrix} -576\pi^4\lambda_1(\lambda_4-2\lambda_5+\lambda_6) \\ -576\pi^4(\lambda_2\lambda_5+\lambda_3(\lambda_4-2\lambda_5+\lambda_6)) \\ -576\pi^4(\lambda_4^2+(\lambda_5+\lambda_6)\lambda_4+(\lambda_5-\lambda_6)^2) \\ -576\pi^4(\lambda_6(3\lambda_7+\lambda_{10})+\lambda_4(5\lambda_7+\lambda_8+\lambda_{10})+\lambda_5(-\lambda_7+\lambda_8+\lambda_9-2\lambda_{10})) \end{pmatrix},$$

$$\begin{pmatrix} 576\pi^4\lambda_1(6\lambda_7-\lambda_8+2\lambda_9-\lambda_{10}) \\ -576\pi^4(\lambda_2(6\lambda_7+\lambda_9)+\lambda_3(-6\lambda_7+\lambda_8-2\lambda_9+\lambda_{10})) \\ -576\pi^4(\lambda_5(6\lambda_7+\lambda_9)+\lambda_4(6\lambda_7+\lambda_8+\lambda_9)+\lambda_6(-6\lambda_7+\lambda_8-2\lambda_9+\lambda_{10})) \\ -576\pi^4(13\lambda_7^2+(11\lambda_8+5\lambda_9-3\lambda_{10})\lambda_7+\lambda_8^2+(\lambda_9-\lambda_{10})^2+\lambda_8(\lambda_9+\lambda_{10})) \end{pmatrix}.$$

The matrix (11.8.4) is upper triangular with ones along the diagonal if and only if

$$\lambda_1^2=-\frac{1}{576\pi^4},\quad \lambda_2^2=-\frac{1}{576\pi^4},\quad \lambda_3=0,\qquad \lambda_4^2=-\frac{1}{576\pi^4},\quad \lambda_5=-\lambda_4,$$
$$\lambda_6=0,\qquad \lambda_7^2=-\frac{1}{576\pi^4},\quad \lambda_8=-2\lambda_7,\quad \lambda_9=-3\lambda_7,\qquad \lambda_{10}=-3\lambda_7.$$

For the choice $\lambda_1=\lambda_2=\lambda_4=\lambda_7=-\frac{i}{24\pi^2}$, we obtain the central connection and Stokes matrices (11.8.2) and (11.8.3). ∎

Theorem 11.8.3. *The central connection matrix of $QH^\bullet(\mathbb{F}_{2k+1})$, computed with respect to an oriented line of slope $\varepsilon>0$ sufficiently small, and a suitable choice of the branch of the Ψ-matrix, equals*

$$C_k=\begin{pmatrix} \frac{1}{2\pi} & -\frac{1}{2\pi} & \frac{1}{2\pi} & -\frac{1}{2\pi} \\ \frac{\gamma}{\pi} & -\frac{\gamma}{\pi} & i+\frac{\gamma}{\pi} & -i-\frac{\gamma}{\pi} \\ \frac{\gamma-2\gamma k-i\pi}{2\pi} & -\frac{\gamma-2\gamma k+i\pi}{2\pi} & \frac{-2\gamma k-i(2\pi k+\pi)+\gamma}{2\pi} & \frac{(2k-1)(\gamma+i\pi)}{2\pi} \\ \gamma\left(-i+\frac{2\gamma}{\pi}\right) & \gamma\left(-i-\frac{2\gamma}{\pi}\right) & \frac{2\gamma(\gamma+i\pi)}{\pi} & -\frac{2(\gamma+i\pi)^2}{\pi} \end{pmatrix}. \tag{11.8.5}$$

This is the matrix associated with the morphism

$$Д^-_{\mathbb{F}_{2k+1}}\colon K_0(\mathbb{F}_{2k+1})_{\mathbb{C}}\to H^\bullet(\mathbb{F}_{2k+1},\mathbb{C}),$$
$$[\mathscr{F}]\mapsto \frac{1}{2\pi}\widehat{\Gamma}^-_{\mathbb{F}_{2k+1}}\cup e^{-\pi i c_1(\mathbb{F}_{2k+1})}\cup \mathrm{Ch}(\mathscr{F}),$$

with respect to

- *an exceptional basis $\mathfrak{E}:=(E_i)_{i=1}^4$ of $K_0(\mathbb{F}_{2k+1})_{\mathbb{C}}$,*
- *the basis $(T_{i,2k+1})_{i=0}^3$ of $H^\bullet(\mathbb{F}_{2k+1},\mathbb{C})$.*

The exceptional basis $\mathfrak{E}$ mutates to the exceptional basis

$$\left([\mathcal{O}],[\mathcal{O}(\Sigma_2^{2k+1})],[\mathcal{O}(\Sigma_4^{2k+1})],[\mathcal{O}(\Sigma_2^{2k+1}+\Sigma_4^{2k+1})]\right), \tag{11.8.6}$$

by application of the following natural transformations:

(1) *action of the braid $\beta_3\beta_2\beta_1\beta_3\beta_2$,*

(2) *action of the element $\widetilde{J}_k\in(\mathbb{Z}/2\mathbb{Z})^4$*

$$\widetilde{J}_k:=\begin{cases}(-1,-1,(-1)^p,(-1)^{p+1}) & \text{if } k=2p,\\ (-1,-1,(-1)^{p+1},(-1)^{p+1}) & \text{if } k=2p+1,\end{cases}$$

(3) *action of the element β_3^k.*

Proof. Equations (9.3.11) and Proposition 4.5.1 imply equation (11.8.5). The matrix associated to $\varPi^{-}_{\mathbb{F}_{2k+1}}$ with respect to the basis (11.8.6) is

$$E_k := \begin{pmatrix} \frac{1}{2\pi} & \frac{1}{2\pi} & \frac{1}{2\pi} & \frac{1}{2\pi} \\ -i+\frac{\gamma}{\pi} & -i+\frac{\gamma}{\pi} & \frac{\gamma}{\pi} & \frac{\gamma}{\pi} \\ \frac{(1-2k)(\gamma-i\pi)}{2\pi} & \frac{-2\gamma k+i(2\pi k+\pi)+\gamma}{2\pi} & \frac{(1-2k)(\gamma-i\pi)}{2\pi} & \frac{-2\gamma k+i(2\pi k+\pi)+\gamma}{2\pi} \\ \frac{2(\gamma-i\pi)^2}{\pi} & \frac{2\gamma(\gamma-i\pi)}{\pi} & \gamma\left(-i+2ik+\frac{2\gamma}{\pi}\right) & \gamma\left(i+2ik+\frac{2\gamma}{\pi}\right) \end{pmatrix}.$$

Set $C'_k := C_k^{\beta_3\beta_2\beta_1\beta_3\beta_2}$. We have

$$(C'_k)^{-1}E_k = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1-k & -k \\ 0 & 0 & -k & -k-1 \end{pmatrix}.$$

It is now easy to see that this is the matrix representing the action of the element $(\widetilde{J}_k, \beta_3^k) \in (\mathbb{Z}/2\mathbb{Z})^4 \rtimes \mathcal{B}_4$: the argument is the same as in Step 3 of the proof of Theorem 10.3.3. ∎

Appendix A

Proof of Theorem 5.1.2

We need some preliminary results.

