DOCUMENTA MATHEMATICA SERIES

A Collection of Manuscripts Written in Honour of Kazuya Kato on the Occasion of His Fiftieth Birthday

Edited by
Spencer Bloch
Ivan Fesenko
Luc Illusie
Masato Kurihara
Shuji Saito
Takeshi Saito
Peter Schneider

DMV
EMS PRESS

Documenta Mathematica

Gegründet 1996 durch die
Deutsche Mathematiker-Vereinigung
Extra Volume

A Collection of Manuscripts
Written in Honour of

Kazuya Kato

on the Occasion of His Fiftieth Birthday

Editors:

S. Bloch, I. Fesenko, L. Illusie, M. Kurihara,
S. Saito, T. Saito, P. Schneider

Documenta Mathematica veröffentlicht Forschungsarbeiten aus allen mathematischen Gebieten und wird in traditioneller Weise referiert.

Artikel können als TEX-Dateien per E-Mail bei einem der Herausgeber eingereicht werden. Hinweise für die Vorbereitung der Artikel können unter der unten angegebenen WWW-Adresse gefunden werden.

Documenta Mathematica publishes research manuscripts out of all mathematical fields and is refereed in the traditional manner.

Manuscripts should be submitted as TEX -files by e-mail to one of the editors. Hints for manuscript preparation can be found under the following WWW-address.

`http://www.mathematik.uni-bielefeld.de/documenta`

ISSN 1431-0635 (Print), ISSN 1431-0643 (Internet)

SPARC Leading Edge

Documenta Mathematica is a Leading Edge Partner of SPARC, the Scholarly Publishing and Academic Resource Coalition of the Association of Research Libraries (ARL), Washington DC, USA.

Address of Technical Managing Editor: Ulf Rehmann, Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, D-33501 Bielefeld,

Documenta Mathematica

Extra Volume: Kazuya Kato's Fiftieth Birthday, 2003

Preface

The editors of this volume wish to express to Kazuya Kato their gratitude and admiration for his seminal contributions to arithmetic algebraic geometry and the deep influence he has been exerting in this area for over twenty years through his students and collaborators throughout the world.

S. Bloch, I. Fesenko, L. Illusie, M. Kurihara, S. Saito, T. Saito, P. Schneider

素数の歌

加藤　和也

素数の歌はとんかからり
耳をすませば聞こえます
楽しい歌が聞こえます

素数の歌はちんかからり
声をあわせて歌います
素数の国の愛の歌

素数の歌はぽんぽろり
素数は夢を見ています
あしたの夢を歌います

Prime Numbers

Kazuya Kato

The song of prime numbers sounds Tonnkarari,
We can hear if we keep our ears open,
We can hear their joyful song.

The song of prime numbers sounds Chinnkarari,
Prime numbers sing together in harmony
The song of love in the land of prime numbers.

The song of prime numbers sounds Ponnporori,
Prime numbers are seeing dreams,
They sing the dreams for the tomorrow.

Sosuu no uta wa tonnkarari
Mimi o sumaseba kikoe masu
Tanoshii uta ga kikoe masu

Sosuu no uta wa chinnkarari
Koe o awasete utai masu
Sosuu no kuni no ai no uta

Sosuu no uta wa ponnporori
Sosuu wa yume o mite imasu
Ashita no yume o utai masu

Alphabetic transcription of Kato's poem

Ramification of Local Fields with Imperfect Residue Fields II

Dedicated to Kazuya Kato on the occasion of his 50th birthday

Ahmed Abbes and Takeshi Saito

Received: October 10, 2002
Revised: August 6, 2003

Abstract. In [1], a filtration by ramification groups and its logarithmic version are defined on the absolute Galois group of a complete discrete valuation field without assuming that the residue field is perfect. In this paper, we study the graded pieces of these filtrations and show that they are abelian except possibly in the absolutely unramified and non-logarithmic case.

2000 Mathematics Subject Classification: primary 11S15, secondary 14G22
Keywords and Phrases: local fields, wild ramification, log structure, affinoid variety.

In the previous paper [1], a filtration by ramification groups and its logarithmic version are defined on the absolute Galois group G_K of a complete discrete valuation field K without assuming that the residue field is perfect. In this paper, we study the graded pieces of these filtrations and show that they are abelian except possibly in the absolutely unramified and non-logarithmic case. Let G_K^j $(j > 0, \in \mathbb{Q})$ denote the decreasing filtration by ramification groups and $G_{K,\log}^j$ $(j > 0, \in \mathbb{Q})$ be its logarithmic variant. We put $G_K^{j+} = \overline{\bigcup_{j'>j} G_K^{j'}}$ and $G_{K,\log}^{j+} = \overline{\bigcup_{j'>j} G_{K,\log}^{j'}}$. In [1], we show that the wild inertia subgroup $P \subset G_K$ is equal to $G_K^{1+} = G_{K,\log}^{0+}$. The main result is the following.

Theorem 1 *Let K be a complete discrete valuation field.*
1. (see Theorem 2.15) *Assume either K has equal characteristics $p > 0$ or K has mixed characteristic and p is not a prime element. Then, for a rational number $j > 1$, the graded piece $Gr^j G_K = G_K^j / G_K^{j+}$ is abelian and is a subgroup of the center of the pro-p-group G_K^{1+}/G_K^{j+}.*
2. (see Theorem 5.12) *For a rational number $j > 0$, the graded piece $Gr^j_{\log} G_K = G^j_{K,\log}/G^{j+}_{K,\log}$ is abelian and is a subgroup of the center of the pro-p-group $G^{0+}_{K,\log}/G^{j+}_{K,\log}$.*

The idea of the proof of 1 is the following. Under some finiteness assumption, denoted by (F), we define a functor $\bar{X}^j$ from the category of finite étale K-algebras with ramification bounded by $j+$ to the category of finite étale schemes over a certain tangent space Θ^j with continuous semi-linear action of G_K. For a finite Galois extension L of K with ramification bounded by $j+$, the image $\bar{X}^j(L)$ has two mutually commuting actions of $G = \mathrm{Gal}(L/K)$ and G_K. The arithmetic action of G_K comes from the definition of the functor $\bar{X}^j$ and the geometric action of G is defined by functoriality. Using these two commuting actions, we prove the assertion. The assumption that p is not a prime element is necessary in the construction of the functor $\bar{X}^j$.
In Section 1, for a rational number $j > 0$ and a smooth embedding of a finite flat O_K-algebra, we define its j-th tubular neighborhood as an affinoid variety. We also define its j-th twisted reduced normal cone.
We recall the definition of the filtration by ramification groups in Section 2.1 using the notions introduced in Section 1. In the equal characteristic case, under the assumption (F), we define a functor $\bar{X}^j$ mentioned above in Section 2.2 using j-th tubular neighborhoods. In the mixed characteristic case, we give a similar but subtler construction using the twisted normal cones, assuming further that the residue characteristic p is not a prime element of K in Section 2.3. Then, we prove Theorem 2.15 in Section 2.4. We also define a canonical surjection $\pi_1^{\mathrm{ab}}(\Theta^j) \to Gr^j G_K$ under the assumption (F).
After some preparations on generalities of log structures in Section 3, we study a logarithmic analogue in Sections 4 and 5. We define a canonical surjection $\pi_1^{\mathrm{ab}}(\Theta^j_{\log}) \to Gr^j_{\log} G_K$ under the assumption (F) and prove the logarithmic part, Theorem 5.12, of the main result in Section 5.2. Among other results, we compare the construction with the logarithmic construction given in [1] in Lemma 4.10. We also prove in Corollary 4.12 a logarithmic version of [1] Theorem 7.2 (see also Corollary 1.16).
In Section 6, assuming the residue field is perfect, we show that the surjection $\pi_1^{\mathrm{ab}}(\Theta^j_{\log}) \to Gr^j_{\log} G_K$ induces an isomorphism $\pi_1^{\mathrm{ab,gp}}(\Theta^j_{\log}) \to Gr^j_{\log} G_K$ where $\pi_1^{\mathrm{ab,gp}}(\Theta^j_{\log})$ denotes the quotient classifying the étale isogenies to $\Theta^j_{\log}$ regarded as an algebraic group.
When one of the authors (T.S.) started studing mathematics, Kazuya Kato, who was his adviser, suggested to read [13] and to study how to generalize it when the residue field is no longer assumed perfect. This paper is a partial

answer to his suggestion. The authors are very happy to dedicate this paper to him for his 51st anniversary.

Notation. Let K be a complete discrete valuation field, O_K be its valuation ring and F be its residue field of characteristic $p > 0$. Let $\bar{K}$ be a separable closure of K, $O_{\bar{K}}$ be the integral closure of O_K in $\bar{K}$, $\bar{F}$ be the residue field of $O_{\bar{K}}$, and $G_K = \mathrm{Gal}(\bar{K}/K)$ be the Galois group of $\bar{K}$ over K. Let π be a uniformizer of O_K and ord be the valuation of K normalized by $\mathrm{ord}\pi = 1$. We denote also by ord the unique extension of ord to $\bar{K}$.

1 Tubular neighborhoods for finite flat algebras

For a semi-local ring R, let $\mathfrak{m}_R$ denote the radical of R. We say that an O_K-algebra R is formally of finite type over O_K if R is semi-local, $\mathfrak{m}_R$-adically complete, Noetherian and the quotient $R/\mathfrak{m}_R$ is finite over F. An O_K-algebra R formally of finite type over O_K is formally smooth over O_K if and only if its factors are formally smooth. We say that an O_K-algebra R is topologically of finite type over O_K if R is π-adically complete, Noetherian and the quotient $R/\pi R$ is of finite type over F. For an O_K-algebra R formally of finite type over O_K, we put $\hat{\Omega}_{R/O_K} = \varprojlim_n \Omega_{(R/\mathfrak{m}_R^n)/O_K}$. For an O_K-algebra R topologically of finite type over O_K, we put $\hat{\Omega}_{R/O_K} = \varprojlim_n \Omega_{(R/\pi^n R)/O_K}$. Here and in the following, Ω denotes the module of differential 1-forms. For a surjection $R \to R'$ of rings, its formal completion is defined to be the projective limit $R^\wedge = \varprojlim_n R/(\mathrm{Ker}(R \to R'))^n$.
In this section, A will denote a finite flat O_K-algebra.

1.1 Embeddings of finite flat algebras

Definition 1.1 *1. Let A be a finite flat O_K-algebra and $\mathbf{A}$ be an O_K-algebra formally of finite type and formally smooth over O_K. We say that a surjection $\mathbf{A} \to A$ of O_K-algebras is an embedding if it induces an isomorphism $\mathbf{A}/\mathfrak{m}_{\mathbf{A}} \to A/\mathfrak{m}_A$.*
2. We define $\mathcal{E}mb_{O_K}$ to be the category whose objects and morphisms are as follows. An object of $\mathcal{E}mb_{O_K}$ is a triple $(\mathbf{A} \to A)$ where:

- *A is a finite flat O_K-algebra.*
- *$\mathbf{A}$ is an O_K-algebra formally of finite type and formally smooth over O_K.*
- *$\mathbf{A} \to A$ is an embedding.*

A morphism $(f, \mathbf{f}) : (\mathbf{A} \to A) \to (\mathbf{B} \to B)$ of $\mathcal{E}mb_{O_K}$ is a pair of O_K-homomorphisms $f : A \to B$ and $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ such that the diagram

$$\begin{array}{ccc} \mathbf{A} & \longrightarrow & A \\ {\scriptstyle \mathbf{f}}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbf{B} & \longrightarrow & B \end{array}$$

is commutative.
3. For a finite flat O_K-algebra A, let $\mathcal{E}mb_{O_K}(A)$ be the subcategory of $\mathcal{E}mb_{O_K}$ whose objects are of the form $(\mathbf{A} \to A)$ and morphisms are of the form $(\mathrm{id}_A, \mathbf{f})$.
4. We say that a morphism $(f, \mathbf{f}) : (\mathbf{A} \to A) \to (\mathbf{B} \to B)$ of $\mathcal{E}mb_{O_K}$ is finite flat if $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ is finite and flat and if the map $\mathbf{B} \otimes_{\mathbf{A}} A \to B$ is an isomorphism.

If $(\mathbf{A} \to A)$ is an embedding, the $\mathbf{A}$-module $\hat{\Omega}_{\mathbf{A}/O_K}$ is locally free of finite rank.

Lemma 1.2 *1. For a finite flat O_K-algebra A, the category $\mathcal{E}mb_{O_K}(A)$ is non-empty.*
2. For a morphism $f : A \to B$ of finite flat O_K-algebras and for embeddings $(\mathbf{A} \to A)$ and $(\mathbf{B} \to B)$, there exists a morphism $(f, \mathbf{f}) : (\mathbf{A} \to A) \to (\mathbf{B} \to B)$ lifting f.
3. For a morphism $f : A \to B$ of finite flat O_K-algebras, the following conditions are equivalent.
(1) The map $f : A \to B$ is flat and locally of complete intersection.
(2) Their exists a finite flat morphism $(f, \mathbf{f}) : (\mathbf{A} \to A) \to (\mathbf{B} \to B)$ of embeddings.

Proof. 1. Take a finite system of generators $t_1, \dots, t_n$ of A over O_K and define a surjection $O_K[T_1, \dots, T_n] \to A$ by $T_i \mapsto t_i$. Then the formal completion $\mathbf{A} \to A$ of $O_K[T_1, \dots, T_n] \to A$, where $\mathbf{A} = \varprojlim_m O_K[T_1, \dots, T_n]/(\mathrm{Ker}(O_K[T_1, \dots, T_n] \to A))^m$, is an embedding.
2. Since $\mathbf{A}$ is formally smooth over O_K and $\mathbf{B} = \varprojlim_n \mathbf{B}/I^n$ where $I = \mathrm{Ker}(\mathbf{B} \to B)$, the assertion follows.
3. (1)⇒(2). We may assume A and B are local. By 1 and 2, there exists a morphism $(f, \mathbf{f}) : (\mathbf{A} \to A) \to (\mathbf{B} \to B)$ lifting f. Replacing $\mathbf{B} \to B$ by the projective limit $\varprojlim_n (\mathbf{A}/\mathfrak{m}_{\mathbf{A}}^n \otimes_{O_K} \mathbf{B}/\mathfrak{m}_{\mathbf{B}}^n)^\wedge \to B/\mathfrak{m}_B^n$ of the formal completion $(\mathbf{A}/\mathfrak{m}_{\mathbf{A}}^n \otimes_{O_K} \mathbf{B}/\mathfrak{m}_{\mathbf{B}}^n)^\wedge \to B/\mathfrak{m}_B^n$ of the surjections $\mathbf{A}/\mathfrak{m}_{\mathbf{A}}^n \otimes_{O_K} \mathbf{B}/\mathfrak{m}_{\mathbf{B}}^n \to B/\mathfrak{m}_B^n$, we may assume that the map $\mathbf{A} \to \mathbf{B}$ is formally smooth. Since $A \to B$ is locally of complete intersection, the kernel of the surjection $\mathbf{B} \otimes_{\mathbf{A}} A \to B$ is generated by a regular sequence $(t_1, \dots, t_n)$. Take a lifting $(\tilde{t}_1, \dots \tilde{t}_n)$ in $\mathbf{B}$ and define a map $\mathbf{A}[[T_1, \dots, T_n]] \to \mathbf{B}$ by $T_i \mapsto t_i$. We consider an embedding $\mathbf{A}[[T_1, \dots, T_n]] \to A$ defined by the composition $\mathbf{A}[[T_1, \dots, T_n]] \to \mathbf{A} \to A$ sending T_i to 0. Replacing $\mathbf{A}$ by $\mathbf{A}[[T_1, \dots, T_n]]$, we obtain a map $(\mathbf{A} \to A) \to (\mathbf{B} \to B)$ such that the map $\mathbf{B} \otimes_{\mathbf{A}} A \to B$ is an isomorphism and $\dim \mathbf{A} = \dim \mathbf{B}$. By Nakayama's lemma, the map $\mathbf{A} \to \mathbf{B}$ is finite. Hence the map $\mathbf{A} \to \mathbf{B}$ is flat by EGA Chap 0_{IV} Corollaire (17.3.5) (ii).
(2)⇒(1). Since $\mathbf{A}$ and $\mathbf{B}$ are regular, $\mathbf{B}$ is locally of complete intersection over $\mathbf{A}$. Since $\mathbf{B}$ is flat over $\mathbf{A}$, B is also flat and locally of complete intersection over A. □

The base change of an embedding by an extension of complete discrete valuation fields is defined as follows.

LEMMA 1.3 *Let K' be a complete discrete valuation field and $K \to K'$ be a morphism of fields inducing a local homomorphism $O_K \to O_{K'}$. Let $(\mathbf{A} \to A)$ be an object of $\mathcal{E}mb_{O_K}$. We define $\mathbf{A}\hat{\otimes}_{O_K}O_{K'}$ to be the projective limit $\varprojlim_n(\mathbf{A}/\mathfrak{m}_{\mathbf{A}}^n \otimes_{O_K} O_{K'})$. Then the $O_{K'}$-algebra $\mathbf{A}\hat{\otimes}_{O_K}O_{K'}$ is formally of finite type and formally smooth over $O_{K'}$. The natural surjection $\mathbf{A}\hat{\otimes}_{O_K}O_{K'} \to A\hat{\otimes}_{O_K}O_{K'}$ defines an object $(\mathbf{A}\hat{\otimes}_{O_K}O_{K'} \to A \otimes_{O_K} O_{K'})$ of $\mathcal{E}mb_{O_{K'}}$.*

Proof. The O_K-algebra $\mathbf{A}$ is finite over the power series ring $O_K[[T_1, \ldots, T_n]]$ for some $n \geq 0$. Hence the $O_{K'}$-algebra $\mathbf{A}\hat{\otimes}_{O_K}O_{K'}$ is finite over $O_{K'}[[T_1, \ldots, T_n]]$ and is formally of finite type over $O_{K'}$. The formal smoothness is clear from the definition. The rest is clear. □.

For an object $(\mathbf{A} \to A)$ of $\mathcal{E}mb_{O_K}$, we let the object $(\mathbf{A}\hat{\otimes}_{O_K}O_{K'} \to A\otimes_{O_K}O_{K'})$ of $\mathcal{E}mb_{O_{K'}}$ defined in Lemma 1.3 denoted by $(\mathbf{A} \to A)\hat{\otimes}_{O_K}O_{K'}$. By sending $(\mathbf{A} \to A)$ to $(\mathbf{A} \to A)\hat{\otimes}_{O_K}O_{K'}$, we obtain a functor $\hat{\otimes}_{O_K}O_{K'} : \mathcal{E}mb_{O_K} \to \mathcal{E}mb_{O_{K'}}$. If K' is a finite extension of K, we have $\mathbf{A}\hat{\otimes}_{O_K}O_{K'} = \mathbf{A} \otimes_{O_K} O_{K'}$.

1.2 TUBULAR NEIGHBORHOODS FOR EMBBEDINGS

Let $(\mathbf{A} \to A)$ be an object of $\mathcal{E}mb_{O_K}$ and I be the kernel of the surjection $\mathbf{A} \to A$. Mimicing [3] Chapter 7, for a pair of positive integers $m, n > 0$, we define an O_K-algebra $\mathcal{A}^{m/n}$ topologically of finite type as follows. Let $\mathbf{A}[I^n/\pi^m]$ be the subring of $\mathbf{A} \otimes_{O_K} K$ generated by $\mathbf{A}$ and the elements f/π^m for $f \in I^n$ and let $\mathcal{A}^{m/n}$ be its π-adic completion. For two pairs of positive integers m, n and m', n', if m' is a multiple of m and if $m'/n' \leq m/n$, we have an inclusion $\mathbf{A}[I^{n'}/\pi^{m'}] \subset \mathbf{A}[I^n/\pi^m]$. It induces a continuous homomorphism $\mathcal{A}^{m'/n'} \to \mathcal{A}^{m/n}$. Then we have the following.

LEMMA 1.4 *Let $(\mathbf{A} \to A)$ be an object of $\mathcal{E}mb_{O_K}$ and $m, n > 0$ be a pair of positive integers. Then,*

1. The O_K-algebra $\mathcal{A}^{m/n}$ is topologically of finite type over O_K. The tensor product $\mathcal{A}_K^{m/n} = \mathcal{A}^{m/n} \otimes_{O_K} K$ is an affinoid algebra over K.

2. The map $\mathbf{A} \to \mathcal{A}^{m/n}$ is continuous with respect to the $\mathfrak{m}_{\mathbf{A}}$-adic topology on $\mathbf{A}$ and the π-adic topology on $\mathcal{A}^{m/n}$.

3. Let m', n' be another pair of positive integers and assume that m' is a multiple of m and $j' = m'/n' \leq j = m/n$. Then, by the map $X^{m/n} = \mathrm{Sp}\ \mathcal{A}_K^{m/n} \to X^{m'/n'} = \mathrm{Sp}\ \mathcal{A}_K^{m'/n'}$ induced by the inclusion $\mathbf{A}[I^{n'}/\pi^{m'}] \subset \mathbf{A}[I^n/\pi^m]$, the affinoid variety $X^{m/n}$ is identified with a rational subdomain of $X^{m'/n'}$.

4. The affinoid variety $X^{m/n} = \mathrm{Sp}\ \mathcal{A}_K^{m/n}$ depends only on the ratio $j = m/n$.

The proof is similar to that of [3] Lemma 7.1.2.

Proof. 1. Since the O_K-algebra $\mathcal{A}^{m/n}$ is π-adically complete, it is sufficient to show that the quotient $\mathbf{A}[I^n/\pi^m]/(\pi)$ is of finite type over F. Since it is finitely generated over $\mathbf{A}/(\pi, I^n)$ and $\mathbf{A}/(\pi, I) = A/(\pi)$ is finite over F, the assertion follows.

2. Since $A/\pi = \mathbf{A}/(\pi, I)$ is of finite length, a power of $\mathfrak{m}_{\mathbf{A}}$ is in (π^m, I^n). Since the image of (π^m, I^n) in $\mathcal{A}^{m/n}$ is in $\pi^m\mathcal{A}^{m/n}$, the assertion follows.
3. Take a system of generators $f_1, \ldots, f_N$ of I^n and define a surjection $\mathbf{A}[I^{n'}/\pi^{m'}][T_1, \ldots, T_N]/(\pi^m T_i - f_i) \to \mathbf{A}[I^n/\pi^m]$ by sending T_i to f_i/π^m. Since it induces an isomorphism after tensoring with K, its kernel is annihilated by a power of π. Hence it induces an isomorphism $\mathcal{A}_K^{m'/n'}\langle T_1, \ldots, T_N\rangle/(\pi^m T_i - f_i, i = 1, \ldots, N) \to \mathcal{A}_K^{m/n}$.
4. Further assume $m/n = m'/n'$ and put $k = m'/m$. Let $f_1, \ldots, f_N \in I^n$ be a system of generators of I^n as above. Then $\mathbf{A}[I^n/\pi^m]$ is generated by $(f_1/\pi^m)^{k_1} \cdots (f_N/\pi^m)^{k_N}, 0 \leq k_i < k$ as an $\mathbf{A}[I^{n'}/\pi^{m'}]$-module. Hence the cokernel of the inclusion $\mathcal{A}^{m'/n'} \to \mathcal{A}^{m/n}$ is annihilated by a power of π and the assertion follows. □

If $\mathbf{A} = O_K[[T_1, \ldots, T_N]]$ and $I = (T_1, \ldots, T_N)$, the ring $\mathcal{A}^{m/1}$ is isomorphic to the π-adic completion of $O_K[T_1/\pi^m, \ldots, T_N/\pi^m]$ and is denoted by $O_K\langle T_1/\pi^m, \ldots, T_N/\pi^m\rangle$. By Lemma 1.4.4, the integral closure $\mathcal{A}^j$ of $\mathcal{A}^{m/n}$ in the affinoid algebra $\mathcal{A}^{m/n} \otimes_{O_K} K$ depends only on $j = m/n$.

Definition 1.5 *Let $(\mathbf{A} \to A)$ be an object of $\mathcal{E}mb_{O_K}$ and $j > 0$ be a rational number. We define $\mathcal{A}^j$ to be the integral closure of $\mathcal{A}^{m/n}$ for $j = m/n$ in the affinoid algebra $\mathcal{A}^{m/n} \otimes_{O_K} K$ and define the j-th tubular neighborhood $X^j(\mathbf{A} \to A)$ to be the affinoid variety* Sp $\mathcal{A}_K^j$.

In the case $\mathbf{A} = O_K[[T_1, \ldots, T_n]]$ and the map $\mathbf{A} \to A = O_K$ is defined by sending T_i to 0, the affinoid variety $X^j(\mathbf{A} \to A)$ is the n-dimensional polydisk $D(0, \pi^j)^n$ of center 0 and of radius π^j. For each positive rational number $j > 0$, the construction attaching the j-th tubular neighboorhood $X^j(\mathbf{A} \to A)$ to an object $(\mathbf{A} \to A)$ of $\mathcal{E}mb_{O_K}$ defines a functor

$$X^j : \mathcal{E}mb_{O_K} \to (\text{Affinoid}/K)$$

to the category of affinoid varieties over K. For $j' \leq j$, we have a natural morphism $X^j \to X^{j'}$ of functors. A finite flat morphism of embeddings induces a finite flat morphism of affinoid varieties.

Lemma 1.6 *Let $j > 0$ be a positive rational number and $(\mathbf{A} \to A) \to (\mathbf{B} \to B)$ be a finite and flat morphism in $\mathcal{E}mb_{O_K}$. Then, the induced map $f^j : X^j(\mathbf{B} \to B) \to X^j(\mathbf{A} \to A)$ is a finite flat map of affinoid varieties.*

Proof. Let I and $J = IB$ be the kernels of the surjections $\mathbf{A} \to A$ and $\mathbf{B} \to B$. Since the map $\mathbf{A} \to \mathbf{B}$ is flat, it induces isomorphisms $\mathbf{B} \otimes_{\mathbf{A}} \mathbf{A}[I^n/\pi^m] \to \mathbf{B}[J^n/\pi^m]$ and $\mathbf{B} \otimes_{\mathbf{A}} \mathcal{A}_K^j \to \mathcal{B}_K^j$. The assertion follows from this immediately. □

For an extension K' of complete discrete valuation field K, the construction of j-th tubular neighborhoods commutes with the base change. More precisely, we have the following. Let K' be a complete discrete valuation field and $K \to K'$ be a morphism of fields inducing a local homomorphism $O_K \to O_{K'}$. Then by sending an affinoid variety Sp $\mathcal{A}_K$ over K to the affinoid variety Sp $\mathcal{A}_K\hat{\otimes}_K K'$ over K', we obtain a functor $\hat{\otimes}_K K'$: (Affinoid/K) $\to$ (Affinoid/K') (see [2] 9.3.6). Let e be the ramification index $e_{K'/K}$ and $j>0$ be a positive rational number. Then the canonical map $\mathbf{A}\to\mathbf{A}\hat{\otimes}_{O_K}O_{K'}$ induces an isomorphism $X^j(\mathbf{A}\to A)\hat{\otimes}_K K'\to X^{ej}((\mathbf{A}\to A)\hat{\otimes}_{O_K}O_{K'})$ of affinoid varieties over K'. In other words, we have a commutative diagram of functors

$$\begin{array}{ccc} X^j:\mathcal{E}mb_{O_K} & \longrightarrow & (\text{Affinoid}/K) \\ {\scriptstyle \hat{\otimes}_{O_K}O_{K'}}\big\downarrow & & \big\downarrow{\scriptstyle \hat{\otimes}_K K'} \\ X^{ej}:\mathcal{E}mb_{O_K} & \longrightarrow & (\text{Affinoid}/K'). \end{array}$$

Lemma 1.7 *For a rational number $j>0$, the affinoid algebra $\mathcal{A}^j_K$ is smooth over K.*

Proof. By the commutative diagram above, it is sufficient to show that there is a finite separable extension K' of K such that the base change $X^j(\mathbf{A}\to A)\otimes_K K' = X^j(\mathbf{A}\otimes_{O_K}O_{K'}\to A\otimes_{O_K}O_{K'})$ is smooth over K'. Replacing K by K' and separating the factors of A, we may assume $A/\mathfrak{m}_A=F$. Then we also have $\mathbf{A}/\mathfrak{m}_{\mathbf{A}}=F$ and an isomorphism $O_K[[T_1,\ldots,T_n]]\to\mathbf{A}$. We define an object $(\mathbf{A}\to O_K)$ of $\mathcal{E}mb_{O_K}$ by sending $T_i\in\mathbf{A}$ to 0. Let I and I' be the kernel of $\mathbf{A}\to A$ and $\mathbf{A}\to O_K$ respectively and put $j=m/n$. Since $\mathbf{A}/(\pi^m,I^n)$ is of finite length, there is an integer $n'>0$ such that $I'^{n'}\subset(\pi^m,I^n)$. Then we have an inclusion $\mathbf{A}[I'^{n'}/\pi^m]\to\mathbf{A}[I^n/\pi^m]$ and hence a map $X^{m/n}(\mathbf{A}\to A)\to X^{m/n'}(\mathbf{A}\to O_K)$. By the similar argument as in the proof of Lemma 1.4.3, the affinoid variety $X^{m/n}(\mathbf{A}\to A)$ is identified with a rational subdomain of $X^{m/n'}(\mathbf{A}\to O_K)$. Since the affinoid variety $X^{m/n'}(\mathbf{A}\to O_K)$ is a polydisk, the assertion follows. □

By Lemma 1.7, the j-th tubular neighborhoods in fact define a functor

$$X^j:\mathcal{E}mb_{O_K}\longrightarrow(\text{smoooth Affinoid}/K)$$

to the category of smooth affinoid varieties over K. Also by Lemma 1.7, $\hat{\Omega}_{\mathcal{A}^j/O_K}\otimes K$ is a locally free $\mathcal{A}^j_K$-module.

An idea behind the definition of the j-th tubular neighborhood is the following description of the valued points. Let $(\mathbf{A}\to A)$ be an object of $\mathcal{E}mb_{O_K}$ and $j>0$ be a rational number. Let $\mathcal{A}^j_K$ be the affinoid algebra defining the affinoid variety $X^j(\mathbf{A}\to A)$ and let $X^j(\mathbf{A}\to A)(\bar{K})$ be the set of $\bar{K}$-valued points. Since a continuous homomorphism $\mathcal{A}^j_K\to\bar{K}$ is determined by the induced map $\mathbf{A}\to O_{\bar{K}}$, we have a natural injection $X^j(\mathbf{A}\to A)(\bar{K})\to Hom_{\text{cont}.O_K\text{-alg}}(\mathbf{A},O_{\bar{K}})$. The surjection $\mathbf{A}\to A$ induces an injection

$$(1.8.0)\qquad Hom_{O_K\text{-alg}}(A,O_{\bar{K}})\longrightarrow X^j(\mathbf{A}\to A)(\bar{K}).$$

For a rational number $j>0$, let $\mathfrak{m}^j$ denote the ideal $\mathfrak{m}^j=\{x\in\bar{K};\text{ord}x\geq j\}$. We naturally identify the set $Hom_{O_K\text{-alg}}(A,O_{\bar{K}}/\mathfrak{m}^j)$ of O_K-algebra homomorphisms with a subset of the set $Hom_{\mathrm{cont}.O_K\text{-alg}}(\mathbf{A}, O_{\bar K}/\mathfrak{m}^j)$ of continuous O_K-algebra homomorphisms.

Lemma 1.8 *Let* $(\mathbf{A} \to A)$ *be an object of* $\mathcal{E}mb_{O_K}$ *and* $j > 0$ *be a rational number. Then by the injection* $X^j(\mathbf{A} \to A)(\bar K) \to Hom_{\mathrm{cont}.O_K\text{-}alg}(\mathbf{A}, O_{\bar K})$ *above, the set* $X^j(\mathbf{A} \to A)(\bar K)$ *is identified with the inverse image of the subset* $Hom_{O_K\text{-}alg}(A, O_{\bar K}/\mathfrak{m}^j)$ *by the projection* $Hom_{cont.O_K\text{-}alg}(\mathbf{A}, O_{\bar K}) \to Hom_{cont.O_K\text{-}alg}(\mathbf{A}, O_{\bar K}/\mathfrak{m}^j)$*. In other words, we have a cartesian diagram*

$$\begin{array}{ccc} X^j(\mathbf{A} \to A)(\bar K) & \longrightarrow & Hom_{\mathrm{cont}.O_K\text{-alg}}(\mathbf{A}, O_{\bar K}) \\ \downarrow & & \downarrow \\ Hom_{O_K\text{-alg}}(A, O_{\bar K}/\mathfrak{m}^j) & \longrightarrow & Hom_{\mathrm{cont}.O_K\text{-alg}}(\mathbf{A}, O_{\bar K}/\mathfrak{m}^j). \end{array} \tag{1.8.1}$$

The arrows are compatible with the natural G_K*-action.*

Proof. Let $j = m/n$. By the definition of $\mathcal{A}^{m/n}$, a continuous morphism $\mathbf{A} \to O_{\bar K}$ is extended to $\mathcal{A}^j_K \to \bar K$, if and only if the image of I^n is contained in the ideal (π^m). Hence the assertion follows. □

For an affinoid variety X over K, let $\pi_0(X_{\bar K})$ denote the set $\varprojlim_{K'/K} \pi_0(X_{K'})$ of geometric connected components, where K' runs over finite extensions of K in $\bar K$. The set $\pi_0(X_{\bar K})$ is finite and carries a natural continuous right action of the absolute Galois group G_K. To get a left action, we let $\sigma \in G_K$ act on $X_{\bar K}$ by σ^{-1}. The natural map $X^j(\mathbf{A} \to A)(\bar K) \to \pi_0(X_{\bar K})$ is compatible with this left G_K-action. Let G_K-(Finite Sets) denote the category of finite sets with a continuous left action of G_K and let (Finite Flat/O_K) be the category of finite flat O_K-algebras. Then, for a rational number $j > 0$, we obtain a sequence of functors

$$\begin{array}{l} (\text{Finite Flat}/O_K) \longleftarrow \mathcal{E}mb_{O_K} \xrightarrow{X^j} \\ \qquad\qquad (\text{smooth Affinoid}/K) \xrightarrow{X \mapsto \pi_0(X_{\bar K})} G_K\text{-(Finite Sets)}. \end{array}$$

We show that the composition $\mathcal{E}mb_{O_K} \to G_K$-(Finite Sets) induces a functor (Finite Flat/O_K) $\to G_K$-(Finite Sets).

Lemma 1.9 *Let* $j > 0$ *be a positive rational number.*

1. Let $(\mathbf{A} \to A)$ *be an embedding. Then, the map* $X^j(\mathbf{A} \to A)(\bar K) \to Hom_{O_K\text{-alg}}(A, O_{\bar K}/\mathfrak{m}^j)$ (1.8.1) *induces a surjection*

$$Hom_{O_K\text{-alg}}(A, O_{\bar K}/\mathfrak{m}^j) \longrightarrow \pi_0(X^j(\mathbf{A} \to A)_{\bar K}). \tag{1.9.1}$$

2. Let $(\mathbf{A} \to A)$ *and* $(\mathbf{A}' \to A)$ *be embeddings. Then, there exists a unique bijection* $\pi_0(X^j(\mathbf{A} \to A)_{\bar K}) \to \pi_0(X^j(\mathbf{A}' \to A)_{\bar K})$ *such that the diagram*

$$\begin{array}{ccc} Hom_{O_K\text{-alg}}(A, O_{\bar K}/\mathfrak{m}^j) & \longrightarrow & \pi_0(X^j(\mathbf{A} \to A)_{\bar K}) \\ \| & & \downarrow \\ Hom_{O_K\text{-alg}}(A, O_{\bar K}/\mathfrak{m}^j) & \longrightarrow & \pi_0(X^j(\mathbf{A}' \to A)_{\bar K}) \end{array} \tag{1.9.2}$$

is commutative.
3. Let $(f, \mathbf{f}) : (\mathbf{A} \to A) \to (\mathbf{B} \to B)$ *be a morphism of* $\mathcal{E}mb_{O_K}$. *Then, the induced map* $\pi_0(X^j(\mathbf{B} \to B)_{\bar{K}}) \to \pi_0(X^j(\mathbf{A} \to A)_{\bar{K}})$ *depends only on* f.
4. Let $(f, \mathbf{f}) : (\mathbf{A} \to O_K) \to (\mathbf{B} \to B)$ *be a finite flat morphism of* $\mathcal{E}mb_{O_K}$. *Then the map* (1.8.0) *induces a surjection*

$$(1.9.3) \qquad Hom_{O_K\text{-}alg}(B, O_{\bar{K}}) \longrightarrow \pi_0(X^j(\mathbf{B} \to B)_{\bar{K}}).$$

Proof. 1. The fibers of the map $Hom_{\text{cont.}O_K\text{-alg}}(\mathbf{A}, O_{\bar{K}}) \to Hom_{\text{cont.}O_K\text{-alg}}(\mathbf{A}, O_{\bar{K}}/\mathfrak{m}^j)$ are $\bar{K}$-valued points of polydisks. Hence the surjection $X^j(\mathbf{A} \to A)(\bar{K}) \to Hom_{\text{cont.}O_K\text{-alg}}(A, O_{\bar{K}}/\mathfrak{m}^j)$ induces a surjection $Hom_{\text{cont.}O_K\text{-alg}}(A, O_{\bar{K}}/\mathfrak{m}^j) \to \pi_0(X^j(\mathbf{A} \to A)_{\bar{K}})$ by Lemma 1.8.
2. By 1 and Lemma 1.2.2, there exists a unique surjection $\pi_0(X^j(\mathbf{A} \to A)_{\bar{K}}) \to \pi_0(X^j(\mathbf{A}' \to A)_{\bar{K}})$ such that the diagram (1.9.2) is commutative. Switching $\mathbf{A} \to A$ and $\mathbf{A}' \to A$, we obtain the assertion.
3. In the commutative diagram

$$\begin{array}{ccc} Hom_{\text{cont.}O_K\text{-alg}}(B, O_{\bar{K}}/\mathfrak{m}^j) & \longrightarrow & \pi_0(X^j(\mathbf{B} \to B)_{\bar{K}}) \\ {\scriptstyle f^*}\downarrow & & \downarrow \\ Hom_{\text{cont.}O_K\text{-alg}}(A, O_{\bar{K}}/\mathfrak{m}^j) & \longrightarrow & \pi_0(X^j(\mathbf{A} \to A)_{\bar{K}}), \end{array}$$

the horizontal arrows are surjective by 1. Hence the assertion follows.
4. The map $f^j : X^j(\mathbf{B} \to B) \to X^j(\mathbf{A} \to O_K)$ is finite and flat by Lemma 1.6. Let $y : X^j(\mathbf{A} \to O_K)(\bar{K})$ be the point corresponding to the map $\mathbf{A} \to O_K$. Then the fiber $(f^j)^{-1}(y)$ is identified with the set $Hom_{O_K\text{-alg}}(B, O_{\bar{K}})$. Since $X^j(\mathbf{A} \to O_K)_{\bar{K}}$ is isomorphic to a disk and is connected, the assertion follows.
□

For a rational number $j > 0$ and a finite flat O_K-algebra A, we put

$$\Psi^j(A) = \varprojlim_{(\mathbf{A} \to A) \in \mathcal{E}mb_{O_K}(A)} \pi_0(X^j(\mathbf{A} \to A)_{\bar{K}}).$$

By Lemmas 1.2.1 and 1.9.2, the projective system in the right is constant. Further by Lemma 1.9.3, we obtain a functor

$$\Psi^j : (\text{Finite Flat}/O_K) \longrightarrow G_K\text{-(Finite Sets)}$$

sending a finite flat O_K-algebra A to $\Psi^j(A)$. Let Ψ : (Finite Flat/O_K) $\to$ G_K-(Finite Sets) be the functor defined by $\Psi(A) = Hom_{O_K\text{-alg}}(A, \bar{K})$. Then, the map (1.9.1) induces a map $\Psi \to \Psi^j$ of functors.

1.3 Stable normalized integral models and their closed fibers

We briefly recall the stable normalized integral model of an affinoid variety and its closed fiber (cf. [1] Section 4). It is based on the finiteness theorem of Grauert-Remmert.

THEOREM 1.10 (Finiteness theorem of Grauert-Remmert, [1] Theorem 4.2) *Let $\mathcal{A}$ be an O_K-algebra topologically of finite type. Assume that the generic fiber $\mathcal{A}_K = \mathcal{A} \otimes_{O_K} K$ is geometrically reduced. Then,*
1. There exists a finite separable extension K' of K such that the geometric closed fiber $\mathcal{A}_{O_{K'}} \otimes_{O_{K'}} \bar{F}$ of the integral closure $\mathcal{A}_{O_{K'}}$ of $\mathcal{A}$ in $\mathcal{A} \otimes_{O_K} K'$ is reduced.
2. Assume further that $\mathcal{A}$ is flat over O_K and that the geometric closed fiber $\mathcal{A} \otimes_{O_K} \bar{F}$ is reduced. Let K' be an extension of complete discrete valuation field over K and π' be a prime element of K'. Then the π'-adic completion of the base change $\mathcal{A} \otimes_{O_K} O_{K'}$ is integrally closed in $\mathcal{A} \otimes_{O_K} K'$.

Let $\mathcal{A}$ be an O_K-algebra topologically of finite type such that $\mathcal{A}_K$ is smooth. If a finite separable extension K' satisfies the condition in Theorem 1.10.1, we say that the integral closure $\mathcal{A}_{O_{K'}}$ of $\mathcal{A}$ in $\mathcal{A}_{K'}$ is a stable normalized integral model of the affinoid variety $X_K = \mathrm{Sp}\ \mathcal{A}_K$ and that the stable normalized integral model is defined over K'. The geometric closed fiber $\bar{X} = \mathrm{Spec}\ \mathcal{A}_{O_{K'}} \otimes_{O_{K'}} \bar{F}$ of a stable normalized integral model is independent of the choice of an extension K' over which a stable normalized integral model is defined, by Theorem 1.10.2. Hence, the scheme $\bar{X}$ carries a natural continuous action of the absolute Galois group $G_K = \mathrm{Gal}(\bar{K}/K)$ compatible with its action on $\bar{F}$.
The construction above defines a functor as follows. Let G_K-$(\mathrm{Aff}/\bar{F})$ denote the category of affine schemes of finite type over $\bar{F}$ with a semi-linear continuous action of the absolute Galois group G_K. More precisely, an object is an affine scheme Y over $\bar{F}$ with an action of G_K compatible with the action of G_K on $\bar{F}$ satisfying the following property: There exist a finite Galois extension K' of K in $\bar{K}$, an affine scheme $Y_{K'}$ of finite type over the residue field F' of K', an action of $\mathrm{Gal}(K'/K)$ on $Y_{K'}$ compatible with the action of $\mathrm{Gal}(K'/K)$ on F' and a G_K-equivariant isomorphism $Y_{K'} \otimes_{F'} \bar{F} \to Y$. Then Theorem 1.10 implies that the geometric closed fiber of a stable normalized integral model defines a functor

$$(\text{smooth Affinoid}/K) \to G_K\text{-}(\mathrm{Aff}/\bar{F}) : X \mapsto \bar{X}.$$

COROLLARY 1.11 *Let $\mathcal{A}$ be an O_K-algebra topologically of finite type such that the generic fiber $\mathcal{A}_K$ is geometrically reduced as in Theorem 1.10. Let $X_K =$ $\mathrm{Sp}\ \mathcal{A}_K$ be the affinoid variety and $X_{\bar{F}}$ be the geometric closed fiber of the stable normalized integral model. Then the natural map $\pi_0(X_{\bar{F}}) \to \pi_0(X_{\bar{K}})$ is a bijection.*

Proof. Replacing $\mathcal{A}$ by its image in $\mathcal{A}_K$, we may assume $\mathcal{A}$ is flat over O_K. Let K' be a finite separable extension of K in $\bar{K}$ such that the stable normalized integral model $\mathcal{A}_{O_{K'}}$ is defined over K'. Then since $\mathcal{A}_{O_{K'}}$ is π-adically complete, the canonical maps $\pi_0(\mathrm{Spec}\mathcal{A}_{O_{K'}}) \to \pi_0(\mathrm{Spec}(\mathcal{A}_{O_{K'}} \otimes_{O_{K'}} F'))$ is bijective. Since the idempotents of $\mathcal{A}_{K'}$ are in $\mathcal{A}_{O_{K'}}$, the canonical maps $\pi_0(\mathrm{Spec}\mathcal{A}_{O_{K'}}) \to \pi_0(\mathrm{Spec}\mathcal{A}_{K'})$ is also bijective. By taking the limit, we obtain the assertion. □

By Corollary 1.11, the functor (smooth Affinoid/K) $\to G_K$-(Finite Sets) sending a smooth affinoid variety X to $\pi_0(X_{\bar K})$ may be also regarded as the composition of the functors

$$\text{(smooth Affinoid}/K) \xrightarrow{X\mapsto \bar X} G_K\text{-(Aff}/\bar F) \xrightarrow{\pi_0} G_K\text{-(Finite Sets)}\cdot$$

Lemma 1.12 *Let $j>0$ be a positive rational number and $(f,\mathbf{f}) : (\mathbf{A}\to O_K)\to(\mathbf{B}\to B)$ be a finite flat morphism of $\mathcal{E}mb_{O_K}$. Let $f^j : X^j(\mathbf{B}\to B)\to X^j(\mathbf{A}\to O_K)$ be the induced map and $\bar f^j : \bar X^j(\mathbf{B}\to B)\to \bar X^j(\mathbf{A}\to O_K)$ be its reduction. Let $y\in X^j(\mathbf{A}\to O_K)(\bar K)$ be the point corresponding to $\mathbf{A}\to A=O_K\to\bar K$ and $\bar y\in \bar X^j(\mathbf{A}\to O_K)$ be its specialization. Then the surjections $(f^j)^{-1}(y)=Hom_{O_K\text{-}alg}(B,O_{\bar K})\to\pi_0(X^{j'}(\mathbf{B}\to B)_{\bar K})$ (1.9.3) and the specialization map $(f^j)^{-1}(y)\to(\bar f^j)^{-1}(\bar y)$ induces a bijection*

(1.12.1) $$\varinjlim_{j'>j}\pi_0(X^{j'}(\mathbf{B}\to B)_{\bar K}) \longrightarrow (\bar f^j)^{-1}(\bar y).$$

Proof. The map $(f^j)^{-1}(y)\to\pi_0(X^{j'}(\mathbf{B}\to B)_{\bar K})$ is a surjection of finite sets by Lemma 1.9.4. Hence there exists a rational number $j'>j$ such that the surjection $\pi_0(X^{j'}(\mathbf{B}\to B)_{\bar K})\to\varinjlim_{j''>j}\pi_0(X^{j''}(\mathbf{B}\to B)_{\bar K})$ is a bijection. Let K' be a finite separable extension such that the surjection $\pi_0(X^{j'}(\mathbf{B}\to B)_{\bar K})\to\pi_0(X^{j'}(\mathbf{B}\to B)_{K'})$ is a bijection and that the stable normalized integral models $\mathcal{B}^j_{O_{K'}}$ of $X^j(\mathbf{B}\to B)$ is defined over K'. Enlarging K' further if necessary, we assume that $e'j$ is an integer where $e'=e_{K'/K}$ is the ramification index. Then the integral model $\mathcal{A}^j_{O_{K'}}$ of $X^j(\mathbf{A}\to O_K)$ is also defined over K'. If $O_K[[T_1,\ldots,T_n]]\to\mathbf{A}$ is an isomorphism such that the kernel of $\mathbf{A}\to O_K$ is generated by $T_1,\ldots,T_n$ and π' is a prime element of K', it induces an isomorphism $O_{K'}\langle T_1/\pi'^{e'j},\ldots,T_n/\pi'^{e'j}\rangle\to\mathcal{A}^j_{O_{K'}}$. Let $\mathcal{A}^j_{O_{K'}}\to O_{K'}$ be the map induced by $\mathbf{A}\to O_K$ and $\mathbf{A}^j_{O_{K'}}$ be the formal completion respect to the surjection $\mathcal{A}^j_{O_{K'}}\to O_{K'}$. If $O_K[[T_1,\ldots,T_n]]\to\mathbf{A}$ is an isomorphism as above, it induces an isomorphism $O_{K'}[[T_1/\pi'^{e'j},\ldots,T_n/\pi'^{e'j}]]\to\mathbf{A}^j_{O_{K'}}$. We put $\mathbf{B}^j_{O_{K'}}=\mathcal{B}^j_{O_{K'}}\otimes_{\mathcal{A}^j_{O_{K'}}}\mathbf{A}^j_{O_{K'}}$. The ring $\mathbf{B}^j_{O_{K'}}$ is finite over $\mathbf{A}^j_{O_{K'}}$ since $\mathcal{B}^j_{O_{K'}}$ is finite over $\mathcal{A}^j_{O_{K'}}$. Enlarging K' further if necessary, we assume that the canonical map $(\bar f^j)^{-1}(\bar y)\to\pi_0(\mathrm{Spec}\ \mathbf{B}^j_{O_{K'}})$ is a bijection.

We show that the surjection $\pi_0(X^{j'}(\mathbf{B}\to B)_{K'})\to\pi_0(\mathrm{Spec}\ \mathbf{B}^j_{O_{K'}})$ is a bijection. For a rational number $j'>0$, let $\mathcal{A}^{j'}_K$ and $\mathcal{B}^{j'}_K$ denote the affinoid K-algebras defining $X^{j'}(\mathbf{A}\to O_K)$ and $X^{j'}(\mathbf{B}\to B)$. We have $\mathcal{B}^{j'}_K=\mathbf{B}\otimes_{\mathbf{A}}\mathcal{A}^{j'}_K$. Since $\pi_0(X^{j'}(\mathbf{B}\to B)_{\bar K})\to\varinjlim_{j''>j}\pi_0(X^{j''}(\mathbf{B}\to B)_{\bar K})$ is a bijection, the injection $\mathcal{B}^{j''}_{\bar K}\to\mathcal{B}^{j'}_{\bar K}$ induce a bijection of idempotents for $j<j''<j'$. Since $\pi_0(X^{j'}(\mathbf{B}\to B)_{\bar K})\to\pi_0(X^{j'}(\mathbf{B}\to B)_{K'})$ is a bijection, the idempotents of $\mathcal{B}^{j'}_{\bar K}$ are in $\mathcal{B}^{j'}_{K'}$. Hence, for $j<j''<j'$, the map $\mathcal{B}^{j''}_{K'}\to\mathcal{B}^{j'}_{K'}$ induces a bijection of idempotents for $j<j''<j'$. Therefore, the map $\mathbf{B}^j_{O_{K'}}\to\mathcal{B}^{j'}_{K'}$

induces a bijection of idempotents by [3] 7.3.6 Proposition. Thus, the map $\pi_0(X^{j'}(\mathbf{B} \to B)_{K'}) \to \pi_0(\mathrm{Spec}\ \mathbf{B}^j_{O_{K'}})$ is a bijection as required. □

For later use in the proof of the commutativity in the logarithmic case, we give a more formal description of the functor (smooth Affinoid/K) $\to G_K$-(Aff/$\bar{F}$) : $X \mapsto \bar{X}$. For this purpose, we introduce a category $\varinjlim_{K'/K}(\mathrm{Aff}/F')$ and an equivalence $\varinjlim_{K'/K}(\mathrm{Aff}/F') \to G_K$-(Aff/$\bar{F}$) of categories. More generally, we define a category $\varinjlim_{K'/K} \mathcal{V}(K')$ in the following setting. Suppose we are given a category $\mathcal{V}(K')$ for each finite separable extension K' of K and a functor $f^* : \mathcal{V}(K'') \to \mathcal{V}(K')$ for each morphism $f : K' \to K''$ of finite separable extension of K satisfying $(f \circ g)^* = g^* \circ f^*$ and $\mathrm{id}^*_{K'} = \mathrm{id}_{\mathcal{V}(K')}$. In the application here, we will take $\mathcal{V}(K')$ to be (Aff/F') for the residue field F'. In Section 4, we will take $\mathcal{V}(K')$ to be $\mathcal{E}mb_{O_{K'}}$. We say that a full subcategory $\mathcal{C}$ of the category (Ext/K) of finite separable extensions in $\bar{K}$ is cofinal if $\mathcal{C}$ is non empty and a finite extension K'' of an extension K' in $\mathcal{C}$ is also in $\mathcal{C}$. We define $\varinjlim_{K'/K} \mathcal{V}(K')$ to be the category whose objects and morphisms are as follows. An object of $\varinjlim_{K'/K} \mathcal{V}(K')$ is a system $((X_{K'})_{K' \in ob(\mathcal{C})}, (\varphi_f)_{f:K' \to K'' \in mor(\mathcal{C})})$ where $\mathcal{C}$ is some cofinal full subcategory of (Ext/K), $X_{K'}$ is an object of $\mathcal{V}(K')$ for each object K' in $\mathcal{C}$ and $\varphi_f : X_{K''} \to f^*(X_{K'})$ is an isomorphism in $\mathcal{V}(K'')$ for each morphism $f : K' \to K''$ in $\mathcal{C}$ satisfying $\varphi_{f \circ f'} = f^*(\varphi_{f'}) \circ \varphi_f$ for morphisms $f' : K' \to K''$ and $f : K'' \to K'''$ in $\mathcal{C}$. For objects $X = ((X_{K'})_{K' \in ob(\mathcal{C})}, (\varphi_f)_{f:K' \to K'' \in mor(\mathcal{C})})$ and $Y = ((Y_{K'})_{K' \in ob(\mathcal{C}')}, (\psi_f)_{f:K' \to K'' \in mor(\mathcal{C}')})$ of the category $\varinjlim_{K'/K} \mathcal{V}(K')$, a morphism $g : X \to Y$ is a system $(g_{K'})_{K' \in ob(\mathcal{C}'')}$, where $\mathcal{C}''$ is some cofinal full subcategory of $\mathcal{C} \cap \mathcal{C}'$ and $g_{K'} : X_{K'} \to Y_{K'}$ is a morphism in $\mathcal{V}(\mathcal{K}')$ such that the diagram

$$\begin{array}{ccc} X_{K''} & \xrightarrow{g_{K'}} & Y_{K''} \\ {\scriptstyle\varphi_f}\downarrow & & \downarrow{\scriptstyle\psi_f} \\ f^*X_{K'} & \xrightarrow{g_{K''}} & f^*Y_{K'} \end{array}$$

is commutative for each morphism $f : K' \to K''$ in $\mathcal{C}''$.

Applying the general construction above, we define a category $\varinjlim_{K'/K}(\mathrm{Aff}/F')$. An equivalence $\varinjlim_{K'/K}(\mathrm{Aff}/F') \to G_K$-(Aff/$\bar{F}$) of categories is defined as follows. Let $X = ((X_{K'})_{K' \in ob(\mathcal{C})}, (f^*)_{f:K' \to K'' \in mor(\mathcal{C})})$ be an object of $\varinjlim_{K'/K}(\mathrm{Aff}/F')$. Let $\mathcal{C}_{\bar{K}}$ be the category of finite extensions of K in $\bar{K}$ which are in $\mathcal{C}$. Then, $X_{\bar{K}} = \varprojlim_{K' \in \mathcal{C}_{\bar{K}}} X_{K'}$ is an affine scheme over $\bar{F}$ and has a natural continuous semi-linear action of the Galois group G_K. By sending X to $X_{\bar{K}}$, we obtain a functor $\varinjlim_{K'/K}(\mathrm{Aff}/F') \to G_K$-(Aff/$\bar{F}$). We can easily verify that this functor gives an equivalence of categories.

The reduced geometic closed fiber defines a functor (smooth Affinoid/K) $\to$ $\varinjlim_{K'/K}(\mathrm{Aff}/F')$ as follows. Let X be a smooth affinoid variety over K. Let $\mathcal{C}_X$

be the full subcategory of (Ext/K) consisting of finite extensions K' such that a stable normalized integral model $\mathcal{A}_{O_{K'}}$ is defined over K'. By Theorem 1.10.1, the subcategory $\mathcal{C}_X$ is cofinal. Further, by Theorem 1.10.2, the system $\bar{X} = (\mathrm{Spec}\ \mathcal{A}_{O_{K'}} \otimes_{O_{K'}} F')_{K' \in ob\mathcal{C}_X}$ defines an object of $\varinjlim_{K'/K}(\mathrm{Aff}/F')$. Thus, by sending X to $\bar{X}$, we obtain a functor (smooth Affinoid$/K$) $\to \varinjlim_{K'/K}(\mathrm{Aff}/\bar{F}')$. By taking the composition with the equivalence of categories, we recover the functor (smooth Affinoid$/K$) $\to G_K$-$(\mathrm{Aff}/\bar{F})$.

1.4 Twisted normal cones

Let $(\mathbf{A} \to A)$ be an object in $\mathcal{E}mb_{O_K}$ and $j > 0$ be a positive rational number. We define $\bar{X}^j(\mathbf{A} \to A)$ to be the geometric closed fiber of the stable normalized integral model of $X^j(\mathbf{A} \to A)$. We will also define a twisted normal cone $\bar{C}^j(\mathbf{A} \to A)$ as a scheme over $A_{\bar{F},\mathrm{red}} = (A \otimes_{O_K} \bar{F})_{\mathrm{red}}$ and a canonical map $\bar{X}^j(\mathbf{A} \to A) \to \bar{C}^j(\mathbf{A} \to A)$.

Let I be the kernel of the surjection $\mathbf{A} \to A$. Then the normal cone $C_{A/\mathbf{A}}$ of Spec A in Spec $\mathbf{A}$ is defined to be the spectrum of the graded A-algebra $\bigoplus_{n=0}^{\infty} I^n/I^{n+1}$. We say that a surjection $R \to R'$ of Noetherian rings is regular if the immersion Spec$R' \to$ Spec$R$ is a regular immersion. If the surjection $\mathbf{A} \to A$ is regular, the conormal sheaf $N_{A/\mathbf{A}} = I/I^2$ is locally free and the normal cone $C_{A/\mathbf{A}}$ is equal to the normal bundle, namely the covariant vector bundle over Spec$A$ defined by the locally free A-module $Hom_A(N_{A/\mathbf{A}}, A)$.

For a rational number j, let $\mathfrak{m}^j$ be the fractional ideal $\mathfrak{m}^j = \{x \in O_{\bar{K}}; \mathrm{ord}(x) \geq j\}$ and put $N^j = \mathfrak{m}^j \otimes_{O_{\bar{K}}} \bar{F}$.

Definition 1.13 *Let $(\mathbf{A} \to A)$ be an object of $\mathcal{E}mb_{O_K}$ and $j > 0$ be a rational number. We define the j-th twisted normal cone $\bar{C}^j(\mathbf{A} \to A)$ to be the reduced part*

$$\left(\mathrm{Spec} \bigoplus_{n=0}^{\infty} (I^n/I^{n+1} \otimes_{O_K} N^{-jn})\right)_{\mathrm{red}}$$

of the spectrum of the $A \otimes_{O_K} \bar{F}$-algebra $\bigoplus_{n=0}^{\infty}(I^n/I^{n+1} \otimes_{O_K} N^{-jn})$.

It is a reduced affine scheme over Spec $A_{\bar{F},\mathrm{red}}$ non-canonically isomorphic to the reduced part of the base change $C_{A/\mathbf{A}} \otimes_{O_K} \bar{F}$. It has a natural continuous semi-linear action of G_K via N^{-jn}. The restriction to the wild inertia subgroup P is trivial and the G_K-action induces an action of the tame quotient $G_K^{\mathrm{tame}} = G_K/P$. If the surjection $\mathbf{A} \to A$ is regular, the scheme $\bar{C}^j(\mathbf{A} \to A)$ is the covariant vector bundle over Spec $A_{\bar{F},\mathrm{red}}$ defined by the $A_{\bar{F},\mathrm{red}}$-module $(Hom_A(I/I^2, A) \otimes_{O_K} N^j) \otimes_{A \otimes_{O_K} \bar{F}} A_{\bar{F},\mathrm{red}}$.

A canonical map $\bar{X}^j(\mathbf{A} \to A) \to \bar{C}^j(\mathbf{A} \to A)$ is defined as follows. Let K' be a finite separable extension of K such that the stable normalized integral model $\mathcal{A}^j_{O_{K'}}$ is defined over K' and that the product je with the ramification index

$e = e_{K'/K}$ is an integer. Then, we have a natural ring homomorphism

$$\bigoplus_{n\geq 0} I^n \otimes_{O_K} \mathfrak{m}_{K'}^{-jen} \longrightarrow \mathcal{A}^j_{O_{K'}} : f \otimes a \mapsto af.$$

Since $I\mathcal{A}^j_{O_{K'}} \subset \mathfrak{m}^{je}_{K'}\mathcal{A}^j_{O_{K'}}$, it induces a map $\bigoplus_n I^n/I^{n+1} \otimes_{O_K} \mathfrak{m}_{K'}^{-jen} \to \mathcal{A}^j_{O_{K'}}/\mathfrak{m}_{K'}\mathcal{A}^j_{O_{K'}}$. Let F' be the residue field of K'. Then by extending the scalar, we obtain a map $\bigoplus_{n=0}^{\infty}(I^n/I^{n+1} \otimes_{O_K} N^{-jn}) \to \mathcal{A}^j_{O_{K'}}/\mathfrak{m}_{K'}\mathcal{A}^j_{O_{K'}} \otimes_{F'} \bar{F}$. By the assumption that $\mathcal{A}^j_{O_{K'}}$ is a stable normalized integral model, we have $\bar{X}^j(\mathbf{A} \to A) = \text{Spec }(\mathcal{A}^j_{O_{K'}}/\mathfrak{m}_{K'}\mathcal{A}^j_{O_{K'}} \otimes_{F'} \bar{F})$. Since $\bar{X}^j(\mathbf{A} \to A)$ is a reduced scheme over $\bar{F}$, we obtain a map $\bar{X}^j(\mathbf{A} \to A) \to \bar{C}^j(\mathbf{A} \to A)$ of schemes over $\bar{F}$.

For a positive rational number $j > 0$, the constructions above define a functor $\bar{C}^j : \mathcal{E}mb_{O_K} \to G_K\text{-}(\text{Aff}/\bar{F})$ and a morphism of functors $\bar{X}^j \to \bar{C}^j$.

Lemma 1.14 *Let $(\mathbf{A} \to A)$ be an object of $\mathcal{E}mb_{O_K}$ and $j > 0$ be a rational number. Then, we have the following.*
1. The canonical map $\bar{X}^j(\mathbf{A} \to A) \to \bar{C}^j(\mathbf{A} \to A)$ is finite.
2. Let $(\mathbf{A} \to A) \to (\mathbf{B} \to B)$ be a morphism in $\mathcal{E}mb_{O_K}$. Then, the canonical maps form a commutative diagram

$$\begin{array}{ccccc} \bar{X}^j(\mathbf{B} \to B) & \longrightarrow & \bar{C}^j(\mathbf{B} \to B) & \longrightarrow & \text{Spec } B_{\bar{F},\text{red}} \\ \downarrow & & \downarrow & & \downarrow \\ \bar{X}^j(\mathbf{A} \to A) & \longrightarrow & \bar{C}^j(\mathbf{A} \to A) & \longrightarrow & \text{Spec } A_{\bar{F},\text{red}}. \end{array}$$

If the morphism $(\mathbf{A} \to A) \to (\mathbf{B} \to B)$ is finite flat, then the right square in the commutative diagram is cartesian.
3. Assume $A = O_K$. Then the surjection $\mathbf{A} \to A$ is regular and the canonical map $N_{A/\mathbf{A}} \to \hat{\Omega}_{\mathbf{A}/O_K} \otimes_{\mathbf{A}} A$ is an isomorphism. The twisted normal cone $\bar{C}^j(\mathbf{A} \to A)$ is equal to the $\bar{F}$-vector space $Hom_{\bar{F}}(\hat{\Omega}_{\mathbf{A}/O_K} \otimes_{\mathbf{A}} \bar{F}, N^j)$. The canonical map $\bar{X}^j(\mathbf{A} \to A) \to \bar{C}^j(\mathbf{A} \to A)$ is an isomorphism.

Proof. 1. Let K' be a finite extension such that the stable normalized integral model $\mathcal{A}^j_{O_{K'}}$ is defined. Let $\mathcal{A}'$ denote the π'-adic completion of the image of the map $\bigoplus_{n\geq 0} I^n \otimes_{O_K} \mathfrak{m}_{K'}^{-jen} \to \mathbf{A} \otimes_{O_K} K'$. Then by the definition and by Lemma 1.3, $\mathcal{A}^j_{O_{K'}}$ is the integral closure of $\mathcal{A}'$ in $\mathcal{A}'_K$. Hence $\mathcal{A}^j_{O_{K'}}/m_{K'}\mathcal{A}^j_{O_{K'}}$ is finite over $\bigoplus_n I^n/I^{n+1} \otimes_{O_K} \mathfrak{m}_{K'}^{-jen}$. Thus the assertion follows.
2. Clear from the definitions.
3. If $A = O_K$, there is an isomorphism $O_K[[T_1, \ldots, T_n]] \to \mathbf{A}$ for some n such that the composition $O_K[[T_1, \ldots, T_n]] \to A$ maps T_i to 0. Then the assertions are clear. □

1.5 Étale covering of tubular neighborhoods

Let A and B be the integer rings of finite étale K-algebras. For a finite flat morphism $(\mathbf{A} \to A) \to (\mathbf{B} \to B)$ of embeddings, we study conditions for the induced finite morphism $X^j(\mathbf{A} \to A) \to X^j(\mathbf{B} \to B)$ to be étale.
Let $X = \mathrm{Sp}\ \mathcal{B}_K$ and $Y = \mathrm{Sp}\ \mathcal{A}_K$ be geometrically reduced affinoid varieties and $\mathcal{A}$ and $\mathcal{B}$ be the maximum integral models. Then a finite map $f : X \to Y$ of affinoid varieties is uniquelly extended to a finite map $\mathcal{A} \to \mathcal{B}$ of integral models.

Proposition 1.15 *Let A and $B = O_L$ be the integer rings of finite separable extensions of K and $(\mathbf{A} \to A) \to (\mathbf{B} \to B)$ be a finite flat morphism of embeddings. Let $j > 1$ be a rational number, π_L a prime element of L and $e = \mathrm{ord}\pi_L$ be the ramification index.*
1. ([1] Proposition 7.3) *Assume $A = O_K$. Suppose that, for each $j' > j$, there exists a finite separable extension K' of K such that the base change $X^{j'}(\mathbf{B} \to B)_{K'}$ is isomorphic to the disjoint union of finitely many copies of $X^{j'}(\mathbf{A} \to A)_{K'}$ as an affinoid variety over $X^{j'}(\mathbf{A} \to A)$. Then there is an integer $0 \leq n < ej$ such that π_L^n annihilates $\Omega_{B/A}$.*
2. ([1] Proposition 7.5) *If there is an integer $0 \leq n < ej$ such that π_L^n annihilates $\Omega_{B/A}$, then the finite flat map $X^j(\mathbf{B} \to B) \to X^j(\mathbf{A} \to A)$ is étale.*

Corollary 1.16 ([1] Theorem 7.2) *Let $A = O_K$ and let B be the integer ring of a finite étale K-algebra. Let $(\mathbf{A} \to A) \to (\mathbf{B} \to B)$ be a finite flat morphism of embeddings. Let $j > 1$ be a rational number. Suppose that, for each $j' > j$, there exists a finite separable extension K' of K such that the base change $X^{j'}(\mathbf{B} \to B)_{K'}$ is isomorphic to the disjoint union of finitely many copies of $X^{j'}(\mathbf{A} \to A)_{K'}$ as in Proposition 1.15.1. Let I be the kernel of the surjection $\mathbf{B} \to B$ and let $N_{B/\mathbf{B}}$ be the B-module I/I^2. Then, we have the following.*
1. The finite map $X^j(\mathbf{B} \to B) \to X^j(\mathbf{A} \to A)$ is étale and is extended to a finite étale map of stable normalized integral models.
2. The finite map $\bar{X}^j(\mathbf{B} \to B) \to \bar{X}^j(\mathbf{A} \to A)$ is étale.
3. The twisted normal cone $\bar{C}^j(\mathbf{B} \to B)$ is canonically isomorphic to the covariant vector bundle defined by the $B_{\bar{F},\mathrm{red}}$-module $(Hom_B(N_{B/\mathbf{B}}, B) \otimes_{O_K} N^j) \otimes_{B_{\bar{F}}} B_{\bar{F},\mathrm{red}}$ and the finite map $\bar{X}^j(\mathbf{B} \to B) \to \bar{C}^j(\mathbf{B} \to B)$ is étale.

Though these statements except Corollary 1.16.3 are proved in [1] Section 7, we present here slightly modified proofs in order to compare with the proofs of the corresponding statements in the logarithmic setting given in Section 4.3. To prove Proposition 1.15, we use the following.

Lemma 1.17 *Let $A = O_L$ be the integer ring of a finite separable extension L, $\mathbf{A} \to A$ be an embedding and let $\mathbf{M}$ be an $\mathbf{A}$-module of finite type. Let $j > 1$ be a rational number and K' be a finite separable extension of K such that the stable normalized integral model $\mathcal{A}^j_{O_{K'}}$ of $X^j(\mathbf{A} \to A)$ is defined over K'. Let e and e' be the ramification indices of L and of K' over K and π_L and π' be*

prime elements of L and K'. Assume that e'/e and $e'j$ are integers. Then, the following conditions are equivalent.
(1) There exists an integer $0 \leq n < ej$ such that the A-module $M = \mathbf{M} \otimes_{\mathbf{A}} A$ is annihilated by π_L^n.
(2) The $\mathcal{A}^j_{O_{K'}}$-module $\mathcal{M}^j = \mathbf{M} \otimes_{\mathbf{A}} \mathcal{A}^j_{O_{K'}}$ is annihilated by $\pi'^{e'j-1}$.

Proof of Lemma 1.17. The image of an element in the kernel I of the surjection $\mathbf{A} \to A$ in $\mathcal{A}^j_{O_{K'}}$ is divisible by $\pi'^{e'j}$. Hence we have a commutative diagram

$$\begin{array}{ccc} \mathbf{A} & \longrightarrow & \mathcal{A}^j_{O_{K'}} \\ \big\downarrow & & \big\downarrow \\ A & \longrightarrow & \mathcal{A}^j_{O_{K'}}/(\pi'^{e'j}). \end{array}$$

We show that the ideals of $\mathcal{A}^j_{O_{K'}}/(\pi'^{e'j})$ generated by the image of $\pi_L \in A$ and by the image of $\pi'^{e'/e} \in \mathcal{A}^j_{O_{K'}}$ are equal. Take a lifting $a \in \mathbf{A}$ of $\pi_L \in A$. Then, the image of a^e is a unit times π and hence is a unit times $\pi'^{e'}$ in $\mathcal{A}^j_{O_{K'}}/(\pi'^{e'j})$. Since $\mathcal{A}^j_{O_{K'}}$ is π'-adically complete, we have $a^e = u\pi'^{e'} + v\pi'^{e'j}$ for some $u \in \mathcal{A}^{j\times}_{O_{K'}}$ and $v \in \mathcal{A}^j_{O_{K'}}$. Since $j > 1$ and $\mathcal{A}^j_{O_{K'}}$ is π'-adically complete, we have $(a/\pi'^{e'/e})^e = u + v\pi'^{e'(j-1)}$ is a unit in $\mathcal{A}^j_{O_{K'}}$. Since $\mathcal{A}^j_{O_{K'}}$ is normal, we have $a/\pi'^{e'/e} \in \mathcal{A}^{j\times}_{O_{K'}}$ and the claim follows.
Assume that the A-module M is isomorphic to $A^r \oplus \bigoplus_{i=1}^s A/(\pi_L^{n_i})$ for integers $0 < n_1 \leq \ldots \leq n_s$. Then, by the commutative diagram above and by the equality $(\pi_L) = (\pi'^{e'/e})$ of the ideals of $\mathcal{A}^j_{O_{K'}}/(\pi'^{e'j})$ proved above, the $\mathcal{A}^j_{O_{K'}}/(\pi'^{e'j})$-module $\mathcal{M}^j/\pi'^{e'j}\mathcal{M}^j$ is isomorphic to $(\mathcal{A}^j_{O_{K'}}/(\pi'^{e'j}))^r \oplus \bigoplus_{i=1}^s \mathcal{A}^j_{O_{K'}}/(\pi'^{\min(e'j, e'n_i/e)})$. The condition (1) is clearly equivalent to that $r = 0$ and $n_s < ej$. We see that the condition (2) is also equivalent to this condition by taking the localization at a prime ideal $\mathcal{A}^j_{O_{K'}}$ of height 1 containing π'. □

Proof of Proposition 1.15. 1. Since $A = O_K$, there is an isomorphism $O_K[[T_1, \ldots, T_n]] \to \mathbf{A}$ such that the composition $O_K[[T_1, \ldots, T_n]] \to A$ maps T_i to 0. For $j > 0$, the affinoid variety $X^j(\mathbf{A} \to A)$ is a polydisk. By the proof of Lemma 1.7, there exist a finite separable extension K' of K of ramification index e', an embedding $(\mathbf{B} \otimes_{O_K} O_{K'} \to B')$ in $\mathcal{E}mb_{O_{K'}}$ isomorphic to the embbedding $(O_{K'}[[S_1, \ldots, S_n]]^N \to O_{K'}^N)$ sending S_i to 0 for some $N > 0$, a positive rational number $\epsilon < j$ and an open immersion $X^j(\mathbf{B} \to B) \otimes_K K' \to X^{e'\epsilon}(\mathbf{B} \otimes_{O_K} O_{K'} \to B')$ as a rational subdomain. The affinoid variety $X^{e'\epsilon}(\mathbf{B} \otimes_{O_K} O_{K'} \to B')$ is the disjoint union of finitely many copies of polydisks. Enlarging K' if necessary, we may assume that $e'j$ and $e'\epsilon$ are integers. We may further assume that there is a rational number $j < j' < j + \epsilon$ such that $e'j'$ is an integer, that the stable normalized integral models $\mathcal{B}^{j'}_{O_{K'}}$ and $\mathcal{B}'^{e'\epsilon}_{O_{K'}}$ of $X^{j'}(\mathbf{B} \to B)$ and of $X^{e'\epsilon}(\mathbf{B} \otimes_{O_K} O_{K'} \to B')$ are defined over K' and that $X^{j'}(\mathbf{B} \to B)_{K'}$ is isomorphic to the disjoint union of copies of $X^{j'}(\mathbf{A} \to A)_{K'}$. Since $e'j'$ is an integer, the stable normalized integral model $\mathcal{A}^{j'}_{O_{K'}}$ of $X^{j'}(\mathbf{A} \to A)$ is also defined over K'. Then we have a commutative diagram

$$\begin{array}{ccccc} \mathbf{A} & & \longrightarrow & & \mathcal{A}^{j'}_{O_{K'}} \\ \downarrow & & & & \downarrow \\ \mathbf{B} & \longrightarrow & \mathcal{B}'^{e'\epsilon}_{O_{K'}} & \longrightarrow & \mathcal{B}^{j'}_{O_{K'}}. \end{array}$$

We consider the modules $\hat{\Omega}_{\mathbf{A}/O_K} = \varprojlim_n \Omega_{(\mathbf{A}/\mathfrak{m}^n_{\mathbf{A}})/O_K)}, \hat{\Omega}_{\mathcal{A}^{j'}_{O_{K'}}/O_{K'}} = \varprojlim_n \Omega_{(\mathcal{A}^{j'}_{O_{K'}}/\pi'^n\mathcal{A}^{j'}_{O_{K'}})/O_{K'}}$ etc as defined in the beginning of Section 1.1. By Lemma 1.4.2, we have a commutative diagram

$$\begin{array}{ccccc} \mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathbf{A}} \hat{\Omega}_{\mathbf{A}/O_K} & & \longrightarrow & & \mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathcal{A}^{j'}_{O_{K'}}} \hat{\Omega}_{\mathcal{A}^{j'}_{O_{K'}}/O_{K'}} \\ \downarrow & & & & \downarrow \\ \mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathbf{B}} \hat{\Omega}_{\mathbf{B}/O_K} & \longrightarrow & \mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathcal{B}'^{e'\epsilon}_{O_{K'}}} \hat{\Omega}_{\mathcal{B}'^{e'\epsilon}_{O_{K'}}/O_{K'}} & \longrightarrow & \hat{\Omega}_{\mathcal{B}^{j'}_{O_{K'}}/O_{K'}}. \end{array}$$

We show that the five $\mathcal{B}^{j'}_{O_{K'}}$-modules are free of rank n and that the five maps are injective. We also show that by identifying the modules with their images in $\hat{\Omega}_{\mathcal{B}^{j'}_{O_{K'}}/O_{K'}}$, we have an inclusion $\pi'^{e'j'}\mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathbf{B}} \hat{\Omega}_{\mathbf{B}/O_K} \subset \pi'^{e'\epsilon}\mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathbf{A}} \hat{\Omega}_{\mathbf{A}/O_K}$ of submodules of $\hat{\Omega}_{\mathcal{B}^{j'}_{O_{K'}}/O_{K'}}$. By the assumption on the covering $X^{j'}(\mathbf{B} \to B)_{K'} \to X^{j'}(\mathbf{A} \to A)_{K'}$, the $\mathcal{A}^{j'}_{O_{K'}}$-algebra $\mathcal{B}^{j'}_{O_{K'}}$ is isomorphic to the product of finitely many copies of $\mathcal{A}^{j'}_{O_{K'}}$. Hence the right vertical map $\mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathcal{A}^{j'}_{O_{K'}}} \hat{\Omega}_{\mathcal{A}^{j'}_{O_{K'}}/O_{K'}} \to \hat{\Omega}_{\mathcal{B}^{j'}_{O_{K'}}/O_{K'}}$ is an isomorphism. The isomorphism $O_K[[T_1, \ldots, T_n]] \to \mathbf{A}$ in the beginning of the proof induces an isomorphism $O_{K'}\langle T_1/\pi'^{e'j}, \ldots, T_n/\pi'^{e'j}\rangle \to \mathcal{A}^{j'}_{O_{K'}}$ and we see that the $\mathbf{A}$-module $\hat{\Omega}_{\mathbf{A}/O_K}$ and the $\mathcal{A}^{j'}_{O_{K'}}$-module $\hat{\Omega}_{\mathcal{A}^{j'}_{O_{K'}}/O_{K'}}$ are free of rank n. Hence $\hat{\Omega}_{\mathcal{B}^{j'}_{O_{K'}}/O_{K'}}$ is also a free $\mathcal{B}^{j'}_{O_{K'}}$-module of rank n. Further by the canonical maps $\mathcal{A}^{j'}_{O_{K'}} \otimes_{\mathbf{A}} \hat{\Omega}_{\mathbf{A}/O_K} \to \hat{\Omega}_{\mathcal{A}^{j'}_{O_{K'}}/O_{K'}}$, the module $\mathcal{A}^{j'}_{O_{K'}} \otimes_{\mathbf{A}} \hat{\Omega}_{\mathbf{A}/O_K}$ is identified with the submodule $\pi'^{e'j'}\hat{\Omega}_{\mathcal{A}^{j'}_{O_{K'}}/O_{K'}}$. Similarly, the $\mathbf{B}$-module $\hat{\Omega}_{\mathbf{B}/O_K}$ and the $\mathcal{B}'^{e'\epsilon}_{O_{K'}}$-module $\hat{\Omega}_{\mathcal{B}'^{e'\epsilon}_{O_{K'}}/O_{K'}}$ are free of rank n and $\mathcal{B}'^{e'\epsilon}_{O_{K'}} \otimes_{\mathbf{B}} \hat{\Omega}_{\mathbf{B}/O_K}$ is identified with the submodule $\pi'^{e'\epsilon}\hat{\Omega}_{\mathcal{B}'^{e'\epsilon}_{O_{K'}}/O_{K'}}$. Since $X^j(\mathbf{B} \to B) \otimes_K K'$ is a rational subdomain of $X^{e'\epsilon}(\mathbf{B} \otimes_{O_K} O_{K'} \to B')$, the map $\mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathcal{B}'^{e'\epsilon}_{O_{K'}}} \hat{\Omega}_{\mathcal{B}'^{e'\epsilon}_{O_{K'}}/O_{K'}} \to \hat{\Omega}_{\mathcal{B}^{j'}_{O_{K'}}/O_{K'}}$ is an injection. Thus, we obtain an inclusion $\pi'^{e'j'}\mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathbf{B}} \hat{\Omega}_{\mathbf{B}/O_K} \subset \pi'^{e'\epsilon}\mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathbf{A}} \hat{\Omega}_{\mathbf{A}/O_K}$ as submodules of $\hat{\Omega}_{\mathcal{B}^{j'}_{O_{K'}}/O_{K'}}$.

Thus the $\mathcal{B}^{j'}_{O_{K'}}$-module $\mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathbf{B}} \Omega_{\mathbf{B}/\mathbf{A}} = \mathrm{Coker}(\mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathbf{A}} \hat{\Omega}_{\mathbf{A}/O_K} \to \mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathbf{B}} \hat{\Omega}_{\mathbf{B}/O_K})$ is annihilated by $\pi'^{e'(j'-\epsilon)}$. Since $0 < j - \epsilon < j' - \epsilon < j$, it suffices to apply Lemma 1.17 (2)⇒(1).

2. Let K' be a finite separable extension such that $e'j$ is an integer and the stable normalized integral models $\mathcal{A}^j_{O_{K'}}$ and $\mathcal{B}^j_{O_{K'}}$ are defined over K'. By the proof of Lemma 1.9.2, we have $\mathcal{B}^j_{O_{K'}} \otimes_{O_{K'}} K' = \mathbf{B} \otimes_{\mathbf{A}} \mathcal{A}^j_{O_{K'}} \otimes_{O_{K'}} K'$ and the map $\mathcal{A}^j_{O_{K'}} \otimes_{O_{K'}} K' \to \mathcal{B}^j_{O_{K'}} \otimes_{O_{K'}} K'$ is finite flat. By Lemma 1.17 (1)⇒(2), the $\mathcal{B}^j_{O_{K'}}$-module $\mathcal{B}^j_{O_{K'}} \otimes_{\mathbf{B}} \Omega_{\mathbf{B}/\mathbf{A}}$ is annihilated by $\pi'^{n'}$ for an integer $0 \leq n' < e'j$. Hence the map $\mathcal{A}^j_{O_{K'}} \otimes_{O_{K'}} K' \to \mathcal{B}^j_{O_{K'}} \otimes_{O_{K'}} K'$ is étale. □

Proof of Corollary 1.16. 1. It follows from Proposition 1.15 that the map $X^j(\mathbf{B} \to B) \to X^j(\mathbf{A} \to A)$ is finite étale. By Lemma 1.12, the fiber $(\bar{f}^j)^{-1}(\bar{y})$ has the same cardinality as the degree of the map $X^j(\mathbf{B} \to B) \to X^j(\mathbf{A} \to A)$ in the notation there. Hence the finite map $X^j(\mathbf{B} \to B)_{O_{K'}} \to X^j(\mathbf{A} \to A)_{O_{K'}}$ of the normalized integral models is étale at a point of $X^j(\mathbf{A} \to A)_{O_{K'}}$ in the closed fiber. Since $X^j(\mathbf{A} \to A)_{O_{K'}}$ is a regular Noetherian scheme, the assertion follows by the purity of branch locus.

2. Clear from 1.

3. Since the surjection $\mathbf{B} \to B$ is regular, the twisted normal cone $\bar{C}^j(\mathbf{B} \to B)$ is canonically isomorphic to the covariant vector bundle defined by the $B_{\bar{F},\mathrm{red}}$-module $(Hom_B(I/I^2, B) \otimes_{O_K} N^j) \otimes_{B_{\bar{F}}} B_{\bar{F},\mathrm{red}}$. We consider the commutative diagram

$$\begin{array}{ccccc} \bar{X}^j(\mathbf{B} \to B) & \longrightarrow & \bar{C}^j(\mathbf{B} \to B) & \longrightarrow & \mathrm{Spec}\ B_{\bar{F},\mathrm{red}} \\ \downarrow & & \downarrow & & \downarrow \\ \bar{X}^j(\mathbf{A} \to O_K) & \longrightarrow & \bar{C}^j(\mathbf{A} \to O_K) & \longrightarrow & \mathrm{Spec}\ \bar{F} \end{array}$$

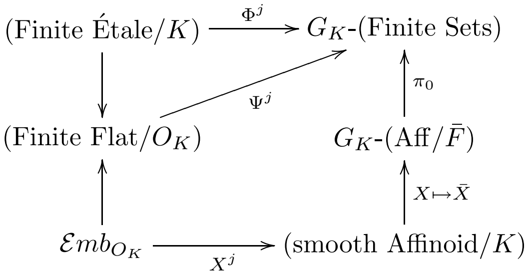
in Lemma 1.14.2. Since the map $(\mathbf{A} \to A) \to (\mathbf{B} \to B)$ is finite and flat, the right square is cartesian. Hence the middle vertical arrow is étale. Since $A = O_K$, the lower left horizontal arrow is an isomorphism by Lemma 1.14.3. By 2, the left vertial arrow is finite étale. Thus the assertion is proved. □

2 Filtration by ramification groups: the non-logarithmic case

2.1 Construction

In this subsection, we rephrase the definition of the filtration by ramification groups given in the previous paper [1] by using the construction in Section 1. The main purpose is to emphasize the parallelism between the non-logarithmic construction recalled here and the logarithmic construction to be recalled in Section 5.1.

Let Φ : (Finite Étale$/K$) $\to$ G_K-(Finite Sets) denote the fiber functor sending a finite étale K-algebra L to the finite set $\Phi(L) = Hom_{K\text{-alg}}(L, \bar{K})$ with the continuous G_K-action. For a rational number $j > 0$, we define a functor Φ^j : (Finite Étale$/K$) $\to$ G_K-(Finite Sets) as the composition of the functor (Finite Étale$/K$) $\to$ (Finite Flat$/O_K$) sending a finite étale K-algebra L to the integral closure O_L of O_K in L and the functor Ψ^j : (Finite Flat$/O_K$) $\to$ G_K-(Finite Sets) defined at the end of Section 1.2. The map (1.9.3) defines a surjection $\Phi \to \Phi^j$ of functors. In [1], we define the filtration by ramification groups on G_K by using the family of surjections $(\Phi \to \Phi^j)_{j>0,\in\mathbb{Q}}$ of functors. The filtration by the ramification groups $G_K^j \subset G_K, j > 0, \in \mathbb{Q}$ is characterized by the condition that the canonical map $\Phi(L) \to \Phi^j(L)$ induces a bijection $\Phi(L)/G_K^j \to \Phi^j(L)$ for each finite étale algebra L over K.
The functor Φ^j is defined by the commutativity of the diagram



We briefly recall how the other arrows in the diagram are defined. The forgetful functor $\mathcal{E}mb_{O_K} \to$ (Finite Flat$/O_K$) sends $(\mathbf{A} \to A)$ to A. The functor X^j : $\mathcal{E}mb_{O_K} \to$ (smooth Affinoid$/K$) is defined by the j-th tubular neighborhood. The functor (smooth Affinoid$/K$) $\to$ G_K-(Aff$/\bar{F}$) sends X to the geometric closed fiber $\bar{X}$ of the stable normalized integral model. The functor π_0 : G_K-(Aff$/\bar{F}$) $\to$ G_K-(Finite Sets) is defined by the set of connected components. They induce a functor Ψ^j : (Finite Flat$/O_K$) $\to$ G_K-(Finite Sets) by Lemma 1.9. The functor Φ^j is defined as the composition of Ψ^j with the functor (Finite Étale$/K$) $\to$ (Finite Flat$/O_K$) sending a finite étale algebra L over K to the integral closure O_L in L of O_K. More concretely, we have

$$\Phi^j(L) = \varprojlim_{(\mathbf{A}\to O_L)\in\mathcal{E}mb_{O_K}(O_L)} \pi_0(\bar{X}^j(\mathbf{A} \to O_L))$$

for a finite étale K-algebra L.
For a rational number $j \geq 0$, we define a functor Φ^{j+} : (Finite Étale$/K$) $\to\to$ G_K-(Finite Sets) by $\Phi^{j+}(L) = \varinjlim_{j'>j} \Phi^{j'}(L)$ for a finite étale K-algebra L. We define a closed normal subgroup G_K^{j+} to be $\overline{\cup_{j'>j} G_K^{j'}}$. Then we have $\Phi^{j+}(L) = \Phi(L)/G_K^{j+}$. The finite set $\Phi^{j+}(L)$ has the following geometric description.

LEMMA 2.1 *Let B be the integer ring of a finite étale algebra L over K and $j > 0$ be a rational number. Let $(f, \mathbf{f}) : (\mathbf{A} \to O_K) \to (\mathbf{B} \to B)$ be a finite* *flat morphism of embeddings. Let* $f^j : X^j(\mathbf{B} \to B) \to X^j(\mathbf{A} \to O_K)$ *and* $\bar{f}^j : \bar{X}^j(\mathbf{B} \to B) \to \bar{X}^j(\mathbf{A} \to O_K)$ *be the canonical maps. Let* $0 \in X^j(\mathbf{A} \to O_K)$ *be the point corresponding to the map* $\mathbf{A} \to O_K$ *and* $\bar{0} \in \bar{X}^j(\mathbf{A} \to O_K)$ *be its specialization. Then the maps* (1.8.0), (1.12.1) *and the specialization map form a commutative diagram*

$$\begin{array}{ccccc} \Phi(L) & \longrightarrow & \Phi^{j+}(L) & \longrightarrow & \Phi^j(L) \\ \downarrow & & \downarrow & & \downarrow \\ (f^j)^{-1}(0) & \longrightarrow & (\bar{f}^j)^{-1}(\bar{0}) & \longrightarrow & \pi_0(\bar{X}^j(\mathbf{B} \to B)) \end{array} \tag{2.1.1}$$

and the vertical arrows are bijections.

Proof. Since the map $(\mathbf{A} \to O_K) \to (\mathbf{B} \to B)$ is finite flat, the map $\mathbf{B} \to B$ induces an isomorphism $\mathcal{B}^j_K \otimes_{\mathcal{A}^j_K} K \to L$. Hence we obtain a bijection $\Phi(L) = Hom_{K\text{-alg}}(L, \bar{K}) \to (f^j)^{-1}(0)$. By Lemma 1.12 and the definition of $\Phi^{j+}(L)$, we have a bijection $\Phi^{j+}(L) \to (\bar{f}^j)^{-1}(\bar{0})$. The bijection $\Phi^j(L) \to \pi_0(\bar{X}^j(\mathbf{B} \to B))$ is clear from the definition of $\Phi^j(L)$. The commutativity is clear. □

For a finite étale algebra L over K and a rational number $j > 0$, we say that the ramification of L is bounded by j if the canonical map $\Phi(L) \to \Phi^j(L)$ is a bijection. Let $A = O_K$ and let $B = O_L$ be the integer ring of a finite étale K-algebra L and $(\mathbf{A} \to A) \to (\mathbf{B} \to B)$ be a finite flat morphism of embeddings. Then, since the map $X^j(\mathbf{B} \to B) \to X^j(\mathbf{A} \to A)$ is finite flat of degree $[L : K]$, the ramification of L is bounded by j if and only if there exists a finite separable extension K' of K such that the affinoid variety $X^j(\mathbf{B} \to B)_{K'}$ is isomorphic to the disjoint union of finitely many copies of $X^j(\mathbf{A} \to A)_{K'}$ over $X^j(\mathbf{A} \to A)_{K'}$. We say that the ramification of L is bounded by $j+$ if the ramification of L is bounded by every rational number $j' > j$. The ramification of L is bounded by $j+$ if and only if the canonical map $\Phi(L) \to \Phi^{j+}(L)$ is a bijection.

Lemma 2.2 *Let* $K \to K'$ *be a map of complete discrete valuation fields inducing a local homomorphism* $O_K \to O_{K'}$ *of integer rings. Assume that a prime element of* K *goes to a prime element of* K' *and that the residue field* F' *of* K' *is a separable extension of the residue field* F *of* K. *Then, for a rational number* $j > 0$, *the map* $G_{K'} \to G_K$ *induces a surjection* $G^j_{K'} \to G^j_K$.

Proof. Let A be the integer ring of a finite étale K-algebra L and $(\mathbf{A} \to A)$ be an object of $\mathcal{E}mb_{O_K}$. By the assumption, the tensor product $A \otimes_{O_K} O_{K'}$ is the integer ring of $L \otimes_K K'$. By the isomorphism $X^j(\mathbf{A} \to A)\hat{\otimes}_K K' \to X^j(\mathbf{A}\hat{\otimes}_{O_K} O_{K'} \to A \otimes_{O_K} O_{K'})$ in Section 1.2 and Theorem 1.10, the natural map $\Phi^j(L \otimes_K K') \to \Phi^j(L)$ is a bijection. Hence the assertion follows. □

Example. Let $K = \mathbb{F}_p(x,y)((\pi))$ and put $L = K[t]/(t^p - t - \frac{x}{\pi^{p^2}}), M = L[t_1, t_2]/(t_1^p - t_1 - \frac{x}{\pi^{p^3}}, t_2^p - t_2 - \frac{y}{\pi^{p^3}})$ and $G = \mathrm{Gal}(M/K) \simeq \mathbb{F}_p^3$. Then we have $G^j = G$ for $j \leq p^2$, $G^j = H = \mathrm{Gal}(M/L) \simeq \mathbb{F}_p^2$ for $p^2 < j \leq p^3$ and $G^j = 1$ for $p^3 < j$.

We put $z = \pi^p t$. Then we have $O_L = O_K[z]/(z^p - \pi^{p(p-1)}z - x)$ and $L = \mathbb{F}_p(z,y)((\pi))$. By putting $s = t_1 - \frac{z}{\pi^{p^2}}$, we also have $M = L[s,t_2]/(s^p - s - \frac{z(-1+\pi^{p(p-1)^2})}{\pi^{p(p^2-p+1)}}, t_2^p - t_2 - \frac{x}{\pi^{p^3}})$. We put $M_1 = L(s) \subset M$. Then we have $H^j = H$ for $j \le p(p^2-p+1)$, $H^j = \mathrm{Gal}(M/M_1) \simeq \mathbb{F}_p$ for $p(p^2-p+1) < j \le p^3$ and $H^j = 1$ for $p^3 < j$.
This example shows that the filtration on the subgroup H induced from the filtration by ramification groups on G is not the filtration by ramification groups on H even after renumbering. It also shows that the "lower numbering" filtration is not equal to the upper numbering filtration defined here even after renumbering.

2.2 Functoriality of the closed fibers of tubular neighborhoods: An equal characteristic case

For a positive rational number $j > 0$, let $(\text{Finite Étale}/K)^{\le j+}$ denote the full subcategory of (Finite Étale/K) consisting of étale K-algebras whose ramification is bounded by $j+$. In this subsection and the following one, we assume the following condition (F) is satisfied.

(F) There exists a perfect subfield F_0 of F such that F is finitely generated over F_0.

Further assuming that p is not a uniformizer of K, we will define a twisted tangent space Θ^j and show that the functor $\bar{X}^j : \mathcal{E}mb_{O_K} \to G_K\text{-}(\mathrm{Aff}/\bar{F})$ induces a functor

$$\bar{X}^j : (\text{Finite Étale}/K)^{\le j+} \to G_K\text{-}(\text{Finite Étale}/\Theta^j).$$

In this subsection, we study the easier case where K is of characteristic p.
Let F_0 be a perfect subfield of F such that F is finitely generated over F_0. We assume K is of characteristic p. Then, F_0 is naturally identified with a subfield of K. We first define a functor

$$(\text{Finite Étale}/K) \to \mathcal{E}mb_{O_K}.$$

In this subsection, A denotes the integer ring of a finite étale K-algebra.

Lemma 2.3 *Let A be the integer ring of a finite étale K-algebra.*
1. Let $(A/\mathfrak{m}_A^n \otimes_{F_0} O_K)^\wedge$ denote the formal completion of $A/\mathfrak{m}_A^n \otimes_{F_0} O_K$ of the surjection $A/\mathfrak{m}_A^n \otimes_{F_0} O_K \to A/\mathfrak{m}_A^n$ sending $a \otimes b$ to ab. Then the projective limit

$$(A\hat{\otimes}_{F_0} O_K)^\wedge = \varprojlim_n (A/\mathfrak{m}_A^n \otimes_{F_0} O_K)^\wedge$$

is an O_K-algebra formally of finite type and formally smooth over O_K.
2. Let $(A\hat{\otimes}_{F_0} O_K)^\wedge \to A$ be the limit of the surjections $(A/\mathfrak{m}_A^n \otimes_{F_0} O_K)^\wedge \to A/\mathfrak{m}_A^n$. Then $((A\hat{\otimes}_{F_0} O_K)^\wedge \to A)$ is an object of $\mathcal{E}mb_{O_K}$.

3. Let $A \to B$ be a morphism of the integer rings of finite étale K-algebras. Then it induces a finite flat morphism $((A\hat{\otimes}_{F_0}O_K)^\wedge \to A) \to ((B\hat{\otimes}_{F_0}O_K)^\wedge \to B)$ of $\mathcal{E}mb_{O_K}$.

Proof. 1. We may assume A is local. Let E be the residue field of A and take a transcendental basis $(\bar{t}_1, \dots, \bar{t}_m)$ of E over the perfect subfield F_0 such that E is a finite separable extension of $F_0(\bar{t}_1, \dots, \bar{t}_m)$. Take a lifting $(t_1, \dots, t_m)$ in A of $(\bar{t}_1, \dots, \bar{t}_m)$ and a prime element $t_0 \in A$. We define a map $F_0[T_0, \dots, T_m] \to A$ by sending T_i to t_i. Then A is finite étale over the completion of the local ring of $F_0[T_0, \dots, T_m]$ at the prime ideal (T_0). Hence there exist an étale scheme X over $\mathbb{A}^{m+1}_{F_0}$, a point ξ of X above (T_0) and an F_0-isomorphism $\varphi : \hat{O}_{X,\xi} \to A$. Let $i : \text{Spec } A \to X \otimes_{F_0} O_K$ be the map defined by φ and $O_K \to A$. Then $(A\hat{\otimes}_{F_0}O_K)^\wedge$ is isomorphic to the coordinate ring of the formal completion of $X \otimes_{F_0} O_K$ along the closed immersion $i : \text{Spec } A \to X \otimes_{F_0} O_K$. Hence $(A\hat{\otimes}_{F_0}O_K)^\wedge$ is formally of finite type and formally smooth over O_K.
2. Since the map $(A\hat{\otimes}_{F_0}O_K)^\wedge \to A$ is surjective, the assertion follows from 1.
3. Since $(B\hat{\otimes}_{F_0}O_K)^\wedge = B \otimes_A (A\hat{\otimes}_{F_0}O_K)^\wedge$, the assertion follows. □

Thus, we obtain a functor (Finite Étale/K) $\to \mathcal{E}mb_{O_K}$ sending a finite étale K-algebra L to $((O_L\hat{\otimes}_{F_0}O_K)^\wedge \to O_L)$. For a rational number $j > 0$, we have a sequence of functors

$$(\text{Finite Étale}/K) \longrightarrow \mathcal{E}mb_{O_K} \longrightarrow (\text{smooth Affinoid}/K) \longrightarrow G_K\text{-}(\text{Aff}/\bar{F}).$$

We also let $\bar{X}^j$ denote the composite functor (Finite Étale/K) $\to G_K$-(Aff/$\bar{F}$). For a finite étale K-algebra L, we have

$$\bar{X}^j(L) = \bar{X}^j((O_L\hat{\otimes}_{F_0}O_K)^\wedge \to O_L).$$

We define an object Θ^j of G_K-(Aff/$\bar{F}$) to be the $\bar{F}$-vector space $\Theta^j = Hom_F(\hat{\Omega}_{O_K/F_0} \otimes_{O_K} F, N^j)$ regarded as an affine scheme over $\bar{F}$ with a natural G_K-action. Let G_K-(Finite Étale/Θ^j) denote the subcategory of G_K-(Aff/$\bar{F}$) whose objects are finite étale schemes over Θ^j and morphisms are over Θ^j.

LEMMA 2.4 *For a rational number $j > 1$, the functor $\bar{X}^j$: (Finite Étale/K) $\to$ G_K-(Aff/$\bar{F}$) induces a functor $\bar{X}^j$: (Finite Étale/K)$^{\leq j+}$ $\to$ G_K-(Finite Étale/Θ^j).*

Proof. The canonical map $\hat{\Omega}_{O_K/F_0} \otimes_{O_K} (O_K\hat{\otimes}_{F_0}O_K)^\wedge \to \hat{\Omega}_{(O_K\hat{\otimes}_{F_0}O_K)^\wedge/O_K}$ is an isomorphism by the definition of $(O_K\hat{\otimes}_{F_0}O_K)^\wedge$. Hence, we obtain isomorphisms $\bar{X}^j(K) \to \bar{C}^j((O_K\hat{\otimes}_{F_0}O_K)^\wedge \to O_K) \to \Theta^j$ by Lemma 1.14.3. We identify $\bar{X}^j(K)$ with Θ^j by this isomorphism. Let L be a finite étale K-algebras whose ramification is bounded by $j+$. Then, by Corollary 1.16, the map $\bar{X}^j(L) \to \bar{X}^j(K) = \Theta^j$ is finite and étale. Thus the assertion is proved. □

The construction in this subsection is independent of the choice of perfect subfield $F_0 \subset F$ by the following Lemma.

Lemma 2.5 *Let K be a complete discrete valuation field of characteristic $p > 0$ satisfying the condition (F). Let F_0 and F_0' be perfect subfields of F such that F is finitely generated over F_0 and F_0'.*
1. There exists a perfect subfield F_0'' of F containing F_0 and F_0'.
2. Assume $F_0 \subset F_0'$. Then F_0' is a finite separable extension of F_0. For the integer ring A of a finite étale algebra over K, the canonical map $(A\hat{\otimes}_{F_0}O_K)^\wedge \to (A\hat{\otimes}_{F_0'}O_K)^\wedge$ is an isomorphism.

Proof. 1. The maximum perfect subfield $\bigcap_n F^{p^n}$ of F contains F_0 and F_0' as subfields.
2. Since F_0' is a perfect subfield of a finitely generated field F over F_0, it is a finite extension of F_0. Since the canonical map $(A\hat{\otimes}_{F_0}O_K)^\wedge \to (A\hat{\otimes}_{F_0'}O_K)^\wedge$ is finite étale and the induced map $(A\hat{\otimes}_{F_0}O_K)^\wedge/\mathfrak{m}_{(A\hat{\otimes}_{F_0}O_K)^\wedge} \to (A\hat{\otimes}_{F_0'}O_K)^\wedge/\mathfrak{m}_{(A\hat{\otimes}_{F_0'}O_K)^\wedge}$ is an isomorphism, the assertion follows. □

2.3 Functoriality of the closed fibers of tubular neighborhoods: A mixed characteristic case

In this subsection, we keep the assumption:

(F) There exists a perfect subfield F_0 of F such that F is finitely generated over F_0.

We do not assume that the characterisic of K is p. Under the assumption (F), there exists a subfield K_0 of K such that $O_{K_0} = O_K \cap K_0$ is a complete discrete valuation ring with residue field F_0. If K is of characteristic 0, the fraction field K_0 of the ring of the Witt vectors $W(F_0) = O_{K_0}$ regarded as a subfield of K satisfies the conditions. If K is of characteristic p, we naturally identify F_0 as a subfield of K and the subfield $F_0((t))$ for any non-zero element $t \in \mathfrak{m}_K$ satisfies the conditions. In this subsection, we take a subfield K_0 of K such that $O_{K_0} = O_K \cap K_0$ is a complete discrete valuation ring with residue field F_0. Here, we do *not* define a functor (Finite Étale/K) $\to \mathcal{E}mb_{O_K}$. Instead, we introduce a new category $\mathcal{E}mb_{K,O_{K_0}}$ and a functor

$$\mathcal{E}mb_{K,O_{K_0}} \to \mathcal{E}mb_{O_K}.$$

In this subsection, A denotes the integer ring of a finite étale K-algebra and π_0 denotes a prime element of the subfield $K_0 \subset K$. For a complete Noetherian local O_{K_0}-algebra R formally smooth over O_{K_0}, we define its relative dimension over O_{K_0} to be the sum $\mathrm{tr.deg}(E/k) + \dim_E \mathfrak{m}_R/(\pi_0, \mathfrak{m}_R^2)$ of the transcendental degree of $E = R/\mathfrak{m}_R$ over k and the dimension $\dim_E \mathfrak{m}_R/(\pi_0, \mathfrak{m}_R^2)$.

Definition 2.6 *Let K be a complete discrete valuation field and K_0 be a subfield of K such that $O_{K_0} = O_K \cap K_0$ is a complete discrete valuation ring with perfect residue field F_0 and that F is finitely generated over F_0.*
1. We define $\mathcal{E}mb_{K,O_{K_0}}$ to be the category whose objects and morphisms are as follows. An object of $\mathcal{E}mb_{K,O_{K_0}}$ is a triple $(\mathbf{A}_0 \to A)$ where:

- *A is the integer ring of a finite étale K-algebra.*
- *$\mathbf{A}_0$ is a complete semi-local Noetherian O_{K_0}-algebra formally smooth of relative dimension* $\mathrm{tr.deg}(F/F_0)+1$ *over O_{K_0}.*
- *$\mathbf{A}_0 \to A$ is a regular surjection of codimension 1 of O_{K_0}-algebras inducing an isomorphism $\mathbf{A}_0/\mathfrak{m}_{\mathbf{A}_0} \to A/\mathfrak{m}_A$.*

A morphism $(f, \mathbf{f}) : (\mathbf{A}_0 \to A) \to (\mathbf{B}_0 \to B)$ is a pair of an O_K-homomorphism $f : A \to B$ and an O_{K_0}-homomorphism $\mathbf{f} : \mathbf{A}_0 \to \mathbf{B}_0$ such that the diagram

$$\begin{array}{ccc} \mathbf{A}_0 & \longrightarrow & A \\ {\scriptstyle \mathbf{f}}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \mathbf{B}_0 & \longrightarrow & B \end{array}$$

is commutative.
2. For the integer ring A of a finite étale K-algebra, we define $\mathcal{E}mb_{K,O_{K_0}}(A)$ to be the subcategory of $\mathcal{E}mb_{K,O_{K_0}}$ whose objects are of the form $(\mathbf{A}_0 \to A)$ and morphisms are of the form $(\mathrm{id}_A, \mathbf{f})$.
3. We say that a morphism $(\mathbf{A}_0 \to A) \to (\mathbf{B}_0 \to B)$ is finite flat if $\mathbf{A}_0 \to \mathbf{B}_0$ is finite flat and the map $\mathbf{B}_0 \otimes_{\mathbf{A}_0} A \to B$ is an isomorphism.

Lemma 2.7 *1. If A is the integer ring of a finite étale K-algebra, then the category $\mathcal{E}mb_{K,O_{K_0}}(A)$ is non-empty.*
2. Let $(\mathbf{A}_0 \to A)$ and $(\mathbf{B}_0 \to B)$ be objects of $\mathcal{E}mb_{K,O_{K_0}}$ and $A \to B$ be an O_K-homomorphism. Then there exists a homomorphism $(\mathbf{A}_0 \to A) \to (\mathbf{B}_0 \to B)$ in $\mathcal{E}mb_{K,O_{K_0}}$ extending $A \to B$.
3. Let $(\mathbf{A}_0 \to A) \to (\mathbf{B}_0 \to B)$ be a morphism of $\mathcal{E}mb_{K,O_{K_0}}$. If a prime element π_0 of K_0 is not a prime element of any factor of A, then the map $(\mathbf{A}_0 \to A) \to (\mathbf{B}_0 \to B)$ is finite and flat.

Proof. 1. We may assume A is local. Take a transcendental basis $(\bar{t}_1, \dots, \bar{t}_m)$ of the residue field E of A over k such that E is a finite separable extension of $k(\bar{t}_1, \dots, \bar{t}_m)$. Take a lifting $(t_1, \dots, t_m)$ in O_K of $(\bar{t}_1, \dots, \bar{t}_m)$ and a prime element t_0 of A. Then A is unramified over the completion of the local ring of $O_{K_0}[T_0, \dots, T_m]$ at the prime ideal (π_0, T_0) by the map sending T_i to t_i. Hence there are an étale scheme X over $\mathbb{A}^{m+1}_{O_{K_0}}$, a point ξ of X above (π_0, T_0) and a regular surjection $\varphi : \hat{O}_{X,\xi} \to A$ of codimension 1. Let $\mathbf{A}_0$ be the O_{K_0}-algebra $\hat{O}_{X,\xi}$. Then $(\mathbf{A}_0 \to A)$ is an object of $\mathcal{E}mb_{K,O_{K_0}}$.
2. Since $\mathbf{A}_0$ is formally smooth over O_{K_0}, it follows from that $\mathbf{B}_0$ is the formal completion of itself with respect to the surjection $\mathbf{B}_0 \to B$.
3. We may assume A and B are local. We show that the map $\mathbf{B}_0 \otimes_{\mathbf{A}_0} A \to B$ is an isomorphism. Let f be a generator of the kernel of $\mathbf{A}_0 \to A$ and consider the class of f in $\mathfrak{m}_{\mathbf{A}_0}/\mathfrak{m}^2_{\mathbf{A}_0}$. We show that the image of the class of f in $\mathfrak{m}_{\mathbf{B}_0}/\mathfrak{m}^2_{\mathbf{B}_0}$ is not 0. Let $t_0 \in \mathbf{A}_0$ and $t'_0 \in \mathbf{B}_0$ be liftings of prime

elements of A and B respectively. By the assumption that π_0 is not a prime element, the surjection $\hat{\Omega}_{\mathbf{A}_0/O_{K_0}} \to \hat{\Omega}_{A/O_{K_0}}$ induces an isomorphism $\hat{\Omega}_{\mathbf{A}_0/O_{K_0}} \otimes_{\mathbf{A}_0} A/\mathfrak{m}_A \to \hat{\Omega}_{A/O_{K_0}} \otimes_A A/\mathfrak{m}_A$. Hence the image of dt_0 is a basis of the kernel of $\hat{\Omega}_{\mathbf{A}_0/O_{K_0}} \otimes_{\mathbf{A}_0} A/\mathfrak{m}_A \to \Omega_{(A/\mathfrak{m}_A)/k}$. Therefore, (π_0, t_0) is a basis of $\mathfrak{m}_{\mathbf{A}_0}/\mathfrak{m}^2_{\mathbf{A}_0}$. Further, by the assumption that π_0 is not a prime element, the kernel of the map $\mathfrak{m}_{\mathbf{A}_0}/\mathfrak{m}^2_{\mathbf{A}_0} \to \mathfrak{m}_A/\mathfrak{m}^2_A$ is generated by the class of π_0. Hence the class of f is a non-zero multiplie of the class of π_0. Similarly (π_0, t'_0) is a basis of $\mathfrak{m}_{\mathbf{B}_0}/\mathfrak{m}^2_{\mathbf{B}_0}$. Thus the image of f in $\mathfrak{m}_{\mathbf{B}_0}/\mathfrak{m}^2_{\mathbf{B}_0}$ is not zero as is claimed. Hence the kernel of $\mathbf{B}_0 \to B$ is also generated by the image of f and the map $\mathbf{B}_0 \otimes_{\mathbf{A}_0} A \to B$ is an isomorphism. Since B is finite over A, $\mathbf{B}_0$ is also finite over $\mathbf{A}_0$ by Nakayama's lemma. Since $\dim \mathbf{A}_0 = \dim \mathbf{B}_0 = 2$, the assertion follows by EGA Chap 0_{IV} Corollaire (17.3.5) (ii). □

COROLLARY 2.8 *Let A be the integer ring of a finite étale K-algebra. If a prime element π_0 of K_0 is not a prime element of any factor of A, then every morphism of $\mathcal{E}mb_{K,O_{K_0}}(A)$ is an isomorphism.*

Proof. If $(\mathbf{A}_0 \to A) \to (\mathbf{A}'_0 \to A)$ is a map, the map $\mathbf{A}_0 \to \mathbf{A}'_0$ is finite flat of degree 1 by Lemma 2.7.3. Hence it is an isomorphism. □

We define a functor $\mathcal{E}mb_{K,O_{K_0}} \to \mathcal{E}mb_{O_K}$.

LEMMA 2.9 *Let $(\mathbf{A}_0 \to A)$ be an object of $\mathcal{E}mb_{K,O_{K_0}}$.*
1. Let $(\mathbf{A}_0/\mathfrak{m}^n_{\mathbf{A}_0} \otimes_{O_{K_0}} O_K)^\wedge$ denote the formal completion of $\mathbf{A}_0/\mathfrak{m}^n_{\mathbf{A}_0} \otimes_{O_{K_0}} O_K$ of the surjection $\mathbf{A}_0/\mathfrak{m}^n_{\mathbf{A}_0} \otimes_{O_{K_0}} O_K \to A/\mathfrak{m}^n_A$ sending $a \otimes b$ to ab. Then the projective limit

$$(\mathbf{A}_0 \hat{\otimes}_{O_{K_0}} O_K)^\wedge = \varprojlim_n (\mathbf{A}_0/\mathfrak{m}^n_{\mathbf{A}_0} \otimes_{O_{K_0}} O_K)^\wedge$$

is an O_K-algebra formally of finite type and formally smooth over O_K.
2. Let $(\mathbf{A}_0 \hat{\otimes}_{O_{K_0}} O_K)^\wedge \to A$ be the limit of the surjections $(\mathbf{A}_0/\mathfrak{m}^n_{\mathbf{A}_0} \otimes_{O_{K_0}} O_K)^\wedge \to A/\mathfrak{m}^n_A$. Then $((\mathbf{A}_0 \hat{\otimes}_{O_{K_0}} O_K)^\wedge \to A)$ is an object of $\mathcal{E}mb_{O_K}$.
3. Let $(\mathbf{A}_0 \to A) \to (\mathbf{B}_0 \to B)$ be a morphism of $\mathcal{E}mb_{K,O_{K_0}}$. Then it induces a morphism $((\mathbf{A}_0 \hat{\otimes}_{O_{K_0}} O_K)^\wedge \to A) \to ((\mathbf{B}_0 \hat{\otimes}_{O_{K_0}} O_K)^\wedge \to B)$ of $\mathcal{E}mb_{O_K}$.

Proof. 1. We may assume A and hence $\mathbf{A}_0$ are local. Let E be the residue field of A and take a transcendental basis $(\bar{t}_1, \ldots, \bar{t}_m)$ of E over k such that E is a finite separable extension of $k(\bar{t}_1, \ldots, \bar{t}_m)$. Take a lifting $(t_1, \ldots, t_m)$ in $\mathbf{A}_0$ of $(\bar{t}_1, \ldots, \bar{t}_m)$. By our assumption, the quotient ring $\mathbf{A}_0/\pi_0\mathbf{A}_0$ is a regular local ring of dimension 1 and hence is a discrete valuation ring. Take a lifting $t_0 \in \mathbf{A}_0$ of a prime element of $\mathbf{A}_0/\pi_0\mathbf{A}_0$. We define a map $O_{K_0}[T_0, \ldots, T_m] \to \mathbf{A}_0$ by sending T_i to t_i. Then $\mathbf{A}_0$ is finite étale over the completion of the local ring of $O_{K_0}[T_0, \ldots, T_m]$ at the prime ideal (T_0, π_0). Hence there exist an étale scheme X over $\mathbb{A}^{m+1}_{O_{K_0}}$, a point ξ of X above (T_0, π_0) and a O_{K_0}-isomorphism $\varphi : \hat{O}_{X,\xi} \to \mathbf{A}_0$. Let $i : \operatorname{Spec} A \to X \otimes_{O_{K_0}} O_K$ be the map defined by φ and $O_K \to A$. Then $(\mathbf{A}_0 \hat{\otimes}_{O_{K_0}} O_K)^\wedge$ is isomorphic to the coordinate ring of the formal completion

of $X \otimes_{O_{K_0}} O_K$ along the closed immersion $i : \mathrm{Spec}\ A \to X \otimes_{O_{K_0}} O_K$. Hence $(\mathbf{A}_0 \hat{\otimes}_{O_{K_0}} O_K)^\wedge$ is formally of finite type and formally smooth over O_K.
2. Since the map $(\mathbf{A}_0 \hat{\otimes}_{O_{K_0}} O_K)^\wedge \to A$ is surjective, the assertion follows from 1.
3. Clear. □

In the rest of this subsection, we put $\mathbf{A} = (\mathbf{A}_0 \hat{\otimes}_{O_{K_0}} O_K)^\wedge$ for an object $(\mathbf{A}_0 \to A)$ of $\mathcal{E}mb_{K,O_{K_0}}$. By Lemma 2.9, we obtain a functor $\mathcal{E}mb_{K,O_{K_0}} \to \mathcal{E}mb_{O_K}$ sending $(\mathbf{A}_0 \to A)$ to $(\mathbf{A} \to A)$. For a rational number $j > 0$, we have a sequence of functors

$$\mathcal{E}mb_{K,O_{K_0}} \longrightarrow \mathcal{E}mb_{O_K} \xrightarrow{X^j} (\text{smooth Affinoid}/K) \longrightarrow G_K\text{-}(\mathrm{Aff}/\bar{F}).$$

We also let $\bar{X}^j$ denote the composite functor $\mathcal{E}mb_{K,O_{K_0}} \to G_K\text{-}(\mathrm{Aff}/\bar{F})$. For an object $(\mathbf{A}_0 \to A)$ of $\mathcal{E}mb_{K,O_{K_0}}$, we have $\bar{X}^j(\mathbf{A}_0 \to A) = \bar{X}^j((\mathbf{A}_0 \hat{\otimes}_{O_{K_0}} O_K)^\wedge \to A)$.

We study the dependence of the construction on the choice of a subfield $K_0 \subset K$, assuming the characteristic of K is 0.

Lemma 2.10 *Let K be a complete discrete valuation field of mixed characteristic satisfying the condition (F). Let K_0 and K_0' be subfields of K such that $O_{K_0} = O_K \cap K_0$ and $O_{K_0'} = O_K \cap K_0'$ are complete discrete valuation rings with perfect residue field F_0 and F_0' and that F is finitely generated over F_0 and F_0'.*
1. There exists a subfield K_0'' of K such that $O_{K_0''} = O_K \cap K_0''$ is a complete discrete valuation ring with perfect residue field and that K_0'' contains K_0 and K_0' as subfields.
2. Assume $K_0 \subset K_0'$. Then K_0' is a finite extension of K_0. For an object $(\mathbf{A}_0 \to A)$ of $\mathcal{E}mb_{K,O_{K_0}}$, the formal completion $\mathbf{A}_0' \to A$ of the surjection $\mathbf{A}_0 \otimes_{O_{K_0}} O_{K_0'} \to A$ defines an object $(\mathbf{A}_0' \to A)$ of $\mathcal{E}mb_{K,O_{K_0'}}$. Further, we have a canonical isomorphism $((\mathbf{A}_0' \hat{\otimes}_{O_{K_0'}} O_K)^\wedge \to A) \to ((\mathbf{A}_0 \hat{\otimes}_{O_{K_0}} O_K)^\wedge \to A)$ in $\mathcal{E}mb_{O_K}$.

Proof. 1. By Lemma 2.5, we may assume the residue fields F_0 and F_0' are the maximum perfect subfields of F. Then both of K_0 and K_0' are finite over the fraction field of $W(F_0)$ regarded as a subfield of K. Hence it is sufficient to take the composition field.
2. By Lemma 2.5.2, the extension K_0' is finite over K_0. The rest is clear from the construction. □

If K is of characteristic p, the construction in this subsection is related to that in the last subsection as follows. Let K_0 be a subfield of K such that $O_{K_0} = O_K \cap K_0$ is a complete discrete valuation ring with perfect residue field F_0 and that F is finitely generated over F_0. Then, if π_0 is a prime element of K_0, we have an isomorphism $F_0((t)) \to K_0$ sending t to π_0. For the integer ring A of a finite étale algebra over K, let $(A \hat{\otimes}_{F_0} O_{K_0})^\wedge$ denote the projective limit of the formal completions $(A/\mathfrak{m}_A^n \otimes_{F_0} O_{K_0})^\wedge$ of the surjections $A/\mathfrak{m}_A^n \otimes_{F_0} O_{K_0} \to A/\mathfrak{m}_A^n$. The surjection $(A \hat{\otimes}_{F_0} O_{K_0})^\wedge \to A$ defines an object

$((A\hat{\otimes}_{F_0}O_{K_0})^\wedge \to A)$ of $\mathcal{E}mb_{K,O_{K_0}}$. Further, we have a canonical isomorphism $((A\hat{\otimes}_{F_0}O_{K_0})^\wedge\hat{\otimes}_{O_{K_0}}O_K)^\wedge \to A) \to ((A\hat{\otimes}_{F_0}O_K)^\wedge \to A)$ in $\mathcal{E}mb_{O_K}$.
In order to define a functor similar to the functor $(\text{Finite Étale}/K)^{\leq j+} \to (\text{Finite Étale}/\Theta^j)$ in Section 2.2, we assume that π_0 is *not* a prime element of K in the rest of this subsection. Note that if p is not a prime element of K and if the condition (F) is satisfied, there exists a subfield $K_0 \subset K$ with residue field F_0 such that a prime element of K_0 is not a prime element of K.
We compute the twisted normal cone $\bar{C}^j((\mathbf{A}_0\hat{\otimes}_{O_{K_0}}O_K)^\wedge \to A)$ for an object $(\mathbf{A}_0 \to A)$ of $\mathcal{E}mb_{K,O_{K_0}}$. Let $N_{A/\mathbf{A}} = I/I^2$ be the conormal module where I is the kernel of the surjection $\mathbf{A} \to A$. We put $\hat{\Omega}_{O_K/O_{K_0}} = \varprojlim_n \Omega_{(O_K/\mathfrak{m}_K^n)/O_{K_0}}$ and let $\tilde{\Omega}_F$ be the F-vector space $\hat{\Omega}_{O_K/O_{K_0}} \otimes_{O_K} F$. Similarly, we put $\hat{\Omega}_{\mathbf{A}/\mathbf{A}_0} = \varprojlim_n \Omega_{(\mathbf{A}/\mathfrak{m}_{\mathbf{A}}^n)/\mathbf{A}_0}$. We also consider the canonical maps $N_{A/\mathbf{A}} \to \hat{\Omega}_{\mathbf{A}/\mathbf{A}_0} \otimes_{\mathbf{A}} A$ and $\hat{\Omega}_{O_K/O_{K_0}} \otimes_{O_K} A \to \hat{\Omega}_{\mathbf{A}/\mathbf{A}_0} \otimes_{\mathbf{A}} A$.

Lemma 2.11 *Assume π_0 is not a prime element of K and let m be the transcendental dimension of F over k. Let $(\mathbf{A}_0 \to A)$ be an object of $\mathcal{E}mb_{K,O_{K_0}}$. Then,*
1. The dimension of the F-vector space $\tilde{\Omega}_F$ is $m+1$.
2. The map $N_{A/\mathbf{A}} \to \hat{\Omega}_{\mathbf{A}/\mathbf{A}_0} \otimes_{\mathbf{A}} A$ is a surjection and the map $\hat{\Omega}_{O_K/O_{K_0}} \otimes_{O_K} A \to \hat{\Omega}_{\mathbf{A}/\mathbf{A}_0} \otimes_{\mathbf{A}} A$ is an isomorphism. They induce an isomorphism $N_{A/\mathbf{A}} \otimes_A A/\mathfrak{m}_A \to \tilde{\Omega}_F \otimes_F A/\mathfrak{m}_A$.
3. Let $(\mathbf{A}_0 \to A) \to (\mathbf{B}_0 \to B)$ be a morphism of $\mathcal{E}mb_{K,O_{K_0}}$ and put $\mathbf{B} = (\mathbf{B}_0\hat{\otimes}_{O_{K_0}}O_K)^\wedge$. Then, the diagram

$$\begin{array}{ccc} N_{A/\mathbf{A}} \otimes_A A/\mathfrak{m}_A & \longrightarrow & \tilde{\Omega}_F \otimes_F A/\mathfrak{m}_A \\ \downarrow & & \downarrow \\ N_{B/\mathbf{B}} \otimes_B B/\mathfrak{m}_B & \longrightarrow & \tilde{\Omega}_F \otimes_F B/\mathfrak{m}_B \end{array}$$

is commutative.

Proof. 1. By the assumption that π_0 is not a prime element of K, we have an exact sequence $0 \to \mathfrak{m}_K/\mathfrak{m}_K^2 \to \tilde{\Omega}_F \to \Omega_{F/k} \to 0$. Since the F-vector space $\Omega_{F/k}$ is of dimension m, the assertion follows.
2. Since the cokernel of the map $N_{A/\mathbf{A}} \to \hat{\Omega}_{\mathbf{A}/\mathbf{A}_0} \otimes_{\mathbf{A}} A$ is $\Omega_{A/\mathbf{A}_0} = 0$, it is a surjection. By the definition of $\mathbf{A}$, the map $\hat{\Omega}_{O_K/O_{K_0}} \otimes_{O_K} \mathbf{A} \to \Omega_{\mathbf{A}/\mathbf{A}_0}$ is an isomorphism. Hence the map $\hat{\Omega}_{O_K/O_{K_0}} \otimes_{O_K} A \to \hat{\Omega}_{\mathbf{A}/\mathbf{A}_0} \otimes_{\mathbf{A}} A$ is also an isomorphism. Then the codimension of the regular surjection $\mathbf{A} \to A$ is $m+1$ and hence $N_{A/\mathbf{A}}$ is free of rank $m+1$. Since the induced map $N_{A/\mathbf{A}} \otimes_A A/\mathfrak{m}_A \to \tilde{\Omega}_F \otimes_F A/\mathfrak{m}_A$ is a surjection of free $A/\mathfrak{m}_A$-modules of rank $m+1$, it is an isomorphism.
3. By the assumption that π_0 is not a prime element of K, every map in $\mathcal{E}mb_{K,O_{K_0}}$ is finite flat by Lemma 2.7.3. Hence the assertion follows. □

For a rational number $j > 0$, let Θ^j be the $\bar{F}$-vector space $\Theta^j = Hom_F(\tilde{\Omega}_F, N^j)$ regarded as an affine scheme over $\bar{F}$.

Corollary 2.12 *Assume that π_0 is not a prime element of K. Let $(\mathbf{A}_0 \to A)$ be an object of $\mathcal{E}mb_{K,O_{K_0}}$ and let $(\mathbf{A} \to A)$ be the image in $\mathcal{E}mb_{K,O_{K_0}}$. Let $j > 0$ be a rational number.*
1. The isomorphism in Lemma 2.11.2 induces an isomorphism $\bar{C}^j(\mathbf{A} \to A) \to \Theta^j \otimes_{\bar{F}} A_{\bar{F},\mathrm{red}}$.
2. Let $(\mathbf{A}_0 \to A) \to (\mathbf{B}_0 \to B)$ be a morphism of $\mathcal{E}mb_{K,O_{K_0}}$. Then the diagram

$$\begin{array}{ccccc} \bar{X}^j(\mathbf{B} \to B) & \longrightarrow & \bar{C}^j(\mathbf{B} \to B) & \longrightarrow & \Theta^j \otimes_{\bar{F}} B_{\bar{F},\mathrm{red}} \\ \downarrow & & \downarrow & & \downarrow \\ \bar{X}^j(\mathbf{A} \to A) & \longrightarrow & \bar{C}^j(\mathbf{A} \to A) & \longrightarrow & \Theta^j \otimes_{\bar{F}} A_{\bar{F},\mathrm{red}} \end{array}$$

is commutative.
3. If the ramification of $A \otimes_{O_K} K$ is bounded by $j+$ and $j > 1$, then the composition $\bar{X}^j(\mathbf{A} \to A) \to \bar{C}^j(\mathbf{A} \to A) \to \Theta^j \otimes_{\bar{F}} A_{\bar{F},\mathrm{red}} \to \Theta^j$ is finite and étale.

Proof. 1. Since the surjection $\mathbf{A} \to A$ is regular, the assertion follows from the isomorphism in Lemma 2.11.2.
2. The left square is commutative by the construction. The commutativity of the right square is a consequence of Lemma 2.11.3.
3. By Lemma 2.7, there exist an embedding $(\mathbf{A}'_0 \to O_K)$ in $\mathcal{E}mb_{K,O_{K_0}}(O_K)$ and a finite flat morphism $(\mathbf{A}'_0 \to O_K) \to (\mathbf{A}_0 \to A)$. Since the ramification is bounded by $j+$, the finite map $\bar{X}^j(\mathbf{A} \to A) \to \bar{C}^j(\mathbf{A} \to A)$ is étale by Corollary 1.16.3. Since $A_{\bar{F},\mathrm{red}}$ is étale over $\bar{F}$, the assertion follows from 1 and 2. □.

For a rational number $j > 0$, we regard Θ^j as an object of G_K-(Aff$/\bar{F}$) with the natural G_K-action. Let G_K-(Finite Étale/Θ^j) denote the subcategory of G_K-(Aff$/\bar{F}$) whose objects are finite étale schemes over Θ^j and morphisms are over Θ^j. Let $\mathcal{E}mb^{\leq j+}_{K,O_{K_0}}$ denote the full subcategory of $\mathcal{E}mb_{K,O_{K_0}}$ consisting of the objects $(\mathbf{A}_0 \to A)$ such that the ramifications of $A \otimes_{O_K} K$ are bounded by $j+$. By Corollary 2.12, the functor $\bar{X}^j : \mathcal{E}mb_{K,O_{K_0}} \to G_K$-(Aff$/\bar{F}$) induces a functor $\bar{X}^j : \mathcal{E}mb^{\leq j+}_{K,O_{K_0}} \to G_K$-(Finite Étale/$\Theta^j$).

We show that the functor $\bar{X}^j : \mathcal{E}mb^{\leq j+}_{K,O_{K_0}} \to G_K$-(Finite Étale/$\Theta^j$) further induces a functor (Finite Étale$/K)^{\leq j+} \to G_K$-(Finite Étale/Θ^j).

Lemma 2.13 *Assume π_0 is not a prime element of K. Let $(f, \mathbf{f}), (g, \mathbf{g}) : (\mathbf{A}_0 \to A) \to (\mathbf{B}_0 \to B)$ be maps in $\mathcal{E}mb_{K,O_{K_0}}$ and $j > 1$ be a rational number. If the ramifications of $A \otimes_{O_K} K$ and $B \otimes_{O_K} K$ are bounded by $j+$ and if $f = g$, then the induced maps*

$$(f, \mathbf{f})_*, (g, \mathbf{g})_* : \bar{X}^j(\mathbf{A}_0 \to A) \longrightarrow \bar{X}^j(\mathbf{B}_0 \to B)$$

are equal.

Proof. By Corollary 2.12, the schemes $\bar{X}^j(\mathbf{A}_0 \to A)$ and $\bar{X}^j(\mathbf{B}_0 \to B)$ are finite étale over Θ^j and the maps $(f,\mathbf{f})_*, (g,\mathbf{g})_* : \bar{X}^j(\mathbf{A}_0 \to A) \to \bar{X}^j(\mathbf{B}_0 \to B)$ are maps over Θ^j. Hence they are determined by the restrictions on the inverse images of a point. The inverse images of the origin $0 \in \Theta^j$ are canonically identified with the sets $Hom_{O_K}(A, \bar{K})$ and $Hom_{O_K}(B, \bar{K})$ respectively by Lemma 2.1. Hence the assertion follows. □

Corollary 2.14 *Assume π_0 is not a prime element of K. Let $j > 1$ be a rational number.*
1. Let L be a finite étale K-algebra with ramification bounded by $j+$. Then the system $\bar{X}^j(\mathbf{A}_0 \to O_L)$ parametrized by the objects $(\mathbf{A}_0 \to O_L)$ of $\mathcal{E}mb_{K,O_{K_0}}(O_L)$ is constant and the limit

$$\bar{X}^j(L) = \varprojlim_{(\mathbf{A}_0 \to O_L) \in \mathcal{E}mb_{K,O_{K_0}}(O_L)} \bar{X}^j(\mathbf{A}_0 \to O_L)$$

is a finite étale scheme over Θ^j.
2. The functor $\bar{X}^j : \mathcal{E}mb^{\leq j+}_{K,O_{K_0}} \to G_K$-(Finite Étale/$\Theta^j$) induces a functor

$$\bar{X}^j : (\text{Finite Étale}/K)^{\leq j+} \longrightarrow G_K\text{-(Finite Étale}/\Theta^j).$$

Proof. 1. By Corollary 2.8 and by the assumption that π_0 is not a prime element, every map in $\mathcal{E}mb_{K,O_{K_0}}(O_L)$ induces an isomorphism. By Lemma 2.7.1, the category $\mathcal{E}mb_{K,O_{K_0}}(O_L)$ is connected. To see that the system is constant, it suffices to apply Lemma 2.13 for $f = g = \mathrm{id}_{O_L}$. The map $\bar{X}^j(L) \to \Theta^j$ is finite étale by Corollary 2.12.3.
2. It is also an immediate consequence of Lemma 2.13. □
By Lemma 2.10 and the canonical isomorphism $\hat{\Omega}_{O_K/O_{K_0}} \otimes_{O_K} F \to \hat{\Omega}_{O_K/O_{K'_0}} \otimes_{O_K} F$, the functor $\bar{X}^j$: (Finite Étale$/K)^{\leq j+} \to G_K$-(Finite Étale/Θ^j) is independent of the choice of subfield K_0 if the characteristic of K is 0. If the characteristic of K is p, it is the same as that defined in Section 2.2.

2.4 Proof of commutativity

Now we are ready to prove the main result. For an integer m prime to p, let I_m be the unique open subgroup of the inertia subgroup $I \subset G_K$ of index m.

Theorem 2.15 *Let K be a complete discrete valuation field. Let $j > 1$ be a rational number and m be the prime-to-p part of the denominator of j. Assume either K has equal characteristics $p > 0$ or K has mixed characteristic and p is not a prime element. Then we have the following.*
1. The graded piece $Gr^j G_K = G^j_K / G^{j+}_K$ is abelian.
2. The commutator $[I_m, G^j_K]$ is a subgroup of G^{j+}_K. In particular, $Gr^j G_K$ is a subgroup of the center of the pro-p-group G^{1+}_K / G^{j+}_K.

Proof. We first prove the case where the condition
(F) There exists a perfect subfield F_0 of F such that F is finitely generated over F_0.
is satisfied. We use the functor $\bar{X}^j$: (Finite Étale$/K)^{\leq j+} \to G_K$-(Finite Étale$/\Theta^j$) defined in Sections 2.2 and 2.3.
Let L be a finite Galois extension of K of ramification bounded by $j+$ and put $G = \mathrm{Gal}(L/K)$. To prove 1, it is sufficient to show that G^j is commutative. By the definition of the functor, the image $\bar{X}^j(L)$ is a finite étale covering of Θ^j with a left action of G_K. We call this action of G_K on $\bar{X}^j(L)$ the arithmetic action. On the other hand, by functoriality, we have a right action of G on $\bar{X}^j(L)$, which commutes with the arithmetic action of G_K. We call this action of G on $\bar{X}^j(L)$ the geometric action. We identify the inverse image in $\bar{X}^j(L)$ of the origin of Θ^j with $\Phi(L)$ as in Lemma 2.1. The arithmetic action of $\sigma \in G_K$ on $\Phi(L) = Hom_K(L, \bar{K})$ is given by $f \mapsto \sigma \circ f$ and the geometric action of $\tau \in G$ is given by $f \mapsto f \circ \tau$. Hence $\Phi(L)$ is a G-torsor and the étale covering $\bar{X}^j(L)$ is also a G-torsor over Θ^j.
The stabilizer in G_K of each connected component of $\bar{X}^j(L)$ with respect to the arithmeric action is equal to G_K^j since $\Phi^j(L)$ is identified with $\pi_0(\bar{X}^j(L))$. Take a connected component $\bar{X}^j(L)_0$ of $\bar{X}^j(L)$. Then, the stabilizer of the intersection $\bar{X}^j(L)_0 \cap \Phi(L)$ in G, with respect to the geometric action, is equal to G^j. Hence the stabilizer of the component $\bar{X}^j(L)_0$ in G, with respect to the geometric action, is also equal to G^j and $\bar{X}^j(L)_0$ is a connected G^j-torsor over Θ^j. Therefore the map $G^j \to \mathrm{Aut}(\bar{X}^j(L)_0/\Theta^j)$ is an isomorphism.
On the other hand, by the assumption that $j > 1$, the group G_K^j is a subgroup of the wild inertia subgroup $G_K^{1+} = P$. Hence the restriction to G_K^j of the arithmetic action on Θ^j is trivial and we get a map $G_K^j \to \mathrm{Aut}(\bar{X}^j(L)_0/\Theta^j)$. Since G_K^j acts on the intersection $\bar{X}^j(L)_0 \cap \Phi(L)$ transitively, the map $G_K^j \to \mathrm{Aut}(\bar{X}^j(L)_0/\Theta^j)$ is surjective. Since the geometric action of G^j and the arithmetic action of G_K^j on $\bar{X}^j(L)_0$ are commutative to each other, the group $G^j \simeq \mathrm{Aut}(\bar{X}^j(L)_0/\Theta^j)$ is commutative. Thus assertion 1 is proved in this case.
We prove assertion 2 assuming the condition (F). We define a canonical map $\pi_1^{\mathrm{ab}}(\Theta^j) \to Gr^j G_K$ as follows. By 1, the image of the functor $\bar{X}^j$: (Finite Étale$/K)^{\leq j+} \to G_K$-(Finite Étale$/\Theta^j$) is in the full subcategory consisting of abelian coverings. Taking the Galois groups, we obtain a map $\pi_1^{\mathrm{ab}}(\Theta^j) \to G_K/G_K^{j+}$ inducing a surjection

$$\pi_1^{\mathrm{ab}}(\Theta^j) \longrightarrow Gr^j G_K.$$

The canonical map $\pi_1^{\mathrm{ab}}(\Theta^j) \to Gr^j G_K$ is compatible with the actions of G_K. The action of G_K on $\pi_1^{\mathrm{ab}}(\Theta^j)$ is induced by that on Θ^j and the action on $Gr^j G_K$ is by conjugation. Since the subgroup I_m acts trivially on Θ^j, it also acts trivially on $\pi_1^{\mathrm{ab}}(\Theta^j)$. Hence, assertion 2 follows in this case by the compatibility of the surjection $\pi_1^{\mathrm{ab}}(\Theta^j) \to Gr^j G_K$ with the G_K-action.
To reduce the general case to the special case proved above, we show the following Lemma.

Lemma 2.16 *Let K be a complete discrete valuation field and K_0 be a subfield of K such that $O_{K_0} = O_K \cap K$ is a complete discrete valuation ring with perfect residue field F_0. Then there exist a filtered family of subextensions $K_\mu \subset K, \mu \in M$ of K_0 satisfying the following conditions:*
For each $\mu \in M$, the intersection $O_{K_\mu} = O_K \cap K_\mu$ is a complete discrete valuation ring and the residue field F_μ is finitely generated over F_0, the residue field F is a separable extension of F_μ and a prime element of K_μ is a prime element of K. The residue field F is equal to the union $\varinjlim_{\mu\in M} F_\mu$.

Proof. Let π_0 be a prime element of K_0. Take a transcendental basis $(\bar{t}_\lambda)_{\lambda\in\Lambda}$ of F over F_0 such that F is separable over $F_0(\bar{t}_\lambda, \lambda \in \Lambda)$. We take liftings $t_\lambda \in O_K, \lambda \in \Lambda$ of $\bar{t}_\lambda$. For a finite subset $\sigma \subset \Lambda$, let $K_{0,\sigma}$ be the fraction field of the completion of the local ring at the prime ideal (π_0) of the ring $O_{K_0}[T_\lambda, \lambda \in \sigma]$ and regard it as a subfield of K. Let $K_{0,\mu} \subset K, \mu \in M_0$ be the family of finite unramified subextensions of $K_{0,\sigma}, \sigma \subset \Lambda$. Let K_0' be the completion of the union $\varinjlim_{\mu\in M_0} K_{0,\mu}$. Then K is a finite totally ramified extension of K_0'. Hence there is an index $\mu_0 \in M_0$ and a finite totally ramified extension K_{μ_0} of K_{0,μ_0} such that K is the composite of K_{μ_0} and K_0'. We put $M = \{\mu \in M_0 : K_{0,\mu_0} \subset K_{0,\mu}\}$. Then the family $K_\mu = K_{\mu_0} K_{0,\mu}, \mu \in M$ satisfies the conditions. □

We complete the proof of Theorem. It is sufficient to show assertion 2. Let $F_0 = \bigcap_n F^{p^n}$ be the maximum perfect subfield of the residue field F. If the characteristic of K is positive, we take a element $\pi_0 \in \mathfrak{m}_K^2, \neq 0$ of K and put $K_0 = F_0((\pi_0)) \subset K$. If the characteristic of K is 0, let K_0 be the fraction field of $W(F_0)$ and regard it as a subfield of K. By the assumption that p is not a prime element of K, a prime element of K_0 is not a prime element of K. Let $K_\mu, \mu \in M$ be a family of subfields of K as in Lemma 2.16. Since K_0 is a subfield of K_μ satisfying the condition (F) and a prime element of K_0 is not a prime element of K_μ, we have $[I_{m,K_\mu}, G^j_{K_\mu}] \subset G^{j+}_{K_\mu}$ for $\mu \in M$.

Since $K' = \varprojlim_{\mu\in M} K_\mu$ is a Henselian discrete valuation field and K is the completion of K', the canonical maps $G_K \to G_{K'} \to \varprojlim_{\mu\in M} G_{K_\mu}$ are isomorphisms. It induces an isomorphism $I_{m,K} \to \varprojlim_\mu I_{m.K_\mu}$. By Lemma 2.2 and by the assumption that the residue field F is separable over F_μ and a prime element of K_μ is a prime element of K, the map $G^j_K \to G^j_{K_\mu}$ is surjective. Hence we have isomorphisms $G^j_K \to \varprojlim_{\mu\in M} G^j_{K_\mu}$ and $G^{j+}_K \to \varprojlim_{\mu\in M} G^{j+}_{K_\mu}$. By taking the limit of $[I_{m,K_\mu}, G^j_{K_\mu}] \subset G^{j+}_{K_\mu}$, we obtain $[I_{m,K}, G^j_K] \subset G^{j+}_K$. □

3 Some generalities on log structures

To study the logarithmic filtration in later sections, we recall and establish some generalities on log structures. More systematic account of a part is given in [10] Section 4. For the basic definitions on log schemes, we refer to [6]. In

this paper, a log structure $M_X \to O_X$ on a scheme X means a Zariski fs-log structure.
We prepare some basic terminologies on log schemes. We call a pair (X, P) of a log scheme X and a chart P on X a charted log scheme. For charted log schemes (X, P) and (S, N), we call a pair (f, φ) of a map $f : X \to S$ of log schemes and a map $N \to P$ of fs-monoid a map $(X, P) \to (S, N)$ of charted log schemes if the diagram

$$\begin{array}{ccc} N & \longrightarrow & \Gamma(S, M_S) \\ \downarrow & & \downarrow \\ P & \longrightarrow & \Gamma(X, M_X) \end{array} \tag{1}$$

is commutative.
For an fs-monoid P, we regard Spec $\mathbb{Z}[P]$ as a log scheme with the log structure defined by the chart $P \to \mathbb{Z}[P]$. For maps $X \to S$ and $Y \to S$ of log schemes, let $X \times_S^{\log} Y$ denote the fibered product in the category of fs-log schemes. If $S = \operatorname{Spec} A$, $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} C$ are affine, $N \to A, P \to B$ and $Q \to C$ are charts and if $(f, \varphi) : (X, P) \to (S, N)$ and $(g, \psi) : (Y, Q) \to (S, N)$ are morphisms of charted log schemes, we have $X \times_S^{\log} Y = \operatorname{Spec} B \otimes_A^{\log} C$ where $B \otimes_A^{\log} C = (B \otimes_A C) \otimes_{\mathbb{Z}[P+Q]} \otimes \mathbb{Z}[P +_N^{\mathrm{sat}} Q]$ and $P +_N^{\mathrm{sat}} Q$ is the saturation of the image of $P + Q$ in the fibered sum $P^{\mathrm{gp}} \oplus_{N^{\mathrm{gp}}} Q^{\mathrm{gp}} = \operatorname{Coker}(\varphi - \psi : N^{\mathrm{gp}} \to P^{\mathrm{gp}} \oplus Q^{\mathrm{gp}})$.

Definition 3.1 *Let $X \to S$ be a morphism of log schemes.*
1. (cf. [7], [11] Theorem 4.6 (iv)) *We say that $X \to S$ is log flat if the following conditions are satisfied:*
For each x, there exist a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & S \end{array}$$

of log schemes, charts P on U and N on V and morphism $(U, P) \to (V, N)$ of charted log schemes such that the underlying map $U \to X$ is a flat surjection to an open neighborhood of x, the underlying map $V \to S$ is flat, the map $N \to P$ is injective and the underlying map $U \to V \otimes_{\mathbb{Z}[N]} \mathbb{Z}[P]$ is flat.
2. We say that $X \to S$ is log locally of complete intersection if the following conditions are satisfied:
For each x, there exist a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & S \end{array}$$

of log schemes such that U is an open neighborhood of x, the map $V \to S$ is log smooth and $U \to V$ is an exact and regular immersion.
3. We say that $X \to S$ is log syntomic if it is log flat and log locally of complete intersection.

For the log syntomic morphisms, the definition here is slightly different from that in [9] (2.5). We introduce the new definition because it is a special case of the general definition due to Illusie and Olsson [5], [11] Definition 4.1 by Lemma 3.3 below. An equivalent statement of Lemma 3.2 in the resp. cases is proved in [6], and in the log flat case in [11] Theorem 4.6. Another proof is given in [10] Section 4.4.

LEMMA 3.2 (cf. [11] Theorem 4.6) *For a morphism $X \to S$ of log schemes, the following conditions are equivalent.*
(1) The map $X \to S$ is log flat (resp. log smooth, log étale).
(2) Let

$$\begin{array}{ccc} X' & \longrightarrow & S' \\ \big\downarrow & & \big\downarrow \\ X & \longrightarrow & S \end{array}$$

be a commutative diagram of log schemes such that $X' \to X \times_S^{\log} S'$ is log étale and $X' \to S'$ is strict. Then the underlying map $X' \to S'$ is flat (resp. smooth, étale).

LEMMA 3.3 *For a morphism $X \to S$ of log schemes, the following conditions are equivalent.*
(1) The map $X \to S$ is log syntomic.
(2) Let

$$\begin{array}{ccc} X' & \longrightarrow & S' \\ \big\downarrow & & \big\downarrow \\ X & \longrightarrow & S \end{array}$$

be a commutative diagram of log schemes such that $X' \to X \times_S^{\log} S'$ is log étale and $X' \to S'$ is strict. Then the underlying map $X' \to S'$ is flat and locally of complete intersection.

To deduce Lemma 3.3 from Lemma 3.2, we introduce some basic constructions on log schemes.

LEMMA 3.4 *Let $f : X \to S$ be a morphism of log schemes and $x \in X$. Then there exist charts P and N on open neighborhoods U of x and $V \supset f(U)$ of $s = f(x)$ and a morphism $(U, P) \to (V, N)$ of charted log schemes such that the map* $\mathrm{Spec}\ \mathbb{Z}[P] \to \mathrm{Spec}\ \mathbb{Z}[N]$ *is log smooth.*

Proof. We put $\bar{M}_S = M_S/O_S^\times$, $\bar{M}_X = M_X/O_X^\times$, $N = \bar{M}_{S,s}$, $P_0 = \bar{M}_{X,x}$ and let $N \to P_0$ be the canonical map. We take charts $N \to \Gamma(V, M_V)$, $P_0 \to \Gamma(U, M_U)$ on open neighborhoods lifting the identities. We define an fs-monoid P to be the inverse image of P_0 by the map $P_0^{\mathrm{gp}} \oplus N^{\mathrm{gp}} \to P_0^{\mathrm{gp}}$ sending (m,n) to $m + f(n)$. Then, shrinking U if necessary, we find a unique map $P \to \Gamma(U, M_X)$ extending the composition $P_0 + N \to \Gamma(X, M_X) + \Gamma(S, M_S) \to \Gamma(X, M_X)$. Thus, we obtain a morphism $(U,P) \to (V,N)$ of charted log schemes. Since the map $N^{\mathrm{gp}} \to P^{\mathrm{gp}}$ is an isomorphism to a direct summand, the map $\operatorname{Spec} \mathbb{Z}[P] \to \operatorname{Spec} \mathbb{Z}[N]$ is log smooth. □

For a morphism $f : N \to P$ of fs-monoids, we define an fs-monoid $(P +_N P)^\sim$ to be the inverse image of P by the map $P^{\mathrm{gp}} \oplus_{N^{\mathrm{gp}}} P^{\mathrm{gp}} \to P^{\mathrm{gp}}$ sending (m, m') to $m + m'$.

LEMMA 3.5 *Let $N \to P$ be a map of fs-monoids and let $(P +_N P)^\sim \subset P^{\mathrm{gp}} \oplus_{N^{\mathrm{gp}}} P^{\mathrm{gp}}$ be as above. Then,*

1. The map $P \times (P^{\mathrm{gp}}/N^{\mathrm{gp}}) \to (P +_N P)^\sim$ sending (m, m') to $(m + m', -m')$ is an isomorphism.

2. The ring homomorphism $\mathbb{Z}[P] \to \mathbb{Z}[(P +_N P)^\sim]$ induced by the map $P \to (P +_N P)^\sim$ of monoids sending m to $(m, 0)$ is faithfully flat.

3. The map $P + P + (P^{\mathrm{gp}}/N^{\mathrm{gp}}) \to (P +_N P)^\sim$ sending $(m, m', m'') \to (m + m'', m' - m'')$ induces an isomorphism $\mathbb{Z}[P \times P \times (P^{\mathrm{gp}}/N^{\mathrm{gp}})]/((m,0,0) - (0,m,m); m \in P) \to \mathbb{Z}[(P +_N P)^\sim]$ of rings.

Proof. 1. The inverse $(P +_N P)^\sim \to P \times (P^{\mathrm{gp}}/N^{\mathrm{gp}})$ is given by $(m, m') \to (m + m', -m')$.

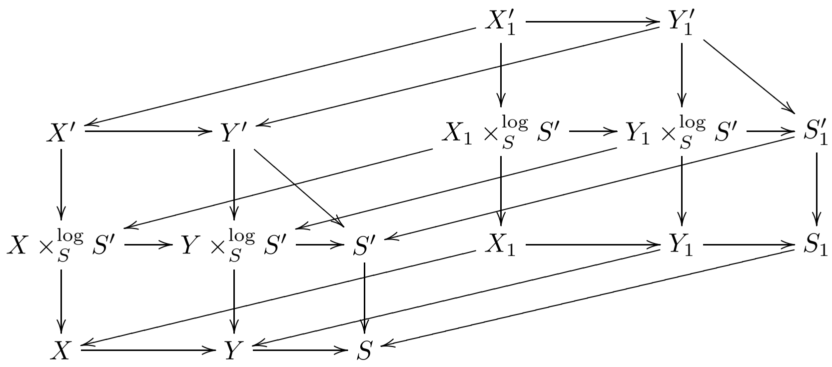
2 and 3. Clear from 1. □

COROLLARY 3.6 *Let $(X, P) \to (S, N)$ be a morphism of charted log schemes and put $S' = S \otimes_{\mathbb{Z}[N]} \mathbb{Z}[P]$ and $X' = X \otimes_{\mathbb{Z}[P]} \mathbb{Z}[(P +_N P)^\sim]$. Then the map $X' \to S'$ is strict, the map $X' \to X \times_S^{\log} S'$ is log étale and $X' \to X$ is faithfully flat.*

Proof. The map $X' \to X \times_S^{\log} S'$ is log étale by the definition of $(P +_N P)^\sim$. The map $X' \to S'$ is strict by Lemma 3.5.1. The map $X' \to X$ is faithfully flat by Lemma 3.5.2. □

Proof of Lemma 3.3. Since the assertion is local on X, we may assume there exist a log smooth scheme Y over S, an exact closed immersion $X \to Y$ over S and a morphism $(Y, P) \to (S, N)$ of charted log schemes as in Lemma 3.4. We put $S_1 = S \otimes^{\log}_{\mathbb{Z}[N]} \mathbb{Z}[P], Y_1 = Y \otimes^{\log}_{\mathbb{Z}[P]} \mathbb{Z}[(P +_N P)^\sim]$ and $X_1 = X \times_Y^{\log} Y_1$.

We show (1)⇒(2). We assume $X \to S$ is log syntomic. We consider the diagram in (2). Since the question is local on X', we may assume there exist a log étale scheme Y' over $Y \times_S S'$ and an isomorphism $X' \to X \times_Y^{\log} Y'$. Shrinking Y', we may assume that the map $Y' \to S'$ is strict. Hence by Lemma 3.2, the underlying map $Y' \to S'$ is smooth. It is sufficient to show that the closed immersion $X' \to Y'$ is a regular immersion. We consider a commutative diagram



by putting $S'_1 = S_1 \times_S^{\log} S', Y'_1 = Y_1 \times_Y^{\log} Y'$ and $X'_1 = X_1 \times_X^{\log} X'$.
Since $Y \to S$ is log smooth, $Y_1 \to S_1$ is strict and $Y_1 \to Y \times_S^{\log} S_1$ is log étale, the underlying map $Y_1 \to S_1$ is smooth by Lemma 3.2. Similarly, since $X \to S$ is log flat, $X_1 \to S_1$ is strict and $X_1 \to X \times_S^{\log} S_1$ is log étale, the underlying map $X_1 \to S_1$ is flat by Lemma 3.2. Since $Y_1 \to Y$ is flat by Lemma 3.5.2 and $X \to Y$ is a regular immersion, the immersion $X_1 \to Y_1$ is a regular immersion. Thus $X_1 \to S_1$ is flat and locally of complete intersection. Since the maps $X_1 \to Y_1 \to S_1$ are strict, the underlying map $X_1 \times_S^{\log} S' \to S'_1$ is flat and locally of complete intersection and the immersion $X_1 \times_S^{\log} S' \to Y_1 \times_S^{\log} S'$ is a regular immersion by EGA IV Propositions (19.3.9)(ii) and (19.3.7). Since $Y'_1 \to Y_1 \times_S^{\log} S'$ is a base change of $Y' \to Y \times_S^{\log} S'$, the map $Y'_1 \to Y_1 \times_S^{\log} S'$ is log étale. Since it is strict, the underlying map $Y'_1 \to Y_1 \times_S^{\log} S'$ is étale by Lemma 3.2. Since $X'_1 \to Y'_1$ is the base change of the regular immersion $X_1 \times_S^{\log} S' \to Y_1 \times_S^{\log} S'$ by the étale map $Y'_1 \to Y_1 \times_S^{\log} S'$, it is also a regular immersion. Since the regular immersion $X'_1 \to Y'_1$ is also the base change of the immersion $X' \to Y'$ by the faithfully flat and strict map $Y_1 \to Y$, the immersion $X' \to Y'$ is a regular immersion as required.
We show (2)⇒(1). We assume the condition (2) is satisfied. It is sufficient to show that the exact closed immersion $X \to Y$ is a regular immersion. By (2), the underlying map $X_1 \to S_1$ is flat and locally of complete intersection and the underlying map $Y_1 \to S_1$ is smooth. Hence the immersion $X_1 \to Y_1$ is a regular immersion by EGA IV Proposition (19.3.7). Since the regular immersion $X_1 \to Y_1$ is the base change of the immersion $X \to Y$ by the strict and faithfully flat map $Y_1 \to Y$, the immersion $X \to Y$ is a regular immersion as required. □

Corollary 3.7 (cf. [11] Corollary 4.12) *Let $f : X \to S$ and $S' \to S$ be morphisms of log schemes and let $f' : X' = X \times_S^{\log} S' \to S'$ be the log base change. Then, if $f : X \to S$ is log flat (resp. log syntomic), the base change $f' : X' \to S'$ is also log flat (resp. log syntomic).*

Proof. Clear from Lemmas 3.2 and 3.3. □

Lemma 3.8 *Let $X \to S$ be a log scheme over S log locally of complete intersection, $Y \to S$ be a log smooth log scheme over S and $X \to Y$ be an exact closed immersion over S. Then,*
1. The immersion $X \to Y$ is a regular immersion.
2. Let $Y' \to S$ be another log smooth log scheme over S and $X \to Y'$ be an exact closed regular immersion over S. Let n and n' be the relative dimensions of Y and of Y' over S and r and r' be the codimensions of the regular immersions $X \to Y$ and of $X \to Y'$ respectively. Then we have $n - r = n' - r'$.

Proof. 1. Since the assertion is local, we may assume there is an exact regular closed immersion $X \to Y'$ into a log smooth scheme Y' over S. By the same argument as in the proof of Lemma 3.4, we may assume that there exist a commutative diagram

$$\begin{array}{ccc} (X,P) & \longrightarrow & (Y,P) \\ \downarrow & & \downarrow \\ (Y',P) & \longrightarrow & (S,N) \end{array}$$

of charted log schemes. We define an fs-monoid $(P +_N P)^\sim \subset P^{\rm gp} \oplus_{N^{\rm gp}} P^{\rm gp}$ as above and put $Y'' = (Y \times^{\log}_S Y') \otimes^{\log}_{\mathbb{Z}[P+P]} \mathbb{Z}[(P +_N P)^\sim]$. Then the projections $Y'' \to Y$ and $Y'' \to Y'$ are log smooth and strict and hence are smooth. Since the immersion $X \to Y'$ is a regular immersion, the immersion $X \to Y''$ is a regular immersion. Since the map $Y'' \to Y$ is also smooth, the immersion $X \to Y$ is also a regular immersion by EGA IV Proposition (19.1.5)(iv)b)⇒a) applied to the immersions $X \times^{\log}_Y Y'' \to Y''$ and $X \to X \times^{\log}_Y Y''$ and by loc.cit (ii). Hence the assertion follows.
2. In the notation above, the relative dimensions of Y'' over Y and Y' are n' and n respectively. Hence the assertion follows. □
If $X \to Y$ is an exact regular immersion of codimension r, and Y is log smooth over S of relative dimension n, we say that $X \to S$ is of relative dimension $n - r$.

Lemma 3.9 *Let X and S be log regular schemes and $f : X \to S$ be a morphism of finite type. Then $f : X \to S$ is log locally of complete intersection.*

Proof. Since the assertion is local, we may assume there is a morphism $(X, P) \to (S, N)$ of charted log schemes as in Lemma 3.4. The map $S' = S \otimes^{\log}_{\mathbb{Z}[N]} \mathbb{Z}[P] \to S$ is log smooth and the map $X \to S$ is factorized as $X \to S' \to S$ where $X \to S'$ is strict. Hence by replacing S by S', we may assume $X \to S$ is strict. Further replacing S by a smooth scheme over S, we may assume $X \to S$ is an exact immersion. It is sufficient to show that the immersion $X \to S$ is a regular immersion.
Since the question is local, we may assume $S = \text{Spec } A$ and $X = \text{Spec } B$ are local. We put $P = \bar{M}_{S,s}$ and take a chart $\alpha : P = \bar{M}_{S,s} \to A$. We

put $\bar{A} = A/\alpha(P - \{1\})$ and $\bar{B} = B \otimes_A \bar{A}$. Since $\bar{A} \to \bar{B}$ is a surjection of regular local rings, the kernel is generated by a regular sequence $(\bar{t}_1, \ldots, \bar{t}_r)$ of $\bar{A}$. We take a lifting $(t_1, \ldots, t_r)$ in the maximal ideal $\mathfrak{m}_A$. We show that $A_i = A/(t_1, \ldots, t_i)$ is log regular of dimension $\dim A - i$ and that $(t_1, \ldots, t_i)$ is a regular sequence. By induction on $i = 1, \ldots, r$, it is sufficient to show the case $i = 1$. Since $t_1 \neq 0$ and A is normal, we have $\dim A_1 = \dim A - 1$. On the other hand, we have $\dim \bar{A}_1 + \text{rank } P^{\text{gp}} = \dim \bar{A} - 1 + \text{rank } P^{\text{gp}}$. Hence, we have $\dim A_1 = \dim \bar{A}_1 + \text{rank } P^{\text{gp}}$ and A_1 is log regular. Thus by induction, A_r is log regular of dimension $\dim A - r$ and $(t_1, \ldots, t_r)$ is a regular sequence. Since $\dim B = \dim \bar{B} + \text{rank } P^{\text{gp}} = \dim \bar{A} - r + \text{rank } P^{\text{gp}} = \dim A_r$ and A_r is normal, the surjection $A_r \to B$ is an isomorphism. Hence the immersion $X \to S$ is a regular immersion of codimension r. □

Let $f : X \to S$ be a map of log schemes such that the map of underlying schemes is locally of finite presentation and $x \in X$. We put $s = f(x)$, $X_s = X \otimes_{\kappa(s)} \kappa(x)$ and define

$$\begin{aligned}\dim_x^{\log} f^{-1}(f(x)) = \\ = \dim O_{X_s,x}/(\alpha(M_{X,x} - O_{X,x}^{\times})) + \text{tr.deg } \kappa(x)/\kappa(s) + \text{rank } \bar{M}_{X,x}^{\text{gp}}/\bar{M}_{S,s}^{\text{gp}}.\end{aligned}$$

Lemma 3.10 *Let $f : X \to S$ be a morphism of log schemes such that the map of underlying schemes is of finite presentation.*
1. Let $(X, P) \to (S, N)$ be a morphism of charted log schemes and let $x \in X$. Regard x as a log scheme with the log structure defined by the chart P. We put $X'_x = (X \times_S x) \otimes_{\mathbb{Z}[P+P]} \mathbb{Z}[(P +_N P)^{\sim}]$ and let $x \to X'_x$ be the section defined by $x \to X$ and the map $(P +_N P)^{\sim} \to P \to \kappa(x)$. Then, we have an equality

$$\dim_x^{\log} f^{-1}(f(x)) = \dim O_{X'_x,x}.$$

2. If $X \to S$ is log flat, the function $\dim_x^{\log} f^{-1}(f(x))$ is a locally constant function of $x \in X$.
3. Assume $X \to S$ is log locally of complete intersection of relative dimension d. If we have an equality $\dim_x^{\log} f^{-1}(f(x)) = d$ for all $x \in X$, the map $X \to S$ is log flat and hence log syntomic.

Proof. 1. By Lemma 3.5.3, X'_x is the closed subscheme of $(X_s \otimes_{\kappa(s)} \kappa(x)) \otimes_{\mathbb{Z}} \mathbb{Z}[P'^{\text{gp}}/N^{\text{gp}}]$ defined by the ideal I generated by $(\alpha(m) \otimes 1) - (1 \otimes \alpha_x(m)) \cdot (m)$ for $m \in P$. The ideal I is generated by $\alpha(m) \otimes 1$ for $m \in P \backslash \text{Ker}(P \to \bar{M}_{X,x})$ and $(m) - (1 \otimes \alpha_x(m))^{-1}(\alpha(m) \otimes 1)$ for $m \in \text{Ker}(P \to \bar{M}_{X,x})$. Hence X'_x is the closed subscheme of $(X_s \otimes_{\kappa(s)} \kappa(x)) \otimes_{\mathbb{Z}} \mathbb{Z}[\bar{M}_{X,x}^{\text{gp}}/\bar{M}_{S,s}^{\text{gp}}]$ defined by the ideal J generated by $\alpha(m) \otimes 1$ for $m \in P \backslash \text{Ker}(P \to \bar{M}_{X,x})$. Thus the assertion follows.
2. Let $S' = S \otimes^{\log}_{\mathbb{Z}[N]} \mathbb{Z}[P], X' = X \otimes^{\log}_{\mathbb{Z}[P]} \mathbb{Z}[(P +_N P)^{\sim}]$ and $f' : X' \to S'$ be the map. Since the map $X' \to X \times^{\log}_S S'$ is log étale, and the composition $X' \to S'$ is strict, the underlying map $X' \to S'$ is flat. Hence the function $\dim_{x'} f'^{-1}(f'(x')) = \dim O_{X'_{f'(x')},x'}$ is locally constant on $x' \in X'$. The function $\dim_x^{\log} f^{-1}(f(x))$ is the pull-back of the locally constant function

$\dim_{x'} f'^{-1}(f'(x'))$ by the section $X \to X'$ induced by the map $(P +_N P)^\sim \to P$. Thus the assertion is proved.
3. Since the question is local, we may further assume that there is an exact regular immersion $X \to Y$ to a log scheme Y log smooth over S. Let n be the relative dimension of Y over S and $r = n - d$ be the codimension of the regular immersion $X \to Y$. We put $Y' = Y \otimes^{\log}_{\mathbb{Z}[P']} \mathbb{Z}[(P' +_N P')^\sim]$. Then we have $X' = X \times^{\log}_Y Y'$. Since $X' \to X$ is faithfully flat by Lemma 3.5.2, it is sufficient to show that the map $X' \to S'$ is flat. Since $Y' \to Y$ is flat, the immersion $X' \to Y'$ is regular of codimension r. The map $Y' \to S'$ is smooth of relative dimension n. Hence the strict map $X' \to S'$ is locally of complete intersection of relative dimension d. By the assumption and the computation above, each fiber of $X' \to S'$ has dimension d. Hence by EGA IV Théorème (11.3.8) d)⇒a), $X' \to S'$ is flat.

Corollary 3.11 *Let $f : X \to S$ be a finite morphism of log regular schemes. Assume $\dim X = \dim S$ and $f^*\bar{M}^{\rm gp}_S \otimes \mathbb{Q} \to \bar{M}^{\rm gp}_X \otimes \mathbb{Q}$ is surjective. Then X is log flat and hence log syntomic over S.*

Proof. By Lemma 3.9, the map $f : X \to S$ is log locally of complete intersection. Further, by the assumption that $X \to S$ is finite and $\dim X = \dim S$, the map $X \to S$ has relative dimension 0. Since $\dim^{\log}_x f^{-1}(f(x)) = 0$ for all $x \in X$, it is sufficient to apply Lemma 3.10 □

For a ring A, we call a Zariski fs-log structure on $X = \mathrm{Spec}\ A$ a log structure on A. We call a ring with a log structure a log ring. If A is a local ring, a log structure on A is defined by a chart $P \to A$. We say that a map $A \to B$ of log rings is a surjection if the underlying ring homomorphism $A \to B$ is surjective and the map $f^*M_Y \to M_X$ is surjective where $f : X = \mathrm{Spec}\ B \to Y = \mathrm{Spec}\ A$ denotes the corresponding map of log schemes and M_X and M_Y denote the log structures. We say that a surjection $A \to B$ of log rings is an exact surjection if the log structure M_X is the pull-back log structure of M_Y. We say that a surjection $A \to B$ is regular if the immersion $\mathrm{Spec}\ B \to \mathrm{Spec}\ A$ of the underlying schemes is a regular immersion. For a map $A \to B$ of log rings, let $\Omega_{B/A}(\log/\log)$ denote the module of logarithmic differential forms, denoted by $\omega_{B/A}$ in [6]. If A and B are local and N and P denote the stalks of the log structures at the closed points, we have

$$\Omega_{B/A}(\log/\log) = \left(\Omega_{B/A} \oplus (B \otimes_{\mathbb{Z}} (P^{\rm gp}/N^{\rm gp}))\right)/(dm - m \otimes m : m \in P).$$

We study formally log smooth maps of complete local Noetherian log rings.

Definition 3.12 (cf. [11] Definition 4.4) *Let A and B be complete local Noetherian rings with log structures and $f : A \to B$ a morphism of log rings such that the underlying ring homomorphism is local.*
1. We say $f : A \to B$ is formally log smooth (resp. formally log étale) if, for a nilpotent exact surjection $R \to R'$ of discrete log A-algebras and a continuous

homomorphism $B \to R'$ of log A-algebras, there exists a (resp. a unique) continuous homomorphism $B \to R$ of log A-algebras lifting $B \to R'$.
2. We put $\hat\Omega_{B/A}(\log/\log) = \varprojlim_n \Omega_{(B/\mathfrak{m}_B^n)/A}(\log/\log)$.

Lemma 3.13 *Let A and B be complete local Noetherian rings with log structures and $f: A \to B$ a morphism of log rings such that the underlying ring homomorphism is local. Assume that the residue field of B is finitely generated over the residue field of A. Then, the following conditions are equivalent.*
(1) B is formally log smooth over A.
(2) There exist a log smooth scheme X over A, a point x of X over the closed point of Spec A *and an étale local homomorphism $B \to \hat O_{X,x}$ over A.*

Proof. It is clear that (2) implies (1). The implication (1)⇒(2) is proved similarly as in the proof of [6] (3.5.1)⇒(3.5.2). □

Corollary 3.14 *Let $A \to B$ be as in Lemma and assume $A \to B$ is log smooth.*
1. The B-module $\hat\Omega_{B/A}(\log/\log)$ *is free of finite rank.*
2. If A is log regular (cf. [8] Definition (2.1))*, then B is also log regular.*

Proof. 1. It follows from Lemma 3.13 (1)⇒(2) and [6] Proposition (3.10).
2. It follows from Lemma 3.13 (1)⇒(2) and [8] Theorem (8.2). □

4 Tubular neighborhoods for finite flat and log flat log algebras

In the rest of the paper, the integer ring O_K is considered as a log ring with its canonical log structure defined by the chart $\mathbb{N} \to O_K$ sending $1 \in \mathbb{N}$ to a prime element. The letter A denotes a finite flat and log flat log O_K-algebra such that the log structure on A_K is trivial. For a finite étale algebra L over K, its integer ring O_L is considered as a log O_K-algebra with its canonical log structure defined by taking the product of the canononical log structures on its factors. The log O_K-algebra O_L is log flat by Corollary 3.11. Hence it is finite flat and log flat and the log structure on L is trivial.

4.1 Log embeddings

Definition 4.1 *1. Let A be a finite flat and log flat log O_K-algebra such that the log structure on A_K is trivial. Let $\mathbf{A}$ be a log O_K-algebra formally of finite type and formally log smooth over O_K. We say that an exact surjection $\mathbf{A} \to A$ of log O_K-algebras is a log embedding if it induces an isomorphism $\mathbf{A}/\mathfrak{m}_{\mathbf{A}} \to A/\mathfrak{m}_A$.*
2. We define $\mathcal{E}mb^{\log}_{O_K}$ to be the category whose objects and morphisms are as follows. An object of $\mathcal{E}mb^{\log}_{O_K}$ is a triple $(\mathbf{A} \to A)$ where:

- *A is a finite flat and log flat log O_K-algebra such that the log structure on A_K is trivial.*

- $\mathbf{A}$ *is a log* O_K*-algebra formally of finite type and formally log smooth over* O_K.
- $\mathbf{A} \to A$ *is a log embedding.*

A morphism $(f, \mathbf{f}) : (\mathbf{A} \to A) \to (\mathbf{B} \to B)$ *is a pair of homomorphisms* $f : A \to B$ *and* $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ *of log* O_K*-algebras such that the diagram*

$$\begin{array}{ccc} \mathbf{A} & \longrightarrow & A \\ {\scriptstyle \mathbf{f}}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbf{B} & \longrightarrow & B \end{array}$$

of log O_K*-algebra homomorphisms is commutative.*
3. For a finite flat and log flat log O_K*-algebra* A *such that the log structure on* A_K *is trivial, let* $\mathcal{E}mb^{\log}_{O_K}(A)$ *be the subcategory of* $\mathcal{E}mb^{\log}_{O_K}$ *whose objects are of the form* $(\mathbf{A} \to A)$ *and morphisms are of the form* $(\mathrm{id}_A, \mathbf{f})$.
4. We say that a morphism $(f, \mathbf{f}) : (\mathbf{A} \to A) \to (\mathbf{B} \to B)$ *of* $\mathcal{E}mb_{O_K}$ *is finite flat and log flat if* $\mathbf{A} \to \mathbf{B}$ *is finite flat and log flat and the map* $\mathbf{B} \otimes^{\log}_{\mathbf{A}} A \to B$ *is an isomorphism of log* O_K*-algebras.*
5. We say that a log embedding $\mathbf{A} \to A$ *is strict if the maps* $O_K \to A$ *and* $O_K \to \mathbf{A}$ *of log rings are strict.*

For a complete semi-local Noetherian log O_K-algebra R such that $R/\mathfrak{m}_R$ is finite over F, we put $\hat{\Omega}_{R/O_K}(\log/\log) = \varprojlim_n \Omega_{(R/\mathfrak{m}_R^n)/O_K}(\log/\log)$. If $(\mathbf{A} \to A)$ is a log embedding, the $\mathbf{A}$-module $\hat{\Omega}_{\mathbf{A}/O_K}(\log/\log)$ is locally free of finite rank. If $(\mathbf{A} \to A)$ is a strict object of $\mathcal{E}mb^{\log}_{O_K}$, by forgetting the log structures, we obtain an object $(\mathbf{A} \to A)^\circ$ of $\mathcal{E}mb_{O_K}$. For an object $(\mathbf{A} \to A)$ of $\mathcal{E}mb_{O_K}$, by putting the pull-back log structures on $\mathbf{A}$ and A from that on O_K, we obtain an object $(\mathbf{A} \to A)^{\log}$ of $\mathcal{E}mb^{\log}_{O_K}$. Thus, we obtain an equivalence of categories between $\mathcal{E}mb_{O_K}$ and the full subcategory of $\mathcal{E}mb^{\log}_{O_K}$ consisting of strict objects.

LEMMA 4.2 *Let* A *be a finite flat and log flat log* O_K*-algebra such that the log structure on* A_K *is trivial. We put* $X = \mathrm{Spec}\ A$ *and* $S = \mathrm{Spec}\ O_K$.
1. For a closed point x *of* $X = \mathrm{Spec}\ A$*, the stalk* $\bar{M}_{X,x}$ *of the sheaf* $\bar{M}_X = M_X/O_X^\times$ *is isomorphic to* $\mathbb{N}$ *and the map* $\bar{M}_{S,s} = \mathbb{N} \to \bar{M}_{X,x} = \mathbb{N}$ *is the multiplication by an integer* $e_x \geq 1$.
2. Let $(\mathbf{A} \to A)$ *be a log embedding. Then, the ring* $\mathbf{A}$ *is regular and the reduced closed fiber* $(\mathbf{A} \otimes_{O_K} F)_{\mathrm{red}}$ *is a regular divisor. The log ring* $\mathbf{A}$ *is log regular and the log structure is defined by the reduced closed fiber* $(\mathbf{A} \otimes_{O_K} F)_{\mathrm{red}}$.
3. A log embedding $(\mathbf{A} \to A)$ *is strict if and only if the map* $O_K \to A$ *is strict.*

Proof. 1. Clear from Lemma 3.10.1.
2. We may assume $\mathbf{A}$ is local and the log structure is defined by a chart $\mathbb{N} \to \mathbf{A}$. Since $\mathbf{A}$ is formally log smooth over O_K, it is log regular by Corollary 3.14.2. Since the stalks of $\bar{M}$ are either $\mathbb{N}$ or 0, the ring $\mathbf{A}$ is regular and the image

$t \in \mathbf{A}$ of $1 \in \mathbb{N}$ defines a regular divisor. Since $\pi/t^{e_x} \in \mathbf{A}^\times$, the assertion follows.
3. We may assume $\mathbf{A}$ is local. Assume the map $O_K \to A$ is strict. Then, in the notation of the proof of 2, we have $e_x = 1$ and $\pi/t \in \mathbf{A}^\times$. Hence the map $O_K \to \mathbf{A}$ is strict. The only if part is obvious. □
To prove the logarithmic version Lemma 4.5 below of Lemma 1.2, we make another definition.

DEFINITION 4.3 *1. Let A be a finite flat and log flat log O_K-algebra such that the log structure on A_K is trivial. Let $\mathbf{A}$ be a log O_K-algebra formally of finite type, formally smooth and formally log smooth over O_K. We say that a surjection $\mathbf{A} \to A$ of log O_K-algebra is a log pre-embedding if it induces an isomorphism $\mathbf{A}/\mathfrak{m}_{\mathbf{A}} \to A/\mathfrak{m}_A$ of underlying F-algebras.*
2. We define $pre\mathcal{E}mb^{\log}_{O_K}$ to be the category whose objects and morphisms are as follows. An object of $\mathcal{E}mb^{\log}_{O_K}$ is a triple $(\mathbf{A} \to A)$ where:

- *A is a finite flat and log flat log O_K-algebra such that the log structure on A_K is trivial.*
- *$\mathbf{A}$ is a log O_K-algebra formally of finite type, formally smooth and formally log smooth over O_K.*
- *$\mathbf{A} \to A$ is a log pre-embedding.*

A morphism $(f, \mathbf{f}) : (\mathbf{A} \to A) \to (\mathbf{B} \to B)$ is a pair of log O_K-homomorphism $f : A \to B$ and $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ such that the diagram

$$\begin{array}{ccc} \mathbf{A} & \longrightarrow & A \\ {\scriptstyle \mathbf{f}}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbf{B} & \longrightarrow & B \end{array}$$

is commutative.
3. For a finite flat and log flat log O_K-algebra A such that the log structure on A_K is trivial, let $pre\mathcal{E}mb^{\log}_{O_K}(A)$ be the subcategory of $pre\mathcal{E}mb^{\log}_{O_K}$ whose objects are of the form $(\mathbf{A} \to A)$ and morphisms are of the form $(\mathrm{id}_A, \mathbf{f})$.

A log pre-embedding $(\mathbf{A} \to A)$ is an embedding together with log structures on $\mathbf{A}$ and on A such that the log ring $\mathbf{A}$ is formally log smooth, that the log ring A is log flat and the log structure on A_K is trivial and that the map $\mathbf{A} \to A$ is a surjection of log O_K-algebras. Hence, by forgetting the log structures, we obtain a functor $pre\mathcal{E}mb^{\log}_{O_K} \to \mathcal{E}mb_{O_K}$.
We also define a functor $pre\mathcal{E}mb^{\log}_{O_K} \to \mathcal{E}mb^{\log}_{O_K}$. For an object $(\mathbf{A} \to A)$ of $pre\mathcal{E}mb^{\log}_{O_K}$, we attach a log embedding $(\mathbf{A}^\sim \to A)$ as follows. First, we consider the case where A is local. Assume the log structure of $\mathbf{A}$ is defined by a chart $P \to \mathbf{A}$. Let $P \to \mathbb{N}$ be the map $P \to \bar{M}_{X,x} = \mathbb{N}$ where x is the closed point of

$X = \operatorname{Spec} A$ and we identify $\bar{M}_{X,x} = \mathbb{N}$ by the unique isomorphism. Let $P^\sim$ be the inverse image of $\mathbb{N}$ by the induced map $P^{\rm gp} \to \bar{M}^{\rm gp}_{X,x} = \mathbb{Z}$. The map $P \to \mathbf{A} \to A$ is extended uniquely to a map $P^\sim \to A$. We define $\mathbf{A}^\sim$ to be the formal completion of the surjection $\mathbf{A} \otimes_{\mathbb{Z}[P]} \mathbb{Z}[P^\sim] \to A$ induced by $P^\sim \to A$. Let $\mathbf{A}^\sim \to A$ be the canonical map. The log ring $\mathbf{A}^\sim$ and the homomorphism $\mathbf{A}^\sim \to A$ are independent of the choice of the chart $P \to \mathbf{A}$ upto a unique isomorphism. In general, we define $\mathbf{A}^\sim$ and $\mathbf{A}^\sim \to A$ by taking the product. By the construction, the canonical map $\mathbf{A} \to \mathbf{A}^\sim$ is formally log étale.

Lemma 4.4 *Let A be a finite flat and log flat log O_K-algebra such that the log structure on A_K is trivial.*
1. The category $pre\mathcal{E}mb^{\log}_{O_K}(A)$ is non-empty.
2. Let $(\mathbf{A} \to A)$ be an object of $pre\mathcal{E}mb^{\log}_{O_K}$ and define $\mathbf{A}^\sim$ and $\mathbf{A}^\sim \to A$ as above. Then $(\mathbf{A}^\sim \to A)$ is an object of $\mathcal{E}mb^{\log}_{O_K}$.

Proof. 1. We may assume A is local. Take a system of generators $t_1, \ldots, t_n$ of A over O_K and a chart $\mathbb{N} \to A$. Let $t_0 \in A$ be the image of $1 \in \mathbb{N}$. We define a surjection $O_K[T_0, \ldots, T_n] \to A$ by sending T_i to t_i and a log structure on $O_K[T_0, \ldots, T_n]$ by the chart $\mathbb{N}^2 \to O_K[T_0, \ldots, T_n]$ sending $(1,0)$ and $(0,1) \in \mathbb{N}^2$ to T_0 and π. Then its formal completion $\mathbf{A} \to A$ is a log pre-embedding.
2. By the definition, the O_K-algebra $\mathbf{A}^\sim$ is formally of finite type over O_K and the surjection $\mathbf{A}^\sim \to A$ is exact. Since the map $\mathbf{A} \to \mathbf{A}^\sim$ is formally log étale, the log O_K-algebra $\mathbf{A}^\sim$ is formally log smooth over O_K. Hence the assertion follows. □
By Lemma 4.4.2, we obtain a functor $pre\mathcal{E}mb^{\log}_{O_K} \to \mathcal{E}mb^{\log}_{O_K}$.

Lemma 4.5 *1. For a finite flat and log flat log O_K-algebra A such that the log structure on A_K is trivial, the category $\mathcal{E}mb^{\log}_{O_K}(A)$ is non-empty.*
2. For a morphism $f : A \to B$ of finite flat and log flat log O_K-algebras such that the log structures on A_K and B_K are trivial and for objects $(\mathbf{A} \to A)$ and $(\mathbf{B} \to B)$ of $\mathcal{E}mb^{\log}_{O_K}$, there exists a morphism $(f, \mathbf{f}) : (\mathbf{A} \to A) \to (\mathbf{B} \to B)$ lifting f.
3. For a morphism $f : A \to B$ of finite flat and log flat log O_K-algebras such that the log structures on A_K and B_K are trivial, the following conditions are equivalent.
(1) The map $f : A \to B$ is log syntomic.
(2) Their exists a finite flat and log flat morphism $(f, \mathbf{f}) : (\mathbf{A} \to A) \to (\mathbf{B} \to B)$ of log embeddings.

Proof. 1. Clear from Lemma 4.4.
2. Since $\mathbf{A}$ is formally log smooth, $\mathbf{B} = \varprojlim_n \mathbf{B}/I^n$ where $I = \operatorname{Ker}(\mathbf{B} \to B)$ and the surjection $\mathbf{B}/I^n \to B$ is exact, the assertion follows.
3. (1)⇒(2). We may assume A and B are local. We take log embeddings $\mathbf{A} \to A$ and $\mathbf{B} \to B$. We define a log embedding $\mathbf{B}' \to B$ by applying an argument similar to the proof of Lemma 4.4.2 to $\varprojlim_n (\mathbf{A}/\mathfrak{m}^n_{\mathbf{A}} \otimes^{\log}_{O_K} \mathbf{B}/\mathfrak{m}^n_{\mathbf{B}})^\wedge \to$

B. Replacing $\mathbf{B} \to B$ by $\mathbf{B}' \to B$, we may assume that there is a map $(\mathbf{A} \to A) \to (\mathbf{B} \to B)$ such that $\mathbf{A} \to \mathbf{B}$ is formally log smooth. Since $A \to B$ is log syntomic, the exact surjection $\mathbf{B} \otimes^{\log}_{\mathbf{A}} A \to B$ is regular by Lemma 3.8.1 and the kernel is generated by a regular sequence $(t_1, \ldots, t_n)$. Take a lifting $(\tilde{t}_1, \ldots \tilde{t}_n)$ in $\mathbf{B}$ and define a map $\mathbf{A}[[T_1, \ldots, T_n]] \to \mathbf{B}$ by sending T_i to t_i. We consider $\mathbf{A}[[T_1, \ldots, T_n]]$ as a log ring with the pull-back log structure by the map $\mathbf{A} \to \mathbf{A}[[T_1, \ldots, T_n]]$. Then the composition $\mathbf{A}[[T_1, \ldots, T_n]] \to \mathbf{A} \to A$ sending T_i to 0 defines a log embedding. Replacing $\mathbf{A}$ by $\mathbf{A}[[T_1, \ldots, T_n]]$, we obtain a map $(\mathbf{A} \to A) \to (\mathbf{B} \to B)$ such that the map $\mathbf{B} \otimes^{\log}_{\mathbf{A}} A \to B$ is an isomorphism and that $\dim \mathbf{A} = \dim \mathbf{B}$. By Nakayama's lemma, the map $\mathbf{A} \to \mathbf{B}$ is finite. Since $\mathbf{A}$ and $\mathbf{B}$ are regular, the map $\mathbf{A} \to \mathbf{B}$ is flat by EGA Chap 0_{IV} Corollaire (17.3.5) (ii). Further by Corollary 3.11, it is log syntomic.
(2)$\Rightarrow$(1). Since $\mathbf{A}$ and $\mathbf{B}$ are log regular and have the same dimension, $\mathbf{B}$ is log syntomic over $\mathbf{A}$ by Corollary 3.11. Hence B is also log syntomic over A by Lemma 3.7.2. □

The base change of a log embedding by an extension of complete discrete valuation fields is defined as follows.

Lemma 4.6 *Let K' be a complete discrete valuation field and $K \to K'$ be a morphism of fields inducing a local homomorphism $O_K \to O_{K'}$. Let $(\mathbf{A} \to A)$ be an object of $\mathcal{E}mb^{\log}_{O_K}$. We define $\mathbf{A}\hat{\otimes}^{\log}_{O_K} O_{K'}$ to be the projective limit $\varprojlim_n (\mathbf{A}/\mathfrak{m}^n_{\mathbf{A}} \otimes^{\log}_{O_K} O_{K'})$. Then the log $O_{K'}$-algebra $\mathbf{A}\hat{\otimes}^{\log}_{O_K} O_{K'}$ is formally of finite type and formally log smooth over $O_{K'}$. The natural surjection $\mathbf{A}\hat{\otimes}^{\log}_{O_K} O_{K'} \to A\hat{\otimes}^{\log}_{O_K} O_{K'}$ defines an object $(\mathbf{A}\hat{\otimes}^{\log}_{O_K} O_{K'} \to A \otimes^{\log}_{O_K} O_{K'})$ of $\mathcal{E}mb^{\log}_{O_{K'}}$.*

Proof. Since $\mathbf{A}\hat{\otimes}^{\log}_{O_K} O_{K'}$ is finite over $\mathbf{A}\hat{\otimes}_{O_K} O_{K'}$, it is formally of finite type over $O_{K'}$. The formal log smoothness is clear from the definition. The rest is clear. □.

Thus we obtain a functor $\hat{\otimes}^{\log}_{O_K} O_{K'} : \mathcal{E}mb^{\log}_{O_K} \to \mathcal{E}mb^{\log}_{O_{K'}}$. If K'' is an extension of complete discrete valuation fields of K', the composition $\mathcal{E}mb^{\log}_{O_K} \to \mathcal{E}mb^{\log}_{O_{K'}} \to \mathcal{E}mb^{\log}_{O_{K''}}$ is the same as $\hat{\otimes}^{\log}_{O_K} O_{K''} : \mathcal{E}mb^{\log}_{O_K} \to \mathcal{E}mb^{\log}_{O_{K''}}$. If K' is a finite extension, we have $\mathbf{A} \otimes^{\log}_{O_K} O_{K'} = \mathbf{A}\hat{\otimes}^{\log}_{O_K} O_{K'}$. If $(\mathbf{A} \to A)$ is strict, we have $(\mathbf{A} \to A) \otimes^{\log}_{O_K} O_{K'} = ((\mathbf{A} \to A)^\circ \otimes_{O_K} O_{K'})^{\log}$.
Similarly as for $\varinjlim_{K'/K} (\mathrm{Aff}/F')$ defined in Section 1.3, we define a category $\varinjlim_{K'/K} \mathcal{E}mb_{O_{K'}}$. We define a functor $\mathcal{E}mb^{\log}_{O_K} \to \varinjlim_{K'/K} \mathcal{E}mb_{O_{K'}}$ as follows.

Lemma 4.7 *Let A be a finite flat and log flat log O_K-algebra. Let $e = e_{A/O_K}$ denote the least common multiple of e_x in Lemma 4.2.1 for the closed points x in $X = \operatorname{Spec} A$. Let K' be a finite separable extension of K of ramification index $e_{K'/K}$. If $e_{K'/K}$ is divisible by e_{A/O_K}, then the log tensor product $A_{O_{K'}} = A \otimes^{\log}_{O_K} O_{K'}$ is strict over $O_{K'}$.*

Proof. We may assume A is local. We put $P = N' = \mathbb{N} \times \mathbb{Z}$ and define maps $\mathbb{N} \to P$ and $\mathbb{N} \to N'$ by sending $1 \in \mathbb{N}$ to $(e_{A/O_K}, 1)$ and to $(e_{O_{K'}/O_K}, 1)$ respectively. There exist morphisms of charts $(\mathbb{N} \to O_K) \to (P \to A)$ and $(\mathbb{N} \to O_K) \to (N' \to O_{K'})$. Since e_{A/O_K} divides $e_{O_{K'}/O_K}$, the saturation $P +^{\rm sat}_{\mathbb{N}} N'$ is isomorphic to $\mathbb{N} \times (\mathbb{Z}/e_{A/O_K}\mathbb{Z}) \times \mathbb{Z}^2$ and the composition $\mathbb{N} \subset N' \to P +^{\rm sat}_{\mathbb{N}} N' \to \mathbb{N}$ is the identity. Hence $A \otimes^{\log}_{O_K} O_{K'}$ is strict over $O_{K'}$. □

Let $(\mathbf{A} \to A)$ be an object of $\mathcal{E}mb^{\log}_{O_K}$ and define $e = e_{A/K}$ as in Lemma 4.7. Let $\mathcal{C}_e$ be the full subcategory of the category (Ext/K) of finite separable extensions of K consisting of the extensions with ramification index divisible by e. If K' is a finite separable extension in $\mathcal{C}_e$, then by Lemmas 4.7 and 4.2.3, the base change $(\mathbf{A} \otimes^{\log}_{O_K} O_{K'} \to A \otimes^{\log}_{O_K} O_{K'})$ is strict and defines an object $(\mathbf{A} \otimes^{\log}_{O_K} O_{K'} \to A \otimes^{\log}_{O_K} O_{K'})^\circ$ of $\mathcal{E}mb_{O_{K'}}$. We consider a system consisting of $(\mathbf{A} \otimes^{\log}_{O_K} O_{K'} \to A \otimes^{\log}_{O_K} O_{K'})^\circ$ for extensions K' in $\mathcal{C}_e$ and isomorphisms $(\mathbf{A} \otimes^{\log}_{O_K} O_{K'} \to A \otimes^{\log}_{O_K} O_{K'})^\circ \otimes_{O_{K'}} O_{K''} \to (\mathbf{A} \otimes^{\log}_{O_K} O_{K''} \to A \otimes^{\log}_{O_K} O_{K''})^\circ$ for K-morphisms $K' \to K''$ of extensions in $\mathcal{C}_e$. Then it defines an object of $\varinjlim_{K'/K} \mathcal{E}mb_{O_{K'}}$. Thus we obtain a functor $\mathcal{E}mb^{\log}_{O_{K'}} \to \varinjlim_{K'/K} \mathcal{E}mb_{O_{K'}}$.

4.2 Tubular neighborhoods for log embeddings

For a rational number $j > 0$, a functor $X^j : \varinjlim_{K'/K} \mathcal{E}mb_{O_{K'}} \to \varinjlim_{K'/K}(\text{smooth Affinoid}/K')$ is defined as the limit of the functors $X^{je_{K'/K}} : \mathcal{E}mb_{O_{K'}} \to (\text{smooth Affinoid}/K')$ defined in Section 1.2. We define a functor $\varinjlim_{K'/K}(\text{smooth Affinoid}/K') \to \varinjlim_{K'/K}(\text{Aff}/F')$ as follows. Let $(X_{K'})_{K' \in ob\mathcal{C}}$ be an object of $(\text{smooth Affinoid}/K')$. Then the extensions K' in $\mathcal{C}$ such that the stable normalized integral model $\mathcal{A}_{O_{K'}}$ is defined over K' form a cofinal full subcategory $\mathcal{C}'$ by Theorem 1.10. For an extension K' in $\mathcal{C}'$, let $\bar{X}_{F'}$ denote the affine scheme $\mathcal{A}_{O_{K'}} \otimes_{O_{K'}} F'$ over the residue field F' of K'. By sending $(X_{K'})_{K' \in ob\mathcal{C}}$ to $(\bar{X}_{F'})_{K' \in ob\mathcal{C}'}$, we obtain a functor $\varinjlim_{K'/K}(\text{smooth Affinoid}/K') \to \varinjlim_{K'/K}(\text{Aff}/F')$. Thus, we have a sequence of functors

$$\mathcal{E}mb^{\log}_{O_K} \longrightarrow \varinjlim_{K'/K} \mathcal{E}mb_{O_K} \longrightarrow \varinjlim_{K'/K}(\text{smooth Affinoid}/K') \longrightarrow \varinjlim_{K'/K}(\text{Aff}/F') \longrightarrow G_K\text{-}(\text{Aff}/\bar{F}).$$

The compositions $X^j_{\log} : \mathcal{E}mb^{\log}_{O_K} \to \varinjlim_{K'/K}(\text{smooth Affinoid}/K')$ and $\bar{X}^j_{\log} : \mathcal{E}mb^{\log}_{O_K} \to G_K\text{-}(\text{Aff}/\bar{F})$ are more concretely described as follows. For an object $(\mathbf{A} \to A)$ of $\mathcal{E}mb^{\log}_{O_K}$ and a finite separable extension K' such that the ramification index $e' = e_{K'/K}$ is divisible by the integer e_{A/O_K} in Lemma 4.7, the base change $(\mathbf{A} \hat{\otimes}^{\log}_{O_K} O_{K'} \to A \otimes^{\log}_{O_K} O_{K'})$ is strict and we define an affinoid variety $X^j_{\log}(\mathbf{A} \to A)_{K'}$ over K' by

$$X^j_{\log}(\mathbf{A} \to A)_{K'} = X^{e'j}((\mathbf{A} \hat{\otimes}^{\log}_{O_K} O_{K'} \to A \otimes^{\log}_{O_K} O_{K'})^\circ).$$

The composite functors $X^j_{\log} : \mathcal{E}mb^{\log}_{O_K} \to \varinjlim_{K'/K}(\text{smooth Affinoid}/K')$ sends an object $(\mathbf{A} \to A)$ of $\mathcal{E}mb^{\log}_{O_K}$ to the system $X^j_{\log}(\mathbf{A} \to A) = (X^j_{\log}(\mathbf{A} \to A)_{K'})_{K'}$ where K' runs over finite separable extensions such that the ramification index $e' = e_{K'/K}$ is divisible by the integer e_{A/O_K}.
By Lemma 1.8 and the universality of $\otimes^{\log}$, we obtain a cartesian diagram

$$\begin{array}{ccc} X^j_{\log}(\mathbf{A} \to A)(\bar{K}) & \longrightarrow & Hom_{\text{cont.log}O_K\text{-alg}}(\mathbf{A}, O_{\bar{K}}) \\ \downarrow & & \downarrow \\ Hom_{\log O_K\text{-alg}}(A, O_{\bar{K}}/\mathfrak{m}^j) & \longrightarrow & Hom_{\text{cont.log}O_K\text{-alg}}(\mathbf{A}, O_{\bar{K}}/\mathfrak{m}^j). \end{array}$$

Here $O_{\bar{K}}/\mathfrak{m}^j$ denotes the limit $O_{K'}/\mathfrak{m}^{je_{K'/K}}$ of fs-log rings where K' runs finite extensions in $\bar{K}$ such that $je_{K'/K}$ is an integer. Similarly as in Section 1.2, the surjection $X^j_{\log}(\mathbf{A} \to A)(\bar{K}) \to \pi_0(X^j_{\log}(\mathbf{A} \to A))_{\bar{K}}$ induces a surjection

$$(4.2.1) \qquad Hom_{\text{cont.log}O_K\text{-alg}}(\mathbf{A}, O_{\bar{K}}/\mathfrak{m}^j) \longrightarrow \pi_0(X^j_{\log}(\mathbf{A} \to A))_{\bar{K}}.$$

The map $\mathbf{A} \to A$ also induces a map

$$(4.2.2) \qquad Hom_{\log O_K\text{-alg}}(A, O_{\bar{K}}) \longrightarrow X^j_{\log}(\mathbf{A} \to A)(\bar{K}).$$

Similarly as Lemma 1.9.4, if $(f, \mathbf{f}) : (\mathbf{A} \to O_K) \to (\mathbf{B} \to B)$ is a finite flat and log flat morphism of $\mathcal{E}mb^{\log}_{O_K}$, the map (4.2.2) induces a surjection

$$(4.2.3) \qquad Hom_{\log O_K\text{-alg}}(B, O_{\bar{K}}) \longrightarrow \pi_0(X^j_{\log}(\mathbf{B} \to B)_{\bar{K}}).$$

Let (Finite Flat and Log Flat/O_K) denote the category of finite flat and log flat log O_K-algebras A such that the log structure on $A \otimes_{O_K} K$ is trivial. We define functors $\Psi_{\log}$ and $\Psi^j_{\log}$: (Finite Flat and Log Flat/O_K) $\to G_K$-(Finite Sets) for a rational number $j > 0$ as in Section 1.2 by sending a finite flat and log flat log O_K-algebra A such that the log structure on $A \otimes_{O_K} K$ to the set $\Psi_{\log}(A) = Hom^{\log}_{O_K}(A, O_{\bar{K}})$ and to the set

$$\Psi^j_{\log}(A) = \varprojlim_{(\mathbf{A} \to A) \in \mathcal{E}mb^{\log}_{O_K}(A)} \pi_0(X^j_{\log}(\mathbf{A} \to A)_{\bar{K}})$$

respectively. As in Section 1.2, the surjection (4.2.1) implies that the projective system in the right hand side is constant. Further it induces a map $\Psi_{\log} \to \Psi^j_{\log}$ of functors.
Similarly, for an object $(\mathbf{A} \to A)$ of $\mathcal{E}mb^{\log}_{O_K}$ and a finite separable extension K' such that the ramification index $e' = e_{K'/K}$ is divisible by the integer e_{A/O_K} and that a stable normalized integral model $\mathcal{A}^j_{O_{K'}}$ of $X^j_{\log}(\mathbf{A} \to A)_{K'}$ is defined over K', an affine scheme $\bar{X}^j_{\log}(\mathbf{A} \to A)_{K'}$ over the residue field F' of K' is defined as the closed fiber $\text{Spec}(\mathcal{A}^j_{O_{K'}} \otimes_{O_{K'}} F')$. The system

$\bar{X}^j_{\log}(\mathbf{A} \to A) = (\bar{X}^j_{\log}(\mathbf{A} \to A)_{K'})_{K'}$ defines an object of $\varinjlim_{K'/K}(\mathrm{Aff}/F')$. By identifying the category $\varinjlim_{K'/K}(\mathrm{Aff}/F')$ with G_K-$(\mathrm{Aff}/\bar{F})$, we obtain the composite functor $\bar{X}^j_{\log} : \mathcal{E}mb^{\log}_{O_K} \to G_K$-$(\mathrm{Aff}/\bar{F})$. For $j > 0$, the functor $\Psi^j_{\log}$: (Finite Flat and Log Flat$/O_K$) $\to G_K$-(Finite Sets) is induced by the composition of the functors

$$\mathcal{E}mb^{\log}_{O_K} \xrightarrow{\bar{X}^j_{\log}} G_K\text{-}(\mathrm{Aff}/\bar{F}) \xrightarrow{\pi_0} G_K\text{-(Finite Sets)}.$$

We also have a functor $\bar{C}^j_{\log} : \mathcal{E}mb^{\log}_{O_K} \to G_K$-$(\mathrm{Aff}/\bar{F})$ and a map of functors $\bar{X}^j_{\log} \to \bar{C}^j_{\log}$. Let $(\mathbf{A} \to A)$ be an object of $\mathcal{E}mb^{\log}_{O_K}$ and $j > 0$ be a rational number. Let K' be a finite separable extension of K such that the ramification index $e' = e_{K'/K}$ is divisible by e_{A/O_K} and by the denominator of j and that $((A \otimes^{\log}_{O_K} O'_K) \otimes_{O_{K'}} F')_{\mathrm{red}}$ is étale over F'. Let I be the kernel of $\mathbf{A} \otimes^{\log}_{O_K} O'_K \to A \otimes^{\log}_{O_K} O'_K$ and we put

$$\bar{C}^j_{\log}(\mathbf{A} \to A)_{K'} = \mathrm{Spec}\left(\bigoplus_{n=0}^{\infty} I^n/I^{n+1} \otimes_{O_{K'}} \mathfrak{m}^{e'jn}_{K'}/m^{e'jn+1}_{K'}\right)_{\mathrm{red}.}$$

Then the system $(\bar{C}^j_{\log}(\mathbf{A} \to A)_{K'})_{K'}$ defines an object $\varinjlim_{K'/K}(\mathrm{Aff}/F')$ and hence an object $\bar{C}^j_{\log}(\mathbf{A} \to A)$ of G_K-$(\mathrm{Aff}/\bar{F})$. It is a scheme over $((A \otimes^{\log}_{O_K} O_{K'}) \otimes_{O_{K'}} \bar{F})_{\mathrm{red}}$ for K' as above. In the following, we put $A_{\log \bar{F},\mathrm{red}} = ((A \otimes^{\log}_{O_K} O_{K'}) \otimes_{O_{K'}} \bar{F})_{\mathrm{red}} = (A \otimes^{\log}_{O_K} \bar{F})_{\mathrm{red}}$. In the right hand side, $\bar{F}$ is regarded as the limit of an fs-log ring with the chart $\mathbb{Q}_{\geq 0} \to \bar{F}$ sending positive rational numbers to 0.

We study relations between X^j and $X^j_{\log}$. Let $(\mathbf{A} \to A)$ be an object of $\mathcal{E}mb_{O_K}$ and $(\mathbf{B} \to B)$ be an object of $\mathcal{E}mb^{\log}_{O_K}$. Let $(\mathbf{A} \to A)^{\log}$ be the object of $\mathcal{E}mb^{\log}_{O_K}$ defined by the pull-back log structures. An O_K-algebra homomorphism $A \to B$ can be lifted to a morphism $(\mathbf{A} \to A)^{\log} \to (\mathbf{B} \to B)$ of $\mathcal{E}mb^{\log}_{O_K}$ by Lemma 4.2. For a rational number $j > 0$, a morphism $(\mathbf{A} \to A)^{\log} \to (\mathbf{B} \to B)$ of $\mathcal{E}mb^{\log}_{O_K}$ induces a morphism $X^j_{\log}(\mathbf{B} \to B) \to X^j_{\log}((\mathbf{A} \to A)^{\log}) = X^j(\mathbf{A} \to A)$ of affinoid varieties.

Let $(\mathbf{A} \to A)$ be a log pre-embedding. We have an embedding $(\mathbf{A} \to A)^{\circ}$, a log embedding $(\mathbf{A}^{\sim} \to A)$ and a canonical map $((\mathbf{A} \to A)^{\circ})^{\log} \to (\mathbf{A}^{\sim} \to A)$ of log embeddings by the construction in Lemma 4.4.2. For a rational number $j > 0$, we have affinoid varieties $X^j((\mathbf{A} \to A)^{\circ})$ and $X^j_{\log}(\mathbf{A}^{\sim} \to A)$ and a map of affinoid varieties $X^j_{\log}(\mathbf{A}^{\sim} \to A) \to X^j((\mathbf{A} \to A)^{\circ})$.

Lemma 4.8 *Let $(\mathbf{A} \to A)$ be an object of pre$\mathcal{E}mb^{\log}_{O_K}$ and $j > 0$ be a positive integers.*

1. The canonical map $X^j_{\log}(\mathbf{A}^{\sim} \to A) \to X^j((\mathbf{A} \to A)^{\circ})$ is an open immersion and $X^j_{\log}(\mathbf{A}^{\sim} \to A)$ is identified with a rational subdomain.

2. Assume A is local and put $S = \text{Spec } O_K, X = \text{Spec } A$ and $\mathbf{X} = \text{Spec } \mathbf{A}$ and let s and x be the closed points of S and of X. We put $P = \bar{M}_{\mathbf{X},x}$ and identify $\bar{M}_{X,x}$ and $\bar{M}_{S,s}$ with $\mathbb{N}$. Let $e = e_{A/O_K}$ be the image of $1 \in \bar{M}_{S,s} = \mathbb{N}$ by the composition $\bar{M}_{S,s} \to \bar{M}_{\mathbf{X},x} \to \bar{M}_{X,x} = \mathbb{N}$ as in Lemma 4.7. Let $m_1, \ldots, m_n$ be a system of generators of the monoid P and $e_1, \ldots, e_n$ be their images by $P \to \mathbb{N} = \bar{M}_{X,x}$. Let $j' \geq j + \max_i e_i/e$ be a rational number strictly greater than 1. Then we have an open immersion $X^{j'}((\mathbf{A} \to A)^\circ) \to X^j_{\log}(\mathbf{A}^\sim \to A)$ of rational subdomains $X^j((\mathbf{A} \to A)^\circ)$.

Proof. 1. We may assume A is local. We use the notation in 2. Let I be the kernel of the surjection $\mathbf{A} \to A$ and J be the kernel of the surjection $\mathbf{A}^\sim \to A$. By renumbering the indices if necessary, we may assume $e_1 = 1$. We take a chart $\varphi : P \to \mathbf{A}$ and put $t_i = \varphi(m_i) \in \mathbf{A}$. We define a monoid $P^\sim$ as in Lemma 4.4.2 and $\tilde{\varphi} : P^\sim \to \mathbf{A}^\sim$ be the extension. The monoid $P^\sim$ is generated by P and $(m_i m_1^{-e_i})^{\pm 1}, i = 2, \ldots, n$. Hence the ring $\mathbf{A}^\sim$ is the completion of the subring generated by $\tilde{\varphi}(m_i m_1^{-e_i})^{\pm 1}$ over $\mathbf{A}$. For $i = 2, \ldots, n$, take liftings $u_i \in \mathbf{A}^\times$ of the image of $\tilde{\varphi}(m_i m_1^{-e_i})$ in $A^\times$. Then, the ideal J is generated by the image of I and $\tilde{\varphi}(m_i m_1^{-e_i}) - u_i, i = 2, \ldots, n$. Hence $X^j_{\log}(\mathbf{A}^\sim \to A)$ is the rational subdomain $X^j((\mathbf{A} \to A)^\circ)$ defined by the conditions $\text{ord}(t_i t_1^{-e_i} - u_i) \geq j$ for $i = 2, \ldots, n$.
2. Similarly as in the proof of Lemma 1.17, we have $\text{ord } t_1 = 1/e$ on $X^{j'}((\mathbf{A} \to A)^\circ)$ by the assumption $j' > 1$. Since $t_i - u_i t_1^{-e_i} \in I$ for $i = 2, \ldots, n$, we have $\text{ord}(t_i - u_i t_1^{e_i}) \geq j' \geq j + e_i/e$ on $X^{j'}((\mathbf{A} \to A)^\circ)$. Hence the assertion follows. □

COROLLARY 4.9 *Let $(\mathbf{A} \to A)$ be a log pre-embedding constucted in the proof of Lemma 4.4.1. Then, for a rational number $j > 0$, we have open immersions*

$$X^{j+1}((\mathbf{A} \to A)^\circ) \longrightarrow X^j_{\log}(\mathbf{A}^\sim \to A) \longrightarrow X^j((\mathbf{A} \to A)^\circ)$$

of rational subdomains.

Proof. The log structure on $\mathbf{A}$ is defined by a chart $\mathbb{N}^2 \to \mathbf{A}$ and we have $e_1 = 1$ and $e_2 = e_{L/K}$ for $m_1 = (1, 0)$ and $m_2 = (0, 1)$ in the notation of Lemma 4.8.2. Hence the assertion follows. □

The affinoid varieties $X^j_{\log}(\mathbf{A} \to A)$ and $\mathcal{Y}^j_{Z,P}$ defined in [1] Section 3.2 are related as follows. Let L be a finite separable extension of K and $A = O_L$ be the integer ring. Let $Z = (z_i)_{i \in I}$ be a finite system of generators of O_L over O_K and $P \subset I$ be a subset such that z_i is a prime element of O_L for some $i \in P$ and z_i is not zero for any $i \in P$. We recall a description of $\mathcal{Y}^j_{Z,P}$ for a rational number $j > 0$. We put $e_i = \text{ord}_L z_i$ and $e = e_{L/K}$ and let π be a prime element of K. Let I_Z be the kernel of the surjection $O_K[T_i; i \in I] \to A$ sending T_i to z_i and $(f_1, \ldots, f_m)$ be a finite set of generators of I_Z. For $i \in P$ and $(i, j) \in P^2$, we take polynomials $g_i, h_{i,j} \in O_K[T_i; i \in I]$ such that the images in O_L are

$u_i = z_i^e/\pi^{e_i}$ and $u_{i,j} = z_j^{e_i}/z_i^{e_j}$. If z_ι is a prime element for $\iota \in P$, then we have

$$\mathcal{Y}^j_{Z,P}(\bar{K}) = \left\{ (x_i)_{i\in I} \in O^I_{\bar{K}} \;\middle|\; \begin{array}{cr} \mathrm{ord} f_l(x_i) \geq j & \text{for } 1 \leq l \leq m \\ \mathrm{ord}(x_\iota^e/\pi^{e_\iota} - g_\iota(x_i)) \geq j & \\ \mathrm{ord}(x_k^{e_\iota}/x_\iota^{e_k} - h_{k,\iota}(x_i)) \geq j & \text{for } k \in P \end{array} \right\}$$

by [1] Lemma 3.9 (2). Furthermore, for $(x_i)_{i\in I} \in \mathcal{Y}^j_{Z,P}(\bar{K})$, we have $x_i/x_\iota^{e_i} \in O^\times_{\bar{K}}$ for $i \in P$.

We define a log structure on $O_K[T_i, i \in I]$ by the chart $M = \mathbb{N} \times \mathbb{N}^P \to O_K[T_i, i \in I]$ sending $(1,0)$ to π and $(0, f_i)$ to T_i where $f_i \in \mathbb{N}^P$ is the i-th standard basis. Let $\mathbf{A}$ be the formal completion of the surjection $O_K[T_i, i \in I] \to A$ sending T_i to z_i.

Lemma 4.10 *Let the notation be as above. Then $(\mathbf{A} \to A)$ is a log pre-embedding and the affinoid variety $X^j_{\log}(\mathbf{A}^\sim \to A)_{\bar{K}}$ defined by the log embedding $(\mathbf{A}^\sim \to A)$ is the same as $\mathcal{Y}^j_{Z,P}$ defined in [1] Section 3.2.*

Proof. It is clear that $(\mathbf{A} \to A)$ is a log pre-embedding. We describe the log O_K-algebra $\mathbf{A}^\sim$. As in Lemma 4.4.2, let $P^\sim \subset P^{\mathrm{gp}} = \mathbb{Z} \times \mathbb{Z}^P$ be the inverse image of $\mathbb{N}$ by the map $\mathbb{Z} \times \mathbb{Z}^P \to \mathbb{Z}$ sending $T_0 = (1,0)$ to e and the standard basis T_i of $\mathbb{Z}^P$ to e_i for $i \in P$. We consider a chart $\mathbb{N} \to O_K$ and a map of monoids $\mathbb{N} \to P^\sim$ sending $1 \in \mathbb{N}$ to a prime element $\pi \in O_K$ and to $T_0 \in P^\sim$. We put $A_{I,P} = O_K \otimes_{\mathbb{Z}[\mathbb{N}]} \mathbb{Z}[P^\sim][T_i, i \in I - P]$ and define a log structure by the chart $P^\sim \to A_{I,P}$. Then, $\mathbf{A}^\sim$ is identified with the formal completion of the natural surjection $A_{I,P} \to A$.

Let K' be a finite separable extension of K containing L as a subfield. We compute the log tensor product $A_{I,P} \otimes^{\log}_{O_K} O_{K'}$. By choosing a numbering, we assume $P = \{1, \ldots, r\} \subset I = \{1, \ldots, m\}$ and z_r is a prime element. Let $T_i, i = 0, \ldots, r$ be the standard basis of $P = \mathbb{N} \times \mathbb{N}^P$ and put $U_i = T_i T_r^{-e_i}$ for $i = 1, \ldots, r-1$ and $U_0 = T_0 T_r^{-e}$. Then the monoid $P^\sim$ is generated by $U_i^{\pm 1}, i = 0, \ldots, r-1$ and T_r and is isomorphic to $\mathbb{Z}^r \times \mathbb{N}$. Let N' be the monoid $\mathbb{N} \times \mathbb{Z}$ with the map $\mathbb{N} \to \mathbb{N} \times \mathbb{Z}$ sending $1 \in \mathbb{N}$ to $(e', 1)$. Let π' be a prime element of K' and $e' = e_{K'/K}$ be the ramification index and define a unit u' of $O_{K'}$ by $\pi = u'\pi'^{e'}$. We consider a chart $N' \to O_{K'}$ sending $U' = (0,1)$ to u' and $T' = (1,0)$ to π'. By the assumption $L \subset K'$, $\bar{e} = e'/e$ is an integer and the saturation $P^\sim +^{\mathrm{sat}}_{\mathbb{N}} N'$ is generated by $U_i^{\pm 1}, i = 1, \ldots, r-1, V^{\pm 1}, U'^{\pm 1}$ and T' where $V = T_r T'^{-\bar{e}}$ and is isomorphic to $\mathbb{Z}^{r+1} \times \mathbb{N}$. Hence $A_{I,P} \otimes^{\log}_{O_K} O_{K'} = O_{K'} \otimes_{\mathbb{Z}[N']} \mathbb{Z}[P^\sim +^{\mathrm{sat}}_{\mathbb{N}} N'][T_{r+1}, \ldots, T_m]$ is isomorphic to $O_{K'}[U_1^{\pm 1}, \ldots, U_{r-1}^{\pm 1}, T_{r+1}, \ldots, T_m, V^{\pm 1}]$. The log structure is the pull-back of that on $O_{K'}$.

The base change $\mathbf{A} \hat{\otimes}^{\log}_{O_K} O_{K'}$ is the formal completion of the surjection $A_{I,P} \otimes^{\log}_{O_K} O_{K'} \to O_L \otimes^{\log}_{O_K} O_{K'}$. We claim that the kernel of the surjection $A_{I,P} \otimes^{\log}_{O_K} O_{K'} \to O_L \otimes^{\log}_{O_K} O_{K'}$ is generated by I_Z and $U_0 - \pi/z_r^e, U_i - z_i/z_r^{e_i}, i = 1, \ldots, r$. The kernel $\mathrm{Ker}(\mathbf{A} \hat{\otimes}^{\log}_{O_K} O_{K'} \to O_L \otimes^{\log}_{O_K} O_{K'})$ is generated by $\mathrm{Ker}(A_{I,P} \to O_L)$

since the surjection $A_{I,P} \to O_L$ is exact. Since $P^\sim$ is generated by $U_0 = T_0T_r^{-e}, U_1, \ldots, U_{r-1}$ and P, the ring $A_{I,P}$ is also generated by $U_0, U_1, \ldots, U_{r-1}$ over $O_K[T_1, \ldots, T_m]$. Hence, $\mathrm{Ker}(A_{I,P} \to O_L)$ is generated by I_Z and $U_0 - \pi/z_r^e, U_i - z_i/z_r^{e_i}, i = 1, \ldots, r$ and the claim is proved.
For an element $(u_1, \ldots, u_{r-1}, v, x_{r+1}, \ldots, x_m) \in O_{\bar K}^{\times r} \times O_{\bar K}^{m-r}$, we put $x_r = v\pi'^{\bar e}$ and $x_i = u_i x_r^{e_i}$ for $i = 1, \ldots, r-1$. Then, the underlying set of $X^j_{\log}(\mathbf{A} \to A)_{\bar K}$ is

$$\left\{ \begin{matrix} (u_1, \ldots, u_{r-1}, v, x_{r+1}, \ldots, x_m) \\ \in O_{\bar K}^{\times r} \times O_{\bar K}^{m-r} \end{matrix} \middle| \begin{matrix} \mathrm{ord} f_l(x_i) \geq j \quad \text{for } 1 \leq l \leq m \\ \mathrm{ord}(v^e/u' - g_r(x_i)) \geq j \\ \mathrm{ord}(u_k - h_k(x_i)) \geq j \quad \text{for } k = 1, \ldots, r \end{matrix} \right\}.$$

Hence the map $X^j_{\log}(\mathbf{A} \to A)_{\bar K} \to \mathcal{Y}^j_{Z,P}$ sending $(u_1, \ldots, u_{r-1}, v, x_{r+1}, \ldots, x_m)$ to $(x_1, \ldots, x_m)$ is an isomorphism. □

4.3 Étale covering of log tubular neighborhoods

Let A and B be the integer rings of finite étale K-algebras. For a finite flat and log flat morphism $(\mathbf{A} \to A) \to (\mathbf{B} \to B)$ of log embeddings, we study conditions for the induced finite morphism $X^j_{\log}(\mathbf{A} \to A) \to X^j_{\log}(\mathbf{B} \to B)$ to be étale.

Proposition 4.11 *Let A and $B = O_L$ be the integer rings of finite separable extensions of K and $(\mathbf{A} \to A) \to (\mathbf{B} \to B)$ be a finite flat and log flat morphism of log embeddings. Let $j > 0$ be a rational number, π_L a prime element of L and $e = \mathrm{ord}\pi_L$ be the ramification index.*
1. Assume $A = O_K$. Suppose that, for each $j' > j$, there exists a finite separable extension K' of K such that $X^{j'}_{\log}(\mathbf{B} \to B)_{K'}$ is isomorphic to the disjoint union of finitely many copies of $X^{j'}_{\log}(\mathbf{A} \to A)_{K'}$ as an affinoid variety over $X^{j'}_{\log}(\mathbf{A} \to A)_{K'}$. Then there is an integer $0 \leq n < ej$ such that π_L^n annihilates $\Omega_{B/A}(\log/\log)$.
2. If there is an integer $0 \leq n < ej$ such that π_L^n annihilates $\Omega_{B/A}(\log/\log)$, then the finite flat map $X^j_{\log}(\mathbf{B} \to B) \to X^j_{\log}(\mathbf{A} \to A)$ is étale.

Corollary 4.12 *Let $A = O_K$ and let B be the integer ring of a finite étale K-algebra and $(\mathbf{A} \to A) \to (\mathbf{B} \to B)$ be a finite flat and log flat morphism of log embeddings. Let $j > 0$ be a rational number. Suppose that, for each $j' > j$, there exists a finite separable extension K' of K such that $X^{j'}_{\log}(\mathbf{B} \to B)_{K'}$ is isomorphic to the disjoint union of finitely many copies of $X^{j'}_{\log}(\mathbf{A} \to A)_{K'}$ as in Proposition 4.11.1. Let I be the kernel of the surjection $\mathbf{B} \to B$ and let $N_{B/\mathbf{B}}$ be the B-module I/I^2. Then we have the following.*
1. The finite map $X^j_{\log}(\mathbf{B} \to B) \to X^j_{\log}(\mathbf{A} \to A)$ is étale and is extended to a finite étale map of stable normalized integral models.
2. The finite map $\bar X^j_{\log}(\mathbf{B} \to B) \to \bar X^j_{\log}(\mathbf{A} \to A)$ is étale.

3. The twisted normal cone $\bar{C}^j_{\log}(\mathbf{B} \to B)$ *is canonically isomorphic to the covariant vector bundle defined by the* $B_{\bar{F},\mathrm{red}}$*-module* $(Hom_B(N_{B/\mathbf{B}}, B) \otimes_{O_K} N^j) \otimes_{B_{\bar{F}}} B_{\log \bar{F},\mathrm{red}}$ *and the finite map* $\bar{X}^j_{\log}(\mathbf{B} \to B) \to \bar{C}^j_{\log}(\mathbf{B} \to B)$ *is étale.*

To prove Proposition 4.11, we use the following.

Lemma 4.13 *Let* $A = O_L$ *be the integer ring of a finite separable extension* L, $\mathbf{A} \to A$ *be a log embedding and let* $\mathbf{M}$ *be an* $\mathbf{A}$*-module of finite type. Let* $j > 0$ *be a rational number and* K' *be a finite separable extension of* K *such that the map* $O_{K'} \to A \otimes^{\log}_{O_K} O_{K'}$ *is strict and the stable normalized integral model* $\mathcal{A}^j_{O_{K'}}$ *of* $X^j_{\log}(\mathbf{A} \to A)$ *is defined over* K'. *Let* e *and* e' *be the ramification indices of* L *and of* K' *over* K *and* π_L *and* π' *be prime elements of* L *and* K'. *Assume that* e'/e *and* $e'j$ *are integers. Then the following conditions are equivalent.*
(1) There exists an integer $0 \le n < ej$ *such that the* A*-module* $M = \mathbf{M} \otimes_{\mathbf{A}} A$ *is annihilated by* π_L^n.
(2) The $\mathcal{A}^j_{O_{K'}}$*-module* $\mathcal{M}^j = \mathbf{M} \otimes_{\mathbf{A}} \mathcal{A}^j_{O_{K'}}$ *is annihilated by* $\pi'^{e'j-1}$.

Proof of Lemma 4.13. The proof is similar to that of Lemma 1.17. The image of an element in the kernel I of the surjection $\mathbf{A} \otimes^{\log}_{O_K} O_{K'} \to A \otimes^{\log}_{O_K} O_{K'}$ in $\mathcal{A}^j_{O_{K'}}$ is divisible by $\pi'^{e'j}$. Hence we have a commutative diagram

$$\begin{array}{ccc} \mathbf{A} & \longrightarrow & \mathcal{A}^j_{O_{K'}} \\ \downarrow & & \downarrow \\ A & \longrightarrow & \mathcal{A}^j_{O_{K'}}/(\pi'^{e'j}) \end{array}$$

of log rings. The image of $\pi_L \in A$ is a unit times $\pi'^{e'/e}$ in $\mathcal{A}^j_{O_{K'}}/(\pi'^{e'j})$. The rest of the proof is the same as that of Lemma 1.17.

Proof of Proposition 4.11. Proof is similar to that of Proposition 1.15.
1. For $j > 0$, the affinoid variety $X^j_{\log}(\mathbf{A} \to A)$ is a polydisk. By the proof of Lemma 1.7, there exist a finite separable extension K' of K of ramification index e', an embedding $(\mathbf{B} \otimes^{\log}_{O_K} O_{K'} \to B')$ in $\mathcal{E}mb_{O_{K'}}$ isomorphic to $(O_{K'}[[T_1, \ldots, T_n]]^N \to O_{K'}^N)$ for some $N > 0$, a positive rational number $\epsilon < j$ and an open immersion $X^j_{\log}(\mathbf{B} \to B)_{K'} \to X^{e'\epsilon}((\mathbf{B} \otimes^{\log}_{O_K} O_{K'} \to B')^\circ)$ as a rational subdomain. The affinoid variety $X^{e'\epsilon}((\mathbf{B} \otimes^{\log}_{O_K} O_{K'} \to B')^\circ)$ is the disjoint union of finitely many copies of polydisks. Enlarging K' if necessary, we may assume that $e'j$ and $e'\epsilon$ are integers. We may further assume that there is a rational number $j < j' < j + \epsilon$ such that $e'j'$ is an integer, that the stable normalized integral models $\mathcal{B}^{j'}_{O_{K'}}$ and $\mathcal{B}'^{e'\epsilon}_{O_{K'}}$ of $X^{j'}_{\log}(\mathbf{B} \to B)_{K'}$ and of $X^{e'\epsilon}((\mathbf{B} \otimes^{\log}_{O_K} O_{K'} \to B')^\circ)_{K'}$ are defined over K' and $X^{j'}_{\log}(\mathbf{B} \to B)_{K'}$ is isomorphic to the disjoint union of copies of $X^{j'}_{\log}(\mathbf{A} \to A)_{K'}$. Since $e'j'$ is an integer, the stable normalized integral model $\mathcal{A}^{j'}_{O_{K'}}$ of $X^{j'}_{\log}(\mathbf{A} \to A)$ is also

defined over K'. We have a commutative diagram

$$\begin{array}{ccccc} \mathbf{A} & & \longrightarrow & & \mathcal{A}^{j'}_{O_{K'}} \\ \downarrow & & & & \downarrow \\ \mathbf{B} & \longrightarrow & \mathcal{B}'^{e'\epsilon}_{O_{K'}} & \longrightarrow & \mathcal{B}^{j'}_{O_{K'}}. \end{array}$$

We consider the modules $\hat{\Omega}_{\mathbf{A}/O_K}(\log/\log) = \varprojlim_n \Omega_{(\mathbf{A}/\mathfrak{m}^n_{\mathbf{A}})/O_K)}(\log/\log), \hat{\Omega}_{\mathcal{A}^{j'}_{O_{K'}}/O_{K'}} = \varprojlim_n \Omega_{(\mathcal{A}^{j'}_{O_{K'}}/\pi'^n\mathcal{A}^{j'}_{O_{K'}})/O_{K'}}$ etc. Since $\mathbf{A}$ is strict over O_K and $\mathbf{B}\otimes^{\log}_{O_K} O_{K'}$ is strict over $O_{K'}$, the canonical maps $\hat{\Omega}_{\mathbf{A}/O_K}(\log/\log) \to \hat{\Omega}_{\mathbf{A}/O_K}$ and $(\mathbf{B}\otimes^{\log}_{O_K} O_{K'})\otimes_{\mathbf{B}} \hat{\Omega}_{\mathbf{B}/O_K}(\log/\log) \to \hat{\Omega}_{(\mathbf{B}\otimes^{\log}_{O_K} O_{K'})/O_{K'}}$ are isomorphisms. Thus, as in the proof of Proposition 1.15, we have a commutative diagram

$$\begin{array}{ccccc} \mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathbf{A}} \hat{\Omega}_{\mathbf{A}/O_K}(\log/\log) & & \longrightarrow & & \mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathcal{A}^{j'}_{O_{K'}}} \hat{\Omega}_{\mathcal{A}^{j'}_{O_{K'}}/O_{K'}} \\ \downarrow & & & & \downarrow \\ \mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathbf{B}} \hat{\Omega}_{\mathbf{B}/O_K}(\log/\log) & \longrightarrow & \mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathcal{B}'^{e'\epsilon}_{O_{K'}}} \hat{\Omega}_{\mathcal{B}'^{e'\epsilon}_{O_{K'}}/O_{K'}} & \longrightarrow & \hat{\Omega}_{\mathcal{B}^{j'}_{O_{K'}}/O_{K'}}. \end{array}$$

We show that the modules are free $\mathcal{B}^{j'}_{O_{K'}}$-modules of rank n, the maps are injective and that we have an inclusion $\pi'^{e'j'}\mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathbf{B}} \hat{\Omega}_{\mathbf{B}/O_K}(\log/\log) \subset \pi'^{e'\epsilon}\mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathbf{A}} \hat{\Omega}_{\mathbf{A}/O_K}(\log/\log)$ as submodules of $\hat{\Omega}_{\mathcal{B}^{j'}_{O_{K'}}/O_{K'}}$. By the assumption on the covering $X^{j'}_{\log}(\mathbf{B}\to B)_{K'} \to X^{j'}_{\log}(\mathbf{A}\to A)_{K'}$, the $\mathcal{A}^{j'}_{O_{K'}}$-algebra $\mathcal{B}^{j'}_{O_{K'}}$ is isomorphic to the product of finitely many copies of $\mathcal{A}^{j'}_{O_{K'}}$. Hence the right vertical map $\mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathcal{A}^{j'}_{O_{K'}}} \hat{\Omega}_{\mathcal{A}^{j'}_{O_{K'}}/O_{K'}} \to \hat{\Omega}_{\mathcal{B}^{j'}_{O_{K'}}/O_{K'}}$ is an isomorphism. Similarly as in the proof of Proposition 1.15.1, by the canonical map $\mathcal{A}^{j'}_{O_{K'}} \otimes_{\mathbf{A}} \hat{\Omega}_{\mathbf{A}/O_K}(\log/\log) \to \hat{\Omega}_{\mathcal{A}^{j'}_{O_{K'}}/O_{K'}}$, the module $\mathcal{A}^{j'}_{O_{K'}} \otimes_{\mathbf{A}} \hat{\Omega}_{\mathbf{A}/O_K}(\log/\log)$ is identified with the submodules $\pi'^{e'j'}\hat{\Omega}_{\mathcal{A}^{j'}_{O_{K'}}/O_{K'}}$ of the free module $\hat{\Omega}_{\mathcal{A}^{j'}_{O_{K'}}/O_{K'}}$. Also by $\mathcal{B}'^{e'\epsilon}_{O_{K'}} \otimes_{\mathbf{B}} \hat{\Omega}_{\mathbf{B}/O_K}(\log/\log) \to \hat{\Omega}_{\mathcal{B}'^{e'\epsilon}_{O_{K'}}/O_{K'}}$, the module $\mathcal{B}'^{e'\epsilon}_{O_{K'}} \otimes_{\mathbf{B}} \hat{\Omega}_{\mathbf{B}/O_K}(\log/\log)$ is identified with the submodule $\pi'^{e'\epsilon}\hat{\Omega}_{\mathcal{B}'^{e'\epsilon}_{O_{K'}}/O_{K'}}$ of the free module $\hat{\Omega}_{\mathcal{B}'^{e'\epsilon}_{O_{K'}}/O_{K'}}$. Hence we obtain an inclusion $\pi'^{e'j'}\mathcal{B}^{j'}_{O_{K'}}\otimes_{\mathbf{B}}\hat{\Omega}_{\mathbf{B}/O_K}(\log/\log) \subset \pi'^{e'\epsilon}\mathcal{B}^{j'}_{O_{K'}}\otimes_{\mathbf{A}}\hat{\Omega}_{\mathbf{A}/O_K}(\log/\log)$ as submodules of $\hat{\Omega}_{\mathcal{B}^{j'}_{O_{K'}}/O_{K'}}$.

Thus the $\mathcal{B}^{j'}_{O_{K'}}$-module $\mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathbf{B}} \Omega_{\mathbf{B}/\mathbf{A}}(\log/\log) = \mathrm{Coker}(\mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathbf{A}} \hat{\Omega}_{\mathbf{A}/O_K}(\log/\log) \to \mathcal{B}^{j'}_{O_{K'}} \otimes_{\mathbf{B}} \hat{\Omega}_{\mathbf{B}/O_K}(\log/\log))$ is annihilated by $\pi'^{e'(j'-\epsilon)}$.

Since $0 < j - \epsilon < j' - \epsilon < j$, applying Lemma 4.13 (2)$\Rightarrow$(1), the assertion is proved.
2. Let K' be a finite separable extension such that $e'j$ is an integer, that $B \otimes^{\log}_{O_K} O_{K'}$ is strict over $O_{K'}$ and that the stable normalized integral models $\mathcal{A}^j_{O_{K'}}$ and $\mathcal{B}^j_{O_{K'}}$ are defined over K'. By Lemma 4.13 (1)$\Rightarrow$(2), the $\mathcal{B}^j_{O_{K'}}$-module $\mathcal{B}^j_{O_{K'}} \otimes_{\mathbf{B}} \Omega_{\mathbf{B}/\mathbf{A}}(\log / \log)$ is annihilated by $\pi'^{n'}$ for an integer $n' < e'j$. The rest of proof is the same as that of Proposition 1.15.2. ☐

Proof of Corollary 4.12. The same as that of Corollary 1.16. ☐

5 Filtration by ramification groups: the logarithmic case

5.1 Construction

In this subsection, we rephrase the definition of the logarithmic filtration by ramification groups given in the previous paper [1] by using the preceeding constructions.

Let Φ : (Finite Étale/K) $\to$ G_K-(Finite Sets) be the fiber functor as in Section 2.1. For a rational number $j > 0$, we define a functor $\Phi^j_{\log}$: (Finite Étale/K) $\to$ G_K-(Finite Sets) as the composition of the functor (Finite Étale/K) $\to$ (Finite Flat and log Flat/O_K) sending a finite étale K-algebra L to the integral closure O_L of O_K in L with the standard log structure and the functor $\Psi^j_{\log}$: (Finite Flat and log Flat/O_K) $\to$ G_K-(Finite Sets) defined in Section 4.2. The map (4.2.3) defines a surjection $\Phi \to \Phi^j_{\log}$ of functors. In [1], we define the logarithmic filtration by ramification groups on G_K by using the family of surjections $(\Phi \to \Phi^j_{\log})_{j>0,\in\mathbb{Q}}$ of functors. The filtration by the log ramification groups $G^j_{K,\log} \subset G_K, j > 0, \in \mathbb{Q}$ is characterized by the condition that the canonical map $\Phi(L) \to \Phi^j_{\log}(L)$ induces a bijection $\Phi(L)/G^j_{K,\log} \to \Phi^j_{\log}(L)$ for each finite étale algebra L over K.

The functor $\Phi^j_{\log}$ is defined by the commutativity of the diagram

$$\begin{array}{ccc}
(\text{Finite Étale}/K) & \xrightarrow{\quad \Phi^j_{\log} \quad} & G_K\text{-(Finite Sets)} \\
\downarrow & \nearrow_{\Psi^j_{\log}} & \uparrow \pi_0 \\
(\text{Finite Flat and Log Flat}/O_K) & & G_K\text{-}(\text{Aff}/\bar{F}) \\
\uparrow & & \uparrow \\
\mathcal{E}mb^{\log}_{O_K} & & \varinjlim_{K'/K} (\text{Aff}/F') \\
\downarrow {\scriptstyle (\ \otimes^{\log} O_{K'})_{K'}} & & \uparrow {\scriptstyle (X_{K'})_{K'} \mapsto (\bar{X}_{F'})_{K'}} \\
\varinjlim_{K'/K} \mathcal{E}mb_{O_{K'}} & \xrightarrow[(X^{e_{K'/K} j})_{K'}]{} & \varinjlim_{K'/K} (\text{smooth Affinoid}/K')
\end{array}$$

We briefly recall how the other arrows in the diagram are defined. The forgetful functor $\mathcal{E}mb^{\log}_{O_K} \to$ (Finite Flat and Log Flat/O_K) sends $(\mathbf{A} \to A)$ to A. The functor $\mathcal{E}mb^{\log}_{O_K} \to \varinjlim_{K'/K} \mathcal{E}mb_{O_{K'}}$ sends a log embedding to the system of strict base changes. The functor $\varinjlim_{K'/K} \mathcal{E}mb_{O_K} \to \varinjlim_{K'/K}$(smooth Affinoid/$K'$) is defined by the system of tubular neighborhoods. The functor $\varinjlim_{K'/K}$(smooth Affinoid/K') $\to \varinjlim_{K'/K}$(Aff/F') is defined by the closed fiber of the stable normalized integral models. The functor $\varinjlim_{K'/K}$(Aff/F') $\to G_K$-(Aff/$\bar{F}$) is the equivalence of category defined in Section 1.3. The functor π_0 is defined by the set of connecteds components. They induce a functor $\Psi^j_{\log}$: (Finite Flat and log flat/O_K) $\to G_K$-(Finite Sets). The functor $\Phi^j_{\log}$ is defined as the composition of $\Psi^j_{\log}$ with the functor sending a finite étale algebra L to the integral closure O_L in L of O_K with the canonical log structure. More concretely, we have

$$\Phi^j_{\log}(L) = \varprojlim_{(\mathbf{A}\to O_L)\in \mathcal{E}mb^{\log}_{O_K}(O_L)} \pi_0(\varinjlim_{K'/K} \bar{X}^{e_{K'/K}j}((\mathbf{A} \otimes^{\log}_{O_K} O_{K'} \to O_L \otimes^{\log}_{O_K} O_{K'})^{\circ}))$$

for a finite étale K-algebra L. This definition agrees with that given in [1] by Lemma 4.10.

For a rational number $j \geq 0$, we define a functor $\Phi^{j+}_{\log}$: (Finite Étale/K) $\to\to$ G_K-(Finite Sets) by $\Phi^{j+}_{\log}(L) = \varinjlim_{j'>j} \Phi^{j'}_{\log}(L)$ for a finite étale K-algebra L. We define a closed normal subgroup $G^{j+}_{K,\log}$ to be $\overline{\cup_{j'>j} G^{j'}_K}$. Then we have $\Phi^{j+}_{\log}(L) = \Phi(L)/G^{j+}_{K,\log}$. Similarly as Lemma 2.1, the finite set $\Phi^{j+}_{\log}(L)$ has the following geometric description.

Lemma 5.1 *Let B be the integer ring with the standard log structure of a finite étale algebra L over K and $j > 0$ be a rational number. Let $(f, \mathbf{f}) : (\mathbf{A} \to O_K) \to (\mathbf{B} \to B)$ be a finite flat and log flat morphism of embeddings. Let $f^j : X^j_{\log}(\mathbf{B} \to B) \to X^j_{\log}(\mathbf{A} \to O_K)$ and $\bar{f}^j : \bar{X}^j_{\log}(\mathbf{B} \to B) \to \bar{X}^j_{\log}(\mathbf{A} \to O_K)$ be the canonical maps. Let $0 \in X^j_{\log}(\mathbf{A} \to O_K)$ be the point corresponding to the map $\mathbf{A} \to O_K$ and $\bar{0} \in \bar{X}^j_{\log}(\mathbf{A} \to O_K)$ be its specialization. Then the maps* (1.8.0), (1.12.1) *and the specialization map form a commutative diagram*

$$\begin{array}{ccccc} \Phi(L) & \longrightarrow & \Phi^{j+}_{\log}(L) & \longrightarrow & \Phi^j_{\log}(L) \\ \downarrow & & \downarrow & & \downarrow \\ (f^j)^{-1}(0) & \longrightarrow & (\bar{f}^j)^{-1}(\bar{0}) & \longrightarrow & \pi_0(\bar{X}^j_{\log}(\mathbf{B} \to B)) \end{array} \tag{5.1.1}$$

and the vertical arrows are bijections.

For a finite étale algebra L over K and a rational number $j > 0$, we say that the log ramification of L is bounded by j if the canonical map $\Phi(L) \to \Phi^j_{\log}(L)$

is a bijection. Let $A = O_K$ and let $B = O_L$ be the integer ring of a finite étale K-algebra L and $(\mathbf{A} \to A) \to (\mathbf{B} \to B)$ be a finite flat and log flat morphism of log embeddings. Then, since the map $X^j_{\log}(\mathbf{B} \to B) \to X^j_{\log}(\mathbf{A} \to A)$ is finite flat of degree $[L : K]$, the ramification of L is bounded by j if and only if there exists a finite separable extension K' of K such that the affinoid variety $X^j_{\log}(\mathbf{B} \to B)_{K'}$ is isomorphic to the disjoint union of finitely many copies of $X^j_{\log}(\mathbf{A} \to A)_{K'}$ over $X^j_{\log}(\mathbf{A} \to A)_{K'}$. We say that the log ramification of L is bounded by $j+$ if the log ramification of L is bounded by every rational number $j' > j$. The log ramification of L is bounded by $j+$ if and only if the canonical map $\Phi(L) \to \Phi^{j+}_{\log}(L)$ is a bijection.

Lemma 5.2 *Let $K \to K'$ be a map of complete discrete valuation fields inducing a local homomorphism $O_K \to O_{K'}$ of integer rings. Assume that the ramification index $e = e_{K'/K}$ is prime to p and that the residue field F' of K' is a separable extension of the residue field F of K. Then, for a rational number $j > 0$, the map $G_{K'} \to G_K$ induces a surjection $G^{ej}_{\log,K'} \to G^j_{\log,K}$.*

Proof. Let A be the integer ring of a finite étale K-algebra L and $(\mathbf{A} \to A)$ be an object of $\mathcal{E}mb_{O_K}$. By the assumption, the log tensor product $A \otimes^{\log}_{O_K} O_{K'}$ is the integer ring of $L \otimes_K K'$. The rest is the same as the proof of Lemma 2.2. □

The two filtrations by ramification groups are related as follows.

Lemma 5.3 *Let K be a complete discrete valuation field and $j > 0$ be a rational number. Then, we have inclusions $G^j_K \supset G^j_{K,\log} \supset G^{j+1}_K$.*

Proof. By Corollary 4.9, there are natural morphisms $\Phi^{j+1} \to \Phi^j_{\log} \to \Phi^j$ of functors. Hence the assertion follows. □

5.2 Functoriality of the closed fibers of log tubular neighborhoods

For a positive rational number $j > 0$, let $(\text{Finite Étale}/K)^{\leq j+}_{\log}$ denote the full subcategory of (Finite Étale/K) consisting of étale K-algebras whose log ramification is bounded by $j+$. At the end of the section, we prove Theorem 5.12. As in the proof of Theorem 2.15, we reduce it to the case where the condition

(F) There exists a perfect subfield F_0 of F such that F is finitely generated over F_0.

is satisfied. Assuming the condition (F), we define a twisted tangent space $\Theta^j_{\log}$ and show that the functor $\bar{X}^j_{\log} : \mathcal{E}mb^{\log}_{O_K} \to G_K\text{-}(\text{Aff}/\bar{F})$ induces a functor

$$\bar{X}^j_{\log} : (\text{Finite Étale}/K)^{\leq j+}_{\log} \to G_K\text{-}(\text{Finite Étale}/\Theta^j_{\log}).$$

In this subsection, L denotes a finite étale K-algebra and $A = O_L$ denotes the integer ring with the canonical log structure.
We assume that the condition (F) is satisfied. Let K_0 be a subfield of K such that $O_{K_0} = O_K \cap K_0$ is a complete discrete valuation ring with perfect residue field F_0 and F is finitely generated over F_0 as in Section 2.3. Let π_0 denote a prime element of O_{K_0}. We consider O_{K_0} as a log ring with the *trivial* log structure. We introduce a new category $\mathcal{E}mb^{\log}_{K,O_{K_0}}$ and a functor $\mathcal{E}mb^{\log}_{K,O_{K_0}} \to \mathcal{E}mb^{\log}_{O_K}$ similarly as in Section 2.3.

Definition 5.4 *Let K be a complete discrete valuation field and K_0 be a subfield of K such that $O_{K_0} = O_K \cap K_0$ is a complete discrete valuation ring with perfect residue field F_0 and that F is finitely generated over F_0. We put $m = \text{tr.deg}(F/F_0)$. We consider O_{K_0} as a log ring with the* trivial *log structure.*
1. We define $\mathcal{E}mb^{\log}_{K,O_{K_0}}$ to be the category whose objects and morphisms are as follows. An object of $\mathcal{E}mb^{\log}_{K,O_{K_0}}$ is a triple $(\mathbf{A}_0 \to A)$ where:

- *A is the integer ring of a finite étale K-algebra with the canonical log structure.*
- *$\mathbf{A}_0$ is a complete semi-local Noetherian log O_{K_0}-algebras formally smooth and formally log smooth of relative dimension $m + 1 = \text{tr.deg}(F/F_0) + 1$ over O_{K_0}.*
- *$\mathbf{A}_0 \to A$ is an exact and regular surjection of codimension 1 of log O_{K_0}-algebras and induces an isomorphism $\mathbf{A}_0/\mathfrak{m}_{\mathbf{A}_0} \to A/\mathfrak{m}_A$ of underlying F_0-algebras.*

A morphism $(f, \mathbf{f}) : (\mathbf{A}_0 \to A) \to (\mathbf{B}_0 \to B)$ is a pair of a log O_K-homomorphism $f : A \to B$ and a log O_{K_0}-homomorphism $\mathbf{f} : \mathbf{A}_0 \to \mathbf{B}_0$ such that the diagram

$$\begin{array}{ccc} \mathbf{A}_0 & \longrightarrow & A \\ {\scriptstyle \mathbf{f}}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbf{B}_0 & \longrightarrow & B \end{array}$$

is commutative.
2. For the integer ring A of a finite étale K-algebra, we define $\mathcal{E}mb^{\log}_{K,O_{K_0}}(A)$ to be the subcategory of $\mathcal{E}mb^{\log}_{K,O_{K_0}}$ whose objects are of the form $(\mathbf{A}_0 \to A)$ and morphisms are of the form $(\text{id}_A, \mathbf{f})$.
3. We say that a morphism $(\mathbf{A}_0 \to A) \to (\mathbf{B}_0 \to B)$ is finite flat and log flat if $\mathbf{A}_0 \to \mathbf{B}_0$ is finite flat and log flat and the canonical map $\mathbf{B}_0 \otimes^{\log}_{\mathbf{A}_0} A \to B$ is an isomorphism.

An object $(\mathbf{A}_0 \to A)$ of $\mathcal{E}mb^{\log}_{K,O_{K_0}}$ is an object $(\mathbf{A}_0 \to A)$ of $\mathcal{E}mb_{K,O_{K_0}}$ together with a log strucure on $\mathbf{A}_0$ such that the log ring $\mathbf{A}_0$ is formally log smooth over O_{K_0} and that the surjection $\mathbf{A}_0 \to A$ is exact.

Lemma 5.5 *1. Let A be the integer ring of a finite étale K-algebra with the canonical log structure. Then, the category $\mathcal{E}mb^{\log}_{K,O_{K_0}}(A)$ is non-empty.*

2. Let $(\mathbf{A}_0 \to A)$ and $(\mathbf{B}_0 \to B)$ be objects of $\mathcal{E}mb^{\log}_{K,O_{K_0}}$ and $A \to B$ be an O_K-homomorphism. Then there exists a homomorphism $(\mathbf{A}_0 \to A) \to (\mathbf{B}_0 \to B)$ in $\mathcal{E}mb^{\log}_{K,O_{K_0}}$ extending $A \to B$.

3. Every morphism in $\mathcal{E}mb^{\log}_{K,O_{K_0}}$ is finite flat and log flat.

Proof. 1. We may assume A is local. Take a transcendental basis $(\bar{t}_1, \ldots, \bar{t}_m)$ of the residue field E of A over F_0 such that E is a finite separable extension of $F_0(\bar{t}_1, \ldots, \bar{t}_m)$. Take a lifting $(t_1, \ldots, t_m)$ in A of $(\bar{t}_1, \ldots, \bar{t}_m)$ and prime elements t_0 of A and π_0 of O_{K_0}. Then A is unramified over the completion of the local ring of $O_{K_0}[T_0, \ldots, T_m]$ at the prime ideal (π_0, T_0) by the map defined by sending T_i to t_i. Hence there are an étale scheme X over $\mathbb{A}^{m+1}_{O_{K_0}}$, a point ξ of X above (π_0, T_0) and a regular immersion $\varphi : \hat{O}_{X,\xi} \to A$ of codimension 1. Let $\mathbf{A}_0$ be the O_{K_0}-algebra $\hat{O}_{X,\xi}$ with the log structure defined by the chart $\mathbb{N} \to \mathbf{A}_0$ sending $1 \in \mathbb{N}$ to T_0. Then $(\mathbf{A}_0 \to A)$ is an object of $\mathcal{E}mb^{\log}_{K,O_{K_0}}$.
2. Since $\mathbf{A}_0$ is formally log smooth over O_{K_0}, it follows from that $\mathbf{B}_0$ is the formal completion of itself with respect to the surjection $\mathbf{B}_0 \to B$.
3. We may assume A and B are local. We show that the map $\mathbf{B}_0 \otimes^{\log}_{\mathbf{A}_0} A \to B$ is an isomorphism. Let f be a generator of the kernel of $\mathbf{A}_0 \to A$. It is sufficient to show that the image of f in $\mathfrak{m}_{\mathbf{B}_0}/\mathfrak{m}^2_{\mathbf{B}_0}$ is not 0. We take charts $\mathbb{N} \to \mathbf{A}_0$ and $\mathbb{N} \to \mathbf{B}_0$ and let $t_0 \in \mathbf{A}_0$ and $t'_0 \in \mathbf{B}_0$ be the images of $1 \in \mathbb{N}$. The charts $\mathbb{N} \to \mathbf{A}_0$ and $\mathbb{N} \to \mathbf{B}_0$ induces isomorphisms $\mathbb{N} \to M_{\mathbf{Y},y}$ and $\mathbb{N} \to M_{\mathbf{X},x}$ where y and x are the closed points of the log schemes $\mathbf{Y} = \operatorname{Spec} \mathbf{A}_0$ and $\mathbf{X} = \operatorname{Spec} \mathbf{X}_0$. The map $\mathbb{N} = M_{\mathbf{Y},y} \to \mathbb{N} = M_{\mathbf{X},x}$ is the multiplication by the ramification index e of $B \otimes_{O_K} K$ over $A \otimes_{O_K} K$.
Since dt_0 is in the kernel of the surjection $\hat{\Omega}_{\mathbf{A}_0/O_{K_0}} \otimes_{\mathbf{A}_0} A/\mathfrak{m}_A \to \Omega_{(A/\mathfrak{m}_A)/F_0}$ and is non-zero, (π_0, t_0) is a basis of $\mathfrak{m}_{\mathbf{A}_0}/\mathfrak{m}^2_{\mathbf{A}_0}$. We put $f = a\pi_0 + bt_0$ in $\mathfrak{m}_{\mathbf{A}_0}/\mathfrak{m}^2_{\mathbf{A}_0}$ for some element a, b in the residue field E of A. Since the surjection $\mathbf{A}_0 \to A$ is regular of codimension 1, either of a and b is not 0. Since the image of t_0 is a basis of $\mathfrak{m}_A/\mathfrak{m}^2_A$ and the image of f is 0, we have $a \neq 0$. Similarly (π_0, t'_0) is a basis of $\mathfrak{m}_{\mathbf{B}_0}/\mathfrak{m}^2_{\mathbf{B}_0}$. Since the map $\mathbb{N} = M_{\mathbf{Y},y} \to \mathbb{N} = M_{\mathbf{X},x}$ is the multiplication by the ramification index e, the image of t_0 is a unit times t'^e_0. Hence the image of f in $\mathfrak{m}_{\mathbf{B}_0}/\mathfrak{m}^2_{\mathbf{B}_0}$ is not zero. Thus the map $\mathbf{B}_0 \otimes^{\log}_{\mathbf{A}_0} A \to B$ is an isomorphism. Since B is finite over A, $\mathbf{B}_0$ is also finite over $\mathbf{A}_0$ by Nakayama's lemma. Since $\dim \mathbf{A}_0 = \dim \mathbf{B}_0 = 2$ the assertion follows by Corollary 3.11. □

Corollary 5.6 *Every morphism in $\mathcal{E}mb^{\log}_{K,O_{K_0}}(A)$ is an isomorphism.*

Proof. If $(\mathbf{A}_0 \to A) \to (\mathbf{A}'_0 \to A)$ is a map, the map $\mathbf{A}_0 \to \mathbf{A}'_0$ is finite flat of degree 1 and is an isomorphism. □

We define a functor $\mathcal{E}mb^{\log}_{K,O_{K_0}} \to \mathcal{E}mb^{\log}_{O_K}$ as follows. Let $(\mathbf{A}_0 \to A)$ be an object of $\mathcal{E}mb^{\log}_{K,O_{K_0}}$. We define an embedding $((\mathbf{A}_0 \hat{\otimes}_{O_{K_0}} O_K)^\wedge \to A)$ by regarding $(\mathbf{A}_0 \to A)$ as an object of $\mathcal{E}mb_{K,O_{K_0}}$. Since the underlying ring of $\mathbf{A}/\mathfrak{m}_{\mathbf{A}}^n \otimes^{\log}_{O_{K_0}} O_K/\mathfrak{m}_K^n$ is $\mathbf{A}/\mathfrak{m}_{\mathbf{A}}^n \otimes_{O_{K_0}} O_K/\mathfrak{m}_K^n$, we define a log structure on $(\mathbf{A}_0 \hat{\otimes}_{O_{K_0}} O_K)^\wedge$ as the limit of those on $\mathbf{A}/\mathfrak{m}_{\mathbf{A}}^n \otimes^{\log}_{O_{K_0}} O_K/\mathfrak{m}_K^n$. We let $(\mathbf{A}_0 \hat{\otimes}^{\log}_{O_{K_0}} O_K)^\wedge$ denote the log ring $(\mathbf{A}_0 \hat{\otimes}_{O_{K_0}} O_K)^\wedge$ with this log structure.

Lemma 5.7 *Let $(\mathbf{A}_0 \to A)$ be an object of $\mathcal{E}mb^{\log}_{K,O_{K_0}}$. Then, $((\mathbf{A}_0 \hat{\otimes}^{\log}_{O_{K_0}} O_K)^\wedge \to A)$ is a log pre-embedding and hence $((\mathbf{A}_0 \hat{\otimes}^{\log}_{O_{K_0}} O_K)^{\wedge\sim} \to A)$ is a log embedding.*

Proof. By the construction, the log O_K-algebra $(\mathbf{A}_0 \hat{\otimes}^{\log}_{O_{K_0}} O_K)^\wedge$ is formally log smooth and $((\mathbf{A}_0 \hat{\otimes}^{\log}_{O_{K_0}} O_K)^\wedge \to A)$ is a log pre-embedding. The rest follows from Lemma 4.4.2. □

In the following, we put $\mathbf{A} = (\mathbf{A}_0 \hat{\otimes}_{O_{K_0}} O_K)^{\wedge\sim}$. We obtain a functor $\mathcal{E}mb^{\log}_{K,O_{K_0}} \to \mathcal{E}mb^{\log}_{O_K}$ sending $(\mathbf{A}_0 \to A)$ to $(\mathbf{A} \to A) = ((\mathbf{A}_0 \hat{\otimes}_{O_{K_0}} O_K)^{\wedge\sim} \to A)$ by Lemma 5.7. For a rational number $j > 0$, we have a sequence of functors

$$\mathcal{E}mb^{\log}_{K,O_{K_0}} \longrightarrow \mathcal{E}mb^{\log}_{O_K} \xrightarrow{X^j_{\log}} \varinjlim_{K'/K} (\text{smooth Affinoid}/K') \longrightarrow G_K\text{-}(\text{Aff}/\bar{F}).$$

We also let $\bar{X}^j_{\log}$ denote the composite functor $\mathcal{E}mb^{\log}_{K,O_{K_0}} \to G_K\text{-}(\text{Aff}/\bar{F})$. Thus, for an object $(\mathbf{A}_0 \to A)$ of $\mathcal{E}mb^{\log}_{K,O_{K_0}}$, we have $\bar{X}^j_{\log}(\mathbf{A}_0 \to A) = \bar{X}^j_{\log}((\mathbf{A}_0 \hat{\otimes}_{O_{K_0}} O_K)^{\wedge\sim} \to A)$.

For a rational number $j > 0$, the composition

$$\mathcal{E}mb^{\log}_{K,O_{K_0}} \longrightarrow \mathcal{E}mb^{\log}_{O_K} \xrightarrow{\bar{C}^j_{\log}} G_K\text{-}(\text{Aff}/\bar{F}).$$

defines a functor $\bar{C}^j_{\log} : \mathcal{E}mb^{\log}_{K,O_{K_0}} \to G_K\text{-}(\text{Aff}/\bar{F})$. We compute the twisted normal cone $\bar{C}^j_{\log}(\mathbf{A} \to A)$ for an object $(\mathbf{A}_0 \to A)$ of $\mathcal{E}mb^{\log}_{K,O_{K_0}}$ and $\mathbf{A} = (\mathbf{A}_0 \hat{\otimes}_{O_{K_0}} O_K)^{\wedge\sim}$. It is a scheme over $(A_{\log \bar{F}})_{\text{red}} = (A \otimes^{\log}_{O_K} \bar{F})_{\text{red}}$. Let $N_{A/\mathbf{A}} = I/I^2$ be the conormal module where I is the kernel of the surjection $\mathbf{A} \to A$. We put $\hat{\Omega}_{O_K/O_{K_0}}(\log) = \varprojlim_n \Omega_{(O_K/\mathfrak{m}_K^n)/O_{K_0}}(\log)$ with respect to the canonical log structure on O_K and the trivial log structure on O_{K_0}. Similarly, we put $\hat{\Omega}_{\mathbf{A}/\mathbf{A}_0}(\log/\log) = \varprojlim_n \Omega_{(\mathbf{A}/\mathfrak{m}_{\mathbf{A}}^n)/\mathbf{A}_0}(\log/\log)$. Since the map $\mathbf{A} \to \mathbf{A}_0$ is strict, we have $\hat{\Omega}_{\mathbf{A}/\mathbf{A}_0}(\log/\log) = \hat{\Omega}_{\mathbf{A}/\mathbf{A}_0}$. Let $\Omega_F(\log)$ be the F-vector space $\Omega_{F/F_0}(\log)$ with respect to the trivial log structure on F_0 and the log structure on F defined by the chart $\mathbb{N} \to F$ sending $1 \in \mathbb{N}$ to 0. The canonical map $\hat{\Omega}_{O_K/O_{K_0}}(\log) \otimes_{O_K} F \to \Omega_F(\log)$ is an isomorphism. We have an exact sequence $0 \to \Omega_{F/F_0} \to \Omega_{F/F_0}(\log) \overset{\text{res}}{\to} F \to 0$. We have canonical maps $N_{A/\mathbf{A}} \to \hat{\Omega}_{\mathbf{A}/\mathbf{A}_0} \otimes_{\mathbf{A}} A$ and $\hat{\Omega}_{O_K/O_{K_0}}(\log) \otimes_{O_K} A \to \hat{\Omega}_{\mathbf{A}/\mathbf{A}_0}(\log/\log) \otimes_{\mathbf{A}} A$. Similarly as Lemma 2.11, we have the following.

Lemma 5.8 *Let* $(\mathbf{A}_0 \to A)$ *be an object of* $\mathcal{E}mb^{\log}_{K,O_{K_0}}$.
1. If m *is the transcendental dimension of* F *over* F_0*, the dimension of the* F*-vector space* $\Omega_F(\log)$ *is* $m+1$.
2. The map $N_{A/\mathbf{A}} \to \Omega_{\mathbf{A}/\mathbf{A}_0} \otimes_{\mathbf{A}} A$ *is a surjection and the map* $\Omega_{O_K/O_{K_0}}(\log) \otimes_{O_K} A \to \Omega_{\mathbf{A}/\mathbf{A}_0} \otimes_{\mathbf{A}} A$ *is an isomorphism. They induce an isomorphism* $N_{A/\mathbf{A}} \otimes_A A/\mathfrak{m}_A \to \Omega_F(\log) \otimes_F A/\mathfrak{m}_A$.
3. Let $(\mathbf{A}_0 \to A) \to (\mathbf{B}_0 \to B)$ *be a morphism of* $\mathcal{E}mb^{\log}_{K,O_{K_0}}$ *and put* $\mathbf{B} = (\mathbf{B}_0 \hat{\otimes}_{O_{K_0}} O_K)^{\wedge\sim}$. *Then, the diagram*

$$\begin{array}{ccc} N_{A/\mathbf{A}} \otimes_A A/\mathfrak{m}_A & \longrightarrow & \Omega_F(\log) \otimes_F A/\mathfrak{m}_A \\ \downarrow & & \downarrow \\ N_{B/\mathbf{B}} \otimes_B B/\mathfrak{m}_B & \longrightarrow & \Omega_F(\log) \otimes_F B/\mathfrak{m}_B \end{array}$$

is commutative.

For a rational number $j > 0$, let $\Theta^j_{\log}$ be the $\bar{F}$-vector space $Hom_F(\Omega_F(\log), N^j)$ regarded as an affine scheme over $\bar{F}$. Similarly as Corollary 2.12, we have the following.

Corollary 5.9 *Let* $(\mathbf{A}_0 \to A)$ *be an object of* $\mathcal{E}mb^{\log}_{K,O_{K_0}}$ *and let* $(\mathbf{A} \to A)$ *be its image in* $\mathcal{E}mb^{\log}_{O_{K_0}}$. *Let* $j > 0$ *be a rational number.*
1. Let $\bar{C}^j_{\log}(\mathbf{A} \to A)$ *be the twisted normal cone. The isomorphism in Lemma 5.8.2 induces an isomorphism* $\bar{C}^j_{\log}(\mathbf{A} \to A) \to \Theta^j_{\log} \otimes_{\bar{F}} (A_{\log \bar{F}})_{\rm red}$.
2. Let $(\mathbf{A}_0 \to A) \to (\mathbf{B}_0 \to B)$ *be a morphism of* $\mathcal{E}mb^{\log}_{K,O_{K_0}}$. *Then the diagram*

$$\begin{array}{ccccc} \bar{X}^j_{\log}(\mathbf{B} \to B) & \longrightarrow & \bar{C}^j_{\log}(\mathbf{B} \to B) & \longrightarrow & \Theta^j_{\log} \otimes_{\bar{F}} (B_{\log \bar{F}})_{\rm red} \\ \downarrow & & \downarrow & & \downarrow \\ \bar{X}^j_{\log}(\mathbf{A} \to A) & \longrightarrow & \bar{C}^j_{\log}(\mathbf{A} \to A) & \longrightarrow & \Theta^j_{\log} \otimes_{\bar{F}} (A_{\log \bar{F}})_{\rm red} \end{array}$$

is commutative.
3. If the ramification of $A \otimes_{O_K} K$ *is bounded by* $j+$*, then the composition* $\bar{X}^j_{\log}(\mathbf{A} \to A) \to \bar{C}^j_{\log}(\mathbf{A} \to A) \to \Theta^j_{\log}$ *is finite and étale.*

For a rational number $j > 0$, we regard $\Theta^j_{\log}$ as an object of G_K-(Aff$/\bar{F}$) with the natural G_K-action. Let G_K-(Finite Étale$/\Theta^j_{\log}$) denote the subcategory of G_K-(Aff$/\bar{F}$) whose objects are finite étale schemes over $\Theta^j_{\log}$ and morphisms are over $\Theta^j_{\log}$. Let $\mathcal{E}mb^{\log,\le j+}_{K,O_{K_0}}$ denote the full subcategory of $\mathcal{E}mb^{\log}_{K,O_{K_0}}$ consisting of the objects $(\mathbf{A}_0 \to A)$ such that the log ramifications of $A \otimes_{O_K} K$ are bounded by $j+$. By Corollary 5.9, the functor $\bar{X}^j_{\log} : \mathcal{E}mb^{\log}_{K,O_{K_0}} \to G_K$-(Aff$/\bar{F}$) induces a functor $\bar{X}^j_{\log} : \mathcal{E}mb^{\log,\le j+}_{K,O_{K_0}} \to G_K$-(Finite Étale$/\Theta^j_{\log}$).

The functor $\bar{X}^j_{\log} : \mathcal{E}mb^{\log, \leq j+}_{K,O_{K_0}} \to G_K$-(Finite Étale/$\Theta^j_{\log}$) further induces a functor $\bar{X}^j_{\log}$: (Finite Étale/$K)^{\leq j+}_{\log} \to G_K$-(Finite Étale/$T^j_{\log}$). In fact, similarly as Lemma 2.13 and Corollary 2.14, we have the following.

Lemma 5.10 *Let $f : A \to B$ be a map over O_K and let $(f, \mathbf{f}), (g, \mathbf{g}) : (\mathbf{A}_0 \to A) \to (\mathbf{B}_0 \to B)$ be maps in $\mathcal{E}mb^{\log}_{K,O_{K_0}}$. If $f = g$, then the induced maps*

$$(f, \mathbf{f})_*, (g, \mathbf{g})_* : \bar{X}^j_{\log}(\mathbf{A}_0 \to A) \longrightarrow \bar{X}^j_{\log}(\mathbf{B}_0 \to B)$$

are equal.

Corollary 5.11 *Let $j > 0$ be a rational number.*
1. Let L be a finite étale K-algebra L such that the log ramification is bounded by $j+$. Then the system $\bar{X}^j_{\log}(\mathbf{A}_0 \to O_L)$ parametrized by the objects $(\mathbf{A}_0 \to O_L)$ of $\mathcal{E}mb^{\log}_{K,O_{K_0}}(O_L)$ is constant and the limit

$$\bar{X}^j_{\log}(L) = \varprojlim_{(\mathbf{A}_0 \to O_L) \in \mathcal{E}mb^{\log}_{K,O_{K_0}}(O_L)} \bar{X}^j_{\log}(\mathbf{A}_0 \to O_L)$$

is a finite étale scheme over $\Theta^j_{\log}$.
2. The functor $\bar{X}^j_{\log} : \mathcal{E}mb^{\log, \leq j+}_{K,O_{K_0}} \to G_K$-(Finite Étale/$\Theta^j_{\log}$) induces a functor

$$\bar{X}^j_{\log} : (\text{Finite Étale}/K)^{\leq j+}_{\log} \to G_K\text{-(Finite Étale}/\Theta^j_{\log}).$$

Using the functor $\bar{X}^j_{\log}$: (Finite Étale/$K)^{\leq j+} \to G_K$-(Finite Étale/Θ^j) defined under the condition (F), we obtain the following theorem by the same argument as the proof of Theorem 2.15.

Theorem 5.12 *Let K be a complete discrete valuation field and let $j > 0$ be a rational number. Let m be the prime-to-p part of the denominator of j and I_m be the subgroup of the inertia group $I \subset G_K$ of index m. Then we have the following.*
1. The graded piece $Gr^j G_K = G^j_{K,\log}/G^{j+}_{K,\log}$ is abelian.
2. The commutator $[I_m, G^j_{K,\log}]$ is a subgroup of $G^{j+}_{K,\log}$. In particular, $Gr^j_{\log} G_K$ is a subgroup of the center of the pro-p-group $G^{0+}_{K,\log}/G^{j+}_{K,\log}$.

Similarly as in the proof of Theorem 2.15, assuming the condition (F), we obtain a canonical surjection

$$\pi_1^{\text{ab}}(\Theta^j_{\log}) \longrightarrow Gr^j_{\log} G_K. \tag{5.12.1}$$

The canonical surjections $\pi_1^{\text{ab}}(\Theta^j_{\log}) \to Gr^j_{\log} G_K$ and $\pi_1^{\text{ab}}(\Theta^j) \to Gr^j G_K$ are related as follows. The exact sequences $0 \to N \to \tilde{\Omega}_F \to \Omega_F \to 0$ and $0 \to \Omega_F \to \Omega_F(\log) \to F \to 0$ induces canonical maps $\Theta^j_{\log} \to \Theta^j$ and $\Theta^{j+1} \to \Theta^j_{\log}$.

Lemma 5.13 *Assume that the condition (F) is satisfied and that p is not a prime element of K. Then, for a rational number $j>0$, we have a commutative diagram*

$$\begin{array}{ccccc}
\pi_1^{\mathrm{ab}}(\Theta^{j+1}) & \longrightarrow & \pi_1^{\mathrm{ab}}(\Theta^{j}_{\log}) & \longrightarrow & \pi_1^{\mathrm{ab}}(\Theta^{j}) \\
\downarrow & & \downarrow & & \downarrow \\
Gr^{j+1}G_K & \longrightarrow & Gr^{j}_{\log}G_K & \longrightarrow & Gr^{j}G_K.
\end{array}$$

Proof. We show the commutativity of the left square. Let L be a finite separable extension of K such that the log ramification is bounded by $j+$ and A be the integer ring of L. By Lemma 5.3, the ramification of L is bounded by $(j+1)+$. Let $(\mathbf{A}_0 \to A)$ be an object of $\mathcal{E}mb^{\log}_{K,O_{K_0}}$. By Lemma 5.7, the surjection $\mathbf{A} = \mathbf{A}_0 \otimes^{\log}_{O_{K_0}} O_K \to A$ defines a log pre-embedding $(\mathbf{A}_0 \otimes^{\log}_{O_{K_0}} O_K \to A)$. By forgetting the log structure, we obtain an embedding $(\mathbf{A} \to A)^\circ$. By applying Lemma 4.4, we obtain a log embedding $(\mathbf{A}^\sim \to A)$. Then, by Lemma 4.8 we have an open immersion $X^{j+1}((\mathbf{A} \to A)^\circ) \to X^j_{\log}(\mathbf{A}^\sim \to A)$ of affinoid subdomains of $X^j((\mathbf{A} \to A)^\circ)$. It induces a map $\bar{X}^{j+1}(L) \to \bar{X}^j_{\log}(L)$. By the functoriality, we obtain a commutative diagram

$$\begin{array}{ccc}
\bar{X}^{j+1}(L) & \longrightarrow & \bar{X}^{j}_{\log}(L) \\
\downarrow & & \downarrow \\
\Theta^{j+1} = \bar{X}^{j+1}(K) & \longrightarrow & \Theta^{j}_{\log} = \bar{X}^{j}_{\log}(K).
\end{array}$$

From this diagram, we deduce the commutativity of the left square. The proof for the right square is similar and omitted. □

6 The perfect residue field case

6.1 The Newton polygon of a polynomial

We recall the notion of Newton polygons and establish some properties. We say that a function $l : [0,n] \to \mathbb{R} \cup \{\infty\}$ is convex if for every $0 \le x \le y \le n$, the graph of l is below the line segment connecting $(x, l(x))$ and $(y, l(y))$. If at least one of $l(x)$ and $l(y)$ is ∞, we define the line segment connecting $(x, l(x))$ and $(y, l(y))$ to be the union $\{(z,\infty) | x < z < y\} \cup \{(x, l(x)), (y, l(y))\}$. For a polynomial $h(T) = \sum_{i=0}^n b_i T^{n-i} \in \bar{K}[T]$ of degree $\le n$, we define its Newton polygon to be the graph of the maximum convex function $l_h : [0,n] \to \mathbb{R} \cup \{\infty\}$ satisfying $l_h(i) \le \mathrm{ord}\, b_i$.
If $b_0 = 1$, the Newton polygon of h and the solutions of the equation $h(T) = 0$ are related as follows. Let $z_1, \ldots, z_n$ be the solution of $h(T) = \prod_{i=1}^n (T - z_i) = 0$ and assume $\mathrm{ord} z_i$ is increasing in i. Then, since $b_i = (-1)^i \sum_{1 \le k_1 < \ldots < k_i \le n} z_{k_1} \cdots z_{k_i}$, the slope of l_h on the interval $(i-1, i)$ is equal to $\mathrm{ord} z_i$. If $l(x) = \infty$, we define the slope of l at x to be ∞.

Lemma 6.1 *Let* $f(T) = \sum_{i=0}^n a_i T^{n-i} \in O_K[T]$ *be a polynomial of degree* n *and* z *be an element of* $\bar{K}^\times$ *such that* $\mathrm{ord} z = \frac{1}{n}$*. We assume* $a_0 = 1$ *and* $\mathrm{ord} a_i \geq 1$ *for* $1 \leq i < n$*. We put*

$$h(T) = \frac{f(z(T+1)) - f(z)}{z^n} = \sum_{i=0}^{n-1} b_i T^{n-i} \in \bar{K}[T]$$

and let $l_h : [0, n] \to \mathbb{R} \cup \{\infty\}$ *be the function defining the Newton polygon of* $h(T)$*. Then, for an integer* $0 < i < n$*, the equality* $l_h(i) = \mathrm{ord} b_i$ *implies* $i = n - p^k$ *for some integer* $k \geq 0$*.*

Proof. For an integer $0 \leq r \leq n$, we put $f_r(T) = a_{n-r} T^r$, $h_r(T) = (f_r(z(T+1)) - f_r(z))/z^n = a_{n-r} z^{-(n-r)}((T+1)^r - 1)$ and let $l_r : [0, n] \to \mathbb{R} \cup \{\infty\}$ denote the function defining the Newton polygon of $h_r(T)$. We have

$$h(T) = \sum_{r=1}^n h_r(T) = \sum_{i=1}^n \left(\sum_{r=i}^n a_{n-r} z^{-(n-r)} \binom{r}{i} \right) T^i.$$

Since $\mathrm{ord} z = \frac{1}{n}$, we have $\mathrm{ord}\ b_{n-i} = \min_{i \leq r \leq n}(\mathrm{ord}\ a_{n-r} z^{-(n-r)} \binom{r}{i})$. Hence l_h is the maximum convex function satisfying $l_h \leq l_r$ for $1 \leq r \leq n$.
We compute the function l_r for $1 \leq r \leq n$. We have $h_r(T) = a_{n-r} z^{-(n-r)} \sum_{i=1}^r \binom{r}{i} T^i$. For an integer $0 < i \leq p^k | r$, we have

$$\mathrm{ord} \binom{r}{i} = \mathrm{ord}\ \frac{r}{i} + \sum_{j=1}^{i-1} \mathrm{ord}\ \frac{r-j}{j} = \mathrm{ord}\ \frac{r}{i} \geq \mathrm{ord}\ \frac{r}{p^k}.$$

The equality holds only for $i = p^k$. Hence, l_r is the maximum convex function satisfying

$$l_r(i) = (\mathrm{ord}\ a_{n-r} - 1) + \frac{r}{n} + \begin{cases} 0 & \text{if } i = n - r \\ \mathrm{ord}\ \dfrac{r}{p^k} & \text{if } i = n - p^k \text{ for an integer } 0 \leq k \leq \mathrm{ord}_p r. \end{cases}$$

Thus, for an integer i satisfying $n - p^{\mathrm{ord}_p r} \leq i \leq n$, the equality $l_h(i) = l_r(i)$ implies $i = n - p^k$ for an integer $1 \leq k \leq \mathrm{ord}_p r$. It also follows that we have $0 = l_h(0) < l_r(n - r) = l_r(n - p^{\mathrm{ord}_p r})$ for $1 \leq r < n$. Hence the equality $l_h(i) = l_r(i)$ implies $i \geq n - p^{\mathrm{ord}_p r}$. Thus the assertion is proved. □

For a polynomial $h(T) \in \bar{K}[T], \neq 0$, let $\mathrm{ord}\ h(T)$ denote the minimum of the valuations of the coefficients. For a rational number u, let π^u denote an element of $\bar{K}^\times$ satisfying $\mathrm{ord} \pi^u = u$. We define a function $\varphi_h : [0, \infty) \to [0, \infty)$ by $\varphi_h(u) = \mathrm{ord}\ h(\pi^u T)$. The function φ_h is continuous, convex and piecewise linear.

Lemma 6.2 *Let* $h(T) = \sum_{i=0}^n b_i T^{n-i} = \prod_{i=1}^n (T - z_i) \in \bar{K}[T]$ *be a monic polynomial of degree* n*. Let* $l_h : [0, n] \to \mathbb{R} \cup \{\infty\}$ *be the function defining the*

Newton polygon of $h(T)$ *and* $\varphi_h : [0,\infty) \to [0,\infty)$ *be the function* $\varphi_h(u) = \mathrm{ord}\ h(\pi^u T)$ *defined above. Then,*
1. The minimum value of the function $l_h(t) + (n-t)u$ *on* $t \in [0,n]$ *is equal to* $\varphi_h(u)$*.*
2. We have an equality

$$\varphi_h(u) = \sum_{i=1}^{n} \min(u, \mathrm{ord}\ z_i).$$

3. If the coefficient of T^r *in* $\overline{\left(\frac{h(\pi^u T)}{\pi^{\varphi_h(u)}}\right)} \in \bar{F}[T]$ *is not zero, then the function* $l_h(t) + (n-t)u$ *attains the minimum value at* $t = r$ *and we have* $l_h(r) = \mathrm{ord} b_r$*.*

Proof. 1. Since the function $l_h(t) + (n-t)u$ defines the Newton polygon of $h(\pi^u T)$, the assertion follows.
2. We put $s_i = \mathrm{ord}\ z_i$. Let $t_0 \in [0,n]$ be the minimum where the function $l_h(t) + (n-t)u$ takes the minimum value. Then t_0 is the maximum such that the function $l_h(t) + (n-t)u$ is strictly decreasing on $[0, t_0]$. Hence t_0 is the cardinality of the set $\{i | s_i < u\}$ and the minimum value of $l_h(t) + (n-t)u$ is given by

$$l_h(t_0) + (n-t_0)u = \sum_{s_i < u} s_i + \sum_{s_i \geq u} u = \sum_{i=1}^{n} \min(s_i, u).$$

Thus the assertion follows from 1.
3. The coefficient of T^r in $\overline{h(\pi^u T)/\pi^{\varphi_h(u)}} \in \bar{F}[T]$ is not zero if and only if the value of the function defining the Newton polygon of $h(\pi^u T)/\pi^{\varphi_h(u)}$ at r is zero and $l_h(r) = \mathrm{ord} b_r$. Hence the assertion follows from 1. □

6.2 The structure of graded pieces

In this subsection, we assume that the residue field F is perfect. Since the residue map $\Omega_F(\log) \to F$ is an isomorphism in this case, we have an isomorphism $\Theta^j_{\log} \to N^j$ of $\bar{F}$-vector spaces of dimension 1. Let $\pi_1^{\mathrm{ab,gp}}(N^j)$ denote the quotient of $\pi_1^{\mathrm{ab}}(N^j)$ classifying the étale isogenies to the algebraic group N^j.

PROPOSITION 6.3 *Let* K *be a complete discrete valuation field with perfect residue field and* $j > 0$ *be a positive rational number. Then,*
1. ([1] Propositions 3.7 (3) and 3.15 (4)) *We have* $G^j_{\log,K} = G^{j+1}_K$*. If* p *is not a prime element of* K*, the horizontal arrows in the diagram of Lemma 5.13 are isomorphism.*
2. The canonical surjection $\pi_1^{\mathrm{ab}}(N^j) \to Gr^j_{\log} G_K$ (5.12.1) *induces an isomorphism* $\pi_1^{\mathrm{ab,gp}}(N^j) \to Gr^j_{\log} G_K$*.*

Contrary to the proof given in [12], we give a proof without using the "lower numbering" filtration or local class field theory.

Before starting proof, we introduce some notations. Let L be a finite separable extension of K and π_L be a prime element of L. Let K_1 be the maximum unramified extension of K in L and let $f(T) \in O_{K_1}[T]$ be the minimal polynomial of π_L over K_1. Since, L is totally ramified over K_1, the polynomial $f(T)$ is an Eisenstein polynomial. We put $n = [L : K_1] = \deg f$.

We put $A = O_L$ and $K_0 = K$ and define an object $(\mathbf{A} \to A)$ of $\mathcal{E}mb^{\log}_{K,O_{K_0}}$ as follows. We define a log structure on $O_{K_1}[T]$ by the chart $\mathbb{N} \to O_{K_1}[T]$ sending 1 to T. We define a log O_{K_0}-algebra $\mathbf{A} = O_{K_1}[[T]]$ to be the formal completion of the surjection $O_{K_1}[T] \to O_L$ sending T to π_L with the induced log structure. Then the surjection $\mathbf{A} \to A$ defines an object $(\mathbf{A} \to A)$ of $\mathcal{E}mb^{\log}_{K,O_{K_0}}$. By Lemma 5.7, it defines a log pre-embedding $(\mathbf{A} \otimes^{\log}_{O_{K_0}} O_K \to A)$. The log ring $\mathbf{A} \otimes^{\log}_{O_{K_0}} O_K$ is the ring $\mathbf{A}$ itself with the log structure defined by the chart $\mathbb{N}^2 \to \mathbf{A}$ sending $(1,0)$ to T and $(0,1)$ to a prime element π of O_K. By forgetting the log structure, we obtain an embedding $(\mathbf{A} \to A)^\circ$. By applying Lemma 4.4, we obtain a log embedding $(\mathbf{A}^\sim \to A)$. The log ring $\mathbf{A}^\sim$ is identified with the formal completion of the surjection $O_K[T, U^{\pm 1}]/(T^n - U\pi) \to A$ of log O_K-algebras sending T to π_L and U to $\pi_L^n/\pi \in A^\times$ with log structure defined by the chart $\mathbb{N} \to O_{K_1}[T, U^{\pm 1}]/(T^n - U\pi)$ sending 1 to T. Let K' be a finite separable extension of K containing the conjugates of K_1 over K and an element z of ord $z = 1/n$. Then, the log tensor product $\mathbf{A}^\sim \otimes^{\log}_{O_K} O_{K'}$ is further identified with the formal completion of the surjection $O_{K_1} \otimes_{O_K} O_{K'}[W^{\pm 1}] = \prod_{\sigma : K_1 \to K'} O_{K'}[W^{\pm 1}] \to A \otimes^{\log}_{O_K} O_{K'}$ of strict log $O_{K'}$-algebras sending W to $(\pi_L \otimes 1)/(1 \otimes z)$. With this identification, the canonical map $\mathbf{A}^\sim \to \mathbf{A}^\sim \otimes^{\log}_{O_K} O_{K'}$ sends T to $(1 \otimes z)W$ and U to $((1 \otimes z)^n/\pi) \cdot W^n$. Further, we identify the affinoid variety $X^j_{\log}(\mathbf{A}^\sim \to A)_{K'}$ as an affinoid subdomain of $\coprod_{\sigma : K_1 \to K'} \mathrm{Sp} K' \langle W^{\pm 1} \rangle$.

Similarly as for $(\mathbf{A} \to A)$, by taking a prime element π of O_K, we define an object $(\mathbf{B} \to O_K)$ of $\mathcal{E}mb^{\log}_{K,O_{K_0}}$ as the formal completion of the surjection $O_{K_0}[S] \to O_K$ sending S to π. By Lemma 4.4, the log ring $\mathbf{B}^\sim$ is identified with the formal completion of the surjection $O_K[V^{\pm 1}] \to A$ of strict log O_K-algebras sending V to 1. With this identification, the canonical map $\mathbf{B} \to \mathbf{B}^\sim$ sends S to πV. Further, we identify the affinoid variety $X^j_{\log}(\mathbf{B}^\sim \to O_K)_K$ with the subdisk $D(1, \pi^j) \subset \mathrm{Sp} K \langle V^{\pm 1} \rangle$.

We define a map $(\mathbf{B} \to O_K) \to (\mathbf{A} \to A)$ of $\mathcal{E}mb^{\log}_{K,O_{K_0}}$ as follows. Since $f(T)$ is an Eisenstein polynomial of degree n, $g(T) = (T^n - f(T))/\pi$ is in $O_{K_1}[T]$ and its image is invertible in $\mathbf{A}$. By sending S to $T^n g(T)^{-1}$, we obtain a map $(\mathbf{B} \to O_K) \to (\mathbf{A} \to A)$ of $\mathcal{E}mb^{\log}_{K,O_{K_0}}$.

The Herbrand functions φ and $\psi : [0, \infty) \to [0, \infty)$ are defined as follows (cf. [4] Appendix). We put $h(T) = f(\pi_L(T+1))/\pi_L^n$ and define φ to be the function φ_h in Lemma 6.2. The function φ is strictly increasing, continuous and piecewise linear. We define $\psi : [0, \infty) \to [0, \infty)$ to be the inverse φ^{-1}. The function ψ is

also strictly increasing, continuous and piecewise linear.
For an embedding $\sigma : K_1 \to \bar{K}$ over K, let $f^\sigma(T) \in O_{\bar{K}}[T]$ denote the image of $f(T)$ by σ. For $w \in \bar{K}$ and a rational number $u > 0$, let $D(w, \pi^u)$ denote the disk with center w and radius π^u.

Lemma 6.4 *Let the notation be as above.*
1. The open immersion $X^{j+1}((\mathbf{A} \to A)^\circ) \subset X^j_{\log}(\mathbf{A}^\sim \to A)$ in Corollary 4.9 is an isomorphism.
2. As affinoid subdomains of $\coprod_{\sigma:K_1\to K'} \mathrm{Sp}K'\langle W^{\pm 1}\rangle$, we have an equality

$$X^j_{\log}(\mathbf{A}^\sim \to A) = \coprod_{\sigma:K_1\to K'} \bigcup_{f^\sigma(z_i^\sigma)=0} D(\frac{z_i^\sigma}{z}, \pi^{\psi(j)}). \tag{2}$$

The log ramification of L is bounded by j if and only if $\psi(j)$ is larger than the slope s_{n-1} of the Newton polygon of h on the interval $(n-2, n-1)$.
3. Let $\sigma : K_1 \to \bar{K}$ be an embedding and $z_i^\sigma \in O_{\bar{K}}$ be a solution of $f^\sigma(T) = 0$. We put

$$h_i^\sigma(T) = -\frac{f^\sigma(z(\pi^{\psi(j)}T + \frac{z_i^\sigma}{z}))}{\pi^j f^\sigma(0)}.$$

Then we have $h_i^\sigma \in O_{\bar{K}}[T]$. Let $\bar{h}_i^\sigma \in \bar{F}[T]$ be the reduction and let $\bar{h}_i^\sigma : \mathbb{A}^1 \to \mathbb{A}^1$ be the map defined by the polynomial $\bar{h}_i^\sigma$. Then the isomorphisms $\times\pi^{\psi(j)} + \frac{z_i^\sigma}{z} : D(0,1) \to D(\frac{z_i^\sigma}{z}, \pi^{\psi(j)}) \subset X^j_{\log}(\mathbf{A}^\sim \to A)$ and $\times\pi^j + 1 : D(0,1) \to D(1, \pi^j)$ induce a commutative diagrams

$$\begin{array}{ccc} \mathbb{A}^1 & \longrightarrow & \overline{D(\frac{z_i^\sigma}{z}, \pi^{\psi(j)})} \subset \bar{X}^j_{\log}(\mathbf{A}^\sim \to A) \\ {\scriptstyle \bar{h}_i^\sigma}\downarrow & & \downarrow \\ \mathbb{A}^1 & \longrightarrow & \overline{D(1, \pi^j)} = \bar{X}^j_{\log}(\mathbf{B}^\sim \to O_K) = N^j. \end{array}$$

Proof. 1. As in Lemma 4.8, we identify $X^j_{\log}(\mathbf{A}^\sim \to A)$ and $X^{j+1}((\mathbf{A} \to A)^\circ)$ as affinoid subdomains of $X^j((\mathbf{A} \to A)^\circ)$. The kernels of the surjections $\mathbf{A} \to A$ and $\mathbf{A}^\sim \to A$ are generated by $f(T)$ and $U^{-1} - g(T) = f(T)/\pi$ respectively. Hence, the affinoid subdomains $X^j_{\log}(\mathbf{A}^\sim \to A)$ and $X^{j+1}((\mathbf{A} \to A)^\circ)$ of $X^j((\mathbf{A} \to A)^\circ)$ are defined by the conditions ord $f(x)/\pi \geq j$ and by ord $f(x) \geq j+1$ respectively. Hence the assertion follows.
2. Since the kernel of surjection $\mathbf{A}^\sim \to A$ is generated by $f(T)/\pi$, the kernel of surjection $\mathbf{A}^\sim \otimes^{\log}_{O_K} O_{K'} \to A \otimes^{\log}_{O_K} O_{K'}$ is generated by $(z^n/\pi) \cdot (f(zW)/z)$. Hence we have

$$X^j_{\log}(\mathbf{A}^\sim \to A)(\bar{K}) = \coprod_{\sigma:K_1\to K'} \{w \in O_{\bar{K}} | \mathrm{ord}\ f^\sigma(zw)/z^n \geq j\}$$

We fix an embedding $\sigma : K_1 \to \bar{K}$ and drop σ in the notation. For $i = 1, \ldots, n$, we put $U_i = \{w | \mathrm{ord}(w - z_i/z) \geq \mathrm{ord}(w - z_k/z)$ for $k = 1, \ldots, n\}$. By the equality above, to prove (2), it is sufficient to show

$$\{w \in O_{\bar K} | \text{ord } f(zw)/z^n \geq j\} \cap U_i \subset D(z_i/z, \pi^{\psi(j)}) \subset \{w \in O_{\bar K} | \text{ord } f(zw)/z^n \geq j\}$$

for each i. Let $w \in O_{\bar K}$. We put $I_1 = \{k : \text{ord}(w - z_i/z) > \text{ord}(w - z_k/z)\}$ and $I_2 = \{k : \text{ord}(w - z_i/z) \leq \text{ord}(w - z_k/z)\}$. For $i \in I_1$, we have $\text{ord}(w - z_k/z) = \text{ord}(z_k - z_i)/z < \text{ord}(w - z_i/z)$ and, for $i \in I_2$, we have $\text{ord}(w - z_k/z) \geq \text{ord}(w - z_i/z)$ with the equality if $x \in U_i$. Since $f(zW)/z^n = \prod_{k=1}^n (W - z_k/z)$, we have an inequality

$$\text{ord}\frac{f(zw)}{z^n} = \sum_{k=1}^n \text{ord}(w - \frac{z_k}{z}) \geq \\ \geq \sum_{k \in I_1} \text{ord}(\frac{z_k}{z} - \frac{z_i}{z}) + \sum_{k \in I_2} \text{ord}(w - \frac{z_i}{z}) = \varphi(\text{ord}(w - \frac{z_i}{z})).$$

We have an equality if $x \in U_i$. Thus the equality (2) is proved. The last assertion follows from the equality (2) and $s_{n-1} = \max_{i \neq k} \text{ord } (z_i/z - z_k/z)$.

3. We show $h_i^\sigma(T) \in O_{\bar K}[T]$. We extend $\sigma : K_1 \to \bar K$ to $\sigma_i : L \to \bar K$ by sending π_L to z_i^σ and put $u = \psi(j)$. Then we have $h_i^\sigma(T) = -h^{\sigma_i}(\pi^u \cdot (z/z_i) \cdot T)/\pi^{\varphi(u)} f(0)$. Since z/z_i and $f(0)/z^n$ are units, we have $h_i^\sigma(T) \in O_{\bar K}[T]$ by the definition of $\varphi(u)$.

We show the commutativity of the diagram. Since $\mathbf{B} \to \mathbf{A}$ sends S to $T^n g(T)^{-1}$, the induced map $\mathbf{B}^\sim \to \mathbf{A}^\sim \otimes^{\log}_{O_K} O_{K'}$ sends V to

$$\frac{T^n}{\pi \cdot g(T)} = \frac{f(T)}{\pi \cdot g(T)} + 1 = \frac{f((1 \otimes z)W)}{\pi \cdot g((1 \otimes z)W)} + 1.$$

We fix $\sigma : K_1 \to K$ and we drop σ in the notation. We define a map $D(z_i/z, \pi^{\psi(j)}) \to D(1, \pi^j)$ by sending w to $(f(zw)/(\pi g(zw))) + 1$. Then, we have a commutative diagram

$$\begin{array}{ccc} D(\frac{z_i}{z}, \pi^{\psi(j)}) & \xrightarrow{\subset} & X^j_{\log}(\mathbf{A}^\sim \to A) \\ \downarrow & & \downarrow \\ D(1, \pi^j) & = & X^j_{\log}(\mathbf{B}^\sim \to O_K). \end{array}$$

The polynomial $g(zW)$ is congruent to the constant $-f(0)/\pi$ modulo the maximal ideal. Hence, by substituting $W = \pi^{\psi(j)}T + z_i/z$, we get the assertion. □

Proof of Proposition 6.3. 1. The equality $G^j_{K,\log} = G^{j+1}_K$ follows from Lemma 6.4.1. The rest is clear.

2. First we show that the map $\pi_1^{\text{ab}}(N^j) \to Gr^j_{\log} G_K$ factors the quotient $\pi_1^{\text{ab,gp}}(N^j)$. By Lemma 6.4.3, it is sufficient to show that the map $\bar h_i^\sigma : \mathbb{A}^1 \to \mathbb{A}^1$ is an isogeny. In other words, it is enough to show that if the coefficient of T^r

in $\bar{h}_i^\sigma$ is non-zero, then r is a power of p. Since $h_i^\sigma(T) = -h^{\sigma_i}(\pi^u \cdot (z/z_i) \cdot T)/\pi^{\varphi(u)} f(0)$ and z/z_i and $f(0)/z^n$ are units, the coefficient of T^r in $\bar{h}_i^\sigma$ is non-zero if and only if the coefficient of T^r in $\overline{h(\pi^u T)/\pi^{\varphi(u)}}$ is non-zero. Let l_h be the function defining the Newton polygon of h. We apply Lemma 6.2 to $h(T) = f(\pi_L(T+1))/\pi_L^n = \sum_{i=0}^{n-1} b_i T^{n-i}$. Then, if the coefficient of T^r in $\overline{h(\pi^u T)/\pi^{\varphi(u)}}$ is non-zero, we have $l_h(r) = \mathrm{ord}\ b_r$. Since $\mathrm{ord}\ z = 1/n$, we may apply Lemma 6.1 to the polynomial $h(T)$. Thus the equality $l_h(r) = \mathrm{ord}\ b_r$ implies that r is a power of p as required.

We show that the surjection $\pi_1^{\mathrm{ab,gp}}(N^j) \to Gr^j_{\log} G_K$ is an isomorphism. By Lemma 5.2, we may replace K by the completion of a maximum unramified extension and assume the residue field F is algebraically closed. To show the isomorphism, it is sufficient to construct every étale isogeny of degree p to N^j from a finite separable extension of K. Recall that every étale isogeny of degree p to N^j is obtained by pulling-back the isogeny $\mathbb{A}^1 \to \mathbb{A}^1$ defined by the polynomial $T^p - T$ by an isomorphism $N^j \to \mathbb{A}^1$.

We show the following Lemma.

Lemma 6.5 *Let $n, m, l \geq 1$ be integers such that $m \leq n$ and $pl \leq n$, m and l are prime to p and that $p^2 | n$. Let π be a prime element of O_K and a, b be element of O_K. We put $m' = n \cdot \mathrm{ord}\ a + m$ and $l' = n \cdot \mathrm{ord}\ b + pl$ and assume $pl' < m' < pl' + n \cdot \mathrm{ord}\ p$ and $pl' < n \cdot \mathrm{ord}\ p \cdot \mathrm{ord}_p(n/p^2)$. Let $f(T)$ be the Eisenstein polynomial*

$$f(T) = T^n - \pi(aT^m - bT^{pl} + 1)$$

and let $z = \pi_L$ be the image of T in $L = K[T]/f(T)$. We put

$$j = \frac{p}{p-1} \cdot \frac{m' - l'}{n} \qquad \text{and} \qquad \pi^j = maz^m \left(\frac{maz^m}{bz^{pl}} \right)^{\frac{1}{p-1}}.$$

Then,

1. The log ramification of the extension $L = K[T]/(f(T))$ is bounded by $j+$.

2. We define a map $(\mathbf{B}^\sim \to O_K) \to (\mathbf{A}^\sim \to O_L)$ as above and consider $X^j_{\log}(\mathbf{A}^\sim \to A)_L$ as an affinoid subdomain of $\mathrm{Sp} O_L\langle W\rangle$ by taking $K' = L$ and z to be the image of T. Let $\mathbb{A}^1 \to \mathbb{A}^1$ be the map defined by the polynomial $T^p - T$. Then, $D(1, \pi^{\psi(j)})$ is a connected component of $X^j_{\log}(\mathbf{A}^\sim \to A)$. Further, the isomorphism $\times \pi^j + 1 : D(0,1) \to D(1, \pi^j)$ induce a commutative diagram

$$\begin{array}{ccc} \mathbb{A}^1 & \longrightarrow & \overline{D(1,\pi^{\psi(j)})} \subset \bar{X}^j_{\log}(\mathbf{A}^\sim \to A) \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \longrightarrow & \overline{D(1,\pi^j)} = \bar{X}^j_{\log}(\mathbf{B}^\sim \to O_K) = N^j. \end{array}$$

Proof. 1. We put $h(T) = f(z(T+1))/z^n$ and let $l_h : [0,n] \to \mathbb{R} \cup \{\infty\}$ be the function defining the Newton polygon of $h(T)$. Let $l_1 : [0,n] \to \mathbb{R} \cup$

$\{\infty\}$ be the linear function characterized by $l_1(n-1) = m'/n$ and $l_1(n-p) = pl'/n$. We claim that we have an equality $l_h = l_1$ on and only on the interval $(n-p, n-1)$. By Lemma 6.1, it is sufficient to show $l_h(n-1) = m'/n, l_h(n-p) = pl'/n$ and $l_h(n-p^2) > l_1(n-p^2)$. By the proof of Lemma 6.1, we have $l_h(n-1) = \min(\text{ord } n, \text{ord } maz^m, \text{ord } plbz^{pl}) = \min(\text{ord } p \cdot \text{ord}_p n, m'/n, \text{ord } p + pl'/n)$. By the assumptions, we have $m' < n \cdot \text{ord } p + pl' < n \cdot \text{ord } p \cdot \text{ord}_p n$ and $l_h(n-1) = m'/n$. Similarly, we have $l_h(n-p) = \min(\text{ord } \binom{n}{p}, m'/n, \text{ord } \binom{pl}{p} bz^{pl}) = \min(\text{ord } p \cdot \text{ord}_p(n/p), m'/n, pl'/n) = pl'/n$ and $l_h(n-p^2) \geq \min(\text{ord } \binom{n}{p^2}, m'/n, pl'/n) = pl'/n \geq l_1(n-p) > l_1(n-p^2)$. Thus the claim is proved.

By Lemma 6.4.2, it is sufficient to show that the slope s_{n-1} of l_h on the interval $(n-2, n-1)$ is $\psi(j)$. By the claim above, we have $s_{n-1} = (l_h(n-1) - l_h(n-p))/(p-1)$ and $\varphi(s_{n-1}) = l_h(n-1) + s_{n-1} = (p \cdot l_h(n-1) - l_h(n-p))/(p-1) = p(m'-l')/(p-1)n = j$. Thus the assertion follows.

2. In Lemma 6.4.3, we put $\pi^{\psi(j)} = (maz^m/bz^{pl})^{1/(p-1)}$ and $\pi^j = maz^m \pi^{\psi(j)}$. Then we have

$$-\frac{f(z(\pi^{\psi(j)}T+1))}{\pi^j f(0)} \equiv -\frac{-\binom{pl}{p} bz^{pl} \pi^{p\psi(j)} T^p + maz^m \pi^{\psi(j)} T}{\pi^j} \equiv T^p - T.$$

Hence the assertion follows. □

We complete the proof of Proposition 6.3.2. By Lemma 6.5, it is sufficient to show the following: For every rational number $j > 0$, there exist integers $n, m', l' > 0$ satisfying the conditions in Lemma 6.5 and, for every non-zero element x of N^j, there exist $a, b \in O_K$ such that ord a is the integral part of m'/n, ord b is the integral part of pl'/n and $x \equiv maz^m (maz^m/bz^{pl})^{1/(p-1)}$. First, we prove the claim for j. Assume p is odd (resp. even). Let $n > 0$ be an integer such that $n(p-1)j/p$ (resp. $n(p-1)j/2p$) and n/p^2 are integers and $(p-1)j/p \in [(p+1)/n, (p-1)n/p^2 \cdot \text{ord } p \cdot \text{ord}_p(n/p^2) - (p+1)/n]$. Then there exist integers l', m' such that $(p-1)j/p = (m'-l')/n$, l' and m' are prime to p, $pl' < m' < pl' + n \cdot \text{ord } p$ and $pl' < n \cdot \text{ord } p \cdot \text{ord}_p(n/p^2)$. Thus the claim is proved for j. Since we may multiply a an arbitrary unit, the claim for x is clear. Hence the assertion is proved. □

References

[1] A. Abbes, T. Saito, *Ramification of local fields with imperfect residue fields*, American J. of Math. 124.5 (2002), 879-920.

[2] S. Bosch, U. Güntzer, and R. Remmert, Non-Archimedean Analysis, Fundamental Principles of Math. Sciences, vol. 261, Springer-Verlag, Berlin, 1984.

[3] A. J. de Jong, *Crystalline Dieudonné module theory via formal and rigid geometry*, Publ. Math. IHES 82 (1995), 5-96.

[4] P. Deligne, *Les corps locaux de characteristic p, limites de corps locaux de characteristic 0*, in J.-N. Bernstein, P. Deligne, D. Kazhdan, M.-F. Vign-

eras, Représentations des groupes réductifs sur un corps local, Travaux en cours, Hermann, Paris (1984), 119-157.

[5] L. Illusie, *Champs toriques et log lissité*, preprint, 2000.

[6] K. Kato, *Logarithmic structure of Fontaine-Illusie*, Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), Johns Hopkins Univ. Press, Baltimore, MD, 1989, pp.191-224.

[7] —–, *Logarithmic structure of Fontaine-Illusie II*, unpublished preprint.

[8] —–, *Toric singularities*, American J. of Math., 116 (1994), pp. 1073–1099.

[9] —–, *Semi-stable reduction and p-adic etale cohomology*, in J.-M. Fontaine (ed.), Périodes p-adiques, Astérisque 223 (1994), pp. 269-293.

[10] K. Kato, T. Saito, *Conductor formula of Bloch*, preprint (revised version), 2003.

[11] M. Olsson, *Logarithmic geometry via algebraic stacks*, to appear in Ann. Sci. Ecole Norm. Sup.

[12] T. Saito, *Ramification groups and local constants*, preprint, Univ. of Tokyo, 1996.

[13] J.-P. Serre, Corps Locaux, Hermann, Paris, 1968.

Ahmed Abbes
LAGA, Institut Galilée
Université Paris-Nord
93430 Villetaneuse, France
abbes@math.univ-paris13.fr

Takeshi Saito
Department of Mathematical Sciences
University of Tokyo
Tokyo 153-8914 Japan
t-saito@ms.u-tokyo.ac.jp

Specialization of the p-adic Polylogarithm to p-th Power Roots of Unity

Dedicated to Professor Kazuya Kato

for his fiftieth birthday

Kenichi Bannai

Received: November 11, 2002
Revised: April 28, 2003

Abstract. The purpose of this paper is to calculate the restriction of the p-adic polylogarithm sheaf to p-th power torsion points.

2000 Mathematics Subject Classification: 14F30,14G20
Keywords and Phrases: p-adic polylogarithm, syntomic cohomology, rigid cohomology

1 Introduction

Fix a rational prime p. The classical polylogarithm sheaf, constructed by Beilinson and Deligne, is a variation of mixed Hodge structures on the projective line minus three points. The p-adic polylogarithm sheaf is its p-adic analogue, and is expected to be the p-adic realization of the motivic polylogarithm sheaf. In our previous paper [Ban1], we explicitly calculated the p-adic polylogarithm sheaf on the projective line minus three points, and calculated its specializations to the d-th roots of unity for *d prime to p*. The purpose of this paper is to extend this calculation to the d-th roots of unity for *d divisible by p*. In particular, we prove that the specialization of the p-adic polylogarithm sheaf to d-th roots of unity is again related to special values of the p-adic polylogarithm function defined by Coleman [Col].

Let $K = \mathbb{Q}_p(\mu_d)$, with ring of integers $\mathcal{O}_K$. Let $\mathbb{G}_m = \operatorname{Spec} \mathcal{O}_K[t, t^{-1}]$ be the multiplicative group over $\mathcal{O}_K$. Denote by $S(\mathbb{G}_m)$ the category of *syntomic coefficients* on $\mathbb{G}_m$. This category is a rough p-adic analogue of the category of variation of mixed Hodge structures. Since p is in general ramified in K, we will use the definition in [Ban2], which is a generalization of the definition in [Ban1] to the case when p is ramified in K.

In order to describe the polylogarithm sheaf, it is first necessary to introduce the logarithmic sheaf $\mathcal{L}og$, which is a pro-object in $S(\mathbb{G}_m)$. The first property we prove for this sheaf is that it satisfies the *splitting principle*, even at roots of unity whose order is divisible by p.

PROPOSITION (= PROPOSITION 5.1) *Let $z \neq 1$ be a d-th root of unity in K, and let $i_z : \operatorname{Spec} \mathcal{O}_K \hookrightarrow \mathbb{G}_m$ be the closed immersion defined by $t \mapsto z$. Then*

$$i_z^* \mathcal{L}og = \prod_{j \geq 0} K(j).$$

Let $\mathbb{U} = \mathbb{G}_m \setminus \{1\}$. In our previous paper, following the method of [HW1] Definition III 2.2, we constructed the polylogarithm extension

$$\mathrm{pol} \in \operatorname{Ext}^1_{S_{\mathrm{syn}}(\mathbb{U})}(K(0), \mathcal{L}og).$$

We first consider the case when z is a d-th root of unity, where d is an integer of the form $d = Np^r$ with $(N, p) = 1$ and $N > 1$. In this case, we have a natural map $i_z : \operatorname{Spec} \mathcal{O}_K \to \mathbb{U}$. Let $i_z^* \mathrm{pol}$ be the image of pol in

$$\operatorname{Ext}^1_{S(\mathcal{O}_K)}(K(0), i_z^* \mathcal{L}og) = \prod_{j \geq 0} \operatorname{Ext}^1_{S(\mathcal{O}_K)}(K(0), K(j))$$

with respect to the pull-back map

$$\operatorname{Ext}^1_{S(\mathbb{U})}(K(0), \mathcal{L}og) \xrightarrow{i_z^*} \operatorname{Ext}^1_{S(\mathcal{O}_K)}(K(0), i_z^* \mathcal{L}og).$$

Our main result is concerned with the explicit shape of $i_z^* \mathrm{pol}$.

For integers $j \geq 1$, let $\mathrm{Li}_j(t)$ be the p-adic polylogarithm function defined by Coleman ([Col] VI, the function denoted $\ell_j(t)$). It is a locally analytic function defined on $\mathbb{P}^1(\mathbb{C}_p) \setminus \{1, \infty\}$ satisfying $\mathrm{Li}_j(0) = 0$. On the open unit disc $\{z \in \mathbb{C}_p \mid |z|_p < 1\}$, the function is given by the usual power series

$$\mathrm{Li}_j(t) = \sum_{n=1}^{\infty} \frac{t^n}{n^j}.$$

To deal with the specialization at points in the open unit disc around one, we also consider the locally analytic function

$$\mathrm{Li}_{j,c}(t) = \mathrm{Li}_j(t) - c^{1-j} \, \mathrm{Li}_j(t^c),$$

where c is an integer > 1.

Our main theorem may be stated as follows:

THEOREM 1 (= THEOREM 7.3) *Let z be a d-th root of unity, where d is an integer of the form $d = Np^r$ with $(N,p) = 1$ and $N > 1$. Then we have*

$$i_z^* \operatorname{pol} = \left((-1)^j \operatorname{Li}_j(z)\right)_{j\geq 1} \in \prod_{j\geq 0} \operatorname{Ext}^1_{S(\mathcal{O}_K)}(K(0), K(j)),$$

where we view $(-1)^j \operatorname{Li}_j(z)$ as elements of $\operatorname{Ext}^1_{S(\mathcal{O}_K)}(K(0), K(j))$ through the isomorphism

$$\operatorname{Ext}^1_{S(\mathcal{O}_K)}(K(0), K(j)) \cong K. \tag{1}$$

REMARK 1 *The above is compatible with the results of Somekawa [So] and also Besser-de Jeu [BdJ] on the calculation of the syntomic regulator.*

REMARK 2 *In [Ban1], we proved that when d is prime to p,*

$$i_z^* \operatorname{pol} = \left((-1)^j \ell_j^{(p)}(z)\right)_{j\geq 1},$$

where $\ell_j^{(p)}(t)$ is a locally analytic function on $\mathbb{P}^1(\mathbb{C}_p) \setminus \{1, \infty\}$, whose expansion on the open unit disc around 0 is given by

$$\ell_j^{(p)}(t) = \sum_{n\geq 1, (n,p)=1} \frac{t^n}{n^j}.$$

The difference between this formula and the formula of the previous theorem comes from the choice of the isomorphism (1). *(See Remark 7.2 for details.)*

For the case when z is a p^r-th root of unity, let $c > 1$ be an integer and let $[c] : \mathbb{G}_m \to \mathbb{G}_m$ be the multiplication by c map induced from $t \mapsto t^c$. We denote by $[c]^*$ the pull back morphism of syntomic coefficients. We define the modified polylogarithm to be

$$\operatorname{pol}_c = \operatorname{pol} - [c]^* \operatorname{pol},$$

which we prove to be an element in $\operatorname{Ext}^1_{S_{\mathrm{syn}}(\mathbb{U}_c)}(K(0), \mathcal{L}og)$ for

$$\mathbb{U}_c = \operatorname{Spec} \mathcal{O}_K \left[t, \frac{t-1}{t^c - 1}\right].$$

We note that this modification, which removes the singularity around one, is standard in Iwasawa theory.
Our theorem in this case is:

THEOREM 2 (= THEOREM 8.3) *Let z be a p^r-th root of unity. Then we have*

$$i_z^* \operatorname{pol}_c = \left((-1)^j \operatorname{Li}_{j,c}(z)\right)_{j\geq 1} \in \prod_{j\geq 0} \operatorname{Ext}^1_{S(\mathcal{O}_K)}(K(0), K(j)),$$

where i_z^* is the pull back of syntomic coefficient by the natural inclusion i_z : $\operatorname{Spec} \mathcal{O}_K \to \mathbb{U}_c$. Again, we view $\operatorname{Li}_{j,c}(z)$ as an element of $\operatorname{Ext}^1_{S(\mathcal{O}_K)}(K(0), K(j))$ through the isomorphism (1).

Contents

Acknowledgement. I would like to wish Professor Kazuya Kato a happy fiftieth birthday, and to thank him for all that he has contributed to mathematics. I would also like to thank the referee for carefully reading this paper, and for giving helpful comments.

Notation Let p be a rational prime. In this paper, we let K be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_K$ and residue field k. We denote by π a generator of the maximal ideal of $\mathcal{O}_K$. We let K_0 the maximal unramified extension of $\mathbb{Q}_p$ in K, and W its ring of integers. We denote by σ the Frobenius morphism on K_0 and W.

2 Review of the p-adic polylogarithm function

In this section, we will review the theory of p-adic polylogarithm functions due to Coleman [Col]. Since we will mainly deal with the value of the p-adic polylogarithm function at units in $\mathcal{O}_{\mathbb{C}_p}$, we will not need the full theory of Coleman integration.

As in [Col], we call any locally analytic homomorphism $\log : \mathbb{C}_p^\times \to \mathbb{C}_p^+$, such that $\frac{d}{dz}\log(1) = 1$, a branch of the logarithm. Throughout this paper, we fix once and for all a branch of the logarithm. Since we will only deal with the values of p-adic analytic functions at points outside the open unit disc where the functions have logarithmic poles, the results of this paper is *independent* of the choice of the branch.

We define the p-adic polylogarithm function $\ell_j^{(p)}(t)$ for $|t| < 1$ by

$$\ell_j^{(p)}(t) = \sum_{(n,p)=1} \frac{t^n}{n^j} \qquad (j \geq 1).$$

By [Col] Proposition 6.2, this function extends to a rigid analytic function on $\mathbb{C}_p \setminus \{z\,;\, |z-1|_p < p^{(p-1)^{-1}}\}$.

PROPOSITION 2.1 ([COL] SECTION VI) *The p-adic polylogarithm function* $\mathrm{Li}_j(t)$ *(Denoted $\ell_j(t)$ in [Col]) is a locally analytic function on* $\mathbb{P}^1(\mathbb{C}_p) \setminus \{1, \infty\}$ *satisfying*

(i) $\mathrm{Li}_0(t) = t/(1-t)$

(ii) $\frac{d}{dt} \mathrm{Li}_{j+1}(t) = \frac{1}{t} \mathrm{Li}_j(t) \qquad (j \geq 0).$

(iii) $\ell_j^{(p)}(t) = \mathrm{Li}_j(t) - p^{-j} \mathrm{Li}_j(t^p) \qquad (j \geq 1).$

DEFINITION 2.2 *(i) For any integer j, we define the function $u_j(t)$ by*

$$u_j(t) = \begin{cases} \frac{1}{j!} \log^j(t) & (j \geq 0) \\ 0 & (j < 0). \end{cases}$$

Note that if z is a root of unity in $\mathbb{C}_p$, then $u_j(z) = 0$ $(j \neq 0)$.

(ii) For any integer $n \geq 1$, we define the function $D_n(t)$ by

$$D_n(t) = \sum_{j=0}^{n-1} (-1)^j \mathrm{Li}_{n-j}(t) u_j(t).$$

If z is a root of unity in $\mathbb{C}_p$, then $D_n(z) = \mathrm{Li}_n(z)$.

To deal with the torsion points of p-th power order, we need modified versions of the above functions.

DEFINITION 2.3 *Let $c > 1$ be an integer prime to p. We let:*

(i) $\ell_{j,c}^{(p)}(z) = \ell_j^{(p)}(z) - c^{1-n} \ell_j^{(p)}(z^c) \quad (j \geq 1).$

(ii) $\mathrm{Li}_{j,c}(z) = \mathrm{Li}_{j,c}(z) - c^{1-n} \mathrm{Li}_{j,c}(z^c) \quad (j \geq 1).$

(iii)

$$D_{n,c}(z) = \sum_{j=0}^{n-1} (-1)^j \mathrm{Li}_{n-j,c}(t) u_j(t).$$

The above functions are locally analytic on the open unit disc around one.

3 The Category of Syntomic Coefficients

In this section, we will review the construction of the category of syntomic coefficients given in [Ban2] §4. Note that since we need to deal with the case when the prime p is ramified in K, the theory of [Ban1] is not sufficient.

Definition 3.1 *A syntomic datum* $\mathfrak{X} = (X, \overline{X}, j, \mathcal{P}_X, \phi_X, \iota)$ *consists of the following:*

(i) A proper smooth scheme $\overline{X}$*, separated an of finite type over* $\mathcal{O}_K$*, and an open immersion* $j : X \hookrightarrow \overline{X}$*, such that the complement* D *is a relative simple normal crossing divisor over* $\mathcal{O}_K$*.*

(ii) A formal scheme $\mathcal{P}_X$ *over* W*.*

(iii) For the formal completion $\overline{\mathcal{X}}$ *of* $\overline{X}$ *with respect to the special fiber, a closed immersion* $\iota : \overline{\mathcal{X}} \to \mathcal{P}_X \otimes_W \mathcal{O}_K$*, such that both* $\mathcal{P}_X$ *and the morphism* ι *are smooth in a neighborhood of* X_k*.*

(iv) A Frobenius map $\phi_X : \mathcal{P}_X \to \mathcal{P}_X$*, which fits into the diagram*

$$\begin{array}{ccccc} \overline{X}_k & \xrightarrow{\iota} & \mathcal{P}_X & \longrightarrow & \operatorname{Spf} W \\ {\scriptstyle F}\downarrow & & {\scriptstyle \phi_X}\downarrow & & {\scriptstyle \sigma^*}\downarrow \\ \overline{X}_k & \xrightarrow{\iota} & \mathcal{P}_X & \longrightarrow & \operatorname{Spf} W, \end{array} \tag{2}$$

where F *is the absolute Frobenius of* $\overline{X}_k$*.*

We will often omit j and ι from the notation and write

$$\mathfrak{X} = (X, \overline{X}, \mathcal{P}_X, \phi_X).$$

Example 3.2 *1. Let* $\mathbb{P}^1$ *be the projective line over* W *with coordinate* t*, and let* $\mathbb{P}^1_{\mathcal{O}_K} = \mathbb{P}^1 \otimes \mathcal{O}_K$*. We let* $\mathbb{G}_m$ *be the syntomic datum given by*

$$\mathbb{G}_m = \left(\mathbb{G}_{m\mathcal{O}_K}, \mathbb{P}^1_{\mathcal{O}_K}, \widehat{\mathbb{P}}^1, \phi\right),$$

where

(a) $\mathbb{G}_{m\mathcal{O}_K}$ *is the multiplicative group over* $\mathcal{O}_K$*, with natural inclusion* $j : \mathbb{G}_{m\mathcal{O}_K} \hookrightarrow \mathbb{P}^1_{\mathcal{O}_K}$*.*

(b) $\widehat{\mathbb{P}}^1$ *is the* p*-adic formal completion of* $\mathbb{P}^1$*.*

(c) $\iota : \widehat{\mathbb{P}}^1_{\mathcal{O}_K} \to \widehat{\mathbb{P}}^1 \otimes \mathcal{O}_K$ *is the identity.*

(d) ϕ *is the Frobenius given by* $\phi(t) = t^p$ *for the coordinate* t *on* $\widehat{\mathbb{P}}^1$*.*

2. We let $\mathbb{U}$ *be the syntomic datum given by*

$$\mathbb{U} = \left(\mathbb{U}_{\mathcal{O}_K}, \mathbb{P}^1_{\mathcal{O}_K}, \widehat{\mathbb{P}}^1, \phi\right),$$

where $\mathbb{U}_{\mathcal{O}_K} = \mathbb{P}^1_{\mathcal{O}_K} \setminus \{0, 1, \infty\}$*, with the natural inclusion* $j : \mathbb{U}_{\mathcal{O}_K} \hookrightarrow \mathbb{P}^1_{\mathcal{O}_K}$.

3. We let $\mathcal{O}_K$ *be the syntomic datum given by*

$$\mathcal{O}_K = (\operatorname{Spec} \mathcal{O}_K, \operatorname{Spec} \mathcal{O}_K, \operatorname{Spf} W, \sigma),$$

where j *and* ι *are the identity.*

Throughout this section, we fix a syntomic datum $\mathfrak{X}$. We will next review the definition of the category of *syntomic coefficients* $S(\mathfrak{X})$ on $\mathfrak{X}$. We will first define the categories $S_{\mathrm{dR}}(\mathfrak{X})$, $S_{\mathrm{rig}}(\mathfrak{X})$ and $S_{\mathrm{vec}}(\mathfrak{X})$. Let $X_K = X \otimes K$ and $\overline{X}_K = \overline{X} \otimes K$.

Definition 3.3 *We define the category* $S_{\mathrm{dR}}(\mathfrak{X})$ *to be the category consisting of objects the triple* $M_{\mathrm{dR}} := (M_{\mathrm{dR}}, \nabla_{\mathrm{dR}}, F^\bullet)$*, where:*

(i) M_{dR} *is a coherent* $\mathcal{O}_{\overline{X}_K}$ *module.*

(ii) $\nabla_{\mathrm{dR}} : M_{\mathrm{dR}} \to M_{\mathrm{dR}} \otimes \Omega^1(\log D_K)$ *is an integrable connection on* M_{dR} *with logarithmic poles along* $D_K = D \otimes K$.

(iii) $F^\bullet$ *is the Hodge filtration, which is a descending exhaustive separated filtration on* M_{dR} *by coherent sub-*$\mathcal{O}_{\overline{X}_K}$ *modules satisfying*

$$\nabla_{\mathrm{dR}}(F^m M_{\mathrm{dR}}) \subset F^{m-1} M_{\mathrm{dR}} \otimes \Omega^1_{\overline{X}_K}(\log D_K).$$

Let $X_k = X \otimes k$ be the special fiber of X and $\mathcal{X}$ the formal completion of X with respect to the special fiber. We denote by $\mathcal{X}_K$ the rigid analytic space over K associated to $\mathcal{X}$ ([Ber1] Proposition (0.2.3)) and by X_K^{an} the rigid analytic space over K associated to X_K (loc. cit. Proposition (0.3.3)). We will use the same notations for $\overline{X}$.

Definition 3.4 *We say that a set* $V \subset \overline{\mathcal{X}}_K$ *is a strict neighborhood of* $\mathcal{X}_K$ *in* X_K^{an}*, if* $V \cup (X_K^{\mathrm{an}} \setminus \mathcal{X}_K)$ *is a covering of* X_K^{an} *for the Grothendieck topology.*

For any abelian sheaf M on X_K^{an}, we let

$$j^\dagger M := \varinjlim_V \alpha_{V*} \alpha_V^* M,$$

where the limit is taken with respect to strict neighborhoods V of $\mathcal{X}_K$ in X_K^{an} with inclusion $\alpha_V : V \hookrightarrow \overline{\mathcal{X}}_K$. If M has a structure of a $\mathcal{O}_{X_K^{\mathrm{an}}}$-module, then $j^\dagger M$ has a structure of a $j^\dagger \mathcal{O}_{X_K^{\mathrm{an}}}$-module.

Definition 3.5 *We define the category $S_{\mathrm{vec}}(\mathfrak{X})$ to be the category consisting of objects the pair $M_{\mathrm{vec}} := (M_{\mathrm{vec}}, \nabla_{\mathrm{vec}})$, where:*

(i) M_{vec} *is a coherent* $j^\dagger\mathcal{O}_{X_K^{\mathrm{an}}}$ *module.*

(ii) $\nabla_{\mathrm{vec}} : M_{\mathrm{vec}} \to M_{\mathrm{vec}} \otimes \Omega^1_{X_K^{\mathrm{an}}}$ *is an integrable connection on* M_{vec}.

Let $p_{\mathrm{dR}} : X_K^{\mathrm{an}} \to \overline{X}_K$ be the natural map.

Definition 3.6 *We define the functor*

$$\mathbf{F}_{\mathrm{dR}} : S_{\mathrm{dR}}(\mathfrak{X}) \to S_{\mathrm{vec}}(\mathfrak{X})$$

by associating to $M_{\mathrm{dR}} := (M_{\mathrm{dR}}, \nabla_{\mathrm{dR}}, F^\bullet)$ the module $j^\dagger(p_{\mathrm{dR}}^* M_{\mathrm{dR}})$ with the connection induced from ∇_{dR}. The functor $\mathbf{F}_{\mathrm{dR}}$ is exact, since it is a composition of exact functors ([Ber1] Proposition 2.1.3 (iii)).

Let $\mathcal{P}_{K_0}$ be the rigid analytic space over K_0 associated to $\mathcal{P}_X$ ([Ber1] (0.2.2)). As in loc. cit. Définitions (1.1.2)(i), we define the *tubular neighborhood* of $\overline{X}_k$ (resp. X_k) in $\mathcal{P}_{K_0}$ by

$$]\overline{X}_k[_{\mathcal{P}} := \mathrm{sp}^{-1}(\overline{X}_k) \qquad \big(\text{resp.} \quad]X_k[_{\mathcal{P}} := \mathrm{sp}^{-1}(X_k)\big),$$

where $\mathrm{sp} : \mathcal{P}_{K_0} \to \mathcal{P}_X$ is the *spécialization* [Ber1] (0.2.2.1). The tubular neighborhoods are rigid analytic spaces over K_0 with structures induced from that of $\mathcal{P}_{K_0}$.

Definition 3.7 *We say that a set $V \subset]\overline{X}_k[_{\mathcal{P}}$ is a strict neighborhood of $]X_k[_{\mathcal{P}}$ in $]\overline{X}_k[_{\mathcal{P}}$, if*

$$V \cup (]\overline{X}_k[_{\mathcal{P}} \backslash]X_k[_{\mathcal{P}})$$

is a covering of $]\overline{X}_k[_{\mathcal{P}}$ for the Grothendieck topology.

For any abelian sheaf M on $]\overline{X}_k[_{\mathcal{P}}$, we let

$$j^\dagger M := \varinjlim_V \alpha_{V*}\alpha_V^* M,$$

where the limit is taken with respect to strict neighborhoods V of $]X_k[_{\mathcal{P}}$ in $]\overline{X}_k[_{\mathcal{P}}$ with inclusion $\alpha_V : V \hookrightarrow]\overline{X}_k[_{\mathcal{P}}$. If M has a structure of a $\mathcal{O}_{]\overline{X}_k[_{\mathcal{P}}}$-module, then $j^\dagger M$ has a structure of a $j^\dagger\mathcal{O}_{]\overline{X}_k[_{\mathcal{P}}}$-module.
The Frobenius map $\phi_X : \mathcal{P}_X \to \mathcal{P}_X$ induces a natural morphism of rigid analytic spaces $\phi_X :]\overline{X}_k[_{\mathcal{P}} \to]\overline{X}_k[_{\mathcal{P}}$.

Definition 3.8 *We define the category $S_{\mathrm{rig}}(\mathfrak{X})$ to be the category consisting of objects the triple $M_{\mathrm{rig}} := (M_{\mathrm{rig}}, \nabla_{\mathrm{rig}}, \Phi_M)$, where:*

(i) M_{rig} *is a coherent* $j^\dagger\mathcal{O}_{]\overline{X}_k[_{\mathcal{P}}}$*-module.*

(ii) $\nabla_{\mathrm{rig}} : M_{\mathrm{rig}} \to M_{\mathrm{rig}} \otimes \Omega^1_{]\overline{X}_k[_{\mathcal{P}}}$ *is an integrable connection on* M_{rig}*.*

(iii) Φ_M *is the Frobenius morphism, which is an isomorphism*

$$\Phi_M : \phi_X^* M_{\mathrm{rig}} \xrightarrow{\cong} M_{\mathrm{rig}}$$

of $j^\dagger \mathcal{O}_{]\overline{X}_k[_{\mathcal{P}}}$*-modules compatible with the connection.*

The map $\iota : \overline{\mathcal{X}} \to \mathcal{P}_X \otimes_W \mathcal{O}_K$ induces a map of rigid analytic spaces

$$p_{\mathrm{rig}} : X_K^{\mathrm{an}} \to]\overline{X}_k[_{\mathcal{P}}. \tag{3}$$

Definition 3.9 *We define the functor*

$$\mathbf{F}_{\mathrm{rig}} : S_{\mathrm{rig}}(\mathfrak{X}) \to S_{\mathrm{vec}}(\mathfrak{X})$$

by associating to the object $M_{\mathrm{rig}} := (M_{\mathrm{rig}}, \nabla_{\mathrm{rig}}, \Phi_M)$ *the object*

$$\mathbf{F}_{\mathrm{rig}}(M_{\mathrm{rig}}) := (p_{\mathrm{rig}}^* M_{\mathrm{rig}}, p_{\mathrm{rig}}^* \nabla_{\mathrm{rig}})$$

in $S_{\mathrm{vec}}(\mathfrak{X})$*. This functor is exact by definition.*

Definition 3.10 *We define the category of syntomic coefficients to be the category* $S(\mathfrak{X})$ *such that:*

(i) The objects of $S(\mathfrak{X})$ *consists of the triple* $\mathcal{M} := (M_{\mathrm{dR}}, M_{\mathrm{rig}}, \mathbf{p})$*, where:*

(a) M_{typ} *is an object in* $S_{\mathrm{typ}}(\mathfrak{X})$ *for* $\mathrm{typ} \in \{\mathrm{dR}, \mathrm{rig}\}$*.*

(b) $\mathbf{p}$ *is an isomorphism*

$$\mathbf{p} : \mathbf{F}_{\mathrm{dR}}(M_{\mathrm{dR}}) \xrightarrow{\cong} \mathbf{F}_{\mathrm{rig}}(M_{\mathrm{rig}})$$

in $S_{\mathrm{vec}}(\mathfrak{X})$*.*

(ii) A morphism $f : \mathcal{M} \to \mathcal{N}$ *in* $S(\mathfrak{X})$ *is given by a pair* $(f_{\mathrm{dR}}, f_{\mathrm{rig}})$*, where* $f_{\mathrm{typ}} : M_{\mathrm{typ}} \to N_{\mathrm{typ}}$ *are morphisms in* $S_{\mathrm{typ}}(\mathfrak{X})$ *for* $\mathrm{typ} \in \{\mathrm{dR}, \mathrm{rig}\}$ *compatible with the comparison isomorphism* $\mathbf{p}$*.*

Example 3.11 *For each integer* $n \in \mathbb{Z}$*, we define the Tate object* $K(n)$ *in* $S(\mathfrak{X})$ *to be the set* $K(n) := (K(n)_{\mathrm{dR}}, K(n)_{\mathrm{rig}}, \mathbf{p})$*, where:*

(i) $K(n)_{\mathrm{dR}}$ *in* $S_{\mathrm{dR}}(\mathfrak{X})$ *is given by the rank one free* $\mathcal{O}_{\overline{X}_K}$*-module generated by* $e_{n,\mathrm{dR}}$*, with connection* $\nabla_{\mathrm{dR}}(e_{n,\mathrm{dR}}) = 0$ *and Hodge filtration*

$$\begin{cases} F^m K(n)_{\mathrm{dR}} = K(n)_{\mathrm{dR}} & m \le -n \\ F^m K(n)_{\mathrm{dR}} = 0 & m > -n. \end{cases}$$

(ii) $K(n)_{\mathrm{rig}}$ *in* $S_{\mathrm{rig}}(\mathfrak{X})$ *is given by the rank one free* $j^\dagger\mathcal{O}_{]\overline{X}_k[_{\mathcal{P}}}$*-module generated by* $e_{n,\mathrm{rig}}$*, with connection* $\nabla_{\mathrm{rig}}(e_{n,\mathrm{rig}}) = 0$ *and Frobenius*

$$\Phi(e_{n,\mathrm{rig}}) := p^{-n} e_{n,\mathrm{rig}}.$$

(iii) $\mathbf{p}$ *is the isomorphism given by* $\mathbf{p}(e_{n,\mathrm{dR}}) = e_{n,rig}$.

EXAMPLE 3.12 (SEE [BAN1] DEFINITION 5.1) *We define the logarithmic sheaf*

$$\mathcal{L}og^{(n)} := (L^{(n)}_{\mathrm{dR}}, L^{(n)}_{\mathrm{rig}}, \mathbf{p})$$

in $S(\mathbb{G}_m)$ *by:*

(i) $L^{(n)}_{\mathrm{dR}}$ *in* $S_{\mathrm{dR}}(\mathbb{G}_m)$ *is given by the rank n free* $\mathcal{O}_{\mathbb{P}^1_K}$*-module*

$$L^{(n)}_{\mathrm{dR}} = \prod_{j=0}^{n} \mathcal{O}_{\mathbb{P}^1_K} e_{j,\mathrm{dR}},$$

with connection $\nabla_{\mathrm{dR}}(e_{j,\mathrm{dR}}) = e_{j+1,\mathrm{dR}} \otimes d\log t$ *for* $0 \leq j \leq n-1$ *and* $\nabla(e_{n,\mathrm{dR}}) = 0$*, and Hodge filtration given by*

$$F^{-m} L^{(n)}_{\mathrm{dR}} = \prod_{j=0}^{m} \mathcal{O}_{\mathbb{P}^1_K} e_{j,\mathrm{dR}}.$$

(ii) $L^{(n)}_{\mathrm{rig}}$ *in* $S_{\mathrm{rig}}(\mathbb{G}_m)$ *is given by the rank n free* $j^\dagger\mathcal{O}_{]\mathbb{P}^1_k[_{\widehat{\mathbb{P}}^1}}$*-module*

$$L^{(n)}_{\mathrm{rig}} = \prod_{j=0}^{n} j^\dagger\mathcal{O}_{]\mathbb{P}^1_k[_{\widehat{\mathbb{P}}^1}} e_{j,\mathrm{rig}},$$

with connection $\nabla_{\mathrm{rig}}(e_{j,\mathrm{rig}}) = e_{j+1,\mathrm{rig}} \otimes d\log t$ *for* $0 \leq j \leq n-1$ *and* $\nabla(e_{n,\mathrm{rig}}) = 0$*, and Frobenius*

$$\Phi(e_{j,\mathrm{rig}}) := p^{-j} e_{j,\mathrm{rig}}.$$

(iii) $\mathbf{p}$ *is the isomorphism given by* $\mathbf{p}(e_{j,\mathrm{dR}}) = e_{j,\mathrm{rig}}$.

4 MORPHISMS OF SYNTOMIC DATA

DEFINITION 4.1 *Define a morphism between syntomic data* $u : \mathfrak{X} \to \mathfrak{Y}$ *to be a pair* $(u_{\mathrm{dR}}, u_{\mathrm{rig}})$ *such that:*

(i) $u_{\mathrm{dR}} : \overline{X} \to \overline{Y}$ *is a morphism of schemes over* $\mathcal{O}_K$.

(ii) $u_{\mathrm{rig}} : \mathcal{P}_X \to \mathcal{P}_Y$ *is a morphism of formal schemes over* W *compatible with the Frobenius, such that the diagram*

$$\begin{array}{ccc} \overline{X} \otimes k & \xrightarrow{\iota} & \mathcal{P}_X \otimes k \\ {\scriptstyle u_{\mathrm{dR}}}\downarrow & & {\scriptstyle u_{\mathrm{rig}}}\downarrow \\ \overline{Y} \otimes k & \xrightarrow{\iota} & \mathcal{P}_Y \otimes k \end{array} \tag{4}$$

is commutative.

Remark 4.2 *Notice that in* (4)*, contrary to [Ban2] Definition 4.2 (iii), we do not impose the commutativity of the diagram*

$$\begin{array}{ccc} \overline{\mathcal{X}} & \xrightarrow{\iota} & \mathcal{P}_X \\ {\scriptstyle u_{\mathrm{dR}}}\downarrow & & {\scriptstyle u_{\mathrm{rig}}}\downarrow \\ \overline{\mathcal{Y}} & \xrightarrow{\iota} & \mathcal{P}_Y. \end{array} \tag{5}$$

Example 4.3 *Let* z *be an element in* $\mathcal{O}_K^\times$*, and let* $\mathbb{G}_m$ *be the syntomic datum defined in Example 3.2.1. We denote by* z_0 *the Teichmüller representative of* z*. In other words,* z_0 *is a root of unity in* W *such that* $z \equiv z_0 \pmod{\pi}$*. Then*

$$i_z = (i_{\mathrm{dR}}, i_{\mathrm{rig}}) : \mathcal{O}_K \to \mathbb{G}_m$$

is a morphism of syntomic data, where $i_{\mathrm{dR}} : \operatorname{Spec} \mathcal{O}_K \to \mathbb{G}_{m\mathcal{O}_K}$ *and* $i_{\mathrm{rig}} : \operatorname{Spf} \mathcal{O}_K \to \widehat{\mathbb{P}}^1_W$ *are morphisms defined respectively by* $t \mapsto z$ *and* $t \mapsto z_0$*.*

Let $u = (u_{\mathrm{dR}}, u_{\mathrm{rig}}) : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of syntomic data. By [Ber1] (2.2.16), we have a functor $u^*_{\mathrm{rig}} : S_{\mathrm{rig}}(\mathfrak{Y}) \to S_{\mathrm{rig}}(\mathfrak{X})$.

Lemma 4.4 *Let* $u : \mathfrak{X} \to \mathfrak{Y}$ *be a morphism of syntomic data, and let* $\mathcal{M} := (M_{\mathrm{dR}}, M_{\mathrm{rig}}, \mathbf{p})$ *be an object in* $S(\mathfrak{Y})$*. Then there exists a canonical and functorial isomorphism*

$$u^*(\mathbf{p}) : \mathbf{F}_{\mathrm{dR}}(u^*_{\mathrm{dR}} M_{\mathrm{dR}}) \to \mathbf{F}_{\mathrm{rig}}(u^*_{\mathrm{rig}} M_{\mathrm{rig}})$$

in $S_{\mathrm{vec}}(\mathfrak{X})$*.*

The above lemma is trivial if we assume the commutativity of (5).
Proof. Let $u_{\mathrm{vec}} : \overline{\mathcal{X}} \to \overline{\mathcal{Y}}$ be the morphism of formal schemes induced from u_{dR}, and denote again by u_{vec} the map induced on the associated rigid analytic space. Then we have

$$\mathbf{F}_{\mathrm{dR}}(u^*_{\mathrm{dR}} M_{\mathrm{dR}}) = u^*_{\mathrm{vec}} \mathbf{F}_{\mathrm{dR}}(M_{\mathrm{dR}}).$$

Let $u_1 := \iota \circ u_{\mathrm{vec}}$ and $u_2 := (u_{\mathrm{rig}} \otimes 1) \circ \iota$ be maps of formal schemes

$$u_1, u_2 : \overline{\mathcal{X}} \to \mathcal{P}_Y \otimes \mathcal{O}_K.$$

Then $u_{\mathrm{vec}}^* \mathbf{F}_{\mathrm{rig}}(M_{\mathrm{rig}}) = u_{1K}^*(M_{\mathrm{rig}} \otimes K)$ and $\mathbf{F}_{\mathrm{rig}}(u_{\mathrm{rig}}^* M_{\mathrm{rig}}) = u_{2K}^*(M_{\mathrm{rig}} \otimes K)$. Since (4) is commutative, u_1 and u_2 coincide on $\overline{X}_k$. Hence by [Ber1] Proposition (2.2.17), we have a canonical isomorphism

$$\epsilon_{1,2} : u_{1K}^*(M_{\mathrm{rig}} \otimes K) \xrightarrow{\cong} u_{2K}^*(M_{\mathrm{rig}} \otimes K). \tag{6}$$

The isomorphism of the lemma is the composition of the isomorphism

$$\mathbf{F}_{\mathrm{dR}}(u_{\mathrm{dR}}^* M_{\mathrm{dR}}) = u_{\mathrm{vec}}^* \mathbf{F}_{\mathrm{dR}}(M_{\mathrm{dR}}) \xrightarrow{\mathbf{p} \cong} u_{\mathrm{vec}}^* \mathbf{F}_{\mathrm{rig}}(M_{\mathrm{rig}}).$$

with $\epsilon_{1,2}$.

DEFINITION 4.5 *Let $u : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of syntomic data. Then*

$$u^* : S(\mathfrak{Y}) \to S(\mathfrak{X})$$

is the functor defined by associating to any object $\mathcal{M} := (M_{\mathrm{dR}}, M_{\mathrm{rig}}, \mathbf{p})$ the object

$$u^*\mathcal{M} = (u_{\mathrm{dR}}^* M_{\mathrm{dR}}, u_{\mathrm{rig}}^* M_{\mathrm{rig}}, u^*(\mathbf{p}))$$

in $S(\mathfrak{X})$.

5 THE SPLITTING PRINCIPLE

Let $\mathcal{L}og^{(n)}$ be the logarithmic sheaf defined in Example 3.12. In this section, we will extend the splitting principle of [Ban1] Proposition 5.2 to the points defined in Example 4.3.

PROPOSITION 5.1 (SPLITTING PRINCIPLE) *Let d be a positive integer, and let $z = \zeta_d$ be a primitive d-th root of unity in K. Let*

$$i_z = (i_{\mathrm{dR}}, i_{\mathrm{rig}}) : \mathcal{O}_K \to \mathbb{G}_m$$

be the morphism of syntomic data of Example 4.3 corresponding to z. Then we have an isomorphism

$$i_z^* \mathcal{L}og^{(n)} \cong \prod_{j=0}^{n} K(j)$$

in $S(\mathcal{O}_K)$.

The proof of the proposition will be given at the end of this section. In order to prove the proposition, it is necessary to explicitly calculate the map $i_z^*(\mathbf{p})$ of Lemma 4.4. For this purpose, we first review the Monsky-Washnitzer interpretation of overconvergent isocrystals and the explicit description of $\epsilon_{1,2}$ of (6) (See [Ber1] §2 and [T] §2 for details).

We assume for now that z is an arbitrary element in $\mathcal{O}_K^\times$. We denote by z_0 the root of unity in W such that $z \equiv z_0 \pmod{\pi}$. Let $A = \Gamma(\mathbb{G}_{m\mathcal{O}_K}, \mathcal{O}_{\mathbb{G}_{m\mathcal{O}_K}}) = \mathcal{O}_K[t, t^{-1}]$. We fix a presentation

$$\mathcal{O}_K[x_1, \cdots, x_n]/I \cong A$$

over $\mathcal{O}_K$, which defines a closed immersion

$$\mathbb{G}_{m\mathcal{O}_K} \hookrightarrow \mathbb{A}^n_{\mathcal{O}_K}.$$

Then the intersections U_λ of $\mathbb{G}^{\mathrm{an}}_{mK}$ with the ball $B(0, \lambda^+) \subset \mathbb{A}^{n\,\mathrm{an}}_K$ for $\lambda \to 1^+$ form a system of strict neighborhoods (Definition 3.4) of $\widehat{\mathbb{G}}_{mK}$ in $\mathbb{G}^{\mathrm{an}}_{mK}$. For $\lambda > 1$, we let $A_\lambda = \Gamma(U_\lambda, \mathcal{O}_{U_\lambda})$. Then $\lim_{\lambda \to 1^+} A_\lambda = A^\dagger \otimes K$, where $A^\dagger$ is the weak completion of A.

Let $M_{\mathrm{vec}} = (M_{\mathrm{vec}}, \nabla_{\mathrm{vec}})$ be an object in $S_{\mathrm{vec}}(\mathbb{G}_m)$. By [Ber1] Proposition 2.2.3, M_{vec} is of the form $j^\dagger(M_0, \nabla_0)$, where M_0 is a coherent module with integrable connection ∇_0 on a strict neighborhood U_λ. Let $M_\lambda = \Gamma(U_\lambda, M_0)$. Then for $\lambda' < \lambda$, the section $\Gamma(U_{\lambda'}, M_0)$ is given by $M_{\lambda'} = M_\lambda \otimes_{A_\lambda} A_{\lambda'}$, and

$$M := \Gamma(\mathbb{G}^{\mathrm{an}}_{mK}, M_{\mathrm{vec}}) = \varinjlim_{\lambda \to 1^+} M_\lambda. \tag{7}$$

M is a projective $A^\dagger \otimes K$-module with integrable connection $\nabla : M \to M \otimes \Omega^1_{A^\dagger \otimes K}$ induced from ∇_0.

Suppose the connection ∇_{vec} is *overconvergent*. By [Ber1] Proposition 2.2.13, for any $\eta < 1$, there exists $\lambda > 1$ such that

$$\left\| \frac{1}{i!} \nabla_\lambda(\partial_t^i)(m) \right\| \eta^i \to 0 \quad (i \to \infty) \tag{8}$$

for any $m \in M_\lambda$. Here, $\nabla_\lambda : M_\lambda \to M_\lambda \otimes \Omega^1_{A_\lambda/K}$ is the connection induced from ∇_0, ∂_t is the derivation by t, and $\| - \|$ is a Banach norm on M_λ.

Let $\mathcal{M} = (M_{\mathrm{dR}}, M_{\mathrm{rig}}, \mathbf{p})$ be an object in $S(\mathbb{G}_m)$. Then

$$M_{\mathrm{vec}} := \mathbf{F}_{\mathrm{rig}}(M_{\mathrm{rig}}) = (M_{\mathrm{rig}} \otimes_{K_0} K, \nabla_{\mathrm{rig}} \otimes_{K_0} K)$$

is an object in $S_{\mathrm{vec}}(\mathbb{G}_m)$. We have

$$i^*_{\mathrm{vec}} \mathbf{F}_{\mathrm{rig}}(M_{\mathrm{rig}}) = M \otimes_{i_{\mathrm{vec}}} K, \qquad \mathbf{F}_{\mathrm{rig}}(i^*_{\mathrm{rig}} M_{\mathrm{rig}}) = M \otimes_{i_{\mathrm{rig}}} K,$$

where M is as in (7), and $i_{\mathrm{vec}}, i_{\mathrm{rig}} : A^\dagger \otimes_{\mathcal{O}_K} K \to K$ are ring homomorphisms given respectively by $t \mapsto z$ and $t \mapsto z_0$. By [Ber1] 2.2.17 Remarque,

$$\epsilon_{1,2} : M \otimes_{i_{\mathrm{vec}}} K \xrightarrow{\cong} M \otimes_{i_{\mathrm{rig}}} K$$

of (6) is given explicitly by the Taylor series

$$\epsilon_{1,2}(m \otimes_{i_{\mathrm{vec}}} 1) = \sum_{i \geq 0} \frac{1}{i!} \nabla(\partial_t^i)(m) \otimes_{i_{\mathrm{rig}}} (z - z_0)^i. \tag{9}$$

The existence of the Frobenius Φ_M on $M_{\rm rig}$ insures that the connection $\nabla_{\rm rig}$ (hence $\nabla_{\rm vec}$) is overconvergent ([Ber1] Theorem 2.5.7). Since $|z - z_0| < 1$, the above series converges by (8).
Next, let

$$\mathcal{L}og^{(n)} := (L^{(n)}_{\rm dR}, L^{(n)}_{\rm rig}, \mathbf{p})$$

be the logarithmic sheaf of Example 3.12. As in (7), we $L = \Gamma(\mathbb{G}^{\rm an}_{mK}, L^{(n)}_{\rm vec})$ for $L^{(n)}_{\rm vec} = L^{(n)}_{\rm rig} \otimes_{K_0} K$. Then

$$L = \prod_{j=0}^{n} (A^\dagger \otimes K) e_j$$

for the basis $e_j = e_{j,{\rm rig}} \otimes 1$, and the connection is given by

$$\nabla(e_j) = e_{j+1} \otimes \frac{dt}{t} \quad (0 \leq j \leq n-1). \tag{10}$$

Let $u_j(t)$ be the function defined in Definition 2.2.

Proposition 5.2 *For integers $i, m \geq 0$, let $a^{(i)}_m$ be elements in $A^\dagger_K$ such that*

$$\nabla(\partial^i_t)(e_0) = \sum_{j=0}^{n} a^{(i)}_j e_j.$$

Then

$$\partial^i_t(u_m) = \sum_{j=0}^{n} a^{(n)}_j u_{m-j}.$$

In particular, we have

$$a^{(i)}_m(z_0) = \partial^i_t(u_m)(z_0). \tag{11}$$

Remark 5.3 *The definition of $a^{(i)}_j$ implies*

$$\nabla(\partial^i_t)(e_m) = \sum_{j=0}^{n-m} a^{(i)}_j e_{m+j}.$$

Proof. We will give the proof by induction on $i \geq 0$. Since $a^{(0)}_0 = 1$, the statement is true for $i = 0$. Suppose for an integer $i \geq 0$, we have

$$\partial^i_t(u_m) = \sum_{j=0}^{n} a^{(i)}_j u_{m-j}. \tag{12}$$

By comparing the definition of $a^{(i+1)}_j$ with the equality

$$\nabla(\partial^{i+1}_t)(e_0) = \nabla(\partial_t) \circ \nabla(\partial^i_t)(e_0) = \sum_{j=0}^{n} \left((\partial_t a^{(i)}_j) e_j + t^{-1} a^{(i)}_j e_{j+1} \right),$$

we obtain the equality

$$a_j^{(i+1)} = \partial_t a_j^{(i)} + t^{-1} a_{j-1}^{(i)}. \tag{13}$$

Similarly, from the hypothesis (12) and $\partial_t u_m = t^{-1} u_{m-1}$, we have

$$\partial_t^{i+1}(u_m) = \partial_t \circ \partial_t^i(u_m) = \sum_{j=0}^{n} \left((\partial_t a_j^{(i)}) u_{m-j} + t^{-1} a_j^{(i)} u_{m-j-1} \right).$$

This together with (13) gives the desired result. (11) follows from the fact that since z_0 is a root of unity, $u_m(z_0) = 0$ unless $m = 0$.

Corollary 5.4 *For any integers $i, m \geq 0$, we have*

$$\nabla(\partial_t^i)(e_m) \otimes_{i_{\mathrm{rig}}} 1 = \sum_{j=0}^{n-m} \left(e_{m+j} \otimes_{i_{\mathrm{rig}}} \partial_t^i(u_j)(z_0) \right).$$

Proof. The assertion follows immediately from Remark 5.3

Proposition 5.5 *We have*

$$\epsilon_{1,2}(e_m \otimes_{i_{\mathrm{vec}}} 1) = \sum_{j=0}^{n-m} \left(e_{m+j} \otimes_{i_{\mathrm{rig}}} u_j(z) \right)$$

for the map $\epsilon_{1,2} : L \otimes_{i_{\mathrm{vec}}} K \to L \otimes_{i_{\mathrm{rig}}} K$ of (9) *associated to L.*

Proof. Since $\log(z_0) = 0$, we have $\partial_t^i(u_j)(z_0) = 0$ for $i < j$. Substituting z to the Taylor expansion of $u_j(t)$ at $t = z_0$ gives the equality

$$u_j(z) = \sum_{i=j}^{\infty} \frac{1}{i!} \partial_t^i(u_j)(z_0)(z - z_0)^i.$$

The proposition now follows from the definition of $\epsilon_{1,2}$ (9) and Corollary 5.4.
Let us now return to the case when $z = \zeta_d$ is a primitive d-th root of unity.
Proof of Proposition 5.1. Since the connection is the only structure preventing $L_{\mathrm{dR}}^{(n)}$ and $L_{\mathrm{rig}}^{(n)}$ from splitting, we have

$$i_{\mathrm{dR}}^* L_{\mathrm{dR}}^{(n)} = \prod_{j=0}^{n} K e_{j,\mathrm{dR}} \qquad i_{\mathrm{rig}}^* L_{\mathrm{rig}}^{(n)} = \prod_{j=0}^{n} K_0 e_{j,\mathrm{rig}}.$$

It is sufficient to prove that the comparison isomorphism $i_z^*(\mathbf{p})$ respects the splitting. The isomorphism

$$\mathbf{p} : i_{\mathrm{dR}}^* L_{\mathrm{dR}}^{(n)} \to L \otimes_{i_{\mathrm{vec}}} K$$

is given by $e_{j,\mathrm{dR}} \mapsto e_{j,\mathrm{rig}}$. Since z is a torsion point, $u_j(z) = 0$ for $j \neq 0$. Hence by Proposition 5.5,

$$\epsilon_{1,2} : L \otimes_{i_{\mathrm{vec}}} K \to L \otimes_{i_{\mathrm{rig}}} K$$

maps $e_{j,\mathrm{rig}} \otimes_{i_{\mathrm{vec}}} 1$ to $e_{j,\mathrm{rig}} \otimes_{i_{\mathrm{rig}}} 1$. Hence $i_z^*(\mathbf{p}) = \epsilon_{1,2} \circ \mathbf{p}$ respects the splitting. We have

$$i_z^* \mathcal{L}og^{(n)} \cong \prod_{j=0}^{n} K(j)$$

in $S(\mathcal{O}_K)$ as desired.

REMARK 5.6 *The calculation of Proposition 5.5 shows that if z is an arbitrary element in $\mathcal{O}_K^\times$, then*

$$i_z^* \mathcal{L}og^{(n)} = (L^{(n)}_{z,\mathrm{dR}}, L^{(n)}_{z,\mathrm{rig}}, \mathbf{p}_z) \in S(\mathcal{O}_K),$$

where

$$L^{(n)}_{z,\mathrm{dR}} = \prod_{j=0}^{n} K e_{j,\mathrm{dR}}, \qquad L^{(n)}_{z,\mathrm{rig}} = \prod_{j=0}^{n} K_0 e_{j,\mathrm{rig}},$$

and

$$\mathbf{p}_z(e_{m,\mathrm{dR}}) = \sum_{j=0}^{n-m} e_{m+j,\mathrm{rig}} \otimes_{K_0} u_j(z).$$

6 THE SPECIALIZATION OF pol TO TORSION POINTS

In this section, we will first introduce the p-adic polylogarithmic extension pol calculated in [Ban1]. Then we will calculate its restriction to d-th roots of unity, where d is an integer of the form $d = Np^r$ with $(N, p) = 1$ and $N > 1$. The case $N = 1$ will be treated in Section 8.

Let $\mathbb{U}$ be the syntomic datum correspoinding to the projective line minus three points, as defined in Definition 3.2. The p-adic polylogarithm sheaf is an extension in $S(\mathbb{U})$ of the trivial object $K(0)$ by the logarithmic sheaf $\mathcal{L}og$ having a certain residue. In our previous paper, we determined the explicit shape of this sheaf.

THEOREM 6.1 ([BAN1] THEOREM 2) *The p-adic polylogarithmic extension $\mathrm{pol}^{(n)}$ is the extension*

$$0 \to \mathcal{L}og^{(n)} \to \mathrm{pol}^{(n)} \to K(0) \to 0$$

in $S(\mathbb{U})$, given explicitly by $\mathrm{pol}^{(n)} := (P^{(n)}_{\mathrm{dR}}, P^{(n)}_{\mathrm{rig}}, \mathbf{p})$, where:

(i) $P^{(n)}_{\mathrm{dR}}$ in $S_{\mathrm{dR}}(\mathbb{U})$ is given by

$$P^{(n)}_{\mathrm{dR}} = \mathcal{O}_{\mathbb{P}^1_K} e_{\mathrm{dR}} \bigoplus L^{(n)}_{\mathrm{dR}},$$

with connection $\nabla_{\mathrm{dR}}(e_{\mathrm{dR}}) = e_{1,\mathrm{dR}} \otimes d\log(t-1)$ and Hodge filtration given by the direct sum.

(ii) $P^{(n)}_{\text{rig}}$ *in* $S_{\text{rig}}(\mathbb{U})$ *is given by*

$$P^{(n)}_{\text{rig}} = j^\dagger \mathcal{O}_{]\mathbb{U}_k[_{\widehat{\mathbb{P}}^1}} e_{\text{rig}} \bigoplus L^{(n)}_{\text{rig}},$$

with connection $\nabla_{\text{rig}}(e_{\text{rig}}) = e_{1,\text{rig}} \otimes d\log(t-1)$ *and Frobenius*

$$\Phi(e_{\text{rig}}) := e_{\text{rig}} + \sum_{j=1}^{n} (-1)^{j+1} \ell^{(p)}_j(t) e_{j,\text{rig}}. \tag{14}$$

(iii) $\mathbf{p}$ *is the isomorphism given by* $\mathbf{p}(e_{\text{dR}}) = e_{\text{rig}} \otimes 1$.

REMARK 6.2 *In [Ban1] Theorem 2, the Frobenius is written as*

$$\Phi(e_{\text{rig}}) := e_{\text{rig}} + \sum_{j=1}^{n} (-1)^{j} \ell^{(p)}_j(t) e_{j,\text{rig}}.$$

This is due to an error in the calculation of the proof. The correct Frobenius is the one given in (14).

Let z be a d-th root of unity, where d is an integer of the form $d = Np^r$ with $(N,p) = 1$ and $N > 1$, and let $z_0 \in W$ such that $z \equiv z_0 \pmod{\pi}$. The purpose of this section is to prove the following theorem.

THEOREM 6.3 *The specialization of the polylogarithm at z is explicitly given as follows:*

(i) $i_z^* P^{(n)}_{\text{dR}} = K e_{\text{dR}} \oplus \bigoplus_{j=0}^n K e_{j,\text{dR}}$ *with the natural Hodge filtration.*

(ii) $i_z^* P^{(n)}_{\text{rig}} = K_0 e_{\text{rig}} \oplus \bigoplus_{j=0}^n K_0 e_{j,\text{rig}}$ *with Frobenius*

$$\Phi(e_{\text{rig}}) := e_{\text{rig}} + \sum_{j=1}^{n} (-1)^{j+1} \ell^{(p)}_j(z_0) e_{j,\text{rig}}.$$

(iii) $\mathbf{p}$ *is the isomorphism given by*

$$\mathbf{p}(e_{\text{dR}}) = e_{\text{rig}} \otimes 1 + \sum_{j=1}^{n} e_{j,\text{rig}} \otimes (-1)^j (D_j(z) - D_j(z_0)),$$

where $D_j(t)$ is the function defined in Definition 2.2.

The proof of the theorem will be given at the end of this section. As in the case of $\mathcal{L}og$, we first consider the Monsky-Washnitzer interpretation of $\text{pol}^{(n)}$. Let $B_K^\dagger = \Gamma(\mathbb{U}_K^{\text{an}}, j^\dagger \mathcal{O}_{\mathbb{U}_K^{\text{an}}})$,

$$P^{(n)}_{\text{vec}} := \mathbf{F}_{\text{rig}}(M_{\text{rig}}) = (M_{\text{rig}} \otimes_{K_0} K, \nabla_{\text{rig}} \otimes_{K_0} K),$$

and $P^{(n)} = \Gamma(\mathbb{U}_K^{\rm an}, P_{\rm vec}^{(n)})$. Then we have

$$P^{(n)} = B_K^\dagger e \bigoplus \prod_{j=0}^n B_K^\dagger e_j$$

where $e = e_{\rm rig} \otimes 1$ and $e_j = e_{j,\rm rig} \otimes 1$, with connection $\nabla(e) = e \otimes d\log(1-t)$ and $\nabla(e_j) = e_{j+1} \otimes d\log t$.

Proposition 6.4 *For integers $i, m > 0$, let $b_m^{(i)}$ be elements in $B_K^\dagger$ such that*

$$\nabla(\partial_t^i)(e) = \sum_{j=1}^n (-1)^j b_j^{(i)} e_j.$$

Then

$$\partial_t^i(D_m) = \sum_{j=1}^n (-1)^{m-j} b_j^{(i)} u_{m-j}.$$

In particular, we have

$$b_m^{(i)}(z_0) = \partial_t^i(D_m)(z_0). \tag{15}$$

Proof. The proof is again by induction on $i > 0$. We first consider the case when $i = 1$. In this case, $b_1^{(1)} = (1-t)^{-1}$. Since $\mathrm{Li}_{m-j}(t)$ and $u_j(t)$ satisfy the differential equations

$$\partial_t(\mathrm{Li}_j(t)) = \frac{1}{t}\mathrm{Li}_{j-1}(t) \quad (j \geq 1) \qquad \partial_t(u_j(t)) = \frac{u_{j-1}}{t} \quad (\forall j),$$

the definition of $D_m(t)$ (Definition 2.2) and the fact that $u_j(t) = 0$ for $j < 0$ implies that:

$$\begin{aligned}
\partial_t(D_m) &= \sum_{j=0}^{m-1} (-1)^j \partial_t(\mathrm{Li}_{m-j}(t) u_j(t)) \\
&= \sum_{j=0}^{m-1} \frac{(-1)^j}{t} \left(\mathrm{Li}_{m-j-1}(t) u_j(t) + \mathrm{Li}_{m-j}(t) u_{j-1}(t)\right) \\
&= \frac{(-1)^{m-1}}{t} \mathrm{Li}_0(t) u_{m-1}(t) = (-1)^{m-1} \frac{u_{m-1}(t)}{1-t} \\
&= (-1)^{m-1} b_1^{(1)}(t) u_{m-1}(t).
\end{aligned}$$

Hence the statement is true for $i = 1$. Suppose for an integer $i \geq 1$, we have

$$\partial_t^i(D_m) = \sum_{j=1}^n (-1)^{m-j} b_j^{(i)} u_{m-j}. \tag{16}$$

By comparing the definition of $b_j^{(i+1)}$ with the equality

$$\nabla(\partial_t^{i+1})(e_0) = \nabla(\partial_t) \circ \nabla(\partial_t^i)(e_0) = \sum_{j=1}^{n} (-1)^j \left((\partial_t b_j^{(i)}) e_j + t^{-1} b_j^{(i)} e_{j+1} \right),$$

we obtain the equality

$$b_j^{(i+1)} = \partial_t b_j^{(i)} - t^{-1} b_{j-1}^{(i)} \qquad (i \geq 1, j > 1). \tag{17}$$

Similarly, from the hypothesis (16) and $\partial_t u_m = t^{-1} u_{m-1}$, we have

$$\begin{aligned} \partial_t^{i+1}(D_m) &= \partial_t \left(\sum_{j=1}^{i} (-1)^{m-j} b_j^{(i)} u_{m-j} \right) \\ &= \sum_{j=1}^{n} (-1)^{m-j} \left((\partial_t b_j^{(i)}) u_{m-j} + t^{-1} b_j^{(i)} u_{m-j-1} \right). \end{aligned}$$

This together with (17) gives the desired result. (15) follows from the fact that since z_0 is a root of unity, $u_m(z_0) = 0$ unless $m = 0$.

Proposition 6.5 *We have*

$$\epsilon_{1,2}(e \otimes_{i_{\mathrm{vec}}} 1) = e \otimes_{i_{\mathrm{rig}}} 1 + \sum_{j=1}^{n} \left(e_j \otimes_{i_{\mathrm{rig}}} (-1)^j (D_j(z) - D_j(z_0)) \right)$$

for the map $\epsilon_{1,2} : P \otimes_{i_{\mathrm{vec}}} K \to P \otimes_{i_{\mathrm{rig}}} K$ *of* (9) *associated to* P.

Proof. Substituting z to the Taylor expansion of $D_j(t)$ at $t = z_0$ gives the equality

$$D_j(z) = \sum_{i=0}^{\infty} \frac{1}{i!} \partial_t^i (D_j)(z_0)(z - z_0)^i.$$

The proposition now follows from the definition of $\epsilon_{1,2}$ and Proposition 6.4.

7 The main result (Case $N > 1$)

The following lemma is well-known.

Lemma 7.1 *There is a canonical isomorphism*

$$\mathrm{Ext}^1_{S(\mathcal{O}_K)}(K(0), K(j)) = K(j)_{\mathrm{dR}} \tag{18}$$

for $j > 0$.

Proof. Suppose $\widetilde{M} = (\widetilde{M}_{\rm dR}, \widetilde{M}_{\rm rig}, \widetilde{\mathbf{p}})$ is an extension of $K(0)$ by $K(j)$ in $S(\mathcal{O}_K)$. We have exact sequences

$$0 \to K(j)_{\rm dR} \to \widetilde{M}_{\rm dR} \to K(0)_{\rm dR} \to 0$$
$$0 \to K(j)_{\rm rig} \to \widetilde{M}_{\rm rig} \to K(0)_{\rm rig} \to 0.$$

Denote by $e_{j,\rm dR}$ and $e_{j,\rm rig}$ the basis of $K(j)_{\rm dR}$ and $K(j)_{\rm rig}$, and let $\widetilde{e}_{0,\rm dR}$ and $\widetilde{e}_{0,\rm rig}$ respectively be the liftings of $e_{0,\rm dR}$ and $e_{0,\rm rig}$ in $\widetilde{M}_{\rm dR}$ and $\widetilde{M}_{\rm rig}$. If we map $\widetilde{e}_{0,\rm dR}$ to $e_{0,\rm dR}$, then we have an isomorphism

$$\widetilde{M}_{\rm dR} \cong K(0)_{\rm dR} \bigoplus K(j)_{\rm dR}$$

in $S_{\rm dR}(\mathcal{O}_K)$. Next, since the quotient of M by $K(j)$ is isomorphic to $K(0)$, the Frobenius and $\widetilde{\mathbf{p}}$ is given by

$$\widetilde{\mathbf{p}}(\widetilde{e}_{0,\rm dR}) = \widetilde{e}_{0,\rm rig} \otimes 1 + e_{j,\rm rig} \otimes a$$
$$\phi^*(\widetilde{e}_{0,\rm rig}) = \widetilde{e}_{0,\rm rig} + c e_{j,\rm rig}$$

for some $a \in K$ and $c \in K_0$. If we take $b \in K_0$ such that $(1 - \sigma/p^j)b = c$, then we have an isomorphism

$$\widetilde{M}_{\rm rig} \cong K(0)_{\rm rig} \bigoplus K(j)_{\rm rig}$$

in $S_{\rm rig}(\mathcal{O}_K)$ given by $\widetilde{e}_{0,\rm rig} \mapsto e_{0,\rm rig} - b e_{j,\rm rig}$. The above shows that we have an isomorphism

$$\widetilde{M} \cong \left(K(0)_{\rm dR} \bigoplus K(j)_{\rm dR},\, K(0)_{\rm rig} \bigoplus K(j)_{\rm rig},\, \mathbf{p}\right)$$

of extensions of $K(0)$ by $K(j)$ in $S(\mathcal{O}_K)$, where $\mathbf{p}$ is the isomorphism given by

$$\begin{aligned}\mathbf{p}(e_{0,\rm dR}) &= \widetilde{e}_{0,\rm rig} \otimes 1 + e_{j,rig} \otimes a \\ &= e_{0,\rm rig} \otimes 1 + e_{j,\rm rig} \otimes (a+b).\end{aligned}$$

The canonical map of the lemma is given by associating to $\widetilde{M}$ the element $(a+b)e_{j,\rm dR}$ in $K(j)_{\rm dR}$.
The inverse of this canonical map is constructed by associating to $we_{j,\rm dR}$ in $K(j)_{\rm dR}$ the extension

$$\left(K(0)_{\rm dR} \bigoplus K(j)_{\rm dR},\, K(0)_{\rm rig} \bigoplus K(j)_{\rm rig},\, \mathbf{p}\right),$$

where

$$\mathbf{p}(e_{0,\rm dR}) = e_{0,\rm rig} \otimes 1 + e_{j,\rm rig} \otimes w.$$

This construction shows that the canonical map is in fact an isomorphism.

Remark 7.2 *Suppose $K = K_0$. Then by [Ban1] Theorem 1 and Example 2.8, we have an isomorphism*

$$\mathrm{Ext}^1_{S(\mathcal{O}_K)}(K(0), K(j)) \xrightarrow{\cong} H^1_{\mathrm{syn}}(\mathcal{O}_K, K(j)) = K(j)_{\mathrm{rig}}. \tag{19}$$

If M is an extension in $S(\mathcal{O}_K)$ corresponding to $ae_{j,\mathrm{dR}}$ in Lemma 7.1, then M maps by (19) *to $((1-p^{-j}\sigma)a)e_{j,\mathrm{rig}}$ in $K(j)_{\mathrm{rig}}$.*

The following theorem is Theorem 1 of the introduction.

Theorem 7.3 *Let z be a torsion point of order $d = Np^r$, where $(N,p) = 1$ and $N > 1$. Then*

$$i_z^* \operatorname{pol}^{(n)} = ((-1)^j \operatorname{Li}_j(z) e_{j,\mathrm{dR}})_{j\geq 1}$$

in

$$\mathrm{Ext}^1_{S(\mathcal{O}_K)}(K(0), i_z^* \mathcal{L}og(1)) = \prod_{j=0}^{n} \mathrm{Ext}^1_{S(\mathcal{O}_K)}(K(0), K(j)),$$

where we view $(-1)^j \operatorname{Li}_j(z) e_{j,\mathrm{dR}}$ as an element in $\mathrm{Ext}^1_{S(\mathcal{O}_K)}(K(0), K(j))$ through the isomorphism of lemma 7.1

Proof. By Theorem 6.3, the image of $i_z^* \operatorname{pol}^{(n)}$ in $\mathrm{Ext}^1_{S(\mathcal{O}_K)}(K(0), K(j))$ is the extension $\widetilde{M} = (M_{\mathrm{dR}}, \widetilde{M}_{\mathrm{rig}}, \widetilde{\mathbf{p}})$ given as follows: M_{dR} is the direct sum

$$M_{\mathrm{dR}} = K(0)_{\mathrm{dR}} \bigoplus K(j)_{\mathrm{dR}},$$

$\widetilde{M}_{\mathrm{rig}}$ is the extension of $K(0)_{\mathrm{rig}}$ by $K(j)_{\mathrm{rig}}$ with the Frobenius given by

$$\Phi(\widetilde{e}_{0,\mathrm{rig}}) = \widetilde{e}_{0,\mathrm{rig}} + (-1)^{j+1} \ell_j^{(p)}(z_0) e_{j,\mathrm{rig}}$$

for the lifting $\widetilde{e}_{0,\mathrm{rig}}$ of $e_{0,\mathrm{rig}}$ in $\widetilde{M}_{\mathrm{rig}}$, and $\widetilde{\mathbf{p}}$ is the isomorphism given by

$$\widetilde{\mathbf{p}}(e_{0,\mathrm{dR}}) = \widetilde{e}_{0,\mathrm{rig}} \otimes 1 + e_{j,\mathrm{rig}} \otimes (-1)^j (\operatorname{Li}_j(z) - \operatorname{Li}_j(z_0)).$$

This implies that, in the notation of Lemma 7.1, we have

$$\begin{aligned} a &= (-1)^j (\operatorname{Li}_j(z) - \operatorname{Li}_j(z_0)) \\ c &= (-1)^{j+1} \ell_j^{(p)}(z_0). \end{aligned}$$

Since z_0 is a root of unity prime to p, the Frobenius acts by $\sigma(z_0) = z_0^p$. Hence the Formula of Propisition 2.1 (iii) gives

$$\ell_j^{(p)}(z_0) = \left(1 - \frac{\sigma}{p^j}\right) \operatorname{Li}_j(z_0).$$

Again, in the notation of Lemma 7.1, we have

$$c = (-1)^{j+1} \operatorname{Li}_j(z_0).$$

Since $a + b = (-1)^j \operatorname{Li}_j(z)$, the construction of the canonical map shows that the image of $i_z^* \operatorname{pol}^{(n)}$ in $\mathrm{Ext}^1_{S(\mathcal{O}_K)}(K(0), K(j))$ maps to $(-1)^j \operatorname{Li}_j(z) e_{j,\mathrm{dR}}$ in $K(j)_{\mathrm{dR}}$.

8 The main result (Case $N = 1$)

In this section, we will consider the specialization of the polylogarithm sheaf to p-th power roots of unity. As mentioned in the introduction, we will consider a slightly modified version of the polylogarithm. Let $c > 1$ be an integer prime to p, and let $\mathbb{U}^0_{c,\mathcal{O}_K} = \operatorname{Spec} \mathcal{O}_K[t, (1-t^c)^{-1}]$. We denote by $\mathbb{U}^0_c$ the syntomic data

$$\mathbb{U}^0_c = (\mathbb{U}^0_{c,\mathcal{O}_K}, \mathbb{P}^1_{\mathcal{O}_K}, \widehat{\mathbb{P}}^1, \phi).$$

The multiplication by $[c]$ map on $\mathbb{G}_{m\mathcal{O}_K}$ defines a morphism of syntomic datum

$$[c] : \mathbb{U}^0_c \to \mathbb{U}.$$

Definition 8.1 *We define the modified p-adic polylogarithmic* $\operatorname{pol}^{(n)}_c$ *by*

$$\operatorname{pol}^{(n)}_c = \operatorname{pol}^{(n)} - [c]^* \operatorname{pol}^{(n)} \in \operatorname{Ext}^1_{S(\mathbb{U}^0_c)}(K(0), \mathcal{L}og^{(n)}).$$

The explicit shape of $\operatorname{pol}^{(n)}$ given in Theorem 6.1 and the definition of the pull-back $[c]^*$ gives the following proposition. Let

$$\theta_c(t) = \frac{1-t^c}{1-t}.$$

Proposition 8.2 *The modified p-adic polylogarithmic* $\operatorname{pol}^{(n)}_c$ *is the extension in* $S(\mathbb{U}^0_c)$*, given explicitly by* $\operatorname{pol}^{(n)}_c := (P^{(n)}_{\mathrm{dR}}, P^{(n)}_{\mathrm{rig}}, \mathbf{p})$*, where:*

(i) $P^{(n)}_{\mathrm{dR}}$ *in* $S_{\mathrm{dR}}(\mathbb{U}^0_c)$ *is given by*

$$P^{(n)}_{\mathrm{dR}} = \mathcal{O}_{\mathbb{P}^1_K} e_{\mathrm{dR}} \bigoplus L^{(n)}_{\mathrm{dR}},$$

with connection $\nabla_{c,\mathrm{dR}}(e_{\mathrm{dR}}) = e_{1,\mathrm{dR}} \otimes d\log\theta_c(t)$ *and Hodge filtration given by the direct sum.*

(ii) $P^{(n)}_{\mathrm{rig}}$ *in* $S_{\mathrm{rig}}(\mathbb{U}^0_c)$ *is given by*

$$P^{(n)}_{\mathrm{rig}} = j^\dagger \mathcal{O}_{]\mathbb{U}^0_{c,k}[_{\widehat{\mathbb{P}}^1}} e_{\mathrm{rig}} \bigoplus L^{(n)}_{\mathrm{rig}},$$

with connection $\nabla_{c,\mathrm{rig}}(e_{\mathrm{rig}}) = e_{1,\mathrm{rig}} \otimes d\log\theta_c(t)$ *and Frobenius*

$$\Phi(e_{\mathrm{rig}}) := e_{\mathrm{rig}} + \sum_{j=1}^n (-1)^{j+1} \ell^{(p)}_{j,c}(t) e_{j,\mathrm{rig}},$$

(iii) $\mathbf{p}$ *is the isomorphism given by* $\mathbf{p}(e_{\mathrm{dR}}) = e_{\mathrm{rig}} \otimes 1$.

Let $\mathbb{U}_{c,\mathcal{O}_K} = \operatorname{Spec} \mathcal{O}_K[t, \theta_c(t)^{-1}]$, and denote by $\mathbb{U}_c$ the syntomic data

$$\mathbb{U}_c = (\mathbb{U}_{c,\mathcal{O}_K}, \mathbb{P}^1_{\mathcal{O}_K}, \widehat{\mathbb{P}}^1, \phi).$$

The explicit shape of $\mathrm{pol}_c^{(n)}$ given in the previous proposition shows that $\mathrm{pol}_c^{(n)}$ is in fact an object in $S(\mathbb{U}_c)$. In particular, we can specialize $\mathrm{pol}_c^{(n)}$ at points on the open unit disc around one.
Similar caluclations as that of Theorem 6.3 with $\ell_j^{(p)}$, $D_j^{(p)}$ and D_j replaced by $\ell_{j,c}^{(p)}$, $D_{j,c}^{(p)}$ and $D_{j,c}$ gives the following theorem, which is Theorem 2 of the introduction.

THEOREM 8.3 *Let z be a p^r-th root of unity, and let $z_0 = 1$. Then the specialization of the modified polylogarithm at z is explicitly given as follows:*

(i) $i_z^* P_{\mathrm{dR}}^{(n)} = K e_{\mathrm{dR}} \oplus \bigoplus_{j=0}^n K e_{j,\mathrm{dR}}$ *with the natural Hodge filtration.*

(ii) $i_z^* P_{\mathrm{rig}}^{(n)} = K e_{\mathrm{rig}} \oplus \bigoplus_{j=0}^n K e_{j,\mathrm{rig}}$ *with Frobenius*

$$\Phi(e_{\mathrm{rig}}) := e_{\mathrm{rig}} + \sum_{j=1}^n (-1)^{j+1} \ell_{j,c}^{(p)}(z_0) e_{j,\mathrm{rig}}.$$

(iii) $\mathbf{p}_c$ *is the isomorphism given by*

$$\mathbf{p}_c(e_{\mathrm{dR}}) = e_{\mathrm{rig}} \otimes 1 + \sum_{j=1}^n e_{j,\mathrm{rig}} \otimes (-1)^j (D_{j,c}(z) - D_{j,c}(z_0)).$$

As a corollary, we obtain the following result.

COROLLARY 8.4 *Let z be a torsion point of order p^r. Then*

$$i_z^* \mathrm{pol}_c^{(n)} = ((-1)^j \operatorname{Li}_j(z) e_{j,\mathrm{dR}})_{j \geq 1}$$

in

$$\operatorname{Ext}^1_{S(\mathcal{O}_K)}(K(0), i_z^* \mathcal{L}og(1)) = \prod_{j=0}^n \operatorname{Ext}^1_{S(\mathcal{O}_K)}(K(0), K(j)),$$

where we view $(-1)^j \operatorname{Li}_{j.c}(z) e_{j,\mathrm{dR}}$ as an element in $\operatorname{Ext}^1_{S(\mathcal{O}_K)}(K(0), K(j))$ *through the isomorphism of lemma 7.1*

REFERENCES

[BC] F. Baldassarri and B. Chiarellotto, Algebraic versus rigid cohomology with logarithmic coefficients. In: *V. Cristante, W. Messing (eds.), Barsotti Symposium in Algebraic Geometry*, Perspectives in Math. 15, Academic Press (1994).

[Ban1] K. Bannai, Rigid syntomic cohomology and p-adic polylogarithms, J. Reine Angew. Math. 529 (2000), 205-237.

[Ban2] K. Bannai, Syntomic cohomology as a p-adic absolute Hodge cohomology, Math. Z. 242 (2002) No. 3, 443-480.

[Ber1] P. Berthelot, Cohomologie rigide et cohomologie rigide à supports propres, Première partie, Prépublication IRMAR 96-03, Université de Rennes (1996).

[Ber2] P. Berthelot, Finitude et pureté cohomologique en cohomologie rigide, Invent. math. 128 (1997), 329-377.

[Bei] A. A. Beilinson, Polylogarithm and Cyclotomic Elements, typewritten preprint, MIT 1989 or 1990.

[BD] A. A. Beilinson and P. Deligne, Motivic Polylogarithm and Zagier's conjecture, preprint, 1992.

[Bes1] A. Besser, Syntomic Regulators and p-adic Integration I: Rigid Syntomic Regulators, Israel. J. Math. 120 (2000), 291-334.

[Bes2] A. Besser, Syntomic Regulators and p-adic Integration II: K_2 of curves, Israel. J. Math. 120 (2000), 335-359.

[BdJ] A. Besser and R. de Jeu, The syntomic regulator for K-theory of fields, preprint 15 December 2001, arXiv:math.AG/0110334v2.

[BGR] S. Bosch, U. Güntzer, and R. Remmert, *Non-archimedean analysis*, Grundlehren der math. Wissenschaften 261, Springer-Verlag (1984).

[Ch] B. Chiarellotto, Weights in rigid cohomology Applications to unipotent F-isocrystals, Ann. Scient. École Norm. Sup. 31 (1998), 683-715.

[Col] R. Coleman, Dilogarithms, Regulators and p-adic L-functions, Inv. math. 69 (1982), 171-208.

[C-LS] J. Y. Etesse and B. Le Stum, Fonctions L associées aux F-isocristaux surconvergents. I. Interprétation cohomologique, Math. Ann. 296 (1993), 557-576.

[D] P. Deligne, Le Groups Fondamental de la Droite Projective Moins Trois Points, In:*Galois Groups over* $\mathbb{Q}$, Proceedings of a workshop held March 23-27, 1987 at the MSRI, Berkeley, California, Springer 1989, 79-297.

[Fon1] J.-M. Fontaine, Modules Galoisiens, modules filtrés et anneaux de Barsotti-Tate, Astérisque 65 (1979), 3-80.

[Fon2] J.-M. Fontaine, Exposé III: Représentations p-adiques semi-stables, Astérisque 223 (1994), 113-184.

[GK] M. Gros, Régulateurs syntomiques et valeurs de fonctions L p-adiques I (avec un appendice par Masato Kurihara), Invent. Math. 99 (1990), 293-320.

[G] M. Gros, Régulateurs syntomiques et valeurs de fonctions L p-adiques II, Invent. Math. 115 (1994), 61-79.

[HW1] A. Huber and J. Wildeshaus, The Classical Polylogarithm, Abstract of a series of lectures given at a workshop on polylogs in Essen, May 1-4 (1997).

[HW2] A. Huber and J. Wildeshaus, Classical Motivic Polylogarithm according to Beilinson and Deligne, Doc. Math. 3 (1998), 27-133.

[MW] P. Monsky and G. Washnitzer, Formal Cohomology: I, Annals of Math 88 (1968), 181-217.

[Ni] W. Nizioł, Cohomology of crystalline smooth sheaves, Comp. Math. 129 (2001), 123-147.

[So] M. Somekawa, Log syntomic regulators and p-adic polylogarithms, K-Theory 17 (1999), 256-294.

[Su] S. Sugimoto, Filtered modules and p-adic polylogarithms, Ph. D. thesis, University of Tokyo (1992).

[T] N. Tsuzuki, On the Gysin isomorphism of rigid cohomology, Hiroshima Math. J. 29 no. 3 (1999), 479-527.

[W] J. Wildeshaus, Realization of Polylogarithms, SLN 1650 (1997).

Kenichi Bannai
Graduate School of Mathematics
Nagoya University
Furo-cho, Chikusa-ku, Nagoya 464-8602
Japan
bannai@math.nagoya-u.ac.jp

Bloch and Kato's Exponential Map: Three Explicit Formulas

To Kazuya Kato on the occasion of his fiftieth birthday

Laurent Berger

Received: September 23, 2002
Revised: March 13, 2003

Abstract. The purpose of this article is to give formulas for Bloch-Kato's exponential map and its dual for an absolutely crystalline p-adic representation V, in terms of the (φ, Γ)-module associated to V. As a corollary of these computations, we can give a very simple and slightly improved description of Perrin-Riou's exponential map, which interpolates Bloch-Kato's exponentials for the twists of V. This new description directly implies Perrin-Riou's reciprocity formula.

2000 Mathematics Subject Classification: 11F80, 11R23, 11S25, 12H25, 13K05, 14F30, 14G20
Keywords and Phrases: Bloch-Kato's exponential, Perrin-Riou's exponential, Iwasawa theory, p-adic representations, Galois cohomology.

Contents

Introduction

In his article [Ka93] on L-functions and rings of p-adic periods, K. Kato wrote:

> *I believe that there exist explicit reciprocity laws for all p-adic representations of* $\mathrm{Gal}(\overline{K}/K)$*, though I can not formulate them. For a de Rham representation V, this law should be some explicit description of the relationship between* $\mathbf{D}_{\mathrm{dR}}(V)$ *and the Galois cohomology of V, or more precisely, some explicit descriptions of the maps* exp *and* $\exp^*$ *of V.*

In this paper, we explain how results of Benois, Cherbonnier-Colmez, Colmez, Fontaine, Kato, Kato-Kurihara-Tsuji, Perrin-Riou, Wach and the author give such an explicit description when V is a crystalline representation of an unramified field.

Let p be a prime number, and let V be a p-adic representation of $G_K = \mathrm{Gal}(\overline{K}/K)$ where K is a finite extension of $\mathbf{Q}_p$. Such objects arise (for example) as the étale cohomology of algebraic varieties, hence their interest in arithmetic algebraic geometry.

Let $\mathbf{B}_{\mathrm{cris}}$ and $\mathbf{B}_{\mathrm{dR}}$ be the rings of periods of Fontaine, and let $\mathbf{D}_{\mathrm{cris}}(V)$ and $\mathbf{D}_{\mathrm{dR}}(V)$ be the invariants attached to V by Fontaine's construction. Bloch and Kato have defined in [BK91, §3], for a de Rham representation V, an "exponential" map,

$$\exp_{K,V} : \mathbf{D}_{\mathrm{dR}}(V)/\operatorname{Fil}^0 \mathbf{D}_{\mathrm{dR}}(V) \to H^1(K,V).$$

It is obtained by tensoring the so-called fundamental exact sequence:

$$0 \to \mathbf{Q}_p \to \mathbf{B}_{\mathrm{cris}}^{\varphi=1} \to \mathbf{B}_{\mathrm{dR}}/\mathbf{B}_{\mathrm{dR}}^+ \to 0$$

with V and taking the invariants under the action of G_K. The exponential map is then the connecting homomorphism $\mathbf{D}_{\mathrm{dR}}(V)/\operatorname{Fil}^0 \mathbf{D}_{\mathrm{dR}}(V) \to H^1(K,V)$.

The reason for their terminology is the following (cf. [BK91, 3.10.1]): if G is a formal Lie group of finite height over $\mathcal{O}_K$, and $V = \mathbf{Q}_p \otimes_{\mathbf{Z}_p} T$ where T is the p-adic Tate module of G, then V is a de Rham representation and $\mathbf{D}_{\mathrm{dR}}(V)/\operatorname{Fil}^0 \mathbf{D}_{\mathrm{dR}}(V)$ is identified with the tangent space $\tan(G(K))$ of $G(K)$. In this case, we have a commutative diagram:

$$\begin{array}{ccc}
\tan(G(K)) & \xrightarrow{\exp_G} & \mathbf{Q} \otimes_{\mathbf{Z}} G(\mathcal{O}_K) \\
{\scriptstyle =}\downarrow & & {\scriptstyle \delta_G}\downarrow \\
\mathbf{D}_{\mathrm{dR}}(V)/\operatorname{Fil}^0 \mathbf{D}_{\mathrm{dR}}(V) & \xrightarrow{\exp_{K,V}} & H^1(K,V),
\end{array}$$

where δ_G is the Kummer map, the upper $\exp_G$ is the usual exponential map, and the lower $\exp_{K,V}$ is Bloch-Kato's exponential map.

The cup product $\cup : H^1(K,V) \times H^1(K,V^*(1)) \to H^2(K,\mathbf{Q}_p(1)) \simeq \mathbf{Q}_p$ defines a perfect pairing, which we can use (by dualizing twice) to define Bloch and Kato's dual exponential map $\exp^*_{K,V^*(1)} : H^1(K,V) \to \mathrm{Fil}^0\,\mathbf{D}_{\mathrm{dR}}(V)$. Kato has given in [Ka93] a very simple formula for $\exp^*_{K,V^*(1)}$, see proposition II.5 below.

When K is an unramified extension of $\mathbf{Q}_p$ and V is a crystalline representation of G_K, Perrin-Riou has constructed in [Per94] a period map $\Omega_{V,h}$ which interpolates the $\exp_{K,V(k)}$ as k runs over the positive integers. It is a crucial ingredient in the construction of p-adic L functions, and is a vast generalization of Coleman's map. Perrin-Riou's constructions were further generalized by Colmez in [Col98].

Let us recall the main properties of her map. For that purpose we need to introduce some notation which will be useful throughout the article. Let $H_K = \mathrm{Gal}(\overline{K}/K(\mu_{p^\infty}))$, let Δ_K be the torsion subgroup of $\Gamma_K = G_K/H_K = \mathrm{Gal}(K(\mu_{p^\infty})/K)$ and let $\Gamma_K^1 = \mathrm{Gal}(K(\mu_{p^\infty})/K(\mu_p))$ so that $\Gamma_K \simeq \Delta_K \times \Gamma_K^1$. Let $\Lambda_K = \mathbf{Z}_p[[\Gamma_K]]$ and $\mathcal{H}(\Gamma_K) = \mathbf{Q}_p[\Delta_K] \otimes_{\mathbf{Q}_p} \mathcal{H}(\Gamma_K^1)$ where $\mathcal{H}(\Gamma_K^1)$ is the set of $f(\gamma_1 - 1)$ with $\gamma_1 \in \Gamma_K^1$ and where $f(T) \in \mathbf{Q}_p[[T]]$ is a power series which converges on the p-adic open unit disk.

Recall that the Iwasawa cohomology groups of V are the projective limits for the corestriction maps of the $H^i(K_n,V)$ where $K_n = K(\mu_{p^n})$. More precisely, if T is any lattice of V then $H^i_{\mathrm{Iw}}(K,V) = \mathbf{Q}_p \otimes_{\mathbf{Z}_p} H^i_{\mathrm{Iw}}(K,T)$ where $H^i_{\mathrm{Iw}}(K,T) = \varprojlim_n H^i(K_n,T)$ so that $H^i_{\mathrm{Iw}}(K,V)$ has the structure of a $\mathbf{Q}_p \otimes_{\mathbf{Z}_p} \Lambda_K$-module (see §II.4 for more details). Roughly speaking, these cohomology groups are where Euler systems live (at least locally).

The main result of [Per94] is the construction, for a crystalline representation V of G_K of a family of maps (parameterized by $h \in \mathbf{Z}$):

$$\Omega_{V,h} : \mathcal{H}(\Gamma_K) \otimes_{\mathbf{Q}_p} \mathbf{D}_{\mathrm{cris}}(V) \to \mathcal{H}(\Gamma_K) \otimes_{\Lambda_K} H^1_{\mathrm{Iw}}(K,V)/V^{H_K},$$

whose main property is that they interpolate Bloch and Kato's exponential map. More precisely, if $h, j \gg 0$, then the diagram:

$$\begin{array}{ccc} \big(\mathcal{H}(\Gamma_K) \otimes_{\mathbf{Q}_p} \mathbf{D}_{\mathrm{cris}}(V(j))\big)^{\Delta=0} & \xrightarrow{\Omega_{V(j),h}} & \mathcal{H}(\Gamma_K) \otimes_{\Lambda_K} H^1_{\mathrm{Iw}}(K,V(j))/V(j)^{H_K} \\ \Xi_{n,V(j)} \big\downarrow & & \mathrm{pr}_{K_n,V(j)} \big\downarrow \\ K_n \otimes_K \mathbf{D}_{\mathrm{cris}}(V) & \xrightarrow[\exp_{K_n,V(j)}]{(h+j-1)!\times} & H^1(K_n,V(j)) \end{array}$$

is commutative where Δ and $\Xi_{n,V}$ are two maps whose definition is rather technical. Let us just say that the image of Δ is finite-dimensional over $\mathbf{Q}_p$ and that $\Xi_{n,V}$ is a kind of evaluation-at-$(\varepsilon^{(n)} - 1)$ map (see §II.5 for a precise definition).

Using the inverse of Perrin-Riou's map, one can then associate to an Euler system a p-adic L-function (see for example [Per95]). For an enlightening survey of this, see [Col00]. If one starts with $V = \mathbf{Q}_p(1)$, then Perrin-Riou's map is the inverse of the Coleman isomorphism and one recovers Kubota-Leopoldt's p-adic L-functions. It is therefore important to be able to construct the maps $\Omega_{V,h}$ as explicitly as possible.

The goal of this article is to give formulas for $\exp_{K,V}$, $\exp^*_{K,V^*(1)}$, and $\Omega_{V,h}$ in terms of the (φ,Γ)-module associated to V by Fontaine. As a corollary, we recover a theorem of Colmez which states that Perrin-Riou's map interpolates the $\exp^*_{K,V^*(1-k)}$ as k runs over the negative integers. This is equivalent to Perrin-Riou's conjectured reciprocity formula (proved by Benois, Colmez and Kato-Kurihara-Tsuji). Our construction of $\Omega_{V,h}$ is actually a slight improvement over Perrin-Riou's (one does not have to kill the Λ_K-torsion, see remark II.14). In addition, our construction should generalize to the case of de Rham representations, to families and to settings other than cyclotomic.

We refer the reader to the text itself for a statement of the actual formulas (theorems II.3, II.6 and II.13) which are rather technical.

This article does not really contain any new results, and it is mostly a re-interpretation of formulas of Cherbonnier-Colmez (for the dual exponential map), and of Benois and Colmez and Kato-Kurihara-Tsuji (for Perrin-Riou's map) in the language of the author's article [Ber02] on p-adic representations and differential equations.

ACKNOWLEDGMENTS. This research was partially conducted for the Clay Mathematical Institute, and I thank them for their support. I would also like to thank P. Colmez and the referee for their careful reading of earlier versions of this article. It is P. Colmez who suggested that I give a formula for Bloch-Kato's exponential in terms of (φ,Γ)-modules.

Finally, it is a pleasure to dedicate this article to Kazuya Kato on the occasion of his fiftieth birthday.

I. PERIODS OF p-ADIC REPRESENTATIONS

Throughout this article, k will denote a finite field of characteristic $p > 0$, so that if $W(k)$ denotes the ring of Witt vectors over k, then $F = W(k)[1/p]$ is a finite unramified extension of $\mathbf{Q}_p$. Let $\overline{\mathbf{Q}}_p$ be the algebraic closure of $\mathbf{Q}_p$, let K be a finite totally ramified extension of F, and let $G_K = \mathrm{Gal}(\overline{\mathbf{Q}}_p/K)$ be the absolute Galois group of K. Let μ_{p^n} be the group of p^n-th roots of unity; for every n, we will choose a generator $\varepsilon^{(n)}$ of μ_{p^n}, with the additional requirement that $(\varepsilon^{(n)})^p = \varepsilon^{(n-1)}$. This makes $\varprojlim_n \varepsilon^{(n)}$ into a generator of $\varprojlim_n \mu_{p^n} \simeq \mathbf{Z}_p(1)$. We set $K_n = K(\mu_{p^n})$ and $K_\infty = \cup_{n=0}^{+\infty} K_n$. Recall that the cyclotomic character $\chi : G_K \to \mathbf{Z}_p^*$ is defined by the relation: $g(\varepsilon^{(n)}) = (\varepsilon^{(n)})^{\chi(g)}$ for all

$g \in G_K$. The kernel of the cyclotomic character is $H_K = \mathrm{Gal}(\overline{\mathbf{Q}}_p/K_\infty)$, and χ therefore identifies $\Gamma_K = G_K/H_K$ with an open subgroup of $\mathbf{Z}_p^*$.

A *p-adic representation* V is a finite dimensional $\mathbf{Q}_p$-vector space with a continuous linear action of G_K. It is easy to see that there is always a $\mathbf{Z}_p$-lattice of V which is stable by the action of G_K, and such lattices will be denoted by T. The main strategy (due to Fontaine, see for example [Fo88b]) for studying p-adic representations of a group G is to construct topological $\mathbf{Q}_p$-algebras B (*rings of periods*), endowed with an action of G and some additional structures so that if V is a p-adic representation, then

$$D_B(V) = (B \otimes_{\mathbf{Q}_p} V)^G$$

is a B^G-module which inherits these structures, and so that the functor $V \mapsto D_B(V)$ gives interesting invariants of V. We say that a p-adic representation V of G is *B-admissible* if we have $B \otimes_{\mathbf{Q}_p} V \simeq B^d$ as $B[G]$-modules.

In the next two paragraphs, we will recall the construction of a number of rings of periods. The relations between these rings are mapped in appendix C.

I.1. p-adic Hodge theory. In this paragraph, we will recall the definitions of Fontaine's rings of periods. One can find some of these constructions in [Fo88a] and most of what we will need is proved in [Col98, III] to which the reader should refer in case of need. He is also invited to turn to appendix C.

Let $\mathbf{C}_p$ be the completion of $\overline{\mathbf{Q}}_p$ for the p-adic topology and let

$$\widetilde{\mathbf{E}} = \varprojlim_{x \mapsto x^p} \mathbf{C}_p = \{(x^{(0)}, x^{(1)}, \cdots) \mid (x^{(i+1)})^p = x^{(i)}\},$$

and let $\widetilde{\mathbf{E}}^+$ be the set of $x \in \widetilde{\mathbf{E}}$ such that $x^{(0)} \in \mathcal{O}_{\mathbf{C}_p}$. If $x = (x^{(i)})$ and $y = (y^{(i)})$ are two elements of $\widetilde{\mathbf{E}}$, we define their sum $x+y$ and their product xy by:

$$(x+y)^{(i)} = \lim_{j \to +\infty} (x^{(i+j)} + y^{(i+j)})^{p^j} \quad \text{and} \quad (xy)^{(i)} = x^{(i)} y^{(i)},$$

which makes $\widetilde{\mathbf{E}}$ into an algebraically closed field of characteristic p. If $x = (x^{(n)})_{n \geq 0} \in \widetilde{\mathbf{E}}$, let $v_{\mathbf{E}}(x) = v_p(x^{(0)})$. This is a valuation on $\widetilde{\mathbf{E}}$ for which $\widetilde{\mathbf{E}}$ is complete; the ring of integers of $\widetilde{\mathbf{E}}$ is $\widetilde{\mathbf{E}}^+$. Let $\widetilde{\mathbf{A}}^+$ be the ring $W(\widetilde{\mathbf{E}}^+)$ of Witt vectors with coefficients in $\widetilde{\mathbf{E}}^+$ and let

$$\widetilde{\mathbf{B}}^+ = \widetilde{\mathbf{A}}^+[1/p] = \{\sum_{k \gg -\infty} p^k [x_k],\ x_k \in \widetilde{\mathbf{E}}^+\}$$

where $[x] \in \widetilde{\mathbf{A}}^+$ is the Teichmüller lift of $x \in \widetilde{\mathbf{E}}^+$. This ring is endowed with a map $\theta : \widetilde{\mathbf{B}}^+ \to \mathbf{C}_p$ defined by the formula

$$\theta\left(\sum_{k \gg -\infty} p^k [x_k]\right) = \sum_{k \gg -\infty} p^k x_k^{(0)}.$$

The absolute Frobenius $\varphi : \widetilde{\mathbf{E}}^+ \to \widetilde{\mathbf{E}}^+$ lifts by functoriality of Witt vectors to a map $\varphi : \widetilde{\mathbf{B}}^+ \to \widetilde{\mathbf{B}}^+$. It's easy to see that $\varphi(\sum p^k[x_k]) = \sum p^k[x_k^p]$ and that φ is bijective.

Let $\varepsilon = (\varepsilon^{(i)})_{i\geq 0} \in \widetilde{\mathbf{E}}^+$ where $\varepsilon^{(n)}$ is defined above, and define $\pi = [\varepsilon]-1$, $\pi_1 = [\varepsilon^{1/p}]-1$, $\omega = \pi/\pi_1$ and $q = \varphi(\omega) = \varphi(\pi)/\pi$. One can easily show that $\ker(\theta : \widetilde{\mathbf{A}}^+ \to \mathcal{O}_{\mathbf{C}_p})$ is the principal ideal generated by ω.

The ring $\mathbf{B}_{\mathrm{dR}}^+$ is defined to be the completion of $\widetilde{\mathbf{B}}^+$ for the $\ker(\theta)$-adic topology:

$$\mathbf{B}_{\mathrm{dR}}^+ = \varprojlim_{n\geq 0} \widetilde{\mathbf{B}}^+/(\ker(\theta)^n).$$

It is a discrete valuation ring, whose maximal ideal is generated by ω; the series which defines $\log([\varepsilon])$ converges in $\mathbf{B}_{\mathrm{dR}}^+$ to an element t, which is also a generator of the maximal ideal, so that $\mathbf{B}_{\mathrm{dR}} = \mathbf{B}_{\mathrm{dR}}^+[1/t]$ is a field, endowed with an action of G_K and a filtration defined by $\mathrm{Fil}^i(\mathbf{B}_{\mathrm{dR}}) = t^i\mathbf{B}_{\mathrm{dR}}^+$ for $i \in \mathbf{Z}$.

We say that a representation V of G_K is *de Rham* if it is $\mathbf{B}_{\mathrm{dR}}$-admissible which is equivalent to the fact that the filtered K-vector space

$$\mathbf{D}_{\mathrm{dR}}(V) = (\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)^{G_K}$$

is of dimension $d = \dim_{\mathbf{Q}_p}(V)$.

Recall that the topology of $\widetilde{\mathbf{B}}^+$ is defined by taking the collection of open sets $\{([\overline{\pi}]^k, p^n)\widetilde{\mathbf{A}}^+\}_{k,n\geq 0}$ as a family of neighborhoods of 0. The ring $\mathbf{B}_{\max}^+$ is defined as being

$$\mathbf{B}_{\max}^+ = \{\sum_{n\geq 0} a_n \frac{\omega^n}{p^n} \text{ where } a_n \in \widetilde{\mathbf{B}}^+ \text{ is sequence converging to } 0\},$$

and $\mathbf{B}_{\max} = \mathbf{B}_{\max}^+[1/t]$. The ring $\mathbf{B}_{\max}$ was defined in [Col98, III.2] where a number of its properties are established. It is closely related to $\mathbf{B}_{\mathrm{cris}}$ but tends to be more amenable (loc. cit.). One could replace ω by any generator of $\ker(\theta)$ in $\widetilde{\mathbf{A}}^+$. The ring $\mathbf{B}_{\max}$ injects canonically into $\mathbf{B}_{\mathrm{dR}}$ and, in particular, it is endowed with the induced Galois action and filtration, as well as with a continuous Frobenius φ, extending the map $\varphi : \widetilde{\mathbf{B}}^+ \to \widetilde{\mathbf{B}}^+$. Let us point out that φ does not extend continuously to $\mathbf{B}_{\mathrm{dR}}$. One also sets $\widetilde{\mathbf{B}}_{\mathrm{rig}}^+ = \cap_{n=0}^{+\infty}\varphi^n(\mathbf{B}_{\max}^+)$.

We say that a representation V of G_K is *crystalline* if it is $\mathbf{B}_{\max}$-admissible or (which is the same) $\widetilde{\mathbf{B}}_{\mathrm{rig}}^+[1/t]$-admissible (the periods of crystalline representations live in finite dimensional F-vector subspaces of $\mathbf{B}_{\max}$, stable by φ, and so in fact in $\cap_{n=0}^{+\infty}\varphi^n(\mathbf{B}_{\max}^+)[1/t]$); this is equivalent to requiring that the F-vector space

$$\mathbf{D}_{\mathrm{cris}}(V) = (\mathbf{B}_{\max} \otimes_{\mathbf{Q}_p} V)^{G_K} = (\widetilde{\mathbf{B}}_{\mathrm{rig}}^+[1/t] \otimes_{\mathbf{Q}_p} V)^{G_K}$$

be of dimension $d = \dim_{\mathbf{Q}_p}(V)$. Then $\mathbf{D}_{\mathrm{cris}}(V)$ is endowed with a Frobenius φ induced by that of $\mathbf{B}_{\max}$ and $(\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)^{G_K} = \mathbf{D}_{\mathrm{dR}}(V) = K \otimes_F \mathbf{D}_{\mathrm{cris}}(V)$ so that a crystalline representation is also de Rham and $K \otimes_F \mathbf{D}_{\mathrm{cris}}(V)$ is a filtered

K-vector space. Note that this definition of $\mathbf{D}_{\mathrm{cris}}(V)$ is compatible with the "usual" one (via $\mathbf{B}_{\mathrm{cris}}$) because $\cap_{n=0}^{+\infty}\varphi^n(\mathbf{B}^+_{\max}) = \cap_{n=0}^{+\infty}\varphi^n(\mathbf{B}^+_{\mathrm{cris}})$.

If V is a p-adic representation, we say that V is *Hodge-Tate*, with Hodge-Tate weights $h_1, \cdots, h_d$, if we have a decomposition $\mathbf{C}_p \otimes_{\mathbf{Q}_p} V \simeq \oplus_{j=1}^d \mathbf{C}_p(h_j)$. We will say that V is positive if its Hodge-Tate weights are negative (the definition of the sign of the Hodge-Tate weights is unfortunate; some people change the sign and talk about geometrical weights). By using the map $\theta : \mathbf{B}^+_{\mathrm{dR}} \to \mathbf{C}_p$, it is easy to see that a de Rham representation is Hodge-Tate and that the Hodge-Tate weights of V are those integers h such that $\mathrm{Fil}^{-h}\,\mathbf{D}_{\mathrm{dR}}(V) \neq \mathrm{Fil}^{-h+1}\,\mathbf{D}_{\mathrm{dR}}(V)$.

To summarize, let us recall that crystalline implies de Rham implies Hodge-Tate. Of course, the significance of these definitions is to be found in geometrical applications. For example, if V is the Tate module of an abelian variety A, then V is de Rham and it is crystalline if and only if A has good reduction.

I.2. (φ, Γ)-modules. The results recalled in this paragraph can be found in [Fo91], and the version which we use here is described in [CC98] and [CC99].

Let $\widetilde{\mathbf{A}}$ be the ring of Witt vectors with coefficients in $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{B}} = \widetilde{\mathbf{A}}[1/p]$. Let $\mathbf{A}_F$ be the completion of $\mathcal{O}_F[\pi, \pi^{-1}]$ in $\widetilde{\mathbf{A}}$ for this ring's topology, which is also the completion of $\mathcal{O}_F[[\pi]][\pi^{-1}]$ for the p-adic topology (π being small in $\widetilde{\mathbf{A}}$). This is a discrete valuation ring whose residue field is $k((\varepsilon - 1))$. Let $\mathbf{B}$ be the completion for the p-adic topology of the maximal unramified extension of $\mathbf{B}_F = \mathbf{A}_F[1/p]$ in $\widetilde{\mathbf{B}}$. We then define $\mathbf{A} = \mathbf{B} \cap \widetilde{\mathbf{A}}$, $\mathbf{B}^+ = \mathbf{B} \cap \widetilde{\mathbf{B}}^+$ and $\mathbf{A}^+ = \mathbf{A} \cap \widetilde{\mathbf{A}}^+$. These rings are endowed with an action of Galois and a Frobenius deduced from those on $\widetilde{\mathbf{E}}$. We set $\mathbf{A}_K = \mathbf{A}^{H_K}$ and $\mathbf{B}_K = \mathbf{A}_K[1/p]$. When $K = F$, the two definitions are the same. Let $\mathbf{B}^+_F = (\mathbf{B}^+)^{H_F}$ as well as $\mathbf{A}^+_F = (\mathbf{A}^+)^{H_F}$ (those rings are not so interesting if $K \neq F$). One can show that $\mathbf{A}^+_F = \mathcal{O}_F[[\pi]]$ and that $\mathbf{B}^+_F = \mathbf{A}^+_F[1/p]$.

If V is a p-adic representation of G_K, let $\mathbf{D}(V) = (\mathbf{B} \otimes_{\mathbf{Q}_p} V)^{H_K}$. We know by [Fo91] that $\mathbf{D}(V)$ is a d-dimensional $\mathbf{B}_K$-vector space with a slope 0 Frobenius and a residual action of Γ_K which commute (it is an étale (φ, Γ_K)*-module*) and that one can recover V by the formula $V = (\mathbf{B} \otimes_{\mathbf{B}_K} \mathbf{D}(V))^{\varphi=1}$.

If T is a lattice of V, we get analogous statements with $\mathbf{A}$ instead of $\mathbf{B}$: $\mathbf{D}(T) = (\mathbf{A} \otimes_{\mathbf{Z}_p} T)^{H_K}$ is a free $\mathbf{A}_K$-module of rank d and $T = (\mathbf{A} \otimes_{\mathbf{A}_K} \mathbf{D}(T))^{\varphi=1}$.

The field $\mathbf{B}$ is a totally ramified extension (because the residual extension is purely inseparable) of degree p of $\varphi(\mathbf{B})$. The Frobenius map $\varphi : \mathbf{B} \to \mathbf{B}$ is injective but therefore not surjective, but we can define a left inverse for φ, which will play a major role in the sequel. We set: $\psi(x) = \varphi^{-1}(p^{-1}\,\mathrm{Tr}_{\mathbf{B}/\varphi(\mathbf{B})}(x))$.

Let us now set $K = F$ (i.e. we are now working in an unramified extension of $\mathbf{Q}_p$). We say that a p-adic representation V of G_F is of *finite height* if $\mathbf{D}(V)$ has a basis over $\mathbf{B}_F$ made up of elements of $\mathbf{D}^+(V) = (\mathbf{B}^+ \otimes_{\mathbf{Q}_p} V)^{H_F}$. A result of Fontaine ([Fo91] or [Col99, III.2]) shows that V is of finite height if and only if $\mathbf{D}(V)$ has a sub-$\mathbf{B}_F^+$-module which is free of finite rank d, and stable by φ. Let us recall the main result (due to Colmez, see [Col99, théorème 1] or also [Ber02, théorème 3.10]) regarding crystalline representations of G_F:

Theorem I.1. *If V is a crystalline representation of G_F, then V is of finite height.*

If $K \neq F$ or if V is no longer crystalline, then it is no longer true in general that V is of finite height, but it is still possible to say something about the periods of V. Every element $x \in \widetilde{\mathbf{B}}$ can be written in a unique way as $x = \sum_{k \gg -\infty} p^k [x_k]$, with $x_k \in \widetilde{\mathbf{E}}$. For $r > 0$, let us set:

$$\widetilde{\mathbf{B}}^{\dagger,r} = \left\{ x \in \widetilde{\mathbf{B}},\ \lim_{k \to +\infty} v_{\mathbf{E}}(x_k) + \frac{pr}{p-1} k = +\infty \right\}.$$

This makes $\widetilde{\mathbf{B}}^{\dagger,r}$ into an intermediate ring between $\widetilde{\mathbf{B}}^+$ and $\widetilde{\mathbf{B}}$. Let us set $\mathbf{B}^{\dagger,r} = \mathbf{B} \cap \widetilde{\mathbf{B}}^{\dagger,r}$, $\widetilde{\mathbf{B}}^{\dagger} = \cup_{r \geq 0} \widetilde{\mathbf{B}}^{\dagger,r}$, and $\mathbf{B}^{\dagger} = \cup_{r \geq 0} \mathbf{B}^{\dagger,r}$. If R is any of the above rings, then by definition $R_K = R^{H_K}$.

We say that a p-adic representation V is *overconvergent* if $\mathbf{D}(V)$ has a basis over $\mathbf{B}_K$ made up of elements of $\mathbf{D}^{\dagger}(V) = (\mathbf{B}^{\dagger} \otimes_{\mathbf{Q}_p} V)^{H_K}$. The main result on the overconvergence of p-adic representations of G_K is the following (cf [CC98, corollaire III.5.2]):

Theorem I.2. *Every p-adic representation V of G_K is overconvergent, that is there exists $r = r(V)$ such that $\mathbf{D}(V) = \mathbf{B}_K \otimes_{\mathbf{B}_K^{\dagger,r}} \mathbf{D}^{\dagger,r}(V)$.*

The terminology "overconvergent" can be explained by the following proposition, which describes the rings $\mathbf{B}_K^{\dagger,r}$. Let e_K be the ramification index of K_∞ / F_∞ and let F' be the maximal unramified extension of F contained in K_∞ (note that F' can be larger than F):

Proposition I.3. *Let $\mathcal{B}_{F'}^{\alpha}$ be the set of power series $f(X) = \sum_{k \in \mathbf{Z}} a_k X^k$ such that a_k is a bounded sequence of elements of F', and such that $f(X)$ is holomorphic on the p-adic annulus $\{p^{-1/\alpha} \leq |T| < 1\}$.*

There exist $r(K)$ and $\pi_K \in \mathbf{B}_K^{\dagger,r(K)}$ such that if $r \geq r(K)$, then the map $f \mapsto f(\pi_K)$ from $\mathcal{B}_{F'}^{e_K r}$ to $\mathbf{B}_K^{\dagger,r}$ is an isomorphism. If $K = F$, then $F' = F$ and one can take $\pi_F = \pi$.

I.3. p-adic representations and differential equations. We shall now recall some of the results of [Ber02], which allow us to recover $\mathbf{D}_{\mathrm{cris}}(V)$ from the (φ, Γ)-module associated to V. Let $\mathcal{H}_{F'}^{\alpha}$ be the set of power series $f(X) =$

$\sum_{k\in\mathbf{Z}} a_k X^k$ such that a_k is a sequence (not necessarily bounded) of elements of F', and such that $f(X)$ is holomorphic on the p-adic annulus $\{p^{-1/\alpha} \leq |T| < 1\}$.

For $r \geq r(K)$, define $\mathbf{B}^{\dagger,r}_{\mathrm{rig},K}$ as the set of $f(\pi_K)$ where $f(X) \in \mathcal{H}^{e_K r}_{F'}$. Obviously, $\mathbf{B}^{\dagger,r}_K \subset \mathbf{B}^{\dagger,r}_{\mathrm{rig},K}$ and the second ring is the completion of the first one for the natural Fréchet topology. If V is a p-adic representation, let

$$\mathbf{D}^{\dagger,r}_{\mathrm{rig}}(V) = \mathbf{B}^{\dagger,r}_{\mathrm{rig},K} \otimes_{\mathbf{B}^{\dagger,r}_K} \mathbf{D}^{\dagger,r}(V).$$

One of the main technical tools of [Ber02] is the construction of a large ring $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$, which contains $\widetilde{\mathbf{B}}^{+}_{\mathrm{rig}}$ and $\widetilde{\mathbf{B}}^{\dagger}$. This ring is a bridge between p-adic Hodge theory and the theory of (φ,Γ)-modules.

As a consequence of the two above inclusions, we have:

$$\mathbf{D}_{\mathrm{cris}}(V) \subset (\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}[1/t] \otimes_{\mathbf{Q}_p} V)^{G_K} \quad \text{and} \quad \mathbf{D}^{\dagger}_{\mathrm{rig}}(V)[1/t] \subset (\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}[1/t] \otimes_{\mathbf{Q}_p} V)^{H_K}.$$

One of the main results of [Ber02] is then (cf. [Ber02, theorem 3.6]):

Theorem I.4. *If V is a p-adic representation of G_K then:* $\mathbf{D}_{\mathrm{cris}}(V) = (\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)[1/t])^{\Gamma_K}$. *If V is positive, then* $\mathbf{D}_{\mathrm{cris}}(V) = \mathbf{D}^{\dagger}_{\mathrm{rig}}(V)^{\Gamma_K}$.

Note that one does not need to know what $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$ looks like in order to state the above theorem. We will not give the rather technical construction of that ring, but recall that $\mathbf{B}^{\dagger,r}_{\mathrm{rig},K}$ is the completion of $\mathbf{B}^{\dagger,r}_K$ for that ring's natural Fréchet topology and that $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ is the union of the $\mathbf{B}^{\dagger,r}_{\mathrm{rig},K}$. Similarly, there is a natural Fréchet topology on $\widetilde{\mathbf{B}}^{\dagger,r}$, $\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$ is the completion of $\widetilde{\mathbf{B}}^{\dagger,r}$ for that topology, and $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}} = \cup_{r\geq 0}\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$. Actually, one can show that $\widetilde{\mathbf{B}}^{+}_{\mathrm{rig}} \subset \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$ for any r and there is an exact sequence (see [Ber02, lemme 2.18]):

$$0 \to \widetilde{\mathbf{B}}^{+} \to \widetilde{\mathbf{B}}^{+}_{\mathrm{rig}} \oplus \widetilde{\mathbf{B}}^{\dagger,r} \to \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}} \to 0,$$

which the reader can take as providing a definition of $\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$.

Recall that if $n \geq 0$ and $r_n = p^{n-1}(p-1)$, then there is a well-defined injective map $\varphi^{-n} : \widetilde{\mathbf{B}}^{\dagger,r_n} \to \mathbf{B}^{+}_{\mathrm{dR}}$, and this map extends (see for example [Ber02, §2.2]) to an injective map $\varphi^{-n} : \widetilde{\mathbf{B}}^{\dagger,r_n}_{\mathrm{rig}} \to \mathbf{B}^{+}_{\mathrm{dR}}$.

The reader who feels that he needs to know more about those constructions and theorem I.4 above is invited to read either [Ber02] or the expository paper [Col01] by Colmez. See also appendix C.

Let us now return to the case when $K = F$ and V is a crystalline representation of G_F. In this case, Colmez's theorem tells us that V is of finite height so that one can write $\mathbf{D}^{\dagger,r}_{\mathrm{rig}}(V) = \mathbf{B}^{\dagger,r}_{\mathrm{rig},F} \otimes_{\mathbf{B}^{+}_F} \mathbf{D}^{+}(V)$ and theorem I.4 above therefore says that $\mathbf{D}_{\mathrm{cris}}(V) = (\mathbf{B}^{\dagger,r}_{\mathrm{rig},F}[1/t] \otimes_{\mathbf{B}^{+}_F} \mathbf{D}^{+}(V))^{\Gamma_F}$.

One can give a more precise result. Let $\mathbf{B}^+_{\mathrm{rig},F}$ be the set of $f(\pi)$ where $f(X) = \sum_{k\geq 0} a_k X^k$ with $a_k \in F$, and such that $f(X)$ is holomorphic on the p-adic open unit disk. Set $\mathbf{D}^+_{\mathrm{rig}}(V) = \mathbf{B}^+_{\mathrm{rig},F} \otimes_{\mathbf{B}^+_F} \mathbf{D}^+(V)$. One can then show (see [Ber03, §II.2]) the following refinement of theorem I.4:

Proposition I.5. *We have* $\mathbf{D}_{\mathrm{cris}}(V) = (\mathbf{D}^+_{\mathrm{rig}}(V)[1/t])^{\Gamma_F}$ *and if* V *is positive then* $\mathbf{D}_{\mathrm{cris}}(V) = \mathbf{D}^+_{\mathrm{rig}}(V)^{\Gamma_F}$.

Indeed if $\mathbf{N}(V)$ denotes, in the terminology of [loc. cit.], the Wach module associated to V, then $\mathbf{N}(V) \subset \mathbf{D}^+(V)$ when V is positive and it is shown in [loc. cit., §II.2] that under that hypothesis, $\mathbf{D}_{\mathrm{cris}}(V) = (\mathbf{B}^+_{\mathrm{rig},F} \otimes_{\mathbf{B}^+_F} \mathbf{N}(V))^{\Gamma_F}$.

I.4. Construction of cocycles. The purpose of this paragraph is to recall the constructions of [CC99, §I.5] and extend them a little bit. In this paragraph, V will be an arbitrary p-adic representation of G_K. Recall that in loc. cit., a map $h^1_{K,V} : \mathbf{D}(V)^{\psi=1} \to H^1(K,V)$ was constructed, and that (when Γ_K is torsion free at least) it gives rise to an exact sequence:

$$0 \longrightarrow \mathbf{D}(V)^{\psi=1}_{\Gamma_K} \xrightarrow{h^1_{K,V}} H^1(K,V) \longrightarrow \left(\frac{\mathbf{D}(V)}{\psi-1}\right)^{\Gamma_K} \longrightarrow 0.$$

We shall extend $h^1_{K,V}$ to a map $h^1_{K,V} : \mathbf{D}^\dagger_{\mathrm{rig}}(V)^{\psi=1} \to H^1(K,V)$. We will first need a few facts about the ring of periods $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ and the modules $\mathbf{D}^{\dagger,r}_{\mathrm{rig}}(V)$.

Lemma I.6. *If* r *is large enough and* $\gamma \in \Gamma_K$ *then*

$$1-\gamma : \mathbf{D}^{\dagger,r}_{\mathrm{rig}}(V)^{\psi=0} \to \mathbf{D}^{\dagger,r}_{\mathrm{rig}}(V)^{\psi=0}$$

is an isomorphism.

Proof. We will first show that $1-\gamma$ is injective. By theorem I.4, an element in the kernel of $1-\gamma$ would have to be in $\mathbf{D}_{\mathrm{cris}}(V)$ and therefore in $\mathbf{D}_{\mathrm{cris}}(V)^{\psi=0}$ which is obviously 0.

We will now prove surjectivity. Recall that by [CC98, II.6.1], if r is large enough and $\gamma \in \Gamma_K$ then $1-\gamma : \mathbf{D}^{\dagger,r}(V)^{\psi=0} \to \mathbf{D}^{\dagger,r}(V)^{\psi=0}$ is an isomorphism whose inverse is uniformly continuous for the Fréchet topology of $\mathbf{D}^{\dagger,r}(V)$.

In order to show the surjectivity of $1-\gamma$ it is therefore enough to show that $\mathbf{D}^{\dagger,r}(V)^{\psi=0}$ is dense in $\mathbf{D}^{\dagger,r}_{\mathrm{rig}}(V)^{\psi=0}$ for the Fréchet topology. For r large enough, $\mathbf{D}^{\dagger,r}(V)$ has a basis in $\varphi(\mathbf{D}^{\dagger,r/p}(V))$ so that

$$\mathbf{D}^{\dagger,r}(V)^{\psi=0} = (\mathbf{B}^{\dagger,r}_K)^{\psi=0} \cdot \varphi(\mathbf{D}^{\dagger,r/p}(V))$$
$$\mathbf{D}^{\dagger,r}_{\mathrm{rig}}(V)^{\psi=0} = (\mathbf{B}^{\dagger,r}_{\mathrm{rig},K})^{\psi=0} \cdot \varphi(\mathbf{D}^{\dagger,r/p}(V)).$$

The fact that $\mathbf{D}^{\dagger,r}(V)^{\psi=0}$ is dense in $\mathbf{D}^{\dagger,r}_{\mathrm{rig}}(V)^{\psi=0}$ for the Fréchet topology will therefore follow from the density of $(\mathbf{B}^{\dagger,r}_K)^{\psi=0}$ in $(\mathbf{B}^{\dagger,r}_{\mathrm{rig},K})^{\psi=0}$. This last statement follows from the facts that by definition $\mathbf{B}_K^{\dagger,r/p}$ is dense in $\mathbf{B}_{\mathrm{rig},K}^{\dagger,r/p}$ and that:

$$(\mathbf{B}_K^{\dagger,r})^{\psi=0} = \oplus_{i=1}^{p-1}[\varepsilon]^i\varphi(\mathbf{B}_K^{\dagger,r/p}) \quad \text{and} \quad (\mathbf{B}_{\mathrm{rig},K}^{\dagger,r})^{\psi=0} = \oplus_{i=1}^{p-1}[\varepsilon]^i\varphi(\mathbf{B}_{\mathrm{rig},K}^{\dagger,r/p}).$$

□

Lemma I.7. *The following maps are all surjective and their kernel is* $\mathbf{Q}_p$*:*

$$1-\varphi : \widetilde{\mathbf{B}}^\dagger \to \widetilde{\mathbf{B}}^\dagger, \quad 1-\varphi : \widetilde{\mathbf{B}}_{\mathrm{rig}}^+ \to \widetilde{\mathbf{B}}_{\mathrm{rig}}^+ \quad \textit{and} \quad 1-\varphi : \widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger \to \widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger.$$

Proof. We'll start with the assertion on the kernel of $1-\varphi$. Since $\widetilde{\mathbf{B}}_{\mathrm{rig}}^+ \subset \widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger$ and $\widetilde{\mathbf{B}}^\dagger \subset \widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger$ it is enough to show that $(\widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger)^{\varphi=1} = \mathbf{Q}_p$. If $x \in (\widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger)^{\varphi=1}$, then [Ber02, prop 3.2] shows that actually $x \in (\widetilde{\mathbf{B}}_{\mathrm{rig}}^+)^{\varphi=1}$, and therefore $x \in (\widetilde{\mathbf{B}}_{\mathrm{rig}}^+)^{\varphi=1} = (\mathbf{B}_{\max}^+)^{\varphi=1} = \mathbf{Q}_p$ by [Col98, III.3].

The surjectivity of $1-\varphi : \widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger \to \widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger$ results from the surjectivity of $1-\varphi$ on the first two spaces since by [Ber02, lemme 2.18], one can write $\alpha \in \widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger$ as $\alpha = \alpha^+ + \alpha^-$ with $\alpha^+ \in \widetilde{\mathbf{B}}_{\mathrm{rig}}^+$ and $\alpha^- \in \widetilde{\mathbf{B}}^\dagger$.

The surjectivity of $1-\varphi : \widetilde{\mathbf{B}}_{\mathrm{rig}}^+ \to \widetilde{\mathbf{B}}_{\mathrm{rig}}^+$ follows from the facts that $1-\varphi : \mathbf{B}_{\max}^+ \to \mathbf{B}_{\max}^+$ is surjective (see [Col98, III.3]) and that $\widetilde{\mathbf{B}}_{\mathrm{rig}}^+ = \cap_{n=0}^{+\infty}\varphi^n(\mathbf{B}_{\max}^+)$.

The surjectivity of $1-\varphi : \widetilde{\mathbf{B}}^\dagger \to \widetilde{\mathbf{B}}^\dagger$ follows from the facts that $1-\varphi : \widetilde{\mathbf{B}} \to \widetilde{\mathbf{B}}$ is surjective (it is surjective on $\widetilde{\mathbf{A}}$ as can be seen by reducing modulo p and using the fact that $\widetilde{\mathbf{E}}$ is algebraically closed) and that if $\beta \in \widetilde{\mathbf{B}}$ is such that $(1-\varphi)\beta \in \widetilde{\mathbf{B}}^\dagger$, then $\beta \in \widetilde{\mathbf{B}}^\dagger$ as we shall see presently.

If $x = \sum_{i=0}^{+\infty} p^i[x_i] \in \widetilde{\mathbf{A}}$, let us set $w_k(x) = \inf_{i\leq k} v_{\mathbf{E}}(x_i) \in \mathbf{R} \cup \{+\infty\}$. The definition of $\widetilde{\mathbf{B}}^{\dagger,r}$ shows that $x \in \widetilde{\mathbf{B}}^{\dagger,r}$ if and only if $\lim_{k\to+\infty} w_k(x) + \frac{pr}{p-1}k = +\infty$. A short computation also shows that $w_k(\varphi(x)) = pw_k(x)$ and that $w_k(x+y) \geq \inf(w_k(x), w_k(y))$ with equality if $w_k(x) \neq w_k(y)$.

It is then clear that

$$\lim_{k\to+\infty} w_k((1-\varphi)x) + \frac{pr}{p-1}k = +\infty \implies \lim_{k\to+\infty} w_k(x) + \frac{p(r/p)}{p-1}k = +\infty$$

and so if $x \in \widetilde{\mathbf{A}}$ is such that $(1-\varphi)x \in \widetilde{\mathbf{B}}^{\dagger,r}$ then $x \in \widetilde{\mathbf{B}}^{\dagger,r/p}$ and likewise for $x \in \widetilde{\mathbf{B}}$ by multiplying by a suitable power of p. □

The torsion subgroup of Γ_K will be denoted by Δ_K. We also set $\Gamma_K^n = \mathrm{Gal}(K_\infty/K_n)$. When $p \neq 2$ and $n \geq 1$ (or $p = 2$ and $n \geq 2$), Γ_K^n is torsion free. If $x \in 1 + p\mathbf{Z}_p$, then there exists $k \geq 1$ such that $\log_p(x) \in p^k\mathbf{Z}_p^*$ and we'll write $\log_p^0(x) = \log_p(x)/p^k$.