Lemma A.1. *For $n \geqslant 0$, and $\delta \in H^2(X, \mathbb{C})$, we have*

$$\langle\langle \tau_n T_\alpha, 1 \rangle\rangle_0(\delta) = \frac{1}{(n+1)!}\left(\int_X T_\alpha \cup \delta^{n+1}\right) + \sum_{\beta \neq 0}\sum_{\nu \geqslant 0} \frac{\mathbf{Q}^\beta e^{\int_\beta \delta}}{\nu!} \langle \tau_{n-\nu} T_\alpha \cup \delta^\nu, 1\rangle^X_{0,2,\beta}.$$

Proof. We have

$$\langle\langle \tau_n T_\alpha, 1 \rangle\rangle_0(\delta) = \frac{\partial}{\partial t^{\alpha,n}}\frac{\partial}{\partial t^{0,0}} \mathcal{F}_0^X \Big|_\delta = \sum_{k=0}^\infty \sum_\beta \frac{\mathbf{Q}^\beta}{k!} \langle \tau_n T_\alpha, 1, \delta \dots, \delta\rangle^X_{0,k+2,\beta}.$$

Two cases occur:

- If $\beta \neq 0$, then for $k \geqslant 0$ we have

$$\langle \tau_n T_\alpha, 1, \delta \dots, \delta\rangle^X_{0,k+2,\beta} = \sum_{\mu+\nu=k} \frac{k!}{\mu!\nu!}\left(\int_\beta \delta\right)^\mu \langle \tau_{n-\nu} T_\alpha \cup \delta^\nu, 1\rangle^X_{0,2,\beta},$$

  by the divisor axiom of Gromov–Witten invariants. Here any invariant with τ_{-r} with $r > 0$ is vanishing.
- If $\beta = 0$, then for $k > 0$ by the divisor axiom we have[1]

$$\langle \tau_n T_\alpha, 1, \delta \dots, \delta\rangle^X_{0,k+2,0} = \langle \tau_{n-k+1} T_\alpha \cup \delta^k, 1, \delta\rangle_{0,3,0} = \left(\int_X T_\alpha \cup \delta^k\right)\delta_{k,n+1}.$$

So, we obtain

$$\begin{aligned}\langle\langle \tau_n T_\alpha, 1 \rangle\rangle_0(\delta) &= \frac{1}{(n+1)!}\left(\int_X T_\alpha \cup \delta^{n+1}\right) \\ &\quad + \sum_{\beta \neq 0}\sum_{k \geqslant 0} \frac{\mathbf{Q}^\beta}{k!} \sum_{\mu+\nu=k} \frac{k!}{\mu!\nu!}\left(\int_\beta \delta\right)^\mu \langle \tau_{n-\nu} T_\alpha \cup \delta^\nu, 1\rangle^X_{0,2,\beta} \\ &= \frac{1}{(n+1)!}\left(\int_X T_\alpha \cup \delta^{n+1}\right) \\ &\qquad + \sum_{\beta \neq 0}\sum_{\nu \geqslant 0} \frac{\mathbf{Q}^\beta e^{\int_\beta \delta}}{\nu!} \langle \tau_{n-\nu} T_\alpha \cup \delta^\nu, 1\rangle^X_{0,2,\beta}.\end{aligned}$$

∎

[1] Here, we use the fact that $\mathcal{L}_1$ is trivial on $\overline{\mathcal{M}}_{0,3}(X, 0)$ and hence has zero Chern class. This follows from the fact that $\overline{\mathcal{M}}_{0,3}(X, 0) \cong X$, and the forgetful morphism $\overline{\mathcal{M}}_{0,4}(X, 0) \to \overline{\mathcal{M}}_{0,3}(X, 0)$ is the projection $X \times \overline{\mathcal{M}}_{0,4} \to X$.

Lemma A.2. *Let $\delta \in H^2(X,\mathbb{C})$. We have*

$$J_X(\delta) = e^{\frac{\delta}{\hbar}} + \sum_{\alpha}\sum_{\beta\neq 0}\sum_{n=0}^{\infty}\sum_{k+p=n} \hbar^{-(n+1)}\frac{\mathbf{Q}^{\beta}e^{\int_\beta \delta}}{p!}\langle \tau_k T_\alpha \cup \delta^p, 1\rangle_{0,2,\beta}^{X} T^{\alpha}.$$

Proof. By Lemma A.1, we have

$$\begin{aligned} J_X(\delta) &= 1 + \sum_{\alpha}\sum_{n=0}^{\infty}\frac{\hbar^{-(n+1)}}{(n+1)!}\left(\int_X T_\alpha \cup \delta^{n+1}\right)T^{\alpha} \\ &\quad + \sum_{\alpha}\sum_{\beta\neq 0}\sum_{n=0}^{\infty}\sum_{k+p=n} \hbar^{-(n+1)}\frac{\mathbf{Q}^{\beta}e^{\int_\beta \delta}}{p!}\langle \tau_k T_\alpha \cup \delta^p, 1\rangle_{0,2,\beta}^{X} T^{\alpha} \\ &= e^{\frac{\delta}{\hbar}} + \sum_{\alpha}\sum_{\beta\neq 0}\sum_{n=0}^{\infty}\sum_{k+p=n} \hbar^{-(n+1)}\frac{\mathbf{Q}^{\beta}e^{\int_\beta \delta}}{p!}\langle \tau_k T_\alpha \cup \delta^p, 1\rangle_{0,2,\beta}^{X} T^{\alpha}. \end{aligned}$$

■

Lemma A.3. *For $\delta \in H^2(X,\mathbb{C})$, we have*

$$\begin{aligned} Z_{\mathrm{top}}(\delta, z)T_\alpha &= e^{z\delta} \cup z^{\mu}z^{c_1(X)}T_\alpha \\ &\quad + \sum_{\beta\neq 0}\sum_{\lambda} e^{\int_\beta \delta}\left\langle \frac{ze^{z\delta}}{1-z\psi} \cup z^{\mu}z^{c_1(X)}T_\alpha, T_\lambda\right\rangle_{0,2,\beta}^{X} T^{\lambda}. \end{aligned} \tag{A.1}$$

Proof. For $\boldsymbol{\tau} \in H^{\bullet}(X,\mathbb{C})$, we have

$$\begin{aligned} \Theta(\boldsymbol{\tau}, z)T_\alpha &= \sum_{\varepsilon}\Theta(\boldsymbol{\tau}, z)_{\alpha}^{\varepsilon}T_\varepsilon \\ &= \sum_{\lambda}\left.\frac{\partial\theta_\alpha}{\partial t^{\lambda}}\right|_{(\boldsymbol{\tau},z)}T^{\lambda} \\ &= \sum_{\lambda}\sum_{p=0}^{\infty} z^p \langle\langle \tau_p T_\alpha, 1, T_\lambda\rangle\rangle_0(\boldsymbol{\tau})|_{\mathbf{Q}=1}T^{\lambda} \\ &= \sum_{\lambda}\sum_{p=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\beta}\frac{z^p}{k!}\langle \tau_p T_\alpha, 1, T_\lambda, \boldsymbol{\tau}, \dots, \boldsymbol{\tau}\rangle_{0,3+k,\beta}^{X}T^{\lambda}. \end{aligned}$$