If K and n are such that Γ_K^n is torsion-free, then we will construct maps $h^1_{K_n,V}$ such that $\mathrm{cor}_{K_{n+1}/K_n} \circ h^1_{K_{n+1},V} = h^1_{K_n,V}$. If Γ_K^n is no longer torsion free, we'll therefore define $h^1_{K_n,V}$ by the relation $h^1_{K_n,V} = \mathrm{cor}_{K_{n+1}/K_n} \circ h^1_{K_{n+1},V}$. In the following proposition, we therefore assume that Γ_K is torsion free (and therefore procyclic), and we let γ denote a topological generator of Γ_K. Recall that if M is a Γ_K-module, it is customary to write M_{Γ_K} for $M/\mathrm{im}(\gamma-1)$.

Proposition I.8. *If $y \in \mathbf{D}^\dagger_{\mathrm{rig}}(V)^{\psi=1}$, then there exists $b \in \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbf{Q}_p} V$ such that $(\gamma-1)(\varphi-1)b = (\varphi-1)y$ and the formula*

$$h^1_{K,V}(y) = \log_p^0(\chi(\gamma)) \left[\sigma \mapsto \frac{\sigma-1}{\gamma-1} y - (\sigma-1)b\right]$$

then defines a map $h^1_{K,V} : \mathbf{D}^\dagger_{\mathrm{rig}}(V)^{\psi=1}_{\Gamma_K} \to H^1(K,V)$ which does not depend either on the choice of a generator γ of Γ_K or on the choice of a particular solution b, and if $y \in \mathbf{D}(V)^{\psi=1} \subset \mathbf{D}^\dagger_{\mathrm{rig}}(V)^{\psi=1}$, then $h^1_{K,V}(y)$ coincides with the cocycle constructed in [CC99, I.5].

Proof. Our construction closely follows [CC99, I.5]; to simplify the notations, we can assume that $\log_p^0(\chi(\gamma)) = 1$. The fact that if we start from a different γ, then the two $h^1_{K,V}$ we get are the same is left as an easy exercise for the reader.

Let us start by showing the existence of $b \in \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbf{Q}_p} V$. If $y \in \mathbf{D}^\dagger_{\mathrm{rig}}(V)^{\psi=1}$, then $(\varphi-1)y \in \mathbf{D}^\dagger_{\mathrm{rig}}(V)^{\psi=0}$. By lemma I.6, there exists $x \in \mathbf{D}^\dagger_{\mathrm{rig}}(V)^{\psi=0}$ such that $(\gamma-1)x = (\varphi-1)y$. By lemma I.7, there exists $b \in \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbf{Q}_p} V$ such that $(\varphi-1)b = x$.

Recall that we define $h^1_{K,V}(y) \in H^1(K,V)$ by the formula:

$$h^1_{K,V}(y)(\sigma) = \frac{\sigma-1}{\gamma-1} y - (\sigma-1)b.$$

Notice that, a priori, $h^1_{K,V}(y) \in H^1(K, \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbf{Q}_p} V)$, but

$$\begin{aligned}(\varphi-1)h^1_{K,V}(y)(\sigma) &= \frac{\sigma-1}{\gamma-1}(\varphi-1)y - (\sigma-1)(\varphi-1)b \\ &= \frac{\sigma-1}{\gamma-1}(\gamma-1)x - (\sigma-1)x \\ &= 0,\end{aligned}$$

so that $h^1_{K,V}(y)(\sigma) \in (\mathbf{B}^\dagger_{\mathrm{rig}})^{\varphi=1} \otimes_{\mathbf{Q}_p} V = V$. In addition, two different choices of b differ by an element of $(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}})^{\varphi=1} \otimes_{\mathbf{Q}_p} V = V$, and therefore give rise to two cohomologous cocycles.

It is clear that if $y \in \mathbf{D}(V)^{\psi=1} \subset \mathbf{D}^{\dagger}_{\mathrm{rig}}(V)^{\psi=1}$, then $h^1_{K,V}(y)$ coincides with the cocycle constructed in [CC99, I.5], as can be seen by their identical construction, and it is immediate that if $y \in (\gamma-1)\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$, then $h^1_{K,V}(y) = 0$. □

Lemma I.9. *We have* $\mathrm{cor}_{K_{n+1}/K_n} \circ h^1_{K_{n+1},V} = h^1_{K_n,V}$.

Proof. The proof is exactly the same as that of [CC99, §II.2] and in any case it is rather easy. □

II. Explicit formulas for exponential maps

Recall that $\exp_{K,V} : \mathbf{D}_{\mathrm{dR}}(V)/\operatorname{Fil}^0 \mathbf{D}_{\mathrm{dR}}(V) \to H^1(K,V)$ is obtained by tensoring the fundamental exact sequence (see [Col98, III.3]):

$$0 \to \mathbf{Q}_p \to \mathbf{B}_{\max}^{\varphi=1} \to \mathbf{B}_{\mathrm{dR}}/\mathbf{B}^+_{\mathrm{dR}} \to 0$$

with V and taking the invariants under the action of G_K (note once again that $\mathbf{B}_{\mathrm{cris}}^{\varphi=1} = \mathbf{B}_{\max}^{\varphi=1}$). The exponential map is then the connecting homomorphism $\mathbf{D}_{\mathrm{dR}}(V)/\operatorname{Fil}^0 \mathbf{D}_{\mathrm{dR}}(V) \to H^1(K,V)$.

The cup product $\cup : H^1(K,V) \times H^1(K,V^*(1)) \to H^2(K,\mathbf{Q}_p(1)) \simeq \mathbf{Q}_p$ defines a perfect pairing, which we use (by dualizing twice) to define Bloch and Kato's dual exponential map $\exp^*_{K,V^*(1)} : H^1(K,V) \to \operatorname{Fil}^0 \mathbf{D}_{\mathrm{dR}}(V)$.

The goal of this chapter is to give explicit formulas for Bloch-Kato's maps for a p-adic representation V, in terms of the (φ,Γ)-module $\mathbf{D}(V)$ attached to V. Throughout this chapter, V will be assumed to be a crystalline representation of G_F.

II.1. Preliminaries on some Iwasawa algebras. Recall that (cf [CC99, III.2] or [Ber02, §2.4] for example) we have maps $\varphi^{-n} : \widetilde{\mathbf{B}}^{\dagger,r_n}_{\mathrm{rig}} \to \mathbf{B}^+_{\mathrm{dR}}$ whose restriction to $\mathbf{B}^+_{\mathrm{rig},F}$ satisfy $\varphi^{-n}(\mathbf{B}^+_{\mathrm{rig},F}) \subset F_n[[t]]$ and which can then characterized by the fact that π maps to $\varepsilon^{(n)}\exp(t/p^n) - 1$.

If $z \in F_n((t)) \otimes_F \mathbf{D}_{\mathrm{cris}}(V)$, then the constant coefficient (i.e. the coefficient of t^0) of z will be denoted by $\partial_V(z) \in F_n \otimes_F \mathbf{D}_{\mathrm{cris}}(V)$. This notation should not be confused with that for the derivation map ∂ defined below.

We will make frequent use of the following fact:

Lemma II.1. *If* $y \in (\mathbf{B}^+_{\mathrm{rig},F}[1/t] \otimes_F \mathbf{D}_{\mathrm{cris}}(V))^{\psi=1}$, *then for any* $m \geq n \geq 0$, *the element* $p^{-m}\operatorname{Tr}_{F_m/F_n} \partial_V(\varphi^{-m}(y)) \in F_n \otimes_F \mathbf{D}_{\mathrm{cris}}(V)$ *does not depend on* m *and we have:*

$$p^{-m}\operatorname{Tr}_{F_m/F_n} \partial_V(\varphi^{-m}(y)) = \begin{cases} p^{-n}\partial_V(\varphi^{-n}(y)) & \text{if } n \geq 1 \\ (1-p^{-1}\varphi^{-1})\partial_V(y) & \text{if } n = 0. \end{cases}$$

Proof. Recall that if $y = t^{-\ell}\sum_{k=0}^{+\infty} a_k\pi^k \in \mathbf{B}^+_{\mathrm{rig},F}[1/t]\otimes_F \mathbf{D}_{\mathrm{cris}}(V)$, then

$$\varphi^{-m}(y) = p^{m\ell}t^{-\ell}\sum_{k=0}^{+\infty}\varphi^{-m}(a_k)(\varepsilon^{(m)}\exp(t/p^m)-1)^k,$$

and that by the definition of ψ, $\psi(y)=y$ means that:

$$\varphi(y) = \frac{1}{p}\sum_{\eta^p=1} y(\eta(1+T)-1).$$

The lemma then follows from the fact that if $m\geq 2$, then the conjugates of $\varepsilon^{(m)}$ under $\mathrm{Gal}(F_m/F_{m-1})$ are the $\eta\varepsilon^{(m)}$, where $\eta^p=1$, while if $m=1$, then the conjugates of $\varepsilon^{(1)}$ under $\mathrm{Gal}(F_1/F)$ are the η, where $\eta^p=1$ but $\eta\neq 1$. □

We will also need some facts about the Iwasawa algebra of Γ_F and some differential operators which it contains. Recall that since F is an unramified extension of $\mathbf{Q}_p$, $\Gamma_F \simeq \mathbf{Z}_p^*$ and that $\Gamma_F^n = \mathrm{Gal}(F_\infty/F_n)$ is the set of elements $\gamma\in\Gamma_F$ such that $\chi(\gamma)\in 1+p^n\mathbf{Z}_p$.

The completed group algebra of Γ_F is $\Lambda_F = \mathbf{Z}_p[[\Gamma_F]] \simeq \mathbf{Z}_p[\Delta_F]\otimes_{\mathbf{Z}_p}\mathbf{Z}_p[[\Gamma_F^1]]$, and we set $\mathcal{H}(\Gamma_F) = \mathbf{Q}_p[\Delta_F]\otimes_{\mathbf{Q}_p}\mathcal{H}(\Gamma_F^1)$ where $\mathcal{H}(\Gamma_F^1)$ is the set of $f(\gamma-1)$ with $\gamma\in\Gamma_F^1$ and where $f(X)\in\mathbf{Q}_p[[X]]$ is convergent on the p-adic open unit disk. Examples of elements of $\mathcal{H}(\Gamma_F)$ are the ∇_i (which are Perrin-Riou's ℓ_i's), defined by

$$\nabla_i = \ell_i = \frac{\log(\gamma)}{\log_p(\chi(\gamma))} - i.$$

We will also use the operator $\nabla_0/(\gamma_n-1)$, where γ_n is a topological generator of Γ_F^n. It is defined (see [Ber02, §4.1]) by the formula:

$$\frac{\nabla_0}{\gamma_n-1} = \frac{\log(\gamma_n)}{\log_p(\chi(\gamma_n))(\gamma_n-1)} = \frac{1}{\log_p(\chi(\gamma_n))}\sum_{i\geq 1}\frac{(1-\gamma_n)^{i-1}}{i},$$

or equivalently by

$$\frac{\nabla_0}{\gamma_n-1} = \lim_{\substack{\eta\in\Gamma_F^n\\ \eta\to 1}}\frac{\eta-1}{\gamma_n-1}\frac{1}{\log_p(\chi(\eta))}.$$

It is easy to see that $\nabla_0/(\gamma_n-1)$ acts on F_n by $1/\log_p(\chi(\gamma_n))$.

Note that "$\nabla_0/(\gamma_n-1)$" is a suggestive notation for this operator but it is not defined as a (meaningless) quotient of two operators.

The algebra $\mathcal{H}(\Gamma_F)$ acts on $\mathbf{B}^+_{\mathrm{rig},F}$ and one can easily check that:

$$\nabla_i = t\frac{d}{dt} - i = \log(1+\pi)\partial - i, \quad \text{where} \quad \partial = (1+\pi)\frac{d}{d\pi}.$$

In particular, $\nabla_0\mathbf{B}^+_{\mathrm{rig},F}\subset t\mathbf{B}^+_{\mathrm{rig},F}$ and if $i\geq 1$, then

$$\nabla_{i-1}\circ\cdots\circ\nabla_0\mathbf{B}^+_{\mathrm{rig},F}\subset t^i\mathbf{B}^+_{\mathrm{rig},F}.$$

Lemma II.2. *If $n \geq 1$, then $\nabla_0/(\gamma_n - 1)(\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0} \subset (t/\varphi^n(\pi))(\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0}$ so that if $i \geq 1$, then:*

$$\nabla_{i-1} \circ \cdots \circ \nabla_1 \circ \frac{\nabla_0}{\gamma_n - 1}(\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0} \subset \left(\frac{t}{\varphi^n(\pi)}\right)^i (\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0}.$$

Proof. Since $\nabla_i = t \cdot d/dt - i$, the second claim follows easily from the first one, which we will now show. By the standard properties of p-adic holomorphic functions, what we need to do is to show that if $x \in (\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0}$, then

$$\frac{\nabla_0}{\gamma_n - 1} x(\varepsilon^{(m)} - 1) = 0$$

for all $m \geq n+1$.

On the one hand, up to a scalar factor, one has for $m \geq n+1$:

$$\frac{\nabla_0}{\gamma_n - 1} x(\varepsilon^{(m)} - 1) = \mathrm{Tr}_{F_m/F_n}\, x(\varepsilon^{(m)} - 1)$$

as can be seen from the fact that

$$\frac{\nabla_0}{\gamma_n - 1} = \lim_{\substack{\eta \in \Gamma_F^n \\ \eta \to 1}} \frac{\eta - 1}{\gamma_n - 1} \cdot \frac{1}{\log_p(\chi(\eta))}.$$

On the other hand, the fact that $\psi(x) = 0$ implies that for every $m \geq 2$, $\mathrm{Tr}_{F_m/F_{m-1}}\, x(\varepsilon^{(m)} - 1) = 0$. This completes the proof. □

Finally, let us point out that the actions of any element of $\mathcal{H}(\Gamma_F)$ and of φ commute. Since $\varphi(t) = pt$, we also see that $\partial \circ \varphi = p\varphi \circ \partial$.

We will henceforth assume that $\log_p(\chi(\gamma_n)) = p^n$, so that $\log^0_p(\chi(\gamma_n)) = 1$, and in addition $\nabla_0/(\gamma_n - 1)$ acts on F_n by p^{-n}.

II.2. Bloch-Kato's exponential map. The goal of this paragraph is to show how to compute Bloch-Kato's map in terms of the (φ, Γ)-module of V. Let $h \geq 1$ be an integer such that $\mathrm{Fil}^{-h}\, \mathbf{D}_{\mathrm{cris}}(V) = \mathbf{D}_{\mathrm{cris}}(V)$.

Recall that we have seen that $\mathbf{D}_{\mathrm{cris}}(V) = (\mathbf{D}^+_{\mathrm{rig}}(V)[1/t])^{\Gamma_F}$ and that by [Ber03, §II.3] there is an isomorphism:

$$\mathbf{B}^+_{\mathrm{rig},F}[1/t] \otimes_F \mathbf{D}_{\mathrm{cris}}(V) = \mathbf{B}^+_{\mathrm{rig},F}[1/t] \otimes_F \mathbf{D}^+_{\mathrm{rig}}(V).$$

If $y \in \mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathbf{D}_{\mathrm{cris}}(V)$, then the fact that $\mathrm{Fil}^{-h}\, \mathbf{D}_{\mathrm{cris}}(V) = \mathbf{D}_{\mathrm{cris}}(V)$ implies by the results of [Ber03, §II.3] that $t^h y \in \mathbf{D}^+_{\mathrm{rig}}(V)$, so that if

$$y = \sum_{i=0}^{d} y_i \otimes d_i \in (\mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathbf{D}_{\mathrm{cris}}(V))^{\psi=1},$$

then

$$\nabla_{h-1} \circ \cdots \circ \nabla_0(y) = \sum_{i=0}^{d} t^h \partial^h y_i \otimes d_i \in \mathbf{D}^+_{\mathrm{rig}}(V)^{\psi=1}.$$

One can then apply the operator $h^1_{F_n,V}$ to $\nabla_{h-1} \circ \cdots \circ \nabla_0(y)$, and the main result of this paragraph is:

Theorem II.3. *If* $y \in (\mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathbf{D}_{\mathrm{cris}}(V))^{\psi=1}$, *then*

$$h^1_{F_n,V}(\nabla_{h-1} \circ \cdots \circ \nabla_0(y)) = (-1)^{h-1}(h-1)! \begin{cases} \exp_{F_n,V}(p^{-n}\partial_V(\varphi^{-n}(y))) & \text{if } n \geq 1 \\ \exp_{F,V}((1-p^{-1}\varphi^{-1})\partial_V(y)) & \text{if } n = 0. \end{cases}$$

Proof. Because the diagram

$$\begin{array}{ccc} F_{n+1} \otimes_F \mathbf{D}_{\mathrm{cris}}(V) & \xrightarrow{\exp_{F_{n+1},V}} & H^1(F_{n+1},V) \\ \downarrow{\scriptstyle \mathrm{Tr}_{F_{n+1}/F_n}} & & \downarrow{\scriptstyle \mathrm{cor}_{F_{n+1}/F_n}} \\ F_n \otimes_F \mathbf{D}_{\mathrm{cris}}(V) & \xrightarrow{\exp_{F_n,V}} & H^1(F_n,V) \end{array}$$

is commutative, it is enough to prove the theorem under the further assumption that Γ_F^n is torsion free. Let us then set $y_h = \nabla_{h-1} \circ \cdots \circ \nabla_0(y)$. Since we are assuming for simplicity that $\log_p^0(\chi(\gamma_n)) = 1$, the cocycle $h^1_{F_n,V}(y_h)$ is defined by:

$$h^1_{F_n,V}(y_h)(\sigma) = \frac{\sigma - 1}{\gamma_n - 1} y_h - (\sigma - 1) b_{n,h}$$

where $b_{n,h}$ is a solution of the equation $(\gamma_n - 1)(\varphi - 1) b_{n,h} = (\varphi - 1) y_h$. In lemma II.2 above, we proved that:

$$\nabla_{i-1} \circ \cdots \circ \nabla_1 \circ \frac{\nabla_0}{\gamma_n - 1} (\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0} \subset \left(\frac{t}{\varphi^n(\pi)}\right)^i (\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0}.$$

It is then clear that if one sets

$$z_{n,h} = \nabla_{h-1} \circ \cdots \circ \frac{\nabla_0}{\gamma_n - 1}(\varphi - 1) y,$$

then

$$\begin{aligned} z_{n,h} &\in \left(\frac{t}{\varphi^n(\pi)}\right)^h (\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0} \otimes_F \mathbf{D}_{\mathrm{cris}}(V) \\ &\subset \varphi^n(\pi^{-h}) \mathbf{D}^+_{\mathrm{rig}}(V)^{\psi=0} \\ &\subset \mathbf{D}^\dagger_{\mathrm{rig}}(V)^{\psi=0}. \end{aligned}$$

Recall that $q = \varphi(\pi)/\pi$. By lemma II.4 (which will be stated and proved below), there exists an element $b_{n,h} \in \varphi^{n-1}(\pi^{-h}) \widetilde{\mathbf{B}}^+_{\mathrm{rig}} \otimes_{\mathbf{Q}_p} V$ such that

$$(\varphi - \varphi^{n-1}(q^h))(\varphi^{n-1}(\pi^h) b_{n,h}) = \varphi^n(\pi^h) z_{n,h},$$

so that $(1-\varphi)b_{n,h} = z_{n,h}$ with $b_{n,h} \in \varphi^{n-1}(\pi^{-h})\widetilde{\mathbf{B}}^+_{\mathrm{rig}} \otimes_{\mathbf{Q}_p} V$.

If we set

$$w_{n,h} = \nabla_{h-1} \circ \cdots \circ \frac{\nabla_0}{\gamma_n - 1} y,$$

then $w_{n,h}$ and $b_{n,h} \in \mathbf{B}_{\max} \otimes_{\mathbf{Q}_p} V$ and the cocycle $h^1_{F_n,V}(y_h)$ is then given by the formula $h^1_{F_n,V}(y_h)(\sigma) = (\sigma - 1)(w_{n,h} - b_{n,h})$. Now $(\varphi - 1)b_{n,h} = z_{n,h}$ and $(\varphi - 1)w_{n,h} = z_{n,h}$ as well, so that $w_{n,h} - b_{n,h} \in \mathbf{B}_{\max}^{\varphi=1} \otimes_{\mathbf{Q}_p} V$.

We can also write $h^1_{F_n,V}(y_h)(\sigma) = (\sigma - 1)(\varphi^{-n}(w_{n,h}) - \varphi^{-n}(b_{n,h}))$. Since we know that $b_{n,h} \in \varphi^{n-1}(\pi^{-h})\mathbf{B}^+_{\max} \otimes_{\mathbf{Q}_p} V$, we have $\varphi^{-n}(b_{n,h}) \in \mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V$.

The definition of the Bloch-Kato exponential gives rise to the following construction: if $x \in \mathbf{D}_{\mathrm{dR}}(V)$ and $\widetilde{x} \in \mathbf{B}_{\max}^{\varphi=1} \otimes_{\mathbf{Q}_p} V$ is such that $x - \widetilde{x} \in \mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V$ then $\exp_{K,V}(x)$ is the class of the cocyle $g \mapsto g(\widetilde{x}) - \widetilde{x}$.

The theorem will therefore follow from the fact that:

$$\varphi^{-n}(w_{n,h}) - (-1)^{h-1}(h-1)! p^{-n} \partial_V(\varphi^{-n}(y)) \in \mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V,$$

since we already know that $\varphi^{-n}(b_{n,h}) \in \mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V$.

In order to show this, first notice that

$$\varphi^{-n}(y) - \partial_V(\varphi^{-n}(y)) \in tF_n[[t]] \otimes_F \mathbf{D}_{\mathrm{cris}}(V).$$

We can therefore write

$$\frac{\nabla_0}{\gamma_n - 1}\varphi^{-n}(y) = p^{-n}\partial_V(\varphi^{-n}(y)) + tz_1$$

and a simple recurrence shows that

$$\nabla_{i-1} \circ \cdots \circ \frac{\nabla_0}{1 - \gamma_n}\varphi^{-n}(y) = (-1)^{i-1}(i-1)! p^{-n}\partial_V(\varphi^{-n}(y)) + t^i z_i,$$

with $z_i \in F_n[[t]] \otimes_F \mathbf{D}_{\mathrm{cris}}(V)$. By taking $i = h$, we see that

$$\varphi^{-n}(w_{n,h}) - (-1)^{h-1}(h-1)! p^{-n}\partial_V(\varphi^{-n}(y)) \in \mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V,$$

since we chose h such that $t^h \mathbf{D}_{\mathrm{cris}}(V) \subset \mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V$. □

We will now prove the technical lemma which was used above:

Lemma II.4. *If $\alpha \in \widetilde{\mathbf{B}}^+_{\mathrm{rig}}$, then there exists $\beta \in \widetilde{\mathbf{B}}^+_{\mathrm{rig}}$ such that*

$$(\varphi - \varphi^{n-1}(q^h))\beta = \alpha.$$

Proof. By [Ber02, prop 2.19] applied to the case $r = 0$, the ring $\widetilde{\mathbf{B}}^+$ is dense in $\widetilde{\mathbf{B}}^+_{\mathrm{rig}}$ for the Fréchet topology. Hence, if $\alpha \in \widetilde{\mathbf{B}}^+_{\mathrm{rig}}$, then there exists $\alpha_0 \in \widetilde{\mathbf{B}}^+$ such that $\alpha - \alpha_0 = \varphi^n(\pi^h)\alpha_1$ with $\alpha_1 \in \widetilde{\mathbf{B}}^+_{\mathrm{rig}}$ (one may also show this directly; the point is that when one completes all the localizations are the same).

The map $\varphi - \varphi^{n-1}(q^h) : \widetilde{\mathbf{B}}^+ \to \widetilde{\mathbf{B}}^+$ is surjective, because $\varphi - \varphi^{n-1}(q^h) : \widetilde{\mathbf{A}}^+ \to \widetilde{\mathbf{A}}^+$ is surjective, as can be seen by reducing modulo p and using the fact that $\widetilde{\mathbf{E}}$ is algebraically closed and that $\widetilde{\mathbf{E}}^+$ is its ring of integers.

One can therefore write $\alpha_0 = (\varphi - \varphi^{n-1}(q^h))\beta_0$. Finally by lemma I.7, there exists $\beta_1 \in \widetilde{\mathbf{B}}^+_{\mathrm{rig}}$ such that $\alpha_1 = (\varphi - 1)\beta_1$, so that $\varphi^n(\pi^h)\alpha_1 = (\varphi - \varphi^{n-1}(q^h))(\varphi^{n-1}(\pi^h)\beta_1)$. □

II.3. Bloch-Kato's dual exponential map. In the previous paragraph, we showed how to compute Bloch-Kato's exponential map for V. We will now do the same for the dual exponential map. The starting point is Kato's formula [Ka93, §II.1], which we recall below (it is valid for any field K):

Proposition II.5. *If V is a de Rham representation, then the map from $\mathbf{D}_{\mathrm{dR}}(V)$ to $H^1(K, \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)$ defined by $x \mapsto [g \mapsto \log(\chi(\overline{g}))x]$ is an isomorphism, and the dual exponential map $\exp^*_{V^*(1)} : H^1(K,V) \to \mathbf{D}_{\mathrm{dR}}(V)$ is equal to the composition of the map $H^1(K,V) \to H^1(K, \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)$ with the inverse of this isomorphism.*

Let us point out that the image of $\exp^*_{V^*(1)}$ is included in $\mathrm{Fil}^0\, \mathbf{D}_{\mathrm{dR}}(V)$ and that its kernel is $H^1_g(K,V)$, the subgroup of $H^1(K,V)$ corresponding to classes of de Rham extensions of $\mathbf{Q}_p$ by V.

Let us now return to a crystalline representation V of G_F. We then have the following formula, which is proved in much more generality (i.e. for de Rham representations) in [CC99, IV.2.1]:

Theorem II.6. *If $y \in \mathbf{D}^\dagger_{\mathrm{rig}}(V)^{\psi=1}$ and $y \in \mathbf{D}^+_{\mathrm{rig}}(V)[1/t]$ (so that in particular $y \in (\mathbf{B}^+_{\mathrm{rig},F}[1/t] \otimes_F \mathbf{D}_{\mathrm{cris}}(V))^{\psi=1}$), then*

$$\exp^*_{F_n,V^*(1)}(h^1_{F_n,V}(y)) = \begin{cases} p^{-n}\partial_V(\varphi^{-n}(y)) & \text{if } n \geq 1 \\ (1 - p^{-1}\varphi^{-1})\partial_V(y) & \text{if } n = 0. \end{cases}$$

Note that by theorem A.3, we know that $\mathbf{D}^\dagger(V)^{\psi=1} \subset \mathbf{D}^+_{\mathrm{rig}}(V)[1/t]$.

Proof. Since the following diagram

$$\begin{array}{ccc} H^1(F_{n+1},V) & \xrightarrow{\exp^*_{F_{n+1},V^*(1)}} & F_{n+1} \otimes_F \mathbf{D}_{\mathrm{cris}}(V) \\ {\scriptstyle \mathrm{cor}_{F_{n+1}/F_n}} \downarrow & & \downarrow {\scriptstyle \mathrm{Tr}_{F_{n+1}/F_n}} \\ H^1(F_n,V) & \xrightarrow{\exp^*_{F_n,V^*(1)}} & F_n \otimes_F \mathbf{D}_{\mathrm{cris}}(V) \end{array}$$

is commutative, we only need to prove the theorem when Γ^n_F is torsion free. We then have (bearing in mind that we are assuming that $\log^0_p(\chi(\gamma_n)) = 1$ for simplicity):

$$h^1_{F_n,V}(y)(\sigma) = \frac{\sigma-1}{\gamma_n-1}y - (\sigma-1)b,$$

where $(\gamma_n-1)(\varphi-1)b = (\varphi-1)y$. Recall that $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} = \cup_{r>0}\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$. Since $b \in \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbf{Q}_p} V$, there exists $m \gg 0$ such that $b \in \widetilde{\mathbf{B}}^{\dagger,r_m}_{\mathrm{rig}} \otimes_{\mathbf{Q}_p} V$. Recall also that we have seen in I.3 that the map φ^{-m} embeds $\widetilde{\mathbf{B}}^{\dagger,r_m}_{\mathrm{rig}}$ into $\mathbf{B}^+_{\mathrm{dR}}$. We can then write

$$h^1(y)(\sigma) = \frac{\sigma-1}{\gamma_n-1}\varphi^{-m}(y) - (\sigma-1)\varphi^{-m}(b),$$

and $\varphi^{-m}(b) \in \mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V$. In addition, $\varphi^{-m}(y) \in F_m((t)) \otimes_F \mathbf{D}_{\mathrm{cris}}(V)$ and $\gamma_n - 1$ is invertible on $t^k F_m \otimes_F \mathbf{D}_{\mathrm{cris}}(V)$ for every $k \neq 0$. This shows that the cocycle $h^1_{F_n,V}(y)$ is cohomologous in $H^1(F_n, \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)$ to

$$\sigma \mapsto \frac{\sigma-1}{\gamma_n-1}(\partial_V(\varphi^{-m}(y)))$$

which is itself cohomologous (since $\gamma_n - 1$ is invertible on $F_m^{\mathrm{Tr}_{F_m/F_n}=0}$) to

$$\begin{aligned}\sigma \mapsto \frac{\sigma-1}{\gamma_n-1}&\left(p^{n-m}\operatorname{Tr}_{F_m/F_n}\partial_V(\varphi^{-m}(y))\right)\\ &= \sigma \mapsto p^{-n}\log(\chi(\overline{\sigma}))p^{n-m}\operatorname{Tr}_{F_m/F_n}\partial_V(\varphi^{-m}(y)).\end{aligned}$$

It follows from this and Kato's formula (proposition II.5) that

$$\begin{aligned}\exp^*_{F_n,V^*(1)}(h^1_{F_n,V}(y)) &= p^{-m}\operatorname{Tr}_{F_m/F_n}\partial_V(\varphi^{-m}(y))\\ &= \begin{cases} p^{-n}\partial_V(\varphi^{-n}(y)) & \text{if } n \geq 1\\ (1-p^{-1}\varphi^{-1})\partial_V(y) & \text{if } n = 0.\end{cases}\end{aligned}$$

□

II.4. Iwasawa theory for p-adic representations. In this specific paragraph, V can be taken to be an arbitrary representation of G_K. Recall that the Iwasawa cohomology groups $H^i_{\mathrm{Iw}}(K,V)$ are defined by $H^i_{\mathrm{Iw}}(K,V) = \mathbf{Q}_p \otimes_{\mathbf{Z}_p} H^i_{\mathrm{Iw}}(K,T)$ where T is any G_K-stable lattice of V, and where

$$H^i_{\mathrm{Iw}}(K,T) = \varprojlim_{\mathrm{cor}_{K_{n+1}/K_n}} H^i(K_n,T).$$

Each of the $H^i(K_n,T)$ is a $\mathbf{Z}_p[\Gamma_K/\Gamma_K^n]$-module, and $H^i_{\mathrm{Iw}}(K_n,T)$ is then endowed with the structure of a Λ_K-module where

$$\Lambda_K = \mathbf{Z}_p[[\Gamma_K]] = \mathbf{Z}_p[\Delta_K] \otimes_{\mathbf{Z}_p} \mathbf{Z}_p[[\Gamma_K^1]].$$

The $H^i_{\mathrm{Iw}}(K,V)$ have been studied in detail by Perrin-Riou, who proved the following (see for example [Per94, §3.2]):

Proposition II.7. *If V is a p-adic representation of G_K, then $H^i_{\mathrm{Iw}}(K,V) = 0$ whenever $i \neq 1$, 2. In addition:*

(1) *the torsion sub-module of* $H^1_{\mathrm{Iw}}(K,V)$ *is a* $\mathbf{Q}_p\otimes_{\mathbf{Z}_p}\Lambda_K$*-module isomorphic to* V^{H_K} *and* $H^1_{\mathrm{Iw}}(K,V)/V^{H_K}$ *is a free* $\mathbf{Q}_p\otimes_{\mathbf{Z}_p}\Lambda_K$*-module whose rank is* $[K:\mathbf{Q}_p]d$*;*
(2) $H^2_{\mathrm{Iw}}(K,V)=\left(V^*(1)^{H_K}\right)^*$.

If $y\in\mathbf{D}(T)^{\psi=1}$ (where T is still a lattice of V), then the sequence of $\{h^1_{F_n,V}(y)\}_n$ is compatible for the corestriction maps, and therefore defines an element of $H^1_{\mathrm{Iw}}(K,T)$. The following theorem is due to Fontaine and is proved in [CC99, §II.1]:

THEOREM II.8. *The map* $y\mapsto\varprojlim_n h^1_{K_n,V}(y)$ *defines an isomorphism from* $\mathbf{D}(T)^{\psi=1}$ *to* $H^1_{\mathrm{Iw}}(K,T)$ *and from* $\mathbf{D}(V)^{\psi=1}$ *to* $H^1_{\mathrm{Iw}}(K,V)$*.*

Notice that $V^{H_K}\subset\mathbf{D}(V)^{\psi=1}$, and it is its $\mathbf{Q}_p\otimes_{\mathbf{Z}_p}\Lambda_K$-torsion submodule. In addition, it is shown in [CC99, §II.3] that the modules $\mathbf{D}(V)/(\psi-1)$ and $H^2_{\mathrm{Iw}}(K,V)$ are naturally isomorphic. One can summarize the above results as follows:

COROLLARY II.9. *The complex of* $\mathbf{Q}_p\otimes_{\mathbf{Z}_p}\Lambda_K$*-modules*

$$0\longrightarrow\mathbf{D}(V)\xrightarrow{1-\psi}\mathbf{D}(V)\longrightarrow 0$$

computes the Iwasawa cohomology of V*.*

There is a natural projection map $\mathrm{pr}_{K_n,V}:H^i_{\mathrm{Iw}}(K,V)\to H^i(K_n,V)$ and when $i=1$ it is of course equal to the composition of:

$$H^1_{\mathrm{Iw}}(K,V)\longrightarrow\mathbf{D}(V)^{\psi=1}\xrightarrow{h^1_{K_n,V}}H^1(K_n,V).$$

II.5. PERRIN-RIOU'S EXPONENTIAL MAP. By using the results of the previous paragraphs, we can give a "uniform" formula for the image of an element $y\in(\mathbf{B}^+_{\mathrm{rig},F}\otimes_F\mathbf{D}_{\mathrm{cris}}(V))^{\psi=1}$ in $H^1(F_n,V(j))$ under the composition of the following maps:

$$\left(\mathbf{B}^+_{\mathrm{rig},F}\otimes_F\mathbf{D}_{\mathrm{cris}}(V)\right)^{\psi=1}\xrightarrow{\nabla_{h-1}\circ\cdots\circ\nabla_0}\mathbf{D}^\dagger_{\mathrm{rig}}(V)^{\psi=1}\xrightarrow{\otimes e_j}$$
$$\mathbf{D}^\dagger_{\mathrm{rig}}(V(j))^{\psi=1}\xrightarrow{h^1_{F_n,V(j)}}H^1(F_n,V(j)).$$

Here e_j is a basis of $\mathbf{Q}_p(j)$ such that $e_{j+k}=e_j\otimes e_k$ so that if V is a p-adic representation, then we have compatible isomorphisms of $\mathbf{Q}_p$-vector spaces $V\to V(j)$ given by $v\mapsto v\otimes e_j$.

Theorem II.10. *If* $y \in (\mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathbf{D}_{\mathrm{cris}}(V))^{\psi=1}$*, and* $h \geq 1$ *is such that* $\mathrm{Fil}^{-h}\,\mathbf{D}_{\mathrm{cris}}(V) = \mathbf{D}_{\mathrm{cris}}(V)$*, then for all* j *with* $h+j \geq 1$*, we have:*

$$h^1_{F_n,V(j)}(\nabla_{h-1}\circ\cdots\circ\nabla_0(y)\otimes e_j) = (-1)^{h+j-1}(h+j-1)! \\ \times \begin{cases} \exp_{F_n,V(j)}(p^{-n}\partial_{V(j)}(\varphi^{-n}(\partial^{-j}y\otimes t^{-j}e_j))) & \text{if } n \geq 1 \\ \exp_{F,V(j)}((1-p^{-1}\varphi^{-1})\partial_{V(j)}(\partial^{-j}y\otimes t^{-j}e_j)) & \text{if } n = 0, \end{cases}$$

while if $h+j \leq 0$*, then we have:*

$$\exp^*_{F_n,V^*(1-j)}(h^1_{F_n,V(j)}(\nabla_{h-1}\circ\cdots\circ\nabla_0(y)\otimes e_j)) = \\ \frac{1}{(-h-j)!}\begin{cases} p^{-n}\partial_{V(j)}(\varphi^{-n}(\partial^{-j}y\otimes t^{-j}e_j)) & \text{if } n \geq 1 \\ (1-p^{-1}\varphi^{-1})\partial_{V(j)}(\partial^{-j}y\otimes t^{-j}e_j) & \text{if } n = 0. \end{cases}$$

Proof. If $h+j \geq 1$, then the following diagram is commutative:

$$\begin{array}{ccc} \mathbf{D}^+_{\mathrm{rig}}(V)^{\psi=1} & \xrightarrow{\otimes e_j} & \mathbf{D}^+_{\mathrm{rig}}(V(j))^{\psi=1} \\ \uparrow{\scriptstyle \nabla_{h-1}\circ\cdots\circ\nabla_0} & & \uparrow{\scriptstyle \nabla_{h+j-1}\circ\cdots\circ\nabla_0} \\ \left(\mathbf{B}^+_{\mathrm{rig},F}\otimes_F \mathbf{D}_{\mathrm{cris}}(V)\right)^{\psi=1} & \xrightarrow{\partial^{-j}\otimes t^{-j}e_j} & \left(\mathbf{B}^+_{\mathrm{rig},F}\otimes_F \mathbf{D}_{\mathrm{cris}}(V(j))\right)^{\psi=1}. \end{array}$$

and the theorem is then a straightforward consequence of theorem II.3 applied to $\partial^{-j}y\otimes t^{-j}e_j$, $h+j$ and $V(j)$ (which are the j-th twists of y, h and V).

If on the other hand $h+j \leq 0$, and Γ^n_F is torsion free, then theorem II.6 shows that

$$\exp^*_{F_n,V^*(1-j)}(h^1_{F_n,V(j)}(\nabla_{h-1}\circ\cdots\circ\nabla_0(y)\otimes e_j)) = \\ p^{-n}\partial_{V(j)}(\varphi^{-n}(\nabla_{h-1}\circ\cdots\circ\nabla_0(y)\otimes e_j))$$

in $\mathbf{D}_{\mathrm{cris}}(V(j))$, and a short computation involving Taylor series shows that

$$p^{-n}\partial_{V(j)}(\varphi^{-n}(\nabla_{h-1}\circ\cdots\circ\nabla_0(y)\otimes e_j)) = \\ (-h-j)!^{-1}p^{-n}\partial_{V(j)}(\varphi^{-n}(\partial^{-j}y\otimes t^{-j}e_j)).$$

Finally, to get the case $n=0$, one just needs to use the corresponding statement of theorem II.6 or equivalently to corestrict. □

Remark II.11. The notation ∂^{-j} is somewhat abusive if $j \geq 1$ as ∂ is not injective on $\mathbf{B}^+_{\mathrm{rig},F}$ (it is surjective as can be seen by "integrating" directly a power series) but the reader can check for himself that this leads to no ambiguity in the formulas of theorem II.10 above.

We will now use the above result to give a construction of Perrin-Riou's exponential map. If $f \in \mathbf{B}^+_{\mathrm{rig},F}\otimes_F \mathbf{D}_{\mathrm{cris}}(V)$, we define $\Delta(f)$ to be the image of

$\oplus_{k=0}^h \partial^k(f)(0)$ in $\oplus_{k=0}^h (\mathbf{D}_{\mathrm{cris}}(V)/(1-p^k\varphi))(k)$. There is then an exact sequence of $\mathbf{Q}_p \otimes_{\mathbf{Z}_p} \Lambda_F$-modules (see [Per94, §2.2] for a proof):

$$0 \longrightarrow \oplus_{k=0}^h t^k \mathbf{D}_{\mathrm{cris}}(V)^{\varphi=p^{-k}} \longrightarrow \left(\mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathbf{D}_{\mathrm{cris}}(V)\right)^{\psi=1} \xrightarrow{1-\varphi}$$
$$(\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0} \otimes_F \mathbf{D}_{\mathrm{cris}}(V) \xrightarrow{\Delta} \oplus_{k=0}^h \left(\frac{\mathbf{D}_{\mathrm{cris}}(V)}{1-p^k\varphi}\right)(k) \longrightarrow 0.$$

If $f \in ((\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0} \otimes_F \mathbf{D}_{\mathrm{cris}}(V))^{\Delta=0}$, then by the above exact sequence there exists

$$y \in (\mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathbf{D}_{\mathrm{cris}}(V))^{\psi=1}$$

such that $f = (1-\varphi)y$, and since $\nabla_{h-1} \circ \cdots \circ \nabla_0$ kills $\oplus_{k=0}^{h-1} t^k \mathbf{D}_{\mathrm{cris}}(V)^{\varphi=p^{-k}}$ we see that $\nabla_{h-1} \circ \cdots \circ \nabla_0(y)$ does not depend upon the choice of such a y unless $\mathbf{D}_{\mathrm{cris}}(V)^{\varphi=p^{-h}} \neq 0$.

Definition II.12. Let $h \geq 1$ be an integer such that $\mathrm{Fil}^{-h}\,\mathbf{D}_{\mathrm{cris}}(V) = \mathbf{D}_{\mathrm{cris}}(V)$ and such that $\mathbf{D}_{\mathrm{cris}}(V)^{\varphi=p^{-h}} = 0$. One deduces from the above construction a well-defined map:

$$\Omega_{V,h} : ((\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0} \otimes_F \mathbf{D}_{\mathrm{cris}}(V))^{\Delta=0} \to \mathbf{D}^+_{\mathrm{rig}}(V)^{\psi=1},$$

given by $\Omega_{V,h}(f) = \nabla_{h-1} \circ \cdots \circ \nabla_0(y)$ where $y \in (\mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathbf{D}_{\mathrm{cris}}(V))^{\psi=1}$ is such that $f = (1-\varphi)y$.

If $\mathbf{D}_{\mathrm{cris}}(V)^{\varphi=p^{-h}} \neq 0$ then we get a map:

$$\Omega_{V,h} : ((\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0} \otimes_F \mathbf{D}_{\mathrm{cris}}(V))^{\Delta=0} \to \mathbf{D}^+_{\mathrm{rig}}(V)^{\psi=1}/V^{G_F=\chi^h}.$$

Theorem II.13. *If V is a crystalline representation and $h \geq 1$ is such that we have* $\mathrm{Fil}^{-h}\,\mathbf{D}_{\mathrm{cris}}(V) = \mathbf{D}_{\mathrm{cris}}(V)$, *then the map*

$$\Omega_{V,h} : ((\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0} \otimes_F \mathbf{D}_{\mathrm{cris}}(V))^{\Delta=0} \to \mathbf{D}^+_{\mathrm{rig}}(V)^{\psi=1}/V^{H_F}$$

which takes $f \in ((\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0} \otimes_F \mathbf{D}_{\mathrm{cris}}(V))^{\Delta=0}$ to $\nabla_{h-1} \circ \cdots \circ \nabla_0((1-\varphi)^{-1}f)$ is well-defined and coincides with Perrin-Riou's exponential map.

Proof. The map $\Omega_{V,h}$ is well defined because as we have seen above, the kernel of $1-\varphi$ is killed by $\nabla_{h-1} \circ \cdots \circ \nabla_0$, except for $t^h \mathbf{D}_{\mathrm{cris}}(V)^{\varphi=p^{-h}}$, which is mapped to copies of $\mathbf{Q}_p(h) \subset V^{H_F}$.

The fact that $\Omega_{V,h}$ coincides with Perrin-Riou's exponential map follows directly from theorem II.10 above applied to those j's for which $h+j \geq 1$, and the fact that by Perrin-Riou's [Per94, théorème 3.2.3], the $\Omega_{V,h}$ are uniquely determined by the requirement that they satisfy the following diagram for $h, j \gg 0$

(see remark II.17 about the signs however):

$$\begin{array}{ccc}
\left(\mathcal{H}(\Gamma_F)\otimes_{\mathbf{Q}_p}\mathbf{D}_{\mathrm{cris}}(V(j))\right)^{\Delta=0} & \xrightarrow{\Omega_{V(j),h}} & \mathcal{H}(\Gamma_F)\otimes_{\Lambda_F} H^1_{\mathrm{Iw}}(F,V(j))/V(j)^{H_F} \\
\Xi_{n,V(j)}\downarrow & & \downarrow \mathrm{pr}_{F_n,V(j)} \\
F_n\otimes_F \mathbf{D}_{\mathrm{cris}}(V) & \xrightarrow[\exp_{F_n,V(j)}]{(h+j-1)!\times} & H^1(F_n,V(j)).
\end{array}$$

Here $\Xi_{n,V(j)}(g)=p^{-n}(\varphi\otimes\varphi)^{-n}(f)(\varepsilon^{(n)}-1)$ where f is such that

$$(1-\varphi)f=g(\gamma-1)(1+\pi)\in(\mathbf{B}^+_{\mathrm{rig},F}\otimes_F\mathbf{D}_{\mathrm{cris}}(V))^{\psi=0}$$

and the φ on the left of $\varphi\otimes\varphi$ is the Frobenius on $\mathbf{B}^+_{\mathrm{rig},F}$ while the φ on the right is the Frobenius on $\mathbf{D}_{\mathrm{cris}}(V)$. Our F_n is Perrin-Riou's H_{n-1}.

Note that by theorem II.8, we have an isomorphism $\mathbf{D}(V)^{\psi=1}\simeq H^1_{\mathrm{Iw}}(F,V)$ and therefore we get a map $\mathcal{H}(\Gamma_F)\otimes_{\Lambda_F}H^1_{\mathrm{Iw}}(F,V)\to\mathbf{D}^\dagger_{\mathrm{rig}}(V)^{\psi=1}$. On the other hand, there is a map

$$\mathcal{H}(\Gamma_F)\otimes_{\mathbf{Q}_p}\mathbf{D}_{\mathrm{cris}}(V(j))\to(\mathbf{B}^+_{\mathrm{rig},F}\otimes_F\mathbf{D}_{\mathrm{cris}}(V))^{\psi=0}$$

which sends $\sum f_i(\gamma-1)\otimes d_i$ to $\sum f_i(\gamma-1)(1+\pi)\otimes d_i$. These two maps allow us to compare the diagram above with the formulas given by theorem II.10. □

Remark II.14. By the above remarks, if V is a crystalline representation and $h\geq 1$ is such that we have $\mathrm{Fil}^{-h}\mathbf{D}_{\mathrm{cris}}(V)=\mathbf{D}_{\mathrm{cris}}(V)$ and $\mathbf{Q}_p(h)\not\subset V$, then the map

$$\Omega_{V,h}:((\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0}\otimes_F\mathbf{D}_{\mathrm{cris}}(V))^{\Delta=0}\to\mathbf{D}^+_{\mathrm{rig}}(V)^{\psi=1}$$

which takes $f\in((\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0}\otimes_F\mathbf{D}_{\mathrm{cris}}(V))^{\Delta=0}$ to $\nabla_{h-1}\circ\cdots\circ\nabla_0((1-\varphi)^{-1}f)$ is well-defined, without having to kill the Λ_F-torsion of $H^1_{\mathrm{Iw}}(F,V)$ which improves upon Perrin-Riou's construction.

Remark II.15. It is clear from theorem II.10 that we have:

$$\Omega_{V,h}(x)\otimes e_j=\Omega_{V(j),h+j}(\partial^{-j}x\otimes t^{-j}e_j)\quad\text{and}\quad\nabla_h\circ\Omega_{V,h}(x)=\Omega_{V,h+1}(x),$$

and following Perrin-Riou, one can use these formulas to extend the definition of $\Omega_{V,h}$ to all $h\in\mathbf{Z}$ by tensoring all $\mathcal{H}(\Gamma_F)$-modules with the field of fractions of $\mathcal{H}(\Gamma_F)$.

II.6. The explicit reciprocity formula. In this paragraph, we shall recall Perrin-Riou's explicit reciprocity formula and show that it follows easily from theorem II.10 above.

There is a map $\mathcal{H}(\Gamma_F)\to(\mathbf{B}^+_{\mathrm{rig},\mathbf{Q}_p})^{\psi=0}$ which sends $f(\gamma-1)$ to $f(\gamma-1)(1+\pi)$. This map is a bijection and its inverse is the Mellin transform so that if $g(\pi)\in(\mathbf{B}^+_{\mathrm{rig},\mathbf{Q}_p})^{\psi=0}$, then $g(\pi)=\mathrm{Mel}(g)(1+\pi)$. See [Per00, B.2.8] for a reference, where Perrin-Riou has also extended Mel to $(\mathbf{B}^\dagger_{\mathrm{rig},\mathbf{Q}_p})^{\psi=0}$. If $f,g\in(\mathbf{B}^\dagger_{\mathrm{rig},\mathbf{Q}_p})^{\psi=0}$ then we define $f*g$ by the formula $\mathrm{Mel}(f*g)=\mathrm{Mel}(f)\,\mathrm{Mel}(g)$. Let

$[-1] \in \Gamma_F$ be the element such that $\chi([-1]) = -1$, and let ι be the involution of Γ_F which sends γ to γ^{-1}. The operator ∂^j on $(\mathbf{B}^+_{\mathrm{rig},\mathbf{Q}_p})^{\psi=0}$ corresponds to Tw_j on Γ_F (Tw_j is defined by $\mathrm{Tw}_j(\gamma) = \chi(\gamma)^j\gamma$). For instance, it is a bijection. We will make use of the facts that $\iota\circ\partial^j = \partial^{-j}\circ\iota$ and that $[-1]\circ\partial^j = (-1)^j\partial^j\circ[-1]$.

If V is a crystalline representation, then the natural maps

$$\mathbf{D}_{\mathrm{cris}}(V) \otimes_F \mathbf{D}_{\mathrm{cris}}(V^*(1)) \longrightarrow \mathbf{D}_{\mathrm{cris}}(\mathbf{Q}_p(1)) \xrightarrow{\mathrm{Tr}_{F/\mathbf{Q}_p}} \mathbf{Q}_p$$

allow us to define a perfect pairing $[\cdot,\cdot]_V : \mathbf{D}_{\mathrm{cris}}(V) \times \mathbf{D}_{\mathrm{cris}}(V^*(1))$ which we extend by linearity to

$$[\cdot,\cdot]_V : (\mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathbf{D}_{\mathrm{cris}}(V))^{\psi=0} \times (\mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathbf{D}_{\mathrm{cris}}(V^*(1)))^{\psi=0} \to (\mathbf{B}^+_{\mathrm{rig},\mathbf{Q}_p})^{\psi=0}$$

by the formula $[f(\pi)\otimes d_1, g(\pi)\otimes d_2]_V = (f*g)(\pi)[d_1,d_2]_V$.

We can also define a semi-linear (with respect to ι) pairing

$$\langle\cdot,\cdot\rangle_V : \mathbf{D}^+_{\mathrm{rig}}(V)^{\psi=1} \times \mathbf{D}^+_{\mathrm{rig}}(V^*(1))^{\psi=1} \to (\mathbf{B}^+_{\mathrm{rig},\mathbf{Q}_p})^{\psi=0}$$

by the formula

$$\langle y_1, y_2\rangle_V = \varprojlim_n \sum_{\tau\in\Gamma_F/\Gamma_F^n} \langle \tau^{-1}(h^1_{F_n,V}(y_1)), h^1_{F_n,V^*(1)}(y_2)\rangle_{F_n,V} \cdot \tau(1+\pi)$$

where the pairing $\langle\cdot,\cdot\rangle_{F_n,V}$ is given by the cup product:

$$\langle\cdot,\cdot\rangle_{F_n,V} : H^1(F_n,V)\times H^1(F_n,V^*(1)) \to H^2(F_n,\mathbf{Q}_p(1)) \simeq \mathbf{Q}_p.$$

The pairing $\langle\cdot,\cdot\rangle_V$ satisfies the relation $\langle\gamma_1x_1,\gamma_2x_2\rangle_V = \gamma_1\iota(\gamma_2)\langle x_1,x_2\rangle_V$. Perrin-Riou's explicit reciprocity formula (proved by Colmez [Col98], Benois [Ben00] and Kato-Kurihara-Tsuji [KKT96]) is then:

Theorem II.16. *If $x_1 \in (\mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathbf{D}_{\mathrm{cris}}(V))^{\psi=0}$ and $x_2 \in (\mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathbf{D}_{\mathrm{cris}}(V^*(1)))^{\psi=0}$, then for every h, we have:*

$$(-1)^h\langle\Omega_{V,h}(x_1), [-1]\cdot\Omega_{V^*(1),1-h}(x_2)\rangle_V = -[x_1,\iota(x_2)]_V.$$

Proof. By the theory of p-adic interpolation, it is enough to prove that if $x_i = (1-\varphi)y_i$ with $y_1 \in (\mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathbf{D}_{\mathrm{cris}}(V))^{\psi=1}$ and $y_2 \in (\mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathbf{D}_{\mathrm{cris}}(V^*(1)))^{\psi=1}$ then for all $j \gg 0$:

$$\left(\partial^{-j}(-1)^h\langle\Omega_{V,h}(x_1), [-1]\cdot\Omega_{V^*(1),1-h}(x_2)\rangle_V\right)(0) = -\left(\partial^{-j}[x_1,\iota(x_2)]_V\right)(0).$$

The above formula is equivalent to:

$$\begin{aligned}(1)\quad (-1)^{h+j}\langle & h^1_{F,V(j)}\Omega_{V(j),h+j}(\partial^{-j}x_1\otimes t^{-j}e_j),\\ & h^1_{F,V^*(1-j)}\Omega_{V^*(1-j),1-h-j}(\partial^jx_2\otimes t^je_{-j})\rangle_{F,V(j)}\\ & = -[\partial_{V(j)}(\partial^{-j}x_1\otimes t^{-j}e_j), \partial_{V^*(1-j)}(\partial^jx_2\otimes t^je_{-j})]_{V(j)}.\end{aligned}$$

By combining theorems II.10 and II.13 with remark II.15 we see that for $j \gg 0$:

$$\begin{aligned}&h^1_{F,V(j)}\Omega_{V(j),h+j}(\partial^{-j}x_1 \otimes t^{-j}e_j)\\ &\quad = (-1)^{h+j-1}\exp_{F,V(j)}((h+j-1)!(1-p^{-1}\varphi^{-1})\partial_{V(j)}(\partial^{-j}y_1 \otimes t^{-j}e_j)),\end{aligned}$$

and that

$$\begin{aligned}&h^1_{F,V^*(1-j)}\Omega_{V^*(1-j),1-h-j}(\partial^{j}x_2 \otimes t^{j}e_{-j})\\ &\quad = (\exp^*_{F,V^*(1-j)})^{-1}(h+j-1)!^{-1}((1-p^{-1}\varphi^{-1})\partial_{V^*(1-j)}(\partial^{j}y_2 \otimes t^{j}e_{-j})).\end{aligned}$$

Using the fact that by definition, if $x \in \mathbf{D}_{\mathrm{cris}}(V(j))$ and $y \in H^1(F, V(j))$ then

$$[x, \exp^*_{F,V^*(1-j)} y]_{V(j)} = \langle \exp_{F,V(j)} x, y\rangle_{F,V(j)},$$

we see that:

$$\begin{aligned}(2)\quad &\langle h^1_{F,V(j)}\Omega_{V(j),h+j}(\partial^{-j}x_1 \otimes t^{-j}e_j),\\ &\qquad h^1_{F,V^*(1-j)}\Omega_{V^*(1-j),1-h-j}(\partial^{j}x_2 \otimes t^{j}e_{-j})\rangle_{F,V(j)}\\ &\qquad = (-1)^{h+j-1}[(1-p^{-1}\varphi^{-1})\partial_{V(j)}(\partial^{-j}y_1 \otimes t^{-j}e_j),\\ &\qquad\qquad (1-p^{-1}\varphi^{-1})\partial_{V^*(1-j)}(\partial^{j}y_2 \otimes t^{j}e_{-j})]_{V(j)}.\end{aligned}$$

It is easy to see that under $[\cdot,\cdot]$, the adjoint of $(1-p^{-1}\varphi^{-1})$ is $1-\varphi$, and that if $x_i = (1-\varphi)y_i$, then

$$\begin{aligned}\partial_{V(j)}(\partial^{-j}x_1 \otimes t^{-j}e_j) &= (1-\varphi)\partial_{V(j)}(\partial^{-j}y_1 \otimes t^{-j}e_j),\\ \partial_{V^*(1-j)}(\partial^{j}x_2 \otimes t^{j}e_{-j}) &= (1-\varphi)\partial_{V^*(1-j)}(\partial^{j}y_2 \otimes t^{j}e_{-j}),\end{aligned}$$

so that (2) implies (1), and this proves the theorem. □

Note that as I. Fesenko pointed out it is better to call the above statement an "explicit reciprocity formula" rather than an "explicit reciprocity law" as the latter terminology is reserved for statements of a more global nature.

Remark II.17. One should be careful with all the signs involved in those formulas. Perrin-Riou has changed the definition of the ℓ_i operators from [Per94] to [Per99] (the new ℓ_i is minus the old ℓ_i). The reciprocity formula which is stated in [Per99, 4.2.3] does not seem (to me) to have the correct sign. On the other hand, the formulas of [Ben00, Col98] do seem to give the correct signs, but one should be careful that [Col98, IX.4.5] uses a different definition for one of the pairings, and that the signs in [CC99, IV.3.1] and [Col98, VII.1.1] disagree. Our definitions of $\Omega_{V,h}$ and of the pairing agree with Perrin-Riou's ones (as they are given in [Per99]).

Appendix A. The structure of $\mathbf{D}(T)^{\psi=1}$

The goal of this paragraph is to prove a theorem which says that for a crystalline representation V, $\mathbf{D}(V)^{\psi=1}$ is quite "small". See theorem A.3 for a precise statement.

Let V be a crystalline representation of G_F and let T denote a G_F-stable lattice of V. The following proposition, which improves slightly upon the results of N. Wach [Wa96], is proved in detail in [Ber03, §II.1]:

Proposition A.1. *If T is a lattice in a positive crystalline representation V, then there exists a unique sub $\mathbf{A}_F^+$-module $\mathbf{N}(T)$ of $\mathbf{D}^+(T)$, which satisfies the following conditions:*

1. *$\mathbf{N}(T)$ is an $\mathbf{A}_F^+$-module free of rank $d = \dim_{\mathbf{Q}_p}(V)$;*
2. *the action of Γ_F preserves $\mathbf{N}(T)$ and is trivial on $\mathbf{N}(T)/\pi\mathbf{N}(T)$;*
3. *there exists an integer $r \geq 0$ such that $\pi^r \mathbf{D}^+(T) \subset \mathbf{N}(T)$.*

Furthermore, $\mathbf{N}(T)$ is stable by φ, and the $\mathbf{B}_F^+$-module $\mathbf{N}(V) = \mathbf{B}_F^+ \otimes_{\mathbf{A}_F^+} \mathbf{N}(T)$ is the unique sub-$\mathbf{B}_F^+$-module of $\mathbf{D}^+(V)$ satisfying the corresponding conditions.

The $\mathbf{A}_F^+$-module $\mathbf{N}(T)$ is called the *Wach module* associated to T.

Notice that $\mathbf{N}(T(-1)) = \pi\mathbf{N}(T) \otimes e_{-1}$. When V is no longer positive, we can therefore define $\mathbf{N}(T)$ as $\pi^{-h}\mathbf{N}(T(-h)) \otimes e_h$, for h large enough so that $V(-h)$ is positive. Using the results of [Ber03, §III.4], one can show that:

Proposition A.2. *If T is a lattice in a crystalline representation V of G_F, whose Hodge-Tate weights are in $[a;b]$, then $\mathbf{N}(T)$ is the unique sub-$\mathbf{A}_F^+$-module of $\mathbf{D}^+(T)[1/\pi]$ which is free of rank d, stable by Γ_F with the action of Γ_F being trivial on $\mathbf{N}(T)/\pi\mathbf{N}(T)$, and such that $\mathbf{N}(T)[1/\pi] = \mathbf{D}^+(T)[1/\pi]$.*

Finally, we have $\varphi(\pi^b\mathbf{N}(T)) \subset \pi^b\mathbf{N}(T)$ and $\pi^b\mathbf{N}(T)/\varphi^*(\pi^b\mathbf{N}(T))$ is killed by q^{b-a}. The construction $T \mapsto \mathbf{N}(T)$ gives a bijection between Wach modules over $\mathbf{A}_F^+$ which are lattices in $\mathbf{N}(V)$ and Galois lattices T in V.

We shall now show that $\mathbf{D}(V)^{\psi=1}$ is not very far from being included in $\mathbf{N}(V)$. Indeed:

Theorem A.3. *If V is a crystalline representation of G_F, whose Hodge-Tate weights are in $[a;b]$, then $\mathbf{D}(V)^{\psi=1} \subset \pi^{a-1}\mathbf{N}(V)$.*

If in addition V has no quotient isomorphic to $\mathbf{Q}_p(a)$, then actually $\mathbf{D}(V)^{\psi=1} \subset \pi^a\mathbf{N}(V)$.

Before we prove the above statement, we will need a few results concerning the action of ψ on $\mathbf{D}(T)$. In lemmas A.5 through A.7, we will assume that the Hodge-Tate weights of V are ≥ 0. In particular, $\mathbf{N}(T) \subset \varphi^*\mathbf{N}(T)$ so that $\psi(\mathbf{N}(T)) \subset \mathbf{N}(T)$.

Lemma A.4. *If $m \geq 1$, then there exists a polynomial $Q_m(X) \in \mathbf{Z}_p[X]$ such that $\psi(\pi^{-m}) = \pi^{-m}(p^{m-1} + \pi Q_m(\pi))$.*

Proof. By the definition of ψ, it is enough to show that if $m \geq 1$, there exists a polynomial $Q_m(X) \in \mathbf{Z}[X]$ such that

$$\frac{1}{p}\sum_{\eta^p=1}\frac{1}{(\eta(1+X)-1)^m} = \frac{p^{m-1} + ((1+X)^p-1)Q_m((1+X)^p-1)}{((1+X)^p-1)^m},$$

which is left as an exercise for the reader (or his students). □

Lemma A.5. *If $k \geq 1$, then $\psi(p\mathbf{D}(T) + \pi^{-(k+1)}\mathbf{N}(T)) \subset p\mathbf{D}(T) + \pi^{-k}\mathbf{N}(T)$. In addition, $\psi(p\mathbf{D}(T) + \pi^{-1}\mathbf{N}(T)) \subset p\mathbf{D}(T) + \pi^{-1}\mathbf{N}(T)$.*

Proof. If $x \in \mathbf{N}(T)$, then one can write $x = \sum \lambda_i \varphi(x_i)$ with $\lambda_i \in \mathbf{A}_F^+$ and $x_i \in \mathbf{N}(T)$, so that $\psi(\pi^{-(k+1)}x) = \sum \psi(\pi^{-(k+1)}\lambda_i)x_i$. By the preceding lemma, $\psi(\pi^{-(k+1)}\lambda_i) \in p\mathbf{A}_F + \pi^{-k}\mathbf{A}_F^+$ whenever $k \geq 1$. The lemma follows easily, and the second claim is proved in the same way. □

Lemma A.6. *If $k \geq 1$ and $x \in \mathbf{D}(T)$ has the property that $\psi(x) - x \in p\mathbf{D}(T) + \pi^{-k}\mathbf{N}(T)$, then $x \in p\mathbf{D}(T) + \pi^{-k}\mathbf{N}(T)$.*

Proof. Let ℓ be the smallest integer ≥ 0 such that $x \in p\mathbf{D}(T) + \pi^{-\ell}\mathbf{N}(T)$. If $\ell \leq k$, then we are done and otherwise lemma A.5 shows that

$$\psi(x) \in p\mathbf{D}(T) + \pi^{-(\ell-1)}\mathbf{N}(T),$$

so that $\psi(x) - x$ would be in $p\mathbf{D}(T) + \pi^{-\ell}\mathbf{N}(T)$ but not $p\mathbf{D}(T) + \pi^{-(\ell-1)}\mathbf{N}(T)$, a contradiction if $\ell > k$. □

Lemma A.7. *We have $\mathbf{D}(T)^{\psi=1} \subset \pi^{-1}\mathbf{N}(T)$.*

Proof. We shall prove by induction that $\mathbf{D}(T)^{\psi=1} \subset p^k\mathbf{D}(T) + \pi^{-1}\mathbf{N}(T)$ for $k \geq 1$. Let us start with the case $k = 1$. If $x \in \mathbf{D}(T)^{\psi=1}$, then there exists some $j \geq 1$ such that $x \in p\mathbf{D}(T) + \pi^{-j}\mathbf{N}(T)$. If $j = 1$ we are done and otherwise the fact that $\psi(x) = x$ combined with lemma A.5 shows that j can be decreased by 1. This proves our claim for $k = 1$.

We will now assume our claim to be true for k and prove it for $k + 1$. If $x \in \mathbf{D}(T)^{\psi=1}$, we can therefore write $x = p^k y + n$ where $y \in \mathbf{D}(T)$ and $n \in \pi^{-1}\mathbf{N}(T)$. Since $\psi(x) = x$, we have $\psi(n) - n = p^k(\psi(y) - y)$ so that $\psi(y) - y \in \pi^{-1}\mathbf{N}(T)$ (this is because $p^k\mathbf{D}(T) \cap \mathbf{N}(T) = p^k\mathbf{N}(T)$). By lemma A.6, this implies that $y \in p\mathbf{D}(T) + \pi^{-1}\mathbf{N}(T)$, so that we can write $x = p^k(py' + n') + n = p^{k+1}y' + (p^k n' + n)$, and this proves our claim.

Finally, it is clear that our claim implies the lemma: if one can write $x = p^k y_k + n_k$, then the n_k will converge for the p-adic topology to a $n \in \pi^{-1}\mathbf{N}(T)$ such that $x = n$. □

Proof of theorem A.3. Clearly, it is enough to show that if T is a G_F-stable lattice of V, then $\mathbf{D}(T)^{\psi=1} \subset \pi^{a-1}\mathbf{N}(T)$. It is also clear that we can twist V as we wish, and we shall now assume that the Hodge-Tate weights of V are in $[0;h]$. In this case, the theorem says that $\mathbf{D}(T)^{\psi=1} \subset \pi^{-1}\mathbf{N}(T)$, which is the content of lemma A.7 above.

Let us now prove that if a positive V has no quotient isomorphic to $\mathbf{Q}_p$, then actually $\mathbf{D}(T)^{\psi=1} \subset \mathbf{N}(T)$. Recall that $\mathbf{N}(T) \subset \varphi^*(\mathbf{N}(T))$, since the Hodge-Tate weights of V are ≥ 0, so that if $e_1, \cdots, e_d$ is a basis of $\mathbf{N}(T)$, then there exists $q_{ij} \in \mathbf{A}_F^+$ such that $e_i = \sum_{j=1}^d q_{ij}\varphi(e_j)$. If $\psi(\sum_{i=1}^d \alpha_i e_i) = \sum_{i=1}^d \alpha_i e_i$, with $\alpha_i \in \pi^{-1}\mathbf{A}_F^+$, then this translates into $\psi(\sum_{i=1}^d \alpha_i q_{ij}) = \alpha_j$ for $1 \leq j \leq d$.

Let $\alpha_{i,n}$ be the coefficient of π^n in α_i, and likewise for $q_{ij,n}$. Since $\psi(1/\pi) = 1/\pi$, the equations $\psi(\sum_{i=1}^d \alpha_i q_{ij}) = \alpha_j$ then tell us that for $1 \leq j \leq d$:

$$\sum_{i=1}^{d} \alpha_{i,-1} q_{ij,0} = \varphi(\alpha_{j,-1}).$$

Since $\mathbf{N}(V)/\pi\mathbf{N}(V) \simeq \mathbf{D}_{\mathrm{cris}}(V)$ as φ-modules by [Ber03, §III.4], the above equations say that 1 is an eigenvalue of φ on $\mathbf{D}_{\mathrm{cris}}(V)$. It is easy to see that if a representation has positive Hodge-Tate weights and $\mathbf{D}_{\mathrm{cris}}(V)^{\varphi=1} \neq 0$, then V has a quotient isomorphic to $\mathbf{Q}_p$. □

Remark A.8. It is proved in [Ber03, III.2] that $\mathbf{D}_{\mathrm{cris}}(V) = (\mathbf{B}^+_{\mathrm{rig},F} \otimes_{\mathbf{B}^+_F} \mathbf{N}(V))^{G_F}$ and that if $\mathrm{Fil}^{-h}\,\mathbf{D}_{\mathrm{cris}}(V) = \mathbf{D}_{\mathrm{cris}}(V)$, then

$$\left(\frac{t}{\pi}\right)^h \mathbf{B}^+_{\mathrm{rig},F} \otimes_{\mathbf{B}^+_F} \mathbf{D}_{\mathrm{cris}}(V) \subset \mathbf{B}^+_{\mathrm{rig},F} \otimes_{\mathbf{B}^+_F} \mathbf{N}(V).$$

In all the above constructions, one could therefore replace $\mathbf{D}^+_{\mathrm{rig}}(V)$ by $\mathbf{B}^+_{\mathrm{rig},F} \otimes_{\mathbf{B}^+_F} \pi^h\mathbf{N}(V)$. For example, the image of the map $\Omega_{V,h}$ is included in $(\pi^h\mathbf{B}^+_{\mathrm{rig},F} \otimes_{\mathbf{B}^+_F} \mathbf{N}(V))^{\psi=1}$ so that we really get a map:

$$\Omega_{V,h} : ((\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0} \otimes_F \mathbf{D}_{\mathrm{cris}}(V))^{\Delta=0} \to (\pi^h\mathbf{B}^+_{\mathrm{rig},F} \otimes_{\mathbf{B}^+_F} \mathbf{N}(V))^{\psi=1}.$$

This slight refinement may be useful in order to prove Perrin-Riou's $\delta_{\mathbf{Z}_p}$ conjecture.

APPENDIX B. LIST OF NOTATIONS

Here is a list of the main notations in the order in which they occur:

I: p, k, $W(k)$, F, K, G_K, μ_{p^n}, $\varepsilon^{(n)}$, K_n, K_∞, H_K, Γ_K, χ, V, T.

I.1: $\mathbf{C}_p$, $\widetilde{\mathbf{E}}$, $\widetilde{\mathbf{E}}^+$, $v_{\mathbf{E}}$, $\widetilde{\mathbf{A}}^+$, $\widetilde{\mathbf{B}}^+$, θ, φ, ε, π, π_1, ω, q, $\mathbf{B}_{\mathrm{dR}}^+$, $\mathbf{B}_{\mathrm{dR}}$, $\mathbf{D}_{\mathrm{dR}}(V)$, $\mathbf{B}_{\max}^+$, $\mathbf{B}_{\max}$, $\mathbf{B}_{\mathrm{cris}}$, $\widetilde{\mathbf{B}}_{\mathrm{rig}}^+$, $\mathbf{D}_{\mathrm{cris}}(V)$, h.

I.2: $\widetilde{\mathbf{A}}$, $\widetilde{\mathbf{B}}$, $\mathbf{A}_F$, $\mathbf{B}$, $\mathbf{B}_F$, $\mathbf{A}$, $\mathbf{B}^+$, $\mathbf{A}^+$, $\mathbf{A}_K$, $\mathbf{B}_K$, $\mathbf{A}_F^+$, $\mathbf{B}_F^+$, $\mathbf{D}(V)$, ψ, $\mathbf{D}^+(V)$, $\widetilde{\mathbf{B}}^{\dagger,r}$, $\mathbf{B}^{\dagger,r}$, $\widetilde{\mathbf{B}}^\dagger$, $\mathbf{B}^\dagger$, $\mathbf{D}^\dagger(V)$, $\mathbf{D}^{\dagger,r}(V)$, e_K, F', π_K.

I.3: $\mathbf{B}_{\mathrm{rig},K}^{\dagger,r}$, $\mathbf{D}_{\mathrm{rig}}^{\dagger,r}(V)$, $\widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger$, $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r}$, r_n, φ^{-n}, $\mathbf{B}_{\mathrm{rig},F}^+$, $\mathbf{D}_{\mathrm{rig}}^+(V)$.

I.4: $h_{K,V}^1$, w_k, Δ_K, Γ_K^n, $\log_p^0$, γ, M_Γ.

II: $\exp_{K,V}$, $\exp_{K,V^*(1)}^*$.

II.1: ∂_V, Λ_F, $\mathcal{H}(\Gamma_F)$, ∇_i, $\nabla_0/(\gamma_n-1)$, ∂.

II.4: T, $H_{\mathrm{Iw}}^i(K,V)$, $\mathrm{pr}_{K,V}$.

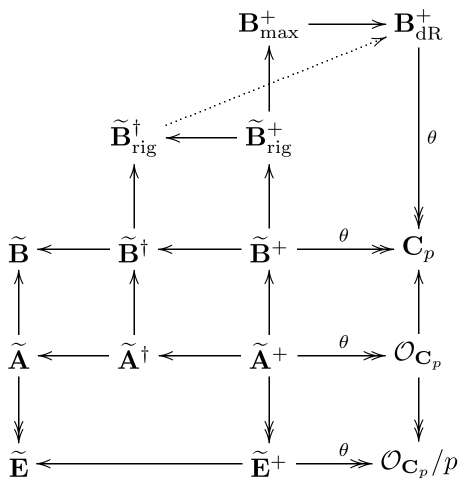
II.5: e_j, Δ, $\Omega_{V,h}$, $\Xi_{n,V}$.

II.6: Mel, Tw_j, $[-1]$, ι, $[\cdot,\cdot]_V$, $\langle\cdot,\cdot\rangle_V$, ℓ_i.

A: T, $\mathbf{N}(V)$.

Appendix C. Diagram of the rings of periods

The following diagram summarizes the relationships between the different rings of periods. The arrows ending with $\longrightarrow\!\!\!\!\!\rightarrow$ are surjective, the dotted arrow $\dashrightarrow$ is an inductive limit of maps defined on subrings ($\varphi^{-n}: \widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r_n} \to \mathbf{B}_{\mathrm{dR}}^+$), and all the other ones are injective.



All the rings with tildes also have versions without a tilde: one goes from the latter to the former by making Frobenius invertible and completing.

The three rings in the leftmost column (at least their tilde-free versions) are related to the theory of (φ, Γ)-modules. The two rings on the top line are related to p-adic Hodge theory. To go from one theory to the other, one goes from one place to the other through all the intermediate rings but as the reader will notice, one has to go "upstream".

Let us finally review the different rings of power series which occur in this article; let $C[r;1[$ be the annulus$\{z \in \mathbf{C}_p,\ p^{-1/r} \leq |z|_p < 1\}$. We then have:

$\mathbf{A}_F^+$	$\mathcal{O}_F[[\pi]]$	$\mathbf{A}_F$	$\mathcal{O}_F\widehat{[[\pi]][\pi^{-1}]}$
$\mathbf{B}_F^+$	$F \otimes_{\mathcal{O}_F} \mathcal{O}_F[[\pi]]$	$\mathbf{B}_F$	$F \otimes_{\mathcal{O}_F} \mathcal{O}_F\widehat{[[\pi]][\pi^{-1}]}$

$\mathbf{A}_F^{\dagger,r}$	Laurent series $f(\pi)$, convergent on $C[r;1[$, and bounded by 1
$\mathbf{B}_F^{\dagger,r}$	Laurent series $f(\pi)$, convergent on $C[r;1[$, and bounded

$\mathbf{B}_{\mathrm{rig},F}^{\dagger,r}$	Laurent series $f(\pi)$, convergent on $C[r;1[$
$\mathbf{B}_{\mathrm{rig},F}^{+}$	$f(\pi) \in F[[\pi]],\ f(\pi)$ converges on the open unit disk $D[0;1[$

References

[Ben00] Benois D.: *On Iwasawa theory of crystalline representations.* Duke Math. J. 104 (2000) 211–267.

[Ber02] Berger L.: *Représentations p-adiques et équations différentielles.* Invent. Math. 148 (2002), 219–284.

[Ber03] Berger L.: *Limites de représentations cristallines.* To appear.

[BK91] Bloch S., Kato K.: *L-functions and Tamagawa numbers of motives.* The Grothendieck Festschrift, Vol. I, 333–400, Progr. Math. 86, Birkhäuser Boston, Boston, MA 1990.

[CC98] Cherbonnier F., Colmez P.: *Représentations p-adiques surconvergentes.* Invent. Math. 133 (1998), 581–611.

[CC99] Cherbonnier F., Colmez P.: *Théorie d'Iwasawa des représentations p-adiques d'un corps local.* J. Amer. Math. Soc. 12 (1999), 241–268.

[Col98] Colmez P.: *Théorie d'Iwasawa des représentations de de Rham d'un corps local.* Ann. of Math. 148 (1998), 485–571.

[Col99] Colmez P.: *Représentations cristallines et représentations de hauteur finie.* J. Reine Angew. Math. 514 (1999), 119–143.

[Col00] Colmez P.: *Fonctions L p-adiques.* Séminaire Bourbaki, 1998/99, Astérisque 266 (2000) Exp. 851.

[Col01] Colmez P.: *Les conjectures de monodromie p-adiques.* Séminaire Bourbaki, 2001/02, Astérisque, Exp. 897.

[Fo88a] Fontaine J-M.: *Le corps des périodes p-adiques.* Périodes p-adiques (Bures-sur-Yvette, 1988), Astérisque 223 (1994) 59–111.

[Fo88b] Fontaine J-M.: *Représentations p-adiques semi-stables.* Périodes p-adiques, (Bures-sur-Yvette, 1988), Astérisque 223 (1994) 113–184.
[Fo91] Fontaine J-M.: *Représentations p-adiques des corps locaux I.* The Grothendieck Festschrift, Vol. II, 249–309, Progr. Math. 87, Birkhäuser Boston, Boston, MA 1990.
[Ka93] Kato K.: *Lectures on the approach to Iwasawa theory for Hasse-Weil L-functions via* $\mathbf{B}_{\mathrm{dR}}$. Arithmetic algebraic geometry, Lecture Notes in Mathematics 1553, Springer-Verlag, Berlin, 1993, 50–63.
[KKT96] Kato K., Kurihara M., Tsuji T.: *Local Iwasawa theory of Perrin-Riou and syntomic complexes.* Preprint, 1996.
[Per94] Perrin-Riou B.: *Théorie d'Iwasawa des représentations p-adiques sur un corps local.* Invent. Math. 115 (1994) 81–161.
[Per95] Perrin-Riou B.: *Fonctions L p-adiques des représentations p-adiques.* Astérisque No. 229 (1995), 198 pp.
[Per99] Perrin-Riou B.: *Théorie d'Iwasawa et loi explicite de réciprocité.* Doc. Math. 4 (1999), 219–273 (electronic).
[Per00] Perrin-Riou B.: *Théorie d'Iwasawa des représentations p-adiques semi-stables.* Mém. Soc. Math. Fr. (N.S.) No. 84 (2001), vi+111 pp.
[Wa96] Wach N.: *Représentations p-adiques potentiellement cristallines.* Bull. Soc. Math. France 124 (1996), 375–400.

Laurent Berger
Harvard University
Department of Mathematics
One Oxford St
Cambridge, MA 02138 USA
laurent@math.harvard.edu

The Additive Dilogarithm

To Kazuya Kato, with fondness and profound respect,

on the occasion of his fiftieth birthday

Spencer Bloch and Hélène Esnault

Received: October 10, 2002
Revised: March 12, 2003

Abstract. A notion of additive dilogarithm for a field k is introduced, based on the K-theory and higher Chow groups of the affine line relative to $2(0)$. Analogues of the K_2-regulator, the polylogarithm Lie algebra, and the ℓ-adic realization of the dilogarithm motive are discussed. The higher Chow groups of 0-cycles in this theory are identified with the Kähler differential forms Ω^*_k. It is hoped that these results will serve as a guide in developing a theory of contravariant motivic cohomology with modulus, modelled on the generalized Jacobians of Rosenlicht and Serre.

2000 Mathematics Subject Classification: Primary 11G55

1. Introduction

In [18] Déf. (5.1.1), Laumon introduces the category of generalized 1-motives over a field k of characteristic 0. Objects in this category are arrows $f : \mathcal{G} \to G$ where $\mathcal{G}$ and G are commutative algebraic groups, with $\mathcal{G}$ assumed formal, torsion free, and G assumed connected. These, of course, generalize the more restricted category of 1-motives introduced by Deligne [8] as a model for the category of mixed Hodge structures of types $\{(0,0),(0,-1),(-1,0),(-1,-1)\}$. Of particular interest for us are motives of the form $\mathbb{Z} \to \mathbb{V}$ which arise in the study of algebraic cycles relative to a "modulus". Here $\mathbb{V} \cong \mathbb{G}_a^n$ is a vector group. The simplest example is

$$\mathrm{Pic}(\mathbb{A}^1, 2\{0\}) \cong \mathbb{G}_a. \tag{1.1}$$

which may be viewed as a degenerate version of the identification $\mathrm{Pic}(\mathbb{A}^1, \{0,\infty\}) \cong \mathbb{G}_m$ obtained by associating to a unit the corresponding Kummer extension of $\mathbb{Z}$ by $\mathbb{Z}(1)$. (For more details, cf. [6], [5], [13], [14], [19].) We expect such generalized motives to play an important role in

the (as yet undefined) contravariant theory of motivic sheaves and motivic cohomology for (possibly singular) varieties.
The polylog mixed motives of Beilinson and Deligne are generalizations to higher weight of Kummer extensions, so it seems natural to look for degenerate, or $\mathbb{G}_a$ versions of these. The purpose of this article is to begin to study an additive version of the dilogarithm motive. We assume throughout that k is a field which for the most part will be taken to be of characteristic 0. Though our results are limited to the dilogarithm, the basic result from cyclic homology

$$Gr^n_\gamma \ker\Big(K_{2n-1}(k[t]/(t^2)) \to K_{2n-1}(k)\Big) \cong k \tag{1.2}$$

suggests that higher polylogarithms exist as well.
In the first part of the article we introduce an additive "Bloch group" $TB_2(k)$ for an algebraically closed field k of characteristic $\neq 2$. In lieu of the 4-term sequence in motivic cohomology associated to the usual Bloch group

$$\begin{aligned} 0 \to H^1_M(\mathrm{Spec}\,(k), \mathbb{Q}(2)) \to B_2(k) \to k^\times \otimes k^\times \otimes \mathbb{Q} \\ \to H^2_M(\mathrm{Spec}\,(k), \mathbb{Q}(2)) \to 0 \end{aligned} \tag{1.3}$$

(with $H^1_M(\mathrm{Spec}\,(k), \mathbb{Q}(2)) \cong K_3(k)_{ind} \otimes \mathbb{Q}$ and $H^2_M(\mathrm{Spec}\,(k), \mathbb{Q}(2)) \cong K_2(k) \otimes \mathbb{Q}$), we find an additive 4-term sequence

$$\begin{aligned} 0 \to TH^1_M(\mathrm{Spec}\,(k), \mathbb{Q}(2)) \to TB_2(k) \to k \otimes k^\times \\ \stackrel{d\log}{\to} TH^2_M(\mathrm{Spec}\,(k), \mathbb{Q}(2)) \to 0 \end{aligned} \tag{1.4}$$

where

$$TH^1_M(\mathrm{Spec}\,(k), \mathbb{Q}(2)) := K_2(\mathbb{A}^1_t, (t^2)) \cong (t^3)/(t^4) \cong k; \tag{1.5}$$

$$TH^2_M(\mathrm{Spec}\,(k), \mathbb{Q}(2)) := K_1(\mathbb{A}^1, (t^2)) \cong \Omega^1_k = \text{absolute Kähler 1-forms};$$

$$d\log(a \otimes b) = a\frac{db}{b}.$$

Our construction should be compared and contrasted with the results of [7]. Cathelineau's group $\beta_2(k)$ is simply the kernel

$$0 \to \beta_2(k) \to k \otimes k^\times \to \Omega^1_k \to 0, \tag{1.6}$$

so there is an exact sequence

$$\begin{array}{ccccccc} 0 \longrightarrow & TH^1_M(\mathrm{Spec}\,(k), \mathbb{Q}(2)) & \longrightarrow & TB_2(k) & \longrightarrow & \beta_2(k) \to 0 \\ & \big\downarrow \cong & & & & \\ & k & & & & \end{array} \tag{1.7}$$

For $a \in k$ we define $\langle a \rangle \in TB_2(k)$ lifting similar elements defined by Cathelineau and satisfying his 4-term infinitesimal version

$$\langle a \rangle - \langle b \rangle + a\langle b/a \rangle + (1-a)\langle (1-b)/(1-a) \rangle = 0; \quad a \neq 0, 1. \tag{1.8}$$

of the classical 5-term dilogarithm relation. Here, the notation $x\langle y \rangle$ refers to an action of $k^\times$ on $TB_2(k)$. Unlike $\beta_2(k)$, this action does *not* extend to a

k-vector space structure on $TB_2(k)$. Thus (1.7) is an exact sequence of $k^\times$-modules, where the kernel and cokernel have k-vector space structures but the middle group does not.

Finally in this section we show the assignment $\langle a\rangle \mapsto a(1-a)$ defines a regulator map $\rho : TB_2(k) \to k$ and the composition

$$TH^1_M(\mathrm{Spec}\,(k), \mathbb{Q}(2)) \hookrightarrow TB_2(k) \xrightarrow{\rho} k \tag{1.9}$$

is an isomorphism.

It seems plausible that $TB_2(k)$ can be interpreted as a Euclidean sissors-congruence group, with $\partial : TB_2(k) \to k \otimes k^\times$ the Dehn invariant and $\rho : TB_2(k) \to k$ the volume. Note the scaling for the $k^\times$-action is appropriate, with $\partial(x\langle y\rangle) = x\partial(\langle y\rangle)$ and $\rho(x\langle y\rangle) = x^3\rho(\langle y\rangle)$. For a careful discussion of Euclidean sissors-congruence and its relation with the dual numbers, the reader is referred to [17] and the references cited there.

In §4 we introduce an extended polylogarithm Lie algebra. The dual co-Lie algebra has generators $\{x\}_n$ and $\langle x\rangle_n$ for $x \in k-\{0,1\}$. The dual of the bracket satisfies $\partial\{x\}_n = \{x\}_{n-1}\cdot\{1-x\}_1$ and $\partial\langle x\rangle_n = \langle x\rangle_{n-1}\cdot\{1-x\}_1 + \langle 1-x\rangle_1\cdot\{x\}_{n-1}$ with $\langle x\rangle_1 = x \in k$. For example, $\partial\langle x\rangle_2 = x\otimes x + (1-x)\otimes(1-x) \in k\otimes k^\times$ is the Cathelineau relation [7]. It seems likely that there exists a representation of this Lie algebra, extending the polylog representation of the sub Lie algebra generated by the $\{x\}_n$, and related to variations of Hodge structure over the dual numbers lifting the polylog Hodge structure.

§5 was inspired by Deligne's interpretation of symbols [9] in terms of line bundles with connections. We indicate how this viewpoint is related to the additive dilogarithm. In characteristic 0, one finds affine bundles with connection, and the regulator map on K_2 linearizes to the evident map $H^0(X,\Omega^1) \to \mathbb{H}^1(X, \mathcal{O} \to \Omega^1)$. In characteristic p, Artin-Schreier yields an exotic flat realization of the additive dilogarithm motive. For simplicity we limit ourselves to calculations mod p. The result is a flat covering T of $\mathbb{A}^1 - \{0,1\}$ which is a torsor under a flat Heisenberg groupscheme $\mathcal{H}_{AS}$. This groupscheme has a natural representation on the abelian groupscheme $\mathbb{V} := \mathbb{Z}/p\mathbb{Z} \oplus \mu_p \oplus \mu_p$. The contraction

$$T \overset{\mathcal{H}_{AS}}{\times} \mathbb{V} \tag{1.10}$$

should, we think, be considered as analogous to the mod ℓ étale sheaf on $\mathbb{A}^1 - \{0,1\}$ with fibre $\mathbb{Z}/\ell\mathbb{Z} \oplus \mu_\ell \oplus \mu_\ell^{\otimes 2}$ associated to the ℓ-adic dilogarithm.

The polylogarithms can be interpreted in terms of algebraic cycles on products of copies of $\mathbb{P}^1 - \{1\}$ ([3], (3.3)), so it seems natural to consider algebraic cycles on

$$(\mathbb{A}^1, 2\{0\}) \times (\mathbb{P}^1 - \{1\}, \{0,\infty\})^n. \tag{1.11}$$

In the final section of this paper, we calculate the Chow groups of 0-cycles on these spaces. Our result:

$$CH_0\Big((\mathbb{A}^1, 2\{0\}) \times (\mathbb{P}^1 - \{1\}, \{0,\infty\})^n\Big) \cong \Omega^n_k, \quad n \geq 0, \tag{1.12}$$

is a "degeneration" of the result of Totaro [23] and Nesterenko-Suslin [21]

$$CH_0\Big((\mathbb{P}^1 - \{1\}, \{0,\infty\})^n\Big) \cong K_n^M(k) = n\text{-th Milnor } K\text{-group}, \tag{1.13}$$

and a cubical version of the simplicial result $SH^n(k,n) \cong \Omega_k^{n-1}$ (see [6]).

We thank Jörg Wildeshaus and Jean-Guillaume Grebet for helpful remarks.

2. Additive Bloch groups

Let k be a field with $1/2 \in k$. In this section, we mimic the construction in [2] §5, replacing the semi-local ring of functions on $\mathbb{P}^1$, regular at 0 and ∞ by the local ring of functions on $\mathbb{A}^1$, regular at 0, and the relative condition on K-theory at 0 and ∞ by the one at $2 \cdot \{0\}$. In particular, as we fix only 0 and ∞ in this theory, we have a $k^\times$-action on the parameter t on $\mathbb{A}^1$ so our groups will be $k^\times$-modules.

Thus let R be the local ring at 0 on $\mathbb{A}^1_k$. One has an exact sequence of relative K-groups

$$K_2(\mathbb{A}^1_k) \to K_2(k[t]/(t^2)) \to K_1(\mathbb{A}^1, (t^2)) \to K_1(\mathbb{A}^1) \to K_1(k[t]/(t^2)). \tag{2.1}$$

Using Van der Kallen's calculation of $K_2(k[t]/(t^2))$ [24] and the homotopy property $K_*(k) \cong K_*(\mathbb{A}^1_k)$, we conclude

$$K_1(\mathbb{A}^1_k, (t^2)) \cong \Omega^1_k. \tag{2.2}$$

Now we localize on $\mathbb{A}^1$ away from 0. Assuming for simplicity that k is algebraically closed, we get

$$\coprod_{k-\{0\}} K_2(k) \to K_2(\mathbb{A}^1, (t^2)) \to K_2(R, (t^2)) \to \coprod_{k-\{0\}} k^\times \to \Omega^1_k \to 0 \tag{2.3}$$

To $a \in (t^2)$ and $b \in R$ we associate the pointy-bracket symbol [20] $\langle a, b\rangle \in K_2(R, (t^2))$ which corresponds to the Milnor symbol $\{1 - ab, b\}$ if $b \neq 0$. These symbols generate $K_2(R, (t^2))$. If the divisors of a and b are disjoint, we get

$$\text{tame}\langle a, b\rangle = a|_{\text{poles of } b} + b|_{ab=1} + b^{-1}|_{\text{poles of } a} \tag{2.4}$$

We continue to assume k algebraically closed. Let $\mathcal{C} \subset K_2(R, (t^2))$ be the subgroup generated by pointy-bracket symbols with $b \in k$. For $a \in (t^2)$ write

$$a(t) = \frac{a_0 t^n + \ldots + a_{n-2}t^2}{t^m + b_1 t^{m-1} + \ldots + b_{m-1}t + b_m}; \quad b_m \neq 0. \tag{2.5}$$

We assume numerator and denominator have no common factors. If α_i are the solutions to the equation $a(t) = \kappa \in k^\times \cup \infty$, then $\sum \alpha_i^{-1} = -b_{m-1}/b_m$. In particular, this is independent of κ. It follows that one has an isomorphism

$$\coprod_{k-\{0\}} k^\times \Big/ \text{tame}(\mathcal{C}) \cong k \otimes_{\mathbb{Z}} k^\times; \quad u|_v \mapsto v^{-1} \otimes u. \tag{2.6}$$

Define

$$TB_2(k) := K_2(R,(t^2))/\mathcal{C} \tag{2.7}$$

$$TH^1_M(k,2) := \text{image}\Big(K_2(\mathbb{A}^1,(t^2)) \to TB_2(k)\Big)$$

$$TH^2_M(k,2) := \Omega^1_k = K_1(R,(t^2))$$

When $\text{char}(k) = 0$, a basic result of Goodwillie [16] yields $\text{gr}^2_\gamma K_2(\mathbb{A}^1,(t^2)) \cong k$, so $TH^1_M(k,2)$ is a quotient of k. We will see (remark 2.6) that in fact $TH^1_M(k,2) \cong k$. The above discussion yields

Proposition 2.1. *Let k be an algebraically closed field of characteristic $\neq 2$. With notations as above, we have an exact sequence*

$$0 \to TH^1_M(k,2) \to TB_2(k) \xrightarrow{\partial} k \otimes k^\times \xrightarrow{\pi} \Omega^1_k \to 0. \tag{2.8}$$

Here $\pi(a \otimes b) = a\frac{db}{b}$ and ∂ is defined via the tame symbol.

Remark 2.2. There is an evident action of the group $k^\times$ on $\mathbb{A}^1$ (multiplying the parameter) and hence on the sequence (2.8). This action extends to a k-vector space structure on all the terms except $TB_2(k)$.

Let $\mathfrak{m} = tR \subset R$. One has the following purely algebraic description of $K_2(R,\mathfrak{m}^2)$ ([22], formula (1.4), and the references cited there).
generators:

$$\langle a,b\rangle; \qquad (a,b) \in (R \times \mathfrak{m}^2) \cup (\mathfrak{m}^2 \times R) \tag{2.9}$$

Relations:

$$\langle a,b\rangle = -\langle b,a\rangle; \quad a \in \mathfrak{m}^2 \tag{2.10}$$

$$\langle a,b\rangle + \langle a,c\rangle = \langle a, b+c-abc\rangle; \quad a \in \mathfrak{m}^2 \text{ or } b,c \in \mathfrak{m}^2 \tag{2.11}$$

$$\langle a,bc\rangle = \langle ab,c\rangle + \langle ac,b\rangle; \quad a \in \mathfrak{m}^2 \tag{2.12}$$

Proposition 2.3. *There is a well-defined and nonzero map*

$$\rho : K_2(R,\mathfrak{m}^2) \to \mathfrak{m}^3/\mathfrak{m}^4 \tag{2.13}$$

defined by

$$\rho\langle a,b\rangle := \begin{cases} -adb & a \in \mathfrak{m}^2 \\ bda & b \in \mathfrak{m}^2. \end{cases} \tag{2.14}$$

Proof. Note first if $a,b \in \mathfrak{m}^2$ then $adb \equiv bda \equiv 0 \mod \mathfrak{m}^4$ so the definition (2.14) is consistent. For $a \in \mathfrak{m}^2$

$$\langle a,b\rangle + \langle b,a\rangle \mapsto -adb + adb = 0, \tag{2.15}$$

so (2.10) holds. For $a \in \mathfrak{m}^2$

$$\langle a,b\rangle + \langle a,c\rangle \mapsto -ad(b+c) \equiv -ad(b+c-abc) \mod \mathfrak{m}^4 \tag{2.16}$$

for $b,c \in \mathfrak{m}^2$

$$\langle a,b\rangle + \langle a,c\rangle \mapsto (b+c)da \equiv (b+c-abc)da \mod \mathfrak{m}^4 \tag{2.17}$$

For $a \in \mathfrak{m}^2$,

$$\langle a, bc\rangle \mapsto -ad(bc) = -abdc - acdb = \rho(\langle ab, c\rangle + \langle ac, b\rangle) \tag{2.18}$$

□

REMARK 2.4. Note that

$$-adb \equiv \log(1-ab)db/b \in \mathfrak{m}^2\Omega^1_R/d\log(1+\mathfrak{m}^4) \cong \mathfrak{m}^3/\mathfrak{m}^4. \tag{2.19}$$

The group $\mathfrak{m}^2\Omega^1_R/d\log(1+\mathfrak{m}^4)$ is the group of isomorphism classes of rank 1 line bundles, trivialized at the order 4 at $\{0\}$, with a connection vanishing at the order 2 at $\{0\}$. Thus the regulator map ρ assigns such a connection to a pointy symbol. Over the field of complex numbers $\mathbb{C}$, one can think of it in terms of "Deligne cohomology" $\mathbb{H}^2(\mathbb{A}^1, j_!\mathbb{Z}(2) \to t^4\mathcal{O} \to t^2\omega)$, and one can, as in [12], write down explicitely an analytic Cech cocycle for this regulator as a Loday symbol.

One has $\rho\langle t^2, x\rangle = -t^2dt \neq 0$, thus ρ is not trivial. Note also, the appearance of $\mathfrak{m}^3$ is consistent with A. Goncharov's idea [15] that the regulator in this context should correspond to the volume of a simplex in hyperbolic 3 space in the sissors-congruence interpretation [17]. In particular, it should scale as the third power of the coordinate.
Proposition 2.3 yields

COROLLARY 2.5. *Let $\mathfrak{m} \subset R$ be the maximal ideal. One has a well-defined map*

$$\rho : TB_2(k) \to \mathfrak{m}^3/\mathfrak{m}^4 \tag{2.20}$$

given on pointy-bracket symbols by

$$\rho\langle a, b\rangle = -a \cdot db; \quad a \in \mathfrak{m}^2,\ b \in R. \tag{2.21}$$

For $x \in TB_2(k)$ and $c \in k^\times$, write $c \star x$ for the image of x under the mapping $t \mapsto c \cdot t$ on polynomials. Then $\rho(c \star x) = c^3 \cdot \rho(x)$.

Proof. The first assertion follows because if $b \in k$, then $db = 0$. The second assertion is clear. □

REMARK 2.6. The map ρ is non-trivial on $TH^1_M(k, 2)$ because $\rho\langle t^2, t\rangle = -t^2dt \neq 0$. Since this group is a $k^\times$-module (remark 2.2) and is a quotient of k by the result of Goodwillie cited above, it follows that

$$TH^1_M(k, 2) \cong (t^3)/(t^4) \cong k \tag{2.22}$$

when $\mathrm{char}(k) = 0$.

3. Cathelineau elements and the entropy functional equation

We continue to assume k is an algebraically closed field of characteristic $\neq 2$. Define for $a \in k - \{0,1\}$

$$\langle a\rangle := \langle t^2, \frac{a(1-a)}{t-1}\rangle \in TB_2(k) \tag{3.1}$$
$$\epsilon(a) := a \otimes a + (1-a)\otimes(1-a) \in k^\times \otimes k$$

Lemma 3.1. *Writing ∂ for the tame symbol as in proposition 2.1, we have $\partial(\langle a\rangle) = 2\epsilon(a)$.*

Proof.

$$\partial(\langle a\rangle) = \mathrm{tame}\Big\{\frac{\frac{1-t}{a(1-a)}+t^2}{\frac{1-t}{a(1-a)}}, \frac{a(1-a)}{t-1}\Big\} = \tag{3.2}$$

$$\frac{a(1-a)}{t-1}\Big|_{t=\frac{1}{a}} + \frac{a(1-a)}{t-1}\Big|_{t=\frac{1}{1-a}} \mapsto a^2\otimes a + (1-a)^2\otimes(1-a) = 2\epsilon(a) \in k^\times \otimes k.$$

□

Lemma 3.2. *We have $\rho(\langle a\rangle) = a(1-a)t^2dt \in (t^3)/(t^4)$.*

Proof. Straightforward from corollary 2.5. □

Lemma 3.3. *Let notations be as in corollary 2.5, Assume k is algebraically closed, and* $\mathrm{char}(k) \neq 2,3$. *Then every element in $TB_2(k)$ can be written as a sum $\sum c_i \star \langle a_i\rangle$. In other words, $TB_2(k)$ is generated as a $k^\times$-module by the $\langle a\rangle$.*

Proof. Define

$$\mathfrak{b} := \mathrm{Image}(\partial : TB_2(k) \to k^\times \otimes k) = \ker(k^\times \otimes k \to \Omega^1_k). \tag{3.3}$$

The k-vectorspace structure $c\cdot(a\otimes b)$ on $k^\times\otimes k$ is defined by $a\otimes cb$. By (2.6) and (2.4), the map $TB_2(k) \to k^\times \otimes k$ is $k^\times$-equivariant.
Let $A \subset TB_2(k)$ be the subgroup generated by the $c \star \langle a\rangle$. $\mathfrak{b}$ is a k-vector space which is generated [7] by the $\epsilon(a)$ so the composition $A \subset TB_2(k) \to \mathfrak{b}$ is surjective. For $c_1, c_2 \in k^\times$ with $c_1 + c_2 \neq 0$ we have $(c_1+c_2)\star\langle a\rangle - c_1\star\langle a\rangle - c_2\star\langle a\rangle \mapsto 0 \in \mathfrak{b}$, so this element lies in $A \cap H^1_M(k,2)$. It is not trivial because

$$\rho((c_1+c_2)\star\langle a\rangle - c_1\star\langle a\rangle - c_2\star\langle a\rangle) = \tag{3.4}$$
$$\Big((c_1+c_2)^3 - c_1^3 - c_2^3\Big)a(1-a)t^2dt =$$
$$3\Big(c_1c_2(c_1+c_2)\Big)a(1-a)t^2dt.$$

Since the equation $\lambda = 3\Big(c_1c_2(c_1+c_2)\Big)a(1-a)$ can be solved in k, one has $A \supset H^1_M(k,2)$. This finishes the proof. □

THEOREM 3.4. *Under the assumptions of lemma 3.3, the group $TB_2(k)$ is generated as a $k^\times$-module by the $\langle a\rangle$. These satisfy relations*

$$\langle a\rangle - \langle b\rangle + a\star\langle b/a\rangle + (1-a)\star\langle (1-b)/(1-a)\rangle = 0. \tag{3.5}$$

Proof. The generation statement is lemma 3.3. Because we factor out by symbols with one entry constant, we get

$$x\star\langle a\rangle = \langle x^2t^2, \frac{a(1-a)}{xt-1}\rangle = \langle t^2, \frac{x^2a(1-a)}{xt-1}\rangle. \tag{3.6}$$

The identity to be established then reads

$$0 = \langle t^2, \frac{a(1-a)}{t-1}\rangle - \langle t^2, \frac{b(1-b)}{t-1}\rangle + \langle t^2, \frac{b(a-b)}{at-1}\rangle + \langle t^2, \frac{(1-b)(b-a)}{(1-a)t-1}\rangle. \tag{3.7}$$

The pointy bracket identity $\langle a,b\rangle + \langle a,c\rangle = \langle a, b+c-abc\rangle$ means we can compute the above sum using "faux" symbols

$$\{t^2, 1-\frac{a(1-a)t^2}{t-1}\}\{t^2, 1-\frac{b(1-b)t^2}{t-1}\}^{-1}\{t^2, 1-\frac{b(a-b)t^2}{at-1}\}\times \{t^2, 1-\frac{(1-b)(b-a)t^2}{(1-a)t-1}\} = \{t^2, X\} \tag{3.8}$$

with

$$X = \frac{(1-t+a(1-a)t^2)(1-at+b(a-b)t^2)(1-(1-a)t+(1-b)(b-a)t^2)}{(1-t+b(1-b)t^2)(1-at)(1-(1-a)t)} \tag{3.9}$$

$$= \frac{(1-at)(1-(1-a)t)(1-bt)(1-(a-b)t)(1-(1-b)t)(1-(b-a)t)}{(1-bt)(1-(1-b)t)(1-at)(1-(1-a)t)}$$

$$= (1-(a-b)t)(1-(b-a)t) = 1-(a-b)^2t^2$$

Reverting to pointy brackets, the Cathelineau relation equals

$$\{t^2, X\} = \{1-(a-b)^2t^2, (a-b)^2\} = \langle t^2, (a-b)^2\rangle = 0 \tag{3.10}$$

since we have killed symbols with one entry constant. □

REMARK 3.5. One can get a presentation for $TB_2(k)$ if one imposes (3.5) and in addition relations of the form

$$\Big((x+y+z+w) - (x+y+z) - (x+y+w) - \dots - (x) - (y) - (z) - (w)\Big)\star\langle a\rangle = 0 \tag{3.11}$$

$$(-1)\star\langle a\rangle = -\langle a\rangle. \tag{3.12}$$

Here $x \in k^\times$ corresponds to $(x) \in \mathbb{Z}[k^\times]$, and the first relation is imposed whenever it makes sense, i.e. whenever all the partial sums are non-zero. The proof uses uniqueness of solutions for the entropy equation [10]. Details are left for the reader.

REMARK 3.6. It is remarkable that a functional equation equivalent to (3.5),

$$\langle a\rangle + (1-a)\star\langle\frac{b}{1-a}\rangle = \langle b\rangle + (1-b)\star\langle\frac{a}{1-b}\rangle \tag{3.13}$$

occurs in information theory, where it is known to have a unique continuous functional solution (up to scale) given by $y\star\langle x\rangle \mapsto -yx\log(x) - y(1-x)\log(1-x)$. If on the other hand, we interpret the torus action $y\star$ as multiplication by $y^p, p \neq 1$, then the unique solution is $\langle x\rangle \mapsto x^p + (1-x)^p - 1$ [10]. Note the regulator map $\rho(y\star\langle x\rangle) = y^3x(1-x)$, so ρ is a solution for $p = 3$. Indeed, $x(1-x) = \frac{1}{3}(x^3 + (1-x)^3 - 1)$. (Again, one uses char $(k) \neq 3$.)

One can check that the functional equation (3.13) is equivalent to (3.5). To see this, one needs the following property of the elements $\langle a\rangle$.

LEMMA 3.7. $\langle a\rangle = -a\star\langle a^{-1}\rangle$.

Proof. We remark again that $TB_2(k) \xrightarrow{\rho\oplus\partial} k\oplus\mathfrak{b}$ is an isomorphism, so it suffices to check the relations on $\epsilon(a)$ and on $\rho(a) = a(1-a)t^2dt$. These become respectively

$$a\otimes a + (1-a)\otimes(1-a) = a^{-1}\otimes -1 + (1-a^{-1})\otimes(1-a) \in k^\times\otimes k \tag{3.14}$$

$$-a^3(a^{-1}(1-a^{-1})) = a(1-a). \tag{3.15}$$

The second relation is trivial. For the first one, one writes

(3.16)
$$\begin{aligned} a\otimes a + (1-a)\otimes(1-a) &= a\otimes a + (-a)\otimes(1-a) + (1-a^{-1})\otimes(1-a) \\ &= a^{-1}\otimes(-a+a-1) + (-1)\otimes(1-a). \end{aligned}$$

Since k is 2-divisible, one has $(-1)\otimes b = 0$. ☐

4. A CONJECTURAL LIE ALGEBRA OF CYCLES

The purpose of this section is to sketch a conjectural algebraic cycle based theory of additive polylogarithms. The basic reference is [4], where a candidate for the Tannakian Lie algebra of the category of mixed Tate motives over a field k is constructed. The basic tool is a differential graded algebra (DGA) $\mathcal{N}$ with a supplementary grading (Adams grading)

$$\mathcal{N}^\bullet = \oplus_{j\geq 0}\mathcal{N}(j)^\bullet \tag{4.1}$$

$$\mathcal{N}(j)^i \subset \text{Codim. } j \text{ algebraic cycles on } (\mathbb{P}^1 - \{1\})^{2j-i}$$

where $\mathcal{N}(j)^i$ consists of cycles which meet the faces (defined by setting coordinates $= 0,\infty$) properly and which are alternating with respect to the action of the symmetric group on the factors and with respect to inverting the coordinates. The product structure is the external product $(\mathbb{P}^1 - \{1\})^{2j_1-i_1} \times (\mathbb{P}^1 - \{1\})^{2j_2-i_2} = (\mathbb{P}^1 - \{1\})^{2j_1-i_1+2j_2-i_2}$ followed by alternating projection, and the boundary map is an alternating sum of restrictions to faces. For full details, cf. op. cit.

We consider an enlarged DGA

$$\widetilde{\mathcal{N}}^\bullet = \oplus_{j\geq 0}\widetilde{\mathcal{N}}(j)^\bullet \tag{4.2}$$

$$\widetilde{\mathcal{N}}(j)^i := \mathcal{N}(j)^i \oplus T\mathcal{N}(j)^i$$

$$T\mathcal{N}(j)^i \subset \text{Codim. } j \text{ algebraic cycles on } \mathbb{A}^1 \times (\mathbb{P}^1 - \{1\})^{2j-i-1}.$$

The same sort of alternation and good position requirements are imposed for the factors $\mathbb{P}^1 - \{1\}$. In addition, we impose a "modulus" condition at the point $0 \in \mathbb{A}^1$. The following definition is tentative, and is motivated by example 4.2 below.

Definition 4.1. Let $Z \subset \mathbb{A}^1 \times (\mathbb{P}^1)^n$ be an irreducible subvariety and let $\pi : \tilde{Z} \to Z$ be the normalization. Let $F_i : y_i = 1$, where y_i are the coordinates on $(\mathbb{P}^1)^n$, and let $F = \cup_{i=1}^n F_i$. We say that Z has modulus $m \geq 1$ if

$$\pi^*(F - m\{0\} \times (\mathbb{P}^1)^n) \text{ is an effective Cartier divisor.} \tag{4.3}$$

Let $Z^0 = Z \cap (\mathbb{A}^1 \times (\mathbb{P}^1 \setminus \{1\})^n)$. The higher Chow groups are computed using face maps $\partial_I(Z^0) \subset \mathbb{A}^1 \times (\mathbb{P}^1 \setminus \{1\})^{n-|I|}$. We say Z^0 has modulus $m \geq 1$ for higher Chow theory of the closure $Z = \overline{Z^0}$ and every component of $\overline{\partial_I(Z^0)}$ have modulus m in the above sense.

If Z satisfies modulus m and X is any subvariety of $(\mathbb{P}^1)^r$ which is not contained in a face, then $Z \times X$ satisfies modulus m on $\mathbb{A}^1 \times (\mathbb{P}^1)^{n+r}$.
When $\dim(Z) = 1$ and Z has modulus m, the differential form $\frac{1}{x^m}\frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_n}{y_n}$ restricted to $\tilde{Z}$ has no poles along $\tilde{Z} \cdot (x = 0)$. (See (6.19).)

Example 4.2. Milnor K-theory of a field can be interpreted in terms of 0-cycles [21], [23]. More generally, a Milnor symbol $\{f_1, \ldots, f_p\}$ over a ring R corresponds to the cycle on $\mathrm{Spec}\,(R) \times (\mathbb{P}^1)^p$ which is just the graph

$$\{(x, f_1(x), \ldots f_p(x)) | x \in \mathrm{Spec}\,(R)\}$$

One would like cycles with modulus to relate to relative K-theory. Assume R is semilocal, and let $J \subset R$ be an ideal. Then we have already used (2.9) that $K_2(R, J)$ has a presentation with generators given by pointy-bracket symbols $\langle a, b\rangle$ with $a \in R$ and $b \in J$ or vice-versa. The pointy-bracket symbol $\langle a, b\rangle$ corresponds to the Milnor symbol $\{1-ab, b\}$ when the latter is defined. Suppose R is the local ring on $\mathbb{A}^1_k$ at the origin, with k a field, and take $J = (s^m)$, where s is the standard parameter. For $a \in J$ and $b \in (s^p)$ for some $p \geq 0$, we see that our definition of cycle with modulus is designed so the cycle $\{(x, 1 - a(x)b(x), b(x))\}$ has modulus at least m.

The modulus condition is compatible with pullback to the faces $t_i = 0, \infty$.

Definition 4.3. $T\mathcal{N}(j)^i$, $-\infty \leq i \leq 2j - 1$, is the $\mathbb{Q}$-vectorspace of codimension j algebraic cycles on $\mathbb{A}^1 \times (\mathbb{P}^1 - \{1\})^{2j-i-1}$ which are in good position for the face maps $t_i = 0, \infty$ and have modulus $2\{0\} \times (\mathbb{P}^1)^{2j-i-1}$. Here, in order to calculate the modulus, we close up the cycle to a cycle on $\mathbb{A}^1 \times (\mathbb{P}^1)^{2j-i-1}$.

Note that a cycle Z of modulus $m \geq 1$ doesn't meet $\{0\} \times (\mathbb{P}^1)^{2j-i-1}$ on $\mathbb{A}^1 \times (\mathbb{P}^1 - \{1\})^{2j-i-1}$.
We have a split-exact sequence of DGA's,

$$0 \to T\mathcal{N}^\bullet \to \widetilde{\mathcal{N}}^\bullet \leftrightarrows \mathcal{N}^\bullet \to 0 \tag{4.4}$$

with multiplication defined so $T\mathcal{N}^\bullet$ is a square-zero ideal. Denote the cohomology groups by

$$\widetilde{H}^i_M(k,j) := H^i(\widetilde{\mathcal{N}}^\bullet(j)); \quad TH^i_M(k,j) := H^i(T\mathcal{N}^\bullet(j)). \tag{4.5}$$

As an example, we will see in section 6 that the Chow groups of 0-cycles in this context compute the Kähler differential forms:

$$TH^j_M(k,j) \cong \Omega^{j-1}_k; j \geq 0. \tag{4.6}$$

(Here $\Omega^0_k = k$.)
One may apply the bar construction to the DGA $\widetilde{\mathcal{N}}^\bullet$ as in [4]. Taking H^0 yields an augmented Hopf algebra (defining TH^0 as the augmentation ideal)

$$0 \to TH^0(B(\widetilde{\mathcal{N}}^\bullet)) \to H^0(B(\widetilde{\mathcal{N}}^\bullet)) \leftrightarrows H^0(\mathcal{N}^\bullet) \to 0. \tag{4.7}$$

The hope would be that the corepresentations of the co-Lie algebra of indecomposables (here $H^{0,+} := \ker(H^0 \to \mathbb{Q})$ denotes the elements of bar degree > 0, cf. op. cit. §2)

$$\widetilde{\mathcal{M}} := H^0(B(\widetilde{\mathcal{N}}^\bullet))^+/(H^0(B(\widetilde{\mathcal{N}}^\bullet))^+)^2 = \mathcal{M} \oplus T\mathcal{M} \tag{4.8}$$

correspond to contravariant motives over $k[t]/(t^2)$. In particular, the work of Cathelineau [7] suggests a possible additive polylogarithm Lie algebra. In the remainder of this section, we will speculate a bit on how this might work.
For a general DGA $A^\bullet$ which is not bounded above, the total grading on the double complex $B(A^\bullet)$ has infinitely many summands (cf. [4], (2.15)). For example the diagonal line corresponding to $H^0(B(A^\bullet))$ has terms ($A^+ := \ker(A^\bullet \to \mathbb{Q})$)

$$A^1,\ (A^+ \otimes A^+)^2,\ (A^+ \otimes A^+ \otimes A^+)^3, \ldots \tag{4.9}$$

When, however, $A^\bullet$ has a graded structure

$$A^i = \oplus_{j\geq 0} A^i(j); \quad dA(j) \subset A(j); \quad A^+ = \oplus_{j>0} A^+(j), \tag{4.10}$$

for each fixed j, only finitely many tensors can occur. For example $H^0(B(\widetilde{\mathcal{N}}^\bullet(1))) = H^1(\widetilde{\mathcal{N}}^\bullet(1)) = k \oplus k^\times$, and $H^0(B(\widetilde{\mathcal{N}}^\bullet(2)))$ is the cohomology along the indicated degree 0 diagonal in the diagram

$$\begin{array}{ccc} \widetilde{\mathcal{N}}^1(1) \otimes \widetilde{\mathcal{N}}^1(1) & \xrightarrow{\delta} & \widetilde{\mathcal{N}}^2(2) \\ \uparrow \partial & \ddots\ \textit{deg. } 0 & \uparrow \partial \\ (\widetilde{\mathcal{N}}^1(1) \otimes \widetilde{\mathcal{N}}^0(1)) \oplus (\widetilde{\mathcal{N}}^0(1) \otimes \widetilde{\mathcal{N}}^1(1)) & \xrightarrow{\delta} & \widetilde{\mathcal{N}}^1(2) \\ & & \uparrow \\ & & \widetilde{\mathcal{N}}^0(2). \end{array} \tag{4.11}$$

In the absence of more information about the DGA $\widetilde{\mathcal{N}}^\bullet$, it is difficult to be precise about the indecomposable space $\widetilde{\mathcal{M}}$. As an approximation, we have

Proposition 4.4. *Let $db(\widetilde{\mathcal{N}}) \subset \widetilde{\mathcal{N}}^1$ be the subspace of elements x with decomposable boundary, i.e. such that there exists $y \in (\widetilde{\mathcal{N}} \otimes \widetilde{\mathcal{N}})^2$ with $\delta(y) = \partial(x) \in \widetilde{\mathcal{N}}^2$. Define*

$$q\widetilde{\mathcal{N}} := db(\widetilde{\mathcal{N}})/(\partial\widetilde{\mathcal{N}}^0 + \delta(\widetilde{\mathcal{N}}^+ \otimes \widetilde{\mathcal{N}}^+)^1) \tag{4.12}$$

Then there exists a natural map, compatible with the grading by codimension of cycles (Adams grading)

$$\phi : \widetilde{\mathcal{M}} \to q\widetilde{\mathcal{N}}. \tag{4.13}$$

Proof. Straightforward. □

As above, we can decompose

$$q\widetilde{\mathcal{N}} = q\mathcal{N} \oplus Tq\mathcal{N}, \tag{4.14}$$

where $q\mathcal{N}(p)$ is a subquotient of the space of codimension p cycles on $(\mathbb{P}^1 - \{1\})^{2p-1}$, and $Tq\mathcal{N}$ is a subquotient of the cycles on $\mathbb{A}^1 \times (\mathbb{P}^1 - \{1\})^{2p-2}$

Example 4.5. The polylogarithm cycle $\{a\}_p$ for $a \in \mathbb{C} - \{0, 1\}$ is defined to be the image under the alternating projection of $(-1)^{p(p-1)/2}$ times the locus in $(\mathbb{P}^1 - \{1\})^{2p-1}$ parametrized in nonhomogeneous coordinates by

$$(x_1, \ldots, x_{p-1}, 1 - x_1, 1 - x_2/x_1, \ldots, 1 - x_{p-1}/x_{p-2}, 1 - a/x_{p-1}) \tag{4.15}$$

(We take $\{a\}_1 = 1 - a \in \mathbb{P}^1 - \{1\}$.) To build a class in $H^0(B(\mathcal{N}^\bullet))^+$ and hence in $\mathcal{M}$ one uses that $\partial\{a\}_n = \{a\}_{n-1} \cdot \{1 - a\}_1$.

The following should be compared with [11], where a similar formula is proposed. The key new point here is that algebraic cycles make it possible to envision this formula in the context of Lie algebras.

Conjecture 4.6. There exist elements $\langle a\rangle_n \in T\mathcal{M}(n)$ (4.8) represented by cycles $Z_n(a)$ of codimension n on $\mathbb{A}^1 \times (\mathbb{P}^1 - \{1\})^{2n-2}$ with $\langle a\rangle_1 = a \in \mathbb{A}^1 - \{0\}$. These cycles should satisfy the boundary condition

$$\partial\langle a\rangle_n = \langle a\rangle_{n-1} \cdot \{1 - a\}_1 + \langle 1 - a\rangle_1 \cdot \{a\}_{n-1} \in \Big(\bigwedge^2 \widetilde{\mathcal{M}}\Big)(n). \tag{4.16}$$

For example, for $n = 2$,

$$\partial\langle a\rangle_2 = a \otimes a + (1 - a) \otimes (1 - a) \tag{4.17}$$
$$\in k \otimes k^\times \cong T\mathcal{M}(1) \otimes \mathcal{M}(1) \subset \Big(\bigwedge^2 \widetilde{\mathcal{M}}\Big)(2)$$

gives Cathelineau's relation [7].

Proposition 4.7. *Assume given elements $\langle a\rangle_n$ satisfying* (4.16). *Let $\widetilde{\mathcal{P}} = \bigoplus_{n=1}^\infty \mathbb{Q}\langle a\rangle_n \oplus \mathbb{Q}\{a\}_n$ be the constant graded sheaf over $\mathbb{A}^1 - \{0, 1\}$. Then $\widetilde{\mathcal{P}}, \partial : \widetilde{\mathcal{P}} \to \bigwedge^2 \widetilde{\mathcal{P}}$ is a sheaf of co-lie algebras.*

Proof. It suffices to show that $\partial^2 = 0$. Using the derivation property of the boundary,

$$(4.18)\quad \partial\partial\langle a\rangle_n = (\partial\langle a\rangle_{n-1})\cdot\{1-a\}_1 - \langle 1-a\rangle_1\cdot\partial\{a\}_{n-1} =$$
$$\Big(\langle a\rangle_{n-2}\cdot\{1-a\}_1+\langle 1-a\rangle_1\cdot\{a\}_{n-2}\Big)\cdot\{1-a\}_1 - \langle 1-a\rangle_1\cdot\{a\}_{n-2}\cdot\{1-a\}_1 =$$
$$0\in\bigwedge^3\widetilde{\mathcal{M}}.$$

□

We can make the definition (independent of any conjecture)

DEFINITION 4.8. The additive polylogarithm sheaf of Lie algebras over $\mathbb{A}^1 - \{0,1\}$ is the graded sheaf of Lie algebras with graded dual the sheaf $\widetilde{\mathcal{P}}, \partial : \widetilde{\mathcal{P}} \to \bigwedge^2\widetilde{\mathcal{P}}$ satisfying (4.16) above.

Of course, $\langle a\rangle_2$ should be closely related to the element $\langle a\rangle \in K_2(R,(t^2))$ (3.1). The cycle

$$(4.19)\qquad \{(t, 1-\frac{t^2a(1-a)}{t-1}, \frac{a(1-a)}{t-1}) \mid t\in\mathbb{A}^1\} \subset \mathbb{A}^1\times(\mathbb{P}^1-\{1\})^2$$

associated to the pointy bracket symbol in (3.1) satisfies the modulus 2 condition but is not in good position with respect to the faces. (It contains $(1,\infty,\infty)$.) It is possible to give symbols equivalent to this one whose corresponding cycle is in good position, but we do not have a canonical candidate for such a cycle, or a candidate whose construction would generalize in some obvious way to give all the $\langle a\rangle_n$.

5. THE ARTIN-SCHREIER DILOGARITHM

The purpose of this section is to present a definition of what one might call an Artin-Schreier dilogarithm in characteristic p. To begin with, however, we take X to be a complex-analytic manifold and sketch certain analogies between the multiplicative and additive theory. We write $\mathcal{O}$ (resp. $\mathcal{O}^\times$, Ω^1) for the sheaf of analytic functions (resp. invertible analytic functions, analytic 1-forms). The reader is urged to compare with [9].

(5.1)

MULTIPLICATIVE		ADDITIVE
$\mathcal{O}^\times \overset{\mathbb{L}}{\otimes} \mathcal{O}^\times$	$\longleftrightarrow$	$\mathcal{O}\otimes\mathcal{O}^\times$
$\mathcal{O}^\times \overset{\mathbb{L}}{\otimes} \mathcal{O}^\times \to \Big(\mathcal{O}^\times(1)\to\Omega^1\Big)[1]$	$\longleftrightarrow$	$\mathcal{O}\otimes\mathcal{O}^\times \to \Big(\mathcal{O}(1)\to\Omega^1\Big)[1]$
K_2	$\longleftrightarrow$	Ω^1
Steinberg rel'n $=$ $a\otimes(1-a)$	$\longleftrightarrow$	Cathelineau rel'n $=$ $a\otimes a+(1-a)\otimes(1-a)$
exponential of dilogarithm $=$ $\exp\Big(\int_0^a \log(1-t)dt/t\Big)$	$\longleftrightarrow$	Shannon entropy function $=$ $\int^a \log(\frac{t}{1-t})dt =$ $a\log a+(1-a)\log(1-a)$.

In the multiplicative (resp. additive) theory, one applies $\mathcal{O}^\times \overset{\mathbb{L}}{\otimes}_{\mathbb{Z}} \bullet$ (resp. $\mathcal{O}\otimes_{\mathbb{Z}}\bullet$) to the exponential sequence (here $\mathbb{Z}(1) := \mathbb{Z}\cdot 2\pi i$)

$$0 \to \mathbb{Z}(1) \to \mathcal{O} \to \mathcal{O}^\times \to 0. \tag{5.2}$$

The regulator maps (5.1), line 2, come from liftings of these tensor products to

$$\begin{array}{ccccc} \mathcal{O}^\times(1) & \longrightarrow & \mathcal{O}^\times \overset{\mathbb{L}}{\otimes} \mathcal{O} & \longrightarrow & \mathcal{O}^\times \otimes \mathcal{O}^\times \\ \| & & \downarrow & & \| \\ \mathcal{O}^\times(1) & \longrightarrow & \mathcal{O}^\times \otimes \mathcal{O} & \longrightarrow & \mathcal{O}^\times \otimes \mathcal{O}^\times \\ \downarrow & & \downarrow & & \\ \Omega^1 & = & \Omega^1. & & \end{array} \tag{5.3}$$

$$\begin{array}{ccccc} \mathcal{O}(1) & \longrightarrow & \mathcal{O} \otimes \mathcal{O} & \longrightarrow & \mathcal{O} \otimes \mathcal{O}^\times \\ \downarrow & & \downarrow & & \\ \Omega^1 & = & \Omega^1 & & \end{array} \tag{5.4}$$

In the multiplicative theory, the regulator map can be viewed as associating to two invertible analytic functions f, g on X a line bundle with connection $\mathcal{L}(f,g)$ on X, [9]. The exponential of the dilogarithm

$$\exp\left(\frac{1}{2\pi i}\int_0^f \log(1-t)\frac{dt}{t}\right) \tag{5.5}$$

determines a flat section trivializing $\mathcal{L}(1-g,g)$. Let $\{U_i\}$ be an analytic cover of X, and let $\log_i f$ be an analytic branch of the logarithm on U_i. Then $\mathcal{L}(f,g)$ is represented by the Cech cocycle

$$\left(g^{\frac{1}{2\pi i}(\log_i f - \log_j f)}, \frac{1}{2\pi i}\log_i f \frac{dg}{g}\right) \tag{5.6}$$

The trivialization comes from the 0-cochain

$$i \mapsto \exp\left(\frac{1}{2\pi i}\int_0^f \log_i(1-t)\frac{dt}{t}\right). \tag{5.7}$$

The additive theory associates to $a\otimes f \in \mathcal{O}\otimes\mathcal{O}^\times$ the class in $\mathbb{H}^1(X, \mathcal{O}(1) \to \Omega^1)$ represented by the cocycle for $\mathcal{O}(1) \to \Omega^1$

$$\Big(a \otimes (\log_i f - \log_j f), \log_i f \cdot da\Big). \tag{5.8}$$

This can be thought of as defining a connection on the affine bundle $\mathcal{A}(a,f)$ associated to the coboundary of $a \otimes f$ in $H^1(X, \mathcal{O}(1))$. The affine bundle itself is canonically trivialized because in the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}(1) & \longrightarrow & \mathcal{O}\otimes\mathcal{O} & \longrightarrow & \mathcal{O}\otimes\mathcal{O}^\times & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & & & \\ & & \Omega^1 & = & \Omega^1 & & & & \end{array} \tag{5.9}$$

the top sequence is split (by multiplication $\mathcal{O}\otimes\mathcal{O}\to\mathcal{O}(1)$). The splitting is not compatible with the vertical arrows, so it does not trivialize the connection. More concretely, $a\otimes f\in\mathcal{O}\otimes\mathcal{O}^\times$ gives the 1-cocycle $(a\otimes(\log_i f-\log_j f),\ \log_i f\cdot da)\in\mathbb{H}^1(X,\mathcal{O}(1)\to\Omega^1)$. Subtracting the coboundary of the 0-cochain $\frac{1}{2\pi i}a\log_i f\otimes 2\pi i$ leaves the cocycle $(0,a\frac{df}{f})$. We have proved:

Proposition 5.1. *The map* $\partial: H^0(X,\mathcal{O}\otimes\mathcal{O}^\times)\to\mathbb{H}^1(X,\mathcal{O}(1)\to\Omega^1)$ *factors*

$$H^0(X,\mathcal{O}\otimes\mathcal{O}^\times)\xrightarrow{a\otimes f\mapsto a\,df/f}H^0(X,\Omega^1)\longrightarrow\mathbb{H}^1(X,\mathcal{O}(1)\to\Omega^1).$$

In particular, for $a\in\mathcal{O}$ *such that* a *and* $1-a$ *are both units, the Cathelineau elements* $\epsilon(a)=a\otimes a+(1-a)\otimes(1-a)$ (3.1) *lift to*

$$a\otimes\log a+(1-a)\otimes\log(1-a)-\frac{1}{2\pi i}\int^a\log\left(\frac{t}{1-t}\right)dt\otimes 2\pi i\in H^0(X,\mathcal{O}\otimes\mathcal{O}).$$

Remark 5.2. The element $a\otimes a\in H^0(X,\mathcal{O}\otimes\mathcal{O}^\times)$ maps to $da=0\in\mathbb{H}^1(X,\mathcal{O}(1)\to\Omega^1)$, but the above construction does not give a canonical trivializing 0- cocycle.

We now suppose X is a smooth variety in characteristic $p>0$, and we consider an Artin-Schreier analog of the above construction. In place of the exponential sequence (5.2) we use the Artin-Schreier sequence of étale sheaves

$$0\longrightarrow\mathbb{Z}/p\longrightarrow\mathbb{G}_a\xrightarrow{1-F}\mathbb{G}_a\longrightarrow 0. \tag{5.10}$$

Here F is the Frobenius map. We replace the twist by $\mathcal{O}_{\text{an}}^\times$ over $\mathbb{Z}$ by the twist over $\mathbb{Z}/p$ by $\mathbb{G}_m/\mathbb{G}_m^p$ to build a diagram (compare (5.3). Here $Z^1\subset\Omega^1$ is the subsheaf of closed forms.)

(5.11)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{G}_m/\mathbb{G}_m^p & \longrightarrow & \mathbb{G}_a\otimes\mathbb{G}_m/\mathbb{G}_m^p & \xrightarrow{(1-F)\otimes 1} & \mathbb{G}_a\otimes\mathbb{G}_m/\mathbb{G}_m^p & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle d\log} & & \big\downarrow{\scriptstyle f\otimes g\mapsto f^p dg/g} & & & & \\ & & Z^1 & = & Z^1 & & & & \end{array}$$

The group $\mathbb{H}^1(\mathbb{G}_m\to Z^1)$ is the group of isomorphism classes of line bundles with integrable connections as usual, and H^1(the subcomplex $\mathbb{G}_m^p\to 0$) is the subgroup of connections corresponding to a Frobenius descent. We get an exact sequence

$$\begin{aligned} 0\to\{\text{line bundle + integrable connection}\}/\{\text{lb + Frobenius descent}\}\\ \to\mathbb{H}^1(X,\mathbb{G}_m/\mathbb{G}_m^p\to Z^1)\to{}_pH^2(X,\mathbb{G}_m). \end{aligned} \tag{5.12}$$

Proposition 5.3. *Let $\iota, C : Z^1 \to \Omega^1$ be the natural inclusion and the Cartier operator, respectively. One has a quasi-isomorphism $(\mathbb{G}_m/\mathbb{G}_m^p \to Z^1) \overset{\iota - C}{\sim} \Omega^1[-1]$. Then the diagram*

$$
\begin{array}{ccc}
H^0(X, \mathbb{G}_a \otimes \mathbb{G}_m/\mathbb{G}_m^p) & \xrightarrow{\partial\ (5.11)} & \mathbb{H}^1(X, \mathbb{G}_m/\mathbb{G}_m^p \to Z^1) \\
{\scriptstyle a\otimes b \mapsto a db/b}\big\downarrow & & \cong \big\downarrow {\scriptstyle \iota - C} \\
H^0(X, \Omega^1) & = & H^0(X, \Omega^1)
\end{array}
$$

is commutative.

Proof. Straightforward from the commutative diagram (with $\mathfrak{b}$ defined to make the columns exact and $\phi(a \otimes b) = a \cdot db/b$).

(5.13)

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathfrak{b} & = & \mathfrak{b} & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{G}_m/\mathbb{G}_m^p & \longrightarrow & \mathbb{G}_a \otimes \mathbb{G}_m/\mathbb{G}_m^p & \xrightarrow{F\otimes 1 - 1} & \mathbb{G}_a \otimes \mathbb{G}_m/\mathbb{G}_m^p & \longrightarrow & 0 \\
 & & \| & & \big\downarrow {\scriptstyle \phi\circ(F\otimes 1)} & & \big\downarrow {\scriptstyle \phi} & & \\
0 & \longrightarrow & \mathbb{G}_m/\mathbb{G}_m^p & \longrightarrow & Z^1 & \xrightarrow{1-C} & \Omega^1 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}
$$

□

Next we want to see what plays the role of the exponential of the dilogarithm or the Shannon entropy function in this Artin-Schreier context. Let $X = \mathrm{Spec}\,(\mathbb{F}_{p^2}[x])$. Begin with Cathelineau's element

$$\epsilon(x) := x \otimes x + (1-x) \otimes (1-x) \in \mathfrak{b}. \tag{5.14}$$

Choose an Artin-Schreier roots $y^p - y = x$ and $\beta^p - \beta = 1$. To simplify we view $\beta \in \mathbb{F}_{p^2}$ as fixed, and we write $\mathbb{F} = \mathbb{F}_{p^2}$. A local lifting of $\epsilon(x)$ on the étale cover $\mathrm{Spec}\,\mathbb{F}(y) \to \mathrm{Spec}\,\mathbb{F}(x)$ is given by

$$\rho(y) := y \otimes x + (\beta - y) \otimes (1-x) \in \Gamma(\mathrm{Spec}\,\mathbb{F}(y), \mathbb{G}_a \otimes \mathbb{G}_m/\mathbb{G}_m^p). \tag{5.15}$$

From diagram (5.13) there should exist a canonical global lifting, i.e. a lifting defined over $\mathrm{Spec}\,\mathbb{F}(x)$. This lifting, call it $\theta(x)$ has the form $\theta(x) = \rho(y) \cdot \delta(y)^{-1}$ for some $\delta(y) \in \Gamma(\mathrm{Spec}\,\mathbb{F}(y), \mathbb{G}_m/\mathbb{G}_m^p)$. We want to calculate $\delta(y)$.

To do this calculation, note

$$\phi\circ(F\otimes 1)(\rho(y)) = y^p dx/x - (\beta-y)^p dx/(1-x) = \frac{(-y^p(1-y^p+y)+(\beta+1-y^p)(y^p-y))dy}{(y^p-y)(1-y^p+y)} = \frac{(\beta(y^p-y)-y)dy}{(y^p-y)(1-y^p+y)} =: \eta(y)\in Z^1. \tag{5.16}$$

Viewed as a meromorphic form on $\mathbb{P}^1_y$, η has simple poles at the points a and $\beta-a$ for $a\in\mathbb{F}_p$. The residue of a form P/Qdy at a point a where Q has a simple zero is given by $P(a)/Q'(a)$. Using this, the residue of η at $a\in\mathbb{F}_p$ is a. The residue at $\beta-a$ is $\frac{\beta\cdot 1-(\beta-a)}{1}=a$. Necessarily, therefore, since η is regular at $y=\infty$ we must have

$$\eta = d\log\Big(\prod_{a=1}^{p-1}\frac{(\beta-(y+a))^a}{(y+a)^a}\Big) \tag{5.17}$$

We conclude

$$\delta(y) = \prod_{a=1}^{p-1}\frac{(\beta-(y+a))^a}{(y+a)^a}. \tag{5.18}$$

Everything is invariant under the automorphism $y\mapsto\beta-y$. Indeed, the equation can be rewritten (of course $\mod \mathbb{F}(y)^{\times p}$)

$$\delta(y) = \prod_{a=1}^{(p-1)/2}\Big[\frac{(\beta-(y+a))(y-a)}{(y+a)(\beta-(y-a))}\Big]^a \tag{5.19}$$

Note $\delta(y)$ depends on y, not just on x. Indeed the product (2.6) can be taken for $0\le a\le p-1$, i.e. for $a\in\mathbb{F}_p$. One gets then

$$\frac{\delta(y+1)}{\delta(y)} \equiv \prod_{a=0}^{p-1}\frac{y+a}{\beta-(y+a)} = \frac{y^p-y}{1-y^p-y} = \frac{x}{1-x} \mod \mathbb{F}(y)^{\times p}. \tag{5.20}$$

The fact that $\rho(y)\delta(y)^{-1}$ is defined over $\mathbb{F}(x)$ says that the Cech boundaries of $\rho(y)$ and $\delta(y)$ coincide. Since the latter is, by definition, the coboundary in $\mathbb{G}_m/\mathbb{G}_m^p$ of $\epsilon(x)=x\otimes x+(1-x)\otimes(1-x)$, it follows that $\delta(y)$ is a 0-cochain for the Galois cohomology

$$\mathbb{H}^*(\mathbb{F}(y)/\mathbb{F}(x),\mathbb{G}_m/\mathbb{G}_m^p\to Z^1)$$

which trivializes the coboundary of $\rho(\epsilon(x))$.

Finally, in this section, we discuss a flat realization of the Artin-Schreier dilogarithm. To see the point, consider the ℓ-adic realization of the usual dilogarithm mixed Tate motivic sheaf over $\mathbb{A}^1-\{0,1\}$. Reducing mod ℓ yields a sheaf with fibre an $\mathbb{F}_\ell$-vector space of dimension 3. The sheaf has a filtration with successive quotients having fibres $\mathbb{Z}/\ell\mathbb{Z},\mu_\ell,\mu_\ell^{\otimes 2}$. The geometric fundamental group acts on the fibre via a Heisenberg type group. We visualize this action as follows:

$$\begin{pmatrix} 1 & \mu_\ell & \mu_\ell^{\otimes 2} \\ 0 & 1 & \mu_\ell \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mu_\ell^{\otimes 2} \\ \mu_\ell \\ \mathbb{Z}/\ell\mathbb{Z} \end{pmatrix} \tag{5.21}$$

Here the notation means that for $g \in \pi_1^{geo}$ the corresponding matrix

$$\begin{pmatrix} 1 & a_{12}(g) & a_{13}(g) \\ 0 & 1 & a_{23}(g) \\ 0 & 0 & 1 \end{pmatrix}$$

has $a_{ij} \in \mathrm{Hom}(\mu_\ell^{\otimes i-1}, \mu_\ell^{\otimes j-1}) = \mu_\ell^{j-i}$.
The essential ingredients here are first the Heisenberg group $\mathcal{H}_\ell$, second the $\mathcal{H}_\ell$-torsor over $\mathbb{A}^1 - \{0,1\}$ corresponding to the kernel of the representation, and third the (standard) representation of $\mathcal{H}_\ell$ on $\mathbb{Z}/\ell\mathbb{Z} \oplus \mu_\ell \oplus \mu_\ell^{\otimes 2}$.
We define an Artin-Schreier Heisenberg group as the non-commutative flat groupscheme $\mathcal{H}_{AS}$ which we could suggestively write

$$\mathcal{H}_{AS} := \begin{pmatrix} 1 & \mathbb{Z}/p\mathbb{Z} & \mu_p \\ 0 & 1 & \mu_p \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.22}$$

More precisely, $\mathcal{H}_{AS}$ is a central extension

$$0 \to \mu_p \to \mathcal{H}_{AS} \to \mu_p \times \mathbb{Z}/p\mathbb{Z} \to 0. \tag{5.23}$$

Let

$$\begin{aligned} b : (\mu_p \times \mathbb{Z}/p\mathbb{Z}) \times (\mu_p \times \mathbb{Z}/p\mathbb{Z}) &\to \mu_p \\ b((\zeta_1, a_1), (\zeta_2, a_2)) &= \zeta_1^{-a_2}\zeta_2^{a_1}. \end{aligned} \tag{5.24}$$

Define $\mathcal{H}_{AS} = \mu_p \times (\mu_p \times \mathbb{Z}/p\mathbb{Z})$ as a scheme, with group structure given by

$$(\zeta_1, \theta_1, a_1) \cdot (\zeta_2, \theta_2, a_2) := (\zeta_1\zeta_2\theta_2^{a_1}, \theta_1\theta_2, a_1 + a_2). \tag{5.25}$$

The commutator pairing on $\mathcal{H}_{AS}$ is given by

$$\Big[(\zeta_1, \theta_1, a_1), (\zeta_2, \theta_2, a_2)\Big] = \Big(b((\theta_1, a_1), (\theta_2, a_2)), 1, 0\Big) = b((\theta_1, a_1), (\theta_2, a_2)) \in \mu_p. \tag{5.26}$$

We fix a solution $\beta^p - \beta = 1$. We define a flat $\mathcal{H}_{AS}$-torsor $T = T_\beta$ over $\mathbb{A}^1_{\mathbb{F}_{p^2}}$ as follows. A local (for the flat topology) section t is determined by
1. A p-th root of $\frac{x}{1-x}$: $w^p \equiv \frac{x}{1-x} \mod \mathbb{F}_{p^2}(x)^{\times p}$.
2. A y satisfying $y^p - y = x$.
3. A p-th root z of $\delta(y) : z^p \equiv \delta(y) \mod \mathbb{F}_{p^2}(x)^{\times p}$ (where $\delta(y)$ is as in (5.18).)
The action of $\mathcal{H}_{AS}$ is given by

$$(\zeta, \theta, a) \star (z, w, y) = (\zeta z w^a, \theta w, y + a). \tag{5.27}$$

Note $(\zeta z w^a)^p = \delta(y)(\frac{x}{1-x})^a = \delta(y+a)$ by (5.20), so the triple on the right lies in T. This is an action because

$$(5.28)\quad (\zeta',\theta',a')\star\Big((\zeta,\theta,a)\star(z,w,y)\Big) = \\ (\zeta',\theta',a')\star\Big((\zeta z w^a,\theta w,y+a)\Big) = \\ \Big(\zeta'\zeta z w^a(\theta w)^{a'},\theta'\theta w,y+a+a'\Big) = (\zeta'\zeta\theta^{a'},\theta'\theta,a+a')\star(z,w,y) = \\ \Big((\zeta',\theta',a')\star(\zeta,\theta,a)\Big)\star(z,w,y).$$

Define $\mathbb{V} := \mu_p\times\mu_p\times\mathbb{Z}/p\mathbb{Z}$. There is an evident action of $\mathcal{H}_{AS}$ on $\mathbb{V}$, viewed as column vectors. We suggest that the contraction $T \overset{\mathcal{H}_{AS}}{\times} \mathbb{V}$ should be thought of as analogous to the mod ℓ étale sheaf on $\mathbb{A}^1 - \{0,1\}$ with fibre $\mathbb{Z}/\ell\mathbb{Z}\oplus\mu_\ell\oplus\mu_\ell^{\otimes 2}$ associated to the ℓ-adic dilogarithm.

6. The additive cubical (higher) Chow groups

In this section, we show that the modulus condition we introduced in definition (4.1) yields additive Chow groups which we can compute in weights (n,n). We assume throughout that k is a field and $\frac{1}{6}\in k$.
One sets

$$(6.1)\quad A = (\mathbb{A}^1, 2\{0\}) \\ B = (\mathbb{P}^1\setminus\{1\},\{0,\infty\}).$$

The coordinates will be x on A and $(y_1,\dots,y_n)$ on B. One considers

$$(6.2)\quad X_n = A\times B^n.$$

The boundary maps $X_{n-1}\hookrightarrow X_n$ defined by $y_i = 0,\infty$ are denoted by $\partial_i^j, i = 1,\dots,n, j=0,\infty$. One denotes by $Y_n\subset X_n$ the union of the faces $\partial_i^j(X_{n-1})$. One defines

$$(6.3)\quad \mathcal{Z}_0(X_n) = \oplus\mathbb{Z}\xi,\ \xi\in X_n\setminus Y_n, \\ \xi \text{ closed point.}$$

For any 1-cycle C in X_n, one denotes by $\nu:\bar{C}\to\mathbb{P}^1\times(\mathbb{P}^1)^n$ the normalisation of its compactification. One defines

$$(6.4)\quad \mathcal{Z}_1(X_n) = \oplus\mathbb{Z}C,\ C\subset X_n \text{ with} \\ \partial_i^j(C)\in\mathcal{Z}_0(X_{n-1}) \text{ and (cf. definition 4.1)} \\ 2\nu^{-1}(\{0\}\times(\mathbb{P}^1)^n)\subset\sum_{i=1}^n \nu^{-1}(\mathbb{P}^1\times(\mathbb{P}^1)^{i-1}\times\{1\}\times(\mathbb{P}^1)^{n-i})$$

One defines

$$(6.5)\quad \partial := \sum_{i=1}^n(-1)^i(\partial_i^0-\partial_i^\infty) : \mathcal{Z}_1(X_n)\to\mathcal{Z}_0(X_{n-1}) \text{ for all } i,j.$$

Further one defines the differential form

(6.6)
$$\psi_n = \frac{1}{x}\frac{dy_1}{y_1} \wedge \ldots \wedge \frac{dy_n}{y_n} \in \Gamma(\mathbb{P}^1 \times (\mathbb{P}^1)^n, \Omega^1_{(\mathbb{P}^1\times(\mathbb{P}^1)^n)/\mathbb{Z}}(\log Y_n)(\{x=0\})).$$

We motivate the choice of this differential form as follows. One considers

(6.7)
$$V_n(t) = (\mathbb{P}^1 \setminus \{0,t\}, \infty) \times (\mathbb{P}^1 \setminus \{0,\infty\}, 1)^n.$$

Its cohomology

(6.8)
$$H^{n+1}(V_n(t)) = H^1 \otimes (H^1)^n = F^1 \otimes (F^1)^n$$

is Hodge-Tate for $t \neq 0$. The generator is given by

(6.9)
$$\omega_{n+1}(t) = (\frac{dx}{x} - \frac{d(x-t)}{(x-t)}) \wedge \frac{dy_1}{y_1} \wedge \ldots \wedge \frac{dy_n}{y_n}.$$

Thus

(6.10)
$$\frac{\omega_{n+1}(t)}{t}|_{t=0} = d(\psi_n).$$

Definition 6.1. We define the additive cubical (higher) Chow groups

$$TH^n_M(k,n) = \mathcal{Z}_0(X_{n-1})/\partial\mathcal{Z}_1(X_n)$$

One has the following reciprocity law

Proposition 6.2. *The map $\mathcal{Z}_0(X_{n-1}) \to \Omega^{n-1}_k$ which associates to a closed point $\xi \in X_{n-1} \setminus Y_{n-1}$ the value* $\mathrm{Trace}(\kappa(\xi)/k)(\psi_{n-1}(\xi))$ *factors through*

$$TH^n_M(k,n) := \mathcal{Z}_0(X_{n-1})/\partial\mathcal{Z}_1(X_n).$$

Proof. Let C be in $\mathcal{Z}_1(X_n)$. Let $\Sigma \subset \bar{C}$ be the locus of poles of $\nu^*\psi_n$. One has the functoriality map

(6.11)
$$\nu^* : \Omega^n_{\mathbb{P}^1\times(\mathbb{P}^1)^{n-1}}(\log Y_{n-1})(\{x=0\}) \to \Omega^{n-1}_{\bar{C}/\mathbb{Z}}(*\Sigma).$$

Thus reciprocity says

(6.12)
$$\sum_{\sigma\in\Sigma} \mathrm{res}_\sigma \nu^*(\psi_n) = 0.$$

Recall that here res means the following. One has a surjection

(6.13)
$$\Omega^n_{\bar{C}/\mathbb{Z}}(*\Sigma) \to \Omega^{n-1}_{k/\mathbb{Z}} \otimes \omega_{\bar{C}/k}(*\Sigma)$$

which yields

(6.14)
$$\Gamma(\bar{C}, \Omega^n_{\bar{C}/\mathbb{Z}}(*\Sigma)) \to \Gamma(\bar{C}, \Omega^{n-1}_{k/\mathbb{Z}} \otimes \omega_{\bar{C}/k}(*\Sigma)) = \Omega^{n-1}_{k/\mathbb{Z}} \otimes \Gamma(\bar{C}, \omega_{\bar{C}/k}(*\Sigma)).$$

By definition, res on $\Omega^{n-1}_{k/\mathbb{Z}} \otimes \omega_{\bar{C}/k}(*\Sigma)$ is $1 \otimes \mathrm{res}$. This explains the reciprocity. Now we analyze $\Sigma \subset \sigma^{-1}(Y_n \cup \{x=0\})$. Let t be a local parameter on $\bar{C}$ in a point σ of $\nu^{-1}(\{x=0\})$.

We write $x = t^m \cdot u$, where $u \in \mathcal{O}^\times_{\bar{C},\sigma}, m \geq 0$. If $m \geq 1$, the assumption we have on $\mathcal{Z}_1$ says that there is at least one i such that $\{t=0\}$ lies in

$\nu^{-1}(\{y_i = 1\})$. Let us order $i = 1, \ldots, n$ such that $\{t = 0\}$ lies in $\nu^{-1}(\{y_i = 1\})$ for $i = 1, 2, \ldots, r$. Thus we write

$$y_i - 1 = t^{m_i} \cdot u_i, m_1 \geq m_2 \geq \ldots \geq 1, u_i \in \mathcal{O}^\times, i = 1, \tag{6.15}$$
$$y_i = t^{p_i} u_i, p_i \geq 0, u_i \in \mathcal{O}^\times, i = r+1, \ldots, n.$$

The assumption we have says

$$2m \leq m_1 + \ldots + m_r. \tag{6.16}$$

One has around the point σ

$$\nu^{-1}(\psi_n)|_\sigma = \frac{u^{-1}}{t^m} \cdot \frac{d(t^{m_1} \cdot u_1)}{1 + t^{m_1} \cdot u_1} \wedge \ldots \wedge \frac{d(t^{m_r} \cdot u_r)}{1 + t^{m_r} \cdot u_r} \wedge_{i=r+1}^{n} \frac{d(t^{p_i} \cdot u_i)}{t^{p_i} \cdot u_i}. \tag{6.17}$$

We analyze the poles of the right hand side. The numerator of this expression is divisible by $t^{(m_1 + \ldots + m_r) - 1}$. Thus the condition for $\nu^{-1}\psi_n$ to be smooth in σ is

$$m + 1 \leq m_1 + \ldots + m_r. \tag{6.18}$$

This is our condition. We have

$$\nu^{-1}\psi_n \text{ smooth in } \nu^{-1}(\{x = 0\}). \tag{6.19}$$

On the other hand, one obviously has

$$\mathrm{res}_{y_i=0}\psi_n = -\mathrm{res}_{y_i=\infty}\psi_n = (-1)^i \psi_{n-1}. \tag{6.20}$$

Thus one concludes

$$\sum_{\sigma \in \Sigma} \mathrm{res}_\sigma \nu^{-1}\psi_n = \sum_{i=1}^{n} (-1)^i \psi_{n-1}(\partial_i^0 - \partial_i^\infty)(C) = 0. \tag{6.21}$$

□

We want to see that the reciprocity map in proposition 6.2 is an isomorphism. Define

$$k \otimes_{\mathbb{Z}} \wedge_{i=1}^{n-1} k^\times \to TH_M^n(k, n) \tag{6.22}$$
$$a \otimes (b_1 \wedge \ldots \wedge b_{n-1}) \mapsto (\frac{1}{a}, b_1, \ldots, b_{n-1}) \text{ for } a \neq 0$$
$$\mapsto 0 \text{ for } a = 0.$$

PROPOSITION 6.3. *Assume $\frac{1}{6} \in k$. Then (6.22) factors through*

$$\Omega_k^{n-1} \to TH_M^n(k, n).$$

Proof. One has the following relations, where $\equiv$ means equivalence modulo $\partial \mathcal{Z}_1(X_{n+1})$:

$$(\frac{1}{x + x'}, y_1, \ldots, y_n) \equiv (\frac{1}{x}, y_1, \ldots, y_n) + (\frac{1}{x'}, y_1, \ldots, y_n) \tag{6.23}$$

$$(x, y_1 z_1, y_2, \ldots, y_n) \equiv (x, y_1, y_2, \ldots, y_n) + (x, z_1, y_2, \ldots, y_n) \tag{6.24}$$

$$(x, -1, y_2, \ldots, y_n) \equiv 0 \in TH_M^n(k, n). \tag{6.25}$$

Note, the last is obviously a consequence of the first two:

$$(2x, -1, y_2, \ldots, y_n) = 2(x, -1, y_2, \ldots, y_n) = (x, 1, y_2, \ldots, y_n) = 0, \tag{6.26}$$

so we need only consider (6.23) and (6.24). Assume first $xx'(x + x') \neq 0$, and define

$$C = (t, y_1 = \frac{(1 - xt)(1 - x't)}{1 - (x + x')t}, y_2, \ldots, y_n) \in \mathcal{Z}_1(X_n). \tag{6.27}$$

Indeed, the expansion of y_1 in $t = 0$ reads $1 + t^2 c_2 +$ (higher order terms), so our modulus condition is fulfilled. Also we have taken $y_i \in k^\times$ so C meets the faces properly. Then one has

$$\partial(C) = (\frac{1}{x}, y_2, \ldots, y_n) + (\frac{1}{x'}, y_2, \ldots, y_n) - (\frac{1}{x + x'}, y_2, \ldots, y_n). \tag{6.28}$$

Similarly, if $x + x' = 0$, then one sets

$$C = (t, y_1 = (1 - \frac{t^2}{x^2}), y_i) \in \mathcal{Z}_1(X_n). \tag{6.29}$$

One has

$$\partial(C) = (x, y_i) + (-x, y_i), \tag{6.30}$$

proving (6.23). Note the proposition for $n = 1$ is a consequence of this identity. To show multiplicativity in the y variables, one uses Totaro's curve [23]. There is a $\mathcal{C} \in \mathcal{Z}_1(B^{n+1})$ with $\partial(\mathcal{C}) = (y_1 z_1, y_2, \ldots, y_n) - (y_1, y_2, \ldots, y_n) - (z_1, y_2, \ldots, y_n)$. One sets $C = (x, \mathcal{C}) \in \mathcal{Z}_1(X_{n+1})$. Here x is fixed and nonzero, so the modulus condition is automatic, and one has $\partial(C) = (x, \partial(\mathcal{C}))$. This proves (6.24).

It remains to verify the Cathelineau relation (cf. [6], [7])

$$(\frac{1}{a}, a, b_2, \ldots, b_n) + (\frac{1}{1 - a}, (1 - a), b_2, \ldots, b_n) \equiv 0. \tag{6.31}$$

In fact, the $b_2, \ldots b_n \in k^\times$ play no role, so we will drop them. One considers the 1-cycle which is given by its parametrization

$$\begin{aligned} Z(a) &= -Z_1(a) + Z_2 \\ Z_1(a) &= (t, 1 + \frac{t}{2}, 1 - \frac{a^2 t^2}{4}) \\ Z_2 &= (\frac{t}{4}, 1 + \frac{t}{6}, 1 - \frac{t^2}{4}), \end{aligned} \tag{6.32}$$

We see immediately that $Z \in \mathcal{Z}_1(X_2)$. One has

$$\partial(Z_1(a)) \tag{6.33}$$
$$= (-2, 1-a^2) - (\frac{2}{a}, 1+\frac{1}{a}) - (-\frac{2}{a}, 1-\frac{1}{a}) =$$
$$(-2, 1-a) + (-2, 1+a) + (\frac{2}{a}, \frac{a-1}{a+1}) =$$
$$(-2, 1-a) + (\frac{2}{a}, a-1) + (-2, 1+a) - (\frac{2}{a}, a+1) =$$
$$(-2, a-1) + (\frac{2}{a}, a-1) + (-2, 1+a) + (-\frac{2}{a}, a+1) =$$
$$(\frac{2}{a-1}, a-1) - (\frac{2}{a+1}, a+1).$$

Setting $a = 1 - 2b$, one obtains

$$\partial(Z_1(a)) = (-\frac{1}{b}, -2b) - (\frac{1}{1-b}, 2(1-b)) = \tag{6.34}$$
$$-(\frac{1}{b}, b) - (\frac{1}{1-b}, 1-b) - (\frac{1}{b}, 2) - (\frac{1}{1-b}, 2) =$$
$$-(\frac{1}{b}, b) - (\frac{1}{1-b}, 1-b) - (1, 2).$$

One has

$$\partial(Z_2) = (-\frac{3}{2}, -8) - (\frac{1}{2}, \frac{4}{3}) - (-\frac{1}{2}, \frac{2}{3}) = \tag{6.35}$$
$$3(-\frac{3}{2}, 2) - (\frac{1}{2}, 2) =$$
$$(-\frac{3}{2}, 2) + (-\frac{3}{4}, 2) - (\frac{1}{2}, 2) =$$
$$(-\frac{1}{2}, 2) - (\frac{1}{2}, 2) = -(1, 2).$$

In conclusion

$$\partial(Z(1-2a)) = (\frac{1}{a}, a, b_i) + (\frac{1}{1-a}, 1-a, b_i). \tag{6.36}$$

□

We now have well defined maps ϕ_n, ψ_n

$$\Omega_k^{n-1} \xrightarrow{\phi_n} TH_M^n(k,n) \xrightarrow{\psi_n} \Omega_k^{n-1} \tag{6.37}$$

which split $TH_M^n(k,n)$. The image of the differential forms consists of all 0-cycles which are equivalent to 0-cycles $\sum m_i p_i$ with $p_i \in X_n(k)$.

THEOREM 6.4. . *Assume* $\frac{1}{6} \in k$. *The above maps identify* $TH_M^n(k,n)$ *with* Ω_k^{n-1}.

Proof. It suffices to show that the class of a give closed point $p \in (\mathbb{A}^1 - \{0\}) \times (\mathbb{P}^1 - \{0, 1, \infty\})^{n-1}$ lies in the image of ϕ_n. Write $\kappa = \kappa(p)$ for the residue field at p. One first applies a Bertini type argument as in [6], Proposition 4.5, to reduce to the case where κ/k is separable. Then we follow the argument in loc.cit. The degree $[\kappa : k] < \infty$, so standard cycle constructions yield a norm map $N : TH^n_M(\kappa, n) \to TH^n_M(k, n)$. We claim the diagram

$$\begin{array}{ccc} \Omega^{n-1}_\kappa & \xrightarrow{\phi_{n,\kappa}} & TH^n_M(\kappa, n) \\ {\scriptstyle \mathrm{Tr}}\downarrow & & \downarrow{\scriptstyle N} \\ \Omega^{n-1}_k & \xrightarrow{\phi_{n,k}} & TH^n_M(k, n) \end{array} \tag{6.38}$$

is commutative, where Tr is the trace on differential forms. Indeed, $\Omega^n_\kappa = \kappa \otimes \Omega^n_k$, so it suffices to check on forms $a d\log(b_1) \wedge \ldots \wedge d\log(b_{n-1})$ with $a \in \kappa$ and $b_i \in k$. But in this situation, we have projection formulas, both for 0-cycles and for differential forms, and it is straightforward to check (ignore the b_i and reduce to $n = 1$)

(6.39)

$$\mathrm{Tr}(a d\log(b_1) \wedge \ldots \wedge d\log(b_{n-1})) = \mathrm{Tr}_{\kappa/k}(a) d\log(b_1) \wedge \ldots \wedge d\log(b_{n-1})$$

$$N(\frac{1}{a}, b_1, \ldots, b_{n-1}) = \begin{cases} (\frac{1}{\mathrm{Tr}_{\kappa/k} a}, b_1, \ldots, b_{n-1}) & \mathrm{Tr}(a) \neq 0 \\ 0 & \mathrm{Tr}(a) = 0 \end{cases}$$

Write $[p]_\kappa$ (resp. $[p]_k$) for the class of $p \in TH^n_M(\kappa, n)$ (resp. $TH^n_M(k, n)$.) One has $[p]_k = N([p]_\kappa)$. Since p is κ-rational, $[p]_\kappa \in \mathrm{Image}(\phi_\kappa)$. Commutativity of (6.38) implies $[p]_k \in \mathrm{Image}(\phi_k)$. It follows that ϕ_k is surjective, proving the theorem. □

References

[1] A. Beilinson, P. Deligne: Interprétation motivique de la conjecture de Zagier reliant polylogarithmes et régulateurs, in *Motives*, Proc. Symp. Pure Math Vol. 55, Part 1, AMS (1994), 97-122.

[2] S. Bloch: Higher regulators, algebraic K-theory, and zeta functions of elliptic curves. CRM Monograph Series, 11. American Mathematical Society, Providence, RI, 2000. x+97 pp.

[3] S. Bloch: Algebraic Cycles and the Lie Algebra of Mixed Tate Motives, JAMS Vol. 4 no. 4 Oct. (1991) 771-791.

[4] S. Bloch, I. Kriz: Mixed Tate motives, Annals of Math 140 (1994), 557-605.

[5] S. Bloch, V. Srinivas: Enriched Hodge Structures, in Proc. Int. Coll. on Algebra, Arithmetic and Geometry, Mumbai 2000, Tata Inst. of Fund. Res. St. in Math. 16 (2002), 171-184.

[6] S. Bloch, H. Esnault: An additive version of higher Chow groups, preprint 2001, 16 pages.

[7] J-L. Cathelineau: Remarques sur les différentielles des polylogarithmes uniformes, Ann. Inst. Fourier, Grenoble 46 (1996) no. 5, 1327-1347.

[8] P. Deligne: Théorie de Hodge III, Publ. Math. IHES 44 (1974), 5-77.
[9] P. Deligne: Le symbole modéré, Inst. Hautes Études Sci. Publ. Math. 73 (1991), 147–181.
[10] Z. Daróczy: Generalized Information Functions, Information and Control 16 (1970) 36-51.
[11] P. Elbaz-Vincent, H. Gangl: On poly(ana)logs I, Compositio 130 (2002), 161-210.
[12] H. Esnault: On the Loday symbol in the Deligne-Beilinson cohomology. K-Theory 3 (1989), no. 1, 1–28.
[13] H. Esnault: Hodge type of subvarieties of $\mathbb{P}^n$ of small degrees. Math. Ann. 288 (1990), No. 3, 549–551.
[14] H. Esnault, V. Srinivas, E. Viehweg: The universal regular quotient of the Chow group of points on projective varieties. Invent. Math. 135 (1999), no. 3, 595–664.
[15] A. Goncharov: Volumes of hyperbolic manifolds and mixed Tate motives. J. Amer. Math. Soc. 12 (1999), no. 2, 569–618.
[16] T. Goodwillie: Relative algebraic K-theory and cyclic homology. Ann. of Math. (2) 124 (1986), no. 2, 347–402.
[17] J-G. Grebet: Aspects infinitésimaux du troisième problème de Hilbert, Thèse.
[18] G. Laumon: Transformation de Fourier généralisée, arXiv:alg-geom/9603004.
[19] M. Levine:Deligne-Beilinson cohomology for singular varieties. Algebraic K-theory, commutative algebra, and algebraic geometry (Santa Margherita Ligure, 1989), 113–146, Contemp. Math., 126, Amer. Math. Soc., Providence, RI, 1992.
[20] J.-L. Loday: Cyclic homology. Grundlehren der Mathematischen Wissenschaften 301. Springer-Verlag, Berlin, 1998. xx+513 pp.
[21] Yu. Nesterenko, A. Suslin Homology of the full linear group over a local ring, and Milnor's K- theory, Math. USSR, Izv. 34, No.1, 121-145 (1990); translation from Izv. Akad. Nauk SSSR, Ser. Mat. 53, No.1 (1989), 121-146.
[22] J. Stienstra: On K_2 and K_3 of truncated polynomial rings. Algebraic K-theory, Proc. Conf., Evanston 1980, Lect. Notes Math. 854 (1981), 409-455.
[23] B. Totaro: Milnor K-theory is the simplest part of algebraic K-theory. K-Theory 6, No.2 (1992), 177-189.
[24] W. Van der Kallen: The relative K_2 of truncated polynomial rings. J. Pure Appl. Algebra 34 (1984), 277-289.

Spencer Bloch
Dept. of Mathematics
University of Chicago
Chicago, IL 60637
USA
bloch@math.uchicago.edu

Hélène Esnault
Mathematik
Universität Essen
Essen
Germany
esnault@uni-essen.de

Equivariant Weierstrass Preparation and Values of L-functions at Negative Integers

Dedicated to Professor Kazuya Kato

David Burns and Cornelius Greither

Received: September 30, 2002
Revised: March 18, 2003

Abstract. We apply an equivariant version of the p-adic Weierstrass Preparation Theorem in the context of possible non-commutative generalizations of the power series of Deligne and Ribet. We then consider CM abelian extensions of totally real fields and by combining our earlier considerations with the known validity of the Main Conjecture of Iwasawa theory we prove, modulo the conjectural vanishing of certain μ-invariants, a (corrected version of a) conjecture of Snaith and the 'rank zero component' of Kato's Generalized Iwasawa Main Conjecture for Tate motives of strictly positive weight. We next use the validity of this case of Kato's conjecture to prove a conjecture of Chinburg, Kolster, Pappas and Snaith and also to compute explicitly the Fitting ideals of certain natural étale cohomology groups in terms of the values of Dirichlet L-functions at negative integers. This computation improves upon results of Cornacchia and Østvær, of Kurihara and of Snaith, and, modulo the validity of a certain aspect of the Quillen-Lichtenbaum Conjecture, also verifies a finer and more general version of a well known conjecture of Coates and Sinnott.

2000 Mathematics Subject Classification: 11R42 11R33 11R70
Keywords and Phrases: Iwasawa theory, values of L-functions, Euler characteristics, Fitting ideals

1. Introduction

In a beautiful series of papers in 1993 Kato formulated and studied a 'Generalized Iwasawa Main Conjecture' for motives over number fields with respect to certain commutative coefficient rings [24, 25, 26]. This conjecture refined the 'Tamagawa number conjectures' previously formulated by Bloch and Kato in [3] and by Fontaine and Perrin-Riou in [20], and led naturally to the subsequent formulation by Flach and the first named author in [9] of a Tamagawa number conjecture for motives over number fields with respect to more general coefficient rings which, in particular, need not be commutative.

The above approach has already led to some remarkable new insights and results in a number of rather different contexts. We recall, for example, that it has led to a universal approach to and refinement of the 'refined Birch and Swinnerton-Dyer Conjectures' for abelian varieties with complex multiplication which were formulated by Gross (cf. [9, Rem. 10 and §3.3, Examples c), d) e)]) and of all of the 'refined abelian Stark conjectures' which were formulated by Gross, by Tate, by Rubin and by Darmon (cf. [6, 7]). At the same time, the approach has led to a natural re-interpretation and refinement of all of the central conjectures of classical Galois module theory (cf. [8, 5, 2]) and has also more recently become the focus of attempts to formulate a natural 'Main Conjecture of non-abelian Iwasawa theory' (see, for example, the forthcoming articles of Huber and Kings and of Weiss and the first named author in this regard). It is certainly a great pleasure, on the occasion of Kato's fiftieth birthday, to offer in this manuscript some additional explicit evidence in support of his Generalized Iwasawa Main Conjecture and, more generally, to demonstrate yet further the enormous depth and significance of the approach that he introduced in [24, 25, 26].

To describe the main results of the present manuscript in some detail we now fix a totally real number field k and a finite Galois extension K of k which is either totally real or a CM field, and we set $G := \mathrm{Gal}(K/k)$. We also fix a rational prime number p and an algebraic closure $\mathbb{Q}_p^c$ of the field of p-adic rationals $\mathbb{Q}_p$.

We recall that if G is abelian, then a key ingredient of Wiles' proof [39] of the Main Conjecture of Iwasawa theory is the construction by Deligne and Ribet of an element of the power series ring $\mathbb{Z}_p[G][[T]]$ which is uniquely characterized by its relation to the p-adic L-series associated to the extension K/k.

In this manuscript we first relax the restriction that G is abelian and discuss the possible existence of elements of the (non-commutative) power series ring $\mathbb{Z}_p[G][[T]]$ which are related in a precise manner to the p-adic Artin L-functions associated to irreducible $\mathbb{Q}_p^c$-valued characters of G. In particular, under a certain natural hypothesis on G (which does not require G to be abelian), and assuming the vanishing of certain μ-invariants, we apply an appropriate version of the Weierstrass Preparation Theorem for the ring $\mathbb{Z}_p[G][[T]]$ to derive relations between two hypothetical generalizations of the power series of Deligne and Ribet.

In the remainder of the manuscript our main aim is to show that, if G is abelian, then the above considerations can be combined with the constructions of Deligne and Ribet and the theorem of Wiles to shed light on a number of interesting questions. To describe these applications we assume for the rest of this introduction that G is abelian.

Our first application is to the 'Wiles Unit Conjecture' which is formulated by Snaith in [34, Conj. 6.3.4]. Indeed, by using the above approach we are able to show that the validity of a slightly amended version of Snaith's conjecture follows directly from the (in general conjectural) vanishing of certain natural μ-invariants, and also to show that the original version of Snaith's conjecture does not hold in general (cf. Remark 5).

To describe the next application we fix an integer r with $r > 1$. Under the aforementioned hypothesis concerning μ-invariants we shall prove the Generalized Iwasawa Main Conjecture of [25, Conj. 3.2.2] for the pair $(h^0(\mathrm{Spec}\,(K))(1-r), \mathfrak{A}_r)$, where $\mathfrak{A}_r$ is a natural ring which annihilates the space $\mathbb{Q} \otimes_{\mathbb{Z}} K_{2r-1}(K)$. If $k = \mathbb{Q}$, then (by a result of Ferrero and Washington in [18]) the appropriate μ-invariants are known to vanish and hence we obtain in this way a much more direct proof of the relevant parts of the main result (Cor. 8.1) of our earlier paper [11]. We remark however that the proofs of all of our results in this area involve a systematic use of the equivariant Iwasawa theory of complexes which was initiated by Kato in [25] and subsequently extended by Nekovář in [30].

As a further application, we combine our result on the Generalized Iwasawa Main Conjecture with certain explicit cohomological computations of Flach and the first named author in [8] to prove (modulo the aforementioned hypothesis on μ-invariants) that the element $\Omega_{r-1}(K/k)$ of $\mathrm{Pic}(\mathbb{Z}[G])$ which is defined by Chinburg, Kolster, Pappas and Snaith in [13] belongs to the kernel of the natural scalar extension morphism $\mathrm{Pic}(\mathbb{Z}[G]) \to \mathrm{Pic}(\mathfrak{A}_r)$.

As a final application we then combine our approach with a development of a purely algebraic observation of Cornacchia and the second named author in [15] to compute explicitly certain Fitting ideals which are of arithmetical interest. To be more precise in this regard we assume that p is odd, we fix a finite set of places S of K which contains all archimedean places and all places which either ramify in K/k or are of residue characteristic p and we write $\mathcal{O}_{K,S}$ for the ring of S-integers of K. Writing τ for the complex conjugation in G we let e_r denote the idempotent $\frac{1}{2}(1+(-1)^r\tau)$ of $\mathbb{Z}_p[G]$. Then, under the aforementioned hypothesis on μ-invariants, we prove that the Fitting ideal of the étale cohomology module $e_r \cdot H^2(\mathrm{Spec}(\mathcal{O}_{K,S})_{\text{ét}}, \mathbb{Z}_p(r))$ over the ring $\mathbb{Z}_p[G]e_r$ can be completely described in terms of the values at $1-r$ of the S-truncated Dirichlet L-functions which are associated to K/k. This result improves upon previous results of Cornacchia and Østvær [16, Thm. 1.2], of Kurihara [28, Cor. 12.5] and of Snaith [34, Thm. 1.6, Thm. 2.4, Thm. 5.2] and also implies a natural analogue of the main result of Solomon in [36] concerning relations between Bernoulli numbers and the structure of certain ideal class groups (cf. Remark 8). We finally remark that, under the assumed validity of a particular case of the Quillen-Lichtenbaum Conjecture, our result verifies a finer and more general version of the well known conjecture formulated by Coates and Sinnott in [14, Conj. 1].

Acknowledgements. The authors are very grateful to Masato Kurihara and Otmar Venjakob for illuminating discussions relating to the results of this manuscript. In addition, they are grateful to the referee for several very helpful remarks.

2. Equivariant Weierstrass Preparation

In this section we discuss a natural generalization of the classical p-adic Weierstrass Preparation Theorem.

We let $\mathfrak{A}$ be a ring, write $\mathrm{rad}(\mathfrak{A})$ for its Jacobson radical and set $\overline{\mathfrak{A}} := \mathfrak{A}/\mathrm{rad}(\mathfrak{A})$. In the sequel we shall say that $\mathfrak{A}$ is *strictly admissible* if it is both separated and complete in the $\mathrm{rad}(\mathfrak{A})$-adic topology and is also such that $\overline{\mathfrak{A}}$ is a skew field. More generally, we shall say that $\mathfrak{A}$ is *admissible* if it is a finite product of strictly admissible rings.

Remark 1. Let G be a finite group and p any prime number. It can be shown that the group ring $\mathfrak{A} = \mathbb{Z}_p[G]$ is admissible if G is the direct product of a p-group and an abelian group (and, in particular therefore, if G is itself abelian). Note also that in any such case the ring $\overline{\mathfrak{A}}$ is a product of finite skew fields and is therefore commutative.

In this manuscript we define the power series ring $\mathfrak{A}[[T]]$ over $\mathfrak{A}$ just as for commutative base rings; in particular, we require that the variable T commutes with all elements of the coefficient ring $\mathfrak{A}$. We observe that, with this definition, an element f of $\mathfrak{A}[[T]]$ is invertible if and only if its constant term $f(0)$ is invertible in $\mathfrak{A}$.

If f is any element of $\mathfrak{A}[[T]]$, then we write $\overline{f}$ for its image under the obvious reduction map $\mathfrak{A}[[T]] \to \overline{\mathfrak{A}}[[T]]$. Assume for the moment that $\mathfrak{A}$ is admissible, with a decomposition $\mathfrak{A} = \prod_{i\in I}\mathfrak{A}_i$ for strictly admissible rings $\mathfrak{A}_i$. If $f = (f_i)_{i\in I}$ is any element of $\mathfrak{A}[[T]]$, then we define the *degree* $\deg(f)$, respectively *reduced degree* $\mathrm{rdeg}(f)$, of f to be the vector $(\deg(f_i))_{i\in I}$, respectively $(\deg(\overline{f}))_{i\in I}$, where by convention we regard the zero element of each ring $\mathfrak{A}_i[[T]]$ and $\overline{\mathfrak{A}_i}[[T]]$ to be of degree $+\infty$. We observe that if $\mathfrak{A} = \mathbb{Z}_p$, then $\mathrm{rdeg}(f)$ is finite if and only if the μ-invariant of the $\mathbb{Z}_p[[T]]$-module $\mathbb{Z}_p[[T]]/(f)$ is zero. By analogy, if $\mathfrak{A}$ is any admissible ring, then we shall write '$\mu_{\mathfrak{A}}(f) = 0$' to express the fact that (each component of) $\mathrm{rdeg}(f)$ is finite.

If $\mathfrak{A}$ is strictly admissible, respectively admissible, then we shall say that an element of $\mathfrak{A}[[T]]$ is a *distinguished polynomial* if it is a monic polynomial all of whose non-leading coefficients are in $\mathrm{rad}(\mathfrak{A})$, respectively if all of its components are distinguished polynomials (of possibly varying degrees).

Proposition 2.1. *('Equivariant Weierstrass Preparation') Let* $\mathfrak{A}$ *be an admissible ring. If* f *is any element of* $\mathfrak{A}[[T]]$ *for which* $\mu_{\mathfrak{A}}(f) = 0$*, then there exists* *a unique distinguished polynomial* f^* *and a unique unit element* u_f *of* $\mathfrak{A}[[T]]$ *such that* $f = f^* \cdot u_f$.

Proof. By direct verification one finds that the argument of [4, Chap.VII, §3, no. 8] extends to the present (non-commutative) context to prove the following 'Generalized Division Lemma': if f is any element of $\mathfrak{A}[[T]]$ for which $\mu_{\mathfrak{A}}(f) = 0$, then each element g of $\mathfrak{A}[[T]]$ can be written uniquely in the form $g = fq + r$ where q and r are elements of $\mathfrak{A}[[T]]$ and each component of $\deg(r)$ is strictly less than the corresponding component of $\mathrm{rdeg}(f)$.
The deduction of the claimed result from this Generalized Division Lemma now proceeds exactly as in [31, V.5.3.3-V.5.3.4]. □

Remark 2. i) If $\mathfrak{A}$ is a discrete valuation ring, then it is (strictly) admissible and Proposition 2.1 is equivalent to the classical Weierstrass Preparation Theorem (cf. [38, Thm. 7.1]).
ii) Shortly after the first version of this manuscript was circulated (in December 2001) we learnt of a recent preprint [37] of Venjakob in which a Weierstrass Preparation Theorem is proved under conditions which are considerably more general than those of Proposition 2.1. We remark that if $\mathfrak{A}$ is strictly admissible, then it can be shown that the result of Proposition 2.1 is indeed equivalent to a special case of the main result of loc. cit.
iii) For any prime p, any $\mathbb{Z}_p$-order $\mathfrak{A}$ which is not admissible and any element f of $\mathfrak{A}[[T]]$ a natural interpretation of the equality '$\mu_{\mathfrak{A}}(f) = 0$' would be that, for each primitive central idempotent ϵ of $\mathfrak{A}$, the element $\epsilon \cdot f$ is not divisible by p (indeed, this interpretation recovers that given above in the case that $\mathfrak{A}$ is admissible). However, under this interpretation the product decomposition of Proposition 2.1 is not always possible. For example, if $\mathfrak{A} = \mathrm{M}_2(\mathbb{Z}_p)$, then the constant series $f := \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ satisfies $\mu_{\mathfrak{A}}(f) = 0$ (in the above sense) and yet cannot be written in the stated form $f^* \cdot u_f$. Indeed, if it did admit such a decomposition, then the $\mathbb{Z}_p$-module $\mathfrak{A}[[T]]/f \cdot \mathfrak{A}[[T]]$ would be finitely generated and this is not true since $f \cdot \mathfrak{A}[[T]]$ is equal to the subset of $\mathrm{M}_2(\mathbb{Z}_p[[T]])$ consisting of those matrices which have both second row entries divisible by p.
iv) In just the same way as Proposition 2.1, one can prove that if $\mathfrak{A}$ is admissible, then every element f of $\mathfrak{A}[[T]]$ for which $\mu_{\mathfrak{A}}(f) = 0$ can be written uniquely in the form $u^f \cdot f_*$ with f_* a distinguished polynomial and u^f a unit of $\mathfrak{A}[[T]]$. However, as the following example shows, the relation between the elements f^* and f_* (and u^f and u_f) is in general far from clear.

EXAMPLE 1. Let a and b be elements of $\mathrm{rad}(\mathfrak{A})$, and set $f := (T - a)(1 + bT)$. Then it is clear that $\mu_{\mathfrak{A}}(f) = 0$, $f^* = T - a$ and $u_f = 1 + bT$. On the other hand it may be shown that $f_* = T - c$ where c is the unique element of $\mathfrak{A}$ which satisfies $T - c \in \mathfrak{A} \cdot f$, and that $u^f = (1 - ab + bc) + bT$. Upon calculating c as a power series in the noncommuting variables a and b, one finds that

$$c = a - baa + aba + bbaaa - baaba - abbaa + ababa + \dots.$$

However, describing c completely is tricky. For example, a convenient option is to use the context-free formal language with four productions $S \to Ta$, $S \to TbSS$, $T \to \epsilon, T \to Tab$. Indeed, it may be shown that the degree n part of the series c is equal to the weighted sum of all words w of length n in this language, counted with weight $(-1)^{e(w)}$ where $e(w)$ denotes the number of times that b occurs in w not immediately preceded by a. For background and a similar example, we refer the reader to [32], in particular Chapter VI.

3. p-adic L-functions

In this section we apply Proposition 2.1 in the context of Iwasawa theory. The main result of this section (Theorem 3.1) was first motivated by the observation that the constructions of Deligne and Ribet which are used by Wiles in [39, p. 501f] can be combined with Proposition 2.1 to shed light upon the 'Wiles Unit Conjecture' formulated by Snaith in [34, Conj. 6.3.4]. In particular, by these means we shall prove that the validity in the relative abelian case of a corrected version of Snaith's conjecture is a direct consequence of the (in general conjectural) vanishing of certain natural μ-invariants.

We first introduce some necessary notation. Throughout this section we fix an odd prime p and a finite group G. We recall that $\mathbb{Q}_p^c$ is a fixed algebraic closure of $\mathbb{Q}_p$, and we write $\mathrm{Irr}_p(G)$ for the set of irreducible $\mathbb{Q}_p^c$-characters of G. For each $\rho \in \mathrm{Irr}_p(G)$ we write $\mathbb{Z}_p(\rho)$ for the extension of $\mathbb{Z}_p$ which is generated by the values of ρ.

Following Fröhlich [21, Chap. II], we now define for each element f of $\mathbb{Z}_p[G][[T]]$ a canonical element $\mathrm{Det}(f)$ of $\mathrm{Map}(\mathrm{Irr}_p(G), \mathbb{Q}_p^c[[T]])$. To do this we fix a subfield N of $\mathbb{Q}_p^c$ which is of finite degree over $\mathbb{Q}_p$ and over which all elements of $\mathrm{Irr}_p(G)$ can be realized, and we write $\mathcal{O}_N$ for the valuation ring of N. For each character $\rho \in \mathrm{Irr}_p(G)$ we choose a finitely generated $\mathcal{O}_N[G]$-module L_ρ which is free over $\mathcal{O}_N$ (of rank n say) and is such that the space $L_\rho \otimes_{\mathcal{O}_N} N$ has character ρ, and we write $r_\rho : G \to \mathrm{GL}_n(\mathcal{O}_N)$ for the associated homomorphism. If now $f = \sum_{i\geq 0} c_i T^i$, then $r_\rho(f) := \sum_{i\geq 0} r_\rho(c_i)T^i$ belongs to $M_n(\mathcal{O}_N[[T]])$ and we define $\mathrm{Det}(f)(\rho) := \det(r_\rho(f)) \in \mathcal{O}_N[[T]]$ (which is indeed independent of the choices of field N and lattice L_ρ). We observe in particular that if ρ is any element of $\mathrm{Irr}_p(G)$ which is of dimension 1, then one has $\mathrm{Det}(f)(\rho) = \rho(f)$.

In the remainder of this manuscript we assume given a finite Galois extension of number fields K/k for which $\mathrm{Gal}(K/k) = G$. We write k_∞ (or k_∞^p if we need to be more precise) for the cyclotomic $\mathbb{Z}_p$-extension of k, and K_∞ (or K_∞^p) for the compositum of K and k_∞.

In the rest of this section we assume that k is totally real and that K is either totally real or a CM field. We also fix a finite set S of non-archimedean places of k which contains all non-archimedean places which ramify in K/k. We write $\mathrm{Irr}_p^+(G)$ for the subset of $\mathrm{Irr}_p(G)$ consisting of those characters which are even (that is, factor through characters of the Galois group of the maximal totally real extension K^+ of k in K). We fix a topological generator γ of $\mathrm{Gal}(k_\infty/k)$ and, with χ_{cyclo} denoting the cyclotomic character, we set $u := \chi_{\mathrm{cyclo}}(\gamma) \in \mathbb{Z}_p^\times$.

We recall that for each $\rho \in \mathrm{Irr}_p^+(G)$ there exists a p-adic L-function $L_{p,S}(-,\rho)$ and an associated element $f_{S,\rho}$ of the quotient field of $\mathbb{Z}_p(\rho)[[T]]$ such that

$$L_{p,S}(1-s,\rho) = f_{S,\rho}(u^s - 1) \tag{1}$$

for almost all $s \in \mathbb{Z}_p$. To be more precise about the denominator of $f_{S,\rho}$ we set $H_\rho := 1$ unless ρ is induced by a multiplicative character of $\mathrm{Gal}(k_\infty/k)$ (that is, ρ is a character of 'type W' in the terminology of Wiles [39]) in which case we set $H_\rho := \rho(\gamma)(1+T) - 1 \in \mathbb{Z}_p(\rho)[[T]]$. Then there exists an element $G_{S,\rho}$ of $\mathbb{Z}_p(\rho)[[T]][\frac{1}{p}]$ such that $f_{S,\rho} = G_{S,\rho} \cdot H_\rho^{-1}$ (cf. [39, Thm. 1.1]). We hope that the reader will not in the sequel be confused by our notation: whenever G occurs without a subscript it denotes a Galois group; whenever G is adorned with a subscript it denotes a power series.

By the classical Weierstrass Preparation Theorem, each series $G_{S,\rho}$ can be decomposed as a product

$$G_{S,\rho} = \pi(\rho)^{\mu(S,\rho)} \cdot G^*_{S,\rho} \cdot U_{S,\rho} \tag{2}$$

where $\pi(\rho)$ is a uniformising parameter of $\mathbb{Z}_p(\rho)$, $\mu(S,\rho)$ is an integer, $G^*_{S,\rho}$ is a distinguished polynomial and $U_{S,\rho}$ is a unit of $\mathbb{Z}_p(\rho)[[T]]$.
We now proceed to describe four natural hypotheses relating to the Weierstrass decompositions (2). The main result of this section will then describe certain relations that exist between these hypotheses.

Hypothesis (μ_p): *For each $\rho \in \mathrm{Irr}_p^+(G)$ one has $\mu(S,\rho) = 0$.*

Remark 3. i) It is a standard conjecture that Hypothesis (μ_p) is always valid (the first statement of this was due to Iwasawa [23]). However, at present the only general result one has in this direction is that Hypothesis (μ_p) is valid for K/k when $k = \mathbb{Q}$ and G is abelian. Indeed, this is proved by Ferrero and Washington in [18].
ii) In this remark we describe a natural Iwasawa-theoretical reinterpretation of Hypothesis (μ_p) in the case that K is totally real. To do this we write S_p for the union of S and the set of places of k which lie above p, and we let $Y(S_p)$ denote the Galois group of the maximal abelian pro-p-extension of K_∞ which is unramified outside the set of places which lie above any element of S_p. (We note that, since p is odd, any pro-p-extension of K_∞ is automatically unramified at all archimedean places.)

LEMMA 1. *If K is totally real, then Hypothesis (μ_p) is valid for K/k if and only if the μ-invariant of the $\mathbb{Z}_p[\mathrm{Gal}(K_\infty/K)]$-module $Y(S_p)$ is* 0.

Proof. We set $L := K(\zeta_p)$ and $\Delta := \mathrm{Gal}(L/K)$ and let $\omega : \Delta \to \mathbb{Z}_p^\times$ denote the Teichmüller character. For each $\mathbb{Z}_p[\Delta]$-module M and integer i we write $M^{(i)}$ for the submodule consisting of those elements m which satisfy $\delta(m) = \omega^i(\delta)\cdot m$ for all $\delta \in \Delta$. (Since $p \nmid |\Delta|$, each such functor $M \mapsto M^{(i)}$ is exact.)
We write Y, respectively Y_L, for the Galois group of the maximal abelian pro-p-extension of K_∞, respectively of L_∞, which is unramified outside all places above p. For each $\rho \in \mathrm{Irr}_p(\mathrm{Gal}(K/k))$ we also write G_ρ for the element of

$\mathbb{Z}_p(\rho)[[T]][\frac{1}{p}]$ which is defined just as $G_{S,\rho}$ but with S taken to be the empty set.

We first observe that $\mu(Y(S_p)) = 0$ if and only if $\mu(Y) = 0$ and that for each $\rho \in \mathrm{Irr}_p(\mathrm{Gal}(K/k))$ one has $\mu(S, \rho) = 0$ if and only if $\mu(G_\rho) = 0$.

We also note that $\mu(Y) = 0$ if and only if $\mu(X_L^{(1)}) = 0$, where X_L denotes the Galois group of the maximal unramified abelian pro-p-extension of L_∞. For the reader's convenience, we briefly sketch the argument. By Kummer duality, for each even integer i the module $Y_L^{(i)}$ is isomorphic to $\mathrm{Hom}(\mathrm{Cl}(L_\infty)^{(1-i)}, \mathbb{Q}_p/\mathbb{Z}_p)(1)$ and hence in turn pseudo-isomorphic to $X_L^{(1-i),\#}(1)$ where $\#$ indicates contragredient action of $\mathrm{Gal}(L_\infty/k)$ [31, (11.1.8), (11.4.3)]. This implies, in particular, that $Y = Y_L^{(0)}$ is pseudo-isomorphic to $X_L^{(1),\#}(1)$, and this in turn implies the claimed result.

We next recall that, as a consequence of [39, Thm. 1.4], one has $\mu(X_L^{(1)}) = 0$ if and only if $\mu(G_{\eta_0}) = 0$, where η_0 denote the trivial character of Δ.

To finish the proof of the lemma, we now need only invoke the inductive property of L-functions and Iwasawa series. Indeed, one has $G_{\eta_0} = G_{\chi_{\mathrm{reg}}}$, where χ_{reg} is the character of the regular representation of $\mathrm{Gal}(K/k)$ (note that this is equal to the induction of η_0 from K to k), and by its very definition, one has $G_{\chi_{\mathrm{reg}}} = \prod_\rho G_\rho^{\deg(\rho)}$, where ρ runs over all elements of $\mathrm{Irr}_p(\mathrm{Gal}(K/k))$. It is therefore clear that $\mu(G_{\eta_0}) = 0$ if and only if for all $\rho \in \mathrm{Irr}_p(\mathrm{Gal}(K/k))$ one has $\mu(G_\rho) = 0$ (or equivalently $\mu(S, \rho) = 0$), as required. □

We continue to introduce further natural hypotheses relating to the decompositions (2).

If f and f' are elements of $\mathbb{Z}_p[G][[T]]$, then we say that f is *right associated*, respectively *left associated*, to f' if there exists a unit element u of $\mathbb{Z}_p[G][[T]]$ such that $f = f' \cdot u$, respectively $f = u \cdot f'$.

Hypothesis (EPS) ('Equivariant Power Series') *Assume that Hypothesis (μ_p) is valid for K/k. Then there exist elements $G_S = G_S(T)$ and $H = H(T)$ of $\mathbb{Z}_p[G][[T]]$ which are each right associated to distinguished polynomials and are such that for all $\rho \in \mathrm{Irr}_p^+(G)$ the quotient* $\mathrm{Det}(G_S)(\rho)/\mathrm{Det}(H)(\rho)$ *is defined and equal to $G_{S,\rho}/H_\rho = f_{S,\rho}$.*

Remark 4. i) If $\mathbb{Z}_p[G]$ is admissible, then Proposition 2.1 (and Remark 2iv)) implies that an element f of $\mathbb{Z}_p[G][[T]]$ is right associated to a distinguished polynomial if and only if it is left associated to a distinguished polynomial and that these conditions are in turn equivalent to an equality $\mu_{\mathbb{Z}_p[G]}(f) = 0$.

ii) The completed group ring $\mathbb{Z}_p[[\mathrm{Gal}(K_\infty/k)]]$ is naturally isomorphic to the power series ring $\mathbb{Z}_p[\mathrm{Gal}(K_\infty/k_\infty)][[T]]$. If $K \cap k_\infty = k$, then this ring can be identified with $\mathbb{Z}_p[G][[T]]$ but in general no such identification is possible.

We write c_ρ for the leading coefficient of H_ρ (so, explicitly, one has $c_\rho = 1$ unless ρ is a non-trivial character of 'type W' in which case $c_\rho = \rho(\gamma)$). We observe that the polynomial $H_\rho^* := c_\rho^{-1} \cdot H_\rho$ is distinguished.

The following hypothesis is directly motivated by the 'Wiles Unit Conjecture' which is formulated by Snaith in [34, Conj. 6.3.4]. (We shall explain the precise connection at the end of this section.)

Hypothesis (EUS) ('Equivariant Unit Series') *There exists a unit element* U_S *of* $\mathbb{Z}_p[G][[T]]$ *which is such that for all* $\rho \in \mathrm{Irr}_p^+(G)$ *one has* $\mathrm{Det}(U_S)(\rho) = c_\rho^{-1} U_{S,\rho}$.

For each character $\rho \in \mathrm{Irr}_p^+(G)$ we next consider the vector space over N which is given by $H^0(\mathrm{Gal}(K_\infty/k_\infty), \mathrm{Hom}_N(L_\rho \otimes_{\mathcal{O}_N} N, Y(S_p) \otimes_{\mathbb{Z}_p} N))$. This space is finite-dimensional (over N) and also equipped with a canonical action of the quotient group $\mathrm{Gal}(K_\infty/k)/\mathrm{Gal}(K_\infty/k_\infty) \cong \mathrm{Gal}(k_\infty/k)$ and hence, in particular, of the automorphism γ. We write $h_{S,\rho}$ for the characteristic polynomial of the endomorphism of the above space which is induced by the action of $\gamma - 1$.

Hypothesis (ECP) ('Equivariant Characteristic Polynomials') *Assume that Hypothesis* (μ_p) *is valid for* K/k. *Then there exist distinguished polynomials* $G_S^* = G_S^*(T)$ *and* $H^* = H^*(T)$ *in* $\mathbb{Z}_p[G][T]$ *which are such that for all* $\rho \in \mathrm{Irr}_p^+(G)$ *the quotient* $\mathrm{Det}(G_S^*)(\rho)/\mathrm{Det}(H^*)(\rho)$ *is defined and equal to* $h_{S,\rho}/H_\rho^*$.

We can now state the main result of this section.

Theorem 3.1. *Assume that Hypothesis* (μ_p) *is valid for* K/k.

i) *If Hypotheses (ECP) and (EUS) are both valid for* K/k, *then Hypothesis (EPS) is valid for* K/k *with* $G_S = G_S^* \cdot U_S$ *and* $H = H^*$.

ii) *If* $\mathbb{Z}_p[G]$ *is admissible and Hypothesis (EPS) is valid for* K/k, *then Hypotheses (EUS) and (ECP) are both valid for* K/k.

iii) *If* K/k *is abelian, then Hypotheses (EPS), (ECP) and (EUS) are all valid for* K/k.

Proof. i) We suppose that Hypotheses (μ_p), (ECP) and (EUS) are all valid for K/k. Under these hypotheses we claim that the series $G_S := G_S^* \cdot U_S \in \mathbb{Z}_p[G][[T]]$ and $H := H^* \in \mathbb{Z}_p[G][T]$ are as described in Hypothesis (EPS). To show this we first observe that for every character $\rho \in \mathrm{Irr}_p^+(G)$ one has

$$\begin{aligned}\mathrm{Det}(G_S)(\rho) \cdot \mathrm{Det}(H)(\rho)^{-1} &= h_{S,\rho} c_\rho^{-1} U_{S,\rho} \cdot (H_\rho^*)^{-1} \\ &= h_{S,\rho} \cdot U_{S,\rho} \cdot H_\rho^{-1} \\ &= (h_{S,\rho} \cdot (G_{S,\rho}^*)^{-1}) \cdot (G_{S,\rho} \cdot H_\rho^{-1}),\end{aligned}$$

where the last equality is a consequence of the decomposition (2) and our assumption that $\mu(S,\rho) = 0$. It is therefore enough to show that for each $\rho \in \mathrm{Irr}_p^+(G)$ one has an equality

$$h_{S,\rho} = G_{S,\rho}^*. \tag{3}$$

Now if ρ is a one-dimensional even character which is of 'type S', then this equality is equivalent to the Main Conjecture of Iwasawa theory as proved by Wiles [39, Thm. 1.3]. In the general case the equality has been verified by

Snaith [34, Thm. 6.2.5] and, for the reader's convenience, we briefly sketch the argument (for details see loc. cit.). One proceeds by reduction to the result of Wiles by means of Brauer's Induction Theorem and the fact that characteristic polynomials and Iwasawa series enjoy the same inflation and induction properties. The only complication in this reduction is caused by the need to twist with characters which are of 'type W' and by the denominator polynomials H_ρ, and this is resolved by using the fact, first observed by Greenberg [22], that the denominator of the Iwasawa series which is attached to each irreducible character ρ of dimension greater than 1 is trivial (this justifies our setting $H_\rho = 1$ for these ρ). We would like to point out that there is actually a very simple argument for this absence of denominator in the case that $K \cap k_\infty = k$. Indeed, in this case, if ρ is any irreducible character of G which has degree greater than 1, then Brauer induction implies that $\rho = \sum_i n_i \operatorname{Ind}_{k_i}^{k}(\rho_i)$ where the k_i are suitable intermediate fields and ρ_i is a one-dimensional character of $\mathrm{Gal}(K/k_i)$ which is of 'type S'. It follows that the denominator of each series f_{ρ_i} is trivial unless ρ_i is itself the trivial character. Further, if χ_0 denotes the one-dimensional trivial representation of G, then one has $0 = \langle\chi_0, \rho\rangle = \sum_i n_i \langle\chi_0, \operatorname{Ind}_{k_i}^{k}(\rho_i)\rangle$. Since the latter sum is equal to the sum of the multiplicities n_i for which ρ_i is trivial, it follows that the denominator of f_ρ is indeed trivial.

ii) We now suppose that $\mathbb{Z}_p[G]$ is admissible and that Hypotheses (μ_p) and (EPS) are both valid for K/k. Recalling Remark 4i) (and Proposition 2.1), we find that the series G_S and H (as given by Hypothesis (EPS)) admit canonical decompositions $G_S = G_S^* \cdot U_S'$ and $H = H^* \cdot V$, where G_S^* and H^* are distinguished polynomials in $\mathbb{Z}_p[G][T]$ and U_S' and V are units of the ring $\mathbb{Z}_p[G][[T]]$. It follows that for each character $\rho \in \mathrm{Irr}_p^+(G)$ one has an equality

$$\frac{G_{S,\rho}}{H_\rho} = \frac{\mathrm{Det}(G_S^*)(\rho)\ \mathrm{Det}(U_S')(\rho)}{\mathrm{Det}(H^*)(\rho)\ \mathrm{Det}(V)(\rho)}.$$

We now recall that $G_{S,\rho} = G_{S,\rho}^* \cdot U_{S,\rho} = h_{S,\rho} \cdot U_{S,\rho}$ (by (2) and (3)) and we set $U_S := U_S' \cdot V^{-1} \in \mathbb{Z}_p[G][[T]]^\times$. Upon clearing denominators in the last displayed formula, we therefore obtain equalities

$$h_{S,\rho} \cdot \mathrm{Det}(H^*)(\rho) \cdot U_{S,\rho} = \mathrm{Det}(G_S^*)(\rho) \cdot H_\rho \cdot \mathrm{Det}(U_S)(\rho),$$

or equivalently

$$h_{S,\rho} \cdot \mathrm{Det}(H^*)(\rho) \cdot c_\rho^{-1} U_{S,\rho} = \mathrm{Det}(G_S^*)(\rho) \cdot H_\rho^* \cdot \mathrm{Det}(U_S)(\rho). \tag{4}$$

LEMMA 2. *Let f be a distinguished polynomial in $\mathbb{Z}_p[G][[T]]$. Then, for each $\rho \in \mathrm{Irr}_p(G)$, the series $\mathrm{Det}(f)(\rho)$ is a distinguished polynomial in $\mathcal{O}_N[[T]]$.*

Proof. It is clear that the series $\mathrm{Det}(f)(\rho)$ is a polynomial in $\mathcal{O}_N[[T]]$ which is distinguished if and only if the polynomial $\mathrm{Det}(f)(\rho)^{p^j}$ is distinguished for any natural number j. In addition, since f is a distinguished polynomial, of degree d say, there exists a natural number j such that $f^{p^j} \equiv T^{dp^j}$ (modulo $p \cdot \mathbb{Z}_p[G][[T]]$). Since $\mathrm{Det}(f^{p^j})(\rho) = \mathrm{Det}(f)(\rho)^{p^j}$ we may therefore assume in the sequel that f is a monic polynomial which satisfies $f \equiv T^d$ (modulo $p \cdot \mathbb{Z}_p[G][[T]]$).

We now use the notation introduced at the beginning of §3 (when defining the map $\mathrm{Det}(f)$). We write $f = T^d + \sum_{i=0}^{d-1} \alpha_i T^i$ where $\alpha_i \in p \cdot \mathbb{Z}_p[G]$ for each integer i with $0 \le i < d$, and for each such i we set $R_i := r_\rho(\alpha_i) \in \mathrm{M}_n(\mathcal{O}_N)$. Upon denoting the $n \times n$ identity matrix by R_d, we obtain an equality $r_\rho(f) = \sum_{i=0}^{d} R_i T^i$ in $\mathrm{M}_n(\mathcal{O}_N[T])$. We observe that each off-diagonal entry of $r_\rho(f)$ belongs to $p \cdot \mathcal{O}_N[T]$ and is of degree strictly less than d, and that each diagonal entry of $r_\rho(f)$ is a monic polynomial which is congruent to T^d modulo $p \cdot \mathcal{O}_N[T]$. From this description it is immediately clear that $\mathrm{Det}(f)(\rho) := \det(r_\rho(f))$ is a monic polynomial which is congruent to T^{nd} modulo $p \cdot \mathcal{O}_N[T]$, and hence that it is distinguished, as claimed. □

Upon applying this lemma with f equal to G_S^* and H^* we deduce that the polynomials $\mathrm{Det}(G_S^*)(\rho)$ and $\mathrm{Det}(H^*)(\rho)$, and hence also $\mathrm{Det}(G_S^*)(\rho) \cdot H_\rho^*$ and $h_{S,\rho} \cdot \mathrm{Det}(H^*)(\rho)$, are distinguished. When combined with the equality (4) and the uniqueness of Weierstrass product decompositions in the ring $\mathcal{O}_N[[T]]$, this observation implies that $h_{S,\rho} \cdot \mathrm{Det}(H^*)(\rho) = \mathrm{Det}(G_S^*)(\rho) \cdot H_\rho^*$, as required by Hypothesis (ECP), and also that $c_\rho^{-1} U_{S,\rho} = \mathrm{Det}(U_S)(\rho)$, as required by Hypothesis (EUS).

iii) We now assume that G is abelian, so that $\mathbb{Z}_p[G]$ is admissible. Following claim ii), it is therefore enough for us to assume that Hypothesis (μ_p) is valid for K/k, and then to prove that Hypothesis (EPS) is valid for K/k.

To verify Hypothesis (EPS) for K/k we first assume that $K \cap k_\infty = k$. In this case, we may use the constructions of Deligne and Ribet which are used by Wiles in [39, p.501f.]. To be explicit, we obtain the elements G_S and H as required by Hypothesis (EPS) by combining in the obvious way the elements $G_{m,c,S}$ and $H_{m,c}$ of loc. cit., where m runs over the 'components' of $\mathbb{Z}_p[G]$ (and for each non-trivial component m we set $H_{m,c} := 1$). The required equalities $\rho(G_S)/\rho(H) = f_{S,\rho}$ (for each $\rho \in \mathrm{Irr}_p^+(G)$) and the fact that $\mu_{\mathbb{Z}_p[G]}(H) = 0$ then follow as direct consequences of the properties of the series $G_{m,c,S}$ and $H_{m,c}$ described by Wiles in loc. cit., and the fact that $\mu_{\mathbb{Z}_p[G]}(G_S) = 0$ follows from the assumed validity of Hypothesis (μ_p) for K/k. From Remark 4i) we therefore deduce that G_S and H are both right associated to distinguished polynomials, as required.

In general one has $K \cap k_\infty \neq k$, and in this case we proceed as follows. There exists an extension K' of k such that $K' \cap k_\infty = k$ and K_∞ is the compositum of K' and k_∞. We observe that K'/k is a finite abelian extension and we set $G' := \mathrm{Gal}(K'/k)$ and $\Gamma := \mathrm{Gal}(k_\infty/k)$. Each character $\rho \in \mathrm{Irr}_p(G)$ can be lifted to a character of $\mathrm{Gal}(K_\infty/k) \cong G' \times \Gamma$ (which we again denote by ρ), and as such it has a unique factorisation $\rho = \psi\kappa$ where $\psi \in \mathrm{Irr}_p(G')$ and κ is a character of Γ which has finite order. As a consequence of the formula [39, (1.4)] one has an equality $G_{S,\psi\kappa}(T) = G_{S,\psi}(\kappa(\gamma)(1+T)-1)$. By the very definition of the polynomials $H_{\psi\kappa}$ and H_ψ one also has an equality $H_{\psi\kappa}(T) = H_\psi(\kappa(\gamma)(1+T)-1)$.

We now write $G_{K',S}(T)$ and $H_{K'}(T)$ for the elements of $\mathbb{Z}_p[G'][[T]]$ which are afforded by Hypothesis (EPS) for the extension K'/k (which we know to be valid by the above argument since K', which sits between k and some K_n, is again either totally real or a CM field). Let $\tilde{G}_S(T) = G_{K',S}(\gamma(1+T)-1)$ and $\tilde{H}(T) = H_{K'}(\gamma(1+T)-1)$; these series lie in $\mathbb{Z}_p[[G' \times \Gamma]][[T]]$. Then applying the character ρ of G (considered as a character of $G' \times \Gamma$ by inflation) to $\tilde{G}_S(T)$ yields $\psi(G_{K',S})(\kappa(\gamma)(1+T)-1)$, similarly for $\tilde{H}(T)$, so applying ρ to the quotient $\tilde{G}_S(T)/\tilde{H}(T)$ gives $f_\psi(\kappa(\gamma)(1+T)-1)$ by Hypothesis (EPS) for K'/k. By the aforementioned formulas, we therefore have $f_\psi(\kappa(\gamma)(1+T)-1) = f_\rho(T)$ and so we are almost done: indeed, it simply suffices to define $G_S(T)$, respectively $H(T)$, to be equal to the image of $\tilde{G}_S(T)$, respectively $\tilde{H}(T)$, under the map $\mathbb{Z}_p[[G' \times \Gamma]][[T]] \to \mathbb{Z}_p[G][[T]]$ which is induced by the epimorphism $G' \times \Gamma \to G$. □

Remark 5. To end this section we now explain the precise connection between Hypothesis (EUS) and the 'Wiles Unit Conjecture' [34, Conj. 6.3.4] of Snaith. To do this we fix an integer n with $n > 1$ and, assuming Hypothesis (EUS) to be valid for K/k, we set $\alpha_{S,n} := U_S(u^n - 1) \in \mathbb{Z}_p[G]^\times$. Then for each $\rho \in \mathrm{Irr}_p^+(G)$ one has an equality

$$\mathrm{Det}(\alpha_{S,n})(\rho) = c_\rho^{-1} U_{S,\rho}(u^n - 1).$$

After taking account of the equalities (2) and (3) one finds that this property of $\alpha_{S,n}$ is closely related to that which should be satisfied by the element $\alpha_{n,K^+/k}$ of $\mathbb{Z}_p[\mathrm{Gal}(K^+/k)]^\times$ whose existence is predicted by [34, Conj. 6.3.4]. However, there are two important differences: in loc. cit. the p-adic L-functions are untruncated and the factors c_ρ are omitted. Whilst, a priori, Hypothesis (EUS) and [34, Conj. 6.3.4] could be simultaneously valid, we now present an explicit example which shows that [34, Conj. 6.3.4] is not valid (because the relevant p-adic L-functions are untruncated). We remark that similarly explicit examples exist to show that [34, Conj. 6.3.4] must also be corrected by the introduction of the factors c_ρ.

Example 2. We set $p := 3$ and $k := \mathbb{Q}$ and we let K denote the composite of the cyclic cubic extension K_1 of $\mathbb{Q}$ which has conductor 7 and the field $K_2 := \mathbb{Q}(\sqrt{5})$. We set $G := \mathrm{Gal}(K/\mathbb{Q})$ (which is cyclic), we write ρ_0 for the nontrivial character of $\mathrm{Gal}(K_2/\mathbb{Q})$ (considered as a character of G) and ρ_1 for any faithful character of G, and we set $S := \{5, 7\}$. Then ρ_0 and ρ_1 have conductors 5 and 35 respectively and Theorem 3.1iii) implies that there exists a unit element U_S of $\mathbb{Z}_p[G][[T]]$ such that, for $i \in \{1, 2\}$, the unit part U_{ρ_i} of the Iwasawa series f_{S,ρ_i} which is associated to $L_{p,S}(-, \rho_i)$ is equal to $\rho_i(U_S)$. We now let $w(T)$ be the power series such that $w(u^n - 1)L_{p,\{5\}}(1-n, \rho_0) = L_{p,S}(-, \rho_0)$ for all natural numbers n. Then $w(u^n - 1) = 1 - \rho_0(7)\omega^{-n}(7)7^{n-1}$ where ω is the 3-adic Teichmüller character. Since $\omega(7) = 1$ it follows that $w(T) = 1 - \rho_0(7) \cdot 7^{-1}(T+1)^a$ with $u^a = 7$. Now $\rho_0(7) = -1$ and so $w(0) = \frac{8}{7} \equiv -1 \pmod 3$; in particular $w(T) \in \mathbb{Z}_p[[T]]^\times$ and so the unit part U'_{ρ_0} of the Iwasawa series $f_{\{5\},\rho_0}$ is equal to $w(T)^{-1}U_{\rho_0}$. In this setting [34, Conj. 6.3.4] predicts the existence (for any given n) of an element α'_n of $\mathbb{Z}_3[G]^\times$ which satisfies both $\rho_0(\alpha'_n) = U'_{\rho_0}(u^n-1)$ and $\rho_1(\alpha'_n) = U_{\rho_1}(u^n-1)$. If such an element existed, then the element $q := U_S(u^n-1)(\alpha'_n)^{-1}$ of $\mathbb{Z}_3[G]^\times$ would satisfy both $\rho_0(q) = w(u^n-1)$ and $\rho_1(q) = 1$. However, the above calculation shows that $w(u^n-1) \equiv -1 \pmod 3$ and so no such element q can exist (indeed, ρ_0 and ρ_1 differ by a character of order 3 and so, for any $q' \in \mathbb{Z}_3[G]$, the elements $\rho_0(q')$ and $\rho_1(q')$ must be congruent modulo the maximal ideal of $\mathbb{Z}_3[\zeta_3]$).

4. Algebraic Preliminaries

In the remainder of this manuscript our aim is to describe certain explicit consequences of Theorem 3.1iii) concerning the values of Dirichlet L-functions at strictly negative integers. However, before doing so, in this section we describe some necessary algebraic preliminaries.

We now let K/k be any finite Galois extension of number fields of group G (which is not necessarily abelian). We fix any rational prime p and a finite set of places T of k which contains all archimedean places, all places which ramify in K/k and all places of residue characteristic p. For any extension E of k we write $\mathcal{O}_{E,T}$ for the ring of T_E-integers in E, where T_E denotes the set of places of E which lie above those in T. We set $U := \mathrm{Spec}(\mathcal{O}_{k,T})$ and we write $G_{k,T}$ for the Galois group of the maximal algebraic extension of k which is unramified outside T. For each non-negative integer n we write K_n for the subextension of K^p_∞ which is of degree p^n over K, and $\pi_n : \mathrm{Spec}(\mathcal{O}_{K_n,T}) \to U$ for the morphism of schemes which is induced by the inclusion $\mathcal{O}_{k,T} \subseteq \mathcal{O}_{K_n,T}$. If $\mathcal{F}$ is any finite $G_{k,T}$-module, then we use the same symbol to denote the associated locally-constant sheaf on the étale site $U_{\text{ét}}$. If $\mathcal{F}$ is any continuous $G_{k,T}$-module which is finitely generated over $\mathbb{Z}_p$, then we let $\mathcal{F}_\infty$ denote the associated pro-sheaf $(\mathcal{F}_n, t_n)_{n\geq 0}$ on $U_{\text{ét}}$ where, for each non-negative integer n, we set $\mathcal{F}_n := \pi_{n,*} \circ \pi_n^*(\mathcal{F}/p^{n+1})$ and the transition morphism t_n is induced by the composite of the trace map $\pi_{n+1,*} \circ \pi_{n+1}^*(\mathcal{F}/p^{n+2}) \to \pi_{n,*} \circ \pi_n^*(\mathcal{F}/p^{n+2})$ and the natural projection $\mathcal{F}/p^{n+2} \to \mathcal{F}/p^{n+1}$.