Consider the contribution coming from the fact that $(k,\beta) = (0,0)$: by the mapping-to-point axiom of Gromov–Witten invariants, we have[2]

$$\begin{aligned} \sum_{\lambda}\sum_{p=0}^{\infty} z^p \langle \tau_p T_\alpha, 1, T_\lambda, \rangle_{0,3,0}^{X}T^{\lambda} &= \sum_{\lambda}\sum_{p=0}^{\infty} z^p\left(\int_X T_\alpha \cup T_\lambda\right)\delta_{0,p}T^{\lambda} \\ &= T_\alpha. \end{aligned}$$

[2]Also here, we use the fact that $\mathcal{L}_1$ is trivial on $\overline{\mathcal{M}}_{0,3}(X,0)$.

By the fundamental class axiom, instead, the contribution from $(k,\beta)\neq(0,0)$ can be rewritten as

$$\sum_{\lambda}\sum_{p=0}^{\infty}\sum_{k=1}^{\infty}\sum_{\beta\neq 0}\frac{z^p}{k!}\langle\tau_{p-1}T_\alpha,T_\lambda,\boldsymbol{\tau},\dots,\boldsymbol{\tau}\rangle^X_{0,2+k,\beta}T^\lambda.$$

Thus, we have recovered the formula

$$\Theta(\boldsymbol{\tau},z)=\mathrm{Id}+\sum_{\lambda}\sum_{p=0}^{\infty}z^{p+1}\langle\!\langle\tau_p(-),T_\lambda\rangle\!\rangle_0(\boldsymbol{\tau})|_{\mathbf{Q}=1}T^\lambda,$$

which was used in [23, Proposition 7.1] to define Θ. At this point the proof is known, and can be found in [27, Proposition 10.2.3]: the parameter $\hbar$ of [27] has to be replaced by our z, and pre-composition with $z^\mu z^{c_1(X)}$ has to be taken into account in order to obtain formula (A.1). ■

We are now ready for the proof of Theorem 5.1.2.

Proof of Theorem 5.1.2. Let us compute the entries of the first row of the matrix

$$\eta\Theta(\delta,z)z^\mu z^{c_1(X)}.$$

By Lemma A.3, we have

$$\begin{aligned}
&\left[\eta\Theta(\delta,z)z^\mu z^{c_1(X)}\right]^1_\alpha\\
&\quad=\eta\big(1,\Theta(\delta,z)z^\mu z^{c_1(X)}T_\alpha\big)\\
&\quad=\eta\Bigg(1,e^{z\delta}\cup z^\mu z^{c_1(X)}T_\alpha\\
&\qquad\qquad+\sum_{\beta\neq0}\sum_{\lambda}e^{\int_\beta\delta}\left\langle\frac{ze^{z\delta}}{1-z\psi}\cup z^\mu z^{c_1(X)}T_\alpha,T_\lambda\right\rangle^X_{0,2,\beta}T^\lambda\Bigg)\\
&\quad=\eta(1,e^{z\delta}\cup z^\mu z^{c_1(X)}T_\alpha)\\
&\qquad\qquad+\eta\left(1,\sum_{\beta\neq0}\sum_{\lambda}e^{\int_\beta\delta}\left\langle\frac{ze^{z\delta}}{1-z\psi}\cup z^\mu z^{c_1(X)}T_\alpha,T_\lambda\right\rangle^X_{0,2,\beta}T^\lambda\right).
\end{aligned}$$

Using the identity of endomorphisms of $H^\bullet(X,\mathbb{C})$

$$z^{-\mu}\circ(h^k\cup)\circ z^\mu=z^{-k}(h^k\cup),\quad h\in H^2(X,\mathbb{C}),\quad k\in\mathbb{N},$$

and the η-skew-symmetry of μ, we can rewrite the first summand as

$$\begin{aligned}
\eta(1,e^{z\delta}\cup z^\mu z^{c_1(X)}T_\alpha)&=\eta(1,z^\mu e^\delta z^{c_1(X)}T_\alpha)\\
&=\eta(z^{-\mu}(1),e^\delta z^{c_1(X)}T_\alpha)\\
&=z^{\frac{\dim_{\mathbb{C}}X}{2}}\int_X e^\delta z^{c_1(X)}T_\alpha.
\end{aligned}$$

For the second summand, notice that

(1) the only nonzero contribution comes from $\lambda = 0$,

(2) for any $\varphi \in H^\bullet(X, \mathbb{C})$ we have

$$\frac{ze^{z\delta}}{1 - z\psi} \cup \varphi = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z^{n+1}}{(n-k)!} \psi^k \delta^{n-k} \varphi,$$

(3) we have

$$z^\mu z^{c_1(X)} T_\alpha = \sum_{\ell=0}^{\infty} \frac{(\log z)^\ell}{\ell!} z^{\frac{2\ell + \deg T_\alpha - \dim X}{2}} c_1(X)^\ell T_\alpha,$$

(4) the Gromov–Witten invariant

$$\langle \tau_k \delta^{n-k} c_1(X)^\ell T_\alpha, 1\rangle_{0,2,\beta}^X$$

is nonzero only if

$$2k + 2(n-k) + 2\ell + \deg T_\alpha = 2 \dim_{\mathbb{C}} X + 2 \int_\beta c_1(X) - 2.$$

So, we obtain

$$\begin{aligned}
&\left\langle \frac{ze^{z\delta}}{1 - z\psi} \cup z^\mu z^{c_1(X)} T_\alpha, 1 \right\rangle_{0,2,\beta}^X \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{\ell=0}^{\infty} \frac{(\log z)^\ell}{\ell!(n-k)!} z^{n+1+\frac{2\ell + \deg T_\alpha - \dim X}{2}} \langle \tau_k \delta^{n-k} c_1(X)^\ell T_\alpha, 1\rangle_{0,2,\beta}^X \\
&= z^{\frac{\dim X}{2}} z^{\int_\beta c_1(X)} \sum_{h=0}^{\infty} \sum_{m+\ell+k=h} \frac{(\log z)^\ell}{\ell! m!} \langle \tau_k \delta^m c_1(X)^\ell T_\alpha, 1\rangle_{0,2,\beta}^X \\
&= z^{\frac{\dim X}{2}} z^{\int_\beta c_1(X)} \sum_{h=0}^{\infty} \sum_{k+p=h} \frac{1}{p!} \langle \tau_k (\delta + \log z \cdot c_1(X))^p T_\alpha, 1\rangle_{0,2,\beta}^X.
\end{aligned}$$