Let Λ be a pro-p ring. (Thus we depart here from the usual convention that Λ has the fixed meaning $\mathbb{Z}_p[[T]]$.) We write $\mathcal{D}(\Lambda)$ for the derived category of bounded complexes of Λ-modules and $\mathcal{D}^{\mathrm{p}}(\Lambda)$, respectively $\mathcal{D}^{\mathrm{p,f}}(\Lambda)$, for the full triangulated subcategory of $\mathcal{D}(\Lambda)$ consisting of those complexes which are perfect, respectively are perfect and have finite cohomology groups.

If $\mathcal{F}$ is any (p-adic) étale sheaf of Λ-modules on U, then we follow the approach of [9, §3.2] to define the complex of compactly supported cohomology $R\Gamma_c(U_{\text{ét}}, \mathcal{F})$ so as to lie in a canonical distinguished triangle in $\mathcal{D}(\Lambda)$

$$R\Gamma_c(U_{\text{ét}}, \mathcal{F}) \longrightarrow R\Gamma(U_{\text{ét}}, \mathcal{F}) \longrightarrow \bigoplus_{v\in T} R\Gamma(\mathrm{Spec}(k_v)_{\text{ét}}, \mathcal{F}). \tag{5}$$

We recall that the approach developed by Kato in [25, §3.1] and by Nekovář in [30] (cf. also [11, Rem. 4.1] in this regard) allows one to extend the definitions of each of these complexes in a natural manner to the case of pro-sheaves of Λ-modules of the form $\mathcal{F}_\infty$ discussed above, and that in this case there is again a canonical distinguished triangle of the form (5). If $p = 2$, then we set $R\Gamma_*(U_{\text{ét}}, -) := R\Gamma_c(U_{\text{ét}}, -)$. If p is odd, then we let $R\Gamma_*(U_{\text{ét}}, -)$ denote either $R\Gamma_c(U_{\text{ét}}, -)$ or $R\Gamma(U_{\text{ét}}, -)$. In each degree i we then set $H^i_*(U_{\text{ét}}, -) := H^i R\Gamma_*(U_{\text{ét}}, -)$.

We next recall that if Λ is any $\mathbb{Z}_p$-order which spans a finite dimensional semisimple $\mathbb{Q}_p$-algebra $\Lambda_{\mathbb{Q}_p}$, then to each object C of $\mathcal{D}^{\text{p,f}}(\Lambda)$ one can associate a canonical element $\chi^{\text{rel}}_\Lambda C$ of the relative algebraic K-group $K_0(\Lambda, \mathbb{Q}_p)$ (cf. [5, Prop. 1.2.1] or [9, §2.8, Rem. 4]). We recall further that the Whitehead group $K_1(\Lambda_{\mathbb{Q}_p})$ of $\Lambda_{\mathbb{Q}_p}$ is generated by elements of the form $[\alpha]$ where α is an automorphism of a finitely generated $\Lambda_{\mathbb{Q}_p}$-module, and we write $\delta_\Lambda : K_1(\Lambda_{\mathbb{Q}_p}) \to K_0(\Lambda, \mathbb{Q}_p)$ for the homomorphism which occurs in the long exact sequence of relative K-theory (as described explicitly in, for example, [5, §1.1]).

PROPOSITION 4.1. *Let $\mathcal{F}$ be a continuous $\mathbb{Z}_p[G_{k,T}]$-module which is both finitely generated and free over $\mathbb{Z}_p$. Then $R\Gamma_*(U_{\text{ét}}, \mathcal{F}_\infty)$ is an object of $\mathcal{D}^{\text{p}}(\mathbb{Z}_p[\text{Gal}(K^p_\infty/k)])$.*

Assume now that $K \cap k^p_\infty = k$. Let ϵ be a central idempotent of $\mathbb{Z}_p[G]$, set $\Lambda := \mathbb{Z}_p[G]\epsilon$, and let θ_∞ be an injective $\mathbb{Z}_p[\text{Gal}(K^p_\infty/k)]$-equivariant endomorphism of $\epsilon \cdot \mathcal{F}_\infty$. If both

ci) *in each degree i the $\mathbb{Z}_p$-module $H^i_*(U_{\text{ét}}, \epsilon \cdot \mathcal{F}_\infty)$ is finitely generated, and*
cii) *in each degree i the endomorphism $H^i_*(U_{\text{ét}}, \theta_\infty) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is bijective,*

then $R\Gamma_*(U_{\text{ét}}, \text{coker}(\theta_\infty))$ is an object of $\mathcal{D}^{\text{p,f}}(\Lambda)$, and in $K_0(\Lambda, \mathbb{Q}_p)$ one has an equality

$$\chi^{\text{rel}}_\Lambda R\Gamma_*(U_{\text{ét}}, \text{coker}(\theta_\infty)) = \sum_{i \in \mathbb{Z}} (-1)^i \delta_\Lambda([H^i_*(U_{\text{ét}}, \theta_\infty) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p]).$$

Proof. For each non-negative integer n we set $\Lambda_n := (\mathbb{Z}/p^{n+1})[\text{Gal}(K_n/k)]$. We also set $\Lambda_\infty := \varprojlim_n \Lambda_n$ where the limit is taken with respect to the natural projection morphisms $\rho_n : \Lambda_{n+1} \to \Lambda_n$. In the sequel we identify Λ_∞ with $\mathbb{Z}_p[\text{Gal}(K^p_\infty/k)]$ in the natural way.

We first note that, for each non-negative integer n, $\mathcal{F}_n$ is the sheaf which is associated to the free Λ_n-module $\Lambda_n \otimes_{\mathbb{Z}_p} \mathcal{F}$ and that t_n is the morphism which is associated to the natural morphism of Λ_{n+1}-modules

$$\Lambda_{n+1} \otimes_{\mathbb{Z}_p} \mathcal{F} \to \Lambda_n \otimes_{\Lambda_{n+1}, \rho_n} (\Lambda_{n+1} \otimes_{\mathbb{Z}_p} \mathcal{F}) \cong \Lambda_n \otimes_{\mathbb{Z}_p} \mathcal{F}.$$

By using results of Flach [19, Thm. 5.1, Prop. 4.2] we may therefore deduce that, for each such n, $R\Gamma_*(U_{\text{ét}}, \mathcal{F}_n)$ is an object of $\mathcal{D}^{\text{p}}(\Lambda_n)$ which is acyclic outside degrees $0, 1, 2, 3$ and is also such that there exists an isomorphism ψ_n in $\mathcal{D}^{\text{p}}(\Lambda_n)$ between $\Lambda_n \otimes^{\mathbb{L}}_{\Lambda_{n+1}, \rho_n} R\Gamma_*(U_{\text{ét}}, \mathcal{F}_{n+1})$ and $R\Gamma_*(U_{\text{ét}}, \mathcal{F}_n)$.

We observe next that Λ_{n+1} is Artinian and that $\ker(\rho_n)$ is a two sided nilpotent ideal. By using the structure theory of [17, Prop. (6.17)] we may thus deduce that for any morphism of finitely generated projective Λ_n-modules $\phi_n : M_n \to N_n$ there exists a morphism of finitely generated projective Λ_{n+1}-modules $\phi_{n+1} : M_{n+1} \to N_{n+1}$ for which one has $M_n = \Lambda_n \otimes_{\Lambda_{n+1}, \rho_n} M_{n+1}$,

$N_n = \Lambda_n \otimes_{\Lambda_{n+1},\rho_n} N_{n+1}$ and $\phi_n = \Lambda_n \otimes_{\Lambda_{n+1},\rho_n} \phi_{n+1}$. This fact allows one to adapt the constructions of Milne in [29, p.264-265] and hence to prove that, for each non-negative integer n, there exists a complex of finitely generated projective Λ_n-modules $C_n^{\cdot}$ with the following properties: $C_n^i = 0$ for $i \notin \{0,1,2,3\}$; $C_n^{\cdot}$ is isomorphic in $\mathcal{D}^{\mathrm{p}}(\Lambda_n)$ to $R\Gamma_*(U_{\text{ét}}, \mathcal{F}_n)$; there exists a Λ_{n+1}-equivariant homomorphism of complexes $\psi'_n : C_{n+1}^{\cdot} \to C_n^{\cdot}$ which is such that the morphism $\Lambda_n \otimes_{\Lambda_{n+1},\rho_n} \psi'_n : \Lambda_n \otimes_{\Lambda_{n+1},\rho_n} C_{n+1}^{\cdot} \to C_n^{\cdot}$ is bijective in each degree and induces ψ_n. In this way we obtain a bounded complex of finitely generated projective Λ_∞-modules $C_\infty^{\cdot} := \varprojlim_{\psi'_n} C_n^{\cdot}$ which represents $R\Gamma_*(U_{\text{ét}}, \mathcal{F}_\infty)$. This proves the first claim of the proposition.

We now assume that $K \cap k_\infty^p = k$ and that θ_∞ is an injective Λ_∞-equivariant endomorphism of the pro-sheaf $\epsilon \cdot \mathcal{F}_\infty$. By adapting the constructions of [29, Chap. VI, Lem. 8.17, Lem. 13.10] (but note that [loc. cit., Lem. 8.17] is incorrect as stated since the morphism ψ need not be a quasi-isomorphism) it may be shown that there exists a Λ_∞-equivariant endomorphism $\theta_\infty^{\cdot}$ of the complex $D_\infty^{\cdot} := \epsilon \cdot C_\infty^{\cdot}$ which induces the morphism $R\Gamma_*(U_{\text{ét}}, \theta_\infty)$. In this way one obtains a canonical short exact sequence of complexes

$$0 \to D_\infty^{\cdot} \to \mathrm{Cone}(\theta_\infty^{\cdot}) \to D_\infty^{\cdot}[1] \to 0 \tag{6}$$

and also an isomorphism in $\mathcal{D}^{\mathrm{p}}(\Lambda_\infty)$ between $\mathrm{Cone}(\theta_\infty^{\cdot})$ and $R\Gamma_*(U_{\text{ét}}, \mathrm{coker}(\theta_\infty))$.

Now Λ_∞ is a free $\mathbb{Z}_p[G]$-module and so $D_\infty^{\cdot}$ is a bounded complex of projective Λ-modules. If also each $\mathbb{Z}_p$-module $H^i(D_\infty^{\cdot})$ is finitely generated, as is implied by condition ci), then by a standard argument (see, for example, the proof of [12, Thm. 1.1, p.447]) it follows that $D_\infty^{\cdot}$ belongs to $\mathcal{D}^{\mathrm{p}}(\Lambda)$. The exact sequence (6) then implies that $\mathrm{Cone}(\theta_\infty^{\cdot})$ also belongs to $\mathcal{D}^{\mathrm{p}}(\Lambda)$. Further, condition cii) now combines with the long exact sequence of cohomology which is associated to (6) to imply that each module $H_*^i(\mathrm{Cone}(\theta_\infty^{\cdot}))$ is finite and hence that $\mathrm{Cone}(\theta_\infty^{\cdot})$ belongs to $\mathcal{D}^{\mathrm{p,f}}(\Lambda)$, as claimed.

It only remains to prove the explicit formula for $\chi_\Lambda^{\mathrm{rel}} R\Gamma_*(U_{\text{ét}}, \mathrm{coker}(\theta_\infty))$. To do this we let $P^{\cdot}$ be a bounded complex of finitely generated projective Λ-modules which is quasi-isomorphic to $D_\infty^{\cdot}$ and $\hat{\theta}^{\cdot} : P^{\cdot} \to P^{\cdot}$ a morphism of complexes which induces $\theta_\infty^{\cdot}$. Condition cii) combines with the argument of [13, Lem. 7.10] to imply we may assume that in each degree i the map $\hat{\theta}^i$ is injective (and so has finite cokernel). It follows that $R\Gamma_*(U_{\text{ét}}, \mathrm{coker}(\theta_\infty))$ is isomorphic in $\mathcal{D}^{\mathrm{p,f}}(\Lambda)$ to the complex $\mathrm{coker}(\hat{\theta}^{\cdot})$ which is equal to $\mathrm{coker}(\hat{\theta}^i)$ in each degree i and for which the differentials are induced by those of $P^{\cdot}$, and hence that $\chi_\Lambda^{\mathrm{rel}} R\Gamma_*(U_{\text{ét}}, \mathrm{coker}(\theta_\infty)) = \chi_\Lambda^{\mathrm{rel}} \mathrm{coker}(\hat{\theta}^{\cdot})$.

We next recall that $\chi_\Lambda^{\mathrm{rel}}$ is additive on exact triangles in $\mathcal{D}^{\mathrm{p,f}}(\Lambda)$ [5, Prop. 1.2.2]. From the short exact sequences of complexes

$$0 \to \mathrm{coker}(\hat{\theta}^i)[-i] \to \tau_i\, \mathrm{coker}(\hat{\theta}^{\cdot}) \to \tau_{i-1}\, \mathrm{coker}(\hat{\theta}^{\cdot}) \to 0$$

(where, for each integer j, τ_j denotes the naive truncation in degree j) we may therefore deduce that $\chi_\Lambda^{\mathrm{rel}} \mathrm{coker}(\hat{\theta}^{\cdot}) = \sum_{i \in \mathbb{Z}} \chi_\Lambda^{\mathrm{rel}}(\mathrm{coker}(\hat{\theta}^i)[-i])$. The claimed formula now follows directly from the fact that for each integer i one has $\chi_\Lambda^{\mathrm{rel}}(\mathrm{coker}(\hat\theta^i)[-i]) = (-1)^i\delta_\Lambda([\hat\theta^i\otimes_{\mathbb{Z}_p}\mathbb{Q}_p])$ and in $K_1(\Lambda_{\mathbb{Q}_p})$ there is an equality

$$[\hat\theta^i\otimes_{\mathbb{Z}_p}\mathbb{Q}_p] = [\hat\theta^{i+1}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p\mid_{B^{i+1}}] + [H^i_*(U_{\text{ét}},\theta_\infty)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p] + [\hat\theta^i\otimes_{\mathbb{Z}_p}\mathbb{Q}_p\mid_{B^i}].$$

Here we write B^i for the submodule of coboundaries of $P^\cdot\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ in degree i, and the displayed equality is a consequence of the natural filtration of $P^i\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ which has graded pieces isomorphic to $B^{i+1}, H^i_*(U_{\text{ét}},\epsilon\cdot\mathcal{F}_\infty)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ and B^i. □

Remark 6. In this remark we assume that G is abelian, but otherwise use the same notation and hypotheses as in the second part of Proposition 4.1. We write Det_Λ for the determinant functor introduced by Knudsen and Mumford in [27], and (both here and in the sequel) we identify any graded invertible Λ-module of the form $(I,0)$ with the underlying invertible Λ-module I.
We recall that the assignment $\chi_\Lambda^{\mathrm{rel}}C\mapsto \mathrm{Det}_\Lambda\, C$ (where C ranges over all objects of $\mathcal{D}^{\mathrm{p,f}}(\Lambda)$) induces a well-defined isomorphism between $K_0(\Lambda,\mathbb{Q}_p)$ and the multiplicative group of invertible Λ-lattices in $\Lambda_{\mathbb{Q}_p}$ (cf. [1, Lem. 2.6]). In particular, in this case the equality at the end of Proposition 4.1 is equivalent to the following equality in $\Lambda_{\mathbb{Q}_p}$

$$\mathrm{Det}_\Lambda\, R\Gamma_*(U_{\text{ét}},\mathrm{coker}(\theta_\infty)) = \prod_{i\in\mathbb{Z}}\det{}_{\Lambda_{\mathbb{Q}_p}}(H^i_*(U_{\text{ét}},\theta_\infty)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)^{(-1)^{i+1}}\cdot\Lambda.$$

(We remark that the exponent $(-1)^{i+1}$ on the right hand side of this formula is not a misprint!)

5. Values of Dirichlet L-functions

In this section we derive certain explicit consequences of Theorem 3.1iii) and Proposition 4.1 concerning the values of Dirichlet L-functions at strictly negative integers.
To this end we continue to use the notation introduced in §3. In particular, we now assume that k is totally real and that K is a CM *abelian* extension of k and we set $G := \mathrm{Gal}(K/k)$. We also fix an odd prime p, algebraic closures $\mathbb{Q}^c$ of $\mathbb{Q}$ and $\mathbb{Q}_p^c$ of $\mathbb{Q}_p$, and we set $G^\wedge := \mathrm{Hom}(G,\mathbb{Q}^{c\times})$ and $G^{\wedge,p} := \mathrm{Hom}(G,\mathbb{Q}_p^{c\times})$. We let τ denote the complex conjugation in G, and for each integer a we write e_a for the idempotent $\frac12(1+(-1)^a\tau)$ of $\mathbb{Z}[\frac12][G]$, and $G^\wedge_{(a)}$ and $G^{\wedge,p}_{(a)}$ for the subsets of $G^\wedge$ and $G^{\wedge,p}$ respectively which consist of those characters ψ satisfying $\psi(\tau) = (-1)^a$. For each element ψ of $G^\wedge$, respectively of $G^{\wedge,p}$, we write e_ψ for the associated idempotent $\frac{1}{|G|}\sum_{g\in G}\psi(g)g^{-1}$ of $\mathbb{Q}^c[G]$, respectively of $\mathbb{Q}_p^c[G]$.
We fix a finite set S of non-archimedean places of k which contains all non-archimedean places which ramify in K/k and, for each $\psi\in G^\wedge$, we write $L_S(s,\psi)$ for the Dirichlet L-function of ψ which is truncated by removing the Euler factors at all places in S.

If r is any integer with $r > 1$, then each function $L_S(s, \psi)$ is holomorphic at $s = 1 - r$ and so we may set

$$L_S(1-r) := \sum_{\psi \in G^\wedge} L_S(1-r, \psi) e_\psi \in \mathbb{C}[G].$$

If $k = \mathbb{Q}$, then this element can be interpreted in terms of higher Bernoulli numbers and is therefore a natural analogue of the classical Stickelberger element. In general, by a result of Siegel [33], one knows that $L_S(1-r)$ belongs to the unit group of the ring $\mathbb{Q}[G]e_r$.

In order to state our next result we assume that $K \cap k^p_\infty = k$. Under this hypothesis we set

$$h_S := \sum_{\rho \in G^{\wedge,p}_{(0)}} h_{S,\rho} e_\rho \in \mathbb{Z}_p[G][[T]][\frac{1}{p}].$$

We also write e for the idempotent $\frac{1}{|G|}\sum_{g \in G} g$ of $\mathbb{Q}_p[G]$ and then set

$$H' := \sum_{\rho \in G^{\wedge,p}_{(0)}} H_\rho e_\rho = Te + (e_0 - e) \in \mathbb{Q}_p[G][T],$$

where the second equality is a consequence of our assumption that $K \cap k^p_\infty = k$. For the purposes of the next result we also assume that K contains a primitive p-th root of unity, and we write ω for the Teichmüller character of G. For each integer b we then let tw_b denote the $\mathbb{Z}_p$-linear automorphism of $\mathbb{Z}_p[G]$ which sends each element g of G to $\omega^b(g) \cdot g$.

Theorem 5.1. *Assume that Hypothesis (μ_p) is valid for K/k, that $K \cap k^p_\infty = k$ and that K contains a primitive p-th root of unity. Then for each integer $r > 1$ one has an equality*

$$L_S(1-r) \cdot \mathbb{Z}_p[G] = \mathrm{tw}_r(H'(u^r - 1)^{-1} h_S(u^r - 1)) \cdot \mathbb{Z}_p[G]e_r.$$

Proof. At the outset we fix an integer $r > 1$ and an embeddding $j : \mathbb{Q}^c \to \mathbb{Q}^c_p$ and, for each $\chi \in G^\wedge$, we set $\rho_\chi := (j \circ \chi) \cdot \omega^r \in G^{\wedge,p}$.

We observe that ω belongs to $G^{\wedge,p}_{(1)}$ and hence that χ belongs to $G^\wedge_{(r)}$ if and only if ρ_χ belongs to $G^{\wedge,p}_{(0)}$. In addition, for all characters $\chi \in G^\wedge_{(r)}$ one has an equality

$$(j \circ \chi)(L_{S_p}(1-r)) = j(L_{S_p}(1-r, \chi)) = L_{p,S}(1-r, \rho_\chi). \tag{7}$$

We now set $Z_S := (H')^{-1} \cdot h_S$. Upon comparing images under each character $\rho \in G^{\wedge,p}$, recalling the equality (3) and noting that in the present case $c_\rho = 1$ for all such ρ, we may deduce that Z_S is equal to the quotient G^*_S/H^* which occurs in Hypothesis (ECP). With U_S denoting the unit element which occurs in Hypothesis (EUS) for K/k, and setting $G_S := G^*_S \cdot U_S$ and $H := H^*$ it therefore follows that

$$\begin{aligned} Z_S \cdot U_S &= (G^*_S \cdot U_S) \cdot (H^*)^{-1} \\ &= G_S \cdot H^{-1}. \end{aligned}$$

From Theorem 3.1i),iii) we know that the series G_S and H satisfy the conditions specified in Hypothesis (EPS). Hence the last displayed formula implies that, for each character $\chi \in G^{\wedge}_{(r)}$, one has

$$\begin{aligned}\rho_\chi(Z_S \cdot U_S) &= \rho_\chi(G_S)\rho_\chi(H)^{-1} \\ &= f_{S,\rho_\chi}.\end{aligned}$$

Upon combining this formula with the equalities (1) and (7) we deduce that

$$\begin{aligned}(j \circ \chi)(L_{S_p}(1-r)) &= \rho_\chi(Z_S(u^r-1) \cdot U_S(u^r-1)) \\ &= (j \circ \chi)(\mathrm{tw}_r(Z_S(u^r-1) \cdot U_S(u^r-1))) \\ &= (j \circ \chi)(\mathrm{tw}_r(Z_S(u^r-1)) \cdot \mathrm{tw}_r(U_S(u^r-1))).\end{aligned}$$

Since this equality is valid for every character χ in $G^{\wedge}_{(r)}$ it implies that the elements $L_{S_p}(1-r)$ and $\mathrm{tw}_r(Z_S(u^r-1))$ of $\mathbb{Q}_p[G]e_r$ differ by the factor $\mathrm{tw}_r(U_S(u^r-1))$ which is a unit of the ring $\mathbb{Z}_p[G]e_r$.
It now only remains for us to show that the elements $L_{S_p}(1-r)$ and $L_S(1-r)$ differ by a unit of $\mathbb{Z}_p[G]e_r$. But $L_{S_p}(1-r) = L_S(1-r)x$ where x is a product of Euler factors of the form $1-\mathrm{N}v^{r-1} \cdot f_v$ where v is a place of k which divides p and does not belong to S, $\mathrm{N}v$ is the absolute norm of v and f_v is the Frobenius automorphism of v in G. Further, since $r > 1$, it is clear that each such element $1-\mathrm{N}v^{r-1} \cdot f_v$ is a unit of $\mathbb{Z}_p[G]$. □

Our next result concerns a special case of Kato's Generalized Iwasawa Main Conjecture. However, before stating this result, it will be convenient to introduce some further notation.
For the remainder of this section we let Σ denote the (finite) set of rational primes ℓ which satisfy either $\ell = 2$ or $K \cap k^{\ell}_{\infty} \neq k$. We also write $\mathbb{Z}_\Sigma$ for the subring of $\mathbb{Q}$ which is generated by the inverses of each element of Σ.
For any extension E of k and any finite set of places V of k we let $\mathcal{O}_{E,V}$ denote the ring of V_E-integers in E, where V_E denotes the set of all places of E which are either archimedean or lie above a place in V. We set $U_k := \mathrm{Spec}(\mathcal{O}_{k,S_p})$ and for each p-adic étale sheaf $\mathcal{F}$ on U_k and each finite Galois extension E/k which is unramified at all non-archimedean places outside S_p we write $\mathcal{F}_E$ for the étale sheaf of $\mathbb{Z}_p[\mathrm{Gal}(E/k)]$-modules $\pi_*\pi^*\mathcal{F}$ on U_k where π denotes the morphism $\mathrm{Spec}(\mathcal{O}_{E,S_p}) \to U_k$ which is induced by the inclusion $\mathcal{O}_{k,S_p} \subseteq \mathcal{O}_{E,S_p}$. We recall that, since π_* is exact, the complexes $R\Gamma(U_{k,\text{ét}}, \mathcal{F}_E)$ and $R\Gamma_c(U_{k,\text{ét}}, \mathcal{F}_E)$ are canonically isomorphic in $\mathcal{D}(\mathbb{Z}_p[\mathrm{Gal}(E/k)])$ to $R\Gamma(\mathrm{Spec}(\mathcal{O}_{E,S_p})_{\text{ét}}, \pi^*\mathcal{F})$ and $R\Gamma_c(\mathrm{Spec}(\mathcal{O}_{E,S_p})_{\text{ét}}, \pi^*\mathcal{F})$ respectively, and in the sequel we shall often use such identifications without explicit comment.
For each integer $r > 1$ we set

$$C_{K,1-r} := R\Gamma_c(U_{k,\text{ét}}, e_r\mathbb{Z}_p(1-r)_K)$$

and we recall that (since $r > 1$) this complex is an object of $\mathcal{D}^{\mathrm{p,f}}(\mathbb{Z}_p[G]e_r)$ (see the upcoming proof of Lemma 3 for further details in this regard). From the equalities of [8, (11),(12)] (with r replaced by $1-r$), it therefore follows that the 'Equivariant Tamagawa Number Conjecture' of [9, Conj. 4(iv)] is for the pair $(h^0(\mathrm{Spec}(K))(1-r), \mathbb{Z}_\Sigma[G]e_r)$ equivalent to asserting that if p does not belong to Σ, then in $\mathbb{Q}_p[G]e_r$ one has an equality

$$\mathrm{Det}^{-1}_{\mathbb{Z}_p[G]e_r} C_{K,1-r} = L_S(1-r)\cdot \mathbb{Z}_p[G] \tag{8}$$

(cf. Remark 6).
Before proceeding, we remark that the above equality is in general strictly finer than the corresponding case of the Generalized Iwasawa Main Conjecture which Kato formulates in [25, Conj. 3.2.2 and 3.4.14]. Indeed, since graded determinants are not used in [25] the central conjecture of loc. cit. is in this case only well defined to within multiplication by elements of $(\mathbb{Q}_p[G]e_r)^\times$ of square 1 which reflect possible re-ordering of the factors in tensor products. For more details in this regard we refer the reader to [loc cit., Rem. 3.2.3(3) and 3.2.6(3),(5)] and [9, Rem. 9]. We recall also that a direct comparison of [9, Conj. 4(iv)] with the central conjecture formulated by Kato in [24, Conj. (4.9)] can be found in [10, §2].

THEOREM 5.2. *Assume that p does not belong to Σ and that Hypothesis (μ_p) is valid for K/k. Then for each integer $r > 1$ the equality (8) is valid. In particular, the Generalized Iwasawa Main Conjecture of Kato is valid for each such pair $(h^0(\mathrm{Spec}(K))(1-r), \mathbb{Z}_p[G]e_r)$.*

Remark 7. i) It is straightforward to describe explicit conditions on K/k which ensure that $\Sigma = \{2\}$. For example, if $[K:k]$ is coprime to the class number of k, then $\Sigma = \{2\}$ whenever the conductor of K/k is not divisible by the square of any prime ideal which divides $[K:k]$.
ii) If $K/\mathbb{Q}$ is abelian, then Hypothesis (μ_p) is known to be valid for all p (Remark 3i)) and so Theorem 5.2 gives an alternative proof of parts of the main result (Cor. 8.1) of [11]. The reader will find that the approach of loc. cit. is considerably more involved than that used here. We remark that, nevertheless, the approach of loc. cit. can be extended to improve upon Theorem 5.2 by showing that [9, Conj. 4(iv)] is valid for the pair $(h^0(\mathrm{Spec}(K))(1-r), \mathbb{Z}[\frac{1}{2}][G]e_r)$ under the assumption that Hypothesis (μ_p) is valid for K/k at all odd p.

Proof of Theorem 5.2. For the purposes of this argument we set $\mathfrak{A} := \mathbb{Z}_p[G]e_r$ and $A := \mathbb{Q}_p[G]e_r$.
We first remark that, when verifying the equality (8), the functorial behaviour of compactly supported étale cohomology and of Dirichlet L-functions under Galois descent allows us to replace K by the extension of K which is generated by a primitive p-th root of unity (cf. [9, Prop. 4.1b)]). We may therefore henceforth assume that K contains a primitive p-th root of unity and is such that $K \cap k_\infty^p = k$. After taking into account the result of Theorem 5.1 it is therefore enough for us to prove that in $\mathbb{Q}_p[G]e_r$ one has an equality

$$\mathrm{Det}^{-1}_{\mathfrak{A}} C_{K,1-r} = \mathrm{tw}_r(H'(u^r-1)^{-1}h_S(u^r-1))\cdot\mathfrak{A}. \tag{9}$$

We now use the notation of Proposition 4.1. We regard γ as a topological generator of $\mathrm{Gal}(K^p_\infty/K) \cong \mathrm{Gal}(k^p_\infty/k)$, we set $\hat\gamma := 1-\gamma \in \mathbb{Z}_p[\mathrm{Gal}(K^p_\infty/k)]$ and we observe that the action of $\hat\gamma$ induces an injective $\mathbb{Z}_p[\mathrm{Gal}(K^p_\infty/k)]$-equivariant endomorphism $\hat\gamma_{1-r}$ of the pro-sheaf $e_r\cdot\mathbb{Z}_p(1-r)_\infty$ on $U_{k,\text{ét}}$.

Lemma 3. *Let T denote the union of S_p and the set of archimedean places of k, and set $U := U_k, \mathcal{F} := \mathbb{Z}_p(1-r), \epsilon := e_r$ and $\theta_\infty := \hat\gamma_{1-r}$. If Hypothesis (μ_p) is valid for K/k, then this data satisfies the conditions ci) and cii) of Proposition 4.1, and in A one has an equality*

$$\prod_{i\in\mathbb{Z}} \det{}_A(H^i_c(U_{k,\text{ét}},\hat\gamma_{1-r})\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)^{(-1)^i} = \det{}_A(\hat\gamma \mid e_0Y(S_p)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p(-r))\cdot\det{}_A(\hat\gamma\mid\mathbb{Q}_p(-r))^{-1}$$

where $\hat\gamma$ acts diagonally on $e_0Y(S_p)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p(-r)$.

Proof. We assume that Hypothesis (μ_p) is valid for K/k, and we recall (from Remark 3ii)) that this is equivalent to asserting that $e_0Y(S_p)$ is a finitely generated $\mathbb{Z}_p$-module.

For each integer i we set $H^i_c(1-r) := H^i_c(U_{\text{ét}}, e_r\cdot\mathbb{Z}_p(1-r)_\infty)$. To verify that condition ci) of Proposition 4.1 is satisfied by the given data and also to prove the claimed equality, it is clearly enough to show that $H^i_c(1-r)$ vanishes if $i\notin\{2,3\}$ and that $H^2_c(1-r)$ and $H^3_c(1-r)$ are canonically isomorphic to $e_0Y(S_p)\otimes_{\mathbb{Z}_p}\mathbb{Z}_p(-r)$ (endowed with the natural diagonal action of γ) and $\mathbb{Z}_p(-r)$ respectively.

To show this we first observe that, since p is odd and k is totally real, for each archimedean place v of k and any non-negative integer n the complex $R\Gamma(\mathrm{Spec}(k_v)_{\text{ét}}, e_r\cdot\mathbb{Z}_p(1-r)_n)$ is acyclic. This implies that our definition of compactly supported cohomology (as in (5)) coincides with that used by Nekovář in [30, (5.3)], and hence that the complex $R\Gamma_c(U_{\text{ét}}, e_r\cdot\mathbb{Z}_p(1-r)_\infty)$ coincides with the complex $R\Gamma_{c,\mathrm{Iw}}(K_\infty/k,\mathbb{Z}_p(1-r))$ which is defined in [loc. cit., (8.5.4)]. To compute $H^i_c(1-r)$ we may therefore use the fact that there are natural isomorphisms of $\mathbb{Z}_p[\mathrm{Gal}(K_\infty/k)]$-modules

$$\begin{aligned}(10)\qquad H^i_c(1-r) &\cong e_r(H^i_c(U_{\text{ét}},\mathbb{Z}_p(1)_\infty\otimes_{\mathbb{Z}_p}\mathbb{Z}_p(-r)))\\ &\cong (\varprojlim_n H^{i,+}_{c,n}(1))\otimes_{\mathbb{Z}_p}\mathbb{Z}_p(-r)\\ &\cong (\varprojlim_n H^{i,+}_{c,n,n}(1))\otimes_{\mathbb{Z}_p}\mathbb{Z}_p(-r)\end{aligned}$$

where, for each non-negative integer n, we set $H^{i,+}_{c,n}(1) := e_0\cdot H^i_c(U_{k,\text{ét}},\mathbb{Z}_p(1)_{K_n})$ and $H^{i,+}_{c,n,n}(1) := e_0\cdot H^i_c(U_{k,\text{ét}},(\mu_{p^{n+1}})_{K_n})$, each limit over the integers $n\geq 0$ is taken with respect to the natural projection maps, $\mathrm{Gal}(K_\infty/k)$ acts diagonally on each tensor product, and the second and third isomorphisms follow as a consequence of [loc. cit., Prop 8.5.5(ii), respectively Lem. (4.2.2)].

Now to compute explicitly each group $H^{i,+}_{c,n}(1)$ for $i\neq 2$ it is enough to combine the long exact sequence of cohomology of the triangle (5) (with $U = U_k$ and

$\mathcal{F} = \mathbb{Z}_p(1)_{K_n}$) together with certain standard results of Kummer theory and class field theory. To describe the result we write

$$\lambda_n : \mathcal{O}^\times_{K_n,T} \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \prod_{w_n} \varprojlim_{m\geq 1} K^\times_{n,w_n}/(K^\times_{n,w_n})^{p^m}$$

for the natural 'diagonal' morphism where on the right hand side w_n runs over all places of K_n which lie above places in T and the limit over m is taken with respect to the natural projection morphisms. Then one finds that $H^{i,+}_{c,n}(1)$ vanishes if $i \notin \{1,2,3\}$, that $H^{3,+}_{c,n}(1)$ identifies with $\mathbb{Z}_p$ and that $H^{1,+}_{c,n}(1)$ is isomorphic to $e_0 \cdot \ker(\lambda_n)$. Upon passing to the inverse limit over n (and by using (10)) one finds that $H^i_c(1-r)$ vanishes if $i \notin \{2,3\}$ and that $H^3_c(1-r)$ is canonically isomorphic to $\mathbb{Z}_p(-r)$.

To proceed we next recall that, for each pair of non-negative integers m and n, the Artin-Verdier Duality Theorem induces a canonical isomorphism in $\mathcal{D}(\mathbb{Z}/p^m\mathbb{Z}[\mathrm{Gal}(K_n/k)])$

$$R\Gamma_c(U_{\text{ét}}, e_r(\mu_{p^m}^{\otimes(1-r)})_{K_n}) \cong \mathrm{Hom}_{\mathbb{Z}/p^m\mathbb{Z}}(R\Gamma(U_{\text{ét}}, e_r(\mu_{p^m}^{\otimes r})_{K_n}), \mathbb{Z}/p^m\mathbb{Z}[-3])$$

where the linear dual is endowed with the contragredient action of $\mathrm{Gal}(K_n/k)$ (cf. [30, Prop. (5.4.3)(i), (2.11)] with $R = \mathbb{Z}/p^m\mathbb{Z}[\mathrm{Gal}(K_n/k)], J = R[0], K = k$, $S = T$ and $X = e_r(\mu_{p^m}^{\otimes(1-r)})_{K_n}[0]$). Now if $K^{\mathrm{ab},n}_{n,T}$ denotes the maximal abelian extension of K_n which is unramified outside T and of exponent dividing p^{n+1}, then the above isomorphism (with $r = 0$ and $m = n+1$) implies that $H^{2,+}_{c,n,n}(1)$ is canonically isomorphic to $e_0 \cdot \mathrm{Gal}(K^{\mathrm{ab},n}_{n,T}/K_n)$. These isomorphisms are compatible with the natural transition morphisms as n varies and hence upon passing to the inverse limit (and using (10)) we obtain a canonical isomorphism between $H^2_c(1-r)$ and $e_0Y(S_p) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(-r)$ (endowed with the diagonal action of $\mathrm{Gal}(K_\infty/k)$), as required.

At this stage we need only verify that condition cii) of Proposition 4.1 is satisfied by the specified data. However, this is so because $\mathrm{coker}(\hat{\gamma}_{1-r})$ is isomorphic to the constant pro-sheaf $e_r\mathbb{Z}_p(1-r)_K$ and all cohomology groups of the complex $C_{K,1-r}$ are finite. To explain the latter fact we recall that, for any given n and r, the above displayed duality isomorphisms are compatible with the natural transition morphisms as m varies and hence (in the case $n = 0$) induce upon passing to the inverse limit a canonical isomorphism in $\mathcal{D}(\mathfrak{A})$

$$C_{K,1-r} \cong R\,\mathrm{Hom}_{\mathbb{Z}_p}(R\Gamma(U_{\text{ét}}, e_r\mathbb{Z}_p(r)_K), \mathbb{Z}_p[-3]). \tag{11}$$

where the linear dual is endowed with the action of $\mathfrak{A}$ which is induced by the contragredient action of G. (The existence of such an isomorphism also follows from the exactness of the central column of [8, diagram (114)] (where L corresponds to our field K) and the fact that each complex $R\Gamma_\Delta(L_w, \mathbb{Z}_p(1-r))^*$ which occurs in that diagram becomes acyclic upon multiplication by e_r.) Now, after taking (11) into account, it is enough for us to prove that all of the groups $H^i(U_{\text{ét}}, e_r\mathbb{Z}_p(r)_K)$ are finite and this follows, for example, as a consequence of the description of [11, Lem. 3.2ii)] and the fact that $e_r(K_{2r-i}(\mathcal{O}_{K,T}) \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ is finite for both $i \in \{1,2\}$. □

We next observe that (as can be verified by explicit computation)

$$\begin{aligned}\det_A(\hat\gamma \mid \mathbb{Q}_p(-r)) &= (1-u^{-r})e_{\omega^{-r}} + (e_r - e_{\omega^{-r}})\\ &= \mathrm{tw}_r(v_r\cdot H'(u^r-1))\end{aligned}$$

where $v_r := u^{-r}e + (e_0 - e)$, and also

$$\begin{aligned}&\det_A(\hat\gamma \mid e_0Y(S_p)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p(-r))\\ =&\sum_{\rho\in G^{\wedge,p}_{(r)}} \det_{\mathbb{Q}_p^c}(1-\gamma \mid e_\rho(Y(S_p)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p^c(-r)))e_\rho\\ =&\sum_{\rho\in G^{\wedge,p}_{(r)}} \det_{\mathbb{Q}_p^c}(1-u^{-r}\gamma \mid e_{\rho\omega^r}(Y(S_p)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p^c))e_\rho\\ =&\ \mathrm{tw}_r(v'_r\cdot h_S(u^r-1)),\end{aligned}$$

where $v'_r := \sum_{\rho\in G^{\wedge,p}_{(0)}} u^{-rd_\rho}e_\rho$ with $d_\rho := \dim_{\mathbb{Q}_p^c}(e_\rho(Y(S_p)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p^c))$ for each $\rho\in G^{\wedge,p}_{(0)}$. We remark that in proving the last displayed equality one uses the fact that for each $\kappa\in G^{\wedge,p}$ the $\mathbb{Q}_p^c[\gamma]$-module $e_\kappa(Y(S_p)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p^c)$ is isomorphic to $H^0(\mathrm{Gal}(K_\infty/k_\infty), \mathrm{Hom}_{\mathbb{Q}_p^c}(\mathbb{Q}_p^c\cdot e_\kappa, Y(S_p)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p^c))$.
Upon combining the last two displayed formulas with the result of Lemma 3, the quasi-isomorphism $C_{K,1-r}\cong R\Gamma_c(U_{k,\text{ét}}, \mathrm{coker}(\hat\gamma_{1-r}))$ and the equality of Remark 6 we find that

$$\mathrm{Det}^{-1}_{\mathfrak{A}}\, C_{K,1-r} = \mathrm{tw}_r(v_r^{-1}v'_r)\,\mathrm{tw}_r(H'(u^r-1)^{-1}h_S(u^r-1))\cdot\mathfrak{A}.$$

The required equality (9) is thus a consequence of the following observation.

Lemma 4. *$v_r^{-1}v'_r$ is a unit of $\mathbb{Z}_p[G]e_0$.*

Proof. We start by making a general observation. For this we set $\mathfrak{B} := \mathbb{Z}_p[G]e_0$, and we let $f_1(T)$ and $f_2(T)$ denote any elements of $\mathfrak{B}[[T]]$ which satisfy $\mu_{\mathfrak{B}}(f_1(T)) = \mu_{\mathfrak{B}}(f_2(T)) = 0$. For $i=1,2$ we write $f_i^*(T)$ and $U_i(T)$ for the distinguished polynomial and unit series which occur in the product decomposition of $f_i(T)$ afforded by Proposition 2.1 (with $\mathfrak{A}=\mathfrak{B}$). We also set $f_{i,r}(T) := f_i(u^r(1+T)-1)\in\mathfrak{B}[[T]]$ and, observing that $\mu_{\mathfrak{B}}(f_{i,r}(T)) = 0$, we write $U_{i,r}(T)$ for the unit series which occurs in the product decomposition of $f_{i,r}(T)$ afforded by Proposition 2.1. Then, by explicit computation, one verifies that the element $(U_1(u^r-1)U_{2,r}(0))(U_2(u^r-1)U_{1,r}(0))^{-1}$ of $\mathfrak{B}^\times$ is equal to $\sum_{\rho\in G^{\wedge,p}_{(0)}} u^{-r\delta_\rho}e_\rho$ where $\delta_\rho := \deg(\rho(f_1^*(T))) - \deg(\rho(f_2^*(T)))$.
We now apply this observation with $f_1(T)$ and $f_2(T)$ equal to the series G_S and H which occur in Hypothesis (EPS). We observe that, in this case, the equality (3) implies that for each $\rho\in G^{\wedge,p}_{(0)}$ one has $\deg(\rho(G_S^*)) - \deg(\rho(H)) = d'_\rho$ where here $d'_\rho := d_\rho - 1$ if ρ is trivial, and $d'_\rho := d_\rho$ otherwise. From the general observation made above we may therefore deduce that the element $v_r^{-1}v'_r = \sum_{\rho\in G^{\wedge,p}_{(0)}} u^{-rd'_\rho}e_\rho$ belongs to $\mathfrak{B}^\times$, as claimed. □

This completes our proof of Theorem 5.2. □

We next use Theorem 5.2 to prove a result concerning the element $\Omega_{r-1}(K/k)$ of $\mathrm{Pic}(\mathbb{Z}[G])$ which is defined by Chinburg, Kolster, Pappas and Snaith in [13, §3]. We recall that it has been conjectured by the authors of loc. cit. that $\Omega_{r-1}(K/k) = 0$.

We write $\rho_{\Sigma,r}$ for the natural scalar extension morphism $\mathrm{Pic}(\mathbb{Z}[G]) \to \mathrm{Pic}(\mathbb{Z}_\Sigma[G]e_r)$. With R denoting either $\mathbb{Z}$ or $\mathbb{Q}$ we also write $\rho_\#$ for the R-linear involution of $R[G]$ which is induced by sending each element of G to its inverse.

COROLLARY 1. *Assume that (if $k \neq \mathbb{Q}$, then) Hypothesis (μ_p) is valid for K/k at each prime $p \notin \Sigma$. Then for each integer $r > 1$ one has an equality $\rho_{\Sigma,r}(\Omega_{r-1}(K/k)) = 0$.*

Proof. The key point we use here is a result of Flach and the first named author. Indeed, the result of [8, Thm. 4.1] implies that $\rho_{\Sigma,r}(\Omega_{r-1}(K/k))$ is equal to the class of the invertible $\mathbb{Z}_\Sigma[G]e_r$-submodule of $\mathbb{Q}[G]e_r$ which is defined by means of the intersection

$$\Big(\bigcap_{p\notin\Sigma} \mathrm{Det}^{-1}_{\mathbb{Z}_p[G]e_r} R\Gamma_c(U_{k,\text{ét}}, e_r\mathbb{Z}_p(1-r)_K)\Big) \otimes_{\mathbb{Z}[G],\rho_\#} \mathbb{Z}[G].$$

(To see this one must recall that the normalisation of the determinant functor which is used in [8] is the inverse of that used here.)

On the other hand, Theorem 5.2 implies that the above intersection is equal to the free $\mathbb{Z}_\Sigma[G]e_r$-module which is generated by the element $\rho_\#(L_S(1-r))$. Hence one has $\rho_{\Sigma,r}(\Omega_{r-1}(K/k)) = 0$, as required. □

Before stating our final result we introduce a little more notation. If V is any finite set of places of k, then for each rational prime ℓ we let V_ℓ denote the union of V and the set of places of k which are either archimedean or of residue characteristic ℓ. We set $\mathbb{Z}' := \mathbb{Z}[\frac{1}{2}]$ and we define a $\mathbb{Z}'[G]$-module by setting

$$H^2(\mathcal{O}_{K,V}, \mathbb{Z}'(r)) := \bigoplus_{\ell\neq 2} H^2(\mathrm{Spec}(\mathcal{O}_{K,V_\ell})_{\text{ét}}, \mathbb{Z}_\ell(r)).$$

We let Σ' denote the set $\{2\}$, respectively Σ, if $k = \mathbb{Q}$, respectively $k \neq \mathbb{Q}$, and we write $\mathbb{Z}_{\Sigma'}$ for the subring of $\mathbb{Q}$ which is generated by the inverses of each element of Σ'. We also write $\mu_{\mathbb{Q}^c}$ for the torsion subgroup of $\mathbb{Q}^{c\times}$.

COROLLARY 2. *Assume that (if $k \neq \mathbb{Q}$, then) Hypothesis (μ_p) is valid for K/k at each prime $p \notin \Sigma$. Then for each integer $r > 1$ one has an equality*

$$\rho_\#(L_S(1-r)) \cdot \mathrm{Ann}_{\mathbb{Z}[G]}(H^0(K, \mu_{\mathbb{Q}^c}^{\otimes r})) \otimes_{\mathbb{Z}} \mathbb{Z}_{\Sigma'} = e_r \cdot \mathrm{Fit}_{\mathbb{Z}'[G]}(H^2(\mathcal{O}_{K,S}, \mathbb{Z}'(r))) \otimes_{\mathbb{Z}'} \mathbb{Z}_{\Sigma'}.$$

Remark 8. i) If T is any subset of S, then the localisation sequence of étale cohomology induces a natural inclusion of $\mathbb{Z}'[G]$-modules $H^2(\mathcal{O}_{K,T}, \mathbb{Z}'(r)) \subseteq H^2(\mathcal{O}_{K,S}, \mathbb{Z}'(r))$. From this we may deduce that $\mathrm{Ann}_{\mathbb{Z}'[G]}(H^2(\mathcal{O}_{K,S}, \mathbb{Z}'(r))) \subseteq$

$\mathrm{Ann}_{\mathbb{Z}'[G]}(H^2(\mathcal{O}_{K,T},\mathbb{Z}'(r)))$ and also if, for example, G is cyclic, that $\mathrm{Fit}_{\mathbb{Z}'[G]}(H^2(\mathcal{O}_{K,S},\mathbb{Z}'(r))) \subseteq \mathrm{Fit}_{\mathbb{Z}'[G]}(H^2(\mathcal{O}_{K,T},\mathbb{Z}'(r)))$. In particular, if $\Sigma' = \{2\}$ (cf. Remark 7i)), and we write $\mathcal{O}_K$ in place of $\mathcal{O}_{K,\emptyset}$ (which, in terms of our current notation, denotes the ring of algebraic integers in K), then the equality of Corollary 2 implies that

$$(12)\quad \rho_\#(L_S(1-r))\cdot \mathrm{Ann}_{\mathbb{Z}[G]}(H^0(K,\mu_{\mathbb{Q}^c}^{\otimes r}))\otimes_{\mathbb{Z}}\mathbb{Z}' \subseteq \mathrm{Ann}_{\mathbb{Z}'[G]}(H^2(\mathcal{O}_K,\mathbb{Z}'(r))),$$

and also if, for example, G is cyclic, that

$$\rho_\#(L_S(1-r))\cdot \mathrm{Ann}_{\mathbb{Z}[G]}(H^0(K,\mu_{\mathbb{Q}^c}^{\otimes r}))\otimes_{\mathbb{Z}}\mathbb{Z}' \subseteq \mathrm{Fit}_{\mathbb{Z}'[G]}(H^2(\mathcal{O}_K,\mathbb{Z}'(r))).$$

We observe that if (as has been famously conjectured by Quillen and Lichtenbaum) the $\mathbb{Z}'[G]$-module $H^2(\mathcal{O}_K,\mathbb{Z}'(r))$ is isomorphic to $K_{2r-2}(\mathcal{O}_K)\otimes_{\mathbb{Z}}\mathbb{Z}'$, then the inclusion (12) is finer than (the image under $-\otimes_{\mathbb{Z}}\mathbb{Z}'$ of) the inclusion

$$\#H^0(\mathbb{Q},\mu_{\mathbb{Q}^c}^{\otimes r})\rho_\#(L_S(1-r))\cdot \mathrm{Ann}_{\mathbb{Z}[G]}(H^0(K,\mu_{\mathbb{Q}^c}^{\otimes r})) \subseteq \mathrm{Ann}_{\mathbb{Z}[G]}(K_{2r-2}(\mathcal{O}_K))$$

which was conjectured in the case $k=\mathbb{Q}$ by Coates and Sinnott in [14, Conj. 1]. We also recall that if $k=\mathbb{Q}$, r is even and K is any abelian extension of $\mathbb{Q}$, then Kurihara has recently used different methods to explicitly compute $\mathrm{Fit}_{\mathbb{Z}'[G]}(H^2(\mathcal{O}_K,\mathbb{Z}'(r)))$ in terms of Stickelberger elements [28, Cor. 12.5 and Rem. 12.6].

ii) In the case that $k=\mathbb{Q}$ and the conductor of K/k is a prime power, the image under multiplication by $e_r - e_{\omega^r}$ of the equality of Corollary 2 has already been proved by Cornacchia and Østvær in [16, Thm. 1.2].

iii) If r is even, then the equality of Corollary 2 can be re-expressed as an equality in which K is replaced by K^+ and the idempotent factor e_r is omitted. In a recent preprint [35], Snaith uses results from [11] to prove a weaker version of the equality of Corollary 2 in this context. More precisely, Snaith's results [loc. cit., Th. 1.6, Th. 5.2] assume that K is totally real, that r is even and that Hypothesis (μ_p) is valid for K/k at all odd primes p, and involve chains of inclusions rather than a precise specification of Fitting ideals (see also Remark 9 in this regard).

iv) In this remark we fix an odd prime p, an embedding $j:\mathbb{Q}^c\to\mathbb{Q}_p^c$ and a character $\chi\in G_{(r)}^{\wedge}$. We set $\mathcal{O} := \mathbb{Z}_p(j\circ\chi)$ and we write $\mathrm{lth}_{\mathcal{O}}(M)$ for the length of any finite $\mathcal{O}$-module M. We let K_χ denote the (cyclic) extension of $\mathbb{Q}$ which corresponds to $\ker(\chi)$. Then the image under the functor $-\otimes_{\mathbb{Z}'[G]}\mathcal{O}$ of the equality of Corollary 2 with $k=\mathbb{Q}$ and $K=K_\chi$ is equivalent to an equality

$$\mathrm{val}_{\mathcal{O}}(j(L_S(1-r,\chi^{-1}))) = \mathrm{lth}_{\mathcal{O}}(H^2(\mathcal{O}_{K,S},\mathbb{Z}'(r))\otimes_{\mathbb{Z}'[G]}\mathcal{O}) - \mathrm{lth}_{\mathcal{O}}(H^0(K,\mu_{\mathbb{Q}^c}^{\otimes r})\otimes_{\mathbb{Z}[G]}\mathcal{O}).$$

In addition, since in this case G is cyclic, for any finite $\mathbb{Z}'[G]$-module N one has $\mathrm{lth}_{\mathcal{O}}(N\otimes_{\mathbb{Z}'[G]}\mathcal{O}) = \mathrm{lth}_{\mathcal{O}}(\mathrm{Hom}_{\mathbb{Z}'[G]}(\mathcal{O},N))$ and so the previous displayed equality provides a natural analogue of the main result (Thm. II.1) of Solomon in [36] concerning the relation between generalised Bernoulli numbers and the structure of certain ideal class groups. (The first named author is very grateful to Masato Kurihara for a most helpful conversation in this regard.)

Proof of Corollary 2. We now fix a prime $p \notin \Sigma'$ and we assume that Hypothesis (μ_p) is valid for K/k (as is known in the case $k = \mathbb{Q}$). We set $\mathfrak{A} := \mathbb{Z}_p[G]e_r$, $\mu(r) := H^0(K, \mu_{\mathbb{Q}^c}^{\otimes r}) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and $C := R\Gamma(U_{k,\text{ét}}, e_r\mathbb{Z}_p(r)_K)$.

In the sequel we shall say that a commutative $\mathbb{Z}_p$-algebra Λ is 'relatively Gorenstein over $\mathbb{Z}_p$' if $\mathrm{Hom}_{\mathbb{Z}_p}(\Lambda, \mathbb{Z}_p)$ (endowed with the natural action of Λ) is a free Λ-module of rank one.

For each bounded object X of $\mathcal{D}(\mathfrak{A})$ we set $X^* := R\,\mathrm{Hom}_{\mathbb{Z}_p}(X, \mathbb{Z}_p)$ which we endow with the action of $\mathfrak{A}$ which is induced by the contragredient action of G. We observe that $\mathfrak{A}$ is relatively Gorenstein over $\mathbb{Z}_p$ and hence that X^* belongs to $\mathcal{D}^{\mathrm{p}}(\mathfrak{A})$, respectively $\mathcal{D}^{\mathrm{p,f}}(\mathfrak{A})$, if and only if X belongs to $\mathcal{D}^{\mathrm{p}}(\mathfrak{A})$, respectively $\mathcal{D}^{\mathrm{p,f}}(\mathfrak{A})$. Now for any $\mathfrak{A}$-module X there exists a canonical isomorphism between the $\mathfrak{A}$-modules $\mathrm{Hom}_{\mathfrak{A}}(X, \mathfrak{A}) \otimes_{\mathbb{Z}_p[G],\rho_\#} \mathbb{Z}_p[G]$ and $\mathrm{Hom}_{\mathbb{Z}_p}(X, \mathbb{Z}_p)$. This in turn implies that for any object X of $\mathcal{D}^{\mathrm{p}}(\mathfrak{A})$ the lattice $\mathrm{Det}_{\mathfrak{A}}^{-1} X^*[-3]$ identifies canonically with $(\mathrm{Det}_{\mathfrak{A}}^{-1} X) \otimes_{\mathbb{Z}_p[G],\rho_\#} \mathbb{Z}_p[G]$. Upon noting that (11) induces an isomorphism in $\mathcal{D}^{\mathrm{p,f}}(\mathfrak{A})$ of the form $C \cong R\Gamma_c(U_{k,\text{ét}}, e_r\mathbb{Z}_p(1-r)_K)^*[-3]$, and recalling that the equality (8) is known to be valid as a consequence of Theorem 5.2 in the case $k \neq \mathbb{Q}$ and as a consequence of [11, Cor. 8.1] in the case $k = \mathbb{Q}$, we deduce that $\mathrm{Det}_{\mathfrak{A}}^{-1} C = \rho_\#(L_S(1-r)) \cdot \mathfrak{A}$. The equality of Corollary 2 will therefore follow if we can show that

$$\mathrm{Det}_{\mathfrak{A}}^{-1} C \cdot \mathrm{Ann}_{\mathfrak{A}}(\mu(r)) = \mathrm{Fit}_{\mathfrak{A}}(H^2(C)). \tag{13}$$

We next observe that, since C belongs to $\mathcal{D}^{\mathrm{p,f}}(\mathfrak{A})$ and is acyclic outside degrees 1 and 2, there exists an exact sequence of $\mathfrak{A}$-modules

$$0 \to H^1(C) \to Q \xrightarrow{d} Q' \to H^2(C) \to 0 \tag{14}$$

which is such that both Q and Q' are finite and of projective dimension at most 1 and there exists an isomorphism ι in $\mathcal{D}^{\mathrm{p,f}}(\mathfrak{A})$ between C and the complex $Q \xrightarrow{d} Q'$ (where the modules are placed in degrees 1 and 2, and the cohomology is identified with $H^1(C)$ and $H^2(C)$ by using the maps in (14)) for which $H^i(\iota)$ is the identity map in each degree i. This implies that $\mathrm{Fit}_{\mathfrak{A}}(Q)$ and $\mathrm{Fit}_{\mathfrak{A}}(Q')$ are invertible ideals of $\mathfrak{A}$ and that $\mathrm{Det}_{\mathfrak{A}}^{-1} C = \mathrm{Fit}_{\mathfrak{A}}(Q)^{-1}\,\mathrm{Fit}_{\mathfrak{A}}(Q')$.

LEMMA 5. *Let R be any reduced commutative $\mathbb{Z}_p$-algebra which is finitely generated, free and relatively Gorenstein over $\mathbb{Z}_p$. If*

$$0 \to A \to P \to P' \to A' \to 0$$

is any exact sequence of finite R-modules in which P and P' are both of projective dimension at most 1 over R, then $\mathrm{Fit}_R(P)$ and $\mathrm{Fit}_R(P')$ are principal ideals of R and one has an equality

$$\mathrm{Fit}_R(A^\vee)\,\mathrm{Fit}_R(P') = \mathrm{Fit}_R(P)\,\mathrm{Fit}_R(A'),$$

where $A^\vee$ denotes the Pontryagin dual $\mathrm{Hom}_{\mathbb{Z}_p}(A, \mathbb{Q}_p/\mathbb{Z}_p)$ (endowed with the natural action of R).

Proof. This is almost covered by the result of [15, Prop. 6]; however, the latter is stated only for rings R of the form $\mathcal{O}[G]$ with G a finite abelian p-group, and involves $\mathrm{Fit}_R(A)$ rather than $\mathrm{Fit}_R(A^\vee)$. In addition, the second named author would like to take this opportunity to point out that the argument in loc. cit. makes the *assumption* that G (which is written as P in loc. cit.) is *cyclic*, which is unfortunately not stated explicitly at the appropriate place, and that the equality of [15, Prop. 6] does not appear to hold in general. We will therefore now quickly adapt the arguments of [15, Prop. 6] to better suit our present purpose.

We first observe that, since R is semilocal, the Fitting ideal of any finite R-module P which is of projective dimension at most 1 is principal, being generated by the determinant of α in any presentation $R^n \xrightarrow{\alpha} R^n \to P \to 0$. We may find the following data, proceeding exactly as in [15, p.456f.]: a nonzerodivisor f of R (it is in fact always possible to take f to be a large enough power of p); a natural number n; short exact sequences $0 \to Q \to \tilde{A} \to A \to 0$ and $0 \to A' \to \tilde{A}' \to Q' \to 0$ in which Q and Q' are both finite and of projective dimension at most 1, and a four term exact sequence

$$0 \to \tilde{A} \to (R/fR)^n \to (R/fR)^n \to \tilde{A}' \to 0.$$

In a similar way one obtains the equalities

$$\mathrm{Fit}_R(P)\,\mathrm{Fit}_R(Q) = f^n R = \mathrm{Fit}_R(P')\,\mathrm{Fit}_R(Q').$$

Now R/fR is Gorenstein of dimension zero, that is: $(R/fR)^\vee \cong R/fR$ as R-modules. Therefore the argument in loc. cit. starting with equation (4) applies to give an equality

$$\mathrm{Fit}_R(\tilde{A}') = \mathrm{Fit}_R(\tilde{A}^\vee).$$

(Note that we do not claim that $\mathrm{Fit}_R(\tilde{A}^\vee) = \mathrm{Fit}_R(\tilde{A})$, as was done in loc. cit.) Applying the result of [15, Lem. 3] to the sequence $0 \to A' \to \tilde{A}' \to Q' \to 0$, respectively to the Pontryagin dual of the sequence $0 \to Q \to \tilde{A} \to A \to 0$, we obtain an equality

$$\mathrm{Fit}_R(\tilde{A}') = \mathrm{Fit}_R(A')\,\mathrm{Fit}_R(Q'),$$

respectively

$$\mathrm{Fit}_R(\tilde{A}^\vee) = \mathrm{Fit}_R(A^\vee)\,\mathrm{Fit}_R(Q^\vee).$$

Lemma 6. $\mathrm{Fit}_R(Q^\vee) = \mathrm{Fit}_R(Q)$.

Proof. Since Q is finite and of projective dimension at most 1 there exists an exact sequence of R-modules of the form

$$0 \to R^n \xrightarrow{\alpha} R^n \to Q \to 0.$$

Now $\mathrm{Hom}_{\mathbb{Z}_p}(Q, \mathbb{Z}_p) = 0$ and $\mathrm{Ext}^1_{\mathbb{Z}_p}(Q, \mathbb{Z}_p)$ is isomorphic to $Q^\vee$ (as Q is finite), $\mathrm{Ext}^1_{\mathbb{Z}_p}(R^n, \mathbb{Z}_p) = 0$ (as R is $\mathbb{Z}_p$-free) and $\mathrm{Hom}_{\mathbb{Z}_p}(R^n, \mathbb{Z}_p)$ is isomorphic to R^n (as R is relatively Gorenstein over $\mathbb{Z}_p$). From the long exact sequence of $\mathrm{Ext}^i_{\mathbb{Z}_p}(-, \mathbb{Z}_p)$-groups which is associated to the above sequence we therefore obtain a further exact sequence of R-modules

$$0 \to R^n \xrightarrow{\alpha^t} R^n \to Q^\vee \to 0,$$

where α^t denotes the transpose of α. By using the two displayed sequences we now compute that $\mathrm{Fit}_R(Q) = \det(\alpha)R = \det(\alpha^t)R = \mathrm{Fit}_R(Q^\vee)$, as claimed. □

The equality of Lemma 5 now follows directly upon combining the equality of Lemma 6 with the four displayed equalities which immediately precede it. □

Upon applying Lemma 5 with $R = \mathfrak{A}$ (which is both reduced and relatively Gorenstein over $\mathbb{Z}_p$) to the exact sequence (14) we obtain an equality

$$\begin{aligned}\mathrm{Det}_{\mathfrak{A}}^{-1} C \cdot \mathrm{Fit}_{\mathfrak{A}}(H^1(C)^\vee) &= \mathrm{Fit}_{\mathfrak{A}}(Q)^{-1}\,\mathrm{Fit}_{\mathfrak{A}}(Q')\,\mathrm{Fit}_{\mathfrak{A}}(H^1(C)^\vee)\\ &= \mathrm{Fit}_{\mathfrak{A}}(H^2(C)).\end{aligned}$$

To deduce the required equality (13) from this equality we now simply observe that the $\mathfrak{A}$-module $H^1(C)^\vee$ is isomorphic to the cyclic $\mathfrak{A}$-module $\mu(r)^\vee$, and hence that $\mathrm{Fit}_{\mathfrak{A}}(H^1(C)^\vee) = \mathrm{Fit}_{\mathfrak{A}}(\mu(r)^\vee) = \mathrm{Ann}_{\mathfrak{A}}(\mu(r)^\vee) = \mathrm{Ann}_{\mathfrak{A}}(\mu(r))$.
This completes our proof of Corollary 2. □

Remark 9. Let Γ be any finite abelian group and ℓ any rational prime. If M is any finite $\mathbb{Z}_\ell[\Gamma]$-module, then $\mathrm{Ann}_{\mathbb{Z}_\ell[\Gamma]}(M) = \mathrm{Ann}_{\mathbb{Z}_\ell[\Gamma]}(M^\vee)$ and $\mathrm{Ann}_{\mathbb{Z}_\ell[\Gamma]}(M)^{n(M)} \subseteq \mathrm{Fit}_{\mathbb{Z}_\ell[\Gamma]}(M) \subseteq \mathrm{Ann}_{\mathbb{Z}_\ell[\Gamma]}(M)$ where $n(M)$ denotes the minimal number of elements required to generate M over $\mathbb{Z}_\ell[\Gamma]$. Hence, if X is any object of $\mathcal{D}^{\mathrm{p,f}}(\mathbb{Z}_\ell[\Gamma])$ which is acyclic outside degrees 0 and 1 and $t_i \in \mathrm{Ann}_{\mathbb{Z}_\ell[\Gamma]}(H^i(X))$ for $i \in \{0,1\}$, then (since $\mathbb{Z}_\ell[\Gamma]$ is both reduced and relatively Gorenstein over $\mathbb{Z}_\ell$) the equality of Lemma 5 implies that

$$t_0^{n(H^0(X)^\vee)} \cdot \mathrm{Det}_{\mathbb{Z}_\ell[\Gamma]} X \subseteq \mathrm{Fit}_{\mathbb{Z}_\ell[\Gamma]}(H^1(X))$$

and also

$$t_1^{n(H^1(X))} \cdot \mathrm{Det}_{\mathbb{Z}_\ell[\Gamma]}^{-1} X \subseteq \mathrm{Fit}_{\mathbb{Z}_\ell[\Gamma]}(H^0(X)^\vee).$$

In particular, if $n(H^0(X)^\vee) = n(H^0(X))$ (which, for example, is the case when $H^0(X) = H^0(K, \mu_{\mathbb{Q}^c}^{\otimes r}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$), then Lemma 5 refines the main algebraic result (Thm. 2.4) of Snaith in [35].

References

[1] W. Bley and D. Burns, Equivariant Tamagawa Numbers, Fitting Ideals and Iwasawa Theory, Compositio Math. 126 (2001), 213-247.

[2] W. Bley and D. Burns, Equivariant epsilon constants, discriminants and étale cohomology, Proc. London Math. Soc. 87 (2003), 545-590

[3] S. Bloch, K. Kato, L-functions and Tamagawa numbers of motives, In: 'The Grothendieck Festschrift' vol. 1, Progress in Math. 86, Birkhäuser, Boston, (1990) 333-400.

[4] N. Bourbaki, Commutative Algebra, Hermann, Paris 1974.

[5] D. Burns, Equivariant Tamagawa Numbers and Galois module theory I, Compositio Math. 129 (2001), 203-237.

[6] D. Burns, Equivariant Tamagawa Numbers and refined abelian Stark Conjectures, J. Math. Sci. Univ. Tokyo 10 (2003), 225-259

[7] D. Burns, On values of equivariant Zeta-functions of curves over finite fields, manuscript submitted for publication.

[8] D. Burns and M. Flach, On Galois structure invariants associated to Tate motives, Amer. J. Math. 120 (1998), 1343-1397.

[9] D. Burns and M. Flach, Tamagawa Numbers for Motives with (non-commutative) coefficients, Documenta Math. 6 (2001), 501-570.

[10] D. Burns and M. Flach, Tamagawa Numbers for Motives with (non-commutative) coefficients, Part II, Amer. J. Math. 125 (2003), 475-512

[11] D. Burns and C. Greither, On the Equivariant Tamagawa Number Conjecture for Tate motives, Invent. Math. 153 (2003), 303-359

[12] T. Chinburg, Galois structure of de Rham cohomology of tame covers of schemes, Ann. of Math. 139 (1994), 443-490.

[13] T. Chinburg, M. Kolster, G. Pappas, and V. Snaith, Galois structure of K-groups of rings of integers, K-theory 14 (1998), 319-369.

[14] J. H. Coates and W. Sinnott, An analogue of Stickelberger's theorem for the higher K-groups, Invent. Math. 24 (1974), 149-161.

[15] P. Cornacchia and C. Greither, Fitting ideals of class groups of real fields with prime power conductor, J. Number Th. 73 (1998), 459-471.

[16] P. Cornacchia and P. A. Østvær, On the Coates-Sinnott conjecture, K-Theory 19 (2000), 195-209.

[17] C. W. Curtis and I. Reiner, Methods of Representation Theory, Vol. 1, Wiley Classics Library, 1981.

[18] B. Ferrero and L. Washington, The Iwasawa invariant μ_p vanishes for abelian number fields, Annals of Math. 109 (1979), 377-395.

[19] M. Flach, Euler characteristics in relative K-groups, Bull. London Math. Soc. 32 (2000), 272-284.

[20] J.-M. Fontaine et B. Perrin-Riou, Autour des conjectures de Bloch et Kato: cohomologie galoisienne et valeurs de fonctions L, In: Motives (Seattle) Proc. Symp. Pure Math. 55, I, (1994) 599-706.

[21] A. Fröhlich, Galois module structure of algebraic integers, Springer, Heidelberg, 1983.

[22] R. Greenberg, On p-adic Artin L-functions, Nagoya J. Math. 89 (1983), 77-87.

[23] K. Iwasawa, On the μ-invariants of $\mathbb{Z}_l$-extensions, In: Number Theory, Algebraic Geometry and Commutative Algebra, in honor of Y. Akizuki, Kinokuniya, Tokyo 1973, 1-11.

[24] K. Kato, Iwasawa theory and p-adic Hodge theory, Kodai Math. J. 16 no 1 (1993) 1-31.

[25] K. Kato, Lectures on the approach to Iwasawa theory of Hasse-Weil L-functions via B_{dR}, Part I, In: Arithmetical Algebraic Geometry (ed. E. Ballico), Lecture Notes in Math. 1553 (1993) 50-163, Springer, New York, 1993.

[26] K. Kato, Lectures on the approach to Iwasawa theory for Hasse-Weil L-functions via B_{dR}, Part II, preprint 1993.

[27] F. Knudsen and D. Mumford, The projectivity of the moduli space of stable curves I: Preliminaries on 'det' and 'Div', Math. Scand. 39 (1976), 19-55.

[28] M. Kurihara, Iwasawa theory and Fitting ideals, J. reine angew. Math. 561 (2003), 39-86

[29] J. S. Milne, Étale Cohomology, Princeton Mathematics Series 17, Princeton University Press, 1980.

[30] J. Nekovář, Selmer Complexes, to appear in Astérisque.

[31] J. Neukirch, A. Schmidt, K. Wingberg, Cohomology of Number Fields, Grundlehren 323, Springer 2000.

[32] A. Salomaa, Formal Languages, Academic Press, 1973.

[33] C. L. Siegel, Über die Fourierschen Koeffizienten von Modulformen, Nachr. Akad. Wiss. Göttingen 3 (1970) 15-56.

[34] V. P. Snaith, Algebraic K-groups as Galois modules, Prog. in Math. 206, Birkhäuser 2002.

[35] V. P. Snaith, Relative K_0, Fitting ideals and the Stickelberger phenomena, preprint, December 2001.

[36] D. Solomon, On the classgroups of imaginary abelian fields, Ann. Inst. Fourier 40 (1990), 467-492.

[37] O. Venjakob, A noncommutative Weierstrass Preparation Theorem and applications to Iwasawa theory, J. reine angew. Math. 559 (2003), 153-191

[38] L. C. Washington, Introduction to Cyclotomic Fields, Graduate Texts in Mathematics 83, Springer, New York 1982.

[39] A. Wiles, The Iwasawa conjecture for totally real fields, Annals of Math. 131 (1990), 493-540.

David Burns
Department of Mathematics
King's College London Strand
London WC2R 2LS
England
david.burns@kcl.ac.uk

Cornelius Greither
Fakultät für Informatik
Universität der
Bundeswehr München,
D-85577 Neubiberg
Germany
greither@informatik.unibw-muenchen.de

Links Between Cyclotomic and GL_2 Iwasawa Theory

John Coates, Peter Schneider, Ramdorai Sujatha

Received: September 26, 2002
Revised: February 26, 2003

Vying with the light
Of the heaven-coursing sun,
Oh, let me search,
That I find it once again,
The Way that was so pure.

Dedicated to Kazuya Kato

Abstract. We study, in the case of ordinary primes, some connections between the GL_2 and cyclotomic Iwasawa theory of an elliptic curve without complex multiplication.

2000 Mathematics Subject Classification: 11G05; 11R23.
Keywords and Phrases: Iwasawa theory, elliptic curves.

1 Introduction

Let F be a finite extension of $\mathbb{Q}$, E an elliptic curve defined over F, and p a prime number such that E has good ordinary reduction at all primes v of F dividing p. Let F^{cyc} denote the cyclotomic $\mathbb{Z}_p$-extension of F, and put $\Gamma = G(F^{cyc}/F)$. A very well known conjecture due to Mazur [15] asserts that the dual of the Selmer group of E over F^{cyc} is a torsion module over the Iwasawa algebra $\Lambda(\Gamma)$ of Γ (the best result in the direction of this conjecture is due to Kato [13], who proves it when E is defined over $\mathbb{Q}$, and F is an abelian

extension of $\mathbb{Q}$). Now let F_∞ denote a Galois extension of F containing F^{cyc} whose Galois group G over F is a p-adic Lie group of dimension > 1. This paper will make some modest observations about the following general problem. To what extent, and in what way, does the arithmetic of E over F^{cyc} influence the arithmetic of E over the much larger p-adic Lie extension F_∞? For example, assuming that G is pro-p and has no element of order p, and that Mazur's conjecture is true for E over F^{cyc}, can one deduce that the dual of the Selmer group of E over F_∞ is a torsion module over the Iwasawa algebra $\Lambda(G)$ of G? In their paper [12] in this volume, Hachimori and Venjakob prove the surprising result that the answer is yes for a wide class of non-abelian extensions F_∞ of F in which G has dimension 2. In the first two sections of this paper, we study the different case in which $F_\infty = F(E_{p^\infty})$ is the field obtained by adjoining to F the coordinates of all p-power division points on E. We assume that E has no complex multiplication, so that G is open in $GL_2(\mathbb{Z}_p)$ by a well known theorem of Serre. In [4], it was shown that, in this case, the dual of the Selmer group of E over F_∞ is $\Lambda(G)$-torsion provided Mazur's conjecture for E over F^{cyc} is true, and, in addition, the μ-invariant of the dual of Selmer of E over F^{cyc} is zero. Although we can do no better than this result as far as showing the dual of E over F_∞ is $\Lambda(G)$-torsion, we do prove some related results which were not known earlier. The main result of this paper is Theorem 3.1, relating the truncated G-Euler characteristic of the Selmer group of E over F_∞ with the Γ-Euler characteristic of E over $F^{\rm cyc}$ (only a slightly weaker form of this result was shown in [4] under the more restrictive assumption that the Selmer group of E over F is finite). We also establish a relation between the μ-invariant of the dual of the Selmer group of E over F_∞ and the μ-invariant of the dual of the Selmer group of E over$F^{\rm cyc}$ (see Propositions 3.12 and 3.13). The proof of this relationship between μ-invariants led us to study in §4 a new invariant attached to a wide class of finitely generated torsion modules for the Iwasawa algebra of any pro-p p-adic Lie group G, which has no element of order p, and which has a closed normal subgroup H such that $\Gamma = G/H$ is isomorphic to $\mathbb{Z}_p$. This new invariant is a refinement of the G-Euler characteristic of such modules, and, in particular, we investigate its behaviour on pseudo-null modules.
Added in proof: Since this paper was written, O. Venjakob, in his Heidelberg Habilitation thesis, has made use of our invariant to prove the existence of an analogue of the characteristic power series of commutative Iwasawa theory for any module in the category $\mathfrak{M}_H(G)$ which is defined at the beginning of §4.

Notation

Let p be a fixed prime number. If A is an abelian group, $A(p)$ will always denote its p-primary subgroup. Throughout G will denote a compact p-adic Lie group, and we write

$$\Lambda(G) = \varprojlim_U \mathbb{Z}_p\,[G/U],$$

where U runs over all open normal subgroups of G, for the Iwasawa algebra of G. All modules we consider will be left modules for $\Lambda(G)$. If W is a compact $\Lambda(G)$-module, we write $\widehat{W} = \mathrm{Hom}_{\mathbb{Z}_p}(W, \mathbb{Q}_p/\mathbb{Z}_p)$ for its Pontrjagin dual. It is a discrete p-primary abelian group, endowed with its natural structure of a left $\Lambda(G)$-module. We shall write $H_i(G, W)$ for the homology groups of W. If W is a finitely generated $\Lambda(G)$-module, then it is well known that, for each $i \geqslant 0$, $H_i(G, W)$ has as its Pontrjagin dual the cohomology group $H^i(G, \widehat{W})$, which is defined with continuous cochains.

When K is a field, $\overline{K}$ will denote a fixed separable closure of K, and G_K will denote the Galois group of $\overline{K}$ over K. We write $G(L/K)$ for the Galois group of a Galois extension L over K. If A is a discrete G_K-module, $H^i(K, A)$ will denote the usual Galois cohomology groups. Throughout, F will denote a finite extension of $\mathbb{Q}$, and E an elliptic curve defined over F, which will always be assumed to have $\mathrm{End}_{\overline{F}}(E) = \mathbb{Z}$. We impose throughout sections 2 and 3 of this paper the hypothesis that E has good ordinary reduction at all primes v of F dividing p. We let S denote any fixed set of places of F such that S contains all primes of F dividing p, and all primes of F where E has bad reduction. We write F_S for the maximal extension of F which is unramified outside S and the archimedean primes of F. For each intermediate field L with $F \subset L \subset F_S$, we put $G_S(L) = G(F_S/L)$. Finally, we shall always assume that our prime p satisfies $p \geq 5$.

2 The fundamental exact sequence

We recall that we always assume that E has good ordinary reduction at all primes v of F dividing p. Let F^{cyc} denote the cyclotomic $\mathbb{Z}_p$-extension of F, and put $\Gamma = G(F^{\mathrm{cyc}}/F)$. By a basic conjecture of Mazur [15], the dual of the Selmer group of E over F^{cyc} is a torsion $\Lambda(\Gamma)$-module. The aim of this section is to analyse the consequences of this conjecture for the study of the Selmer group of E over the field generated by all the p-power division points on E.

If L is any intermediate field with $F \subset L \subset F_S$, we recall that the Selmer group $\mathcal{S}(E/L)$ is defined by

$$\mathcal{S}(E/L) = \mathrm{Ker}(H^1(L, E_{p^\infty}) \to \prod_w H^1(L_w, E(\overline{L}_w))),$$

where E_{p^∞} denotes the Galois module of all p-power division points on E. Here w runs over all non-archimedean valuations of L, and L_w denotes the union of the completions at w of all finite extensions of $\mathbb{Q}$ contained in L. As usual, it is more convenient to view $\mathcal{S}(E/L)$ as a subgroup of $H^1(G_S(L), E_{p^\infty})$. For $v \in S$, we define $J_v(L) = \varinjlim J_v(K)$, where the inductive limit is taken with respect to the restriction maps as K ranges over all finite extensions of F contained in L, and where, for such a finite extension K of F, we define

$$J_v(K) = \bigoplus_{w|v} H^1(K_w, E(\overline{K}_w))(p).$$

Since $L \subset F_S$, we then have $\mathcal{S}(E/L) = \mathrm{Ker}\ \lambda_S(L)$, where

$$\lambda_S(L)\ :\ H^1(G_S(L), E_{p^\infty}) \to \bigoplus_{v\in S} J_v(L) \tag{1}$$

denotes the evident localization map. We shall see that, when L is an infinite extension of F, the question of the surjectivity of $\lambda_S(L)$ is a basic one. We write

$$X(E/L) = \mathrm{Hom}(\mathcal{S}(E/L), \mathbb{Q}_p/\mathbb{Z}_p) \tag{2}$$

for the compact Pontrjagin dual of the discrete module $\mathcal{S}(E/L)$. We shall be primarily concerned with the case in which L is Galois over F, in which case both $\mathcal{S}(E/L)$ and $X(E/L)$ have a natural left action of $G(L/F)$, which extends to a left action of the whole Iwasawa algebra $\Lambda(G(L/F))$. It is easy to see that $X(E/L)$ is a finitely generated $\Lambda(G(L/F))$-module.
We are going to exploit the following well-known lemma (see [17, Lemmas 4 and 5] and also [12, §7] for an account of the proof in a more general setting).

Lemma 2.1 *Assume that $X(E/F^{\mathrm{cyc}})$ is $\Lambda(\Gamma)$-torsion. Then the localization map $\lambda_S(F^{\mathrm{cyc}})$ is surjective, i.e. we have the exact sequence of Γ-modules*

$$0 \to \mathcal{S}(E/F^{\mathrm{cyc}}) \to H^1(G_S(F^{\mathrm{cyc}}), E_{p^\infty}) \overset{\lambda_S(F^{\mathrm{cyc}})}{\to} \bigoplus_{v\in S} J_v(F^{\mathrm{cyc}}) \to 0. \tag{3}$$

Moreover, we also have

$$H^2(G_S(F^{\mathrm{cyc}}), E_{p^\infty}) = 0. \tag{4}$$

We now consider the field $F_\infty = F(E_{p^\infty})$, which always contains F^{cyc} by the Weil pairing. We write

$$G = G(F_\infty/F), \quad H = G(F_\infty/F^{\mathrm{cyc}}), \tag{5}$$

so that $G/H = \Gamma$. By Serre's theorem, G is an open subgroup of $\mathrm{Aut}(T_p(E)) = GL_2(\mathbb{Z}_p)$, where, as usual, $T_p(E) = \varprojlim E_{p^n}$. The following is the principal result of this section.

Theorem 2.2 *Assume that (i) $p \geq 5$, (ii) E has good ordinary reduction at all primes v of F dividing p, and (iii) $X(E/F^{\mathrm{cyc}})$ is $\Lambda(\Gamma)$-torsion. Then we have the exact sequence*

(6)

$$0 \to \mathcal{S}(E/F_\infty)^G \to W_\infty^G \to \bigoplus_{v\in S} J_v(F_\infty)^G \to H^1(G, \mathcal{S}(E/F_\infty)) \to H^1(G, W_\infty) \to 0,$$

where $W_\infty = H^1(G_S(F_\infty), E_{p^\infty})$.

In fact, we expect $\lambda_S(F_\infty)$ to be surjective for all prime numbers p. If $p \geq 5$, it is shown in [4] that $H^i(G, W_\infty) = 0$ for all $i \geq 2$, and that $H^i(G, J_v(F_\infty)) = 0$ for all $i \geq 1$ and all $v \in S$. Thus if $\lambda_S(F_\infty)$ is surjective for a prime number $p \geq 5$, the exact sequence (6) follows. However, what is surprising about Theorem 2.2 is that we can establish it without knowing the surjectivity of $\lambda_S(F_\infty)$ (in our present state of knowledge [4], to prove the surjectivity of $\lambda_S(F_\infty)$ we must assume hypotheses *(i)* and *(ii)* of Theorem 2.2, replace *(iii)* by the stronger hypothesis that $X(E/F^{\mathrm{cyc}})$ is a finitely generated $\mathbb{Z}_p$-module, and, in addition, assume that G is pro-p). Finally, we mention that if hypotheses *(i)* and *(ii)* of Theorem 2.2 hold, and if also G is pro-p, then $X(E/F_\infty)$ is $\Lambda(G)$-torsion if and only if $\lambda_S(F_\infty)$ is surjective.
We now proceed to establish Theorem 2.2 via a series of lemmas. For these lemmas, we assume that the hypotheses *(i), (ii)* and *(iii)* of Theorem 2.2 are valid.

Lemma 2.3 *We have the exact sequence*

$$0 \to \mathcal{S}(E/F_\infty)^H \to H^1(G_S(F_\infty), E_{p^\infty})^H \overset{\rho_S(F_\infty)}{\to} \bigoplus_{v \in S} J_v(F_\infty)^H \to 0,$$

where $\rho_S(F_\infty)$ is induced by the localization map $\lambda_S(F_\infty)$.

Proof All we have to show is that $\rho_S(F_\infty)$ is surjective. We clearly have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{S}(E/F_\infty)^H & \longrightarrow & H^1(G_S(F_\infty), E_{p^\infty})^H & \xrightarrow[\rho_S(F_\infty)]{} & \bigoplus_{v\in S} J_v(F_\infty)^H & & \\
 & & \uparrow & & \uparrow & & \uparrow \gamma_S(F^{\mathrm{cyc}}) & & \\
0 & \longrightarrow & \mathcal{S}(E/F^{\mathrm{cyc}}) & \longrightarrow & H^1(G_S(F^{\mathrm{cyc}}), E_{p^\infty}) & \xrightarrow[\lambda_S(F^{\mathrm{cyc}})]{} & \bigoplus_{v\in S} J_v(F^{\mathrm{cyc}}) & \longrightarrow & 0,
\end{array}$$

where $\gamma_S(F^{\mathrm{cyc}})$ is induced by restriction, and where the surjectivity of $\lambda_S(F^{\mathrm{cyc}})$ is given by Lemma 2.1, thanks to our assumption that $X(E/F^{\mathrm{cyc}})$ is $\Lambda(\Gamma)$-torsion. Hence it suffices to prove the surjectivity of $\gamma_S(F^{\mathrm{cyc}})$. This is essentially contained in the proof of [4, Lemma 6.7], but we give the detailed proof as it is only shown there that $\gamma_S(F^{\mathrm{cyc}})$ has finite cokernel. As is well known and easy to see, there are only finitely many primes of F^{cyc} above each non-archimedean prime of F. Hence we have

$$\mathrm{Coker}(\gamma_S(F^{\mathrm{cyc}})) = \bigoplus_{w|S} \mathrm{Coker}(\gamma_w(F^{\mathrm{cyc}})),$$

where w runs over all primes of F^{cyc} lying above primes in S, and where, as $H^2(F_{\infty,w}, E) = 0$,

$$\mathrm{Coker}(\gamma_w(F^{\mathrm{cyc}})) = H^2(\Omega_w, E(F_{\infty,w}))(p);$$

here Ω_w denotes the decomposition group in H of some fixed prime of F_∞ lying above w. Now Ω_w is a p-adic Lie group with no elements of order p as $p \geq 5$, and so Ω_w has finite p-cohomological dimension equal to its dimension as a p-adic Lie group. Moreover, a simple local analysis (see [2] or [4]) shows that Ω_w has dimension at most 1 when w does not divide p, and dimension 2 when w does divide p. We claim that we always have

$$H^2(\Omega_w, E_{p^\infty}) = 0. \tag{7}$$

This is plain from the above remarks when v does not divide p. When v divides p, Ω_w has p-cohomological dimension equal to 2, and, thus $H^2(\Omega_w, E_{p^\infty})$ is a divisible group. On the other hand, as E has good reduction at w, it is well-known (see [4] for a direct argument, or [8] for a general result) that $H^2(\Omega_w, E_{p^\infty})$ is finite, whence (7) follows. We now finish the proof using (7). Assume first that w does not divide p. Then (see [2, Lemma 3.7]) we have

$$\mathrm{Coker}(\gamma_w(F^{\mathrm{cyc}})) = H^2(\Omega_w, E_{p^\infty}),$$

and so we obtain the surjectivity of $\gamma_w(F^{\mathrm{cyc}})$ from (7). Suppose now that w does divide p. Then it follows from the results of [3] that

$$\mathrm{Coker}(\gamma_w(F^{\mathrm{cyc}})) = H^2(\Omega_w, \tilde{E}_{w,p^\infty}),$$

where $\tilde{E}_{w,p^\infty}$ denotes the image of E_{p^∞} under reduction modulo w. But as Ω_w has p-cohomological dimension equal to 2, the vanishing of $H^2(\Omega_w, \tilde{E}_{w,p^\infty})$ is an immediate consequence of (7). This completes the proof of Lemma 2.3. □

Lemma 2.4 *We have* $H^i(H, H^1(G_S(F_\infty), E_{p^\infty})) = 0$ *for all* $i \geq 1$.

Proof We have

$$H^m(G_S(F_\infty), E_{p^\infty}) = 0, \quad (m \geq 2). \tag{8}$$

Indeed, (8) is obvious for $m > 2$ as $G_S(F_\infty)$ has p-cohomological dimension equal to 2, and it is a consequence of Iwasawa's work on the cyclotomic $\mathbb{Z}_p$-extension of number fields when $m = 2$ (see [2, Theorem 2.10]). In view of (8), the Hochschild-Serre spectral sequence gives

$$H^{i+1}(G_S(F^{\mathrm{cyc}}), E_{p^\infty}) \to H^i(H, H^1(G_S(F_\infty), E_{p^\infty})) \to H^{i+2}(H, E_{p^\infty}). \tag{9}$$

The group on the left of (9) vanishes (for $i = 1$, we use Lemma 2.1). Now H has p-cohomological dimension 3, and so $H^{i+2}(H, E_{p^\infty})$ is zero for $i > 1$, and divisible for $i = 1$. On the other hand, it is known [6] that $H^k(H, E_{p^\infty})$ is finite for all $k \geq 0$, whence, in particular, we must have $H^3(H, E_{p^\infty}) = 0$. Thus (9) gives Lemma 2.4 as required. □

Lemma 2.5 *We have* $H^1(H, \mathcal{S}(E/F_\infty)) = 0$.

Proof Let A_∞ denote the image of $\lambda_S(F_\infty)$. Hence, in view of Lemma 2.4 with $i = 1$, we have the exact sequence

$$0 \to \mathcal{S}(E/F_\infty)^H \to H^1(G_S(F_\infty), E_{p^\infty})^H \to A_\infty^H \to H^1(H, \mathcal{S}(E/F_\infty)) \to 0.$$

But the surjectivity of $\rho_S(F_\infty)$ in Lemma 2.3 shows that

$$A_\infty^H = \bigoplus_{v \in S} J_v(F_\infty)^H,$$

whence it is clear that $H^1(H, \mathcal{S}(E/F_\infty)) = 0$, as required. □

Remark 2.6 By a similar argument to that given in the proof of [2, Theorem 3.2], we see that $H^i(H, J_v(F_\infty)) = 0$ for all $i \geq 1$, for all primes $p \geq 5$, and all finite places v of F. Thus we deduce from Lemma 2.4 that the surjectivity of $\lambda_S(F_\infty)$ implies that $H^i(H, \mathcal{S}(E/F_\infty)) = 0$ for all $i \geq 1$. Unfortunately, we cannot at present prove the surjectivity of $\lambda_S(F_\infty)$ assuming only the hypotheses *(i), (ii)* and *(iii)* of Theorem 2.2.

Lemma 2.7 *We have isomorphisms*

$$H^1(\Gamma, H^1(G_S(F_\infty), E_{p^\infty})^H) \simeq H^1(G, H^1(G_S(F_\infty), E_{p^\infty}))$$
$$H^1(\Gamma, \mathcal{S}(E/F_\infty)^H) \simeq H^1(G, \mathcal{S}(E/F_\infty)).$$

Proof This is immediate from Lemmas 2.4 and 2.5, and the usual inflation-restriction exact sequence for H^1. □

Lemma 2.8 *We have* $H^1(\Gamma, \bigoplus_{v \in S} J_v(F_\infty)^H) = 0$.

Proof To simplify notation, let us put $K = F^{\text{cyc}}$. Since the map

$$\gamma_S(K) \ : \ \bigoplus_{v \in S} J_v(K) \to \bigoplus_{v \in S} J_v(F_\infty)^H$$

is surjective by Lemma 2.3, and since Γ has p-cohomological dimension equal to 1, it suffices to show that

$$H^1(\Gamma, \bigoplus_{v \in S} J_v(K)) = 0. \quad (10)$$

It is well-known that (10) is valid, but we sketch a proof now for completeness. For each place v of F, let w be a fixed place of K above v, and let $\Gamma_v \subset \Gamma$ denote the decomposition group of w over v, which is an open subgroup of Γ. As usual, it follows from Shapiro's lemma that

$$H^1(\Gamma, J_v(K)) \simeq H^1(\Gamma_v, H^1(K_w, E)(p)),$$

and so we must prove that

$$H^1(\Gamma_v, H^1(K_w, E)(p)) = 0. \tag{11}$$

We begin by noting that, for each algebraic extension L of F_v, we have

$$H^2(L, E_{p^\infty}) = 0. \tag{12}$$

When L is a finite extension of F_v, Tate local duality shows that $H^2(L, E_{p^\infty})$ is dual to $H^0(L, T_p(E))$, and this latter group is zero because the torsion subgroup of $E(L)$ is finite. Clearly (12) is now true for all algebraic extensions L of F_v by passing to the inductive limit over all finite extensions of F_v contained in L. Suppose now that v does not divide p. Then

$$H^1(K_w, E_{p^\infty}) \simeq H^1(K_w, E)(p).$$

In view of (12), the Hochschild-Serre spectral sequence for K_w/F_v shows that we have the exact sequence

$$H^2(F_v, E_{p^\infty}) \to H^1(\Gamma_v, H^1(K_w, E_{p^\infty})) \to H^3(\Gamma_v, E_{p^\infty}(K_v)).$$

But the group on the left is zero again by (12), and the group on the right is zero because Γ_v has p-cohomological dimension equal to 1. This proves (11) in this case. Suppose next that v divides p. As K_w is a deeply ramified p-adic field, it follows from [3] that

$$H^1(K_w, E)(p) \simeq H^1(K_w, \tilde{E}_{w,p^\infty}),$$

where $\tilde{E}_{w,p^\infty}$ denotes the image of E_{p^∞} under reduction modulo w. As $\tilde{E}_{w,p^\infty}$ is a quotient of E_{p^∞}, and as the Galois group of $\overline{F_v}$ over K_w has p-cohomological dimension at most 2, we conclude from (12) that

$$H^2(K_w, \tilde{E}_{w,p^\infty}) = 0. \tag{13}$$

In fact, the Galois group of $\overline{F}_v$ over K_w has p-cohomological dimension 1, so that (13) also follows directly from this fact. In view of (13), the Hochschild-Serre spectral sequence for K_w/F_v yields the exact sequence

$$H^2(F_v, \tilde{E}_{w,p^\infty}) \to H^1(\Gamma_v, H^1(K_w, \tilde{E}_{w,p^\infty}) \to H^3(\Gamma_v, \tilde{E}_{w,p^\infty}).$$

The group on the right is zero because Γ_v has p-cohomological dimension equal to 1. By Tate local duality, the dual of the group on the left is $H^0(F_v, T_p(\hat{E}_w))$, where $T_p(\hat{E}_w) \varprojlim \hat{E}_{w,p^n}$, and $\hat{E}_{w,p^n}$ denotes the kernel of multiplication by p^n on the formal group $\hat{E}_w$ of E at w. But again $H^0(F_v, T_p(\hat{E}_w)) = 0$ because the torsion subgroup of $E(F_v)$ is finite, and so we have proven (11) in this case. This completes the proof of Lemma 2.8. □

PROOF OF THEOREM 2.2 We can now prove Theorem 2.2. Put $W_\infty = H^1(G_S(F_\infty), E_{p^\infty})$. Taking Γ-cohomology of the exact sequence of Lemma 2.3, and using Lemma 2.8, we obtain the long exact sequence

$$0 \to \mathcal{S}(E/F_\infty)^G \to W_\infty^G \to \bigoplus_{v\in S} J_v(F_\infty)^G \to$$
$$\to H^1(\Gamma, \mathcal{S}(E/F_\infty)^H) \to H^1(\Gamma, W_\infty^H) \to 0.$$

Theorem 2.2 now follows immediately from Lemma 2.7. □

The following is a curious consequence of the arguments in this section, and we have included it because a parallel result has a striking application to the work of Hachimori and Venjakob [12].

PROPOSITION 2.9 *Assume that (i) $p \geq 5$, (ii) E has good ordinary reduction at all primes v of F dividing p, (iii) $X(E/F^{\text{cyc}})$ is $\Lambda(\Gamma)$ -torsion, and (iv) G is pro-p. Then $X(E/F_\infty)$ is $\Lambda(G)$-torsion if and only if $H^2(H, \mathcal{S}(E/F_\infty)) = 0$.*

PROOF Since G is pro-p, it is well-known (see for example, Theorem 4.12 of [2]) that $X(E/F_\infty)$ is $\Lambda(G)$-torsion if and only if $\lambda_S(F_\infty)$ is surjective. We have already observed in Remark 2.6 that the surjectivity of $\lambda_S(F_\infty)$ implies that $H^i(H, \mathcal{S}(E/F_\infty)) = 0$ for all $i \geq 1$. Conversely, assume that $H^2(H, \mathcal{S}(E/F_\infty)) = 0$. As in the proof of Lemma 2.5, let A_∞ denote the image of $\lambda_S(F_\infty)$. Taking the H-cohomology of the exact sequence

$$0 \to \mathcal{S}(E/F_\infty) \to H^1(G_S(F_\infty), E_{p^\infty}) \to A_\infty \to 0,$$

we conclude from Lemma 2.4 that

$$H^1(H, A_\infty) \simeq H^2(H, \mathcal{S}(E/F_\infty)).$$

Hence our hypothesis implies that $H^1(H, A_\infty) = 0$. Now let $B_\infty = \text{Coker}(\lambda_S(F_\infty))$. Taking H-cohomology of the exact sequence

$$0 \to A_\infty \to \bigoplus_{v\in S} J_v(F_\infty) \to B_\infty \to 0,$$

and using Lemma 2.3 and the fact mentioned in Remark 2.6 that $H^1(H, J_v(F_\infty)) = 0$ for all $v \in S$, we conclude that

$$B_\infty^H = H^1(H, A_\infty).$$

Hence $B_\infty^H = 0$. But as H is pro-p and B_∞ is a p-primary discrete H-module, it follows that $B_\infty = 0$. Thus $\lambda_S(F_\infty)$ is surjective, and this completes the proof of Proposition 2.5. □

We conclude this section by proving a result relating the so-called μ-invariants of the $\Lambda(G)$-module $X(E/F_\infty)$ and the $\Lambda(\Gamma)$-module $X(E/F^{\text{cyc}})$. Let us assume for the rest of this section that G is pro-p. Let W be any finitely generated

$\Lambda(G)$-module. We write $W(p)$ for the submodule of all elements of W which are annihilated by some power of p, and we then define

$$W_f = W/W(p). \tag{14}$$

We recall that the homology groups $H_i(G,W)$ are the Pontrjagin duals of the cohomology groups $H^i(G,\widehat{W})$, where $\widehat{W} = \mathrm{Hom}_{\mathbb{Z}_p}(W,\mathbb{Q}_p/\mathbb{Z}_p)$ is the discrete p-primary Pontrjagin dual of W (see [11]). As is explained in [11], the $H^i(G,W)$ are finitely generated $\mathbb{Z}_p$-modules, and thus the $H_i(G,W(p))$ are finite groups for all $i \geq 0$. Now the μ-invariant of W, which we shall denote by $\mu_G(W)$, can be defined in various equivalent fashions (see [22], [11]) in terms of the structure theory of the $\Lambda(G)$-module $W(p)$. However, for us it will be more convenient to use the description of $\mu_G(W)$ in terms of the Euler characteristics which is proven in [11], namely

$$p^{\mu_G(W)} = \prod_{i\geq 0} \#(H_i(G,W(p)))^{(-1)^i}. \tag{15}$$

As usual, we shall denote the right hand side of (15) by $\chi(G,W(p))$. We shall use the analogous notation and results for the μ-invariants of finitely generated $\Lambda(\Gamma)$-modules. For the remainder of this section, we always assume the hypotheses (i)-(iv) of Proposition 2.9.

Lemma 2.10 *Both $H_0(H,X(E/F_\infty)_f)$ and $H_1(H,X(E/F_\infty)_f)$ are finitely generated torsion $\Lambda(\Gamma)$-modules, and $H_1(H,X(E/F_\infty)_f)$ is annihilated by some power of p.*

Proof For simplicity, put $X = X(E/F_\infty)$. By duality, the restriction map on cohomology induces a Γ-homomorphism

$$\alpha : X_H = H_0(H,X) \to X(E/F^{\mathrm{cyc}}). \tag{16}$$

Thanks to the basic results of [5], it is shown in [4, Lemma 6.7], that $\mathrm{Ker}(\alpha)$ is a finitely generated $\mathbb{Z}_p$-module, and that $\mathrm{Coker}(\alpha)$ is finite. As $X(E/F^{\mathrm{cyc}})$ is assumed to be $\Lambda(\Gamma)$-torsion, it follows that $H_0(H,X)$ is a finitely generated torsion $\Lambda(\Gamma)$-module. Now if we take H-cohomology of the exact sequence

$$0 \to X(p) \to X \to X_f \to 0, \tag{17}$$

and recall that $H_1(H,X) = 0$ by Lemma 2.5, we obtain the exact sequence of $\Lambda(\Gamma)$-modules

$$0 \to H_1(H,X_f) \to H_0(H,X(p)) \to H_0(H,X) \to H_0(H,X_f) \to 0. \tag{18}$$

The right hand end of (18) shows that $H_0(H,X_f)$ is $\Lambda(\Gamma)$-torsion, and the left hand end shows that $H_1(H,X_f)$ is finitely generated over $\Lambda(\Gamma)$ and annihilated by a power of p, because these two properties clearly hold for $H_0(H,X(p))$. This completes the proof of Lemma 2.10. □

REMARK 2.11 Put $X = X(E/F_\infty)$, and continue to assume that hypotheses (i)-(iv) of Proposition 2.9 hold. As H is pro-p, Nakayama's lemma also shows that $X_f = X/X(p)$ is finitely generated as a $\Lambda(H)$-module if and only if $(X_f)_H = H_0(H, X_f)$ is a finitely generated $\mathbb{Z}_p$-module. As $H_0(H, X_f)$ is a finitely generated torsion $\Lambda(\Gamma)$-module, it follows that X_f is finitely generated as a $\Lambda(H)$-module if and only if $\mu_\Gamma(H_0(H, X_f)) = 0$.

PROPOSITION 2.12 *Assume that (i) $p \geq 5$, (ii) E has good ordinary reduction at all primes v of F dividing p (iii) G is pro-p, and (iv) $X(E/F^{\text{cyc}})$ is $\Lambda(\Gamma)$-torsion. Then we have*

$$\mu_G(X(E/F_\infty)) = \mu_\Gamma(X(E/F^{\text{cyc}})) + \delta + \epsilon, \tag{19}$$

where, writing $X = X(E/F_\infty)$,

$$\delta = \Sigma_{i=0}^{1} (-1)^{i+1}\mu_\Gamma(H_i(H, X_f)), \quad \epsilon = \Sigma_{i=1}^{3}(-1)^i\mu_\Gamma(H_i(H, X(p))). \tag{20}$$

PROOF As each module in the exact sequence (18) is $\Lambda(\Gamma)$-torsion, it follows (see [11, Prop. 1.9]) that the alternating sum of the μ_Γ-invariants taken along (18) is zero. Moreover, the μ_Γ-invariants of the two middle terms in (18) can be calculated as follows. Firstly, as $\text{Ker}(\alpha)$ and $\text{Coker}(\alpha)$ are finitely generated $\mathbb{Z}_p$-modules, it follows from (16) that

$$\mu_\Gamma(H_0(H, X)) = \mu_\Gamma(X(E/F^{\text{cyc}})). \tag{21}$$

Secondly, for $i = 1, 2, 3, 4$, the Hochschild-Serre spectral sequence yields the short exact sequence

$$0 \to H_0(\Gamma, H_i(H, X(p))) \to H_i(G, X(p)) \to H_1(\Gamma, H_{i-1}(H, X(p))) \to 0. \tag{22}$$

Also, we have $H_4(H, X(p)) = 0$ because H has p-homological dimension equal to 3. It follows easily that

$$\chi(G, X(p)) = \prod_{i=0}^{3} \chi(\Gamma, H_i(H, X(p)))^{(-1)^i}, \tag{23}$$

whence by (15) for both the group G and the group Γ, we obtain

$$\mu_G(X) = \Sigma_{i=0}^{3}(-1)^i\mu_\Gamma(H_i(H, X(p))). \tag{24}$$

Proposition 2.12 now follows immediately (21) and (24) and from the fact that the alternating sum of the μ_Γ-invariants along (18) is 0. □

We now give a stronger form of (19) when we impose the additional hypothesis that $X(E/F_\infty)$ is $\Lambda(G)$-torsion.

PROPOSITION 2.13 *Assume that (i) $p \geq 5$, (ii) E has good ordinary reduction at all primes v of F dividing p, (iii) G is pro-p, (iv) $X(E/F^{\text{cyc}})$ is $\Lambda(\Gamma)$-torsion, and (v) $X(E/F_\infty)$ is $\Lambda(G)$-torsion. Then $H_i(H, X_f)$ $(i = 1, 2)$, where*

$X_f = X(E/F_\infty)/X(E/F_\infty)(p)$, is a finitely generated $\Lambda(\Gamma)$-module which is killed by a power of p. Moreover,

$$\mu_G(X(E/F_\infty)) = \mu_\Gamma(X(E/F^{\mathrm{cyc}})) + \Sigma_{i=0}^2 \, (-1)^{i+1}\mu_\Gamma(H_i(H, X_f)). \tag{25}$$

Corollary 2.14 *Assume hypotheses (i)-(iv) of Proposition 2.13 and replace (v) by the hypothesis that $X(E/F_\infty)/X(E/F_\infty)(p)$ is finitely generated over $\Lambda(H)$. Then*

$$\mu_G(X(E/F_\infty)) = \mu_\Gamma(X(E/F^{\mathrm{cyc}})). \tag{26}$$

To deduce the corollary, we first note that the hypothesis that X_f is finitely generated over $\Lambda(H)$ implies that X_f is $\Lambda(G)$-torsion, whence X is also $\Lambda(G)$-torsion. Secondly, the $H_i(H, X_f)$ are finitely generated $\mathbb{Z}_p$-modules once X_f is finitely generated over $\Lambda(H)$, and hence their μ_Γ-invariants are zero. Thus (26) then follows from (25). We remark that, in all cases known to date in which we can prove $X(E/F_\infty)$ is $\Lambda(G)$-torsion, one can show that X_f is finitely generated over $\Lambda(H)$, but we have no idea at present of how to prove this latter assertion in general.

Proof of Proposition 2.13 Again put $X = X(E/F_\infty)$. Since we are now assuming that X is $\Lambda(G)$-torsion, or equivalently that $\lambda_S(F_\infty)$ is surjective, we have already remarked (see Remark 2.6) that

$$H_i(H, X) = 0 \text{ for all } i \geq 1. \tag{27}$$

We next claim that

$$H_3(H, X_f) = 0. \tag{28}$$

Indeed, as H has p-cohomological dimension 3, and multiplication by p is injective on X_f, it follows that multiplication by p must also be injective on $H_3(H, X_f)$. On the other hand, taking H-homology of the exact sequence (17), we see that $H_3(H, X_f)$ injects into the torsion group $H_2(H, X(p))$ because $H_3(H, X) = 0$. Thus (28) follows. Moreover, using (27) and (28), we conclude from the long exact sequence of H-cohomology of (17) that

$$H_1(H, X(p)) = H_2(H, X_f), \quad H_i(H, X(p)) = 0 \ (i = 2, 3). \tag{29}$$

Thus (25) now follows from (19), completing the proof of Proposition 2.13. □

3 The truncated Euler characteristic

We assume throughout this section our three standard hypotheses: (i) $p \geq 5$, (ii) E has good ordinary reduction at all primes v of F dividing p, and (iii) $X(E/F^{\mathrm{cyc}})$ is $\Lambda(\Gamma)$-torsion. Since G has dimension 4 as a p-adic Lie group and has no element of order p, G has p-cohomological dimension equal to

4. We begin by defining the notion of the truncated G-Euler characteristic $\chi_t(G,A)$ of a discrete p-primary G-module A. The main aim of this section is to compute the truncated G-Euler characteristic of $\mathcal{S}(E/F_\infty)$ in terms of the Γ-Euler characteristic of $\mathcal{S}(E/F^{\mathrm{cyc}})$. A Birch-Swinnerton-Dyer type formula for the latter Euler characteristic is well-known (see Schneider [19], Perrin-Riou [18]), and we shall recall this at the end of this section.
If D is a discrete p-primary Γ-module, we have

$$H^0(\Gamma,D)=D^\Gamma,\quad H^1(\Gamma,D)\cong D_\Gamma,$$

and hence there is the obvious map

$$\phi_D\ :\ H^0(\Gamma,D)\to H^1(\Gamma,D) \tag{30}$$

given by $\phi_D(x)=$ residue class of x in D_Γ. If f is any homomorphism of abelian groups, we define

$$q(f)=\#(\mathrm{Ker}(f))/\#(\mathrm{Coker}(f)), \tag{31}$$

saying $q(f)$ is finite if both $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$ are finite. We say D has finite Γ-Euler characteristic if $q(\phi_D)$ is finite, and we then define $\chi(\Gamma,D)=q(\phi_D)$. Suppose now that A is a discrete p-primary G-module. As in the previous section, we write $H=G(F_\infty/F^{\mathrm{cyc}})$, so that H is a closed normal subgroup of G with $G/H=\Gamma$. Parallel to (30), we define a map

$$\xi_A\ :\ H^0(G,A)\to H^1(G,A) \tag{32}$$

by $\xi_A=\eta\circ\phi_{A^H}$, where η is the inflation map from $H^1(\Gamma,A^H)$ to $H^1(G,A)$. We say that A has *finite truncated G-Euler characteristic* if $q(\xi_A)$ is finite, and we then define the truncated G-Euler characteristic of A by

$$\chi_t(G,A)=q(\xi_A). \tag{33}$$

Of course, if the $H^i(G,A)$ are finite for all $i\geqslant 0$, and zero for $i\geqslant 2$, then the truncated G-Euler characteristic of A coincides with the usual G-Euler characteristic of A. However, we shall be interested in G-modules A, which arise naturally in arithmetic, and for which we can often show that the truncated Euler characteristic is finite and compute it, without being able to prove anything about the $H^i(G,A)$ for $i\geqslant 2$. In fact (see the remarks immediately after Theorem 2.2), it is conjectured that in the arithmetic example we consider when $A=S(E/F_\infty)$, we always have $H^i(G,A)=0$ for $i\geqslant 2$.
If v is a finite prime of F, we write $L_v(E,s)$ for the Euler factor at v of the complex L-function of E over F. In particular, when E has bad reduction (the only case we shall need) $L_v(E,s)$ is 1, $(1-(Nv)^{-s})^{-1}$, and $(1+(Nv)^{-s})^{-1}$, according as E has additive, split multiplicative, or non-split multiplicative reduction at v. Also, if u and v are non-zero elements of $\mathbb{Q}_p$, $u\sim v$ means that u/v is a p-adic unit. The following is the main result of this section. As usual, j_E denotes the j-invariant of the elliptic curve E.

Theorem 3.1 *Assume that (i) $p \geq 5$, (ii) E has good ordinary reduction at all places v of F dividing p, and (iii) $X(E/F^{\mathrm{cyc}})$ is $\Lambda(\Gamma)$-torsion. Then $\chi_t(G, \mathcal{S}(E/F_\infty))$ is finite if and only if $\chi(\Gamma, \mathcal{S}(E/F^{\mathrm{cyc}}))$ is finite. Moreover, when $\chi(\Gamma, \mathcal{S}(E/F^{\mathrm{cyc}}))$ is finite, we have*

$$\chi_t(G, \mathcal{S}(E/F_\infty)) \sim \chi(\Gamma, \mathcal{S}(E/F^{\mathrm{cyc}})) \times \prod_{v \in \mathcal{M}} L_v(E,1)^{-1},$$

where $\mathcal{M}$ consists of all places v of F where the j-invariant of E is non-integral,

The following is a special case of Theorem 3.1 (see [2, Theorem 1.15] for a weaker result in this direction). Assuming hypotheses (i) and (ii) of Theorem 3.1, it is well-known (see [6] or Greenberg [10]) that both $X(E/F^{\mathrm{cyc}})$ is $\Lambda(\Gamma)$-torsion and $\chi(\Gamma, \mathcal{S}(E/F^{\mathrm{cyc}}))$ is finite when $\mathcal{S}(E/F)$ is finite.

Corollary 3.2 *Assume that (i) $p \geq 5$, (ii) E has good ordinary reduction at all places v of F dividing p, and (iii) $\mathcal{S}(E/F)$ is finite. Then $\chi_t(G, \mathcal{S}(E/F_\infty))$ is finite and*

$$\chi_t(G, \mathcal{S}(E/F_\infty)) \sim \chi(\Gamma, \mathcal{S}(E/F^{\mathrm{cyc}})) \times \prod_{v \in \mathcal{M}} L_v(E,1)^{-1},$$

We now give the proof of Theorem 3.1 via a series of lemmas. For these lemmas, we assume that hypotheses (i), (ii) and (iii) of Theorem 3.1 are valid. We let S' be the subset of S consisting of all places v of F such that $\mathrm{ord}_v(j_E) < 0$. We then define

$$\mathcal{S}'(E/F^{\mathrm{cyc}}) = \mathrm{Ker}(H^1(G_S(F^{\mathrm{cyc}}), E_{p^\infty}) \to \bigoplus_{v \in S \setminus S'} J_v(F^{\mathrm{cyc}})). \tag{34}$$

We remark that we could also define $\mathcal{S}'(E/F_\infty)$ analogously, but in fact $\mathcal{S}'(E/F_\infty) = \mathcal{S}(E/F_\infty)$ as $J_v(F_\infty) = 0$ for v in S' (see [2, Lemma 3.3]).

Lemma 3.3 *We have the exact sequence of Γ-modules*

$$0 \to \mathcal{S}(E/F^{\mathrm{cyc}}) \to \mathcal{S}'(E/F^{\mathrm{cyc}}) \to \bigoplus_{v \in S'} J_v(F^{\mathrm{cyc}}) \to 0.$$

Proof This is clear from the commutative diagram with exact rows (cf. Lemma 2.1)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{S}(E/F^{\mathrm{cyc}}) & \longrightarrow & H^1(G_S(F^{\mathrm{cyc}}), E_{p^\infty}) & \xrightarrow{\lambda_S(F^{\mathrm{cyc}})} & \bigoplus_{v \in S} J_v(F^{\mathrm{cyc}}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{S}'(E/F^{\mathrm{cyc}}) & \longrightarrow & H^1(G_S(F^{\mathrm{cyc}}), E_{p^\infty}) & \longrightarrow & \bigoplus_{v \in S \setminus S'} J_v(F^{\mathrm{cyc}}) & \longrightarrow & 0 \end{array}$$

where all the vertical arrows are the natural maps, the first is the natural inclusion, the middle map is the identity and the right vertical map is the natural projection. □

LEMMA 3.4 *We have that $\chi(\Gamma, \mathcal{S}'(E/F^{\text{cyc}}))$ is finite if and only if $\chi(\Gamma, \mathcal{S}(E/F^{\text{cyc}}))$ is finite. Moreover, when $\chi(\Gamma, \mathcal{S}(E/F^{\text{cyc}}))$ is finite, we have*

$$\chi(\Gamma, \mathcal{S}'(E/F^{\text{cyc}})) \sim \chi(\Gamma, \mathcal{S}(E/F^{\text{cyc}})) \times \prod_{v \in \mathcal{M}} L_v(E,1)^{-1}$$

and $\mathcal{M}$ is the set of places with non-integral j-invariant as before.

PROOF The lemma is very well known (see, for example, [4, Lemma 5.6 and Corollary 5.8] for a special case), and so we only sketch the proof. Indeed, by the multiplicativity of the Euler characteristic in exact sequences, we see that all follows from Lemma 3.3 and the fact that

$$\chi(\Gamma, \bigoplus_{v \in S'} J_v(F^{\text{cyc}})) \sim \prod_{v \in \mathcal{M}} L_v(E,1)^{-1},$$

(see [4, Lemmas 5.6 and 5.11]). □

LEMMA 3.5 *Let $f : A \to B$ be a homomorphism of p-primary Γ-modules with both* $\text{Ker}(f)$ *and* $\text{Coker}(f)$ *finite. If*

$$g \ : \ A^{\Gamma} \to B^{\Gamma}, \quad h \ : \ A_{\Gamma} \to B_{\Gamma}$$

denote the two maps induced by f, then $q(g)$ and $q(h)$ are finite, and $q(g) = q(h)$.

PROOF This follows easily from breaking up the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A^{\Gamma} & \longrightarrow & A & \xrightarrow{\gamma-1} & A & \longrightarrow & A_{\Gamma} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & B^{\Gamma} & \longrightarrow & B & \xrightarrow{\gamma-1} & B & \longrightarrow & B_{\Gamma} & \longrightarrow & 0,
\end{array}$$

where γ is a topological generator of Γ, into two commutative diagrams of short exact sequences and applying the snake lemma to each of these diagrams. □

The heart of the proof of Theorem 3.1 is to apply Lemma 3.5 to the map

$$f \ : \ \mathcal{S}'(E/F^{\text{cyc}}) \to \mathcal{S}(E/F_{\infty})^H \tag{35}$$

given by the restriction. We therefore need

LEMMA 3.6 *If f is given by* (35), *both* $\text{Ker}(f)$ *and* $\text{Coker}(f)$ *are finite.*

PROOF Put $S'' = S \setminus S'$. As $J_v(F_{\infty}) = 0$ for $v \in S'$, we have the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{S}(E/F_\infty)^H & \longrightarrow & H^1(G_S(F_\infty), E_{p^\infty})^H & \longrightarrow & \bigoplus_{v\in S''} J_v(F_\infty)^H & & \\ & & f\big\uparrow & & \big\uparrow & & \big\uparrow \gamma''_S(F^{\text{cyc}}) & & \\ 0 & \longrightarrow & \mathcal{S}'(E/F^{\text{cyc}}) & \longrightarrow & H^1(G_S(F^{\text{cyc}}), E_{p^\infty}) & \longrightarrow & \bigoplus_{v\in S''} J_v(F^{\text{cyc}}) & \longrightarrow & 0. \end{array}$$

Now it is known that $H^i(H, E_{p^\infty})$ is finite for all $i \geq 0$ (see [6, Appendix A.2.6]). Hence Lemma 3.6 will follow from this diagram provided we show that the kernel of $\gamma_{S''}(F^{\text{cyc}})$ is finite (it is here that we will use the fact that S'' contains no place of potential multiplicative reduction for E). Now

$$\text{Ker}(\gamma_{S''}(F^{\text{cyc}})) = \bigoplus_w \text{Ker}(\gamma_w(F^{\text{cyc}})),$$

where

$$\text{Ker}(\gamma_w(F^{\text{cyc}})) = H^1(\Omega_w, E(F_{\infty,w}))(p);$$

here w runs over all primes of F^{cyc} lying above primes in S'', and Ω_w denotes the decomposition group in H of some fixed prime of F_∞ above w. Then (see [2, Lemma 3.7]) we have

$$\text{Ker}(\gamma_w(F^{\text{cyc}})) = H^1(\Omega_w, E_{p^\infty}). \tag{36}$$

But E has potential good reduction at w, and F^{cyc}_w contains the unique unramified $\mathbb{Z}_p$ extension of F_w. Hence, as $p \geq 5$, it follows from Serre-Tate [20] that Ω_w must be a finite group of order prime to p, and thus (36) shows that $\gamma_w(F^{\text{cyc}})$ is injective in this case. Suppose next that w does divide p. Then it follows from the results of [3] that

$$\text{Ker}(\gamma_w(F^{\text{cyc}})) = H^1(\Omega_w, \tilde{E}_{w,p^\infty}), \tag{37}$$

where $\tilde{E}_{w,p^\infty}$ denotes the image of E_{p^∞} under reduction modulo w. But it is known (see either [8] or [4, Lemma 5.25]) that the cohomology groups $H^i(\Omega_w, \tilde{E}_{w,p^\infty})$ are finite for all $i \geq 0$. Hence $\text{Ker}(\gamma_w(F^{\text{cyc}}))$ is finite, and the proof of Lemma 3.6 is complete. □

We can now finish the proof of Theorem 3.1. Consider the Γ-modules

$$A = \mathcal{S}'(E/F^{\text{cyc}}), \quad B = \mathcal{S}(E/F_\infty)^H,$$

and the map given by (35). By Lemma 2.7, we have

$$H^0(\Gamma, B) = H^0(G, \mathcal{S}(E/F_\infty)), \quad H^1(\Gamma, B) = H^1(G, \mathcal{S}(E/F_\infty)),$$

and we clearly have the commutative diagram

$$\begin{array}{ccc} H^0(\Gamma, A) & \xrightarrow{\ g\ } & H^0(\Gamma, B) = H^0(G, \mathcal{S}(E/F_\infty)) \\ \phi_A\big\downarrow & & \xi_B\big\downarrow \\ H^1(\Gamma, A) & \xrightarrow{\ h\ } & H^1(\Gamma, B) = H^1(G, \mathcal{S}(E/F_\infty)), \end{array}$$

here g and h are induced by f, and ϕ_A and ξ_B are defined by (30) and (32), respectively. By Lemmas 3.5 and 3.6, we know that $q(g)$ and $q(h)$ are finite, whence it is plain from the diagram that $q(\phi_A)$ is finite if and only if $q(\xi_B)$ is finite. Now it is a basic property of the q-function, that q of a composition of two maps is the product of the $q's$ of the individual maps. Hence, as $\xi_B \circ g = \phi_A \circ h$, we conclude that $q(\xi_B)q(g) = q(\phi_A)q(h)$, when $q(\phi_A)$ or equivalently $q(\xi_B)$ is finite. But $q(g) = q(h)$ by Lemma 3.5, and so $q(\phi_A) = q(\xi_B)$. Applying Lemma 3.4 to compute $q(\phi_A)$ in terms of $\chi(\Gamma, \mathcal{S}(E/F^{\text{cyc}}))$, the proof of Theorem 3.1 is now complete. □

We now briefly recall the value of $\chi(\Gamma, \mathcal{S}(E/F^{\text{cyc}}))$ determined by Perrin-Riou [18] and Schneider [19], and use Theorem 3.1 to discuss several numerical examples. We need the following notation. If A is an abelian group, $A(p)$ will denote its p-primary subgroup. If w is any finite prime of F, we put $c_w = [E(F_w) : E_0(F_w)]$, where $E_0(F_w)$ denotes the subgroup of points in $E(F_w)$ with non-singular reduction modulo w. Put

$$\tau_p(E) = |\Pi_w c_w|_p^{-1},$$

where the p-adic valuation is normalized so that $|p|_p = p^{-1}$. For each place v of F dividing p, let k_v be the residue field of v, and let $\widetilde{E}_v$ over k_v be the reduction of E modulo v. We say that a prime v of F dividing p is *anomalous* if $\widetilde{E}_v(k_v)(p) \neq 0$. We always continue to assume that $p \geqslant 5$, and that E has good ordinary reduction at all primes v of F dividing p. Write $\text{Ш}(E/F)$ for the Tate-Shafarevich group of E over F.

CASE 1. We assume that $E(F)$ is finite, and $\text{Ш}(E/F)(p)$ is finite, or equivalently $\mathcal{S}(E/F)$ is finite. Then it is shown in [18], [19] that $H^i(\Gamma, \mathcal{S}(E/F^{\text{cyc}}))$ $(i = 0, 1)$ is finite, and

$$\chi(\Gamma, \mathcal{S}(E/F^{\text{cyc}})) = \frac{\#(\text{Ш}(E/F)(p))}{\#(E(F)(p))^2} \times \tau_p(E) \times \prod_{v|p} \#(\widetilde{E_v}\ (k_v)(p))^2.$$

EXAMPLE 1. Take $F = \mathbb{Q}$ and let E be the elliptic curve

$$X_1(11) : y^2 + y = x^3 - x^2.$$

Kolyvagin's theorem tells us that both $E(\mathbb{Q})$ and $\text{Ш}(E/\mathbb{Q})$ are finite, since the complex L-function of E does not vanish at $s1$. The conjecture of Birch and Swinnerton-Dyer predicts that $\text{Ш}(E/\mathbb{Q}) = 0$, but the numerical verification of this via Heegner points does not seem to exist in the literature. Now it is well known that $\mathcal{S}(E/\mathbb{Q}^{\text{cyc}}) = 0$ for $p = 5$, and that $\mathcal{S}(E/\mathbb{Q}^{\text{cyc}}) = 0$ for a good ordinary prime $p > 5$ provided $\text{Ш}(E/\mathbb{Q})(p) = 0$ (see [6]). Hence, assuming $\text{Ш}(E/\mathbb{Q})(p) = 0$ when $p > 5$, we conclude that

$$\chi(\Gamma, \mathcal{S}(E/\mathbb{Q}^{\text{cyc}})) = 1$$

for all good ordinary primes $p \geqslant 5$. Our knowledge of $X(E/F_\infty)$ is extremely limited. We know (see [2]) that $X(E/F_\infty)\bigotimes_{\mathbb{Z}_p} \mathbb{Q}_p$ has infinite dimension over $\mathbb{Q}_p$ for all primes $p \geqslant 5$. Take $S\{p, 11\}$, and note that 11 is the only prime where E has non-integral j-invariant and $L_{11}(E, S) = (1 - 11^{-s})^{-1}$. First, assume that $p = 5$. Theorem 3.1 then tells us that

$$\chi_t(G, \mathcal{S}(E/F_\infty)) = 5.$$

But it is shown in [4] that the map $\lambda_S(G_\infty)$ is surjective when $p = 5$. Hence we have also $H^i(G, \mathcal{S}(E/F_\infty)) = 0$ for $i = 2, 3, 4$ and so we have the stronger result that $\mathcal{S}(E/F_\infty)$ has finite G-Euler characteristic, and $\chi(G, \mathcal{S}(E/F_\infty)) = 5$. Next assume that p is a good ordinary prime > 5, and that $\text{Ш}(E/\mathbb{Q})(p) = 0$. Then Theorem 3.1 gives

$$\chi_t(G, \mathcal{S}(E/F_\infty)) = 1.$$

However, it has not been proven yet that $\lambda_S(F_\infty)$ is surjective for a single good ordinary prime $p > 5$, and so we know nothing about the $H^i(G, \mathcal{S}(E/F_\infty))$ for $i = 2, 3, 4$. But we can deduce a little more from the above evaluation of the truncated Euler characteristic. For $E = X_1(11)$, Serre has shown that $GGL_2(\mathbb{Z}_p)$ for all $p \geqslant 7$, whence it follows by a well known argument that $H^i(G, E_{p^\infty}) = 0$ for all $i \geqslant 1$. But it is proven in [2] (see Lemmas 4.8 and 4.9) that the order of $H^1(G, \mathcal{S}(E/F_\infty))$ must divide the order of $H^3(G, E_{p^\infty})$. Hence finally we conclude that

$$H^0(G, \mathcal{S}(E/F_\infty)) = H^1(G, \mathcal{S}(E/F_\infty)) = 0$$

for all good ordinary primes $p > 5$ such that $\text{Ш}(E/\mathbb{Q})(p) = 0$.

Case 2. We assume now that $E(F)$ has rank $g \geq 1$, and that $\text{Ш}(E/F)(p)$ is finite. We write

$$< \ , \ >_{F,p} : E(F) \times E(F) \to \mathbb{Q}_p$$

for the canonical p-adic height pairing (see [16], [18], [19]), which exists because of our hypotheses that E has good ordinary reduction at all places v of F dividing p. If $P_1, \cdots, P_g$ denote a basis of $E(F)$ modulo torsion, we define

$$R_p(E/F) = \det < P_i, P_j > .$$

It is conjectured that we always have $R_p(E/F) \neq 0$, but this is unknown. Recalling that we are assuming that $\text{Ш}(E/F)(p)$ is finite, the principal result of [18], [19] is that firstly $\chi(\Gamma, \mathcal{S}(E/F^{\text{cyc}}))$ is finite if and only if $R_p(E/F) \neq 0$, and secondly, when $R_p(E/F) \neq 0$, we have

$$\chi(\Gamma, \mathcal{S}(E/F^{\text{cyc}})) = p^{-g} \mid \rho_p \mid_p^{-1} .$$

where

$$\rho_p = \frac{R_p(E/F) \times \#(\text{Ш}(E/F)(p))}{\sharp(E(F)(p))^2} \times \tau_p(E) \times \prod_{v \mid p} \#(\widetilde{E}_v(k_v)(p))^2.$$

The work of Mazur and Tate [16] gives a very efficient numerical method for calculating $R_p(E/F)$ up to a p-adic unit, and so the right hand side of this formula can be easily computed in simple examples, provided we know the order of $\text{Ш}(E/F)(p)$.

EXAMPLE 2. Take $F = \mathbb{Q}$, and let E be the elliptic curve of conductor 37.

$$E \ : \ y^2 + y = x^3 - x.$$

It is well-known that $E(\mathbb{Q})$ is a free abelian group of rank 1 generated by $P = (0, 0)$. Moreover, the complex L-function of E over $\mathbb{Q}$ has a simple zero at $s = 1$, and so $\text{Ш}(E/\mathbb{Q})$ is finite by Kolyvagin's theorem. In fact, the conjecture of Birch and Swinnerton-Dyer predicts that $\text{Ш}(E/\mathbb{Q}) = 0$, but again the numerical verification of this via Heegner points does not seem to exist in the literature. We put $h_p(P) =< P, P >_{\mathbb{Q},p}$. The supersingular primes for E less than 500 are 2, 3, 17, 19, 257, 311. The ordinary primes for E less than 500 which are anamolous are 53, 127, 443. The following values for $h_p(P)$ for p ordinary with $p < 500$ have been calculated by C. Wuthrich. We have

$$\mid h_p(P) \mid_p = p^{-2} \text{ for } p = 13,\ 67 \text{ and } \mid h_p(P) \mid_p = p \text{ for } p = 53,\ 127,\ 443,$$

and $\mid h_p(P) \mid_p = p^{-1}$ for the remaining primes. As $c_{37} = 1$, we conclude from the above formula that $\chi(\Gamma, \mathcal{S}(E/\mathbb{Q}^{\text{cyc}})) = 1$ for all ordinary primes $p < 500$, with the exception of $p = 13,\ 67$, where we have $\chi(\Gamma, \mathcal{S}(E/\mathbb{Q}^{\text{cyc}})) = p$; here we are assuming that $\text{Ш}(E/\mathbb{Q})(p) = 0$. Now 37 is the only prime where the j-invariant of E is non-integral, and $L_{37}(E, s) = (1+37^{-s})^{-1}$. Hence we conclude from Theorem 3.1 that $\chi_t(G, \mathcal{S}(E/F_\infty))$ is finite and

$$\chi_t(G, \mathcal{S}(E/F_\infty)) = \chi(\Gamma, \mathcal{S}(E/\mathbb{Q}^{\text{cyc}})).$$

We point out that we do not know that $\lambda_S(F_\infty)$ is surjective for a single ordinary prime p for this curve.

4 AN ALGEBRAIC INVARIANT

In this last section, we attach a new invariant to a wide class of modules over the Iwasawa algebra of a compact p-adic Lie group G satisfying the following conditions: (i) G is pro-p, (ii) G has no element of order p, and (iii) G has a closed normal subgroup H such that G/H is isomorphic to $\mathbb{Z}_p$. We always put

$$\Gamma = G/H, \tag{38}$$

and write $Q(\Gamma)$ for the quotient field of the Iwasawa algebra $\Lambda(\Gamma)$ of Γ. We assume that G satisfies these hypotheses for the rest of this section. By [14], G has finite p-cohomological dimension, which is equal to the dimension d of

G as a p-adic Lie group. We are interested in the following full subcategory of the category of all finitely generated torsion $\Lambda(G)$-modules. Let $\mathfrak{M}_H(G)$ denote the category whose objects are all $\Lambda(G)$-modules which are finitely generated over $\Lambda(H)$ (such modules are automatically torsion $\Lambda(G)$-modules, because $\Lambda(G)$ is not finitely generated as a $\Lambda(H)$-module). Perhaps somewhat unexpectedly, it is shown in [4] that many $\Lambda(G)$-modules which arise in the GL_2-Iwasawa theory of elliptic curves belong to $\mathfrak{M}_H(G)$ (specifically, in the notation of §2, $X(E/F_\infty)$ is a $\Lambda(G)$-module in $\mathfrak{M}_H(G)$ when we assume that (i) $p \geqslant 5$, (ii) E has good ordinary reduction at all primes v of F dividing p, (iii) G is pro-p, and (iv) $X(E/F^{cyc})$ is a finitely generated $\mathbb{Z}_p$-module; here $H = G(F_\infty/F^{cyc})$).

If f and g are any two non-zero elements of $Q(\Gamma)$, we write $f \sim g$ if fg^{-1} is a unit in $\Lambda(\Gamma)$. To each M is $\mathfrak{M}_H(G)$, we now attach a non-zero element f_M of $Q(\Gamma)$, which is canonical in the sense that it is well-defined up to the equivalence relation $\sim$. As M is a finitely generated $\Lambda(H)$-module, all of the homology groups

$$H_i(H, M) \qquad (i \geqslant 0) \tag{39}$$

are finitely generated $\mathbb{Z}_p$-modules. On the other hand, as M is a $\Lambda(G)$-module, these homology groups have a natural structure as $\Lambda(\Gamma)$-modules, and they must therefore be torsion $\Lambda(\Gamma)$-modules. Let g_i in $\Lambda(\Gamma)$ be a characteristic power series for the $\Lambda(\Gamma)$-module $H_i(H, M)$, and define

$$f_M = \prod_{i \geqslant 0} g_i^{(-1)^i}. \tag{40}$$

This product is, of course, finite because $H_i(H, M) = 0$ for $i \geqslant d$, and it is well defined up to $\sim$, because each g_i is well defined up to multiplication by a unit in $\Lambda(\Gamma)$.

Lemma 4.1 *(i) If $M(p)$ denotes the submodule of M in $\mathfrak{M}_H(G)$ consisting of all elements of p-power order, then $f_M \sim f_{M/M(p)}$; (ii) If we have an exact sequence of modules in $\mathfrak{M}_H(G)$,*

$$0 \to M_1 \to M_2 \to M_3 \to 0, \tag{41}$$

then $f_{M_2} \sim f_{M_1} \cdot f_{M_3}$.

Proof (i) is plain from the long exact sequence of H-homology derived from the exact sequence

$$0 \to M(p) \to M \to M/M(p) \to 0,$$

and the fact that the $H_i(H, M(p))(i \geqslant 0)$ are killed by any power of p which annihilates $M(p)$, and are therefore finite. Similarly, (ii) follows from the exact sequence of H-homology derived from (41), and the classical fact that the $\Lambda(\Gamma)$-characteristic series is multiplicative in exact sequences. $\square$

One of the main reasons we are interested in the invariant f_M is its link with the G-Euler characteristic of M. If g is any element of $\Lambda(\Gamma)$, we write $g(0)$ as usual for the image of g under the augmentation map from $\Lambda(\Gamma)$ to $\mathbb{Z}_p$. Similarly, if g is any non-zero element of $Q(\Gamma)$, then the fact that $\Lambda(\Gamma)$ is a unique factorization domain allows us to write $g = h_1/h_2$, where h_1 and h_2 are relatively prime elements of $\Lambda(\Gamma)$. We say that $g(0)$ is defined if $h_2(0) \neq 0$, and then put $g(0) = h_1(0)/h_2(0)$. If $M \in \mathfrak{M}_H(G)$, we say that M has finite G-Euler characteristic if the $H_i(G, M)$ are finite for all $i \geqslant 0$, and, when this is the case, we define

$$\chi(G, M) = \prod_{i \geqslant 0} \#(H_i(G, M))^{(-1)^i}. \tag{42}$$

LEMMA 4.2 *Assume that M in $\mathfrak{M}_H(G)$ has finite G-Euler characteristic. Then $f_M(0)$ is defined and non-zero, and we have*

$$\chi(G, M) = |f_M(0)|_p^{-1}. \tag{43}$$

PROOF As Γ has cohomological dimension 1, for all $i \geqslant 1$ the Hochschild-Serre spectral sequence yields an exact sequence

$$0 \to H_0(\Gamma, H_i(H, M)) \to H_i(G, M) \to H_1(\Gamma, H_{i-1}(H, M)) \to 0. \tag{44}$$

The finiteness of the $H_i(G, M)$ ($i \geqslant 0$) therefore implies that, for all $i \geqslant 0$, we have $g_i(0) \neq 0$, where, as above, g_i denotes a characteristic power series for $H_i(H, M)$. Moreover, by a classical formula for $\Lambda(\Gamma)$-modules, we then have

$$|g_i(0)|_p^{-1} = \frac{\#(H_0(\Gamma, H_i(H, M)))}{\#(H_1(\Gamma, H_i(H, M)))} \tag{45}$$

for all $i \geqslant 0$. The formula (43) is now plain from (44) and (45), and the proof of the lemma is complete. $\square$

Let

$$\pi_\Gamma : \Lambda(G) \to \Lambda(\Gamma) \tag{46}$$

be the natural ring homomorphism.

LEMMA 4.3 *Let g be a non-zero element of $\Lambda(G)$ such that $N = \Lambda(G)/\Lambda(G)g$ belongs to $\mathfrak{M}_H(G)$. Then $H_i(H, N) = 0$ for all $i > 0$. Moreover, $f_N \sim \pi_\Gamma(g)$ lies in $\Lambda(\Gamma)$, and $f_N(0) \neq 0$ if and only if N has finite G-Euler characteristic.*

Proof We have the exact sequence of $\Lambda(G)$-modules

$$0 \to \Lambda(G) \xrightarrow{\cdot g} \Lambda(G) \to N \to 0, \tag{47}$$

where $\cdot g$ denotes multiplication on the right by g. Since $H_i(G, \Lambda(G)) = 0$ for $i \geqslant 1$, we conclude that $H_i(G, N) = 0$ for $i > 1$. But, for any compact $\Lambda(G)$-module R, the Hochschild-Serre spectral sequence provides an injection of $H_0(\Gamma, H_i(H, R))$ into $H_i(G, R)$, and so the vanishing of $H_i(G, R)$ implies the vanishing of the compact Γ-module $H_i(H, R)$. It follows that $H_i(H, \Lambda(G)) = 0$ for $i \geqslant 1$, and that $H_i(H, N) = 0$ for $i > 1$. Put $h = \pi_\Gamma(g)$. Taking H-homology of (47). we obtain the exact sequence of $\Lambda(\Gamma)$-modules

$$0 \to H_1(H, N) \to \Lambda(\Gamma) \xrightarrow{\cdot h} \Lambda(\Gamma) \to H_0(H, N) \to 0, \tag{48}$$

where $\cdot h$ now denotes right multiplication by h. Note that $H_0(H, N)$ is $\Lambda(\Gamma)$-torsion as N is in $\mathfrak{M}_H(G)$, and so we must have $h \neq 0$. But then multiplication by h in $\Lambda(\Gamma)$ is injective, and so $H_1(H, N) = 0$, and it is then clear than $f_N \sim h$. Finally, returning to the G-homology of (47), we obtain the exact sequence

$$0 \to H_1(G, N) \to \mathbb{Z}_p \xrightarrow{\cdot h(0)} \mathbb{Z}_p \to H_0(G, N) \to 0, \tag{49}$$

and so N has finite G-Euler characteristic if and only if $h(0) \neq 0$. This completes the proof of Lemma 4.3. □

We recall (see [22]) that a finitely generated torsion $\Lambda(G)$-module M is defined to be *pseudo-null* if $\mathrm{Ext}^1_{\Lambda(G)}(M, \Lambda(G)) = 0$. If M lies in $\mathfrak{M}_H(G)$, it is an important fact that M is pseudo-null as a $\Lambda(G)$-module if and only if M is $\Lambda(H)$-torsion. Since G as an extension of Γ by H necessarily splits, and hence is a semi-direct product, this is proven in [23, Prop. 5.4] provided H is uniform. However, by a well-known argument [22, Prop. 2.7], it then follows in general for our G, since H must always contain an open subgroup which is uniform.

Lemma 4.4 *Let M be a module in $\mathfrak{M}_H(G)$. If G is isomorphic to $\mathbb{Z}_p^r$, for some integer $r \geqslant 1$, then $f_M \sim 1$ if and only if M is pseudo-null as a $\Lambda(G)$-module.*

Proof We assume $r \geq 2$, since pseudo-null modules are finite when $r = 1$. Let M be a $\Lambda(H)$-torsion module in $\mathfrak{M}_H(G)$. We recall that $f_M = \prod_{i \geqslant 0} g_i^{(-1)^i}$, where g_i in $\Lambda(\Gamma)$ is a characteristic power series for the torsion $\Lambda(\Gamma)$-module $H_i(H, M)$. By (i) of Lemma 4.1, we may assume that M has no p-torsion. Thus the assumptions of Lemma 2 of [9] when applied to M as a $\Lambda(H)$-module are satisfied, and we conclude that there exists a closed subgroup J of H such that $H/J \xrightarrow{\sim} \mathbb{Z}_p$ and M is finitely generated as a $\Lambda(J)$-module. Analogously to (44), the Hochschild-Serre spectral sequence gives rise to the exact sequences of finitely generated torsion $\Lambda(\Gamma)$-modules, for all $i \geqslant 1$,

$$0 \to H_0(H/J, H_i(J, M)) \to H_i(H, M) \to H_1(H/J, H_{i-1}(J, M)) \to 0. \tag{50}$$

Hence, if $g_{j,i}$ denotes the characteristic power series of $H_j(H/J, H_i(J,M))$ as a torsion $\Lambda(\Gamma)$-module, we obtain

$$g_0 \sim g_{0,0}, \quad g_i \sim g_{1,i-1} \cdot g_{0,i} \quad (i \geqslant 1).$$

Thus

$$f_M \sim g_{0,0} \times \prod_{i \geqslant 1} (g_{1,i-1} \cdot g_{0,i})^{(-1)^i} \sim \prod_{i \geqslant o} (g_{0,i}/g_{1,i})^{(-1)^i}.$$

On the other hand, letting $\overline{h}$ denote a topological generator of H/J, we have the exact sequence of $\Lambda(C)$-modules

$$0 \to H_1(H/J, H_i(J,M)) \to H_i(J,M) \xrightarrow{\overline{h}-1} H_i(J,M) \to H_0(H/J, H_i(J,M)) \to 0.$$

But $H_i(J,M)$ is a finitely generated $\mathbb{Z}_p$-module, because J was chosen so that M is a finitely generated $\Lambda(J)$-module. Hence the above exact sequence is in fact an exact sequence of finitely generated torsion $\Lambda(C)$-modules. By the multiplicativity of the characteristic power series along exact sequences, it follows that $g_{0,i} \sim g_{1,i}$.

Conversely, let $f_M \sim 1$. Under our hypotheses on G, the Iwasawa algebra $\Lambda(G)$ is a commutative regular local ring. By the classical structure theorem for finitely generated modules, M is pseudo-isomorphic to $\mathfrak{E}(M)$ where

$$\mathfrak{E}(M) = \bigoplus_{i=1}^{m} \Lambda(G)/\Lambda(G)g_i$$

with g_i non-zero for $i = 1, \cdots, m$. By Lemma 4.3 and the first part of the proof above, we see that $f_M \sim \pi_\Gamma(g_1 \cdots g_m)$. As f_M is assumed to be a unit in $\Lambda(\Gamma)$, we conclude that $\pi_\Gamma(g_1 \cdots g_m)$ is also a unit in $\Lambda(\Gamma)$, and hence $\pi_\Gamma(g_1 \cdots g_m)$ does not belong to the maximal ideal of $\Lambda(G)$. But, as the residue field of $\Lambda(G)$ is $\mathbb{F}_p$, and $\Lambda(G)$ is local, we see that $g_1 \cdots g_m$ is a unit in $\Lambda(G)$ and hence so is each g_1 $(i = 1, \cdots, m)$. Thus $\mathfrak{E}(M)$ is zero and M is pseudo-null, thereby completing the proof of the lemma. □

Lemma 4.5 *Assume that G is a direct product $C \times H$, where C is isomorphic to Γ and H has dimension ≥ 1. Suppose that M in $\mathfrak{M}_H(G)$ is finitely generated as a $\mathbb{Z}_p$-module, then $f_M \sim 1$.*

Proof We remark that a finitely generated $\mathbb{Z}_p$-module M in $\mathfrak{M}_H(G)$ is necessarily pseudo-null since H has dimension ≥ 1.

We fix a topological generator c of C end let $g_i(T)$ denote the characteristic polynomial of $c-1$ on the finite dimensional vector space $H_i(H,M) \underset{\mathbb{Z}_p}{\otimes} \mathbb{Q}_p$. We will show that

$$\prod_{i \geqslant 0} g_i^{(-1)^{(i)}} = 1.$$

Note first of all that, because M is finitely generated as a $\mathbb{Z}_p$-module, it is $\Lambda(H)$-torsion, and so we have (see [11])

$$\sum_{i\geqslant 0}(-1)^i \deg(g_i) = 0.$$

We now choose a finite extension $L/\mathbb{Q}_p$ which is a splitting field for all the polynomials g_i. Then g_i also can be viewed as the characteristic polynomial of $c-1$ on the L-vector space

$$H_i(H,M)\otimes_{\mathbb{Z}_p} L = H_i(H, M\otimes_{\mathbb{Z}_p} L).$$

Since our assertion is multiplicative in short exact sequences we may argue by induction with respect to the dimension of $M \underset{\mathbb{Z}_p}{\otimes} L$. As the H-action and the C-action commute, the H-action preserves all C-eigenspaces. This reduces us to the case where c acts on $M \underset{\mathbb{Z}_p}{\otimes} L$ by a single eigenvalue λ. Then c certainly acts on each $H_i(H,M) \underset{\mathbb{Z}_p}{\otimes} L$ by the same eigenvalue λ, and we obtain

$$\prod_{i\geqslant 0} g_i^{(-1)^{(i)}} = (T-\lambda)^{\sum(-1)^i \deg(g_i)} = (T-\lambda)^0 = 1.$$

□

When M is a finitely generated $\mathbb{Z}_p$-module which has finite G-Euler characteristic, then Lemmas 4.4 and 4.5 prove that $\chi(G,M)=1$. This is a slightly stronger version of the main theorem of [21] for a group $G = C\times H$ as in Lemma 4.5.

The previous two results might lead one to believe that the invariant f_M is a unit for all pseudo-null modules M in $\mathfrak{M}_H(G)$. However, the following two examples illustrate that this is not the case.

Example 3 (see [8], §5). Let K be any finite extension of $\mathbb{Q}_p$ which contains μ_p if p is odd and μ_4 if $p=2$. Take E to be any Tate elliptic curve over K, and write

$$G = G(K(E_{p^\infty})/K),\ \ H = G(K(E_{p^\infty})/K(\mu_{p^\infty})).$$

Thus G is a p-adic Lie group of dimension 2, H is a closed normal subgroup of G which is isomorphic to $\mathbb{Z}_p$, and $\Gamma G/H$ is isomorphic to $\mathbb{Z}_p$. If M is a $\mathbb{Z}_p$-module on which G_K-acts, $M(n)$, as usual will denote the n-fold Tate twist of M by $T_p(\mu) = \varprojlim \mu_{p^n}$. We now consider the G-module

$$W = T_p(E).$$

As is shown in [8] (see [8], formula (48)), we then have

$$H_0(H,W) = \mathbb{Z}_p,\ \ H_1(H,W) = \mathbb{Z}_p(-1).$$

To compute the corresponding f_W in $Q(\Gamma)$, let $\chi : G_K \to \mathbb{Z}_p^*$ be the cyclotomic character of G_K, i.e. $\sigma(\zeta) = \zeta^{\chi(\sigma)}$ for all σ in G_k and all ζ in μ_{p^∞}. Fix a topological generator γ of Γ, and identify $\Lambda(\Gamma)$ with the ring $\mathbb{Z}_p\big[[T]\big]$ of formal power series in T with coefficients in $\mathbb{Z}_p$ by mapping γ to $1+T$. Then we see immediately that

$$f_W = \frac{T}{T+1-\chi(\gamma)^{-1}}.$$

In particular, f_W does not belong to $\Lambda(\Gamma)$ in this example. We note also that, since G has dimension > 1, W is certainly a pseudo-null $\Lambda(G)$-module because W is a finitely generated $\mathbb{Z}_p$-module (see [24], Prop. 3.6).

Example 4. We now assume that $G = C \times H$, and we take a $\Lambda(H)$-module M of the form

$$M = \Lambda(H)/\Lambda(H)g, \tag{51}$$

where g is any non-zero element of $\Lambda(H)$. To make M into a $\Lambda(G)$-module, it suffices to give a continuous action of C on M, which commutes with the H-action. Fix a topological generator c of C. If z is any element of $\Lambda(H)$, we write $z(0)$ for the image of z under the augmentation map in $\mathbb{Z}_p$. We now take units w and u in $\Lambda(H)$ satisfying

$$gw = ug, \quad w(0) \equiv u(0) \equiv 1 \bmod p. \tag{52}$$

We let c act on M by

$$c.(z + \Lambda(H)g) = zw + \Lambda(H)g. \tag{53}$$

This is well-defined by the first condition in (52), and extends to a continuous action of C by the second condition in (52). Conversely, let M be any $\Lambda(G)$-module which is of the form (51) as a $\Lambda(H)$-module with $g \neq 0$. It is then easy to see that the action of c on M is necessarily given by (53), where u and w are units in $\Lambda(H)$ satisfying (52), because $\Lambda(H)$ is a local ring. As discussed before Lemma 4.4, this $\Lambda(G)$-module M is pseudo-null because it is plainly $\Lambda(H)$-torsion. To compute the invariant f_M of M, we consider the exact sequence of $\Lambda(G)$-modules

$$0 \to \Lambda(H) \xrightarrow{\cdot g} \Lambda(H) \to M \to 0, \tag{54}$$

where c acts on the first copy of $\Lambda(H)$ by right multiplication by u, and on the second by right multiplication by w. Taking H-coinvariants of (54), we obtain that $H_i(H, M) = 0$ for $i \geq 1$, and the exact sequence of $\Lambda(C)$-modules

$$0 \to H_1(H, M) \to \mathbb{Z}_p \xrightarrow{\cdot g(0)} \mathbb{Z}_p \to H_0(H, M) \to 0; \tag{55}$$

here c acts on the first copy of $\mathbb{Z}_p$ by multiplication by $u(0)$, and on the second by multiplication by $w(0)$. Thus

$$f_M(T) = (T - w(0) + 1)/(T - u(0) + 1). \tag{56}$$

This is not a unit in $\Lambda(\Gamma)$ provided $w(0) \neq u(0)$.

To obtain concrete examples of such modules M with $w(0) \neq u(0)$, we now assume that H contains a subgroup which is a semi-direct product of the following form. Let H_0 and C_0 be two closed subgroups of H which are isomorphic to $\mathbb{Z}_p$, and which are such that

$$xhx^{-1} = h^{\phi(x)} \quad (x \in C_0,\ h \in H_0),$$

where $\phi : C_0 \hookrightarrow \mathrm{Aut}(H_0) = {\mathbb{Z}_p}^*$ is a continuous injective group homomorphism. We note that this hypothesis is valid for any H which is open in $SL_n(\mathbb{Z}_p)$ ($n \geq 2$), or more generally for any compact H which is open in the group of $\mathbb{Q}_p$-points of a split semi-simple algebraic group over $\mathbb{Q}_p$. Now fix a toplogical generator h_0 of H_0, and any element γ of C_0 with $\gamma \neq 1$. Define

$$g = h_0 - 1,\ w = \gamma + p^r,$$

where r is any integer ≥ 1. We then have

$$gw = ug,\ \text{where } u = \gamma \cdot \frac{h_0^{\phi(\gamma)} - 1}{h_0 - 1} + p^r.$$

Hence

$$w(0) = 1 + p^r, \quad u(0) = \phi(\gamma) + p^r,$$

and so $u(0) \neq w(0)$ because ϕ is injective. Moreover, as $w(0)$ and $u(0)$ are not equal to 1, we see that M has finite G-Euler characteristic, which by Lemma 4.2 is given by

$$\chi(G, M) = \ \left| \frac{\phi(\gamma) - 1 + p^r}{p^r} \right|_p .$$

This therefore gives a new class of pseudo-null $\Lambda(G)$-modules, which are not finitely generated over $\mathbb{Z}_p$, and which have a non-trivial G-Euler characteristic.

Finally, we interpret this example as a statement about K-theory classes. Writing $K_0(\mathfrak{M}_H(G))$ for the Grothendieck group of $\mathfrak{M}_H(G)$, it is clear from Lemma 4.1 that the map $M \mapsto f_M$ induces a homomorphism from $K_0(\mathfrak{M}_H(G))$ to $Q(\Gamma)^*/\Lambda(\Gamma)^*$. Now let M be any $\Lambda(G)$-module of the form (51) as a $\Lambda(H)$-module with $g \neq 0$. It follows from the computation of f_M given by (56) that the class of M in $K_0(\mathfrak{M}_H(G))$ is non-zero.

Let $\mathcal{C}^o(G)$ denote the abelian category of finitely generated torsion $\Lambda(G)$-modules which contains $\mathfrak{M}_H(G)$ as a full subcategory. Thus there is a natural map

$$i : K_0(\mathfrak{M}_H(G)) \to K_0(\mathcal{C}^0(G))$$

on the Grothendieck groups. Again, let M be a $\Lambda(G)$-module of the form (51). Writing $[M]$ for the class of M in $K_0(\mathfrak{M}_H(G))$, we have just seen that $[M]$ is not zero provided $w(0) \neq u(0)$. However, we now show that $i[M] = 0$. Let

$Q(G)$ denote the skew-field of fractions of $\Lambda(G)$, and let $\Omega(G)$ be the abelian group defined by

$$\Omega(G) := Q(G)^*/\Lambda(G)^*[Q(G)^*, Q(G)^*], \tag{57}$$

where for any ring A, A^* denotes the multiplicative group of units in A and $[A^*, A^*]$ is the commutator subgroup. It is well-known [1] that the localisation sequence yields an isomorphism

$$\phi : K_0(\mathcal{C}^0(G)) \simeq \Omega(G)$$

which sends the class of any module of the form $(\Lambda(G)/\Lambda(G)z)$, $z \neq 0$ in $\Lambda(G)$, to the class of z in $\Omega(G)$. Hence (54) shows that $\phi(i([M]))$ is the coset of $(c-u)^{-1}(c-w)$. We prove that this coset is trivial in $\Omega(G)$. Indeed, since c is in the centre of G, by (52), we get

$$g(c-w) = (c-u)g. \tag{58}$$

Hence in $\Omega(G)$, we have

$$\begin{aligned} g(c-u)^{-1}(c-w) \quad &= g(c-w)(c-u)^{-1} \\ &= (c-u)g(c-u)^{-1} \quad \text{by (58)} \\ &= (c-u)(c-u)^{-1}g \\ &= g. \end{aligned}$$

Since $\Omega(G)$ is a group, it follows that the class of $(c-u)^{-1}(c-w)$ in $\Omega(G)$ is trivial, as required.

References

[1] H. Bass, Algebraic K-theory, W.A. Benjamin, Inc., (1968).

[2] J. Coates, *Fragments of the GL_2 Iwasawa theory of elliptic curves without complex multiplication*, in Arithmetic Theory of Elliptic curves, ed. C. Viola, Lecture Notes in Math. 1716 (1997), 1–50.

[3] J. Coates, R. Greenberg, *Kummer theory for abelian varieties over local fields*, Invent. Math. 124 (1996), 124–178.

[4] J. Coates, S. Howson, *Euler characteristics and elliptic curves II*, Journal of Math. Society of Japan, 53 (2001), 175–235.

[5] J. Coates, R. Sujatha, *Euler-Poincaré characteristics of abelian varieties*, C.R. Acad. Sci. Paris, t.329, Série I (1999), 309–313.

[6] J. Coates, R. Sujatha, Galois cohomology of elliptic curves, TIFR Lecture Notes Series, Narosa Publishing house (2000).

[7] J. Coates, P. Schneider, R. Sujatha, *Modules over Iwasawa algebras*, Journal of the Inst. of Math Jussieu, 2 (2003), 73–108.

[8] J. Coates, R. Sujatha, J.-P. Wintenberger, *On the Euler-Poincaré characteristics of finite dimensional p-adic Galois representations*, Publ. Math. IHES, 93 (2001), 107–143.

[9] R. Greenberg, *On the structure of certain Galois groups*, Invent. Math. 47 (1978), 85–99.

[10] R. Greenberg, *Iwasawa theory for Elliptic curves*, in Arithmetic Theory of Elliptic curves, ed. C. Viola, Lecture Notes in Math. 1716 (1997), 1–50.

[11] S. Howson, *Euler characteristics as invariants of Iwasawa modules*, Proc. London Math. Soc. 85 (2002), 634–658.

[12] Y. Hachimori, O. Venjakob, *Completely faithful Selmer groups over Kummer extensions*, this volume.

[13] K. Kato, *p-adic Hodge theory and values of zeta functions of modular forms*, To appear.

[14] M. Lazard, *Groupes analytiques p-adiques*, Publ. Math. IHES 26 (1965), 389–603.

[15] B. Mazur, *Rational points of abelian varieties in towers of number fields*, Invent. Math. 18 (1992), 183–266.

[16] B. Mazur, J. Tate, *The p-adic sigma function*, Duke Math. J. 62 (1991), 663-688.

[17] B. Perrin-Riou, *Fonctions L p-adiques, théorie d'Iwasawa, et points de Heegner*; Bull. Soc. Math. France 115 (1987), 399-456.

[18] B. Perrin-Riou, *Théorie d'Iwasawa et hauteurs p-adiques*, Invent. Math., 109 (1992), 137–185.

[19] P. Schneider, *p-adic height pairings II*, Invent. Math., 79 (1985), 329–374.

[20] J.-P. Serre, J. Tate, *Good reduction of abelian varieties*, Ann. of Math., 88 (1968), 492–517.

[21] B. Totaro, *Euler characteristics for p-adic Lie groups*; Publ. Math. IHES 90 (1999), 169–225.

[22] O. Venjakob, *On the structure theory of the Iwasawa algebra of a p-adic Lie group*, J. Eur. Math. Soc. 4 (2002), 271–311.

[23] O. Venjakob, *A non-commutative Weierstrass preparation theorem and applications to Iwasawa theory*, Jour. reine und angew. Math. 559 (2003), 153–191.

[24] Y. Ochi, O. Venjakob, *On the structure of Selmer groups over p-adic Lie extensions*; J. Alg. Geom. 11 (2002), 547–576.

John Coates
DPMMS, University of Cambridge
Centre for Mathematical Sciences
Wilberforce Road
Cambridge CB3 0WB, England
j.h.coates@dpmms.cam.ac.uk

Peter Schneider
Mathematisches Institut
Westfälische Wilhelms-Universität
Einsteinstr. 62
D-48149 Münster, Germany
pschnei@math.uni-muenster.de

Ramdorai Sujatha
School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road
Mumbai 400 005, India
sujatha@math.tifr.res.in

Stable Maps of Curves

Robert F. Coleman

Received: October 31, 2002
Revised: February 21, 2003

Abstract. Let $h: X \to Y$ be a finite morphism of smooth connected complete curves over $\mathbf{C}_p$. We show h extends to a finite morphism between semi-stable models of X and Y.

2000 Mathematics Subject Classification: 11G20
Keywords and Phrases: Stable models, wide opens

Let p be a prime. It is known that if C is a smooth proper curve over a complete subfield K of C_p, there exists a finite extension L of K in C_p and a model of the base extension of C to L over the ring of integers, R_L, of L whose reduction modulo the maximal ideal has at worst ordinary double points as singularities. In fact, if $g(C)$, the genus of C, is at least 2 or $g(C) = 1$ and C has a model with good reduction, there is a minimal such model, which is called the stable model. Indeed, if L' is any complete extension of L in C_p, the base extension of a stable model over R_L is the stable model over $R_{L'}$.

Liu and Lorenzini showed [L-L; Proposition 4.4(a)] that a finite morphism of curves extends to a morphism of stable models, but the extension is not in general finite. E.g., Edixhoven has show that the natural map from $X_0(p^2)$ to $X_0(p)$ does not in general extend to a finite morphism of stable models [E] (see also [C-M]). However, we show,

Theorem. *Suppose $h: X \to Y$ is a finite morphism of smooth connected complete curves over C_p. Then there are semi-stable models $\mathcal{X}$ and $\mathcal{Y}$ of X and Y over the ring of integers of C_p such that h extends to a finite morphism from $\mathcal{X}$ to $\mathcal{Y}$.*

(We work over C_p to avoid having to worry about base extensions and because reduced affinoids over C_p have reduced reductions.)

In fact, when X has a stable model (i.e., $g(X) \geq 2$ or $g(X) = 1$ and X has good reduction) and either X/Y is Galois, or the model has irreducible reduction and Y has a stable model, one can take $\mathcal{X}$ to be the stable model for X. In the latter case $\mathcal{Y}$ will be the stable model for Y, which, a fortiori, will have irreducible reduction.

Abbes has informed us that this result also follows from results of Raynaud (in particular Proposition 5 of [R] (and its corollary)).

We say such a morphism is semi-stable, and stable if it is the minimal object in the category of semi-stable morphisms from X to Y (which may not exist).

Terminology and Notation

If Z is a rigid space $A(Z)$ will denote the ring of analytic functions on Z and $A^0(Z)$ the subring of functions whose spectral norm is bounded by 1.

If X is an affinoid over C_p, $\overline{X} = \ \mathrm{Spec} A^0(X)/\mathfrak{m}A^0(X)$, where $\mathfrak{m}$ is the maximal ideal of R_p and if $x \in \overline{X}(\bar{F}_p)$, R_x will denote the corresponding residue class in X. (Residue classes are also called formal fibers.)

By a regular singular point on a curve we mean a singular point which is an ordinary double point. If C is a curve, let $S(C)$ denote the set of irregular singular points on C.

1. Wide Opens

In this section, we review and extend the results on wide open spaces discussed in [RLC].

A (smooth one-dimensional) wide open is a rigid space conformal to $C - D$ where C is a smooth complete curve and D is a finite disjoint union of affinoid disks in C, which contains at least one in each connected component. A wide open disk is the complement of one affinoid disk in P^1 (it is conformal to $B(0,1)$) and a wide open annulus is conformal to the complement of two disjoint such disks (it is conformal to $A(r,1)$ where $r \in |C_p|$, $0 < r < 1$).

An underlying affinoid Z of a wide open W is an affinoid subdomain Z of W such that $W \backslash Z$ is a finite disjoint union of annuli none of which is contained in an affinoid subdomain of W. An end of W is an element of the inverse limit of of the set of connected components of $W \backslash Z$ where Z ranges over subaffinoids of W.

We slightly modify the definition of basic wide open given in [RLC] and say a wide open W is basic if it has an underlying affinoid Z such that $\overline{Z}$ is irreducible and has at worst regular singular points.

Suppose X is a smooth one dimensional affinoid over C_p and $x \in \overline{X}$. Because $A^0(R_x)$ is the completion of $A^0(X)$ at x, we have,

Lemma 1.1. *Then x is a smooth point of $\overline{X}$ if and only if R_x is a wide open disk and a regular singular point if and only if R_x is a wide open annulus.*

We have the following generalization of Proposition 3.3(ii) of [RLC],

Lemma 1.2. *The residue class, R_x, is a connected wide open and its ends can naturally be put in 1-1 correspondence to the branches of $\overline{X}$ through x.*

Proof. That R_x is connected is a consequence of Satz 6 of [B].

Theorem A-1 of [pAI] and its proof naturally generalizes to SEMI-DAGGER ALGEBRAS. These are quotients of the rings of series $\sum_{I,J} a_{I,J} x^I y^J$ in $K[x_1, \ldots, x_n, y_1, \ldots, y_m]$, where K is a complete non-Archimedean valued field, such that there exists $r_{\in R} > 1$ so that

$$\lim_{s(I,J)\to\infty} |a_{I,J}| r^{s(J)} = 0,$$

where I and J range over $Z^n_{\geq 0}$ and $Z^m_{\geq 0}$ and $s(M)$, where M is a multi-index, is the sum of its entries. What this implies in our context is that if R is an affinoid over C_p whose reduction is equal to the normalization of $\overline{X}$ and S is the set of points of $\overline{R}$ above T, the singular points of $\overline{X}$, then the rings

$$\lim_{\substack{\to \\ M}} A(R\backslash M) \quad \text{and} \quad \lim_{\substack{\to \\ N}} A(X\backslash N)$$

are isomorphic, where M ranges over the subaffinoids of $\bigcup_{s\in S} R_s$ and N ranges over the subaffinoids of $\bigcup_{x\in T} R_x$. Since $\bigcup_{s\in S} R_s$ is a union of wide open disks which correspond to the set, B, of branches of $\overline{X}$ through points in T, this, in turn, implies that there exists a subaffinoid N of $\bigcup_{x\in T} R_x$ such that $\bigcup_{x\in \overline{T}} R_x - N$ is a finite union of wide open annuli which correspond to the elements of B. One can now glue affinoid disks to $\bigcup_{x\in T} R_x$ to make a smooth complete curve, using [B; Satz 6.1] and the direct image theorem of [K], as in the proof of Proposition 3.3 of [RLC]. The result follows. ∎

It follows from Lemmas 3.1 and 3.2 of [RLC] that,

Lemma 1.3. *if A and B are disjoint wide open annuli or disjoint affinoids in a smooth curve C over C_p, then A is disconnected from B in C.*

If W is a wide open space

$$H^1_{DR}(W) = \Omega^1_W / \mathrm{d}A(W),$$

where Ω^1_W is the $A(W)$-module of rigid analytic differentials on W. It follows from Theorem 4.2 of [RLC] that $H^1_{DR}(W)$ is finite dimensional over C_p.

Lemma 1.4. *Suppose $f\colon W \to V$ is a finite morphism of wide opens. Then, if W is a disk or annulus, the same is true for V.*

Proof. Suppose W is a disk and f has degree d. Then V has only one end. Suppose ω is a differential on V. Then $f^*\omega = \mathrm{d}g$ for some function $g \in A(W)$ since $\dim H^1_{DR}(W) = 0$, in this case. Hence $\omega = \mathrm{d}\ \mathrm{Tr}\,(g/d)$. Thus $H^1_{DR}(V) = 0$. Let C be a proper curve obtained by glueing a wide open disk D to V along the end, as in the proof of Proposition 3.3 of [RLC]. From the Meyer-Vietoris long exact sequence, we see that C has genus zero and as $B := D\backslash V$ is an affinoid disk $V = C\backslash B$ is a wide open disk.

The argument in the case where W is an annulus is similar, except one has to use residues. ∎

Proposition 1.5. *Suppose $f: X \to Y$ is a finite map of smooth one dimensional affinoids over C_p. Then, if the reduction of X has only regular singular points, the same is true of Y.*

Proof. We know the map f induces a finite morphism $\bar{f}: \bar{X} \to \bar{Y}$. Let y be a point of $\bar{Y}$. Let $x \in \bar{X}$ such that $\bar{f}(x) = y$. Then f restricts to a finite morphism $R_x \to R_y$. But R_x is a disk or annulus. It follows from Lemma 1.2 that R_y is a wide open and hence by Lemma 1.4 is a disk or annulus, as well. Hence y is either smooth or regular singular by Lemma 1.1. ∎

This implies the well known result that if $h: X \to Y$ is a finite morphism of curves and X has good reduction so does Y. In fact, it implies the result of Lorenzini-Liu, [L-L; Corollary 4.10], that, in this case if $g(Y) \geq 1$, h extends to a finite morphism between the unique models of good reduction. It also implies that if X has a stable model with irreducible reduction, so does Y.

Lemma 1.6. *Suppose $\phi: X \to Y$ is a non-constant rigid morphism of smooth one dimension affinoids over C_p. Suppose G is a finite group acting on X such that $\phi^\sigma = \phi$ for $\sigma \in G$ and $\overline{X} = \bigcup_{\sigma \subset G} V^\sigma$ where V is an irreducible component of $\overline{X}$. Then ϕ surjects onto an affinoid subdomain of Y.*

Proof. Let $x \in X$. Because ϕ is non-constant we can find an element f of $A(Y)$ such that $f(\phi(x)) = 0$ and $|\phi^* f|_X = 1$. We can and will replace Y with the affinoid subdomain $\{y \in Y: |f(y)| \leq 1\}$. Then $\overline{\phi}|_V$ is non-constant. Since $\overline{\phi X} = \overline{\phi} V$ and V is irreducible, $\overline{\phi}$ factors through the inclusion of an irreducible component S of $\overline{Y}$. Let S^0 be the complement in S of the other irreducible components of $\overline{Y}$. Then, $Z = \mathrm{red}^{-1}\mathrm{S}^0$ is an affinoid subdomain of Y whose reduction is S^0. Let X' be the affinoid subdomain of X, $\phi^{-1}Z$. This is just X minus a finite number of residue classes stable under the action of G so its reduction is the union of the G-conjugates of $V' = V \backslash \overline{\phi}^{-1}(S \backslash S^0)$ which is irreducible. Suppose $s \in \overline{\phi X} \cap S^0$. We claim that $R_s \subset \phi(X)$.

Suppose $y_0 \in R_s \backslash \phi(X)$. Since the class group of Z is torsion, there exists an $h \in A(Z)$ such that y_0 is the only zero of h. Because $\overline{Z}$ is irreducible, we can also suppose $|h|_Z = 1$. Since $h(y_0) = 0$, it follows that $|h(y)| < 1$ for $y \in R_s$. Let $g = \phi^* h \in A^0(X')$. If $y_0 \notin \phi(X)$, $1/g \in A(X')$ but $|1/g|_{X'} = |c| > 1$, for some $c \in C_p$, since s is in the image of $\overline{\phi}_{\overline{X}'}$. However,

$$|g(1/cg)|_{X'} = |c^{-1}| < 1 = |g|_{X'}|(1/cg)|_{X'}.$$

This implies $\overline{g}_V = 0$ or $\overline{(1/cg)}_V = 0$, but as $X' = \bigcup_{\sigma \in G} V'^\sigma$, this implies the contradiction that $\overline{g} = 0$ or $\overline{(1/cg)} = 0$.

We will finish the proof by showing $X = X'$.

Let Y' be the affinoid obtained by glueing in disks to $\mathrm{red}^{-1}\mathrm{S}$ at the ends of the wide open $\mathrm{red}^{-1}\mathrm{S}\backslash\mathrm{S}^0$ corresponding to irreducible components of $\overline{Y}$ distinct from S. The reduction of this affinoid is naturally isomorphic to S. Then as ϕ factors through the inclusion of $\mathrm{red}^{-1}\mathrm{S}$ in Y we naturally obtain a morphism $\phi': X \to Y'$. Since by construction, for each point $s \in S \backslash S^0$, there is a point in the residue class of Y' not in the image of X the above argument implies no point in this residue class is in the image of X and so $\phi'(X)$ in the reduction inverse in Y' of S^0. As this latter is naturally isomorphic to Z, $X = X'$. ∎

Suppose W is a wide open annulus. If $\sigma: W \to W$ is a rigid analytic morphism, define $\rho(\sigma)$ by

$$\rho(\sigma)\ \mathrm{Res}\omega = \ \mathrm{Res}\, \sigma^*\omega.$$

The restriction of ρ to the group of rigid analytic automorphisms of W is a homomorphism from $Aut(W)$ onto $\{\pm 1\}$. We say σ is ORIENTATION PRESERVING if $\rho(\sigma) = 1$.

LEMMA 1.7. *Suppose G is a finite group of rigid automorphisms of the wide open annulus $W = A(r, 1)$ of order m. Then there is a rigid morphism $\phi: W \to V$ of degree m such that $A(W)^G = \phi^* A(V)$, where $V = A(r^m, 1)$ if G is orientation preserving and $V = A(B(0, 1))$ if not.*

Proof. First, if $\sigma \in G$

$$\sigma^* T = c_\sigma T^{\rho(\sigma)} h_\sigma(T),$$

where $h_\sigma(T) \in A(W)$, $|h_\sigma(t) - 1| < 1$, for $t \in W$, and $c_\sigma \in C_p$,

$$|c_\sigma| = \begin{cases} 1 & \text{if } \rho(\sigma) = 1 \\ r & \text{if } \rho(\sigma) = -1 \end{cases}.$$

Let $G^o = \mathrm{Ker}\, \rho$ and $n = |G^o|$. Let $S = \prod_{\tau \in G^o} \tau^* T$. Then

$$S(T) = T^n g(T),$$

where $|g(t)| = 1$. Let $\alpha: W \to A(r^n, 1)$, be the map

$$t \mapsto S(t).$$

It is easy to see this map has degree n and $R := \alpha^* A(A(r^n, 1)) \subseteq A(W)^G$. In particular, R is an integral domain and its fraction field is K^G where K is the fraction field of $A(W)$. Since, R and $A(W)$ are Dedekind domains, it follows that $R = A(W)^{G_0}$. If G is orientation preserving, $G = G^o$ and taking $\phi = \alpha$ completes the proof, in this case.

Suppose now G is not orientation preserving. Then G/G^o has order 2. Using, the result of the last paragraph we can replace W with $A(r^n, 1)$ and assume G^o is trivial. Let $G = \{1, \sigma\}$ and

$$U(T) = T + \sigma^* T = T + c_\sigma T^{-1} h_\sigma(T).$$

Now, if we define $\phi: W \to B(0, 1)$ to be the morphism

$$t \mapsto U(t),$$

we can apply the same argument, as above, to complete the proof. ∎

Remark. *One can show:*

Proposition. *Suppose p is odd G is a finite group of automorphisms of $A(r,R)$. Then there is a natural homomorphsm of G into $\mathrm{Aut}\, G_m^{\overline{F}_p}$ whose kernel is the unique p-Sylow subgroup of G^o. Moreover, the exact sequence,*

$$1 \to G^o \to G \to G/G^o \to 1,$$

splits.

For example: Suppose $p = 3$, $1 > |r| > |27|$ and $V = A(r,1)$. Let s be the parameter on A^1. Then the integral closure of $A(V)$ in the splitting field of $X^3 + sX = s$ over $K(V)$ is the ring of analytic functions on an annulus W which is an étale Galois cover of V. If G is the Galois group, $G = G^o \cong S_3$.

2. Semi-stable Coverings

A semi-stable covering of a curve C is a finite admissible covering $\mathcal{D}$ of C by connected wide opens such that
(i) if $U \neq V \in \mathcal{D}$, $U \cap V$ is a finite collection of disjoint wide open annuli,
(ii) if $T, U, V \in \mathcal{D}$ are pairwise distinct, $T \cap U \cap V = \emptyset$.
(iii) for $U \in \mathcal{D}$, if

$$U^u = U \backslash (\bigcup_{\substack{V \in \mathcal{D} \\ V \neq U}} V),$$

U^u is a non-empty affinoid whose reduction is irreducible and has at worst regular singular points.

In particular, if $U \in \mathcal{D}$, U is a basic wide open and U^u is an underlying affinoid of U. We let $E(U)$ denote the set of connected components of $U \backslash U^u$. These are all wide open annuli.

Proposition 2.1. *Semi-stable models of C whose reductions have at least two components correspond to semi-stable covers of C.*

Proof. Suppose $\mathcal{C}$ is a semi-stable model for C whose reduction $\overline{\mathcal{C}}$ has at least two components. Let $I_{\mathcal{C}}$ denote the set of irreducible components of $\overline{\mathcal{C}}$. If $Z \in I_{\mathcal{C}}$ let $Z^0 = Z \backslash \bigcup_{\substack{A \in I_{\mathcal{C}} \\ A \neq Z}} A$ and $W_Z := \mathrm{red}^{-1}\mathrm{Z}$. As every singular point of $\overline{\mathcal{C}}$ is regular it follows from Lemma 1.2 that W_Z is a basic wide open with underlying affinoid $\mathrm{red}^{-1}\mathrm{Z}^0$ and $\{W_Z : Z \in I_{\mathcal{C}}\}$ is a semi-stable cover.

Conversely, suppose $\mathcal{D}$ is a semi-stable cover of C. For $U, V \in \mathcal{D}$, let $Z_U = \mathrm{Spf}\, A^0(Z)$ and $Z_{U,V} = \mathrm{Spf}\,(U \cap V)$. Then the formal schemes Z_U glue by means of the glueing data

$$Z_{U,V} \to Z_U \coprod Z_V$$

into a model $\mathcal{S}_{\mathcal{D}}$ of C. ∎

If we have semi-stable coverings $\mathcal{D}_X$ and $\mathcal{D}_Y$ such that for every $W, V \in \mathcal{D}_X$, $h(W) \in \mathcal{D}_Y$ and there exist $W', V' \in \mathcal{D}_X$ such that $h(W) \cap h(V) = h(W' \cap V')$, then f extends to a finite morphism from $\mathcal{S}_{\mathcal{D}_X}$ to $\mathcal{S}_{\mathcal{D}_Y}$. We say h induces a finite morphism of semi-stable covers from $\mathcal{D}_X$ to $\mathcal{D}_Y$.

3. Proof of Theorem

First, let $h' : X' \to Y$ be the Galois closure of h with Galois group G. Let $\mathcal{D}$ be a semistable cover X' of such that $Y \notin \mathcal{C}$ where

$$\mathcal{C} = \{h'(U) : U \in \mathcal{D}\}.$$

Then we claim $\mathcal{C}$ is a semi-stable cover of Y. Clearly $\mathcal{C}$ is a finite admissible open cover. By Lemma 1.7, if $W \in \mathcal{D}$ and $A \in E(W)$, $h'(A)$ is a wide open disk or annulus. Since $h'(W) \neq Y$, $h'(A)$ cannot be a disk for all $A \in E(W)$. It follows by a glueing argument, as in the proof of Lemma 1.2, that $h'(W)$ is a connected wide open. Now suppose $U, V \in \mathcal{D}$ and $h'(W) \neq h'(V)$. We must show $h'(W) \cap h'(V)$ is a finite union of disjoint annuli. First, we remark that $h'(W^u)$ and $h'(V^u)$ are disjoint affinoids in Y, using Lemma 1.6. Suppose A is a component of $W \cap V$ so $A \in E(W) \cap E(V)$. Suppose (x_n) is a sequence of points in A. If $x_n \to W^u$, $h'(x_n) \to h'(W^u)$ and if $x_n \to V^u$, $h'(x_n) \to h'(V^u)$. It follows that $h'(A)$ is an annulus. Also, we know that if U is a connected component of $h'(W) \cap h'(V)$, $U = \bigcup_{\sigma \in S} h'(A_\sigma)$ where the A_σ are in $E(W) \cap E(V^\sigma)$, for some subset S of G. Now it follows from Lemma 1.3 that if S has more than one element and $\sigma \in S$ there must be a $\tau \in S$ such that $\tau \neq \sigma$ and $A_\sigma \cap A_\tau \neq \emptyset$. Then $A_\sigma \cup A_\tau$ is an annulus, arguing as in the proof of Corollary 3.6a of [RLC] ($A_\sigma \cup A_\tau \neq Y$ since $W^u \cap (A_\sigma \cup A_\tau) = \emptyset$). The fact that A_σ and A_τ are connected to both W^u and V^u implies $A_\tau = A_\sigma \cup A_\tau = A_\sigma$. We conclude that all the A_σ equal U, for $\sigma \in S$ and so U is a wide open annulus.

Suppose $U, V, W \in \mathcal{D}$ are such that $h'(U), h'(V), h'(W)$ are distinct. If $y \in h'(U) \cap h'(V) \cap h'(W)$, there exist $\sigma, \tau \in G$ and $x \in U \cap V^\sigma \cap W^\tau$ such that $y = h'(x)$. But this implies U, V^σ, W^τ are not distinct which in turn implies $h'(U), h'(V), h'(W)$ are not distinct.

We must show for $U \in \mathcal{D}$, $h'(U)^u$, which equals

$$h'(U) \backslash (\bigcup_{\substack{V \in \mathcal{C} \\ V \neq h'(U)}} V),$$

is an affinoid whose reduction is irreducible and only has regular singular points. Now,

$$h'(U)^u = h'(\bigcup_{\sigma \in G} (U^\sigma \backslash \bigcup_{\substack{A \in E(U^\sigma) \cap E(V) \\ V \in \mathcal{D}, V \neq U^\tau, \tau \in G}} A)$$

and also

$$= h'(U^u) \cup \bigcup_{\substack{A \in E(U) \cap E(U^\sigma) \\ \sigma_{\neq 1} \in G,}} h'(A).$$

It follows from the first equality that $h'(U)^u$ is an affinoid using Lemma 1.6 and Proposition 3.3 of [RLC]. Its reduction is irreducible as the reduction U^u is, and from Proposition 1.5 it has at worst only regular singular points. Finally, since all the $h'(A)$ are disks or annuli by Lemma 1.7 whose ends are connected to $h'(U^u)$, $h(U)^u$ is an affinoid and these $h'(A)$ must, by Lemma 1.1, be smooth or regular singular classes of $h'(U)^u$, and thus, in particular, $h'(U)^u$ must have irreducible reduction. Thus $\mathcal{C}$ is a semi-stable cover and clearly h' induces a finite map of semi-stable covers from $\mathcal{D}$ to $\mathcal{C}$.

We also know X'/X is Galois and if $r\colon X' \to X$ is the corresponding morphism,

$$\mathcal{E} := \{r(U)\colon U \in \mathcal{D}\}$$

does not contain X so is a semi-stable cover of X and r induces a finite map of semi-stable covers from $\mathcal{D}$ to $\mathcal{E}$. It follows that h induces a finite map of semi-stable covers from $\mathcal{E}$ to $\mathcal{C}$ and hence extends to the corresponding semi-stable models.

Now we must explain how we can find a cover $\mathcal{D}$ of X' with the required properties. If X' has a stable model $\mathcal{X}$, then $\mathcal{X}$ is preserved by G. Let D be a wide open disk in X' such that $D^\sigma \cap D = \emptyset$ for all $\sigma_{\neq 1} \in G$ and B an affinoid ball in D. Let $\mathcal{X}'$ be the minimal semi-stable refinement of X such that no two elements of $\{D^\sigma\colon \sigma \in G\}$ are contained in the same residue class. Let $E = \bigcup_{\sigma \in G} B^\sigma$. Then we can take for $\mathcal{D}$,

$$\{(\mathrm{red}^{-1}\mathrm{Z})\backslash\mathrm{E}\colon \mathrm{Z} \text{ is an irreducible component of } \overline{\mathcal{X}}'\} \cup \{\mathrm{D}^\sigma\colon \sigma \in \mathrm{G}\}.$$

If $g(X') \leq 1$ and the set of ramified points $S \subset X'$ contains at least $3 - 2g(X')$ elements we do the same thing starting with the minimal semi-stable model with the property that S injects into the smooth points of the reduction of this model. The remaining cases are easier.

References

[B] Bosch, S., Eine bemerkenswerte Eigenschaft der formellen Fasern affinoider Räume, Math. Ann. 229 (1977) 25-45.

[pAI] Coleman, R., Torsion points on curves and p-adic Abelian integrals, Ann. Math. 121 (1985), 111-168.

[RLC] ________, Reciprocity Laws on Curves, Compositio, 72, (1989) 205-235.

[C-M] ________ and W. McCallum, Stable Reduction of Fermat Curves and Local Components of Jacobi Sum Hecke Characters, vvJ. reine angew. Math. 385 (1988) 41-101.

[E] Edixhoven, S., Stable models of modular curves and applications, PhD thesis.

[K] Kiehl, R., Der Endlichkeitssatz fúr eigenliche Abbildungen in der nichtarchimedischen Funktionentheorie, Math. 2, (1967) 191-214.

[L-L] Liu, Q. and D. Lorenzini, Models of curves and finite covers, Compositio, 118, (1999) 62-102.

[R] Raynaud, M., p-groupes et reduction semi-stable des courbes, Grothendieck Festschrift III, Birkhäuser (1990) 179-197.

Robert F. Coleman
Department of Mathematics
University of California
California, USA 94530
coleman@math.berkeley.edu

Two-Variable Zeta Functions and Regularized Products

Dedicated to Kazuya Kato

Christopher Deninger

Received: October 17, 2002
Revised: April 3, 2003

Abstract. In this paper we prove a regularized product expansion for the two-variable zeta functions of number fields introduced by van der Geer and Schoof. The proof is based on a general criterion for zeta-regularizability due to Illies. For number fields of non-zero unit rank our method involves a result of independent interest about the asymptotic behaviour of certain oscillatory integrals in the geometry of numbers. We also explain the cohomological motivation for the paper.

2000 Mathematics Subject Classification: 11M38; 11M41; 11H41; 14G40
Keywords and Phrases: zeta function, zeta regularization, oscillatory integral, metrized lattice

1 Introduction

In his paper [P] Pellikaan studied an interesting two-variable zeta function for algebraic curves over finite fields. Using notions from Arakelov theory of arithmetic curves, van der Geer and Schoof were led to introduce an analogous zeta function for number fields [GS].
In [LR] Lagarias and Rains investigated this two-variable zeta function thoroughly for the special case of the rational number field. They also made some comments on the general case.
In earlier work we introduced a conjectural cohomological formalism to express Dedekind and more general zeta functions as regularized determinants of a

certain operator Θ on cohomology. In this framework it is not unreasonable to assume that the second variable w of the two-variable zeta function corresponds to an operator Θ_w depending on w. These heuristics which are explained in the last section suggest a formula for the two-variable zeta function as a regularized product.

The main contribution of the paper is to prove this formula for the two-variable zeta function of any number field, Theorem 5.2. Our method is based on a powerful criterion of Illies for zeta-regularizability [I1], [I2]. We refer to section 5 for a short review of the relevant facts from the theory of regularization.

We also treat the much easier case of curves over finite fields. For number fields, our approach requires us to determine the asymptotic behaviour for $\operatorname{Re} s \to \infty$ of certain oscillatory integrals over spaces of lattices Γ. The function to be integrated is a_Γ^{-s} where a_Γ is the minimal length among the non-zero vectors in Γ. This is an interesting problem already for real quadratic fields in which case Don Zagier found a solution. The general case is treated in section 4.

The treatment in [GS] and [LR] of the two-variable zeta function for general number fields is somewhat brief. Also, the precise analogy with Pellikaan's original zeta function is not written down. In the first two sections we therefore give a more detailed exposition of these topics. After this, some readers might wish to read the last section which motivated the paper.

I would like to thank Don Zagier very much for his help in the real quadratic case which was a great inspiration for me. I am also grateful to Eva Bayer and Georg Illies for useful remarks and to the CRM in Montreal for its support. Finally I would like to thank the referees for their careful reading of the paper and their comments.

2 Background on two-variable zeta functions for curves over finite fields

Consider an algebraic curve X over the finite field $\mathbb{F}_q$ with $q = p^r$ elements. Let $|X|$ be the set of closed points of X and for $x \in |X|$ set
$\deg x = (\mathbb{F}_q(x) : \mathbb{F}_q)$. The zeta function of X is defined by the Euler product

$$Z_X(T) = \prod_{x \in |X|} (1 - T^{\deg x})^{-1} \quad \text{in } \mathbb{Z}[|T|] \, .$$

For a divisor $D = \sum_{x \in |X|} n_x \cdot x$ with $n_x \in \mathbb{Z}$ we set $\deg D = \sum n_x \deg x$. Then we have

$$Z_X(T) = \sum_{D \geq 0} T^{\deg D} \tag{1}$$

where the sum runs over all effective divisors i.e. those with $n_x \geq 0$ for all $x \in |X|$. Let $CH^1(X)$ denote the divisor class group of X and for $\mathcal{D} = [D]$ set

$$h^i(\mathcal{D}) := h^i(D) = \dim H^i(X, \mathcal{O}(D)) \, .$$

Summing over divisor classes in (1), one gets the formula:

$$Z_X(T) = \sum_{\mathcal{D}} \frac{q^{h^0(\mathcal{D})} - 1}{q-1} T^{\deg \mathcal{D}} \ . \tag{2}$$

Here it is enough to sum over $\mathcal{D}$'s with $\deg \mathcal{D} := \deg D \geq 0$. In [P] § 3 Pellikaan had the idea to replace q by a variable u in this formula. His two-variable zeta function is defined by

$$Z_X(T,u) = \sum_{\mathcal{D}} \frac{u^{h^0(\mathcal{D})} - 1}{u-1} T^{\deg \mathcal{D}} \ . \tag{3}$$

Reconsidering classical proofs he obtained the following properties in the case where X is smooth projective and geometrically irreducible:

$$Z_X(T,u) = \frac{P_X(T,u)}{(1-T)(1-uT)} \quad \text{with } P_X(T,u) \in \mathbb{Z}[T,u] \tag{4}$$

$$P_X(T,u) = \sum_{i=0}^{2g} P_i(u) T^i \quad \text{with } P_i(u) \in \mathbb{Z}[u] \ , \text{ where} \tag{5}$$

$$P_0(u) = 1, P_{2g}(u) = u^g \ , \ \deg P_i(u) \leq 1 + \frac{i}{2} \quad \text{and } g \text{ is the genus of } X \ . \tag{6}$$

The two-variable zeta function enjoys the functional equation

$$Z_X(T,u) = u^{g-1} T^{2g-2} Z_X\left(\frac{1}{Tu}, u\right) \ . \tag{7}$$

In terms of the $P_i(u)$ it reads:

$$P_{2g-i}(u) = u^{g-i} P_i(u) \ . \tag{8}$$

For example, for $X = \mathbb{P}^1$ one has $P_X(T,u) = 1$ and for X an elliptic curve $P_X(T,u) = 1 + (|X(\mathbb{F}_q)| - 1 - u)T + uT^2$.
Recently Naumann [N] proved the interesting fact that the polynomial $P_X(T,u)$ is irreducible in $\mathbb{C}[T,u]$.
In [GS] § 7, van der Geer and Schoof consider the following variant of Pellikaan's zeta function. They show that for complex s and t in $\operatorname{Re} s < 0, \operatorname{Re} t < 0$ the series

$$\zeta_X^{GS}(s,t) = \sum_{\mathcal{D} \in CH^1(X)} q^{sh^0(\mathcal{D}) + th^1(\mathcal{D})} \tag{9}$$

defines a holomorphic function with a meromorphic continuation to $\mathbb{C} \times \mathbb{C}$. The explicit relation with $Z_X(T,u)$ is not stated in [GS], so we give it here:

Proposition 2.1

$$\begin{aligned} \zeta_X^{GS}(s,t) &= (q^{s+t} - 1) q^{t(g-1)} Z_X(q^{-t}, q^{s+t}) \\ &= (q^{s+t} - 1) q^{s(g-1)} Z_X(q^{-s}, q^{s+t}) \ . \end{aligned}$$

PROOF Using the Riemann–Roch theorem one obtains, c.f. [GS] proof of prop. 5:

$$\zeta_X^{GS}(s,t) = q^{t(g-1)} \sum_{0 \le \deg \mathcal{D} \le 2g-2} q^{(s+t)h^0(\mathcal{D})} q^{-t \deg \mathcal{D}} + h \left(\frac{q^{sg}}{1-q^s} + \frac{q^{tg}}{1-q^t} \right) .$$

Here h is the order of $CH^1(X)^0$, the group of degree zero divisor classes on X. This gives the meromorphic continuation to $\mathbb{C} \times \mathbb{C}$. On the other hand according to [P], p. 181 setting $u = q^{s+t}, T = q^{-t}$ we have:

$$\begin{aligned}(q^{s+t}-1)Z_X(q^{-t}, q^{s+t}) \quad &= \sum_{0 \le \deg \mathcal{D} \le 2g-2} q^{(s+t)h^0(\mathcal{D})} q^{-t \deg \mathcal{D}} \\ &\quad + h \left(\frac{q^{sg+t(1-g)}}{1-q^s} - \frac{1}{1-q^{-t}} \right) .\end{aligned}$$

This implies the first equality in the proposition. The second follows from the functional equation (7) of $Z_X(T,u)$. □

In particular the second relation in the proposition shows that for $s + t = 1$ we have

$$\zeta_X^{GS}(s, 1-s) = (q-1)q^{s(g-1)}\zeta_X(s) \quad \text{where } \zeta_X(s) = Z_X(q^{-s}) \tag{10}$$

as stated in [GS] proposition 5. Note that for $\zeta_X^{GS}(s,t)$ the functional equation takes the simple form:

$$\zeta_X^{GS}(s,t) = \zeta_X^{GS}(t,s) . \tag{11}$$

In the number field case, Lagarias and Rains introduced the substitution $t = w - s$. Thus we define here as well

$$\zeta_X(s,w) = \zeta_X^{GS}(s, w-s) = (q^w - 1)q^{-s(1-g)} Z_X(q^{-s}, q^w) . \tag{12}$$

This meromorphic function of s and w satisfies the functional equation

$$\zeta_X(s,w) = \zeta_X(w-s, w) \tag{13}$$

and for $w = 1$ we have:

$$\zeta_X(s,1) = (q-1)q^{-s(1-g)}\zeta_X(s) . \tag{14}$$

The rest of this section contains observations of a tentative nature which are not necessary for the sequel. It is unknown whether $Z_X(T,u)$ has a natural cohomological interpretation. The properties of $Z_X(T,u)$ are compatible with the following conjectural setup. Let K be a field of characteristic zero containing $\mathbb{Q}(u)$. For varieties over finite fields there might exist a cohomology theory QH^i consisting of finite-dimensional K-vector spaces with the following property: The q-linear Frobenius Fr_q acting on a variety $X/\mathbb{F}_q$ should induce a K-linear map Fr_q^* such that we have:

$$Z_X(T,u) = \prod_{i=0}^{2} \det_K(1 - T\mathrm{Fr}_q^* \,|\, QH^i(X))^{(-1)^{i+1}} . \tag{15}$$

We get the correct denominator in (4) if

$$QH^0(X) = K \; , \; \mathrm{Fr}_q^* = \mathrm{id} \quad \text{and} \quad QH^2(X) \cong K \text{ with } \mathrm{Fr}_q^* = u \cdot \mathrm{id}$$

and

$$QH^i(X) = 0 \quad \text{for } i > 2 \; .$$

Then $P(T,u)$ would be the characteristic polynomial of Fr_q^* on $QH^1(X)$ and therefore we would have

$$\dim_K QH^1(X) = 2g \; .$$

The functional equation (7) would be a consequence of Poincaré duality – a perfect Fr_q^*-equivariant pairing of K-vector spaces:

$$QH^i(X) \times QH^{2-i}(X) \longrightarrow QH^2(X) \cong K \; .$$

Moreover Poincaré duality would imply

$$\det(\mathrm{Fr}_q^* \,|\, QH^1(X)) = u^g \; .$$

For an elliptic curve $X/\mathbb{F}_q$, comparing the logarithmic derivatives of (4) and (15) at $T = 0$ gives

$$\sum_{i=0}^{2} (-1)^i \mathrm{Tr}(\mathrm{Fr}_q^* \,|\, QH^i(X)) = |X(\mathbb{F}_q)| \; . \tag{16}$$

However, if in (16) we replace Fr_q^* by its power $\mathrm{Fr}_q^{\nu *}$ we do not obtain $|X(\mathbb{F}_{q^\nu})|$ for $\nu \geq 2$.

3 Background on two-variable zeta functions of number fields

We begin by collecting some notions from one-dimensional Arakelov theory following [GS].

For a number field $k/\mathbb{Q}$ let $\mathfrak{o}_k$ be its ring of integers. By $\mathfrak{p}$ we denote the prime ideals in $\mathfrak{o}_k$ and by v the infinite places of k. Consider the "arithmetic curve"

$$X_k = \mathrm{spec}\, \mathfrak{o}_k \cup \{v \,|\, \infty\} \; .$$

The elements of the group

$$Z^1(X_k) = \bigoplus_{\mathfrak{p}} \mathbb{Z} \cdot \mathfrak{p} \oplus \bigoplus_{v \,|\, \infty} \mathbb{R} \cdot v$$

are called Arakelov divisors. Define a map

$$\mathrm{div} : k^* \longrightarrow Z^1(X_k)$$

by the formula

$$\mathrm{div}\,(f) = \sum_{\mathfrak{p}} \mathrm{ord}_{\mathfrak{p}}(f)\mathfrak{p} - \sum_v e_v \log |f|_v v \; .$$

Here $|f|_v = |\sigma_v(f)|$ for any embedding σ_v in the class v and $e_v = 1$ if v is real and $e_v = 2$ if v is complex. The cokernel of div is called the Arakelov Chow group $CH^1(X_k)$ of X_k.

With the evident topologies the groups $k^*, Z^1(X_k)$ and $CH^1(X_k)$ become locally compact topological groups. The counting measure on $\bigoplus_{\mathfrak{p}} \mathbb{Z} \cdot \mathfrak{p}$ and the Lebesgue measure on $\bigoplus_{v \mid \infty} \mathbb{R} \cdot v$ induce Haar measures dD on $Z^1(X_k)$ and $d\mathcal{D}$ on $CH^1(X_k)$.

For an Arakelov divisor

$$D = \sum_{\mathfrak{p}} \nu_{\mathfrak{p}} \cdot \mathfrak{p} + \sum_v x_v \cdot v \quad \text{in } Z^1(X_k)$$

define a fractional ideal in k by the formula

$$I(D) = \prod_{\mathfrak{p}} \mathfrak{p}^{-\nu_{\mathfrak{p}}} \; .$$

The infinite components of D determine a norm $\| \; \|_D$ on $k \otimes \mathbb{R} = \bigoplus_v k_v$ by the formula

$$\|(z_v)\|_D^2 = \sum_v |z_v|^2 \|1\|_v^2 \; .$$

Here $\|1\|_v^2 = e^{-2x_v}$ if v is real and $\|1\|_v^2 = 2e^{-x_v}$ if v is complex.

For $f \in k \hookrightarrow k \otimes \mathbb{R}$ we then have

$$\|f\|_D^2 = \sum_{v \,\mathrm{real}} |f|_v^2 e^{-2x_v} + 2 \sum_{v \,\mathrm{complex}} |f|_v^2 e^{-x_v} \; . \tag{17}$$

The embedding $I(D) \hookrightarrow k \otimes \mathbb{R}$ and the norm $\| \; \|_D$ turn $I(D)$ into a metrized lattice. The lattices $I(D)$ and $I(D')$ are isometric (by an $\mathfrak{o}_k$-linear isometry) if and only if $[D] = [D']$ in $CH^1(X_k)$.

Let κ be the Arakelov divisor with zeroes at the infinite components and $I(\kappa) = \mathfrak{d}^{-1}$, where $\mathfrak{d} = \mathfrak{d}_{k/\mathbb{Q}}$ is the different of $k/\mathbb{Q}$.

In the number field case, van der Geer and Schoof replace the order $q^{h^i(\mathcal{D})}$ of $H^i(X, \mathcal{O}(D))$ for $X/\mathbb{F}_q$ by the Theta series:

$$k^0(\mathcal{D}) = \sum_{f \in I(D)} e^{-\pi \|f\|_D^2} \tag{18}$$

and

$$k^1(\mathcal{D}) = k^0([\kappa] - \mathcal{D}) \tag{19}$$

for $\mathcal{D} = [D]$ in $CH^1(X_k)$. For quadratic number fields the behaviour of $k^0(\mathcal{D})$ is studied in some detail in [F].

According to [GS] proposition 1, the Poisson summation formula gives the Riemann–Roch type formula

$$k^0(\mathcal{D})k^1(\mathcal{D})^{-1} = \mathcal{N}(\mathcal{D})d_k^{-1/2} \; . \tag{20}$$

Here $d_k = |d_{k/\mathbb{Q}}|$ is the absolute value of the discriminant of $k/\mathbb{Q}$ and

$$\mathcal{N} : CH^1(X_k) \longrightarrow \mathbb{R}_+^*$$

is the Arakelov norm induced by the map

$$\mathcal{N} : Z^1(X_k) \longrightarrow \mathbb{R}_+^* \; , \; \mathcal{N}(D) = \prod_{\mathfrak{p}} N\mathfrak{p}^{\nu_{\mathfrak{p}}} \prod_v e^{x_v} \; .$$

Let $Z^1(X_k)^0$ be the kernel of this map and set

$$CH^1(X_k)^0 = Z^1(X_k)^0 / \mathrm{div}\,(k^*) \; .$$

This is a compact topological group which fits into the exact sequence

$$0 \longrightarrow CH^1(X_k)^0 \longrightarrow CH^1(X_k) \xrightarrow{\mathcal{N}} \mathbb{R}_+^* \longrightarrow 1 \; . \tag{21}$$

Let $d^0\mathcal{D}$ be the Haar measure on $CH^1(X_k)^0$ with

$$\mathrm{vol}\,(CH^1(X_k)^0) = h_k R_k \tag{22}$$

where $h_k = |CH^1(\mathrm{spec}\,\mathfrak{o}_k)|$ is the class number of k and R_k is the regulator. Then we have

$$d\mathcal{D} = d^0\mathcal{D}\frac{dt}{t} \; . \tag{23}$$

For t in $\mathbb{R}_+^*$ consider the Arakelov divisor, where $n = (k : \mathbb{Q})$

$$D_t = n^{-1} \sum_{v \text{ real}} \log t \cdot v + n^{-1} \sum_{v \text{ complex}} 2 \log t \cdot v \; .$$

Setting $\mathcal{D}_t = [D_t]$ we have $\mathcal{N}(\mathcal{D}_t) = t$, so that the homomorphism $t \mapsto \mathcal{D}_t$ provides a splitting of (21).

We need the following estimates:

Proposition 3.1 *For every number field k and every $R \geq 0$ there are positive constants c_1, c_2, α such that uniformly in $\mathcal{D} \in CH^1(X_k)^0$ and $|w| \leq R$ we have the estimates*

a) $|k^0(\mathcal{D} + \mathcal{D}_t)^w - 1| \leq c_1 |w| \exp(-\pi n t^{-2/n})$ *for all* $0 < t \leq \sqrt{d_k}$.

b) $|k^0(\mathcal{D} + \mathcal{D}_t)^w - t^w d_k^{-w/2}| \leq c_2 |w| \exp(-\alpha t^{2/n})$ *for all* $t \geq \sqrt{d_k}$.

PROOF According to [GS] corollary 1 there is a constant $\beta > 0$ depending only on the field k such that for all $\mathcal{D}$ in $CH^1(X_k)^0$ and all $0 < t \le d_k^{1/2}$ we have

$$0 < k^0(\mathcal{D} + \mathcal{D}_t) - 1 \le \beta \exp(-\pi n t^{-2/n}) \; . \tag{24}$$

We may assume that $R \ge 1$. For every $-\frac{1}{2} \le x \le \frac{1}{2}$ and $|w| \le R$ setting

$$(1+x)^w = 1 + wx + wx^2 \vartheta(x,w) \tag{25}$$

we have

$$|\vartheta(x,w)| \le e^{2R} \; . \tag{26}$$

Namely, writing

$$(1+x)^w = e^{w \log(1+x)} = e^{wx(1+\eta x)}$$

we have $\eta = -\frac{1}{2} + \frac{x}{3} - \frac{x^2}{4} + - \ldots$ and hence $|\eta| \le 1$. Expanding $e^{wx(1+\eta x)}$ as a Taylor series and estimating gives inequality (26). For the moment we only need the following consequence of (26):

$$|(1+x)^w - 1| \le x|w|\Big(1 + \frac{1}{2} e^{2R}\Big) \quad \text{for } 0 \le x \le 1/2 \text{ and } |w| \le R \ge 1 \; . \tag{27}$$

If $\varepsilon = \varepsilon(k) > 0$ is sufficiently small, (24) implies that

$$x = k^0(\mathcal{D} + \mathcal{D}_t) - 1$$

lies in $(0, 1/2)$ for all $0 < t \le \varepsilon$ and all $\mathcal{D}$. Using (24) and (27) we therefore find a constant c_1' such that a) holds for all $0 < t \le \varepsilon$. By compactness of

$$CH^1(X_k)^0 \times \{|w| \le R\} \times [\varepsilon, \sqrt{d_k}]$$

and continuity of $\frac{1}{w}(k^0(\mathcal{D} + \mathcal{D}_t)^w - 1)$ as a function of $\mathcal{D}, w$ and t there is a constant c_1'' such that a) holds in $\varepsilon \le t \le \sqrt{d_k}$. Thus we get the estimate a) by taking $c_1 = \max(c_1', c_1'')$. The estimate b) follows from a) using the Riemann–Roch formula (20) and observing that $\mathcal{N}([\kappa]) = d_k$. □

The two-variable zeta function of van der Geer and Schoof is defined by an integral analogous to the series (9)

$$\zeta_{X_k}^{GS}(s,t) = \int_{CH^1(X_k)} k^0(\mathcal{D})^s k^1(\mathcal{D})^t \, d\mathcal{D} \quad \text{in } \operatorname{Re} s < 0, \operatorname{Re} t < 0 \; . \tag{28}$$

According to [GS] proposition 6, this integral defines a holomorphic function in $\operatorname{Re} s < 0, \operatorname{Re} t < 0$. This also follows from the considerations below.
We refer the reader to the introduction of [LR] for further motivation to consider this two-variable zeta function.

Making the substitution $\mathcal{D} \mapsto [\kappa] - \mathcal{D}$ in the integral we find the formula

$$\zeta^{GS}_{X_k}(s,t) = \int_{CH^1(X_k)} k^0(\mathcal{D})^t k^1(\mathcal{D})^s \, d\mathcal{D} \quad \text{in } \operatorname{Re} s < 0, \operatorname{Re} t < 0 \, . \tag{29}$$

We will use the Lagarias–Rains variables s and $w = t + s$ and concentrate on the function

$$\zeta_{X_k}(s,w) = \zeta^{GS}_{X_k}(s, w-s) = \int_{CH^1(X_k)} k^0(\mathcal{D})^{w-s} k^1(\mathcal{D})^s \, d\mathcal{D} \tag{30}$$

$$\overset{(20)}{=} d_k^{s/2} \int_{CH^1(X_k)} k^0(\mathcal{D})^w \mathcal{N}(\mathcal{D})^{-s} \, d\mathcal{D} \, . \tag{31}$$

It is holomorphic in the region $\operatorname{Re} w < \operatorname{Re} s < 0$.
Most of the following proposition is stated in [GS] and proved in [LR] Appendix using references to Ch. XIII of Serge Lang's book on algebraic number theory. Below we will write down the direct proof which is implicit in [GS].

Proposition 3.2 *The function $\zeta_{X_k}(s,w)$ has a meromorphic continuation to $\mathbb{C}^2$ and it satisfies the functional equation*

$$\zeta_{X_k}(s,w) = \zeta_{X_k}(w-s,w) \, .$$

Moreover the function

$$w^{-1} s(w-s) \zeta_{X_k}(s,w)$$

is holomorphic in $\mathbb{C}^2$. More precisely, the integral

$$J(s,w) = \int_0^{\sqrt{d_k}} \int_{CH^1(X_k)^0} w^{-1}(k^0(\mathcal{D}+\mathcal{D}_t)^w - 1) d^0\mathcal{D} \, t^{-s} \frac{dt}{t}$$

defines an entire function in $\mathbb{C}^2$ and we have the formula

$$\zeta_{X_k}(s,w) = w\left(d_k^{s/2} J(s,w) + d_k^{(w-s)/2} J(w-s,w)\right) - \left(\frac{1}{s} + \frac{1}{w-s}\right) h_k R_k \, .$$

Recall that $\operatorname{vol} CH^1(X_k)^0 = h_k R_k$. *Finally, for $w = 1$ one has*

$$\zeta_{X_k}(s,1) = |\mu(k)| d_k^{s/2} 2^{-r_1/2} \hat{\zeta}_k(s) \, . \tag{32}$$

Here $\hat{\zeta}_k(s)$ is the completed Dedekind zeta function of k

$$\hat{\zeta}_k(s) = \zeta_k(s) \Gamma_{\mathbb{R}}(s)^{r_1} \Gamma_{\mathbb{C}}(s)^{r_2}$$

where we have set

$$\Gamma_{\mathbb{R}}(s) = 2^{-1/2} \pi^{-s/2} \Gamma(s/2) \quad \text{and} \quad \Gamma_{\mathbb{C}}(s) = (2\pi)^{-s} \Gamma(s) \, .$$

Thus $\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1) = \Gamma_{\mathbb{C}}(s)$. Here r_1 and r_2 are the numbers of real resp. complex places of k. Moreover $\mu(k)$ is the group of roots of unity in k.

REMARKS 1 Formula (32) coincides with the corresponding formula in [GS] proposition 6 after correcting two small misprints in that paper: We have $\sqrt{|\Delta|}^s$ instead of $\sqrt{|\Delta|}^{s/2}$ in [GS] proposition 6 and $2^{-1}\pi^{-s/2}\ldots$ instead of $2\pi^{-s/2}\ldots$ in the third equality on p. 388 of [GS].
2 The reason for our normalization of $\Gamma_{\mathbb{R}}(s)$ comes from the theory of zeta-regularization, c.f. section 5.

PROOF We write the integral representation (31) for $\zeta_{X_k}(s,w)$ as a sum of two contributions:

$$\zeta_{X_k}(s,w) = I(s,w) + II(s,w) \tag{33}$$

where

$$I(s,w) = d_k^{s/2} \int_0^{\sqrt{d_k}} \int_{CH^1(X_k)^0} k^0(\mathcal{D}+\mathcal{D}_t)^w d^0\mathcal{D}\, t^{-s}\frac{dt}{t}$$

and

$$II(s,w) = d_k^{s/2} \int_{\sqrt{d_k}}^{\infty} \int_{CH^1(X_k)^0} k^0(\mathcal{D}+\mathcal{D}_t)^w d^0\mathcal{D}\, t^{-s}\frac{dt}{t}\ .$$

The estimate in proposition 3.1 a) shows that the first integral defines a holomorphic function of (s,w) in the region $\{\mathrm{Re}\, s < 0\} \times \mathbb{C}$. Here and in the following we use the following well known fact. Consider a function $f(\mathbf{s},x)$ holomorphic in several complex variables $\mathbf{s}$ and μ-integrable in x which locally in $\mathbf{s}$ is bounded by integrable functions of x. Then the integral $\int f(\mathbf{s},x)d\mu(x)$ is holomorphic in $\mathbf{s}$.
Writing $I(s,w)$ in the form

$$I(s,w) = d_k^{s/2} \int_0^{\sqrt{d_k}} \int_{CH^1(X_k)^0} (k^0(\mathcal{D}+\mathcal{D}_t)^w - 1) d^0\mathcal{D}\, t^{-s}\frac{dt}{t} - \frac{h_k R_k}{s} \tag{34}$$

the same estimate gives its meromorphic continuation to $\mathbb{C}^2$. Note that, even divided by w the first term is holomorphic in $\mathbb{C}^2$.
Using Riemann–Roch (20) a short calculation shows that for $\mathrm{Re}\, s > \mathrm{Re}\, w$ we have

$$II(s,w) = I(w-s,w)\ . \tag{35}$$

In particular the integral (31) defines a holomorphic function in $\mathrm{Re}\, w < \mathrm{Re}\, s < 0$ as asserted earlier. Using (34) we find the formula:

$$II(s,w) = d_k^{(w-s)/2} \int_0^{\sqrt{d_k}} \int_{CH^1(X_k)^0} (k^0(\mathcal{D}+\mathcal{D}_t)^w - 1) d^0\mathcal{D}\, t^{-(w-s)}\frac{dt}{t} - \frac{h_k R_k}{w-s} \tag{36}$$

which gives the meromorphic continuation of $II(s,w)$ to $\mathbb{C}^2$: Again, even after division by w the first term is holomorphic in $\mathbb{C}^2$. This implies the assertions of the proposition except for formula (32) which requires a lemma that will be useful in the next section as well: □

Lemma 3.3 *In the region* $\mathrm{Re}\, s > \mathrm{Re}\, w, \mathrm{Re}\, s > 0$ *the following integral representation holds, the integral defining a holomorphic function even after division by* w*:*

$$\zeta_{X_k}(s,w) = d_k^{s/2} \int_{CH^1(X_k)} (k^0(\mathcal{D})^w - 1)\mathcal{N}\mathcal{D}^{-s} d\mathcal{D} \, . \tag{37}$$

Proof of formula (32) Using (37) we find for $w = 1 < \mathrm{Re}\, s$ that

$$|\mu(k)|^{-1} d_k^{-s/2} \zeta_{X_k}(s,1) = |\mu(k)|^{-1} \int_{CH^1(X_k)} (k^0(\mathcal{D}) - 1)\mathcal{N}\mathcal{D}^{-s} d\mathcal{D} \, .$$

Now on p. 388 of [GS] this integral is shown to equal

$$(2^{-1}\pi^{-s/2}\Gamma(s/2))^{r_1}((2\pi)^{-s}\Gamma(s))^{r_2}\zeta_k(s)$$

c.f. remark 1 above. □

Proof of the lemma The estimate in proposition 3.1, b) shows that the following formula is valid in the region $\mathrm{Re}\, s > \mathrm{Re}\, w, \mathrm{Re}\, s > 0$:

$$II(s,w) = d_k^{s/2} \int_{\sqrt{d_k}}^{\infty} \int_{CH^1(X_k)^0} (k^0(\mathcal{D} + \mathcal{D}_t)^w - 1) d^0\mathcal{D}\, t^{-s} \frac{dt}{t} + \frac{h_k R_k}{s} \, . \tag{38}$$

Note here that the double integral with integrand $1 - t^w d_k^{-w/2}$ is absolutely convergent when $\mathrm{Re}\, s > \mathrm{Re}\, w, \mathrm{Re}\, s > 0$.
The integral in formula (38) defines a holomorphic function in this region even after division by w. As the integral in formula (34) for $w^{-1}I(s,w)$ gives a holomorphic function in $\mathbb{C}^2$ the assertion follows by adding equations (34) and (38). □

Remark For $k = \mathbb{Q}$ a more elaborate version of the lemma is given in [LR] Theorem 2.2.