Putting this all together, we obtain

$$\begin{aligned}
&\left[\eta \Theta(\delta, z) z^\mu z^{c_1(X)}\right]_\alpha^1 \\
&= z^{\frac{\dim X}{2}} \Bigg(\int_X e^\delta z^{c_1(X)} T_\alpha \\
&\qquad + \sum_{\beta \neq 0} e^{\int_\beta \delta} z^{\int_\beta c_1(X)} \sum_{h=0}^{\infty} \sum_{k+p=h} \frac{1}{p!} \langle \tau_k (\delta + \log z \cdot c_1(X))^p T_\alpha, 1\rangle_{0,2,\beta}^X \Bigg) \\
&= z^{\frac{\dim X}{2}} \int_X T_\alpha \cup J_X(\delta + \log z \cdot c_1(X)) \Big|_{\substack{\mathbf{Q}=1 \\ \hbar = 1}}.
\end{aligned}$$

The last equality follows by Lemma A.2. This completes the proof. ∎

Appendix B

Coefficients $\mathcal{A}_j^{(i)}$ and $\mathcal{B}_j^{(i)}$

The coefficients $\mathcal{A}_j^{(i)}$ and $\mathcal{B}_j^{(i)}$, introduced in equations (11.4.4) and (11.4.5), are

$$
\begin{aligned}
\mathcal{A}_1^{(1)}(m,n) &= -\frac{8}{9} m n^2 A_{0,0} B_{0,0}, \\
\mathcal{A}_2^{(1)}(m,n) &= \frac{8}{9} n^2 A_{0,0} B_{0,0}, \\
\mathcal{A}_3^{(1)}(m,n) &= \frac{8}{9} m A_{0,0} B_{0,0}, \\
\mathcal{A}_1^{(2)}(m,n) &= \frac{4}{9} n \big(4mn A_{0,0} B_{0,0} H_m + 6mn A_{0,0} B_{0,0} H_n - 4mn A_{0,1} B_{0,0} \\
&\quad - 3mn A_{0,0} B_{0,1} - 4mn A_{0,0} B_{0,0} \psi^{(0)}(m+n+1) \\
&\quad - 4m A_{0,0} B_{0,0} - 2n A_{0,0} B_{0,0}\big), \\
\mathcal{A}_2^{(2)}(m,n) &= -\frac{4}{9} n \big(4n A_{0,0} B_{0,0} H_m + 6n A_{0,0} B_{0,0} H_n \\
&\quad - 4n A_{0,0} B_{0,0} \psi^{(0)}(m+n+1) - 4n A_{0,1} B_{0,0} \\
&\quad - 3n A_{0,0} B_{0,1} - 4 A_{0,0} B_{0,0}\big), \\
\mathcal{A}_3^{(2)}(m,n) &= -\frac{4}{9} \big(6m A_{0,0} B_{0,0} H_n + 4m A_{0,0} B_{0,0} H_m \\
&\quad - 4m A_{0,0} B_{0,0} \psi^{(0)}(m+n+1) - 4m A_{0,1} B_{0,0} \\
&\quad - 3m A_{0,0} B_{0,1} - 2 A_{0,0} B_{0,0}\big), \\
\mathcal{A}_1^{(3)}(m,n) &= -\frac{2}{9} \big(24mn^2 A_{0,0} B_{0,0} H_m H_n - 24mn^2 A_{0,1} B_{0,0} H_n \\
&\quad - 12mn^2 A_{0,0} B_{0,1} H_m - 8mn^2 A_{0,0} B_{0,0} H_m \psi^{(0)}(m+n+1) \\
&\quad - 18mn^2 A_{0,0} B_{0,0} H_n \psi^{(0)}(m+n+1) - 16mn A_{0,0} B_{0,0} H_m \\
&\quad - 12mn A_{0,0} B_{0,0} H_n - 12n^2 A_{0,0} B_{0,0} H_n + 12mn^2 A_{0,1} B_{0,1} \\
&\quad + 9mn^2 A_{0,0} B_{n,2} + 5mn^2 A_{0,0} B_{0,0} \psi^{(0)}(m+n+1)^2 \\
&\quad + 4n^2 A_{0,0} B_{0,0} \psi^{(0)}(m+n+1) \\
&\quad + 5mn^2 A_{0,0} B_{0,0} \psi^{(1)}(m+n+1) \\
&\quad + 8mn^2 A_{0,1} B_{0,0} \psi^{(0)}(m+n+1) \\
&\quad + 9mn^2 A_{0,0} B_{0,1} \psi^{(0)}(m+n+1) \\
&\quad + 16mn A_{0,1} B_{0,0} + 6mn A_{0,0} B_{0,1} \\
&\quad + 12mn A_{0,0} B_{0,0} \psi^{(0)}(m+n+1) \\
&\quad + 2m A_{0,0} B_{0,0} + 6n^2 A_{0,0} B_{0,1} + 8n A_{0,0} B_{0,0}\big),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}_2^{(3)}(m,n) = \ & \frac{2}{9}\big(24n^2 A_{0,0} B_{0,0} H_m H_n - 12n^2 A_{0,0} B_{0,1} H_m \\
& - 8n^2 A_{0,0} B_{0,0} H_m \psi^{(0)}(m+n+1) \\
& - 18n^2 A_{0,0} B_{0,0} H_n \psi^{(0)}(m+n+1) \\
& - 16n A_{0,0} B_{0,0} H_m - 24n^2 A_{0,1} B_{0,0} H_n - 12n A_{0,0} B_{0,0} H_n \\
& + 5n^2 A_{0,0} B_{0,0} \psi^{(0)}(m+n+1)^2 \\
& + 5n^2 A_{0,0} B_{0,0} \psi^{(1)}(m+n+1) \\
& + 8n^2 A_{0,1} B_{0,0} \psi^{(0)}(m+n+1) \\
& + 9n^2 A_{0,0} B_{0,1} \psi^{(0)}(m+n+1) \\
& + 12n A_{0,0} B_{0,0} \psi^{(0)}(m+n+1) \\
& + 12n^2 A_{0,1} B_{0,1} + 9n^2 A_{0,0} B_{n,2} \\
& + 16n A_{0,1} B_{0,0} + 6n A_{0,0} B_{0,1} + 2A_{0,0} B_{0,0}\big), \\
\mathcal{A}_3^{(3)}(m,n) = \ & \frac{2}{9}\big(24m A_{0,0} B_{0,0} H_m H_n - 24m A_{0,1} B_{0,0} H_n \\
& - 8m A_{0,0} B_{0,0} H_m \psi^{(0)}(m+n+1) \\
& - 18m A_{0,0} B_{0,0} H_n \psi^{(0)}(m+n+1) \\
& - 12m A_{0,0} B_{0,1} H_m - 12 A_{0,0} B_{0,0} H_n + 9m A_{0,0} B_{n,2} \\
& + 5m A_{0,0} B_{0,0} \psi^{(0)}(m+n+1)^2 + 4A_{0,0} B_{0,0} \psi^{(0)}(m+n+1) \\
& + 8m A_{0,1} B_{0,0} \psi^{(0)}(m+n+1) + 9m A_{0,0} B_{0,1} \psi^{(0)}(m+n+1) \\
& + 5m A_{0,0} B_{0,0} \psi^{(1)}(m+n+1) + 12m A_{0,1} B_{0,1} + 6A_{0,0} B_{0,1}\big), \\
\mathcal{A}_1^{(4)}(m,n) = & -\frac{2}{9}\big(-18mn^2 A_{0,0} H_m B_{n,2} - 2mn^2 A_{0,0} B_{0,0} H_m \psi^{(0)}(m+n+1)^2 \\
& - 6mn^2 A_{0,0} B_{0,0} H_n \psi^{(0)}(m+n+1)^2 \\
& - 6n^2 A_{0,0} B_{0,0} H_n \psi^{(0)}(m+n+1) \\
& + 12mn^2 A_{0,0} B_{0,0} H_m H_n \psi^{(0)}(m+n+1) \\
& - 12mn^2 A_{0,1} B_{0,0} H_n \psi^{(0)}(m+n+1) \\
& - 6mn^2 A_{0,0} B_{0,1} H_m \psi^{(0)}(m+n+1) \\
& - 2mn^2 A_{0,0} B_{0,0} H_m \psi^{(1)}(m+n+1) \\
& - 6mn^2 A_{0,0} B_{0,0} H_n \psi^{(1)}(m+n+1) \\
& + 24mn A_{0,0} B_{0,0} H_m H_n - 24mn A_{0,1} B_{0,0} H_n \\
& - 12mn A_{0,0} B_{0,1} H_m - 8mn A_{0,0} B_{0,0} H_m \psi^{(0)}(m+n+1) \\
& - 12mn A_{0,0} B_{0,0} H_n \psi^{(0)}(m+n+1) - 4m A_{0,0} B_{0,0} H_m \\
& - 12n A_{0,0} B_{0,0} H_n + 18mn^2 A_{0,1} B_{n,2} \\
& + mn^2 A_{0,0} B_{0,0} \psi^{(0)}(m+n+1)^3
\end{aligned}
$$