Proposition 3.2 and formula (32) in particular suggest that a better definition of a two variable zeta function might be the following

$$\zeta(X_k, s, w) = w^{-1} \frac{2^{r_1/2}}{|\mu(k)|} d_k^{-s/2} \zeta_{X_k}(s,w) \, .$$

This is a meromorphic function on $\mathbb{C}^2$ which satisfies the equations

$$\zeta(X_k, w - s, w) = d_k^{s - w/2} \zeta(X_k, s, w) \quad \text{and} \quad \zeta(X_k, s, 1) = \hat{\zeta}_k(s) \, . \tag{39}$$

In section 5 we will see that $\zeta(X_k, s, w)$ is the "$\frac{1}{2\pi}$-zeta regularized version" of $\zeta_{X_k}(s, w)$. We also consider an entire version of this function which in the one variable case and in [LR] is called the ξ-function. Because of our different normalization we give it another name which is suggested by the cohomological arguments in section 6.

DEFINITION 3.4 *The two-variable L-function of X_k is defined by the formula*

$$\begin{aligned} L(H^1(X_k), s, w) &= \frac{s}{2\pi}\frac{s-w}{2\pi}\zeta(X_k, s, w) \\ &= \frac{1}{4\pi^2}\frac{s(s-w)}{w}\frac{2^{r_1/2}}{|\mu(k)|}d_k^{-s/2}\zeta_{X_k}(s, w) . \end{aligned}$$

According to proposition 3.2 it is holomorphic in $\mathbb{C}^2$ and satisfies the functional equation

$$L(H^1(X_k), w-s, w) = d_k^{s-w/2} L(H^1(X_k), s, w) .$$

PROPOSITION 3.5 *For any $k/\mathbb{Q}$ and every fixed w the entire function $L(H^1(X_k), s, w)$ of s has order at most one.*

PROOF Proposition 3.2 implies the formula

$$L(H^1(X_k), s, w) = s(s-w)(T(s, w) + d_k^{\frac{w}{2}-s}T(w-s, w)) + \frac{d_k^{-s/2}}{4\pi^2}\frac{2^{r_1/2}}{|\mu(k)|}h_k R_k$$

where $T(s, w)$ is the entire function in $\mathbb{C}^2$ defined by the integral

$$T(s, w) = \frac{1}{4\pi^2}\frac{2^{r_1/2}}{|\mu(k)|}\int_0^{\sqrt{d_k}}\int_{CH^1(X_k)^0} w^{-1}(k^0(\mathcal{D}+\mathcal{D}_t)^w - 1)d^0\mathcal{D}\, t^{-s}\frac{dt}{t} .$$

Using the estimate in proposition 3.1, a) we find for some $c(w) > 0$:

$$\begin{aligned} |T(s, w)| &\leq c(w)\int_0^{\sqrt{d_k}} \exp(-\pi n t^{-2/n})t^{-\mathrm{Re}\, s}\frac{dt}{t} \\ &= c(w)d_k^{-\mathrm{Re}\, s/2}\int_1^{\infty}\exp(-\pi n d_k^{-1/n}t^{2/n})t^{\mathrm{Re}\, s}\frac{dt}{t} . \end{aligned}$$

For $\mathrm{Re}\, s \leq 1$ the latter integral is bounded. For $\mathrm{Re}\, s > 1$ we have

$$\begin{aligned} |T(s, w)| &\leq c(w)d_k^{-\mathrm{Re}\, s/2}\int_0^{\infty}\exp(-\pi n d_k^{-1/n}t^{2/n})t^{\mathrm{Re}\, s}\frac{dt}{t} \\ &= \frac{nc(w)}{2}(\pi n)^{-\frac{n\mathrm{Re}\, s}{2}}\Gamma\left(\frac{n\mathrm{Re}\, s}{2}\right) \\ &= O\left(\exp\left(\frac{n}{2}\mathrm{Re}\, s\right)\log(\mathrm{Re}\, s)\right) \end{aligned}$$

where the O-constant depends on w. Hence for all $s \in \mathbb{C}$ we have

$$|T(s,w)| = O\left(\exp\left(\frac{n}{2}|s|\log|s|\right)\right) .$$

Thus for every $\varepsilon > 0$ the required estimate holds:

$$|L(H^1(X_k), s, w)| = O(\exp(|s|^{1+\varepsilon})) \quad \text{for } s \in \mathbb{C} .$$

□

Remark For $k = \mathbb{Q}$ Lagarias and Rains prove that $L(H^1(X_{\mathbb{Q}}), s, w)$ is entire of order at most one as a function of two variables, [LR] Theorem 4.1. They also mention that this assertion holds for general k as well.

4 An oscillatory integral in the geometry of numbers

Recall that an Arakelov divisor D in $Z^1(X_k)$ may be viewed as the lattice $(I(D), \| \, \|_D)$. Two divisors define the same class $\mathcal{D}$ in $CH^1(X_k)$ if and only if the corresponding metrized lattices are isometric by an $\mathfrak{o}_k$-linear isometry. In particular the following numbers are well defined for $\mathcal{D} = [D]$:

$$\begin{aligned} a(\mathcal{D}) &= \min\{\|f\|_D^2 \,|\, 0 \neq f \in I(D)\} \\ b(\mathcal{D}) &= \min\{\|f\|_D^2 \,|\, f \in I(D) \text{ such that } \|f\|_D^2 > a(\mathcal{D})\} \\ \nu(\mathcal{D}) &= |\{f \in I(D) \,|\, \|f\|_D^2 = a_{\mathcal{D}}\}| . \end{aligned}$$

By definition $b(\mathcal{D}) > a(\mathcal{D}) > 0$ are positive real numbers and $\nu(\mathcal{D})$ is a positive integer – the so called kissing number of the lattice class.
These numbers arise naturally in the study of theta functions: Ordering terms, we may write

$$\begin{aligned} k^0(\mathcal{D} + \mathcal{D}_t) &= \sum_{f \in I(D)} \exp(-\pi t^{-2/n} \|f\|_D^2) \\ &= 1 + \nu(\mathcal{D}) e^{-\pi t^{-2/n} a(\mathcal{D})} + \ldots \end{aligned}$$

Here the next term is $e^{-\pi t^{-2/n} b(\mathcal{D})}$ with its multiplicity.

Proposition 4.1 *On $CH^1(X_k)$ the function a is continuous whereas b and ν are only upper semicontinuous. In particular a, b and ν are measurable. We have $b(\mathcal{D}) \leq 4a(\mathcal{D})$ for all $\mathcal{D}$, and ν is locally bounded. On $CH^1(X_k)^0$ the functions a, b, ν are bounded.*

Points of discontinuity for b and ν arise as follows. Already for $k = \mathbb{Q}(\sqrt{2})$ there exist convergent sequences $\mathcal{D}_n \to \mathcal{D}$ even in $CH^1(X_k)^0$ such that

$b(\mathcal{D}_n) \to a(\mathcal{D})$. Thus at the point $\mathcal{D}$ we have $\lim_{n\to\infty} b(\mathcal{D}_n) < b(\mathcal{D})$ and also the multiplicity ν jumps up.

Proof Fix an element $f \in I(D)$ with $\|f\|_D^2 = a_{\mathcal{D}}$. Then $\|2f\|_D^2 = 4a_{\mathcal{D}}$. Thus $b_{\mathcal{D}} \le 4a_{\mathcal{D}}$. The continuity properties may be checked locally. So let us fix a class $\mathcal{D}^0 = [D^0]$ in $CH^1(X_k)$ and write:

$$D^0 = \sum_{\mathfrak{p}} \nu^0_{\mathfrak{p}} \cdot \mathfrak{p} + \sum_v x^0_v \cdot v \quad \text{in } Z^1(X_k)\,.$$

Let V be an open neighborhood of $x^0 = (x^0_v)_{v\,|\,\infty}$ in $\bigoplus_{v\,|\,\infty}\mathbb{R}$ and consider the continuous map:

$$V \longrightarrow CH^1(X_k)\,,\; x \longmapsto \mathcal{D}_x = [D_x] \quad \text{where } D_x = \sum_{\mathfrak{p}} \nu^0_{\mathfrak{p}} \cdot \mathfrak{p} + \sum_{v\,|\,\infty} x_v \cdot v\,.$$

For V small enough this map is a homeomorphism of V onto an open neighborhood U of $\mathcal{D}^0$ in $CH^1(X_k)$. Fix some $R > 0$ such that for all x in V we have

$$R^{-1} \le e^{x_v} \le R \text{ if } v \text{ is real} \quad \text{and} \quad R^{-2} \le e^{x_v} \le R^2 \text{ if } v \text{ is complex}\,.$$

It follows that for $x \in V$ and all $f \in I(D_x) = I(D^0)$ we have the estimate

$$R^{-2}\|f\|^2 \le \|f\|^2_{D_x} = \sum_{v \text{ real}} |f|_v^2 e^{-2x_v} + 2 \sum_{v \text{ complex}} |f|_v^2 e^{-x_v} \le 2R^2\|f\|^2\,. \tag{40}$$

Here

$$\|f\| = \Big(\sum_{v\,|\,\infty} |f|_v^2\Big)^{1/2}$$

is the Euclidean norm in $k \otimes \mathbb{R}$ applied to the element $f \in k \subset k \otimes \mathbb{R}$.
Since $I(D^0)$ is discrete in $k \otimes \mathbb{R}$ it follows that for any $C > 0$ the set

$$\mathcal{F}_C = \{f \in I(D^0) \,|\, 0 < \|f\|^2_{D_x} \le C \quad \text{for some } x \in V\}$$

is finite. If V is bounded it also follows that the map $\mathcal{D} \mapsto a(\mathcal{D})$ is bounded on U and so is b since $b(\mathcal{D}) \le 4a(\mathcal{D})$. Thus for large enough $C > 0$ the finite subset $\mathcal{F} = \mathcal{F}_C \subset I(D)$ has the following properties: For all $x \in V$ we have:

$$\begin{aligned} a(\mathcal{D}_x) &= \min\{\|f\|^2_{D_x} \,|\, f \in \mathcal{F}\} \\ b(\mathcal{D}_x) &= \min\{\|f\|^2_{D_x} \,|\, f \in \mathcal{F} \text{ such that } \|f\|^2_{D_x} > a(\mathcal{D}_x)\} \\ \nu(\mathcal{D}_x) &= |\{f \in \mathcal{F} \,|\, \|f\|^2_{D_x} = a(\mathcal{D}_x)\}|\,. \end{aligned} \tag{41}$$

The functions $x \mapsto \|f\|^2_{D_x}$ for $f \in \mathcal{F}$ being continuous it is now clear that $a(\mathcal{D})$ is continuous near $\mathcal{D}^0$, hence everywhere since $\mathcal{D}^0$ was arbitrary. (This fact is already mentioned in [GS].)

To check upper semicontinuity of b and ν at $\mathcal{D}^0$ let $\mathcal{F}'$ be the subset of $\mathcal{F}$ consisting of all f with $\|f\|^2_{D^0} > a(\mathcal{D}_0)$. For small enough V we then have

$$\|f\|^2_{D_x} > a(\mathcal{D}_x) \quad \text{for all } f \in \mathcal{F}' \text{ and } x \in V \tag{42}$$

since both sides are continuous in x. It follows that

$$b(\mathcal{D}_x) \le \min\{\|f\|^2_{D_x} \,|\, f \in \mathcal{F}'\} =: \mu(x)\ .$$

Then μ is continuous and $\mu(x^0) = b(\mathcal{D}^0)$. Hence, for every $\varepsilon > 0$ there exists an open neighborhood V' of x^0 in V such that

$$\mu(V') \subset (\mu(x^0) - \varepsilon, \mu(x^0) + \varepsilon) = (b(\mathcal{D}^0) - \varepsilon, b(\mathcal{D}^0) + \varepsilon)\ .$$

Thus $b(\mathcal{D}_x) \le b(\mathcal{D}^0) + \varepsilon$ for all $x \in V'$ and hence $b(\mathcal{D}) \le b(\mathcal{D}^0) + \varepsilon$ for all $\mathcal{D}$ in a neighborhood (the image of V') of $\mathcal{D}^0$ in $CH^1(X_k)$. Hence b is upper semicontinuous at $\mathcal{D}^0$.
As for ν, the representation (41) shows that $\nu(\mathcal{D}_x) \le |\mathcal{F}|$ for all $x \in V$. Hence ν is a locally bounded function on $CH^1(X_k)$.
With notations as above we have by (41) that

$$\nu(\mathcal{D}_x) \le |\mathcal{F} \setminus \mathcal{F}'| = \nu(\mathcal{D}^0) \quad \text{for all } x \in V\ .$$

This implies that ν is upper semicontinuous at $\mathcal{D}^0$. □

The following theorem shows that on $CH^1(X_k)^0$ the function $a = a(\mathcal{D})$ acquires a unique global minimum at $\mathcal{D} = 0$. We also describe $a(\mathcal{D})$ explicitly in a neighborhood of $\mathcal{D} = 0$.
Set

$$a_{\min} = \min\{a(\mathcal{D}) \,|\, \mathcal{D} \in CH^1(X_k)^0\} > 0$$

and

$$b_{\inf} = \inf\{b(\mathcal{D}) \,|\, \mathcal{D} \in CH^1(X_k)^0\}\ .$$

Theorem 4.2 *Set $n = (k : \mathbb{Q})$ and let the notations be as above.*
1 $a_{\min} = n$.
2 *For $\mathcal{D} \in CH^1(X_k)^0$ we have $a(\mathcal{D}) = a_{\min}$ if and only if $\mathcal{D} = 0$.*
3 *For the representative $D = 0$ of $\mathcal{D} = 0$ and $f \in \mathfrak{o}_k = I(0)$ we have $\|f\|^2_0 = a(0) = a_{\min}$ if and only if $f \in \mu(k)$.*
4 *For $\mathcal{D} \in CH^1(X_k)^0$ with $I(D)$ non-principal there is the estimate*

$$a(\mathcal{D}) \ge \sqrt[n]{4}\, a_{\min} = n \sqrt[n]{4}\ .$$

5 *For every open neighborhood U of $\mathcal{D} = 0$ in $CH^1(X_k)^0$ there is a positive ε such that $a(\mathcal{D}) < a_{\min} + \varepsilon$ for some $\mathcal{D} \in CH^1(X_k)^0$ implies $\mathcal{D} \in U$.*
6 *There is a neighborhood U of $\mathcal{D} = 0$ in $CH^1(X_k)^0$ with the following properties: Every $\mathcal{D} \in U$ has the form $\mathcal{D} = [D]$ with $D = \sum_{v \,|\, \infty} x_v \cdot v$. For $f \in I(D) = \mathfrak{o}_k$ we have:*

$$\|f\|^2_D = a(\mathcal{D}) \quad \textit{if and only if } f \in \mu(k)\ .$$

Moreover:

$$a(\mathcal{D}) = \sum_{v \text{ real}} e^{-2x_v} + 2 \sum_{v \text{ complex}} e^{-x_v}$$

and $\nu(\mathcal{D}) = |\mu(k)|$.
7 *We have* $b_{\inf} > a_{\min}$.

PROOF The main tool is the inequality between the arithmetic and the geometric mean. This inequality was already used in [GS]. Let $\| \; \|_v = | \; |_v^{e_v}$ be the normalized absolute value at the infinite place v.
1 For $\mathcal{D} = [D]$ in $CH^1(X_k)^0$ and $f \in I(D)$ we have

$$\begin{aligned} \|f\|_D^2 &= \sum_{v \text{ real}} (\|f\|_v e^{-x_v})^2 + \sum_{v \text{ complex}} \|f\|_v e^{-x_v} + \sum_{v \text{ complex}} \|f\|_v e^{-x_v} \\ &\overset{(a)}{\geq} n \left(\prod_v \|f\|_v \right)^{2/n} \left(\prod_v e^{-x_v} \right)^{2/n} = n|N(f)|^{2/n} \left(\prod_v e^{x_v} \right)^{-2/n} \\ &= n(|N(f)|/N(I(D)))^{2/n} . \end{aligned}$$

Here (a) is the arithmetic-geometric mean inequality and we have used that

$$1 = \mathcal{N}(\mathcal{D}) = \prod_{\mathfrak{p}} N\mathfrak{p}^{\nu_{\mathfrak{p}}} \prod_v e^{x_v} = N(I(D))^{-1} \prod_v e^{x_v} .$$

Now $I(D)$ divides (f) and for $f \neq 0$ we therefore have

$$|N(f)|/N(I(D)) \geq 1 .$$

It follows that $\|f\|_D^2 \geq n$, so that $a(\mathcal{D}) \geq n$ and therefore $a_{\min} \geq n$. On the other hand for $D = 0$ and $f \in \mu(k)$ we have $\|f\|_0^2 = r_1 + 2r_2 = n$. Therefore $a(0) = n$ and hence $a_{\min} = n$.

2 We have seen that $a(0) = a_{\min}$. Now assume that $a(\mathcal{D}) = a_{\min}$. Then there is some $f \in I(D)$ with $\|f\|_D^2 = n$. It follows that $|N(f)| = N(I(D))$ hence that $I(D) = (f)$ is principal and that we have equality in (a) above. Now in the arithmetic-geometric mean inequality, equality is achived precisely if all terms are equal. Thus there is a positive real ξ such that

$$\xi = (\|f\|_v e^{-x_v})^2 \quad \text{for real } v \text{ and } \xi = \|f\|_v e^{-x_v} \text{ for complex } v .$$

Hence

$$\xi^n = \xi^{r_1} \xi^{2r_2} = \left(\prod_v \|f\|_v e^{-x_v} \right)^2 = (|N(f)| N(I(D))^{-1})^2 = 1$$

since $\mathcal{N}(\mathcal{D}) = 1$ and $|N(f)| = N(I(D))$ as observed above. Thus $\xi = 1$ and therefore

$$\|f\|_v = e^{x_v} \quad \text{for all } v \,|\, \infty . \tag{43}$$

It follows that

$$\begin{aligned} \operatorname{div} f^{-1} &= \sum_{\mathfrak{p}} \operatorname{ord}_{\mathfrak{p}} f^{-1} \cdot \mathfrak{p} - \sum_{v} \log \|f^{-1}\|_v \cdot v \\ &= \sum_{\mathfrak{p}} \operatorname{ord}_{\mathfrak{p}} I(D)^{-1} \cdot \mathfrak{p} + \sum_{v} \log \|f\|_v \cdot v \\ &= \sum_{\mathfrak{p}} \nu_{\mathfrak{p}} \cdot \mathfrak{p} + \sum_{v} x_v \cdot v = D\,. \end{aligned}$$

Hence $\mathcal{D} = [D] = 0$ in $CH^1(X_k)^0$ and 2 is proved.

3 For $D = 0$ we have $I(D) = \mathfrak{o}_k$. For $f \in I(D) = \mathfrak{o}_k$ the equation $\|f\|_0^2 = a(0) = n$ implies $\|f\|_v = 1$ for all $v \,|\, \infty$ by (43). Since $\|f\|_{\mathfrak{p}} \le 1$ for all finite primes $\mathfrak{p}$ it follows by a theorem of Kronecker that f is a root of unity.

4 If $I(D)$ is non-principal and $0 \neq f \in I(D)$, then we have $(f) = I(D) \cdot \mathfrak{a}$ for some integral ideal $\mathfrak{a} \neq \mathfrak{o}_k$. Hence $|N(f)| \ge 2N(I(D))$ and 4 follows from the above estimate for $\|f\|_D^2$.

5 Let a_U be the minimum of the continuous function $a = a(\mathcal{D})$ on the compact set $CH^1(X_k)^0 \smallsetminus U$. For $\mathcal{D} \neq 0$ we have $a(\mathcal{D}) > a_{\min}$ by 2. Hence $\varepsilon := a_U - a_{\min} > 0$. It is clear that $a(\mathcal{D}) < a_{\min} + \varepsilon$ implies that $\mathcal{D} \in U$.

6 As in the proof of proposition 4.1 there exists an open neighborhood V' of $x^0 = 0$ in $\{x \in \bigoplus_{v \,|\, \infty} \mathbb{R} \,|\, \sum x_v = 0\}$ such that firstly the map

$$V' \longrightarrow CH^1(X_k)^0\,,\ x \longmapsto \mathcal{D}_x = [D_x] \text{ where } D_x = \sum_{v \,|\, \infty} x_v \cdot v$$

is a homeomorphism onto an open neighborhood U' of $\mathcal{D} = 0$ in $CH^1(X_k)^0$. In particular $I(D_x) = \mathfrak{o}_k$ for all $x \in V'$. Secondly there is a finite subset $\mathcal{F} \supset \mu(k)$ of $\mathfrak{o}_k$ such that for all $x \in V'$ we have:

$$\begin{aligned} a(\mathcal{D}_x) &= \min\{\|f\|_{D_x}^2 \,|\, f \in \mathcal{F}\} \\ b(\mathcal{D}_x) &= \min\{\|f\|_{D_x}^2 \,|\, f \in \mathcal{F} \text{ such that } \|f\|_{D_x}^2 > a(\mathcal{D}_x)\} \end{aligned} \tag{44}$$

and

$$\nu(\mathcal{D}_x) = |\{f \in \mathcal{F} \,|\, \|f\|_{D_x}^2 = a(\mathcal{D}_x)\}|\,.$$

Now, according to 3 we have

$$\|f\|_{D_0}^2 = a(\mathcal{D}_0) \quad \text{for } f \in \mu(k)$$

and

$$\|f\|_{D_0}^2 > a(\mathcal{D}_0) \quad \text{for } f \in \mathcal{F} \smallsetminus \mu(k)\,.$$

Choose some $\varepsilon > 0$, such that $\|f\|_{D_0}^2 - a(\mathcal{D}_0) \ge 2\varepsilon$ for all $f \in \mathcal{F} \smallsetminus \mu(k)$. By a continuity argument we may find an open neighborhood $0 \in V \subset V'$ such that for all $x \in V$ we have

$$\|f\|_{D_x}^2 - a(\mathcal{D}_x) < \varepsilon \quad \text{if } f \in \mu(k)$$

and

$$\|f\|^2_{D_x} - a(\mathcal{D}_x) \geq \varepsilon \quad \text{if } f \in \mathcal{F} \smallsetminus \mu(k) \ .$$

As $\|f\|^2_{D_x} = \|1\|^2_{D_x}$ for all $f \in \mu(k)$ it follows that for $x \in V$ we have

$$\|f\|^2_{D_x} = a(\mathcal{D}_x) \quad \text{if and only if } f \in \mu(k) \ .$$

Moreover $\nu(\mathcal{D}_x) = |\mu(k)|$ and

$$a(\mathcal{D}_x) = \|1\|^2_{D_x} = \sum_{v \text{ real}} e^{-2x_v} + 2 \sum_{v \text{ complex}} e^{-x_v} \ .$$

Therefore, in 6 we may take U to be the image of V in $CH^1(X_k)^0$.

7 Assume that $b_{\mathrm{inf}} = a_{\min}$ and let $(\mathcal{D}_n)$ be a sequence of $\mathcal{D}_n \in CH^1(X_k)^0$ with $b(\mathcal{D}_n) \to a_{\min}$. Since $CH^1(X_k)^0$ is compact we may assume that $(\mathcal{D}_n)$ is convergent, $\mathcal{D}_n \to \mathcal{D}_0$. Because of $a_{\min} \leq a(\mathcal{D}_n) \leq b(\mathcal{D}_n)$ it follows that $a(\mathcal{D}_n) \to a_{\min}$. On the other hand since a is continuous we have $a(\mathcal{D}_n) \to a(\mathcal{D}_0)$. Hence $a(\mathcal{D}_0) = a_{\min}$ and by 2 this implies that $\mathcal{D}_0 = 0$. Thus we have $\mathcal{D}_n \to 0$ and $b(\mathcal{D}_n) \to a_{\min}$.
Let V, U and $\mathcal{F}$ be as in the proof of 6. Then for $f \in \mathcal{F}$ and $x \in V$ we have

$$\|f\|^2_{D_x} > a(\mathcal{D}_x) \quad \text{if and only if } f \notin \mu(k) \ .$$

By (44) this gives

$$b(\mathcal{D}_x) = \min\{\|f\|^2_{D_x} \mid f \in \mathcal{F} \smallsetminus \mu(k)\} \quad \text{for all } x \in V \ .$$

In particular $b(\mathcal{D}_x)$ is a continuous function of $x \in V$ and therefore $b\,|_U$ is continuous. Let $\tilde{U} \subset U$ be a compact neighborhood of $\mathcal{D} = 0$ in $CH^1(X_k)^0$. Then there is some $\tilde{\mathcal{D}} \in \tilde{U}$ with $b(\mathcal{D}) \geq b(\tilde{\mathcal{D}}) > a(\tilde{\mathcal{D}}) \geq a_{\min}$ for all $\mathcal{D}$ in $\tilde{U}$. On the other hand, for n large enough we have $\mathcal{D}_n \in \tilde{U}$ and hence $b(\mathcal{D}_n) \geq b(\tilde{\mathcal{D}}) > a_{\min}$. Hence $b(\mathcal{D}_n)$ cannot converge to $a_{\min}$, Contradiction. □

REMARK Since $\mu(k)$ acts isometrically on $(I(D), \| \,\|_D)$ and since $\nu(0) = |\mu(k)|$ the minimal value of the function $\nu = \nu(\mathcal{D})$ is $|\mu(k)|$. As ν is upper semi-continuous it follows that the set of $\mathcal{D}$ in $CH^1(X_k)$ resp. $CH^1(X_k)^0$ with $\nu(\mathcal{D}) = |\mu(k)|$ is open. It should be possible to show that the complements have measure zero.

In the following we will deal with the asymptotic behaviour of certain functions defined at least in $\mathrm{Re}\, s > 0$ as $\mathrm{Re}\, s$ tends to infinity. For such functions f and g we will write

$$f \sim g \quad \text{to signify that } \lim_{\mathrm{Re}\, s \to \infty} f(s)/g(s) = 1 \ .$$

The following theorem is the main result of the present section:

Theorem 4.3 *For a number field $k/\mathbb{Q}$ let $r = r_1 + r_2 - 1$ be the unit rank. Then the entire function*

$$C(s) = \int_{CH^1(X_k)^0} \nu(\mathcal{D}) a(\mathcal{D})^{-s} d^0\mathcal{D}$$

has the following asymptotic behaviour as $\operatorname{Re} s \to \infty$

$$C(s) \sim |\mu(k)| \alpha_k s^{-r/2} n^{-s} .$$

Here we have set:

$$\alpha_k = (\pi n)^{r/2} 2^{-r_1/2} \sqrt{2/n} .$$

Proof If $r = 0$ then $\alpha_k = 1$ and $CH^1(X_k)^0 = CH^1(\operatorname{spec} \mathfrak{o}_k)$ is the class group of k. Hence $C(s)$ is a finite Dirichlet series. For $k = \mathbb{Q}$ we have $C(s) = \nu(0)a(0)^{-s} = |\mu(\mathbb{Q})| = 2$. For k imaginary quadratic the main contribution as $\operatorname{Re} s \to \infty$ comes from the term corresponding to $\mathcal{D} = 0$ which is $\nu(0)a(0)^{-s} = |\mu(k)| 2^{-s}$. These assertions follow from theorem 4.2 parts 1 and 2 (or 4) and 3.

Now assume that $r \geq 1$. The function $\nu = \nu(\mathcal{D})$ is measurable and bounded on $CH^1(X_k)^0$ by proposition 4.1. The function $a = a(\mathcal{D})$ is continuous and $CH^1(X_k)^0$ is compact. Hence $C(s)$ is an entire function of s. We will compare $C(s)$ with certain integrals over unbounded domains which can be evaluated explicitly in terms of Γ-functions. It is not obvious that these integrals converge. For this we require the following lemma where for $x \in \mathbb{R}^N$ we set $\|x\|_\infty = \max |x_i|$.

After a series of auxiliary results the proof of theorem 4.3 is concluded after the proof of corollary 4.3.4 below.

Lemma 4.3.1 *Assume $N \geq 2$ and consider the hyperplane $H_N = \{x \mid \sum x_i = 0\}$ in $\mathbb{R}^N$. For every x in H_N we have*

$$\max x_i \geq (N-1)^{-1} \|x\|_\infty \quad \textit{and} \quad \min x_i \leq -(N-1)^{-1} \|x\|_\infty .$$

Proof We may assume that $x_1 \leq \ldots \leq x_N$, so that $x_1 = \min x_i$ and $x_N = \max x_i$. As $x \in H_N$ we have $x_1 \leq 0 \leq x_N$. It is clear that $\|x\|_\infty = \max(-x_1, x_N)$.

If $\|x\|_\infty = x_N$ the first estimate is clear. If $\|x\|_\infty = -x_1$ then

$$(N-1) \max x_i = (N-1)x_N \geq x_N + x_{N-1} + \ldots + x_2 = -x_1 = \|x\|_\infty .$$

Hence the first estimate holds in this case as well. The second estimate follows by replacing x with $-x$. □

We can now evaluate a certain class of integrals which are useful for our purposes.

PROPOSITION 4.3.2 *For $N \geq 2$ let $d\lambda$ be the Lebesgue measure on H_N. Fix positive real numbers $c_1, \ldots, c_N$ and positive integers $\nu_1, \ldots, \nu_N$. Then for* $\mathrm{Re}\, s > 0$ *we have the following formula where* $q = 1/\sum_{i=1}^N \nu_i^{-1}$

$$I := \int_{H_N} \Big(\sum_{i=1}^N c_i e^{-\nu_i x_i} \Big)^{-s} d\lambda = \frac{q}{\nu_1 \cdots \nu_N} \Big(\prod_{i=1}^N c_i^{q/\nu_i} \Big)^{-s} \Gamma(s)^{-1} \prod_{i=1}^N \Gamma(qs/\nu_i) \; .$$

PROOF First we show that the integral exists. Using lemma 4.3.1 and the fact that $\min(x_i) \leq 0$ for $x \in H_N$, we find with $c = \min(c_i)$:

$$\sum_{i=1}^N c_i e^{-\nu_i x_i} \geq c\, e^{-\min(x_i)} \geq c \exp((N-1)^{-1} \|x\|_\infty) \quad \text{for } x \in H_N \; . \tag{45}$$

Thus the function

$$\Big(\sum_{i=1}^N c_i e^{-\nu_i x_i} \Big)^{-\mathrm{Re}\, s}$$

is integrable over H_N. In order to evaluate the integral we recall the Mellin transform of a (suitable) function h on $\mathbb{R}_+^*$:

$$(Mh)(s) = \int_0^\infty h(t) t^s \frac{dt}{t} \quad \text{for } \mathrm{Re}\, s \geq 1$$

and the convolution of two L^1-functions h_1 and h_2 on $\mathbb{R}_+^*$:

$$(h_1 * h_2)(t) = \int_0^\infty h_1(t_1) h_2(t t_1^{-1}) \frac{dt_1}{t_1} \; .$$

For suitable h_1 and h_2 Fubini's theorem implies the basic formula

$$M(h_1 * h_2) = (Mh_1) \cdot (Mh_2) \quad \text{for } \mathrm{Re}\, s \geq 1 \; .$$

For $t > 0$ let $d\mu$ be the image of Lebesgue measure under the exponential isomorphism:

$$\{x \in \mathbb{R}^N \mid \sum x_i = \log t\} \xrightarrow{\sim} \{(t_1, \cdots, t_N) \in (\mathbb{R}_+^*)^N \mid t_1 \cdots t_N = t\} \; .$$

The N-fold convolution of L^1-functions $h_1, \ldots, h_N$ on $\mathbb{R}_+^*$ is given by the formula

$$(h_1 * \ldots * h_N)(t) = \int_{t_1 \cdots t_N = t} h_1(t_1) \cdots h_N(t_N) \, d\mu \; .$$

Note that convolution is associative.
We may rewrite I as follows

$$I = \int_{t_1 \cdots t_N = 1} \Big(\sum_{i=1}^N c_i t_i^{\nu_i} \Big)^{-s} d\mu \; .$$

Thus

$$\begin{aligned}
\Gamma(s)\cdot I &= \int_0^\infty \Big(\int_{t_1\cdots t_N=1} \exp\Big(-t\sum_{i=1}^N c_i t_i^{\nu_i}\Big)d\mu\Big)t^s\frac{dt}{t} \qquad (46)\\
&= \int_0^\infty \Big(\int_{t_1\cdots t_N=t^{1/q}} \exp\Big(-\sum_{i=1}^N c_i t_i^{\nu_i}\Big)d\mu\Big)t^s\frac{dt}{t}\\
&= qM(e^{-c_1t^{\nu_1}} * \ldots * e^{-c_N t^{\nu_N}})(qs)\\
&= qM(e^{-c_1t^{\nu_1}})(qs)\cdots M(e^{-c_N t^{\nu_N}})(qs)\\
&= q\prod_{i=1}^N \nu_i^{-1}c_i^{-qs/\nu_i}\Gamma(qs/\nu_i)\;.
\end{aligned}$$

□

We may now use the complex Stirling asymptotics

$$\Gamma(s)\sim\sqrt{2\pi}e^{-s}e^{\left(s-\frac{1}{2}\right)\log s} \quad \text{for } |s|\to\infty \text{ in } -\pi<\arg s<\pi \qquad (47)$$

to draw the following consequence of proposition 4.3.2.

Corollary 4.3.3 *Let $k/\mathbb{Q}$ be a number field of degree n with unit rank $r = r_1+r_2-1\geq 1$. Then we have the following asymptotic formula for* $\mathrm{Re}\, s\to\infty$*, the integral being defined for* $\mathrm{Re}\, s>0$*:*

$$\int_{\sum_{v\,|\,\infty} x_v=0}\Big(\sum_{v \text{ real}} e^{-2x_v} + 2\sum_{v \text{ complex}} e^{-x_v}\Big)^{-s} d\lambda \sim \alpha_k s^{-r/2}n^{-s}\;.$$

Here

$$\alpha_k = (\pi n)^{r/2}2^{-r_1/2}\sqrt{2/n}\;.$$

Proof Applying proposition 4.3.2 with $N = r_1+r_2$ and the obvious choices of c_i's and ν_i's the integral is seen to equal:

$$n^{-1}2^{1-r_1}2^{-2sr_2/n}\Gamma(s)^{-1}\Gamma(s/n)^{r_1}\Gamma(2s/n)^{r_2}\;.$$

Applying the Stirling asymptotics gives the result after some calculation. □

Corollary 4.3.4 *Assumptions as in corollary 4.3.3. For any $\varepsilon>0$ set*

$$V_\varepsilon = \Big\{x\in\bigoplus_{v\,|\,\infty}\mathbb{R}\,|\,\sum_{v\,|\,\infty}x_v=0 \quad \textit{and} \quad \|x\|_\infty<\varepsilon\Big\}\;.$$

Then we have the asymptotic formula for $\mathrm{Re}\, s\to\infty$*:*

$$\int_{V_\varepsilon}\Big(\sum_{v \text{ real}} e^{-2x_v} + 2\sum_{v \text{ complex}} e^{-x_v}\Big)^{-s} d\lambda \sim \alpha_k s^{-r/2}n^{-s}\;.$$

PROOF Set $f(x) = \sum_{v \text{ real}} e^{-2x_v} + 2\sum_{v \text{ complex}} e^{-x_v}$. For $x \in \bigoplus_{v \mid \infty} \mathbb{R}$ with $\sum_{v \mid \infty} x_v = 0$ we have by lemma 4.3.1 that:

$$f(x) \geq \exp(r^{-1}\|x\|_\infty) . \tag{48}$$

Choose $R \geq 2r \log 2n$. For $\|x\|_\infty \geq R$ and $\alpha \geq 0$ we find

$$\exp\Big(-\frac{\alpha}{r}\|x\|_\infty\Big) \leq (2n)^{-\alpha} \exp\Big(-\frac{\alpha}{2r}\|x\|_\infty\Big) .$$

For $\mathrm{Re}\, s \geq 1$ this implies that

$$\Big| \int_{\substack{\sum x_v = 0 \\ \|x\|_\infty > R}} f(x)^{-s} d\lambda \Big| \leq \gamma (2n)^{-\mathrm{Re}\, s} \tag{49}$$

where

$$\gamma = \int_{\substack{\sum x_v = 0 \\ \|x\|_\infty > R}} \exp\Big(-\frac{1}{2r}\|x\|_\infty\Big) d\lambda < \infty .$$

By the arithmetic-geometric mean inequality we see that in $\{\sum x_v = 0\}$ the function $f(x)$ has global minimum equal to n. We have $f(0) = n$ and $f(x) > n$ for all $x \neq 0$, c.f. the proof of theorem 4.2, 1. Choose $R \geq 2r \log 2n$ such that $R \geq \varepsilon$. Let $a_{\varepsilon,R}$ be the minimum of f in the compact set $S_{\varepsilon,R}$ of x with $\sum x_v = 0$ and $\varepsilon \leq \|x\|_\infty \leq R$. Then we have $a_{\varepsilon,R} > n$ and

$$\Big| \int_{S_{\varepsilon,R}} f(x)^{-s} d\lambda \Big| \leq \mathrm{vol}\,(S_{\varepsilon,R}) a_{\varepsilon,R}^{-\mathrm{Re}\, s} \quad \text{for } \mathrm{Re}\, s \geq 0 . \tag{50}$$

Using corollary 4.3.3 and the estimates (49) and (50) we find successively:

$$\alpha_k s^{-r/2} n^{-s} \sim \int_{\sum x_v = 0} f(x)^{-s} d\lambda \sim \int_{\substack{\sum x_v = 0 \\ \|x\|_\infty \leq R}} f(x)^{-s} d\lambda \sim \int_{\substack{\sum x_v = 0 \\ \|x\|_\infty \leq \varepsilon}} f(x)^{-s} d\lambda .$$

□

We can now conclude the proof of theorem 4.3. Let $\varepsilon > 0$ be so small that the image of V_ε in $CH^1(X_k)^0$ under the map $x \mapsto \mathcal{D}_x = [D_x]$ with $D_x = \sum_{v \mid \infty} x_v \cdot v$ is a homeomorphism onto its image U_ε. Moreover $\varepsilon > 0$ should be so small that U_ε is contained in a neighborhood U as in theorem 4.2, 6. Then we have

$$\begin{aligned} \int_{U_\varepsilon} \nu(\mathcal{D}) a(\mathcal{D})^{-s} d^0\mathcal{D} &= |\mu(k)| \int_{V_\varepsilon} \Big(\sum_{v \text{ real}} e^{-2x_v} + 2 \sum_{v \text{ complex}} e^{-x_v}\Big)^{-s} d\lambda \quad (51) \\ &\sim |\mu(k)| \alpha_k s^{-r/2} n^{-s} \quad \text{for } \mathrm{Re}\, s \to \infty \end{aligned}$$

by corollary 4.3.4. By theorem 4.2, 1 and 2 (or 5) the minimum a_{U_ε} of $a = a(\mathcal{D})$ on the compact set $CH^1(X_k)^0 \smallsetminus U_\varepsilon$ satisfies $a_{U_\varepsilon} > n$. Moreover $\nu = \nu(\mathcal{D})$ is bounded, $\leq d$ say. Together with the estimate

$$\Big| \int_{CH^1(X_k)^0 \smallsetminus U_\varepsilon} \nu(\mathcal{D}) a(\mathcal{D})^{-s} d^0\mathcal{D} \Big| \leq d \,\mathrm{vol}\,(CH^1(X_k)^0) a_{U_\varepsilon}^{-\mathrm{Re}\, s} \quad \text{for } \mathrm{Re}\, s \geq 0$$

the asymptotics (51) now imply the assertion of theorem 4.3. □

Remark 4.4 Using the asymptotic development of the Γ-function instead of (47) one can improve the assertion of theorem 4.3. For example, the same proof shows that for any $\varphi \in (0, \pi/2)$ we have

$$C(s) = |\mu(k)|\alpha_k s^{-r/2} n^{-s}(1 + O(s^{-1})) \quad \text{as } \operatorname{Re} s \to \infty$$

in the angular domain $|\arg s| < \varphi$. The O-constant depends on φ.

5 The two-variable zeta function as a regularized product

In this section we first review a theorem of Illies about the zeta-regularizability of entire functions of finite order.
We then apply his criterion to prove that $L(H^1(X_k), s, w)$ and $\zeta(X_k, s, w)$ are zeta-regularized as functions of s.
There are many instances where one would like to give a sense to a non-convergent product of distinct non-zero complex numbers a_ν given with multiplicities $m_\nu \in \mathbb{Z}$. Sometimes the process of zeta regularization helps. Fix arguments $-\pi < \arg a_\nu \leq \pi$ and assume that the Dirichlet series $D(u) = \sum m_\nu a_\nu^{-u}$ converges for $\operatorname{Re} u \gg 0$ with a meromorphic continuation to $\operatorname{Re} u > -\varepsilon$ for some $\varepsilon > 0$. If D is holomorphic at $u = 0$ we may define the zeta-regularized product

$$\prod\nolimits^{(m_\nu)} a_\nu := \exp(-D'(0)) \; .$$

If all $m_\nu = 1$, one sets $\prod a_\nu = \prod^{(1)} a_\nu$. In this way one obtains for example $\prod_{\nu=1}^\infty \nu = \sqrt{2\pi}$. For a finite sequence of a_ν, m_ν the zeta-regularized product $\prod^{(m_\nu)} a_\nu$ exists and equals the ordinary product $\prod a_\nu^{m_\nu}$.
For complex s with $s \neq a_\nu$ for all ν one may ask whether $\prod^{(m_\nu)} (s - a_\nu)$ exists. In favourable instances it will define a meromorphic function in $\mathbb{C}$ whose zeroes and poles are precisely the numbers a_ν with their multiplicity m_ν. On the other hand if we are given a meromorphic function $f(s)$ whose zeroes and poles are the numbers a_ν with multiplicity m_ν we may ask whether $\prod^{(m_\nu)} (s - a_\nu)$ exists and defines a meromorphic function in $\mathbb{C}$ and how it compares to $f(s)$. Sometimes it is also useful to introduce a scaling factor $\alpha > 0$ and compare $f(s)$ with $\prod^{(m_\nu)} \alpha(s - a_\nu)$. In the case where we have

$$f(s) = \prod\nolimits^{(m_\nu)} \alpha(s - a_\nu) \; ,$$

the function f is called "α-zeta regularized".
A much more thorough discussion of these problems and other regularization procedures (δ-regularization) may be found in Illies' papers [I1], [I2] and his references.
We now describe the precise technical result from Illies' work that we will use.

For φ_1, φ_2 in $(0, \pi)$ define the open sets

$$\mathcal{W}r_{\varphi_1,\varphi_2} = \{s \in \mathbb{C}^* \mid -\varphi_2 < \arg s < \varphi_1\}$$

and

$$\mathcal{W}l_{\varphi_1,\varphi_2} = \mathbb{C}^* \setminus \overline{\mathcal{W}r_{\varphi_1,\varphi_2}} \; .$$

A meromorphic function in $\mathbb{C}$ is said to be of finite order if it is the quotient of two entire functions of finite order.

THEOREM 5.1 (ILLIES) *Let f be a meromorphic function of finite order in $\mathbb{C}$ such that almost all zeroes and poles lie in some $\mathcal{W}l_{\varphi_1,\varphi_2}$. We assume that for some $0 < p \le \infty$ and any $p' < p$ we have*

$$f(s) - 1 = O(|s|^{-p'}) \quad \text{in } \mathcal{W}r_{\varphi_1,\varphi_2} \text{ as } |s| \to \infty \; .$$

Then the following two assertions hold:
A *Setting $m(\rho) = \mathrm{ord}_{s=\rho} f(s)$, for any scaling factor $\alpha > 0$ the Dirichlet series*

$$\xi(u,s) = \sum_{\rho \in \mathcal{W}l_{\varphi_1,\varphi_2}} m(\rho)[\alpha(s-\rho)]^{-u}$$

is uniformly convergent to a holomorphic function in $\mathrm{Re}\, u \gg 0$ and $|s| \ll 1$. Here we have chosen $-\pi < \arg(s-\rho) < \pi$ which is possible for small enough $|s|$. The function $\xi(u,s)$ has a holomorphic continuation to any region of the form

$$\{\mathrm{Re}\, u > -p\} \times G$$

where G is an arbitary simply connected domain which does not contain zeroes or poles of f.
B *We have an equality of meromorphic functions in $\mathbb{C}$*

$$\begin{aligned} f(s) &= \exp\Big(-\frac{\partial \xi}{\partial u}(0,s)\Big) \prod_{\rho \notin \mathcal{W}l_{\varphi_1,\varphi_2}} [\alpha(s-\rho)]^{m(\rho)} \\ &= \textstyle\coprod_\rho^{(m(\rho))} \alpha(s-\rho) \; . \end{aligned}$$

REMARK According to B the function f equals the zeta-regularized determinant (scaled by α) of its divisor. In fact f is the δ-regularized determinant of its divisor for any regularization sequence δ as in [I2] Definition 3.4 but we do not need this stronger statement.

PROOF The result generalizes [I2] Corollary 8.1 and is proved in the same way using [I2] Theorem 5 and Proposition 3.3. The latter results are stated for the case where *all* zeroes and poles of f lie in $\mathcal{W}l_{\varphi_1,\varphi_2}$. Using translation invariance of regularization as indicated in [I2] Example 2) after Definition 4.1 gives the general case. In the thesis [I1] more details can be found: Theorem 4.1 is a special case of [I1] Korollar 2.7.1 and translation invariance is discussed in [I1] Definition 2.2.3 and Korollar 2.2.4. □

We can now state our main theorem.

Theorem 5.2 *For $k/\mathbb{Q}$ and any fixed complex number w the functions $\zeta(X_k, s, w)$ and $L(H^1(X_k), s, w)$ of s are $\frac{1}{2\pi}$-zeta regularized.*

In the function field case the corresponding but much simpler result is this

Theorem 5.3 *Let $X/\mathbb{F}_q$ be a smooth projective and geometrically irreducible curve. Then for any fixed w the meromorphic function $Z_X(q^{-s}, q^w)$ of s and the entire function $P_X(q^{-s}, q^w)$ are α-zeta regularized for any $\alpha > 0$.*

Remarks a) In theorem 5.2, contrary to theorem 5.3 the zeta- and L-functions are α-zeta regularized for $\alpha = 1/2\pi$ only. This has to do with our normalizations of $\zeta(X_k, s, w)$ which in turn is suggested by the choice of Γ-factors for $\hat{\zeta}_k(s)$. For further discussions of this point, see the remark at the end of section 5 in [D2].
b) Comparing 5.2 and 5.3 we see that

$$\zeta(X, s, w) := Z_X(q^{-s}, q^w) = (q^w - 1)^{-1} q^{s(1-g)} \zeta_X(s, w)$$

corresponds to

$$\zeta(X_k, s, w) = w^{-1} \frac{2^{r_1/2}}{|\mu(k)|} d_k^{-s/2} \zeta_{X_k}(s, w)$$

in the following sense: For every fixed w both functions of s are obtained by the process of $\frac{1}{2\pi}$-zeta regularization from the zeroes and poles of the analogous functions $(q^w - 1)^{-1} \zeta_X(s, w)$ and $w^{-1} \zeta_{X_k}(s, w)$. Note also that we have

$$\zeta(X, s, 1) = \zeta_X(s) \quad \text{and} \quad \zeta(X_k, s, 1) = \hat{\zeta}_k(s) \, .$$

Proof of theorem 5.3 In view of formulas (4) and (6) this can be deduced from [D2] 2.7 Lemma which evaluates $\prod_{\nu \in \mathbb{Z}} \alpha(s + \nu)$ for $\alpha \in \mathbb{C}^*$. Alternatively the theorem follows without difficulty from theorem 5.1. □

For the proof of theorem 5.2 we first need a refinement of the estimate given in Proposition 3.1 a).

LEMMA 5.4 *For any number field $k/\mathbb{Q}$ and every $R \geq 0$ there is a constant $c = c_k(R)$ such that setting*

$$g(t, \mathcal{D}, w) = w^{-1}(k^0(\mathcal{D} + \mathcal{D}_t)^w - 1 - w\nu(\mathcal{D})e^{-\pi t^{-2/n} a(\mathcal{D})})$$

we have

$$|g(t, \mathcal{D}, w)| \leq c \exp(-\pi t^{-2/n} \min(2a(\mathcal{D}), b(\mathcal{D})))$$

uniformly in $\mathcal{D} \in CH^1(X_k)^0$ and $0 < t \leq \sqrt{d_k}$ and $|w| \leq R$.

PROOF For $\mathcal{D} = [D]$ we have:

$$k^0(\mathcal{D} + \mathcal{D}_t) = \sum_{f \in I(D)} e^{-\pi t^{-2/n} \|f\|_D^2} .$$

Hence

$$\delta(t, \mathcal{D}) := k^0(\mathcal{D} + \mathcal{D}_t) - 1 - \nu(\mathcal{D})e^{-\pi t^{-2/n} a(\mathcal{D})}$$

is positive. We claim that there is a constant γ depending only on k such that for all $\mathcal{D} \in CH^1(X_k)^0$ and $0 < t \leq d_k^{1/2}$ we have the estimate

$$0 < \delta(t, \mathcal{D}) \leq \gamma e^{-\pi t^{-2/n} b(\mathcal{D})} . \tag{52}$$

This is seen as follows:

$$\begin{aligned}
\delta(t, \mathcal{D})e^{\pi t^{-2/n} b(\mathcal{D})} &= \sum_{\|f\|_D^2 \geq b(\mathcal{D})} e^{-\pi t^{-2/n}(\|f\|_D^2 - b(\mathcal{D}))} \\
&\leq \sum_{\|f\|_D^2 \geq b(\mathcal{D})} e^{-\pi d_k^{-1/n}(\|f\|_D^2 - b(\mathcal{D}))} \quad \text{since } t \leq \sqrt{d_k} \\
&\leq e^{\pi d_k^{-1/n} b(\mathcal{D})} k^0(\mathcal{D} + \mathcal{D}_{\sqrt{d_k}}) = f(\mathcal{D}) .
\end{aligned}$$

Hence for γ we may choose the supremum of the bounded function f on $CH^1(X_k)^0$, c.f. Proposition 4.1.

Since the left hand side of the estimate in lemma 5.4 is bounded and since $a = a(\mathcal{D})$ is bounded on $CH^1(X_k)^0$ it suffices to prove the desired estimate for all $0 < t \leq \varepsilon$, where $\varepsilon > 0$ is small. We choose $0 < \varepsilon \leq \sqrt{d_k}$ such that for all $0 < t \leq \varepsilon$ and $\mathcal{D} \in CH^1(X_k)^0$ we have

$$0 < x = x(t, \mathcal{D}) = k^0(\mathcal{D} + \mathcal{D}_t) - 1 \leq 1/2 .$$

This is possible by (24) or (52). We may assume that $R \geq 1$. Using inequality (26), we find:

$$\begin{aligned}
k^0(\mathcal{D} + \mathcal{D}_t)^w &= (1 + x)^w = 1 + wx + wx^2\vartheta \\
&= 1 + w\nu(\mathcal{D})e^{-\pi t^{-2/n} a(\mathcal{D})} + w\psi
\end{aligned}$$

where
$$|\vartheta| = |\vartheta(t,w,\mathcal{D})| \le e^{2R}$$
and
$$\psi = \delta(t,\mathcal{D}) + x^2\vartheta \,.$$
From (52) we get
$$\begin{aligned} |\psi| &\le \gamma e^{-\pi t^{-2/n} b(\mathcal{D})} + e^{2R} e^{-2\pi t^{-2/n} a(\mathcal{D})} (\nu(\mathcal{D}) + \gamma e^{-\pi t^{-2/n}(b(\mathcal{D})-a(\mathcal{D}))})^2 \\ &\le \gamma e^{-\pi t^{-2/n} b(\mathcal{D})} + e^{2R} e^{-2\pi t^{-2/n} a(\mathcal{D})} (\nu(\mathcal{D}) + \gamma)^2 \,. \end{aligned}$$
This gives the required estimate in $0 < t \le \varepsilon$ for $c = e^{2R} \max_{\mathcal{D}}(\nu(\mathcal{D}) + \gamma)$. □

Proof of theorem 5.2 According to lemma 3.3 we may write the function $\zeta(X_k, s, w)$ as follows in the region $\mathrm{Re}\, s > \mathrm{Re}\, w, \mathrm{Re}\, s > 0$
$$\begin{aligned} \zeta(X_k,s,w) &= w^{-1} \frac{2^{r_1/2}}{|\mu(k)|} \int_{CH^1(X_k)} (k^0(\mathcal{D})^w - 1) \mathcal{N}\mathcal{D}^{-s} d\mathcal{D} \\ &= \frac{2^{r_1/2}}{|\mu(k)|} \int_0^\infty \int_{CH^1(X_k)^0} w^{-1}(k^0(\mathcal{D}+\mathcal{D}_t)^w - 1) d^0\mathcal{D}\, t^{-s} \frac{dt}{t} \\ &= \frac{2^{r_1/2}}{|\mu(k)|} \int_0^\infty \int_{CH^1(X_k)^0} (\nu(\mathcal{D}) e^{-\pi t^{-2/n} a(\mathcal{D})} + g(t,\mathcal{D},w)) d^0\mathcal{D}\, t^{-s} \frac{dt}{t} \,. \end{aligned}$$
The first term leads to the following meromorphic function in $\mathbb{C}$
$$\begin{aligned} A(s) &:= \int_0^\infty \int_{CH^1(X_k)^0} \nu(\mathcal{D}) e^{-\pi t^{-2/n} a(\mathcal{D})} d^0\mathcal{D}\, t^{-s} \frac{dt}{t} \\ &= \frac{n}{2} \pi^{-\frac{ns}{2}} \Gamma\Big(\frac{ns}{2}\Big) \int_{CH^1(X_k)^0} \nu(\mathcal{D}) a(\mathcal{D})^{-\frac{ns}{2}} d^0\mathcal{D} \,. \end{aligned}$$
Setting
$$f_w(s) = 1 + A(s)^{-1} \int_0^\infty \int_{CH^1(X_k)^0} g(t,\mathcal{D},w) d^0\mathcal{D}\, t^{-s} \frac{dt}{t}$$
we may write
$$\zeta(X_k,s,w) = \frac{2^{r_1/2}}{|\mu(k)|} A(s) f_w(s) \,. \tag{53}$$
We will now show that for any fixed $w \in \mathbb{C}$ and $\alpha > 0$ the function $f_w(s)$ is α-zeta regularized and that its $\xi(u,s)$-function in the sense of theorem 5.1 A has a holomorphic continuation to any region $\mathbb{C} \times G$ where $G \subset \mathbb{C}$ is any simply connected domain disjoint from the zeroes and poles of f_w. First note that f_w is meromorphic of finite order (≤ 1) since this is true for $s \mapsto \zeta(X_k,s,w)$ by proposition 3.5 and clear for $\frac{2^{r_1/2}}{|\mu(k)|} A(s)$.

Let $0 < \varphi < \pi/2$ be any angle such that

$$\varphi\ \tan\varphi < \log\min(2, b_{\mathrm{inf}}/n)\ .$$

Note here that because of theorem 4.2, 1 and 7 we have $b_{\mathrm{inf}} > a_{\mathrm{min}} = n$.
We will now show that for any $p' > 0$ and every $w \in \mathbb{C}$ we have the estimate

$$f_w(s) - 1 = O(|s|^{-p'}) \quad \text{in } \overline{\mathcal{W}r_{\varphi,\varphi}} \text{ as } |s| \to \infty\ . \tag{54}$$

It follows in particular that f_w has only finitely many zeroes or poles in $\overline{\mathcal{W}r_{\varphi,\varphi}}$. Hence the conclusions of Illies' theorem 5.1 apply to f_w and $\varphi_1 = \varphi_2 = \varphi$ with $p = \infty$.
In order to prove (54) we have to show the following estimates for any fixed w and $p' > 0$ as $\mathrm{Re}\, s \to \infty$ in $\overline{\mathcal{W}r_{\varphi,\varphi}}$:

$$A(s)^{-1} \int_0^{\sqrt{d}_k} \int_{CH^1(X_k)^0} g(t, \mathcal{D}, w) d^0\mathcal{D}\ t^{-s} \frac{dt}{t} = O(|s|^{-p'}) \tag{55}$$

and

$$A(s)^{-1} \int_{\sqrt{d}_k}^{\infty} \int_{CH^1(X_k)^0} g(t, \mathcal{D}, w) d^0\mathcal{D}\ t^{-s} \frac{dt}{t} = O(|s|^{-p'})\ . \tag{56}$$

We begin with (55). By lemma 5.4 we have for $0 < t \le \sqrt{d}_k$ and $\mathcal{D} \in CH^1(X_k)^0$:

$$\begin{aligned} |g(t, \mathcal{D}, w)| &\le c \exp(-\pi t^{-2/n} \min(2a(\mathcal{D}), b(\mathcal{D}))) \\ &\le c \exp(-\pi t^{-2/n} \min(2a_{\mathrm{min}}, b_{\mathrm{inf}}))\ . \end{aligned}$$

Hence there are constants c_1, c_2 depending on w such that for $\mathrm{Re}\, s \to \infty$ we have

$$\begin{aligned} |A(s)^{-1} \int_0^{\sqrt{d}_k} \cdots| &\le c_1 |A(s)|^{-1} (\pi \min(2a_{\mathrm{min}}, b_{\mathrm{inf}}))^{-\frac{n\mathrm{Re}\, s}{2}} \Gamma\Big(\frac{n\mathrm{Re}\, s}{2}\Big) \\ &\le c_2 \frac{\Gamma\left(\frac{n\mathrm{Re}\, s}{2}\right)}{|\Gamma\left(\frac{ns}{2}\right)|} |s|^{r/2} \Big(\frac{n}{\min(2n, b_{\mathrm{inf}})}\Big)^{\frac{n\mathrm{Re}\, s}{2}}\ . \end{aligned} \tag{57}$$

For the second inequality we have used theorem 4.3 which was the main result of section 4 and the fact that $a_{\mathrm{min}} = n$. Now the Stirling asymptotics (47) shows that for any $\varphi \in (0, \pi/2)$ there is a constant c_φ such that for all $z \in \mathbb{C}$ with $\mathrm{Re}\, z \ge 1/2$ and $z \in \overline{\mathcal{W}r_{\varphi,\varphi}}$ we have the estimate

$$\frac{\Gamma(\mathrm{Re}\, z)}{|\Gamma(z)|} \le c_\varphi \exp((\mathrm{Re}\, z)\varphi\ \tan\varphi)\ . \tag{58}$$

Namely, setting $z = re^{i\alpha}$ with $|\alpha| \le \varphi$ we have as $r \to \infty$

$$\begin{aligned} \frac{\Gamma(\mathrm{Re}\, z)}{|\Gamma(z)|} &\sim \exp((r\cos\alpha - \frac{1}{2}) \log\cos\alpha + r\alpha\sin\alpha) \\ &\le \exp(r\alpha\sin\alpha) = \exp((\mathrm{Re}\, z)\alpha\tan\alpha) \le \exp((\mathrm{Re}\, z)\varphi\tan\varphi)\ . \end{aligned}$$

Thus we can proceed with the estimate (57), obtaining

$$|A(s)^{-1}\int_0^{\sqrt{d_k}}\cdots| \le c_3|s|^{r/2}\exp\Big(\frac{n\mathrm{Re}\, s}{2}(\varphi\tan\varphi - \log\min(2, b_{\mathrm{inf}}/n))\Big)\,. \quad (59)$$

By our choice of φ, the second term converges exponentially fast to zero as $\mathrm{Re}\, s \to \infty$. Since $|s| \le (\mathrm{Re}\, s)(1+\tan^2\varphi)^{1/2}$ in $\overline{\mathcal{W}r_{\varphi,\varphi}}$ the estimate (55) follows for all $p' > 0$.

Next, we prove (56). We have

$$\begin{aligned}|g(t,\mathcal{D},w)| &\le |w^{-1}(k^0(\mathcal{D}+\mathcal{D}_t)^w - 1)| + \nu(\mathcal{D})e^{-\pi t^{-2/n}a(\mathcal{D})}\\ &\le |w^{-1}(k^0(\mathcal{D}+\mathcal{D}_t)^w - t^w d_k^{-w/2})| + |w^{-1}(t^w d_k^{-w/2} - 1)| + \nu(\mathcal{D})e^{-\pi t^{-2/n}a(\mathcal{D})}\,.\end{aligned}$$

For $\mathcal{D} \in CH^1(X_k)^0$ and $t \ge \sqrt{d_k}$ it follows from proposition 3.1 b) and the boundedness of ν on $CH^1(X_k)^0$ that we have

$$\begin{aligned}|g(t,\mathcal{D},w)| &\le c_4\exp(-\alpha t^{2/n}) + |w^{-1}(t^w d_k^{-w/2} - 1)| + c_5 e^{-\pi n t^{-2/n}}\\ &\le c_6 t^M\,.\end{aligned}$$

Here $M = \max(\mathrm{Re}\, w, 1)$ will do, and the constants c_i depend on w. Observe that for $w = 0$ the middle term becomes a logarithm in t which is absorbed in t^M since $M \ge 1 > 0$.

Using the estimate and theorem 4.3 we find for $\mathrm{Re}\, s > M$ in $\overline{\mathcal{W}r_{\varphi,\varphi}}$:

$$\begin{aligned}|A(s)^{-1}\int_{\sqrt{d_k}}^{\infty}\cdots| &\le c_7|A(s)|^{-1}|\int_{\sqrt{d_k}}^{\infty} t^{M-s}\frac{dt}{t}|\\ &\le c_8|\Gamma\Big(\frac{ns}{2}\Big)|^{-1}|s|^{r/2}(\pi n)^{\frac{n\mathrm{Re}\, s}{2}}|M-s|^{-1}d_k^{-\frac{\mathrm{Re}\, s}{2}}\\ &\le c_9|e^{-\frac{ns}{2}\log\frac{ns}{2}}||s|^{(r+1)/2}(\pi e n)^{\frac{n\mathrm{Re}\, s}{2}}|M-s|^{-1}d_k^{-\frac{\mathrm{Re}\, s}{2}}\end{aligned}$$

by the Stirling asymptotics. Together with the estimate

$$|e^{-\frac{ns}{2}\log\frac{ns}{2}}| \le e^{-\frac{n\mathrm{Re}\, s}{2}\log|\frac{ns}{2}|}e^{\frac{n\mathrm{Re}\, s}{2}\varphi\tan\varphi}$$

this implies the desired estimate (56) for any $p' > 0$.

Having thus proved (54), theorem 5.1 implies that f_w is α-zeta regularized for any $\alpha > 0$. Now, equation (53) together with formula (39) gives

$$\zeta(X_k, s, w) = \hat{\zeta}_k(s)\frac{f_w(s)}{f_1(s)}\,. \quad (60)$$

It has been known for quite some time that $\hat{\zeta}_k(s)$ is $\frac{1}{2\pi}$-zeta regularized and there are different ways to see this. For example $\zeta_k(s)$ is α-zeta regularized for any positive $\alpha > 0$ by [I2] corollary 8.1 which applies to very general Dirichlet series. Furthermore the Γ-factors $\Gamma_{\mathbb{R}}(s)$ and $\Gamma_{\mathbb{C}}(s)$ are $\frac{1}{2\pi}$-zeta regularized as follows from a formula essentially due to Lerch, [D2] (2.7.1):

$$\coprod_{\nu=0}^{\infty}\alpha(z+\nu)=\alpha^{\frac{1}{2}-z}\Big(\frac{1}{\sqrt{2\pi}}\Gamma(z)\Big)^{-1}. \tag{61}$$

It follows from (60) that $s\mapsto\zeta(X_k,s,w)$ is $\frac{1}{2\pi}$-zeta regularized for any w. Hence theorem 5.2 is proved. □

Incidentally, we may deduce the following corollary from the proof of 5.2:

COROLLARY 5.5 *For any number field $k/\mathbb{Q}$ the entire function*

$$\tilde{C}(s)=2^{r_1/2}\sqrt{n/2}\int_{CH^1(X_k)^0}\frac{\nu(\mathcal{D})}{|\mu(k)|}\Big(\frac{a(\mathcal{D})}{n}\Big)^{-\frac{ns}{2}}d^0\mathcal{D}$$

is α-zeta regularized for every positive α.

1. PROOF (works only for $\alpha=1/2\pi$) According to the above and formula (53) for $w=1$, the function

$$\frac{\hat{\zeta}_k(s)}{f_1(s)}=\frac{2^{r_1/2}}{|\mu(k)|}\frac{n}{2}\pi^{-\frac{ns}{2}}\Gamma\Big(\frac{ns}{2}\Big)\int_{CH^1(X_k)^0}\nu(\mathcal{D})a(\mathcal{D})^{-\frac{ns}{2}}d^0\mathcal{D} \tag{62}$$

is $\frac{1}{2\pi}$-zeta regularized. It follows from formula (61) that we have

$$\pi^{-\frac{ns}{2}}\Gamma\Big(\frac{ns}{2}\Big)=n^{\frac{ns}{2}}\sqrt{2/n}\Big(\coprod_{\nu=0}^{\infty}\frac{1}{2\pi}(s+\frac{2\nu}{n})\Big)^{-1}.$$

Up to a $\frac{1}{2\pi}$-zeta regularized function the term $\pi^{-\frac{ns}{2}}\Gamma(ns/2)$ in formula (62) can therefore be replaced by $n^{\frac{ns}{2}}\sqrt{2/n}$. This gives the assertion. □

2. PROOF It follows from remark 4.4 that for any $\varphi\in(0,\pi/2)$ we have

$$(s/2\pi)^{r/2}\tilde{C}(s)=1+O(s^{-1})\quad\text{in }\overline{\mathcal{W}r_{\varphi,\varphi}}\text{ as }\operatorname{Re}s\to\infty.$$

By theorem 5.1 this function is therefore even α-zeta regularized for every $\alpha>0$. □

Let us check the corollary for $k=\mathbb{Q}$ and k imaginary quadratic. For $k=\mathbb{Q}$ the function equals 1 which is regularized. For k imaginary quadratic the function reduces to the integral, which in this case is a finite Dirichlet series over ideal classes. Because of $\nu(0)=|\mu(k)|$ and $a(0)=n$ this Dirichlet series starts with a constant term 1. Now [I2] Corollary 8.1, resp. its proof shows that such a finite Dirichlet series is α-zeta regularized for any $\alpha>0$.

6 The cohomological motivation

In this section we explain how theorem 5.2 fits into the speculative cohomological setting of [D3]. For every number field $k/\mathbb{Q}$ there should exist complex topological cohomology spaces $H^i(X_k, \mathcal{C})$ together with an $\mathbb{R}$-action Φ^t. The infinitesimal generator Θ of this $\mathbb{R}$-action should exist. We expect that

$$H^0(X_k, \mathcal{C}) = \mathbb{C} \quad \text{with } \Theta = 0$$

and

$$H^2(X_k, \mathcal{C}) \xrightarrow{\sim} \mathbb{C} \quad \text{with } \Theta = \mathrm{id}\,.$$

The space $H^1(X_k, \mathcal{C})$ should be infinite dimensional and decompose in a suitable sense into the eigenspaces of Θ, the eigenvalues being the zeroes of $\hat{\zeta}_k(s)$. In degrees greater than two the cohomologies should vanish.

The zeta-regularized determinant $\det_\infty(\varphi)$ of a diagonalizable operator φ is defined as the zeta-regularized product of its eigenvalues with their multiplicities. See [D2] for more precise definitions. The relation between $\hat{\zeta}_k(s)$ and cohomology is expected to be:

$$\hat{\zeta}_k(s) = \prod_{i=0}^{2} \det_\infty\Big(\frac{1}{2\pi}(s \cdot \mathrm{id} - \Theta) \,|\, H^i(X_k, \mathcal{C})\Big)^{(-1)^{i+1}} . \tag{63}$$

From this and the above it follows that we would have

$$\begin{aligned} L(H^1(X_k), s) \quad &:= \quad \frac{s}{2\pi}\frac{s-1}{2\pi}\hat{\zeta}_k(s) \\ &= \quad \det_\infty\Big(\frac{1}{2\pi}(s \cdot \mathrm{id} - \Theta) \,|\, H^1(X_k, \mathcal{C})\Big) . \end{aligned} \tag{64}$$

Formulas (63) and (64) would imply in particular that $\hat{\zeta}_k(s)$ and $L(H^1(X_k), s)$ are $\frac{1}{2\pi}$-zeta regularized and this turned out to be true, [D1] § 4, [SchS], [JL], [I1].

How to incorporate the two-variable zeta function into this picture? One natural idea is to assume that there is an operator Θ_w on $H^\bullet(X_k, \mathcal{C})$ for every $w \in \mathbb{C}$ deforming $\Theta_1 = \Theta$ and such that the two variable zeta-function equals

$$\prod_{i=0}^{2} \det_\infty\Big(\frac{1}{2\pi}(s \cdot \mathrm{id} - \Theta_w) \,|\, H^i(X_k, \mathcal{C})\Big)^{(-1)^{i+1}} . \tag{65}$$

The equation (32)

$$\hat{\zeta}_k(s) = \frac{2^{r_1/2}}{|\mu(k)|} d_k^{-s/2} \zeta_{X_k}(s, 1)$$

and formula (63) suggest that the function

$$\frac{2^{r_1/2}}{|\mu(k)|} d_k^{-s/2} \zeta_{X_k}(s, w) \tag{66}$$

might be equal to (65). However the function (66) is identically zero for $w = 0$ and this is incompatible with (65). Namely the zeroes of (65) come from factors of the form $\frac{1}{2\pi}(s-\lambda)$ where $\lambda \in \operatorname{spec}(\Theta_w)$ if the zeta-regularized products exist in the sense recalled in section 5. The easiest modification of (66) which takes this point into account is to consider instead of (66) the function

$$w^{-1}\frac{2^{r_1/2}}{|\mu(k)|}d_k^{-s/2}\zeta_{X_k}(s,w)$$

i.e. $\zeta(X_k, s, w)$. Thus the following equation is suggested

$$\zeta(X_k,s,w) = \prod_{i=0}^{2} \det_{\infty}\Big(\frac{1}{2\pi}(s\cdot \mathrm{id} - \Theta_w)\,|\, H^i(X_k,\mathcal{C})\Big)^{(-1)^{i+1}} . \tag{67}$$

It would imply that $\zeta(X_k, s, w)$ is $\frac{1}{2\pi}$-zeta regularized. This was proved in theorem 5.2.

The poles of $s \mapsto \zeta(X_k, s, w)$ lie at $s = 0$ and $s = w$. For $w \neq 0$ they have order one. For $w = 0$ there is a double pole at $s = 0$. According to (67) the poles of $\zeta(X_k, s, w)$ are accounted for by the eigenvalues of Θ_w on $H^0(X_k, \mathcal{C})$ and $H^2(X_k, \mathcal{C})$. On $H^0(X_k, \mathcal{C}) = \mathbb{C}$ it is natural to expect $\Theta_w = 0$ for all w. It follows that on $H^2(X_k, \mathcal{C}) \cong \mathbb{C}$ we *must* have $\Theta_w = w \cdot \mathrm{id}$. Then (67) implies the formula

$$\frac{s}{2\pi}\frac{s-w}{2\pi}\zeta(X_k,s,w) = \det_{\infty}\Big(\frac{1}{2\pi}(s\cdot \mathrm{id} - \Theta_w)\,|\, H^1(X_k,\mathcal{C})\Big) .$$

This is the reason why we denoted the left hand side by $L(H^1(X_k), s, w)$ in definition 3.4.

Having explained the motivation behind theorem 5.2 let us discuss the speculative formula (67) a little further. The functional equation (39) for $\zeta(X_k, s, w)$ says in particular that $\rho \mapsto w-\rho$ is an involution on the set of zeroes resp. poles of $s \mapsto \zeta(X_k, s, w)$. Under (67) this is compatible with the expected Poincaré duality

$$\cup : H^i(X_k,\mathcal{C}) \times H^{2-i}(X_k,\mathcal{C}) \longrightarrow H^2(X_k,\mathcal{C}) \cong \mathbb{C}$$

if we assume that Θ_w is a derivation with respect to $\cup$-product. It looks like Θ_w was the infinitesimal generator of an $\mathbb{R}$-action Φ_w^t on cohomology which respects cup product. It could be interesting to check whether there is a symplectic structure in the distribution of the low lying zeroes of $s \mapsto L(H^1(k), s, w)$ as in the work of Katz and Sarnak [S].

In contrast to Θ the operators Θ_w for $w < 0$ will not commute with the Hodge $*$-operator as in [D3] § 3 since this would force the zeroes of $\zeta(X_k, s, w)$ to lie on the line $\mathrm{Re}\, s = \frac{w}{2}$ which is not the case for $w < 0$ by the investigations of Lagarias and Rains, [LR] § 7.

From calculations in the function field case, I do not expect the operators Θ_w for different w to commute. One possibility seems to be that $[\Theta_{w_1}, \Theta_{w_2}] = (w_1 - w_2)\mathrm{id}$.

References

[D1] C. Deninger, Local L-factors of motives and regularized determinants. Invent. Math. 107 (1992), 135–150

[D2] C. Deninger, Motivic L-functions and regularized determinants. Proc. Symp. Pure Math. 55, 1 (1994), 707–743

[D3] C. Deninger, Some analogies between number theory and dynamical systems on foliated spaces. Doc. Math. J. DMV. Extra Volume ICM I (1998), 23–46

[F] P. Francini, The size function h^0 for quadratic number fields. Journal de Théorie des Nombres de Bordaux 13 (2001), 125–135

[GS] G. van der Geer and R. Schoof, Effectivity of Arakelov divisors and the theta divisor of a number field. Selecta Math., New Series 6 (2000), 377–398

[I1] G. Illies, Regularisierte Produkte, Spurformeln und Cramér-Funktionen. Schriftenreihe des Math. Inst. Univ. Münster, 3. series, Heft 22 (1998)

[I2] G. Illies, Regularized products and determinants. Commun. Math. Phys. 220 (2001), 69–94

[JL] J. Jorgenson, S. Lang, Basic analysis of regularized series and products, LNM 1564, Springer 1994

[LR] J.C. Lagarias, E. Rains, On a two-variable zeta function for number fields. ArXiv: math.NT/0104176v5, 7 July 2002

[N] N. Naumann, On the irreducibility of the two variable zeta-function for curves over finite fields. ArXiv:math.AG/0209092, 2002

[P] R. Pellikaan, On special divisors and the two variable zeta function of algebraic curves over finite fields. In: R. Pellikaan, M. Perret, S.G. Vladut, eds.: Arithmetic, Geometry and Coding theory, Walter de Gruyter, Berlin 1996, 175–184

[S] P. Sarnak, L-functions. Doc. Math. J. DMV. Extra volume ICM I (1998), 453–465

[SchS] M. Schröter, C. Soulé, On a result of Deninger concerning Riemann's zeta function. Proc. Symp. Pure Math. 55, 1 (1994), 745–747

Christopher Deninger
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster
Germany
deninger@math.uni-muenster.de

Analysis on Arithmetic Schemes. I

Ivan Fesenko

Received: October 10, 2002
Revised: March 30, 2003

Dedicated to Kazuya Kato

Abstract. A shift invariant measure on a two dimensional local field, taking values in formal power series over reals, is introduced and discussed. Relevant elements of analysis, including analytic duality, are developed. As a two dimensional local generalization of the works of Tate and Iwasawa a local zeta integral on the topological Milnor K_2^t-group of the field is introduced and its properties are studied.

2000 Mathematics Subject Classification: 28B99, 28C99, 28E05, 28-99, 42-99, 11S99, 11M99, 11F99, 11-99, 19-99.

The functional equation of the twisted zeta function of algebraic number fields was first proved by E. Hecke (for a recent exposition see [N, Ch. VII]). J. Tate [T] and, for unramified characters without local theory, K. Iwasawa [I1–I2] lifted the zeta function to a zeta integral defined on an adelic space. Their method of proving the functional equation and deriving finiteness of several number theoretical objects is a generalization of one of the proofs of the functional equation by B. Riemann. The latter in the case of rational numbers uses an appropriate theta formula derived from a summation formula which itself follows from properties of Fourier transform. Hence the functional equation of the zeta integral reflects symmetries of the Fourier transform on the adelic object and its quotients, and the right mixture of their multiplicative and additive structures. With slight modification the approach of Tate and Iwasawa for characteristic zero can be extended to a uniform treatment of any characteristic, e.g. [W2]. Earlier, the functional field case was treated similarly to Hecke's method by E. Witt in 1936 (cf. [Rq, sect. 7.4]), and it was also established using essentially harmonic analysis on finite rings by H.L. Schmid and O. Teichmüller [ST] in 1943 (for a modern exposition see e.g. [M,3.5]); those works were not widely known.

Two components are important for a two dimensional generalization of the works of Iwasawa and Tate (both locally and globally). The first component is appropriate objects from the right type of higher class field theory. In the local case these are so called topological Milnor K^t-groups ([P2–P4], [F1–F3], [IHLF]), for the local class field theory see [Ka1], [P3–P4], [F1–F3], [IHLF]. The second component is an appropriate theory of measure, integration and generalized Fourier transform on objects associated to arithmetic schemes of higher dimensions. In the local case such objects are a higher local field and its K^t-group. It is crucial that the additive and multiplicative groups of the field are not locally compact in dimension > 1. The bare minimum of the theory for two dimensional local fields is described in part 1 of this work. Unlike the dimension one case, this theory on the additive group of the field is not enough to immediately define a local zeta integral: for the zeta integral one uses the topological K_2^t-group of the field whose closed subgroups correspond (via class field theory) to abelian extensions of the field. A local zeta integral on them with its properties is introduced and discussed in part 2.

We concentrate on the dimension two case, but as usual, the two dimensional case should lead relatively straightforward to the general case.

We briefly sketch the contents of each part, see also their introductions. Of course, throughout the text we use ideas and constructions of the one dimensional theory, whose knowledge is assumed.

Two dimensional local fields are self dual objects in appropriate sense, c.f. section 3. In part 1, in the absence of integration theory on non locally compact abelian groups and harmonic analysis on them, we define a new shift invariant measure

$$\mu\colon \mathcal{A} \longrightarrow \mathbb{R}\,((X))$$

on appropriate ring $\mathcal{A}$ of subsets of a two dimensional local field. This measure is not countably additive, but very close to such. The variable X can be viewed as an infinitesimally small positive element of $\mathbb{R}\,((X))$ with respect to the standard two dimensional topology, see section 1. The main motivation for this measure comes from nonstandard mathematics, but the exposition in this work does not use the latter. Elements of measure theory and integration theory are developed in sections 4–7 to the extent required for this work. We have basically all analogues of the one dimensional theory. Section 9 presents a generalization of the Fourier transform and a proof of a double transform formula (or transform inverse formula) for functions in a certain space. We comment on the case of formal power series over archimedean local fields in section 11.

In part 2, using a covering of a so called topological Milnor K_2^t-group of a two dimensional local field (which is the central object in explicit local class field theory) by the product of the group of units with itself we introduce integrals over the K_2^t-group. In this part we use three maps

$$\mathfrak{o}\colon T = \mathcal{O}^\times \times \mathcal{O}^\times \longrightarrow F \times F, \qquad \mathfrak{r}, \mathfrak{t}\colon T \longrightarrow K_2^t(F),$$

which in some sense generalize the one dimensional map $E \setminus \{0\} \longrightarrow E$ on the module, additive and multiplicative level structures respectively. In section 17 we define the main object – a local zeta integral $\zeta(f,\chi)$ associated to a function $f\colon F\times F \longrightarrow \mathbb{C}$ continuous on T and quasi-character $\chi\colon K_2^t(F) \longrightarrow \mathbb{C}^\times$

$$\zeta(f,\chi) = \int_T f(\alpha)\,\chi_{\mathfrak{t}}(\alpha)\,|\mathfrak{t}(\alpha)|_2^{-2}\,d\mu_T(\alpha),$$

for the notations see section 17. The definition of the zeta integral is the result of several trials which included global tests. Its several first properties in analogy with the one dimensional case are discussed and proved. A local functional equation for appropriate class of functions is proved in section 19. In the ramified case one meets new difficulties in comparison to the dimension one case, they are discussed in section 21. Zeta integrals for formal power series fields over archimedean local fields are introduced in section 23, their values are not really new in comparison to the dimension one case; this agrees with the well known fact that the class field theory for such fields degenerate and does not really use Milnor K_2.

For an extension of this work to adelic and K-delic theory on arithmetic schemes and applications to zeta integral and zeta functions of arithmetic schemes see [F4].

One of stimuli for this work was a suggestion by K. Kato in 1995 to collaborate on a two dimensional version of the work of Iwasawa and Tate, another was a talk of A.N. Parshin at the Münster conference in 1999, see [P6]. I am grateful to A.N. Parshin, K. Kato, T. Chinburg, J. Tate, M. Kurihara, P. Deligne, S.V. Vostokov, J.H. Coates, R. Hill, I.B. Zhukov, V. Snaith, S. Bloch for useful remarks on this part of the project or encouragement. This work was partially supported by the American Institute of Mathematics and EPSRC.

1. Measure and integration on higher local fields

We start with discussion of self duality of higher local fields which distinguishes them among other higher dimensional local objects. Then we introduce a nontrivial shift invariant measure on higher dimensional local fields, which may be viewed as a higher dimensional generalization of Haar measure on one dimensional fields. We define and study properties of integrals of a certain class of functions on the field. Self duality leads to a higher dimensional transform on appropriate space of functions. One of the key properties, a double transform formula for a class of functions in a space which generalizes familiar one dimensional spaces, is proved in section 9.

For various results about higher local fields see sections of [IHLF].

1. In higher dimensions sequential aspects of the topology are fundamental: for example, in dimensions greater than one the multiplication is not always continuous but is sequentially continuous. Certainly, in the one dimensional case one does not see the difference between the two.

In a two dimensional local field $\mathbb{R}((X))$ a sequence of series $\left(\sum a_{i,n}X^i\right)_n$ is a fundamental sequence if there is i_0 such that for all n coefficients $a_{i,n}$ are zero for all $i < i_0$ and for every i the sequence $(a_{i,n})_n$ is a fundamental sequence; similarly one defines convergence of sequences. Every fundamental sequence converges. Now consider on $\mathbb{R}((X))$ the so called sequential saturation topology (sequential topology for short): a set is open if and only if every sequence which converges to any element of it has almost all its elements in the set.

The sequence $(aX)^n$, $n \in \mathbb{N}$, tends to zero for every $a \in \mathbb{R}$. Hence, if a norm on $\mathbb{R}((X))$ takes values in an extension of $\mathbb{R}$ and is the usual module on the coefficients, then one deduces that $|aX| < 1$ and so $|X|$ is smaller than any positive real number. In other words, for two dimensional fields the local parameter X plays the role of an infinitesimally small element, and therefore it is very natural to use hyperconstructions of nonstandard mathematics. Nevertheless, to simplify the reading the following text does not contain anything nonstandard.

2. Let F be a two dimensional local field with local parameters t_1, t_2 (t_2 is a local parameter with respect to the discrete valuation of rank 1) with finite last residue field having q elements. Denote its ring of integers by O and the group of units by U; denote by R the set of multiplicative representatives of the last finite residue field. The multiplicative group $F^\times$ is the product of infinite cyclic groups generated by t_2 and t_1 and the group of units U.

Denote by $\mathcal{O}$ the ring of integers of F with respect to the discrete valuation of rank one (so t_2 generates the maximal ideal $\mathcal{M}$ of $\mathcal{O}$). There is a projection $p: \mathcal{O} \longrightarrow \mathcal{O}/\mathcal{M} = E$ where E is the (first) residue field of F.

Fractional ideals of F are of two types: principal $t_2^i t_1^j O$ and nonprincipal $\mathcal{M}^i$.

For the definition of the topology on F see [sect. 1 part I of IHLF]. We shall view F endowed with the sequential saturation topology (see [sect. 6 part I of IHLF]), so sequentially continuous maps from F are the same as continuous maps. If the reader prefers to use the former topology on F, then the word "continuous" should everywhere below be replaced with "sequentially continuous".

3. For a field $L((t))$ denote by $\operatorname{res}_i = \operatorname{res}_{t^i}: L((t)) \longrightarrow L$ the linear map $\sum a_j t^j \to a_i$. Similarly define $\operatorname{res} = \operatorname{res}_{-1}: L\{\{t\}\} \longrightarrow L$ in the case where L is a complete discrete valuation field of characteristic zero (for the definition of $L\{\{t\}\}$ see [Z2]).

For a one dimensional local field L denote by ψ_L a character of the additive group L with conductor O_L. Introduce

$$\psi' = \psi_L \circ \operatorname{res}_0: L((t)) \longrightarrow \mathbb{C}^\times, \quad \psi' = \psi_L \circ (\pi_L^{-1} \operatorname{res}): L\{\{t\}\} \longrightarrow \mathbb{C}^\times,$$

where π_L is a prime element of L. The conductor of ψ' is the ring O of the corresponding field: i.e. $\psi'(O) = 1 \neq \psi'(t_1^{-1}O)$. An arbitrary two dimensional local field F of mixed characteristic has a finite extension of type $L\{\{t\}\}$ which is the compositum of F and a finite extension of $\mathbb{Q}_p$ [sect. 3 of Z2], so the restriction of ψ' on F has conductor O_F.

Thus, in each case we have a character

$$\psi_0 : F \longrightarrow \mathbb{C}^\times$$

with conductor O_F.

The additive group F is self-dual:

LEMMA. *The group of continuous characters on F consists of characters of the form $\alpha \mapsto \psi(a\alpha)$ where a runs through all elements of F.*

This assertion is easy, at least for the usual topology on F in the positive characteristic case where F is dual to appropriately topologized $\Omega^2_{F/\mathbb{F}_q}$ (c.f. [P1], [Y]) which itself is noncanonically dual to F. Since we are working with a stronger, sequential saturation topology, we indicate a short proof of duality.

Proof. Given a continuous character ψ', there are i, j such that $\psi'(t_2^i t_1^j O) = 1$, and so we may assume that the conductor of ψ' is O. In the equal characteristic case the restriction of ψ' on $\mathcal{O}$ induces a continuous character on E which by the one dimensional theory is a "shift" of ψ_E, hence there is $a_0 \in \mathcal{O}$ such that $\psi_1(\alpha) = \psi'(\alpha) - \psi(a_0\alpha)$ is trivial on $\mathcal{O}$. Similarly by induction, there is $a_i \in t_2^i\mathcal{O}$ such that $\psi_{i+1}(\alpha) = \psi_i(\alpha) - \psi(a_i\alpha)$ is trivial on $t_2^{-i}\mathcal{O}$. Then $\psi'(\alpha) = \psi(a\alpha)$ with $a = \sum_0^{+\infty} a_i$.

In the mixed characteristic case it suffices to consider the case of $\mathbb{Q}_p\{\{t\}\}$. The restriction of ψ' on $t^i\mathbb{Q}_p$ is of the form $\alpha \mapsto \psi(a_i\alpha)$ with $a_i \in t^{-i-1}p\mathbb{Z}_p$, and $a_i \to 0$ when $i \to +\infty$; hence $\psi'(\alpha) = \psi(a\alpha)$ for $a = \sum_{-\infty}^{+\infty} a_i$.

REMARK. Equip characters of F with the topology of uniform convergence on compact subsets (with respect to the sequential topology) of F; an example of such a set in the equal characteristic case is $\{\sum_{i \geqslant i_0} a_i t_2^i : a_i \in W_i\}$ where W_i are compact subsets of E. It is easy to verify that the map $a \mapsto (\alpha \mapsto \psi(a\alpha))$ is a homeomorphism between F and its continuous characters.

4. To introduce a measure on F we first specify a nice class of measurable sets.

DEFINITION. A distinguished subset is of the form $\alpha + t_2^i t_1^j O$. Denote by $\mathcal{A}$ the minimal ring in the sense of [H] containing all distinguished sets.

It is easy to see that if the intersection of two distinguished sets is nonempty, then it equals to one of them. This implies that an element A of $\mathcal{A}$ can be written as $\bigcup_i A_i \setminus (\bigcup_j B_j)$ with distinguished disjoint sets A_i and distinguished disjoint sets B_j such that $\bigcup B_j \subset \bigcup A_i$ (moreover, one can even arrange that each B_j is a subset of some A_i). One easily checks that if A is also of the similar form $\bigcup_l C_l \setminus (\bigcup_k D_k)$ then $A = \bigcup(A_i \bigcap C_l) \setminus \big(\bigcup(A_i \bigcap B_j \bigcap C_l \bigcap D_k)\big)$.

Every element of $\mathcal{A}$ is a disjoint (maybe infinite countable) union of some distinguished subsets. Distinguished sets are closed but not open. For example, sets $O \setminus t_2\mathcal{O} = \bigcup_{j\geqslant 0} t_1^j U$ and $\mathcal{O}^\times = \bigcup_{j\in\mathbb{Z}} t_1^j U$ do not belong to $\mathcal{A}$.

Alternatively, $\mathcal{A}$ is the minimal ring which contains sets $\alpha + t_2^i p^{-1}(S)$, $i \in \mathbb{Z}$, where S is a compact open subset of E.

DEFINITION–LEMMA. There is a unique measure μ on F with values in $\mathbb{R}((X))$ which is a shift invariant finitely additive measure on $\mathcal{A}$ such that $\mu(\emptyset) = 0$,

$$\mu(t_2^i t_1^j O) = q^{-j} X^i.$$

The proof immediately follows from the properties of the distinguished sets. The measure μ depends on the choice of t_2 but not on the choice of t_1.

We get $\mu(t_2^i p^{-1}(S)) = X^i \mu_E(S)$ where μ_E is the normalized Haar measure on E such that $\mu_E(O_E) = 1$.

REMARKS.

1. This measure can be viewed as induced (in appropriate sense) from a measure which takes values in hyperreals ${}^*\mathbb{R}$. A field of hyperreal numbers ${}^*\mathbb{R}$ introduced by Robinson [Ro1] (for a modern exposition see e.g. [Go1]) is a minimal field extension of $\mathbb{R}$ which contains infinitesimally small elements and on which one has all reasonable analogues of analytic constructions. If one fixes a positive infinitesimal $\omega^{-1} \in {}^*\mathbb{R}$, then a surjective homomorphism from the fraction field of approachable polynomials $\mathbb{R}[X]^{ap}$ to $\mathbb{R}((X))$, and $\omega^{-1} \mapsto X$ determines an isomorphism of a subquotient of ${}^*\mathbb{R}$ onto $\mathbb{R}((X))$.

The first variant of this work employed hyperobjects, and its conceptual value ought to be emphasized.

2. The measure μ takes values in $\mathbb{R}((X))$ (for its topology see section 1) which has the total ordering: $\sum_{n\geqslant n_0} a_n X^n > 0$, $a_{n_0} \in \mathbb{R}^\times$, iff $a_{n_0} > 0$. Notice two unusual properties: $a_n = 1 - q^{-n}$ is smaller than $1 - X$, but the limit of (a_n) is 1; not every subset bounded from below has an infimum. Thus, the standard concepts in real valued (or Banach spaces valued) measure theory, e.g. the outer measure, do not seem to be useful here. In particular, the integral to be defined below will possess some unusual properties like those in section 8.

3. The set $O \in \mathcal{A}$ of measure 1 is the disjoint union of $t_2 O \in \mathcal{A}$ of measure X, $t_2 t_1^{-j} O \setminus t_2 t_1^{-j+1} O$ of measure $q^j(1-q^{-1})X$ for $j > 0$, and $t_1^l O \setminus t_1^{l+1} O$ of measure $q^{-l}(1-q^{-1})$ for $l \geqslant 0$. Since $\sum_{j>0} q^j$ diverges, the measure μ is not σ-additive. It is σ-additive for those sets of countably many disjoint sets A_n in the algebra $\mathcal{A}$ which "don't accumulate at break points" from one horizontal line $t_2^m O$ to $t_2^{m+1} O$, i.e. for which the series $\mu(A_n)$ absolutely converges (see section 6).

5. For $A \in \mathcal{A}$, $\alpha \in F^\times$ one has $\alpha A \in \mathcal{A}$ and $\mu(\alpha A) = |\alpha|\mu(A)$, where $|\ |$ is a two dimensional module: $|0| = 0$, $\quad |t_2^i t_1^j u| = q^{-j} X^i \quad$ for $u \in U$. The module is a generalization of the usual module on locally compact fields. For example, every $\alpha \in F$ can be written as a convergent series $\sum \alpha_{i,j}$ with $\alpha_{i,j} \in t_2^i t_1^j O$ and $|\alpha_{i,j}| \to 0$. This simplifies convergence conditions in previous use, e.g. [Z2].

6. We introduce a space R_F of complex valued functions on F and their integrals against the measure μ.

Definition. Call a series $\sum c_n$, $c_n \in \mathbb{C}((X))$, absolutely convergent if it converges and if $\sum |\operatorname{res}_{X^i}(c_n)|$ converges for every i.

For an absolutely convergent series $\sum c_n$ and every subsequence n_m the series $\sum c_{n_m}$ absolutely converges and the limit does not depend on the terms order.

Lemma. *Suppose that a function $f\colon F \longrightarrow \mathbb{C}$ can be written as $\sum c_n \, char_{A_n}$ with countably many disjoint distinguished A_n, $c_n \in \mathbb{C}$, where $char_C$ is the characteristic function of C, and suppose that the series $\sum c_n \mu(A_n)$ absolutely converges. If f has a second presentation of the same type $\sum d_m \, char_{B_m}$, then $\sum c_n \mu(A_n) = \sum d_m \mu(B_m)$.*

Proof. Notice that if $\bigcup A_n = \bigcup B_m$ for distinguished disjoint sets, then for every n either A_n equals to the union of some of B_m, or the union of A_n and possibly several other A's equals to one of B_m. It remains to use the following property: if a distinguished set C is the disjoint union of distinguished sets C_n and $\sum \mu(C_n)$ absolutely converges, then for every $a \in F$ and integer i condition $C \supset a + t_2^i \mathcal{O}$ implies $C_n \supset a + t_2^i \mathcal{O}$ for all n; therefore $\mu(C) = \sum \mu(C_n)$.

Definition. Define R_F as the vector space generated by functions f as in the previous lemma and by functions which are zero outside finitely many points. For an f as in the previous lemma define its integral

$$\int f d\mu = \sum c_n \mu(A_n),$$

and for f which are zero outside finitely many points define its integral as zero.

The previous definition implies that the integral is an additive function.

Example. Let a function f satisfy the following conditions:

there is i_0 such that $f(\beta) = 0$ for all $\beta \in t_2^i t_1^j U$, all $i \neq i_0$;

there is $k(j)$ such that for every $\alpha \in t_2^{i_0} t_1^j U$ $f(\alpha) = f(\alpha + \beta)$ for all $\beta \in t_2^{i_0} t_1^{k(j)} O$.

For every $\theta \in R^\times$ write the set $\theta t_2^{i_0} t_1^j + t_2^{i_0} t_1^{j+1} O$ as a disjoint union of finitely many $c_{\theta,j,l} + t_2^{i_0} t_1^{k(j)} O$, $l \in L_{\theta,j}$; then

$$\int f d\mu = \Big(\sum_{j \in \mathbb{Z}} \sum_{\theta \in R^\times} \sum_{l \in L_{\theta,j}} f(c_{\theta,j,l}) \, q^{-k(j)} \Big) X^{i_0}.$$

If the series in the brackets absolutely converges, then $f \in R_F$.
Redenote f as f_{i_0}. Then $\sum_{i\geqslant i_1} f_i$ also belongs to R_F.

For another example see section 8.

7. Some important for harmonic analysis functions do not belong to R_F, for example, the function $\alpha \mapsto \psi(a\alpha)\, char_A(\alpha)$ for, say, $A = \mathcal{O} \in \mathcal{A}$, $a \notin \mathcal{O}$. In the one dimensional case all such functions do belong to the analogue of R_F.

Recall that in the one dimensional case the integral over an open compact subgroup of a nontrivial character is zero. This leads to the following natural

Definition. Denote $\psi(C) = 0$ is ψ takes more than one value on a distinguished set C and $=$ the value of ψ if it is constant on C. For a distinguished set A and $a \in F^\times$ define

$$\int \psi(a\alpha)\, char_A(\alpha)\, d\mu(\alpha) = \mu(A)\,\psi(aA).$$

So, if $A \subset \mathcal{O}$ and is the preimage of a shift of a compact open subgroup of the residue field E with respect to p, and if $a \in \mathcal{O} \setminus t_2\mathcal{O}$, then the previous definition agrees with the property of a character on E, mentioned above.

Lemma. *For a function $f = \sum c_n\, \psi(a_n\alpha)\, char_{A_n}(\alpha)$ with finite set $\{a_n\}$ and with countably many disjoint distinguished A_n such that the series $\sum c_n\mu(A_n)$ absolutely converges the sum $\sum c_n \int \psi(a_n\alpha)\, char_{A_n}(\alpha)\, d\mu(\alpha)$ does not depend on the choice of c_n, a_n, A_n.*

Proof. To show correctness, one can reduce to sets on which $|f|$ is constant, then use a simple fact that if a distinguished set C is the disjoint union of distinguished sets C_n, and $\psi(a\alpha)\, char_C(\alpha) = \sum d_n\psi(b_n\alpha)\, char_{C_n}(\alpha)$ with $|d_n| = 1$, absolutely convergent series $\sum d_n\, \mu(C_n)$ and finitely many distinct b_n, then $\psi(aC) = \sum d_n\, \psi(b_nC_n)\, \mu(C_n)$.

Definition. Put

$$\int f\, d\mu = \sum c_n \int \psi(a_n\alpha)\, char_{A_n}(\alpha)\, d\mu(\alpha).$$

Denote by R'_F the space generated by functions $f(\alpha)\,\psi(a\alpha)$ with $f \in R_F$, $a \in F$, and functions which are zero outside a point; all of them are integrable. This space will be enough for the purposes of this work.

Remark. Here is a slightly different approach to the extension of R_F: Suppose that a function $f\colon F \to \mathbb{C}$ is zero outside a distinguished subgroup A of F. Suppose that there are finitely many $a_1, \dots, a_m \in A$ such that the function $g(x) = \sum_i f(a_i + x)$ belongs to R_F. Then *define*

$$\int f = \frac{1}{m} \int g\, d\mu.$$

First of all, this is well defined: if $h(x) = \sum_j f(b_j + x)$ belongs to R_F for $b_1, \dots, b_n \in A$, then $\sum_{i=1}^m h(a_i+x) = \sum_{j=1}^n g(b_j+x)$, and from $h(a_i+x) \in R_F$ and shift invariant property, and the similar property for shifts of g one gets $m \int h\, d\mu = n \int g\, d\mu$.

Second, for two functions $f_1, f_2 \colon A \to \mathbb{C}$ if $\sum_i f_1(a_i+x), \sum_j f_2(b_j+x) \in R_F$, then $\sum_{i,j} f_l(a_i + b_j + x) \in R_F$ for $l = 1, 2$ and so $\int f_1 + f_2 = \int f_1 + \int f_2$. Denote by R''_F the space of all such f.

Third, this definition is compatible with all the properties of R_F in the previous section: if f already belongs to R_F then $\int f = \int f\, d\mu$. Also, $\int f(x) = \int f(a+x)$ for $a \in A$.

Finally, this definition is compatible with the preceding definitions of this section: of course, for nontrivial characters on a distinguished subgroup one has $g = 0$. The space R'_F is a subspace of R''_F: for a function $f = \sum c_n\, \psi(a\alpha)\, char_{A_n}(\alpha)$, such that $\psi(aA_n) = 0$, i.e. $A_n + a^{-1}t_1^{-1}O = A_n$, for every n, choose a_i in $a^{-1}t_1^{-1}O$.

For $f \in R'_F$ we have

$$\int f(\alpha + a)\, d\mu(\alpha) = \int f(\alpha)\, d\mu(\alpha)$$

and using section 5

$$\int f(\alpha)\, d\mu(\alpha) = |a| \int f(a\alpha)\, d\mu(\alpha).$$

For a subset S of F put $\int_S f d\mu = \int f\, char_S d\mu$.

Example. We have $\int_{t_2^i t_1^j O} \psi(a\alpha)\, d\mu(\alpha) = q^{-j} X^i$ if $a \in t_2^{-i} t_1^{-j} O$ and $= 0$ otherwise (since then $\psi(a\alpha)$ is a nontrivial character on $t_2^i t_1^j O$). Hence

$$\int_{t_2^i t_1^j U} \psi(a\alpha)\, d\mu(\alpha) = \begin{cases} 0 & \text{if } a \notin t_2^{-i} t_1^{-j-1} O, \\ -q^{-1-j} X^i & \text{if } a \in t_2^{-i} t_1^{-j-1} U, \\ q^{-j}(1 - q^{-1}) X^i & \text{if } a \in t_2^{-i} t_1^{-j} O. \end{cases}$$

8. Example. If two functions $f, h \colon \mathcal{O} \longrightarrow \mathbb{C}$ are constant on $t_2\mathcal{O} \setminus \{0\}$ and their restriction to $\mathcal{O} \setminus t_2\mathcal{O}$ coincide, then

$$\int_{\mathcal{O}} f\, d\mu = \int_{\mathcal{O}} h\, d\mu.$$

Indeed, if $(f - h)|_{t_2\mathcal{O}\setminus\{0\}} = c$, then

$$\int_{\mathcal{O}} (f-h) d\mu = \int_{O} (f-h) d\mu = \int_{O} c d\mu - \int_{O \setminus t_2\mathcal{O}} c d\mu = c - c\,(1-q^{-1}) \sum_{j \geqslant 0} q^{-j} = 0.$$

From the previous we deduce $\int_{t_2\mathcal{O}} c\, d\mu = 0$, and therefore, similarly,

$$\int_F c\, char_{\mathcal{O}}\, d\mu = \int_{\mathcal{O}} c\, d\mu = 0, \quad \int_{\mathcal{O}\setminus t_2\mathcal{O}} c\, d\mu = 0.$$

Of course, the sets $\mathcal{O}, \mathcal{O} \setminus t_2\mathcal{O}$ are not in $\mathcal{A}$.

REMARK. One has $\int_{Z_l} d\mu = \sum_{-l\leqslant j\leqslant l} q^{-j}$ where $Z_l = \bigcup_{-l\leqslant j\leqslant l} t_1^j U$, and $\mathcal{O}\setminus t_2\mathcal{O}$ is the "limit" of Z_l when $l \to \infty$. Compare with "equality" $\sum_{n\in\mathbb{Z}} z^n = 0$ used by L. Euler. One of interpretations of it is to view z as a complex variable, then for every integer m the sum of analytic continuations of two rational functions $\sum_{n\in\mathbb{Z},n<m} z^n$ and $\sum_{n\in\mathbb{Z},n\geqslant m} z^n$ is zero. Euler's equality can be applied to show equivalence of the Riemann–Roch theorem and the functional equation of the zeta function of a one dimensional global field of positive characteristic (cf. [Rq, sect. 4.3.3]).

From the definitions we immediately get

LEMMA. *Suppose that $g\colon E \longrightarrow \mathbb{C}$ is integrable over E with respect to the normalized Haar measure μ_E as in section 4. Then the function $g\circ p$ extended by zero outside $\mathcal{O}$ is in R_F and*

$$\int_{\mathcal{O}} g\circ p\, d\mu = \int_E g\, d\mu_E.$$

One can say that the Haar measure μ_E equals $p_*(\mu')$ where the measure μ' on $\mathcal{O}$ is induced by μ by extending functions on $\mathcal{O}$ by zero to F.

9. DEFINITION. For $f \in R_F$ introduce the transform function

$$\mathcal{F}(f)(\beta) = \int_F f(\alpha)\,\psi(\alpha\beta)\, d\mu(\alpha).$$

In particular, $\mathcal{F}(char_O) = char_O$.

DEFINITION. Denote by Q_F the subspace of R_F consisting of functions f with support in $\mathcal{O}$ and such that $f|_{\mathcal{O}} = g\circ p|_{\mathcal{O}}$ for a function $g\colon E \longrightarrow \mathbb{C}$ which belongs to the Bruhat space generated by characteristic functions of shifts of proper fractional ideals of E.

THEOREM. *Given $f \in Q_F$, the function $\mathcal{F}(f)$ belongs to Q_F and we have a double transform formula*

$$\mathcal{F}^2(f)(\alpha) = f(-\alpha).$$

Proof. First note that $\psi(\alpha) = \psi_E \circ p(\alpha)$ for all $\alpha \in \mathcal{O}$ where ψ_E is an appropriate character on E with conductor O_E.

Using sections 7 and 8 we shall verify that

if $\beta \notin \mathcal{O}$ then $\mathcal{F}(f)(\beta) = 0$;

if $\beta \in \mathcal{O}$ then $\mathcal{F}(f)(\beta) = \mathcal{F}(g)\circ p\,(\beta)$ (where $\mathcal{F}(g)$ denotes the Fourier transform of g with respect to ψ_E and μ_E).

For $\beta \notin \mathcal{O}$ the definitions imply

$$\mathcal{F}(f)(\beta) = \int_{\mathcal{O}} f(\alpha)\,\psi(\alpha\beta)\, d\mu(\alpha) = 0.$$

For $\beta \in \mathcal{O} \setminus \mathcal{O}^\times$

$$\mathcal{F}(f)(\beta) = \int_{\mathcal{O}} f(\alpha)\, d\mu(\alpha) = \int_{\mathcal{O}} g \circ p(\alpha)\, d\mu(\alpha)$$
$$= \int_E g(\overline{\alpha})\, d\mu_E(\overline{\alpha}) = \mathcal{F}(g)(0) = \mathcal{F}(f)(0).$$

For $\beta \in \mathcal{O}^\times$

$$\mathcal{F}(f)(\beta) = \int_F f(\alpha)\, \psi(\alpha\beta)\, d\mu(\alpha) = \int_{\mathcal{O}} f(\alpha)\, \psi(\alpha\beta)\, d\mu(\alpha)$$
$$= \int_{\mathcal{O}} g \circ p(\alpha)\, \psi_E \circ p(\alpha\beta)\, d\mu(\alpha) = \int_E g(\overline{\alpha})\, \psi_E(\overline{\alpha}\overline{\beta})\, d\mu_E(\overline{\alpha}) = \mathcal{F}(g)(p(\beta)).$$

It remains to use the one dimensional double transform formula.

As a corollary, we deduce that if a function $f\colon F \longrightarrow \mathbb{C}$ with support in $\mathcal{O}^\times$ coincides on $\mathcal{O}^\times$ with a function $h \in Q_F$, then

$$\mathcal{F}^2(f)(\alpha) = \begin{cases} f(-\alpha) & \text{if } \alpha \in \mathcal{O}^\times \bigcup (F \setminus \mathcal{O}), \\ h(0) & \text{if } \alpha \in \mathcal{O} \setminus \mathcal{O}^\times. \end{cases}$$

Remark. It is more natural to integrate $\mathbb{C}((X))$-valued functions on F: for a function $f = \sum_{i \geqslant i_0} f_i X^i \colon F \longrightarrow \mathbb{C}((X))$, with $f_i \in R'_F$ define

$$\int_F f d\mu = \sum X^i \int_F f_i d\mu.$$

Similarly to the previous text one checks the correctness of the definition and properties of the integral. In particular, the previous theorem remains true for functions $f = \sum_{i \geqslant i_0} f_i X^i$, f_i in the space $Q'_F = \{\alpha \mapsto \sum_{n \geqslant n_0} g_n(t_2^{-n}\alpha)\}$ where g_n belong to Q_F (or even just "lifts" of functions on E for which the double transform formula holds).

10. Introduce the product measure $\mu_{F\times F} = \mu_F \otimes \mu_F$ on $F \times F$ and define spaces $R_{F\times F} = R_F \otimes R_F$, $R'_{F\times F} = R'_F \otimes R'_F$ and $Q_{F\times F} = Q_F \otimes Q_F$.

For a function $f \in Q_{F\times F}$ define its transform

$$\mathcal{F}(f)(\beta_1, \beta_2) = \int_{F\times F} f(\alpha_1, \alpha_2)\, \psi(\alpha_1\beta_1 + \alpha_2\beta_2)\, d\mu(\alpha_1)\, d\mu(\alpha_2).$$

From section 9 we deduce $\mathcal{F}^2(f)(\alpha_1, \alpha_2) = f(-\alpha_1, -\alpha_2)$ for $f \in Q_{F\times F}$.

11. In the case of two dimensional local fields of type $F = K((t))$, where K is an archimedean local field, O, U are not defined. Define a character $\psi\colon K((t)) \longrightarrow \mathbb{C}^\times$ to be the composite of the $\mathrm{res}_0\colon K((t)) \longrightarrow K$ and the archimedean character $\psi_K(\alpha) = \exp(2\pi i \operatorname{Tr}_{K/\mathbb{R}}(\alpha))$ on K. The role of distinguished sets is played by $A = a + t^iD + t^{i+1}K[[t]]$ where D is an open ball in K. In this case if the intersection of two distinguished sets is nonempty then it equals either to one of them, or to a smaller distinguished set. The measure is the shift invariant additive measure μ on the ring generated by distinguished sets and such that $\mu(A) = \mu_K(D)X^i$. It can be extended to $\mu(a + t^iC + t^{i+1}K[[t]]) = \mu_K(C)X^i$, where $C \subset K$ is a Lebesgue measurable set.

The module is $|\sum_{i\geqslant i_0} a_it^i| = |a_{i_0}|_K X^{i_0}$, where the module on the real K is the absolute value and on the complex K is the square of the absolute value.

The fields of this type are not used in the global theory, but we briefly sketch the analogues of the previous constructions.

The definitions of spaces R_F and R'_F follow the general pattern of sections 6 and 7: for disjoint distinguished sets A_n such that $\sum c_n\,\mu(A_n)$ absolutely converges put

$$\int \sum c_n\, char_{A_n}\, d\mu = \sum c_n\, \mu(A_n).$$

For a function $f = \sum c_n\, \psi(a_n\alpha)\, char_{A_n}(\alpha)$ with finite set $\{a_n\}$ and with countably many disjoint distinguished A_n such that the series $\sum c_n\,\mu(A_n)$ absolutely converges put

$$\int f\, d\mu = \sum c_n \int_{A_n} \psi(a_n\alpha)\, d\mu(\alpha),$$

where

$$\int_A \psi(c\alpha)\, d\mu(\alpha) = \psi(ca)\, X^i \int_D \psi_K(c_{-i}\beta)\, d\mu_K(\beta), \quad c_{-i} = \mathrm{res}_{t^{-i}}\, c,$$

A, a, D, i are defined at the beginning of this section.

The analogues of the definitions and results of sections 8 and 9 hold. The transform of a function $f \in R_F$ is defined by the same formula

$$\mathcal{F}(f)(\beta) = \int_F f(\alpha)\, \psi(\alpha\beta)\, d\mu(\alpha).$$

The space Q_F consists of functions $f \in R_F$ such that $f = g \circ p$ for a function g in the Schwartz space on K, where $p = \mathrm{res}_{t^0}$. The double transform formula is $\mathcal{F}^2(f)(\alpha) = f(-\alpha)$ for $f \in Q_F$.

Remark. It is interesting to investigate if further extensions of this sort of measure theory and harmonic analysis (they seem to be more powerful than applications of Wiener's measure) would be useful for mathematical description of Feynman integrals and integration on loop spaces.

12. In general, for the global theory, we need to work with normalized measures corresponding to shifted characters. Similar to the one dimensional case, if we start with a character ψ with conductor aO then the dual to μ measure μ' on F is normalized such that $|a|\mu(O)\mu'(O) = 1$. The double transform formula of section 9 holds.

For example, if $\mu' = \mu$ then $\mu(O) = |a|^{-1/2}$. If $a\mathcal{O} = t_2^i\mathcal{O}$ then in the last formula of section 8 one should insert the coefficient $X^{-i/2}$ on the right hand side. If $f_0(\alpha) = f(a\alpha)$ belongs to Q_F or Q'_F then $\mathcal{F}(f) = |a|^{1/2}\,\mathcal{F}_0(f_0)$ where $\mathcal{F}_0$ is the transform with respect to character $\psi_0(\alpha) = \psi(a\alpha)$ of conductor O and measure μ_0 such that $\mu_0(O) = 1$. In particular, $\mathcal{F}(char_O) = |a|^{-1/2}char_{aO}$.

13. Generally, given an integral domain A with principal ideal $P = tA$ and projection $A \longrightarrow A/P = B$ and a shift invariant measure on B, similar to the previous one defines a measure and integration on A. For example, the analogue of the ring $\mathcal{A}$ is the minimal ring which contains sets $\alpha + t^i p^{-1}(S)$, where S is from a class of measurable subsets of B; the space of integrable functions is generated as a vector space by functions $\alpha \to g \circ p(t^{-i}\alpha)$ extended by zero outside t^iA, where g is an integrable function on B. Such a measure and integration is natural to call lifts of the corresponding measure and integration on the base B.

In particular, if $B = \mathbb{A}_k$ for a global field k of positive characteristic, then one has a lift μ_1 of the measure $\mu_{\mathbb{A}_k}$, a lift μ_2 of the counting measure of k (so $\mu_2(a + tA) = 1$), and a lift μ_3 of the measure on $\mathbb{A}_k/k$. If the measures are normalized such that $\mu_{\mathbb{A}_k} = \mu_k \otimes \mu_{\mathbb{A}_k/k}$ (of course, μ_k is a counting measure), then $\mu_1 = \mu_2 \otimes \mu_3$.

REMARKS.

1. T. Satoh [S], A.N. Parshin [P6], and M. Kapranov [Kp] suggested two other very different approaches to define a measure on two dimensional fields (notice that Satoh's work was an attempt to use Wiener's measure). The work [P6] suggested to use an ind-pro description for a generalization of Haar measure to two dimensional local fields, see also [Kp]. The works [P6] and [Kp] don't introduce a measure, and deal with distributions and functions; local zeta integrals are not discussed.

After this part had been written, the author was informed by A.N. Parshin that the formula $\mu(t_2^i t_1^j O) = q^{-j}X^i$ was briefly discussed in a short message [P5], however, it led to unresolved at that stage paradoxes.

2. In this work we do not need a general theory of analysis on non locally compact abelian groups, since the case of higher dimensional local fields and natural adelic objects composed of them is enough for applications in number theory. Undoubtedly, there is a more general theory of harmonic analysis on non locally compact groups than presented above, see also the next remark.

3. The existence of Haar measure on locally compact abelian groups can be proved in the shortest and most elegant conceptual way by viewing it as

induced from the counting measure on a covering hyperfinite abelian group (e.g. [Gor1]); the same is also true for the Fourier transform [Gor2].

Conceptually, the integration theory of this work is supposed to be induced by integration on hyper locally compact abelian groups (or hyper hyper finite groups). This would also give an extension of this theory to a more general class of non locally compact groups than those of this part.

2. DIMENSION TWO LOCAL ZETA INTEGRAL

Using the theory of the previous part we explain how to integrate over topological K-groups of higher local fields, define a local zeta integral and discuss its properties and new phenomena of the dimension two situation. Using a covering of a so called topological Milnor K_2^t-group of a two dimensional local field (which is the central object in explicit local class field theory) by the product of the group of units with itself we introduce integrals over the K_2^t-group. For this we use additional maps $\mathfrak{r}, \mathfrak{t}, \mathfrak{o}$ whose meaning can only be finally clarified in constructions of the global part [F4]. In section 17 we define the main object – a higher dimensional local zeta integral associated to a function on the field and a continuous character of the K_2^t-group, followed by concrete examples. Several first properties of the zeta integral, in analogy with the one dimensional case, are discussed and proved. In the case of formal power series over archimedean local fields zeta integrals introduced in section 23 are not really new in comparison to one dimensional integrals; this agrees with the fact that the class field theory for such fields does not really require K_2^t.

14. On $F^\times$ one has the induced from $F \oplus F$ topology with respect to

$$F^\times \longrightarrow F \oplus F, \quad \alpha \mapsto (\alpha, \alpha^{-1}),$$

and the shift-invariant measure

$$\mu_{F^\times} = (1 - q^{-1})^{-1} \, \mu / |\ |.$$

Explicit higher class field theory describes abelian extensions of F via closed subgroups in the topological Milnor K-group $K_2^t(F) = K_2(F)/\Lambda_2(F)$ where $\Lambda_2(F)$ is the intersection of all open neighborhoods of zero in the strongest topology on $K_2(F)$ in which the subtraction in $K_2(F)$ and the map

$$F^\times \times F^\times \longrightarrow K_2(F), \quad (a, b) \mapsto \{a, b\}$$

are sequentially continuous. For more details see [IHLF, sect. 6 part I] and [F3]. Denote by $UK_2^t(F)$ the subgroup generated by symbols $\{u, \alpha\}$, $u \in U, \alpha \in F^\times$.

15. Introduce a module on $K_2^t(F)$:

$$|\ |_2\colon K_2^t(F) \longrightarrow \mathbb{R}_+^\times, \quad \alpha \mapsto q^{-v(\alpha)},$$

where $v\colon K_2^t(F) \longrightarrow K_0(\mathbb{F}_q) = \mathbb{Z}$ is the composite of two boundary homomorphisms.

Definition. Introduce a subgroup

$$T = \mathcal{O}^\times \times \mathcal{O}^\times = \{(t_1^j u_1, t_1^l u_2) : j, l \in \mathbb{Z}, u_i \in U\}$$

of $F^\times \times F^\times$. The closure of T in $F \times F$ equals $\mathcal{O} \times \mathcal{O}$.

A specific feature of the two dimensional theory is that one needs to use auxiliary maps $\mathfrak{o}, \mathfrak{o}'$ to modify the integral in such a way that later on, in the adelic work in [F4] one gets the right factors for transformed functions and their zeta integrals.

Definition. Introduce a map (morphism of multiplicative structures)

$$\mathfrak{o}'\colon T \longrightarrow F \times F, \quad (t_1^j u_1, t_1^l u_2) \mapsto (t_1^{2j} u_1, t_1^{2l} u_2)$$

and denote by $\mathfrak{o}$ the bijection $\mathfrak{o}'(T) \longrightarrow T$.

For a complex valued continuous function f whose domain includes T and is a subset of $F \times F$ form $f \circ \mathfrak{o}\colon \mathfrak{o}'(T) \longrightarrow \mathbb{C}$, then extend it by continuity to the closure of $\mathfrak{o}'(T)$ in $F \times F$ and by zero outside the closure, denote the result by $f_\mathfrak{o}\colon F \times F \to \mathbb{C}$.

To be able to apply the transform $\mathcal{F}$ introduce an extension $\mathrm{e}(g)$ of a function $g = f_\mathfrak{o}\colon F \times F \longrightarrow \mathbb{C}$ as the continuous extension on $F \times F$ of

$$\mathrm{e}(g)(\alpha_1, \alpha_2) = g(\alpha_1, \alpha_2) + \sum_{1 \leqslant i \leqslant 3} g((\alpha_1, \alpha_2)\,\nu_i), \quad (\alpha_1, \alpha_2) \in F^\times \times F^\times,$$

where $\nu_1 = (t_1^{-1}, t_1^{-1})$, $\nu_2 = (t_1^{-1}, 1)$, $\nu_3 = (1, t_1^{-1})$.

For example, if $f = char_{(t_1^j O, t_1^l O)}$ then $f_\mathfrak{o}(\alpha_1, \alpha_2) = 1$ for $(\alpha_1, \alpha_2) \in F^\times \times F^\times$ only if $\alpha_1 \in t_1^{2k} U, \alpha_2 \in t_1^{2m} U$ for $k \geqslant j, m \geqslant l$, hence $f_\mathfrak{o}$ is not a continuous function; but $\mathrm{e}(f_\mathfrak{o}) = char_{(t_1^{2j} O, t_1^{2l} O)}$ is.

Definition. For a function $f\colon T \longrightarrow \mathbb{C}$ such that $\mathrm{e}(f_\mathfrak{o}) \in R_{F \times F}$ define its transform

$$\widehat{f} = \mathcal{F}(\mathrm{e}(f_\mathfrak{o})) \circ \mathfrak{o}'\colon T \longrightarrow \mathbb{C}.$$

16. Due to the well known useful equality

$$\{1-\alpha, 1-\beta\} = \{\alpha, 1+\alpha\beta/(1-\alpha)\} + \{1-\beta, 1+\alpha\beta/(1-\alpha)\},$$

and a topological argument we have surjective maps ([IHLF, sect. 6 part I]):

$$\mathfrak{r}\colon T \longrightarrow K_2^t(F), \quad \mathfrak{r}\big((t_1^j, t_1^l)(u_1,u_2)\big) = \min(j,l)\,\{t_1,t_2\} + \{t_1,u_1\} + \{u_2,t_2\},$$
$$\mathfrak{t}\colon T \longrightarrow K_2^t(F), \quad \mathfrak{t}\big((t_1^j, t_1^l)(u_1,u_2)\big) = (j+l)\,\{t_1,t_2\} + \{t_1,u_1\} + \{u_2,t_2\}.$$

For $(\alpha_1,\alpha_2)\in T$ we have

$$|\mathfrak{t}(\alpha_1,\alpha_2)|_2 = |\alpha_1|\,|\alpha_2| = |(\alpha_1,\alpha_2)|.$$

The homomorphism $\mathfrak{t}$ plays in important role in the study of topological Milnor K-groups of higher local fields ([P3], [F3]). If we denote $H = \{(\alpha,\beta) : \alpha,\beta \in \langle t_1\rangle U, \alpha\beta^{-1}\in U\}$, then $\mathfrak{r}(H) = K_2^t(F)$. Note that if $K_2^t(F)$ is replaced by the usual $K_2(F)$ then none of the previous maps is surjective. The topology of $K_2^t(F)$ is the quotient topology of the sequential saturation of the multiplicative topology on H (cf. [F3], [IHLF, sect. 6 part I]).

The maps $\mathfrak{r},\mathfrak{t}$ depend on the choice of t_1,t_2 but the induced map to $K_2^t(F)/\ker(\partial)$ (see section 18 for the definition) does not depend on the choice of t_2. The homomorphism $\mathfrak{t}$ (which depends on the choice of t_1,t_2) is much more convenient to use for integration on $K_2^t(F)$ than the canonically defined map $F^\times\times F^\times \longrightarrow K_2^t(F)$.

For a function $h\colon K_2^t(F)\longrightarrow \mathbb{C}$ denote by $h_\mathfrak{r}\colon F\times F\longrightarrow\mathbb{C}$ ($h_\mathfrak{t}\colon F\times F\longrightarrow\mathbb{C}$) the composite $h\circ\mathfrak{r}$ (resp. the composite $h\circ\mathfrak{t}$) extended by zero outside T.

17. Let $\chi\colon K_2^t(F)\longrightarrow\mathbb{C}^\times$ be a continuous homomorphism. Similarly to the one dimensional case one easily proves that $\chi = \chi_0|\ |_2^s$ ($s\in\mathbb{C}$), where χ_0 is a lift of a character of $UK_2^t(F)$ such that $\chi_0(\{t_1,t_2\}) = 1$. The group $V = 1+t_1O$ of principal units has the property: every open subgroup contains V^{p^n} for sufficiently large n, c.f. [Z1, Lemma 1.6]. This property and the definitions imply that $|\chi_0(K_2^t(F))| = 1$. Put $s = s(\chi)$.

Definition. For a function f such that $char_{\mathfrak{o}'(T)}\, f_\mathfrak{o}\in R_{F\times F}$ introduce

$$\int_T f\,d\mu_T = (1-q^{-1})^{-2}\int_{\mathfrak{o}'(T)} f_\mathfrak{o}\,d\mu_{F\times F}.$$

Notice that $\int_T f\,d\mu_T = (1-q^{-1})^{-2}\int_{F\times F} char_T\, f\,|\ |\,d\mu_{F\times F}$.

For a function $f\colon F\times F\longrightarrow\mathbb{C}$ continuous on T and a continuous quasi-character $\chi\colon K_2^t(F)\longrightarrow\mathbb{C}^\times$ introduce a zeta function

$$\zeta(f,\chi) = \int_T f(\alpha)\,\chi_\mathfrak{t}(\alpha)\,|\mathfrak{t}(\alpha)|_2^{-2}\,d\mu_T(\alpha)$$

as an element of $\mathbb{C}((X))$ if the integral converges.

From the definitions we obtain $\zeta(f,\chi)$

$$= (1-q^{-1})^{-2} \sum_{j,l\in\mathbb{Z}} (q^{-s})^{j+l} \int_{U\times U} f(t_1^j u_1, t_1^l u_2)\, \chi_0\big(\mathfrak{t}(u_1,u_2)\big)\, d\mu(u_1)\, d\mu(u_2).$$

If for fixed j,l the value $f(t_1^j u_1, t_1^l u_2)$ is constant $= f_0(j,l)$, then we get

$$\zeta(f,\chi) = (1-q^{-1})^{-2} \sum_{j,l\in\mathbb{Z}} (q^{-s})^{j+l}\, f_0(j,l) \int_{U\times U} \chi_0\big(\mathfrak{t}(u_1,u_2)\big)\, d\mu(u_1)\, d\mu(u_2).$$

Examples.

1. Let $f = char_{OK_2^t(F)_{\mathfrak{r}}}$ where $OK_2^t(F) = v^{-1}(\{0,1,\dots\})$. So $f_0(j,l)=1$ if $j,l\geqslant 0$ and $f_0(j,l)=0$ otherwise. Then

$$\zeta(f, |\ |_2^s) = \left(\frac{1}{1-q^{-s}}\right)^2$$

which converges for $\mathrm{Re}(s)>0$, and has a single valued meromorphic continuation to the whole complex place. Note that for this specific choice of f the local zeta integral *does not involve* X.

2. More generally, for a function $f\in Q_{F\times F}$ the local zeta integral $\zeta(f,\chi)$ converges for $\mathrm{Re}(s)>0$ to a rational function in q^s and therefore has a meromorphic continuation to the whole complex plane.

To show this, one can assume that $f(\alpha_1,\alpha_2) = char_{A_1}(\alpha_1) char_{A_2}(\alpha_2)$ with distinguished sets A_1, A_2, each of which is either of type $t_1^l(a+t_1^r O)$, $a\in U$, $r>0$, or of type $t_1^m O$. For example, if A_1 of the first type and A_2 of the second type, then

$$\zeta(f,\chi) = (1-q^{-1})^{-2} \sum_{j\geqslant m} (q^{-s})^{j+l} \int_{(a+t_1^r O)\times U} \chi_0\big(\mathfrak{t}(u_1,u_2)\big)\, d\mu(u_1)\, d\mu(u_2),$$

which for $\mathrm{Re}(s)>0$ converges to $(1-q^{-1})^{-2}(q^{-s})^{m+l}(1-q^{-s})^{-1}\, c$ where $c = \int_{(a+t_1^r O)\times U} \chi_0\big(\mathfrak{t}(u_1,u_2)\big)\, d\mu(u_1)\, d\mu(u_2)$, and therefore equals to this rational function of q^s on the whole plane.

3. For $f\in Q_{F\times F}$ and $(\alpha_1,\alpha_2)\in T$ define, in analogy with A. Weil's definition in [W1]

$$W(f)(\alpha_1,\alpha_2) = f(\alpha_1,\alpha_2) - f(t_1^{-1}\alpha_1,\alpha_2) - f(\alpha_1, t_1^{-1}\alpha_2) + f(t_1^{-1}\alpha_1, t_1^{-1}\alpha_2).$$

Then from the definitions and the properties of μ we get

$$\zeta(W(f), |\ |_2^s) = (1-q^{-s})^2\, \zeta(f, |\ |_2^s).$$

In particular, let $g = W(f)$, $f|_T = \big(char_{OK_2^t(F)}\big)_{\mathfrak{r}}$. Then

$$\zeta(g, |\ |_2^s) = 1.$$

18. Let $\partial\colon K_2(F) \longrightarrow K_1(E)$ be the boundary homomorphism. Note that $\partial(\Lambda_2(F)) = 1$. Define

$$\lambda\colon K_1(E) \longrightarrow K_2^t(F)/\ker(\partial), \quad \alpha \mapsto \{\tilde\alpha, t_2\}$$

where $\tilde\alpha \in \mathcal{O}$ is a lifting of $\alpha \in E^\times$. Then $\partial(\lambda(\alpha)) = \alpha$.

Let $f\colon K_2^t(F) \longrightarrow \mathbb{C}$ be a continuous function which factorizes through the quotient $K_2^t(F)/\ker(\partial)$. Let $\chi\colon K_2^t(F) \longrightarrow \mathbb{C}^\times$ be a weakly ramified continuous quasi-character, i.e. $\chi_0(\ker(\partial)) = 1$, so χ is induced by a (not necessarily unramified) quasi-character of the first residue field E of F. Then using Lemma in section 8 one immediately deduces that

$$\zeta(f_{\mathfrak{r}}, \chi) = \zeta_E(f \circ \lambda, \chi \circ \lambda)^2$$

where ζ_E is the one dimensional zeta integral on E which corresponds to the normalized Haar measure on E.

Similarly, if F is a mixed characteristic field, and K is the algebraic closure of $\mathbb{Q}_p$ in F, then one has a residue map $\mathfrak{d}\colon K_2^t(F) \longrightarrow K_1(K)$, see e.g. [Ka2, sect. 2], the map $\mathfrak{d}$ is $-$res defined there.

Assume that a prime element of K is a t_2-local parameter of F, then we can identify $F = K\{\{t_1\}\}$. Define

$$\mathfrak{l}\colon K_1(K) \longrightarrow K_2^t(F), \quad \alpha \mapsto \{t_1, \alpha\},$$

then $\mathfrak{d}(\mathfrak{l}(\alpha) = \alpha$.

Let $f\colon K_2^t(F) \longrightarrow \mathbb{C}$ and a quasi-character χ have analogous to the above properties but with respect to $\mathfrak{d}$. Then similarly one has

$$\zeta(f_{\mathfrak{r}}, \chi) = \zeta_K(f \circ \mathfrak{l}, \chi \circ \mathfrak{l})^2.$$

Thus, the two dimensional zeta integral for F links zeta integrals for both E and K, local fields of characteristic p and 0.

19. Put $\widehat{\chi} = \chi^{-1}|\ |_2^2$.

Proposition. *Let $g, h \in Q_{F\times F}$ be continuous functions, and let $\chi\colon K_2^t(F) \longrightarrow \mathbb{C}^\times$ be a continuous quasi-character. For $0 < \operatorname{Re} s(\chi) < 2$ (and therefore for all s) one has a local functional equation*

$$\zeta(g, \chi)\,\zeta(\widehat{h}, \widehat{\chi}) = \zeta(\widehat{g}, \widehat{\chi})\,\zeta(h, \chi).$$

Proof. One proof is similar to Example 2 in the previous section and reduces the verification to the case where both g and h are of the simple form $char_{A_1} char_{A_2}$. We indicate another, similar to dimension one, with one new feature – a factor $k(\beta)$ which corresponds to the transform $\widehat{f}$ involving the map $\mathfrak{o}$.

For $\alpha = (\alpha_1, \alpha_2)$ denote $\alpha^{-1} = (\alpha_1^{-1}, \alpha_2^{-1})$, $\alpha\beta = (\alpha_1\beta_1, \alpha_2\beta_2)$, $|\alpha| = |\alpha_1|\,|\alpha_2|$, $\mu(\alpha) = \mu_{F\times F}(\alpha_1, \alpha_2)$, $\psi(\alpha) = \psi(\alpha_1)\psi(\alpha_2)$. Put

$$k(\beta) = \psi(\beta) + q^{-2}\psi(\nu_1^{-1}\beta) + q^{-1}\psi(\nu_2^{-1}\beta) + q^{-1}\psi(\nu_3^{-1}\beta),$$

ν_i defined in section 15. We will use $|\mathfrak{to}(\alpha)|_2^2 = |\alpha|$ for $\alpha \in \mathfrak{o}'(T)$.

We have

$$\begin{aligned}
&\zeta(g,\chi)\,\zeta(\widehat{h},\widehat{\chi}) = \iint_{T\times T} g(\alpha)\,\widehat{h}(\beta)\,\chi_{\mathfrak{t}}(\alpha\beta^{-1})\,|\beta|^2\,|\alpha|^{-2}\,|\beta|^{-2}\,d\mu_T(\alpha)\,d\mu_T(\beta)\\
&= \int_{F\times F}\int_{F\times F} g_{\mathfrak{o}}(\alpha)\,\mathcal{F}(\mathrm{e}(h_{\mathfrak{o}}))(\beta)\,\chi_{\mathfrak{to}}(\alpha\beta^{-1})\,|\alpha|^{-1}\,d\mu(\alpha)\,d\mu(\beta)\\
&= \iiint_{F^6} g_{\mathfrak{o}}(\alpha)\;\mathrm{e}(h_{\mathfrak{o}})(\gamma)\,\psi(\beta\gamma)\,\chi_{\mathfrak{to}}(\alpha\beta^{-1})\,|\alpha|^{-1}\,d\mu(\alpha)\,d\mu(\beta)\,d\mu(\gamma)\\
&= \iiint_{F^6} g_{\mathfrak{o}}(\alpha)\,h_{\mathfrak{o}}(\gamma)\,\psi(\beta\gamma)\,\chi_{\mathfrak{to}}(\alpha\beta^{-1})\,|\alpha|^{-1}\,d\mu(\alpha)\,d\mu(\beta)\,d\mu(\gamma)\\
&+ \sum_{1\leqslant i\leqslant 3}\iiint_{F^6} g_{\mathfrak{o}}(\alpha)\,h_{\mathfrak{o}}(\gamma)\,\psi(\beta\nu_i^{-1}\gamma)\,\chi_{\mathfrak{to}}(\alpha\beta^{-1})\,|\alpha|^{-1}\,d\mu(\alpha)\,d\mu(\beta)\,d\mu(\nu_i^{-1}\gamma)\\
&= \iiint_{F^6} g_{\mathfrak{o}}(\alpha)\,h_{\mathfrak{o}}(\gamma)\,k(\beta\gamma)\,\chi_{\mathfrak{to}}(\alpha\beta^{-1})\,|\alpha|^{-1}\,d\mu(\alpha)\,d\mu(\beta)\,d\mu(\gamma)\\
&\qquad(\beta\to\gamma^{-1}\beta)\\
&= \iiint_{F^6} g_{\mathfrak{o}}(\alpha)\,h_{\mathfrak{o}}(\gamma)\,\chi_{\mathfrak{to}}(\alpha\gamma)\,|\alpha\gamma|^{-1}\,d\mu(\alpha)\,d\mu(\gamma)\,k(\beta)\,\chi_{\mathfrak{to}}(\beta^{-1})\,d\mu(\beta)
\end{aligned}$$

(due to $\mathfrak{to}$ the integrals are actually taken over $\mathfrak{o}'(T)$ where one can apply the Fubini property). The symmetry in α, γ implies the local functional equation.

REMARK. Undoubtedly, one can extend this proposition to a larger class of continuous functions $g, h\colon F\times F \longrightarrow \mathbb{C}$.

20. Similar to section 12, we need to take care of zeta integrals which involve normalized measures corresponding to shifted characters. The zeta integral should be more correctly written as $\zeta(f, \chi|\ |_2^s, \mu)$ since it depends on the normalization of the measure μ. Above we used the normalized measure μ such that $\mu(O) = 1$ (and the conductor of ψ is O). More generally, let the conductor of a character ψ' be $t_1^d O$, so $\psi'(\alpha) = \psi_0(t_1^{-d}u\alpha)$ for some unit u, ψ_0 was defined in section 3. Put $\psi(\alpha) = \psi_0(t_1^{-2d}u\alpha)$ and use ψ and $\psi(\alpha_1, \alpha_2)$ for the transform $\mathcal{F}$ of functions in Q_F and $Q_{F\times F}$ as in sections 9–10. Then self dual measures μ on F and $F\times F$ with respect to ψ satisfy $\mu(O) = q^d$, $\mu(O,O) = q^{2d}$.

Let $f = char_{c\{t_1,t_2\}+OK_2^t(F)_{\mathfrak{r}}}$, so $f(t_1^j u_1, t_1^l u_2) = 1$ if $j, l \geqslant c$ and $= 0$ otherwise. Now $\mathcal{F}(\mathrm{e}(f_{\mathfrak{o}}))(t_1^j u_1, t_1^l u_2) = q^{2d-4c}$ if $j, l \geqslant 2d - 2c$ and $= 0$ otherwise; and $\widehat{f}(t_1^j u_1, t_1^l u_2) = q^{2d-4c}$ if $j, l \geqslant d - c$ and $= 0$ otherwise. Notice that $\widehat{\widehat{f}} = f$

on T. We get

$$\zeta(f,|\,|_2^s,\mu) = q^{2d-2cs}\left(\frac{1}{1-q^{-s}}\right)^2,$$

$$\zeta(\widehat{f},|\,|_2^{2-s},\mu) = q^{2d}\,q^{2d-4c}\,q^{-2(d-c)(2-s)}\left(\frac{1}{1-q^{s-2}}\right)^2 = q^{2s(d-c)}\left(\frac{1}{1-q^{s-2}}\right)^2,$$

$$\zeta(f,|\,|_2^s,\mu)\,\zeta(\widehat{f},|\,|_2^{2-s},\mu)^{-1} = q^{-2d(s-1)}\left(\frac{1}{1-q^{-s}}\right)^2\left(\frac{1}{1-q^{s-2}}\right)^{-2}.$$

The local constant for $|\,|_2^s,\mu$ is $q^{-2d(s-1)}$. This calculation will be quite useful in [F4].

21. The previous constructions of the maps $\mathfrak{o}',\mathfrak{r},\mathfrak{t},\mathrm{e}$ are suitable for unramified characters, but are not good enough for ramified characters.

Let a weakly ramified character χ of $K_2^t(F)$ be induced by a character $\overline{\chi}$ of the field E with conductor $1+\overline{t_1}^{\,r}O_E$, $r>0$. By analogy with the one dimensional case it seems reasonable to calculate the zeta integral $\zeta(f,\chi)$ for $f\colon T\longrightarrow\mathbb{C}$ which is defined by $(t_1^j u_1, t_1^l u_2)\mapsto 1$ if and only if $j\geqslant 0, l=0, u_2\in 1+t_1^r O$. Using the dimension one calculation in [T] one easily obtains

$$\zeta(f,\chi) = \frac{1}{1-q^{-s}}\,(1-q^{-1})^{-1}\,q^{-r},$$
$$\zeta(\widehat{f},\widehat{\chi}) = \frac{1}{1-q^{s-2}}\,(1-q^{-1})^{-1}\,q^{-(s+1)r/2-\delta}\,\rho_0(\overline{\chi}),$$

where $\rho_0(\overline{\chi})$ is the root number as in [T, 2.5], $\delta=s/2$ if r is odd and $=0$ if r is even.

Notice that unlike the case of unramified characters in section 20, $\widehat{\widehat{f}}$ differs from f on T due to the difference between $\mathrm{e}(\widehat{f}_{\mathfrak{o}})$ and $\mathcal{F}(\mathrm{e}(f_{\mathfrak{o}}))$; so for the current choice of $\mathfrak{o}$ and e as above there is no canonical local constant associated to χ.

This may be related to the well known difficulty of constructing a general higher ramification theory which is still not available. It is expected that one can refine $\mathfrak{o}$, e, $\mathfrak{r},\mathfrak{t}$ to get canonical local constants for ramified characters as well.

22. One can slightly modify the definition of the zeta integral by extending T to the set

$$T' = \{(t_2^m t_1^j u_1, t_2^n t_1^l u_2)\}, \qquad \text{where } j,l\in\mathbb{Z},\, u_i\in U,\, m,n\geqslant 0,\, mn=0.$$

The set T' will be useful in the study of global zeta integrals in [F4].

Introduce a set $\Gamma=\{(t_2^m,1): m\geqslant 0\}\bigcup\{(1,t_2^m): m\geqslant 1\}$, then $T'=T\times\Gamma$. Extend $\mathfrak{r},\mathfrak{t}$ symmetrically to T' by $(m>0)$

$$\mathfrak{r}(t_2^m t_1^j u_1, t_1^l u_2) = l\,\{t_1,t_2\}+\{t_1,u_1\}+\{u_2,t_2\},$$
$$\mathfrak{t}(t_2^m t_1^j u_1, t_1^l u_2) = \{t_1,u_1\}+\{u_2,t_2\}.$$

Extend $\mathfrak{o}'$ to $\mathfrak{o}': T' \longrightarrow F^\times \times F^\times$ by $(t_2^m t_1^j u_1, t_2^n t_1^l u_2) \mapsto (t_2^m t_1^{2j} u_1, t_2^n t_1^{2l} u_2)$ and denote by $\mathfrak{o}$ the inverse bijection $\mathfrak{o}'(T') \longrightarrow T'$.

For a function $f: T' \longrightarrow \mathbb{C}$ define $f_\mathfrak{o}: F \times F \longrightarrow \mathbb{C}$ as the composite $f \circ \mathfrak{o}$ extended by zero outside $\mathfrak{o}'(T')$. Introduce

$$\int_{T'} f \, d\mu_{T'} = (1 - q^{-1})^{-2} \int_{F\times F} f_\mathfrak{o} \, d\mu_{F\times F}.$$

Let a function $f: F \times F \longrightarrow \mathbb{C}$ be continuous on T' and let

$$f(t_2^m \alpha_1, \alpha_2) = f(\alpha_1, t_2^m \alpha_2) = f(\alpha_1, \alpha_2)$$

for every $(\alpha_1, \alpha_2) \in T$, $m > 0$. Then for a continuous quasi-character $\chi: K_2^t(F) \longrightarrow \mathbb{C}^\times$ such that $\chi(UK_2^t(F)) = 1$ we have

$$\int_{T'} f(\alpha)\, \chi_\mathfrak{t}(\alpha)\, |\mathfrak{t}(\alpha)|_2^{-2} \, d\mu_{T'}(\alpha) = \zeta(f, \chi).$$

This follows from the following observation: the function $f(\alpha)\,\chi_\mathfrak{t}(\alpha)|\mathfrak{t}(\alpha)|_2^{-2}$ on the set $(t_2^m t_1^j u_1, t_1^l u_2)$, $m > 0$, depends on $t_1^l u_2$ only, and therefore one can write the integral $\int_{T'} f(\alpha)\,\chi_\mathfrak{t}(\alpha)|\mathfrak{t}(\alpha)|_2^{-2}\, d\mu_{T'}(\alpha)$ as the sum of $\zeta(f,\chi)$ and the sum of the product of two integrals one of which is the integral of a constant function over $t_2^m \mathcal{O} \setminus t_2^{m+1}\mathcal{O}$, and hence is zero by Example 8.

23. We describe elements of the theory for $K_2^t(K((t)))$ where K is an archimedean local field. Class field theory for such fields is described in [P2] and in [KS]. In this case class field theory does not really match nicely the structure of the K_2^t-group of the field, and this is reflected in the constructions of this section.

Introduce the sequential topology on $L = K((t))$ as in section 1. Define

$$K_2^t(L) = K_2(L)/\Lambda_2(L), \qquad \Lambda_2(L) = \Lambda_2'(L) + \{K^\times, 1 + tK[[t]]\} + \{t, 1 + tK[[t]]\},$$

$\Lambda_2'(L)$ is the intersection of all neighborhoods of zero in the strongest topology on $K_2(L)$ in which the subtraction in $K_2(L)$ and the map

$$L^\times \times L^\times \longrightarrow K_2(L), \quad (a,b) \mapsto \{a, b\}$$

are sequentially continuous. Then $K_2^t(L)$ is generated by $\{a, t\}$, $a \in K^\times$ and $\{-1,-1\}$ (which is zero if $\sqrt{-1} \in K$). We have $\partial(\Lambda_2(L)) = 1$ where ∂ is the boundary map. ∂ induces $K_2^t(L) \longrightarrow K_1(K)$.

Introduce a module on $K_2^t(L)$: $|\alpha|_2 = |\partial(\alpha)|_K$.

A continuous homomorphism $\chi: K_2^t(L) \longrightarrow \mathbb{C}^\times$ can be written as $\chi = \chi_0 | \;|_2^s$ ($s \in \mathbb{C}$), where $\chi_0(\{K^\times, t_2\}) = 1$ and $|\chi_0(K_2^t(L))| = 1$ (so χ_0 is determined by its value on $\{-1,-1\}$; this symbol corresponds via class field theory to the totally ramified extension $K((t^{1/2}))/K((t)))$.

Let

$$\mathfrak{r}: K^\times \longrightarrow K_2^t(L), \quad \alpha \mapsto \{\alpha, t\}$$
$$\mathfrak{t}: T = K^\times \times K^\times \longrightarrow K_2^t(L), \quad (\alpha,\beta) \mapsto \{\alpha\beta, t\}$$

(totally ramified characters with respect to L/K are ignored).

For a function $f: L \times L \longrightarrow \mathbb{C}$ continuous on T and rapidly decaying at $(\pm\infty, \pm\infty)$ and a continuous quasi-character $\chi: K_2^t(L) \longrightarrow \mathbb{C}^\times$ introduce a zeta function

$$\zeta(f,\chi) = \int_{K\times K} f(\alpha,\beta)\, \chi_{\mathfrak{t}}(\alpha,\beta)\, \frac{d\mu(\alpha,\beta)}{|\alpha|_K\, |\beta|_K}.$$

Define $\lambda: K_1(K) \longrightarrow K_2^t(L)/\ker(\partial)$ by $\alpha \mapsto \{\tilde{\alpha}, t\}$. Let $f: K_2^t(L) \longrightarrow \mathbb{C}$ factorize through $K_2^t(L)/\ker(\partial)$. Let $\chi: K_2^t(L) \longrightarrow \mathbb{C}^\times$ be a continuous quasi-character such that $\chi_0(\ker(\partial)) = 1$ (i.e. χ is an unramified character with respect to L/K). Then

$$\zeta(g,\chi) = \zeta_1(f\circ\lambda, \chi\circ\lambda)^2,$$

where $g(\alpha,\beta) = f_{\mathfrak{r}}(\alpha)\, f_{\mathfrak{r}}(\beta)$, and ζ_1 is the one dimensional zeta integral.

REMARK. The well known technique in dimension one which extends the local integrals associated to representations of the multiplicative group to representations of algebraic groups is likely to give a similar extension of this work.

BIBLIOGRAPHY

[F1] I. Fesenko, Class field theory of multidimensional local fields of characteristic 0, with the residue field of positive characteristic, Algebra i Analiz (1991); English transl. in St. Petersburg Math. J. 3(1992), 649–678.

[F2] I. Fesenko, Multidimensional local class field theory. II, Algebra i Analiz 3 issue 5(1991), 168–189; English transl. in St. Petersburg Math. J. 3(1992), 1103–1126.

[F3] I. Fesenko, Sequential topologies and quotients of Milnor K-groups of higher local fields, Algebra i Analiz, vol. 13(2001), issue 3, 198–221; English transl. in St. Petersburg Math. J. 13(2002), 485–501.

[F4] I. Fesenko, Analysis on arithmetic schemes. II, www.maths.nott.ac.uk/personal/ibf/ao2.ps

[Gol] R. Goldblatt, Lectures on the hyperreals, Springer, 1998.

[Gor1] E. Gordon, Nonstandard analysis and locally compact abelian groups, Acta Appl. Math. 25(1991), 221–239.

[Gor2] E. Gordon, Nonstandard methods in commutative harmonic analysis, AMS, Transl. Math. Monogr. 164, 1997.

[H] P.R. Halmos, Measure theory, D. Van Nostrand Comp., 1951.

[I1] K. Iwasawa, A note on functions, Proceed. IMC 1950, p. 322.

[I2] K. Iwasawa, Letter to J. Dieudonne, Adv. Stud. Pure Math., vol. 21, 1992, 445–450.

[IHLF] I. Fesenko and M. Kurihara (eds.) Invitation to higher local fields, Geometry and Topology Monographs, vol. 3, Geometry and Topology Publications, Warwick 2000, www.maths.warwick.ac.uk/gt/gtmcontents3.html

[Ka1] K. Kato, A generalization of local class field theory by using K-groups, I, II, III, J. Fac. Sci. Univ. Tokyo, Sect. 1A, 26(1979), 303–376; 27(1980), 603–683; 29(1982), 31–43.

[Ka2] K. Kato, Residue homomorphism in Milnor K-theory, Adv. Stud. Pure Math., vol. 2, 1983, 153–172.

[KS] K. Kato and Sh. Saito, Two dimensional class field theory, Adv. Stud. Pure Math., vol. 2, 1983, 103–152.

[Kp] M. Kapranov, Semiinfinite symmetric powers, math.QA/0107089.

[M] C. Moreno, Algebraic curves over finite fields, Cambr. Univ. Press 1991.

[N] J. Neukirch, Algebraic number theory, Springer 1999.

[P1] A.N. Parshin, On the arithmetic of two dimensional schemes, 1, Repartitions and residues, Math. USSR Izv., 10(1976), 695–747.

[P2] A.N. Parshin, Abelian coverings of arithmetic schemes, Soviet Math. Dokl., 19(1978), 1438–1442.

[P3] A.N. Parshin, Local class field theory, Trudy Mat. Inst. Steklov 165(1984), 143–170; Engl. transl. in Proc. Steklov Inst. Math. 1985, no.3, 157–185.

[P4] A.N. Parshin, Galois cohomology and the Brauer group of local fields, Trudy Mat. Inst. Steklov, 183(1990), 159–169; Engl. transl. in Proc. Steklov Inst. Math. 1991, no. 4, 191–201.

[P5] A.N. Parshin, E-message to I.B. Zhukov, 1995.

[P6] A.N. Parshin, Higher dimensional local fields and L-functions, 199–213, in [IHLF].

[Ro1] A. Robinson, Non-standard analysis, reprint of the 2nd (1974) edit., with a foreword by W.A. Luxemburg, Princeton Landmarks in Math., Princeton Univ. Press, 1996.

[Ro2] A. Robinson, Nonstandard arithmetic, Bull. Amer. Math. Soc., 73(1967), 818–843.

[Rq] P. Roquette, Class field theory in characteristic p, its origin and development, Class field theory – its centenary and prospect, Adv. Stud. Pure Math. 30, 549–631, Math. Soc. Japan, Tokyo 2001.

[St] T. Satoh, Wiener measures on certain Banach spaces over non-Archimedean local fields, Comp. Math., 93(1994), 81–108.

[ST] H.L. Schmid and O. Teichmüller, Ein neuer Beweis für die Funktionalgleichung der L-Reihen, Abh. Math. Sem. Hansischen Univ. 15(1943), 85–96.

[T] J. Tate, Fourier analysis in number fields and Hecke's zeta function, thesis, Princeton Univ., 1950; reproduced in Algebraic number theory, eds. J. Cassels and A. Fröhlich, Academic Press, 1967.

[W1] A. Weil, Fonctions zeta et distributions, Semin. Bourbaki, 312 (1965/1966); reproduced in Collect. Papers, vol. III, 158–163, Springer 1979.

[W2] A. Weil, Basic number theory, Springer 1967.

[Y] A. Yekutieli, An explicit construction of the Grothendieck residue complex, Asterisque 208, 1992.

[Z1] I.B. Zhukov, Milnor and topological K-groups of multidimensional complete fields, Algebra i Analiz 9(1997), issue 1, 98–146; Engl. transl. in St. Petersburg Math. J. 9(1998), 69–105.

[Z2] I.B. Zhukov, Higher dimensional local fields, 5–18, in [IHLF].

Ivan Fesenko
School of Mathematics
Univ. of Nottingham
Nottingham NG7 2RD
England
ibf@maths.nott.ac.uk

Presque C_p-Représentations

A Kazuya Kato,

à l'occasion de son cinquantième anniversaire

Jean-Marc Fontaine

Received: October 25, 2002
Revised: April 3, 2003

Abstract. Let $\overline{\mathbb{Q}}_p$ be an algebraic closure of $\mathbb{Q}_p$ and C its p-adic completion. Let K be a finite extension of $\mathbb{Q}_p$ contained in $\overline{\mathbb{Q}}_p$ and set $G_K = \mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$. A $\mathbb{Q}_p$-representation (resp. a C-representation) of G_K is a finite dimensional $\mathbb{Q}_p$-vector space (resp. C-vector space) equipped with a linear (resp. semi-linear) continuous action of G_K. A banach representation of G_K is a topological $\mathbb{Q}_p$-vector space, whose topology may be defined by a norm with respect to which it is complete, equipped with a linear and continuous action of G_K. An almost C-representation of G_K is a banach representation X which is almost isomorphic to a C-representation, i.e. such that there exists a C-representation W, finite dimensional sub-$\mathbb{Q}_p$-vector spaces V of X and V' of W stable under G_K and an isomorphism $X/V \to W/V'$. The almost C-representations of G_K form an abelian category $\mathcal{C}(G_K)$. There is a unique additive function $dh : \mathrm{Ob}\mathcal{C}(G_K) \to \mathbb{N} \times \mathbb{Z}$ such that $dh(W) = (\dim_C W, 0)$ if W is a C-representation and $dh(V) = (0, \dim_{\mathbb{Q}_p} V)$ if V is a $\mathbb{Q}_p$-representation. If X and Y are objects of $\mathcal{C}(G_K)$, the $\mathbb{Q}_p$-vector spaces $\mathrm{Ext}^i_{\mathcal{C}(G_K)}(X,Y)$ are finite dimensional and are zero for $i \notin \{0,1,2\}$. One gets $\sum_{i=0}^2 (-1)^i \dim_{\mathbb{Q}_p} \mathrm{Ext}^i_{\mathcal{C}(G_K)}(X,Y) = -[K : \mathbb{Q}_p] h(X) h(Y)$. Moreover, there is a natural duality between $\mathrm{Ext}^i_{\mathcal{C}(G_K)}(X,Y)$ and $\mathrm{Ext}^{2-i}_{\mathcal{C}(G_K)}(Y, X(1))$.

2000 Mathematics Subject Classification: Primary: 11F80, 11S20 11S25. Secondary:11G25, 11S31, 14F30, 14G22.
Mots clés: Corps p-adiques, représentations galoisiennes p-adiques, cohomologie galoisienne, espaces de Banach p-adiques.
Keywords and Phrases: p-adic fields, p-adic Galois representations, Galois cohomology, p-adic Banach spaces.

TABLE

1 – INTRODUCTION

1.1 – PROLOGUE

Cet article est le premier d'une série en préparation consacrée à l'étude de certains phénomènes que l'on rencontre lorsque l'on étudie les représentations p-adiques associées aux motifs des variétés algébriques sur les corps p-adiques. Considérons d'abord la situation suivante : On se donne un complété K d'un corps de nombres, on choisit une clôture algébrique $\overline{K}$ de K, on pose $G_K = \mathrm{Gal}(\overline{K}/K)$ et on note C le complété de $\overline{K}$.
Soient A une variété abélienne sur K de dimension g et t_A son espace tangent. C'est un K-espace vectoriel de dimension g. Il est commode de le voir comme un groupe vectoriel, i.e. de poser $t_A(R) = R \otimes_K t_A$ pour toute K-algèbre R.

Supposons d'abord K archimédien. On a donc $K = \mathbb{R}$ ou $\mathbb{C}$, $\overline{K} = C = \mathbb{C}$, $G_K = \{1, \tau\}$, avec τ la conjugaison complexe ou $G_K = \{1\}$. L'exponentielle est définie partout et on a une suite exacte

$$0 \to H_1(A(\mathbb{C}), \mathbb{Z}) \to t_A(\mathbb{C}) \to A(\mathbb{C}) \to 0$$

Si maintenant K est une extension finie de $\mathbb{Q}_p$, c'est au contraire le logarithme qui est partout défini et on a un diagramme commutatif

$$\begin{array}{ccccccccc} 0 & \to & A_{\mathrm{tor}}(\overline{K}) & \to & A(\overline{K}) & \to & t_A(\overline{K}) & \to & 0 \\ & & \| & & \cap & & \cap & & \\ 0 & \to & A_{\mathrm{tor}}(C) & \to & A(C) & \to & t_A(C) & \to & 0 \end{array}$$

Le groupe $A(\overline{K})$ est muni d'une topologie naturelle : notons $\mathcal{O}_K$ (resp. $\mathcal{O}_{\overline{K}}$) l'anneau des entiers de K (resp. $\overline{K}$) ; si $\mathcal{A}$ est un modèle propre (pas nécessairement lisse) de A sur $\mathcal{O}_K$, on a $A(\overline{K}) = \mathcal{A}(\mathcal{O}_{\overline{K}})$; on prend la topologie la moins fine rendant continues toutes les applications $\mathcal{A}(\mathcal{O}_{\overline{K}}) \to \mathcal{A}(\mathcal{O}_{\overline{K}}/p^n)$, pour

$n \in \mathbb{N}$ (avec la topologie discrète sur $\mathcal{A}(\mathcal{O}_{\overline{K}}/p^n)$). On vérifie que cette topologie est indépendante du choix du modèle, fait de $A(\overline{K})$ un groupe topologique induisant la topologie discrète sur $A_{tor}(\overline{K})$ et la topologie naturelle sur $t_A(\overline{K})$ et que $A(C)$ s'identifie au séparé complété de $A(\overline{K})$ pour cette topologie.
On a $A_{\text{tor}}(\overline{K}) = A_{p-\text{tor}}(\overline{K}) \oplus A_{p'-\text{tor}}(\overline{K})$ où $A_{p-\text{tor}}(\overline{K})$ (resp. $A_{p'-\text{tor}}(\overline{K})$) désigne le sous-groupe de p-torsion (resp. de p'-torsion) de $A(\overline{K})$. L'exponentielle est définie localement, ce qui veut dire qu'il existe un réseau Λ de t_A (i.e. un sous-$\mathcal{O}_K$-module libre de rang g de t_A) et un homomorphisme continu G_K-équivariant $\exp : \mathcal{O}_C \otimes_{\mathcal{O}_K} \Lambda \to A(C)$ vérifiant $\log(\exp(x)) = x$, pour tout $x \in \mathcal{O}_C \otimes_{\mathcal{O}_K} \Lambda$; on a $\exp(\mathcal{O}_{\overline{K}} \otimes_{\mathcal{O}_K} \Lambda) \subset A(\overline{K})$. Ceci permet en particulier de définir une section s de l'inclusion de $A_{p'-\text{tor}}(\overline{K})$ dans $A(C)$: si $x \in A(C)$, $p^n \log(x) \in \mathcal{O}_C \otimes_{\mathcal{O}_K} \Lambda$, pour n assez grand et $s(x) = p^{-n}\pi(x^{p^n} / \exp(p^n \log(x)))$, où $\pi : A_{\text{tor}}(\overline{K}) \to A_{p'-\text{tor}}(\overline{K})$ est la projection canonique (on a noté multiplicativement $A(C)$ et additivement $A_{p'-\text{tor}}(\overline{K})$). Si $A^{(p)}(C)$ désigne le noyau de s, on a $A(C) = A_{p'-\text{tor}}(\overline{K}) \oplus A^{(p)}(C)$ et $A(\overline{K}) = A_{p'-\text{tor}}(\overline{K}) \oplus A^{(p)}(\overline{K})$, en posant $A^{(p)}(\overline{K}) = A(\overline{K}) \cap A^{(p)}(C)$. Cette décomposition est compatible avec la topologie et $A^{(p)}(C)$ est le complété de $A^{(p)}(\overline{K})$ pour la topologie induite.

Le premier fait remarquable est que l'*on peut retrouver $A^{(p)}(\overline{K})$ (et donc aussi $A^{(p)}(C)$) en tant que groupe topologique muni d'une action de G_K à partir de la seule connaissance de $A_{p-\text{tor}}(\overline{K})$* (et donc également $A(\overline{K})$ et $A(C)$ à partir de $A_{\text{tor}}(\overline{K})$):
Notons $U_C^+ = 1 + m_{\mathcal{O}_C}$ le groupe des unités de l'anneau des entiers $\mathcal{O}_C$ de C qui sont congrues à 1 modulo l'idéal maximal. Le logarithme p-adique définit une suite exacte

$$0 \to (\mathbb{Q}_p/\mathbb{Z}_p)(1) \to U_C^+ \to C \to 0$$

Le module de Tate $T_p(A)$, limite projective des $A_{p^n}(\overline{K})$, pour $n \in \mathbb{N}$ est un $\mathbb{Z}_p$-module libre de rang $2g$ et en tensorisant la suite exacte ci-dessus avec $T_p(A)(-1)$, on obtient une autre suite exacte

$$0 \to A_{p-\text{tor}}(\overline{K}) \to U_C^+(-1) \otimes_{\mathbb{Z}_p} T_p(A) \to C(-1) \otimes_{\mathbb{Z}_p} T_p(A) \to 0$$

Proposition 1.1 (cf. §8.4). — *Soit A une variété abélienne sur K. Il existe une et une seule application additive continue G_K-équivariante*

$$\xi : A^{(p)}(\overline{K}) \to U_C^+(-1) \otimes_{\mathbb{Z}_p} T_p(A)$$

induisant l'identité sur $A_{p-\text{tor}}(\overline{K})$. Cette application est injective et identifie $A^{(p)}(\overline{K})$ au plus grand sous-groupe de $U_C^+(-1) \otimes_{\mathbb{Z}_p} T_p(A)$ sur lequel l'action de G_K est discrète. L'application $\bar{\xi} : t_A(\overline{K}) \to C(-1) \otimes T_p(A)$, déduite de ξ par passage au quotient, est $\overline{K}$-linéaire.

On remarque en outre que $\bar{\xi}$ induit une application injective ξ_1 de t_A sur $(C(-1) \otimes (T_p(A))^{G_K}$. L'application analogue pour la variété abélienne duale A' fournit par dualité une application surjective $\xi_0 : (C \otimes_{\mathbb{Z}_p} T_p(A))^{G_K} \to t^*_{A'}$

(où $t^*_{A'}$ est le K-espace vectoriel dual de l'espace tangent de A'). Grâce à un théorème de Tate ([Se67a], prop.4), pour toute représentation p-adique V, on a $\dim_K(C(-1)\otimes_{\mathbb{Q}_p} V)^{G_K} + \dim_K(C\otimes_{\mathbb{Q}_p} V)^{G_K} \le \dim_{\mathbb{Q}_p} V$. Pour des raisons de dimension, ξ_0 et ξ_1 sont donc des isomorphismes. On retrouve ainsi la décomposition de Hodge-Tate pour les variétés abéliennes.

En fait, on a plus que cela : $A^{(p)}$ a une structure naturelle de groupe rigide analytique. On peut retrouver cette structure à partir de $A_{p-\mathrm{tor}}(\overline{K})$: U_C^+ a une structure naturelle de groupe rigide analytique sur C : c'est le groupe multiplicatif des éléments inversibles de l'anneau sous-jacent au disque unité fermé. D'où une structure analytique sur $U_C^+(-1)\otimes T_p(A) \simeq (U_C^+)^{2g}$. Et $A^{(p)}(C)$ s'identifie à un sous-groupe fermé.

Au groupe $A_{p-\mathrm{tor}}(\overline{K})$ muni de l'action de G_K, on peut donc associer deux objets intéressants
(A) *le groupe topologique* $A^{(p)}(C)$ *muni de son action de* G_K,
(B) *le groupe analytique rigide* $A^{(p)}$ [Attention, on ne peut pas obtenir trop : ce n'est pas la variété analytique rigide A^{rig} associée à A ; dans le cas de bonne réduction par exemple, c'est seulement la fibre générique du schéma en groupes formel (pas nécessairement connexe) $\widehat{\mathcal{A}}$ associé au groupe de Barsotti-Tate $(\mathcal{A}_{p^n})_{n\in\mathbb{N}}$ où $\mathcal{A}$ est le modèle de Néron de A].
Lorsque A a bonne réduction, on peut encore associer un troisième objet, qui est
(C) un *faisceau de groupes abéliens pour la topologie* plate (pour les généralisations, ce sera mieux de considérer la topologie *syntomique lisse sur* $SpfW$), à savoir le faisceau représenté par $\widehat{\mathcal{A}}$.

Ces trois constructions se généralisent aux motifs. Pour fixer les idées, soient X une variété algébrique propre et lisse sur K, $m\in\mathbb{N}$ et $i\in\mathbb{Z}$. Au "motif" $M = H^m(X)(i)$, on peut associer $M_{p-\mathrm{tor}}(\overline{K}) = H^m_{\text{ét}}(X_{\overline{K}}, (\mathbb{Q}_p/\mathbb{Z}_p)(i))$. A partir de ce module galoisien, on peut construire des objets du type (A), (B) et, dans le cas de bonne réduction, (C) qui vivent dans de jolies catégories. Pour (C), on ne sait le faire qu'à isogénie près, sauf si $0\le m\le p-2$. En fait pour fabriquer un objet du type (A) ou (B) il suffit de partir d'un groupe abélien de p-torsion Λ avec action linéaire et continue de G_K qui est de cotype fini. Si on travaille seulement à isogénie près, il suffit de partir d'une représentation p-adique V de G_K qui peut être quelconque. Pour pouvoir faire (C), il faut la supposer cristalline.
Le but du présent article est d'introduire et étudier la catégorie des jolis objets du type (A) à isogénie près. L'étude des objets des types (B) et (C) sera faite ailleurs. Signalons dès à présent que celle des objets de type (B) repose sur le travail fondamental de Colmez sur *les Espaces de Banach de dimension finie* [Co02], travail qui joue déjà un rôle crucial ici (§4).

1.2 – Conventions, notations

Dans toute la suite de l'article, K est une extension finie de $\mathbb{Q}_p$, $\overline{K}$ une clôture algébrique fixée de K et $G_K = \mathrm{Gal}(\overline{K}/K)$.
Un *banach* est un espace de Banach p-adique à équivalence de normes près. Autrement dit, c'est un $\mathbb{Q}_p$-espace vectoriel topologique V qui contient un sous-$\mathbb{Z}_p$-module $\mathcal{V}$ qui est séparé et complet pour la topologie p-adique et est tel que $V = \varinjlim_{n\in\mathbb{N}} p^{-n}\mathcal{V}$ (avec la topologie correspondante. Un tel $\mathcal{V}$ s'appelle un *réseau de V*. Pour qu'un autre sous-$\mathbb{Z}_p$-module $\mathcal{V}'$ de V soit aussi un réseau, il faut et il suffit qu'il existe $r, s \in \mathbb{N}$ tels que $p^r\mathcal{V} \subset \mathcal{V}' \subset p^s\mathcal{V}$.
Une *représentation banachique de G_K* (ou seulement une *représentation banachique* s'il n'y a pas de risque de confusion sur G_K) est un banach muni d'une action linéaire et continue de G_K. Avec comme morphismes les applications $\mathbb{Q}_p$-linéaires continues G_K-équivariantes, les représentations banachiques forment une catégorie additive $\mathbb{Q}_p$-linéaire $\mathcal{B}(G_K)$.
Tout comme la catégorie des banach , la catégorie $\mathcal{B}(G_K)$ a une structure de *catégorie exacte* au sens de [Qu73] (§2, cf. aussi [La83], §1.0) : Un morphisme $f : X \to Y$ de $\mathcal{B}(G_K)$ est un *épimorphisme strict* (ou *admissible*) (resp. un *monomorphisme strict*) (ou *admissible*) si et seulement si l'application sous-jacente est surjective (resp. si elle induit un homéomorphisme de X sur un fermé de Y). Un *morphisme strict* est un morphisme qui peut s'écrire $f_2 \circ f_1$ avec f_1 un épimorphisme strict et f_2 un monomorphisme strict.
Si $f : X \to Y$ est un morphisme strict, le noyau et le conoyau de l'application sous-jacente sont de façon naturelle des représentations banachiques, les morphismes $\mathrm{Ker}\ f \to X$ et $Y \to \mathrm{Coker}\ f$ sont stricts et l'application $\mathrm{Coim}\ f \to \mathrm{Im}\ f$ est un isomorphisme.
Une *suite exacte courte* de $\mathcal{B}(G_K)$ est une suite

$$O \to S' \xrightarrow{f} S \xrightarrow{g} S'' \to 0$$

où g est un épimorphisme strict et f un noyau de G.
Disons qu'une sous-catégorie strictement pleine $\mathcal{D}$ de $\mathcal{B}(G_K)$ est *stricte* si elle contient 0, est stable par somme directe et si tout morphisme de $\mathcal{D}$ est strict (en tant que morphisme de $\mathcal{B}(G_K)$) et a son noyau et son conoyau dans $\mathcal{B}(G_K)$. Une telle catégorie est abélienne.
Appelons *représentation p-adique (de G_K)* toute représentation banachique qui est de dimension finie sur $\mathbb{Q}_p$. Ces représentations forment une sous-catégorie stricte $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ de $\mathcal{B}(G_K)$.

1.3 – C-représentations et B_{dR}^+-représentations

Par continuité le groupe G_K opère sur le corps C complété de $\overline{K}$ pour la topologie p-adique. Une *C-représentation (de G_K)* est un C-espace vectoriel de dimension finie muni d'une action semi-linéaire continue de G_K. Avec comme morphismes les applications C-linéaires G_K-équivariantes, ces représentations forment une catégorie abélienne K-linéaire que nous notons $\mathrm{Rep}_C(G_K)$.

Tout C-espace vectoriel de dimension finie muni de sa topologie naturelle est un banach. Toute C-représentation est donc de manière naturelle une représentation banachique.

Théorème A (cf. §3.3). — *Le foncteur d'oubli*

$$\mathrm{Rep}_C(G_K) \to \mathcal{B}(G_K)$$

est pleinement fidèle.

Autrement dit, si W_1 et W_2 sont deux C-représentations de G_K, toute application $\mathbb{Q}_p$-linéaire continue, G_K-équivariante, de W_1 dans W_2 est C-linéaire. Ceci nous permet d'identifier $\mathrm{Rep}_C(G_K)$ à une sous-catégorie pleine de $\mathcal{B}(G_K)$ qui, bien sûr, est exacte. Il en est de même de la sous-catégorie pleine $\mathrm{Rep}_C^{\mathrm{triv}}(G_K)$ de $\mathrm{Rep}_C(G_K)$ dont les objets sont les *C-représentations triviales*, i.e. les représentations W telles qu'il existe un entier d et un isomorphisme de C^d sur W. On remarque qu'une C-représentation W est triviale si et seulement si W est engendré en tant que C-espace vectoriel par W^{G_K}.

Nous allons avoir besoin de plonger $\mathrm{Rep}_C(G_K)$ dans une catégorie un peu plus grande.
Choisissons un générateur t de $\mathbb{Z}_p(1)$ (noté additivement). Rappelons (cf., par exemple [Fo00], §3.1 et 3.2) que le corps B_{dR} des périodes p-adiques est une $\overline{K}$-algèbre, contenant $\mathbb{Z}_p(1)$, munie d'une topologie et d'une action semi-linéaire continue de G_K.
Ce corps a aussi une structure naturelle de corps complet pour une valuation discrète (la topologie définie par cette valuation est plus fine que la topologie canonique). Son corps résiduel est C et t est une uniformisante. On note B_{dR}^+ l'anneau de la valuation (qui est aussi ouvert et fermé dans B_{dR} pour la topologie canonique). Pour tout $m \in \mathbb{N}$, on pose $B_m = B_{dR}^+/t^m B_{dR}^+$ (on a donc $B_0 = 0$ et $B_1 = C$).
Pour tout $m \in \mathbb{N}$, B_m, muni de la topologie induite par la topologie canonique de B_{dR}, est un banach (et sur $B_1 = C$, cette topologie coïncide avec la topologie p-adique sur C). Inversement, la topologie canonique sur $B_{dR}^+ = \varprojlim_{m\in\mathbb{N}} B_m$ est la topologie de la limite projective avec la topologie de banach sur chaque B_m.
Un B_{dR}^+-module de longueur finie n'est autre qu'un B_{dR}^+-module de type fini annulé par une puissance de t, i.e. c'est un B_m-module de type fini pour m assez grand. Pour tout B_{dR}^+-module W de longueur finie, on note $d_1(W)$ sa longueur. On a donc $d_1(B_m) = m$.
On appelle *B_{dR}^+-représentation (de G_K)* tout B_{dR}^+-module de longueur finie muni d'une action semi-linéaire et continue de G_K. Ces représentations forment de manière évidente une catégorie abélienne K-linéaire que nous notons $\mathrm{Rep}_{B_{dR}^+}(G_K)$. La catégorie $\mathrm{Rep}_C(G_K)$ s'identifie à la sous-catégorie pleine de $\mathrm{Rep}_{B_{dR}^+}(G_K)$ dont les objets sont ceux qui sont annulés par t. Le théorème A s'étend à $\mathrm{Rep}_{B_{dR}^+}(G_K)$.

Théorème A' (cf. §3.3). — *Le foncteur d'oubli*

$$\mathrm{Rep}_{B_{dR}^+}(G_K) \to \mathcal{B}(G_K)$$

est pleinement fidèle.

1.4 – Presque-C-représentations

Soient X_1 et X_2 deux représentations banachiques. On dit que X_1 et X_2 sont *presqu'isomorphes* s'il existe des sous-$\mathbb{Q}_p$-espaces vectoriels de dimension finie V_1 de X_1 et V_2 de X_2 stables par G_K et un isomorphisme $X_1/V_1 \to X_2/V_2$ dans $\mathcal{B}(G_K)$. On a ainsi défini une relation d'équivalence sur les objets de $\mathcal{B}(G_K)$. Une *presque C-représentation (de G_K)* est une représentation banachique qui est presqu'isomorphe à une C-représentation triviale. On note $\mathcal{C}(G_K)$ la sous-catégorie pleine de $\mathcal{B}(G_K)$ dont les objets sont les presque C-représentations.

Théorème B (cf. §5.1). — *La catégorie $\mathcal{C}(G_K)$ est une sous-catégorie stricte de $\mathcal{B}(G_K)$. En outre, il existe des fonctions additives*

$$d : \mathrm{Ob}\ \mathcal{C}(G_K) \to \mathbb{N} \text{ et } h : \mathrm{Ob}\mathcal{C}(G_K) \to \mathbb{Z}$$

caractérisées par $d(C) = 1,\ h(C) = 0$ et

$$d(V) = 0\ ,\ h(V) = \dim_{\mathbb{Q}_p} V \text{ pour toute représentation } p\text{-adique } V \text{ de } G_K$$

En fait $\mathcal{C}(G_K)$ contient toutes les C-représentations et même les B_{dR}^+-représentations :

Théorème C (cf. §5.5). — *Soit W une B_{dR}^+-représentation et soit d la longueur du B_{dR}^+-module sous-jacent. Alors W est presqu'isomorphe à C^d. On a $d(W) = d$ et $h(W) = 0$.*

1.5 – Extensions

Disons qu'une suite exacte courte de $\mathcal{B}(G_K)$

$$0 \to S' \to S \to S'' \to 0$$

est *presque scindée* s'il existe un sous-$\mathbb{Q}_p$-espace vectoriel de dimension finie V de S' stable par G_K tel que la suite

$$0 \to S'/V \to S/V \to S'' \to 0$$

est scindée.

THÉORÈME D (cf. §5.2, 5.4 et 5.5). — *Soit*

$$0 \to S' \to S \to S'' \to 0$$

une suite exacte courte de $\mathcal{B}(G_K)$.
i) Si S' *et* S'' *sont des presque* C*-représentations, pour que* S *soit une presque-*C*-représentation, il faut et il suffit que la suite soit presque scindée.*
ii) Si S' *et* S'' *sont des* B_{dR}^+*-représentations, pour que* S *soit une* B_{dR}^+*-représentation, il faut et il suffit que la suite soit presque scindée.*

1.6 – LE PLAN

Faisons une dernière convention : Si $\mathcal{C}$ est une catégorie abélienne, si X et Y sont deux objets de $\mathcal{C}$ et si $n \in \mathbb{N}$, on note $\mathrm{Ext}^n_{\mathcal{C}}(X,Y)$ le groupe des *classes de* n*-extensions de Yoneda*. Si $\mathcal{C} = \mathrm{Rep}_E(G)$ est une catégorie de représentations (semi)-linéaires d'un groupe G à coefficients dans un anneau commutatif E, on écrit aussi $\mathrm{Ext}^n_{E[G]}(X,Y)$.
Expliquons maintenant, pour terminer cette introduction, comment cet article est organisé.

– L'objet du §2, est une étude détailléee des B_{dR}^+-représentations de G_K. Celle-ci repose de façon essentielle sur l'article de Tate sur les groupes p-divisibles [Ta67] et sur la classification de Sen des C-représentations [Sen80]. Ces travaux ont été repris et poursuivis dans [Fo00] que l'on utilise abondamment. On introduit les *petites représentations* pour lesquelles on peut tout écrire explicitement et auxquelles on peut se ramener dans la plupart des cas parce que toute B_{dR}^+-représentation devient petite après changement de base fini. On détermine les groupes d'extensions dans la catégorie $\mathrm{Rep}_{B_{dR}^+}(G_K)$.
Ces calculs, qui seront utiles dans la suite, ne sont pas difficiles. Ils reposent essentiellement sur la description explicite des groupes proalgébriques associés aux catégories tannakiennes sous-jacentes. Ils sont plutôt fastidieux et nous recommandons au lecteur de passer rapidement sur le §2 en première lecture.

– Dans le §3, on établit les théorèmes de pleine fidélité (th. A et A' ci-dessus). Puis, on montre comment construire toutes les extensions d'une C-représentation – ou plus généralement d'une B_{dR}^+-représentation – W par une représentation p-adique V.
Rappelons [Fo88a] que B_{cris} est une sous-$\mathbb{Q}_p$-algèbre de B_{dR} stable par G_K et munie d'un Frobenius φ qui est un endomorphisme de $\mathbb{Q}_p$-algèbres commutant à l'action de G_K. Si B_e désigne la sous-$\mathbb{Q}_p$-algèbre de B_{cris} formée des b tels que $\varphi(b) = b$, on dispose (cf. par exemple, [FPR94], prop. 3.1.1) d'une suite exacte

$$0 \to \mathbb{Q}_p \to B_e \to B_{dR}/B_{dR}^+ \to 0$$

où la flèche $B_e \to B_{dR}/B_{dR}^+$ est le composé de l'inclusion de $B_e \subset B_{cris}$ dans B_{dR} avec la projection sur B_{dR}/B_{dR}^+.

En tensorisant avec V, on obtient une suite exacte

$$0 \to V \to B_e \otimes_{\mathbb{Q}_p} V \to (B_{dR}/B_{dR}^+) \otimes_{\mathbb{Q}_p} V \to 0$$

d'où une application[(1)]

$$\delta_{W,V} : \mathrm{Hom}(W, (B_{dR}/B_{dR}^+) \otimes_{\mathbb{Q}_p} V) \to \mathrm{Ext}^1_{\mathcal{B}(G_K)}(W, V)$$

Le point essentiel est que $\delta_{W,V}$ est un isomorphisme. Pour le prouver, on commence par utiliser beaucoup de cohomologie galoisienne et en particulier, la théorie du corps de classes local pour montrer que $\mathrm{Ext}^1_{\mathcal{B}(G_K)}(W, V)$ est de dimension finie et calculer sa dimension. On constate alors que la source et le but de $\delta_{W,V}$ ont la même dimension et il est facile de vérifier que $\delta_{W,V}$ est injective.
Dans le cas où W est une C-représentation, on peut remplacer B_e par $B_e \cap t^{-1}B_{dR}^+$ qui est une extension de $C(-1)$ par $\mathbb{Q}_p$ que l'on peut décrire très simplement : avec les notations du §1.1, on a $B_e \cap t^{-1}B_{dR}^+ = U(-1)$ où U est le $\mathbb{Q}_p$-espace vectoriel des suites $(u^{(n)})_{n\in\mathbb{N}}$ d'éléments de U_C^+ vérifiant $(u^{(n+1)})^p = u^{(n)}$ pour tout n.
Remarquons au passage que cela nous fournit un procédé pour construire – au moins théoriquement et modulo la construction des représentations de dimension finie sur $\mathbb{Q}_p$ – tous les objets de $\mathcal{C}(G_K)$. Si S est l'un d'entre eux, on peut en effet trouver une C-représentation W (que l'on peut même choisir triviale) et une représentation p-adique V telles que S soit isomorphe au quotient d'une extension de W par V par une autre représentation p-adique V'.

– Dans le §4, on commence par énoncer, dans un langage un peu différent, l'un des résultats essentiels de l'article de Colmez [Co02] sur les *Espaces de Banach de dimension finie*. On introduit ce que nous appelons les *espaces de Banach-Colmez effectifs* qui sont des espaces de Banach munis d'une structure de limite inductive de limite projective d'objets en groupes commutatifs dans la catégorie des espaces rigides analytiques sur C vérifiant certaines propriétés. Un tel groupe a une dimension et une hauteur qui sont des entiers naturels. On a $C = \varinjlim_{n\in\mathbb{N}} p^{-n}\mathcal{O}_C$ ce qui permet de munir C d'une structure d'espace de Banach-Colmez (la structure analytique sur $\mathcal{O}_C$ est la structure habituelle du disque unité fermé) de dimension 1 et de hauteur 0. Tout $\mathbb{Q}_p$-espace vectoriel de dimension finie h a une structure naturelle d'espace de Banach-Colmez de dimension 0 et hauteur h. Le résultat de Colmez peut alors s'exprimer en disant essentiellement que, si S est un espace de Banach-Colmez effectif de dimension 1 et de hauteur h et si $f : S \to C$ est un morphisme (d'espaces de Banach-Colmez effectifs) dont l'image n'est pas de dimension finie, alors f est surjectif et son noyau est de dimension 0 et de hauteur h.

(1) Ici $\mathrm{Hom}(W, (B_{dR}/B_{dR}^+) \otimes_{\mathbb{Q}_p} V)$ désigne le groupe des applications $\mathbb{Q}_p$-linéaires continues G_K-équivariantes – ou, cela revient au même, des applications B_{dR}^+-linéaires G_K-équivariantes – de W dans $(B_{dR}/B_{dR}^+) \otimes_{\mathbb{Q}_p} V$.

On utilise ensuite les résultats du §3 pour montrer
i) que, si E est une représentation banachique de G_K extension de C par une représentation p-adique de dimension h, alors E est munie d'une structure d'espace de Banach-Colmez effectif de dimension 1 et hauteur h,
ii) que si $\eta : E \to C$ est une application $\mathbb{Q}_p$-linéaire continue G_K-équivariante, alors elle est induite par un morphisme dans la catégorie des espaces de Banach-Colmez effectifs.
Le résultat de Colmez implique alors que si l'image de η n'est pas de dimension finie, η est surjective et son noyau est un $\mathbb{Q}_p$-espace vectoriel de dimension h.

– Le théorème de structure pour la catégorie $\mathcal{C}(G_K)$ (th.B ci-dessus) est une conséquence essentiellement formelle de ce résultat, comme on le montre au début du §5. La suite de ce paragraphe consiste surtout à prouver que toute extension de B_{dR}^+-représentations est *presque scindée*, que toute B_{dR}^+-représentation est presqu'isomorphe à une C-représentation triviale et que, dans la catégorie des représentations banachiques, toute extension presque scindée de B_{dR}^+-représentations est encore une B_{dR}^+-représentation. On y fait un grand usage de l'étude des B_{dR}^+-représentations faite au §1. On utilise aussi certaines représentations p-adiques spécifiques pour construire explicitement certains presque-scindages et certains presqu'isomorphismes.

– Dans le §6, on calcule les groupes d'extensions dans la catégorie des presque-C-représentations. Les résultats du §5 permettent de ramener ces calculs soient à ceux des groupes d'extensions dans la catégorie des B_{dR}^+-représentations, calculs déjà faits au §1, soient à ceux des groupes d'extensions dans la catégorie des représentations p-adiques, lesquels se ramènent aux calculs de cohomologie galoisienne continue de Tate.
Soient X et Y des presque-C-représentations. On montre (th.6.1) que les $\mathbb{Q}_p$-espaces vectoriels $\mathrm{Ext}^n_{\mathcal{C}(G_K)}(X,Y)$ sont de dimension finie, nulle si $n \geq 3$ et que

$$\sum_{n=0}^{2}(-1)^n \dim_{\mathbb{Q}_p} \mathrm{Ext}^n_{\mathcal{C}(G_K)}(X,Y) = -[K:\mathbb{Q}_p]h(X)h(Y)$$

On construit (prop.6.8) une application naturelle $\mathrm{Ext}^2_{\mathcal{C}(G_K)}(X,X(1)) \to \mathbb{Q}_p$ et on montre (prop.6.9) que, pour $0 \leq n \leq 2$, l'application

$$\mathrm{Ext}^n_{\mathcal{C}(G_K)}(X,Y) \times \mathrm{Ext}^{2-n}_{\mathcal{C}(G_K)}(Y,X(1)) \to \mathrm{Ext}^2_{\mathcal{C}(G_K)}(X,X(1)) \to \mathbb{Q}_p$$

définit une dualité parfaite.

– Dans le §7, on étudie la catégorie des *presque-C-représentations de G_K à presqu'isomorphismes près*, i.e. la catégorie déduite de $\mathcal{C}(G_K)$ en rendant inversible les presqu'isomorphismes. C'est une catégorie abélienne semi-simple qui a une seule classe d'isomorphisme d'objets simples, celle de C. Le corps gauche $\mathcal{D}_K$ des endomorphismes de C dans cette catégorie est assez gros. Il contient K et on dispose d'une suite exacte courte de K-espaces vectoriels

$$0 \to K \to \mathcal{D}_K \to ((C \otimes_{\mathbb{Q}_p} C_f)(-1))^{G_K} \to K \to 0$$

où C_f désigne la réunion des sous-$\mathbb{Q}_p$-espaces vectoriels de dimension finie, stables par G_K, de C. Mais la structure multiplicative de $\mathcal{D}_K$ reste assez mystérieuse.

– Dans le §8, on fait le lien entre cet article et le prologue de cette introduction. On décrit quelques objets universels que l'on peut associer à la cohomologie étale des variétés algébriques sur K, en particulier dans le cas des variétés abéliennes.
On définit aussi l'espace tangent t_V d'une représentation p-adique quelconque et l'exponentielle de Bloch-Kato

$$\exp_{BK} : t_V \to H^1_{\mathrm{cont}}(K, V)$$

qui généralisent de façon évidente ces notions, maintenant classiques, pour des représentations de de Rham. On montre que l'image $H^1_e(K, V)$ de $\exp_{BK}$ est le sous-groupe de $H^1_{\mathrm{cont}}(K, V) = \mathrm{Ext}^1_{\mathbb{Q}_p[G_K]}(\mathbb{Q}_p, V)$ formé des classes d'extensions qui proviennent, via l'inclusion de $\mathbb{Q}_p$ dans B^+_{dR}, d'une extension de B^+_{dR} par V.

Une partie de ce travail a été fait pendant un séjour à l'*University of Sydney* en Australie, une autre pendant un séjour au *Korea Institute for Advanced Study* de Séoul en République de Corée. J'ai plaisir à remercier ces institutions - et tout particulièrement Mark Kisin et Minhyong Kim - pour leur accueil chaleureux.
Je voudrais enfin remercier le rapporteur pour sa lecture minutieuse et ses remarques pertinentes.

2 – – Etude des B^+_{dR}-représentations de G_K

Soit K_∞ la $\mathbb{Z}_p$-extension cyclotomique de K contenue dans $\overline{K}$, i.e. l'unique $\mathbb{Z}_p$-extension de K contenue dans le sous-corps de $\overline{K}$ engendré sur K par les racines de l'unité d'ordre une puissance de p. Soient $H_K = \mathrm{Gal}(\overline{K}/K)$ et $\Gamma_K = G_K/H_K$.
Dans le §2.1, nous introduisons un anneau de séries formelles $K_\infty[[\underline{t}]]$ qui est un sous-anneau de $(B^+_{dR})^{H_K}$ stable par Γ_K et construisons une équivalence entre la catégorie des B^+_{dR}-représentations de G_K et celle des $K_\infty[[\underline{t}]]$-*représentations de* Γ_K.
Dans le §2.2, on montre que cette dernière catégorie a une structure de catégorie tannakienne K-linéaire et est équippée d'un foncteur fibre à valeurs dans les

K_∞-espaces vectoriels. On décrit le groupe pro-algébrique associé à ce foncteur-fibre. On en déduit quelques résultats sur les groupes d'extensions dans cette catégorie.
Si Y est une $K_\infty[[\underline{t}]]$-représentation de Γ_K, la théorie de Sen permet de définir un endomorphisme ∇_0 du K_∞-espace vectoriel sous-jacent. Si les valeurs propres de ∇_0 sont dans K et suffisamment petites, on peut utiliser l'exponentielle pour donner une description terre à terre de Y et de la B_{dR}^+-représentation qui lui est associée (on dit qu'une telle B_{dR}^+-représentation est *petite*). L'étude des petites représentations est l'objet du §2.3.
Si W est une B_{dR}^+-représentation de G_K, il existe une extension finie L de K contenue dans $\overline{K}$ telle que W est petite en tant que représentation de $G_L = \mathrm{Gal}(\overline{K}/L)$. Une notion de changement de base permet alors de ramener le calcul des groupes d'extensions dans la catégorie des B_{dR}^+-représentations au cas des petites représentations. C'est ce qu'on fait dans le §2.4.
Dans le §2.5 enfin, on calcule ces groupes d'extensions. On montre que si W_1 et W_2 sont des B_{dR}^+-représentations, les $\mathrm{Ext}^i(W_1,W_2)$ sont des K-espaces vectoriels de dimension finie, nuls si $i \geq 3$ et $\sum_{i=0}^2 \dim_K \mathrm{Ext}^i(W_1,W_2) = 0$ (th.2.14). On dispose en outre (prop.2.16) d'une dualité parfaite

$$\mathrm{Ext}^i(W_1,W_2) \times \mathrm{Ext}^{2-i}(W_2,W_1(1)) \to K$$

En outre (prop. 2.15), si W est un objet simple de la catégorie des B_{dR}^+-représentations, les $\mathrm{Ext}^i(C,W)$ sont tous nuls sauf dans les cas suivants :
(i) $W = C$ et $i = 0$ ou 1, et chacun de ces deux K-espaces vectoriels est de dimension 1,
(ii) $W = C(1)$ et $i = 1$ ou 2, chacun de ces deux K-espaces vectoriels étant encore de dimension 1.

2.1 – B_{dR}^+-représentations et $K_\infty[[\underline{t}]]$-représentations ; structures tannakiennes

Le générateur choisi t de $\mathbb{Z}_p(1)$ correspond à une suite $(\varepsilon^{(n)})_{n\in\mathbb{N}}$ où $\varepsilon^{(n)} \in \overline{K}$ est une racine primitive p^n-ième de l'unité et où $(\varepsilon^{(n+1)})^p = \varepsilon^{(n)}$ pour tout n. Si $p \neq 2$, on note π_t l'unique uniformisante du corps $\mathbb{Q}_p[\varepsilon^{(1)}]$ telle que $(\pi_t)^{p-1}+p = 0$ et $v_p(\varepsilon^{(1)} - 1 - \pi_t) \geq \frac{2}{p-1}$; si $p = 2$, on pose $\pi_t = 2\varepsilon^{(2)}$. Alors $\underline{t} = t/\pi_t$ est un élément de $(B_{dR}^+)^{H_K}$ qui est encore une uniformisante de B_{dR}^+. L'anneau $K_\infty[[\underline{t}]]$ des séries formelles en l'indéterminée $\underline{t}$ s'identifie à un sous-anneau de $(B_{dR}^+)^{H_K}$ stable par Γ_K et, pour tout $m \in \mathbb{N}$, l'image de cet anneau dans $(B_m)^{H_K} = (B_{dR}^+/t^m B_{dR}^+)^{H_K} = (B_{dR}^+/\underline{t}^m B_{dR}^+)^{H_K} = (B_{dR}^+)^{H_K}/\underline{t}^m (B_{dR}^+)^{H_K}$ s'identifie à $K_\infty[[\underline{t}]]/\underline{t}^m$ et est dense dans $(B_m)^{H_K}$ (cf [Fo00], §3).
Soit W une B_{dR}^+-représentation. Alors W^{H_K} est un $(B_{dR}^+)^{H_K}$-module muni d'une action semi-linéaire de Γ_K. Notons W^f la réunion des sous-K-espaces vectoriels de dimension finie de W^{H_K} stables par Γ_K. C'est un sous-$K_\infty[[\underline{t}]]$-module stable par Γ_K de W^{H_K}.

Proposition 2.1 (cf. [Fo00], th.3.5 et 3.6 [(2)]). — *Pour toute B_{dR}^+-représentation W de G_K, W^{H_K} est un $(B_{dR}^+)^{H_K}$-module de longueur finie, W^f un $K_\infty[[\underline{t}]]$-module de longueur finie et les applications naturelles*

$$B_{dR}^+ \otimes_{(B_{dR}^+)^{H_K}} W^{H_K} \to W \text{ et } B_{dR}^+ \otimes_{K_\infty[[\underline{t}]]} W^f \to W$$

sont des isomorphismes.

Autrement dit, la correspondance $W \mapsto W^f$ est de façon évidente un foncteur de $\mathrm{Rep}_{B_{dR}^+}(G_K)$ dans la catégorie $\mathrm{Rep}_{K_\infty[[\underline{t}]]}(\Gamma_K)$ des $K_\infty[[\underline{t}]]$-*représentations de* Γ_K (i.e. des $K_\infty[[\underline{t}]]$-modules de longueur finie munis d'une action linéaire et continue de Γ_K [(3)])). Ce foncteur est une équivalence de catégorie et le foncteur $Y \mapsto Y_{dR} := B_{dR}^+ \otimes_{K_\infty[[\underline{t}]]} Y$ est un quasi-inverse.

Si Y_1 et Y_2 sont deux $K_\infty[[\underline{t}]]$-représentations de G_K, le produit tensoriel $Y_1 \otimes Y_2 := Y_1 \otimes_{K_\infty} Y_2$ (attention : on ne prend pas le produit tensoriel au dessus de $K_\infty[[\underline{t}]]$!) est muni d'une action naturelle de Γ_K. On en fait une $K_\infty[[\underline{t}]]$-représentation de Γ_K en posant, $\underline{t}(y_1 \otimes y_2) = \underline{t}y_1 \otimes y_2 + y_1 \otimes \underline{t}y_2$ quels que soient $y_1 \in Y_1$ et $y_2 \in Y_2$.
De la même façon, le K_∞-espace vectoriel $Hom(Y_1, Y_2) := \mathcal{L}_{K_\infty}(Y_1, Y_2)$ des applications K_∞-linéaires de Y_1 dans Y_2 est muni d'une action naturelle de Γ_K. On en fait une $K_\infty[[\underline{t}]]$-représentation de Γ_K en posant $(\underline{t}\eta)(y) = \underline{t}\eta(y) - \eta(\underline{t}y)$ pour tout $\eta \in Hom(Y_1, Y_2)$ et $y \in Y_1$.
On voit que l'on a ainsi muni $\mathrm{Rep}_{K_\infty[[\underline{t}]]}(\Gamma_K)$ d'une structure de catégorie tannakienne sur K, dont l'objet unité est K_∞ muni de l'action tautologique de $\Gamma_K = \mathrm{Gal}(K_\infty/K)$.
Par transport de structure, $\mathrm{Rep}_{B_{dR}^+}(G_K)$ devient une catégorie tannakienne dont l'objet-unité est $B_{dR}^+ \otimes_{K_\infty[[\underline{t}]]} K_\infty = C$. Si W_1 et W_2 sont deux objets de $\mathrm{Rep}_{B_{dR}^+}(G_K)$, $W_1 \otimes W_2 = B_{dR}^+ \otimes_{K_\infty[[\underline{t}]]} (W_1^f \otimes W_2^f)$ et $\mathcal{H}\wr\Updownarrow(W_1, W_2) = B_{dR}^+ \otimes_{K_\infty[[\underline{t}]]} Hom(W_1^f, W_2^f)$. On prendra garde à ne pas confondre $W_1 \otimes W_2$ avec $W_1 \otimes_{B_{dR}^+} W_2$ et $Hom(W_1, W_2)$ avec $\mathcal{L}_{B_{dR}^+}(W_1, W_2)$; ces dernières structures ne font d'ailleurs pas de $\mathrm{Rep}_{B_{dR}^+}(G_K)$ une catégorie tannakienne. Toutefois, si W_1 et W_2 sont des C-représentations, $W_1 \otimes W_2 = W_1 \otimes_C W_2$ et $Hom(W_1, W_2) = \mathcal{L}_C(W_1, W_2)$.

Exercice : Si $m, n \in \mathbb{N}$, avec $m \geq n$, alors $B_m \otimes B_n \simeq \oplus_{i=0}^{n-1} B_{m+n-1-2i}(i)$.

---

(2) On prendra garde que dans [Fo00] les B_{dR}^+-représentations de G_K et les $K_\infty[[\underline{t}]]$-représentations de Γ_K considérées ne sont pas supposées de longueur finie, comme on le fait ici.

(3) Attention que $K_\infty[[\underline{t}]]$ n'est pas complet pour la topologie induite par celle de B_{dR}^+ ; et que K_∞ et les $K_\infty[[\underline{t}]]/(\underline{t}^r)$ ne sont pas des banach.

2.2 – CONNEXIONS ET GROUPES PRO-ALGÉBRIQUES

Soient E un corps de caractéristique 0 et $E[[\underline{t}]]$ l'anneau des séries formelles en une indéterminée $\underline{t}$ à coefficients dans E. Notons $\Omega^{\log}_{E[[\underline{t}]]/E}$ le module des *E-différentielles continues logarithmiques de* $E[[\underline{t}]]$, autrement dit le $E[[\underline{t}]]$-module libre de rang 1 de base $d\underline{t}/\underline{t}$ (base qui ne change pas si l'on remplace $\underline{t}$ par $\lambda\underline{t}$, avec $\lambda \in E^*$).
Si Y est un $E[[\underline{t}]]$-module de longueur finie, une *connexion* ∇ *sur* Y est une application E-linéaire de Y dans $Y \otimes_{E[[\underline{t}]]} \Omega^{\log}_{E[[\underline{t}]]/E}$ vérifiant la règle de Leibniz. Se donner une application E-linéaire ∇ de Y dans $Y \otimes_{E[[\underline{t}]]} \Omega^{\log}_{E[[\underline{t}]]/E}$ revient à se donner une application E-linéaire $\nabla_0 : Y \to Y$ (on pose $\nabla(y) = \nabla_0(y) \otimes d\underline{t}/\underline{t}$) et ∇ est alors une connexion si et seulement si ∇_0 vérifie $\nabla_0(\underline{t}y) = \underline{t}(y + \nabla_0(y))$ pour tout $y \in Y$.

PROPOSITION 2.2 (cf [Fo00], prop.3.7 et 3.8). — *i) Pour toute $K_\infty[[\underline{t}]]$-représentation Y de Γ_K, il existe une et une seule connexion ∇ sur Y qui a la propriété que, pour tout $y \in Y$, il existe un sous-groupe ouvert $\Gamma_{K,y}$ de Γ_K tel que, pour tout $\gamma \in \Gamma_{K,y}$,*

$$\gamma(y) = \exp(\log \chi(\gamma).\nabla_0)(y)$$

ii) Deux $K_\infty[[\underline{t}]]$-représentations Y_1 et Y_2 de Γ_K sont isomorphes si et seulement s'il existe une application $K_\infty[[\underline{t}]]$-linéaire $\varphi : Y_1 \to Y_2$ qui commute à l'action de ∇_0.

Remarquons que, si E est un corps de caractéristique 0, se donner un $E[[\underline{t}]]$-module de longueur finie muni d'une connexion ∇ revient à se donner un E-espace vectoriel de dimension finie muni de deux endomorphismes $\underline{t}$ et ∇_0, avec $\underline{t}$ nilpotent, vérifiant la relation $\nabla_0\underline{t} - \underline{t}\nabla_0 = \underline{t}$. C'est la raison pour nous pour introduire la catégorie qui suit : on fixe une clôture algébrique $\overline{E}$ de E et on se donne un sous-ensemble S du groupe additif de $\overline{E}$ stable par $\mathrm{Gal}(\overline{E}/E)$ et par la translation $\alpha \mapsto \alpha + 1$. On note $\mathcal{C}_{S,E}$ la catégorie suivante :
– un objet est un E-espace vectoriel muni de deux endomorphismes ∇_0 et $\underline{t}$ vérifiant :
i) les valeurs-propres de ∇_0 dans $\overline{E}$ sont dans S,
ii) l'endomorphisme $\underline{t}$ est nilpotent,
iii) on a $\nabla_0\underline{t} - \underline{t}\nabla_0 = \underline{t}$;
– un morphisme est une application E-linéaire qui commute à ∇_0 et $\underline{t}$.

On obtient ainsi une catégorie abélienne E-linéaire. La sous-catégorie pleine $\mathcal{C}^f_{S,E}$ de $\mathcal{C}_{S,E}$ dont les objets sont ceux qui sont de dimension finie sur E s'identifie à la sous-catégorie pleine de la catégorie des $E[[\underline{t}]]$-modules de longueur finie munis d'une connection ∇ dont les objets sont ceux pour lesquels les valeurs propres de l'endomorphisme ∇_0 du E-espace vectoriel sous-jacent sont dans S.
Lorsque S est *un sous-groupe de* $\overline{E}$, cette catégorie a une structure de catégorie tannakienne neutre sur E :

– le E-espace vectoriel sous-jacent au produit tensoriel $X_1 \otimes X_2$ de X_1 et X_2 est le produit tensoriel des E-espaces vectoriels sous-jacents, avec $\nabla_0(x_1 \otimes x_2) = \nabla_0 x_1 \otimes x_2 + x_1 \otimes \nabla_0 x_2$ et $\underline{t}(x_1 \otimes x_2) = \underline{t}x_1 \otimes x_2 + x_1 \otimes \underline{t}x_2$,
– le E-espace vectoriel sous-jacent au Hom interne $Hom(X_1, X_2)$ de X_1 et X_2 est $\mathcal{L}_E(X_1, X_2)$, avec $\nabla_0(\eta)(x) = \nabla_0(\eta(x)) - \eta(\nabla_0 x)$ et $\underline{t}(\eta)(x) = \underline{t}(\eta(x)) - \eta(\underline{t}x)$,
– l'objet unité est E avec $\nabla_0 = \underline{t} = 0$.
On prendra garde que si X_1 et X_2 sont deux objets de $\mathcal{C}^f_{S,E}$, le $E[[\underline{t}]]$-module sous-jacent à $X_1 \otimes X_2$ (resp. $Hom(X_1, X_2)$) n'est pas isomorphe en général à $X_1 \otimes_{E[[\underline{t}]]} X_2$ (resp. $\mathcal{L}_{E[[\underline{t}]]}(X_1, X_2)$).

Toujours lorsque S est un sous-groupe de $\overline{E}$, la catégorie $\mathcal{C}^f_{S,E}$ est neutre et s'identifie donc à la catégorie des représentations E-linéaires de dimension finie du groupe pro-algébrique sur E qui est le groupe $\mathbb{C}_{S,E}$ des $\otimes$-automorphismes du foncteur fibre tautologique qui, à tout objet de $\mathcal{C}^f_{S,E}$, associe le E-espace vectoriel sous-jacent. La sous-catégorie pleine de $\mathcal{C}^f_{S,E}$ dont les objets sont ceux sur lesquels $\underline{t} = 0$ s'identifie à la catégorie des représentations E-linéaires de dimension finie d'un quotient $\mathbb{T}_{S,E}$ de $\mathbb{C}_{S,E}$. En écrivant $\nabla_0 = \nabla_0^n + \nabla_0^{ss}$, avec $\nabla_0^n \nabla_0^{ss} = \nabla_0^{ss} \nabla_0^n$, ∇_0^n nilpotent et ∇_0^{ss} semi-simple, on peut identifier (cf. par exemple [Fo00], §2.4) $\mathbb{T}_{S,E}$ au produit du groupe additif sur E par le groupe de type multiplicatif $\mathbb{T}^m_{S,E}$ dont le groupe des caractères $\mathrm{Hom}_{\overline{E}}(\mathbb{T}^m_{S,E} \times \overline{E}, \mathbb{G}_{m,\overline{E}})$ est S (avec l'action de $\mathrm{Gal}(\overline{E}/E)$ induite par l'action sur $\overline{E}$).
La projection de $\mathbb{C}_{S,E}$ sur $\mathbb{T}_{S,E}$ admet une section canonique : elle s'obtient en associant à tout objet X de $\mathcal{C}_{S,E}$, la représentation de $\mathbb{T}_{S,E}$ dont le E-espace vectoriel sous-jacent est X muni de la même action de ∇_0, mais ou l'on fait agir $\underline{t}$ par 0.
Se donner une action du groupe additif $\mathbb{G}_{a,E}$ sur un E-espace vectoriel revient à se donner un endomorphisme nilpotent de cet espace. L'action de $\underline{t}$ induit ainsi une action de $\mathbb{G}_{a,E}$ sur tout objet X de $\mathcal{C}_{S,E}$, ce qui définit un morphisme de $\mathbb{G}_{a,E}$ dans $\mathbb{C}_{S,E}$. On voit que ce morphisme identifie $\mathbb{G}_{a,E}$ au noyau de la projection de $\mathbb{C}_{S,E}$ sur $\mathbb{T}_{S,E}$, donc que $\mathbb{C}_{S,E}$ est le produit semi-direct du groupe pro-algébrique commutatif $\mathbb{T}_{S,E}$ par le sous-groupe invariant $\mathbb{G}_{a,E}$.
Pour voir l'action de $\mathbb{T}_{S,E}$ sur $\mathbb{G}_{a,E}$, on commence par vérifier que la relation $\nabla_0 \underline{t} - \underline{t}\nabla_0 = \underline{t}$ équivaut en fait à $\nabla_0^n \underline{t} = \underline{t}\nabla_0^n$ et $\nabla_0^{ss}\underline{t} - \underline{t}\nabla_0^{ss} = \underline{t}$. Ceci implique que le sous-groupe de $\mathbb{T}_{S,E}$ isomorphe à $\mathbb{G}_{a,E}$ opère trivialement sur $\mathbb{G}_{a,E}$. Soit $\mathbb{H}_E$ le sous-groupe de $\mathbb{G}L_{2,E}$ formé des matrices de la forme $\begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix}$. L'inclusion de $\mathbb{Z}$ dans S fournit un morphisme $\mathbb{T}_{S,E} \to \mathbb{T}^m_{S,E} \to \mathbb{T}^m_{\mathbb{Z},E} = \mathbb{G}_{m,E}$ et on vérifie que $\mathbb{C}_{S,E}$ s'identifie au produit fibré $\mathbb{H} \times_{\mathbb{G}_{m,E}} \mathbb{T}_{S,E}$.

Pour tout objet X de $\mathcal{C}_{S,E}$, notons $X\{-1\}$ l'objet de $\mathcal{C}_{S,E}$ qui a le même E-espace vectoriel sous-jacent, avec la même action de $\underline{t}$, la nouvelle action de ∇_0 étant $x \mapsto (\nabla_0 - 1)(x)$.

PROPOSITION 2.3. — *Soient* E *un corps de caractéristique* 0 *et* $\overline{E}$ *une clôture algébrique de* E*. Soient* S *et* S' *deux sous-ensembles de* $\overline{E}$ *stables par* $\mathrm{Gal}(\overline{E}/E)$ *et par la translation* $\alpha \mapsto \alpha + 1$.
i) Si X *est un objet de* $\mathcal{C}_{S,E}$*, les* $\mathrm{Ext}^n(E, X)$ *s'identifient, canoniquement et fonctoriellement, aux groupes de cohomologie du complexe*

$$(C_X) \qquad X \xrightarrow{d^0} X\{-1\} \oplus X \xrightarrow{d^1} X\{-1\} \to 0 \to 0 \to 0 \dots$$

où le premier terme est placé en degré 0*,* $d^0(x) = (\underline{t}x, \nabla_0 x)$ *et* $d^1(y,z) = \underline{t}z - \nabla_0 y$.
ii) Si $S \subset S'$ *et si* X_1 *et* X_2 *sont deux objets de* $\mathcal{C}_{S,E}$*, l'application naturelle* $\mathrm{Ext}^n_{\mathcal{C}_{S,E}}(X_1, X_2) \to \mathrm{Ext}^n_{\mathcal{C}_{S',E}}(X_1, X_2)$ *est bijective pour tout* $n \in \mathbb{N}$.
iii) Si $S \cap S' = \emptyset$*, si* X *est un objet de* $\mathcal{C}_{S,E}$ *et* X' *un objet de* $\mathcal{C}_{S',E}$*, on a* $\mathrm{Ext}^n_{\mathcal{C}_{S \cup S',E}}(X, X') = 0$ *pour tout* n.
iv) Si S *est un sous-groupe de* $\overline{E}$ *et si* X_1 *et* X_2 *sont deux objets de* $\mathcal{C}^f_{S,E}$ *et si* $n \in \mathbb{N}$*, alors* $\mathrm{Ext}^n_{\mathcal{C}_{S,E}}(X_1, X_2)$ *est un* E*-espace vectoriel de dimension finie, nul si* $n \geq 3$ *qui s'identifie (canoniquement et fonctoriellement) à* $\mathrm{Ext}^n_{\mathcal{C}_{S,E}}(E, Hom(X_1, X_2))$.
En outre, $\sum_{n=0}^{2} (-1)^n \dim_E \mathrm{Ext}^n_{\mathcal{C}_{S,E}}(X_1, X_2) = 0$.

Preuve : Remarquons que les $\mathrm{Ext}^n_{\mathcal{C}_{S,E}}(E, X)$ sont les foncteurs dérivés du foncteur $\Gamma : \mathcal{C}_{S,E} \to \underline{\mathrm{Vect}}_E$ qui envoie X sur $\mathrm{Hom}_{\mathcal{C}_{S,E}}(E, X)$. L'assertion (i) peut alors se voir :
– soit en notant $\mathcal{C}_{S,\underline{t}=0,E}$ la sous-catégorie pleine de $\mathcal{C}_{S,E}$ formée des objets sur lesquels $\underline{t} = 0$, en remarquant que $\Gamma = \Gamma_2 \circ \Gamma_1$ où $\Gamma_1 : \mathcal{C}_{S,E} \to \mathcal{C}_{S,\underline{t}=0,E}$ est le foncteur qui envoie X sur $X_{\underline{t}=0}$ tandis que Γ_2 est la restriction à $\mathcal{C}_{S,\underline{t}=0,E}$ de Γ, et que le complexe (C_X) est le complexe simple associé au complexe double

$$\begin{array}{ccccccccccccc}
0 & \to & X & \xrightarrow{\underline{t}} & X\{-1\} & \to & 0 & \to & \dots & \to & 0 & \to & \dots \\
 & & \downarrow \nabla_0 & & \downarrow \nabla_0 & & & & & & & & \\
0 & \to & X & \xrightarrow{\underline{t}} & X\{-1\} & \to & 0 & \to & \dots & \to & 0 & \to & \dots
\end{array}$$

– soit en remarquant que la correspondance $X \mapsto C_X$ est un foncteur exact de la catégorie $\mathcal{C}_{S,E}$ dans la catégorie des complexes bornés à gauche de E-espaces vectoriels, qui définit donc un δ-foncteur (ou foncteur cohomologique) dont le H^0 est ce que l'on veut et dont on vérifie facilement qu'il est effaçable.
Le fait que $\mathrm{Ext}^n_{\mathcal{C}_{S,E}}(X_1, X_2)$ s'identifie à $\mathrm{Ext}^n_{\mathcal{C}_{S,E}}(E, Hom(X_1, X_2))$ est un résultat standard valable dans n'importe quelle catégorie tannakienne. Les autres assertions résultent immédiatement de (i). □

Le groupe $\mathrm{Gal}(\overline{E}/E) \times \mathbb{Z}$ agit sur le sous-ensemble S de $\overline{E}$ stable par $\mathrm{Gal}(\overline{E}/E)$ et par la translation $\alpha \to \alpha + 1$ (qui définit l'action du générateur 1 de $\mathbb{Z}$ sur S). La proposition ci-dessous montre que, si $\mathcal{O}(S)$ désigne l'ensemble des orbites de $\overline{E}$ sous l'action de $\mathrm{Gal}(\overline{E}/E) \times \mathbb{Z}$, on a une décomposition canonique de tout objet X de $\mathcal{C}_{S,E}$

$$X = \oplus_{A \in \mathcal{O}(S)} X_{(A)}$$

où $X_{(A)}$ est le plus grand sous-objet de X pour lequel toutes les valeurs propres de ∇_0 sont dans A.

Pour tout objet X de $\mathcal{C}_{S,E}$ et tout $i \in \mathbb{Z}$, notons $X_{(i)}$ le sous-espace caractéristique correspondant à la valeur-propre i de ∇_0. Il est muni d'une filtration croissante par des sous E-espaces vectoriels

$$0 = X_{(i,-1)} \subset X_{(i,0)} \subset \dots X_{(i,n-1)} \subset X_{(i,n)} \subset \dots \subset X_{(i)}$$

définie inductivement en posant $X_{(i,n)} = \{x \in X \mid \nabla_0(x) - ix \in X_{(i,n-1)}\}$, pour tout $n \in \mathbb{N}$. Si X est dans $\mathcal{C}^f_{S,E}$, on an a $X_{(i)} = X_{(i,n)}$ pour n suffisamment grand. L'assertion suivante est immédiate :

Proposition 2.4. — *Soit S un sous-ensemble de $\overline{E}$ contenant 0, stable par $\mathrm{Gal}(\overline{E}/E)$ et par la translation $\alpha \to \alpha + 1$. Soit X un objet de $\mathcal{C}^f_{S,E}$ et soit $X_{(\mathbb{Z})} = \sum_{i \in \mathbb{Z}} X_{(i)}$. Alors $X_{(\mathbb{Z})}$ est stable par ∇_0 et $\underline{t}$. C'est le plus grand sous-objet de X qui est dans $\mathcal{C}^f_{\mathbb{Z},E}$ et c'est un facteur direct de X. Pour tout $n \in \mathbb{N}$, l'application naturelle $\mathrm{Ext}^n_{\mathcal{C}_{S,E}}(E, X_{(\mathbb{Z})}) \to \mathrm{Ext}^n_{\mathcal{C}_{S,E}}(E, X)$ est un isomorphisme*

□

Remarquons également que le complexe $(C_{X_{(\mathbb{Z})}})$ se décompose en une somme directe de complexes

$$(C_{X,i}) \qquad X_{(i)} \to X_{(i+1)} \oplus X_{(i)} \to X_{(i+1)} \to 0 \to 0 \to 0 \dots$$

avec $d^0(x) = (\underline{t}x, \nabla_0(x))$ et $d^1(y,z) = \underline{t}z - (\nabla_0 - 1)(y)$.

Proposition 2.5. — *Soit X un objet de $\mathcal{C}_{S,E}$.*
i) Les $\mathrm{Ext}^n_{\mathcal{C}_{S,E}}(E, X)$ s'identifient canoniquement et fonctoriellement aux groupes de cohomologie du complexe

$$(C_{X,0}) \qquad X_{(0)} \to X_{(1)} \oplus X_{(0)} \to X_{(1)} \to 0 \to 0 \to 0 \dots$$

(avec $d^0(x) = (\underline{t}x, \nabla_0(x))$ et $d^1(y,z) = \underline{t}z - (\nabla_0 - 1)(y)$).
ii) Soit E' l'objet de $\mathcal{C}_{S,E}$ dont le E-espace vectoriel sous-jacent est E lui-même, avec $\underline{t} = 0$ et $\nabla_0 = \mathrm{id}_E$. Alors $\mathrm{Ext}^2_{\mathcal{C}_{S,E}}(E, E')$ est un E-espace vectoriel de dimension 1 et, pour $0 \le n \le 2$, le cup-produit induit une dualité parfaite

$$\mathrm{Ext}^n_{\mathcal{C}_{S,E}}(E, X) \times \mathrm{Ext}^{2-n}_{\mathcal{C}_{S,E}}(X, E') \to \mathrm{Ext}^2_{\mathcal{C}_{S,E}}(E, E')$$

Preuve : L'assertion (i) résulte de ce que, pour $i \neq 0$, le complexe $C_{X,i}$ est acyclique puisque ∇_0 est bijectif sur $X_{(i)}$ et $\nabla_0 - 1$ est bijectif sur $X_{(i+1)}$. Compte-tenu de (i), le fait que $\mathrm{Ext}^2_{\mathcal{C}_{S,E}}(E, E')$ est de dimension 1 est immédiat. Le reste de l'assertion (ii) résulte de ce que, si $X' = \mathcal{H}om(X, E')$, le complexe $C_{X',0}$ s'identifie au dual, convenablement décalé, du complexe $C_{X,0}$. □

Soit $\varepsilon \in H^1(C_{X,0})$. Il est facile de décrire explicitement une extension Y de E par X dont la classe est ε : si $(b,c) \in Z^1(C_{X,0})$ représente ε, alors, en tant que E-espace vectoriel $Y = X \oplus E$. Si $(x,\lambda) \in Y$, on a $\underline{t}(x,\lambda) = (\underline{t}x + \lambda b, 0)$ et $\nabla_0(x,\lambda) = (\nabla_0(x) + \lambda c, 0)$. En particulier, on voit que cette extension est scindée, en tant que suite exacte de $E[[\underline{t}]]$-modules, si et seulement si l'on peut choisir le représentant (b,c) pour que $b = 0$.

Si X_1 et X_2 sont des objets de $\mathcal{C}^f_{S,E}$, on note $\mathrm{Ext}^1_{\mathcal{C}_{S,E},0}(X_1,X_2)$ le sous-E-espace vectoriel de $\mathrm{Ext}^1_{\mathcal{C}_{S,E}}(X_1,X_2)$ qui classifie les extensions de X_1 par X_2 qui sont scindées en tant qu'extensions de $E[[\underline{t}]]$-modules. Rappelons que le $E[[\underline{t}]]$-module sous-jacent à $\mathcal{H}\wr\Updownarrow(X_1,X_2)$ ne s'identifie pas en général à $\mathcal{L}_{E[[\underline{t}]]}(X_1,X_2)$. Cependant, on vérifie sans peine que lorsque l'on identifie $\mathrm{Ext}^1_{\mathcal{C}_{S,E}}(X_1,X_2)$ à $\mathrm{Ext}^1_{\mathcal{C}_{S,E}}(E,\mathcal{H}\wr\Updownarrow(X_1,X_2))$, le sous-$E$-espace vectoriel $\mathrm{Ext}^1_{\mathcal{C}_{S,E},}(X_1,X_2)$ s'identifie à $\mathrm{Ext}^1_{\mathcal{C}_{S,E},0}(E,\mathcal{H}\wr\Updownarrow(X_1,X_2))$. Le calcul de ce Ext^1 se ramène donc au calcul de $\mathrm{Ext}^1_{\mathcal{C}_{S,E},0}(E,X)$ pour X objet de $\mathcal{C}^f_{S,E}$. Mais ce qu'on vient de faire nous montre que, si $X_{(0),\underline{t}=0}$ désigne le noyau de l'application $X_{(0)} \to X_{(1)}$ induite par la multiplication par $\underline{t}$, ce groupe s'identifie au H^1 du sous-complexe de $C_{X,0}$

$$X_{(0),\underline{t}=0} \xrightarrow{\nabla_0} X_{(0),\underline{t}=0}$$

On a donc :

Proposition 2.6. — *Soit X un objet de $\mathcal{C}^f_{S,E}$. On a une suite exacte*

$$0 \to \mathrm{Hom}_{\mathcal{C}_{S,E}}(E,X) \to X_{(0),\underline{t}=0} \xrightarrow{\nabla_0} X_{(0),\underline{t}=0} \to \mathrm{Ext}^1_{\mathcal{C}_{S,E},0}(E,X) \to 0$$

et $\dim_E \mathrm{Ext}^1_{\mathcal{C}_{S,E},0}(E,X) = \dim_E \mathrm{Hom}_{\mathcal{C}_{S,E}}(E,X)$.

□

Le corps $E((\underline{t}))$ des séries formelles en $\underline{t}$ à coefficients dans E a une structure naturelle d'objet de $\mathcal{C}_{\mathbb{Z},E}$, avec $\nabla_0(\sum a_i\underline{t}^i) = \sum i a_i \underline{t}^i$.Tout idéal fractionnaire de $E[[\underline{t}]]$, en particulier $E[[\underline{t}]]$ lui-même, est stable par $\underline{t}$ et ∇_0.

Proposition 2.7. — *Soit X un objet de $\mathcal{C}^f_{S,E}$. On a une suite exacte*

$$0 \to \mathrm{Hom}_{\mathcal{C}_{S,E}}(E[[\underline{t}]],X) \to X_{(0)} \xrightarrow{\nabla_0} X_{(0)} \to \mathrm{Ext}^1_{\mathcal{C}_{S,E}}(E[[\underline{t}]],X) \to 0$$

En particulier les E-espaces vectoriels $\mathrm{Hom}_{\mathcal{C}_{S,E}}(E[[\underline{t}]],X)$ *et* $\mathrm{Ext}^1_{\mathcal{C}_{S,E}}(E[[\underline{t}]],X)$ *ont la même dimension finie et* $\mathrm{Hom}_{\mathcal{C}_{S,E}}(E[[\underline{t}]],X) = X_{(0,0)}$.

Preuve : Se donner une application $E[[\underline{t}]]$-linéaire de $E[[\underline{t}]]$ dans X revient à se donner l'image x de 1 dans X qui peut être n'importe quel élément. L'application ainsi définie commute à ∇_0 si et seulement si $\nabla_0(x) = 0$. D'où $\mathrm{Hom}_{\mathcal{C}_{S,E}}(E[[\underline{t}]],X) = \mathrm{Ker}\ (X_{(0)} \xrightarrow{\nabla_0} X_{(0)}) = X_{(0,0)}$.

Soit Y une extension de $E[[\underline{t}]]$ par X. L'élément $1 \in E[[\underline{t}]]$ est dans le sous-espace propre associé à la valeur-propre 0 de ∇_0 et on peut choisir un relèvement e de 1 dans le sous-espace caractéristique de Y associé à la valeur propre 0. On doit avoir $a = \nabla_0(e) \in X_{(0)}$. Si l'on change le relèvement de 1 en choisissant $e' = e + x$, avec $x \in X_{(0)}$, on a $\nabla_0(e') = a + \nabla_0(x)$ et l'image de a dans Coker $(X_{(0)} \xrightarrow{\nabla_0} X_{(0)})$ ne dépend pas du choix du relèvement. On a ainsi défini une application de $\mathrm{Ext}^1_{\mathcal{C}_{S,E}}(E[[\underline{t}]], X)$ dans ce conoyau. On vérifie sans peine que c'est un isomorphisme. □

Proposition 2.8. — *Soit X un objet de $\mathcal{C}^f_{S,E}$. Notons $\overline{X}_{(-1)}$ le plus grand quotient de $X_{(-1)}$ sur lequel l'action de ∇_0 est semi-simple. Alors $\mathrm{Hom}_{\mathcal{C}_{S,E}}(X, E((\underline{t}))/E[[\underline{t}]])$ s'identifie au E-espace vectoriel des applications E-linéaires de $\overline{X}_{(-1)}$ dans le sous-E-espace vectoriel de $E((\underline{t}))/E[[\underline{t}]]$ engendré par l'image de $\underline{t}^{-1}$. C'est un E-espace vectoriel de dimension finie égale à celle de $X_{(-1,0)}$.*

Preuve : La deuxième assertion résulte de la première puisqu'alors la dimension de $\mathrm{Hom}_{\mathcal{C}_{S,E}}(X, E((\underline{t}))/E[[\underline{t}]])$ est égale à celle du plus grand quotient de $X_{(-1)}$ sur lequel l'action de ∇_0 est semi-simple et que cette dimension est aussi celle du sous-espace propre associé à la valeur propre -1.
Montrons la première assertion. Comme les valeurs propres de ∇_0 agissant sur $E((\underline{t}))/E[[\underline{t}]]$ sont dans $\mathbb{Z}$, on a

$$\mathrm{Hom}_{\mathcal{C}_{S,E}}(X, E((\underline{t}))/E[[\underline{t}]]) = \mathrm{Hom}_{\mathcal{C}_{S,E}}(X_{\mathbb{Z}}, E((\underline{t}))/E[[\underline{t}]])$$

et on peut supposer que $X = X_{(\mathbb{Z})}$. Comme l'action de ∇_0 sur $E((\underline{t}))/E[[\underline{t}]]$ est semi-simple, tout morphisme de X dans $E((\underline{t}))/E[[\underline{t}]]$ se factorise à travers le plus grand quotient de X sur lequel l'action de ∇_0 est semi-simple ; on peut donc supposer que l'action de ∇_0 sur X est semi-simple et écrire $X = \oplus_{i\in\mathbb{Z}} X_{(i)}$, où les $X_{(i)}$ sont des E-espaces vectoriels de dimension finie presque tous nuls, ∇_0 étant la multiplication par i dans $X_{(i)}$ et $\underline{t}$ étant une application E-linéaire qui envoie $X_{(i)}$ dans $X_{(i+1)}$.
Pour tout $m \in \mathbb{Z}$, $F^m X = \oplus_{i\geq m} X_{(i)}$ est un sous-objet de X. Les $(F^m X)_{m\in\mathbb{Z}}$ définissent une filtration décroissante, exhaustive et séparée de X par des sous-objets de $\mathcal{C}_{S,E}$. Pour tout m, le quotient $F^m X/F^{m+1}X$ s'identifie à $X_{(m)}$, avec $\nabla_0 =$ la multiplication par m et $\underline{t} = 0$.
Par dévissage, il suffit d'établir le lemme suivant :

Lemme 2.9. — *Soit $m \in \mathbb{Z}$ et soit X un E-espace vectoriel de dimension finie, muni d'une structure d'objet de $\mathcal{C}_{S,E}$ définie par $\nabla_0(x) = mx$ et $\underline{t}x = 0$, pour tout $x \in X$.*
i) Si $m = -1$, alors $\mathrm{Hom}_{\mathcal{C}_{S,E}}(X, E((\underline{t}))/E[[\underline{t}]])$ s'identifie au E-espace vectoriel des applications E-linéaires de X dans le E-espace vectoriel engendré par l'image t_{-1} de $\underline{t}^{-1}$ dans $E((\underline{t}))/E[[\underline{t}]])$,
ii) si $m \neq -1$, on a $\mathrm{Hom}_{\mathcal{C}_{S,E}}(X, E((\underline{t}))/E[[\underline{t}]]) = 0$,
iii) si $m \neq 0$, on a $\mathrm{Ext}^1_{\mathcal{C}_{S,E}}(X, E((\underline{t}))/E[[\underline{t}]]) = 0$.

Preuve : On voit que le noyau de $\underline{t}$ dans $E((\underline{t}))/E[[\underline{t}]]$ est aussi le sous-espace propre associé à la valeur-propre -1 et que c'est le E-espace vectoriel de dimension 1 de base t_{-1}. Les assertions (i) et (ii) sont alors évidentes.
Soit maintenant Y un objet de $\mathcal{C}_{S,E}$, extension de $E((\underline{t}))/E[[\underline{t}]]$ par X. Pour tout entier $r \geq 1$, soit Z_r le sous $E[[\underline{t}]]$-module de $E((\underline{t}))/E[[\underline{t}]]$ engendré par l'image t_{-r} de $\underline{t}^{-r}$ et soit Y_r l'image inverse de Z_r dans Y. Alors Y est la réunion croissante des Y_r et il suffit de vérifier que, pour tout entier $r > \sup\{0, -m\}$, la suite

$$0 \to X \to Y_r \to Z_r \to 0$$

admet un unique scindage. Comme $\nabla_0(t_{-r}) = -rt_{-r}$ et, comme $-r$ n'est pas valeur-propre de ∇_0 agissant sur X, il existe un unique relèvement e_{-r} de t_{-r} dans Y_r tel que $\nabla_0(e_{-r}) = -re_{-r}$ et il suffit de vérifier que $\underline{t}^r e_r = 0$. A priori, on a $\underline{t}^r e_{-r} = x \in X$, mais on doit avoir $\nabla_0(\underline{t}^r e_{-r}) = r\underline{t}^r e_{-r} + \underline{t}^r \nabla_0(e_{-r}) = 0$, ce qui implique x=0 puisque $\nabla_0(x) = mx$ est différent de 0 si $x \neq 0$. D'où (iii). □

2.3 – Petites représentations

Soit $\mathcal{C}$ une catégorie abélienne. *Une sous-catégorie épaisse* de $\mathcal{C}$ est une sous-catégorie pleine stable par sous-objet, quotient, somme directe (c'est donc encore une catégorie abélienne) et extension.
Pour tout sous-groupe S de $\overline{K}$ stable sous l'action de G_K, introduisons la sous-catégorie pleine $\mathrm{Rep}_{B^+_{dR},S}(G_K)$ de $\mathrm{Rep}_{B^+_{dR}}(G_K)$ dont les objets sont les W tels que les valeurs-propres de ∇_0 - vu comme un endomorphisme du K_∞-espace vectoriel sous-jacent à W^f - sont dans S. C'est une sous-catégorie tannakienne épaisse de $\mathrm{Rep}_{B^+_{dR}}(G_K)$.

Soit $\mathcal{O}_K$ l'anneau des entiers de K. On dit qu'un élément $\alpha \in \overline{K}$ est *K-petit* s'il est dans K et si $\alpha \log \chi(g) \in 2p\mathcal{O}_K$ pour tout $g \in G_K$. Les éléments K-petits forment un idéal fractionnaire $\mathfrak{a}_K$ de l'anneau des entiers $\mathcal{O}_K$ de K contenant $\mathcal{O}_K$. On dit qu'une B^+_{dR}-représentation W de G_K est *petite* si c'est un objet de $\mathrm{Rep}_{B^+_{dR},\mathfrak{a}_K}(G_K)$. Pour tout $\alpha \in \overline{K}$, il existe une extension finie L de K contenue dans $\overline{K}$ telle que α est L-petit ; pour toute extension finie L' de L contenue dans $\overline{K}$, α est alors a fortiori L'-petit.
Soit X un objet de la catégorie $\mathcal{C}^f_{\mathfrak{a}_K,K}$. L'endomorphisme $\underline{t}$ permet de munir le K-espace vectoriel sous-jacent à X d'une structure de $K[[\underline{t}]]$-module de longueur finie. On fait agir G_K sur X via son quotient Γ_K en posant, pour tout $\gamma \in \Gamma_K$ et tout $x \in X$,

$$\gamma(x) = \exp(\log \chi(\gamma).\nabla_0)(x)$$

Cette action est semi-linéaire relativement à l'action naturelle de Γ_K sur $K[[\underline{t}]]$, ce qui nous permet, par extension des scalaires, de munir le B^+_{dR}-module de type fini $R_{dR}(X) = B^+_{dR} \otimes_{K[[\underline{t}]]} X$ d'une action de G_K qui en fait une B^+_{dR}-représentation de G_K. On peut considérer la correspondance R_{dR} comme un foncteur de $\mathcal{C}^f_{\mathfrak{a}_K,K}$ dans $\mathrm{Rep}_{B^+_{dR}}(G_K)$.

Proposition 2.10. — *Le foncteur* $R_{dR} : \mathcal{C}^f_{\mathfrak{a}_K,K} \to \mathrm{Rep}_{B^+_{dR}}(G_K)$ *est pleinement fidèle et son image essentielle est la catégorie* $\mathrm{Rep}_{B^+_{dR},\mathfrak{a}_K}(G_K)$ *des petites* B^+_{dR}*-représentations de* G_K.

Preuve : Pour tout objet X de $\mathcal{C}^f_{\mathfrak{a}_K,K}$, il est clair que $X \subset (R_{dR}(X))^f$ et on en déduit que $(R_{dR}(X))^f$ s'identifie à $K_\infty[[\underline{t}]] \otimes_{K[[\underline{t}]]} X = K_\infty \otimes_K X$, l'application ∇_0 sur $(R_{dR}(X))^f$ étant l'extension par K_∞-linéarité de l'application ∇_0 sur X. En particulier, $R_{dR}(X)$ est un objet de $\mathrm{Rep}_{B^+_{dR},\mathfrak{a}_K}(G_K)$.

Lemme 2.11. — *Soit* W *un objet de* $\mathrm{Rep}_{B^+_{dR},\mathfrak{a}_K}(G_K)$, ∇_0 *le* K_∞*-endomorphisme de* W^f *qui lui est associé (cf.* prop.2.2*) et soit*

$$W^f_K = \{x \in W^f \mid \gamma(x) = \exp(\log\chi(\gamma).\nabla_0)(x),\ \text{pour tout } \gamma \in \Gamma_K\}\ .$$

L'application K_∞*-linéaire* $K_\infty \otimes_K W^f_K = K_\infty[[\underline{t}]] \otimes_{K[[\underline{t}]]} W^f_K \to W^f$*, déduite par extension des scalaires de l'inclusion de* W^f_K *dans* W^f*, est un isomorphisme*

Le lemme implique la proposition puisqu'on voit que le foncteur $W \mapsto W^f_K$ est un quasi-inverse du foncteur R_{dR}.

Prouvons le lemme : Soient d'abord W une B^+_{dR}-représentation arbitraire et $\delta \in \Gamma_K$. Si $y \in W^f$ et si $\gamma \in \Gamma_K$ est suffisamment petit, on a $\gamma(y) = \delta\gamma\delta^{-1}(y) = \delta(\exp(\log\chi(\gamma).\nabla_0)(\delta^{-1}(y))) = \exp(\log\chi(\gamma).\delta\nabla_0\delta^{-1})(y)$. L'unicité de ∇_0 implique donc que $\nabla_0\delta = \delta\nabla_0$.
Supposons maintenant que W est un objet de $\mathrm{Rep}_{B^+_{dR},\mathfrak{a}_K}(G_K)$. Soient $y_1, y_2, \ldots, y_h$ des éléments de W^f qui l'engendrent en tant que $K_\infty[[\underline{t}]]$-module. Pour chaque y_i, choisissons un Γ_{K,y_i} comme dans la proposition 2.2, soit Γ'_K l'intersection des Γ_{K,y_i} et $K' = K_\infty^{\Gamma'_K}$.
Soient $p^n = (\Gamma_K : \Gamma'_K)$ et γ_0 un générateur topologique de Γ_K, de sorte que
a) le corps K' est l'unique extension de degré p^n de K contenue dans K_∞,
b) tout élément de Γ_K s'écrit d'une manière et d'une seule sous la forme $\gamma_0^{\tau(\gamma)}$ avec $\tau(\gamma) \in \mathbb{Z}_p$,
c) si $\gamma \in \Gamma_K$, alors $\gamma \in \Gamma'_K$ si et seulement si p^n divise $\tau(\gamma)$.
Pour $1 \le i \le h$ et tout $\gamma \in \Gamma'_K$, on a $\gamma(y_i) = \exp(\log\chi(\gamma).\nabla_0)(y_i)$. Soit X' le sous-$K'[[\underline{t}]]$-module de W^f engendré par les $\gamma_0^m y_i$, pour $0 \le m < p^n$ et $1 \le i \le h$. C'est un sous-K'-espace vectoriel de dimension finie de W^f stable par Γ_K. Comme $\delta\nabla_0 = \nabla_0\delta$, pour tout $\delta \in \Gamma_K$, on a $\gamma(x) = \exp(\log\chi(\gamma).\nabla_0)(x)$ pour tout $\gamma \in \Gamma'_K$ et tout $x \in X'$.
Par construction, X' contient une base $\{e_1, e_2, \ldots, e_d\}$ de W^f sur K_∞. Si $x \in W^f$, on peut écrire $x = \sum_{i=1}^d a_i e_i$, avec les $a_i \in K_\infty$. Pour tout $\gamma \in \Gamma'_K$, on a $\gamma(x) = \sum \gamma(a_i)\gamma(e_i)$; si $x \in X'$, on a aussi $\gamma(x) = \sum a_i\gamma(e_i)$, d'où on déduit que $a_i \in K'$ pour tout i ; donc $\{e_1, e_2, \ldots, e_d\}$ est aussi une base de X' sur K' et l'application naturelle

$$K_\infty[[\underline{t}]] \otimes_{K'[[\underline{t}]]} X' = K_\infty \otimes_{K'} X' \to W^f$$

est un isomorphisme.
Considérons l'automorphisme $\eta = \exp(\log \chi(\gamma_0).\nabla_0)$ du K'-espace vectoriel X'. Pour tout $\gamma \in \Gamma_K$, on a $\eta \circ \gamma = \gamma \circ \eta$
Pour tout $\gamma \in \Gamma_K$, notons $\rho(\gamma) \in GL_d(K')$ la matrice dont la j-ième colonne est formée des composantes de $\gamma(e_j)$ sur la base $\{e_1, e_2, \ldots, e_d\}$. Cette matrice commute avec la matrice A de l'endomorphisme η dans la même base. Si, pour tout $\gamma \in \Gamma_K$, on pose $\rho_0(\gamma) = \rho(\gamma)A^{-\tau(\gamma)}$, on voit que $\rho_0(\gamma)$ ne dépend que de l'image de γ dans $\mathrm{Gal}(K'/K)$ et que l'application de $\mathrm{Gal}(K'/K)$ dans $GL_d(K')$ ainsi définie est un 1-cocycle. La trivialité de $H^1(\mathrm{Gal}(K'/K), GL_d(K'))$ implique que, quitte à changer la base, on peut supposer que $\rho_0 = 1$. Mais alors les e_j sont dans W_K^f et le même argument que celui que l'on a utilisé pour prouver qu'ils forment une base de X' sur K' montre qu'ils forment aussi une base de W_K^f sur K. Le lemme en résulte. □

Soit $\mathcal{O}_K$ l'anneau des entiers de K. Pour tout $\alpha \in \overline{K}$ qui est K-petit, on note $\chi^{(\alpha)} : G_K \to \mathcal{O}_K^*$ l'homomorphisme continu défini par $\chi^{(\alpha)}(g) = \exp(\alpha \log(\chi(g))$. On note $\mathcal{O}_K\{\alpha\}$ le $\mathcal{O}_K$-module libre de rang 1 qui est $\mathcal{O}_K$ lui-même sur lequel on fait agir G_K via le caractère $\chi^{(\alpha)}$. Enfin pour tout $\mathcal{O}_K$-module M muni d'une action $\mathcal{O}_K$-linéaire de G_K, on pose $M\{\alpha\} = M \otimes_{\mathcal{O}_K} \mathcal{O}_K\{\alpha\}$. Pour tout $i \in \mathbb{Z}$, $\chi^i = \chi^{(i)}(\omega)^i$, où $\omega : G_K \to \mathbb{Z}_p^*$ est le caractère d'ordre fini qui donne l'action de G_K sur les racines $2p$-ièmes de 1 ; il en résulte que $C\{i\}$ est isomorphe, non canoniquement, à $C(i)$.
Si α est K-petit, comme $C\{\alpha\}$ est de dimension 1 sur C, $C\{\alpha\}$ est un objet simple de la catégorie des B_{dR}^+-représentations ; comme l'opérateur $\nabla_0(W)$ est la multiplication par α, $C\{\alpha\}$ est une petite B_{dR}^+-représentation.

Proposition 2.12. — *L'application qui à $\alpha \in \mathfrak{a}_K$ associe la classe d'isomorphisme de $C\{\alpha\}$ définit une bijection de l'ensemble des éléments K-petits sur celui des classes d'isomorphismes d'objets simples de la catégorie des petites B_{dR}^+-représentations de G_K.*

Preuve : C'est une conséquence immédiate de la proposition précédente. □

Remarque : Notons $\overline{\mathfrak{a}}_K$ l'ensemble des orbites de $\mathfrak{a}_K$ sous l'action de $\mathbb{Z}$ (où 1 agit par la translation $\alpha \to \alpha+1$). La décomposition canonique de tout objet de $\mathcal{C}_{S,E}$ en somme directe indexée par les orbites de S sous l'action de $\mathrm{Gal}(\overline{E}/E)$ induit, via le foncteur R_{dR}, lorsque l'on prend $S = \mathfrak{a}_K$, une décomposition canonique de toute petite représentation

$$W = \oplus_{A \in \overline{\mathfrak{a}}_K} W_A$$

où (avec les notations du lemme 2.11) $W_A = R_{dR}((W_K^f)_{(A)})$ est la plus grande sous-B_{dR}^+-représentation de W telle que les valeurs propres de ∇_0 sur $W_A)^f$ sont dans A. En particulier $W_{\mathbb{Z}}$ correspond à la plus grande représentation pour laquelle les valeurs-propres de ∇_0 sont dans $\mathbb{Z}$.

2.4 – Changement de base

Soit L une extension finie de K (la discussion qui suit reste valable en remplaçant K par n'importe quel corps). Soit $\mathcal{C}$ une catégorie tannakienne sur K. Rappelons (cf. par exemple [DM82], p.155) que l'on peut définir la catégorie tannakienne $\mathcal{C}_L$ sur L déduite de $\mathcal{C}$ par l'extension des scalaires $K \to L$: Un objet de $\mathcal{C}_L$ peut être vu comme un couple (X, ρ) formé d'un objet X de $\mathcal{C}$ et d'un homomorphisme de K-algèbres $\rho : L \to \mathrm{End}_{\mathcal{C}}(X)$; les morphismes sont les morphismes des objets de $\mathcal{C}$ sous-jacents qui sont L-linéaires. La structure tannakienne se définit facilement ; on peut fabriquer l'objet-unité $(1_{\mathcal{C}_L}, \rho_0)$ de $\mathcal{C}_L$ en prenant $1_{\mathcal{C}_L} = (1_{\mathcal{C}})^d$ où $d = [L : K]$ et pour $\rho_0 : L \to \mathcal{M}_d(K)$, anneau des matrices carrées à d lignes et d colonnes à coefficients dans K, un plongement arbitraire. Le foncteur de *restriction des scalaires*

$$\mathrm{Res} : \mathcal{C}_L \to \mathcal{C}$$

est le foncteur qui envoie (X, ρ) sur X. C'est un foncteur additif exact et fidèle qui a un adjoint à gauche, le foncteur d'*extension des scalaires*

$$\mathrm{Ext} : \mathcal{C} \to \mathcal{C}_L$$

qui envoie W sur $W_L = (1_{\mathcal{C}_L} \otimes X, \rho_0 \otimes \mathrm{id})$.
Si X et Y sont deux objets de $\mathcal{C}$, pour tout $i \in \mathbb{N}$, l'application naturelle $L \otimes_K \mathrm{Ext}^i_{\mathcal{C}}(X, Y) \to \mathrm{Ext}^i_{\mathcal{C}_L}(X_L, Y_L)$ est un isomorphisme. Lorsque l'extension L/K est galoisienne, le groupe $\mathrm{Gal}(L/K)$ agit sur $\mathrm{Ext}^i_{\mathcal{C}_L}(X_L, Y_L)$ et $\mathrm{Ext}^i_{\mathcal{C}}(X, Y)$ s'identifie à $\mathrm{Ext}^i_{\mathcal{C}_L}(X_L, Y_L)^{\mathrm{Gal}(L/K)}$.

Notons $B^+_{dR}[G_K]$ le B^+_{dR}-module libre de base les $g \in G_K$. On le munit d'une structure d'anneau non commutatif contenant B^+_{dR} en décrétant que la multiplication de deux éléments de G_K est celle qui est donnée par la loi de groupe et que $g.b = g(b).g$, si $b \in B^+_{dR}$ et $g \in G_K$ (ce n'est donc pas l'algèbre de groupe usuelle, l'anneau B^+_{dR} n'est pas contenu dans le centre de $B^+_{dR}[G_K]$). La catégorie des B^+_{dR}-représentations de G_K s'identifie, de manière évidente à une sous-catégorie pleine de la catégorie des $B^+_{dR}[G_K]$-modules à gauche.
Soit L une extension finie de K contenue dans $\overline{K}$. L'anneau $B^+_{dR}[G_L]$ s'identifie à un sous-anneau de $B^+_{dR}[G_K]$. La restriction des scalaires de $B^+_{dR}[G_L]$ à $B^+_{dR}[G_K]$ et l'extension des scalaires dans l'autre sens induisent des foncteurs adjoints à gauche l'un de l'autre

$$R^L_K : \mathrm{Rep}_{B^+_{dR}}(G_K) \to \mathrm{Rep}_{B^+_{dR}}(G_L) \quad \text{et} \quad I^K_L : \mathrm{Rep}_{B^+_{dR}}(G_L) \to \mathrm{Rep}_{B^+_{dR}}(G_K)$$

On remarque que, si $g_1, g_2, \ldots, g_d$ désigne un système de représentants des classes à gauche de G_K suivant G_L, et si X est un $B^+_{dR}[G_L]$-module à gauche, tout élément de $I^K_L X = B^+_{dR}[G_K] \otimes_{B^+_{dR}[G_L]} X$ s'écrit d'une manière et d'une seule sous la forme $\sum_{i=1}^d g_i \otimes x_i$ avec les $x_i \in X$.

Soit $\mathrm{Rep}_{B^+_{dR}}(G_K)_L$ la catégorie déduite de $\mathrm{Rep}_{B^+_{dR}}(G_K)$ par l'extension des scalaires $K \to L$. On dispose de foncteurs

$$\Phi_{L/K} : \mathrm{Rep}_{B^+_{dR}}(G_K)_L \to \mathrm{Rep}_{B^+_{dR}}(G_L) \text{ et } \Psi_{L/K} : \mathrm{Rep}_{B^+_{dR}}(G_L) \to \mathrm{Rep}_{B^+_{dR}}(G_K)_L$$

Si (W,ρ) est un objet de $\mathrm{Rep}_{B^+_{dR}}(G_K)_L$, $\Phi_{L/K}(W,\rho)$ est le plus grand sous-B^+_{dR}-module de W sur lequel les deux structures données de L-espace vectoriel coïncident :

$$\Phi_{L/K}(W,\rho) = \{w \in W \mid \rho(\lambda)(w) = \lambda w \text{ pour tout } \lambda \in L\}$$

qui est bien sûr stable par G_L. Si X est un objet de $\mathrm{Rep}_{B^+_{dR}}(G_L)$, alors $\Psi_{L/K}(X) = (I_{L/K}X, \rho)$, avec $\rho(\lambda)(\sum g_i \otimes x_i) = \sum g_i \otimes \lambda x_i$.

Proposition 2.13. — *Le foncteur $\Phi_{L/K} : \mathrm{Rep}_{B^+_{dR}}(G_K)_L \to \mathrm{Rep}_{B^+_{dR}}(G_L)$ induit une $\otimes$-équivalence entre ces deux catégories tannakiennes et $\Psi_{L/K}$: $\mathrm{Rep}_{B^+_{dR}}(G_L) \to \mathrm{Rep}_{B^+_{dR}}(G_K)_L$ est un quasi-inverse.*

Preuve : C'est immédiat. □

Remarque : Dans cette correspondance, l'extension des scalaires correspond à la restriction de l'action du groupe de Galois tandis que la restriction des scalaires correspond à l'induction : on a des identifications évidentes de foncteurs

$$\Phi_{L/K} \circ \mathrm{Ext} = R^L_K \text{ et } \mathrm{Res} \circ \Psi_{L/K} = I^K_L$$

Corollaire. — *Soit L une extension finie galoisienne de G_K. Soient W' et W'' deux B^+_{dR}-représentations de G_K. Pour tout $i \in \mathbb{N}$, $\mathrm{Ext}^i_{B^+_{dR}[G_K]}(W'',W')$ s'identifie à $(\mathrm{Ext}^i_{B^+_{dR}[G_L]}(W'',W'))^{\mathrm{Gal}(L/K)}$; on a*

$$\dim_K \mathrm{Ext}^i_{B^+_{dR}[G_K]}(W'',W') = \dim_L \mathrm{Ext}^i_{B^+_{dR}[G_L]}(W'',W').$$

Preuve : Par définition, $\mathrm{Ext}^i_{B^+_{dR}[G_L]}(W'',W') = \mathrm{Ext}^i_{B^+_{dR}[G_L]}(R^L_K W'', R^L_K W')$. D'où, avec des notations évidentes,

$$\mathrm{Ext}^i_{B^+_{dR}[G_L]}(W'',W') = \mathrm{Ext}^i_{B^+_{dR}[G_L]}(\Phi_{L/K}(\mathrm{Ext}(W'')), \Phi_{L/K}(\mathrm{Ext}(W'))) =$$
$$\mathrm{Ext}^i_{B^+_{dR}[G_K]_L}(\mathrm{Ext}(W''), \mathrm{Ext}(W')) = L \otimes_K \mathrm{Ext}^i_{B^+_{dR}[G_K]}(W'',W')$$

et la proposition s'en déduit. □

2.5 – Calcul des Ext^n dans la catégorie des B^+_{dR}-représentations

Soit W une B_{dR}-représentation. Pour tout $i \in \mathbb{Z}$, posons $W_{(i,-1)} = 0$, et définissons inductivement, pour tout $n \in \mathbb{N}$ le K-espace vectoriel $W_{(i,n)}$ par

$$W_{(i,n)} = \{w \in W \mid g(x) - \chi^{(i)}(g)(x) \in W_{(i,n-1)} \text{ pour tout } g \in G_K\}$$

Posons aussi $W_{(i)} = \cup_{n\in\mathbb{N}} W_{(i,n)}$. On voit tout de suite que l'application naturelle de $K_\infty \otimes_K W_{(i)}$ dans W est injective et identifie $K_\infty \otimes_K W_{(i)}$ (resp. $K_\infty \otimes_K W_{(i,0)}$) au sous-espace caractéristique (resp. propre) de W^f associé à la valeur-propre i. En particulier $W_{(\mathbb{Z})} = \oplus_{i\in\mathbb{Z}} W_{(i)}$ a une structure naturelle d'objet de la catégorie $\mathcal{C}^f_{\mathbb{Z},K}$ et, avec les notations du §2.2, on a $(W_\mathbb{Z})_{(i)} = W_{(i)}$ et $(W_\mathbb{Z})_{(i,n)} = W_{(i,n)}$; la B^+_{dR}-représentation $R_{dR}(W_{(\mathbb{Z})}) = B^+_{dR} \otimes_{K[[\underline{t}]]} W_{(\mathbb{Z})}$ s'identifie à la plus grande sous-représentation de W qui est dans l'image essentielle de la restriction à $\mathcal{C}^f_{\mathbb{Z},K}$ du foncteur R_{dR} ; c'est un facteur direct de W.

Remarque : On peut trouver plus naturel de considérer les sous-K-espaces vectoriels $W_{i,n}$ pour $i \in \mathbb{Z}$ et $n \geq -1$ définis inductivement par $W_{i,-1} = 0$ et $W_{i,n} = \{w \in W \mid g(x) - \chi^i(g)(x) \in W_{i,n-1} \text{ pour tout } g \in G_K\}$. Si π_t, est comme au §2.1, de sorte que $t = \pi_t \underline{t}$, on voit que $W_{i,n} = \pi_t^i . W_{(i,n)}$ et $W_i = \pi_t^i W_{(i)}$; en particulier, $\dim_K W_{i,n} = \dim_K W_{(i,n)}$ et $\dim_K W_i = \dim_K W_{(i)}$. Si $W_\mathbb{Z} = \oplus_{i\in\mathbb{Z}} W_i$, $W_\mathbb{Z}$ est un sous-$K[[t]]$-module de B^+_{dR} stable par G_K et l'application naturelle $B^+_{dR} \otimes_{K[[t]]} W_\mathbb{Z} \to W$ est injective et a même image que $B^+_{dR} \otimes_{K[[\underline{t}]]} W_{(\mathbb{Z})}$. On observe que W_i n'est fixe par $H_K = \mathrm{Gal}(\overline{K}/K_\infty)$ que si et seulement si $W_i = W_{(i)}$, ce qui équivaut à $\pi_t^i \in K$.

Si W_1 et W_2 sont des B^+_{dR}-représentations, on note $\mathrm{Ext}^1_{B^+_{dR}[G_K],0}(W_1, W_2)$ le sous-groupe de $\mathrm{Ext}^1_{B^+_{dR}[G_K]}(W_1, W_2)$ qui classifie les extensions qui sont scindées en tant qu'extensions de B^+_{dR}-modules.

Théorème 2.14. — *A) Soient W_1 et W_2 deux B^+_{dR}-représentations. Les K-espaces vectoriels $\mathrm{Ext}^i_{B^+_{dR}[G_K]}(W_1, W_2)$ sont de dimension finie, nuls pour $i \geq 3$ et*

$$\sum_{i=0}^{2} (-1)^i \dim_K \mathrm{Ext}^i_{B^+_{dR}[G_K]}(W_1, W_2) = 0 \ .$$

B) Soit γ_0 un générateur topologique de $\Gamma_K = \mathrm{Gal}(\mathrm{K}_\infty/\mathrm{K})$ et soit W une B^+_{dR}-représentation.
i) Les $\mathrm{Ext}^n_{B^+_{dR}[G_K]}(C, W)$ s'identifient, canoniquement et fonctoriellement, aux groupes de cohomologie du complexe

$$W_{(0)} \to W_{(1)} \oplus W_{(0)} \to W_{(1)} \to 0 \to 0 \ldots$$

avec $d^0(x) = (\underline{t}x, (\gamma_0 - 1)(x))$ et $d^1(y, z) = \underline{t}z - (\chi^{(-1)}(\gamma_0)\gamma_0 - 1)(y)$;
ii) on a une suite exacte

$$0 \to \mathrm{Hom}_{B^+_{dR}[G_K]}(C, W) \to W_{(0),\underline{t}=0} \xrightarrow{\gamma_0 - 1} W_{(0),\underline{t}=0} \to \mathrm{Ext}^1_{B^+_{dR}[G_K],0}(C, W) \to 0$$

En particulier, $\mathrm{Hom}_{B^+_{dR}[G_K]}(C, W)$ et $\mathrm{Ext}^1_{B^+_{dR}[G_K],0}(C, W)$ sont des K-espaces vectoriels de même dimension finie égale à celle du noyau de la multiplication par $\underline{t}$ sur $W_{(0,0)}$;

iii) on a une suite exacte

$$0 \to \mathrm{Hom}_{B^+_{dR}[G_K]}(B^+_{dR}, W) \to W_{(0)} \xrightarrow{\gamma_0 - 1} W_{(0)} \to \mathrm{Ext}^1_{B^+_{dR}[G_K]}(B^+_{dR}, W) \to 0$$

En particulier $\mathrm{Hom}_{B^+_{dR}[G_K]}(B^+_{dR}, W) = W_{(0,0)}$ *et* $\mathrm{Ext}^1_{B^+_{dR}[G_K]}(B^+_{dR}, W)$ *est un* K*-espace vectoriel de dimension finie égale à celle de* $W_{(0,0)}$ *;*
iv) le K*-espace vectoriel* $\mathrm{Hom}_{B^+_{dR}[G_K]}(W, B_{dR}/B^+_{dR})$ *s'identifie au dual du plus grand quotient de* $W_{(-1)}$ *sur lequel l'action de* ∇_0 *est semi-simple ; il est de dimenson finie égale à celle de* $W_{(-1,0)}$.

Preuve : On peut (cor. à la prop.2.13) remplacer K par une extension finie galoisienne, ce qui nous permet de supposer que les B^+_{dR}-représentations qui interviennent sont petites. Comme la catégorie des petites représentations est une sous-catégorie épaisse de celle de toutes les B^+_{dR}-représentations, on peut calculer les Ext^i dans la catégorie des petites représentations et utiliser l'équivalence entre cette catégorie et $\mathcal{C}^f_{\mathfrak{a}_K,K}$ (prop.2.10).
Montrons (B i). Si $W = R_{dR}(X)$, d'après la proposition 2.5, les $\mathrm{Ext}^i_{\mathcal{C}_{S,E}}(C, W)$ sont les groupes de cohomologie du complexe

$$(C_{X,0}) \qquad X_{(0)} \to X_{(1)} \oplus X_{(0)} \to X_{(1)} \to 0 \to 0 \to 0 \ldots$$

les différentielles étant $x \mapsto (\underline{t}x, \nabla_0(x))$ et $(y,z) \mapsto \underline{t}z - (\nabla_0 - 1)(y))$ que l'on peut réécrire $x \mapsto (\underline{t}x, Nx)$ et $(y,z) \mapsto \underline{t}z - Ny$, si l'on pose $N = \nabla_0$ sur $W_{(0)}$ et $N = \nabla_0 - 1$ sur $W_{(-1)}$. On remarque que N est nilpotent sur chacun de ces deux K-espaces vectoriels. On remarque aussi que $W_{(0)} = X_{(0)}$ et $W_{(1)} = X_{(1)}$. Choisissons un relèvement $g_0 \in G_K$ de γ_0 et posons $c = \log \chi(g_0)$. C'est un élément non nul de l'idéal de $\mathbb{Z}_p$ engendré par $2p$ qui ne dépend pas du choix du relèvement. Sur $W_{(0)}$ comme sur $W_{(1)}$, on a $\exp(cN) = Nu$, où $u = c + \frac{c^2}{2!}N + \frac{c^3}{3!}N^2 + \ldots + \frac{c^n}{n!}N^{n-1} + \ldots$ est un automorphisme qui commute à N. On a aussi $(\underline{t}N)(x) = (N\underline{t})(x)$ pour tout $x \in W_{(0)}$.
Sur $W_{(0)}$, on a $(\gamma_0 - 1)(x) = \exp(cN)(x) - x = Nu(x)$; sur $W_{(1)}$, on a $(\chi^{(-1)}(\gamma_0)\gamma_0 - 1)(y) = (\exp(-c)\exp(c(\mathrm{id} + N)(y) - y = Nu(y)$. On a alors un diagramme commutatif de complexes

$$\begin{array}{ccccccccc} X_{(0)} & \to & X_{(1)} \oplus X_{(0)} & \to & X_{(1)} & \to & 0 & \to & 0 & \ldots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ W_{(0)} & \to & W_{(1)} \oplus W_{(0)} & \to & W_{(1)} & \to & 0 & \to & 0 & \ldots \end{array}$$

où les flèches verticales sont successivement $a \mapsto a$, $(b,c) \mapsto (b, u(c))$ et $d \mapsto u(d)$. Comme ce sont des isomorphismes, ces deux complexes sont isomorphes. D'où (B i).
De la même manière, les assertions (B ii), (B iii) et (B iv) sont la traduction dans le langage des B^+_{dR}-représentations des propositions 2.6, 2.7 et 2.8.
Comme C est l'objet-unité de la catégorie tannakienne $\mathrm{Rep}_{B^+_{dR}}(G_K)$, le K-espace vectoriel $\mathrm{Ext}^i_{B^+_{dR}[G_K]}(W_1, W_2)$ s'identifie à $\mathrm{Ext}^i_{B^+_{dR}[G_K]}(C, \mathcal{H}om(W_1, W_2))$

(attention c'est le hom interne pour la structure tannakienne déduite par transport de structure de celle de $\mathcal{C}^f_{\mathfrak{a}_K,K}$) et l'assertion (A) résulte de (B i). □

Rappelons que $B_m = B^+_{dR}/t^m B^+_{dR} = B^+_{dR}/\underline{t}^m B^+_{dR}$. On a $B_0 = 0$, $B_1 = C$ et pour tout m une suite exacte

$$0 \to B_m(1) \to B_{m+1} \to C \to 0$$

En particulier B_2 est une extension de C par $C(1)$. On remarque que la B^+_{dR}-représentation B_m est indécomposable (c'est déjà un B^+_{dR}-module indécomposable), que $(B_m)^f = K_\infty[[\underline{t}]]/\underline{t}^m$ et que les valeurs propres de ∇_0 sont les entiers i vérifiant $0 \leq i \leq m-1$, $(B_m)_{(i,0)} = (B_m)_{(i)}$ étant le K-espace vectoriel de dimension 1 engendré par l'image de $\underline{t}^i$.

Notons $\mathbb{Z}_p[\log t]$ l'anneau des polynômes à coefficients dans $\mathbb{Z}_p$ en une indéterminée $\log t$. On munit cet anneau d'une action de G_K compatible avec sa structure de $\mathbb{Z}_p$-algèbre en posant $g(\log t) = \log(\chi(g)) + \log t$. Cette action se factorise à travers $\Gamma_K = \mathrm{Gal}(K_\infty/K)$. Remarquons que cet anneau peut être défini intrinsèquement : si $t' = at$, avec a une unité p-adique, on identifie $\mathbb{Z}_p[\log t]$ à $\mathbb{Z}_p[\log t']$ en posant $\log t' = \log a + \log t$. Remarquons aussi que si l'on prolonge le logarithme à $\overline{K}^*$ en convenant que $\log p = 0$, on peut identifier $\log t$ à $\log \underline{t}$ (on a $t = \pi_t \underline{t}$ et $\log(\pi_t) = 0$). Pour tout $m \in \mathbb{N}$, on note T_m le sous-$\mathbb{Z}_p$-module de $\mathbb{Z}_p[\log t]$ formé des polynômes de degré $< m$. On pose $C_m = C \otimes_{\mathbb{Z}_p} T_m$. C'est donc une C-représentation de dimension m.

On voit que $(C_m)^f$ s'identifie à $K_\infty \otimes_{\mathbb{Z}_p} T_m$, que la seule valeur-propre de ∇_0 est 0 (en particulier C_m est une petite représentation) et $(C_m)_{(0)} = K \otimes_{\mathbb{Z}_p} T_m$, l'opérateur ∇_0 étant la dérivation par rapport à $\log t$. En particuler C_m est indécomposable.

Pour tout $m \in \mathbb{N}$, on a une suite exacte

$$0 \to C_m \to C_{m+1} \to C \to 0$$

la projection de C_{m+1} sur C étant l'application $\sum_{i=0}^m c_i(\log t)^i \mapsto c_m$.

Proposition 2.15. — *Soit W un objet simple de la catégorie des B^+_{dR}-représentations.*

A – On a $\mathrm{Ext}^1_{C[G_K]}(C,W) = \mathrm{Ext}^1_{B^+_{dR}[G_K]}(C,W) = 0$ sauf si l'on est dans l'un des deux cas suivants (qui s'excluent mutuellement) :

i) $W \simeq C$; alors $\mathrm{Ext}^1_{C[G_K]}(C,C) = \mathrm{Ext}^1_{B^+_{dR}[G_K]}(C,C)$ est un K-espace vectoriel de dimension 1, avec pour base la classe de $C_2 = C \oplus C\log t$.

ii) $W \simeq C(1)$; alors $\mathrm{Ext}^1_{C[G_K]}(C,C(1)) = 0$ et $\mathrm{Ext}^1_{B^+_{dR}[G_K]}(C,C(1))$ est un K-espace vectoriel de dimension 1, avec pour base la classe de B_2.

B – On a $\mathrm{Ext}^2_{B^+_{dR}[G_K]}(C,W) = 0$ sauf si $W \simeq C(1)$; alors $\mathrm{Ext}^2_{B^+_{dR}[G_K]}(C,C(1))$ est un K-espace vectoriel de dimension 1 admettant comme base la classe c_{fond} de la 2-extension

$$0 \to C(1) \to B_2 \xrightarrow{d} C_2 \to C \to 0$$

(où d est le composé de la projection canonique de B_2 sur C avec l'inclusion de C dans C_2). La classe de

$$0 \to C(1) \to C_2(1) \xrightarrow{d'} B_2 \to C \to 0$$

(où d' est le composé de la projection canonique de $C_2(1)$ sur $C(1)$ avec l'inclusion de $C(1)$ dans B_2) est $-c_{\text{fond}}$.

Preuve : On a $W_{(0)} = 0$ sauf si $W \simeq C$ et $W_{(1)} = 0$ sauf si $W \simeq C(1)$.
L'assertion (B) du théorème précédent implique donc que $\operatorname{Ext}^n_{B^+_{dR}[G_K]}(C,W) = 0$ pour tout $n \in \mathbb{N}$ si W n'est ni isomorphe à C, ni isomorphe à $C(1)$. Comme $\operatorname{Ext}^1_{C[G_K]}(C,W) \subset \operatorname{Ext}^1_{B^+_{dR}[G_K]}(C,W)$, on a aussi $\operatorname{Ext}^1_{C[G_K]}(C,W) = 0$.
Si $W = C$, les $\operatorname{Ext}^n_{B^+_{dR}[G_K]}(C,C)$, sont les groupes de cohomologie du complexe

$$K \xrightarrow{0} K \to 0 \to 0 \to \dots$$

Par conséquent, $\operatorname{Ext}^2_{B^+_{dR}[G_K]}(C,C) = 0$ et $\operatorname{Ext}^1_{B^+_{dR}[G_K]}(C,C)$ est un K-espace vectoriel de dimension 1. Comme l'extension

$$0 \to C \to C_2 \to C \to 0$$

est non scindée, la classe de C_2 engendre ce K-espace vectoriel. Comme C_2 est une C-représentation, cette classe appartient à $\operatorname{Ext}^1_{C[G_K]}(C,C)$ et $\operatorname{Ext}^1_{C[G_K]}(C,C) = \operatorname{Ext}^1_{B^+_{dR}[G_K]}(C,C)$.
Si $W = C(1)$, l'assertion (B) du théorème 2.14 implique la nullité de $\operatorname{Ext}^1_{C[G_K]}(C,C(1))$, tandis que les $\operatorname{Ext}^n_{B^+_{dR}[G_K]}(C,C(1))$, sont les groupes de cohomologie du complexe

$$0 \to K.\underline{t} \xrightarrow{0} K.\underline{t} \to 0 \to 0 \to \dots$$

Les K-espaces vectoriels $\operatorname{Ext}^1_{B^+_{dR}[G_K]}(C,C(1))$ et $\operatorname{Ext}^2_{B^+_{dR}[G_K]}(C,C(1))$ sont donc tous deux de dimension 1. Comme l'extension

$$0 \to C(1) \to B_2 \to C \to 0$$

est non scindée, sa classe engendre $\operatorname{Ext}^1_{B^+_{dR}[G_K]}(C,C(1))$.
Mais cette suite exacte induit la suite exacte

$$\operatorname{Ext}^1_{B^+_{dR}[G_K]}(C,B_2) \to \operatorname{Ext}^1_{B^+_{dR}[G_K]}(C,C) \to \operatorname{Ext}^2_{B^+_{dR}[G_K]}(C,C(1))$$

Or $\operatorname{Ext}^1_{B^+_{dR}[G_K]}(C,B_2)$, premier groupe de cohomologie du complexe

$$K \to K\underline{t} \oplus K \to K\underline{t} \to 0 \to 0 \to \dots$$

avec $d^0(x) = (x\underline{t}, 0)$ et $d^1(y,z) = z\underline{t}$, est nul. L'application

$$\mathrm{Ext}^1_{B^+_{dR}[G_K]}(C,C) \to \mathrm{Ext}^2_{B^+_{dR}[G_K]}(C,C(1))$$

est donc injective. La classe de C_2 dans $\mathrm{Ext}^1_{B^+_{dR}[G_K]}(C,C)$ étant non nulle, elle s'envoie sur un élément non nul de $\mathrm{Ext}^2_{B^+_{dR}[G_K]}(C,C(1))$. C'est précisément c_{fond}.
Notons $\theta : B_2 \to C$ la projection canonique. Posons $B_{2,2} = B_2 \otimes_{\mathbb{Z}_p} T_2$, de sorte que tout élément de $B_{2,2}$ s'écrit de manière unique sous la forme $b_0 + b_1 \log t$ avec $b_0, b_1 \in B_2$ et que $C_2(1)$ s'identifie à la sous-B^+_{dR}-représentation de $B_{2,2}$ formée des éléments qui peuvent s'écrire sous la forme $c_0 t + c_1 t \log t$, avec $c_0, c_1 \in C$. On définit des applications

$$\begin{array}{ccc}
\alpha : B_2 \oplus C_2(1) \to B_2 & \text{par} & \alpha(b, c_0t + c_1t\log t) = b + c_1 t \\
\alpha' : B_2 \oplus C_2(1) \to C_2(1) & \text{par} & \alpha'(b, c_0t + c_1t\log t) = c_0t + c_1t\log t \\
\beta : B_{2,2} \to C_2 & \text{par} & \beta(b_0 + b_1 \log t) = \theta(b_0) + \theta(b_1)\log\ t \\
\beta' : B_{2,2} \to B_2 & \text{par} & \beta'(b_0 + b_1\log t) = b_1 \\
d^{-1} : C(1) \to B_2 \oplus C_2(1) & \text{par} & d^{-1}(ct) = (ct, -ct) \\
d^0 : B_2 \oplus C_2(1) \to B_{2,2} & \text{par} & d^0(b, c_0t + c_1t\log t) = b + c_0t + c_1\log t \\
d^1 : B_{2,2} \to C & \text{par} & d^1(b_0 + b_1\log t) = \theta(b_1)
\end{array}$$

On vérifie que le diagramme

$$\begin{array}{ccccccccccc}
0 & \to & C(1) & \to & B_2 & \to & C_2 & \to & C & \to & 0 \\
 & & \| & & \uparrow \alpha & & \uparrow \beta & & \| & & \\
0 & \to & C(1) & \xrightarrow{d^{-1}} & B_2 \oplus C_2(1) & \xrightarrow{d^0} & B_{2,2} & \xrightarrow{d^1} & C & \to & 0 \\
 & & \downarrow -\mathrm{id} & & \downarrow \alpha' & & \downarrow \beta' & & \| & & \\
0 & \to & C(1) & \to & C_2(1) & \to & B_2 & \to & C & \to & 0
\end{array}$$

dont les lignes sont exactes, est commutatif. La dernière assertion de la proposition en résulte. □

Pour toute B^+_{dR}-représentation W, on a

$$\mathrm{Ext}^2_{B^+_{dR}[G_K]}(W,W(1)) = \mathrm{Ext}^2_{B^+_{dR}[G_K]}(C, Hom(W,W(1))$$

(où $Hom(W,W(1))$ est le hom interne dans la catégorie tannakienne des B^+_{dR}-représentations). La flèche naturelle $Hom(W,W(1)) \to C(1)$ nous donne donc une application de $\mathrm{Ext}^2_{B^+_{dR}[G_K]}(W,W(1))$ dans $\mathrm{Ext}^2_{B^+_{dR}[G_K]}(C,C(1)) = K.c_{\mathrm{fond}}$. Pour tout $\varepsilon \in \mathrm{Ext}^2_{B^+_{dR}[G_K]}(W,W(1))$, on note $c_{K,W}(\varepsilon).c_{\mathrm{fond}}$ son image. On a ainsi défini une application K-linéaire $c_{K,W} : \mathrm{Ext}^2_{B^+_{dR}[G_K]}(W,W(1)) \to K$.

Proposition 2.13. — *Soient W_1 et W_2 des B_{dR}^+-représentations. Pour $n = 0,1,2$ l'application bilinéaire*

$$\mathrm{Ext}^n_{B_{dR}^+[G_K]}(W_1,W_2)\times\mathrm{Ext}^{2-n}_{B_{dR}^+[G_K]}(W_2,W_1(1)) \to \mathrm{Ext}^2_{B_{dR}^+[G_K]}(W_1,W_1(1)) \to K$$

est une dualité parfaite de K-espaces vectoriels.

Preuve : Quitte à remplacer W_2 par $Hom(W_1,W_2)$ (au sens tannakien), on peut supposer $W_1 = C$. On est donc ramené à prouver que pour $n = 0,1,2$ et pour toute C-représentation W, l'accouplement

$$\mathrm{Ext}^n_{B_{dR}^+[G_K]}(C,W) \times \mathrm{Ext}^{2-n}_{B_{dR}^+[G_K]}(W,C(1)) \to \mathrm{Ext}^2_{B_{dR}^+[G_K]}(C,C(1))$$

est non dégénéré. Par dévissage, on se ramène au cas où W est simple.
– Si W n'est isomorphe ni à C ni à $C(1)$, les $\mathrm{Ext}^n_{B_{dR}^+[G_K]}(C,W)$ sont nuls ainsi que les $\mathrm{Ext}^n_{B_{dR}^+[G_K]}(W,C(1)) = \mathrm{Ext}^n_{B_{dR}^+[G_K]}(C,W^*(1))$ et il n'y a rien à prouver.
– Supposons $W = C$. Pour $n = 0$, on a $\mathrm{Hom}_{B_{dR}^+[G_K]}(C,C) = K$, $\mathrm{Ext}^2_{B_{dR}^+[G_K]}(C,C(1)) = K.c_{\mathrm{fond}}$ et c'est évident. Pour $n = 1$, $\mathrm{Ext}^1_{B_{dR}^+[G_K]}(C,C)$ est le K-espace vectoriel de dimension 1 engendré par la classe c_2 de C_2, tandis que $\mathrm{Ext}^1_{B_{dR}^+[G_K]}(C,C(1))$ est le K-espace vectoriel de dimension 1 engendré par la classe b_2 de B_2 et cela résulte de ce que, d'après la partie (B) de la proposition précédente, le générateur c_{fond} de $\mathrm{Ext}^2_{B_{dR}^+[G_K]}(C,C(1))$ est le cup-produit de c_2 avec b_2. Pour $n = 2$, on a $\mathrm{Ext}^2_{B_{dR}^+[G_K]}(C,C) = \mathrm{Hom}_{B_{dR}^+[G_K]}(C,C(1)) = 0$ et c'est évident.
– Supposons alors $W = C(1)$. Pour $n = 0$, on a $\mathrm{Hom}_{B_{dR}^+[G_K]}(C,C(1)) = \mathrm{Ext}^2_{B_{dR}^+[G_K]}(C(1),C(1)) = \mathrm{Ext}^2_{B_{dR}^+[G_K]}(C,C) = 0$ et c'est évident. Pour $n = 1$, $\mathrm{Ext}^1_{B_{dR}^+[G_K]}(C,C(1))$ est le K-espace vectoriel de dimension 1 engendré par la classe b_2 de B_2, tandis que $\mathrm{Ext}^1_{B_{dR}^+[G_K]}(C(1),C(1)) \simeq \mathrm{Ext}^1_{B_{dR}^+[G_K]}(C,C)$ est le K-espace vectoriel de dimension 1 engendré par la classe c_2' de $C_2(1)$ et cela résulte de ce que, d'après la partie (B) de la proposition précédente, le générateur $-c_{\mathrm{fond}}$ de $\mathrm{Ext}^2_{B_{dR}^+[G_K]}(C,C(1))$ est le cup-produit de b_2 avec c_2'. Pour $n = 2$ enfin, $\mathrm{Ext}^2_{B_{dR}^+[G_K]}(C,C(1)) = K.c_{\mathrm{fond}}$, tandis que $\mathrm{Hom}_{B_{dR}^+[G_K]}(C(1),C(1)) = K.\mathrm{id}_{C(1)}$ et c'est évident. □

La proposition 2.15 implique aussi le résultat suivant, également très facile à vérifier directement :

Proposition 2.17. — *A) Soient W_1 et W_2 deux C-représentations. Pour tout $i \in \mathbb{N}$, le K-espace vectoriel $\mathrm{Ext}^i_{C[G_K]}(W_1,W_2)$ s'identifie, canoniquement et fonctoriellement, à $\mathrm{Ext}^i_{C[G_K]}(C,W_1^* \otimes_C W_2)$.*

B) Pour toute C-représentation W, on a $\mathrm{Ext}^i_{C[G_K]}(C,W) = 0$ *si* $i \geq 2$, *tandis que* $\mathrm{Hom}_{C[G_K]}(C,W)$ *et* $\mathrm{Ext}^1_{C[G_K]}(C,W)$ *sont des K-espaces vectoriels de même dimension finie. On dispose d'une suite exacte*

$$0 \to \mathrm{Hom}_{C[G_K]}(C,W) \to W_{(0)} \to W_{(0)} \to \mathrm{Ext}^1_{C[G_K]}(C,W) \to 0$$

3 – Quelques calculs d'homomorphismes et d'extensions

Comme au paragraphe précédent, K_∞ est la $\mathbb{Z}_p$-extension cyclotomique de K, $H_K = \mathrm{Gal}(\overline{K}/K_\infty)$ et $\Gamma_K = \mathrm{Gal}(K_\infty/K)$.

3.1 – Cohomologie continue

Si G est un groupe topologique et si M est un groupe topologique abélien muni d'une action linéaire et continue de G, on sait définir les groupes de *cohomologie continue* $H^m_{\mathrm{cont}}(G,M)$ (cf. [Ta76]). Lorsque $G = \mathrm{Gal}(E^s/E)$ où E^s est la clôture séparable d'un corps E, on écrit aussi $H^m_{\mathrm{cont}}(E,M) = H^m_{\mathrm{cont}}(G,M)$.
Si S est un banach muni d'une action linéaire et continue de G et si $\mathcal{S}$ est un réseau de S stable par G, on a $H^m_{\mathrm{cont}}(G,S) = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^m_{\mathrm{cont}}(G,\mathcal{S})$.
Si V est une représentation p-adique, les $H^n_{\mathrm{cont}}(K,V)$ sont des K-espaces vectoriels de dimension finie, nuls pour $n \notin \{0,1,2\}$ et

$$\sum_{n=0}^{2} (-1)^n \dim_K H^n_{\mathrm{cont}}(K,V) = -[K:\mathbb{Q}_p].\dim_{\mathbb{Q}_p} V$$

En outre, on a un isomorphisme canonique de $H^2_{\mathrm{cont}}(K,\mathbb{Q}_p(1))$ sur $\mathbb{Q}_p$ et le cup-produit induit, pour $n = 0,1,2$, une dualité parfaite

$$H^n_{\mathrm{cont}}(K,V) \times H^{2-n}_{\mathrm{cont}}(K,V^*(1)) \to \mathbb{Q}_p$$

(ces résultats bien connus se voient par passage à la limite à partir des résultats analogues pour la cohomologie des G_K-modules finis de p-torsion, cf [Se94], chap.II, th.2 et 5).
Par ailleurs, $H^n_{\mathrm{cont}}(K,V)$ s'identifie à $\mathrm{Ext}^n_{\mathbb{Q}_p[G_K]}(\mathbb{Q}_p,V)$ (ce sont des δ-foncteurs effaçables qui coïncident en degré 0). Si V_1 et V_2 sont deux représentations p-adiques, $\mathrm{Ext}^n_{\mathbb{Q}_p[G_K]}(V_1,V_2)$ s'identifie donc à $H^n_{\mathrm{cont}}(K,V_1^* \otimes V_2)$.

Rappelons (§2.5) que, pour toute B^+_{dR}-représentation W, on a posé $W_{(0,0)} = W^{G_K}$.

PROPOSITION 3.1. — *Soit W une B_{dR}^+-représentation. Alors $H^0(K,W)$ et $H^1_{\rm cont}(K,W)$ sont des K-espaces vectoriels de même dimension finie. On a $H^0(K,W) = W_{(0,0)} = \mathrm{Hom}_{B_{dR}^+[G_K]}(B_{dR}^+,W)$ tandis que $H^1_{\rm cont}(K,W)$ s'identifie à $\mathrm{Ext}^1_{B_{dR}^+[G_K]}(B_{dR}^+,W)$.*
Si W est une C-représentation, $H^0(K,W)$ s'identifie aussi à $\mathrm{Hom}_{C[G_K]}(C,W)$ et $H^1_{\rm cont}(K,W)$ à $\mathrm{Ext}^1_{C[G_K]}(C,W)$.

Preuve : L'application qui à $\eta \in \mathrm{Hom}_{B_{dR}^+[G_K]}(B_{dR}^+,W)$ associe $\eta(1)$ identifie ce K-espace vectoriel à $W^{G_K} = H^0(K,W) = W_{(0,0)}$.
Soit maintenant E un B_{dR}^+-module de type fini, muni d'une action linéaire et continue de G_K, extension de B_{dR}^+ par W. L'extension est scindée en tant qu'extension de B_{dR}^+-modules et si $\hat{1}$ est un relèvement de 1 dans E, l'application $\varepsilon : G_K \to W$ qui à g associe $g(\hat{1}) - \hat{1}$ est un 1-cocycle continu de G_K à valeurs dans W. Si l'on change le relèvement, on change ε par un cobord et la classe de ε dans $H^1_{\rm cont}(K,W)$ ainsi définie ne dépend que de la classe de E dans $\mathrm{Ext}^1_{B_{dR}^+[G_K]}(B_{dR}^+,W)$. On vérifie que l'application de $\mathrm{Ext}^1_{B_{dR}^+[G_K]}(B_{dR}^+,W)$ dans $H^1_{\rm cont}(K,V)$ ainsi définie est bien un isomorphisme.
Le cas où W est une C-représentation se traite de la même manière. □

Rappelons le résultat fondamental de Tate :

PROPOSITION 3.2 ([Ta67], prop.9, cf.aussi [Fo00], th.1.8). — *Soient M une extension finie de K_∞, $\mathcal{O}_M$ l'anneau de ses entiers et $tr_{M/K_\infty} : M \to K_\infty$ la trace. Alors l'idéal maximal $\mathfrak{m}_{K_\infty}$ de l'anneau des entiers de K_∞ est contenu dans $tr_{M/K_\infty}(\mathcal{O}_M)$.* □

Appelons *presque B_{dR}^+-représentation* toute représentation banachique qui est presqu'isomorphe à une B_{dR}^+-représentation.

PROPOSITION 3.3. — *Soit $\mathcal{S}$ un $\mathbb{Z}_p$-module séparé et complet pour la topologie p-adique muni d'une action linéaire et continue de G_K.*
i) Si n est un entier ≥ 3, on a $H^n_{cont}(K,\mathcal{S}) = 0$;
ii) Si $n \in \{0,1,2\}$ et si $\mathcal{S}$ est un réseau stable par G_K d'une presque B_{dR}^+-représentation S, il existe $s \in \mathbb{N}$ tel que p^s annule $H^n_{cont}(K,\mathcal{S})_{\rm tor}$ et $H^n_{cont}(K,\mathcal{S})/H^n_{cont}(K,\mathcal{S})_{\rm tor}$ est un $\mathbb{Z}_p$-module libre de rang fini ;
iii) Pour $n = 2$, ce rang est nul si S est une B_{dR}^+-représentation.

Preuve : Comme la p-dimension cohomologique de G_K est 2 ([Se94], chap.II, §5), si $n \leq 3$, on a $H^n_{cont}(K,\mathcal{S}/p\mathcal{S}) = H^n(K,\mathcal{S}/p\mathcal{S}) = 0$. Ceci nous permet, si $f : G_K^n \to \mathcal{S}$ est un n-cocycle continu de G_K à valeurs dans $\mathcal{S}$, de construire, de proche en proche, une suite $(u_r)_{r\in\mathbb{N}}$ de $(n-1)$-cochaines continues telle que $f = d(\sum_{r\in\mathbb{N}} p^r u_r)$, d'où (i).
Soient $\mathcal{S}'$ et $\mathcal{S}''$ deux réseaux de S. Le fait qu'ils sont commensurables implique que (ii) est vraie pour $\mathcal{S}'$ si et seulement si elle est vraie pour $\mathcal{S}''$. Par ailleurs, soit

$$0 \to S_1 \to S_2 \to S_3 \to 0$$

une suite exacte courte de presque-B_{dR}^+-représentations et soit $\mathcal{S}_2$ un réseau stable par G_K de S_2. L'image $\mathcal{S}_3$ de $\mathcal{S}_2$ dans S_3 et $\mathcal{S}_1 = S_1 \cap \mathcal{S}_2$ sont des réseaux stables par G_K respectivement de S_3 et de S_1 ; la suite exacte courte

$$0 \to \mathcal{S}_1 \to \mathcal{S}_2 \to \mathcal{S}_3 \to 0$$

induit une suite exacte longue de cohomologie, d'où l'on déduit que, si la propriété (ii) est vraie pour deux des trois réseaux $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$, elle est vraie pour le troisième. Comme S est une presque B_{dR}^+-représentation, on peut trouver un isomorphisme $S/V \simeq W/V'$, avec W une B_{dR}^+-représentation et V et V' des $\mathbb{Q}_p$-représentations. On peut donc, pour prouver (ii), supposer que $S = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathcal{S}$ est soit une $\mathbb{Q}_p$-représentation, soit une B_{dR}^+-représentation.
Dans le premier cas, comme $\mathcal{S}$ est un $\mathbb{Z}_p$-module de type fini, les $H^n_{\mathrm{cont}}(K, \mathcal{S})$ sont de type fini : cela résulte formellement (cf. par exemple, [NSW00], cor.2.3.9) du fait que, si M est un $\mathbb{Z}_p$-module fini avec action linéaire discrète de G_K, les groupes $H^n(K, M)$ sont finis (cf. par exemple [Se94], chap. II,prop.14). Ils vérifient donc (ii).
Dans le second cas, par dévissage, on se ramène au cas où S *est une* C*-représentation, ce que nous supposons désormais.* On peut, pour prouver l'assertion, remplacer K par une extension finie galoisienne convenable, ce qui nous permet de supposer que S est K-petite (cf. §2.3). Toujours par dévissage, on peut en outre supposer qu'elle est simple.
La fin de la preuve repose sur les techniques "presque-étales" classiques de Tate [Ta67] et Sen [Se80] :

Lemme 3.4. — *Soient* $\pi \in \mathfrak{m}_{K_\infty}$ *un élément non nul et* $f : H_K \to \mathcal{O}_C$ *un* 1*-cocycle continu. Il existe* $b \in \mathcal{O}_C$ *tel que le* 1*-cocycle continu* $\pi f - db$ *est à valeurs dans* $\pi^2\mathcal{O}_C$.

Preuve : Comme f est continu, il existe un sous-groupe ouvert invariant H'_K de H_K tel que $f'(H_K) \subset \pi^2\mathcal{O}_C$. La condition de cocycle implique que, si $g \in H_K$ et $h \in H'_K$, alors $f(gh) \equiv f(g) \pmod{\pi^2\mathcal{O}_C}$.
Soit $M = \overline{K}^{H'_K}$; c'est une extension finie galoisienne de K_∞ de groupe de Galois $J = H_K/H'_K$. D'après la proposition 3.2, on peut trouver c dans l'anneau des entiers de M tel que $tr_{M/K_\infty}(c) = \pi$. Soit T un système de représentants de J dans H_K. Posons

$$b = -\sum_{g \in T} f(g)g(c) \in \mathcal{O}_C \ .$$

Pour tout $h \in H_K$, on a

$$h(b) = -\sum_{g\in T} h(f(g))hg(c) = -\sum_{g\in T} f(hg).hg(c) + f(h)(\sum_{g \in T} g(c)) \ .$$

Pour tout $g \in H_K$, notons $t(g)$ le représentant, dans T, de son image dans J. Comme $f(g) \bmod \pi^2\mathcal{O}_C$ ne dépend que de l'image de g dans J, on a $\sum_{g\in T} f(hg)hg(c) \equiv \sum_{g \in T} f(t(hg)).t(hg) \equiv b \pmod{\pi^2\mathcal{O}_C}$. Par conséquent, pour tout $h \in H_K$, on a $h(b) - b \equiv \pi f(h) \pmod{\pi^2\mathcal{O}_C}$. □

Lemme 3.5. — *Pour tout* $\pi \in \mathfrak{m}_{K_\infty}$, *on a* $\pi.H^1_{cont}(K_\infty, \mathcal{O}_C) = 0$.

Preuve : On peut supposer π non nul. Le lemme précédent permet de construire une suite d'éléments $(b_n)_{n\in\mathbb{N}}$ dans $\mathcal{O}_C$ et une suite de 1-cocycles continus de H_K à valeurs dans $\mathcal{O}_C$ tels que $f_0 = f$ et $\pi^2 f_n = \pi f_{n-1} - db_{n-1}$ pour tout $n \geq 1$. Si $b = \sum_{n=0}^{+\infty} \pi^n b_n$, on a $db = \sum_{n=0}^{+\infty} \pi^n db_n = \pi f$. □

Terminons alors la preuve de la proposition 3.3 : D'après la proposition 2.1, le C-espace vectoriel S admet une base formée d'éléments fixés par H_K, i.e. est isomorphe, en tant que C-représentation de H_K à C^d. Il existe donc un réseau $\mathcal{S}_0$ de S, stable par H_K qui est isomorphe à $\mathcal{O}_C^d$. D'après le lemme précédent $H^1_{cont}(K_\infty, \mathcal{S}_0)$ est tué par p et comme $\mathcal{S}$ est commensurable à $\mathcal{S}_0$, il existe un entier s_1 tel que p^{s_1} annule $H^1_{cont}(K_\infty, \mathcal{S})$.
Mais le fait que Γ_K et H_K sont de p-dimension cohomologique 1 implique que l'on a une suite exacte

$$0 \to H^1_{\text{cont}}(\Gamma_K, \mathcal{S}^{H_K}) \to H^1_{\text{cont}}(K, \mathcal{S}) \to H^1_{\text{cont}}(K_\infty, \mathcal{S})^{G_K} \to 0$$

et que $H^2_{\text{cont}}(K, \mathcal{S}) = H^1_{\text{cont}}(\Gamma_K, H^1_{\text{cont}}(K_\infty, \mathcal{S}))$. Il suffit donc pour achever la démonstration d'établir le résultat suivant :

Lemme 3.6. — *Soit S une C-représentation et soit $\mathcal{T}$ un réseau stable par Γ_K de $T = S^{H_K}$. Il existe un entier s_0 tel que p^{s_0} annule $H^1_{\text{cont}}(\Gamma_K, \mathcal{T})_{\text{tor}}$ et les $\mathbb{Z}_p$-modules $\mathcal{T}^{\mathcal{G}_\mathcal{K}}$ et $H^1_{\text{cont}}(\Gamma_K, \mathcal{T})/\mathcal{H}^\infty_{\text{cont}}(-_\mathcal{K}, \mathcal{T})_{\text{tor}}$ sont de type fini.*

Preuve : Si γ est un générateur topologique de Γ_K, on a une suite exacte

$$0 \to \mathcal{T}^{-_\mathcal{K}} \to \mathcal{T} \xrightarrow{\gamma-1} \mathcal{T} \to \mathcal{H}^\infty_{\text{cont}}(-_\mathcal{K}, \mathcal{T}) \to \prime$$

Comme on a supposé S simple et K-petite, il existe (prop.2.12) un élément K-petit $\alpha \in K$ tel que $S = C\{\alpha\}$. Si L est le complété de K_∞ et si $u = \exp(\alpha \log \chi(\gamma))$, T s'identifie alors à L, γ agissant sur L par $c \mapsto u.\gamma(c)$.
Si $\alpha = 0$, $T = L$ avec son action naturelle. Soit $tr : K_\infty \to K$ l'application qui envoie $x \in K_N$ sur $\frac{1}{p^N} tr_{K_N/K}(x)$ (où K_N désigne l'unique extension de K de degré p^N contenue dans K_∞). Alors ([Ta67],prop.6) cette application est continue et se prolonge par continuité en une application encore notée tr de L dans K. Si $L_0 = \text{Ker } tr$, on a $L = K \oplus L_0$. Quitte à changer de réseau, on peut supposer que $\mathcal{T} = \mathcal{O}_\mathcal{K} \oplus \mathcal{T}_\prime$ où $\mathcal{T}_\prime$ est un réseau de L_0. Sur $\mathcal{O}_K$, l'opérateur $\gamma - 1$ est nul et on a donc $\mathcal{O}_K^{\Gamma_K} = H^1_{\text{cont}}(\Gamma_K, \mathcal{O}_K) = \mathcal{O}_K$ et c'est un $\mathbb{Z}_p$-module libre de rang fini. Sur L_0, l'opérateur $\gamma - 1$ est bijectif avec un inverse continu (*loc.cit.*, prop.7,b). On en déduit que $\mathcal{T}_\prime^{-_\mathcal{K}} = \prime$ tandis qu'il existe un entier s_0 tel que p^{s_0} annule $H^1_{\text{cont}}(\Gamma_K, \mathcal{T}_\prime)$.
Sinon, l'opérateur $\gamma - u^{-1}$ est bijectif sur L, avec un inverse continu (*loc.cit.*, prop7,c) ; il en est de même de $u.\gamma - 1$ et on en déduit que $\mathcal{T}^{-_\mathcal{K}} = \prime$ tandis qu'il existe un entier s_0 tel que p^{s_0} annule $H^1_{\text{cont}}(\Gamma_K, \mathcal{T})$. □

Corollaire. — *Pour toute presque-B_{dR}^+-représentation S, les $H^n_{\mathrm{cont}}(K,S)$ sont des $\mathbb{Q}_p$-espaces vectoriels de dimension finie, nuls si $n \geq 3$. Si S est une B_{dR}^+-représentation, on a aussi $H^2_{\mathrm{cont}}(K,S)=0$.*

3.2 – Calcul de $\mathrm{Ext}^1_{\mathcal{B}(G_K)}(S,V)$ via le corps de classes

Pour tout anneau commutatif A et tout A-module M, on note M^{*_A} le A-module des applications A-linéaires de M dans A.

Proposition 3.7. — *Soient V une représentation p-adique et S une presque B_{dR}^+-représentation. Il existe une dualité parfaite de $\mathbb{Q}_p$-espaces vectoriels*

$$\mathrm{Ext}^1_{\mathcal{B}(G_K)}(S,V) \times H^1_{\mathrm{cont}}(K, S \otimes_{\mathbb{Q}_p} V^{*_{\mathbb{Q}_p}}(1)) \to \mathbb{Q}_p$$

canonique et fonctorielle en S et V.

En fait, on va utiliser la théorie du corps de classes local (plus précisément la dualité de Tate) pour construire une telle dualité.

Lemme 3.8. — *Il existe une suite croissante*

$$\mathcal{S}_1 \subset \mathcal{S}_2 \subset \ldots \subset \mathcal{S}_n \subset \mathcal{S}_{n+1} \subset \ldots$$

de sous-$\mathbb{Z}_p$-modules de type fini de S, stables par G_K, telle que la réunion $\mathcal{S}_\infty$ des $\mathcal{S}_n$ est séparée pour la topologie p-adique et que l'inclusion de $\mathcal{S}_\infty$ dans S induise un homéomorphisme de $\widehat{\mathcal{S}}_\infty = \varprojlim \mathcal{S}_\infty/p^m\mathcal{S}_\infty$ sur un réseau $\mathcal{S}$ de S.

Preuve : Par dévissage on se ramène au cas où S est soit une représentation p-adique (auquel cas, c'est trivial, il suffit de choisir un réseau $\mathcal{S}$ stable par G_K et de prendre $\mathcal{S}_n = \mathcal{S}$ pour tout n), soit une C-représentation. Dans ce dernier cas, il existe une extension finie galoisienne L de K contenue dans $\overline{K}$ telle que S est petite en tant que B_{dR}^+-représentation de G_L. Avec les notations du §2.3, S_L^f est un sous-L-espace vectoriel stable par G_L de S tel que l'application naturelle $C \otimes_L S_L^f \to S$ (cf. lemme 2.11) est un isomorphisme.
Pour toute extension finie E de K, notons $\mathcal{O}_E$ l'anneau de ses entiers. On peut trouver un sous-$\mathcal{O}_L$-module $\mathcal{S}_L^f$ de S_L^f qui est un réseau de S_L^f et est stable par G_K. Il suffit alors de choisir une suite $L = L_1 \subset L_2 \subset \ldots L_n \subset L_{n+1} \subset \ldots$ d'extensions finies galoisiennes de K contenues dans $\overline{K}$ telles que $\overline{K} = \cup_{n\geq 1} L_n$ et de prendre pour $\mathcal{S}_n$ le sous-$\mathcal{O}_{L_n}$-module de S engendré par $\mathcal{S}_L^f$. □

Lemme 3.9. — *Choisissons unréseau $\mathcal{V}$ de V stable par G_K et des $\mathcal{S}_n$ et $\mathcal{S}$ comme dans le lemme précédent. La flèche naturelle*

$$\mathrm{Ext}^1_{\mathbb{Z}_p[G_K],\mathrm{cont}}(\mathcal{S}_\infty,\mathcal{V}) \to \varprojlim_{n\in\mathbb{N}} \mathrm{Ext}^1_{\mathbb{Z}_p[G_K],\mathrm{cont}}(\mathcal{S}_n,\mathcal{V})$$

est un isomorphisme.

Preuve : Soit $(\varepsilon_n)_{n\in\mathbb{N}} \in \varprojlim \mathrm{Ext}^1_{\mathbb{Z}_p[G_K],\mathrm{cont}}(\mathcal{S}_n,\mathcal{V})$. Pour chaque n, choisissons une extension $\mathcal{E}_n$ de $\mathcal{S}_n$ par $\mathcal{V}$ représentant ε_n. Comme l'image de ε_{n+1} dans $\mathrm{Ext}^1_{\mathbb{Z}_p[G_K],\mathrm{cont}}(\mathcal{S}_n,\mathcal{V})$ est ε_n, on peut trouver une application $\mathbb{Z}_p$-linéaire continue G_K-équivariante $f_n : \mathcal{E}_n \to \mathcal{E}_{n+1}$ qui induit l'identité sur $\mathcal{V}$ et l'inclusion naturelle $\mathcal{S}_n \subset \mathcal{S}_{n+1}$ sur les quotients. Alors la limite inductive des $\mathcal{E}_n$, avec les f_n comme applications de transition, est une extension de $\mathcal{S}_\infty$ par $\mathcal{V}$ dont la classe dans $\mathrm{Ext}^1_{\mathbb{Z}_p[G_K],\mathrm{cont}}(\mathcal{S}_\infty\mathcal{V})$ a pour image $(\varepsilon_n)_{n\in\mathbb{N}}$, ce qui prouve la surjectivité.
Soit maintenant $\mathcal{E}$ une extension de $\mathcal{S}_\infty$ par $\mathcal{V}$ dont la classe est dans le noyau de la flèche qui nous intéresse. Cela veut dire, que si $\mathcal{E}_n$, désigne l'image inverse de $\mathcal{S}_n$ dans $\mathcal{E}$, l'ensemble X_n des sections continues G_K-équivariantes de la projection de $\mathcal{E}_n$ sur $\mathcal{S}_n$ est non vide. Mais, si $x \in X_n$, les autres éléments sont de la forme $x+\eta$, avec η un élément du $\mathbb{Z}_p$-module de type fini $\mathrm{Hom}_{\mathbb{Z}_p[G_K]}(\mathcal{S}_n,\mathcal{V})$. Autrement dit X_n est un espace homogène principal sous ce groupe compact et devient ainsi un espace topologique compact non vide. L'application naturelle $X_{n+1} \to X_n$ est continue et la limite projective des X_n est donc non vide. Mais un élément de cette limite définit une section $\mathbb{Z}_p$-linéaire continue de la projection de $\mathcal{E}$ sur $\mathcal{S}_\infty$, on a donc $\varepsilon = 0$ et l'application est bien injective. □

Preuve de la proposition 3.7 : Choisissons $\mathcal{V}$, $\mathcal{S}$ et des $\mathcal{S}_n$ comme ci-dessus. Avec des notations évidentes, $\mathrm{Ext}^1_{\mathcal{B}(G_K)}(S,V)$ s'identifie à $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathrm{Ext}^1_{\mathbb{Z}_p[G_K],\mathrm{cont}}(\mathcal{S},\mathcal{V})$. Comme $\mathcal{S}$ s'identifie au séparé complété pour la topologie p-adique de $\mathcal{S}_\infty$, la flèche naturelle

$$\mathrm{Ext}^1_{\mathbb{Z}_p[G_K],\mathrm{cont}}(\mathcal{S},\mathcal{V}) \to \mathrm{Ext}^1_{\mathbb{Z}_p[G_K],\mathrm{cont}}(\mathcal{S}_\infty,\mathcal{V})$$

est un isomorphisme. D'après le lemme précédent, $\mathrm{Ext}^1_{\mathbb{Z}_p[G_K],\mathrm{cont}}(\mathcal{S},\mathcal{V})$ s'identifie donc à $\varprojlim \mathrm{Ext}^1_{\mathbb{Z}_p[G_K],\mathrm{cont}}(\mathcal{S}_n,\mathcal{V})$. Pour tout $\mathbb{Z}_p$-module $\mathcal{M}$, notons $\mathcal{M}\check{}$ le $\mathbb{Z}_p$-module des applications $\mathbb{Z}_p$-linéaires de $\mathcal{M}$ dans $\mathbb{Q}_p/\mathbb{Z}_p$.
Pour tout $n \in \mathbb{N}$ fixé, $\mathcal{S}_n$ et $\mathcal{V}$ sont des $\mathbb{Z}_p$-modules libres de rang fini et

$$\begin{aligned}\mathrm{Ext}^1_{\mathbb{Z}_p[G_K],\mathrm{cont}}(\mathcal{S}_n,\mathcal{V}) &= \mathrm{Ext}^1_{\mathbb{Z}_p[G_K],\mathrm{cont}}(\mathbb{Z}_p,(\mathcal{S}_n)^{*\mathbb{Z}_p}\otimes_{\mathbb{Z}_p}\mathcal{V})\\ &= H^1_{\mathrm{cont}}(K,(\mathcal{S}_n)^{*\mathbb{Z}_p}\otimes_{\mathbb{Z}_p}\mathcal{V})\end{aligned}$$

Par la dualité de Tate (cf. par exemple, [Se94], chap. II, th.2), ce dernier groupe s'identifie à $H^1(K,((\mathcal{S}_n)^{*\mathbb{Z}_p}\otimes_{\mathbb{Z}_p}\mathcal{V})\check{}(1))\check{}$.
Comme $((\mathcal{S}_n)^{*\mathbb{Z}_p}\otimes_{\mathbb{Z}_p}\mathcal{V})\check{} = \mathcal{S}_n\otimes_{\mathbb{Z}_p}\mathcal{V}\check{}$, si l'on pose $\mathcal{M}_n = H^1(K,\mathcal{S}_n\otimes\mathcal{V}\check{}(1))$, on a $\mathrm{Ext}^1_{\mathbb{Z}_p[G_K],\mathrm{cont}}(\mathcal{S}_n,\mathcal{V}) = \mathcal{M}_n\check{}$.
On a alors $\mathrm{Ext}^1_{\mathbb{Z}_p[G_K],\mathrm{cont}}(\mathcal{S}_\infty,\mathcal{V}) = \varprojlim_{n\in\mathbb{N}}\mathcal{M}_n\check{} = (\varinjlim_{\mathbb{N}\in\mathbb{N}}\mathcal{M}_n)\check{}$. Mais $\varinjlim_{n\in\mathbb{N}}\mathcal{M}_n = H^1(K,\varinjlim_{n\in\mathbb{N}}\mathcal{S}_n\otimes\mathcal{V}\check{}(1)) = H^1(K,\mathcal{S}_\infty\otimes\mathcal{V}\check{}(1))$.
Posons $\mathcal{T} = \mathcal{S}\otimes_{\mathbb{Z}_p}\mathcal{V}^{*\mathbb{Z}_p}(1)$. C'est un $\mathbb{Z}_p$-module séparé et complet pour la topologie p-adique, avec action semi-linéaire continue de G_K et on a une suite exacte

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{T} & \to & \mathcal{T}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p & \to & \mathcal{T}\otimes_{\mathbb{Z}_p}(\mathbb{Q}_p/\mathbb{Z}_p) & \to & 0\\ & & \| & & \| & & \| & & \\ 0 & \to & \mathcal{S}\otimes_{\mathbb{Z}_p}\mathcal{V}^{*\mathbb{Z}_p}(1) & \to & S\otimes_{\mathbb{Q}_p}V^{*\mathbb{Q}_p}(1) & \to & \mathcal{S}_\infty\otimes_{\mathbb{Z}_p}\mathcal{V}\check{}(1) & \to & 0\end{array}$$

Comme la topologie sur $\mathcal{S}_\infty \otimes \mathcal{V}^\vee(1)$ est la topologie discrète, toute section ensembliste de la projection de $S \otimes V^{*\mathbb{Q}_p}(1)$ sur $\mathcal{S}_\infty \otimes \mathcal{V}^\vee(1)$ est automatiquement continue et [Ta76] la suite exacte courte ci-dessus induit une suite exacte longue

$$0 \to (H^1_{\mathrm{cont}}(K, \mathcal{S} \otimes \mathcal{V}^{*\mathbb{Z}_p}(1)))_{\mathrm{tor}} \to H^1_{\mathrm{cont}}(K, \mathcal{S} \otimes \mathcal{V}^{*\mathbb{Z}_p}(1)) \to$$
$$H^1_{\mathrm{cont}}(K, S \otimes V^{*\mathbb{Q}_p}(1)) \to H^1(K, \mathcal{S}_\infty \otimes \mathcal{V}^\vee(1)) \to H^2_{\mathrm{cont}}(K, \mathcal{S} \otimes \mathcal{V}^{*\mathbb{Z}_p}(1))_{\mathrm{tor}} \to 0$$

Mais $H = H^1_{\mathrm{cont}}(K, S \otimes V^{*\mathbb{Q}_p}(1))$ est un $\mathbb{Q}_p$-espace vectoriel de dimension finie (cor. à la prop.3.3). L'image de $H^1_{\mathrm{cont}}(K, \mathcal{S} \otimes \mathcal{V}^{*\mathbb{Z}_p}(1))$ dans H est un réseau $\mathcal{H}$ de H. Si l'on pose $N = H^2_{\mathrm{cont}}(K, \mathcal{S} \otimes \mathcal{V}^{*\mathbb{Z}_p}(1))_{\mathrm{tor}}$, on a donc des suites exactes

$$0 \to H/\mathcal{H} \to H^1(K, \mathcal{S}_\infty \otimes \mathcal{V}^\vee(1)) \to N \to 0 \qquad \text{et}$$
$$0 \to N^\vee \to \mathrm{Ext}^1_{\mathbb{Z}_p[G_K],\mathrm{cont}}(\mathcal{S}_\infty, \mathcal{V}) \to (H/\mathcal{H})^\vee \to \mathrm{Ext}^1(N, \mathbb{Q}_p/\mathbb{Z}_p)$$

puisque $H^1(K, \mathcal{S}_\infty \otimes \mathcal{V}^\vee(1))^\vee = \mathrm{Ext}^1_{\mathbb{Z}_p[G_K]\mathrm{cont}}(\mathcal{S}_\infty, \mathcal{V})$. Il existe (prop. 3.3) un entier s tel que p^s annule N ; le noyau et le conoyau de $\mathrm{Ext}^1_{\mathbb{Z}_p[G_K]\mathrm{cont}}(\mathcal{S}_\infty, \mathcal{V}) \to (H/\mathcal{H})^\vee$ sont donc tués par p^s. Alors

$$\mathrm{Ext}^1_{\mathcal{B}(G_K)}(S, V) = \mathbb{Q}_p \otimes \mathrm{Ext}^1_{\mathbb{Z}_p[G_K]\mathrm{cont}}(\mathcal{S}, \mathcal{V})$$

s'identifie au $\mathbb{Q}_p$-espace vectoriel de dimension finie $H^{*\mathbb{Q}_p}$.
Enfin, il est clair que l'accouplement ainsi défini est indépendant des choix faits et est fonctoriel en S et en V. □

Corollaire. — *Soient V une représentation p-adique et W une B^+_{dR}-représentation de G_K. Alors $\mathrm{Ext}^1_{\mathcal{B}(G_K)}(W, V)$ est un K-espace vectoriel de dimension finie égale à la dimension du sous-espace propre associé à la valeur propre -1 de l'opérateur $\nabla_0(W \otimes_{\mathbb{Q}_p} V^{*\mathbb{Q}_p})$.*
Il existe une dualité parfaite de K-espaces vectoriels

$$\mathrm{Ext}^1_{\mathcal{B}(G_K)}(W, V) \times H^1_{\mathrm{cont}}(K, W \otimes_{\mathbb{Q}_p} V^{*\mathbb{Q}_p}(1)) \to K$$

canonique et fonctorielle en W et V.

Preuve : Soit H un K-espace vectoriel de dimension finie. L'application, qui à $\eta \in H^{*K}$ associe $\mathrm{tr}_{K/\mathbb{Q}_p} \circ \eta \in H^{*\mathbb{Q}_p}$, induit un isomorphisme du $\mathbb{Q}_p$-espace vectoriel sous-jacent à H^{*K} sur $H^{*\mathbb{Q}_p}$. La deuxième partie de l'assertion résulte donc de la proposition précédente ; comme $H^1_{\mathrm{cont}}(K, W \otimes_{\mathbb{Q}_p} V^{*\mathbb{Q}_p}(1))$ s'identifie à $\mathrm{Ext}^1_{B^+_{dR}[G_K]}(B^+_{dR}, W \otimes_{\mathbb{Q}_p} V^{*\mathbb{Q}_p}(1))$ (prop.3.1), la première résulte de l'assertion (iii) du théorème 2.14. □

3.3 – Théorèmes de pleine fidélité

Rappelons (§1.6) que $B_e = \{b \in B_{cris} \mid \varphi(b) = b\}$ contient $\mathbb{Q}_p$ et que la suite

$$0 \to \mathbb{Q}_p \to B_e \to B_{dR}/B^+_{dR} \to 0$$

est exacte.
Pour tout $i \in \mathbb{Z}$, on note $\mathrm{Fil}^i B_{dR}$ l'idéal fractionnaire de B_{dR}^+, puissance i-ième de l'idéal maximal ; c'est donc le sous-B_{dR}^+-module libre de rang 1 de B_{dR} engendré par t^i. On pose aussi $\mathrm{Fil}^i B_e = B_e \cap \mathrm{Fil}^i B_{dR}$ de sorte que $\mathrm{Fil}^i B_e = 0$ si $i > 0$ et que, pour $m \geq 0$, on dispose d'une suite exacte

$$0 \to \mathbb{Q}_p \to \mathrm{Fil}^{-m} B_e \to B_m(-m) \to 0$$

(en particulier, $\mathrm{Fil}^0 B_e = \mathbb{Q}_p$). Pour tout $m \in \mathbb{N}$, si l'on pose $U_m = (\mathrm{Fil}^{-m} B_m)(m)$, U_m s'identifie au sous-$\mathbb{Q}_p$-espace vectoriel de $B_{cris} \cap B_{dR}^+$ formé des b tels que $\varphi b = p^m b$ et la suite

$$0 \to \mathbb{Q}_p(m) \to U_m \to B_m \to 0$$

est exacte.

Proposition 3.10. — *Soit V une représentation p-adique extension non triviale de $\mathbb{Q}_p(1)$ par $\mathbb{Q}_p$.*
i) Le C-espace vectoriel $V_C = C \otimes_{\mathbb{Q}_p} V$ s'identifie à $C \oplus C(1)$. Il existe une et une seule application $\mathbb{Q}_p$-linéaire G_K-équivariante de V dans C qui prolonge l'inclusion de $\mathbb{Q}_p$ dans C mais cette application ne se relève pas en une application $\mathbb{Q}_p$-linéaire G_K-équivariante de V dans B_2 ;
ii) on a $\dim_K (B_2(-1) \otimes_{\mathbb{Q}_p} V)^{G_K} = 1$,
iii) la représentation V n'est pas de de Rham.

Preuve : La suite exacte

$$0 \to \mathbb{Q}_p(2) \to U_2 \to B_2 \to 0$$

induit un carré commutatif

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{B}(G_K)}(V, B_2) & \xrightarrow{\delta_V} & \mathrm{Ext}^1_{\mathcal{B}(G_K)}(V, \mathbb{Q}_p(2)) \\ \downarrow \alpha & & \downarrow \beta \\ \mathrm{Hom}_{\mathcal{B}(G_K)}(\mathbb{Q}_p, B_2) & \xrightarrow{\delta_{\mathbb{Q}_p}} & \mathrm{Ext}^1_{\mathcal{B}(G_K)}(\mathbb{Q}_p, \mathbb{Q}_p(2)) \end{array}$$

On a $\mathrm{Ext}^1_{\mathcal{B}(G_K)}(\mathbb{Q}_p, \mathbb{Q}_p(2)) = \mathrm{Ext}^1_{\mathbb{Q}_p[G_K]}(\mathbb{Q}_p, \mathbb{Q}_p(2)) = H^1_{\mathrm{cont}}(K, \mathbb{Q}_p(2))$. Par ailleurs $H^0_{cont}(K, \mathbb{Q}_p(2)) = H^2_{\mathrm{cont}}(K, \mathbb{Q}_p(2)) = 0$ (le second parce que c'est le dual de $H^0_{\mathrm{cont}}(K, \mathbb{Q}_p(-1))$ qui est nul). Comme

$$\sum_{n=0}^2 (-1)^n \dim_{\mathbb{Q}_p} H^n_{\mathrm{cont}}(K, \mathbb{Q}_p(2)) = -[K : \mathbb{Q}_p],$$

la dimension du $\mathbb{Q}_p$-espace vectoriel $\mathrm{Ext}^1_{\mathcal{B}(G_K)}(\mathbb{Q}_p, \mathbb{Q}_p(2))$ est égale au degré de K sur $\mathbb{Q}_p$.
On a $K_0 \cap U_2 = 0$ (puisque φ est le Frobenius absolu sur K_0 et la multiplication par p^2 sur U_2). Comme $B_{cris}^{G_K} = K_0$ [Fo88a], on a $\mathrm{Hom}_{\mathcal{B}(G_K)}(\mathbb{Q}_p, U_2) = U_2^{G_K} = 0$ et l'application $\delta_{\mathbb{Q}_p}$ est injective. Mais $\mathrm{Hom}_{\mathcal{B}(G_K)}(\mathbb{Q}_p, B_2) = B_2^{G_K} = K$ (*loc.cit.*) a la même dimension sur $\mathbb{Q}_p$ que $\mathrm{Ext}^1_{\mathcal{B}(G_K)}(\mathbb{Q}_p, \mathbb{Q}_p(2))$ et $\delta_{\mathbb{Q}_p}$ est un isomorphisme.

La suite exacte

$$0 \to \mathbb{Q}_p \to V \to \mathbb{Q}_p(1) \to 0$$

induit une suite exacte

$$\mathrm{Ext}^1_{\mathbb{Q}_p[G_K]}(V, \mathbb{Q}_p(2)) \xrightarrow{\beta} \mathrm{Ext}^1(\mathbb{Q}_p, \mathbb{Q}_p(2)) \to \mathrm{Ext}^2(\mathbb{Q}_p(1), \mathbb{Q}_p(2)) \\ \to \mathrm{Ext}^2_{\mathbb{Q}_p[G_K]}(V, \mathbb{Q}_p(2))$$

On a $\mathrm{Ext}^2_{\mathbb{Q}_p[G_K]}(V, \mathbb{Q}_p(2)) = H^2_{\mathrm{cont}}(K, V^*(2)) = 0$ car c'est le dual de $H^0_{\mathrm{cont}}(K, V(-1)) = V(-1)^{G_K}$, qui est nul puisque $V(-1)$ est une extension non triviale de $\mathbb{Q}_p$ par $\mathbb{Q}_p(-1)$. En revanche $\mathrm{Ext}^2_{\mathbb{Q}_p[G_K]}(\mathbb{Q}_p(1), \mathbb{Q}_p(2)) = H^2_{\mathrm{cont}}(K, \mathbb{Q}_p(1))$ est non nul et β n'est pas surjective. Par conséquent α ne l'est pas non plus.
La suite exacte de C-représentations

$$0 \to C \to V_C \to C(1) \to 0$$

est scindée (prop.2.15), ce scindage est unique puisqu'il n'y a pas de morphisme non trivial de $C(1)$ dans C et V_C s'identifie bien à $C \oplus C(1)$. En particulier, le K-espace vectoriel $\mathrm{Hom}_{\mathcal{B}(G_K)}(V, C) = \mathrm{Hom}_{C[G_K]}(V_C, C)$ est de dimension 1. On a $\mathrm{Hom}_{\mathcal{B}(G_K)}(\mathbb{Q}_p(1), C) = C(-1)^{G_K} = 0$ tandis que $\mathrm{Hom}_{\mathcal{B}(G_K)}(\mathbb{Q}_p, C) = C^{G_K} = K$. On en déduit que l'application $\mathrm{Hom}_{\mathcal{B}(G_K)}(V, C) \to \mathrm{Hom}_{\mathcal{B}(G_K)}(\mathbb{Q}_p, C)$ est bijective. Il existe donc bien un unique $\eta : V \to C$ qui prolonge l'inclusion de $\mathbb{Q}_p$ dans C.
On a une suite exacte de K-espaces vectoriels

$$0 \to \mathrm{Hom}_{\mathcal{B}(G_K)}(\mathbb{Q}_p(1), B_2) \to \mathrm{Hom}_{\mathcal{B}(G_K)}(V, B_2) \xrightarrow{\alpha} \mathrm{Hom}_{\mathcal{B}(G_K)}(\mathbb{Q}_p, B_2)$$

Comme $\dim_K \mathrm{Hom}(\mathbb{Q}_p, B_2) = 1$ et comme α n'est pas surjective, α est nulle et la première flèche est un isomorphisme. Donc η ne se relève pas en un homomorphisme de V dans B_2,d'où (i). En outre, on a $\dim_K \mathrm{Hom}_{\mathcal{B}(G_K)}(V, B_2) = \dim_K \mathrm{Hom}_{\mathcal{B}(G_K)}(\mathbb{Q}_p(1), B_2) = 1$.
On a $(B_2(-1) \otimes_{\mathbb{Q}_p} V)^{G_K} = \mathrm{Hom}_{\mathcal{B}(G_K)}(V^*(1), B_2)$ qui est bien de dimension 1 en appliquant (i) à $V^*(1)$, qui,comme V, est aussi une extension non scindée de $\mathbb{Q}_p(1)$ par $\mathbb{Q}_p$, d'où (ii).
Pour tout entier $i \neq 0$, on a $\mathrm{Hom}_{\mathcal{B}(G_K)}(\mathbb{Q}_p, C(i)) = 0$. On en déduit que, si $i \notin \{0, 1\}$, alors $\mathrm{Hom}_{\mathcal{B}(G_K)}(V, C(i)) = 0$. Comme $\mathrm{Fil}^i B_{dR} / \mathrm{Fil}^{i+1} B_{dR} = C(i)$, il en résulte que $\mathrm{Hom}_{\mathcal{B}(G_K)}(V, B_{dR}) = \mathrm{Hom}_{\mathcal{B}(G_K)}(V, B^+_{dR})$ et que l'application $\mathrm{Hom}_{\mathcal{B}(G_K)}(V, B^+_{dR}) \to \mathrm{Hom}_{\mathcal{B}(G_K)}(V, B_2)$ est injective. On a donc

$$\dim_K \mathrm{Hom}_{\mathcal{B}(G_K)}(V, B_{dR}) \leq \dim_K \mathrm{Hom}_{\mathcal{B}(G_K)}(V, B_2) = 1$$

alors que, si V était de de Rham, on aurait $\dim_K \mathrm{Hom}_{\mathcal{B}(G_K)}(V, B_{dR}) = 2$. □

Remarque : Cette proposition résulte aussi facilement du fait que, dans la dualité de Tate entre $H^1_{\mathrm{cont}}(K, \mathbb{Q}_p(-1))$ et $H^1_{\mathrm{cont}}(K, \mathbb{Q}_p(2))$ le $H^1_f(K, \mathbb{Q}_p(-1))$

est l'orthogonal du $H^1_f(K,\mathbb{Q}_p(2))$ ([BK90, prop.3.8). Comme toute extension de $\mathbb{Q}_p$ par $\mathbb{Q}_p(2)$ est cristalline ([PR88], th.1.5), on a $H^1_f(K,\mathbb{Q}_p(2)) = H^1_{\mathrm{cont}}(K,\mathbb{Q}_p(2))$, donc $H^1_f(K,\mathbb{Q}_p(-1)) = 0$ et V ne peut pas être de de Rham. Les deux autres assertions s'en déduisent immédiatement en utilisant la nullité des $H^r_{\mathrm{cont}}(K,C(i))$ pour $i \neq 0$.

On sait ([Fo00], th.6.1) que, si W_1 et W_2 sont deux C-représentations, toute application $\mathbb{Q}_p$-linéaire continue G_K-équivariante de W_1 dans W_2 est C-linéaire. Ce résultat s'étend aux B^+_{dR}-représentations :

THÉORÈME 3.11. — *Soient W_1 et W_2 deux B^+_{dR}-représentations. L'inclusion naturelle* $\mathrm{Hom}_{B^+_{dR}[G_K]}(W_1,W_2) \to \mathrm{Hom}_{\mathcal{B}(G_K)}(W_1,W_2)$ *est bijective tandis que l'application* $\mathrm{Ext}^1_{B^+_{dR}[G_K]}(W_1,W_2) \to \mathrm{Ext}^1_{\mathcal{B}(G_K)}(W_1,W_2)$ *est injective.*

Ce théorème nous permet d'identifier $\mathrm{Rep}_{B^+_{dR}}(G_K)$ à une sous-catégorie pleine de $\mathcal{B}(G_K)$.

Preuve : D'après le corollaire à la proposition 2.13, pour tout $i \in \mathbb{N}$ et pour toute extension finie galoisienne L de K contenue dans $\overline{K}$, on a $\mathrm{Ext}^i_{\mathrm{Rep}_{B^+_{dR}}(G_L)}(W_1,W_2) = (\mathrm{Ext}^i_{B^+_{dR}[G_K]}(W_1,W_2))^{\mathrm{Gal}(L/K)}$. Pour prouver le théorème on peut donc remplacer K par une telle extension, ce qui nous permet de supposer que W_1 et W_2 sont petites.
Par dévissage, on est ramené à prouver ces deux assertions lorsque W_1 et W_2 sont simples et sont donc des C-espaces vectoriels de dimension 1. La première assertion a été prouvée dans [Fo00] (cor.6.3, où elle est en fait établie pour des C-espaces vectoriels de dimension quelconque).
Prouvons la deuxième. A isomorphisme près, on peut supposer que $W_1 = C\{\alpha\}$, avec α un élément K-petit convenable (prop. 2.12). Quitte à tordre l'action de G_K sur W_1 et W_2 par $\chi^{-(\alpha)}$, on peut supposer que $W_1 = C$. Il résulte de la proposition 2.15 que l'on a $\mathrm{Ext}^1_{B^+_{dR}[G_K]}(W_1,W_2) = 0$ sauf si W_2 est isomorphe à C ou à $C(1)$. On peut donc supposer que $W_2 = C$ ou $W_2 = C(1)$.
Dans le premier cas, si W est une extension non triviale de C par C dans la catégorie $\mathrm{Rep}_{B^+_{dR}}(G_K)$, W est isomorphe au C-espace vectoriel C_2 (prop.2.12). S'il existait une section G_K-équivariante de la projection de C_2 sur C, il existerait en particulier un relèvement v dans C_2 de 1 fixe par G_K, mais alors l'application $c \mapsto cv$ de C dans C_2 serait une section C-linéaire G_K-équivariante, ce qui contredit le fait que la suite

$$0 \to C \to C_2 \to C \to 0$$

n'est pas scindée en tant que suite de C-représentations de G_K (prop.2.12).
Dans le second cas, si W est une extension non triviale de C par $C(1)$ dans $\mathrm{Rep}_{B^+_{dR}}(G_K)$, W est isomorphe à B_2 (prop.2.12). Mais il ne peut exister de section G_K-équivariante de la projection de B_2 sur C car, sinon, pour toute représentation p-adique V de G_K, l'application $\mathrm{Hom}_{\mathcal{B}(G_K)}(V,B_2) \to \mathrm{Hom}_{\mathcal{B}(G_K)}(V,C)$ serait surjective, ce qui contredit la proposition 3.10. $\square$

Corollaire. — *Soient W une B_{dR}^+-représentation et V une représentation p-adique de G_K. On a* $\mathrm{Hom}_{\mathcal{B}(G_K)}(W, V) = 0$

Preuve : En effet si $f \in \mathrm{Hom}_{\mathcal{B}(G_K)}(W, V)$, le composé φ de f avec l'inclusion naturelle de V dans $C \otimes_{\mathbb{Q}_p} V$ est B_{dR}^+-linéaire. Si f n'était pas nulle, φ ne le serait pas non plus et l'image de φ serait un sous-C-espace vectoriel non nul de $C \otimes_{\mathbb{Q}_p} V$ et ne serait donc pas de dimension finie sur $\mathbb{Q}_p$.

3.4 – Construction "explicite" des extensions de W par V

On pose $U = U_1$. Soit R l'ensemble des suites $(x^{(n)})_{n \in \mathbb{N}}$ d'éléments de l'anneau des entiers $\mathcal{O}_C$ de C vérifiant $(x^{(n+1)})^p = x^{(n)}$ pour tout n. Rappelons [Fo88a] que R est muni d'une structure d'anneau de valuation complet de caractéristique $p > 0$ et que c'est un anneau parfait (son corps des fractions est même algébriquement clos) ; si $W(R)$ désigne l'anneau des vecteurs de Witt à coefficients dans R, l'anneau $W(R)[1/p]$ s'identifie à un sous-anneau de $B_{cris}^+ \subset B_{cris} \cap B_{dR}^+$.
Soit $\mathfrak{m}_R$ l'idéal maximal de R. Alors $U_R = 1 + \mathfrak{m}_R$ est un sous-groupe du groupe multiplicatif R^* des éléments inversibles de R. C'est de façon naturelle un $\mathbb{Q}_p$-espace vectoriel topologique. C'est en fait un banach : pour tout $a \neq 0$ dans $\mathfrak{m}_R$, le sous-$\mathbb{Z}_p$-module $1 + aR$ de U_R est un réseau. Rappelons (cf. par exemple, [CF00], prop.1.3) que, pour tout $u \in U_R$, si [u] désigne son représentant de Teichmüller dans $W(R)$, alors la série $\log[u] = \sum_{n=1}^{+\infty} (-1)^{n+1}([u]-1)^n/n$ converge dans B_{cris}^+. L'image de U_R par cette application est U ; on obtient ainsi un isomorphisme (de banach) de U_R sur U.

Soit alors V une représentation p-adique de G_K et soit $V_C = C \otimes_{\mathbb{Q}_p} V$. En tensorisant par $V(-1)$ la suite exacte

$$0 \to \mathbb{Q}_p(1) \to U \to C \to 0$$

on obtient une autre suite exacte courte de $\mathcal{B}(G_K)$

$$0 \to V \to U(-1) \otimes V \to V_C(-1) \to 0$$

Si W est une C-représentation, en appliquant le foncteur $\mathrm{Hom}_{\mathcal{B}(G_K)}(-, W)$, on obtient un opérateur de cobord $\delta : \mathrm{Hom}_{\mathcal{B}(G_K)}(W, V_C(-1)) \to \mathrm{Ext}^1_{\mathcal{B}(G_K)}(W, V)$.

Proposition 3.12. — *Soient W une C-représentation et V une représentation p-adique de G_K. On a* $\mathrm{Hom}_{\mathcal{B}(G_K)}(W, U(-1) \otimes_{\mathbb{Q}_p} V) = 0$ *et l'application*

$$\delta : \mathrm{Hom}_{\mathcal{B}(G_K)}(W, V_C(-1)) \to \mathrm{Ext}^1_{\mathcal{B}(G_K)}(W, V)$$

définie ci-dessus est un isomorphisme.

Nous allons en fait prouver un résultat plus général. Notons, avec une définition évidente, $\mathcal{IB}(\mathcal{G}_\mathcal{K})$ la catégorie des *limites inductives de représentations banachiques.* Pour tout entier $m \in \mathbb{N}$, $\mathrm{Fil}^{-m} B_{dR}/B_{dR}^+$ s'identifie à $B_m(-m)$

de sorte que B_{dR}/B^+_{dR} est une limite inductive d'objets de $\mathrm{Rep}_{B^+_{dR}}(G_K)$: c'est la réunion croisssante des $B_m(-m)$, pour $m \in \mathbb{N}$.
Si maintenant V est une représentation p-adique et W une B^+_{dR}-représentation, on a $\mathrm{Hom}_{\mathcal{IB}(\mathcal{G}_\mathcal{K})}(W,(B_{dR}/B^+_{dR})\otimes_{\mathbb{Q}_p} V) = \mathrm{Hom}_{\mathcal{B}(G_K)}(W, B_m(-m)\otimes_{\mathbb{Q}_p} V)$ pour m assez grand (il suffit que t^m annule W ; en particulier, on peut prendre $m = 1$ si W est une C-représentation). Par conséquent si l'on considère la suite exacte

$$0 \to V \to B_e \otimes_{\mathbb{Q}_p} V \to (B_{dR}/B^+_{dR}) \otimes_{\mathbb{Q}_p} V \to 0$$

en appliquant le foncteur $\mathrm{Hom}_{\mathcal{IB}(\mathcal{G}_\mathcal{K})}(-, W)$, on obtient une flèche

$$\delta : \mathrm{Hom}_{\mathcal{IB}(\mathcal{G}_\mathcal{K})}(W,(B_{dR}/B^+_{dR})\otimes_{\mathbb{Q}_p} V) \to \mathrm{Ext}^1_{\mathcal{B}(G_K)}(W,V)$$

(parce que, si m est un entier suffisamment grand pour que
$\mathrm{Hom}_{\mathcal{IB}(\mathcal{G}_\mathcal{K})}(W,(B_{dR}/B^+_{dR})\otimes V) = \mathrm{Hom}_{\mathcal{B}(G_K)}(W, B_m(-m)\otimes V)$,
on peut remplacer la suite exacte précédente par

$$0 \to V \to \mathrm{Fil}^{-m} B_e \otimes_{\mathbb{Q}_p} V \to B_m(-m) \otimes_{\mathbb{Q}_p} V \to 0$$

qui est une suite exacte courte de $\mathcal{B}(G_K)$).
La proposition 3.12 est un cas particulier de l'énoncé suivant :

Proposition 3.13. — *Soient W une B^+_{dR}-représentation et V une représentation p-adique de G_K. On a* $\mathrm{Hom}_{\mathcal{B}(G_K)}(W, B_e \otimes_{\mathbb{Q}_p} V) = 0$. *Si m est un entier tel que t^m annulle W, alors* $\mathrm{Hom}_{\mathcal{IB}(\mathcal{G}_\mathcal{K})}(W,(B_{dR}/B^+_{dR})\otimes_{\mathbb{Q}_p} V) = \mathrm{Hom}_{\mathcal{B}(G_K)}(W, B_m(-m)\otimes_{\mathbb{Q}_p} V)$. *L'application*

$$\delta : \mathrm{Hom}_{\mathcal{IB}(\mathcal{G}_\mathcal{K})}(W,(B_{dR}/B^+_{dR})\otimes_{\mathbb{Q}_p} V) \to \mathrm{Ext}^1_{\mathcal{B}(G_K)}(W,V)$$

définie ci-dessus est un isomorphisme.

Preuve : On a déjà vu la deuxième assertion. Vérifions les deux autres. Comme $\mathrm{Hom}_{\mathcal{B}(G_K)}(W,V) = \mathrm{Hom}_{\mathcal{B}(G_K)}(V^{*\mathbb{Q}_p}\otimes_{\mathbb{Q}_p} W, \mathbb{Q}_p) = 0$ (cor. au th.3.11), on a un diagramme commutatif

$$\begin{array}{ccccccc}
0 & \to & \mathrm{Hom}(W, B_e\otimes V) & \to & \mathrm{Hom}(W, B_{dR}/B^+_{dR}\otimes V) & \to & \mathrm{Ext}^1(W,V) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \mathrm{Hom}(V^*\otimes W, B_e) & \to & \mathrm{Hom}(V^*\otimes W, B_{dR}/B^+_{dR}) & \to & \mathrm{Ext}^1(V^*\otimes W, \mathbb{Q}_p)
\end{array}$$

(on a posé $V^* = V^{*\mathbb{Q}_p}$) où les lignes sont exactes et les flèches verticales sont des isomorphismes. Quitte à remplacer W par $V^*\otimes_{\mathbb{Q}_p} W$, on peut supposer $V = \mathbb{Q}_p$.
Soit $f : W \to B_e$. Si f était non nulle, le composé de f avec la projection de B_e sur B_{dR}/B^+_{dR} serait une application $\overline{f} : W \to B_{dR}/B^+_{dR}$ non nulle et serait donc B^+_{dR}-linéaire (th.3.11). Son image serait un sous-B^+_{dR}-module non nul de B_{dR}/B^+_{dR}, donc serait de la forme $B_m(-m)$ pour un entier $m \geq 1$ convenable,

et f serait une application de W dans $\mathrm{Fil}^{-m}B_e$. Le noyau W' de $\overline{f}$ serait une B_{dR}^+-représentation et comme $\mathrm{Hom}_{\mathcal{B}(G_K)}(W', \mathbb{Q}_p) = 0$ (appliquer de nouveau le cor. au th.3.11), $\overline{f}$ induirait en fait une section de la projection de $\mathrm{Fil}^{-m}B_e$ sur $B_m(-m)$. Par restriction à $C(-1)$, on en déduirait une section de la projection de $U(-1)$ sur $C(-1)$. En tensorisant par $\mathbb{Q}_p(1)$ cela fournirait une section de la projection de U sur C. On aurait donc un isomorphisme de U sur $\mathbb{Q}_p(1) \oplus C$, ce qui contredit le fait que $U^{G_K} = 0$, tandis que $(\mathbb{Q}_p(1) \oplus C)^{G_K} = C^{G_K} = K$. Donc $f = 0$ et δ est injective.
On sait (th.3.11) que $\mathrm{Hom}_{\mathcal{B}(G_K)}(W, B_{dR}/B_{dR}^+) = \mathrm{Hom}_{B_{dR}^+[G_K]}(W, B_{dR}/B_{dR}^+)$. On sait (th.2.14, (iv)) que ce K-espace vectoriel est de dimension finie égale à la dimension du sous-espace propre associé à la valeur-propre -1 de l'opérateur $\nabla_0(W)$. Comme c'est aussi la dimension de $\mathrm{Ext}^1_{\mathcal{B}(G_K)}(W, V)$, l'injectivité de δ implique que c'est un isomorphisme. □

Remarques : i) Posons $V^* = V^{*\mathbb{Q}_p}$ et $X = W \otimes_{\mathbb{Q}_p} V^*(1)$. On a une dualité naturelle entre les K-espaces vectoriels

$$\mathrm{Hom}_{\mathcal{B}(G_K)}(W, (B_{dR}/B_{dR}^+) \otimes V) = \mathrm{Hom}_{B_{dR}^+[G_K]}(W, (B_{dR}/B_{dR}^+) \otimes V) =$$
$$\mathrm{Hom}_{B_{dR}^+[G_K]}(X, (B_{dR}/B_{dR}^+)(1)) = \mathrm{Hom}_{B_{dR}^+[G_K]}(X, B_{dR}/\mathrm{Fil}^1 B_{dR})$$

et $H^1_{\mathrm{cont}}(K, W \otimes V^*(1)) = \mathrm{Ext}^1_{B_{dR}^+[G_K],0}(C, X)$:
Soient $f \in \mathrm{Hom}_{B_{dR}^+[G_K]}(X, B_{dR}/\mathrm{Fil}^1 B_{dR})$ et $\varepsilon \in \mathrm{Ext}^1_{B_{dR}^+[G_K],0}(C, X)$. Soit F une B_{dR}^+-représentation, extension de C par $B_{dR}/\mathrm{Fil}^1 B_{dR}$ dont la classe est l'image par f de ε. Comme cette extension est scindée en tant qu'extension de B_{dR}^+-modules, le noyau F_0 de la multiplication par t dans F est une C-représentation extension de C par C. Il existe donc (propr.2.15) un unique $\lambda \in K$ tel que la classe de F_0 est λ fois la classe de l'extension C_2 et cette dualité est l'application $(f, \varepsilon) \mapsto \lambda$.
Compte-tenu de cette dualité, la proposition 3.7 définit un isomorphisme

$$\mathrm{Ext}^1_{\mathcal{B}(G_K)}(W, V) \to \mathrm{Hom}_{\mathcal{B}(G_K)}(W, (B_{dR}/B_{dR}^+) \otimes_{\mathbb{Q}_p} V)$$

Il ne devrait pas être difficile de vérifier que δ est l'inverse (au signe près ?) de cette application.
ii) La construction de δ fonctionne encore lorsque l'on remplace K par un corps toujours de caractéristique 0 et complet pour une valuation discrète, à corps résiduel toujours parfait de caractéristique p, mais infini. Il est raisonnable de penser que la proposition 3.13 reste vraie dans ce contexte. Si c'est le cas, une grande partie des résultats de cet article s'étendent à un tel corps K.

3.5 – Extensions presque scindées

Soit

$$0 \to S' \to S \to S'' \to 0 \tag{1}$$

une suite exacte courte de $\mathcal{B}(G_K)$. Un *presque supplémentaire* de S' dans S est un sous-$\mathbb{Q}_p$-espace vectoriel fermé E de S stable par G_K tel que $S = S' + E$ et $V = S' \cap E$ est de dimension finie sur $\mathbb{Q}_p$. On a alors un diagramme commutatif

$$\begin{array}{ccccccccc} 0 & \to & V & \to & E & \to & S'' & \to & 0 \\ & & \cap & & \cap & & \| & & \\ 0 & \to & S' & \to & S & \to & S'' & \to & 0 \end{array}$$

qui identifie S à $S' \oplus_V E$, somme amalgamée de S' et de E, au dessous de V et la suite exacte

$$0 \to S'/V \to S/V \to S'' \to 0$$

est scindée. On dit que la suite exacte (1) est *presque scindée* s'il existe un presque supplémentaire de S' dans S.

Soient $f : V \to W'$ un morphisme de $\mathcal{B}(G_K)$ avec V une représentation p-adique et W' une B_{dR}^+-représentation. On note $E(f)$ le B_{dR}^+-module somme amalgamée de W' et de $B_{dR} \otimes_{\mathbb{Q}_p} V$ au dessous de $B_{dR}^+ \otimes_{\mathbb{Q}_p} V$, l'application de $B_{dR}^+ \otimes V$ dans W' étant déduite de f par extension des scalaires. On a un diagramme commutatif

$$\begin{array}{ccccccccc} 0 & \to & B_{dR}^+ \otimes V & \to & B_{dR} \otimes V & \to & (B_{dR}/B_{dR}^+) \otimes V & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & W' & \to & E(f) & \to & (B_{dR}/B_{dR}^+) \otimes V & \to & 0 \end{array}$$

dont les lignes sont exactes. En outre, si, pour tout $m \in \mathbb{N}$, on note $E_m(f)$ l'image inverse de $B_m(-m) \otimes V$ dans $E(f)$, chaque $E_m(f)$ est une B_{dR}^+-représentation et $E(f)$ est la réunion croissante de ces $E_m(f)$.

Proposition 3.14. — *Soit*

$$0 \to W' \to W \to W'' \to 0$$

une suite exacte courte presque scindée de $\mathcal{B}(G_K)$*. Si* W' *et* W'' *sont des* B_{dR}^+*-représentations, il en est de même de* W*.*

Preuve : Par hypothèse, on a, dans $\mathcal{B}(G_K)$, un diagramme commutatif

$$\begin{array}{ccccccccc} 0 & \to & V & \to & E & \to & W'' & \to & 0 \\ & & \cap & & \cap & & \| & & \\ 0 & \to & W' & \to & W & \to & W'' & \to & 0 \end{array}$$

dont les lignes sont exactes et où V est de dimension finie sur $\mathbb{Q}_p$.
D'après la proposition 3.13, on a un diagramme commutatif

$$\begin{array}{ccccccccc} 0 & \to & V & \to & E & \to & W'' & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & V & \to & B_e \otimes V & \to & (B_{dR}/B_{dR}^+) \otimes V & \to & 0 \end{array}$$

dont les lignes sont exactes, comme le sont celles du diagramme commutatif

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{Q}_p & \to & B_e & \to & B_{dR}/B^+_{dR} & \to & 0 \\ & & \cap & & \cap & & \| & & \\ 0 & \to & B^+_{dR} & \to & B_{dR} & \to & B_{dR}/B^+_{dR} & \to & 0 \end{array}$$

Si f désigne l'inclusion de V dans W', on obtient alors un troisième diagramme commutatif

$$\begin{array}{ccccccccc} 0 & \to & V & \to & E & \to & W'' & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & V & \to & B_e \otimes V & \to & (B_{dR}/B^+_{dR}) \otimes V & \to & 0 \\ & & \cap & & \cap & & \| & & \\ 0 & \to & B^+_{dR} \otimes V & \to & B_{dR} \otimes V & \to & (B_{dR}/B^+_{dR}) \otimes V & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & W' & \to & E(f) & \to & (B_{dR}/B^+_{dR}) \otimes V & \to & 0 \end{array}$$

dont les lignes sont tout autant exactes (la deuxième ligne de flèches verticales se déduit du diagramme précédent en tensorisant avec V).
Comme W est la somme amalgamée de W' et de E au dessous de V, on en déduit un diagramme commutatif dont les lignes sont exactes

$$\begin{array}{ccccccccc} 0 & \to & W' & \to & W & \to & W'' & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & W' & \to & E(f) & \to & (B^+_{dR}/B_{dR}) \otimes V & \to & 0 \end{array}$$

et W s'identifie au B^+_{dR}-module $E(f) \times_{(B_{dR}/B^+_{dR})\otimes V} W''$ (qui est bien de longueur finie : si W'' est tué par t^m, on a aussi $W = E_m(f) \times_{B_m(-m)\otimes V} W''$). $\square$

Remarque : On verra au §5 que, réciproquement, toute suite exacte courte de $\mathrm{Rep}_{B^+_{dR}}(G_K)$ est presque scindée.

4 – Structures analytiques

Le but de ce paragraphe est d'établir le théorème suivant :

Théorème 4.1. — *Soit E une représentation banachique extension d'une C-représentation de dimension 1 par une $\mathbb{Q}_p$-représentation V. Si $\eta : E \to C$ est un morphisme de représentations banachiques tel que $\eta(E) \neq \eta(V)$, alors η est surjective et son noyau est un $\mathbb{Q}_p$-espace vectoriel de dimension finie égale à celle de V.*

Pour cela, nous allons dabord définir *les espaces de Banach-Colmez* : ce sont des espaces de Banach p-adiques munis d'uns structure analytique sur C d'un type particulier. On montre ensuite que E et C sont munis d'une structure naturelle d'espaces de Banach-Colmez et que η est analytique. Le résultat

fondamental de Colmez [Co02] dans son travail sur ce qu'il appelle les *Espaces de Banach de dimension finie* permet alors de conclure.
Pour décrire ces espaces, nous adoptons ici un point de vue un peu plus algébrique (ou géométrique ?) que celui de Colmez : Les espaces de Banach-Colmez sont associés à certains objets en groupes commutatifs dans la catégorie des *variétés pro-analytiques rigides sur* C . Nous reviendrons sur l'étude de ces espaces dans un travail ultérieur [FP]. Remarquons dès à présent qu'une bonne partie de la théorie esquissée ci-dessous peut se développer en remplaçant C par n'importe quel corps valué complet de caractéristique 0 à corps résiduel parfait de caractéristique $p > 0$. En particulier, comme le suggère l'analyticité de l'application η du théorème 4.1, la théorie des espaces de Banach-Colmez sur K se ramène essentiellement à celle des presque C-représentations de G_K.

4.1 – Banach analytiques

Dans ce texte, une *C-algèbre* est un anneau commutatif A muni d'un homomorphisme de C dans A. Une *C-algèbre de Banach* est une C-algèbre normée complète. Si A est une telle algèbre, on note $\mathrm{Spm}_C A$ *le spectre maximal de* A, i.e. l'ensemble des sections continues $s : A \to C$ du morphisme structural. Si $f \in A$ et $s \in \mathrm{Spm}_C A$, on pose $f(s) = s(f)$.
On munit C de la valeur absolue normalisée par $|p| = p^{-1}$. Une *C-algèbre spectrale* est une C-algèbre de Banach A telle que la norme est la norme spectrale, i .e. telle que, pour tout $f \in A$, $||f|| = sup_{s\in \mathrm{Spm}_C A}|f(s)|$.
Par exemple, pour tout $d \in \mathbb{N}$, l'algèbre de Tate $C\{X_1, X_2, \dots, X_d\}$ des *séries formelles restreintes* en les X_i (i.e. des séries $\sum a_{i_1,i_2,\dots,i_d} X_1^{i_1} X_2^{i_2}, \dots, X_d^{i_d}$ telles que, pour tout $\varepsilon > 0$, $|a_{i_1,i_2,\dots,i_d}| < \varepsilon$ pour presque tout d-uple $i_1, i_2, \dots, i_d$), est une algèbre spectrale avec la norme

$$||\sum a_{i_1,i_2,\dots,i_d} X_1^{i_1} X_2^{i_2}, \dots, X_d^{i_d}|| = \sup |a_{i_1,i_2,\dots,i_d}|$$

Pour toute C-algèbre spectrale A, si $f \in A$ est non nul, il existe $s \in \mathrm{Spm}_C A$ tel que $f(s) \neq 0$. Si $f \in A$ et $m \in \mathbb{N}$, on a $||f^m|| = ||f||^m$; en particulier, A est réduite.
Soient A une C-algèbre spectrale. On note $\mathcal{O}_A$ la boule-unité fermée de A, i.e. la sous-$\mathcal{O}_C$-algèbre des $f \in A$ tels que $||f|| \leq 1$. On a des identifications $C \otimes_{\mathcal{O}_C} \mathcal{O}_A = \mathcal{O}_A[1/p] = A$ et $\mathcal{O}_A$ est un réseau de A. Pour $s, s' \in \mathrm{Spm}_C A$, on pose $d(s, s') = \sup_{f\in\mathcal{O}_A} |f(s) - f(s')|$. Cela fait de $\mathrm{Spm}_C A$ un espace (ultra-)métrique complet.
Avec comme morphismes les homomorphismes continus de C-algèbres, les C-algèbres spectrales forment une catégorie. La *catégorie des variétés spectrales affines sur* C est la catégorie opposée. Si A est une C-algèbre spectrale, on parle abusivement de la variété spectrale affine $\mathcal{S} = \mathrm{Spm}_C A$ et A s'appelle l'algèbre affine de la variété. Suivant un usage établi, le symbole $\mathrm{Spm}_C A$ désigne donc, suivant le contexte, soit la variété spectrale affine, soit son spectre maximal, i.e. l'espace topologique $\mathrm{Hom}^{\mathrm{cont}}_{C-\text{algèbres}}(A, C)$. On remarque que le foncteur

d'oubli évident de la catégorie des variétés spectrales affines sur C dans celle des espaces topologiques est fidèle.
Pour $i = 1, 2$, soit $\mathrm{Spm}_C A_i$ une variété spectrale sur C et $\mathcal{S}_i$ son spectre maximal. On dit qu'une application $f : \mathcal{S}_1 \to \mathcal{S}_2$ est *analytique* s'il existe un homomorphisme continu de C-algèbres, nécessairement unique, $f^* : A_2 \to A_1$ tel que $f(s) = s \circ f^*$, pour tout $s \in \mathcal{S}_1$. Ceci nous permet d'identifier les morphismes d'une variété spectrale affine dans une autre à l'ensemble des applications analytiques du spectre maximal de la première dans celui de la seconde.
La catégorie des variétés spectrales affines sur C admet des limites projectives finies. Par exemple, soient $\mathcal{S} = \mathrm{Spm}_C A$, $\mathcal{S}_1 = \mathrm{Spm}_C A_1$ et $\mathcal{S}_2 = \mathrm{Spm}_C A_2$ trois variétés affines spectrales sur C et $\mathcal{S}_1 \to \mathcal{S}$ et $\mathcal{S}_2 \to \mathcal{S}$ des morphismes. Le produit fibré $\mathcal{S}_1 \times_{\mathcal{S}} \mathcal{S}_2$ est muni d'une structure de variété spectrale affine sur C d'algèbre affine le séparé complété $A_1 \hat{\otimes}_A A_2$ de $A_1 \otimes_A A_2$ pour la norme

$$||f|| = \sup_{(s_1,s_2)\in\mathcal{S}_1\times_{\mathcal{S}}\mathcal{S}_2} |f(s_1, s_2)|$$

Un *groupe spectral commutatif affine sur* C est un objet en groupes commutatifs dans la catégorie des variétés spectrales affines sur C. C'est donc une variété spectrale affine $\mathrm{Spm}_C A$ sur C dont le spectre maximal $\mathcal{S}$ est équippé d'une loi de composition $\mathcal{S} \times \mathcal{S} \to \mathcal{S}$, associative, commutative et unitaire, induite par un morphisme de C-algèbres spectrales, le *co-produit* $m^* : A \to A\hat{\otimes}_C A$. En particulier la loi de compostion est continue et $\mathcal{S}$ est un groupe topologique.

Soit S un banach. Une *C-structure analytique*(1) *sur* S est la donnée d'un couple $(\mathrm{Spm}_C A, \alpha)$ formé d'un groupe spectral commutatif affine $\mathrm{Spm}_C A$ sur C et d'un homomorphisme de groupes continu α du spectre maximal $\mathcal{S}$ de A dans S induisant un homémomorphisme de $\mathcal{S}$ sur un réseau de S. Dans la pratique, on utilise cet homomorphisme pour identifier $\mathcal{S}$ a un réseau de S. Un *banach analytique sur* C est un triplet $(S, \mathrm{Spm}_C A, \alpha)$ formé d'un banach S et d'une C-structure analytique $(\mathrm{Spm}_C A, \alpha)$ sur S. S'il n'y a pas de risque de confusion, on dit *banach analytique* au lieu de *banach analytique sur* C.
Soient $(S_1, \mathrm{Spm}_C A_1, \alpha_1)$ et $(S_2, \mathrm{Spm}_C A_2, \alpha_2)$ des banach analytiques ; posons $\mathcal{S}_1 = \mathrm{Spm}_C A_1$ et $\mathcal{S}_2 = \mathrm{Spm}_C A_2$. Un *morphisme de banach analytiques* η : $(S_1, \mathrm{Spm}_C A_1, \alpha_1) \to (S_2, \mathrm{Spm}_C A_2, \alpha_2)$ est une application $\mathbb{Q}_p$-linéaire de S_1 dans S_2 qui est *analytique* i.e. qui a la vertu qu'il existe un entier m et un homomorphisme continu $\nu : A_2 \to A_1$ de C-algèbres tels que $\eta(p^m \mathcal{S}_1) \subset \mathcal{S}_2$ et que la restriction de $p^m \eta$ à $\mathcal{S}_1$ est l'application $f \mapsto f \circ \nu$ (remarquer que, pour m fixé, l'application ν est uniquement déterminée par η). Si cette propriété est vraie pour m elle est vraie pour tout entier supérieur. Les applications analytiques de $(S_1, \mathrm{Spm}_C A_1, \alpha_1)$ dans $(S_2, \mathrm{Spm}_C A_2, \alpha_2)$ forment un sous-$\mathbb{Q}_p$-espace vectoriel de l'espace des applications $\mathbb{Q}_p$-linéaires continues de S_1 dans S_2.

(1) Dans [Fo02] et [FP] on donne une définition un peu plus générale afin de pouvoir faire des quotients par des $\mathbb{Z}_p$-modules de type fini. Nous n'en avons pas besoin ici.

Les banach analytiques forment une catégorie additive et même $\mathbb{Q}_p$-linéaire ; ce n'est pas une catégorie abélienne mais elle admet des limites projectives finies. Si S est un banach, on dit que deux C-structures analytiques $(\mathrm{Spm}_C A_1, \alpha_1)$ et $(\mathrm{Spm}_C A_2, \alpha_2)$ sur S sont équivalentes si l'identité sur S est analytique dans les deux sens (i.e. est un morphisme de banach analytiques de $(S, \mathrm{Spm}_C A_1, \alpha_1)$ dans $(S, \mathrm{Spm}_C A_2, \alpha_2)$, aussi bien que de $(S, \mathrm{Spm}_C A_2, \alpha_2)$ dans $(S, \mathrm{Spm}_C A_1, \alpha_1)$). L'identité sur S est alors un isomorphisme de ces deux banach analytiques que l'on utilise pour les identifier. Autrement dit, on peut aussi bien voir un banach analytique comme un banach muni d'une classe d'équivalence de C-structures analytiques. On parlera souvent abusivement du banach analytique S, la classe d'équivalence de structures anaytiques étant sous-entendue.

4.2 – Banach analytiques constants et vectoriels

Soit V un $\mathbb{Q}_p$-espace vectoriel de dimension finie. Choisissons un réseau $\mathcal{V}$ de V ; soit $\mathcal{F}_{\mathrm{cont}}(\mathcal{V}, C)$ la C-algèbre des fonctions continues sur $\mathcal{V}$ à valeurs dans C. On la munit de la norme spectrale

$$||(f)|| = \sup_{s \in \mathcal{V}} |f(s)|$$

Alors $\mathcal{V} = \mathrm{Spm}_C \mathcal{F}_{\mathrm{cont}}(\mathcal{V}, C)$, munie de l'inclusion de $\mathcal{V}$ dans V, est une structure analytique sur V dont la classe d'équivalence ne dépend pas du choix du réseau ; on l'appelle la *structure analytique constante* et on note V^c le banach analytique ainsi associé à V.

On obtient ainsi un foncteur $V \mapsto V^c$ de la catégorie des $\mathbb{Q}_p$-espaces vectoriels de dimension finie dans celle des banach analytiques. *Ce foncteur est pleinement fidèle*, i.e. si V_1 et V_2 sont deux $\mathbb{Q}_p$-espaces vectoriels de dimension finie, toute application $\mathbb{Q}_p$-linéaire de V_1 dans V_2 non seulement est continue, mais elle est analytique.

Les *banach analytiques constants* sont les $\mathbb{Q}_p$-espaces vectoriels de dimension finie munis de la structure analytique constante. On appelle alors *hauteur* de V^c (ou de V) la dimension de V sur $\mathbb{Q}_p$.

Soit $\mathcal{W}$ un $\mathcal{O}_C$-module libre de rang fini. Notons $\mathcal{W}'$ le $\mathcal{O}_C$-module dual. Notons $\mathcal{O}_C\{\mathcal{W}'\}$ la complétion p-adique de l'algèbre $\mathrm{Sym}_{\mathcal{O}_C} \mathcal{W}'$ et posons $C\{\mathcal{W}'\} = C \otimes_{\mathcal{O}_C} \mathcal{O}_C\{\mathcal{W}'\} = \mathcal{O}_C\{\mathcal{W}'\}[1/p]$. L'algèbre topologique $C\{\mathcal{W}'\}$ est une C-algèbre spectrale ; le choix d'une base $\{e_1, e_2, \ldots, e_d\}$ de $\mathcal{W}$ permet de l'identifier à l'algèbre des séries formelles restreintes $C\{X_1, X_2, \ldots, X_d\}$ (où $\{X_1, X_2, \ldots, X_d\}$ est la base de $\mathcal{W}'$ duale de $\{e_1, e_2, \ldots, e_d\}$). Son spectre maximal s'identifie à $\mathcal{W}$, ce qui nous permet de considérer $\mathcal{W} = \mathrm{Spm}_C C\{\mathcal{W}'\}$ comme un groupe spectral commutatif affine.

Si maintenant W est un C-espace vectoriel de dimension finie, le choix d'un *$\mathcal{O}_C$-réseau* de W, i.e. d'un sous-$\mathcal{O}_C$-module libre $\mathcal{W}$ de W qui engendre W comme C-espace vectoriel, définit une structure analytique $\mathrm{Spm}_C C\{\mathcal{W}'\}$ sur

W. On voit que la classe d'équivalence de cette structure est indépendante du choix de $\mathcal{W}$, ce qui nous permet de considérer W comme le banach sous-jacent à un banach analytique que l'on note W^{an}.
Par exemple, pour $W = C$, on écrit aussi $\mathbb{G}_a^{\mathrm{an}} = C^{\mathrm{an}}$ et on l'appelle le *groupe additif* (sous-entendu dans la catégorie des banach analytiques sur C). L'élément 1 est une base du $\mathcal{O}_C$-réseau $\mathcal{O}_C$ de C et $\mathcal{O}_C$ s'identifie à $\mathrm{Spm}_C C\{X\}$, avec $m^*X = X\hat{\otimes}1 \oplus 1\hat{\otimes}X$.
L'application qui à W associe W^{an} est, de manière évidente, un foncteur de la catégorie des C-espaces vectoriels de dimension finie dans celle des banach analytiques.

Proposition 4.2. — *Le foncteur $W \mapsto W^{\mathrm{an}}$ de la catégorie des C-espaces vectoriels de dimension finie dans celle des banach analytiques est pleinement fidèle.*

Autrement dit, si W_1 et W_2 sont deux C-espaces vectoriels de dimension finie, toute application $\mathbb{Q}_p$-linéaire de W_1 dans W_2 qui est analytique est C-linéaire. Ce résultat est à rapprocher du théorème de pleine fidélité pour les C-représentations (cf. th.3.11).

Preuve : Si W est un C-espace vectoriel de dimension d, le choix d'une base de W définit un isomorphisme de W^{an} sur $(\mathbb{G}_a^{\mathrm{an}})^d$. Ceci nous ramène au cas où $W_1 = W_2 = C$. Si η est un endomorphisme analytique de C, il existe un entier m tel que $p^m\eta$ provient d'un endomorphisme continu ν de la C-algèbre $C\{X\}$ qui commute au coproduit. Se donner ν revient à se donner $\nu(X)$ qui doit être un élément $f \in \mathcal{O}_C\{X\}$ vérifiant $m^*(f) = f\hat{\otimes}1 + 1\hat{\otimes}f$. Mais ceci implique que f appartient au C-espace vectoriel engendré par X, comme on le voit par exemple en utilisant l'inclusion de $\mathcal{O}_C\{X\} \subset C\{X\}$ dans $C[[X]]$, celle de $C\{X\hat{\otimes}1, 1\hat{\otimes}X\}$ dans $C[[X\hat{\otimes}1, 1\hat{\otimes}X]]$ et le fait que le résultat correspondant est vrai pour le groupe formel additif sur un corps de caractéristique 0. On a donc $\nu(X) = \lambda X$ pour un $\lambda \in \mathcal{O}_C$ convenable. Mais alors l'endomorphisme de C induit par ν est la multiplication par λ et η est la multiplication par $p^{-m}\lambda$ qui est bien C-linéaire. □

On a donc une équivalence entre la catégorie des C-espaces vectoriels de dimension finie et celle des *banach analytiques vectoriels*, i.e. la sous-catégorie pleine de la catégorie des banach analytiques qui sont isomorphes à un W^{an} pour un C-espace vectoriel W de dimension finie convenable.

4.3 – Espaces de Banach-Colmez

Les Espaces de Banach-Colmez *présentables* sont les banach analytiques qui peuvent s'écrire comme une extension d'un banach analytique vectoriel par un banach analytique constant. Dans la définition qui suit et qui suffit pour ce que nous faisons ici, il semble que l'on ne considère que des extensions d'un type particulier. En fait ([Co02], [FP]) toutes les extensions d'un banach analytique vectoriel par un banach analytique constant sont de ce type là.

Rappelons (§3.4) que le groupe multiplicatif $U_R = 1 + \mathfrak{m}_R$ a une structure naturelle de banach. On a une structure analytique naturelle sur U_R : Soit $\pi \in R$ tel que $(\pi^{(0)}) = -p$; alors $\mathcal{U}_R = 1 + \pi R$ est un réseau de U_R. Pour tout $n \in \mathbb{N}$, soit $\mathcal{U}_n = 1 + \pi^{(n)}\mathcal{O}_C$ le sous-groupe du groupe multiplicatif $\mathcal{O}_C^*$ de $\mathcal{O}_C$ formé des a tels que $|a-1| \leq p^{-n}$. L'application de $\mathcal{O}_C$ dans $\mathcal{U}_n$ qui envoie y sur $1+\pi^{(n)}y$ est un homéomorphisme et on l'utilise pour munir $\mathcal{U}_n$ d'une structure de groupe spectral commutatif affine ; autrement dit on écrit $\mathcal{U}_\backslash = \mathrm{Spm}_C C\{Y_n\}$ et la loi de groupe est donnée par $m^*Y_n = Y_n\hat{\otimes}1+1\hat{\otimes}Y_n+\pi^{(n)}Y_n\hat{\otimes}Y_n$. On envoie $\mathcal{U}_{n+1}$ sur $\mathcal{U}_n$ via le morphisme analytique défini par l'homomorphisme continu de C-algèbres $C\{Y_n\} \to C\{Y_{n+1}\}$ qui envoie $1+\pi^{(n)}Y_n$ sur $(1+\pi^{(n+1)}Y_{n+1})^p$. Alors $\mathcal{U}_R$ s'identifie à la limite projective des $\mathcal{U}_n$ et on le munit de la structure de groupe spectral commutatif affine induite, i.e. on pose $\mathcal{U}_R = \mathrm{Spm}_C A_U$ où A_U est le complété pour la norme spectrale de $\varinjlim_{n\in\mathbb{N}} C\{Y_n\}$. D'où une C-structure analytique sur U_R. On note U^{an} le banach analytique $(U, \mathrm{Spm}_C A_U, \alpha_U)$ où $\alpha_U : \mathcal{U}_R \to U$ est l'application $u \mapsto \log[u]$.
La suite exacte

$$0 \to \mathbb{Q}_p(1) \to U \to C \to 0$$

induit une suite exacte de banach analytiques

$$0 \to \mathbb{Q}_p(1)^c \to U^{an} \to C^{an} \to 0 \tag{1}$$

En effet, l'application de U sur C qui envoie $\log[u]$ sur $\log(u^{(0)})$ est analytique (elle est induite par l'homomorphisme continu de C-algèbres $C\{X\} \to A_U$ qui envoie X sur $\log(1-pY_0)/p$). L'inclusion de $\mathbb{Q}_p(1)$ dans U est induite par l'unique homomorphisme continu de la C-algèbre A_U dans celle des fonctions continues de $\mathbb{Z}_p(1)$ dans C qui envoie $1+\pi^{(n)}Y_n$ sur la fonction $\varepsilon = (\varepsilon^{(m)})_{m\in\mathbb{N}} \mapsto \varepsilon^{(n)}$.

Soit S le banach sous-jacent à un banach analytique $S^{an} = (S, \mathrm{Spm}_C A, \alpha)$ et soit V un $\mathbb{Q}_p$-espace vectoriel de dimension finie. Le choix d'une base de V définit un isomorphisme de $S \otimes_{\mathbb{Q}_p} V$ sur S^d qui hérite donc de la structure analytique produit. La structure analytique ainsi définie sur $S \otimes V$ ne dépend pas du choix de la base et on note $S^{an} \otimes V$ le banach analytique ainsi défini. Par exemple, pour tout $\mathbb{Q}_p$-espace vectoriel V de dimension fini, V^c s'identifie à $\mathbb{Q}_p^c \otimes V$ aussi bien qu'à $\mathbb{Q}_p(1)^c \otimes V(-1)$ et $V_C(-1)^{an}$ à $C^{an} \otimes V(-1)$. En tensorisant la suite exacte (1) avec $V(-1)$, on obtient une suite exacte de banach analytiques

$$0 \to V^c \to U^{an} \otimes V(-1) \to V_C(-1)^{an} \to 0$$

Pour tout triplet (V, W, f) formé d'un $\mathbb{Q}_p$-espace vectoriel de dimension fini V, d'un C-espace vectoriel de dimension finie W et d'une application C-linéaire $f : W \to V_C(-1)$, on pose $E_{V,W,f} = (U \otimes_{\mathbb{Q}_p} V) \times_{V_C(-1)} W$; c'est le banach sous-jacent au banach analytique $E^{an}_{V,W,f}$ défini comme le produit fibré de $U^{an} \otimes$

$V(-1)$ avec W^{an} au-dessus de $V_C(-1)^{an}$. On a donc un diagramme commutatif

$$\begin{array}{ccccccccc} 0 & \to & V^c & \to & E^{an}_{V,W,f} & \to & W^{an} & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & V^c & \to & U^{an}\otimes V(-1) & \to & V_C(-1)^{an} & \to & 0 \end{array}$$

de banach analytiques dont les lignes sont exactes.
Décrivons un peu plus explicitement la structure analytique : Rappelons que t est un générateur du $\mathbb{Z}_p$-module $\mathbb{Z}_p(1)$. Choisissons une base $\{v_1, v_2, \ldots, v_h\}$ de V sur $\mathbb{Q}_p$; les $v'_i = v_i \otimes t^{-1}$ forment une base de $V(-1)$ sur $\mathbb{Q}_p$. Choisissons une base $\{e_1, e_2, \ldots, e_d\}$ de W sur C, de manière que si $f(e_j) = \sum_{i=1}^h c_{ij} v'_i$, alors les $c_{ij} \in \mathcal{O}_C$. Notons $\mathcal{V}$ le sous-$\mathbb{Z}_p$-module de V engendré par les v_i et $\mathcal{W}$ le sous-$\mathcal{O}_C$-module de W engendré par les e_j.
Soit $\mathcal{U} = \alpha_U(\mathcal{U}_R)$. Soit $A^0_{\mathcal{U}\otimes\mathcal{V}(-\infty)}$ le quotient de l'anneau des polynômes à coefficients dans C en les variables $Y_{i,n}$, pour $1 \le i \le h$ et $n \in \mathbb{N}$, par l'idéal engendré par les $1 + \pi^{(n)} Y_{i,n} - (1 + \pi^{(n+1)} Y_{i,n+1})^p$. On l'identifie à une sous-algèbre de la C-algèbre des fonctions sur $\mathcal{U} \otimes \mathcal{V}(-\infty)$ en identifiant $Y_{r,n}$ à la fonction qui envoie $\sum u_i \otimes v'_i$ sur $\frac{u_r^{(n)}-1}{\pi^{(n)}}$. On note $A_{\mathcal{U}\otimes\mathcal{V}(-\infty)}$ le complété de cette algèbre pour la norme $||f|| = \sup_{s\in\mathcal{U}\otimes\mathcal{V}(-\infty)} |f(s)|$. C'est une algèbre spectrale dont le spectre maximal s'identifie au réseau $\mathcal{U}\otimes\mathcal{V}(-1)$ de $U \otimes V(-1)$ et définit la structure analytique sur ce banach.
Par ailleurs

$$W^{\mathrm{an}} = (W, \mathrm{Spm}_C C\{Z_1, Z_2, \ldots, Z_d\}, \alpha_{\mathcal{W}}) \ \text{ et}$$
$$V_C(-1)^{\mathrm{an}} = (V_C(-1), \mathrm{Spm}_C C\{X_1, X_2, \ldots, X_h\}, \alpha_{\mathcal{V}})$$

où l'algèbre de séries formelles restreintes $C\{Z_1, Z_2, \ldots, Z_d\}$ (resp. $C\{X_1, X_2, \ldots, X_h\}$) s'identifie à l'unique algèbre spectrale de fonctions continues sur $\mathcal{W}$ (resp. $\mathcal{V}_C(-1)$) telle que $Z_s(\sum_{j=1}^s \lambda_j e_j) = \lambda_s$ (resp. $X_r(\sum_{i=1}^h \mu_i \otimes v'_i) = \mu_r$) et où $\alpha_{\mathcal{W}}$ et $\alpha_{\mathcal{V}}$ sont les applications évidentes.
En outre f est induit par l'unique homomorphisme continu de C-algèbres

$$C\{X_1, X_2, \ldots, X_h\} \to C\{Z_1, Z_2, \ldots, Z_d\}$$

qui envoie X_i sur $\sum_{j=1}^d c_{ij} Z_j$ tandis que la projection de $U^{\mathrm{an}} \otimes V(-1)$ sur $V_C(-1)^{\mathrm{an}}$ est induite par l'unique homomorphisme continu de C-algèbres $C\{X_1, X_2, \ldots, X_h\} \to \mathcal{F}_{\mathrm{cont}}(\mathcal{V}, C)$ qui envoie X_r sur $\log(1 - pY_{r,0})/p$. Enfin $E^{\mathrm{an}}_{V,W,f} = (E_{V,W,f}, \mathrm{Spm}_C A_{\mathcal{V},\mathcal{W},f}, \alpha)$ où $A_{\mathcal{V},\mathcal{W},f}$ est le séparé complété du produit tensoriel de $A_{\mathcal{U}\otimes\mathcal{V}(-\infty)}$ et de $C\{Z_1, Z_2, \ldots, Z_d\}$ au-dessus de $C\{X_1, X_2, \ldots, X_h\}$ pour la norme évidente, son spectre maximal est $\mathcal{U} \otimes \mathcal{V}(-1)) \times_{\mathcal{O}_C\otimes\mathcal{V}(-1)} \mathcal{W}$ et α est l'application évidente. Remarquons que $A_{\mathcal{V},\mathcal{W},f}$ est aussi le séparé complété, pour la norme évidente, de la C-algèbre engendrée par des éléments $(Y_{i,n})_{1\le i\le h, n\in\mathbb{N}}$ et des éléments $Z_1, Z_2, \ldots, Z_d$ avec les relations $1 + \pi^{(n)} Y_{i,n} = (1 + \pi^{(n+1)} Y_{i,n+1})^p$ et $\log(1 - pY_{i,0})/p = \sum_{j=1}^d c_{ij} Z_j$.

Quant à la loi de groupe elle est caractérisée par $m^*Y_{n,i} = Y_{n,i}\hat{\otimes}Y_{n,i}$ et $m^*Z_j = Z_j\hat{\otimes}1 + 1\hat{\otimes}Z_j$.

Une *présentation* d'un banach analytique $E^{\rm an}$ consiste en la donnée d'un quadruplet (W,V,f,ι) formé d'un C-espace vectoriel de dimension finie W, d'un $\mathbb{Q}_p$-espace vectoriel de dimension finie V, d'une application C-linéaire $f : W \to V_C(-1)$ et d'un isomorphisme de banach analytiques $\iota : E^{\rm an}_{W,V,f} \to E^{\rm an}$. On appelle *dimension de la présentation* la dimension du C-espace vectoriel W et *hauteur de la présentation* la dimension du $\mathbb{Q}_p$-espace vectoriel V.
Enfin, on appelle *espace de Banach-Colmez présentable (sur C)* tout banach analytique qui admet une présentation.
Les principaux résultats du travail de Colmez sur les *Espaces de Banach de dimension finie* [Co02] peuvent se réinterpréter (cf. [FP], voir aussi[Fo02]) en disant que la sous-catégorie pleine $\mathcal{BC}_{\mathcal{C}}^+$ de la catégorie des banach analytiques dont les objets sont les espaces de Banach-Colmez présentables s'identifie de façon naturelle à une sous-catégorie pleine d'une catégorie abélienne, la catégorie $\mathcal{BC}_{\mathcal{C}}$ des *espaces de Banach-Colmez (sur C)*, tout objet de $\mathcal{BC}_{\mathcal{C}}$ étant isomorphe au quotient d'un objet de $\mathcal{BC}_{\mathcal{C}}^+$ par un autre. Il existe en outre des fonctions additives $d : \mathrm{Ob}\ \mathcal{BC}_{\mathcal{C}} \to \mathbb{N}$ et $h : \mathrm{Ob}\ \mathcal{BC}_{\mathcal{C}} \to \mathbb{Z}$ uniquement déterminées par le fait que si $E^{\rm an}$ est un banach analytique muni d'une présentation, alors $d(E^{\rm an})$ (resp. $h(E^{\rm an})$) est la dimension (resp. la hauteur) de la présentation.
Nous verrons aussi que toute presque C-représentation de G_K est munie de façon naturelle d'une structure d'espace de Banach-Colmez. Pour le moment, nous n'avons pas besoin de ces résultats. Mais nous allons utiliser le résultat crucial de [Co02] (prop.5.19 et cor.5.11, voir aussi [FP]) qui peut s'énoncer ainsi :

Proposition 4.3 (lemme de Colmez). — *Soit $E^{\rm an}$ un banach analytique admettant une présentation (W,V,ρ,ι) de dimension 1. Soit $\eta : E \to C$ un morphisme analytique tel que $\eta(E) \neq f(\iota(V))$. Alors η est surjective et son noyau est un $\mathbb{Q}_p$-espace vectoriel de dimension finie égale à la hauteur de la présentation.*

Remarque : C'est essentiellement la version forte du lemme fondamental de [CF00] ; le lemme fondamental ([CF00],§2) énoncé sous une autre forme, disait seulement que l'application η est surjective.

4.4 – Une application aux presque-C-représentations

Soit E une représentation banachique de G_K extension d'une C-représentation W par une représentation p-adique V. D'après la proposition 3.12, on dispose d'une application C-linéaire G_K-équivariante $f : W \to V_C(-1)$ et d'un diagramme commutatif

$$\begin{array}{ccccccccc} 0 & \to & V & \to & E & \to & W & \to & 0 \\ & & \| & & \downarrow & & \downarrow f & & \\ 0 & \to & V & \to & U(-1)\otimes_{\mathbb{Q}_p} V & \to & V_C(-1) & \to & 0 \end{array}$$

Ceci nous permet d'identifier E au banach sous-jacent à $E^{\mathrm{an}}_{W,V,f}$. Comme C est le banach sous-jacent à $E^{\mathrm{an}}_{C,0,0}$, cela donne un sens à l'énoncé suivant :

Proposition 4.4. — *Soient V un sous-$\mathbb{Q}_p$-espace vectoriel de dimension finie de C stable par G_K et E une représentation banachique extension d'une C-représentation W de dimension 1 par V. Si $\eta : E \to C$ est une application $\mathbb{Q}_p$-linéaire continue G_K-équivariante induisant l'identité sur V, alors η est analytique.*

Preuve : Reprenons les conventions et notations du §4.3 avec $d = 1$. Posons $e = e_1$, $Z = Z_1$, $c_i = c_{i1}$ et $\mathcal{E} = (\mathcal{U} \otimes \mathcal{V}(-1)) \times_{\mathcal{O}_C \otimes \mathcal{V}(-1)} \mathcal{W}$. Quitte à remplacer e par pe, on peut supposer que $c_i \in p\mathcal{O}_C$ pour tout i. Posons encore $A = A_{\mathcal{V},\mathcal{W},f}$ et notons $\mathcal{O}_A$ la boule unité de A. Soit R^A l'ensemble des suites $x = (x^{(n)})_{n\in\mathbb{N}}$ d'éléments de $\mathcal{O}_A$ vérifiant $(x^{(n+1)})^p = x^{(n)}$ pour tout n. On en fait un anneau commutatif unitaire en posant

$$(x+y)^{(n)} = \lim_{m \mapsto +\infty} (x^{(n+m)} + y^{(n+m)})^{p^m} \text{ et } (xy)^{(n)} = x^{(n)} y^{(n)}$$

C'est un anneau de caractéristique p. Il contient l'anneau R formé des x tels que $x^{(n)} \in \mathcal{O}_C$ pour tout n comme sous-anneau. Il est parfait (i.e. le Frobenius $x \mapsto x^p$ est bijectif).

Soit $W(R^A)$ l'anneau des vecteurs de Witt à coefficients dans R^A. Comme R^A est parfait, c'est un anneau séparé et complet pour la topologie p-adique, sans p-torsion. Pour tout $x \in R^A$, notons $[x] = (x, 0, 0, \ldots, 0, \ldots) \in W(R^A)$ son représentant de Teichmüller. Pour tout $(x_0, x_1, \ldots, x_n, \ldots) \in W(R^A)$, on a $(x_0, x_1, \ldots, x_n, \ldots) = \sum_{n=0}^{\infty} p^n [x_n^{p^{-n}}]$.

L'application $\theta : W(R^A) \to \mathcal{O}_A$ qui envoie $(x_0, x_1, x_2, \ldots, x_n, \ldots)$ sur $\sum_{n=0}^{\infty} p^n x_n^{(n)}$ est un homomorphisme d'anneaux.

Lemme 4.5. — *Soit $\pi = (\pi^{(n)})_{n\in\mathbb{N}} \in R$ un élément vérifiant $\pi^{(0)} = -p$. Pour tout $x = (x^{(n)})_{n\in\mathbb{N}} \in R^A$, posons $||x|| = ||x^{(0)}||$. Pour que x appartienne à l'idéal engendré par π, il faut et il suffit que $||x|| \leq |p|$.*

Preuve : Si $x = \pi y$ avec $y \in R^A$, on a $||x|| = ||x^{(0)}|| = ||-py^{(0)}|| = |p| ||y^{(0)}|| \leq |p|$ et la condition est nécessaire. Réciproquement, si $||x|| \leq |p|$, cela veut dire que $||x^{(0}|| \leq |p|$, donc que, pour tout $n \in \mathbb{N}$, $||x^{(n)}|| \leq (|p|)^{p^{-n}} = |\pi^{(n)}|$ puisque $||x^{(0)}|| = (||x^{(n)}||)^{p^n}$. L'élément $y^{(n)} = x^{(n)}/\pi^{(n)}$ de A vérifie donc $||y||^{(n)} \leq 1$. Donc $y = (y^{(n)})_{n\in\mathbb{N}} \in \mathcal{O}_A$ et $x = \pi y$. □

Lemme 4.6. — *Soient π comme ci-dessus et $\xi = [\pi] + p$. Dans $W(R^A)$, la multiplication par ξ est injective et le noyau de θ est l'idéal principal engendré par ξ.*

Preuve : Remarquons d'abord que la multiplication par π est injective dans R^A. En effet, si $x = (x^{(n)})_{n\in\mathbb{N}} \in R^A$ est non nul, $x^{(0)} \neq 0$. On a $\pi x = (\pi^{(n)} x^{(n)})_{n\in\mathbb{N}}$ et $\pi^{(0)} x^{(0)} = -px^{(0)}$ est non nul, puisque p est inversible dans C et A est une C-algèbre.

Soit maintenant $x \in W(R^A)$ non nul. Si r est le plus grand entier tel que p^r divise x, on peut écrire $x = p^r y$ avec $y = (y_0, y_1, \ldots, y_n, \ldots) \in W(R^A)$ et $y_0 \neq 0$. Alors $\xi y = (\pi y_0, \ldots,)$ est non nul puisque $\pi y_0 \neq 0$ et donc aussi $\xi x = p^r \xi y$, puisque $W(R^A)$ est sans p-torsion.
On a $\theta(\xi) = \pi^{(0)} + p = 0$ et l'idéal engendré par ξ est bien contenu dans le noyau de θ. Pour prouver la réciproque, comme $W(R^A)$ est séparé et complet pour la topologie p-adique, il suffit de vérifier que $\mathrm{Ker}\ \theta \subset (\xi) + p\mathrm{Ker}\ \theta$. Soit $x = (x_0, x_1, \ldots, x_n, \ldots) \in \mathrm{Ker}\ \theta$. On a $x_0^{(0)} + p(\sum_{n=1}^{\infty} p^{n-1} x_n^{(n)}) = 0$ donc $x_0^{(0)} \in p\mathcal{O}_A$ et $||x|| = ||x^{(0)}|| \leq |p|$. D'après le lemme précédent, il existe $y_0 \in R^A$ tel que $x_0 = \pi y_0$. Si $z = (z_0, z_1, \ldots, z_n, \ldots) = x - \xi[y_0]$, on a $z_0 = 0$ donc $z = pz'$ avec $z' \in W(R^A)$. Mais $\theta(pz') = 0$, donc aussi $\theta(z') = 0$ et on a bien $x \in (\xi) + p\mathrm{Ker}\ \theta$. □

Notons $\mathcal{O}_{A^\theta}$ le sous-anneau de $\mathcal{O}_A$ image de $W(R^A)$ par θ et $\mathcal{O}_{B_2^A}$ le quotient de $W(R^A)$ par l'idéal $(\mathrm{Ker}\ \theta)^2$ qui est aussi l'idéal principal engendré par ξ^2. L'anneau $\mathcal{O}_{B_2^A}$ est donc une extension de $\mathcal{O}_{A^\theta}$ par un idéal de carré nul qui est le $\mathcal{O}_{A^\theta}$-module libre de base l'image $\bar{\xi}$ de ξ.
Pour $1 \leq i \leq h$, soit $U_i^{(n)} = 1 + \pi^{(n)} Y_{i,n}$. Alors $U_i = (U_i^{(n)})_{n \in \mathbb{N}} \in R^A$. On a $U_i^{(0)} - 1 = \exp(c_i Z) - 1$ qui, comme $c_i Z$, appartient à $p\mathcal{O}_A$. On a $(U_i - 1)^{(0)} \equiv U_i^{(0)} - 1 \bmod p\mathcal{O}_A$ et $||U_i - 1|| \leq |p|$. D'après le lemme 4.5, il existe $V_i \in R^A$ tel que $U_i - 1 = \pi V_i$.
Soit $[U_i] = (U_i, 0, 0, \ldots, 0, \ldots)$ le représentant de Teichmüller de U_i dans $W(R^A)$. Il existe $\alpha_i \in W(R^A)$ tel que $[U_i] - 1 = [U_i - 1] + p\alpha_i$. On peut donc écrire $[U_i] - 1 = [\pi V_i] + p\alpha_i = [\pi][V_i] + p\alpha_i = \xi[V_i] + p(\beta_i - V_i)$. L'anneau $\mathcal{O}_{B_2^A}$ est séparé et complet pour la topologie p-adique et l'idéal engendré par $\bar{\xi}$ est de carré nul ; on en déduit une structure d'idéal à puissances divisées sur l'idéal engendré par $\bar{\xi}$ et p. Par conséquent, si l'on note $\tilde{U}_i$ l'image de $[U_i]$ dans $\mathcal{O}_{B_2^A}$, la série $\log \tilde{U}_i = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(\tilde{U}_i - 1)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} (n-1)! \gamma_n (\tilde{U}_i - 1)$ converge dans cet anneau.
Celui-ci est sans p-torsion et $\mathcal{O}_{B_2^A}$ s'identifie à un sous-anneau de $B_2^A = \mathcal{O}_{B_2^A}[1/p]$. Ce dernier est une extension de l'anneau $A^\theta = \mathcal{O}_{A^\theta}[1/p]$ - que l'on peut voir comme un sous-anneau de A - par le A^θ-module libre de rang 1 de base $\bar{\xi}$.
Notons $\mathcal{O}_{B_2}$ le quotient de l'anneau $W(R)$ par l'idéal engendré par ξ^2. On sait ([Fo88a], §1.5) que B_2 s'identifie à $\mathcal{O}_{B_2}[1/p]$ et que l'image $\bar{t}$ de t dans B_2 est un élément non nul de $\mathcal{O}_{B_2}\bar{\xi} = \mathcal{O}_C \bar{\xi}$; il existe donc $c_0 \in \mathcal{O}_C$ non nul tel que $\bar{t} = c_0 \bar{\xi}$. On voit donc que B_2^A est de façon naturelle une B_2-algèbre, que ξ et $\bar{t}$ engendre le même idéal dans B_2^A et que l'on a un diagramme commutatif

$$\begin{array}{ccccccccc} 0 & \to & C(1) & \to & B_2 & \to & C & \to & 0 \\ & & \cap & & \downarrow & & \cap & & \\ 0 & \to & A^\theta(1) & \to & B_2^A & \to & A^\theta & \to & 0 \end{array}$$

dont les lignes sont exactes. En particulier l'application $B_2 \to B_2^A$ est injective et B_2^A est une B_2-algèbre fidèlement plate.

Pour tout B_2-module W, on pose $W^A = B_2^A \otimes_{B_2} W$. Si W est annulé par t, c'est un C-espace vectoriel et $W^A = A^\theta \otimes_C W$. Remarquons que, si $s \in \mathcal{E}$, alors s induit un homomorphisme de $\mathcal{O}_C$-algèbres de $\mathcal{O}_A$ dans $\mathcal{O}_C$, donc un morphisme de R-algèbres de R^A dans R, un morphisme de $W(R)$-algèbres de $W(R^A)$ dans $W(R)$, donc un morphisme de B_2-algèbres $s_{B_2} : B_2^A \to B_2$. Pour tout B_2-module W, on note encore $s_{B_2} : W^A \to W$ l'application définie par $s_{B_2}(b \otimes w) = s_{B_2}(b)w$ si $b \in B_2^A$ et $w \in W$.

Lemme 4.7. — *Soient W un B_2-module de type fini et W_1 un sous-B_2-module de W. Soit $\alpha \in W^A$. Pour que $\alpha \in W_1^A$, il faut et il suffit que $s_{B_2}(\alpha) \in W_1$ pour tout $s \in \mathcal{E}$.*

Preuve : On peut trouver $m_1 \leq m_2 \leq m$ et $n_1 \leq n$ dans $\mathbb{N}$ et des éléments $(a_i)_{1 \leq i \leq m}$ et $(b_j)_{1 \leq j \leq n}$ dans W tels que, avec des notations évidentes,

$$W = (\bigoplus_{i=1}^{m} B_2 a_i) \oplus (\bigoplus_{j=1}^{n} C b_j) \text{ et } W_1 = (\bigoplus_{i=1}^{m_1} B_2 a_i) \oplus (\bigoplus_{i=m_1+1}^{m_2} C\bar{t} a_i) \oplus (\bigoplus_{j=1}^{n_1} C b_j)$$

On peut alors écrire $\alpha = \sum_{i=1}^{n} \lambda_i \otimes a_i + \sum_{j=1}^{n} \mu_j \otimes b_j$, avec les $\lambda_i \in B_2^A$ et les $\mu_j \in A^\theta$ uniquement déterminés. Pour tout $s \in \mathcal{E}$, on a $s_{B_2}(\alpha) = \sum s_{B_2}(\lambda_i) a_i + \sum s(\mu_j) b_j$; c'est un élément de W_1 si et seulement si $\theta(s_{B_2}(\lambda_i)) = 0$ pour $m_1 < i \leq m_2$, $s_{B_2}(\lambda_i) = 0$ pour $m_2 < i \leq m$ et $s(\mu_j) = 0$ pour $n_1 < j \leq n$. L'assertion résulte alors de ce que, comme A est spectrale, pour tout $\mu \in A$ non nul, il existe $s \in \mathcal{E}$ tel que $s(\mu) \neq 0$, ce qui implique aussi que, si $\lambda \in B_2^A$ vérifie $\theta(\lambda) \neq 0$, alors il existe $s \in \mathcal{E}$ tel que $\theta(s_{B_2}(\lambda)) \neq 0$ et que, si $\lambda \in B_2^A$ est non nul, alors il existe $s \in \mathcal{E}$ tel que $s_{B_2}(\lambda) \neq 0$. □

Fin de la preuve de la proposition 4.4 : On a $E \subset U(-1) \otimes_{\mathbb{Q}_p} V = U \otimes_{\mathbb{Q}_p} V(-1)$. L'inclusion $U \subset B_2$ permet d'identifier E à un sous-$\mathbb{Q}_p$-espace vectoriel du B_2-module $W_1 = B_2 \otimes_{\mathbb{Q}_p} V(-1)$ et on a une suite exacte

$$0 \to V_C \to W_1 \to V_C(-1) \to 0$$

Soit V_C^0 le noyau de la projection $\varpi : V_C \to C$ induite par l'inclusion de V dans C (on a $\varpi(\sum \lambda_i \otimes v_i) = \sum \lambda_i v_i$). C'est un sous-$C$-espace vectoriel de V_C et le quotient $W_2 = W_1 / V_C^0$ est un B_2-module extension de $V_C(-1)$ par C.
L'application composée $E \to U \otimes V(-1) \to W_1 \to W_2$ est injective et on a un diagramme commutatif

$$\begin{array}{ccccccccc} 0 & \to & V & \to & E & \to & W & \to & 0 \\ & & \cap & & \downarrow \varphi & & \downarrow f & & \\ 0 & \to & C & \to & W_1 & \to & V_C(-1) & \to & 0 \end{array}$$

dont les lignes sont exactes. Soit $E_C = C \oplus_V E$ la somme amalgamée de C et de E au-dessous de V. L'application $\varphi : E \to W_1$ s'étend de manière unique en une application $\mathbb{Q}_p$-linéaire continue G_K-équivariante $\varphi_C : E_C \to W_1$ qui

est l'identité sur C. L'image W_2 de φ_C est l'image inverse dans W_1 du sous-C-espace vectoriel $f(W)$ de $V_C(-1)$ et est donc un sous-B_2-module de W_1. En outre, φ_C est un homéomorphisme de E_C sur W_2.
L'application $\eta : E \to C$ s'étend de manière unique en une application $\mathbb{Q}_p$-linéaire continue G_K-équivariante de E_C sur C qui est l'identité sur C. Par transport de structure, on en déduit un morphisme $\eta'_C : W_2 \to C$ de $\mathcal{B}(G_K)$ qui est l'identité sur C. D'après le théorème de pleine fidélité (th.3.11), cette application est B_2-linéaire. Autrement dit, W_2 est en fait un C-espace vectoriel de dimension 2 somme directe de C et du noyau N de η'_C. On a donc une décomposition en somme directe $W_2^A = C^A \oplus N^A = A^\theta \oplus N^A$.
Considérons l'élément $\alpha_1 = \sum_{i=1}^h \log \tilde{U}_i \otimes v'_i$ de W_1^A et notons α_2 son image dans W_2^A. Pour tout $s \in \mathcal{E}$, $s_{B_2}(\alpha_1)$ appartient à l'image de E dans W_1, donc $s_{B_2}(\alpha_2) \in W_2$. D'après le lemme 4.7, ceci implique que $\alpha_2 \in W_2^A$ et on peut écrire $\alpha_2 = \beta_2 + \gamma_2$, avec $\beta_2 \in A^\theta \subset A$ et $\gamma_2 \in N^A$. Quitte à remplacer η par $p^m\eta$, et donc aussi f par $p^m f$, avec m entier suffisamment grand, on peut supposer que $\beta_2 \in \mathcal{O}_A$.
On voit alors que $\eta(\mathcal{E}) \subset \mathcal{O}_C$ et que, pour tout $s \in \mathcal{E}$, on a $\eta(s) = s \circ \nu$, où $\nu : C\{X\} \to A$ est l'unique homomorphisme continu de C-algèbres qui envoie X sur β_2. □

Montrons alors le théorème 4.1 : Soit V' le noyau de la restriction de η à V. Quitte à remplacer E par E/V', on peut supposer $V' = 0$ et utiliser la restriction de η à V pour identifier V à un sous-$\mathbb{Q}_p$-espace vectoriel de C. D'après la proposition précédente, η est analytique. Il suffit alors d'appliquer le lemme de Colmez (prop.4.3). □

COROLLAIRE. — *Soit V un sous-$\mathbb{Q}_p$-espace vectoriel de dimension finie, stable par G_K, de C. Soit W une C-représentation de dimension 1. Si $g : W \to C/V$ est un morphisme non nul de représentations banachiques, alors g est surjective et son noyau est un $\mathbb{Q}_p$-espace vectoriel de dimension finie égale à celle de V.*

En effet, si $E = C \times_{C/V} W$, on a un diagramme commutatif

$$\begin{array}{ccccccccc} 0 & \to & V & \to & E & \to & W & \to & 0 \\ & & \| & & \downarrow \eta & & \downarrow g & & \\ 0 & \to & V & \to & C & \to & C/V & \to & 0 \end{array}$$

dont les lignes sont exactes. Comme $g \neq 0$, $\eta(V) \neq \eta(E)$; mais alors η est surjectif donc aussi g et le noyau de g s'identifie au noyau de η et sa dimension sur $\mathbb{Q}_p$ est bien égale à celle de V. □

5 – LA CATÉGORIE DES PRESQUE C-REPRÉSENTATIONS

5.1 – LE THÉORÈME DE STRUCTURE

On reprend les notations et conventions des §1.2 et 1.4.

Théorème 5.1. — *La catégorie* $\mathcal{C}(G_K)$ *est une sous-catégorie stricte de* $\mathcal{B}(G_K)$. *En outre, il existe deux fonctions additives sur les objets de* $\mathcal{C}(G_K)$

$$d : \mathrm{Ob}\ \mathcal{C}(G_K) \to \mathbb{N} \text{ et } h : \mathrm{Ob}\ \mathcal{C}(G_K) \to \mathbb{Z}$$

uniquement déterminées par $d(C) = 1$, $h(C) = 0$ *et* $d(V) = 0$, $h(V) = \dim_{\mathbb{Q}_p} V$ *pour toute représentation* p*-adique* V.

Remarque : Nous appellerons $d(X)$ la *dimension de* X et $h(X)$ sa *hauteur.* Il est parfois commode d'utiliser la fonction additive $dh : \mathrm{Ob}\ \mathcal{C}(G_K) \to \mathbb{N} \times \mathbb{Z}$ définie par $dh(X) = (d(X), h(X))$.

Preuve : Appelons *présentation* d'un objet X de $\mathcal{B}(G_K)$ un quadruplet formé d'une C-représentation triviale W, de sous-$\mathbb{Q}_p$-espaces vectoriels de dimension finie V de X et V' de W, stables par G_K, et d'un isomorphisme $\alpha : X/V \to W/V'$ (dans $\mathcal{B}(G_K)$). Un objet de $\mathcal{B}(G_K)$ admet donc une présentation si et seulement s'il est dans $\mathcal{C}(G_K)$. Appelons *objet présenté* un quintuplet $\underline{X} = (X, W, V, V', \alpha)$, où X est un objet de $\mathcal{C}(G_K)$ et (W, V, V', α) une présentation de X. On écrit aussi $\underline{X} = (\alpha : X/V \simeq W/V')$. On pose $dh(\underline{X}) = (\dim_C W, \dim_{\mathbb{Q}_p} V - \dim_{\mathbb{Q}_p} V')$; on appelle aussi X l'objet sous-jacent à $\underline{X}$.
Un *morphisme* d'objets présentés est un morphisme des objets sous-jacents. On dit qu'un morphisme $f : \underline{X} \to \underline{Y}$ est *admissible* s'il est strict et s'il existe des présentations $\underline{N}$ du noyau N, $\underline{J}$ du conoyau J et $\underline{I}$ de l'image I de f telles que $dh(\underline{X}) = dh(\underline{N}) + dh(\underline{I})$ et $dh(\underline{Y}) = dh(\underline{I}) + d(h\underline{J})$.
Le théorème équivaut au résultat suivant :

Proposition 5.2. — *Tout morphisme d'objets présentés est admissible.*

Preuve : Si $\underline{X} = (X, W, V, V', \alpha)$ est un objet présenté et si U est une sous-représentation de X de dimension finie, on note $\underline{X}/U$ l'objet présenté $(X/U, W, \tilde{V}, \tilde{V}', \tilde{\alpha})$, où $\tilde{V} = U + V/U$, $\tilde{V}' = \{w \in W \mid w \bmod V' \in \alpha(U + V)\}$ et $\tilde{\alpha}$ est l'application déduite de α par passage au quotient.

Lemme 5.3. — *Soient* $f : \underline{X} \to \underline{Y}$ *un morphisme d'objets présentés,* U *une sous-représentation de dimension finie de* X*,* U' *une sous-représentation de dimension finie de* Y *contenant* $f(U)$ *et* $\tilde{f} : \underline{X}/U \to \underline{Y}/U'$ *le morphisme d'objets présentés déduit de* f *par passage aux quotients. Alors* f *est admissible si et seulement si* $\tilde{f}$ *est admissible.*

Preuve : Exercice. □

Lemme 5.4. — *Soient* $m, h, d \in \mathbb{N}$. *Soient* E *une représentation banachique extension d'une* C*-représentation triviale* X *de dimension* m *par une représentation* p*-adique* V *de dimension* h*,* W *une* C*-représentation triviale de dimension* d *et* $f : E \to W$ *un morphisme de* $\mathcal{C}(G_K)$. *Alors* f *est strict et il existe des présentations* $\underline{N}$ *du noyau* N*,* $\underline{J}$ *du conoyau* J *et* $\underline{I}$ *de l'image* I *de* f *telles que* $dh(\underline{N}) + dh(\underline{I}) = (m, h)$ *et* $dh(\underline{I}) + dh(\underline{J}) = (d, 0)$.

Montrons d'abord comment la proposition résulte des lemmes 5.3 *et* 5.4 : Soit f : $(X_1, W_1, V_1, V_1', \alpha_1) \to (X_2, W_2, V_2, V_2', \alpha_2)$ un morphisme d'objets présentés. Pour $i \in \{1, 2\}$, soit $\underline{W}_i = (W_i, W_i, 0, 0, \mathrm{id}_{W_i})$. Posons $E_1 = W_1 \times_{X_2/f(V_1)} X_2$. On a une suite exacte

$$0 \to f(V_1) \to E_1 \to W_1 \to 0$$

et on pose $\underline{E}_1 = (E_1, W_1, f(V_1), 0, \text{proj.can.})$. Posons $F_1 = E_1 \times_{X_2/V_2} W_2$. On a des suites exactes

$$0 \to V_2' \to F_1 \to E_1 \to 0$$

et, si l'on note V' l'image inverse de $f(V_1)$ dans F_1,

$$0 \to V' \to F_1 \to W_1 \to 0$$

et on pose $\underline{F}_1 = (F_1, W_1, V', 0, \text{proj.can.})$.
On a alors un diagramme commutatif

$$\begin{array}{ccccccccccc} \underline{X}_1 & \to & \underline{X}_1/V_1 & \leftarrow & \underline{W}_1 & \leftarrow & \underline{E}_1 & = & \underline{E}_1 & \leftarrow & \underline{F}_1 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \underline{X}_2 & \to & \underline{X}_2/f(V_1) & = & \underline{X}_2/f(V_1) & \leftarrow & \underline{X}_2 & \to & \underline{X}_2/V_2 & \leftarrow & \underline{W}_2 \end{array}$$

La flèche verticale de droite est admissible d'après le lemme 5.4. En appliquant cinq fois de suite le lemme 5.3, on en déduit que toutes les flèches verticales le sont. □

Prouvons alors le lemme 5.4 : On va procéder par induction sur d, le cas $d = 0$ étant trivial. Supposons donc $d \geq 1$.
Considérons une suite finie

$$W = F^0W \supset F^1W \supset \ldots \supset F^iW \supset F^{i+1}W \supset \ldots \supset F^rW$$

de sous-C-espaces vectoriels de W, stables par G_K, avec F^iW de codimension i. Pour chaque entier i, notons F^iE l'image inverse par f de F^iW, $f_i : F^iE \to F^iW$ la restriction de f, et, lorsque $i < r$, $gr_iW = F^iW/F^{i+1}W$ et $\overline{f}_i : F^iE \to gr^iW$ le composé de f_i avec la projection de F^iW sur gr^iW.
On se propose de construire une telle suite, ainsi que de se donner pour chaque i, une suite exacte de $\mathcal{C}(G_K)$

$$0 \to V_i \to F^iE \to X_i \to 0$$

où V_i est une représentation p-adique de dimension h et X_i une C-représentation triviale de dimension $m - i$ (ceci implique a posteriori que l'on aura $r \leq m$).
On procède inductivement :
a) On doit avoir $F^0W = W$ et $F^0E = E$; on prend $V_0 = V$ et $X_0 = X$.
b) Soit $i \geq 0$ et supposons F^iW, F^iE, V_i et X_i construits :

– si $f_i(F^iE)$ est de dimension finie sur $\mathbb{Q}_p$, on prend $r = i$ (remarquer que c'est nécessairement le cas si $i = m$ ou si $i = d$) ;
– sinon, soient $e_1, e_2, \dots, e_{d-i}$ une base de F^iW formée d'éléments fixes par G_K. Pour $s = 1, 2, \dots, d-i$, soit H_s l'hyperplan de de F^iW engendré par les e_j avec $j \neq s$. Comme $f_i(F^iE)$ s'injecte dans $\oplus_{s=1}^{d-i} F^iW/H_s$, il existe un entier s tel que l'image de $f_i(F^iE)$ dans F^iW/H_s n'est pas de dimension finie. On choisit pour $F^{i+1}W$ un tel H_s. Avec les notations qui précèdent, on a $\overline{f}_i(F^iE) \neq \overline{f}_i(V_i)$. Décomposons X_i en une somme directe $X_i = \oplus_{j=1}^{m-i} L_j$ de C-droites stables par G_K et notons E_j l'image inverse de L_j dans F^iE. On a $F^iE = \sum_{j=1}^{m-i} E_j$, donc $\overline{f}_i(F^iE) = \sum \overline{f}_i(E_j)$ et il existe j tel que $\overline{f}_i(E_j) \neq \overline{f}_i(V_i)$. Quitte à changer la numérotation, on peut supposer $j = 1$. Le choix d'une base de $L_1^{G_K}$ et d'une base de $(gr^iW)^{G_K}$ permet d'identifier le sous-espace fermé E_1 de F^iE à une extension de C par V_i et le quotient gr^iW à C. En appliquant le théorème 4.1 à la restriction g_i de $\overline{f}_i$ à E_1, on voit que g_i *est surjective* et que le noyau V_{i+1} de g_i est une représentation p-adique de dimension h.
Soit X_{i+1} la C-représentation de dimension $m - i - 1$ qui est le quotient de X_i par L_1. Dans $\mathcal{C}(G_K)$, on a une suite exacte

$$0 \to E_1 \to F^iE \to X_{i+1} \to 0$$

Comme g_i est surjective, $\overline{f}_i$ l'est aussi et est donc un épimorphisme strict. Comme $F^{i+1}E$ est le noyau de $\overline{f}_i$, on a un diagramme commutatif

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & V_{i+1} & \to & F^{i+1}E & \to & X_{i+1} & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & E_1 & \to & F^iE & \to & X_{i+1} & \to & 0 \\
 & & \downarrow{\scriptstyle g_i} & & \downarrow{\scriptstyle \overline{f}_i} & & & & \\
 & & \mathrm{gr}^iW & = & \mathrm{gr}^iW & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

dont les lignes et les colonnes sont exactes, ce qui achève la construction au rang $i + 1$.
Soient V'_r le noyau de la restriction de f_r à V_r et $V''_r = f_r(V_r)$. Si $h' = \dim_{\mathbb{Q}_p} V'_r$ et $h'' = \dim_{\mathbb{Q}_p} V''_r$, on a $h = h' + h''$.
Comme $f_r(F^rE)$ est de dimension finie sur $\mathbb{Q}_p$, l'application f_r induit un morphisme de $\mathcal{C}(G_K)$ de la C-représentation X_r sur la représentation p-adique $f_r(F^rE)/V''_r$. D'après le corollaire au théorème 3.11, ce morphisme est nul et on a $f_r(F^rE) = V''_r$.
En particulier, on peut considérer f_r comme un épimorphisme strict de F^rE sur V''_r. Par construction le noyau (en tant qu'application linéaire) de f est aussi le noyau N de f_r et a donc une structure naturelle d'objet de $\mathcal{C}(G_K)$.

Dans cette dernière catégorie, on a un diagramme commutatif

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & V'_r & \to & N & \to & W_r & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & V_r & \to & F^rE & \to & W_r & \to & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & V''_r & = & V''_r & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

dont les lignes et les colonnes sont exactes. Ceci nous permet aussi de munir N d'une structure d'objet de $\mathcal{C}(G_K)$ muni d'une présentation $\underline{N}$ vérifiant $dh(\underline{N}) = (m-r,h')$.
Par construction l'application de E dans W/F^rW composée de f avec la projection de W sur W/F^rW est surjective et, si I désigne l'image ensembliste de l'application f, on a donc une suite exacte

$$0 \to V''_r \to I \to W/F^rW \to 0$$

Si l'on munit I de la topologie induite par celle de W, on voit que I est fermé dans W et que la suite exacte ci-dessus donne à I une structure d'objet de $\mathcal{C}(G_K)$ muni d'une présentation $\underline{I}$ vérifiant $dh(\underline{I}) = (r,h'')$.
De même, comme f induit un isomorphisme de E/F^rE sur W/F^rW, le conoyau (au sens des $\mathbb{Q}_p$-espaces vectoriels) J de f s'identifie à celui de f_r donc au quotient de F^rW par V''_r et est donc aussi muni d'une présentation $\underline{J}$ vérifiant $dh(\underline{J}) = (d-r,-h'')$.
Dans $\mathcal{C}(G_K)$, on a des suites exactes courtes

$$0 \to N \to E \to I \to 0$$

$$\text{et } 0 \to I \to W \to J \to 0$$

ce qui montre que f est strict et, comme $(m,h) = (m-r,h') + (r,h'')$ et $(d,0) = (r,h'') + (d-r,-h'')$ que f est admissible. □

5.2 – Toute suite exacte courte de presque C-représentations est presque scindée

Proposition 5.5. — *Soit W une C-représentation triviale et X un sous-objet de W dans $\mathcal{C}(G_K)$. Alors il existe un presque supplémentaire W' de X dans W qui est un sous-C-espace vectoriel.*

Preuve : Comme W est une C-représentation triviale, tout sous-C-espace vectoriel de W stable par G_K est encore une C-représentation triviale.

Soit W' un sous-C-espace vectoriel de W stable par G_K tel que l'application composée $X \subset W \to W/W'$ est surjective et qui est minimal pour cette propriété. Alors X est une extension de $W'' = W/W'$ par $X' = X \cap W'$. Si H est un hyperplan de W' stable par G_K, on a un diagramme commutatif

$$\begin{array}{ccccccccc} 0 & \to & X' & \to & X & \to & W'' & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & W'/H & \to & W/H & \to & W'' & \to & 0 \end{array}$$

dont les lignes sont exactes. Comme l'application $X \to W/H$ n'est pas surjective, $X' \to W'/H$ ne l'est pas non plus. comme $W'/H \simeq C$, l'image I de cette application est isomorphe à un sous-objet de C dans la catégorie $\mathcal{B}(G_K)$, distinct de C ; on a donc $d(I) = 0$ et I est un $\mathbb{Q}_p$-espace vectoriel de dimension finie. Si l'on choisit une base $\{e_1, e_2, \ldots, e_r\}$ de W' sur C formée d'éléments fixes par G_K et si, pour $1 \leq i \leq r$, on note H_i l'hyperplan de W' de base les e_j avec $j \neq i$, la projection X'_i de X' sur W'/H_i est un $\mathbb{Q}_p$-espace vectoriel de dimension finie. Comme X' s'injecte dans la somme directe des X'_i, c'est aussi un $\mathbb{Q}_p$-espace vectoriel de dimension finie. □

Corollaire. — *Soit*

$$0 \to S' \to S \to S'' \to 0$$

une suite exacte courte de représentations banachiques avec S' et S'' des presque-C-représentations. Pour que S soit une presque C-représentation, il faut et il suffit que cette suite soit presque scindée.

Preuve : Il est clair que la condition est suffisante. Montrons qu'elle est nécessaire : Si S est une presque C-représentation, on peut trouver un isomorhisme $S/V_0 \simeq W/V$ avec W une C-représentation triviale et V_0 et V des représentations p-adiques de dimension finie. Quitte à remplacer S par S/V_0 et S' et S'' par les quotients correspondants, on peut supposer $V_0 = 0$. Quitte à remplacer S par W et S' par le produit fibré $S' \times_S W$, on peut supposer $S = W$. D'après la proposition, il existe un sous-C-espace vectoriel W' de W tel que $S = W = W' + S'$ tandis que $V' = S' \cap W'$ est un $\mathbb{Q}_p$-espace vectoriel de dimension finie. Autrement dit W' est un presque-supplémentaire de S' dans S et la suite est presque scindée.

5.3 – Toute suite exacte courte de B_{dR}^+-représentations est presque scindée

Pour prouver cette affirmation, nous aurons besoin de pouvoir tordre l'action de G_K par des caractères à valeurs dans K^*. C'est pourquoi nous allons établir un resultat apparemment plus fort.

Si W est une B_{dR}^+-représentation et Y un sous-$\mathbb{Q}_p$-espace vectoriel fermé de X, stable par G_K, un *K-presque supplémentaire de Y dans X* est un presque supplémentaire qui est un sous-K-espace vectoriel. On dit qu'une suite exacte courte

$$0 \to W' \to W \to W'' \to 0$$

de B^+_{dR}-représentations est *K-presque scindée* si W' admet un K-presque supplémentaire dans W. Il revient au même de dire qu'il existe un sous-K-espace vectoriel de dimension finie V de W' stable par G_K et une section K-linéaire G_K-équivariante de la projection de W/V sur W''.

Théorème 5.6. — *Soit*

$$(*) \qquad 0 \to W' \to W \to W'' \to 0$$

une suite exacte courte de $\mathcal{B}(G_K)$ avec W' et W'' des B^+_{dR}-représentations de G_K. Les assertions suivantes sont équivalentes :
i) la suite (1) *est presque scindée,*
ii) la suite (1) *est K-presque scindée,*
iii) la représentation W est une B^+_{dR}-représentation.

Commençons par établir quelques lemmes.

Lemme 5.7. — *Soit*

$$\begin{array}{lccccccccc}
 & & & & & (1) & & (2) & & \\
 & & & & & 0 & & 0 & & \\
 & & & & & \downarrow & & \downarrow & & \\
(3) & 0 & \to & Y' & \to & X & \to & X'' & \to & 0 \\
 & & & \| & & \downarrow & & \downarrow & & \\
(4) & 0 & \to & Y' & \to & Y & \to & Y'' & \to & 0 \\
 & & & & & \downarrow & & \downarrow & & \\
 & & & & & Z & = & Z & & \\
 & & & & & \downarrow & & \downarrow & & \\
 & & & & & 0 & & 0 & &
\end{array}$$

un diagramme commutatif de B^+_{dR}-représentations dont les lignes et les colonnes sont exactes.
i) Si les suites (2) et (4) sont K-presque scindées, alors (1) l'est aussi,
ii) Si (1) et (3) sont K-presque scindées, alors (4) l'est aussi.

Preuve : i) Soient E'' un K-presque supplémentaire de X'' dans Y'' et F un K-presque supplémentaire de Y' dans Y. L'image inverse de E'' dans F est un K-presque supplémentaire de Y dans X.
ii) Soient E_1 un K-presque supplémentaire de X dans Y et E_2 un K-presque supplémentaire de Y' dans X. Alors $E_1 + E_2$ somme est un K-presque supplémentaire de Y dans X. □

Lemme 5.8. — *Supposons qu'il existe une extension finie L de K contenue dans $\overline{K}$ telle que $(*)$ est L-presque scindée en tant que suite exacte de B^+_{dR}-représentation de G_L. Alors $(*)$ est K-presque scindée.*

Preuve : Soient $g_1, g_2, \ldots, g_n$ des représentants dans G_K des classes à gauche de G_K suivant G_L. Par hypothèse, il existe un sous-L-espace vectoriel V de W' stable par G_L et de dimension finie et une section L-linéaire $s_0 : W'' \to W/V$

de la projection de W/V sur W'' qui commute à l'action de G_L. Quitte à remplacer V par $\sum_{i=1}^n g_i(V)$, on peut supposer que V est stable par G_K. On voit que l'application $s : W'' \to W/V$ définie par

$$s(x) = \frac{1}{n}\sum_{i=1}^{n} g_i(s_0(g_i^{-1}(x))$$

est une section K-linéaire G_K-équivariante de la projection de W sur W''. □

Lemme 5.9. — *Pour tout entier $m \geq 1$ la suite exacte*

$$0 \to B_{m-1}(1) \to B_m \to C \to 0$$

est K-presque scindée.

Preuve : Soient T_K le module de Tate d'un groupe formel de Lubin-Tate pour K (cf., par exemple [Se67b], §3.3) et $V_K = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_K$. C'est un K-espace vectoriel de dimension 1 sur lequel G_K opère via un caractère dont la restriction à l'inertie est l'inverse de celui donné par la théorie du corps de classes. La représentation V_K est de Hodge-Tate ([Se89], chap.III, §1 et appendice) : si W_0 désigne le noyau de l'application $C \otimes_{\mathbb{Q}_p} V_K \to C \otimes_K V_K$, la suite exacte

$$0 \to W_0 \to C \otimes_{\mathbb{Q}_p} V_K \to C \otimes_K V_K \to 0$$

est canoniquement scindée, $C \otimes_K V_K \simeq C(1)$ et $W_0 \simeq C^{[K:\mathbb{Q}_p]-1}$. Le choix d'un isomorphisme de $C \otimes_K V_K$ sur $C(1)$ définit une application K-linéaire injective G_K-équivariante $\iota_0 : V_K \to C(1)$. Celle-ci se relève de façon unique en une application K-linéaire G_K-équivariante $\iota : V_K \to \mathrm{Fil}^1 B_{dR}$: cela résulte de ce que la représentation V_K est de de Rham, mais aussi, plus simplement, de ce que, pour $n = 0, 1$,

$$\begin{gathered}\mathrm{Ext}^n(V_K, C(m)) = H^n_{\mathrm{cont}}(K, V_K^* \otimes_{\mathbb{Q}_p} C(m)) \simeq \\ H^n_{\mathrm{cont}}(K, C(m-1) \oplus C(m)^{[K:\mathbb{Q}_p]-1}) = 0\end{gathered}$$

pour tout entier $m \geq 2$, ce qui montre que l'application

$$\mathrm{Hom}_{\mathcal{C}(G_K)}(V_K, B_{m+1}) \to \mathrm{Hom}_{\mathcal{C}(G_K)}(V_K, B_m)$$

est bijective. En tensorisant la suite exacte

$$0 \to \mathbb{Q}_p \to U(-1) \to C(-1) \to 0$$

avec V_K, on obtient une suite exacte

$$0 \to V_K \to U(-1) \otimes_{\mathbb{Q}_p} V_K \to C(-1) \otimes_{\mathbb{Q}_p} V_K \to 0$$

Notons U_K l'image inverse de $C(-1) \otimes_K V_K$ dans $U(-1) \otimes_K V_K$.

Soit $\hat{\iota} : U_K \to B_{dR}^+$ le composé de l'inclusion de U_K dans $U(-1) \otimes V_K$ avec l'application $ut^{-1} \otimes v \mapsto ut^{-1}\iota(v)$ et soit $\iota_C(-1) : C(-1)\otimes_K \to C$ l'application $ct^{-1} \mapsto c\iota(v) \otimes t^{-1}$. Dans $\mathcal{C}(G_K)$ a un diagramme commutatif

$$\begin{array}{ccccccccc} 0 & \to & V_K & \to & U_K & \to & C \otimes_K V_K & \to & 0 \\ & & \downarrow \iota & & \downarrow \hat{\iota} & & \downarrow \iota_C(-1) & & \\ 0 & \to & \mathrm{Fil}^1 B_{dR}^+ & \to & B_{dR}^+ & \to & C & \to & 0 \end{array}$$

dans lequel toutes les applications sont K-linéaires. Comme la flèche verticale de droite est un isomorphisme, pour tout entier $m \geq 1$, l'image de U_K dans $B_m = B_{dR}^+/\mathrm{Fil}^m B_{dR}$ est un K-presque supplémentaire de $B_{m-1}(1)$ dans B_m. □

Lemme 5.10. — *La suite*

$$0 \to C \to C_2 \to C \to 0$$

est K-presque scindée.

Preuve : Choisissons une extension non triviale V_1 de $\mathbb{Q}_p(1)$ par $\mathbb{Q}_p$. Rappelons (prop.3.10) que $C \otimes_{\mathbb{Q}_p} V_1$ s'identifie à $C \oplus C(1)$ de sorte que la suite exacte

$$0 \to C \otimes_{\mathbb{Q}_p} V_1 \to B_2(-1) \otimes_{\mathbb{Q}_p} V_1 \to C(-1) \otimes_{\mathbb{Q}_p} V_1 \to 0$$

peut se réécrire

$$0 \to C \oplus C(1) \to B_2(-1) \otimes_{\mathbb{Q}_p} V_1 \to C(-1) \oplus C \to 0$$

Soit $\widehat{W}_1$ l'image inverse de C dans $B_2(-1) \otimes_{\mathbb{Q}_p} V_1$ Le quotient W_1 de $\widehat{W}_1$ par $C(1)$ est une B_{dR}^+-représentation, extension de C par C.
Pour $n = 0, 1$, on a $H^n_{\mathrm{cont}}(K, C(1)) = 0$, et la suite exacte

$$0 \to C(1) \to \widehat{W}_1 \to W_1 \to 0$$

induit un isomorphisme $\widehat{W}_1^{G_K} \to W_1^{G_K}$. Si l'extension W_1 de C par C était scindée, on aurait donc $\dim_K \widehat{W}_1^{G_K} = 2$, ce qui, comme $\widehat{W}_1 \subset B_2(-1) \otimes_{\mathbb{Q}_p} V_1$ contredit le fait que $\dim_K (B_2(-1) \otimes_{\mathbb{Q}_p} V_1)^{G_K} = 1$ (prop.3.10). La proposition 2.15 implique alors que $W_1 \simeq C_2$ et il suffit de vérifier que la suite exacte

$$0 \to C \to W_1 \to C \to 0$$

est K-presque scindée.
D'après le lemme 5.9, la suite exacte

$$0 \to C(1) \to B_2 \to C \to 0$$

est K-presque-scindée. Si U_K est un K-presque supplémentaire de $C(1)$ dans B_2, alors $U_K(-1) \otimes_{\mathbb{Q}_p} V_1$ est un presque K-supplémentaire de $C \otimes_{\mathbb{Q}_p} V_1$ dans $B_2 \otimes_{\mathbb{Q}_p} V_1$, donc $\widehat{E}_1 = (U_K(-1) \otimes_{\mathbb{Q}_p} V_1) \cap \widehat{W}_1$ est un presque K-supplémentaire de $C \otimes_{\mathbb{Q}_p} V_1$ dans $\widehat{W}_1$ et l'image de $\widehat{E}_1$ dans W_1 est un presque-K-supplémentaire de C dans W_1. □

Lemme 5.11. — *Pour tout entier $r \geq 2$, la suite*

$$0 \to C_{r-1} \to C_r \to C \to 0$$

est K-presque scindée.

Preuve : Pour $r = 2$, c'est le lemme précédent et la démonstration pour $r \geq 3$ est très voisine. Rappelons (§2.5) que, pour tout $m \in \mathbb{N}$, $C_m = C \otimes_{\mathbb{Z}_p} T_m$ où T_m est le $\mathbb{Z}_p$-espace vectoriel des polynômes de degré $< m$ en l'indéterminée $\log t$. En particulier $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_1 = \mathbb{Q}_p$ et on a une suite eaxcte

$$0 \to \mathbb{Q}_p \otimes T_{r-2} \to \mathbb{Q}_p \otimes T_{r-1} \to \mathbb{Q}_p \to 0$$

(où $(\log t)^{r-2}$ s'envoie sur 1).
Soit V_1 comme dans la preuve du lemme précédent. Comme le $\mathbb{Q}_p$-espace vectoriel $\mathrm{Ext}^2_{\mathbb{Q}_p[G_K]}(\mathbb{Q}_p(1), \mathbb{Q}_p)) = H^2_{\mathrm{cont}}(G_K, \mathbb{Q}_p(-1))$ est le dual de $H^0(G_K, \mathbb{Q}_p(2))$ (cf. §3.1), il est nul ; par dévissage, on en déduit que $\mathrm{Ext}^2_{\mathbb{Q}_p[G_K]}(\mathbb{Q}_p(1), \mathbb{Q}_p \otimes T_{r-2})) = 0$. Par conséquent, l'application naturelle

$$\mathrm{Ext}^1_{\mathbb{Q}_p[G_K]}(\mathbb{Q}_p(1), \mathbb{Q}_p \otimes T_{r-1}) \to \mathrm{Ext}^1_{\mathbb{Q}_p[G_K]}(\mathbb{Q}_p(1), \mathbb{Q}_p)$$

est surjective, ce qui nous permet de choisir une extension V de $\mathbb{Q}_p(1)$ par $\mathbb{Q}_p \otimes T_{r-1}$ qui relève V_1, i.e. qui est munie d'une identification de $V/(\mathbb{Q}_p \otimes T_{r-2})$ à V_1.
La suite exacte

$$0 \to \mathbb{Q}_p \otimes T_{r-1} \to V \to \mathbb{Q}_p(1) \to 0$$

induit, par extension des scalaires à C, une suite exacte de C-représentations

$$0 \to C_{r-1} \to V_C \to C(1) \to 0$$

On a $\mathrm{Hom}_{C[G_K]}(C(1), C) = \mathrm{Ext}^1_{C[G_K]}(C(1), C) = 0$, on en déduit par dévissage que $\mathrm{Hom}_{C[G_K]}(C(1), C_{r-1}) = \mathrm{Ext}^1_{C[G_K]}(C(1), C_{r-1}) = 0$, la suite exacte précédente est canoniquement scindée ce qui nous permet d'identifier V_C à $C_{r-1} \oplus C(1)$ et $V_C(-1)$ à $C_{r-1}(-1) \oplus C$. La suite exacte

$$0 \to C \otimes_{\mathbb{Q}_p} V \to B_2(-1) \otimes_{\mathbb{Q}_p} V \to C(-1) \otimes_{\mathbb{Q}_p} V \to 0$$

peut donc se réécrire

$$0 \to C_{r-1} \oplus C(1) \to B_2(-1) \otimes_{\mathbb{Q}_p} V \to C_{r-1}(-1) \oplus C \to 0$$

Soient $\widehat{W}$ l'image inverse de C dans $B_2(-1) \otimes_{\mathbb{Q}_p} V$ et W le quotient de $\widehat{W}$ par $C(1)$. C'est une B^+_{dR}-représentation extension de C par C_{r-1}. Cette extension est non scindée car le quotient de W par C_{r-2} s'identifie à l'extension, non scindée, W_1 de C par C considérée dans la preuve du lemmme précédent. L'assertion (Bi) du théorème 2.14 montre que le K-espace vectoriel $\mathrm{Ext}^1_{B^+_{dR}[G_K]}(C, C_{r-1})$ est de dimension 1. Il en résulte que W est isomorphe à C_r et il suffit de vérifier que la suite exacte

$$0 \to C_{r-1} \to W \to C \to 0$$

est presque scindée.
Si U_K est comme dans la preuve du lemme précédent, $U_K(-1) \otimes_{\mathbb{Q}_p} V$ est un presque K-supplémentaire de $C \otimes_{\mathbb{Q}_p} V$ dans $B_2 \otimes_{\mathbb{Q}_p} V$, donc $\widehat{E} = (U_K(-1) \otimes_{\mathbb{Q}_p} V) \cap \widehat{W}$ est un presque K-supplémentaire de $C \otimes_{\mathbb{Q}_p} V$ dans $\widehat{W}$ et l'image de $\widehat{E}$ dans W est un presque-K-supplémentaire de C_{r-1} dans W. □

Pour $m, r \in \mathbb{N}$, posons $B_{m,r} = B_m \otimes_{\mathbb{Z}_p} T_r$. C'est une B^+_{dR}-représentation et, si $m \geq 1$, en tensorisant avec T_r la suite exacte

$$0 \to B_{m-1}(1) \to B_m \to C \to 0$$

on obtient une autre suite exacte

$$0 \to B_{m-1,r}(1) \to B_{m,r} \to C_r \to 0$$

Lemme 5.12. — *Soient m et n des entiers > 0 et soit $B'_{m,r}$ le noyau du composé de la projection de $B_{m,r}$ sur C_r avec la projection de C_r sur C. Les suites exactes*

$$0 \to B_{m-1,r}(1) \to B_{m,r} \to C_r \to 0$$
$$0 \to B'_{m,r} \to B_{m,r} \to C \to 0$$

sont K-presque scindées.