$$
\begin{aligned}
&+ n^2 A_{0,0} B_{0,0} \psi^{(0)}(m+n+1)^2 \\
&+ 2mn^2 A_{0,1} B_{0,0} \psi^{(0)}(m+n+1)^2 \\
&+ 3mn^2 A_{0,0} B_{0,1} \psi^{(0)}(m+n+1)^2 \\
&+ 3mn^2 A_{0,0} B_{0,0} \psi^{(1)}(m+n+1) \psi^{(0)}(m+n+1) \\
&+ 3n^2 A_{0,0} B_{0,1} \psi^{(0)}(m+n+1) \\
&+ 6mn^2 A_{0,1} B_{0,1} \psi^{(0)}(m+n+1) \\
&+ 9mn^2 A_{0,0} B_{n,2} \psi^{(0)}(m+n+1) \\
&+ n^2 A_{0,0} B_{0,0} \psi^{(1)}(m+n+1) \\
&+ mn^2 A_{0,0} B_{0,0} \psi^{(2)}(m+n+1) \\
&+ 2mn^2 A_{0,1} B_{0,0} \psi^{(1)}(m+n+1) \\
&+ 3mn^2 A_{0,0} B_{0,1} \psi^{(1)}(m+n+1) + 12mn A_{0,1} B_{0,1} \\
&+ 4mn A_{0,0} B_{0,0} \psi^{(0)}(m+n+1)^2 \\
&+ 2m A_{0,0} B_{0,0} \psi^{(0)}(m+n+1) \\
&+ 4n A_{0,0} B_{0,0} \psi^{(0)}(m+n+1) \\
&+ 8mn A_{0,1} B_{0,0} \psi^{(0)}(m+n+1) \\
&+ 6mn A_{0,0} B_{0,1} \psi^{(0)}(m+n+1) \\
&+ 4mn A_{0,0} B_{0,0} \psi^{(1)}(m+n+1) \\
&+ 4m A_{0,1} B_{0,0} + 9n^2 A_{0,0} B_{n,2} + 6n A_{0,0} B_{0,1} + 2A_{0,0} B_{0,0} \big),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}_2^{(4)}(m,n) = \frac{2}{9} \big(&-18n^2 A_{0,0} H_m B_{n,2} - 2n^2 A_{0,0} B_{0,0} H_m \psi^{(0)}(m+n+1)^2 \\
&- 6n^2 A_{0,0} B_{0,0} H_n \psi^{(0)}(m+n+1)^2 \\
&+ 12n^2 A_{0,0} B_{0,0} H_m H_n \psi^{(0)}(m+n+1) \\
&- 12n^2 A_{0,1} B_{0,0} H_n \psi^{(0)}(m+n+1) \\
&- 6n^2 A_{0,0} B_{0,1} H_m \psi^{(0)}(m+n+1) \\
&- 2n^2 A_{0,0} B_{0,0} H_m \psi^{(1)}(m+n+1) \\
&- 6n^2 A_{0,0} B_{0,0} H_n \psi^{(1)}(m+n+1) \\
&+ 24n A_{0,0} B_{0,0} H_m H_n - 12n A_{0,0} B_{0,1} H_m \\
&- 8n A_{0,0} B_{0,0} H_m \psi^{(0)}(m+n+1) \\
&- 12n A_{0,0} B_{0,0} H_n \psi^{(0)}(m+n+1) \\
&- 4A_{0,0} B_{0,0} H_m - 24n A_{0,1} B_{0,0} H_n \\
&+ n^2 A_{0,0} B_{0,0} \psi^{(0)}(m+n+1)^3 \\
&+ 2n^2 A_{0,1} B_{0,0} \psi^{(0)}(m+n+1)^2
\end{aligned}
$$

$$
\begin{aligned}
&+ 3n^2 A_{0,0} B_{0,1} \psi^{(0)}(m+n+1)^2 \\
&+ 3n^2 A_{0,0} B_{0,0} \psi^{(1)}(m+n+1) \psi^{(0)}(m+n+1) \\
&+ 6n^2 A_{0,1} B_{0,1} \psi^{(0)}(m+n+1) \\
&+ 9n^2 A_{0,0} B_{n,2} \psi^{(0)}(m+n+1) \\
&+ n^2 A_{0,0} B_{0,0} \psi^{(2)}(m+n+1) + 2n^2 A_{0,1} B_{0,0} \psi^{(1)}(m+n+1) \\
&+ 3n^2 A_{0,0} B_{0,1} \psi^{(1)}(m+n+1) \\
&+ 4n A_{0,0} B_{0,0} \psi^{(0)}(m+n+1)^2 + 2A_{0,0} B_{0,0} \psi^{(0)}(m+n+1) \\
&+ 8n A_{0,1} B_{0,0} \psi^{(0)}(m+n+1) + 6n A_{0,0} B_{0,1} \psi^{(0)}(m+n+1) \\
&+ 4n A_{0,0} B_{0,0} \psi^{(1)}(m+n+1) + 18n^2 A_{0,1} B_{n,2} \\
&+ 12n A_{0,1} B_{0,1} + 4A_{0,1} B_{0,0}\big),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}_3^{(4)}(m,n) = -\frac{2}{9}\big(&18m A_{0,0} H_m B_{n,2} + 2m A_{0,0} B_{0,0} H_m \psi^{(0)}(m+n+1)^2 \\
&+ 6m A_{0,0} B_{0,0} H_n \psi^{(0)}(m+n+1)^2 \\
&- 12m A_{0,0} B_{0,0} H_m H_n \psi^{(0)}(m+n+1) \\
&+ 6A_{0,0} B_{0,0} H_n \psi^{(0)}(m+n+1) \\
&+ 12m A_{0,1} B_{0,0} H_n \psi^{(0)}(m+n+1) \\
&+ 6m A_{0,0} B_{0,1} H_m \psi^{(0)}(m+n+1) \\
&+ 2m A_{0,0} B_{0,0} H_m \psi^{(1)}(m+n+1) \\
&+ 6m A_{0,0} B_{0,0} H_n \psi^{(1)}(m+n+1) \\
&- 18m A_{0,1} B_{n,2} - m A_{0,0} B_{0,0} \psi^{(0)}(m+n+1)^3 \\
&- A_{0,0} B_{0,0} \psi^{(0)}(m+n+1)^2 - 2m A_{0,1} B_{0,0} \psi^{(0)}(m+n+1)^2 \\
&- 3m A_{0,0} B_{0,1} \psi^{(0)}(m+n+1)^2 \\
&- 3m A_{0,0} B_{0,0} \psi^{(1)}(m+n+1) \psi^{(0)}(m+n+1) \\
&- 3A_{0,0} B_{0,1} \psi^{(0)}(m+n+1) - 6m A_{0,1} B_{0,1} \psi^{(0)}(m+n+1) \\
&- 9m A_{0,0} B_{n,2} \psi^{(0)}(m+n+1) - A_{0,0} B_{0,0} \psi^{(1)}(m+n+1) \\
&- m A_{0,0} B_{0,0} \psi^{(2)}(m+n+1) - 2m A_{0,1} B_{0,0} \psi^{(1)}(m+n+1) \\
&- 3m A_{0,0} B_{0,1} \psi^{(1)}(m+n+1) - 9A_{0,0} B_{n,2}\big),
\end{aligned}
$$

$$
\mathcal{B}_1^{(1)}(m,n) = \frac{2}{9} n A_{0,0} B_{0,0}(m-n),
$$

$$
\mathcal{B}_2^{(1)}(m,n) = -\frac{2}{9} n A_{0,0} B_{0,0},
$$

$$
\mathcal{B}_3^{(1)}(m,n) = \frac{2}{9} A_{0,0} B_{0,0},
$$

$$
\begin{aligned}
\mathcal{B}_1^{(2)}(m,n) = {} & -\frac{1}{9}(m-n)\big(4nA_{0,0}B_{0,0}H_m + 6nA_{0,0}B_{0,0}H_n \\
& - 4nA_{0,0}B_{0,0}\psi^{(0)}(m+n+1) - 4nA_{0,1}B_{0,0} \\
& - 3nA_{0,0}B_{0,1} - 2A_{0,0}B_{0,0}\big), \\
\mathcal{B}_2^{(2)}(m,n) = {} & \frac{1}{9}\big(4nA_{0,0}B_{0,0}H_m + 6nA_{0,0}B_{0,0}H_n \\
& - 4nA_{0,0}B_{0,0}\psi^{(0)}(m+n+1) - 4nA_{0,1}B_{0,0} \\
& - 3nA_{0,0}B_{0,1} - 2A_{0,0}B_{0,0}\big), \\
\mathcal{B}_3^{(2)}(m,n) = {} & \frac{1}{9}\big(-4A_{0,0}B_{0,0}H_m - 6A_{0,0}B_{0,0}H_n \\
& + 4A_{0,0}B_{0,0}\psi^{(0)}(m+n+1) + 4A_{0,1}B_{0,0} + 3A_{0,0}B_{0,1}\big), \\
\mathcal{B}_1^{(3)}(m,n) = {} & \frac{1}{18}\big(-24n^2A_{0,0}B_{0,0}H_mH_n + 12n^2A_{0,0}B_{0,1}H_m \\
& + 8n^2A_{0,0}B_{0,0}H_m\psi^{(0)}(m+n+1) \\
& + 18n^2A_{0,0}B_{0,0}H_n\psi^{(0)}(m+n+1) + 16nA_{0,0}B_{0,0}H_m \\
& + 24mnA_{0,0}B_{0,0}H_mH_n - 24mnA_{0,1}B_{0,0}H_n \\
& - 12mnA_{0,0}B_{0,1}H_m - 6mA_{0,0}B_{0,0}H_n \\
& - 8mnA_{0,0}B_{0,0}H_m\psi^{(0)}(m+n+1) \\
& - 18mnA_{0,0}B_{0,0}H_n\psi^{(0)}(m+n+1) \\
& - 8mA_{0,0}B_{0,0}H_m + 24n^2A_{0,1}B_{0,0}H_n \\
& - 5n^2A_{0,0}B_{0,0}\psi^{(0)}(m+n+1)^2 \\
& - 5n^2A_{0,0}B_{0,0}\psi^{(1)}(m+n+1) \\
& - 8n^2A_{0,1}B_{0,0}\psi^{(0)}(m+n+1) \\
& - 9n^2A_{0,0}B_{0,1}\psi^{(0)}(m+n+1) \\
& + 12mnA_{0,1}B_{0,1} + 9mnA_{0,0}B_{n,2} \\
& + 5mnA_{0,0}B_{0,0}\psi^{(0)}(m+n+1)^2 \\
& - 8nA_{0,0}B_{0,0}\psi^{(0)}(m+n+1) \\
& + 5mnA_{0,0}B_{0,0}\psi^{(1)}(m+n+1) \\
& + 8mnA_{0,1}B_{0,0}\psi^{(0)}(m+n+1) \\
& + 9mnA_{0,0}B_{0,1}\psi^{(0)}(m+n+1) \\
& + 6mA_{0,0}B_{0,0}\psi^{(0)}(m+n+1) + 8mA_{0,1}B_{0,0} + 3mA_{0,0}B_{0,1} \\
& - 12n^2A_{0,1}B_{0,1} - 9n^2A_{0,0}B_{n,2} - 16nA_{0,1}B_{0,0} + 2A_{0,0}B_{0,0}\big), \\
\mathcal{B}_2^{(3)}(m,n) = {} & \frac{1}{18}\big(-24nA_{0,0}B_{0,0}H_mH_n + 12nA_{0,0}B_{0,1}H_m \\
& + 8nA_{0,0}B_{0,0}H_m\psi^{(0)}(m+n+1)
\end{aligned}
$$