Preuve : On peut supposer $m \geq 2$. On peut (lemme 5.9) trouver un K-presque supplémentaire U_K de $B_{m-1}(1)$ dans B_m ; si $V_K = B_{m-1}(1) \cap U_K$, on a un diagramme commutatif

$$\begin{array}{ccccccccc}
0 & \to & V_K \otimes T_r & \to & U_K \otimes T_r & \to & C_r & \to & 0 \\
 & & \cap & & \cap & & \| & & \\
0 & \to & B_{m-1,r}(1) & \to & B_{m,r} & \to & C_r & \to & 0
\end{array}$$

dont les lignes sont exactes et $U_K \otimes_{\mathbb{Z}_p} T_r$ est un K-presque supplémentaire de $B_{m-1,r}(1)$ dans $B_{m,r}$.
On a un diagramme commutatif de B^+_{dR}-représentations

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & B_{m-1,1}(1) & \to & B'_{m,r} & \to & C_{r-1} & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & B_{m-1,1}(1) & \to & B_{m,r} & \to & C_r & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & C & = & C & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

dont les lignes et les colonnes sont exactes. On vient de voir que

$$0 \to B_{m-1,r}(1) \to B_{m,r} \to C_r \to 0$$

est K-presque scindée et

$$0 \to C_{r-1} \to C_r \to C \to 0$$

l'est aussi (lemme 5.11). Donc (lemme 5.7, (i))

$$0 \to B'_{m,r} \to B_{m,r} \to C \to 0$$

l'est aussi. □

Prouvons alors le théorème : L'implication (ii)⇒(i) est triviale et on sait (prop.3.14) que (i)⇒(iii). Il suffit donc de prouver que, si

$$0 \to W' \to W \to W'' \to 0$$

est une suite exacte courte de $\mathcal{B}(G_K)$, alors W' admet un K-presque supplémentaire dans W.

Quitte à remplacer K par une extension finie convenable (ce qui est possible grâce au lemme 5.8), on peut supposer que W est petite, ce qui implique que W' et W'' le sont aussi. La décomposition $W = \oplus_{A \in \overline{\mathfrak{a}}_K} W_A$ de W suivant les orbites de $\mathfrak{a}_K$ sous l'action de $\mathbb{Z}$ (cf. remarque à la fin du §2.3) induit une décomposition en somme directe de la suite exacte, ce qui nous permet de supposer que tous les W_A sauf l'un d'entre eux sont nuls. Autrement dit, il existe $\alpha \in \mathfrak{a}_K$ tel que les valeurs propres de ∇_0 agissant sur W^f sont dans $\alpha + \mathbb{Z}$. Quitte à tordre l'action de W sur G_K par le caractère $\chi^{(-\alpha)}$ qui est à valeurs dans K^*, on peut supposer que $\alpha = 0$, i.e. que W est un objet de $\mathrm{Rep}_{B^+_{dR},\mathbb{Z}}(G_K)$.

On procède alors par récurrence sur la longeur d'' de la B^+_{dR}-représentation W'' qui, puisque W'' est petite, est aussi sa longeur en tant que B^+_{dR}-module.

Si $d'' = 1$, W'' objet simple de la catégorie $\mathrm{Rep}_{B^+_{dR},\mathbb{Z}}(G_K)$ est isomorphe à $C(i)$ pour $i \in \mathbb{Z}$ un entier convenable. Quitte à tordre l'action de G_K sur W par le caractère χ^{-i}, on peut supposer que $W'' = C$.

Avec les notations du §2.5 on a $W_{(\mathbb{Z})} = \oplus_{i \in \mathbb{Z}} W_{(i)}$, $W = B^+_{dR} \otimes_{K[[\underline{t}]]} W_{(\mathbb{Z})}$. En outre $W_{(\mathbb{Z})}$ est stable par ∇_0, de même que chaque $W_{(i)}$ et la restriction de ∇_0 à $W_{(0)}$ est nilpotente. On voit aussi que $1 \in C$ a un relèvement e dans $W_{(0)}$. Il existe alors des entiers m et r tels que $t^m e = \nabla_0^r e = 0$. Il est alors facile de voir qu'il existe un unique homomorphisme de B^+_{dR}-représentations $\eta : B_{m,r} \to C$ qui envoie $1 \otimes (\log t)^{r-1}$ sur e. On a alors un diagramme commutatif

$$\begin{array}{ccccccccc} 0 & \to & B'_{m,r} & \to & B_{m,r} & \to & C & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & W' & \to & W & \to & C & \to & 0 \end{array}$$

dont les lignes sont exactes. D'après le lemme 5.12, celle du haut est K-presque scindée ; celle du bas l'est a fortiori.

On peut maintenant appliquer l'hyopthèse de récurrence et supposer $d'' \geq 2$. Il existe donc une suite exacte courte non triviale de B_{dR}^+-représentations

$$0 \to W_1'' \to W'' \to W_2'' \to 0$$

Si W_1 désigne l'image inverse de W_1'' dans W, on a un diagramme commutatif

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & W' & \to & W_1 & \to & W_1'' & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & W' & \to & W & \to & W'' & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & W_2'' & = & W_2'' & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

de B_{dR}^+-représentations dont les lignes et les colonnes sont exactes. Par hypothèse de récurrence, les suites exactes

$$0 \to W_1 \to W \to W_2'' \to 0$$
et
$$0 \to W' \to W_1 \to W_1'' \to 0$$

sont K-presque scindées. L'assertion (ii) du lemme 5.7 implique que

$$0 \to W' \to W \to W'' \to 0$$

l'est aussi. □

5.4 – Toute B_{dR}^+-représentation est presque triviale

Théorème 5.13. — *Soient $d \in \mathbb{N}$ et W une B_{dR}^+-représentation de G_K de longueur d en tant que B_{dR}^+-module. Alors W est presque isomorphe à C^d.*

Preuve : Soit

$$0 \to W' \to W \to W'' \to 0$$

une suite exacte courte de B_{dR}^+-représentations. Comme cette suite est presque scindée, il existe un sous-$\mathbb{Q}_p$-espace vectoriel de dimension finie V, stable par G_K, tel que $W/V \simeq (W'/V) \oplus W''$ et W est presqu'isomorphe à $W' \oplus W''$. Si W' et W'' sont presque triviales, il en est donc de même de W et il suffit de prouver le théorème lorsque W est un objet simple de la catégorie des B_{dR}^+-représentations.

Supposons d'abord que W est petite. On peut supposer (prop.2.5) que $W = C\{\alpha\}$, avec α un élément K-petit convenable.

Il s'agit de montrer que W est presqu'isomorphe à C. On peut supposer $\alpha \notin \{0,-1\}$ (dans le premier cas, parce qu'alors, il n'y a rien à prouver, dans le second, parce que, si le théorème est vrai pour $\alpha = 1$, cela veut dire qu'il existe des sous-$\mathbb{Q}_p$-espaces vectoriels de dimension finie V de $C(1)$ et V' de C, stables par G_K et un isomorphisme $C(1)/V \to C/V'$; en tensorisant par $\mathbb{Q}_p(-1)$, on obtient un isomorphisme $C/V(-1) \to C(-1)/V'(-1)$, donc $C(-1)$ et C sont presqu'isomorphes).
Appelons *K-représentation* la donnée d'un K-espace vectoriel de dimension finie muni d'une action linéaire et continue de G_K. Autrement dit, c'est une représentation p-adique muni d'un plongement de K dans la $\mathbb{Q}_p$-algèbre de ses endomorphismes. Le sous-K-espace vectoriel $K\{\alpha\}$ de $C\{\alpha\}$ est une K-représentation. Soit V une K-représentation, extension non triviale de $K\{\alpha+1\}$ par K (muni de l'action triviale de G_K). En tensorisant, au dessus de K, la suite exacte

$$0 \to K \to V \to K\{\alpha+1\} \to 0$$

avec C, on obtient une suite exacte de C-représentations

$$0 \to C \to C \otimes_K V \to C\{\alpha+1\} \to 0$$

Cette suite est scindée. En effet, comme $\alpha \neq -1$, $C\{\alpha+1\}$ est un objet simple de $\mathrm{Rep}_C(G_K)$ qui n'est pas isomorphe à C et il n'y a donc pas de C-représentation extension non triviale de $C\{\alpha+1\}$ par C (prop.2.15). Ceci nous permet en particulier d'étendre l'inclusion $K \subset C$ en un K-plongement de V dans C et de définir un plongement $\iota : W \to C(-1)\otimes_K V$. Comme $C(-1)\otimes_K V$ est un facteur direct de $C(-1) \otimes_{\mathbb{Q}_p} V = V_C(-1)$, on peut aussi voir ι comme un homomorphisme non nul de W dans $V_C(-1)$.
La suite exacte

$$0 \to V \to C \to C/V \to 0$$

induit une suite exacte

$$\mathrm{Hom}_{\mathcal{B}(G_K)}(W,C) \to \mathrm{Hom}_{\mathcal{B}(G_K)}(W,C/V) \to \mathrm{Ext}^1_{\mathcal{B}(G_K)}(W,V) \to \mathrm{Ext}^1_{\mathcal{B}(G_K)}(W,C)$$

Comme W est un objet simple de $\mathrm{Rep}_{B^+_{dR}}(G_K)$ qui n'est isomorphe ni à C ni à $C(-1)$, pour $i = 0,1$, on a $\mathrm{Ext}^i_{\mathcal{B}(G_K)}(W,C) = \mathrm{Ext}^i_{B^+_{dR}[G_K]}(W,C) = 0$ (prop.2.15) et l'application $\mathrm{Hom}_{\mathcal{B}(G_K)}(W,C/V) \to \mathrm{Ext}^1_{\mathcal{B}(G_K)}(W,V)$ est bijective. Comme $\mathrm{Ext}^1_{\mathcal{B}(G_K)}(W,V)$ s'identifie à $\mathrm{Hom}_{\mathcal{B}(G_K)}(W,V_C(-1))$ (prop.3.12), l'application ι induit un morphisme non nul de W dans C/V. D'après le corollaire au théorème 4.1, cette application est surjective et son noyau est de dimension finie sur $\mathbb{Q}_p$. Par conséquent, W et C sont presqu'isomorphes.

Passons maintenant au cas général. Choisissons une extension finie L de K contenue dans $\bar{K}$ telle que W soit petite en tant que B^+_{dR}-représentation de $G_L = \mathrm{Gal}(\overline{K}/K)$. Soit S un facteur simple de W (en tant que B^+_{dR}-représentation de G_L).

Pour tout objet X de $\mathcal{B}(G_L)$, on note $IX = \mathbb{Q}_p[G_K]\otimes_{\mathbb{Q}_p[G_L]}X$ l'objet de $\mathcal{B}(G_K)$ induit. Si X est une C-représentation de G_L, IX est de façon naturelle une C-représentation de G_K. La simplicité de W implique que W est isomorphe à un facteur direct de IS. On fixe un plongement de W dans IS.
D'après la première partie, S est presqu'isomorphe à C. On peut donc trouver une représentation banachique X de G_L extension de S par une représentation p-adique V_0 de dimension finie et un morphisme surjectif $X \to C$ de $\mathcal{B}(G_L)$ dont le noyau V_1 est de dimension finie sur $\mathbb{Q}_p$. Les suites exactes

$$0 \to V_0 \to X \to S \to 0 \quad \text{et} \quad 0 \to V_1 \to X \to C \to 0$$

induisent des suites exacte

$$0 \to IV_0 \to IX \to IS \to 0 \quad \text{et} \quad 0 \to IV_1 \to IX \to IC \to 0$$

de $\mathcal{B}(G_K)$ et IC est une C-représentation triviale de G_K. Soit E l'image inverse de W dans X. Alors W est presqu'isomorphe à E qui est presqu'isomorphe à son image Y dans IC et il suffit, pour achever la démonstration, d'établir le lemme suivant :

Lemme 5.14. — *Soit $d = \dim_C W$. L'image Y de E dans IC est isomorphe à une extension d'une C-représentation triviale de dimension d par une représentation p-adique de dimension finie.*

Preuve : C'est essentiellement la même que celle de la proposition 5.1 Pour plus de clarté, nous la reproduisons avec les modifications nécessaires.
Soit W' un sous-C-espace vectoriel de IC stable par G_K tel que l'application composée $Y \subset IC \to IC/W'$ est surjective et qui est minimal pour cette propriété. Alors Y est une extension de $W'' = IC/W'$ par $Y' = Y \cap W'$. Si H est un hyperplan de W' stable par G_K, comme l'application composée $Y \subset IC \to IC/H$ n'est pas surjective, l'application $j_H : Y' \subset W' \to W'/H$ ne l'est pas non plus.
Comme S est presqu'isomorphe à C, c'est un objet de $\mathcal{C}(G_L)$. Considérés comme des objets de $\mathcal{B}(G_L)$, IS, W, E, Y et Y' sont des objets de $\mathcal{C}(G_L)$. Comme $W'/H \simeq C$, l'image de j_H est isomorphe, dans la catégorie $\mathcal{C}(G_L)$, à un sous-objet de C distinct de C et est donc de dimension finie sur $\mathbb{Q}_p$. Si l'on choisit une base $\{e_1, e_2, \ldots, e_r\}$ de W' sur C formée d'éléments fixes par G_K et si, pour $1 \leq i \leq r$, on note H_i l'hyperplan de W' de base les e_j avec $j \neq i$, la projection Y'_i de Y' sur W'/H_i est un $\mathbb{Q}_p$-espace vectoriel de dimension finie. Comme Y' s'injecte dans la somme directe des Y'_i, c'est aussi un $\mathbb{Q}_p$-espace vectoriel de dimension finie. Par conséquent Y s'identifie à une extension de la C-représentation triviale W'' par la représentation p-adique de dimension finie Y'. Pour vérifier que $\dim_C W'' = d$, il suffit de restreindre l'action de G_K à G_L et cela résulte de ce que le théorème est déjà prouvé pour les petites représentations. □

COROLLAIRE. — *Soit X une représentation banachique. Les assertions suivantes sont équivalentes :*
i) la représentation X est presqu'isomorphe à une C-représentation triviale,
ii) la représentation X est presqu'isomorphe à une C-représentation,
iii) la représentation X est presqu'isomorphe à une B_{dR}^+-représentation.

6 – CALCUL DES GROUPES D'EXTENSIONS

Rappelons que, si W' et W'' sont deux objets d'une catégorie abélienne $\mathcal{C}$, pour tout $n \in \mathbb{N}$, on note $\mathrm{Ext}^n_{\mathcal{C}}(W'', W')$ le groupe des classes de n-extensions de Yoneda de W'' par W'.
Les objectifs essentiels de ce paragraphe sont :
i) La preuve du théorème suivant :

THÉORÈME 6.1. — *Soient X et Y deux objets de $\mathcal{C}(G_K)$. Pour tout $n \in \mathbb{N}$, le $\mathbb{Q}_p$-espaces vectoriel* $\mathrm{Ext}^n_{\mathcal{C}(G_K)}(X, Y) = 0$ *est de dimension finie et est nul si $n \geq 3$. On a*

$$\sum_{i=0}^{2} \dim_{\mathbb{Q}_p} \mathrm{Ext}^n_{\mathcal{C}(G_K)}(X, Y) = -[K : \mathbb{Q}_p]h(X)h(Y)$$

ii) La construction d'une dualité entre $\mathrm{Ext}^n_{\mathcal{C}(G_K)}(X, Y)$ et $\mathrm{Ext}^{2-n}_{\mathcal{C}(G_K)}(Y, X(1))$ (prop.6.8 et 6.10 ci-dessous).

6.1 – PROPRIÉTÉS D'INVARIANCE DES Ext^n

On dit qu'un complexe de $\mathcal{C}(G_K)$

$$(X) \qquad \ldots \to X^{i-1} \xrightarrow{\delta_X^{i-1}} X^i \xrightarrow{\delta_X^i} X^{i+1} \to \ldots$$

est *presque trivial* si, pour tout $i \in \mathbb{Z}$ l'image de δ_X^i est un $\mathbb{Q}_p$-espace vectoriel de dimension finie.
Par ailleurs, pour toute catégorie abélienne $\mathcal{A}$ et tout entier $n \geq 1$, on note $C_n(\mathcal{A})$ la catégorie dont les objets sont les complexes (X) de $\mathcal{A}$ vérifiant $X^i = 0$ si $i < 0$ ou si $i \geq n$ et les flèches les morphismes de complexes.

PROPOSITION 6.2. — *Soit n un entier ≥ 1.*
i) Si (X) est un complexe de $C_n(\mathcal{C}(G_K))$, il existe un couple $((E), \eta)$ formé d'un complexe (E) presque trivial de $C_n(\mathcal{C}(G_K))$ et d'un morphisme $\eta : (E) \to (X)$ qui est un quasi-isomorphisme.
ii) Pour tout morphisme $\alpha : (X) \to (Y)$ de $C_n(\mathcal{C}(G_K))$, on peut trouver un diagramme commutatif de $C_n(\mathcal{C}(G_K))$

$$\begin{array}{ccc} (E) & \to & (X) \\ \downarrow & & \downarrow \\ (F) & \to & (Y) \end{array}$$

où les flèches horizontales sont des quasi-isomorphismes et où (E) et (F) sont presque triviaux.

LEMME 6.3. — *Soient* (X) *un complexe de* $\mathcal{C}(G_K)$ *et* $i \in \mathbb{Z}$. *Il existe un sous-complexe* (Y) *de* (X) *tel que*
i) l'inclusion de (Y) *dans* (X) *induit un quasi-isomorphisme,*
ii) on a $Y^j = X^j$ *pour tout entier* $j \notin \{i-1, i\}$,
iii) le $\mathbb{Q}_p$*-espace vectoriel image de l'application* $Y^{i-1} \to Y^i$ *est de dimension finie.*

Preuve du lemme : Avec des notations évidentes, on a, dans $\mathcal{C}(G_K)$, des suites exactes courtes

$$0 \to Z^i \to X^i \to B^{i+1} \to 0 \text{ et } 0 \to B^i \to Z^i \to H^i \to 0$$

D'après le corollaire à la proposition 5.5, elles sont presque scindées et on peut trouver des sous-objets X_0^i de X^i et Z_1^i de Z^i tels que les applications $X_0^i \to B^{i+1}$ et $Z^i \to H^i$ sont surjectives et que leurs noyaux respectifs Z_0^i et B_1^i sont de dimension finie sur $\mathbb{Q}_p$. Soit alors $Y^i = Z_1^i + X_0^i \subset X^i$. On a une suite exacte

$$0 \to Z_0^i + Z_1^i \to Y^i \to B^{i+1} \to 0$$

Si B_2^i désigne le noyau de la restriction à $Z_0^i + Z_1^i$ de la projection de Z^i sur H^i et si $\overline{Z}_0^i = Z_0^i / Z_0^i \cap Z_1^i$, on a un diagramme commutatif

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & B_1^i & \to & Z_1^i & \to & H^i & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & B_2^i & \to & Z_0^i + Z_1^i & \to & H^i & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \overline{Z}_0^i & = & \overline{Z}_0^i & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

dont les lignes sont exactes. Commme $\overline{Z}_0^i$ et B_1^i sont de dimension finie sur $\mathbb{Q}_p$, il en est de même de B_2^i et il suffit de prendre pour Y^{i-1} l'image inverse de B_2^i dans X^{i-1}. $\square$

Prouvons maintenant la proposition : Montrons d'abord (i) : Le lemme nous permet de construire une suite décroissante de sous-complexes $(X) = (X_n) \supset (X_{n-1}) \supset \ldots (X_r) \supset \ldots (X_2) \supset (X_1)$ de X tel que l'inclusion induit un quasi-isomorphisme et que l'image de X_r^{i-1} dans X_r^i soit de dimension finie sur $\mathbb{Q}_p$ pour $i \geq r$. Il suffit de prendre $(E) = (X_1)$.
ii) D'après (i), on peut trouver des morphismes de $C_n(\mathcal{C}(G_K))$

$$(E_0) \to (X) \text{ et } (F) \to (Y)$$

qui sont des quasi-isomorphismes, avec (E_0) et (F) presque triviaux. Si l'on prend pour (E) le produit fibré $(E_0) \times_{(Y)} (F)$, on voit que (E) est encore presque trivial et que l'on a un diagramme comme on veut. $\square$

Proposition 6.4. — *Soient V' et V'' des représentations p-adiques de G_K. Pour tout $n \in \mathbb{N}$, la flèche* $\mathrm{Ext}^n_{\mathbb{Q}_p[G_K]}(V'', V') \to \mathrm{Ext}^n_{\mathcal{C}(G_K)}(V'', V')$ *est bijective.*

Preuve : C'est clair si $n = 0$ ou 1. Le cas $n \geq 2$ résulte de la proposition précédente grâce au fait qu'un complexe de $\mathcal{C}(G_K)$ dont les groupes de cohomologie sont de dimension finie sur $\mathbb{Q}_p$ et qui est presque trivial est un complexe de $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$. Soyons un peu plus explicite :
Soit

$$0 \to V' \to X^0 \to X^1 \to \ldots \to X^{i-1} \to X^i \to \ldots \to X^{n-1} \to V'' \to 0$$

une suite exacte de $\mathcal{C}(G_K)$ représentant un élément ε de $\mathrm{Ext}^n_{\mathcal{C}(G_K)}(V'', V')$. Si

$$E^0 \to E^1 \to \ldots X^{i-1} \to E^i \to \ldots \to E^{n-1}$$

est un sous-complexe de

$$X^0 \to X^1 \to \ldots \to X^{i-1} \to X^i \to \ldots \to X^{n-1}$$

qui lui est quasi-isomorphe et est presque trivial,

$$0 \to V' \to E^0 \to E^1 \to \ldots \to E^{i-1} \to E^i \to \ldots \to E^{n-1} \to V'' \to 0$$

est une suite exacte courte de $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ dont la classe dans $\mathrm{Ext}^n_{\mathbb{Q}_p[G_K]}(V'', V')$ a pour image ε dans $\mathrm{Ext}^n_{\mathcal{C}(G_K)}(V'', V')$. L'application est bien surjective. L'assertion (ii) de la proposition 6.2 permet de montrer que, si deux complexes de $C_n(\mathrm{Rep}_{\mathbb{Q}_p}(G_K))$ sont quasi-isomorphes dans $\mathcal{C}(G_K)$, alors ils sont aussi quasi-isomorphes dans $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$, ce qui montre l'injectivité. □

Proposition 6.5. — *Soient W' et W'' des B^+_{dR}-représentations de G_K. Pour tout $n \in \mathbb{N}$, la flèche* $\mathrm{Ext}^n_{B^+_{dR}[G_K]}(W'', W') \to \mathrm{Ext}^n_{\mathcal{C}(G_K)}(W'', W')$ *est bijective.*

Preuve : C'est clair pour $n = 0$ et la conjonction du corollaire à la prop.5.5 et du th.5.6 montre que c'est vrai pour $n = 1$. Supposons donc $n \geq 2$. La proposition 6.5 résulte de la proposition 6.2 et du fait que tout objet de $C_n(\mathcal{C}(G_K))$ qui est presque trivial et dont les groupes de cohomologie sont des B^+_{dR}-représentations est quasi-isomorphe à un complexe de $C_n(\mathrm{Rep}_{B^+_{dR}}(G_K))$. Soyons plus précis. Pour prouver la surjectivité, il s'agit de vérifier que si

$$0 \to W' \to X^0 \to \ldots \to X^i \to X^{i+1} \to \ldots X^{n-1} \to W'' \to 0$$

est une suite exacte de $\mathcal{C}(G_K)$, avec W' et W'' des objets de $\mathrm{Rep}_{B^+_{dR}}(G_K)$, alors il existe un complexe de $C_n(\mathrm{Rep}_{B^+_{dR}}(G_K))$ qui est quasi-isomorphe à

$$(X) \qquad X^0 \to \ldots \to X^i \to X^{i+1} \to \ldots X^{n-1}$$

La proposition 6.2 nous permet de supposer que (X) est presque trivial. On a donc des suites exactes courtes

$$\begin{array}{c} 0 \to W' \to X^0 \to B^1 \to 0 \\ 0 \to B^i \to X^i \to B^{i+1} \to 0 \qquad \text{pour } 1 \leq i \leq n-2 \\ 0 \to B^{n-1} \to X^{n-1} \to W'' \to 0 \end{array}$$

où les B^i sont de dimension finie sur $\mathbb{Q}_p$. Ceci implique que, pour $1 \leq i \leq n-2$ le $\mathbb{Q}_p$-espace vectoriel X^i est aussi de dimension finie.
Pour $1 \leq i \leq n-1$, posons $B^i_C = C \otimes_{\mathbb{Q}_p} B^i$. On a

$$\mathrm{Ext}^1_{\mathcal{C}(G_K)}(B^1, W') = H^1_{\mathrm{cont}}(K, (B^1)^{*_{\mathbb{Q}_p}} \otimes_{\mathbb{Q}_p} W') = H^1_{\mathrm{cont}}(K, (B^1_C)^{*_C} \otimes_C W') = \mathrm{Ext}^1_{B^+_{dR}[G_K],0}(B^1_C, W')$$

groupe des extensions de B^1_C par W' qui sont scindées en tant que suites exactes de B^+_{dR}-modules, de sorte que, si E^0 désigne une extension de B^1_C par W' dont la classe est celle de X^0, on a un diagramme commutatif

$$\begin{array}{ccccccccc} 0 & \to & W' & \to & X^0 & \to & B^1 & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & W' & \to & E^0 & \to & B^1_C & \to & 0 \end{array}$$

dont les lignes sont exactes.
Pour $1 \leq i \leq n-1$, posons $E^i = C \otimes_{\mathbb{Q}_p} X^i$ de sorte que l'on a un diagramme commutatif

$$\begin{array}{ccccccccc} 0 & \to & B^i & \to & X^i & \to & B^{i+1} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & B^i_C & \to & E^i & \to & B^{i+1}_C & \to & 0 \end{array}$$

dont les lignes sont exactes.
Enfin, notons E^{n-1} la somme amalgamée de B^{n-1}_C et de X^{n-1} au-dessous de B^{n-1}. On a un un diagramme commutatif

$$\begin{array}{ccccccccc} 0 & \to & B^{n-1} & \to & X^{n-1} & \to & W'' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & B^{n-1}_C & \to & E^{n-1} & \to & W'' & \to & 0 \end{array}$$

dont les lignes sont exactes. D'après la proposition 3.14, E^{n-1}, extension presque scindée de W'' par B^{n-1}_C, est une B^+_{dR}-représentation.
Le complexe

$$(E) \qquad E^0 \to \ldots \to E^i \to E^{i+1} \to \ldots E^{n-1}$$

est formé de B^+_{dR}-représentations et on a un morphisme naturel $(E) \to (X)$ qui induit un quasi-isomorphisme.

Pour prouver l'injectivité, il suffit de montrer que si

$$\begin{array}{ccccccccccccccccc} 0 & \to & W' & \to & X^0 & \to & \dots & \to & X^i & \to & \dots & \to & X^{n-1} & \to & W'' & \to & 0 \\ & & \| & & \downarrow & & & & \downarrow & & & & \downarrow & & \| & & \\ 0 & \to & W' & \to & Y^0 & \to & \dots & \to & Y^i & \to & \dots & \to & Y^{n-1} & \to & W'' & \to & 0 \end{array}$$

est un diagramme commutatif de $\mathcal{C}(G_K)$ dont les lignes sont exactes et si W' et W'' sont des B_{dR}^+-représentations, alors, avec des notations évidentes, on peut trouver un carré commutatif

$$\begin{array}{ccc} (X) & \to & (E) \\ \downarrow & & \downarrow \\ (Y) & \to & (F) \end{array}$$

dans $C_n(\mathcal{C}(G_K))$ tel que (E) et (F) sont dans $C_n(B_{dR}^+)$ et que les flèches horizontales induisent un quasi-isomorphisme. La proposition 6.2 nous permet de supposer que (X) et (Y) sont presque triviaux. Si l'on prend pour (E) le complexe défini à partir de (X) comme plus haut et pour (F) le complexe associé à (Y) par la même recette, la flèche de (X) dans (Y) induit, de manière évidente une flèche de (E) dans (F) qui convient. □

6.2 – Le cas où X est de dimension finie

Proposition 6.6. — *Soient $n \in \mathbb{N}$, X et Y des presque C-représentations. Pour tout $\varepsilon \in \mathrm{Ext}^n_{\mathcal{C}(G_K)}(X,Y)$, il existe un sous-$\mathbb{Q}_p$-espace vectoriel de dimension finie V de Y, stable par G_K tel que ε appartient à l'image de $\mathrm{Ext}^n_{\mathcal{C}(G_K)}(X,V)$.*

Preuve : Soit

$$0 \to Y \to E^0 \to E^1 \to \dots E^{n-1} \to X \to 0$$

une suite exacte de $\mathcal{C}(G_K)$ représentant ε. Si N désigne l'image de $E^0 \to E^1$, on a une suite exacte de $\mathcal{C}(G_K)$

$$0 \to Y \to E^0 \to N \to 0$$

qui (corollaire à la proposition 5.5) est presque scindée. Si F^0 désigne un sous-objet de E^0 qui s'envoie surjectivement sur N tel que $V = F^0 \cap Y$ est de dimension finie, la suite exacte

$$0 \to V \to F^0 \to E^1 \to \dots E^{n-1} \to X \to 0$$

définit un élément de $\mathrm{Ext}^n_{\mathcal{C}(G_K)}(X,V)$ dont l'image dans $\mathrm{Ext}^n_{\mathcal{C}(G_K)}(X,Y)$ est ε. □

Soit V une représentation p-adique fixée. Les $(\mathrm{Ext}^n_{\mathcal{C}(G_K)}(V,-))_{n\in\mathbb{N}}$ et les $H^n_{\mathrm{cont}}(K, V^* \otimes_{\mathbb{Q}_p} -))_{n\in\mathbb{N}}$ forment des δ-foncteurs de la catégorie $\mathcal{C}(G_K)$ dans celle des groupes abéliens, on a $\mathrm{Hom}_{\mathcal{C}(G_K)}(V,-) = H^0_{\mathrm{cont}}(K, V^* \otimes_{\mathbb{Q}_p} -)$ et, pour $n \geq 1$, $\mathrm{Ext}^n_{\mathcal{C}(G_K)}(V,-)$ est effaçable. On a donc des morphismes naturels de foncteurs $\mathrm{Ext}^n_{\mathcal{C}(G_K)}(V,-) \to H^n_{\mathrm{cont}}(K, V^* \otimes_{\mathbb{Q}_p} -)$.

Proposition 6.7. — *Soient V une représentation p-adique de G_K et S une presque-C-représentation.*
i) Toute représentation banachique E extension de V par S est une presque C-représentation.
ii) Pour tout $n \in \mathbb{N}$, le $\mathbb{Q}_p$-espace vectoriel $\mathrm{Ext}^n_{\mathcal{C}(G_K)}(V,S)$ est de dimension finie, nulle si $n \geq 3$.
iii) Pour tout $n \in \mathbb{N}$, l'application naturelle
$$\mathrm{Ext}^n_{\mathcal{C}(G_K)}(V,S) \to H^n_{\mathrm{cont}}(K, V^* \otimes_{\mathbb{Q}_p} S)$$
est un isomorphisme.

Preuve : a) Quitte à remplacer S par $V^* \otimes_{\mathbb{Q}_p} S$, on peut supposer que $V = \mathbb{Q}_p$. Mais (i) signifie que l'inclusion $\mathrm{Ext}^1_{\mathcal{C}(G_K)}(\mathbb{Q}_p, S) \to \mathrm{Ext}^1_{\mathcal{B}(G_K)}(\mathbb{Q}_p, S)$ est une bijection. Comme $\mathrm{Ext}^1_{\mathcal{B}(G_K)}(\mathbb{Q}_p, S)$ s'identifie à $H^1_{\mathrm{cont}}(K,S)$, (i) résulte de (iii). Compte tenu du corollaire à la proposition 3.3, (iii) implique aussi (ii).

b) *Prouvons (ii) lorsque S est une représentation p-adique* : D'après la proposition 6.4, on a $\mathrm{Ext}^n_{\mathcal{C}(G_K)}(\mathbb{Q}_p, S) = \mathrm{Ext}^n_{\mathbb{Q}_p[G_K]}(\mathbb{Q}_p, S)$. Il suffit donc de vérifier que, pour $n \geq 1$, la restriction à la catégorie des représentations p-adiques du foncteur $H^n_{\mathrm{cont}}(K, -)$ est effaçable. Ces foncteurs sont nuls pour $n \geq 3$ et c'est évident pour $n = 1$. Pour $n = 2$, il s'agit de vérifier que l'on peut plonger toute représentation p-adique S de G_K dans une autre $\widehat{S}$ de manière que la flèche $H^2_{\mathrm{cont}}(K,S) \to H^2_{\mathrm{cont}}(K,\widehat{S})$ soit nulle. Par dualité cela revient à vérifier que toute représentation p-adique V_2 (prendre $V_2 = S^*(1)$) est isomorphe au quotient d'une représentation p-adique $\widehat{V}_2$ telle que l'application $\widehat{V}_2^{G_K} \to V_2^{G_K}$ est nulle.
Soient $V_1 = V_2^{G_K}$ et $V_3 = V_2/V_1$. Pour tout entier $r \in \mathbb{Z}$, $H^2_{\mathrm{cont}}(K, V_3^* \otimes V_1(r))$ s'identifie au dual de $(V_3 \otimes V_1^*)(1-r))^{G_K}$ et est nul pour presque tout r. Choisissons r ainsi. Choisissons aussi une extension non triviale V_0 de $\mathbb{Q}_p$ par $\mathbb{Q}_p(r)$.
Dans le carré commutatif
$$\begin{array}{ccc} \mathrm{Ext}^1_{\mathcal{B}(G_K)}(V_2, V_1(r)) & \to & \mathrm{Ext}^1_{\mathcal{B}(G_K)}(V_1, V_1(r)) \\ \| & & \| \\ H^1_{\mathrm{cont}}(K, V_2^* \otimes V_1(r)) & \to & H^1_{\mathrm{cont}}(K, V_1^* \otimes V_1(r)) \end{array}$$
la flèche inférieure est surjective, donc aussi la flèche supérieure. On peut donc construire un diagramme commutatif, dont les lignes et les colonnes sont exactes
$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & V_1(r) & \to & V_1 \otimes V_0 & \to & V_1 & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & V_1(r) & \to & \widehat{V}_2 & \to & V_2 & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & V_3 & = & V_3 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

On a alors $\widehat{V}_2^{G_K} = (V_1 \otimes V_0)^{G_K} = 0$.

c) *Prouvons (iii) lorsque S est une C-représentation* : Le groupe $H^1_{\mathrm{cont}}(K, S)$ classifie aussi bien les extensions de $\mathbb{Q}_p$ par S dans la catégorie des représentations banachiques que les extensions de C par S dans la catégorie des C-représentations. Ceci implique que pour toute représentation banachique E extension de $\mathbb{Q}_p$ par S, on peut trouver une C-représentation E_C et un diagramme commutatif

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & S & \to & E & \to & \mathbb{Q}_p & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & S & \to & E_C & \to & C & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & C/\mathbb{Q}_p & = & C/\mathbb{Q}_p & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

dont les lignes et les colonnes sont exactes. Mais alors S et $C/\mathbb{Q}_p$ sont des presque-C-représentations, donc aussi E qui s'identifie au noyau du morphisme $E_C \to C/\mathbb{Q}_p$. La flèche $\mathrm{Ext}^1_{\mathcal{C}(G_K)}(\mathbb{Q}_p, S) \to H^1_{\mathrm{cont}}(K, S)$ est donc bien un isomorphisme.
On a $H^2_{\mathrm{cont}}(K, S) = 0$ (cor. à la prop.3.3). Pour finir de prouver (c), il suffit donc de vérifier que $\mathrm{Ext}^2_{\mathcal{B}(G_K)}(\mathbb{Q}_p, S) = 0$. Soit

$$0 \to S \to X^0 \to X^1 \to \mathbb{Q}_p \to 0$$

une suite exacte presque triviale (cf. prop.6.2) de $\mathcal{C}(G_K)$ représentant un élément $\varepsilon \in \mathrm{Ext}^2_{\mathcal{B}(G_K)}(\mathbb{Q}_p, S)$. Alors $V = dX^0$ est de dimension finie et on a deux suites exactes courtes :

$$0 \to S \to X^0 \to V \to 0$$
$$0 \to V \to X^1 \to \mathbb{Q}_p \to 0$$

Comme $\mathrm{Ext}^1_{\mathcal{C}(G_K)}(V, S) = H^1_{\mathrm{cont}}(K, V^* \otimes S) = \mathrm{Ext}^1_{C[G_K]}(C \otimes V, S)$ on peut plonger X^0 dans un C-espace vectoriel X^0_C extension de $C \otimes V$ par S. Les deux suites exactes

$$0 \to S \to X^0_C \to C \otimes V \to 0$$
$$0 \to C \otimes V \to C \otimes X^1 \to C \to 0$$

définissent un élément $\varepsilon_C \in \mathrm{Ext}^2_{\mathcal{C}(G_K)}(C, S)$. Comme ce groupe est nul (prop.2.17), il existe un diagramme commutatif de C-représentations

$$\begin{array}{ccccccccccc} 0 & \to & S & \to & X^0_C & \to & C \otimes X^1 & \to & C & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & S & \to & Y^0 & \to & Y^1 & \to & C & \to & 0 \end{array}$$

dont les lignes sont exactes et qui a la vertu que la projection de Y^1 sur C admet une section s. Si Y_0^1 désigne l'image inverse de $\mathbb{Q}_p$ dans Y^1, ε est la classe de la 2-extension

$$0 \to S \to Y^0 \to Y_0^1 \to \mathbb{Q}_p \to 0$$

La restriction de s à $\mathbb{Q}_p$ fournit une trivialisation de cette 2-extension et $\varepsilon = 0$.

d) *Fin de la preuve* : Si

$$0 \to Y' \to Y \to Y'' \to 0$$

est une suite exacte de presque-C-représentations et si le résultat est vrai pour Y' et Y, il l'est aussi pour Y''. S'il est vrai pour Y' et Y'', il l'est aussi pour Y. La proposition s'en déduit puisque le fait que S soit une presque C-représentation implique que l'on peut trouver des suites exactes de $\mathcal{C}(G_K)$

$$0 \to V' \to W \to X \to 0 \text{ et } 0 \to V'' \to S \to X \to 0$$

avec V', V'' des représentations p-adiques et W une C-représentation. □

6.3 – Le cas où Y est de dimension finie

Si V est une représentation p-adique, $\mathrm{Ext}^2_{\mathcal{C}(G_K)}(V, V(1)) = H^2_{\mathrm{cont}}(K, V^* \otimes V(1))$ s'envoie dans $H^2_{\mathrm{cont}}(K, \mathbb{Q}_p(1)) = \mathbb{Q}_p$.

Proposition 6.8. — *Soient V une représentation p-adique et S une presque-C-représentation.*
i) Les $\mathbb{Q}_p$-espaces vectoriels $\mathrm{Ext}^n_{\mathcal{C}(G_K)}(S, V(1))$ sont de dimension finie, nulle si $n \geq 3$.
ii) Pour $n = 0, 1, 2$, l'application bilinéaire

$$\mathrm{Ext}^{2-n}_{\mathcal{C}(G_K)}(V, S) \times \mathrm{Ext}^n_{\mathcal{C}(G_K)}(S, V(1)) \to \mathrm{Ext}^2_{\mathcal{C}(G_K)}(V, V(1)) \to \mathbb{Q}_p$$

est non dégénérée.

Preuve : Quitte à remplacer S par $V^* \otimes S$, on peut supposer $V = \mathbb{Q}_p$.
Remarquons d'abord que la proposition est vraie si S est une représentation p-adique : compte-tenu de la proposition précédente, cela résulte des résultats classiques sur la cohomologie continue (§3.1).
Si W est une C-représentation, on a $\mathrm{Hom}_{\mathcal{C}(G_K)}(W, \mathbb{Q}_p(1)) = 0$ (cor. au th.3.11) et $\mathrm{Ext}^2_{\mathcal{C}(G_K)}(\mathbb{Q}_p, W) = H^2_{\mathrm{cont}}(K, W) = 0$ (prop.6.7). Si en outre $\mathrm{Hom}_{\mathcal{C}(G_K)}(W, C) = 0$, on a aussi $\mathrm{Ext}^1_{\mathcal{C}(G_K)}(W, \mathbb{Q}_p(1)) = 0$ (pro.3.12) et $\mathrm{Ext}^1_{\mathcal{C}(G_K)}(\mathbb{Q}_p, W) = H^1_{\mathrm{cont}}(K, W) = 0$ (prop.3.1).
La presque-C-représentation $S(-1)$ est presqu'isomorphe à une C-représentation triviale. Il existe donc un entier d, des sous-$\mathbb{Q}_p$-espaces vectoriels de dimension finie V' de S et V'' de $C(1)^d$, stables par G_K et un isomorphisme

$S/V' \simeq C(1)/V''$. Posons $X = S/V'$ que l'on identifie via cet isomorphisme à $C(1)^d/V''$.
On a un diagramme commutatif

$$\begin{array}{ccccccccc} \ldots \to & \mathrm{Ext}^n(X, \mathbb{Q}_p(1)) & \to & \mathrm{Ext}^n(C(1)^d, \mathbb{Q}_p(1)) & \to & \mathrm{Ext}^n(V'', \mathbb{Q}_p(1)) & \to \ldots \\ & \downarrow & & \downarrow & & \downarrow & \\ \ldots \to & \mathrm{Ext}^{2-n}(\mathbb{Q}_p, X)^* & \to & \mathrm{Ext}^{2-n}(\mathbb{Q}_p, C(1)^d)^* & \to & \mathrm{Ext}^{2-n}(\mathbb{Q}_p, V'')^* & \to \ldots \end{array}$$

dont les lignes sont exactes. Or $\mathrm{Ext}^n(V'', \mathbb{Q}_p(1)) \to \mathrm{Ext}^{2-n}(\mathbb{Q}_p, V'')^*$ est un isomorphisme pour tout n et $\mathrm{Ext}^n(C(1)^d, \mathbb{Q}_p(1)) = \mathrm{Ext}^{2-n}(\mathbb{Q}_p, C(1)^d)$ pour $n = 0, 1$. On en déduit que $\mathrm{Hom}(X, \mathbb{Q}_p(1)) = \mathrm{Ext}^2(\mathbb{Q}_p, X) = 0$ et que $\mathrm{Ext}^1(X, \mathbb{Q}_p(1)) \to \mathrm{Ext}^1(\mathbb{Q}_p, X)^*$ est un isomorphisme.
On a aussi le diagramme commutatif

$$\begin{array}{ccccccccc} \ldots \to & \mathrm{Ext}^n(X, \mathbb{Q}_p(1)) & \to & \mathrm{Ext}^n(S, \mathbb{Q}_p(1)) & \to & \mathrm{Ext}^n(V', \mathbb{Q}_p(1)) & \to \ldots \\ & \downarrow & & \downarrow & & \downarrow & \\ \ldots \to & \mathrm{Ext}^{2-n}(\mathbb{Q}_p, X)^* & \to & \mathrm{Ext}^{2-n}(\mathbb{Q}_p, S)^* & \to & \mathrm{Ext}^{2-n}(\mathbb{Q}_p, V')^* & \to \ldots \end{array}$$

dont les lignes sont aussi exactes. Mais $\mathrm{Hom}(X, \mathbb{Q}_p(1)) = \mathrm{Ext}^2(\mathbb{Q}_p, X) = 0$. Comme $\mathrm{Hom}(V'', \mathbb{Q}_p(1)) \to \mathrm{Ext}^2(\mathbb{Q}_p, V'')^*$ et $\mathrm{Ext}^1(X, \mathbb{Q}_p(1)) \to \mathrm{Ext}^1(\mathbb{Q}_p, X)^*$ sont des isomorphismes, on en déduit que $\mathrm{Hom}(S, \mathbb{Q}_p(1)) \to \mathrm{Ext}^2(\mathbb{Q}_p, S)$ est aussi un isomorphisme.
Il suffit alors pour achever la preuve de vérifier que, pour $n = 1, 2$, le foncteur contravariant $F^n : \mathcal{C}(G_K)^{\mathrm{op}} \to \underline{\mathrm{Vect}}_{\mathbb{Q}_p}$ qui envoie S sur $\mathrm{Ext}^{2-n}(\mathbb{Q}_p, S)^*$ est effaçable, i.e. que pour tout S, on peut trouver un épimorphisme $X \to S$ tel que l'application $F^n(X) \to F^n(S)$ est nulle. Par dévissage, on voit qu'il suffit de le prouver lorsque S est de dimension finie sur $\mathbb{Q}_p$ ou lorsque $S = C(1)$. Dans le premier cas cela provient de ce que l'on sait déjà que l'application $\mathrm{Ext}^n(S, \mathbb{Q}_p(1)) \to F^n(S)$ est un isomorphisme. Dans le second de ce que $F^n(S) = 0$. □

6.4 – Preuve du théorème 6.1

Disons que la propriété $P(X, Y)$ est vraie si le théorème 6.1 est vrai pour X et Y. On voit que si

$$0 \to X' \to X \to X'' \to 0 \text{ et } 0 \to Y' \to Y \to Y'' \to 0$$

sont des suites exacte courtes de $\mathcal{C}(G_K)$, alors
– si deux des trois propriétés $P(X', Y), P(X, Y), P(X'', Y)$ sont vraies, la troisième aussi,
– si deux des trois propriétés $P(X, Y'), P(X, Y), P(X, Y'')$ sont vraies, la troisième aussi.
En utilisant le fait que pour toute presque C-représentation X, il existe un isomorphisme $X/V \simeq W/V'$ avec V et V' des $\mathbb{Q}_p$-représentations et W une C-représentation, on est ramené à prouver que $P(X, Y)$ est vrai dans chacun des quatre cas suivants :

i) X et Y sont des $\mathbb{Q}_p$-représentations,
ii) X et Y sont des C-représentations,
iii) X est une $\mathbb{Q}_p$-représentation et Y est une C-représentation
iv) X est une C-représentation et Y est une $\mathbb{Q}_p$-représentation.
Dans le cas (ii), on a $\mathrm{Ext}^n_{\mathcal{C}(G_K)}(X,Y) = \mathrm{Ext}^n_{B^+_{dR}[G_K]}(X,Y)$ (prop.6.5) et c'est l'assertion (a) du théorème 2.14.
Dans les trois autres cas, les propositions 6.7 et 6.8 impliquent que les $\mathrm{Ext}^n_{\mathcal{C}(G_K)}(X,Y)$ sont de dimension finie, nulle si $n \geq 3$.
Dans les cas (i) et (iii), on a en outre $\mathrm{Ext}^n_{\mathcal{C}(G_K)}(X,Y) = H^n_{\mathrm{cont}}(K, X^* \otimes Y)$ (prop.6.7). L'égalité

$$\sum_{i=0}^{2} \dim_{\mathbb{Q}_p} \mathrm{Ext}^n_{\mathcal{C}(G_K)}(X,Y) = -[K:\mathbb{Q}_p]h(X)h(Y)$$

résulte alors
– dans le cas (i), de ce que (§3.1)

$$\sum_{n=0}^{2}(-1)^n \dim_{\mathbb{Q}_p} H^n_{\mathrm{cont}}(K, X^* \otimes Y) = -[K:\mathbb{Q}_p].\dim_{\mathbb{Q}_p}(X^* \otimes Y)$$

– dans le cas (iii), de ce que, comme $X^* \otimes Y$ est une C-représentation, $H^0(K, X^* \otimes Y)$ et $H^1_{\mathrm{cont}}(K, X^* \otimes Y)$ sont de même dimension (prop.3.1) tandis que $H^2_{\mathrm{cont}}(K, X^* \otimes Y) = 0$ (cor.à la prop.3.3).
Enfin, la proposition 6.8 permet de déduire le cas (iv) du cas (iii). □

6.5 – Dualité

Pour toute représentation p-adique V, on note c_V la flèche naturelle

$$\mathrm{Ext}^2_{\mathcal{C}(G_K)}(V, V(1)) \simeq H^2_{\mathrm{cont}}(K, V^* \otimes V(1)) \to H^2_{\mathrm{cont}}(K, \mathbb{Q}_p(1)) = \mathbb{Q}_p$$

Soient X une presque C-représentation et $\varepsilon \in \mathrm{Ext}^2_{\mathcal{C}(G_K)}(X, X(1))$. Choisissons un complexe presque trivial de $C_2(\mathcal{C}(G_K))$

$$X^0 \xrightarrow{d} X^1$$

représentant ε. Alors dX^0 est de dimension finie sur $\mathbb{Q}_p$, la suite exacte

$$0 \to X(1) \to X^0 \to dX^0 \to 0$$

est presque scindée et on peut choisir un sous-$\mathbb{Q}_p$-espace vectoriel de dimension finie Y^0 de X^0, stable par G_K, tel que la restriction de d à Y^0 soit surjective. Notons V le sous-$\mathbb{Q}_p$-espace vectoriel de X défini par $V(1) = X(1) \cap \mathrm{Ker}\, d\,|_{Y^0}$ et Y^1 l'image inverse de V dans X^1. On a un diagramme commutatif de presque-C-représentations

$$\begin{array}{ccccccccc} 0 & \to & V(1) & \to & Y^0 & \to & Y^1 & \to & V & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & X(1) & \to & X^0 & \to & X^1 & \to & X & \to & 0 \end{array}$$

dont les lignes sont exactes, les objets de la première étant tous de dimension finie sur $\mathbb{Q}_p$.

Proposition 6.9. — *Soient X une presque C-représentation et $\varepsilon \in \mathrm{Ext}^2_{\mathcal{C}(G_K)}(X, X(1))$. Choisissons un diagramme*

$$\begin{array}{ccccccccccc} 0 & \to & V(1) & \to & Y^0 & \to & Y^1 & \to & V & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & X(1) & \to & X^0 & \to & X^1 & \to & X & \to & 0 \end{array}$$

comme ci-dessus et soit $\varepsilon_V \in \mathrm{Ext}^2_{\mathcal{C}(G_K)}(V, V(1))$ l'élément défini par la ligne supérieure. Alors $c_V(\varepsilon_V)$ ne dépend pas des choix faits. Si l'on pose $c_X(\varepsilon) = c_V(\varepsilon_V)$, l'application $c_X : \mathrm{Ext}^2_{\mathcal{C}(G_K)}(X, X(1)) \to \mathbb{Q}_p$ ainsi définie est $\mathbb{Q}_p$-linéaire.

Lemme 6.10. — *Soit*

$$\begin{array}{ccccccccccc} 0 & \to & V_1(1) & \to & Y_1^0 & \to & Y_1^1 & \to & V_1 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & V_2(1) & \to & Y_2^0 & \to & Y_2^1 & \to & V_2 & \to & 0 \end{array}$$

un diagramme commutatif de représentation p-adiques dont les lignes sont exactes. Pour $i = 1, 2$, soit $\varepsilon_{V_i} \in \mathrm{Ext}^2_{\mathbb{Q}_p[G_K]}(V_i, V_i(1))$ l'élément défini par la i-ième ligne. On a, avec des conventions évidentes

$$c_{V_2}(\varepsilon_{V_2}) = c_{V_1}(\varepsilon_{V_1})$$

Preuve : Pour $i = 1, 2$, on a, avec des notations évidentes $V_i^* \otimes V_i = sl(V_i) \oplus \mathbb{Q}_p$ et $c_{V_i}(\varepsilon_{V_i})$ est la classe de

$$0 \to \mathbb{Q}_p(1) \to E_i^0 \to E_i^1 \to \mathbb{Q}_p \to 0$$

où E_i^0 est le quotient de $V_i^* \otimes Y_i^0$ par $sl(V_i)(1)$ tandis que E_i^1 est l'image inverse de $\mathbb{Q}_p$ dans $V_i^* \otimes Y_i^1$.
On voit que le noyau de l'application composée naturelle

$$V_2^* \otimes (V_2(1) \oplus Y_1^0) = (V_2^* \otimes V_2(1)) \oplus (V_2^* \otimes Y_1^0) \to \mathbb{Q}_p(1) \oplus (V_1^* \otimes Y_1^0) \to E_1^0$$

contient le noyau de la projection $V_2^* \otimes (V_2(1) \oplus Y_1^0) \to V_2^* \otimes Y_2^0 \to E_2^0$ d'où, par passage au quotient, une application de E_2^0 dans E_1^0.
On a $E_1^1 \subset V_1^* \otimes Y_1^1 \subset V_1^* \otimes Y_2^1$. On voit que l'image de l'application composée $E_2^1 \subset V_2^* \otimes Y_2^1 \to V_1^* \otimes Y_2^1$ est contenue dans E_2^1, d'où une application de E_2^1 dans E_1^1. On vérifie alors que le diagramme

$$\begin{array}{ccccccccccc} 0 & \to & \mathbb{Q}_p(1) & \to & E_1^0 & \to & E_1^1 & \to & \mathbb{Q}_p & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & \mathbb{Q}_p(1) & \to & E_2^0 & \to & E_2^1 & \to & \mathbb{Q}_p & \to & 0 \end{array}$$

est commutatif et on a donc $c_{V_1}(\varepsilon_{V_1}) = c_{V_2}(\varepsilon_{V_2})$. □

Prouvons maintenant la proposition : Commençons par vérifier que, le complexe presque trivial

$$X^0 \xrightarrow{d} X^1$$

étant fixé, $c_V(\varepsilon_V)$ ne dépend pas du choix de la représentation p-adique $Y^0 \subset X^0$ telle que $dY^0 = dX^0$: Si Y_1^0 et Y_2^0 sont deux choix, on peut quitte à remplacer Y_2^0 par $Y_1^0 + Y_2^0$, supposer qe $Y_1^0 \subset Y_2^0$. Si, pour $i = 1, 2$, on pose $V_i(1) = X(1) \cap Y_i^0$, on a $V_1 \subset V_2$ et on dispose d'un diagramme commutatif

$$\begin{array}{ccccccccc} 0 & \to & V_1(1) & \to & Y_1^0 & \to & Y_1^1 & \to & V_1 & \to & 0 \\ & & \cap & & \cap & & \cap & & \cap & & \\ 0 & \to & V_2(1) & \to & Y_2^0 & \to & Y_2^1 & \to & V_2 & \to & 0 \end{array}$$

dont les lignes sont exactes. D'après le lemme précédent, on a bien, avec des notations évidentes $c_{V_1}(\varepsilon_{V_1}) = c_{V_2}(\varepsilon_{V_2})$.

Pour achever la démonstration, il suffit de montrer que si

$$\begin{array}{ccccccccccc} 0 & \to & V(1) & \to & Y^0 & \to & Y^1 & \to & V & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & X(1) & \to & X_1^0 & \to & X_1^1 & \to & X & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & X(1) & \to & X_2^0 & \to & X_2^1 & \to & X & \to & 0 \end{array}$$

est un diagramme commutatif de presque C-représentations, dont les lignes sont exactes, la première étant constituée de représentations p-adiques, alors il existe un diagramme commutatif

$$\begin{array}{ccccccccccc} 0 & \to & V_2(1) & \to & Y_2^0 & \to & Y_2^1 & \to & V_2 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & X(1) & \to & X_2^0 & \to & X_2^1 & \to & X & \to & 0 \end{array}$$

dont les lignes sont exactes, la première étant formée de représentations p-adiques, telle que $c_{V_2}(\varepsilon_{V_2}) = c_V(\varepsilon_V)$. Il suffit de prendre $V_2 = V$ et pour Y_2^0 (resp. Y_2^1) l'image de Y^0 (resp. Y^1) dans X_2^0 (resp. X_2^1). On a alors un diagramme commutatif

$$\begin{array}{ccccccccccc} 0 & \to & V(1) & \to & Y^0 & \to & Y^1 & \to & V & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & V_2(1) & \to & Y_2^0 & \to & Y_2^1 & \to & V_2 & \to & 0 \end{array}$$

et, dans $\mathrm{Ext}^2_{\mathbb{Q}_p[G_K]}(V, V(1))$, les deux 2-extensions considérées définissent le même élément. Donc $c_{V_2}(\varepsilon_{V_2}) = c_V(\varepsilon_V)$. □

Si X et Y sont deux presque-C-représentations, on dispose alors de deux applications bilinéaires

$$\mathrm{Ext}^n_{\mathcal{C}(G_K)}(X, Y) \times \mathrm{Ext}^{2-n}_{\mathcal{C}(G_K)}(Y, X(1)) \to \mathbb{Q}_p$$

La première envoie (a, b) sur $c_X(a \cup b)$ et la deuxième sur $c_Y(b \cup a)$ (on a identifié, de manière évidente, $\mathrm{Ext}^n_{\mathcal{C}(G_K)}(X(1), Y(1))$ à $\mathrm{Ext}^n_{\mathcal{C}(G_K)}(X, Y)$).

Proposition 6.11. — *Soient X et Y des presque C-représentations et soit $n \in \{0,1,2\}$.*
i) Si $a \in \mathrm{Ext}^2_{\mathcal{C}(G_K)}(X,Y)$ et $b \in \mathrm{Ext}^2_{\mathcal{C}(G_K)}(Y,X(1))$, on a $c_Y(b \cup a) = (-1)^n c_X(a \cup b)$.
ii) L'application bilinéaire

$$\mathrm{Ext}^n_{\mathcal{C}(G_K)}(X,Y) \times \mathrm{Ext}^{2-n}_{\mathcal{C}(G_K)}(Y,X(1)) \to \mathrm{Ext}^2_{\mathcal{C}(G_K)}(X,X(1)) \xrightarrow{c_X} \mathbb{Q}_p$$

qui envoie (a,b) sur $c_X(a \cup b)$, est non dégénérée.

Preuve : Montrons (i) :
Supposons d'abord $n = 0$, de sorte que a est un morphisme de X dans Y et que b est la classe d'une 2-extension

$$0 \to X(1) \to E^0 \xrightarrow{d} E^1 \to Y \to 0$$

avec $Z = d(E^0)$ de dimension finie. Si $E^1_X = E^1 \times_Y X$, le cup-produit $a \cup b$ est la classe de la 2-extension

$$0 \to X(1) \to E^0 \to E^1_X \to X \to 0$$

Si $E^0_Y = Y(1) \oplus_{X(1)} E^0$ (où $X(1) \to Y(1)$ est $a \otimes \mathrm{id}_{\mathbb{Z}_p(1)}$), le cup-produit $b \cup a$ est la classe de

$$0 \to Y(1) \to E^0_Y \to E^1 \to Y \to 0$$

Choisissons $F^0 \subset E^0$ de dimension finie telle que $d(F^0) = d(E^0)$, posons $V_1(1) = F^0 \cap X(1)$ et notons F^1 l'image inverse de V_1 dans E^1_X. On a deux diagrammes commutatifs

$$\begin{array}{ccccccccc} 0 & \to & V_1(1) & \to & F^0 & \to & Z & \to & 0 \\ & & \cap & & \cap & & \| & & \\ 0 & \to & X(1) & \to & E^0 & \to & Z & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & Y(1) & \to & E^0_Y & \to & Z & \to & 0 \end{array} \qquad \begin{array}{ccccccccc} 0 & \to & Z & \to & F^1 & \to & V_1 & \to & 0 \\ & & \| & & \downarrow & & \cap & & \\ 0 & \to & Z & \to & E^1_X & \to & X & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & Z & \to & E^1 & \to & Y & \to & 0 \end{array}$$

dont les lignes sont exactes. Soient $V_2 = a(V_1)$, G^0 l'image de F^0 dans E^0_Y et G^1 l'image de F^1 dans E^1. On a un diagramme commutatif

$$\begin{array}{ccccccccccc} 0 & \to & V_1(1) & \to & F^0 & \to & F^1 & \to & V_1 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & V_2(1) & \to & G^0 & \to & G^1 & \to & V_2 & \to & 0 \end{array}$$

dont les lignes sont exactes. Notons ε_1 (resp. ε_2) la classe de la 2-extension de V_1 par $V_1(1)$ (resp. V_2 par $V_2(1)$) définie par la première (resp. la seconde). On voit que $c_X(a \cup b) = c_{V_1}(\varepsilon_1)$ tandis que $c_Y(b \cup a) = c_{V_2}(\varepsilon_2)$. D'après le lemme 6.10, on a bien $c_{V_1}(\varepsilon_1) = c_{V_2}(\varepsilon_2)$.

Supposons maintenant $n = 1$. Soient

$$0 \to Y \to E^1 \to X \to 0 \ \text{ et } \ 0 \to X(1) \to E^0 \to Y \to 0$$

deux suites exactes courtes représentant a et b. Le choix d'un presque-scindage de chacune d'elles nous donnent des diagrammes commutatifs de presque-C-représentations

$$\begin{array}{ccccccccc} 0 & \to & V_2 & \to & G^1 & \to & V_1 & \to & 0 \\ & & \| & & \cap & & \cap & & \\ 0 & \to & V_2 & \to & F^1 & \to & X & \to & 0 \\ & & \cap & & \cap & & \| & & \\ 0 & \to & Y & \to & E^1 & \to & X & \to & 0 \end{array} \qquad \begin{array}{ccccccccc} 0 & \to & V_1(1) & \to & G^0 & \to & V_2 & \to & 0 \\ & & \| & & \cap & & \cap & & \\ 0 & \to & V_1(1) & \to & F^0 & \to & Y & \to & 0 \\ & & \cap & & \cap & & \| & & \\ 0 & \to & X(1) & \to & E_0 & \to & Y & \to & 0 \end{array}$$

dont les lignes sont exactes, avec V_1 et V_2 de dimension finie sur $\mathbb{Q}_p$ (on choisit les F^i, puis on pose $V_2 = F^1 \cap Y$, $V_1(1) = F^0 \cap X(1)$, $G^1 = F^1 \times_X V_1$ et $G^0 = F^0 \times_Y V_2$). On dispose alors d'un diagramme commutatif

$$\begin{array}{ccccccccccc} 0 & \to & V_1(1) & \to & G^0 & \to & G^1 & \to & V_1 & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & V_1(1) & \to & F^0 & \to & E^1 \times_X V_1 & \to & V_1 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & X(1) & \to & E^0 & \to & E^1 & \to & X & \to & 0 \end{array}$$

dont les lignes sont exactes. Si l'on note $a' \in \mathrm{Ext}^1_{\mathbb{Q}_p[G_K]}(V_1, V_2)$ la classe de G^1 et $b' \in \mathrm{Ext}^1_{\mathbb{Q}_p[G_K]}(V_2, V_1(1))$ celle de G^0, on en déduit que $c_X(a \cup b) = c_{V_1}(a' \cup b')$. Un calcul similaire montre que $c_Y(b \cup a) = c_{V_2}(b' \cup a')$.
Mais, si $V = V_1^* \otimes V_2$, on a

$$\mathrm{Ext}^1_{\mathbb{Q}_p[G_K]}(V_1, V_2) = H^1_{\mathrm{cont}}(K, V) \text{ et } \mathrm{Ext}^1_{\mathbb{Q}_p[G_K]}(V_2, V_1(1)) = H^1_{\mathrm{cont}}(K, V^*(1))$$

Lorsque l'on identifie $V \otimes V^*(1)$ à $V^*(1) \otimes V$, on a $b \cup a = -a \cup b$. Par conséquent $c_Y(b \cup a)$, qui est l'image de $b' \cup a'$ dans $H^2(K, \mathbb{Q}_p(1)) = \mathbb{Q}_p$ est l'opposé de $c_X(a \cup b)$ qui est l'image de $a' \cup b'$.

Pour $n = 2$, la preuve est entièrement analogue au cas $n = 0$.

Montrons alors (ii) : Pour X fixé, les $F^n_X(Y) = \mathrm{Ext}^{2-n}_{\mathcal{C}(G_K)}(Y, X(1))^*$ forment un δ-foncteur. On voit que l'application

$$\mathrm{Ext}^n(X, Y) \to F^n_X(Y)$$

induite par l'accouplement $(a, b) \to c_X(a \cup b)$ est un δ-foncteur.
De même, pour Y fixé, les $G^n_Y(X) = \mathrm{Ext}^{2-n}_{\mathcal{C}(G_K)}(Y, X(1))^*$ forment un δ-foncteur contravariant tandis que l'application

$$\mathrm{Ext}^n(X, Y) \to F^n_X(Y)$$

induite par l'accouplement $(a,b) \to c_Y(b \cup a) = -(1)^n c_X(a \cup b)$ est aussi un δ-foncteur.
Mais, toute presque-C-représentation X est presque isomorphe à C^d pour un entier d convenable, et toute presque C-représentation Y est presqu'isomorphe à un $C(1)^{d'}$ pour un entier d' convenable. Par dévissage, on est alors ramené à vérifier la proposition dans chacun des trois cas suivants
a) X est une $\mathbb{Q}_p$-représentation,
b) Y est une $\mathbb{Q}_p$-représentation,
c) $X = C(1)$ et $Y = C$.
Le cas (a) n'est autre que la proposition 6.8 et l'assertion (i) ramène le cas (b) au cas (a). Dans le cas (c), on a $\mathrm{Ext}^n_{\mathcal{C}(G_K)}(C(1),C) = \mathrm{Ext}^{2-n}_{\mathcal{C}(G_K)}(C,C(2)) = 0$ et il n'y a rien à démontrer. □

7 – PRESQUE C-REPRÉSENTATIONS À PRESQU'ISOMORPHISMES PRÈS

Nous renvoyons par exemple à [Iv87], §XI et à [KS90], §1.6 pour tout ce qui concerne la notion de localisation dans une catégorie additive.
Disons qu'un morphisme f de presque-C-représentations est un *presqu'isomorphisme* si son noyau et son conoyau sont tous deux de dimension finie sur $\mathbb{Q}_p$. Les presqu'isomorphismes forment un système multiplicatif dans la catégorie $\mathcal{C}(G_K)$. On peut donc parler de la catégorie additive $\mathcal{C}_{PI}(G_K)$ localisée de $\mathcal{C}(G_K)$ par rapport aux presqu'isomorphismes.

PROPOSITION 7.1. — *La catégorie quotient $\mathcal{C}_{PI}(G_K)$ des* presque-C-représentations à presqu'isomorphismes près *est une catégorie abélienne semi-simple. Tout objet simple de $\mathcal{C}_{PI}(G_K)$ est isomorphe à C.*

Preuve : Dire que deux objets de $\mathcal{C}_{PI}(G_K)$ sont isomorphes revient à dire qu'ils sont presqu'isomorphes en tant qu'objets de $\mathcal{C}(G_K)$. C'est le cas si et seulement s'ils ont la même dimension. Par conséquent, pour tout objet S de $\mathcal{C}_{PI}(G_K)$, il existe un unique $d \in \mathbb{N}$ tel que S est isomorphe à C^d. On voit aussi que tout endomorphisme non nul de C vu comme objet de $\mathcal{C}_{PI}(G_K)$ est un automorphisme. La proposition en résulte. □

Déterminer complètement - à équivalence de catégories près - la catégorie $\mathcal{C}_{PI}(G_K)$ revient à donc à déterminer le corps gauche $\mathcal{D}_K$ des endomorphismes de C dans cette catégorie. On va décrire sa structure en tant que K-espace vectoriel.
Notons $\mathcal{V}$ l'ensemble des sous-$\mathbb{Q}_p$-espaces vectoriels de dimension finie de C stables par G_K et C_f la réunion (filtrante) des $V \in \mathcal{V}$.

Proposition 7.2. — *Soit* $\mathcal{D}_K = \mathrm{End}_{\mathcal{C}_{PI}(G_K)}(C)$.
i) En tant que K*-espace vectoriel,* $\mathcal{D}_K$ *s'identifie à* $\varinjlim_{V\in\mathcal{V}} \mathrm{Hom}_{\mathcal{C}(G_K)}(C, C/V)$.
ii) Pour tout $V \in \mathcal{V}$*, on dispose d'une suite exacte de* K*-espaces vectoriels*

$$0 \to K \to \mathrm{Hom}_{\mathcal{C}(G_K)}(C, C/V) \to (C \otimes_{\mathbb{Q}_p} V(-1))^{G_K} \to K$$

et l'application de droite est surjective dès que V *est assez grand.*
iii) On dispose d'une suite exacte de K*-espaces vectoriels*

$$0 \to K \to \mathcal{D}_K \to (C \otimes_{\mathbb{Q}_p} C_f((-1))^{G_K} \to K \to 0$$

Preuve : La première assertion paraît claire. Si l'on veut être rigoureux, il est peut-être nécessaire de procéder ainsi : Un élément de $\mathcal{D}_K$ est la classe d'un couple

$$C \xleftarrow{\pi} E \xrightarrow{\varphi} C$$

de morphismes de $\mathcal{C}(G_K)$, où π est un presqu'isomorphisme. Ceci implique (th.5.1) que $d(E) = 1$, que π est surjective et que son noyau V' est de dimension finie sur $\mathbb{Q}_p$. Ceci permet de voir E comme une extension de C par V' et π comme la projection de E sur C. Si $V = \varphi(V')$, l'application φ induit par passage au quotient un morphisme $f : C \to C/V$. Si maintenant $C \xleftarrow{\pi_1} E \xrightarrow{\varphi_1} C$ et $C \xleftarrow{\pi_2} E \xrightarrow{\varphi_2} C$ sont des diagrammes de $\mathcal{C}(G_K)$ avec π_1 et π_2 des presqu'isomorphismes, ils définissent, par la recette précédente des sous-$\mathbb{Q}_p$-espaces vectoriels de dimension finie, stables par G_K, de C et des morhismes $f_1 : C \to C/V_1$ et $f_2 : C \to C/V_2$. On voit facilement que la classe de (π_1, φ_1) est égale à celle de (π_2, φ_2) si et seulement le carré

$$\begin{array}{ccc} C & \xrightarrow{f_1} & C/V_1 \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle \text{proj.can.}} \\ C/V_2 & \xrightarrow[\text{proj.can.}]{} & C/(V_1 + V_2) \end{array}$$

est commutatif. D'où (i).
Soit $V \in \mathcal{V}$. Comme $\mathrm{Hom}_{\mathcal{C}(G_K)}(C, V) = 0$ (cor. au th.3.11) et $\mathrm{End}_{\mathcal{C}(G_K)}(C) = K$ (th.3.11), la suite exacte

$$0 \to V \to C \to C/V \to 0$$

induit une suite exacte

$$0 \to K \to \mathrm{Hom}_{\mathcal{C}(G_K)}(C, C/V) \to \mathrm{Ext}^1_{\mathcal{C}(G_K)}(C, V)) \to \mathrm{Ext}^1_{\mathcal{C}(G_K)}(C, C)$$

Rappelons (prop.3.12) que $\mathrm{Ext}^1_{\mathcal{C}(G_K)}(C, V)$ s'identifie à $\mathrm{Hom}_{\mathcal{C}(G_K)}(C, C \otimes_{\mathbb{Q}_p} V(-1)) \simeq (C \otimes_{\mathbb{Q}_p} V(-1))^{G_K}$. Comme $\mathrm{Ext}^1_{\mathcal{C}(G_K)}(C, C)$ est un K-espace vectoriel de dimension 1 de base la classe c_2 de C_2, l'application $K \to \mathrm{Ext}^1_{\mathcal{C}(G_K)}(C, C)$

qui envoie λ sur λc_2 identifie K à $\mathrm{Ext}^1_{\mathcal{C}(G_K)}(C,C)$. On a donc bien une suite exacte

$$0 \to K \to \mathrm{Hom}_{\mathcal{C}(G_K)}(C, C/V) \to (C \otimes_{\mathbb{Q}_p} V(-1))^{G_K} \to K$$

Dans le cas particulier où V est une extension non triviale de $\mathbb{Q}_p(1)$ par $\mathbb{Q}_p$, on voit (prop.3.10) que l'application $(C \otimes_{\mathbb{Q}_p} V(-1))^{G_K} \to K$ est un isomorphisme. Si $V_1 \subset V_2$ sont dans $\mathcal{V}$, le diagramme

$$\begin{array}{ccccccccc} 0 \to & & \to \mathrm{Hom}_{\mathcal{C}(G_K)}(C, C/V_1) & \to & (C \otimes_{\mathbb{Q}_p} V_1(-1))^{G_K} & \to & K & \to 0 \\ & \| & \downarrow & & \downarrow & & \| & \\ 0 \to & K & \to \mathrm{Hom}_{\mathcal{C}(G_K)}(C, C/V_2) & \to & (C \otimes_{\mathbb{Q}_p} V_2(-1))^{G_K} & \to & K & \to 0 \end{array}$$

est commutatif. Par conséquent, pour que l'application $(C \otimes_{\mathbb{Q}_p} V(-1))^{G_K} \to K$ soit surjective, il suffit que V contienne une extension non triviale de $\mathbb{Q}_p(1)$ par $\mathbb{Q}_p$.
Compte-tenu de la commutativité du diagramme ci-dessus, l'assertion (iii) résulte de (ii) par passage à la limite. □

Remarques : i) J'ignore si l'inclusion $(C \otimes_{\mathbb{Q}_p} C_f(-1))^{G_K} \subset (C \otimes_{\mathbb{Q}_p} C(-1))^{G_K}$ est stricte. ii) On comparera ce théorème avec le résultat analogue dans le contexte des espaces de Banach-Colmez ([Co02], th.9.5).
ii) Pour toute presque-C-représentation X de dimension 1, le K-espace vectoriel $\mathcal{D}_K$ s'identifie à $\mathrm{Hom}_{\mathcal{C}_{PI}(G_K)}(C, X)$. Notons $\mathcal{V}_X$ l'ensemble des sous-$\mathbb{Q}_p$-espaces vectoriels de dimension finie de X stables par G_K et X_f la réunion (filtrante) des $V \in \mathcal{V}_X$. Le même raisonnement que précédemment montre que $\mathcal{D}_K = \varinjlim_{V \in \mathcal{V}_X} \mathrm{Hom}_{\mathcal{C}(G_K)}(C, X/V)$. Si X n'est isomorphe ni à C, ni à $C(1)$, pour tout $V \in \mathcal{V}_X$, l'application $\mathrm{Hom}_{\mathcal{C}(G_K)}(C, X/V) \to \mathrm{Ext}^1_{\mathcal{C}(G_K)}(C, V) \simeq (C \otimes_{\mathbb{Q}_p} V(-1))^{G_K}$ est un isomorphisme. Par passage à la limite, on en déduit un isomorphisme de K-espaces vectoriels

$$\mathcal{D}_K \to (C \otimes_{\mathbb{Q}_p} X_f)^{G_K} .$$

8 – Extensions universelles

8.1 – B^+_{dR}–représentations triviales

Proposition 8.1. — *Soit W une B^+_{dR}-représentation de G_K. Les propriétés suivantes sont équivalentes :*
i) l'application B^+_{dR}-linéaire $B^+_{dR} \otimes_K W^{G_K} \to W$ déduite par extension des scalaires de l'inclusion de W^{G_K} dans W est surjective,
ii) le sous-$\overline{K}$-espace vectoriel W^{disc} de W formé des $w \in W$ dont le fixateur $G_w = \{g \in G_K \mid g(w) = w\}$ est ouvert dans G_K est dense dans W,

iii) il existe une suite d'entiers $r_1, r_2, \ldots, r_m, \ldots$ naturels, presque tous nuls, et un isomorphisme de B_{dR}^+-représentations $\oplus_{m\geq 1} B_m^{r_m} \simeq W$.

Preuve : Pour tout $\overline{K}$-espace vectoriel X muni d'une action semi-linéaire discrète de G_K, l'application $\overline{K}$-linéaire

$$\overline{K} \otimes_K X^{G_K} \to X$$

déduite par extension des scalaires de l'inclusion de X^{G_K} dans X est un isomorphisme : lorsque X est de dimension finie, cela résulte de ce que, pour tout $d \in \mathbb{N}$, $H^1(G_K, GL_d(\overline{K}))$ est trivial. Le cas général s'en déduit par passage à la limite.
Ceci implique que W^{disc} s'identifie à $\overline{K} \otimes_K W^{G_K}$. L'équivalence de (i) et (ii) résulte alors de ce que $\overline{K}$ est dense dans B_{dR}^+ (c'est un résultat de Colmez, cf. [Fo88a], appendice).
L'implication (iii)⇒(i) est immédiate. Montrons la réciproque : Pour tout $m \in \mathbb{N}$, notons $F_m W^{G_K}$ le sous-K-espace vectoriel de W^{G_K} formé des x tels que $t^m x = 0$. On a $F_{m-1} W^{G_K} \subset F_m W^{G_K}$ pour tout m, $F_0 W^{G_K} = 0$ et $F_m W^{G_K} = W^{G_K}$ pour m assez grand. Pour chaque $m \geq 1$, choisissons des $(e_{m,i})_{1\leq i\leq r_m}$ dans $F_m W^{G_K}$ qui relèvent une base de $F_m W^{G_K}/F_{m-1} W^{G_K}$. Si W vérifie (i), l'application $\oplus_{m\leq 1} B_m^{r_m} \to W$ qui envoie $(b_{m,i})_{m\geq 1, 1\leq i\leq r_m}$ sur $\sum b_{m,i} e_{m,i}$ est un isomorphisme. □

On dira qu'une B_{dR}^+-représentation W est *triviale* si elle vérifie les propriétés équivalentes de la proposition précédente. Lorsque W est une C-représentation, on retrouve la définition donnée au §1.3.

8.2 – Extensions universelles par des représentations p-adiques

Reprenons les notations du §3.4. Pour toute représentation p-adique V, posons $E_e(V) = B_e \otimes_{\mathbb{Q}_p} V$ et, pour tout $m \in \mathbb{N}$, $E_m(V) = \mathrm{Fil}^{-m} B_e \otimes_{\mathbb{Q}_p} V$. On a des suites exactes

$$0 \to V \to E_e(V) \to (B_{dR}/B_{dR}^+) \otimes_{\mathbb{Q}_p} V \to 0$$
$$\text{et} \quad 0 \to V \to E_m(V) \to B_m(-m) \otimes_{\mathbb{Q}_p} V \to 0$$

Chaque $E_m(V)$ est une presque C-représentation, extension d'une *B_m-représentation* (i.e. une B_{dR}^+-représentation tuée par t^m) par V tandis que $E_e(V)$ est la réunion croissante des $E_m(V)$. En particulier $E_e(V)$ a une structure naturelle de $\mathbb{Q}_p$-espace vectoriel topologique (c'est une limite inductive de banach) et l'action de G_K est continue.
La proposition 3.13 signifie que $E_m(V)$ est l'*extension universelle d'une B_m-représentation par V*, i.e. qu'étant donnée une suite exacte

$$0 \to V \to E \to W \to 0$$

de $\mathcal{B}(G_K)$, avec W une B_m-représentation, il existe un et un seul morphisme de $\mathcal{B}(G_K)$ (ou un morphisme de B_m-représentations, cela revient au même d'après le théorème 3.11)

$$f : W \to B_m(-m) \otimes_{\mathbb{Q}_p} V$$

tel que $E = E_m(V) \times_{B_m(-m)\otimes V} W$. Autrement dit, $E_m(V)$ est solution du *problème universel des extensions de B_m-représentations par V* : étant donnée une telle extension il existe une et une seule application $\mathbb{Q}_p$-linéaire continue G_K-équivariante de E dans $E_m(V)$ qui est l'identité sur V. Par passage à la limite on voit que $E_e(V)$ est une limite inductive de représentations banachiques caractérisée à isomorphisme unique près par le fait qu'étant donné une représentation banachique E, extension d'une B_{dR}^+-représentation par V, il existe une et une seule application $\mathbb{Q}_p$-linéaire continue G_K-équivariante de E dans $E_e(V)$ qui est l'identité sur V.

Pour toute représentation p-adique V de G_K, on appelle *espace tangent de V* le K-espace vectoriel $t_V = ((B_{dR}/B_{dR}^+) \otimes_{\mathbb{Q}_p} V)^{G_K}$. Soit $m \in \mathbb{Z}$. Posons $\mathrm{Fil}^m t_V = 0$ si $m \geq 0$ et $\mathrm{Fil}^m t_V = ((\mathrm{Fil}^m B_{dR}/B_{dR}^+) \otimes_{\mathbb{Q}_p} V)^{G_K}$ sinon. Si $\mathrm{Gr}^m t_V = \mathrm{Fil}^m t_V/\mathrm{Fil}^{m+1} t_V$, on a $\mathrm{Gr}^m t_V = 0$ pour $m \geq 0$ tandis que, pour $m < 0$, $\mathrm{Gr}^m t_V$ s'identifie à un sous-K-espace vectoriel de $(C(m) \otimes_{\mathbb{Q}_p} V)^{G_K}$. Par conséquent, $\mathrm{Gr}\ t_V = \oplus_{m\in\mathbb{Z}} \mathrm{Gr}^m t_V$ s'identifie à un sous-K-espace vectoriel de $D_{HT}(V) = \oplus_{m\in\mathbb{Z}} (C(m) \otimes_{\mathbb{Q}_p} V)^{G_K}$. Comme $D_{HT}(V)$ est un K-espace vectoriel de dimension finie inférieure ou égale à la dimension h de V sur $\mathbb{Q}_p$, il en est de même de t_V. Lorsque V est de de Rham, c'est-à-dire lorsque le K-espace vectoriel $D_{dR}(V) = (B_{dR} \otimes_{\mathbb{Q}_p} V)^{G_K}$ est de dimension h, on voit, pour des raisons de dimension, que la suite

$$0 \to (B_{dR}^+ \otimes_{\mathbb{Q}_p} V)^{G_K} \to D_{dR}(V) \to t_V \to 0$$

est exacte, ce qui fait que l'on retrouve la définition habituelle de l'espace tangent ([FPR94], §2.2).

Pour toute K-algèbre Λ, notons $t_V(\Lambda)$ le Λ-module $\Lambda \otimes_K t_V$. L'inclusion de t_V dans $(B_{dR}/B_{dR}^+) \otimes V$ induit une application $\overline{K}$-linéaire $t_V(\overline{K}) \to (B_{dR}/B_{dR}^+) \otimes_{\mathbb{Q}_p} V$ et une application B_{dR}^+-linéaire $t_V(B_{dR}^+) \to (B_{dR}/B_{dR}^+) \otimes_{\mathbb{Q}_p} V$. On voit que la première est injective - ce qui nous permet d'identifier $t_V(\overline{K})$ à un sous-$\overline{K}$-espace vectoriel de $(B_{dR}/B_{dR}^+) \otimes V$ stable par G_K - et que l'image de la seconde, que nous notons $\widehat{t}_V(\overline{K})$ s'identifie à l'adhérence de $t_V(\overline{K})$ dans $(B_{dR}/B_{dR}^+) \otimes V$; c'est aussi la plus grande sous-B_{dR}^+-représentation triviale de $(B_{dR}/B_{dR}^+) \otimes V$.

On note $E_+(V)$ (resp. $E_{\mathrm{disc}}(V)$) l'image inverse de $\widehat{t}_V(\overline{K})$ (resp. $t_V(\overline{K})$) dans $E_e(V)$ de sorte que l'on a un diagramme commutatif dont les lignes sont exactes

$$\begin{array}{ccccccccc}
0 & \to & V & \to & E_{\mathrm{disc}}(V) & \to & t_V(\overline{K}) & \to & 0 \\
 & & \| & & \cap & & \cap & & \\
0 & \to & V & \to & E_+(V) & \to & \widehat{t}_V(\overline{K}) & \to & 0 \\
 & & \| & & \cap & & \cap & & \\
0 & \to & V & \to & E_e(V) & \to & (B_{dR}/B_{dR}^+) \otimes V & \to & 0
\end{array}$$

On voit aussi que $E_+(V)$ est, en un sens évident, *l'extension universelle* d'une B_{dR}^+-représentation triviale par V.

Enfin, en prenant les éléments fixe par G_K, chacune de ces trois suites exactes donne la même suite exacte

$$0 \to V^{G_K} \to E_e(V)^{G_K} \to t_V \to H^1_{\mathrm{cont}}(K,V)$$

L'application $\exp_{BK} : t_V \to H^1_{\mathrm{cont}}(K,V)$ ainsi définie n'est autre, lorsque V est une représentation de de Rham, que *l'exponentielle de Bloch-Kato* ([BK90], def.3.10) et son image est le sous-K-espace vectoriel $H^1_e(K,V)$.

Exercice : Soit $m \in \mathbb{Z}$. Alors, si $m \leq 0$, $E_+(\mathbb{Q}_p(m)) = \mathbb{Q}_p(m)$. Si $m > 0$, $E_+(\mathbb{Q}_p(m))$ est une extension de B_m par $\mathbb{Q}_p(m)$ et s'identifie au sous-espace de B_{cris} formé des b tels que $\varphi(b) = p^m b$.

Remarque : Soient V une représentation p-adique et posons

$$\mathrm{Hom}(B^+_{dR}, (B_{dR}/B^+_{dR}) \otimes_{\mathbb{Q}_p} V) = \varprojlim_{m\in\mathbb{N}} \varinjlim_{n\in\mathbb{N}} \mathrm{Hom}_{\mathcal{C}(G_K)}(B_m, B_n(-n) \otimes_{\mathbb{Q}_p} V)$$

On a $\mathrm{Hom}(B^+_{dR}, (B_{dR}/B^+_{dR}) \otimes_{\mathbb{Q}_p} V) = \mathrm{Hom}_{\mathcal{C}(G_K)}(B_m, B_n(-n) \otimes_{\mathbb{Q}_p} V)$ pour m et n suffisamment grands et le théorème de pleine fidélité implique que tout $f \in \mathrm{Hom}(B^+_{dR}, (B_{dR}/B^+_{dR}) \otimes_{\mathbb{Q}_p} V)$ est une application B^+_{dR}-linéaire. Elle est donc déterminée par l'image de 1 qui doit être fixe par G_K. Ce K-espace vectoriel s'identifie donc à l'espace tangent t_V.
En utilisant l'inclusion de $\mathbb{Q}_p$ dans B^+_{dR}, on obtient en appliquant le foncteur $\mathrm{Hom}(B^+_{dR}, --)$ et le foncteur $\mathrm{Hom}(\mathbb{Q}_p, --)$ à la suite exacte

$$0 \to V \to B_e \otimes V \to (B_{dR}/B^+_{dR}) \otimes V \to 0$$

un carré commutatif

$$\begin{array}{ccc} t_V & \xrightarrow{\simeq} & \mathrm{Ext}^1(B^+_{dR}, V) \\ \| & & \downarrow \\ t_V & \xrightarrow{\exp_{BK}} & \mathrm{Ext}^1_{\mathcal{C}(G_K)}(\mathbb{Q}_p, V) \end{array}$$

Donc $H^1_e(K,V)$ *s'identifie au sous-groupe de* $H^1(K,V) = \mathrm{Ext}^1_{\mathcal{C}(G_K)}(\mathbb{Q}_p, V)$ *formé des extensions de* $\mathbb{Q}_p$ *par* V *qui proviennent d'une extension de* B^+_{dR} *par* V.

Appelons $K_0[\varphi, N]$-*module* (ou (φ, N)-*module sur* k) la donnée d'un K_0-espace vectoriel D muni de deux applications $\varphi : D \to D$, $N : D \to D$, la première semi-linéaire relativement au Frobenius absolu σ agissant sur K_0, la deuxième K_0-linéaire, vérifiant $N\varphi = p\varphi N$. Rappelons ([Fo86a],§3) que B_{st} est une sous-K_0-algèbre de B_{dR} contenant B_{cris}, stable par G_K, que l'endomorphisme φ de B_{cris} s'étend en un endomorphisme, encore noté φ,toujours G_K équivariant, de B_{st} et que B_{st} est équippé d'une B_{cris}-dérivation $N : B_{st} \to B_{st}$ également G_K-équivariante et vérifiant $N\varphi = p\varphi N$.

Pour toute représentation p-adique V de G_K, $D_{st}(V) = (B_{st} \otimes_{\mathbb{Q}_p} V)^{G_K}$ est de façon naturelle un $K_0[\varphi, N]$-module. Sa dimension en tant que K_0-espace vectoriel est inférieure ou égale à la dimension de V sur $\mathbb{Q}_p$; on dit que V est *semi-stable* si l'on a l'égalité ([Fo86b], §5).

Appelons *B_e-représentation (de G_K)* la donnée d'un B_e-module libre de rang fini muni d'une action semi-linéaire continue de G_K.

Proposition 8.2. — *Soient V_1 et V_2 deux représentations p-adiques de même dimension sur $\mathbb{Q}_p$. Supposons V_1 semi-stable. Les conditions suivantes sont équivalentes :*
i) les (φ, N)-modules $D_{st}(V_1)$ et $D_{st}(V_2)$ sont isomorphes,
ii) les B_e-représentations $E_e(V_1) \to E_e(V_2)$ sont isomorphes.

Remarquons que (i) implique que V_2 est aussi semi-stable.

Preuve : Pour $i = 1, 2$, $B_{st} \otimes_{\mathbb{Q}_p} V_i$ est muni d'une structure de (φ, N)-module en posant $\varphi(b \otimes v) = \varphi(b) \otimes v$ et $N(b \otimes v) = N(b) \otimes v$ si $b \in B_{st}$ et $v \in V_i$ et $D_{st}(V_i)$ est le sous-(φ, N)-module $(B_{st} \otimes V_i)^{G_K}$ de $B_{st} \otimes V_i$. On a $B_{st} \otimes_{\mathbb{Q}_p} V_i = B_{st} \otimes_{B_e} (B_e \otimes_{\mathbb{Q}_p} V_i) = B_{st} \otimes_{B_e} E_e(V_i))$. Comme $\varphi(b) = b$ et $Nb = 0$ pour tout $b \in B_e$, on a $\varphi(x) = x$ et $N(x) = 0$ pour tout $x \in E_e(V_i)$. Comme l'action de φ sur B_{st} est un endomorphisme de la structure d'anneau tandis que celle de N est une dérivation, on a

$$\varphi(b \otimes x) = \varphi(b) \otimes x \text{ et } N(b \otimes x) = Nb \otimes x \text{ quelque soient } b \in B_{st} \text{ et } x \in E_e(V_i)$$

Par conséquent, toute application B_e-linéaire G_K-équivariant bijective de $E_e(V_1)$ sur $E_e(V_2)$ induit, par extension des scalaires à B_{st} un isomorphisme de B_{st}-modules de $B_{st} \otimes_{\mathbb{Q}_p} V_1$ sur $B_{st} \otimes_{\mathbb{Q}_p} V_2$ compatible avec l'action de G_K, celle de φ et celle de N. En prenant les invariants sous G_K, on en déduit un isomorphisme (de (φ, N)-modules) de $D_{st}(V_1)$ sur $D_{st}(V_2)$.
Réciproquement, si les (φ, N)-modules $D_{st}(V_1)$ et $D_{st}(V_2)$ sont isomorphes, on a $\dim_{K_0} D_{st}(V_2) = \dim_{K_0} D_{st}(V_1) = \dim_{\mathbb{Q}_p} V_1 = \dim_{\mathbb{Q}_p} V_2$ et V_2 est aussi semi-stable. Pour $i = 1, 2$, $B_{st} \otimes_{\mathbb{Q}_p} V_i$ s'identifie donc à $B_{st} \otimes_{K_0} D_{st}(V_i)$ ([Fo86b], th.5.3.5) et on dispose donc d'un isomorphisme de B_{st}-modules de $B_{st} \otimes_{\mathbb{Q}_p} V_1$ sur $B_{st} \otimes_{\mathbb{Q}_p} V_2$ compatible avec l'action de G_K, celle de φ et celle de N. En prenant la partie sur laquelle $\varphi = 1$ et $N = 0$, on trouve un isomorphisme de $E_e(V_1)$ sur $E_e(V_2)$. □

Remarque : Il serait intéressant d'étudier plus en détail les B_e-représentations. En particulier, on peut se demander si le foncteur d'oubli de la catégorie des B_e-représentations dans celle des $\mathbb{Q}_p$-espaces vectoriels topologiques avec action linéaire et continue de G_K n'est pas pleinement fidèle. On a en tout cas le résultat suivant (en se fatigant un peu plus, on devrait pouvoir éviter d'avoir à remplacer K_0 par K_0' et G_K par $G_{K'}$) :

Proposition 8.3. — *Soient V_1 et V_2 deux représentations p-adiques. Supposons V_1 semi-stable et $\dim_{\mathbb{Q}_p} V_2 \leq \dim_{\mathbb{Q}_p} V_1$. Supposons qu'il existe une application $\mathbb{Q}_p$-linéaire G_K-équivariante injective $E_e(V_1) \to E_e(V_2)$. Alors*
i) les $\mathbb{Q}_p$-espaces vectoriels V_1 et V_2 ont la même dimension et V_2 est semi-stable ;
ii) il existe une extension finie non ramifiée K_0' de K_0 telle que les $K_0'[\varphi, N]$-modules $K_0' \otimes_{K_0} D_{st}(V_1)$ et $K_0' \otimes_{K_0} D_{st}(V_2)$ sont isomorphes ;
iii) il existe un sous-groupe ouvert $G_{K'}$ de G_K contenant le groupe d'inertie et une application B_e-linéaire bijective $G_{K'}$-équivariante de $E_e(V_1)$ dans $E_e(V_2)$.

Preuve : Soit h le ppcm des dénominateurs des pentes de $D_{st}(V_1)$ (vu comme un φ-isocristal). Rappelons les propriétés dont nous allons avoir besoin du sous-anneau B_{st}^h de B_{st} introduit dans [Fo00], §5.5 : Notons P_0 le corps des fractions des vecteurs de Witt à coefficients dans le corps résiduel de $\overline{K}$; c'est un sous-corps de B_{cris}, stable par G_K et par l'action de φ (qui opère via le Frobenius absolu) ; l'homomorphisme de $\mathbb{Q}_p$-algèbres $P_0 \otimes_{\mathbb{Q}_p} B_e \to B_{cris}$ est injectif et identifie $P_0 \otimes_{\mathbb{Q}_p} B_e$ à un sous-anneau de B_{cris} noté B_0. Soit t_h un élément de B_{cris} vérifiant $t_h \in \mathrm{Fil}^1 B_{dR}$, $\varphi^n t_h \in B_{dR}^+$ pour $1 \leq n \leq h-1$ et $\varphi^h(t_h) = pt_h$. Alors t_h est inversible dans B_{cris}, on a $g(t_h)/t_h \in P_0$ pour tout $g \in G_K$ et la sous-B_0-algèbre de B_{cris} engendrée par t_h et $1/t_h$ s'identifie à l'algèbre des polynômes de Laurent en t_h à coefficients dans B_0. Notons u l'élément de B_{st} noté $\log[\pi]$ dans *loc.cit.* ; on a $g(u) - u \in \mathbb{Z}_p(1) = \mathbb{Z}_p t$, pour tout $g \in G_K$, $\varphi(u) = pu$ et $Nu = -1$. Alors B_{st}^h s'identifie à l'anneau des polynômes en u à coefficients dans B_0, donc aussi à $P_0[t_h, 1/t_h, u] \otimes_{\mathbb{Q}_p} B_e$. Enfin le fait que $h\alpha \in \mathbb{Z}$ pour toute pente α de $D_{st}(V_1)$ implique que $D_{st}(V_1) = (B_{st}^h \otimes_{\mathbb{Q}_p} V_1)^{G_K}$. Pour $i = 1, 2$, on a $B_e \otimes_{\mathbb{Q}_p} V_i = E_e(V_i)$ et on dispose donc d'une application $\mathbb{Q}_p$-linéaire injective G_K-équivariante $B_e \otimes_{\mathbb{Q}_p} V_1 \hookrightarrow B_e \otimes_{\mathbb{Q}_p} V_2$. D'où par extension des scalaires une application $P_0[t_h, 1/t_h, u]$-linéaire injective $B_{st}^h \otimes_{\mathbb{Q}_p} V_1 \hookrightarrow B_{st}^h \otimes_{\mathbb{Q}_p} V_2$. Comme le sous-anneau $P_0[t_h, 1/t_h, u]$ de B_{st} est stable par G_K, cette application est aussi G_K-équivariante. Pour les mêmes raisons, elle commute aussi à l'action de φ^h et à celle de N. En prenant les invariants sous Galois, on obtient une application K_0-linéaire injective

$$D_{st}(V_1) \to (B_{st}^h \otimes_{\mathbb{Q}_p} V_2)^{G_K} \subset D_{st}(V_2)$$

Comme $\dim_{K_0} D_{st}(V_2) \leq \dim_{K_0}$, on en déduit que V_1 et V_2 ont la même dimension sur $\mathbb{Q}_p$, que V_2 est semi-stable et que l'application ci-dessus est une bijection. D'où (i).
Ceci implique aussi qu'il existe un isomorphisme de $K_0[\varphi^h, N]$-modules de $D_{st}(V_1)$ sur $D_{st}(V_2)$.
Supposons d'abord que h divise $[K_0 : \mathbb{Q}_p]$ et montrons qu'alors il existe aussi un isomorphisme de $K_0[\varphi, N]$-modules de $D_{st}(V_1)$ sur $D_{st}(V_2)$. Cela résulte de la conjonction des deux faits suivants (on a noté $\mathbb{Q}_{p^h}$ l'unique extension de $\mathbb{Q}_p$ de degré h contenue dans K_0) :
a) la flèche évidente $\mathbb{Q}_{p^h} \otimes_{\mathbb{Q}_p} \mathrm{Hom}_{K_0[\varphi,N]}(D_1, D_2) \to \mathrm{Hom}_{K_0[\varphi^h,N]}(D_1, D_2)$ est surjective (bien sûr elle est aussi injective) ;

b) si L est un sous-$\mathbb{Q}_p$-espace vectoriel de $\mathcal{L}_{\mathbb{Q}_{p^h}}(D_1, D_2)$ tel que le sous-$\mathbb{Q}_{p^h}$-espace vectoriel de $\mathcal{L}_{\mathbb{Q}_{p^h}}(D_1, D_2)$ engendré par L contient un isomorphisme, alors L aussi.

Pour le premier, on remarque que, si $f \in \mathrm{Hom}_{K_0[\varphi^h,N]}(D_1, D_2)$, l'application $\theta(f) : D_1 \to D_2$ définie par $\theta(f) = \sum_{i=0}^{h-1} \varphi^i f \varphi^{-i}$ est en fait dans $\mathrm{Hom}_{K_0[\varphi,N]}(D_1, D_2)$; si $b_1, b_2, \ldots, b_h$ est une base de $\mathbb{Q}_{p^h}$ sur $\mathbb{Q}_p$, la matrice des $(\sigma^i(b_r))_{0 \le i < h, 1 \le r \le h}$ est inversible, il existe donc $a_1, a_2, \ldots, a_h \in \mathbb{Q}_{p^h}$ tels que

$$\sum a_r b_r = 1 \text{ et } \sum a_r \sigma^i(b_r) = 0 \text{ pour} 1 \le i < h$$

On a $f = \sum_{r=1}^{h} a_r \theta(b_r f)$.

L'assertion (b) qui n'utilise que le fait que le corps $\mathbb{Q}_p$ a une infinité d'éléments, que $\mathbb{Q}_{p^h}$ est une extension finie sur $\mathbb{Q}_p$ et que D_1 et D_2 sont de dimension finie sur $\mathbb{Q}_{p^h}$ est bien connue (cf., par exemple [Fo00], lemme 2.7).

Dans le cas général enfin, il suffit de prendre pour K_0' la plus petite extension non ramifiée de K_0 contenue dans $\overline{K}$ telle que h divise $[K_0' : \mathbb{Q}_p]$. Ceci termine la preuve de (ii). L'assertion (iii) résulte alors de la proposition précédente où l'on remplace K par l'extension finie non ramifiée $K\mathbb{Q}_{p^h}$. □

8.3 – Extensions universelles par des représentations de torsion

Appelons *groupe abélien de cotype fini* tout groupe abélien de torsion Λ tel que le $\widehat{\mathbb{Z}}$-module des homomorphismes de Λ dans $\mathbb{Q}/\mathbb{Z}$ est de type fini. Si Λ est un groupe de p-torsion, il est de cotype fini si et seulement si d'une part sa partie divisible Λ_{div} est isomorphe à $(\mathbb{Q}_p/\mathbb{Z}_p)^h$ pour un entier h convenable et d'autre part $\Lambda/\Lambda_{\mathrm{div}}$ est un groupe fini.

Une *représentation de cotype fini (de G_K)* est un groupe abélien de cotype fini muni d'une action linéaire discrète de G_K.

Soit Λ une représentation de cotype fini. On pose

$$T_p(\Lambda) = \varprojlim_{n \in \mathbb{N}} \Lambda_{p^n} \text{ et } V_p(\Lambda) = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_p(\Lambda)$$

Alors $T_p(\Lambda)$ est un $\mathbb{Z}_p$-module libre de rang fini, avec action linéaire et continue de G_K et $V_p(\Lambda)$ est une représentation p-adique de G_K. Le quotient $V_p(\Lambda)/T_p(\Lambda)$ s'identifie à la partie divisible $\Lambda_{(p),\mathrm{div}}$ du sous-groupe de p-torsion $\Lambda_{(p)}$ de Λ.

Pour ? désignant un symbole qui est $e, +,$ disc ou un entier $m \ge 0$, $E_?(V_p(\Lambda))$ est un $\mathbb{Q}_p$-espace vectoriel topologique avec action linéaire et continue de G_K contenant $T_p(\Lambda)$ comme un sous-groupe fermé stable par G_K. Le quotient $E_?(V_p(\Lambda))/T_p(\Lambda)$ est donc un $\mathbb{Z}_p$-module topologique avec action linéaire et continue de G_K, contenant $\Lambda_{(p),\mathrm{div}}$ comme sous-groupe discret. On pose alors

$$E_?(\Lambda) = \Lambda \oplus_{\Lambda_{(p),\mathrm{div}}} (E_?(V_p(\Lambda))/T_p(\Lambda))$$

somme amalgamée de Λ et de $E_?(V)$ au-dessous de $\Lambda_{(p),\mathrm{div}}$. Bien sûr, cette construction n'a d'intérêt que pour un groupe de p-torsion puisque, si $\Lambda_{(p')}$

désigne le sous-groupe de Λ formé des éléments d'ordre premier à p, on a $E_?(\Lambda) = \Lambda_{(p')} \oplus E_?(\Lambda_{(p)})$.

Posons aussi $t_\Lambda = t_{V_p(\Lambda)}$ et $\widehat{t}_\Lambda(\overline{K}) = \widehat{t}_{V_p(\Lambda)}(\overline{K})$. On a un diagramme commutatif

$$\begin{array}{ccccccccc}
0 & \to & \Lambda & \to & E_{\mathrm{disc}}(\Lambda) & \to & t_\Lambda(\overline{K}) & \to & 0 \\
 & & \| & & \cap & & \cap & & \\
0 & \to & \Lambda & \to & E_+(\Lambda) & \to & \widehat{t}_\Lambda(\overline{K}) & \to & 0 \\
 & & \| & & \cap & & \cap & & \\
0 & \to & \Lambda & \to & E_e(\Lambda) & \to & (B_{dR}/B_{dR}^+) \otimes V_p(\Lambda) & \to & 0 \\
 & & \| & & \cup & & \cup & & \\
0 & \to & \Lambda & \to & E_{m(\Lambda)} & \to & B_m(-m) \otimes V_p(\Lambda) & \to & 0
\end{array}$$

dont les lignes sont exactes.

Proposition 8.4. — *Soit Λ une représentation de cotype fini de G_K. Alors $E_{\mathrm{disc}}(\Lambda)$ est le plus grand sous-groupe stable par G_K de $E_e(\Lambda)$ sur lequel l'action de G_K est discrète tandis que $E_+(\Lambda)$ est l'adhérence de $E_{\mathrm{disc}}(\Lambda)$ dans $E_e(\Lambda)$.*

Preuve : Pour tout $m \in \mathbb{N}$, notons $E_{m,\mathrm{disc}}(\Lambda)$ le plus grand sous-groupe stable par G_K de $E_m(\Lambda)$ sur lequel l'action de G_K est discrète. Comme l'action de G_K est discrète sur Λ, c'est l'image inverse du plus grand sous-groupe de $B_m(-m) \otimes V_p(\Lambda)$ sur lequel l'action de G_K est discrète ; ce sous-groupe s'identifie à $\overline{K} \otimes_K (B_m(-m) \otimes V_p(\Lambda))^{G_K}$ (prop.8.1). Par passage à la limite, on en déduit que le plus grand sous-groupe de $E_e(\Lambda)$ stable par G_K sur lequel l'action de G_K est discrète est l'image inverse de $\overline{K} \otimes_K ((B_{dR}/B_{dR}^+) \otimes V_p(\Lambda))^{G_K} = t_\Lambda(\overline{K})$. La deuxième assertion est évidente. □

On a encore des propriétés d'universalité :

Proposition 8.5. — *Soient Λ une représentation de cotype fini de G_K et E un groupe topologique abélien muni d'une action linéaire et continue de G_K, extension d'une B_{dR}^+-représentation W par Λ. Alors il existe une et une seule application linéaire continue G_K-équivariante $f : E \to E_e(\Lambda)$ qui est l'identité sur Λ. Son image est contenue dans $E_+(\Lambda)$ (resp. $E_m(\Lambda)$) si et seulement si la B_{dR}^+-représentation W est triviale (resp. annulée par t^m).*

Preuve : Pour tout groupe topologique abélien M notons $V_p(M)$ le groupe des homomorphismes (de groupes) continus de $\mathbb{Q}_p$ dans M. On a bien $V_p(\Lambda) = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_p(\Lambda)$. Si M est un $\mathbb{Q}_p$-espace vectoriel, on a $V_p(M) = M$. On a $V_p(E_e(\Lambda)) = E_e(V_p(\Lambda))$ et on dispose d'une suite exacte courte

$$0 \to V_p(\Lambda) \to V_p(E) \to W \to 0$$

Toute application $f : E \to E_e(V)$ induit donc une application $V_p(f) : V_p(E) \to E_e(V_p(\Lambda))$ et $V_p(f)$ est l'identité sur $V_p(\Lambda)$ si f est l'identité sur Λ.
Inversement, on a $E_e(\Lambda) = \Lambda \oplus_{\Lambda_{(p),\mathrm{div}}} (E_e(V_p(\Lambda))/T_p(\Lambda))$ et E s'identifie à $\Lambda \oplus_{\Lambda_{(p),\mathrm{div}}} ((V_p(E))/T_p(\Lambda))$, ce qui fait que toute flèche $V_p(E) \to E_e(V_p(\Lambda))$ qui

est l'identité sur $V_p(\Lambda)$ induit une flèche $E \to E_e(\Lambda)$. La proposition résulte alors des propriétés universelles mises en évidence au paragraphe précédent. □

8.4 – Applications aux variétés abéliennes, aux groupes de Barsotti-Tate et aux motifs

Proposition 8.6. — *Soient A une variété abélienne sur K, t_A son espace tangent, t'_A le K-espace vectoriel dual de l'espace tangent de la variété abélienne duale, $T_p(A) = \varprojlim_{n\in\mathbb{N}} A_{p^n}(\overline{K})$, $V = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_p(A)$ et $V_C = C \otimes_{\mathbb{Q}_p} V$. On a des isomorphismes canoniques et fonctoriels*
i) $V_C = t_A(C)(1) \oplus t'_A(C)$,
ii) $t_A \simeq t_V = t_{A_{p-\mathrm{tor}}(\overline{K})} = t_{A_{\mathrm{tor}}(\overline{K})}$,
iii) $A(\overline{K}) \simeq E_{\mathrm{disc}}(A_{\mathrm{tor}}(\overline{K}))$ et $A(C) \simeq E_+(A_{\mathrm{tor}}(\overline{K}))$.

Remarque : La première assertion n'est autre que la décomposition de Hodge-Tate pour les variétés abéliennes dont on obtient ainsi une nouvelle preuve (mais c'est loin d'être la plus simple !).

Preuve : Le logarithme est défini partout sur $A(C)$ et induit une suite exacte courte

$$0 \to A_{\mathrm{tor}}(\overline{K}) \to A(C) \to t_A(C) \to 0$$

Mais $t_A(C)$ est une C-représentation triviale de dimension la dimension g de A. D'après la proposition précédente, il existe des applications $\mathbb{Q}_p$-linéaires continues G_K-équivariantes uniques de $\eta : A(C) \to E_+(C)$ et $\bar\eta : t_A(C) \to t_V(C)$ telles que le diagramme

$$\begin{array}{ccccccccc} 0 & \to & A_{\mathrm{tor}}(\overline{K}) & \to & A(C) & \to & t_A(C) & \to & 0 \\ & & \| & & \downarrow \eta & & \downarrow \bar\eta & & \\ 0 & \to & A_{\mathrm{tor}}(\overline{K}) & \to & E_+(A_{\mathrm{tor}}(\overline{K})) & \to & \widehat{t}_V(\overline{K}) & \to & 0 \end{array}$$

est commutatif.
Les applications η et $\bar\eta$ sont injectives. En effet, sinon comme le noyau W de η s'identifie au noyau de $\bar\eta$ qui est une application B^+_{dR}-linéaire, ce serait une sous-B^+_{dR}-représentation non nulle de $t_A(C) \simeq C^g$ et contiendrait donc une sous représentation isomorphe à C ; en prenant les invariants sous G_K, on voit que $A(K)$ contiendrait un sous-groupe isomorphe à K ; mais ceci contredit le fait que $A(K)$ est compact (si $\mathcal{A}$ est un modèle propre de A sur $\mathcal{O}_K$, on a $A(K) = \mathrm{Hom}_{\mathcal{O}_K-\text{schémas}}(\mathrm{Spec}\mathcal{O}_K, \mathcal{A}) = \varprojlim_{n\in\mathbb{N}} \mathrm{Hom}_{\mathcal{O}_K-\text{schémas}}(\mathrm{Spec}(\mathcal{O}_K/p^n\mathcal{O}_K), \mathcal{A})$ et chacun de ces ensembles est fini).
En particulier, $(B_{dR}/B^+_{dR}) \otimes_{\mathbb{Q}_p} V \supset \widehat{t}_V(\overline{K})$ contient une sous-B^+_{dR}-représentation isomorphe à C^g. Celle-ci est contenue dans le noyau de la multiplication par t dans $B_{dR}/B^+_{dR}) \otimes_{\mathbb{Q}_p} V$ qui est $V_C(-1)$, d'où une application injective $t_A(C) \to V_C(-1)$. La transposée de l'application analogue pour la variété abélienne duale nous donne une application surjective $V_C \to t'_A(C)$ ou encore de $V_C(-1)$ dans $t_A(C)(-1)$. Comme il n'y a pas d'application $\mathbb{Q}_p$-linéaire continue

G_K-équivariante non nulle de $t_A(C) \simeq C^g$ dans $t'_A(C)(-1) \simeq C(-1)^g$, l'application composée $t_A(C) \to V_C(-1) \to t_A(C)(-1)$ est nulle. Pour des raisons de dimension, la suite

$$0 \to t_A(C) \to V_C(-1) \to t'_A(C)(-1) \to 0$$

est exacte. Comme, dans la catégorie des C-représentations, il n'y a ni homomorphisme ni extension non trivial(e) de $C(-1)$ dans (par) C, cette suite est scindée de manière unique et $V_C(-1)$ s'identifie, canoniquement et fonctoriellement à $t_A(C) \oplus t'_A(C)(-1)$, d'où (i) en tensorisant par $\mathbb{Q}_p(1)$.
Ceci implique que, pour tout entier $m \notin \{0,-1\}$, on a $V_C(m)^{G_K} = 0$, donc que $t_V = ((B_{dR}/B^+_{dR}) \otimes V)^{G_K} = (V_C(-1))^{G_K} = t_A$ puisque $(t'_A(C)(-1))^{G_K} = 0$, D'où (ii).
En outre, $\widehat{t_V}(\overline{K})$ est l'adhérence de $t_V(\overline{K})$ dans $V_C(-1)$ qui est une C-représentation et est donc $t_V(C)$. L'application $\overline{\eta}$ est donc un isomorphisme et η fournit l'isomorphisme cherché de $A(C)$ sur $E_+(A_{\rm tor}(\overline{K}))$. Enfin $A(\overline{K})$ est la réunion des $A(L)$ pour L parcourant les extensions finies galoisiennes de K contenues dans $\overline{K}$ et $\eta(A(\overline{K}))$ est bien contenu dans $E_{\rm disc}(A_{\rm tor}(\overline{K}))$. Comme l'application $A(\overline{K}) \to t_A(\overline{K})$ est surjective, les lignes du diagramme commutatif

$$\begin{array}{ccccccccc} 0 & \to & A_{\rm tor}(\overline{K}) & \to & A(\overline{K}) & \to & t_A(\overline{K}) & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & A_{\rm tor}(\overline{K}) & \to & E_{\rm disc}(A_{\rm tor}(\overline{K})) & \to & t_V(\overline{K}) & \to & 0 \end{array}$$

sont exactes. Comme les flèches verticales de gauche et de droite sont des isomorphismes, celle du milieu l'est aussi et fournit l'isomorphisme cherché entre $A(\overline{K})$ et $E_{\rm disc}(A_{\rm tor}(\overline{K}))$. □

La proposition 1.1 énoncée dans l'introduction est alors claire : Si ξ est comme dans l'énoncé, l'application $\overline{\xi} : t_A(\overline{K}) \to C(-1) \otimes_{\mathbb{Q}_p} V_p(A)$ induite est encore continue et induit donc une application $\mathbb{Q}_p$-linéaire continue G_K-équivariante du complété $t_A(C)$de $t_A(\overline{K})$ dans $C(-1) \otimes_{\mathbb{Q}_p} V_p(A)$. Cette dernière application est C-linéaire (th.3.11). Le reste résulte des deux propositions précédentes. □

On dispose de résultats analogues pour les groupes de Barsotti-Tate : soit $J = (J_{p^n})_{n\in\mathbb{N}}$ un tel groupe sur l'anneau des entiers $\mathcal{O}_K$ de K ; soient h sa hauteur et d sa dimension. Notons $\widehat{J}$ le groupe formel associé : si A_n désigne l'algèbre affine de J_{p^n}, alors A_n est un $\mathcal{O}_K$-module libre de rang p^{nh}. Alors $\widehat{J} = \mathrm{Spf} A_{\widehat{J}}$, avec $A_{\widehat{J}} = \varprojlim A_n$. C'est une algèbre formellement lisse ; l'algèbre affine de la composante connexe de l'élément neutre est une algèbre de séries formelles en d-variables à coefficients dans $\mathcal{O}_K$. On peut considérer le groupe $\widehat{J}(\mathcal{O}_{\overline{K}})$ (resp. $\widehat{J}(\mathcal{O}_C)$) des homomorphismes continus de la $\mathcal{O}_K$-algèbre $A_{\widehat{J}}$ dans $\mathcal{O}_{\overline{K}}$ (resp. $\mathcal{O}_C$). Le sous-groupe de torsion $\widehat{J}_{tor}(\mathcal{O}_{\overline{K}})$ de $\widehat{J}(\mathcal{O}_{\overline{K}})$ est aussi celui de $\widehat{J}(\mathcal{O}_C)$ et c'est l'union $J(\mathcal{O}_{\overline{K}})$ des $J_{p^n}(\mathcal{O}_{\overline{K}}) = \mathrm{Hom}_{\mathcal{O}_K-\text{algèbres}}(A_n, \mathcal{O}_{\overline{K}})$. En tant que groupe, il est isomorphe à $(\mathbb{Q}_p/\mathbb{Z}_p)^h$. L'espace tangent t_J de J (ou de $\widehat{J}$ si l'on préfère) est le K-espace vectoriel de dimension d dual du K-espace vectoriel des formes différentielles invariantes sur $A_{\widehat{J}}$.

Remarque : La proposition 8.6 s'étend aussi, de façon évidente (aussi bien pour son énoncé que pour sa démonstration), aux 1-*motifs sur K* et aux *groupes de Barsotti-Tate sur $\mathcal{O}_K$*.

Soit maintenant M un *motif* sur K. Pour fixer les idées supposons soit que M est un 1-motif soit que $M = H^m(X)(i)$ où X est une variété propre et lisse sur K, $m \in \mathbb{N}$ et $i \in \mathbb{Z}$. La seule chose importante est de savoir définir le groupe $M_{tor}(\overline{K})$ des points de torsion de M à valeurs dans $\overline{K}$ qui doit être une représentation de cotype fini. Pour M un 1-motif c'est la définition usuelle ([De75], §10), pour $M = H^m(X)(i)$, on pose $M_{\mathrm{tor}}(\overline{K}) = H^m_{ét}(X_{\overline{K}}, \mathbb{Q}/\mathbb{Z})(i)$.
On peut alors associer à M les objets suivants, tous construits à partir de $M_{\mathrm{tor}}(\overline{K})$:
– pour ℓ nombre premier , *la réalisation ℓ-adique de M* est le $\mathbb{Q}_\ell$-espace vectoriel $H_\ell(M)$ des homomorphismes continus de $\mathbb{Q}_\ell$ dans $M_{\mathrm{tor}}(\overline{K})$,
– *l'espace tangent t_M de M* est le K-espace vectoriel $t_{M_{\mathrm{tor}}(\overline{K})} = t_{H_p(M)}$,
– *le groupe $M(\overline{K})$ des points de M à valeurs dans $\overline{K}$* est $E_{\mathrm{disc}}(M_{\mathrm{tor}}(\overline{K}))$,
– *le complété $\widehat{M}(\overline{K})$ de ce groupe des points* est le groupe topologique $E_+(M_{\mathrm{tor}(\overline{K})})$,
– *la presque-C-représentation* $E_+(M)$ définie par $E_+(M) = V_p(\widehat{M}(\overline{K})) = E_+(H_p(M))$.

Alors $M(\overline{K})$ s'identifie au sous-groupe de $\widehat{M}(\overline{K})$ formé des points sur lesquels l'action de G_K est discrète ; c'est un sous-groupe dense de $\widehat{M}(\overline{K})$.
Lorsque M est un 1-motif, on a $\widehat{t}_M(\overline{K}) = t_M(C)$ (en particulier, c'est une C-représentation. Ceci est du au fait que les poids de la représentation $H_p(M)$ sont tous ≤ 1, ce qui ne reste pas vrai en général.

Pour toute extension L de K contenue dans $\overline{K}$, posons

$$M(L) = M(\overline{K})^{\mathrm{Gal}(\overline{K}/L)} = \widehat{M}(\overline{K})^{\mathrm{Gal}(\overline{K}/L)}$$

(lorsque M est une variété abélienne c'est bien le groupe des points de la variété abélienne à valeurs dans L). La suite exacte

$$0 \to M_{\mathrm{tor}}(\overline{K}) \to M(\overline{K}) \to t_M(\overline{K}) \to 0$$

induit une suite exacte

$$0 \to M_{\mathrm{tor}}(K) \to M(K) \to t_M \to H^1_e(K, M_{\mathrm{tor}}(\overline{K})) \to 0$$

en notant $H^1_e(K, M_{\mathrm{tor}}(\overline{K}))$ l'image de t_M dans $H^1(K, M_{\mathrm{tor}}(\overline{K}))$ (remarquer que, si $M_{p'-\mathrm{tor}}(\overline{K})$ est le sous-groupe de $M_{\mathrm{tor}}(\overline{K})$ formé des points d'ordre premier à p, on a $H^1_e(K, M_{p'-\mathrm{tor}}(\overline{K})) = 0$).

Enfin la représentation p-adique $H_p(M)$ est de de Rham et la suite exacte

$$0 \to H_p(M) \to E_+(M) \to \widehat{t}_M(\overline{K}) \to 0$$

induit une suite exacte

$$0 \to H_p(M)^{G_K} \to E_+(M)^{G_K} \to t_M \xrightarrow{\exp_{BK}} H^1_e(K, H_p(M)) \to 0$$

où $\exp_{BK}$ est l'exponentielle de Bloch-Kato.

9 – Principales définitions

1.2 : Banach, réseau, représentation banachique, catégorie exacte, morphisme strict, sous-catégorie stricte, représentation p-adique.
1.3 : C-représentation, C-représentation triviale, B^+_{dR}-représentation.
1.4 : Représentations banachiques presqu'isomorphes, presque C-représentation.
1.5 : Extension presque scindée.
2.1 : $K_\infty[[\underline{t}]]$-représentation de Γ_K.
2.3 : K-petit, petite représentation.
3.1 : Cohomologie continue, presque B^+_{dR}-représentation.
3.5 : Presque supplémentaire, presque scindée.
4.1 : C-algèbre de Banach, spectre maximal, variété spectrale affine, application analytique, groupe spectral commutatif affine, C-structure analytique, banach analytique.
4.2 : Banach anlytique constant, banach analytique vectoriel.
4.3 : Espace de Banach-Colmez présentable.
5.1 : Dimension, hauteur, présentation.
5.3 : K-presque supplémentaire, suite K-presque scindée.
6.1 : Complexe presque trivial.
7.1 : Presqu'isomorphisme.
8.1 : B^+_{dR}-représentation triviale, extensions universelles par une représentation p-adique.
8.2 : Espace tangent d'une représentation p-adique, exponentielle de Bloch-Kato, (φ, N)-module, représentation semi-stable, B_e-représentation.
8.3 : Représentation de cotype fini.
8.4 : Espace tangent d'un motif, groupe des points d'un motif à valeurs dans $\overline{K}$.

10 – Principales notations

1.2 : K, $\overline{K}$, G_K, $\mathcal{B}(G_K)$,
1.3 : C, $\mathrm{Rep}_C(G_K)$, $\mathrm{Rep}_C^{\mathrm{triv}}(G_K)$, t, B_{dR}, B^+_{dR}, B_m, $\mathrm{Rep}_{B^+_{dR}}(G_K)$,

1.4 : $\mathcal{C}(G_K)$, d, h,
1.6 : $\mathrm{Ext}^n_{\mathcal{C}}(X,Y)$, $\mathrm{Ext}^n_{E[G]}(X,Y)$,
2.1 : K_∞, H_K, Γ_K, π_t, $\underline{t}$, W^f, $\mathrm{Rep}_{K_\infty[[\underline{t}]]}(\Gamma_K)$,
2.2 : $\Omega^{\log}_{E[[\underline{t}]]/E}$, ∇, ∇_0, $\mathcal{C}_{S,E}$, $\mathcal{C}^f_{S,E}$, $\mathbb{C}_{S,E}$, $\mathbb{T}_{S,E}$, $\mathbb{C}^m_{S,E}$, $X\{-1\}$, (C_X), $\mathcal{O}(S)$, $X_{(i)}$, $X_{(i,n)}$, $X_{(\mathbb{Z})}$, $(C_{X,0})$, $\mathrm{Ext}^1_{\mathcal{C}_{S,E},0}(X_1,X_2)$,
2.3 : $\mathrm{Rep}_{B^+_{dR},S}(G_K)$, $\mathcal{O}_K$, $\mathfrak{a}_K$, $R_{dR}(X)$, $\chi^{(\alpha)}$, $M\{\alpha\}$,
2.5 : $W_{(i,n)}$, $W_{(i)}$, $W_{\mathbb{Z}}$, $\mathrm{Ext}^1_{B^+_{dR}[G_K],0}(W_1,W_2)$, $\log t$, T_m, C_m, c_{fond},
3.1 : $H^m_{\mathrm{cont}}(E,M)$,
3.2 : M^{*A}, $\mathcal{M}\check{}$,
3.3 : B_e, B_{cris}, K_0, φ, $\mathrm{Fil}^i B_{dR}$, $\mathrm{Fil}^i B_e$, U_m,
3.4 : U, R, $W(R)$, B^+_{cris}, U_R, $\mathcal{IB}(G_K)$,
3.5 : $E(f)$, $E_m(f)$,
4.1 : $\mathrm{Spm}_C A$, $\mathcal{O}_A$,
4.2 : V^c, W^{an},
4.3 : U^{an}, $E^{\mathrm{an}}_{V,W,f}$,
7.1 : $\mathcal{C}_{PI}(G_K)$, $\mathcal{D}_K$,
8.2 : $E_e(V)$, $E_m(V)$, t_V, $\widehat{t}_V(\overline{K})$, $E_+(V)$, $E_{\mathrm{disc}}(V)$, $H^1_e(K,V)$, B_{st},
8:3 : $T_p(\Lambda)$, $V_p(\Lambda)$, $E_?(\Lambda)$, t_Λ, $\widehat{t}(\overline{K})$,
8:4 : $M_{\mathrm{tor}}(\overline{K})$, $H_l(M)$, t_M, $M(\overline{K})$, $\widehat{M}(\overline{K})$, $E_+(M)$.

11 – Bibliographie

[BK90] S. Bloch and K. Kato, *L-Functions and Tamagawa numbers of motives*, in the Grothendieck Festschrift, vol I, Birkhäuser, Boston (1990), 332–400.

[Co02] P. Colmez, *Espaces de Banach de dimension finie*, Journal of the Inst. of Math. Jussieu 1 (2002), 331–439.

[CF00] P. Colmez et J.-M. Fontaine, *Construction des représentations semi-stables*, Inv.Math. 140 (2000), 1–43.

[De75] P. Deligne *Théorie de Hodge III*, Pub. Math. I.H.E.S. 44 (1975), 5–78.

[DM82] P. Deligne and J. Milne, *Tannakian categories* in Hodge Cycles, Motives and Shimura Varieties, Lect. Notes Math. 900, Springer, Berlin (1982), 101–228.

[Fo88a] J.-M. Fontaine, *Le corps des périodes p-adiques*, avec un appendice par Pierre Colmez, in Périodes p-adiques, Astérisque 223, Société mathématique de France, Paris (1994), 59–111.

[Fo88b] J.-M. Fontaine, *Représentations p-adiques semi-stables*, in Périodes p-adiques, Astérisque 223, Société mathématique de France, Paris (1994), 59–111.

[Fo00] J.-M. Fontaine, *Arithmétique des représentations galoisiennes p-adiques*, prépublication, Orsay 2000-24.

[Fo02] J.-M. Fontaine, *Analyse p-adique et représentations galoisiennes*, International Congress of Mathematicians 2002, vol.2, Higher Education Press, Beijing (2002), 139–148.

[FP] J.-M. Fontaine et J. Plût, *Espaces de Banach-Colmez*, en préparation.

[FPR94] J.-M. Fontaine et B. Perrin-Riou, *Autour des conjectures de Bloch et Kato : cohomologie galoisienne et valeurs de fonctions L*, in Motives, Proc. of Symposia in Pure Math. 55-1, Amer. Math. Soc., Providence (1994), 599–706.

[Iv87] B. Iversen, *Cohomology of sheaves*, Springer, Berlin (1987).

[KS90] M. Kashiwara and P. Schapira, *Sheaves on Manifolds*, Springer, Berlin (1990).

[La83] G. Laumon, *Sur la catégorie dérivée de la catégorie des $\mathcal{D}$-modules filtrés*, in Algebraic Geometry Proceedings, Tokyo-Kyoto 1982, Lect. Notes Math. 1016, Springer, Berlin (1983), 151–237.

[NSW00] J. Neukirch, A. Schmidt and K. Vingberg, *Cohomology of Number Fields*, Springer, Berlin (2000).

[PR88] B. Perrin-Riou,*Représentations p-adiques ordinaires*, in Périodes p-adiques, Astérisque 223, Société mathématique de France, Paris (1994), 185–207.

[Qu73] D. Quillen, *Higher algebraic K-Theory : I*, in Algebraic K-Theory I, Lect. Notes Math. 341, Springer, Berlin (1973), 77–139.

[Sen80] S. Sen, *Continuous Cohomology and p-adic Galois Representations,* Invent. Math. 62 (1980), 89–116.

[Se67a] J.-P. Serre, *Sur les groupes de Galois attachés aux groupes p-divisibles,* Proc. of a conference on local fields, Nuffic Summer School at Driebergen, Springer, Berlin (1967), 118–131.

[Se67b] J.-P. Serre, *Local Class Field Theory*, in Algebraic Number Theory (J.W.S. Cassels and A. Fröhlich ed.), Academic Press, London (1967), 129–161.

[Se89] J.-P. Serre, *Abelian l-adic Representations and Elliptic Curves*, Reprint, Addison-Wesley, Redwood City (1989).

[Se94] J.-P. Serre, *Cohomologie Galoisienne (5-ième éd.)* Lect. Notes Math. 5, Springer, Berlin (1994).

[Ta67] J. Tate, *p-divisible groups,* Proc. of a conference on local fields, Nuffic Summer School at Driebergen, Springer, Berlin (1967), 158–183.

[Ta76] J. Tate, *Relations between K_2 and Galois cohomology,* Invent. Math. 36 (1976), 257–274.

Jean-Marc Fontaine
Institut Universitaire de France
et UMR 8628 du CNRS,
Mathématique
Université de Paris-Sud
Bâtiment 425
F-91405 ORSAY Cedex
fontaine@math.u-psud.fr

Coleman Power Series for K_2 and p-Adic Zeta Functions of Modular Forms

dedicated to Professor Kazuya Kato

Takako Fukaya[1]

Received: November 30, 2002
Revised: July 3, 2003

Abstract. For a usual local field of mixed characteristic $(0, p)$, we have the theory of Coleman power series [Co]. By applying this theory to the norm compatible system of cyclotomic elements, we obtain the p-adic Riemann zeta function of Kubota-Leopoldt [KL]. This application is very important in cyclotomic Iwasawa theory.

In [Fu1], the author defined and studied Coleman power series for K_2 for certain class of local fields. The aim of this paper is following the analogy with the above classical case, to obtain p-adic zeta functions of various cusp forms (both in one variable attached to cusp forms, and in two variables attached to ordinary families of cusp forms) by Amice-Vélu, Vishik, Greenberg-Stevens, and Kitagawa,... by applying the K_2 Coleman power series to the norm compatible system of Beilinson elements defined by Kato [Ka2] in the projective limit of K_2 of modular curves.

2000 Mathematics Subject Classification: Primary 11F85; Secondary 11G55, 19F27
Keywords and Phrases: p-adic zeta function, modular form, algebraic K-theory, Euler system.

Contents

[1]Supported by the JSPS Research Fellowship for Young Scientists

1. Introduction

1.1. Let p be any prime number. For a complete discrete valuation field H of mixed characteristic $(0,p)$, with perfect residue field, we have the theory of Coleman power series, as reviewed in section 2. One of the important applications of this theory is the construction of the p-adic Riemann zeta function of Kubota and Leopoldt [KL], by applying the theory to the norm compatible system of cyclotomic elements.
In the paper [Fu1], we have obtained "K_2-version of Coleman power series" for a certain class of local fields. Following the analogy with the case of the usual Coleman power series above, the aim of the present paper is to show that by applying the theory of K_2 Coleman power series to the norm compatible system of Beilinson elements in the projective limit of K_2 of modular curves defined by Kato [Ka2], we obtain p-adic zeta functions of various cusp forms, both in one variable attached to cusp forms (cf.Amice-Vélu [AV], Vishik [Vi]), and two variables attached to universal family of ordinary cusp forms (cf. Greenberg-Stevens [GS], Kitagawa [Ki]).

1.2. We describe our result briefly reviewing the classical result on the p-adic Riemann zeta function.
Let denote by $\zeta_{p^n} \in \overline{\mathbb{Q}_p}$ a primitive p^n-th root of unity and assume $\zeta_{p^{n+1}}^p = \zeta_{p^n}$ for all $n \geq 1$. We write $Q(\mathbb{Z}_p[[G_\infty]])$ for the total quotient ring of the completed group ring $\mathbb{Z}_p[[G_\infty]] = \varprojlim_n \mathbb{Z}_p[(\mathbb{Z}/p^n\mathbb{Z})^\times]$ and $G_\infty = \mathbb{Z}_p^\times$ is regarded as the Galois group $\mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p)$ associated to the cyclotomic p extension of $\mathbb{Q}_p$ via the cyclotomic character.
Iwasawa [Iw] discovered a relationship between the norm compatibles system of cyclotomic elements $(1-\zeta_{p^n})_n \in \varprojlim_n \mathbb{Q}_p(\zeta_{p^n})^\times$ and the p-adic Riemann zeta function $\zeta_{p\text{-adic}} \in Q(\mathbb{Z}_p[[G_\infty]])$ of Kubota and Leopoldt [KL]. The relation of these two appears in the theory of the usual Coleman power series as follows. The theory of Coleman power series for the multiplicative group induces a map $\mathcal{C}$ and $\mathcal{C}$ sends $(1-\zeta_{p^n})_n \in \varprojlim_n \mathbb{Q}_p(\zeta_{p^n})^\times$ to $\zeta_{p\text{-adic}} \in Q(\mathbb{Z}_p[[G_\infty]])$:

$$\mathcal{C} : \varprojlim_n \mathbb{Q}_p(\zeta_{p^n})^\times \xrightarrow{\text{via Coleman power series}} Q(\mathbb{Z}_p[[G_\infty]]),$$

$$\mathcal{C}((1-\zeta_{p^n})_n) = \zeta_{p\text{-adic}}.$$

The purpose of this paper is, by pursuing the analogy with this work, to obtain p-adic zeta functions in one variable attached to cusp forms, and in two variables attached to ordinary families of modular forms, whose existences are already known (for one-variable zeta functions cf. Amice-Vélu [AV], Vishik [Vi],..., and

two-variable zeta functions associated to ordinary families of cusp forms cf. Greenberg-Stevens [GS], Kitagawa [Ki],...).
Let

$$\mathsf{H} = (\varprojlim_n (\mathbb{Z}/p^n\mathbb{Z}[[q]][\frac{1}{q}]))[\frac{1}{p}],$$

where q is an indeterminate. This is a complete discrete valuation field of mixed characteristic $(0,p)$ whose residue field k is an imperfect field satisfying $[\mathsf{k} : \mathsf{k}^p] = p$. As reviewed in section 2, for H, we have a theory of K_2 Coleman power series [Fu1].
Let N be a positive integer which is prime to p. Let us denote by $Y(Np^n, p^n)$ the modular curve corresponding to the subgroup $\Gamma(Np^n, p^n) = \{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z})\ ;\ \alpha \equiv 1(Np^n), \beta \equiv 0(Np^n), \gamma \equiv 0(p^n), \delta \equiv 1(p^n)\}$ whose total constant field is $\mathbb{Q}(\zeta_{p^n})$.
In his paper [Ka2], Kato discovered a norm compatible system of Beilinson elements belonging to $\varprojlim_n K_2(Y(Np^n, p^n))$. We study the image of this Beilinson-Kato system under a map $\mathcal{C}_N$ below which is defined by using our K_2 Coleman power series following the analogy with the classical map $\mathcal{C}$. We call this image a universal zeta modular form (see section 4).

$$\mathcal{C}_N : \varprojlim_n K_2(Y(Np^n, p^n)) \xrightarrow{\text{via } K_2 \text{ Coleman power series}} Q(O_{\mathsf{H}}[[G_\infty^{(1)} \times G_\infty^{(2)}]])$$

$$\mathcal{C}_N(\text{Beilinson-Kato system}) = \text{the universal zeta modular form.}$$

(Precisely we will define the universal zeta modular form as an element obtained from this image with a suitable modification, cf. section 4.) Here $G_\infty^{(1)} \cong G_\infty^{(2)} \cong G_\infty$, $G_\infty^{(1)}$ is a group of diamond operators acting on the space of p-adic modular forms (refer to sections 3 and 4), and $G_\infty^{(2)} = \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p)$. Further $Q(O_{\mathsf{H}}[[G_\infty^{(1)} \times G_\infty^{(2)}]])$ denotes the total quotient ring of the completed group ring $O_{\mathsf{H}}[[G_\infty^{(1)} \times G_\infty^{(2)}]]$.
Theorem 6.2 which is one of our two main results will state that the above universal zeta modular form produces p-adic zeta functions (in one variable) of eigen cusp forms which are not necessarily ordinary.
Theorem 7.3 which is the other main theorem asserts roughly the following.

Theorem 1.3 (cf. Theorem 7.3). *We assume $p \geq 5$. Let $\mathfrak{h}_{Np^\infty}^{\mathrm{ord}}$ be the ordinary part of the ring of Hecke operators of level Np^∞ acting on the space of the p-adic cusp forms of level Np^∞ (cf. section 3). The universal zeta modular form above produces, by the method in section 7, a p-adic zeta function in two variables*

$$L_{p\text{-}adic}^{\mathrm{ord,univ}} \in (\mathfrak{h}_{Np^\infty}^{\mathrm{ord}}/\mathcal{I}_{Np^\infty}^{\mathrm{ord}})[[G_\infty^{(2)}]][\frac{1}{a}]$$

which displays property (1.1) below. Here $\mathcal{I}_{Np^\infty}^{\mathrm{ord}} \subset \mathfrak{h}_{Np^\infty}^{\mathrm{ord}}$ is a certain ideal (see3.7 in section 3), and $a \in \mathfrak{h}_{Np^\infty}^{\mathrm{ord}}[[G_\infty^{(2)}]]$ is a certain non-zerodivisor.

Let f be an ordinary p-stabilized newform of tame conductor N (for the definition of an ordinary p-stabilized newform, see 7.2.1 in section 7) of weight $k \geq 2$. Attached to f, we have a ring homomorphism κ_f :

$$\kappa_f : \mathfrak{h}^{\mathrm{ord}}_{Np^\infty}/\mathcal{I}^{\mathrm{ord}}_{Np^\infty} \longrightarrow \overline{\mathbb{Z}_p}\ ;\ T(n) \mapsto a_n(f) \quad (n \geq 1),$$

with $a_n(f)$ such that $T(n)f = a_n(f)f$. Suppose κ_f satisfies some "suitable" condition. Then κ_f induces a homomorphism which is also denoted by κ_f :

$$\kappa_f : (\mathfrak{h}^{\mathrm{ord}}_{Np^\infty}/\mathcal{I}^{\mathrm{ord}}_{Np^\infty})[[G_\infty]][\frac{1}{a}] \longrightarrow Q(\overline{\mathbb{Z}_p}[[G^{(2)}_\infty]]),$$

and concerning the image $L^{\mathrm{ord,univ}}_{p\text{-}adic}(\kappa_f)$ of $L^{\mathrm{ord,univ}}_{p\text{-}adic}$ under this homomorphism κ_f, we have

$$L^{\mathrm{ord,univ}}_{p\text{-}adic}(\kappa_f) = p\text{-}adic\ zeta\ function\ of\ f \in (O_M[[G^{(2)}_\infty]]) \otimes_{O_M} M. \tag{1.1}$$

Here M is the finite extension $\mathbb{Q}_p(a_n(f); n \geq 1)$ of $\mathbb{Q}_p$.

For the precise statement, see Theorem 7.3 in section 7.

The above $L^{\mathrm{ord,univ}}_{p\text{-adic}}$ is essentially the two-variable p-adic zeta function associated to ordinary families of cusp forms which has been already given by Greenberg-Stevens [GS], Kitagawa [Ki],..., by another method. The significance of our p-adic zeta function is that the coefficients in the p-adic zeta function belong to the ring of Hecke operators as above. Hence our $L^{\mathrm{ord,univ}}_{p\text{-adic}}$ is a p-adic zeta function associated with the universal family of ordinary cusp forms. By another method, Ochiai ([Oc]) has also constructed this kind of two-variable p-adic zeta function.
The author found that Panchishkin [Pa1], [Pa2] gave a new way of the construction of p-adic zeta functions of modular forms by using something similar to our universal zeta modular form at almost the same time as the author gave talks in the conferences in the autumn of 2000 as described in the proceedings [Fu2], [Fu3] in Japanese. Our aim is to obtain p-adic zeta functions of modular forms by applying K_2 Coleman power series to the norm compatible system of Beilinson elements in K_2 of modular curves.

1.4. The organization of this paper is as follows.
In section 2, we review the theory of Coleman power series both in the classical case and in the case for K_2 [Fu1].
In section 3, we review the theory of p-adic modular forms (cf. [Hi1]).
In section 4, we define and study a "universal zeta modular form" which is obtained from the image of Beilinson-Kato system under $\mathcal{C}_N$ appearing in 1.2. In our construction, p-adic properties of p-adic zeta functions are deduced from the p-adic properties of the universal zeta modular form and the relation between the universal zeta modular form and special values of zeta functions of modular forms.
In section 5, we review the theory of p-adic zeta functions of modular forms.

In section 6, we prove our theorem (Theorem 6.2) on the construction of one-variable p-adic zeta functions of eigen cusp forms which are not necessarily ordinary.
In section 7, we prove our theorem (Theorem 7.3) on the construction of p-adic zeta functions in two variables, which are attached to universal families of ordinary cusp forms.

The author would like to express her sincere gratitude to Professor Kazuya Kato for his guidance and constant encouragement. She has learned a great deal from him throughout her graduate course. She is very happy to dedicate this paper to his celebrated 50th birthday.

She is very thankful to Professor John Coates who gave her comments and constant encouragement for this study.

In this paper, for a complete discrete valuation field L, O_L denotes the ring of integers of L.
For a ring R, $Q(R)$ denotes the total quotient ring of R.
We also fix once and for all an embedding of $\overline{\mathbb{Q}}$ into $\overline{\mathbb{Q}_p}$.

2. Review of Coleman power series for K_2

In this section we give a brief review of the theory of Coleman power series both in the usual case (in 2.1 – 2.2) and our K_2-version case (in 2.3 – 2.5).

2.1. We review the classical case of the usual Coleman power series. The existence of Coleman power series were discovered by Coates and Wiles [CW] and almost immediately, Coleman [Co] generalized their approach by an alternative method. The theory of Coleman power series has been obtained for general Lubin-Tate groups, but here we review the theory only for the formal multiplicative group.
Let H be a complete discrete valuation field of mixed characteristic $(0, p)$ with perfect residue field k. We assume that H is absolutely unramified, i.e. p is a prime element of O_H. We denote by $O_H[[\varepsilon - 1]] = \varprojlim_n O_H[\varepsilon^{\pm 1}]/(\varepsilon - 1)^n$ the coordinate ring of the formal completion of the multiplicative group over O_H, and by $O_H((\varepsilon - 1))$ the Laurent series ring $O_H[[\varepsilon - 1]][1/(\varepsilon - 1)]$. Let σ denote the Frobenius automorphism of O_H. We extend σ to an endomorphism of $O_H((\varepsilon - 1))$ by putting $\sigma(\varepsilon) = \varepsilon$. We define a ring homomorphism

$$\varphi : O_H((\varepsilon - 1)) \longrightarrow O_H[[\varepsilon - 1]][\frac{1}{\varepsilon^p - 1}]$$

by $\varphi(f)(\varepsilon) = (\sigma f)(\varepsilon^p)$, and

$$\mathrm{N} : O_H((\varepsilon - 1))^\times \longrightarrow O_H((\varepsilon - 1))^\times$$

to be the norm operator induced by the homomorphism φ. We write $(O_H((\varepsilon - 1))^\times)^{\mathrm{N}=1}$ for the group of all units f in $O_H((\varepsilon - 1))$ which satisfy $\mathrm{N}(f) = f$. Now let ζ_{p^n} denote a primitive p^n-th root of unity, and assume $\zeta_{p^{n+1}}^p = \zeta_{p^n}$

for all $n \geq 1$. Put $H_n = H(\zeta_{p^n})$. The aim in this case is to study $\varprojlim_n H_n^\times$, where the projective limit is taken with respect to the norm maps in the tower of fields H_n $(n = 1, 2, 3, \dots)$.

Theorem 2.2 (Coleman [Co]). *We have an isomorphism*

$$\Psi : (O_H((\varepsilon - 1))^\times)^{\mathrm{N}=1} \xrightarrow{\cong} \varprojlim_n H_n^\times$$

given by $\Psi(f(\varepsilon)) = ((\sigma^{-n} f)(\zeta_{p^n}))_{n=1,2,3,\dots}$.

2.3. Now we review our case of K_2-version of Theorem 2.2. (See [Fu1] for more details.)

Let H be a complete discrete valuation field of characteristic 0, whose residue field k is an imperfect field of characteristic p satisfying $[\mathsf{k} : \mathsf{k}^p] = p$. We assume that H is absolutely unramified. We fix once and for all a p-base b of k, and a lifting q of b to H (all of our subsequent constructions depend on these choices). We define $\sigma : O_\mathsf{H} \to O_\mathsf{H}$ to be the unique ring homomorphism satisfying $\sigma(q) = q^p$, and the action of σ on k is given by raising to the p-th power. For simplicity, let us write

$$S = O_\mathsf{H}[[\varepsilon - 1]], \quad S' = O_\mathsf{H}((\varepsilon - 1)) = S[\frac{1}{\varepsilon - 1}].$$

We extend σ to an endomorphism of S' by putting $\sigma(\varepsilon) = \varepsilon$. We then define a ring homomorphism

$$\varphi : S' \longrightarrow S[\frac{1}{\varepsilon^p - 1}]$$

by $\varphi(f)(\varepsilon) = (\sigma f)(\varepsilon^p)$. For any ring A, let $K_2(A)$ denote Quillen's K_2 group of A ([Qu]). Since $S[1/(\varepsilon^p - 1)]$ is a free S'-module of rank p^2 via φ, we have the K_2 norm map (see [Qu], §4, Transfer maps)

$$K_2(S[\frac{1}{\varepsilon^p - 1}]) \longrightarrow K_2(S').$$

The composition of this with

$$K_2(S') \longrightarrow K_2(S[\frac{1}{\varepsilon^p - 1}])$$

induced by the inclusion map $S' \hookrightarrow S[1/(\varepsilon^p - 1)]$, gives rise to a K_2 norm map

$$\mathrm{N} : K_2(S') \longrightarrow K_2(S').$$

We consider the following tower of fields above H. We take a p^n-th root q^{1/p^n} of q in a fixed algebraic closure $\overline{\mathsf{H}}$ of H and assume that $(q^{1/p^{n+1}})^p = q^{1/p^n}$ for all $n \geq 1$. We define

$$\mathsf{H}_n = \mathsf{H}(\zeta_{p^n}, q^{1/p^n}).$$

Moreover, we define a ring homomorphism

$$\theta_n : O_\mathsf{H} \longrightarrow O_{\mathsf{H}(q^{1/p^n})}$$

by specifying that $\theta_n(q) = q^{1/p^n}$, and that the induced map $\mathsf{k} \longrightarrow \mathsf{k}(b^{1/p^n})$ on the residue fields is the isomorphism $x \mapsto x^{1/p^n}$. For $n \geq 1$, θ_n induces a ring homomorphism

$$h_n : S' \longrightarrow \mathsf{H}_n$$

given by $h_n(\sum_{m=-r}^{\infty} a_m(\varepsilon - 1)^m) = \sum_{m=-r}^{\infty} \theta_n(a_m)(\zeta_{p^n} - 1)^m$. Thus we obtain a map

$$\Psi_n : K_2(S') \longrightarrow K_2(\mathsf{H}_n) \quad (n \geq 1)$$

which is induced by h_n. In order to state our theorem, we need to introduce certain completions $\hat{K}_2$ of our K_2-groups (see 2.5 below for the definition). All of the above homomorphisms give rise to corresponding maps between the completed K_2-groups, which we can check easily from the definition of the completions in 2.5 below and which we denote by the same symbol. We also write $\hat{K}_2(S')^{\mathrm{N}=1}$ for the subgroup of elements f in $\hat{K}_2(S')$ which satisfy $\mathrm{N}(f) = f$.

Instead of $\varprojlim_n H_n^{\times}$, we study the projective limit $\varprojlim_n \hat{K}_2(\mathsf{H}_n)$ with respect to the norm maps in the tower of fields H_n ($n = 1, 2, 3, \dots$).

Theorem 2.4 ([Fu1], Theorem 1.5). *We have an isomorphism*

$$\Psi : \hat{K}_2(S')^{\mathrm{N}=1} \underset{\cong}{\longrightarrow} \varprojlim_n \hat{K}_2(\mathsf{H}_n)$$

given by $\Psi(f) = (\Psi_n(f) : n = 1, 2, 3, \dots)$.

2.5. We describe the completions of the K_2 groups appearing in Theorem 2.4. We introduce the completions $\hat{K}_2(A)$ in the following two cases by which the completions in Theorem 2.4 follows.

(i) $A = S'$.

(ii)A is a complete discrete valuation field L.

Let $r \geq 1$.

In the case of (i), let $U^{(r)} = 1 + (p, (\varepsilon - 1))^r S$, a subgroup of $S^{\times}$, where $(p, (\varepsilon - 1))$ is the ideal of S generated by the elements in ( ).

In the case of (ii), let $U^{(r)} = U_L^{(r)}$, where $U_L^{(r)}$ is the r-th unit group of L, i.e. $1 + m_L^r \subset O_L^{\times}$ for the maximal ideal m_L of L.

We define a subgroup $\mathcal{U}^{(r)} K_2(A)$ of $K_2(A)$ for a ring A of the type (i) or (ii), as the one which is generated by $\{a, A^{\times}\}$ for all $a \in U^{(r)} \subset A^{\times}$, and define

$$\hat{K}_2(A) = \varprojlim_r K_2(A)/\mathcal{U}^{(r)} K_2(A).$$

3. Review of p-adic modular forms

In this section, we briefly review the necessary facts for us on the theory of p-adic modular forms. We follow Hida [Hi1] to which we refer for more details. (See also Katz [Katz1], [Katz2], etc.)

3.1. Let L be a finite extension of $\mathbb{Q}_p$ in $\overline{\mathbb{Q}_p}$, and we take a finite extension L_0 of $\mathbb{Q}$ which is dense in L under the p-adic topology. For $k \geq 0$ and for $M \geq 1$, let $M_k(X_1(M); L_0)$ be the space of modular forms for $\Gamma_1(M)$ of weight k with Fourier coefficients in L_0. We define

$$M_k(X_1(M); L) = M_k(X_1(M); L_0) \otimes_{L_0} L.$$

Similarly we define the space of cusp forms $S_k(X_1(M); L_0)$ and $S_k(X_1(M); L)$. As in [Hi1], §1, it has been known that the spaces $M_k(X_1(M); L)$, $S_k(X_1(M); L)$ are independent of the choice of a subfield L_0 in the evident sense.

Now for $j \geq 0$ we put

$$M^j(X_1(M); L) = \bigoplus_{k=0}^{j} M_k(X_1(M); L),$$

$$S^j(X_1(M); L) = \bigoplus_{k=1}^{j} S_k(X_1(M); L),$$

which are embedded in $L[[q]]$ via the summation of q-expansions, and

$$M^j(X_1(M); O_L) = M^j(X_1(M); L) \cap O_L[[q]],$$

$$S^j(X_1(M); O_L) = S^j(X_1(M); L) \cap O_L[[q]].$$

We define $\overline{M}(X_1(M); O_L)$ as the closure of $\bigcup_{j\geq 1} M^j(X_1(M); O_L)$ in $O_L[[q]]$ for the p-adic topology, and $\overline{S}(X_1(M); O_L)$ to be the closure of $\bigcup_{j\geq 1} S^j(X_1(M); O_L)$ in $O_L[[q]]$ for the p-adic topology.

For an integer $N \geq 1$ which is prime to p, it has been proven that $\overline{M}(X_1(Np^t); O_L)$ and $\overline{S}(X_1(Np^t); O_L)$ are independent of the choice of $t \geq 1$, as in Hida [Hi1], §1, Cor. 1.2 (i), and (1.19a), respectively. For simplicity we put

$$\overline{M}_{Np^\infty} = \overline{M}(X_1(Np^t); \mathbb{Z}_p), \quad \overline{S}_{Np^\infty} = \overline{S}(X_1(Np^t); \mathbb{Z}_p)$$

for any $t \geq 1$.

We introduced the above notation in a general situation for our later use, however in the rest of this section, we always take $L = \mathbb{Q}_p$.

3.2. We review the definition of the rings of Hecke operators $\mathcal{H}_{Np^\infty}$ and $\mathfrak{h}_{Np^\infty}$ acting on $\overline{M}_{Np^\infty}$ and $\overline{S}_{Np^\infty}$, respectively.

For $t \geq 1$ and $j \geq 1$, let $\mathcal{H}^j(X_1(Np^t); \mathbb{Z}_p)$ (resp. $\mathfrak{h}^j(X_1(Np^t); \mathbb{Z}_p)$) be the $\mathbb{Z}_p$-subalgebra of $\mathbb{Z}_p$-endomorphism ring of $M^j(X_1(Np^t); \mathbb{Z}_p)$ (resp. $S^j(X_1(Np^t); \mathbb{Z}_p)$) generated over $\mathbb{Z}_p$ by $T(n)$ ($n \geq 1$).

We put

$$\mathcal{H}(X_1(Np^t); \mathbb{Z}_p) = \varprojlim_j \mathcal{H}^j(X_1(Np^t); \mathbb{Z}_p),$$

$$\mathfrak{h}(X_1(Np^t); \mathbb{Z}_p) = \varprojlim_j \mathfrak{h}^j(X_1(Np^t); \mathbb{Z}_p),$$

where the inverse limits are taken by natural homomorphisms given by the restriction of operators.
As in Hida [Hi1], §1, (1.15a) and (1.19a), respectively, $\mathcal{H}(X_1(Np^t);\mathbb{Z}_p)$ and $\mathfrak{h}(X_1(Np^t);\mathbb{Z}_p)$ do not depend on $t \geq 1$. For simplicity we put

$$\mathcal{H}_{Np^\infty} = \mathcal{H}(X_1(Np^t);\mathbb{Z}_p), \quad \mathfrak{h}_{Np^\infty} = \mathfrak{h}(X_1(Np^t);\mathbb{Z}_p)$$

for any $t \geq 1$. The rings $\mathcal{H}_{Np^\infty}$ and $\mathfrak{h}_{Np^\infty}$ act on $\overline{M}_{Np^\infty}$ and $\overline{S}_{Np^\infty}$, respectively. The ring $\mathfrak{h}_{Np^\infty}$ is, in fact, a quotient of $\mathcal{H}_{Np^\infty}$ by the annihilator in $\mathcal{H}_{Np^\infty}$ of $\overline{S}_{Np^\infty}$.

3.3. By the action of $\mathcal{H}_{Np^\infty}$ on $\overline{M}_{Np^\infty}$ in 3.2, we have a canonical map

$$\mathfrak{i} : \overline{M}_{Np^\infty} \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{H}_{Np^\infty},\mathbb{Z}_p) \ ; \ f \mapsto (T(n) \mapsto a_1(T(n)f))$$

($f \in \overline{M}_{Np^\infty}$) where $a_1(T(n)f)$ is the coefficient of q in the q-expansion of $T(n)f \in \overline{M}_{Np^\infty}$.
This map will play an important role later in the construction of our two-variable p-adic zeta function $L^{\mathrm{ord,univ}}_{p\text{-adic}}$.

3.4. For $j \geq 1$ let $e^j = \lim_{n\longrightarrow\infty} T(p)^{n!}$ in $\mathcal{H}^j(X_1(Np^t);\mathbb{Z}_p)$ or $\mathfrak{h}^j(X_1(Np^t);\mathbb{Z}_p)$, and $e = \varprojlim_j e^j$. Then $e^2 = e$.

We denote by $\mathcal{H}^{\mathrm{ord}}_{Np^\infty}$ the ordinary part $e \cdot \mathcal{H}_{Np^\infty}$ of $\mathcal{H}_{Np^\infty}$ and by $\mathfrak{h}^{\mathrm{ord}}_{Np^\infty}$ the ordinary part $e \cdot \mathfrak{h}_{Np^\infty}$ of $\mathfrak{h}_{Np^\infty}$.
Let $P^{\mathrm{ord}}_{Np^\infty} \subset \mathcal{H}^{\mathrm{ord}}_{Np^\infty}$ (resp. $p^{\mathrm{ord}}_{Np^\infty} \subset \mathfrak{h}^{\mathrm{ord}}_{Np^\infty}$) be the annihilator of the old forms (resp. old cusp forms) of level $N'p^t$ for all N' such that $N'|N$ and $N' < N$. For the precise definition of $P^{\mathrm{ord}}_{Np^\infty}$ (resp. $p^{\mathrm{ord}}_{Np^\infty}$), see [Hi1], §3. In the case $N = 1$, we have

$$P^{\mathrm{ord}}_{p^\infty} = \mathcal{H}^{\mathrm{ord}}_{p^\infty}, \quad p^{\mathrm{ord}}_{p^\infty} = \mathfrak{h}^{\mathrm{ord}}_{p^\infty}.$$

On $P^{\mathrm{ord}}_{Np^\infty}$ and $p^{\mathrm{ord}}_{Np^\infty}$, Hida showed Proposition 3.6 below which is important for us. Preceding it, we set up notation.

3.5. We define a group $G^{(1)}_\infty$ which is endowed with an isomorphism to $\mathbb{Z}_p^\times$ and which acts on the space $\overline{M}_{Np^\infty}$ in the following way.
Firstly for $x \in (\mathbb{Z}/Np^t\mathbb{Z})^\times$ we denote by $\langle x \rangle$ the endomorphism $\begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix}^*$ on $M_k(X_1(Np^t);\mathbb{Q})$ induced by the action of $\begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix} \in GL_2(\mathbb{Z}/Np^t\mathbb{Z})$ on $X(Np^t, Np^t)$. (The action of $GL_2(\mathbb{Z}/Np^t\mathbb{Z})$ on $X(Np^t, Np^t)$ induces an endomorphism on $M_k(X_1(Np^t);\mathbb{Q})$ by the fact that $M_k(X_1(Np^t);\mathbb{Q})$ may be regarded as the fixed part of $M_k(X(Np^t, Np^t))$ by the group $\{\begin{pmatrix} u & v \\ w & x \end{pmatrix} \in GL_2(\mathbb{Z}/Np^t\mathbb{Z}) \ ; \ u \equiv 1(Np^t), w \equiv 0(Np^t)\}$, where $M_k(X(Np^t, Np^t))$ is the space of modular forms on $X(Np^t, Np^t)$ of weight k as, for example, in [Ka2], §3 (3.3.1), and §4.)

We also use the same notation $\langle x\rangle$ for the endomorphism on $M_k(X_1(Np^t);\mathbb{Q}_p)$ induced by the above.
For $a \in \mathbb{Z}_p^\times$, let $g_a^{(1)} \in G_\infty^{(1)}$ denote the element corresponding to a under the given isomorphism. For $f = \sum_k f_k \in \bigcup_j M^j(X_1(Np^t);\mathbb{Q}_p)$ with $f_k \in M_k(X_1(Np^t);\mathbb{Q}_p)$ and $t \geq 1$, we define the action of $g_a^{(1)} \in G_\infty^{(1)}$ as

$$g_a^{(1)} \cdot f = \sum_k a^{k-2}\langle a'\rangle f_k,$$

where $a' \in (\mathbb{Z}/Np^t\mathbb{Z})^\times$ is the element such that $a' \equiv a(p^t)$ and $a' \equiv 1(N)$. This action of $G_\infty^{(1)}$ may be extended to $\overline{M}_{Np^\infty}$.
We remark that the relation between this action of $G_\infty^{(1)}$ and the action of $\mathbb{Z}_p^\times$ in Hida [Hi1], §3, (3.1) is

$$f \mid a = a^2 g_a^{(1)} \cdot f,$$

where $f \mid a$ denotes the image of f under the action of a in the meaning of Hida.
We put $\Lambda = \mathbb{Z}_p[[G_\infty^{(1)}]]$. By the above action of $G_\infty^{(1)}$ on $\overline{M}_{Np^\infty}$, we have a ring homomorphism

$$\Lambda \longrightarrow \mathcal{H}_{Np^\infty}.$$

We see that via this homomorphism $\mathfrak{h}_{Np^\infty}$, $\mathcal{H}^{\mathrm{ord}}_{Np^\infty}$, and $\mathfrak{h}^{\mathrm{ord}}_{Np^\infty}$ become also Λ-algebras, and $P^{\mathrm{ord}}_{Np^\infty}$, and $p^{\mathrm{ord}}_{Np^\infty}$ are Λ-modules.

Proposition 3.6 (Hida [Hi1] Corollary 3.3). *We assume $p \geq 5$.*
(1) *The rings $\mathcal{H}^{\mathrm{ord}}_{Np^\infty}$ and $\mathfrak{h}^{\mathrm{ord}}_{Np^\infty}$ are finitely generated projective modules over Λ.*
(2) *The ideal $P^{\mathrm{ord}}_{Np^\infty}$ (resp. $p^{\mathrm{ord}}_{Np^\infty}$) is a finitely generated projective module over Λ. Moreover the intersection of $P^{\mathrm{ord}}_{Np^\infty}$ (resp. $p^{\mathrm{ord}}_{Np^\infty}$) and the nilradical of $\mathcal{H}^{\mathrm{ord}}_{Np^\infty}$ (resp. $\mathfrak{h}^{\mathrm{ord}}_{Np^\infty}$) is null.*

Proof. For the proof, see [Hi1]. □

3.7. We define an ideal

$$\mathcal{I}^{\mathrm{ord}}_{Np^\infty} \subset \mathfrak{h}^{\mathrm{ord}}_{Np^\infty}$$

to be the annihilator of $p^{\mathrm{ord}}_{Np^\infty} \subset \mathfrak{h}^{\mathrm{ord}}_{Np^\infty}$. Then the natural map

$$p^{\mathrm{ord}}_{Np^\infty} \otimes_\Lambda Q(\Lambda) \longrightarrow (\mathfrak{h}^{\mathrm{ord}}_{Np^\infty}/\mathcal{I}^{\mathrm{ord}}_{Np^\infty}) \otimes_\Lambda Q(\Lambda)$$

is an isomorphism, where the both hands sides are semisimple algebras over $Q(\Lambda)$ (cf. [Hi1], §3).

4. Universal zeta modular form

In this section, we define and study a"universal zeta modular form" which is obtained from the norm compatible system of Beilinson elements defined by Kato [Ka2], via K_2 Coleman power series. The p-adic properties of p-adic zeta functions of modular forms are deduced from the p-adic property of the universal zeta modular form and the relations between the universal zeta modular form and zeta values. In 4.1, we define a map

$$\mathcal{C}_N : \varprojlim_n K_2(Y(Np^n, p^n)) \to Q(O_{\mathsf{H}}[[G_\infty^{(1)} \times G_\infty^{(2)}]])$$

(see 4.1 for the details) by using K_2 Coleman power series, and Proposition 4.4 shows that

$$\mathcal{C}_N : \text{Beilinson-Kato system} \mapsto \text{the universal zeta modular form}$$

$$\text{(reviewed in 4.2)} \qquad\qquad \text{(defined in 4.3)}.$$

In 4.5, we explain the properties of the universal zeta modular form concerning the relation with special values of zeta functions of cusp forms.
In what follows,

$$\mathsf{H} = (\varprojlim_n (\mathbb{Z}/p^n\mathbb{Z}[[q]][\frac{1}{q}]))[\frac{1}{p}].$$

We fix a system $(\zeta_{p^n})_{n\geq 1}$ of primitive p^n-th roots of unity which satisfy $\zeta_{p^{n+1}}^p = \zeta_{p^n}$ for all $n \geq 1$. For $q \in \mathsf{H}$ and $n \geq 1$, we fix p^n-th roots q^{1/p^n} of q in $\overline{\mathsf{H}}$ which satisfy $(q^{1/p^{n+1}})^p = q^{1/p^n}$ for all $n \geq 1$. Let N denote a positive integer such that $(N, p) = 1$.

4.1. By using K_2 Coleman power series, we define a map

$$\mathcal{C}_N : \varprojlim_n K_2(Y(Np^n, p^n)) \longrightarrow Q(O_{\mathsf{H}}[[G_\infty^{(1)} \times G_\infty^{(2)}]]),$$

where the left hand side is the inverse limit of K_2 of modular curves (cf. 4.1.1) taken with respect to the norm maps, and on the right hand side, the group $G_\infty^{(1)}$ is as in section 2 (we will review this in 4.3) and the group $G_\infty^{(2)}$ is the Galois group $\mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p)$.
We put $S = O_{\mathsf{H}}[[\varepsilon - 1]]$ and $S' = O_{\mathsf{H}}[[\varepsilon - 1]][1/(\varepsilon - 1)]$. Let $G_\infty = \mathbb{Z}_p^\times$. The map $\mathcal{C}_N$ is defined as the following composition:

$$\begin{aligned}
\mathcal{C}_N : &\varprojlim_n K_2(Y(Np^n, p^n)) \\
&\xrightarrow{\mathcal{E}_N} \varprojlim_n \hat{K}_2(\mathsf{H}_n)[[G_\infty]] \\
&\xrightarrow[\cong]{\mathrm{Col}} \hat{K}_2(S')^{\mathrm{N}=1}[[G_\infty]] \\
&\xrightarrow{d\log} \Omega_S^2(\log)[[G_\infty]] = \frac{S}{\varepsilon - 1} \cdot d\log(q) \wedge d\log(\varepsilon)[[G_\infty]] \\
&\xrightarrow[\cong]{} \frac{S}{\varepsilon - 1}[[G_\infty]] \to Q(O_{\mathsf{H}}[[G_\infty^{(1)} \times G_\infty^{(2)}]]).
\end{aligned} \tag{4.1}$$

We explain each term and each arrow in the composition (4.1).

4.1.1. For $M_1, M_2 \geq 1$ such that $M_1 + M_2 \geq 5$, let $Y(M_1, M_2)$ be the modular curve over $\mathbb{Q}$, which represents the functor

$$S \mapsto \text{(the set of isomorphism classes of pairs } (E,\ \iota) \text{ where } E \text{ is an elliptic curve over } S \text{ and } \iota \text{ is an injective homomorphism } \mathbb{Z}/M_1\mathbb{Z} \times \mathbb{Z}/M_2\mathbb{Z} \to E \text{ of group schemes over } S).$$

For $M \geq 3$ such that $M_1|M$ and $M_2|M$, we have

$$Y(M_1, M_2) = G\backslash Y(M, M), \tag{4.2}$$

where G is the group $\{\begin{pmatrix} u & v \\ w & x \end{pmatrix} \in GL_2(\mathbb{Z}/M\mathbb{Z})\ ;\ u \equiv 1(M_1), v \equiv 0(M_1), w \equiv 0(M_2), x \equiv 1(M_2)\}$.
We define $Y(M_1, M_2)$ for $M_1, M_2 \geq 1$, $M_1 + M_2 < 5$, by (4.2).

4.1.2. We explain $A[[G_\infty]]$ for an abelian group A. For a set J, $\mathbb{Z}[J]$ denotes a free $\mathbb{Z}$-module on the set J. We define $G_n = (\mathbb{Z}/p^n\mathbb{Z})^\times$, $A[G_n] = A \otimes_{\mathbb{Z}} \mathbb{Z}[G_n]$, and $A[[G_\infty]] = \varprojlim_n A[G_n]$.

4.1.3. Let

$$\Omega^1_{S/\mathbb{Z}}(\log) = (\Omega^1_{S/\mathbb{Z}} \oplus S \otimes_{\mathbb{Z}} S'^{\times})/\mathcal{N},$$

where $\Omega^1_{S/\mathbb{Z}}$ is the module of the absolute differential forms and $\mathcal{N}$ is the S-submodule of the direct sum which is generated by elements $(-da, a \otimes a)$ for $a \in S \cap S'^{\times}$. In $\Omega^1_{S/\mathbb{Z}}(\log)$, we denote the class $(0, 1 \otimes a)$ for $a \in S'^{\times}$ by $d\log(a)$. For $r \geq 1$, let $\Omega^r_{S/\mathbb{Z}}(\log) = \bigwedge^r_S \Omega^1_{S/\mathbb{Z}}(\log)$, and define

$$\Omega^r_S(\log) = \varprojlim_n \Omega^r_{S/\mathbb{Z}}(\log)/p^n\Omega^r_{S/\mathbb{Z}}(\log).$$

Then we have $\Omega^1_S(\log)$ is a free S-module and

$$\Omega^1_S(\log) = S \cdot d\log(q) \oplus S \cdot d\log(\varepsilon - 1), \quad \Omega^2_S(\log) = S \cdot d\log(q) \wedge d\log(\varepsilon - 1),$$

$$\Omega^r_S(\log) = 0 \text{ for } \ r \geq 3.$$

4.1.4. We explain the definition of the map $\mathcal{E}_N$ in (4.1) (cf. [Fu1], §6). This map is induced by the following map for $n \geq 1$ satisfying $Np^n + p^n \geq 5$

$$K_2(Y(Np^n, p^n)) \longrightarrow K_2(\mathsf{H}_n)[G_n]\ \ ;\ \ x \mapsto \sum_{u \in G_n} (\sum_{w \in \mathbb{Z}/p^n\mathbb{Z}} x_{(u,w)})g_u.$$

Here $x_{(u,w)} \in K_2(\mathsf{H}_n)$ is the pull-back of x under the following composition:

$$\mathrm{Spec}(\mathsf{H}_n) \to \mathrm{Spec}(\mathsf{H}_n(q^{1/N})) \to Y(Np^n, p^n), \tag{4.3}$$

where $q^{1/N} \in \overline{\mathsf{H}}$ is a N-th root of q.

The first map in (4.3) is given by the homomorphism

$$\mathsf{H}_n(q^{1/N}) \to \mathsf{H}_n \ \ ; \quad \sum_{i=-\infty}^{\infty} a_i q^{i/Np^n} \mapsto \sum_{i=-\infty}^{\infty} a_i q^{i/p^n}.$$

We define the second map of (4.3). Let $\mathfrak{E}_q$ be the elliptic curve over O_{H} which is obtained from the Tate curve over $\mathbb{Z}[[q]][1/q]$ with q-invariant q. For each $m \geq 1$, we have ${}_m\mathfrak{E}_q(O_{\overline{\mathsf{H}}}) = \{q^{a/m}\zeta_m^b \bmod q^{\mathbb{Z}} \ ; \ a, b \in \mathbb{Z}\}$, where ${}_m\mathfrak{E}_q = \mathrm{Ker}(m : \mathfrak{E}_q \to \mathfrak{E}_q)$. Now we define the second map of (4.3) by the open immersion corresponding to

$$(\mathfrak{E}_q \otimes_{O_{\mathsf{H}}} \mathsf{H}_n(q^{1/N}),\ q^{u'/Np^n} \bmod q^{\mathbb{Z}}, q^{w/p^n}\zeta_{p^n} \bmod q^{\mathbb{Z}}) \quad \text{over } \mathsf{H}_n(q^{1/N}).$$

Here $u' \in (\mathbb{Z}/Np^n\mathbb{Z})^\times$ is the element such that $u' \equiv u(p^n)$ and $u' \equiv 1(N)$.

4.1.5. The second arrow of (4.1), which is an isomorphism, is by Theorem 2.4 in section 2 on K_2 Coleman power series.

4.1.6. We explain the map $d\log$ in (4.1). It is the map induced by the map $d\log : \hat{K}_2(S') \longrightarrow \Omega^2_S(\log)$ characterized by $\{\alpha_1, \alpha_2\} \mapsto d\log(\alpha_1) \wedge d\log(\alpha_2)$, where $\alpha_1, \alpha_2 \in S'^{\times}$ and $\{\alpha_1, \alpha_2\} \in \hat{K}_2(S')$ is the symbol. (The group $\hat{K}_2(S')$ is topologically generated by the symbols. Refer to [Fu1], §4, 4.11.)

4.1.7. We explain the last arrow in (4.1). We firstly define a map

$$S[[G_\infty]] \longrightarrow O_{\mathsf{H}}[[G_\infty^{(1)} \times G_\infty^{(2)}]] \tag{4.4}$$

to be the O_{H}-homomorphism associated to

$$\varepsilon^a g_u \mapsto \begin{cases} u^{-1} g_u^{(1)} g_{u^{-1}a}^{(2)} & \text{if } (a,p) = 1 \\ 0 & \text{if } (a,p) \neq 1, \end{cases}$$

for $a \in \mathbb{Z}$ and $u \in \mathbb{Z}_p^\times$. Here for $u \in \mathbb{Z}_p^\times$, $g_u^{(1)} \in G_\infty^{(1)}$ denotes the corresponding element, and $g_u^{(2)} \in G_\infty^{(2)}$ denotes the corresponding element to u via the cyclotomic character $\chi_{\mathrm{cyclo}} : G_\infty^{(2)} \xrightarrow{\cong} \mathbb{Z}_p^\times$. Next for an integer d' which is prime to p, let $\nu_{d'} : \frac{S}{\varepsilon - 1} \to \frac{S}{\varepsilon - 1}$ be the O_{H}-homomorphism given by sending $f(\varepsilon)/(\varepsilon - 1)$ ($f(\varepsilon) \in S$) to $f(\varepsilon^{d'})/(\varepsilon^{d'} - 1)$. It follows from the definition that the image $(1 - d'\nu_{d'})(\frac{S}{\varepsilon-1})$ is contained in S. Now the last map in (4.1) is defined as the composition

$$\frac{S}{\varepsilon - 1}[[G_\infty]] \xrightarrow{1-d'\nu_{d'}} S[[G_\infty]] \xrightarrow{(4.4)} O_{\mathsf{H}}[[G_\infty^{(1)} \times G_\infty^{(2)}]]$$
$$\xrightarrow{\cdot(1-d' g_{d'}^{(2)})^{-1}} Q(O_{\mathsf{H}}[[G_\infty^{(1)} \times G_\infty^{(2)}]]),$$

where $1 - d'\nu_{d'}$ is applied only for the coefficient $\frac{S}{\varepsilon-1}$ of G_∞. It is easily seen that the map $\mathcal{C}_N$ is independent of the choice of d'.

4.2. Let c and d be integers satisfying $(c, 6Np) = 1$ and $(d, 6p) = 1$. We review the norm compatible system of Beilinson elements defined by Kato in [Ka2]

$$({}_{c,d}z_{Np^n,p^n})_n \in \varprojlim_n K_2(Y(Np^n, p^n)), \tag{4.5}$$

where the projective limit is taken with respect to the norm maps, in the form which is enough for us here (the details are found in [Ka2]).

For $n \geq 1$ satisfying $Np^n + p^n \geq 5$, the element (4.5) is given as

$${}_{c,d}z_{Np^n,p^n} = \{{}_cg_{Np^n,0}, {}_dg_{0,p^n}\},$$

where ${}_cg_{Np^n,0} \in \mathcal{O}(Y(Np^n, 1))^\times$ and ${}_dg_{0,p^n} \in \mathcal{O}(Y(1, p^n))^\times$ are Siegel units, and we introduce their properties which are necessary for us here (see, for example, [Ka2] for the details).

For integers M and c such that $M \geq 3$ and $(c, 6M) = 1$, we have an element ${}_c\theta_E$ of $\mathcal{O}(E \setminus {}_cE)^\times$ which has the following property. Here E is the universal elliptic curve over $Y(M, M)$, ${}_cE = \mathrm{Ker}(c : E \to E)$, and $\mathcal{O}(E \setminus {}_cE)^\times$ is the affine ring. For $\tau \in \mathfrak{H}$ and $z \in \mathbb{C} \setminus c^{-1}(\mathbb{Z}\tau + \mathbb{Z})$, the value at z of ${}_c\theta_E$, on the elliptic curve $\mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$, is

$$q^{(1/12)(c^2-1)}(-t)^{(1/2)(c-c^2)} \cdot \gamma_q(t)^{c^2}\gamma_q(t^c)^{-1},$$

where $q = \exp(2\pi i\tau)$, $t = \exp(2\pi iz)$, and

$$\gamma_q(t) = \prod_{j\geq 0}(1 - q^j t)\prod_{j\geq 1}(1 - q^j t^{-1}).$$

Now the Siegel unit ${}_cg_{\alpha,\beta}$ for $(\alpha, \beta) = (a/M, b/M) \in (((1/M)\mathbb{Z})/\mathbb{Z})^2 \setminus \{(0, 0)\}$ $(a, b \in \mathbb{Z})$ may be defined by

$${}_cg_{\alpha,\beta} = \iota^*_{\alpha,\beta}({}_c\theta_E) \in \mathcal{O}(Y(M, M))^\times.$$

Here

$$\iota_{\alpha,\beta} = ae_1 + be_2 : Y(M, M) \to E \setminus {}_cE$$

with the canonical basis (e_1, e_2) of E over $Y(M, M)$. In the case $\alpha = 0$, ${}_cg_{0,\beta} \in \mathcal{O}(Y(1, M))^\times$ and in the case $\beta = 0$, ${}_cg_{\alpha,0} \in \mathcal{O}(Y(M, 1))^\times$.)

In [Ka2] (cf. [Sc]), it was shown that ${}_{c,d}z_{Np^n,p^n}$ $(n \geq 1)$ form a projective system with respect to the norm maps.

In the paper [Ka2], Kato always used norm compatible systems $({}_{c,d}z_{Mp^n,M'p^n})_n \in \varprojlim_n K_2(Y(Mp^n, M'p^n))$ $(M, M' \geq 1,\ (M + M')p^n \geq 5)$ satisfying the condition that $M|M'$ in application. However clearly the system $({}_{c,d}z_{Np^n,p^n})_n \in \varprojlim_n K_2(Y(Np^n, p^n))$ which we use does not satisfy this condition. When Kato used a system, for example, to construct a p-adic zeta function of an eigen cusp form f, he considered the "f^*-component" of the system, where f^* is the dual cusp form of f (see 6.5.1 in section 6 for the definition of the dual cusp form, and for the meaning of "component", refer to section 6). But our method needs to study the "f-component" of the system. So we must slightly modify his system in application.

4.3. We define a"universal zeta modular form"

$$z^{\mathrm{univ}}_{Np^\infty} \in \overline{M}_{Np^\infty}[[G^{(1)}_\infty \times G^{(2)}_\infty]][\frac{1}{g}](\subset O_{\mathsf{H}}[[G^{(1)}_\infty \times G^{(2)}_\infty]][\frac{1}{g}])$$

which yields special values at $s = r$ ($r \in \mathbb{Z}, 1 \leq r \leq k-1$) of the zeta functions of modular forms of weight $k \geq 2$ and level Np^t for $t \geq 0$. Here $\overline{M}_{Np^\infty}$ is as in section 3. Moreover $G^{(1)}_\infty \cong G^{(2)}_\infty \cong G_\infty = \mathbb{Z}_p^\times$, and $G^{(1)}_\infty$ is , as before, the group acting on the space of p-adic modular forms $\overline{M}_{Np^\infty}$ whose action is characterized by $g^{(1)}_a f = a^{k-2}\langle a' \rangle f$ for $f \in M_k(X_1(Np^t); \mathbb{Q}_p)$, $a \in \mathbb{Z}_p^\times$, and $a' \in (\mathbb{Z}/Np^t\mathbb{Z})^\times$ such that $a' \equiv a(p^t)$ and $a' \equiv 1(N)$. Here $\langle a' \rangle$ is as in 3.5. The group $G^{(2)}_\infty$ is the Galois group $\mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p)$.
We define $z^{\mathrm{univ}}_{Np^\infty}$ as an element of $O_{\mathsf{H}}[[G^{(1)}_\infty \times G^{(2)}_\infty]][1/g]$ and in 4.5.5, we will prove that it belongs to the subspace $\overline{M}_{Np^\infty}[[G^{(1)}_\infty \times G^{(2)}_\infty]][1/g]$.
Firstly we define $F_{N,1}, F_{N,2} \in \mathsf{H}[[G_\infty]] = \varprojlim_n \mathsf{H}[G_n]$ to be

$$F_{N,1} = (\sum_{\substack{i \geq 1 \\ (i,p)=1}} \sum_{j \geq 1} q^{Nij})(g_i - g_{-i}) + \varprojlim_n (\sum_{a \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \zeta_{\equiv a(p^n)}(0) \cdot g_a),$$

$$F_{N,2} = (\sum_{\substack{i \geq 1, i \equiv 1(N) \\ (i,p)=1}} \sum_{j \geq 1} q^{ij} \cdot g_i - \sum_{\substack{i \geq 1, i \equiv -1(N) \\ (i,p)=1}} \sum_{j \geq 1} q^{ij} \cdot g_{-i})$$
$$+ \varprojlim_n (\sum_{\substack{a \in (\mathbb{Z}/Np^n\mathbb{Z})^\times \\ a \equiv 1(N)}} \zeta_{\equiv a(Np^n)}(0) \cdot g_a).$$

Here for $a \in \mathbb{Z}_p^\times$ or $a \in (\mathbb{Z}/p^n\mathbb{Z})^\times$, g_a represents the corresponding element of G_∞ or G_n, respectively. For $M, m \in \mathbb{Z}$, $M \geq 1$, and $a \in \mathbb{Z}/M\mathbb{Z}$, $\zeta_{\equiv a(M)}(m)$ is the evaluation at $s = m$ of the partial Riemann zeta function

$$\zeta_{\equiv a(M)}(s) = \sum_{\substack{j \geq 1 \\ j \equiv a \bmod M}} j^{-s},$$

and $\sum_{\substack{a \in (\mathbb{Z}/Np^n\mathbb{Z})^\times \\ a \equiv 1(N)}} \zeta_{\equiv a(Np^n)}(0) \cdot g_a$ belongs to $H[G_n]$.
We define the product $F_{N,1} \cdot F_{N,2} \in \mathsf{H}[[G_\infty \times G_\infty]]$ naturally (by the rule $xg_a \cdot yg_b \mapsto xyg_{a,1}g_{b,2}$ with $x, y \in \mathsf{H}$, $a, b \in \mathbb{Z}_p^\times$, where $g_{a,1}$ (resp, $g_{b,2}$) means the corresponding element of the first (resp. the second) G_∞).
Now we define the universal zeta modular form $z^{\mathrm{univ}}_{Np^\infty}$ to be the image of $F_{N,1} \cdot F_{N,2}$ under the isomorphism of rings over H

$$\mathsf{H}[[G_\infty \times G_\infty]] \to \mathsf{H}[[G^{(1)}_\infty \times G^{(2)}_\infty]] \; ; \tag{4.6}$$

$$xg_{a,1}g_{b,2} \mapsto xg^{(1)}_b g^{(2)}_{ab^{-1}} \quad (x \in \mathsf{H}, a, b \in \mathbb{Z}_p^\times)$$

$$(F_{N,1} \cdot F_{N,2} \mapsto z^{\mathrm{univ}}_{Np^\infty}).$$

For integers c, d' such that $(c,p)=1$ and $(d',p)=1$, we have

$$(1-c^{-1}g^{(1)}_{c^{-1}}g^{(2)}_{c})(1-d'g^{(2)}_{d'})z^{\mathrm{univ}}_{Np^\infty}\in O_{\mathsf{H}}[[G^{(1)}_\infty\times G^{(2)}_\infty]]. \tag{4.7}$$

This follows from the fact that

$$(1-c^{-1}g_{c^{-1}})\cdot\varprojlim_n \sum_{\substack{a\in(\mathbb{Z}/Np^n\mathbb{Z})^\times\\ a\equiv 1(N)}}\zeta_{\equiv a(Np^n)}(0)\cdot g_a\in O_{\mathsf{H}}[[G_\infty]].$$

By (4.7), we obtain that $z^{\mathrm{univ}}_{Np^\infty}\in O_{\mathsf{H}}[[G^{(1)}_\infty\times G^{(2)}_\infty]][1/g]$ for a non-zero divisor $g\in\mathbb{Z}[[G^{(1)}_\infty\times G^{(2)}_\infty]]$.

Thus the universal zeta modular form is something like a product of two "Λ-adic Eisenstein series" in the sense of Hida in [Hi3], Chapter 7, §7.1.

The following proposition describes the relation between the norm compatible system of Beilinson elements and the universal zeta modular form.

PROPOSITION 4.4 . *Let c and d be as in* 4.2. *We further assume that $c\equiv 1$ mod N. Then we have*

$$\mathcal{C}_N(({}_{c,d}z_{Np^n,p^n})_n)=(c^2-cg^{(1)}_{c^{-1}}g^{(2)}_{c})(d^2-dg^{(2)}_{d})\cdot z^{\mathrm{univ}}_{Np^\infty}.$$

Proof. Firstly we consider a composition $\mathcal{C}'_{N,d'}$ determined by the relation that $(1-d'g^{(2)}_{d'})\cdot\mathcal{C}_N=(4.6)\circ\mathcal{C}'_{N,d'}$, where d' is, as in 4.1.7, an integer which is prime to p. Namely, $\mathcal{C}'_{N,d'}$ is as follows:

$$\begin{aligned}\mathcal{C}'_{N,d'}:\varprojlim_n K_2(Y(Np^n,p^n))&\to\frac{S}{\varepsilon-1}[[G_\infty]]\\ &\xrightarrow{s} O_{\mathsf{H}}[[G_\infty]][[G_\infty]],\end{aligned}$$

where the first arrow is the composition of the first four maps in (4.1), and the map s is defined by the composition

$$s:\frac{S}{\varepsilon-1}[[G_\infty]]\xrightarrow{1-d'\nu_{d'}}S[[G_\infty]]\to O_{\mathsf{H}}[[G_\infty]][[G_\infty]].$$

Here the second map is the O_{H}-homomorphism associated to

$$\varepsilon^a g_u\mapsto\begin{cases}u^{-1}g_{a,1}g_{u,2} & \text{if } (a,p)=1\\ 0 & \text{if } (a,p)\neq 1,\end{cases}$$

for $a\in\mathbb{Z}$ and $u\in\mathbb{Z}_p^\times$.

For the proof of Proposition 4.4, by the definition of $z^{\mathrm{univ}}_{Np^\infty}$, our task is to show that under the map $\mathcal{C}'_{N,d'}$, the norm compatible system of Beilinson elements $({}_{c,d}z_{Np^n,p^n})_n$ is sent to $(c^2-cg_{c^{-1},2})(d^2-dg_{d,1})(1-d'g_{d',1})\cdot F_{N,1}\cdot F_{N,2}$, where $F_{N,1}, F_{N,2}$ are as in the definition of $z^{\mathrm{univ}}_{Np^\infty}$.

We prove the above assertion by showing the result of the computation of the image of $({}_{c,d}z_{Np^n,p^n})_n$ under each step in the composition defining $\mathcal{C}'_{N,d'}$.

Step 1. Firstly we compute the image of ${}_{c,d}z_{Np^n,p^n}=\{{}_cg_{Np^n,0},{}_dg_{0,p^n}\}$ under $K_2(Y(Np^n,p^n))\to K_2(\mathsf{H}_n(q^{1/N}))$ which is given by the pull-back by the latter map of (4.3). Directly from the definition, we have that the image is $\{A_1, B_1\} \in K_2(\mathsf{H}_n(q^{1/N}))$. Here the element $A_1 \in \mathsf{H}_n(q^{1/N})^\times$ is obtained from ${}_c\theta_E$, where E is the universal elliptic curve over $Y(Np^n, Np^n)$, by putting $t = q^{u'/Np^n}$ with $u' \in (\mathbb{Z}/Np^n\mathbb{Z})^\times$ such that $u' \equiv u(p^n)$ and $u' \equiv 1(N)$. The element $B_1 \in \mathsf{H}_n(q^{1/N})^\times$ is obtained from ${}_d\theta_E$, where E is the universal elliptic curve over $Y(p^n, p^n)$ by putting $t = q^{w/p^n}\zeta_{p^n}$. From this, it is easy to have the following result:

$$\begin{aligned}
&\mathcal{E}_N(({}_{c,d}z_{Np^n,p^n})_n) \\
&= (\{ \prod_{u_n \in (\mathbb{Z}/p^n\mathbb{Z})^\times} q^{(N/12)(c^2-1)}(-q^{u'_n/p^n})^{(1/2)(c-c^2)}\gamma_{q^N}(q^{u'_n/p^n})^{c^2}\gamma_{q^N}(q^{cu'_n/p^n})^{-1}, \\
&\quad \prod_{w_n \in \mathbb{Z}/p^n\mathbb{Z}} q^{(N/12)(d^2-1)}(-q^{Nw'_n/p^n}\zeta_{p^n})^{(1/2)(d-d^2)} \\
&\quad\quad \gamma_{q^N}(q^{Nw'_n/p^n}\zeta_{p^n})^{d^2}\gamma_{q^N}(q^{dNw'_n/p^n}\zeta_{p^n}^d)^{-1}\}g_{u_n})_n \\
&\quad\quad\quad \in \varprojlim_n \hat{K}_2(\mathsf{H}_n)[G_n] = (\varprojlim_n \hat{K}_2(\mathsf{H}_n))[[G_\infty]],
\end{aligned}$$

where for $u_n \in (\mathbb{Z}/p^n\mathbb{Z})^\times$, u'_n is an integer such that $u'_n \equiv 1(N)$ and $u'_n \equiv u_n(p^n)$, and for $w_n \in \mathbb{Z}/p^n\mathbb{Z}$, w'_n is an integer such that $w'_n \equiv w_n(p^n)$.
Step 2. From the definition, we obtain that under the notation in the composition (4.1)

$$\mathrm{Col} \circ \mathcal{E}_N(({}_{c,d}z_{Np^n,p^n})_n) \in \hat{K}_2(S')^{\mathrm{N}=1}[[G_\infty]]$$

coincides with the image of (A_2, B_2) with $A_2 \in O_{\mathsf{H}}^\times[[G_\infty]]$ and $B_2 \in S'^\times$ given below under the natural map

$$O_{\mathsf{H}}^\times[[G_\infty]] \times S'^\times \longrightarrow \hat{K}_2(S')[[G_\infty]]\ ;\ (x_u g_u,\ y) \mapsto \{x_u, y\}g_u.$$

The elements A_2, B_2 are as follows:

$$\begin{aligned}
A_2 = \varprojlim_n (&\prod_{u_n \in (\mathbb{Z}/p^n\mathbb{Z})^\times} (-q^{(-c^2\zeta_{\equiv u'_n(Np^n)}(-1)+\zeta_{\equiv cu'_n(Np^n)}(-1))})g_{u_n}) \\
&\cdot \prod_{\substack{i\geq 1, i\equiv 1(N)\\ (i,p)=1}} (1-q^i)^{c^2}g_i \cdot \prod_{\substack{i\geq 1, i\equiv -1(N)\\ (i,p)=1}} (1-q^i)^{c^2}g_{-i} \\
&\cdot \prod_{\substack{i\geq 1, i\equiv 1(N)\\ (i,p)=1}} (1-q^i)^{-1}g_{c^{-1}i} \cdot \prod_{\substack{i\geq 1, i\equiv -1(N)\\ (i,p)=1}} (1-q^i)^{-1}g_{-c^{-1}i},
\end{aligned}$$

where u'_n is as before.

$$B_2 = \varprojlim_n \Big(\prod_{w_n \in \mathbb{Z}/p^n\mathbb{Z}} (-q^{N(-d^2\zeta_{\equiv w_n(p^n)}(-1)+\zeta_{\equiv dw_n(p^n)(-1)})}\Big) \cdot \prod_{i\geq 1}(1-q^{Ni}\varepsilon)^{d^2} \cdot \prod_{i\geq 1}(1-q^{Ni}\varepsilon^{-1})^{d^2} \cdot \prod_{i\geq 1}(1-q^{Ni}\varepsilon^{d})^{-1} \cdot \prod_{i\geq 1}(1-q^{Ni}\varepsilon^{-d})^{-1} \cdot (1-\varepsilon)^{d^2}(1-\varepsilon^d)^{-1}\varepsilon^{(1/2)(d-d^2)}.$$

Step 3. By the definition of the map $d\log$, we find easily that

$$d\log \circ \mathrm{Col} \circ \mathcal{E}_N(({}_{c,d}z_{Np^n,p^n})_n) \in \Omega^2_S(\log)[[G_\infty]]$$

coincides with the image of (A_3, B_3) with $A_3 \in O_{\mathsf{H}}[[G_\infty]]$ and $B_3 \in \frac{S}{\varepsilon-1}$ given below under the map

$$O_{\mathsf{H}}[[G_\infty]] \times \frac{S}{\varepsilon - 1} \longrightarrow \Omega^2_S(\log)[[G_\infty]] \ ;$$

$$(xg_u, y) \mapsto xy \cdot d\log(q) \wedge d\log(\varepsilon) \cdot g_u$$

for $x \in O_{\mathsf{H}}$, $u \in \mathbb{Z}_p^\times$, $y \in \frac{S}{\varepsilon-1}$.
The elements A_3, B_3 are as follows:

$$A_3 = \Big(\sum_{\substack{i\geq 1, i\equiv 1(N)\\(i,p)=1}} \sum_{j\geq 1} iq^{ij}\Big)(c^2\cdot g_i - g_{c^{-1}i}) + \Big(\sum_{\substack{i\geq 1, i\equiv -1(N)\\(i,p)=1}} \sum_{j\geq 1} iq^{ij}\Big)(c^2\cdot g_{-i} - g_{-c^{-1}i}) + \varprojlim_n\Big(\sum_{\substack{a\in(\mathbb{Z}/Np^n\mathbb{Z})^\times\\a\equiv 1(N)}} \zeta_{\equiv a(Np^n)}(-1)(c^2\cdot g_a - g_{c^{-1}a})\Big).$$

$$B_3 = d^2\Big(\sum_{i\geq 1}\sum_{j\geq 1} q^{Nij}\varepsilon^j - \sum_{i\geq 1}\sum_{j\geq 1} q^{Nij}\varepsilon^{-j}\Big) - d\Big(\sum_{i\geq 1}\sum_{j\geq 1} q^{Nij}\varepsilon^{dj} - \sum_{i\geq 1}\sum_{j\geq 1} q^{Nij}\varepsilon^{-dj}\Big) + d^2\frac{\varepsilon}{1-\varepsilon} - d\frac{\varepsilon^d}{1-\varepsilon^d} + \frac{1}{2}(d-d^2).$$

Step 4. By the definition of the map s, we see that

$$s\circ d\log \circ \mathrm{Col} \circ \mathcal{E}_N(({}_{c,d}z_{Np^n,p^n})_n) \in O_{\mathsf{H}}[[G_\infty]][[G_\infty]]$$

coincides with the image of $((1-d'g_{d'})B_4, A_4)$ with $A_4 \in O_{\mathsf{H}}[[G_\infty]]$ and $B_4 \in Q(O_{\mathsf{H}}[[G_\infty]])$ $((1-d'g_{d'})B_4 \in O_{\mathsf{H}}[[G_\infty]])$ given below under the natural O_{H}-homomorphism

$$O_{\mathsf{H}}[[G_\infty]] \times O_{\mathsf{H}}[[G_\infty]] \longrightarrow O_{\mathsf{H}}[[G_\infty]][[G_\infty]] \ ;$$
$$(xg_a, yg_b) \mapsto xyg_{a,1}g_{b,2} \quad (a, b \in \mathbb{Z}_p^\times).$$

The elements A_4, B_4 are as follows:

$$\begin{aligned} A_4 = &\Big(\sum_{\substack{i\geq 1, i\equiv 1(N)\\ (i,p)=1}} \sum_{j\geq 1} q^{ij}\Big)(c^2\cdot g_i - c\cdot g_{c^{-1}i}) \\ &-\Big(\sum_{\substack{i\geq 1, i\equiv -1(N)\\ (i,p)=1}} \sum_{j\geq 1} q^{ij}\Big)(c^2\cdot g_{-i} - c\cdot g_{-c^{-1}i}) \\ &+\varprojlim_n\Big(\sum_{\substack{a\in(\mathbb{Z}/Np^n\mathbb{Z})^\times\\ a\equiv 1(N)}} \zeta_{\equiv a(Np^n)}(0)(c^2\cdot g_a - c\cdot g_{c^{-1}a})\Big). \end{aligned}$$

$$\begin{aligned} B_4 = d^2\Big(&\sum_{i\geq 1}\sum_{\substack{j\geq 1\\ (j,p)=1}} q^{Nij}g_j - \sum_{i\geq 1}\sum_{\substack{j\geq 1\\ (j,p)=1}} q^{Nij}g_{-j}\Big) \\ &- d\Big(\sum_{i\geq 1}\sum_{\substack{j\geq 1\\ (j,p)=1}} q^{Nij}g_{dj} - \sum_{i\geq 1}\sum_{\substack{j\geq 1\\ (j,p)=1}} q^{Nij}g_{-dj}\Big) \\ &+ d^2\varprojlim_n\Big(\sum_{a_n\in(\mathbb{Z}/p^n\mathbb{Z})^\times} \zeta_{\equiv a_n(p^n)}(0)\cdot g_{a_n}\Big) \\ &- d\varprojlim_n\Big(\sum_{a_n\in(\mathbb{Z}/p^n\mathbb{Z})^\times} \zeta_{\equiv a_n(p^n)}(0)\cdot g_{da_n}\Big). \end{aligned}$$

By comparing A_4 with F_1 and B_4 with F_2, we obtain the assertion of Proposition 4.4. □

4.5. We prove that $z^{\mathrm{univ}}_{Np^\infty}$ which has been defined as an element of $O_{\mathsf{H}}[[G^{(1)}_\infty \times G^{(2)}_\infty]][[1/g]$, in fact, belongs to the subspace $\overline{M}_{Np^\infty}[[G^{(1)}_\infty \times G^{(2)}_\infty]][1/g]$. We further show the relation between $z^{\mathrm{univ}}_{Np^\infty}$ and special values of zeta functions of cusp forms. Preceding this, in 4.5.1 – 4.5.4, we review the zeta modular forms in [Ka2], which were defined basing on the works of Shimura [Sh], and whose period integrals yield special values of the zeta functions of cusp forms. In 4.5.5, we show that $z^{\mathrm{univ}}_{Np^\infty}$ is contained in the subspace $\overline{M}_{Np^\infty}[[G^{(1)}_\infty \times G^{(2)}_\infty]][1/g]$, and then in 4.5.6, we describe the relation between $z^{\mathrm{univ}}_{Np^\infty}$ and the zeta modular forms reviewed in 4.5.1 – 4.5.4.

4.5.1. We review some Eisenstein series appearing, for example, in [Ka2], §3. For $M_1, M_2 \geq 1$ such that $M_1 + M_2 \geq 5$, as before, let $M_j(X(M_1, M_2))$ be the space of modular forms on $X(M_1, M_2)$ of weight $j \geq 1$.

Let $M \geq 3$, and $x, y \in ((1/M)\mathbb{Z})/\mathbb{Z}$. We review the q-expansions of Eisenstein series

$$F^{(j)}_{x,y}\ ((j,x,y) \neq (2,0,0)),\ E^{(j)}_{x,y}\ (j \neq 2),\ \tilde{E}^{(2)}_{x,y} \in M_j(X(M,M)),$$

following Kato [Ka2], §4. (In the case $x = 0$, these modular forms are, in fact, elements of $M_j(X(1,M))$ and in the case $y = 0$, they are, in fact, elements of $M_j(X(M,1))$.) For $\gamma \in \mathbb{Q}/\mathbb{Z}$, we define

$$\zeta(\gamma, s) = \sum_{\substack{m \in \mathbb{Q}, m>0 \\ m \bmod \mathbb{Z} \equiv \gamma}} m^{-s}, \qquad \zeta^*(\gamma, s) = \sum_{m=1}^{\infty} \exp(2\pi i \gamma m) \cdot m^{-s}.$$

For (i) $F = F^{(j)}_{x,y}$ $((j,x,y) \neq (2,0,0))$, (ii) $F = E^{(j)}_{x,y}$ $(j \neq 2)$, or (iii) $F = \tilde{E}^{(2)}_{x,y}$, we write $F = \sum_{\substack{m \in \mathbb{Q} \\ m \geq 0}} a_m q^m$ $(q = \exp(2\pi i \tau))$.

In the case of (i), we assume that $(j,x,y) \neq (2,0,0)$. Then a_m for $m > 0$ can be obtained from the equation

$$\sum_{\substack{m \in \mathbb{Q} \\ m>0}} a_m m^{-s} = \zeta(x, s-j+1)\zeta^*(y,s) + (-1)^j \zeta(-x, s-j+1)\zeta^*(-y,s).$$

In the case $j \neq 1$, $a_0 = \zeta(x, 1-j)$.
In the case $j = 1$, $a_0 = \zeta(x,0)$ if $x \neq 0$, and $a_0 = (1/2)(\zeta^*(y,0) - \zeta^*(-y,0))$ if $x = 0$.

In the case of (ii), we assume that $j \neq 2$. Then a_m for $m > 0$ can be obtained from the equation

$$\sum_{\substack{m \in \mathbb{Q} \\ m>0}} a_m m^{-s} = \zeta(x,s)\zeta^*(y, s-j+1) + (-1)^j \zeta(-x,s)\zeta^*(-y, s-j+1).$$

In the case $j \neq 1$, $a_0 = 0$ if $x \neq 0$, and $a_0 = \zeta^*(y, 1-j)$ if $x = 0$.
In the case $j = 1$, $a_0 = \zeta(x,0)$ if $x \neq 0$, and $a_0 = (1/2)(\zeta^*(y,0) - \zeta^*(-y,0))$ if $x = 0$.

In the case of (iii), the a_m for $m > 0$ can be obtained from the equation

$$\sum_{\substack{m \in \mathbb{Q} \\ m>0}} a_m m^{-s} = \zeta(x,s)\zeta^*(y,s-1) + \zeta(-x,s)\zeta^*(-y,s-1) - 2\zeta(s)\zeta(s-1).$$

If $x \neq 0$, $a_0 = 0$, and if $x = 0$, $a_0 = \zeta^*(y,-1) - \zeta(-1)$.

4.5.2. We review the zeta modular forms in [Ka2], §§4 and 5, which were defined basing on the works of Shimura [Sh]. These zeta modular forms yield special values of zeta functions of modular forms by period integrals (concerning this, refer to [Ka2], §5).
Let k, r, m, n be integers such that $k \geq 2$, $1 \leq r \leq k-1$, $1 \leq m \leq n$, and $N(p^n + p^m) \geq 5$. Further for an integer M, let prime(M) denote the set of all of the prime divisors of M.
In the case $r \neq 2$, the zeta modular forms are as follows:

$$z^{(k,r)}_{Np^n,Np^n}(k,r,k-1)$$
$$= (r-1)!^{-1}\cdot(Np^n)^{k-r-2}(Np^n)^{-r}\cdot F^{(k-r)}_{1/Np^n,0}\cdot E^{(r)}_{0,1/Np^n}\in M_k(X(Np^n,Np^n)),$$

$$z^{(k,r)}_{1,Np^m,Np^n}(k,r,k-1,0(1),\mathrm{prim}(Np))$$
$$=\mathrm{Tr}_{Np^m}(z^{(k,r)}_{Np^n,Np^n}(k,r,k-1,0(1),\mathrm{prim}(Np)))\in M_k(X_1(Np^m);\mathbb{Q}(\zeta_{Np^n})).$$

We remark that $M_k(X_1(Np^m);\mathbb{Q}(\zeta_{Np^n}))$ may be regarded as the fixed part of $M_k(X(Np^n,Np^n))$ by the group $\{\begin{pmatrix} u & v \\ w & x\end{pmatrix}\in GL_2(\mathbb{Z}/Np^n\mathbb{Z})\ ;\ u\equiv 1(Np^m), w\equiv 0(Np^m), ux-vw\equiv 1(Np^n)\}$. In the above

$$\mathrm{Tr}_{Np^m}: M_k(X(Np^n,Np^n))\longrightarrow M_k(X_1(Np^m);\mathbb{Q}(\zeta_{Np^n}))$$

denotes the trace map.
Let c and d be integers such that $(c,Np)=1$ and $(d,p)=1$. In the case $r=2$, the zeta modular forms are as follows:

$$\begin{aligned}{}_{c,d}z^{(k,2)}_{Np^n,Np^n}&(k,2,k-1)\\ &=(Np^n)^{k-4}(Np^n)^{-2}c^2d^2\\ &\quad\cdot(F^{(k-2)}_{1/Np^n,0}-c^{2-k}\cdot F^{(k-2)}_{c/Np^n,0})\cdot(\tilde{E}^{(2)}_{0,1/Np^n}-\tilde{E}^{(2)}_{0,d/Np^n})\\ &\qquad\in M_k(X(Np^n,Np^n))),\end{aligned}$$

$$\begin{aligned}{}_{c,d}z^{(k,2)}_{1,Np^m,Np^n}&(k,2,k-1,0(1),\mathrm{prim}(Np))\\ &=\mathrm{Tr}_{Np^m}({}_{c,d}z^{(k,2)}_{Np^n,Np^n}(k,2,k-1,0(1),\mathrm{prim}(Np)))\\ &\qquad\in M_k(X_1(Np^m);\mathbb{Q}(\zeta_{Np^n})).\end{aligned}$$

The above zeta modular forms provide the value at $s=r$ of the zeta functions of modular forms of weight k by period integrals.
In our method, modular forms whose q-expansions belong to $\mathbb{Z}_{(p)}[[q]]$ or $\mathbb{Q}[[q]]$ are important. So we analyze zeta modular forms from this viewpoint. Firstly for $j\in\mathbb{Z}$, $j\geq 1$, and $a\in((1/M)\mathbb{Z})/\mathbb{Z}$ satisfying $(j,a)\neq(2,0)$, directly from the definition we have

$$\sum_{x\in((1/M)\mathbb{Z})/\mathbb{Z}} F^{(j)}_{a,x}\in M_j(X_1(M);\mathbb{Q}).$$

LEMMA 4.5.3 . *We assume that $r\neq 2$.*

(1) *In* $M_k(X_1(Np^n);\mathbb{Q}(\zeta_{Np^n}))$*, we have*

$$\begin{aligned}
&z^{(k,r)}_{1,Np^n,Np^n}(k,r,k-1,0(1),\mathrm{prim}(Np))\\
&\quad= (r-1)!^{-1}\cdot(Np^n)^{k-r-2}(Np^n)^{-r}\cdot \sum_{x\in((1/Np^n)\mathbb{Z})/\mathbb{Z}} F^{(k-r)}_{1/Np^n,x}\cdot E^{(r)}_{0,1/Np^n}\\
&\quad= (r-1)!^{-1}\cdot(Np^n)^{k-r-2}(Np^n)^{-2}\\
&\qquad\cdot(\sum_{a\in\mathbb{Z}/Np^n\mathbb{Z}}(\sum_{x,y\in((1/Np^n)\mathbb{Z})/\mathbb{Z}} F^{(k-r)}_{1/Np^n,x}\cdot F^{(r)}_{a/Np^n,y})\cdot\zeta^a_{Np^n}.
\end{aligned}$$

(2) *Let*

$$\mathrm{Tr}_{Np^n,Np^m}: M_k(X_1(Np^n);\mathbb{Q}(\zeta_{Np^n}))\longrightarrow M_k(X_1(Np^m);\mathbb{Q}(\zeta_{Np^n}))$$

be the trace map. In $M_k(X_1(Np^m);\mathbb{Q}(\zeta_{Np^n}))$*, we have*

$$\begin{aligned}
&z^{(k,r)}_{1,Np^m,Np^n}(k,r,k-1,0(1),\mathrm{prim}(Np))\\
&\quad= \mathrm{Tr}_{Np^n,Np^m}(z^{(k,r)}_{1,Np^n,Np^n}(k,r,k-1,0(1),\mathrm{prim}(Np)))\\
&\quad= (r-1)!^{-1}\cdot(Np^m)^{k-r-2}(Np^n)^{-2}\\
&\qquad\cdot(\sum_{a\in\mathbb{Z}/Np^n\mathbb{Z}} T(p)^{n-m}\\
&\qquad\quad(\sum_{x\in((1/Np^m)\mathbb{Z})/\mathbb{Z}}\ \sum_{y\in((1/Np^n)\mathbb{Z})/\mathbb{Z}}(F^{(k-r)}_{1/Np^m,x}\cdot F^{(r)}_{a/Np^n,y}))\cdot\zeta^a_{Np^n}).
\end{aligned}$$

Here $T(p)=U(p)$ *is the Hecke operator on the space* $M_k(X_1(Np^n);\mathbb{Q})$.

(3) *Let*

$$\mathrm{tr}_{Np^m,p^n}: M_k(X_1(Np^m);\mathbb{Q}(\zeta_{Np^n}))\longrightarrow M_k(X_1(Np^m);\mathbb{Q}(\zeta_{p^n}))$$

be the trace map. In $M_k(X_1(Np^m);\mathbb{Q}(\zeta_{p^n}))$*, we have*

$$\begin{aligned}
&\mathrm{tr}_{Np^n,p^n}(z^{(k,r)}_{1,Np^m,Np^n}(k,r,k-1,0(1),\mathrm{prim}(Np)))\\
&\quad= (r-1)!^{-1}\cdot(Np^m)^{k-r-2}(Np^n)^{-2}\cdot N\\
&\qquad\cdot\prod_{\substack{l:\mathrm{prime}\\ l|N}}(1-l^{-r}T(l)\nu_{l^{-1}})\\
&\qquad(\sum_{a\in\mathbb{Z}/p^n\mathbb{Z}} T(p)^{n-m}\\
&\qquad\quad(\sum_{x\in((1/Np^m)\mathbb{Z})/\mathbb{Z}}\ \sum_{y\in((1/Np^n)\mathbb{Z})/\mathbb{Z}}(F^{(k-r)}_{1/Np^m,x}\cdot F^{(r)}_{Na/Np^n,y}))\cdot\zeta^a_{p^n}),
\end{aligned}\tag{4.8}$$

where for $x\in(\mathbb{Z}/p^n\mathbb{Z})^\times$, ν_x *is the corresponding element of* $\mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q})$ *via the cyclotomic character.*

Proof. (1) The first equality is direct from the definition. The equality ($r \neq 2$)

$$E^{(r)}_{0,1/Np^n} = (Np^n)^{r-2} \cdot \sum_{a\in\mathbb{Z}/Np^n\mathbb{Z}} \sum_{y\in((1/Np^n)\mathbb{Z})/\mathbb{Z}} F^{(r)}_{a/Np^n,y} \cdot \zeta^a_{Np^n}$$

which can be obtained by computation, shows the second equality.

(2) (3) The results are immediate from the definitions. □

The following Lemma 4.5.4 describes the case $r = 2$.

Lemma 4.5.4 . *We use the same notation as in Lemma* 4.5.3*.*

(1) *In* $M_k(X_1(Np^n);\mathbb{Q}(\zeta_{Np^n}))$*, we have*

$$\begin{aligned}
&{}_{c,d}z^{(k,2)}_{1,Np^n,Np^n}(k,2,k-1,0(1),\mathrm{prim}(Np))\\
&\quad = (Np^n)^{k-4}(Np^n)^{-2}c^2d^2\\
&\quad \cdot \sum_{\substack{a\in\mathbb{Z}/Np^n\mathbb{Z}\\ a\neq 0}} \sum_{x,y\in((1/Np^n)\mathbb{Z})/\mathbb{Z}} (F^{(k-2)}_{1/Np^n,x} - c^{2-k}\cdot F^{(k-2)}_{c/Np^n,x})\\
&\qquad \cdot (F^{(2)}_{a/Np^n,y}\cdot\zeta^a_{Np^n} - F^{(2)}_{a/Np^n,y}\cdot\zeta^{da}_{Np^n}).
\end{aligned}$$

(2) *In* $M_k(X_1(Np^m);\mathbb{Q}(\zeta_{Np^n}))$*, we have*

$$\begin{aligned}
&{}_{c,d}z^{(k,2)}_{1,Np^m,Np^n}(k,2,k-1,0(1),\mathrm{prim}(Np))\\
&\quad = (Np^m)^{k-4}(Np^n)^{-2}c^2d^2\\
&\quad \cdot T(p)^{n-m}\Big(\sum_{\substack{a\in\mathbb{Z}/Np^n\mathbb{Z}\\ a\neq 0}} \sum_{x\in((1/Np^m)\mathbb{Z})/\mathbb{Z}} \sum_{y\in((1/Np^n)\mathbb{Z})/\mathbb{Z}}\\
&\qquad ((F^{(k-2)}_{1/Np^m,x} - c^{2-k}\cdot F^{(k-2)}_{c/Np^m,x})\cdot(F^{(2)}_{a/Np^n,y}\cdot\zeta^a_{Np^n} - F^{(2)}_{a/Np^n,y}\cdot\zeta^{da}_{Np^n}))\Big).
\end{aligned}$$

(3) *In* $M_k(X_1(Np^m);\mathbb{Q}(\zeta_{p^n}))$*, we have*

$$\begin{aligned}
&\mathrm{tr}_{Np^n,p^n}({}_{c,d}z^{(k,2)}_{1,Np^m,Np^n}(k,2,k-1,0(1),\mathrm{prim}(Np)))\\
&\quad = (Np^m)^{k-4}(Np^n)^{-2}c^2d^2\cdot N\\
&\qquad \cdot \prod_{\substack{l:\mathrm{prime}\\ l\mid N}} (1 - l^{-r}T(l)\nu_{l^{-1}})\\
&\quad (T(p)^{n-m}\Big(\sum_{\substack{a\in\mathbb{Z}/p^n\mathbb{Z}\\ a\neq 0}} \sum_{x\in((1/Np^m)\mathbb{Z})/\mathbb{Z}} \sum_{y\in((1/Np^n)\mathbb{Z})/\mathbb{Z}} \qquad (4.9)\\
&\qquad ((F^{(k-2)}_{1/Np^m,x} - c^{2-k}\cdot F^{(k-2)}_{c/Np^m,x})\cdot(F^{(2)}_{Na/Np^n,y}\cdot\zeta^a_{p^n} - F^{(2)}_{Na/Np^n,y}\cdot\zeta^{da}_{p^n}))\Big)).
\end{aligned}$$

Proof. (1) The equality follows from the equality

$$\tilde{E}^{(2)}_{0,1/Np^n} - \tilde{E}^{(2)}_{0,d/Np^n} = \sum_{\substack{a\in\mathbb{Z}/Np^n\mathbb{Z}\\ a\neq 0}} \sum_{y\in((1/Np^n)\mathbb{Z})/\mathbb{Z}} (F^{(2)}_{a/Np^n,y}\cdot\zeta^a_{Np^n} - F^{(2)}_{a/Np^n,y}\cdot\zeta^{da}_{Np^n})$$

which can be obtained by computation.