$$
\begin{aligned}
&+ 18nA_{0,0}B_{0,0}H_n\psi^{(0)}(m+n+1)\\
&+ 8A_{0,0}B_{0,0}H_m + 6A_{0,0}B_{0,0}H_n + 24nA_{0,1}B_{0,0}H_n\\
&- 5nA_{0,0}B_{0,0}\psi^{(0)}(m+n+1)^2 - 6A_{0,0}B_{0,0}\psi^{(0)}(m+n+1)\\
&- 8nA_{0,1}B_{0,0}\psi^{(0)}(m+n+1) - 9nA_{0,0}B_{0,1}\psi^{(0)}(m+n+1)\\
&- 5nA_{0,0}B_{0,0}\psi^{(1)}(m+n+1) - 12nA_{0,1}B_{0,1} - 9nA_{0,0}B_{n,2}\\
&- 8A_{0,1}B_{0,0} - 3A_{0,0}B_{0,1}\big),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{B}_3^{(3)}(m,n) = \frac{1}{18}\big(&24A_{0,0}B_{0,0}H_mH_n - 8A_{0,0}B_{0,0}H_m\psi^{(0)}(m+n+1)\\
&- 18A_{0,0}B_{0,0}H_n\psi^{(0)}(m+n+1) - 12A_{0,0}B_{0,1}H_m\\
&- 24A_{0,1}B_{0,0}H_n + 5A_{0,0}B_{0,0}\psi^{(0)}(m+n+1)^2\\
&+ 8A_{0,1}B_{0,0}\psi^{(0)}(m+n+1)\\
&+ 9A_{0,0}B_{0,1}\psi^{(0)}(m+n+1) + 5A_{0,0}B_{0,0}\psi^{(1)}(m+n+1)\\
&+ 9A_{0,0}B_{n,2} + 12A_{0,1}B_{0,1}\big),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{B}_1^{(4)}(m,n) = \frac{1}{18}\big(&-n^2A_{0,0}B_{0,0}\psi^{(0)}(m+n+1)^3\\
&+ mnA_{0,0}B_{0,0}\psi^{(0)}(m+n+1)^3\\
&+ 2mA_{0,0}B_{0,0}\psi^{(0)}(m+n+1)^2\\
&- 3nA_{0,0}B_{0,0}\psi^{(0)}(m+n+1)^2\\
&+ 2n^2H_mA_{0,0}B_{0,0}\psi^{(0)}(m+n+1)^2\\
&- 2mnH_mA_{0,0}B_{0,0}\psi^{(0)}(m+n+1)^2\\
&+ 6n^2H_nA_{0,0}B_{0,0}\psi^{(0)}(m+n+1)^2\\
&- 6mnH_nA_{0,0}B_{0,0}\psi^{(0)}(m+n+1)^2\\
&- 2n^2A_{0,1}B_{0,0}\psi^{(0)}(m+n+1)^2\\
&+ 2mnA_{0,1}B_{0,0}\psi^{(0)}(m+n+1)^2\\
&- 3n^2A_{0,0}B_{0,1}\psi^{(0)}(m+n+1)^2\\
&+ 3mnA_{0,0}B_{0,1}\psi^{(0)}(m+n+1)^2\\
&- 4mH_mA_{0,0}B_{0,0}\psi^{(0)}(m+n+1)\\
&+ 8nH_mA_{0,0}B_{0,0}\psi^{(0)}(m+n+1)\\
&- 6mH_nA_{0,0}B_{0,0}\psi^{(0)}(m+n+1)\\
&+ 6nH_nA_{0,0}B_{0,0}\psi^{(0)}(m+n+1)\\
&- 12n^2H_mH_nA_{0,0}B_{0,0}\psi^{(0)}(m+n+1)\\
&+ 12mnH_mH_nA_{0,0}B_{0,0}\psi^{(0)}(m+n+1)
\end{aligned}
$$

$$
\begin{aligned}
&- 3n^2\psi^{(1)}(m+n+1)A_{0,0}B_{0,0}\psi^{(0)}(m+n+1) \\
&+ 3mn\psi^{(1)}(m+n+1)A_{0,0}B_{0,0}\psi^{(0)}(m+n+1) \\
&+ 4mA_{0,1}B_{0,0}\psi^{(0)}(m+n+1) - 8nA_{0,1}B_{0,0}\psi^{(0)}(m+n+1) \\
&+ 12n^2H_nA_{0,1}B_{0,0}\psi^{(0)}(m+n+1) \\
&- 12mnH_nA_{0,1}B_{0,0}\psi^{(0)}(m+n+1) \\
&+ 3mA_{0,0}B_{0,1}\psi^{(0)}(m+n+1) - 3nA_{0,0}B_{0,1}\psi^{(0)}(m+n+1) \\
&+ 6n^2H_mA_{0,0}B_{0,1}\psi^{(0)}(m+n+1) \\
&- 6mnH_mA_{0,0}B_{0,1}\psi^{(0)}(m+n+1) \\
&- 6n^2A_{0,1}B_{0,1}\psi^{(0)}(m+n+1) \\
&+ 6mnA_{0,1}B_{0,1}\psi^{(0)}(m+n+1) \\
&- 9n^2A_{0,0}B_{n,2}\psi^{(0)}(m+n+1) \\
&+ 9mnA_{0,0}B_{n,2}\psi^{(0)}(m+n+1) \\
&+ 4H_mA_{0,0}B_{0,0} + 12mH_mH_nA_{0,0}B_{0,0} - 24nH_mH_nA_{0,0}B_{0,0} \\
&- 6H_nA_{0,0}B_{0,0} + 2m\psi^{(1)}(m+n+1)A_{0,0}B_{0,0} \\
&- 3n\psi^{(1)}(m+n+1)A_{0,0}B_{0,0} \\
&+ 2n^2H_m\psi^{(1)}(m+n+1)A_{0,0}B_{0,0} \\
&- 2mnH_m\psi^{(1)}(m+n+1)A_{0,0}B_{0,0} \\
&+ 6n^2H_n\psi^{(1)}(m+n+1)A_{0,0}B_{0,0} \\
&- 6mnH_n\psi^{(1)}(m+n+1)A_{0,0}B_{0,0} \\
&- n^2\psi^{(2)}(m+n+1)A_{0,0}B_{0,0} + mn\psi^{(2)}(m+n+1)A_{0,0}B_{0,0} \\
&- 12mH_nA_{0,1}B_{0,0} + 24nH_nA_{0,1}B_{0,0} \\
&- 2n^2\psi^{(1)}(m+n+1)A_{0,1}B_{0,0} \\
&+ 2mn\psi^{(1)}(m+n+1)A_{0,1}B_{0,0} \\
&- 4A_{0,1}B_{0,0} - 6mH_mA_{0,0}B_{0,1} \\
&+ 12nH_mA_{0,0}B_{0,1} - 3n^2\psi^{(1)}(m+n+1)A_{0,0}B_{0,1} \\
&+ 3mn\psi^{(1)}(m+n+1)A_{0,0}B_{0,1} + 3A_{0,0}B_{0,1} + 6mA_{0,1}B_{0,1} \\
&- 12nA_{0,1}B_{0,1} + 9nA_{0,0}B_{n,2} + 18n^2H_mA_{0,0}B_{n,2} \\
&- 18mnH_mA_{0,0}B_{n,2} - 18n^2A_{0,1}B_{n,2} + 18mnA_{0,1}B_{n,2}\big),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{B}_2^{(4)}(m,n) = \frac{1}{18}\big(&-12A_{0,0}B_{0,0}H_mH_n + 18nA_{0,0}H_mB_{n,2} \\
&+ 2nA_{0,0}B_{0,0}H_m\psi^{(0)}(m+n+1)^2 \\
&+ 6nA_{0,0}B_{0,0}H_n\psi^{(0)}(m+n+1)^2
\end{aligned}
$$

$$
\begin{aligned}
&+ 4A_{0,0}B_{0,0}H_m\psi^{(0)}(m+n+1) \\
&- 12nA_{0,0}B_{0,0}H_mH_n\psi^{(0)}(m+n+1) \\
&+ 6A_{0,0}B_{0,0}H_n\psi^{(0)}(m+n+1) \\
&+ 12nA_{0,1}B_{0,0}H_n\psi^{(0)}(m+n+1) \\
&+ 6nA_{0,0}B_{0,1}H_m\psi^{(0)}(m+n+1) \\
&+ 2nA_{0,0}B_{0,0}H_m\psi^{(1)}(m+n+1) \\
&+ 6nA_{0,0}B_{0,0}H_n\psi^{(1)}(m+n+1) \\
&+ 6A_{0,0}B_{0,1}H_m + 12A_{0,1}B_{0,0}H_n \\
&- nA_{0,0}B_{0,0}\psi^{(0)}(m+n+1)^3 \\
&- 2A_{0,0}B_{0,0}\psi^{(0)}(m+n+1)^2 - 2nA_{0,1}B_{0,0}\psi^{(0)}(m+n+1)^2 \\
&- 3nA_{0,0}B_{0,1}\psi^{(0)}(m+n+1)^2 \\
&- 3nA_{0,0}B_{0,0}\psi^{(1)}(m+n+1)\psi^{(0)}(m+n+1) \\
&- 4A_{0,1}B_{0,0}\psi^{(0)}(m+n+1) - 3A_{0,0}B_{0,1}\psi^{(0)}(m+n+1) \\
&- 6nA_{0,1}B_{0,1}\psi^{(0)}(m+n+1) - 9nA_{0,0}B_{n,2}\psi^{(0)}(m+n+1) \\
&- 2A_{0,0}B_{0,0}\psi^{(1)}(m+n+1) - nA_{0,0}B_{0,0}\psi^{(2)}(m+n+1) \\
&- 2nA_{0,1}B_{0,0}\psi^{(1)}(m+n+1) - 3nA_{0,0}B_{0,1}\psi^{(1)}(m+n+1) \\
&- 18nA_{0,1}B_{n,2} - 6A_{0,1}B_{0,1}\big),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{B}_3^{(4)}(m,n) = \frac{1}{18}\big(&-18A_{0,0}H_mB_{n,2} - 2A_{0,0}B_{0,0}H_m\psi^{(0)}(m+n+1)^2 \\
&- 6A_{0,0}B_{0,0}H_n\psi^{(0)}(m+n+1)^2 \\
&+ 12A_{0,0}B_{0,0}H_mH_n\psi^{(0)}(m+n+1) \\
&- 12A_{0,1}B_{0,0}H_n\psi^{(0)}(m+n+1) \\
&- 6A_{0,0}B_{0,1}H_m\psi^{(0)}(m+n+1) \\
&- 2A_{0,0}B_{0,0}H_m\psi^{(1)}(m+n+1) \\
&- 6A_{0,0}B_{0,0}H_n\psi^{(1)}(m+n+1) + A_{0,0}B_{0,0}\psi^{(0)}(m+n+1)^3 \\
&+ 2A_{0,1}B_{0,0}\psi^{(0)}(m+n+1)^2 + 3A_{0,0}B_{0,1}\psi^{(0)}(m+n+1)^2 \\
&+ 3A_{0,0}B_{0,0}\psi^{(1)}(m+n+1)\psi^{(0)}(m+n+1) \\
&+ 6A_{0,1}B_{0,1}\psi^{(0)}(m+n+1) \\
&+ 9A_{0,0}B_{n,2}\psi^{(0)}(m+n+1) + A_{0,0}B_{0,0}\psi^{(2)}(m+n+1) \\
&+ 2A_{0,1}B_{0,0}\psi^{(1)}(m+n+1) + 3A_{0,0}B_{0,1}\psi^{(1)}(m+n+1) \\
&+ 18A_{0,1}B_{n,2}\big).
\end{aligned}
$$

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In the third and final part of the paper, as an application, we show how to use the new analytic tools, introduced in the previous parts, in order to study the quantum differential equations of Hirzebruch surfaces. For Hirzebruch surfaces diffeomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, this analysis reduces to the simpler quantum differential equation of $\mathbb{P}^1$. For Hirzebruch surfaces diffeomorphic to the blow-up of $\mathbb{P}^2$ in one point, the quantum differential equation is integrated via Laplace $(1, 2; 1/2, 1/3)$-multitransforms of solutions of the quantum differential equations of $\mathbb{P}^1$ and $\mathbb{P}^2$, respectively. This leads to explicit integral representations for the Stokes bases of solutions of the quantum differential equations, and finally to the proof of the Dubrovin conjecture for all Hirzebruch surfaces.

MEMOIRS OF THE EUROPEAN MATHEMATICAL SOCIETY

Giordano Cotti

Cyclic Stratum of Frobenius Manifolds, Borel–Laplace (α, β)-Multitransforms, and Integral Representations of Solutions of Quantum Differential Equations

In the first part of this paper, we introduce the notion of *cyclic stratum* of a Frobenius manifold M. This is the set of points of the extended manifold $\mathbb{C}^* \times M$ at which the unit vector field is a cyclic vector for the isomonodromic system defined by the flatness condition of the extended deformed connection. The study of the geometry of the complement of the cyclic stratum is addressed. We show that at points of the cyclic stratum, the isomonodromic system attached to M can be reduced to a scalar differential equation, called the *master differential equation* of M. In the case of Frobenius manifolds coming from Gromov–Witten theory, namely quantum cohomologies of smooth projective varieties, such a construction reproduces the notion of quantum differential equation.

In the second part of the paper, we introduce two multilinear transforms, called *Borel–Laplace (α, β)-multitransforms*, on spaces of Ribenboim formal power series with exponents and coefficients in an arbitrary finite-dimensional $\mathbb{C}$-algebra A. When A is specialized to the cohomology of smooth projective varieties, the integral forms of the Borel–Laplace (α, β)-multitransforms are used in order to rephrase the Quantum Lefschetz theorem. This leads to explicit Mellin–Barnes integral representations of solutions of the quantum differential equations for a wide class of smooth projective varieties, including Fano complete intersections in projective spaces.

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