(2) (3) The results are immediate from the definitions.

□

4.5.5. We prove that $z^{\mathrm{univ}}_{Np^\infty} \in \overline{M}_{Np^\infty}[[G^{(1)}_\infty \times G^{(2)}_\infty]][1/g]$.
For any $j \in \mathbb{Z}$, we define an isomorphism of rings

$$\chi^j : \mathbb{Z}_p[[G_\infty]] \underset{\cong}{\longrightarrow} \mathbb{Z}_p[[G_\infty]]\ ;\ g_a \mapsto a^j g_a \qquad (a \in \mathbb{Z}_p^\times),$$

and for $a_1, a_2 \in \mathbb{Z}$, we define an isomorphism of rings over O_{H}

$$O_{\mathsf{H}}[[G^{(1)}_\infty \times G^{(2)}_\infty]] \xrightarrow{(\chi^{a_1},\chi^{a_2})} O_{\mathsf{H}}[[G^{(1)}_\infty \times G^{(2)}_\infty]]\ ;$$

$$x\cdot g^{(1)}_{b_1}\cdot g^{(2)}_{b_2} \mapsto x b_1^{a_1} b_2^{a_2}\cdot g^{(1)}_{b_1}\cdot g^{(2)}_{b_2}$$

for $x \in O_{\mathsf{H}}$, $b_1, b_2 \in \mathbb{Z}_p^\times$.
Let c and d' be integers which are prime to p. We put

$$_{c,d'}z^{\mathrm{univ}}_{Np^\infty} = (1 - c^{-1}g^{(1)}_{c^{-1}}g^{(2)}_c)(1 - d' g^{(2)}_{d'})z^{\mathrm{univ}}_{Np^\infty} \in O_{\mathsf{H}}[[G^{(1)}_\infty \times G^{(2)}_\infty]].$$

Let a_1 and a_2 be integers such that $0 \le a_2 \le a_1$. We regard $((1 - c^{a_2-a_1-1}g^{(1)}_{c^{-1}}g^{(2)}_c)(1 - d'^{a_2+1}g^{(2)}_{d'}))^{-1} \cdot {}_{c,d'}z^{\mathrm{univ}}_{Np^\infty}(\chi^{a_1},\chi^{a_2})$ as an element of $\mathsf{H}[[G^{(1)}_\infty \times G^{(2)}_\infty]]$ and write $z^{\mathrm{univ}}_{Np^\infty}(\chi^{a_1},\chi^{a_2})$ for it. Then directly from the definitions, $z^{\mathrm{univ}}_{Np^\infty}(\chi^{a_1},\chi^{a_2})$ coincides with the image of the product

$$\begin{aligned}
&((\sum_{\substack{i\ge 1\\ (i,p)=1}}\sum_{j\ge 1} i^{a_2}q^{Nij})(g_{i,1} - (-1)^{a_2}g_{-i,1}) + \varprojlim_n(\sum_{a\in(\mathbb{Z}/p^n\mathbb{Z})^\times}\zeta_{\equiv a(p^n)}(-a_2)\cdot g_{a,1}))\\
&\cdot((\sum_{\substack{i\ge 1, i\equiv 1(N)\\ (i,p)=1}}\sum_{j\ge 1} i^{a_1-a_2}q^{ij}\cdot g_{i,2} - (-1)^{a_1-a_2}\sum_{\substack{i\ge 1, i\equiv -1(N)\\ (i,p)=1}}\sum_{j\ge 1} i^{a_1-a_2}q^{ij}\cdot g_{-i,2})\\
&+\varprojlim_n(\sum_{\substack{a\in(\mathbb{Z}/Np^n\mathbb{Z})^\times\\ a\equiv 1(N)}}\zeta_{\equiv a(Np^n)}(-a_1+a_2)\cdot g_{a,2}))
\end{aligned}$$

under the map (4.6). Hence concerning the image $z^{\mathrm{univ}}_{Np^\infty}(\chi^{a_1},\chi^{a_2})|_{(n,n)}$ of $z^{\mathrm{univ}}_{Np^\infty}(\chi^{a_1},\chi^{a_2})$ under the projection $\mathsf{H}[[G^{(1)}_\infty \times G^{(2)}_\infty]] \to \mathsf{H}[G^{(1)}_n \times G^{(2)}_n]$, we find that

$$z^{\text{univ}}_{Np^\infty}(\chi^{a_1},\chi^{a_2})|_{(n,n)} = (Np^n)^{a_1-a_2-1}N^{-1}(p^n)^{a_2-1} \sum_{\substack{b_1\in(\mathbb{Z}/Np^n\mathbb{Z})^\times \\ b_1\equiv 1(N)}} \sum_{b_2\in(\mathbb{Z}/p^n\mathbb{Z})^\times} \Big(\sum_{x,y((1/Np^n)\mathbb{Z})/\mathbb{Z}} F^{(a_1-a_2+1)}_{b_1/Np^n,x}\cdot F^{(a_2+1)}_{Nb_2/Np^n,y}\Big)\cdot g^{(1)}_{b_1}\cdot g^{(2)}_{b_1^{-1}b_2} \tag{4.10}$$

with $F^{(a_1-a_2+1)}_{b_1/Np^n,x}$ and $F^{(a_2+1)}_{Nb_2/Np^n,y}$ in 4.5.1. From this, we obtain that

$$z^{\text{univ}}_{Np^\infty}(\chi^{a_1},\chi^{a_2})|_{(n,n)} \in M_{a_1+2}(X_1(Np^n);\mathbb{Q})[G^{(1)}_n\times G^{(2)}_n](\subset \mathsf{H}[G^{(1)}_n\times G^{(2)}_n])$$

and hence

$$_{c,d'}z^{\text{univ}}_{Np^\infty}(\chi^{a_1},\chi^{a_2})|_{(n,n)} \in M_{a_1+2}(X_1(Np^n);\mathbb{Z}_{(p)})[G^{(1)}_n\times G^{(2)}_n](\subset O_{\mathsf{H}}[G^{(1)}_n\times G^{(2)}_n]).$$

Here $M_{a_1+2}(X_1(Np^n);\mathbb{Z}_{(p)}) = M_{a_1+2}(X_1(Np^n);\mathbb{Q})\cap\mathbb{Z}_{(p)}[[q]]$. The latter fact implies that $z^{\text{univ}}_{Np^\infty}\in\overline{M}_{Np^\infty}[[G^{(1)}_\infty\times G^{(2)}_\infty]][1/g]$.

4.5.6. We see the relation between $z^{\text{univ}}_{Np^\infty}$ and special values of zeta functions of cusp forms. This relation is described by the relation between the universal zeta modular form $z^{\text{univ}}_{Np^\infty}$ and the zeta modular forms reviewed in 4.5.2 – 4.5.4. This relation will play an important role in sections 6 and 7.
Let

$$\cdot\zeta_{p^n} : \mathbb{Q}[G^{(2)}_n] \longrightarrow \mathbb{Q}(\zeta_{p^n}) \tag{4.11}$$

be the $\mathbb{Q}$-linear map given by the action of $G^{(2)}_n$ on ζ_{p^n} such that $g^{(2)}_a\mapsto\zeta^a_{p^n}$ $(a\in(\mathbb{Z}/p^n\mathbb{Z})^\times)$. We consider the image of $z^{\text{univ}}_{Np^\infty}(\chi^{a_1},\chi^{a_2})|_{(n,n)}$ under the map

$$\cdot\zeta_{p^n} : M_{a_1+2}(X_1(Np^n);\mathbb{Q})[G^{(1)}_n\times G^{(2)}_n] \to M_{a_1+2}(X_1(Np^n);\mathbb{Q}(\zeta_{p^n}))[G^{(1)}_n] \tag{4.12}$$

induced by the map (4.11). By the caluculation until now, we see that the above image is

$$(Np^n)^{a_1-a_2-1}N^{-1}(p^n)^{a_2-1} \sum_{\substack{b_1\in(\mathbb{Z}/Np^n\mathbb{Z})^\times \\ b_1\equiv 1(N)}} \sum_{b_2\in(\mathbb{Z}/p^n\mathbb{Z})^\times} \Big(\sum_{x,y\in((1/Np^n)\mathbb{Z})/\mathbb{Z}} F^{(a_1-a_2+1)}_{b_1/Np^n,x}\cdot F^{(a_2+1)}_{Nb_2/Np^n,y}\Big)\cdot \zeta^{b_1^{-1}b_2}_{p^n}\cdot g^{(1)}_{b_1}. \tag{4.13}$$

We assume $a_2\neq 1$. Putting $a_1=k-2$, $a_2=r-1$ in (4.13), $m=n$ in (4.8) in Lemma 4.5.3 (3), and comparing (4.13) with (4.8), we see that the element (4.13) is closely related to the element (4.8) in Lemma 4.5.3 (3). Roughly speaking, for an eigen cusp form f, "f-component" of $\sum_{x\in(\mathbb{Z}/p^n\mathbb{Z})^\times}\psi(x)\nu_x\cdot(4.8)$, with a character $\psi:(\mathbb{Z}/p^n\mathbb{Z})^\times\to\overline{\mathbb{Q}}^\times$ $(n\geq 0)$, yields $L_{(Np)}(f,\psi,r)$ by period integrals. Here $L_{(Np)}(f,\psi,s)$ denotes the function obtained from

$L(f,\psi,s) = \sum_{i=1}^{\infty} a_i(f)\psi(i)i^{-s}$, where $a_i(f)$ is given by $T(i)f = a_i(f)f$, by removing prime(Np) factors of $L(f,\psi,s)$. We will see in section 6 that $\sum_{x\in(\mathbb{Z}/p^n\mathbb{Z})^\times} \psi(x)\nu_x \cdot (4.13)$ yields $L_{(p)}(f,\psi,r)$. For the details, see section 6.
In the case $a_2 = 1$, the above statements must be modified as follows.
The image of $(1-c^{-a_1}g_{c^{-1}}^{(1)}g_c^{(2)})(1-g_d^{(2)})z_{Np^\infty}^{\mathrm{univ}}(\chi^{a_1},\chi)|_{(n,n)}$ under the map (4.12) is closely related to the element in Lemma 4.5.4 (3).

5. Review of p-adic zeta functions of modular forms

In this section, we review the result of Amice-Vélu [AV] and Vishik [Vi] (Theorem 5.5) which concerns the existence and the characterizing properties of p-adic zeta functions of modular forms.

As referred above, in the rest of this paper, N denotes a positive integer which is prime to p.

5.1. Let

$$f = \sum_{n\geq 1} a_n(f)q^n \in M_k(X(1,Np^t)) \otimes \mathbb{C}$$

be a normalized eigen cusp form of weight $k \geq 2$ of level Np^t for some $t \geq 0$. We assume that the conductor of f is divisible by N. We further assume that t is the smallest integer ≥ 0 such that $f \in M_k(X(1,Np^t)) \otimes \mathbb{C}$. Set $K = \mathbb{Q}(a_n(f); n \geq 1)$. We take a prime λ of K which is above p, and let K_λ be the completion of K by λ.
Suppose that there exists an element $\alpha \in \overline{K_\lambda}^\times$ satisfying $v_p(\alpha) < k-1$ for the additive valuation v_p of $\overline{K_\lambda}$ normalized by $v_p(p) = 1$, and

$$1 - \alpha p^{-s} \quad | \quad (p\text{-factor of } L(f,s))^{-1} \qquad \text{in } \overline{\mathbb{Q}_p}[p^{-s}],$$

where $L(f,s) = \sum_{n\geq 1} a_n(f)n^{-s}$ is the complex zeta function of f. Then the p-adic zeta function of f may be defined for each α satisfying the above conditions. We fix such α and suppress α in the notation of p-adic zeta functions. We will review the characterizing properties of a p-adic zeta function in 5.5.

5.2. We give a review of the space $\mathrm{H}_{K_\lambda,k-1}$ to which the p-adic zeta function of f belongs. We first set up the notation. For the natural decomposition $\mathbb{Z}_p^\times = \mathbb{F}_p^\times \times (1+p\mathbb{Z}_p)$ in the case $p \neq 2$ (resp. $\mathbb{Z}_2^\times = \{\pm 1\} \times (1+4\mathbb{Z}_2)$), let u be a topological generator of the second component $1+p\mathbb{Z}_p$ (resp. $1+4\mathbb{Z}_2$). We denote $\mathbb{F}_p^\times$ (resp. $\{\pm\}$) by Δ. As before $G_\infty^{(2)}$ is the Galois group $\mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p)$ which is endowed with an isomorphism to $\mathbb{Z}_p^\times$ via the cyclotomic character. Now for a finite extension L of $\mathbb{Q}_p$ and for a positive integer d, we define

$$\mathrm{H}_{L,d} = \{\sum_{\substack{n\geq 0\\ a\in\Delta}} c_{n,a} \cdot g_a^{(2)} \cdot (g_u^{(2)} - 1)^n \in L[\Delta][[g_u^{(2)} - 1]] \ ; \qquad \lim_{n\longrightarrow\infty} |c_{n,a}|_p n^{-d} = 0 \text{ for all } a \in \Delta\}.$$

Here $|\ |_p$ is the multiplicative valuation of L normalized by $|p|_p = 1/p$. The space $\mathrm{H}_{L,d}$ is independent of the choice of u in the evident sense.
We have

$$O_L[[G_\infty^{(2)}]] \otimes_{O_L} L \subset \mathrm{H}_{L,1}, \quad \text{and} \quad \mathrm{H}_{L,i} \subset \mathrm{H}_{L,j} \ \text{ for } 1 \leq i \leq j.$$

Here the first inclusion is given by the natural map.
We put

$$\mathrm{H}_{L,\infty} = \bigcup_{d \geq 1} \mathrm{H}_{L,d},$$

then $\mathrm{H}_{L,\infty}$ is a ring.
For any positive integer d and for any subset U of $\mathbb{Z}$, we define a map

$$i_U : \mathrm{H}_{L,d} \longrightarrow \prod_{j \in U} L[[G_\infty^{(2)}]] = \prod_{j \in U} \varprojlim_n L[G_n^{(2)}]$$

$$\sum_{\substack{n \geq 0 \\ a \in \Delta}} c_{n,a} \cdot g_a^{(2)} \cdot (g_u^{(2)} - 1)^n \mapsto (\sum_{\substack{n \geq 0 \\ a \in \Delta}} c_{n,a} \cdot a^j g_a^{(2)} \cdot (u^j g_u^{(2)} - 1)^n)_j.$$

It is known that for any $d \geq 1$, the map $i_{\mathbb{Z}}$ is injective. Moreover for any different d integers $r_1, \ldots, r_d$, the map $i_{\{r_1,\ldots,r_d\}}$ is already injective:

$$i_{\{r_1,\ldots,r_d\}} : \mathrm{H}_{L,d} \hookrightarrow \prod_{j \in \{r_1,\ldots,r_d\}} L[[G_\infty^{(2)}]] (\subset \prod_{j \in \mathbb{Z}} L[[G_\infty^{(2)}]]).$$

Concerning the above injection, we have a proposition (cf. [AV]).

Proposition 5.3 . *Let*

$$\mathbb{H}_{L,d} \subset \prod_{j \in \{1,\ldots,d\}} \varprojlim_n L[G_n^{(2)}] = \prod_{j \in \{1,\ldots,d\}} L[[G_\infty^{(2)}]]$$

be the subspace consisting of elements

$$\mu = (\mu_j)_j = ((\mu_{j,n})_n)_j \in \prod_{j \in \{1,\ldots,d\}} \varprojlim_n L[G_n^{(2)}]$$

satisfying conditions (i) *and* (ii) *below. Then the map*

$$i_{\{1,\ldots,d\}} : \mathrm{H}_{L,d} \hookrightarrow \prod_{j \in \{1,\ldots,d\}} L[[G_\infty^{(2)}]]$$

induces a bijection from $\mathrm{H}_{L,d}$ *onto* $\mathbb{H}_{L,d}$.
(i) *For any* $j = 1, \ldots, d$,

$$\lim_{n \longrightarrow \infty} p^{dn} \mu_{j,n} = 0.$$

(ii) *For* $n \geq 1$, *let* $\phi_n : (\mathbb{Z}/p^n\mathbb{Z})^\times \longrightarrow \mathbb{Z}_p^\times$ *be a lifting, namely it is a map such that the composition* $(\mathbb{Z}/p^n\mathbb{Z})^\times \xrightarrow{\phi_n} \mathbb{Z}_p^\times \xrightarrow{\text{proj.}} (\mathbb{Z}/p^n\mathbb{Z})^\times$ *coincides with the identity map. For* $j \in \mathbb{Z}$, *let* $X_{\phi_n}^j : L[G_n^{(2)}] \to L[G_n^{(2)}]$ *be the* L*-homomorphism* *induced by* $g_a^{(2)} \mapsto \phi_n(a)^j g_a^{(2)}$ *for any* $a \in (\mathbb{Z}/p^n\mathbb{Z})^\times$. *Now for* $1 \le i \le d$ *and for any* ϕ_n *satisfying the above condition,*

$$\lim_{n \longrightarrow \infty} p^{(d-i+1)n} \sum_{j=0}^{i-1} (-1)^{i-j-1} \binom{i-1}{j} \mu_{j+1,n}(X_{\phi_n}^{-j}) = 0.$$

Here $\mu_{j,n}(X_{\phi_n}^{-j}) \in L[G_n^{(2)}]$ *is the image of* $\mu_{j,n}$ *under* $X_{\phi_n}^{-j}$.

Proof. For the proof, see [AV]. □

In section 6, we will construct the p-adic zeta function of f as an element of $\prod_{1 \le j \le k-1} K_\lambda(\alpha)[[G_\infty^{(2)}]]$, and we will prove that it is contained in $\mathbb{H}_{K_\lambda(\alpha),k-1}$ by using Proposition 5.3.

5.4. In this subsection we give a preliminary discussion to introduce Theorem 5.5 concerning the existence and the characterizing properties of p-adic zeta functions.

In the rest of this section, we assume that $a_p(f) \neq 0$.

5.4.1. As in [Ka2], §6, we define $S(f)$ to be

$$S(f) = (M_k(X_1(Np^t); \mathbb{Q}) \otimes_{\mathbb{Q}} K)/(T(n) \otimes 1 - 1 \otimes a_n(f) \ ; \ n \ge 1),$$

which is the quotient of $M_k(X_1(Np^t); \mathbb{Q}) \otimes_{\mathbb{Q}} K$ by the K-subspace generated by $T(n) \otimes 1 - 1 \otimes a_n(f)$ for $n \ge 1$. This $S(f)$ is a one dimensional K-vector space.

5.4.2. We define $V_K(f)$ to be the quotient of $H^1(Y(1, Np^t)(\mathbb{C}), \ \mathrm{Sym}_{\mathbb{Z}}^{k-2}(R^1\lambda_*(\mathbb{Z})) \otimes_{\mathbb{Z}} K)$ by the K-subspace generated by the images of $T(n) \otimes 1 - 1 \otimes a_n(f)$ for $n \ge 1$. Here $\lambda : E \to Y(1, Np^t)$ is the universal elliptic curve. This $V_K(f)$ is a two dimensional K-vector space.

5.4.3. We put $V_{\mathbb{C}}(f) = V_K(f) \otimes_K \mathbb{C}$, and let

$$\mathrm{per}_f : S(f) \longrightarrow V_{\mathbb{C}}(f)$$

be the one induced by the period map (cf. for example, [Ka2], §5, 5.4)

$$\mathrm{per}_{1,Np^t} : M_k(X(1, Np^t)) \otimes \mathbb{C} \longrightarrow H^1(Y(1, Np^t)(\mathbb{C}), \mathrm{Sym}_{\mathbb{Z}}^{k-2}(R^1\lambda_*(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{C}).$$

5.4.4. For the $\mathbb{C}$-linear map

$$\iota : V_{\mathbb{C}}(f) \longrightarrow V_{\mathbb{C}}(f)$$

induced by the complex conjugation on $Y(1, Np^t)(\mathbb{C})$ and $E(\mathbb{C})$, and for $x \in V_{\mathbb{C}}(f)$, we define

$$x^+ = \frac{1}{2}(1 + \iota)(x), \quad x^- = \frac{1}{2}(1 - \iota)(x).$$

Now we take an element $\gamma \in V_K(f)$ such that $\gamma^+ \neq 0$, $\gamma^- \neq 0$. For $\omega \in S(f)$ and for the above $\gamma \in V_K(f)$, we define $\Omega(\omega, \gamma)_+, \Omega(\omega, \gamma)_- \in \mathbb{C}$ as

$$\mathrm{per}_f(\omega) = \Omega(\omega, \gamma)_+ \cdot \gamma^+ + \Omega(\omega, \gamma)_- \cdot \gamma^-.$$

5.4.5. For $x \in (\mathbb{Z}/Np^t\mathbb{Z})^\times$, let $\langle x \rangle$ be as in 3.5 in section 3.
Let

$$\epsilon_f : (\mathbb{Z}/Np^t\mathbb{Z})^\times \longrightarrow \overline{\mathbb{Q}}^\times$$

be the character defined by $\langle x \rangle f = \epsilon_f(x) \cdot f$ for $x \in (\mathbb{Z}/Np^t\mathbb{Z})^\times$.

5.4.6. As before, let $\chi_{\text{cyclo}} : G_\infty^{(2)} \xrightarrow{\cong} \mathbb{Z}_p^\times$ be the cyclotomic character. For $j \in \mathbb{Z}$, we regard

$$\chi_{\text{cyclo}}^j : G_\infty^{(2)} \longrightarrow \mathbb{Z}_p^\times \quad ; \quad g_a^{(2)} \mapsto a^j \quad (a \in \mathbb{Z}_p^\times)$$

also as a character $\mathbb{Z}_p^\times \to \mathbb{Z}_p^\times \; ; \; a \mapsto a^j$.
For $\mu = \sum_{\substack{n \geq 0 \\ a \in \Delta}} c_{n,a} \cdot g_a^{(2)} \cdot (g_u - 1)^n \in \mathrm{H}_{L,d}$ and a continuous character $\psi : \mathbb{Z}_p^\times \to \overline{\mathbb{Q}_p}^\times$, we define

$$\mu(\psi) = \sum_{\substack{n \geq 0 \\ a \in \Delta}} c_{n,a} \cdot \psi(a) \cdot (\psi(u) - 1)^n.$$

Let α be as in 5.1.

THEOREM 5.5 ([AV],[Vi]). *Let $h = \min\{n \in \mathbb{Z} \; ; \; n \geq 1, v_p(\alpha) < n\} (\leq k-1)$. For $\gamma \in V_K(f)$ such that $\gamma^+ \neq 0$, $\gamma^- \neq 0$, and for a non-zero $\omega \in S(f)$, we have a function*

$$L_{p\text{-}adic}(f)_{\omega,\gamma} \in \mathrm{H}_{K_\lambda,h} \subset \mathrm{H}_{K_\lambda,k-1},$$

characterized by the properties (i) *and* (ii) *below. In particular, if $v_p(\alpha) = 0$, $L_{p\text{-}adic}(f)$ belongs to the subspace*

$$L_{p\text{-}adic}(f)_{\omega,\gamma} \in O_{K_\lambda}[[G_\infty^{(2)}]] \otimes_{O_{K_\lambda}} K_\lambda \subset \mathrm{H}_{K_\lambda,1}.$$

(i) *Let $\psi : (\mathbb{Z}/p^n\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times$ be a character with conductor p^n $(n \geq 1)$. We put $\pm = (-1)^{k-r-1}\psi(-1)\epsilon_f(-1)$. Then for any integer r such that $1 \leq r \leq k-1$, we have*

$$\begin{aligned} &L_{p\text{-}adic}(f)_{\omega,\gamma}(\chi_{\text{cyclo}}^r \psi^{-1}) \\ &\quad = (r-1)! \cdot p^{nr}\alpha^{-n} \cdot G(\psi, \zeta_{p^n})^{-1} \cdot (2\pi i)^{k-r-1} \cdot \frac{1}{\Omega(\omega,\gamma)_\pm} \cdot L(f, \psi, r), \end{aligned}$$

where $G(\psi, \zeta_{p^n})$ denotes the Gauss sum $\sum_{x \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \psi(x)\zeta_{p^n}^x$ and $L(f, \psi, r)$ is the evaluation at $s = r$ of the function $L(f, \psi, s) = \sum_{i=1}^\infty a_i(f)\psi(i)i^{-s}$.

(ii) *We put $\pm = (-1)^{k-r-1}\epsilon_f(-1)$. For any integer r such that $1 \leq r \leq k-1$, we have*

$$\begin{aligned} &L_{p\text{-}adic}(f)_{\omega,\gamma}(\chi_{\text{cyclo}}^r) \\ &\quad = (r-1)! \cdot (2\pi i)^{k-r-1} \cdot \frac{1}{\Omega(\omega,\gamma)_\pm} \\ &\qquad \cdot (1 - p^{r-1}\alpha^{-1})(1 - \epsilon_f(p)p^{k-r-1}\alpha^{-1}) \cdot L(f, r). \end{aligned}$$

REMARK 5.5.1 . (1) *In fact, both hands sides of the above equality belong to* $\overline{\mathbb{Q}}$.

(2) *The function in Theorem* 5.5 *can be characterized by only property* (i).

The function $L_{p\text{-adic}}(f)_{\omega,\gamma}$ in Theorem 5.5 is called the p-adic zeta function of f.

5.6. We have a canonical isomorphism

$$\mathrm{H}_{K_\lambda,d} \cong \mathrm{H}_{K_\lambda,d}/(g_{-1}^{(2)} - 1) \times \mathrm{H}_{K_\lambda,d}/(g_{-1}^{(2)} + 1).$$

For $x \in \mathrm{H}_{K_\lambda,d}$, we call its image in $\mathrm{H}_{K_\lambda,d}/(g_{-1}^{(2)} - 1)$ (resp. $\mathrm{H}_{K_\lambda,d}/(g_{-1}^{(2)} + 1)$) under the above isomorphism the $+$-part (resp. $-$-part) of x. Moreover we call $\mathrm{H}_{K_\lambda,d}/(g_{-1}^{(2)} - 1)$ (resp. $\mathrm{H}_{K_\lambda,d}/(g_{-1}^{(2)} + 1)$) the $+$-part (resp. $-$-part) of $\mathrm{H}_{K_\lambda,d}$.

5.7. As in [Ka2], §6, we define $\delta(f, k-1, 0(1)) \in V_K(f)$ to be the image of $\delta(k, k-1, 0(1)) \in H^1(Y(1, Np^t)(\mathbb{C}), \mathrm{Sym}_{\mathbb{Z}}^{k-2}(R^1\lambda_*(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q})$ in [Ka2], §5, 5.4. It is known that $\delta(f, k-1, 0(1))^+ = 0$, and if $L(f, k-1) \neq 0$, $\delta(f, k-1, 0(1)) = \delta(f, k-1, 0(1))^- \neq 0$. In what follows, in the case $L(f, k-1) \neq 0$, we take $\delta(f, k-1, 0(1)) \in V_K(f)$ as $\gamma \in V_K(f)$, we consider $(-1)^k \cdot \epsilon_f(-1)$-part of the p-adic zeta function of f, and we suppress γ in the notation of p-adic zeta functions.

6. THE RESULT ON ONE-VARIABLE p-ADIC ZETA FUNCTION

Let the notation and the setting be as in section 5. Suppose f is an eigen cusp form of weight $k \geq 2$ and of level Np^t with $t \geq 1$ which satisfies the condition in 5.1. In 6.1, for a certain subspace A of $\overline{M}_{Np^\infty}[[G_\infty^{(1)} \times G_\infty^{(2)}]] \otimes_{\mathbb{Z}_p[[G_\infty^{(1)} \times G_\infty^{(2)}]]} Q(\mathbb{Z}_p[[G_\infty^{(1)} \times G_\infty^{(2)}]])$ to which the universal zeta modular form $z_{Np^\infty}^{\mathrm{univ}}$ belongs, we define a map "to take f-component"

$$\mathfrak{L}_{f,\{k-2\},\{0,\dots,k-2\},t} : A \longrightarrow \prod_{r \in \{1,\dots,k-1\}} K_\lambda[[G_\infty^{(2)}]][G_t^{(1)}].$$

The main theorem (Theorem 6.2) of this section is, roughly speaking, that if $L(f, k-1) \neq 0$,

$$\mathfrak{L}_{f,\{k-2\},\{0,\dots,k-2\},t} \ : \ z_{Np^\infty}^{\mathrm{univ}} \ \mapsto \ p\text{-adic zeta function of } f$$

(see Theorem 6.2 for the precise statement).

6.1. We define the subspace A and the map $\mathfrak{L}_{f,\{k-2\},\{0,\dots,k-2\},t}$ in a more general forms $M[[G_\infty^{(1)} \times G_\infty^{(2)}]]_{I_1,I_2}$ (see 6.1.4) and $\mathfrak{L}_{f,I_1,I_2,i}$ (see 6.1.6), respectively. (For simplicity we assumed that $t \geq 1$ in the above, but in fact, we can treat also the case $t = 0$, as seen below.) We begin with some preliminaries. Let f, K, and the other setting be as in 5.1 in section 5.

6.1.1. As in 5.1, we fix α under the notation there. We first consider the case that $t = 0$. We denote the p-factor of $L(f, s)$ by $(1 - \alpha p^{-s})^{-1}(1 - \beta p^{-s})^{-1}$ with $\beta \in \overline{K_\lambda}^\times$.

Let

$$f_\alpha = f - \beta \cdot \varphi_q(f) \in M_k(X(1, Np)) \otimes \mathbb{C},$$

where $\varphi_q(f) = \sum_{n\geq 1} a_n(f) q^{pn}$ with $f = \sum_{n \geq 1} a_n(f) q^n$. We have

$$T(p) f_\alpha = \alpha \cdot f_\alpha,$$

$$L(f_\alpha, s) = (1 - \beta p^{-s}) L(f, s),$$

and $\epsilon_{f_\alpha} : (\mathbb{Z}/Np\mathbb{Z})^\times \longrightarrow \overline{\mathbb{Q}}^\times$ is the one induced by $(\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\epsilon_f} \overline{\mathbb{Q}}^\times$. This f_α is an eigen cusp form but is not a newform.

6.1.2. For f in 5.1, we put $\mathfrak{f} = f$ in the case that $t \geq 1$ with t in 5.1, and $\mathfrak{f} = f_\alpha$ in the case $t = 0$, where f_α is as in 6.1.1. Moreover let Np^m denote the level of $\mathfrak{f}$. (Namely $m = t$ in the case that $\mathfrak{f} = f$ and $m = 1$ in the case that $\mathfrak{f} = f_\alpha$ as above.) Further put

$$L = \begin{cases} K_\lambda & t \geq 1 \\ K_\lambda(\alpha) & t = 0. \end{cases}$$

6.1.3. Let $j, s \in \mathbb{Z}$, and let $M^j(X_1(Np^s); \mathbb{Q})$ be as in section 3. The space $(\cup_{j\geq 2, s\geq 1} M^j(X_1(Np^s); \mathbb{Q}) \otimes_{\mathbb{Q}} L)/(T(n) \otimes 1 - 1 \otimes a_n(\mathfrak{f})\ ;\ n \geq 1)$ is a one dimensional L-vector space in which the class of f is a base. We define a map $\mathrm{pr}_\mathfrak{f}$ by the following composition

$$\begin{aligned} \mathrm{pr}_\mathfrak{f} : & \bigcup_{j\geq 2\ s\geq 1} M^j(X_1(Np^s); \mathbb{Q}) \\ & \to (\bigcup_{j\geq 2, s\geq 1} M^j(X_1(Np^s); \mathbb{Q}) \otimes_{\mathbb{Q}} L)/(T(n) \otimes 1 - 1 \otimes a_n(\mathfrak{f})\ ;\ n \geq 1) \\ & \to L, \end{aligned} \tag{6.1}$$

where the first map is the natural projection and the second map is by sending the class of f to 1.

6.1.4. Let $I_1, I_2 \subset \mathbb{Z}$ be subsets. We denote by $M[[G_\infty^{(1)} \times G_\infty^{(2)}]]_{I_1, I_2}$ the $\mathbb{Z}_p[[G_\infty^{(1)} \times G_\infty^{(2)}]]$-submodule of $\overline{M}_{Np^\infty}[[G_\infty^{(1)} \times G_\infty^{(2)}]]$ defined in the following way:

$$\begin{aligned} & M[[G_\infty^{(1)} \times G_\infty^{(2)}]]_{I_1, I_2} \\ & \quad := \{x \in \overline{M}_{Np^\infty}[[G_\infty^{(1)} \times G_\infty^{(2)}]]\ ; \\ & \qquad x(\chi^{a_1}, \chi^{a_2})|_{(n,n)} \in (\bigcup_{j\geq 2,\ s\geq 1} M^j(X_1(Np^s); \mathbb{Z}_p))[G_n^{(1)} \times G_n^{(2)}] \\ & \qquad \text{for any } (a_1, a_2) \in I_1 \times I_2 \text{ and } n \geq 1\}. \end{aligned}$$

6.1.5. Let c, d' be integers which are prime to p. By the result in 4.5.5, we find that for any $k \geq 2$,

$$\begin{aligned} {}_{c,d'}z^{\mathrm{univ}}_{Np^\infty} =& (1 - c^{-1} g^{(1)}_{c^{-1}} g^{(2)}_{c})(1 - d' g^{(2)}_{d'}) z^{\mathrm{univ}}_{Np^\infty} \\ &\in M[[G^{(1)}_\infty \times G^{(2)}_\infty]]_{\{k-2\},\{0,\ldots,k-2\}}. \end{aligned} \tag{6.2}$$

6.1.6. Let i be a positive integer. We define a map "to take $\mathfrak{f}$-component"

$$\mathfrak{L}_{f,I_1,I_2,i} : M[[G^{(1)}_\infty \times G^{(2)}_\infty]]_{I_1,I_2} \longrightarrow \prod_{(a_1,a_2)\in I_1\times I_2} L[[G^{(2)}_\infty]][G^{(1)}_i]$$

as follows:

$$\mathfrak{L}_{f,I_1,I_2,i} = \prod_{(a_1,a_2)\in I_1\times I_2} \mathfrak{L}_{f,a_1,a_2,i}$$

for

$$\mathfrak{L}_{f,a_1,a_2,i} : M[[G^{(1)}_\infty \times G^{(2)}_\infty]]_{I_1,I_2} \longrightarrow L[[G^{(2)}_\infty]][G^{(1)}_i].$$

The map $\mathfrak{L}_{f,a_1,a_2,i}$ is defined as the composition

$$\begin{aligned} \mathfrak{L}_{f,a_1,a_2,i} : M[[G^{(1)}_\infty \times G^{(2)}_\infty]]_{I_1,I_2} &\xrightarrow{(\chi^{a_1},\chi^{a_2})} M[[G^{(1)}_\infty \times G^{(2)}_\infty]]_{\{0\},\{0\}} \\ &\xrightarrow{\text{proj.}} M[[G^{(2)}_\infty]][G^{(1)}_i]_{\{0\},\{0\}} \\ &\xrightarrow{\mathrm{pr}_{\mathfrak{f}}} L[[G^{(2)}_\infty]][G^{(1)}_i], \end{aligned}$$

where $M[[G^{(2)}_\infty]][G^{(1)}_i]_{\{0\},\{0\}}$ denotes the image of $M[[G^{(1)}_\infty \times G^{(2)}_\infty]]_{\{0\},\{0\}}$ under the natural projection $\overline{M}_{Np^\infty}[[G^{(1)}_\infty \times G^{(2)}_\infty]] \to \overline{M}_{Np^\infty}[[G^{(2)}_\infty]][G^{(1)}_i]$, and the last map is given by $\mathrm{pr}_{\mathfrak{f}}$ (6.1) for each coefficients of $G^{(2)}_n$ and by taking $\varprojlim_n$.

For $x \in M[[G^{(1)}_\infty \times G^{(2)}_\infty]]_{I_1,I_2} \otimes_{\mathbb{Z}_p[[G^{(1)}_\infty \times G^{(2)}_\infty]]} Q(\mathbb{Z}_p[[G^{(1)}_\infty \times G^{(2)}_\infty]])$, if there is a non-zerodivisor $g \in \mathbb{Z}_p[[G^{(1)}_\infty \times G^{(2)}_\infty]]$ such that $g(\chi^{a_1}, \chi^{a_2})$ is invertible in $\mathbb{Q}_p[[G^{(1)}_\infty \times G^{(2)}_\infty]]$ for all $(a_1, a_2) \in I_1 \times I_2$ and $gx \in M[[G^{(1)}_\infty \times G^{(2)}_\infty]]_{I_1,I_2}$, then we define

$$\begin{aligned} \mathfrak{L}_{f,I_1,I_2,i}(x) = \mathfrak{L}_{f,I_1,I_2,i}(gx) \cdot &\prod_{(a_1,a_2)\in I_1\times I_2} g(a_1,a_2)^{-1} \\ \in &\prod_{(a_1,a_2)\in I_1\times I_2} L[[G^{(2)}_\infty]][G^{(1)}_i]. \end{aligned}$$

6.1.7. By (6.2), we find that $\mathfrak{L}_{f,\{k-2\},\{0,\ldots,k-2\},m}(z^{\mathrm{univ}}_{Np^\infty}) \in \prod_{a_2\in\{0,\ldots,k-2\}} L[[G^{(2)}_\infty]][G^{(1)}_m]$ can be defined in the sense at the end of 6.1.6. In what follows, by putting $a_2 = r-1$, we write $\prod_{r\in\{1,\ldots,k-1\}} L[[G^{(2)}_\infty]][G^{(1)}_m]$ instead of $\prod_{a_2\in\{0,\ldots,k-2\}} L[[G^{(2)}_\infty]][G^{(1)}_m]$. Furthermore for $L[[G^{(2)}_\infty]]$, we define $+$ and $-$ parts in the same way as in 5.6, and we define the $*$-part of $\prod_{r\in\{1,\ldots,k-1\}} L[[G^{(2)}_\infty]][G^{(1)}_m]$ with $* = +$ or $* = -$ by

$\prod_{r\in\{1,\dots,k-1\}} L[[G_\infty^{(2)}]]^{*\cdot(-1)^r}[G_m^{(1)}]$, where $L[[G_\infty^{(2)}]]^{*\cdot(-1)^r}$ is the $*\cdot(-1)^r$-part of $L[[G_\infty^{(2)}]]$.

We state our main theorem in this section.
In the situation of Theorem 5.5, we take the class of f as $\omega \in S(\mathfrak{f})$, and suppress the ω in the notation of p-adic zeta functions appearing below. Since we will assume $L(\mathfrak{f}, k-1) \neq 0$, we take $\delta(\mathfrak{f}, k-1, 0(1)) \in V_K(\mathfrak{f}) \setminus \{0\}$ as γ, and suppress γ in the notation of p-adic zeta functions, as referred in 5.7.

Theorem 6.2 . *Put* $\pm = (-1)^k \epsilon_f(-1)$. *Let* $h = \min\{n \in \mathbb{Z}\ ;\ n \geq 1, v_p(\alpha) < n\} (\leq k-1)$, *as in Theorem* 5.5. *Then we have*

$$\mathfrak{L}_{f,\{k-2\},\{0,\dots,k-2\},m}(z_{Np^\infty}^{\text{univ}})^{\pm} \in \prod_{r\in\{1,\dots,k-1\}} L[[G_\infty^{(2)}]][G_m^{(1)}] \tag{6.3}$$

is contained in the subspace

$$i_{\{1,\dots,k-1\}}(\mathrm{H}_{L,h})[G_m^{(1)}].$$

Here $\mathfrak{L}_{f,\{k-2\},\{0,\dots,k-2\},m}(z_{Np^\infty}^{\text{univ}})^{\pm}$ *represents the* $\pm$*-part. Moreover if* $v_p(\alpha) = 0$, (6.3) *belongs to* $i_{\{1,\dots,k-1\}}(O_L[[G_\infty^{(2)}]] \otimes_{O_L} L)[G_m^{(1)}]$.
Concerning the relation with p-adic zeta function, we have the following result. Suppose $L(\mathfrak{f}, k-1) \neq 0$. *In the rest of this theorem, we identify an element of* $\mathrm{H}_{L,h}$ *with its image under* $i_{\{1,\dots,k-1\}} : \mathrm{H}_{L,h} \hookrightarrow \prod_{r\in\{1,\dots,k-1\}} L[[G_\infty^{(2)}]]$.

(1) *In the case* $\mathfrak{f} = f$, *we have*

$$\begin{aligned}&\mathfrak{L}_{f,\{k-2\},\{0,\dots,k-2\},m}(z_{Np^\infty}^{\text{univ}})^{\pm} \\ &\quad = \alpha^m \sum_{a\in(\mathbb{Z}/p^m\mathbb{Z})^\times} L_{p\text{-}adic}(f)^{\pm} \epsilon_f(a'^{-1}) \cdot g_a^{(1)} \in \prod_{r\in\{1,\dots,k-1\}} L[[G_\infty^{(2)}]][G_m^{(1)}].\end{aligned}$$

Here $a' \in (\mathbb{Z}/Np^m\mathbb{Z})^\times$ *is the element such that* $a' \equiv 1(N)$ *and* $a' \equiv a(p^m)$.

(2) *In the case* $\mathfrak{f} = f_\alpha$, *we have*

$$\begin{aligned}&\mathfrak{L}_{f,\{k-2\},\{0,\dots,k-2\},1}(z_{Np^\infty}^{\text{univ}})^{\pm} \\ &\quad = \alpha \sum_{a\in(\mathbb{Z}/p\mathbb{Z})^\times} L_{p\text{-}adic}(f_\alpha)^{\pm} \cdot g_a^{(1)} \in \prod_{r\in\{1,\dots,k-1\}} L[[G_\infty^{(2)}]][G_1^{(1)}].\end{aligned}$$

(3) *In the above, we only considered the* $(-1)^k\epsilon_f(-1)$*-parts of* $L_{p\text{-}adic}(\mathfrak{f})$, *and we put the assumption that* $L(\mathfrak{f}, k-1) \neq 0$ *which always holds in the case* $k \geq 3$. *However we can obtain by the method in* 6.7 *below, the whole* $L_{p\text{-}adic}(\mathfrak{f})$, *including the* $(-1)^{k-1}\epsilon_f(-1)$*-part, without the assumption that* $L(\mathfrak{f}, k-1) \neq 0$.

Remark 6.2.1 . *In Theorem* 6.2 (2), *we present the p-adic zeta function of* f_α *instead of the p-adic zeta function of* f. *By the characterizing property of p-adic zeta functions in Theorem* 5.5, *their p-adic zeta functions are a multiple of the other by a non-zero constant.*

The rest of this section is devoted to the proof of Theorem 6.2. In 6.3, we prove that $\mathfrak{L}_{f,\{k-2\},\{0,\dots,k-2\},m}(z^{\mathrm{univ}}_{Np^\infty})^\pm$ is contained in $i_{\{1,\dots,k-1\}}(\mathrm{H}_{L,h})[G^{(1)}_m]$. Then in 6.4 – 6.7, we prove that $\mathfrak{L}_{f,\{k-2\},\{0,\dots,k-2\},m}(z^{\mathrm{univ}}_{Np^\infty})$ displays the characterizing property (i) of the p-adic zeta function in Theorem 5.5. (Cf. Remark 5.5.1 in section 5.)

6.3. We show that $\mathfrak{L}_{f,\{k-2\},\{0,\dots,k-2\},m}(z^{\mathrm{univ}}_{Np^\infty})^\pm$ belongs to $i_{\{1,\dots,k-1\}}(\mathrm{H}_{L,h})$ $[G^{(1)}_m]$.

First, we give a preliminary discussion which will be important also for the proof on the characterizing properties of p-adic zeta functions.

6.3.1. We define a homomorphism

$$\varphi_q : \mathsf{H} \longrightarrow \mathsf{H}$$

by $\varphi_q(\sum_{i=-\infty}^{\infty} a_i q^i) = \sum_{i=-\infty}^{\infty} a_i q^{pi}$ ($a_i \in \mathbb{Q}_p$, the valuation of a_i is bounded below, and $a_i \to 0$ when $i \to -\infty$). Let

$$\mathrm{Tr}_q : \mathsf{H} \longrightarrow \mathsf{H}$$

be the trace map associated to φ_q.

We use the same symbol Tr_q

$$\mathsf{H}[G^{(1)}_{a_1} \times G^{(2)}_{a_2}] \longrightarrow \mathsf{H}[G^{(1)}_{a_1} \times G^{(2)}_{a_2}] \qquad (a_1, a_2 \geq 1)$$

for the map induced by Tr_q on the coefficients.

For an element x of $L[[G^{(1)}_\infty \times G^{(2)}_\infty]]$ and for positive integers a_1, a_2, we denote by $x|_{(a_1,a_2)}$ the image of x under the natural projection

$$L[[G^{(1)}_\infty \times G^{(2)}_\infty]] \to L[G^{(1)}_{a_1} \times G^{(2)}_{a_2}].$$

PROPOSITION 6.3.2 . *Let a_1, a_2 be integers such that $0 \leq a_2 \leq a_1$. Then for all positive integers n and m such that $n \geq m$, we have*

$$\mathrm{Tr}_q^{n-m}(z^{\mathrm{univ}}_{Np^\infty}(\chi^{a_1}, \chi^{a_2})|_{(m,n)}) \in M_{a_1+2}(X_1(Np^m); \mathbb{Q})[G^{(1)}_m \times G^{(2)}_n],$$

$$\mathrm{Tr}_q^{n-m}({}_{c,d'}z^{\mathrm{univ}}_{Np^\infty}(\chi^{a_1}, \chi^{a_2})|_{(m,n)}) \in M_{a_1+2}(X_1(Np^m); \mathbb{Z}_{(p)})[G^{(1)}_m \times G^{(2)}_n],$$

where $z^{\mathrm{univ}}_{Np^\infty}(\chi^{a_1}, \chi^{a_2})$ is as in 4.5.5.

Proof. Clearly $*|_{(m,n)}$ is the image of $*|_{(n,n)}$ under the projection $\mathsf{H}[G^{(1)}_n \times G^{(2)}_n] \to \mathsf{H}[G^{(1)}_m \times G^{(2)}_n]$. By (4.10), we know $z^{\mathrm{univ}}_{Np^\infty}(\chi^{a_1}, \chi^{a_2})|_{(n,n)}$ which is an element of $M_{a_1+2}(X_1(Np^n); \mathbb{Q})[G^{(1)}_n \times G^{(2)}_n]$ precisely, and hence we can calculate $\mathrm{Tr}_q^{n-m}(z^{\mathrm{univ}}_{Np^\infty}(\chi^{a_1}, \chi^{a_2}))|_{(m,n)}$. By this calculation, we find that the element $\mathrm{Tr}_q^{n-m}(z^{\mathrm{univ}}_{Np^\infty}(\chi^{a_1}, \chi^{a_2}))|_{(m,n)}$ belongs to $M_{a_1+2}(X_1(Np^m); \mathbb{Q})[G^{(1)}_m \times G^{(2)}_n]$. The assertion for ${}_{c,d'}z^{\mathrm{univ}}_{Np^\infty}$ follows from this. □

In 6.3.3 – 6.3.6, we show that the element $\mathfrak{L}_{f,\{k-2\},\{0,\dots,k-2\},m}(z^{\mathrm{univ}}_{Np^\infty})$ belongs to $i_{\{1,\dots,k-1\}}(\mathrm{H}_{L,h} \otimes_{\mathbb{Z}_p[[G^{(2)}_\infty]]} \mathbb{Z}_p[[G^{(2)}_\infty]][1/a])[G^{(1)}_m] = i_{\{1,\dots,k-1\}}(\mathrm{H}_{L,h}[1/a])[G^{(1)}_m]$

($\subset$ $(i_{\{1,\ldots,k-1\}}(\mathrm{H}_{L,h}) \otimes_{\mathbb{Z}_p[[G^{(2)}_\infty]]} Q(\mathbb{Z}_p[[G^{(2)}_\infty]]))[G^{(1)}_m]$) with a certain non-zerodivisor $a \in \mathbb{Z}_p[[G^{(2)}_\infty]]$ which satisfies that $a(\chi^r)$ $(r = 1, \ldots, k-1)$ are invertible in $\mathbb{Q}_p[[G^{(2)}_\infty]]$. In 6.3.7 – 6.3.12, we prove that $\mathfrak{L}_{f,\{k-2\},\{0,\ldots,k-2\},m}(z^{\mathrm{univ}}_{Np^\infty})^\pm$ is, in fact, contained in $i_{\{1,\ldots,k-1\}}(\mathrm{H}_{L,h})[G^{(1)}_m]$.

6.3.3. We prove that

$$\mathfrak{L}_{f,\{k-2\},\{0,\ldots,k-2\},m}({}_{c,d'}z^{\mathrm{univ}}_{Np^\infty}) \in i_{\{1,\ldots,k-1\}}(\mathrm{H}_{L,h})[G^{(1)}_m], \tag{6.4}$$

which means that $\mathfrak{L}_{f,\{k-2\},\{0,\ldots,k-2\},m}(z^{\mathrm{univ}}_{Np^\infty}) \in i_{\{1,\ldots,k-1\}}(\mathrm{H}_{L,h}[1/a])[G^{(1)}_m]$ with a non-zerodivisor $a \in \mathbb{Z}_p[[G^{(2)}_\infty]]$ (which satisfies that $a(\chi^r)$ $(r = 1, \ldots, k-1)$ are invertible in $\mathbb{Q}_p[[G^{(2)}_\infty]]$). It follows directly from the definition that the projection $M_k(X_1(Np^n);\mathbb{Q}) \longrightarrow S(\mathfrak{f})$ $(n \geq m)$ commutes with the Hecke operator $T(p) = U(p) = \mathrm{Tr}_q$. By this and by the fact that the action of $T(p)$ on $S(\mathfrak{f})$ coincides with the multiplication by α, we obtain

$$\begin{aligned}\mathfrak{L}_{f,\{k-2\},\{r-1\},m}&({}_{c,d'}z^{\mathrm{univ}}_{Np^\infty})|_{(m,n)} \\ &= \alpha^{m-n} \cdot \mathrm{pr}_\mathfrak{f}(\mathrm{Tr}_q^{n-m}({}_{c,d'}z^{\mathrm{univ}}_{Np^\infty}(\chi^{k-2},\chi^{r-1})|_{(m,n)})).\end{aligned} \tag{6.5}$$

Therefore for the proof of (6.4) it is enough to show that

$$\begin{aligned}\prod_{r\in\{1,\ldots,k-1\}} \varprojlim_n (\alpha^{-n} \cdot \mathrm{pr}_\mathfrak{f}(\mathrm{Tr}_q^{n-m}({}_{c,d'}z^{\mathrm{univ}}_{Np^\infty}(\chi^{k-2},\chi^{r-1})|_{(m,n)}))) \\ \in i_{\{1,\ldots,k-1\}}(\mathrm{H}_{L,h})[G^{(1)}_m].\end{aligned} \tag{6.6}$$

We denote the r-component $(r \in \{1, \ldots, k-1\})$ of the left hand side of (6.6) by

$$\mu_r(\mathfrak{f}) = (\mu_{r,n}(\mathfrak{f}))_n \in \varprojlim_n L[G^{(2)}_n][G^{(1)}_m] = L[[G^{(1)}_\infty]][G^{(1)}_m].$$

In order to prove the assertion (6.6), by Proposition 5.3 in section 5, it is sufficient to show the following two assertions:
One is to show that

$$\lim_{n\longrightarrow\infty} p^{hn}\mu_{r,n}(\mathfrak{f}) = 0 \quad \text{for all } r \in \{1, \ldots, k-1\}. \tag{6.7}$$

The other is to prove that for any $d \in \mathbb{Z}$ such that $h \leq d \leq k-1$, $(\mu_r(\mathfrak{f}))_{r=1,\ldots,d}$ satisfy the condition (ii) in Proposition 5.3.

6.3.4. We check that $(\mu_r(\mathfrak{f}))_{r=1,\ldots,k-1}$ satisfy (6.7) above.
By Proposition 6.3.2 which shows

$$\mathrm{Tr}_q^{n-m}({}_{c,d'}z^{\mathrm{univ}}_{Np^\infty}(\chi^{k-2},\chi^{r-1})|_{(m,n)}) \in M_k(X_1(Np^m);\mathbb{Z}_{(p)})[G^{(1)}_m][G^{(2)}_n]$$

for all $n \geq m$ and by the fact that $M_k(X_1(Np^m);\mathbb{Z}_{(p)})$ is a finitely generated $\mathbb{Z}_{(p)}$-module, we have that the image of $M_k(X_1(Np^m);\mathbb{Z}_{(p)})$ under $\mathrm{pr}_\mathfrak{f} : M_k(X_1(Np^m);\mathbb{Q}) \longrightarrow L$ is contained in $a \cdot O_L$ for some $a \in L^\times$.

From this, p^{nh} times $\mu_{r,n}(\mathfrak{f})$ is contained in $p^{nh}\cdot\alpha^{-n}a\cdot O_L[G_m^{(1)}][G_n^{(2)}]$. As $v_p(\alpha)<h$, we obtain $\lim_{n\to\infty}p^{nh}\cdot\alpha^{-n}a\cdot O_L=0$ which implies $\lim_{n\to\infty}p^{nh}\mu_{r,n}(\mathfrak{f})=0$ as desired.

6.3.5. We show that $(\mu_r(\mathfrak{f}))_{r=1,\dots,d}$ satisfy the condition (ii) of Proposition 5.3 for any $d\in\mathbb{Z}$ such that $h\le d\le k-1$.
We use the notation in Proposition 5.3 (ii). We write ${}_{c,d'}z^{\mathrm{univ}}_{Np^\infty}(\chi^{k-2},\chi^jX_{\phi_n}^{-j})|_{(m,n)}$ for $({}_{c,d'}z^{\mathrm{univ}}_{Np^\infty}(\chi^{k-2},\chi^j)|_{(m,n)})(\mathrm{id},X_{\phi_n}^{-j})\in M_k(X_1(Np^n);\mathbb{Z}_{(p)})[G_m^{(1)}][G_n^{(2)}]$.
We can prove that

$$\sum_{j=0}^{i-1}(-1)^{i-j-1}\binom{i-1}{j}\mu(\mathfrak{f})_{j+1,n}(X_{\phi_n}^{-j})$$
$$=\sum_{j=0}^{i-1}(-1)^{i-j-1}\binom{i-1}{j}\cdot\alpha^{-n}\cdot\mathrm{pr}_{\mathfrak{f}}(\mathrm{Tr}_q^{n-m}({}_{c,d'}z^{\mathrm{univ}}_{Np^\infty}(\chi^{k-2},\chi^jX_{\phi_n}^{-j})|_{(m,n)}))$$

in Proposition 5.3 coincides with

$$\alpha^{-n}\cdot\mathrm{pr}_{\mathfrak{f}}(\mathrm{Tr}_q^{n-m}(\sum_{j=0}^{i-1}(-1)^{i-j-1}\binom{i-1}{j}\cdot({}_{c,d'}z^{\mathrm{univ}}_{Np^\infty}(\chi^{k-2},\chi^jX_{\phi_n}^{-j})|_{(m,n)}))). \tag{6.8}$$

Moreover we have

$$\sum_{j=0}^{i-1}(-1)^{i-j-1}\binom{i-1}{j}\cdot({}_{c,d'}z^{\mathrm{univ}}_{Np^\infty}(\chi^{k-2},\chi^jX_{\phi_n}^{-j})|_{(m,n)})$$
$$\in M_k(X_1(Np^n);p^{n(i-1)}\mathbb{Z}_{(p)})[G_m^{(1)}][G_n^{(2)}]$$

which follows from the general argument that

$$\sum_{j=0}^{i-1}(-1)^{i-j-1}\binom{i-1}{j}\cdot x(\chi^jX_{\phi_n}^{-j})|_n\in p^{n(i-1)}\mathbb{Z}_{(p)}[G_n]$$

for any $x\in\mathbb{Z}_{(p)}[[G_\infty]]$. Here $x(\chi^jX_{\phi_n}^{-j})|_n$ represents $x(\chi^j)|_n(X_{\phi_n}^{-j})$ for the image $x(\chi^j)|_n\in\mathbb{Z}_{(p)}[G_n]$ of $x(\chi^j)\in\mathbb{Z}_{(p)}[[G_\infty]]$ under the projection. By this, we find (6.8) is contained in $\alpha^{-n}\cdot a\cdot p^{n(i-1)}O_L[G_m^{(1)}][G_n^{(2)}]$ with $a\in L^\times$ in 6.3.4. As $v_p(\alpha)<h\le d$, we obtain

$$\lim_{n\longrightarrow\infty}p^{(d-i+1)n}\sum_{j=0}^{i-1}(-1)^{i-j-1}\binom{i-1}{j}\mu(\mathfrak{f})_{j+1,n}(X_{\phi_n}^{-j})=0,$$

which says that $(\mu_r(\mathfrak{f}))_{r=1,\dots,d}$ $(h\le d\le k-1)$ satisfy the condition (ii) of Proposition 5.3, as desired.

The above arguments conclude our claim (6.4) in 6.3.3.

6.3.6. The argument in 6.3.4 shows that if $v_p(\alpha)=0$,

$$\mu_r(\mathfrak{f})\in(O_L[[G_\infty^{(2)}]]\otimes_{O_L}L)[G_m^{(1)}].$$

This and the argument in 6.3.5 show that in this case, $\mathfrak{L}_{f,\{k-2\},\{0,\dots,k-2\},m}(z_{Np^\infty}^{\mathrm{univ}})$ is contained in $i_{\{1,\dots,k-1\}}((O_L[[G_\infty^{(2)}]] \otimes_{O_L} L)[1/a])[G_m^{(1)}]$.

Now we show that $\mathfrak{L}_{f,\{k-2\},\{0,\dots,k-2\},m}(z_{Np^\infty}^{\mathrm{univ}})^\pm \in i_{\{1,\dots,k-1\}}(\mathrm{H}_{L,h})[G_m^{(1)}]$. The main line of the proof is roughly as follows. Let M be an integer ≥ 1 which is prime to Np. By using ${}_{c,d'}z_{M,Np^\infty}$ which is similar with the element ${}_{c,d'}z_{Np^\infty}^{\mathrm{univ}}$ in 4.5.5 and which is defined in 6.3.7 and 6.3.8, we will construct a function $\mathcal{L}_{\phi,f,\{k-2\},\{0,\dots,k-2\},m}({}_{c,d'}z_{M,Np^\infty}) \in i_{\{1,\dots,k-1\}}(\mathrm{H}_{L(\phi),h})[G_m^{(1)}]$ in 6.3.9. Here ϕ denotes a character $(\mathbb{Z}/M\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times$. Assertions in 6.3.11 and 6.3.12 say that $\mathcal{L}_{\phi,f,\{k-2\},\{0,\dots,k-2\},m}({}_{c,d'}z_{M,Np^\infty})$ is, in fact, a function which is a multiple of our $\mathfrak{L}_{f,\{k-2\},\{0,\dots,k-2\},m}(z_{Np^\infty}^{\mathrm{univ}})$ by a non-zerodivisor in $\mathbb{Z}_p[[G_\infty^{(2)}]]$ which is prime to $a \in \mathbb{Z}_p[[G_\infty]]$ in 6.3.3.

6.3.7. Let M be a positive integer such that $(M, Np) = 1$. We define an element

$$z_{M,Np^\infty} \in \overline{M}_{MNp^\infty}[[G_{Mp^\infty}{}^{(1)} \times G_{Mp^\infty}{}^{(2)}]][\frac{1}{g'}]$$

with a certain non-zerodivisor

$$g' \in \mathbb{Z}[[G_{Mp^\infty}{}^{(1)} \times G_{Mp^\infty}{}^{(2)}]] \subset \overline{M}_{MNp^\infty}[[G_{Mp^\infty}{}^{(1)} \times G_{Mp^\infty}{}^{(2)}]],$$

which is similar to the universal zeta modular form $z_{Np^\infty}^{\mathrm{univ}}$. In fact, in the case $M = 1$, we have $z_{1,Np^\infty} = z_{Np^\infty}^{\mathrm{univ}}$. Here

$$G_{Mp^\infty}{}^{(1)} \cong G_{Mp^\infty}{}^{(2)} \cong G_{Mp^\infty} = (\mathbb{Z}/M\mathbb{Z})^\times \times \mathbb{Z}_p^\times,$$

the group $G_{Mp^\infty}{}^{(1)}$ is the one acting on the space $\overline{M}_{MNp^\infty}$ in the following way. For $a = (a_1, a_2) \in (\mathbb{Z}/M\mathbb{Z})^\times \times \mathbb{Z}_p^\times$, the action of the corresponding element $\mathfrak{g}_a^{(1)} \in G_{Mp^\infty}{}^{(1)}$ on $f = \sum_k f_k \in \bigcup_j M^j(X_1(MNp^t); \mathbb{Q}_p)$ with $f_k \in M_k(X_1(MNp^t); \mathbb{Q}_p)$ and $t \geq 1$ is given as

$$\mathfrak{g}_a^{(1)} \cdot f = \sum_k a_2^{k-2} \langle a' \rangle f_k,$$

where $a' \in (\mathbb{Z}/MNp^t\mathbb{Z})^\times$ is the element such that $a' \equiv a(Mp^t)$ and $a' \equiv 1(N)$. The group $G_{Mp^\infty}{}^{(2)}$ is the Galois group $\mathrm{Gal}(\mathbb{Q}_p(\zeta_{Mp^\infty})/\mathbb{Q}_p)$ which is endowed with an isomorphism to $(\mathbb{Z}/M\mathbb{Z})^\times \times \mathbb{Z}_p^\times$ via the cyclotomic character.
The element z_{M,Np^∞} is the image of the product $F'_{N,1} \cdot F'_{N,2} \in \mathsf{H}[[G_{Mp^\infty}]][[G_{Mp^\infty}]]$ with $F'_{N,1}, F'_{N,2} \in \mathsf{H}[[G_{Mp^\infty}]]$ below under the isomorphism of rings over H

$$\mathsf{H}[[G_{Mp^\infty}]][[G_{Mp^\infty}]] \to \mathsf{H}[[G_{Mp^\infty}{}^{(1)}]][[G_{Mp^\infty}{}^{(2)}]] \ ; \ x\mathfrak{g}_{a,1}\mathfrak{g}_{b,2} \mapsto x\mathfrak{g}_b^{(1)}\mathfrak{g}_{ab^{-1}}^{(2)}$$

$(x \in \mathsf{H}, a, b \in \mathbb{Z}_p^\times \times (\mathbb{Z}/M\mathbb{Z})^\times)$, where $\mathfrak{g}_a \in G_{Mp^\infty}$ is the corresponding element to a. Here

$$F'_{N,1} = (\sum_{\substack{i\ge 1\\(i,Mp)=1}} \sum_{j\ge 1} q^{Nij})(\mathfrak{g}_i - \mathfrak{g}_{-i}) + \varprojlim_n (\sum_{a\in(\mathbb{Z}/Mp^n\mathbb{Z})^\times} \zeta_{\equiv a(Mp^n)}(0)\cdot \mathfrak{g}_a),$$

$$F'_{N,2} = (\sum_{\substack{i\ge 1, i\equiv 1(N)\\(i,Mp)=1}} \sum_{j\ge 1} q^{ij}\cdot \mathfrak{g}_i - \sum_{\substack{i\ge 1, i\equiv -1(N)\\(i,Mp)=1}} \sum_{j\ge 1} q^{ij}\cdot \mathfrak{g}_{-i}) + \varprojlim_n (\sum_{\substack{a\in(\mathbb{Z}/NMp^n\mathbb{Z})^\times\\ a\equiv 1(N)}} \zeta_{\equiv a(NMp^n)}(0)\cdot \mathfrak{g}_a).$$

Originally z_{M,Np^∞} belongs to $\mathsf{H}[[G_{Mp^\infty}{}^{(1)} \times G_{Mp^\infty}{}^{(2)}]]$, but we can prove that $z_{M,Np^\infty} \in \overline{M}_{MNp^\infty}[[G_{Mp^\infty}{}^{(1)} \times G_{Mp^\infty}{}^{(2)}]][1/g']$. This follows from the following lemma which can be proven in the same way as for $z^{\mathrm{univ}}_{Np^\infty}$.
For $n \ge 1$, we write G_{Mp^n} for the group $(\mathbb{Z}/M\mathbb{Z})^\times \times G_n$.

Lemma 6.3.8 . (1) *Let c, d' be integers which are prime to p. Then*

$${}_{c,d'}z_{M,Np^\infty} := (1 - c^{-1}\mathfrak{g}^{(1)}_{c^{-1}}\mathfrak{g}^{(2)}_{c})(1-d'\mathfrak{g}^{(2)}_{d'})z_{M,Np^\infty} \in O_{\mathsf{H}}[[G_{Mp^\infty}{}^{(1)} \times G_{Mp^\infty}{}^{(2)}]].$$

(2) *For $i \in \mathbb{Z}$, let us denote by $\chi^i : G_{Mp^\infty} \to G_{Mp^\infty}$ the map induced by χ^i on the component $\mathbb{Z}_p^\times$. Let a_1, a_2 be integers such that $0 \le a_2 \le a_1$. We define $z_{M,Np^\infty}(\chi^{a_1}, \chi^{a_2})$ in the same way as for $z^{\mathrm{univ}}_{Np^\infty}$ in section 4. Concerning the image of the natural projection, we have*

$$z_{M,Np^\infty}(\chi^{a_1},\chi^{a_2})|_{(n,n)} \in M_{a_1+2}(X_1(MNp^n);\mathbb{Q})[G^{(1)}_{Mp^n} \times G^{(2)}_{Mp^n}] \quad (\subset \mathsf{H}[G^{(1)}_{Mp^n} \times G^{(2)}_{Mp^n}]),$$

$${}_{c,d'}z_{M,Np^\infty}(\chi^{a_1},\chi^{a_2})|_{(n,n)} \in M_{a_1+2}(X_1(MNp^n);\mathbb{Z}_{(p)})[G^{(1)}_{Mp^n} \times G^{(2)}_{Mp^n}] \quad (\subset O_{\mathsf{H}}[G^{(1)}_{Mp^n} \times G^{(2)}_{Mp^n}]).$$

(3) *Let a_1 and a_2 be as in (2). For any integers n, m such that $n \ge m \ge 1$, we have*

$$\mathrm{Tr}_q^{n-m}(z_{M,Np^\infty}(\chi^{a_1},\chi^{a_2})|_{(m,n)}) \in M_{a_1+2}(X_1(MNp^m);\mathbb{Q})[G^{(1)}_{Mp^m}][G^{(2)}_{Mp^n}].$$

$$\mathrm{Tr}_q^{n-m}({}_{c,d'}z_{M,Np^\infty}(\chi^{a_1},\chi^{a_2})|_{(m,n)}) \in M_{a_1+2}(X_1(MNp^m);\mathbb{Z}_{(p)})[G^{(1)}_{Mp^m}][G^{(2)}_{Mp^n}].$$

6.3.9. Let $\phi : (\mathbb{Z}/M\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times$ be a character whose conductor is M. Let $\mathfrak{f}_\phi$ be the modular form given by

$$\mathfrak{f}_\phi = \sum_{n\ge 1} a_n(\mathfrak{f})\phi(n)q^n.$$

It is an eigen cusp form of level MNp^m with zeta function $L(\mathfrak{f}_\phi, s) = L(\mathfrak{f}, \phi, s)$.

In the same manner as in 6.3.3 (6.5), we can show that

$$(\alpha^{-n}\phi(p)^{-n}\cdot \mathrm{pr}_{\mathfrak{f}_\phi}(\mathrm{Tr}_q^{n-m}({}_{c,d'}z_{M,Np^\infty}(\chi^{k-2},\chi^{r-1})|_{(m,n)})))_n \tag{6.9}$$

belongs to $\varprojlim_n L(\phi)[G^{(2)}_{Mp^n}][G^{(1)}_{Mp^m}]$, where $L(\phi)$ is the field generated over L by the values of ϕ. Here remark that $a_p(\mathfrak{f}_\phi) = a_p(\mathfrak{f})\phi(p) = \alpha\phi(p)$. We consider the image of the element (6.9) under the $L(\phi)$-homomorphism given by

$$L(\phi)[[G^{(2)}_{Mp^\infty}]][G^{(1)}_{Mp^m}] \to L(\phi)[[G^{(2)}_\infty]][G^{(1)}_m]\ ;\ x\mathfrak{g}^{(2)}_a\mathfrak{g}^{(1)}_b \mapsto x\phi(ab^2)g^{(2)}_a g^{(1)}_b \tag{6.10}$$

($x \in L(\phi), \mathfrak{g}_* \in G_{Mp^\infty}, g_* \in G_\infty$). We write $\mathcal{L}_{\phi,f,\{k-2\},\{r-1\},m}({}_{c,d'}z_{M,Np^\infty}) \in L(\phi)[[G^{(2)}_\infty]][G^{(1)}_m]$ for this image, and we put

$$\mathcal{L}_{\phi,f,\{k-2\},\{0,\dots,k-2\},m}({}_{c,d'}z_{M,Np^\infty}) = \prod_{r\in\{1,\dots,k-1\}} \mathcal{L}_{\phi,f,\{k-2\},\{r-1\},m}({}_{c,d'}z_{M,Np^\infty}) \in \prod_{r\in\{1,\dots,k-1\}} L(\phi)[[G^{(2)}_\infty]][G^{(1)}_m].$$

Concerning $\mathcal{L}_{\phi,f,\{k-2\},\{0,\dots,k-2\},m}({}_{c,d'}z_{M,Np^\infty})$, we have the following propositions, which are crucial to our purpose.

Proposition 6.3.10 . *The element $\mathcal{L}_{\phi,f,\{k-2\},\{0,\dots,k-2\},m}({}_{c,d'}z_{M,Np^\infty})$ is contained in the subspace $i_{\{1,\dots,k-1\}}(\mathrm{H}_{L(\phi),h})[G^{(1)}_m]$.*

Proof. We can prove Proposition 6.3.10 in the same manner as in 6.3.1 – 6.3.6. □

In the rest of 6.3, we identify an element of $\mathrm{H}_{L(\phi),h}$ and its image under $i_{\{1,\dots,k-1\}} : \mathrm{H}_{L(\phi),h} \hookrightarrow \prod_{r\in\{1,\dots,k-1\}} L(\phi)[[G^{(2)}_\infty]]$.

Proposition 6.3.11 . *Assume $L(\mathfrak{f}_\phi, k-1) = L(\mathfrak{f},\phi,k-1) \neq 0$ and $\phi(-1) = 1$. Then we have*

$$\begin{aligned}
&\mathcal{L}_{\phi,f,\{k-2\},\{0,\dots,k-2\},m}({}_{c,d'}z_{M,Np^\infty})^{\pm} \\
&\quad = x\cdot\Big(\prod_{r\in\{1,\dots,k-1\}}\Big(\prod_{\substack{l:\text{prime}\\ l|M}}(1-a_l(\mathfrak{f})l^{-r}g^{(2)}_{l^{-1}} + \epsilon_{\mathfrak{f}}(l)l^{k-1-2r}g^{(2)}_{l^{-2}})\\
&\qquad\cdot(1-c^{r-k}\epsilon_{\mathfrak{f}}(c^{-1})\phi(c^{-1})g^{(2)}_c)(1-d'^{r}\phi(d')g^{(2)}_{d'}))\\
&\qquad\cdot\alpha^m\cdot\mathfrak{L}_{f,\{k-2\},\{0,\dots,k-2\},m}(z^{\mathrm{univ}}_{Np^\infty}))^{\pm}\\
&\qquad\quad \in \mathrm{H}_{L(\phi),h}[G^{(1)}_m]
\end{aligned}$$

for some $x \in L(\phi)^\times$, where $\pm = (-1)^k\epsilon_{\mathfrak{f}}(-1)$.

Proof. An element of $\mathrm{H}_{L(\phi),h}[1/e](\subset \mathrm{H}_{L(\phi),h}\otimes_{\mathbb{Z}_p[[G^{(2)}_\infty]]} Q(\mathbb{Z}_p[[G^{(2)}_\infty]]))$ can be characterized by specializations $\mathrm{H}_{L(\phi),h}[1/e] \to \overline{L}$ induced by $\psi\circ\chi^{r_i} : G^{(2)}_\infty \to \overline{\mathbb{Q}}^\times$ for different h integers r_i $(i = 1,\dots,h)$ and all but finitely many Dirichlet characters ψ. So we can prove Proposition 6.3.11 by comparing the images of both hands sides under such specializations. The image of the right hand side under specializations will be studied in 6.4 – 6.6 below. The image of the left hand side may be obtained in the same way as for the right hand side, but we omit the details. □

6.3.12. We finish proving that the elements in Theorem 6.2 are contained in $i_{\{1,\ldots,k-1\}}(\mathrm{H}_{L,h})[G_m^{(1)}]$.
We have

$$\begin{aligned}
&\mathfrak{L}_{f,\{k-2\},\{0,\ldots,k-2\},m}({}_{c,d'}z_{Np^\infty}^{\mathrm{univ}}) \\
&\quad = \Big(\prod_{r\in\{1,\ldots,k-1\}} (1-c^{r-k}\epsilon_{\mathfrak{f}}(c^{-1})g_c^{(2)})(1-d'^r g_{d'}^{(2)})\Big) \\
&\qquad \cdot \mathfrak{L}_{f,\{k-2\},\{0,\ldots,k-2\},m}(z_{Np^\infty}^{\mathrm{univ}}) \\
&\qquad\quad \in \mathrm{H}_{L,h}[G_m^{(1)}](\subset \mathrm{H}_{L,h}\otimes_{\mathbb{Z}_p[[G_\infty^{(2)}]]} Q(\mathbb{Z}_p[[G_\infty^{(2)}]])[G_m^{(1)}]),
\end{aligned}$$

which follows directly from the definition.
Since $(1-a_l(\mathfrak{f})g_{l^{-1}}^{(2)}+\epsilon_{\mathfrak{f}}(l)l^{k-1}g_{l^{-2}}^{(2)})$ for all of the prime numbers l which are prime to Np do not have a common divisor, by Propositions 6.3.10 and 6.3.11, it is sufficient to show the following assertion.
There exist c, d', M, and ϕ which satisfy the above given conditions and the following condition. For some characters $\psi_1, \psi_2 : (\mathbb{Z}/p^n\mathbb{Z})^\times \longrightarrow \overline{\mathbb{Q}}^\times$ with conductor divisible by p, $\phi(c) = \psi_1(c) \neq 1$ and $\phi(d') = \psi_2(d') \neq 1$, and $\phi(-1) = 1$ hold. We can take such elements, therefore we obtain the desired result.

6.4. We prove that our elements in Theorem 6.2 (1) and (2) satisfy the characterizing properties of $L_{p\text{-adic}}(\mathfrak{f})$ in Theorem 5.5. In fact, the difference between Theorem 6.2 (1) and (2) comes from the fact that we take $\mathrm{pr}_{\mathfrak{f}}$ instead of "pr_f". We treat the cases Theorem 6.2 (1) and (2) together.
We use the notation in Theorem 5.5.

6.4.1. As referred earlier, for $L[[G_\infty^{(2)}]]$, we define + and − parts in the same way as in 5.6, and for $* = +$ or $-$, we define the $*$-part of $L[[G_\infty^{(2)}]][G_m^{(1)}]$ by $L[[G_\infty^{(2)}]]^*[G_m^{(1)}]$.
In both cases of Theorem 6.2 (1) and (2), we have

$$\begin{aligned}
&\mathfrak{L}_{f,\{k-2\},\{0,\ldots,k-2\},m}(z_{Np^\infty}^{\mathrm{univ}}) \\
&\quad = \prod_{r\in\{1,\ldots,k-1\}} \varprojlim_n (\alpha^{m-n}\cdot \mathrm{pr}_{\mathfrak{f}}(\mathrm{Tr}_q^{n-m}(z_{Np^\infty}^{\mathrm{univ}}(\chi^{k-2},\chi^{r-1})|_{(m,n)}))) \\
&\qquad \in \prod_{r\in\{1,\ldots,k-1\}} \varprojlim_n L[G_n^{(2)}][G_m^{(1)}] = \prod_{r\in\{1,\ldots,k-1\}} L[[G_\infty^{(2)}]][G_m^{(1)}].
\end{aligned} \tag{6.11}$$

This follows from the equality (6.5). We will prove the assertion that the image of the coefficient belonging to $\varprojlim_n L[G_n^{(2)}] = L[[G_\infty^{(2)}]]$ of $g_1^{(1)}$ in

$$(\prod_{\substack{l:\text{prime}\\ l|N}}(1 - a_l(\mathfrak{f})l^{-r}g_{l^{-1}}^{(2)})\cdot \varprojlim_n(\alpha^{-n}\cdot \mathrm{pr}_{\mathfrak{f}}(\mathrm{Tr}_q^{n-m}(z_{Np^\infty}^{\mathrm{univ}}(\chi^{k-2},\chi^{r-1})|_{(m,n)}))))^{\pm\cdot(-1)^r}$$

$$\in \varprojlim_n L[G_n^{(2)}][G_m^{(1)}] = L[[G_\infty^{(2)}]][G_m^{(1)}], \tag{6.12}$$

where $\pm = (-1)^k\epsilon_{\mathfrak{f}}(-1)$ and $(\ )^{\pm\cdot(-1)^r}$ represents the $\pm\cdot(-1)^r$-part of the element in ( ), under the map $\psi^{-1} : L[[G_\infty^{(2)}]] \to \overline{L}$ coincides with the image of

$$(\prod_{\substack{l:\text{prime}\\ l|N}}(1 - a_l(\mathfrak{f})g_{l^{-1}}^{(2)})\cdot L_{p\text{-adic}}(\mathfrak{f}))^{\pm}$$

under $\chi_{\text{cyclo}}^r\psi^{-1} : \mathrm{H}_{L,h} \to \overline{L}$.
By the following facts, this assertion deduces Theorem 6.2.
Firstly we have the equality (6.11).
Secondly $\prod_{\substack{l:\text{prime}\\ l|N}}(1 - a_l(\mathfrak{f})g_{l^{-1}}^{(2)}) \in O_L[[G_\infty^{(2)}]]\otimes_{O_L} L$ is a non-zerodivisor of $\mathrm{H}_{L,h}$.
Finally by the results in 4.5.5, we have

$$(\mathrm{Tr}_q^{n-m}(z_{Np^\infty}^{\mathrm{univ}}(\chi^{k-2},\chi^{r-1})|_{(m,n)}))_n = \sum_{a\in(\mathbb{Z}/p^m\mathbb{Z})^\times}\langle a'^{-1}\rangle x\cdot g_a^{(1)}$$

$$\in M_k(X_1(Np^m);\mathbb{Q})[G_m^{(1)}][[G_\infty^{(2)}]], \tag{6.13}$$

where x is the coefficient of $g_1^{(1)}$ in the left hand side, and $a' \in (\mathbb{Z}/Np^m\mathbb{Z})^\times$ is the element such that $a' \equiv a(p^m)$ and $a' \equiv 1(N)$. Hence the coefficient in $\varprojlim_n L[G_n^{(2)}] = L[[G_\infty^{(2)}]]$ of $g_a^{(1)}$ in (6.12) is $\epsilon_{\mathfrak{f}}(a'^{-1})$ times the coefficient of $g_1^{(1)}$ in (6.12). (Remark that in the case of Theorem 6.2 (2), $\epsilon_{\mathfrak{f}}(a') = 1$ holds.)
Thus we prove the assertion above.

6.4.2. We use the same notation as in Theorem 5.5. We consider the following composition

$$\psi_{\zeta_{p^n}} : L[[G_\infty^{(2)}]] \xrightarrow{\text{proj.}} L[G_n^{(2)}] \to \overline{L}, \tag{6.14}$$

where the second map is defined by

$$\sum_{u\in(\mathbb{Z}/p^n\mathbb{Z})^\times} a_u g_u^{(2)} \mapsto \sum_{x\in(\mathbb{Z}/p^n\mathbb{Z})^\times}\psi(x)\sum_{u\in(\mathbb{Z}/p^n\mathbb{Z})^\times} a_u\zeta_{p^n}^{ux} \qquad (a_u \in L).$$

Lemma 6.4.3 . *Let μ be an element of $L[[G_\infty^{(2)}]]$. Then we have*

$$\mu(\psi^{-1}) = \psi_{\zeta_{p^n}}(\mu)\cdot G(\psi,\zeta_{p^n})^{-1} \in \overline{L}.$$

Proof. We can prove the lemma by direct computation. □

6.4.4. We assume $\psi(-1) = (-1)^{k-r}\epsilon_{\mathfrak{f}}(-1)$. By Lemma 6.4.3, our task is showing that the image of the coefficient of $g_1^{(1)}$ in (6.12) under the map (6.14) is

$$(r-1)! \cdot p^{nr} \cdot \alpha^{-n} \cdot (2\pi i)^{k-r-1} \cdot \frac{1}{\Omega(f, \delta(\mathfrak{f}, k-1, 1(0)))_-} \cdot L_{(Np)}(\mathfrak{f}, \psi, r).$$

6.5. In order to prove our claim in 6.4.4, we need to review the result in [Ka2].

6.5.1. As in [Ka2], let

$$z_{Np^n}(\mathfrak{f}, r, k-1, 0(1), \mathrm{prim}(Np)) \in S(\mathfrak{f}) \otimes \mathbb{Q}(\zeta_{Np^n}) \quad (r \neq 2),$$

$$_{c,d}z_{Np^n}(\mathfrak{f}, 2, k-1, 0(1), \mathrm{prim}(Np)) \in S(\mathfrak{f}) \otimes \mathbb{Q}(\zeta_{Np^n})$$

be the images of $z_{1,Np^m,Np^n}^{(k,r)}(k, r, k-1, 0(1), \mathrm{prim}(Np))$ in the case $r \neq 2$, and $_{c,d}z_{1,Np^m,Np^n}^{(k,2)}(k, 2, k-1, 0(1), \mathrm{prim}(Np))$ in the case $r = 2$, both of which are elements of $M_k(X_1(Np^m); \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_{Np^n})$, respectively, in 4.5.2 under the projection

$$M_k(X_1(Np^m); \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_{Np^n}) \longrightarrow S(\mathfrak{f}) \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_{Np^n}). \tag{6.15}$$

Let $\mathfrak{f}^* = \sum_{n \geq 1} \overline{a_n}(\mathfrak{f}) q^n$ denote the dual cusp form of $\mathfrak{f}$. Here $\overline{a_n}(\mathfrak{f})$ are the complex conjugates of $a_n(\mathfrak{f})$. This is also a normalized eigen cusp form. For $n \geq 1$ such that $(n, Np) = 1$, it is known that

$$(T(n)\langle n^{-1}\rangle)(\mathfrak{f}^*) = a_n(\mathfrak{f})\mathfrak{f}^*. \tag{6.16}$$

For $x \in (\mathbb{Z}/Np^n\mathbb{Z})^\times$, let us denote by ν_x the corresponding element to x of $\mathrm{Gal}(\mathbb{Q}_p(\zeta_{Np^n})/\mathbb{Q}_p)$ via the cyclotomic character.

PROPOSITION 6.5.2 . *Assume $r \neq 2$. Let $\psi : (\mathbb{Z}/Np^n\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times$ be a character. We put $\pm = (-1)^{k-r-1}\psi(-1)$. Then we have*

$$\sum_{x \in (\mathbb{Z}/Np^n\mathbb{Z})^\times} \psi(x) \cdot \mathrm{per}_{\mathfrak{f}}((\langle x^{-1}\rangle \otimes \nu_x)(z_{Np^n}(\mathfrak{f}, r, k-1, 0(1), \mathrm{prim}(Np))))^\pm$$
$$= L_{(Np)}(\mathfrak{f}^*, \psi, r) \cdot (2\pi i)^{k-r-1} \cdot \delta(\mathfrak{f}, k-1, 0(1))^\pm,$$

where $L_{(Np)}(\mathfrak{f}^*, \psi, s)$ denotes the function obtained from $L(\mathfrak{f}^*, \psi, s)$ by removing prime(Np) *factors.*

Proof. Remark that the definitions of the actions of Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_{Np^n})/\mathbb{Q})$ are different between in [Ka2] and here: The action of σ_x in [Ka2], §6, Theorem 6.6 on $S(\mathfrak{f}) \otimes \mathbb{Q}(\zeta_{Np^n})$ is equal to the action of $\langle x^{-1}\rangle \otimes \nu_x$ in our notation. By this relation we see that the above equation is equivalent to 6.6 in [Ka2] which can be deduced from the work of Shimura [Sh]. □

In the case $r = 2$, Proposition 6.5.2 must be modified as

$$\sum_{x \in (\mathbb{Z}/Np^n\mathbb{Z})^\times} \psi(x) \cdot \mathrm{per}_{\mathfrak{f}}((\langle x^{-1} \rangle \otimes \nu_x)({}_{c,d}z_{Np^n}(\mathfrak{f}, 2, k-1, 0(1), \mathrm{prim}(Np))))^{\pm}$$
$$= c^2 d^2 \cdot (1 - c^{2-k}\psi(c)^{-1})(1 - \psi(d)^{-1}\epsilon_{\mathfrak{f}}(d))$$
$$\cdot L_{(Np)}(\mathfrak{f}^*, \psi, 2) \cdot (2\pi i)^{k-3} \cdot \delta(\mathfrak{f}, k-1, 0(1))^{\pm}.$$

The above proposition induces the following corollary.

Corollary 6.5.3 . *Assume $r \neq 2$. Let ψ be as in Proposition* 6.5.2. *We put $\pm = (-1)^{k-r-1}\psi(-1)\epsilon_{\mathfrak{f}}(-1)$. Then we have*

$$\sum_{x \in (\mathbb{Z}/Np^n\mathbb{Z})^\times} \psi(x) \cdot \mathrm{per}_{\mathfrak{f}}((1 \otimes \nu_x)(z_{Np^n}(\mathfrak{f}, r, k-1, 0(1), \mathrm{prim}(Np))))^{\pm}$$
$$= L_{(Np)}(\mathfrak{f}, \psi, r) \cdot (2\pi i)^{k-r-1} \cdot \delta(\mathfrak{f}, k-1, 0(1))^{\pm}.$$

Proof. By the definition of $\epsilon_{\mathfrak{f}}$, we have the equality

$$\sum_{x \in (\mathbb{Z}/Np^n\mathbb{Z})^\times} \psi(x)$$
$$\cdot \mathrm{per}_{\mathfrak{f}}((\langle x^{-1} \rangle \otimes \nu_x)(z_{Np^n}(\mathfrak{f}, r, k-1, 0(1), \mathrm{prim}(Np))))^{(-1)^{k-r-1}\psi(-1)}$$
$$= \sum_{x \in (\mathbb{Z}/Np^n\mathbb{Z})^\times} \psi(x)\epsilon_{\mathfrak{f}}(x^{-1})$$
$$\cdot \mathrm{per}_{\mathfrak{f}}((1 \otimes \nu_x)(z_{Np^n}(\mathfrak{f}, r, k-1, 0(1), \mathrm{prim}(Np))))^{(-1)^{k-r-1}\psi(-1)}.$$

By Proposition 6.5.2 and by this, we see that the left hand side of the equation in Corollary 6.5.3 is equal to

$$L_{(Np)}(\mathfrak{f}^*, \psi \cdot \epsilon_{\mathfrak{f}}, r) \cdot (2\pi i)^{k-r-1} \cdot \delta(\mathfrak{f}, k-1, 0(1))^{\pm}.$$

By (6.16) and by the fact that $\epsilon_{\mathfrak{f}}(n) = \epsilon_{\mathfrak{f}^*}(n^{-1})$ for $n \in \mathbb{Z}$ such that $(n, Np) = 1$, we have

$$L_{(Np)}(\mathfrak{f}^*, \psi \cdot \epsilon_{\mathfrak{f}}, s) = L_{(Np)}(\mathfrak{f}, \psi, s).$$

This shows the result. □

With a suitable modification, we have a similar result with Corollary 6.5.3, in the case $r = 2$.

As a corollary to Corollary 6.5.3, we obtain the following Corollary 6.5.4 which is important for the proof of our Theorem 6.2.
For $r \neq 2$, let $A(k, r)$ be the element of $M_k(X_1(Np^m); \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_{p^n})$ obtained from the right hand side of (4.8) in Lemma 4.5.3 (3) in section 4 by replacing "$a \in \mathbb{Z}/p^n\mathbb{Z}$" in the right hand side of (4.8) by "$a \in (\mathbb{Z}/p^n\mathbb{Z})^\times$". Namely,

$$A(k,r) = (r-1)!^{-1} \cdot (Np^m)^{k-r-2}(Np^n)^{-2} \cdot N \cdot \prod_{\substack{l:\text{prime}\\ l|N}} (1 - l^{-r}T(l)\nu_{l^{-1}}) \Big(\sum_{a \in (\mathbb{Z}/p^n\mathbb{Z})^\times} T(p)^{n-m} \Big(\sum_{x \in ((1/Np^m)\mathbb{Z})/\mathbb{Z}} \sum_{y \in ((1/Np^n)\mathbb{Z})/\mathbb{Z}} (F^{(k-r)}_{1/Np^m,x} \cdot F^{(r)}_{Na/Np^n,y})\Big) \cdot \zeta^a_{p^n}\Big).$$

We define $A(\mathfrak{f}, r)$ to be the image of $A(k,r)$ under the projection (6.15).

COROLLARY 6.5.4 . *Assume $r \neq 2$. Let $\psi : (\mathbb{Z}/p^n\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times$ be a character which does not factor through $(\mathbb{Z}/p^{n-1}\mathbb{Z})^\times$ and which satisfies $\psi(-1) = (-1)^{k-r}\epsilon_\mathfrak{f}(-1)$. Then we have*

$$\sum_{x \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \psi(x) \cdot \mathrm{per}_\mathfrak{f}((1 \otimes \nu_x)(A(\mathfrak{f}, r)))^- = L_{(Np)}(\mathfrak{f}, \psi, r) \cdot (2\pi i)^{k-r-1} \cdot \delta(\mathfrak{f}, k-1, 0(1))^-.$$

Proof. We take ψ as a character which factors through $(\mathbb{Z}/p^n\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times$ and does not factor through $(\mathbb{Z}/p^{n-1}\mathbb{Z})^\times$, and apply Corollary 6.5.3. Then the part of the sum $\sum_{x \in (\mathbb{Z}/N\mathbb{Z})^\times}$ is equivalent to take the trace map tr_{Np^n,p^n} in Lemma 4.5.3 (3). Concerning the problem of changing a, since ψ in Corollary 6.5.4 is primitive, the iterated sum over $\sum_{x \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \psi(x)\zeta^{ax}_{p^n}$ for $(a,p) > 1$ become zero. □

In the case $r = 2$, Corollary 6.5.4 must be modified as follows. We take ${}_{c,d}A(k,2)$ as the element which is obtained from (4.9) by replacing "$a \in \mathbb{Z}/p^n\mathbb{Z}, a \neq 0$" in the right hand side of (4.9) by "$a \in (\mathbb{Z}/p^n\mathbb{Z})^\times$". We also define ${}_{c,d}A(\mathfrak{f}, 2)$ to be the image of ${}_{c,d}A(k,2)$ under the projection (6.15). Then we have

$$\sum_{x \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \psi(x) \cdot \mathrm{per}_\mathfrak{f}((1 \otimes \nu_x)({}_{c,d}A(\mathfrak{f}, 2)))^- = c^2d^2 \cdot ((1 - c^{2-k}\epsilon_\mathfrak{f}(c)^{-1}\psi(c)^{-1})(1 - \psi(d)^{-1})) L_{(Np)}(\mathfrak{f}, \psi, 2) \cdot (2\pi i)^{k-3} \cdot \delta(\mathfrak{f}, k-1, 0(1))^-.$$

6.6. Now we prove our claim in 6.4.4.
Since $T(l)\mathfrak{f} = a_l(\mathfrak{f})\mathfrak{f}$, (6.12) coincides with

$$\Big(\varprojlim_n \Big(\alpha^{-n} \cdot \mathrm{pr}_\mathfrak{f}\Big(\prod_{\substack{l:\text{prime}\\ l|N}} (1 - T(l)l^{-r}g^{(2)}_{l^{-1}})(\mathrm{Tr}^{n-m}_q(z^{\mathrm{univ}}_{Np^\infty}(\chi^{k-2}, \chi^{r-1})|_{(m,n)}))\Big)\Big)\Big)^{\pm \cdot (-1)^r}.$$

So to analyze (6.12), we consider the image of

$$\prod_{\substack{l:\text{prime}\\ l|N}} (1 - T(l)l^{-r}g^{(2)}_{l^{-1}})(\mathrm{Tr}_q^{n-m}(z^{\mathrm{univ}}_{Np^\infty}(\chi^{k-2}, \chi^{r-1})|_{(m,n)})) \\ \in M_k(X_1(Np^m); \mathbb{Q})[G^{(1)}_m \times G^{(2)}_n] \tag{6.17}$$

under the map

$$\cdot\zeta_{p^n} : M_k(X_1(Np^m); \mathbb{Q})[G^{(1)}_m \times G^{(2)}_n] \longrightarrow M_k(X_1(Np^m); \mathbb{Q}(\zeta_{p^n}))[G^{(1)}_m] \tag{6.18}$$

defined in the similar way as (4.12).

In the case $r \neq 2$, by comparing (4.8) in Lemma 4.5.3 (3) and equation (4.13) in 4.5.6 with $a_1 = k-2$ and $a_2 = r-1$, we see that the coefficient of $g^{(1)}_1 \in G^{(1)}_m$ in the image of (6.17) under the map (6.18) is

$$(r-1)! \cdot p^{nr} \cdot A(k, r)$$

with $A(k, r)$ in Corollary 6.5.4. Thus under the composition

$$L[[G^{(2)}_\infty]][G^{(1)}_m] \xrightarrow{\text{proj.}} L[G^{(2)}_n][G^{(1)}_m] \xrightarrow{\cdot\zeta_{p^n}} L(\zeta_{p^n})[G^{(1)}_m] \\ \xrightarrow{\cdot f} (L(\zeta_{p^n}) \cdot f)[G^{(1)}_m], \tag{6.19}$$

the coefficient of $g^{(1)}_1$ in the element (6.12) is sent to

$$\alpha^{-n} \cdot (r-1)! \cdot p^{nr} \cdot A(\mathfrak{f}, r).$$

Hence by Corollary 6.5.4 and by comparing the definition of the map (6.14) and the map in Corollary 6.5.4, we obtain that under the assumption in 6.4.4, $\mathrm{per}_{\mathfrak{f}}(f)^-$ times the image in question in 6.4.4 equals

$$\alpha^{-n} \cdot (r-1)! \cdot p^{nr} \cdot L_{(Np)}(\mathfrak{f}, \psi, r) \cdot (2\pi i)^{k-r-1} \cdot \delta(\mathfrak{f}, k-1, 0(1))^-.$$

Since $L(\mathfrak{f}, k-1) \neq 0$, we have $\delta(\mathfrak{f}, k-1, 0(1)) = \delta(\mathfrak{f}, k-1, 0(1))^- \neq 0$. As $\mathrm{per}_{\mathfrak{f}}(f)^- = \Omega(f, \delta(\mathfrak{f}, k-1, 1(0)))_- \cdot \delta(\mathfrak{f}, k-1, 0(1))^-$, we find that the claim in 6.4.4 is true in the case $r \neq 2$.

In the case $r = 2$, in the same way as for $r \neq 2$, we can show that the coefficient of $g^{(1)}_1$ in the image of $c^2d^2 \cdot (1 - c^{2-k}\epsilon_{\mathfrak{f}}(c^{-1})g^{(2)}_c)(1 - g^{(2)}_d)$ times element (6.17) under the map (6.18) is $p^{2n} \cdot {}_{c,d}A(k, 2)$ with ${}_{c,d}A(k, 2)$ just after Corollary 6.5.4. Hence the image of the coefficient of $g^{(1)}_1$ in $c^2d^2 \cdot (1 - c^{2-k}\epsilon_{\mathfrak{f}}(c^{-1})g^{(2)}_c)(1 - g^{(2)}_d)$ times (6.12) under the composition (6.19) is $\alpha^{-n} \cdot p^{2n} \cdot {}_{c,d}A(\mathfrak{f}, 2)$. From this, in the same way as for $r \neq 2$, we can show that the claim in 6.4.4 is true for $r = 2$.

The above arguments conclude our desired result that the left hands sides of the equations in Theorem 6.2 (1) and (2) satisfy the characterizing property of the right hands sides of them.

6.7. We explain that by our method, we can obtain the whole p-adic zeta function $L_{p\text{-adic}}(\mathfrak{f})$ without assuming that $L(\mathfrak{f}, k-1) \neq 0$.

Similar with $z^{\text{univ}}_{Np^\infty}$, elements z_{M,Np^∞} in 6.3.7 may be obtained via K_2 Coleman power series from the system of Beilinson elements $({}_{c,d}z_{MNp^n,Mp^n})_n \in \varprojlim_n K_2(Y(MNp^n, Mp^n))$, with $c, d \in \mathbb{Z}$ such that $(c, 6MNp) = (d, 6Mp) = 1$ and $c \equiv 1(N)$. In this case, we take the field $\mathsf{H}(\zeta_M)$ instead of H.
We define $\mathcal{L}_{\phi,f,\{k-2\},\{0,\dots,k-2\},m}(z_{M,Np^\infty}) \in \prod_{r\in\{1,\dots,k-1\}} L(\phi)[[G^{(2)}_\infty]][G^{(1)}_m]$ by replacing ${}_{c,d'}z_{M,Np^\infty}$ by z_{M,Np^∞} in (6.9) in the construction of the function $\mathcal{L}_{\phi,f,\{k-2\},\{0,\dots,k-2\},m}({}_{c,d'}z_{M,Np^\infty})$. Then in the same way as for Theorem 6.2, one can show that if $L(\mathfrak{f}_\phi, k-1) = L(\mathfrak{f}, \phi, k-1) \neq 0$,

$$\mathcal{L}_{\phi,f,\{k-2\},\{0,\dots,k-2\},m}(z_{M,Np^\infty})^{\pm\cdot\phi(-1)}$$
$$= x \cdot \sum_{a\in(\mathbb{Z}/p^m\mathbb{Z})^\times} ((\prod_{r\in\{1,\dots,k-1\}} (\prod_{\substack{l:\text{prime}\\ l|M}} (1 - a_l(\mathfrak{f})l^{-r}g^{(2)}_{l^{-1}} + \epsilon_{\mathfrak{f}}(l)l^{k-1-2r}g^{(2)}_{l^{-2}})))$$
$$\cdot L_{p\text{-adic}}(\mathfrak{f}))^{\pm\cdot\phi(-1)}\epsilon_{\mathfrak{f}}(a'^{-1})g^{(1)}_a \in \prod_{r\in\{1,\dots,k-1\}} L(\phi)[[G^{(2)}_\infty]][G^{(1)}_m]$$

with some $x \in L(\phi)^\times$, and $\pm = (-1)^k\epsilon_{\mathfrak{f}}(-1)$.
Moreover by replacing the pair (M, Np) by (N, p), we define a function. Let $\phi : (\mathbb{Z}/N\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times$ be a character whose conductor is N. Now we define $\mathcal{L}_{\phi,f,\{k-2\},\{0,\dots,k-2\},m}(z_{N,p^\infty}) \in \prod_{r\in\{1,\dots,k-1\}} L(\phi)[[G^{(2)}_\infty]][G^{(1)}_m]$ to be the image of

$$\prod_{r\in\{1,\dots,k-1\}} (\alpha^{-n}\phi(p)^{-n} \cdot \mathrm{pr}_{\mathfrak{f}_\phi}(\mathrm{Tr}^{n-m}_q({}_{c,d'}z_{N,p^\infty}(\chi^{k-2}, \chi^{r-1})|_{(m,n)})))_n$$
$$\in \prod_{r\in\{1,\dots,k-1\}} L(\phi)[[G_{Np^\infty}]][G^{(1)}_{Np^m}]$$

under an $L(\phi)$-homomorphism given by

$$L(\phi)[[G^{(2)}_{Np^\infty}]][G^{(1)}_{Np^m}] \to L(\phi)[[G^{(2)}_\infty]][G^{(1)}_m]\ ;\ x\mathfrak{g}^{(2)}_a\mathfrak{g}^{(1)}_b \mapsto x\phi(ab^2)g^{(2)}_a g^{(1)}_b$$

$(x \in L(\phi), \mathfrak{g}_* \in G_{Np^\infty}, g_* \in G_\infty)$.
One can show that if $L(\mathfrak{f}_\phi, k-1) = L(\mathfrak{f}, \phi, k-1) \neq 0$,

$$\mathcal{L}_{\phi,f,\{k-2\},\{0,\dots,k-2\},m}(z_{N,p^\infty})^{\pm\cdot\phi(-1)}$$
$$= x \cdot \sum_{a\in(\mathbb{Z}/p^m\mathbb{Z})^\times} (((\prod_{r\in\{1,\dots,k-1\}} (\prod_{\substack{l:\text{prime}\\ l|N}} (1 - a_l(\mathfrak{f})l^{-r}g^{(2)}_{l^{-1}}))$$
$$\cdot L_{p\text{-adic}}(\mathfrak{f}))^{\pm\cdot\phi(-1)})\epsilon_{\mathfrak{f}}(a'^{-1})g^{(1)}_a \in \prod_{r\in\{1,\dots,k-1\}} L(\phi)[[G^{(2)}_\infty]][G^{(1)}_m]$$

with some $x \in L(\phi)^\times$, and $\pm = (-1)^k\epsilon_{\mathfrak{f}}(-1)$.

From the above equalities, we find that by taking various ϕ which satisfy $L(\mathfrak{f},\phi,k-1)\neq 0$, the whole (i.e. including the $(-1)^{k-1}\epsilon_{\mathfrak{f}}(-1)$-part of) $L_{p\text{-adic}}(\mathfrak{f})$ can be obtained by our method without assuming $L(\mathfrak{f},k-1)\neq 0$.

7. The result on two-variable p-adic zeta function

For a module A over $\mathbb{Z}_p[[G_\infty^{(1)}\times G_\infty^{(2)}]]$, we put $A_Q = A\otimes_{\mathbb{Z}_p[[G_\infty^{(1)}\times G_\infty^{(2)}]]} Q(\mathbb{Z}_p[[G_\infty^{(1)}\times G_\infty^{(2)}]])$. In 7.1, for a certain subspace B of $\overline{M}_{Np^\infty}[[G_\infty^{(1)}\times G_\infty^{(2)}]]_Q$ to which the universal zeta modular form $z_{Np^\infty}^{\mathrm{univ}}$ belongs, we define a map (in 7.1.3)

$$\mathrm{L}_N : B \longrightarrow (((\mathfrak{h}_{Np^\infty}^{\mathrm{ord}}/\mathcal{I}_{Np^\infty}^{\mathrm{ord}})\otimes_\Lambda Q(\Lambda))[[G_\infty^{(2)}]])_Q.$$

The main theorem (Theorem 7.3) of this section is that

$$\mathrm{L}_N\ :\ z_{Np^\infty}^{\mathrm{univ}}\ \mapsto \text{a “universal ordinary } p\text{-adic zeta function”},$$

where the universal ordinary p-adic zeta function is a p-adic zeta function in two variables associated to the universal family of ordinary cusp forms (see Theorem 7.3 for the details).

7.1. We define the subspace B and the map L_N.

7.1.1. We put $\Lambda = \mathbb{Z}_p[[G_\infty^{(1)}]]$ as in 3.5. Let us define a subspace $\overline{\mathsf{m}}_\Lambda$ of $\overline{M}_{Np^\infty}[[G_\infty^{(1)}]]$ in the following way:

$$\overline{\mathsf{m}}_\Lambda = \{x\in \overline{M}_{Np^\infty}[[G_\infty^{(1)}]]\ \ ;\ \ g_{a^{-1}}^{(1)}\cdot x_{n,b} = x_{n,ab}\quad \text{for all } a,b\in(\mathbb{Z}/p^n\mathbb{Z})^\times\}.$$

Here $x_{n,a}\in\overline{M}_{Np^\infty}$ are defined by $x|_n = \sum_{a\in(\mathbb{Z}/p^n\mathbb{Z})^\times} x_{n,a}g_a^{(1)}\in\overline{M}_{Np^\infty}[G_n^{(1)}]$ with the image $x|_n$ of x under the projection $\overline{M}_{Np^\infty}[[G_\infty^{(1)}]]\to\overline{M}_{Np^\infty}[G_n^{(1)}]$.

Proposition 7.1.2 . (1) *We have*

$$z_{Np^\infty}^{\mathrm{univ}}\in\overline{\mathsf{m}}_\Lambda[[G_\infty^{(2)}]][\frac{1}{g}](\subset\overline{M}_{Np^\infty}[[G_\infty^{(1)}\times G_\infty^{(2)}]][\frac{1}{g}]),$$

where g is as before.
(2) *Let* $\mathfrak{i}:\overline{M}_{Np^\infty}\longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{H}_{Np^\infty},\mathbb{Z}_p)$ *be as in* 3.3. *We denote by the same symbol* $\mathfrak{i}$ *the map*

$$\begin{aligned}\mathfrak{i}:\overline{M}_{Np^\infty}[[G_\infty^{(1)}]] &\longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{H}_{Np^\infty},\mathbb{Z}_p)[[G_\infty^{(1)}]]\\ &= \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{H}_{Np^\infty},\Lambda)\end{aligned}$$

induced by $\mathfrak{i}$ *in* 3.3. *Then*

$$\mathfrak{i}(\overline{\mathsf{m}}_\Lambda)\subset\mathrm{Hom}_\Lambda(\mathcal{H}_{Np^\infty},\Lambda).$$

Proof. (1)The results in 4.5.5 deduce the assertion.
(2) The result follows directly from the definition. □

7.1.3. We define a map

$$\overline{\mathfrak{m}}_\Lambda \longrightarrow (\mathfrak{h}^{\mathrm{ord}}_{Np^\infty}/\mathcal{I}^{\mathrm{ord}}_{Np^\infty}) \otimes_\Lambda Q(\Lambda) \tag{7.1}$$

which induces

$$\mathrm{L}_N : \overline{\mathfrak{m}}_\Lambda[[G^{(2)}_\infty]][\frac{1}{g}] \longrightarrow ((\mathfrak{h}^{\mathrm{ord}}_{Np^\infty}/\mathcal{I}^{\mathrm{ord}}_{Np^\infty}) \otimes_\Lambda Q(\Lambda))[[G^{(2)}_\infty]][\frac{1}{g}]$$

for a non-zerodivisor $g \in \mathbb{Z}_p[[G^{(1)}_\infty \times G^{(2)}_\infty]]$.
The map (7.1) is defined as the following composition:

$$\begin{aligned} \overline{\mathfrak{m}}_\Lambda &\xrightarrow{\mathrm{i}} \mathrm{Hom}_\Lambda(\mathcal{H}_{Np^\infty}, \Lambda) \xrightarrow{\mathrm{proj.}} \mathrm{Hom}_\Lambda(\mathcal{H}^{\mathrm{ord}}_{Np^\infty}, \Lambda) \\ &\longrightarrow \mathrm{Hom}_{Q(\Lambda)}(P^{\mathrm{ord}}_{Np^\infty} \otimes_\Lambda Q(\Lambda), Q(\Lambda)) \\ &\longrightarrow \mathrm{Hom}_{Q(\Lambda)}(p^{\mathrm{ord}}_{Np^\infty} \otimes_\Lambda Q(\Lambda), Q(\Lambda)) \\ &\longrightarrow (\mathfrak{h}^{\mathrm{ord}}_{Np^\infty}/\mathcal{I}^{\mathrm{ord}}_{Np^\infty}) \otimes_\Lambda Q(\Lambda). \end{aligned} \tag{7.2}$$

The second arrow is by the natural projection. The third arrow is the evident one. The fourth arrow is given as follows. By the definition of $P^{\mathrm{ord}}_{Np^\infty}$ and $p^{\mathrm{ord}}_{Np^\infty}$, it follows that the natural map $P^{\mathrm{ord}}_{Np^\infty} \to p^{\mathrm{ord}}_{Np^\infty}$ is surjective. The fourth arrow is given as the unique section $p^{\mathrm{ord}}_{Np^\infty} \otimes_\Lambda Q(\Lambda) \to P^{\mathrm{ord}}_{Np^\infty} \otimes_\Lambda Q(\Lambda)$ as algebras over $Q(\Lambda)$ of the surjective homomorphism between semisimple algebras $P^{\mathrm{ord}}_{Np^\infty} \otimes_\Lambda Q(\Lambda) \to p^{\mathrm{ord}}_{Np^\infty} \otimes_\Lambda Q(\Lambda)$ which is induced by the above surjective map. Finally the last arrow in (7.2) is defined in the following way. By Proposition 3.6, $p^{\mathrm{ord}}_{Np^\infty} \otimes_\Lambda Q(\Lambda) \cong (\mathfrak{h}^{\mathrm{ord}}_{Np^\infty}/\mathcal{I}^{\mathrm{ord}}_{Np^\infty}) \otimes_\Lambda Q(\Lambda)$ are finitely generated semisimple algebras over $Q(\Lambda)$. Hence we have an isomorphism

$$(\mathfrak{h}^{\mathrm{ord}}_{Np^\infty}/\mathcal{I}^{\mathrm{ord}}_{Np^\infty}) \otimes_\Lambda Q(\Lambda) \cong \mathrm{Hom}_{Q(\Lambda)}((\mathfrak{h}^{\mathrm{ord}}_{Np^\infty}/\mathcal{I}^{\mathrm{ord}}_{Np^\infty}) \otimes_\Lambda Q(\Lambda), Q(\Lambda)) \quad ; \tag{7.3}$$

$$a \mapsto (x \mapsto \mathrm{Tr}(a \cdot x)),$$

where Tr is the trace map

$$\mathrm{Tr} : \mathfrak{h}^{\mathrm{ord}}_{Np^\infty}/\mathcal{I}^{\mathrm{ord}}_{Np^\infty} \otimes_\Lambda Q(\Lambda) \longrightarrow Q(\Lambda).$$

This map gives the last map of (7.2).

7.1.4. We define a universal ordinary p-adic zeta function

$$L^{\mathrm{ord,univ}}_{p\text{-adic}} \in (\mathfrak{h}^{\mathrm{ord}}_{Np^\infty}/\mathcal{I}^{\mathrm{ord}}_{Np^\infty})[[G^{(2)}_\infty]][\frac{1}{hg}],$$

where $h \in \mathfrak{h}^{\mathrm{ord}}_{Np^\infty}$ is a certain non-zerodivisor and g is as before, as

$$L^{\mathrm{ord,univ}}_{p\text{-adic}} = \mathrm{L}_N(z^{\mathrm{univ}}_{Np^\infty}).$$

From the definition, one can see that universal ordinary p-adic zeta function $L^{\mathrm{ord,univ}}_{p\text{-adic}}$ is an element of $(1/h)(\mathfrak{h}^{\mathrm{ord}}_{Np^\infty}/\mathcal{I}^{\mathrm{ord}}_{Np^\infty})[[G^{(2)}_\infty]][1/g]$, which is contained in $(\mathfrak{h}^{\mathrm{ord}}_{Np^\infty}/\mathcal{I}^{\mathrm{ord}}_{Np^\infty})[[G^{(2)}_\infty]][1/hg]$.

7.2. We review basic facts and results of Hida about ordinary eigen cusp forms.

7.2.1. Let $k \geq 2$ and $m \geq 1$. Let $f \in M_k(X(1, Np^m)) \otimes \mathbb{C}$ be a normalized eigen cusp form. We recall that f is called ordinary if the eigenvalue of $T(p)$ on f is a p-adic unit. Further f is called an ordinary p-stabilized newform of tame conductor N when f is ordinary, the conductor of f is divisible by N, and $m \geq 1$ (cf. [GS]). We call Np^m the level of an ordinary p-stabilized newform f of tame conductor N if m is the smallest (positive) integer such that $f \in M_k(X(1, Np^m)) \otimes \mathbb{C}$.
An ordinary p-stabilized newform of tame conductor N of level Np^m is either it is already a newform of level Np^m or it is an ordinary eigen cusp form f_α obtained from a newform f of level N when α is a p-adic unit by the method in 6.1.1 (cf.[GS]).

7.2.2. We call a ring homomorphism $\kappa : \mathfrak{h}^{\mathrm{ord}}_{Np^\infty} \longrightarrow \overline{\mathbb{Z}_p}$ satisfying the following conditions an N-primitive arithmetic point (cf. [GS]). For an integer i such that $i \geq 0$ and a Dirichlet character ψ of conductor p^n $(n \geq 0)$, let $\mathfrak{P}_{i,\psi}$ be the kernel of the $\mathbb{Z}_p$-homomorphism $\Lambda \to \overline{\mathbb{Z}_p}$ induced by $\psi \circ \chi^i : G^{(1)}_\infty \to \overline{\mathbb{Z}_p}^\times$. The condition for an N-primitive arithmetic point is that it factors through

$$\kappa : \mathfrak{h}^{\mathrm{ord}}_{Np^\infty} \to \mathfrak{h}^{\mathrm{ord}}_{Np^\infty}/(\mathfrak{P}_{i,\psi} + \mathcal{I}^{\mathrm{ord}}_{Np^\infty}) \to \overline{\mathbb{Z}_p} \tag{7.4}$$

for some $i \geq 0$ and some ψ. Here $\mathcal{I}^{\mathrm{ord}}_{Np^\infty}$ is the ideal defined in 3.7 in section 3.

7.2.3. In the case $p \geq 5$, Hida [Hi2], §1, Corollary 1.3 proved that for an N-primitive arithmetic point κ which factors as (7.4), we have a unique ordinary p-stabilized newform $f = \sum_{n \geq 0} a_n(f) q^n$ of weight $i + 2$ of tame conductor N such that $\kappa(T(n)) = a_n(f)$.

Theorem 7.3 . *We assume $p \geq 5$. The universal ordinary p-adic zeta function*

$$L^{\mathrm{ord,univ}}_{p\text{-}adic} \in (\mathfrak{h}^{\mathrm{ord}}_{Np^\infty}/\mathcal{I}^{\mathrm{ord}}_{Np^\infty})[[G^{(2)}_\infty]][\frac{1}{hg}]$$

defined in 7.1.4 *displays property* (7.6) *below. Let*

$$\kappa : \mathfrak{h}^{\mathrm{ord}}_{Np^\infty}/\mathcal{I}^{\mathrm{ord}}_{Np^\infty} \longrightarrow \overline{\mathbb{Z}_p}$$

be an N-primitive arithmetic point. We write $f_\kappa = \sum_{n \geq 1} a_n(f_\kappa) q^n$ for the ordinary p-stabilized newform of tame conductor N attached in the sense of 7.2.3 *with κ. We denote the weight and level of f_κ by k and Np^m $(m \geq 1)$, respectively. We always have that $\kappa(g) \in \overline{\mathbb{Z}_p}[[G^{(2)}_\infty]]$ is a non-zero divisor. We assume that $\kappa(h) \neq 0$. Then κ induces the following homomorphism which is also denoted by κ :*

$$\kappa : (\mathfrak{h}^{\mathrm{ord}}_{Np^\infty}/\mathcal{I}^{\mathrm{ord}}_{Np^\infty})[[G^{(2)}_\infty]][\frac{1}{hg}] \longrightarrow Q(\overline{\mathbb{Z}_p}[[G^{(2)}_\infty]]). \tag{7.5}$$

Now if $L(f_\kappa, k-1) \neq 0$, then concerning the image $L^{\mathrm{ord,univ}}_{p\text{-}adic}(\kappa)$ of $L^{\mathrm{ord,univ}}_{p\text{-}adic}$ under (7.5)*, we have*

$$L^{\mathrm{ord,univ}}_{p\text{-}adic}(\kappa)^\pm = p^{m-1}(p-1) \cdot \kappa(T(p))^m \cdot L_{p\text{-}adic}(f_\kappa)^\pm(\chi) \in O_M[[G^{(2)}_\infty]] \otimes_{O_M} M. \tag{7.6}$$

Here concerning (one-variable) p-adic zeta function $L_{p\text{-}adic}(f_\kappa)$, we take the class of f_κ as $\omega \in S(f_\kappa)$ in the situation of Theorem 5.5. Moreover $\pm = (-1)^k\kappa(\langle -1\rangle)$, $L^{\mathrm{ord,univ}}_{p\text{-}adic}(\kappa)^\pm$ and $L_{p\text{-}adic}(f_\kappa)^\pm$ are the $\pm$-parts of $L^{\mathrm{ord,univ}}_{p\text{-}adic}(\kappa)$ and $L_{p\text{-}adic}(f_\kappa)$, respectively, and M is the finite extension $\mathbb{Q}_p(a_n(f_\kappa); n \geq 1)$ of $\mathbb{Q}_p$.

Remark 7.3.1 . (1) *Recall that as in Theorem 6.2, even for $p = 2, 3$, the p-adic zeta function $L_{p\text{-}adic}(f)$ of each ordinary p-stabilized newform f was constructed from $z^{\mathrm{univ}}_{Np^\infty}$.*

(2) *By the argument in 4.5.5, we obtain ${}_{c,d'}z^{\mathrm{univ}}_{Np^\infty} \in \overline{M}_{Np^\infty}[[G^{(1)}_\infty \times G^{(2)}_\infty]]$. From this and from the fact that $\kappa((1 - c^{-1}g^{(1)}_{c^{-1}}g^{(2)}_c)(1 - d'g^{(2)}_{d'})) \in \overline{\mathbb{Z}_p}[[G^{(2)}_\infty]]$ is a non-zerodivisor for any N-primitive arithmetic point κ, we find that $\kappa(g) \in \overline{\mathbb{Z}_p}[[G^{(2)}_\infty]]$ is a non-zerodivisor for any N-primitive arithmetic point κ.*

(3) *In the above, we put the assumption that $L(f_\kappa, k-1) \neq 0$. (As referred before, $L(f_\kappa, k-1) = 0$ occurs only in the case $k = 2$.) Moreover we only considered the $(-1)^k\kappa(\langle -1\rangle)$-parts of $L_{p\text{-}adic}(f_\kappa)$. However as explained briefly in 7.7 later, by using z_{M,Np^∞}, z_{N,p^∞}, and ϕ in 6.7, with some device, we can construct a two-variable p-adic zeta function which can provide the $(-1)^{k+1}\kappa(\langle -1\rangle)$-part of $L_{p\text{-}adic}(f_\kappa)$ for κ satisfying some conditions (see 7.7 for this condition) even though $L(f_\kappa, k-1) = 0$.*

We prove Theorem 7.3 by using Theorem 6.2 and the argument in the proof of it.
We use the notation in Theorem 7.3 and we assume $\kappa(h) \neq 0$.

7.4. For integers c, d' which are prime to p, we put

$$ {}_{c,d'}L^{\mathrm{ord,univ}}_{p\text{-adic}} = (1 - c^{-1}g^{(1)}_{c^{-1}}g^{(2)}_c)(1 - d'g^{(2)}_{d'})L^{\mathrm{ord,univ}}_{p\text{-adic}} \in (\mathfrak{h}^{\mathrm{ord}}_{Np^\infty}/\mathcal{I}^{\mathrm{ord}}_{Np^\infty})[\frac{1}{h}][[G_\infty]]. $$

One can see that ${}_{c,d'}L^{\mathrm{ord,univ}}_{p\text{-adic}} = \mathrm{L}_N({}_{c,d'}z^{\mathrm{univ}}_{Np^\infty})$. We also put

$$ {}_{c,d'}L_{p\text{-adic}}(f_\kappa) = ((1 - c^{-k}\epsilon_{f_\kappa}(c^{-1})g^{(2)}_c)(1 - g^{(2)}_{d'})) \cdot L_{p\text{-adic}}(f_\kappa). $$

In 7.5 – 7.6, we will prove that under an N-primitive arithmetic point κ satisfying the condition in Theorem 7.3,

$$ {}_{c,d'}L^{\mathrm{ord,univ}}_{p\text{-adic}}(\kappa) = p^{m-1}(p-1) \cdot \kappa(T(p))^m \cdot {}_{c,d'}L_{p\text{-adic}}(f_\kappa)(\chi). \tag{7.7} $$

The result in Theorem 7.3 will follow from this.

7.5. We consider some consequences of Theorem 6.2 for the proof of (7.7).

7.5.1. Since f_κ is ordinary, $L_{p\text{-adic}}(f_\kappa)$ may be characterized by the specialization property in Theorem 5.5 (i) for only one r among $1, \ldots, k-1$ under the notation there.
We have that $S(f_\kappa)_M := M_k(X_1(Np^n); M)/(T(n) - \kappa(T(n))\ ;\ n \geq 1)$ is a one dimensional M-vector space with a base f_κ. We define

$\mathrm{pr}_{f_\kappa} : M_k(X_1(Np^n);M) \longrightarrow M$ as the composition $M_k(X_1(Np^n);M) \xrightarrow{\text{proj.}} S(f_\kappa)_M \longrightarrow M$, where the second map is the one given by sending the class of f_κ to 1. (In the case that f_κ is a newform, the map pr_{f_κ} coincides with the map $\mathrm{pr}_{\mathfrak{f}}$ in section 6. In the case $f_\kappa = f_\alpha$ for some newform f of conductor N and α as before, in section 6, we took f as the base of $S(f_\kappa)_M$. So if we use pr_{f_κ} in Theorem 6.2 instead of $\mathrm{pr}_{\mathfrak{f}}$, we obtain the p-adic zeta function of f_κ which takes the class of f_κ as ω in the notation in Theorem 5.5.) By Theorem 6.2, we obtain

$$(\varprojlim_n \mathrm{pr}_{f_\kappa}({}_{c,d'}z^{\mathrm{univ}}_{Np^\infty}(\chi^{k-2},\chi^{r-1})|_{(m,n)}))^{\pm\cdot(-1)^r}$$
$$= \alpha^m \cdot \sum_{a\in(\mathbb{Z}/p^m\mathbb{Z})^\times} i_{\{r\}}({}_{c,d'}L_{p\text{-adic}}(f_\kappa)^\pm)\epsilon_{f_\kappa}(a'^{-1})g_a^{(1)} \in \varprojlim_n M[G_n^{(2)}][G_m^{(1)}], \tag{7.8}$$

with $\pm = (-1)^k\kappa(\langle -1\rangle)$, where $a' \in (\mathbb{Z}/Np^m\mathbb{Z})^\times$ is the element such that $a' \equiv 1(N)$ and $a' \equiv a(p^m)$ for each a.

7.5.2. Let $\epsilon_p : G_\infty^{(1)} \longrightarrow \mathbb{Z}_p^\times$ denote the character such that the restriction of κ to $G_\infty^{(1)}$ coincides with $\epsilon_p \circ \chi^{k-2}$. Then $\epsilon_p(g_a^{(1)}) = \epsilon_{f_\kappa}(a')$ for any $a \in (\mathbb{Z}/p^m\mathbb{Z})^\times$, and $a' \in (\mathbb{Z}/Np^m\mathbb{Z})^\times$ is the element such that $a' \equiv a(p^m)$ and $a' \equiv 1(N)$. Clearly the image of (7.8) under $\epsilon_p : M[[G_\infty^{(2)}]][G_m^{(1)}] \to M[[G_\infty^{(2)}]]$ is

$$p^{m-1}(p-1)\cdot\alpha^m\cdot i_{\{r\}}({}_{c,d'}L_{p\text{-adic}}(f_\kappa)^\pm).$$

As f_κ is ordinary, by Theorem 6.2, we know $L_{p\text{-adic}}(f_\kappa) \in O_M[[G_\infty^{(2)}]] \otimes_{O_M} M$, and hence we have

$$i_{\{r\}}({}_{c,d'}L_{p\text{-adic}}(f_\kappa)^\pm) = {}_{c,d'}L_{p\text{-adic}}(f_\kappa)^\pm(\chi^r) \in O_M[[G_\infty^{(2)}]] \otimes_{O_M} M.$$

7.5.3. By the commutative diagram

$$\begin{array}{ccc} M_k(X_1(Np^n);\mathbb{Z}_{(p)})[G_m^{(1)}] & \xrightarrow{\epsilon_p} & M_k(X_1(Np^n);\mathbb{Z}_{(p)}[\epsilon_p]) \\ \downarrow{\scriptstyle \mathrm{pr}_{f_\kappa}} & & \downarrow{\scriptstyle \mathrm{pr}_{f_\kappa}} \\ M[G_m^{(1)}] & \xrightarrow{\epsilon_p} & M, \end{array}$$

and (7.8), it follows that

$$\mathrm{pr}_{f_\kappa}({}_{c,d'}z^{\mathrm{univ}}_{Np^\infty}(\epsilon_p\circ\chi^{k-2},\chi^{r-1}))^{\pm\cdot(-1)^r} = p^{m-1}(p-1)\cdot\alpha^m\cdot {}_{c,d'}L_{p\text{-adic}}(f_\kappa)^\pm(\chi^r)$$
$$\in O_M[[G_\infty^{(2)}]] \otimes_{O_M} M. \tag{7.9}$$

Here we denote by $\mathrm{pr}_{f_\kappa}({}_{c,d'}z^{\mathrm{univ}}_{Np^\infty}(\epsilon_p\circ\chi^{k-2},\chi^{r-1}))^{\pm\cdot(-1)^r}$ the inverse limit $(\varprojlim_n \mathrm{pr}_{f_\kappa}({}_{c,d'}z^{\mathrm{univ}}_{Np^\infty}(\epsilon_p\circ\chi^{k-2},\chi^{r-1})|_n))^{\pm\cdot(-1)^r}$ with ${}_{c,d'}z^{\mathrm{univ}}_{Np^\infty}(\epsilon_p\circ\chi^{k-2},\chi^{r-1})|_n \in M_k(X_1(Np^n);\mathbb{Z}_{(p)}[\epsilon_p])[G_n^{(2)}]$.
Furthermore by (6.13), we have

$${}_{c,d'}z^{\mathrm{univ}}_{Np^\infty}(\epsilon_p\circ\chi^{k-2},\chi^{r-1})|_n \in M_k(X_1(Np^n),\epsilon_p;\mathbb{Z}_{(p)}[\epsilon_p])[G_n^{(2)}],$$

where $M_k(X_1(Np^n), \epsilon_p; \mathbb{Z}_{(p)}[\epsilon_p])$ is the sub $\mathbb{Z}_{(p)}[\epsilon_p]$-space of $M_k(X_1(Np^n); \mathbb{Z}_{(p)}[\epsilon_p])$ on which $G_\infty^{(1)}$ acts via $\epsilon_p \circ \chi^{k-2}$.

7.6. Now we study ${}_{c,d'}L_{p\text{-adic}}^{\mathrm{ord,univ}}$.

7.6.1. The following composition of Λ-homomorphisms

$$\begin{aligned}\mathfrak{h}_{Np^\infty}^{\mathrm{ord}}/\mathcal{I}_{Np^\infty}^{\mathrm{ord}} &\to p_{Np^\infty}^{\mathrm{ord}} \otimes_\Lambda Q(\Lambda) \\ &\to P_{Np^\infty}^{\mathrm{ord}} \otimes_\Lambda Q(\Lambda) \to \mathcal{H}_{Np^\infty}^{\mathrm{ord}} \otimes_\Lambda Q(\Lambda)\end{aligned}$$

induces the Λ-homomorphism

$$\mathrm{Hom}_\Lambda(\mathcal{H}_{Np^\infty}^{\mathrm{ord}}, \Lambda) \longrightarrow \mathrm{Hom}_\Lambda(\mathfrak{h}_{Np^\infty}^{\mathrm{ord}}/\mathcal{I}_{Np^\infty}^{\mathrm{ord}}, \frac{1}{h'}\Lambda) \tag{7.10}$$

with some non-zerodivisor $h' \in \Lambda$. It is easy to see that the composition of (7.10) and

$$\mathrm{Hom}_\Lambda(\mathfrak{h}_{Np^\infty}^{\mathrm{ord}}/\mathcal{I}_{Np^\infty}^{\mathrm{ord}}, \frac{1}{h'}\Lambda) \to (\mathfrak{h}_{Np^\infty}^{\mathrm{ord}}/\mathcal{I}_{Np^\infty}^{\mathrm{ord}}) \otimes_\Lambda Q(\Lambda),$$

which is given by (7.3), coincides with the composition in (7.2). From this, we see $(h')^{-1} \subset (h)^{-1}$.

The Λ-homomorphism $\overline{\mathsf{m}}_\Lambda[[G_\infty^{(2)}]] \xrightarrow{\mathrm{i}} \mathrm{Hom}_\Lambda(\mathcal{H}_{Np^\infty}, \Lambda)[[G_\infty^{(2)}]]$ and (7.10) give a Λ-homomorphism

$$\overline{\mathsf{m}}_\Lambda[[G_\infty^{(2)}]] \longrightarrow \mathrm{Hom}_\Lambda(\mathfrak{h}_{Np^\infty}^{\mathrm{ord}}/\mathcal{I}_{Np^\infty}^{\mathrm{ord}}, \Lambda[\frac{1}{h'}])[[G_\infty^{(2)}]],$$

and this Λ-homomorphism induces

$$\begin{aligned}&\overline{\mathsf{m}}_{\Lambda/\mathfrak{P}_{k-2,\epsilon_p}}[[G_\infty^{(2)}]] \\ &\quad\to \mathrm{Hom}_{\Lambda/\mathfrak{P}_{k-2,\epsilon_p}}(\mathfrak{h}_{Np^\infty}^{\mathrm{ord}}/(\mathfrak{P}_{k-2,\epsilon_p} + \mathcal{I}_{Np^\infty}^{\mathrm{ord}}),\ \Lambda[\frac{1}{h'}]/\mathfrak{P}_{k-2,\epsilon_p})[[G_\infty^{(2)}]],\end{aligned} \tag{7.11}$$

where $\overline{\mathsf{m}}_{\Lambda/\mathfrak{P}_{k-2,\epsilon_p}}$ denotes the image of $\overline{\mathsf{m}}_\Lambda$ under the map $\overline{M}_{Np^\infty}[[G_\infty^{(1)}]] \xrightarrow{\epsilon_p \circ \chi^{k-2}} \overline{M}_{Np^\infty} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\epsilon_p]$. The map (7.11) is well-defined as $\kappa(h) \neq 0$.

We put $L = \Lambda[1/h']/\mathfrak{P}_{k-2,\epsilon_p}$. Now the result of Hida [Hi2], Corollary 1.3 affirms that $\mathrm{Hom}_{\Lambda/\mathfrak{P}_{k-2,\epsilon_p}}(\mathfrak{h}_{Np^\infty}^{\mathrm{ord}}/(\mathfrak{P}_{k-2,\epsilon_p} + \mathcal{I}_{Np^\infty}^{\mathrm{ord}}),\ \Lambda[1/h']/\mathfrak{P}_{k-2,\epsilon_p})$ is contained in $S_k^{\mathrm{ord},N}(X_1(Np^m), \epsilon_p; L)$, where $S_k^{\mathrm{ord},N}(X_1(Np^m), \epsilon_p; L)$ is the sub L-space of $e \cdot M_k(X_1(Np^m), \epsilon_p; L)$ with $e = \varprojlim_n T(p)^{n!}$, consisting of cusp forms with conductor divisible by N. Hence (7.11) induces a homomorphism

$$\overline{\mathsf{m}}_{\Lambda/\mathfrak{P}_{k-2,\epsilon_p}}[[G_\infty^{(2)}]] \to S_k^{\mathrm{ord},N}(X_1(Np^m), \epsilon_p; L)[[G_\infty^{(2)}]]. \tag{7.12}$$

7.6.2. We define $\mathsf{m}_\Lambda[[G_\infty^{(2)}]] = \overline{\mathsf{m}}_\Lambda[[G_\infty^{(2)}]] \cap M[[G_\infty^{(1)} \times G_\infty^{(2)}]]_{\{k-2\},\{0,\dots,k-2\}} (\subset \overline{M}_{Np^\infty}[[G_\infty^{(1)} \times G_\infty^{(2)}]])$ with $\overline{\mathsf{m}}_\Lambda$ in 7.1.1 and $M[[G_\infty^{(1)} \times G_\infty^{(2)}]]_{\{k-2\},\{0,\dots,k-2\}}$ in 6.1.4. We see that ${}_{c,d'}z_{Np^\infty}^{\mathrm{univ}} \in \mathsf{m}_\Lambda[[G_\infty^{(2)}]]$.

We define $\mathsf{m}_{\Lambda/\mathfrak{P}_{k-2,\epsilon_p}}[[G_\infty^{(2)}]]$ to be the image of of $\mathsf{m}_\Lambda[[G_\infty^{(2)}]]$ under the map $\overline{M}_{Np^\infty}[[G_\infty^{(1)}\times G_\infty^{(2)}]] \xrightarrow{(\epsilon_p\circ\chi^{k-2},\mathrm{id})} \overline{M}_{Np^\infty}\otimes_{\mathbb{Z}_p}\mathbb{Z}_p[\epsilon_p][[G_\infty^{(2)}]]$. Then the map (7.12) induces a map

$$\mathsf{j}:\mathsf{m}_{\Lambda/\mathfrak{P}_{k-2,\epsilon_p}}[[G_\infty^{(2)}]] \to S_k^{\mathrm{ord},N}(X_1(Np^m),\epsilon_p;L)[[G_\infty^{(2)}]]. \tag{7.13}$$

By definition, the following diagram is commutative for each $n\geq 1$:

$$\begin{array}{ccc} \mathsf{m}_{\Lambda/\mathfrak{P}_{k-2,\epsilon_p}}[G_n^{(2)}] & \xrightarrow{\ \mathsf{j}\ } & S_k^{\mathrm{ord},N}(X_1(Np^m),\epsilon_p;L)[G_n^{(2)}] \\ \downarrow{\scriptstyle \mathrm{pr}_{f_\kappa}} & & \downarrow{\scriptstyle \mathrm{pr}_{f_\kappa}} \\ M & \xrightarrow{\ \mathrm{id}\ } & M. \end{array} \tag{7.14}$$

We write $\mathrm{pr}_{f_\kappa}(\mathfrak{j}({}_{c,d'}z_{Np^\infty}^{\mathrm{univ}}(\epsilon_p\circ\chi^{k-2},\mathrm{id})))$ for the inverse limit $\varprojlim_n(\mathrm{pr}_{f_\kappa}(\mathfrak{j}({}_{c,d'}z_{Np^\infty}^{\mathrm{univ}}(\epsilon_p\circ\chi^{k-2},\chi^{r-1}))|_n))$. By the commutative diagram (7.14), we obtain

$$\mathrm{pr}_{f_\kappa}({}_{c,d'}z_{Np^\infty}^{\mathrm{univ}}(\epsilon_p\circ\chi^{k-2},\mathrm{id})) = \mathrm{pr}_{f_\kappa}(\mathfrak{j}({}_{c,d'}z_{Np^\infty}^{\mathrm{univ}}(\epsilon_p\circ\chi^{k-2},\mathrm{id}))).$$

Hence by this and by (7.9), if we prove the assertion in 7.6.3 below, Theorem 7.3 follows.

7.6.3. The assertion in question is as follows.
We have

$$\mathrm{pr}_{f_\kappa}(\mathfrak{j}({}_{c,d'}z_{Np^\infty}^{\mathrm{univ}}(\epsilon_p\circ\chi^{k-2},\mathrm{id}))) = {}_{c,d'}L_{p\text{-adic}}^{\mathrm{ord,univ}}(\kappa).$$

We prove this assertion. As an element of the right hand side of (7.11), $\mathfrak{j}({}_{c,d'}z_{Np^\infty}^{\mathrm{univ}}(\epsilon_p\circ\chi^{k-2},\mathrm{id}))$ is the element

$$x\mapsto \mathrm{Tr}_{k,\epsilon_p}({}_{c,d'}L_{p\text{-adic}}^{\mathrm{ord,univ}}\cdot x) \qquad (x\in\mathfrak{h}_{Np^\infty}^{\mathrm{ord}}/(\mathfrak{P}_{k-2,\epsilon_p}+\mathcal{I}_{Np^\infty}^{\mathrm{ord}})),$$

where $\mathrm{Tr}_{k,\epsilon_p}:\mathrm{Hom}_{\Lambda/\mathfrak{P}_{k-2,\epsilon_p}}(\mathfrak{h}_{Np^\infty}^{\mathrm{ord}}/(\mathfrak{P}_{k-2,\epsilon_p}+\mathcal{I}_{Np^\infty}^{\mathrm{ord}}),\Lambda/\mathfrak{P}_{k-2,\epsilon_p})$ is the trace map as $\Lambda/\mathfrak{P}_{k-2,\epsilon_p}$-modules. By the result of Hida [Hi2], Corollary 1.3, we find that

$$\mathrm{Tr}_{k,\epsilon_p} = \sum_{\kappa'} f_{\kappa'},$$

where $f_{\kappa'}$ runs over all ordinary p-stabilized newform $\in S_k^{\mathrm{ord},N}(X_1(Np^m),\epsilon_p;L)$ of tame conductor N attached to $\kappa':\mathfrak{h}_{Np^\infty}^{\mathrm{ord}}/\mathcal{I}_{Np^\infty}^{\mathrm{ord}}\to\overline{\mathbb{Z}_p}$ satisfying $\kappa'|_{G_\infty^{(1)}} = \epsilon_p\circ\chi^{k-2}$.
By the definition of ${}_{c,d'}L_{p\text{-adic}}^{\mathrm{ord,univ}}$, we obtain

$$\mathfrak{j}({}_{c,d'}z_{Np^\infty}^{\mathrm{univ}}(\epsilon_p\circ\chi^{k-2},\mathrm{id})) = {}_{c,d'}L_{p\text{-adic}}^{\mathrm{ord,univ}}(\sum_{\kappa'} f_{\kappa'}) = \sum_{\kappa'} {}_{c,d'}L_{p\text{-adic}}^{\mathrm{ord,univ}}(\kappa')f_{\kappa'},$$

where in $\sum_{\kappa'}$, κ' runs as above. This concludes the assertion.

Therefore Theorem 7.3 is proven.

7.7. We briefly explain how to obtain a two-variable p-adic zeta function attached to the universal family of cusp forms which can provide $(-1)^{k+1}\kappa(\langle -1\rangle)$-part of $L_{p\text{-adic}}(f_\kappa)$ for κ satisfying a certain condition, even though $L(f_\kappa, k-1)=0$.
We use the notation in 6.3.7 – 6.3.12 and in 6.7. As before, let M be a positive integer which is prime to Np.
We consider the image of z_{M,Np^∞} under the $\overline{M}_{MNp^\infty}$-homomorphism which is similar to (6.10) and which is given as follows

$$\overline{M}_{MNp^\infty}[[G_{Mp^\infty}{}^{(1)} \times G_{Mp^\infty}{}^{(2)}]] \to \overline{M}_{MNp^\infty}[[G_\infty^{(1)} \times G_\infty^{(2)}]] \ \ ; \tag{7.15}$$

$$x\mathfrak{g}_a^{(2)}\mathfrak{g}_b^{(1)} \mapsto x\phi(ab^2)g_a^{(2)}g_b^{(1)} \qquad (x \in \overline{M}_{MNp^\infty}, \mathfrak{g}_* \in G_{Mp^\infty}, g_* \in G_\infty).$$

We write $z_{M,Np^\infty,\phi}$ for the image of z_{M,Np^∞} under the above map.
Now by replacing N by NM in the definition of $\overline{\mathfrak{m}}_\Lambda$, we define $\overline{\mathfrak{m}}_\Lambda$. Then for this $\overline{\mathfrak{m}}_\Lambda$, the following which is similar to the result in Proposition 7.1.2 (1) holds. Namely we have

$$z_{M,Np^\infty,\phi} \in \overline{\mathfrak{m}}_\Lambda[[G_\infty^{(2)}]][\frac{1}{g''}](\subset \overline{M}_{NMp^\infty}[[G_\infty^{(1)} \times G_\infty^{(2)}]][\frac{1}{g''}]),$$

where g'' is the image of g' in 6.3.7 under the homomorphism (7.15). We study the image of $z_{M,Np^\infty,\phi}$ under the map

$$\overline{\mathfrak{m}}_\Lambda[[G_\infty^{(2)}]][\frac{1}{g''}] \longrightarrow ((\mathfrak{h}^{\mathrm{ord}}_{Np^\infty}/\mathcal{I}^{\mathrm{ord}}_{Np^\infty}) \otimes_\Lambda Q(\Lambda))[[G_\infty^{(2)}]][\frac{1}{g''}] \tag{7.16}$$

which is defined as L_N. We denote this image by

$$L^{\mathrm{ord}}_{p\text{-adic},M,\phi} \in (\mathfrak{h}^{\mathrm{ord}}_{NMp^\infty}/\mathcal{I}^{\mathrm{ord}}_{NMp^\infty})[[G_\infty^{(2)}]][\frac{1}{h'g''}],$$

where $h' \in \mathfrak{h}^{\mathrm{ord}}_{NMp^\infty}$ is a certain non-zerodivisor. This function $L^{\mathrm{ord}}_{p\text{-adic},M,\phi}$ displays the property (7.17) below. For an N-primitive arithmetic point $\kappa : \mathfrak{h}^{\mathrm{ord}}_{Np^\infty}/\mathcal{I}^{\mathrm{ord}}_{Np^\infty} \to \overline{\mathbb{Z}_p}$, let $\kappa_\phi : \mathfrak{h}^{\mathrm{ord}}_{NMp^\infty}/\mathcal{I}^{\mathrm{ord}}_{NMp^\infty} \to \overline{\mathbb{Z}_p}$ be the NM-primitive arithmetic point characterized by $\kappa_\phi(T(n)) = \kappa(T(n))\phi(n)$. We always have that $\kappa_\phi(g'') \in \overline{\mathbb{Z}_p}[[G_\infty^{(2)}]]$ is a non-zero divisor. We assume that $\kappa_\phi(h') \neq 0$. Then κ_ϕ induces the following homomorphism which is also denoted by κ_ϕ :

$$\kappa_\phi : (\mathfrak{h}^{\mathrm{ord}}_{Np^\infty}/\mathcal{I}^{\mathrm{ord}}_{Np^\infty})[[G_\infty^{(2)}]][\frac{1}{h'g''}] \longrightarrow Q(\overline{\mathbb{Z}_p}[[G_\infty^{(2)}]]).$$

Now if $L(f_\kappa, \phi, k-1) \neq 0$, we have

$$L^{\mathrm{ord}}_{p\text{-adic},M,\phi}(\kappa_\phi)^{\pm\cdot\phi(-1)}$$
$$= x\cdot\Big(\Big(\prod_{\substack{l:\mathrm{prime}\\ l|M}}(1-a_l(\mathfrak{f})g^{(2)}_{l^{-1}}+\epsilon_{\mathfrak{f}}(l)l^{k-1}g^{(2)}_{l^{-2}})\Big)\cdot L_{p\text{-adic}}(f_\kappa)\Big)^{\pm\cdot\phi(-1)}(\chi) \tag{7.17}$$
$$\in O_{M(\phi)}[[G^{(2)}_\infty]]\otimes_{O_{M(\phi)}} M(\phi)$$

with some $x\in M(\phi)^\times$, and $\pm=(-1)^k\epsilon_{\mathfrak{f}}(-1)$.
By using the arguments in 6.3.7 – 6.3.12 and 6.7, we can prove the above assertion in the same way as for Theorem 6.2. Furthermore, by using z_{N,p^∞}, we can produce another two-variable p-adic zeta function attached to universal family of ordinary cusp forms in the same manner.
In this way, by taking various ϕ, we can construct two-variable p-adic zeta functions which can provide not only $(-1)^k\kappa(\langle -1\rangle)$-part but also $(-1)^{k+1}\kappa(\langle -1\rangle)$-part of $L_{p\text{-adic}}(f_\kappa)$ for κ satisfying the condition that $\kappa_\phi(h')\neq 0$, without assuming $L(f_\kappa,k-1)\neq 0$.

REFERENCES

[AV] AMICE, Y. and VÉLU, J., *Distributions p-adiques associées aux séries de Hecke*, Astérisque 24-25(1975) 119–131.

[Co] COLEMAN, R., *Division values in local fields*, Invent. Math. 53 (1979) 91–116.

[CW] COATES, J. and WILES, A., *On p-adic L-functions and Elliptic Units*, J. Austral. Math. Soc (Series A) 26 (1978) 1–25.

[Fu1] FUKAYA, T., *The theory of Coleman power series for K_2*, Journal of Alg. Geometry 12 (2003) 1–80.

[Fu2] FUKAYA, T., *The theory of Coleman power series for K_2*, Proceedings from Tsudajuku University in Japanese 21 (2001) 75-85.

[Fu3] FUKAYA, T., *K_2-version of Coleman power series and p-adic zeta functions of modular forms*, Shurikaisekikenkyusho Kokyuroku (in Japanese) from RIMS (Kyoto University) 1200 (2001) 48-59.

[GS] GREENBERG, R. and STEVENS, G., *p-adic L functions and p-adic periods of modular forms*, Invent. Math. 111 (1993) 407–447.

[Hi1] HIDA, H., *Iwasawa modules attached to congruences of cusp forms*, Ann. Sci. École Norm. Sup. 19 (1986) 231–273.

[Hi2] HIDA, H., *Galois representations into $GL_2(\mathbb{Z}_p[[X]])$ attached to ordinary cusp forms*, Invent. Math. 85 (1986) 545–613.

[Hi3] HIDA, H., *Elementary theory of L-functions and Eisenstein series*, London Math. Soc. Student Texts 26, Cambridge Univ. Press (1993).

[Iw] IWASAWA, K., *On some modules in the theory of cyclotomic fields*, J. Math. Soc. Japan 16 (1964) 42–82.

[Ka1] KATO, K., *Lectures on the approach to Iwasawa theory for Hasse-Weil L-functions via B_{dR}*, Lecture Notes in Math. 1553, Springer (1993) 50–163.

[Ka2] KATO, K., *p-adic Hodge theory and values of zeta functions of modular forms*, preprint.

[Katz1] KATZ N.M., *p-adic interpolation of real analytic Eisenstein series*, Ann. of Math., 104 (1976) 459–571.

[Katz2] KATZ N.M., *Higher congruences between modular forms*, Ann. of Math., 101 (1975) 332–367.

[Ki] KITAGAWA, K., *On standard p-adic L-functions of families of elliptic cusp forms*, Contemp. Math., 165 (1991) 81–110.

[KL] KUBOTA, T. and LEOPOLDT, H.W, *Eine p-adische Theorie der Zetawerte*, J. Reine Angew. Math. 214/215 (1964), 328–339.

[Mi] MILNOR, J., *Algebraic K-theory and quadratic forms*, Invent. Math. 9 (1970) 318–344.

[Oc] OCHIAI, T., Doctoral Thesis, University of Tokyo (2001).

[Pa1] PANCHISHKIN *On the Siegel-Eisenstein measure and its applications*, Israel Journal of Mathematics 120 B (2000), 467–509.

[Pa2] PANCHISHKIN *A new method of constructing p-adic L-functions associated with modular forms*, Moscow Mathematical Journal 2 (2001) 1-16.

[Qu] QUILLEN, D., *Higher algebraic K-theory* I, Lecture Notes in Math. 342, Springer (1973) 179–198.

[Sc] SCHOLL, J., *An introduction to Kato's Euler systems*, London Math. Soc. Lecture Note Ser. 254, Cambridge Univ. Press (1998) 379–460.

[Sh] SHIMURA, G. *The special values of the zeta functions associated with cusp forms*, Comm. Pure Appl. Math. 29 (1976) 783–804.

[Vi] VISHIK, M., M., *Non-archimedean measures connected with Dirichlet series*, Math. USSR, Sbornik 28 (1976) 216–228.

Takako Fukaya
Keio University
4-1-1 Hiyoshi, Kohoku-ku
Yokohama, 223-8521
Japan
takakof@hc.cc.keio.ac.jp

Completely Faithful Selmer Groups over Kummer Extensions

Dedicated to Professor Kazuya Kato

Yoshitaka Hachimori and Otmar Venjakob[1]

Received: September 30, 2002
Revised: January 23, 2003

Abstract. In this paper we study the Selmer groups of elliptic curves over Galois extensions of number fields whose Galois group $G \cong \mathbb{Z}_p \rtimes \mathbb{Z}_p$ is isomorphic to the semidirect product of two copies of the p-adic numbers $\mathbb{Z}_p$. In particular, we give examples where its Pontryagin dual is a faithful torsion module under the Iwasawa algebra of G. Then we calculate its Euler characteristic and give a criterion for the Selmer group being trivial. Furthermore, we describe a new asymptotic bound of the rank of the Mordell-Weil group in these towers of number fields.

2000 Mathematics Subject Classification: Primary 11G05, 14K15; Secondary 16S34, 16E65.
Keywords and Phrases: Selmer groups, elliptic curves, Euler characteristics, p-adic analytic groups.

[1] Both authors thank Department of Pure Mathematics and Mathematical Statistics, Cambridge, for its hospitality during part of this research; the first author was supported by JSPS Research Fellowships for Young Scientists while the second author has been supported by the EU Research Training Network "Arithmetical Algebraic Geometry".

1. Introduction

Throughout this paper, let p be a fixed odd prime number. For an elliptic curve E over $\mathbb{Q}$ with good ordinary reduction at p, Mazur's Main Conjecture predicts that the Mazur-Swinnerton-Dyer p-adic L-function $\mathcal{L}_{MSD}$ associated with E can be interpreted as an element of the Iwasawa-algebra $\Lambda = \mathbb{Z}_p[\![\mathrm{Gal}(\mathbb{Q}_{cyc}/\mathbb{Q})]\!]$ of the cyclotomic $\mathbb{Z}_p$-extension $\mathbb{Q}_{cyc}$ of $\mathbb{Q}$ and is a generator of the characteristic ideal of the Pontryagin dual $X_f(\mathbb{Q}_{cyc})$ of the Selmer group of E over $\mathbb{Q}_{cyc}$

$$char(X_f(\mathbb{Q}_{cyc})) = (\mathcal{L}_{MSD}).$$

Kato [K] has proved a partial result towards it showing that, for some $m \geq 0$, the function $p^m\mathcal{L}_{MSD}$ lies in Λ and is divided by the algebraic L-function of $X_f(\mathbb{Q}_{cyc})$. In particular, up to a power of p, the p-adic L-function $\mathcal{L}_{MSD}$ annihilates $X_f(\mathbb{Q}_{cyc})$ modulo pseudo-null modules: "$\mathcal{L}_{MSD}\ X_f(\mathbb{Q}_{cyc}) \equiv 0$." Moreover, if $X_f(\mathbb{Q}_{cyc})$ does not contain any pseudo-null submodule, then $\mathcal{L}_{MSD}X_f(\mathbb{Q}_{cyc}) = 0$. Thus, in classical Iwasawa theory the p-adic L-function is closely related to the annihilator ideal $Ann_\Lambda(X_f(\mathbb{Q}_{cyc}))$ of $X_f(\mathbb{Q}_{cyc})$.

Now, the challenging aim of noncommutative Iwasawa theory is to find and eventually prove a main conjecture over certain field extensions k_∞ of some number field k whose Galois group $G = G(k_\infty/k)$ is a (non-abelian) p-adic Lie group, e.g. over the field $k_\infty = k(E_{p^\infty})$ which arises by adjoining to k all p-power division points E_{p^∞}. If there should exist some p-adic L-function adapted to this situation, it would thus be natural to expect that it has the property of annihilating the dual of the Selmer group $X_f(k_\infty)$ over k_∞. One could even hope that investigating the global annihilator ideal

$$Ann_{\Lambda(G)}(X_f(k_\infty)) := \{\lambda \in \Lambda(G) | \lambda x = 0 \text{ for all } x \in X_f(k_\infty)\}$$

gives some hints for candidates of such a hypothetic L-function in this noncommutative setting, where $\Lambda(G) = \mathbb{Z}_p[\![G]\!]$ denotes the Iwasawa-algebra of G. This question, which motivated the present paper, was already posed by Harris in [Ha2], whereas Coates, Schneider and Sujatha [CSS1] defined a characteristic ideal of $X_f(k_\infty)$ in case $Ann_{\Lambda(G)}(X_f(k_\infty))$ is not zero.

The first main result of this article however tells that in general, over arbitrary p-adic Lie-extensions, such a link between global annihilator elements and p-adic L-functions is *not* possible (but we should stress that this result is no obstruction to the existence of p-adic L-functions in which we nevertheless still believe). Indeed, we prove that $X_f(k_\infty)$ over some infinite Kummer extension k_∞ of k is a finitely generated $\Lambda(G)$-torsion module, but with vanishing global annihilator ideal, i.e. though any single element of $X_f(k_\infty)$ is annihilated by some element of Λ there is no "global" $\lambda \in \Lambda$ which annihilates the whole dual of the Selmer group. In our example, the Galois group $G = G(k_\infty/k)$ is isomorphic to the semidirect product of two copies of the p-adic integers $\mathbb{Z}_p$.

Before stating the precise result we recall that a Λ-module M is called *faithful* if $Ann_\Lambda(M) = 0$ and *bounded* otherwise. These notions extend to objects of the quotient category Λ-mod$/\mathcal{C}$ of Λ-mod by the full subcategory $\mathcal{C}$ of pseudo-null modules and an object $\mathcal{M}$ of this latter category is called *completely faithful* if

all its non-zero subquotient objects are faithful.
Now assume that the number field k contains the pth roots of unity and that E is an elliptic curve over a k which has good ordinary reduction at all places above p. Further, assume $G = G(k_\infty/k) \cong H \rtimes \Gamma$ where both H and Γ are isomorphic to $\mathbb{Z}_p$ and Γ acts *non-trivially* on H, i.e. G is non-abelian.

Theorem (Theorem 3.7). *Assume $X_f(k_\infty)$ is non-zero and finitely generated as a $\Lambda(H)$-module. Then, it is a faithful torsion $\Lambda(G)$-module which is not pseudo-null. Even more, its image in the quotient category is completely faithful and cyclic.*

The purely algebraic fact that every $\Lambda(G)$-module - whether pseudo-null or not - which is finitely generated over $\Lambda(H)$ has a completely faithful, cyclic image in the quotient category has been proved in [V3].
We should mention that e.g. for $p = 5$, the elliptic curve $E = X_1(11)$ of conductor 11 which is defined by the equation

$$y^2 + y = x^3 - x^2,$$

the assumptions of the theorem hold for $k = \mathbb{Q}(\mu_5)$ and $k_\infty = k_{cyc}(\sqrt[5^\infty]{11})$. Indeed, we prove that $X_f(k_\infty)$ is free of rank 4 as $\Lambda(H)$-module where $H = G(k_\infty/k_{cyc})$ (theorem 6.2). Unfortunately, it is still not known even in a single example of an elliptic curve without complex multiplication whether over the "GL_2"-extension $k(E_{p^\infty})$ of k the dual of the Selmer group is bounded or faithful.
The above result suggests that it is worth considering Iwasawa theory over the specified type of extensions whose Galois group is isomorphic to a semidirect product $\mathbb{Z}_p \rtimes \mathbb{Z}_p$: This is the easiest non-commutative case and some questions are attackable for the associated group algebra which can be identified with a certain skew power series ring (cf. [V3]). Also our second main result, which describes the Euler characteristic of the Selmer group, confirms that this example will serve as a good test candidate for further developments in noncommutative Iwasawa theory. A formula for this Euler characteristic was calculated over $\mathbb{Z}_p$-extensions by Perrin-Riou and Schneider and over the "GL_2"-extension by Coates and Howson [CH].
Let $\rho_p(E/k)$ be the p-Birch-Swinnerton-Dyer constant (see section 4 for the definition). We assume that k contains the pth roots of unity and that k_∞ is a Galois extension of k containing the cyclotomic $\mathbb{Z}_p$ -extension k_{cyc} and such that $G(k_\infty/k) \cong \mathbb{Z}_p \rtimes \mathbb{Z}_p$.

Theorem (Theorem 4.1). *Assume* (i) $p \geq 5$, (ii) *E has good ordinary reduction at all primes above p and* (iii) $\mathrm{Sel}_{p^\infty}(E/k)$ *is finite. Then the G-Euler characteristic $\chi(G, \mathrm{Sel}_{p^\infty}(E/k_\infty))$ is defined and*

$$\chi(G, \mathrm{Sel}_{p^\infty}(E/k_\infty)) = \rho_p(E/k) \times \prod_{v \,\in\, \mathfrak{M}} |L_v(E,1)|_p,$$

where $L_v(E,1)$ is the local Euler-factor of the L-function of E evaluated at 1 *and $\mathfrak{M}$ denotes a certain set of places of k which is defined in section* 4.

We note that under the assumptions of the theorem $X_f(k_\infty)$ is Λ-torsion. In section 4 we also treat the case when k does not contain μ_p. This result follows from the explicit calculations of the local and global Galois cohomology, see Theorem 4.2 as well as subsections 4.3 and 4.2. We also calculate the "truncated" G-Euler characteristics introduced by Coates-Schneider-Sujatha ([CSS2]) under some milder conditions (Theorem 4.10).
We keep the assumption that k_∞ is a Galois extension of k which contains all p-power roots of unity and whose Galois group is isomorphic to $\mathbb{Z}_p \rtimes \mathbb{Z}_p$. Then - as Coates and Sujatha pointed out to us - another striking phenomenon in comparison with the GL_2-theory is the fact that the validity of Mazur's conjecture (i.e. that assuming E has good ordinary reduction at all primes above p the dual of Selmer $X_f(k_{cyc})$ over the cyclotomic $\mathbb{Z}_p$ -extension is $\Lambda(\Gamma)$-torsion where $\Gamma = G(k_{cyc}/k)$) implies the torsionness of $X_f(k_\infty)$ over $\Lambda(G)$ *unconditionally*; in particular, the vanishing of the μ-invariant of $X_f(k_{cyc})$ has not to be assumed, see theorem 2.8. As a consequence one obtains a quite general asymptotic bound for the rank of the Mordell-Weil group. Let α be any non-zero element of k which is not a root of unity and let k_n be the field obtained by adjoining to k the p^nth root of unity and the p^nth root of α.

Theorem (Corollary 2.9). *Assume that* (i) *E has good ordinary reduction at all primes ν of k dividing p, and* (ii) *$X_f(k_{cyc})$ is $\Lambda(\Gamma)$-torsion. Then there exists a constant $C > 0$ such that the rank of $E(k_n)$ is at most $C \cdot p^n$ for all $n \geq 0$.*

The following special case is an example of the deep unconditional results which follows from Kato's work. Assume now that E is defined over the rational numbers $\mathbb{Q}$ and that α is any non-zero element of the maximal abelian extension $\mathbb{Q}^{ab}$ of $\mathbb{Q}$ which is not a root of unity. Then there exists a constant C such that

$$\mathrm{rk}_{\mathbb{Z}}\, E(\mathbb{Q}(\mu_{p^n},\, \sqrt[p^n]{\alpha})) \leq C \cdot p^n$$

for all $n \geq 0$.
For the sake of completeness we also discuss other properties of the Selmer group such as having non-zero pseudo-null submodules (theorem 2.6), being (non-) trivial (see subsection 4.6, in particular proposition 4.12) or having non-vanishing μ-invariants (corollary 5.2 and an example in section 6). In section 5 we study the behavior of the μ-invariant under isogeny and we compare the μ-invariants of the duals of Selmer over k_∞ and k_{cyc}.
We hope that these results for the "false Tate curves" are indications of what might be true in general for non-abelian p-adic Lie extensions.

Acknowledgments. We are most grateful to John Coates. It was his kind invitation of both of us to DPMMS and his inspiring questions which gave the impulse to this work. Also we would like to express our warmest thanks to both him and R. Sujatha for suggesting several improvements of our results and keeping us fully informed on their joint work. We would like to thank Kazuo Matsuno for reading parts of the manuscript.

2. Non-existence of pseudo-null submodules

We consider an elliptic curve E over a number field k. Let S be a finite set of places of k containing all places S_p above p, all places S_{bad} at which E has bad reduction and all places S_∞ above infinity. Then we write k_S for the maximal outside S unramified extension of k and denote by $G_S(L) = G(k_S/L)$ the Galois group of k_S over L for any intermediate extension $k_S|L|k$.
Throughout the whole paper we assume that E has good reduction at all places in S_p.
The main object under consideration in this article, the p-Selmer group, is classically defined as

$$\begin{aligned}\mathrm{Sel}_{p^\infty}(E/L) &:= \ker(H^1(L, E_{p^\infty}) \to \bigoplus_w H^1(L_w, E(\overline{L_w}))_{p^\infty}) \\ &\cong \ker(H^1(G_S(L), E_{p^\infty}) \to \bigoplus_{w\, \epsilon\, S(L)} H^1(L_w, E(\overline{L_w}))_{p^\infty}).\end{aligned}$$

Here, L is a finite extension of k and, in the first line, w runs through all places of L while, in the second line, $S(L)$ denotes the set of all places of L lying above some place of S. As usual, L_w denotes the completion of L at the place w and for any field K we fix an algebraic closure $\bar{K}$. For infinite extensions K of k, $\mathrm{Sel}_{p^\infty}(E/K)$ is defined to be the direct limit of $\mathrm{Sel}_{p^\infty}(E/L)$ over all finite intermediate extensions L.
Now, let k_∞ be a Galois extension of k contained in k_S such that its Galois group $G := G(k_\infty/k)$ is a pro-p p-adic Lie group of cohomological p-dimension $\mathrm{cd}_p G = 2$. With other words, the set $S_{ram}(k_\infty/k)$ of all places which ramify in $k_\infty|k$ is contained in S. Note that G is soluble, because its Lie algebra over $\mathbb{Q}_p$ is 2-dimensional, and has no element of finite order. The last fact implies that the Iwasawa algebra, i.e. the completed group algebra

$$\Lambda(G) := \mathbb{Z}_p[[G]]$$

of G is a Noetherian ring *without zero-divisors* and thus has a skewfield $Q(G)$ of fractions by Goldie's theorem. Moreover, $\Lambda(G)$ is an Auslander regular ring (see [V1] for the definition and the proof of this property) of global dimension $d = \mathrm{cd}_p(G) + 1 = 3$. For Auslander regular rings there exists a nice dimension theory for modules over it which coincides with the Krull dimension of the support if Λ is commutative. For a detailed treatment we refer the reader to [V1]. We recall that a Λ-module M is called *pseudo-null* if $\mathrm{E}^0 M = \mathrm{E}^1 M = 0$ where we use the following

Notation 2.1. For a Λ-module M,

$$\mathrm{E}^i(M) := \mathrm{Ext}^i_\Lambda(M, \Lambda)$$

for any integer i and $\mathrm{E}^i(M) = 0$ for $i < 0$ by convention.

Also, by the rank $\mathrm{rk}_{\Lambda(G)}M$ of a (left) $\Lambda(G)$-module M we denote its dimension over $Q(G)$ after extension of scalars

$$\mathrm{rk}_\Lambda M := \dim_{Q(G)} Q(G) \otimes_{\Lambda(G)} M.$$

Now, the Selmer group $\mathrm{Sel}_{p^\infty}(E/k_\infty)$ bears a natural structure as an discrete (left) G-module. For some purposes it is more convenient to deal with (left) compact G-modules, thus we take the Pontryagin duals $-^\vee$ and set

$$\begin{aligned}
X_\nu &:= \begin{cases} \mathrm{H}^1(k_{\infty,\nu}, E_{p^\infty})^\vee & \text{for } \nu \in S \setminus S_p, \\ \mathrm{H}^1(k_{\infty,\nu}, (\widetilde{E_\nu})_{p^\infty})^\vee & \text{for } \nu \in S_p, \end{cases} \\
\mathbb{U}_S &:= \bigoplus_S \mathrm{Ind}_G^{G_\nu} X_\nu, \\
X_S &:= \mathrm{H}^1(G_S(k_\infty), E_{p^\infty})^\vee \text{ and} \\
X_f &:= (\mathrm{Sel}_{p^\infty}(E/k_\infty))^\vee.
\end{aligned}$$

Here we define $\widetilde{E_\nu}$ to be the reduction of E at the prime ν. It is well known that $\mathbb{U}_S$, X_S and X_f are all finitely generated (compact) $\Lambda(G)$-modules.
The following two conditions will be crucial for our considerations
Assumption WL_S*:* $\mathrm{H}^2(G_S(k_\infty), E_{p^\infty}) = 0$.
The validity of this assumption is the statement of a generalized *weak Leopoldt conjecture* for E, k_∞ and S.
Assumption SEQ_S*:* The "defining sequence" for the Selmer group is exact, i.e. also *left* exact:

$$0 \to \mathbb{U}_S \to X_S \to X_f \to 0.$$

Note that the (dual of) $\mathbb{U}_S$ is indeed isomorphic to the local conditions occurring in the above definition of the Selmer group by the work of Coates-Greenberg [CG] and by Mattuck's theorem (see [V2, §4] for details).
We will show in section 7.1 that if $E(k_\infty)_{p^\infty}$ is finite and X_f a torsion $\Lambda(G)$-module, then both assumptions hold and, in particular, are independent of S. On the other hand, if k is totally imaginary and both conditions hold for some S (e.g. $S = \Sigma := S_p \cup S_{bad} \cup S_{ram}(k_\infty/k) \cup S_\infty$), then - as we will see below - the rank of X_f is equal to

$$\mathrm{rk}_{\Lambda(G)} X_f = \sum_{S_p^s} [k_\nu : \mathbb{Q}_p], \tag{2.1}$$

where S_p^s denotes the set of places above p at which E has good supersingular reduction. In particular, if E has good *ordinary* reduction at all places over p, then the dual of its Selmer group X_f must be a $\Lambda(G)$-torsion module assuming WL_S and SEQ_S for some S. We refer the reader to theorem 2.8 at the end of this section for a further discussion about cases in which the equation (2.1) holds.

Remark 2.2. If the cyclotomic $\mathbb{Z}_p$-extension k_{cyc} of k is contained in k_∞, then assumption WL_S would be a consequence of the vanishing of $\mathrm{H}^2(G_S(k_{cyc}), E_{p^\infty})$, which is conjecturally true, see e.g. [P3, section 1.3.3].

Indeed, as G is a Poincaré group of cohomological dimension 2 with quotient $\Gamma = G(k_{cyc}/k) \cong \mathbb{Z}_p$ a Poincaré group of dimension 1, it follows from [NSW, thm. 3.7.4] that $H = G(k_\infty/k_{cyc})$, which is as p-adic Lie group without element of order p also a Poincaré group, has cohomological dimension $\mathrm{cd}_p H = 1$. Now the Hochschild-Serre spectral sequence supplies a surjection $\mathrm{H}^2(G_S(k_{cyc})) \twoheadrightarrow \mathrm{H}^2(G_S(k_\infty))^{\mathrm{H}}$ which implies the claim. We should mention that the vanishing over k_{cyc} was shown by Kato [K] for abelian extensions k of $\mathbb{Q}$ for elliptic curves which are defined over $\mathbb{Q}$ (and hence modular).

In order to avoid frequent repetition we define two further assumptions. The first one concerns the *base* field.
Assumption BASE*:*

k contains the pth root of unity μ_p.

We write $G_\nu \subseteq G$ and $T_\nu \subseteq G_\nu$ for the decomposition group and inertia group at a place ν, respectively. We shall denote by S_p^{ord} the set of places in S_p at which E has good ordinary reduction. The second assumption concerns the *dimensions* of these local groups.
Assumption DIM_S :

a) $\dim G_\nu = 2$ for all finite places $\nu \in S_{bad} \cup S_{ram}(k_\infty/k)$ and $\dim G_\nu \geq 1$ for all $\nu \in S \setminus S_p$.
b) $\dim G_\nu = 2$ for all $\nu \in S_p^{ord}$.
c) $\dim T_\nu = 2$ for all $\nu \in S_p^{ord}$.

Part c) implies

c') $\widetilde{E}_{p^\infty}(k_{\infty,\nu})$ is finite for all $\nu \in S_p^{ord}$.

Indeed, $\widetilde{E}_{p^\infty}(k_{\infty,\nu}) \cong \widetilde{E}_{p^\infty}(\kappa_{\infty,\nu})$, where $\kappa_{\infty,\nu}$ denotes the residue class field of $k_{\infty,\nu}$ which is finite if DIM_S c) holds. But an projective variety over a finite field κ has only finitely many κ- rational points.
Note also that for sets of places $S' \supseteq S \supseteq \Sigma$, the condition $\mathrm{DIM}_{S'}$ implies DIM_S and in particular DIM_Σ.
To recover properties of X_f we first have to consider the local modules X_ν.

Proposition 2.3. (i) *X_ν is a $\Lambda(G_\nu)$-torsion module for every ν in $S \setminus S_p$ and assuming* DIM_S *a) it holds $X_\nu = 0$ for all $\nu \in S_{bad}$.*

(ii) *Let $\nu \in S_p^{ord}$. Then one has $\mathrm{rk}_{\Lambda(G_\nu)} X_\nu = [k_\nu : \mathbb{Q}_p]$. If we assume* DIM_S *b), then there is an exact sequence of $\Lambda(G_\nu)$-modules*

$$0 \to X_\nu \to R_\nu \to \mathrm{E}^2\mathrm{E}^1 X_\nu \to 0,$$

where R_ν is a reflexive, hence torsionfree $\Lambda(G_\nu)$-module.Furthermore, for the projective dimension of X_ν it holds that $\mathrm{pd}_{\Lambda(G_\nu)} X_\nu \leq 1$ and $\mathrm{E}^1\mathrm{E}^1 X_\nu = 0$. If, in addition, DIM_S *c') holds, then $\mathrm{E}^2\mathrm{E}^1 X_\nu = 0$ vanishes, too.*

(iii) *For all $\nu \in S_p^s$, the module X_ν is obviously trivial.*

Proof. For $\nu \nmid p$ the module X_ν is torsion by [OV2, thm. 4.1] and even vanishes if $\dim(G_\nu) = 2$ by prop. 4.5 (loc.cit.). Now let ν be in S_p^{ord}. The statement concerning the rank is again thm. 4,1 (loc.cit.). It is easily seen using the diagram of [OV1, lem. 4.5, rem. 3], that $\mathrm{E}^i X_\nu \cong \mathrm{E}^{i+2}(\widetilde{E_{\nu p^\infty}}(k_{\infty,\nu})^\vee) = 0$ for $i \geq 2$ because $\mathrm{pd}\Lambda(G_\nu) = 3$ by assumption DIM_S b). Thus $\mathrm{pd}_{\Lambda(G_\nu)} X_\nu \leq 1$ using [V1, 6.3,6.4] and hence the module $\mathrm{E}^1\mathrm{E}^1 X_\nu$ coincides with $\mathrm{tor}_{\Lambda(G_\nu)} X_\nu = 0$ (see [V1, §2]) while the short exact sequence of the statement is taken from [V2, prop. 3.4]. Now assume that DIM_S c') holds. Then $\mathrm{E}^2\mathrm{E}^1 X_\nu = 0$ by [V2, lem. 3.1, prop. 3.4] (Note that the additional condition in an earlier version of lemma 3.1 (loc.cit.) in the case $\mathrm{cd}_p(G) = 2$ is superfluous, since in any case $\mathrm{pd} X_\nu \leq 1$ by the above). ☐

It follows immediately that $\mathrm{rk}_{\Lambda(G)} \mathbb{U}_S = \sum_{S_p^{ord}} [k_\nu : \mathbb{Q}_p]$, and under assumptions DIM_Σ a) and DIM_Σ b) that $\mathrm{pd}_{\Lambda(G)} \mathbb{U}_\Sigma \leq 1$ and that $\mathbb{U}_\Sigma$ is torsionfree where $\Sigma = S_p \cup S_{bad} \cup S_{ram}(k_\infty/k) \cup S_\infty$ as above.
With respect to the global modules we have the following

Proposition 2.4.
(i) *Assume* WL_S. *Then the projective dimension of* X_S *is at most one:* $\mathrm{pd}_{\Lambda(G)} X_S \leq 1$, *and, if* k *is totally imaginary, its rank is* $\mathrm{rk}_{\Lambda(G)} X_S = [k : \mathbb{Q}]$.
(ii) *Assuming* DIM_Σ *a), b),* WL_Σ *and* SEQ_Σ *the projective dimension of* X_f *is less or equal to two:* $\mathrm{pd}_{\Lambda(G)} X_f \leq 2$.

Proof. As in the proof of proposition 2.3 we obtain immediately that

$$\mathrm{E}^i X_S \cong \mathrm{E}^{i+2}(E_{p^\infty}(k_\infty)^\vee) = 0$$

for $i \geq 2$ which implies that the projective dimension of X_S is less or equal to 1. The statement about the rank is well known, see (sub)section 7.3 for a sketch of the proof. Since both $\mathrm{pd} X_S$, $\mathrm{pd}\mathbb{U}_S \leq 1$, it follows by homological algebra that $\mathrm{pd} X_f \leq 2$. ☐

Remark 2.5. Let k be totally imaginary. Then we obtain from the results above that assumption SEQ_S for some S implies the following equality: $\mathrm{rk}_{\Lambda(G)} X_f = \sum_{S_p^s} [k_\nu : \mathbb{Q}_p]$, where S_p^s denotes the set of places above p at which E has good *supersingular* reduction. On the other hand, if we assume DIM_Σ a), DIM_Σ b) and WL_Σ, then it follows easily from the long exact Poitou-Tate sequence that condition SEQ_Σ is equivalent to the validity of this rank formula. Indeed, the latter condition forces the kernel of $\mathbb{U}_\Sigma \to X_\Sigma$ to be torsion. But since $\mathbb{U}_\Sigma$ is a torsionfree $\Lambda(G)$-module, the kernel must be zero (see[V2, prop. 4.32, 4.33]).

Theorem 2.6.
(i) [OV1, thm 4.6] *Assume* WL_S. *Then* X_S *does not contain any non-zero pseudo-null submodule.*
(ii) *Assume* DIM_S *a), b), c'),* WL_S *and* SEQ_S *for some* $S \supseteq \Sigma$. *Then* X_f *does not contain any non-zero pseudo-null submodule.*

For the proof of (ii) we need the following characterization on the non-existence of pseudo-null submodules:

Lemma 2.7. [OV1, prop 2.4 1(b)] *A finitely generated* $\Lambda(G)$*-module* M *has zero maximal pseudo-null submodule if and only if* $\mathrm{E}^i\mathrm{E}^i M = 0$ *for all* $i \geq 2$. *In particular, if* $\mathrm{pd}_{\Lambda(G)} M \leq 2$, *this is equivalent to* $\mathrm{E}^2\mathrm{E}^2 M = 0$.

Proof of the theorem. The proof of (ii) is analogous to that of [OV1, thm 5.2]. Since some calculations are different we nevertheless give it completely: Since $\mathrm{pd}_{\Lambda(G)} X_f \leq 2$ it suffices by lemma 2.7 to show that $\mathrm{E}^2\mathrm{E}^2 X_f = 0$ vanishes. We consider the long exact $\mathrm{E}^\bullet$-sequence associated with the sequence in condition SEQ_S:

$$\mathrm{E}^1 X_S \to \bigoplus_{S_p^{ord}} \mathrm{Ind}_G^{G_\nu} \mathrm{E}^1 X_\nu \to \mathrm{E}^2 X_f \to \mathrm{E}^2 X_S = 0,$$

where the last identity follows from proposition 2.4 while the compatibility of Ind and $\mathrm{E}^\cdot$ is the content of [OV1, lem 5.5]. Splitting this into short exact sequences we obtain

$$0 \to B \to \bigoplus_{S_p^{ord}} \mathrm{Ind}_G^{G_\nu} \mathrm{E}^1 X_\nu \to \mathrm{E}^2 X_f \to 0 \qquad \text{and}$$

$$0 \to C \to \mathrm{E}^1 X_S \to B \to 0,$$

where the modules B and C are defined by exactness. Again via the long exact $\mathrm{E}^\bullet$-sequence and using lemma 2.7 with (i) we obtain

$$0 = \bigoplus_{S_p^{ord}} \mathrm{Ind}_G^{G_\nu} \mathrm{E}^1\mathrm{E}^1 X_\nu \to \mathrm{E}^1 B \to \mathrm{E}^2\mathrm{E}^2 X_f \to \bigoplus_{S_p^{ord}} \mathrm{Ind}_G^{G_\nu} \mathrm{E}^2\mathrm{E}^1 X_\nu = 0 \qquad \text{and}$$

$$0 = \mathrm{E}^0 C \to \mathrm{E}^1 B \to \mathrm{E}^1\mathrm{E}^1 X_S,$$

where the vanishing of the local modules follows from proposition 2.3. Also note that $C \subseteq \mathrm{E}^1 X_S$ is a $\Lambda(G)$-torsion module, hence $\mathrm{E}^0 C = 0$. We conclude that the pseudo-null module $\mathrm{E}^2\mathrm{E}^2 X_f$ is contained in the pure module $\mathrm{E}^1\mathrm{E}^1 X_S$ (see [V1, propb 3.5 (v)(a)]) and thus zero. □

For the rest of this section we assume BASE and that k_∞ contains the cyclotomic $\mathbb{Z}_p$-extension k_{cyc} of k. As before we put $\Gamma = G(k_{cyc}/k)$, $H = G(k_\infty/k)$ and recall that both groups are isomorphic to $\mathbb{Z}_p$.

We are very grateful to John Coates and Sujatha for pointing out to us that an analogue of their proposition 2.9 in [CSS2] also holds in our situation. In fact the following result is even stronger since their vanishing condition "$H^2(H, \mathrm{Sel}_{p^\infty}(E/k_\infty)) = 0$" is always satisfied in this situation because now H has p-cohomological dimension one.

Theorem 2.8. *Assume* $\mathrm{rk}_{\Lambda(\Gamma)} X_f(k_{cyc}) = \sum_{S_p^s} [k_\nu : \mathbb{Q}_p]$. *Then*

$$\mathrm{rk}_{\Lambda(G)} X_f(k_\infty) = \sum_{S_p^s} [k_\nu : \mathbb{Q}_p].$$

In particular, if E has good ordinary reduction at all primes ν of k dividing p and $X_f(k_{cyc})$ is $\Lambda(\Gamma)$-torsion, then $X_f(k_\infty)$ is $\Lambda(G)$-torsion.

The striking point of this result (in ordinary case) is that one does not have to assume the vanishing of the μ-invariant of $X_f(k_{cyc})$ as we did in our earlier version of this theorem and as all results in this direction in the GL_2-case did until the work of Coates and Sujatha [CSS2].

Examples in which the assumption of the Theorem holds arise by the results of Kato, if k is abelian over $\mathbb{Q}$ and E is defined over $\mathbb{Q}$. Alternatively, by the (strong) Nakayama lemma, $X_f(k_{cyc})$ is $\Lambda(\Gamma)$-torsion in the good ordinary case, if $\mathrm{Sel}_{p^\infty}(E/k)$ is finite (and k is arbitrary).

Proof. First note that the assumption implies the validity of the weak Leopoldt conjecture $\mathrm{WL}_S(k_{cyc})$ over k_{cyc} and thus, by remark 2.2, the weak Leopoldt conjecture $\mathrm{WL}_S(k_\infty)$ over k_∞. Thus it is easily seen that the lemmas 2.3-2.5 as well as remark 2.6 (loc.cit.) hold also in our situation. In fact their proofs are even easier due to the smaller p-cohomological dimension of G and H. Thus by literally the same proof as that of prop. 2.9 (loc.cit.) one derives SEQ_S, i.e. the surjectivity of the defining sequence of $X_f(k_\infty)$. Now the claim follows by remark 2.5.
We give a second proof: First, $\mathrm{rk}_{\Lambda(G)} X_f(k_\infty) \geq r := \sum_{S_p^s}[k_\nu : \mathbb{Q}_p]$ is shown easily. Next, since the kernel and cokernel of the natural restriction $\mathrm{Sel}_{p^\infty}(E/k_{cyc}) \to \mathrm{Sel}_{p^\infty}(E/k_\infty)^H$ is $\Lambda(\Gamma)$-torsion (see the proof of Theorem 3.1), $\mathrm{rk}_{\Lambda(\Gamma)}(X_f(k_\infty)_H) = r$. By Lemma 7.3 below, we have $\mathrm{rk}_{\Lambda(G)} X_f(k_\infty) \leq r$. This shows the Theorem. □

One consequence of this result is the following asymptotic bound of the Mordell-Weil rank. Let α be any non-zero element of k which is not a root of unity and let k_n be the field obtained by adjoining to k the p^nth root of unity and the p^nth root of α. We are interested in the $\mathbb{Z}$-ranks of the Mordell-Weil group $E(k_n)$ when n varies.

Corollary 2.9. *Assume that* (i) *E has good ordinary reduction at all primes ν of k dividing p, and* (ii) *$X_f(k_{cyc})$ is $\Lambda(\Gamma)$-torsion. Then there exists a constant $C > 0$ such that the rank of $E(k_n)$ is at most $C \cdot p^n$ for all $n \geq 0$.*

Proof. In the next section we will see that $k_\infty = \bigcup_n k_n$ is a Galois extension of k with Galois group G isomorphic to the semidirect product of two copies of $\mathbb{Z}_p$. Thus the theorem implies that $X_f(k_\infty)$ is a $\Lambda(G)$-torsion module. We denote by G_n the normal subgroup of G which consists precisely of the p^nth powers of elements of G. Then its index in G is p^{2n} and, since G is uniform, G_n is nothing else than the lower p-central series, see [DSMS, thm. 3.6]. Now [Ha1, thm. 1.10] (see also [Ha3]) or [Ho1, thm. 2.22] prove the existence of some constant C such that $\mathrm{rk}_{\mathbb{Z}_p} X_f(k_\infty)_{G_n} \leq C \cdot p^n$ for all $n \geq 0$. Since G_n is contained in the normal subgroup $G'_n := G(k_\infty/k_n)$ of G this gives also a bound for $\mathrm{rk}_{\mathbb{Z}} E(k_n) \leq \mathrm{rk}_{\mathbb{Z}_p} X_f(k_n) \leq X_f(k_\infty)_{G'_n}$, because the cokernel of the natural map $X_f(k_\infty)_{G'_n} \to X_f(k_n)$ is finite by lemma 3.12. □

Combined with one of Kato's deepest results one obtains the following striking and general estimate which was suggested to us by John Coates: Assume now that E is defined over the rational numbers $\mathbb{Q}$ and that α is any non-zero element of the maximal abelian extension $\mathbb{Q}^{ab}$ of $\mathbb{Q}$ which is not a root of unity. Taking as base field the abelian extension $k = \mathbb{Q}(\mu_p, \alpha)$ of $\mathbb{Q}$, Kato's work on Euler systems tells us that $X_f(k_{cyc})$ is a torsion $\Lambda(G)$-module. Thus the corollary applies: there exists a constant C (depending on E and α but not on

n) such that

$$\mathrm{rk}_{\mathbb{Z}}\, E(\mathbb{Q}(\mu_{p^n},\, \sqrt[p^n]{\alpha})) \leq C \cdot p^n$$

for all $n \geq 0$.

3. Completely faithful Selmer groups

Throughout this section, we assume BASE for k. We consider the following k_∞ in this section: k_∞ is a Galois extension of k unramified outside a finite set of primes of k containing S_p. Further we assume k_∞ contains k_{cyc} and $H := \mathrm{Gal}(k_\infty/k_{cyc})$ is isomorphic to $\mathbb{Z}_p$.
In this section, we study the case when $X_f(k_\infty) = \mathrm{Sel}_{p^\infty}(E/k_\infty)^\vee$ for an elliptic curve E/k is finitely generated over $\Lambda(H)$. The remarkable fact is the completely faithfulness over $\Lambda(G)$ if G is non-abelian (Theorem 3.7).
One of the examples of k_∞ is a "false Tate curve" extension. We collect some facts on such k_∞ in subsection 3.3.

3.1. $\Lambda(H)$-structure of $X_f(k_\infty)$. Let E/k be an elliptic curve which has good ordinary reduction at all primes above p. Denote by $P_0 = P_0(k_\infty/k_{cyc})$ the set of all primes of k_{cyc} which are not lying above p and ramified in k_∞/k_{cyc}. Note this is a finite set. Put

$$P_1(k_\infty/k_{cyc}, E) := \{u \in P_0 |\ E/k_{cyc} \text{ has split multiplicative reduction at } u\},$$
$$P_2(k_\infty/k_{cyc}, E) := \{u \in P_0 |\ E \text{ has good reduction at } u \text{ and } E(k_{cyc,u})_{p^\infty} \neq 0\}.$$

Let $\Gamma = \mathrm{Gal}(k_{cyc}/k)$. We prove the following.

Theorem 3.1. *Let $p \geq 5$. Assume E has good ordinary reduction at p. Then,*

(i) *$X_f(k_\infty)$ is finitely generated over $\Lambda(H)$ if and only if $X_f(k_{cyc})$ is finitely generated over $\mathbb{Z}_p$, in other words, $X_f(k_{cyc})$ is $\Lambda(\Gamma)$-torsion and its μ-invariant vanishes.*

(ii) *When $X_f(k_\infty)$ is finitely generated over $\Lambda(H)$, then $X_f(k_\infty)$ is $\Lambda(H)$-torsionfree of rank $\lambda + m_1 + 2m_2$, where $\lambda := \mathrm{rank}_{\mathbb{Z}_p} X_f(k_{cyc})$, $m_i = \sharp P_i$ $(i = 1, 2)$. More precisely, there exists an injective $\Lambda(H)$-homomorphism*

$$X_f(k_\infty) \hookrightarrow \Lambda(H)^{\lambda + m_1 + 2m_2}$$

with finite cokernel.

Remark 3.2. By [V3], (ii) implies that $X_f(k_\infty)$ has no non-trivial pseudo-null submodule. This gives another proof of Theorem 2.6 in special cases. We remark that we did not assume E is ordinary at p nor that X_f is finitely generated over $\Lambda(H)$ in Theorem 2.6 while we do not need the Assumptions DIM_S a), b) and c') in the above theorem.

We note that $\Lambda(H)$ is isomorphic to $\mathbb{Z}_p[[X]]$. Let $H_n := H^{p^n}$ for $n \geq 0$ and F_n the intermediate field of k_∞/k_{cyc} corresponding to H_n. To prove the Theorem, we need the following usual fundamental diagram:

$$
(3.2)\quad
\begin{array}{ccccccc}
0\longrightarrow & \mathrm{Sel}_{p^\infty}(E/F_n) & \longrightarrow & H^1(k_S/F_n, E_{p^\infty}) & \underset{\lambda_{F_n}}{\longrightarrow} & \bigoplus_{u\,\in\, S_{cyc}} J'_u(F_n) \\
 & \downarrow r'_n & & \downarrow g'_n & & \downarrow \oplus h'_{n,u} \\
0\longrightarrow & \mathrm{Sel}_{p^\infty}(E/k_\infty)^{H_n} & \longrightarrow & H^1(k_S/k_\infty, E_{p^\infty})^{H_n} & \longrightarrow & \bigoplus_{u\,\in\, S_{cyc}} J'_u(k_\infty)^{H_n}.
\end{array}
$$

Here, S is a finite set of primes of k containing $S_p \cup S_{\mathrm{bad}} \cup S_{\mathrm{ram}}$, where S_{ram} is the set of all primes which are ramified in k_∞/k. We denote by S_{cyc} the set of primes of k_{cyc} above S. For a prime u of k_{cyc}, put

$$J'_u(F_n) := \bigoplus_{u_n|u} H^1(F_{n,u_n}, E(\overline{F_{n,u_n}}))_{p^\infty}$$

and put $J'_u(k_\infty) := \varinjlim_{F_n} J'_u(F_n)$.

By Nakayama's lemma, $X_f(k_\infty)$ is finitely generated over $\Lambda(H)$ if and only if $X_f(k_\infty)_H$ is finitely generated over $\mathbb{Z}_p$. From the above diagram for $n=0$ (note that $H_0 = H$ and $F_0 = k_{cyc}$), we see that $\mathrm{Ker}(r'_0) \subset \mathrm{Ker}(g'_0)$ and $\mathrm{Coker}(r'_0)$ is a subquotient of $\mathrm{Ker}(\bigoplus h'_{0,u})$. Both are cofinitely generated over $\mathbb{Z}_p$, as we will see in Lemma 3.3 and 3.4. Thus, we have $\mathrm{Sel}_{p^\infty}(E/k_\infty)^H$ is cofinitely generated over $\mathbb{Z}_p$ if and only if so is $\mathrm{Sel}_{p^\infty}(E/k_{cyc})$. This implies Theorem 3.1 (i).

For Theorem 3.1 (ii), we first have $X_f(F_n) = \mathrm{Sel}_{p^\infty}(E/F_n)^\vee$ is finitely generated over $\mathbb{Z}_p$ since so is $X_f(k_{cyc})$ by (i) (cf. [HM] Theorem 3.1). Then the map λ_{F_n} is surjective (cf. [HM] Prop. 2.3, note that F_n is the cyclotomic $\mathbb{Z}_p$-extension of some field). Thus, from (3.2), we obtain the exact sequences

(3.3)

$$0 \to \mathrm{Ker}(r'_n) \to \mathrm{Ker}(g'_n) \to \bigoplus_{u\,\in\, S_{cyc}} \mathrm{Ker}(h'_{u,n}) \to \mathrm{Coker}(r'_n) \to \mathrm{Coker}(g'_n),$$

$$(3.4)\quad 0 \to \mathrm{Ker}(r'_n) \to \mathrm{Sel}_{p^\infty}(E/F_n) \to \mathrm{Sel}_{p^\infty}(E/k_\infty)^{H_n} \to \mathrm{Coker}(r'_n) \to 0.$$

By the inflation-restriction exact sequence we have

$$\mathrm{Ker}(g'_n) = H^1(H_n, E(k_\infty)_{p^\infty}) \text{ and } \mathrm{Coker}(g'_n) \hookrightarrow H^2(H_n, E(k_\infty)_{p^\infty}).$$

We have $H^2(H_n, E(k_\infty)_{p^\infty}) = 0$ because $\mathrm{cd}_p(H_n) = 1$.

Lemma 3.3. *$\sharp H^1(H_n, E(k_\infty)_{p^\infty})$ is finite and bounded for all n. Hence, $\sharp\mathrm{Ker}(g'_n)$ and $\sharp\mathrm{Ker}(r'_n)$ are finite and bounded for all n.*

Proof. Since $H^1(H_n, E(k_\infty)_{p^\infty}) \cong (E(k_\infty)_{p^\infty})_{H_n}$, Lemma follows from the facts that $E(k_\infty)_{p^\infty}$ is cofinitely generated and $(E(k_\infty)_{p^\infty})^{H_n} = E(F_n)_{p^\infty}$ is finite. The latter fact is a Theorem of Imai[I]. □

By Shapiro's lemma, we have

$$\mathrm{Ker}(h'_{n,u}) = \bigoplus_{u_n|u} H^1(H_{n,w}, E(k_{\infty,w}))_{p^\infty}.$$

Here, we choose w a prime of k_∞ above u_n and $H_{n,w}$ denotes the decomposition group of w in H_n. We will prove later the following.

Lemma 3.4. (i) *Let u be a prime of k_{cyc} such that $u \nmid p$. Let u_n and w be primes above u of F_n and k_∞ respectively such that $w|u_n|u$. Then $H^1(H_{n,w}, E(k_{\infty,w}))_{p^\infty} \cong H^1(H_{n,w}, E(k_{\infty,w})_{p^\infty})$ and*

$$H^1(H_{n,w}, E(k_{\infty,w})_{p^\infty}) \cong \begin{cases} \mathbb{Q}_p/\mathbb{Z}_p & \text{if } u \in P_1(k_\infty/k_{cyc}, E), \\ (\mathbb{Q}_p/\mathbb{Z}_p)^2 & \text{if } u \in P_2(k_\infty/k_{cyc}, E), \\ 0 & \text{otherwise} \end{cases}$$

as an abelian group.

(ii) *If $u|p$, then $\sharp H^1(H_{n,w}, E(k_{\infty,w}))_{p^\infty}$ is finite and bounded for all n.*

Note that the number of primes of F_n dividing p such that

$$H^1(H_{n,w}, E(k_{\infty,w}))_{p^\infty} \neq 0$$

is bounded if n varies, because $H^1(H_{n,w}, E(k_{\infty,w}))_{p^\infty} = 0$ if u splits completely. By this fact and Lemma 3.4, we have $\oplus_u \mathrm{Ker}(h'_{n,u}) \cong (\mathbb{Q}_p/\mathbb{Z}_p)^{t_n} \oplus D_n$ where

$$t_n = \sum_{u \in P_1} \sum_{u_n|u} 1 + \sum_{u \in P_2} \sum_{u_n|u} 2$$

and $\sharp D_n$ is finite and bounded for n. Since the kernel and cokernel of the map $\oplus_u \mathrm{Ker}(h'_{u,n}) \to \mathrm{Coker}(r'_n)$ are finite, we have that

$$\mathrm{Coker}(r'_n) \cong (\mathbb{Q}_p/\mathbb{Z}_p)^{t_n} \oplus D'_n \tag{3.5}$$

where $\sharp D'_n$ is finite and bounded. Next, we need the following which is a result of Matsuno [M] on finite $\Lambda(\Gamma)$-submodules of Selmer groups.

Lemma 3.5 (Matsuno [M]). *Let F be a totally imaginary algebraic number field and $\Gamma = \mathrm{Gal}(F_{cyc}/F)$. Let E be an elliptic curve over F which has good ordinary reduction at all primes above p. If the dual of the Selmer group $X_f(F_{cyc})$ is $\Lambda(\Gamma)$-torsion and its μ-invariant vanishes, then it is $\mathbb{Z}_p$-torsionfree.*

Combining this with [HM] Theorem 3.1, we have the following.

Lemma 3.6. *Under the assumptions of the Theorem, $\mathrm{Sel}_{p^\infty}(E/F_n) \cong (\mathbb{Q}_p/\mathbb{Z}_p)^{e_n}$ where*

$$e_n = p^n\lambda + \sum_{u \in P_1} \sum_{u_n|u} (p^n/d_n(u) - 1) + 2 \sum_{u \in P_2} \sum_{u_n|u} (p^n/d_n(u) - 1).$$

Here, we put $d_n(u) = \min(p^n, [H : H_w])$ where w is a prime of k_∞ above u and H_w is the decomposition group of w in H.

Proof. By [HM] Theorem 3.1,

$$\mathrm{corank}_{\mathbb{Z}_p} \mathrm{Sel}_{p^\infty}(E/F_n) = p^n\lambda + \sum_{u \in P_1} \sum_{u_n|u} (e(u_n) - 1) + 2 \sum_{u \in P_2} \sum_{u_n|u} (e(u_n) - 1)$$

where $e(u_n)$ is the ramification index of $u_n|u$. For $u \nmid p$, the decomposition group of $u_n|u$ coincides with its inertia group. Thus,

$$e(u_n) = [H_w : (H_n \cap H_w)] = p^n/d_n(u).$$

The cofreeness of $\mathrm{Sel}_{p^\infty}(E/F_n)$ follows from Lemma 3.5. □

Thus, from (3.4), we have

$$\mathrm{Sel}_{p^\infty}(E/k_\infty)^{H_n} \cong (\mathbb{Q}_p/\mathbb{Z}_p)^{s_n} \oplus D''_n \tag{3.6}$$

where

$$s_n = \mathrm{corank}_{\mathbb{Z}_p}\mathrm{Sel}_{p^\infty}(E/F_n) + \mathrm{corank}_{\mathbb{Z}_p}\mathrm{Coker}(r'_n),$$

and $\sharp D''_n$ is finite and bounded for n, because $\sharp D'_n$ in (3.5) is bounded and $\mathrm{Sel}_{p^\infty}(E/F_n)$ is cotorsion-free. By (3.5) and Lemma 3.6, we have

$$s_n = p^n\lambda + \sum_{u\,\epsilon\,P_1}\sum_{u_n|u}(p^n/d_n(u)) + 2\sum_{u\,\epsilon\,P_2}\sum_{u_n|u}(p^n/d_n(u)) = p^n(\lambda + m_1 + 2m_2)$$

since we see that $d_n(u) = \sharp\{u_n|u\}$.

From the well known structure theory of modules over $\Lambda(H)(\cong \mathbb{Z}_p[[X]])$, we see that $X_f(k_\infty)$ is pseudo-isomorphic to $\Lambda(H)^{\lambda+m_1+2m_2}$ by (3.6). Since $X_f(F_n)$ is $\mathbb{Z}_p$-torsionfree by Lemma 3.5, we have $X_f(k_\infty) = \varprojlim X_f(F_n)$ is also $\mathbb{Z}_p$-torsionfree. Therefore it can not have non-trivial finite $\Lambda(H)$-submodules. This proves the Theorem.

Finally, we give a proof of Lemma 3.4. The first assertion of (i) is proved by a standard argument (cf. [CH] §5.1 (59)). If u is unramified in k_∞/k, then u splits completely, so $H_{n,w} = 0$. Thus, $H^1(H_{n,w}, E(k_{\infty,w})_{p^\infty}) = 0$. Note that the type of reduction at any prime does not change in k_∞/k_{cyc} since $p \geq 5$. Assume u is not contained in $P_1 \cup P_2$. Then we have $E(F_{n,u_n})_{p^\infty} = 0$ (cf. [HM] Prop. 5.1 (i),(iii); note that $\mu_p \subseteq F_{n,u_n}$). Thus $H^1(H_{n,w}, E(k_{\infty,w})_{p^\infty}) = 0$. Assume $u \,\epsilon\, P_2$. Then $E(F_{n,u_n})_{p^\infty} \cong (\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus 2}$ (cf. [HM] Prop. 5.1 (i)), so we have $H^1(H_{n,w}, E(k_{\infty,w})_{p^\infty}) = \mathrm{Hom}(H_{n,w}, E(k_{\infty,w})_{p^\infty}) \cong (\mathbb{Q}_p/\mathbb{Z}_p)^2$. Next, assume $u \,\epsilon\, P_1$. Then, $E(F_{n,u_n})_{p^\infty} \cong \mathbb{Q}_p/\mathbb{Z}_p \oplus$(finite group) (cf. [HM] Prop. 5.1 (ii)). We have $E(k_\infty)_{p^\infty} \cong E_{p^\infty}$ because k_∞ is the maximal tame p-extension. Thus we have

$$H^1(H_{n,w}, E(k_{\infty,w})_{p^\infty}) \cong (E(k_{\infty,w})_{p^\infty})_{H_{n,w}} \cong \mathbb{Q}_p/\mathbb{Z}_p.$$

We prove Lemma 3.4 (ii). If u splits completely, $H^1(H_{n,w}, E(k_{\infty,w})_{p^\infty}) = 0$. If u is finitely decomposed, then $H_{n,w} \cong \mathbb{Z}_p$. Since F_n is a deeply ramified extension, we have by Coates-Greenberg([CG])

$$H^1(H_{n,w}, E)_{p^\infty} \cong H^1(H_{n,w}, \tilde{E}_u(\kappa_{\infty,w})_{p^\infty})$$

where $\tilde{E}_u$ is the reduction at u of E and $\kappa_{\infty,w}$ is the residue field of $k_{\infty,w}$. Thus we have $H^1(H_{n,w}, E)_{p^\infty}$ is finite and its order is bounded for n by the same argument of Lemma 3.3 because of the facts that $\tilde{E}_u(\kappa_{\infty,w})_{p^\infty}$ is cofinitely generated and that $\tilde{E}_u(\kappa_{n,u_n})_{p^\infty}$ is finite where κ_{n,u_n} is the residue field of F_{n,u_n}.

3.2. Completely faithfulness of $X_f(k_\infty)$. Henceforth we assume that G is *non-abelian.* In [V3], some properties of $\Lambda(G)$-modules for this specific group $G \cong \mathbb{Z}_p \rtimes \mathbb{Z}_p$, in particular the global annihilator ideal $Ann_{\Lambda(G)}M$ of a $\Lambda(G)$-torsion module M, were studied. Recall that a module is called *faithful* if its annihilator ideal is identical zero. Furthermore, an object $\mathcal{M}$ of the quotient category Λ-mod$/\mathcal{C}$ of the category of finitely generated Λ-modules by the Serre

subcategory $\mathcal{C}$ of pseudo-null modules is *faithful*, by definition, if every lift M ($Q(M) \cong \mathcal{M}$) of $\mathcal{M}$ is a faithful Λ-module. If this condition holds for every non-zero subquotient, then $\mathcal{M}$ is called *completely faithful*.
The following result is a direct consequence of theorem 6.3 (loc.cit.) and theorem 3.1:

Theorem 3.7. *Suppose that G is non-abelian. If X_f is non-zero and finitely generated as a $\Lambda(H)$-module, then X_f is a faithful, but torsion $\Lambda(G)$-module which is not pseudo-null. Even more, its image in the quotient category is completely faithful and thus cyclic.*

Recall that here the cyclicity in the quotient category means that there exists a cyclic submodule C of X_f with pseudo-null cokernel, see [CSS1, lem 2.7]. The following implication is arithmetically by no means obvious:

Corollary 3.8. *Under the assumptions of the theorem the Pontryagin dual $Ш(E/k_\infty)(p)^\vee$ of the (p-primary part of the) Tate-Shafarevich group contains a cyclic submodule with pseudo-null cokernel.*

Proof. Subobjects of completely faithful objects are again completely faithful. □

3.3. The "false Tate curve" case. The typical examples of k_∞ in previous subsections which we keep in our mind are the extensions of the type

$$k_\infty = k_{cyc}(\alpha^{p^{-\infty}})$$

where k_{cyc} denotes the cyclotomic $\mathbb{Z}_p$-extension of k and α is in k^* which is not any root of unity. (We call this the "false Tate curve case".) Then by Kummer theory, the Galois group $G = G(k_\infty/k)$ is isomorphic to the semi-direct product $G = H \rtimes \Gamma$ of $H = G(k_\infty/k_{cyc}) \cong \mathbb{Z}_p$ and $\Gamma = G(k_{cyc}/k) \cong \mathbb{Z}_p$ the latter group acting on the prior by the cyclotomic character, see [V3].
In this subsection, we collect some facts on k_∞.
First we consider DIM_S. Before we determine the dimensions of the decomposition groups we would like to remark that in the actual situation

$$\mathrm{DIM}_S \text{ b)} \Rightarrow \mathrm{DIM}_S \text{ c)} \ \Rightarrow \mathrm{DIM}_S \text{ c')}.$$

Indeed, if $\dim T_\nu(k_\infty/k_{cyc})$ were finite, hence zero, $k_{\infty,\nu}$ would be the compositum of the $\mathbb{Z}_p$ -extensions $k_{cyc,\nu}$ and k_ν^{nr} which denotes the maximal unramified extension of k_ν inside $k_{\infty,\nu}$. With other words, G_ν would be an 2-dimensional abelian subgroup of G, a contradiction.
For $\alpha \in k^* \setminus \mu$ we write S_α for the set of finite places of k which divide (α) and set as before $k_\infty = k_{cyc}(\alpha^{p^{-\infty}})$.

Lemma 3.9.
(i) *If $S = S_\alpha \cup S_p \cup S_\infty$, then k_∞ is outside S unramified, i.e. contained in k_S. In other words $S_{ram}(k_\infty/k)$ is contained in $S_\alpha \cup S_p$.*
(ii) *Let $\nu \in S_p$. Then $\dim G_\nu = 2$. If, in addition, $\alpha \in \mathbb{Q}^*$, $k = \mathbb{Q}(\mu_p)$ and α is not contained in $(\mathbb{Q}_p{}^*)^p$, then the extension $k_\infty|\mathbb{Q}$ is totally ramified at p.*

(iii) *Assume that α is not a pth power in k_{cyc} and let $\nu \in S_\alpha \setminus S_p$. Then, for all places ω_∞ of k_∞ lying above ν the local extension $k_{\infty,\omega_\infty}|k_{cyc,\omega}$, where ω denotes the place of k_{cyc} induced by ω_∞, is a totally ramified $\mathbb{Z}_p$ -extension, i.e. ω is almost totally ramified in $k_\infty|k_{cyc}$. The number of primes which are over k_{cyc} conjugate to ω_∞ equals the maximal power of p which divides $\nu(\alpha)$, where ν is normalized such that $\nu(k_\nu) = \mathbb{Z}$. In particular, $\dim G_\nu = \dim G = 2$ and the places of $S_\alpha \setminus S_p$ decompose only into finitely many ones at k_∞.*

Remark 3.10. Assume that for some $\nu \in S_\alpha \setminus S_p$ it holds $\nu(\alpha) < p$. Then α is not a pth power in k_{cyc}. Indeed, by [B, lem. 6] $k(\sqrt[p]{\alpha})|k$ ramifies totally at ν, thus cannot be contained in k_{cyc}.

Proof. [B, lem. 5] tells us that k_∞ is outside S unramified. In order to prove the first statement of (ii) it suffices to show that if $k(\alpha^{p^{-n}})$ is contained in k_{cyc} for all $n \geq 0$, then α is a root of unity. Using the long exact cohomology sequence for the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_{p^n} & \longrightarrow & k^*_{cyc} & \xrightarrow{p^n} & (k^*_{cyc})^{p^n} & \longrightarrow & 1 \\ & & \| & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \mu_{p^n} & \longrightarrow & \mu_{p^\infty} & \xrightarrow{p^n} & \mu_{p^\infty} & \longrightarrow & 1 \end{array}$$

and Hilbert's theorem 90 one easily sees that the canonical map $\mu(k)(p) \twoheadrightarrow (k^*_{cyc})^{p^n} \cap k^*/(k^*)^{p^n}$ is surjective. Now, if α is contained in $(k^*_{cyc})^{p^n} \cap k^*$ there exist $\zeta_n \in \mu(k)(p) = \mu_{p^{n_0}}$ and $b_n \in (k^*)^{p^n}$ such that $\alpha = \zeta_n \cdot b_n$ and hence $\alpha^{p^{n_0}} \in (k^*)^{p^n}$. Since this holds for all $n \geq 0$, the element $\alpha^{p^{n_0}}$ must be in $\bigcap_n (k^*)^{p^n} = \mu_q$, the roots of unity of order prime to p in k, thus α is a root of unity as we had to show.

Now we consider the local extensions $K = \mathbb{Q}_p(\mu_{p^n})$ and $L = K(\alpha^{p^{-n}})$ of $\mathbb{Q}_p$. Since the extension $\mathbb{Q}_p(\alpha^{p^{-1}})/\mathbb{Q}_p$ is not Galois, no pth root of α can be contained in the cyclic extension $K/\mathbb{Q}_p$. Hence, it follows from Kummer theory that the degree of L over K is $[L : K] = p^n$, i.e. $[L : \mathbb{Q}_p] = [\mathbb{Q}(\mu_{p^n}, \alpha^{p^{-n}}) : \mathbb{Q}](= (p-1)p^{2n-1})$ and in particular p does not split in $k(\mu_{p^n}, \alpha^{p^{-n}})$. Since the maximal abelian quotient G^{ab} of $G = G(L/\mathbb{Q}_p) \cong G(L/K) \rtimes G(K/\mathbb{Q}_p)$ is isomorphic to

$$G^{ab} \cong G(L/K)_{G(K/\mathbb{Q}_p)} \oplus G(K/\mathbb{Q}_p) = G(K/\mathbb{Q}_p)$$

(note that $G(L/K) \cong \mathbb{Z}/p^n(1)$ has no non-zero $G(K/\mathbb{Q}_p)$-invariant quotient because $G(K/\mathbb{Q}_p)$ acts via the cyclotomic character on $G(L/K)$), the only cyclic extensions of $\mathbb{Q}_p$ in L are contained in K and cannot be unramified. Hence p is totally ramified in $k(\mu_{p^n}, \alpha^{p^{-n}})$ for all n and the second statement of (ii) follows.

Finally, we prove (iii): It follows from [B, lem. 6] that for sufficiently large n the extension $k_n(\alpha^{p^{-n}})|k_n$, where $k_n := k(\mu_{p^n})$, is non-trivial and ramified at $\omega_n =$

$\omega_{|k_n}$ and thus not contained in k_{cyc}. Since $k_{cyc,\omega}$ is the maximal unramified p-extension of k_ν, the local extension $k_{\infty,\omega_\infty}|k_{cyc,\omega}$ must be a totally ramified $\mathbb{Z}_p$ -extension. Let H_ν denote the decomposition group of $H = G(k_\infty/k_{cyc})$ at ω_∞ and set $L = (k_\infty)^{H_\nu}$. For sufficiently large n the extensions $k_{cyc}|k_n$ and $k_n(\alpha^{p^{-n}})|k_n$ are linearly disjoint and thus

$$[L : k_{cyc}] = \frac{[k_n(\alpha^{p^{-n}}) : k_n]}{[k_{n,\omega_n}(\alpha^{p^{-n}}) : k_{n,\omega_n}]} = \frac{p^n}{[k_{n,\omega_n}(\alpha^{p^{-n}}) : k_{n,\omega_n}]},$$

by assumption and Kummer theory. On the other hand, since $k_{n,\omega_n}(\alpha^{p^{-n}})|k_{n,\omega_n}$ has no unramified intermediate extension, the order of α in $k^*_{n,\omega_n}/(k^*_{n,\omega_n})^{p^n}$, which is by Kummer theory the same as the degree $[k_{n,\omega_n}(\alpha^{p^{-n}}) : k_{n,\omega_n}]$, is equal to the order of $\omega_n(\alpha)$ in $\mathbb{Z}/p^n$ (Note that $k^*_{n,\nu}/(k^*_{n,\nu})^{p^n} \cong \mathbb{Z}/p^n \times \mu_{p^n}$, where we assume without lost of generality that $\mu_{p^{n+1}} \not\subseteq k_{n,\nu}$, and that the subgroups of μ_{p^n} correspond to the unramified extensions of $k_{n,\nu}$ of exponent dividing p^n). Since $k_{cyc}|k$ is unramified at ν, $\nu(\alpha) = \omega_n(\alpha)$ and thus the claim follows. □

Put

$$M_E = \prod_{l,\ \nu|l \text{ for some } \nu\ \epsilon\ S_{bad}} l$$

and note that M_E is prime to p under our general assumption. The lemma above now implies

Lemma 3.11. *For all $\alpha \,\epsilon\, \mathbb{Z} \setminus \{0\}$ such that M_E divides α, $k_\infty = k_{cyc}(\alpha^{p^{-\infty}})$ is contained in k_S and the assumption* DIM$_S$ *holds with respect to $S = S_\alpha \cup S_p \cup S_\infty \supseteq \Sigma$.*

Proof. Condition DIM$_S$ b) follows from (ii) of lemma 3.9. By definition S_{bad} is contained in S_α. Since α is a rational number it follows easily from Kummer theory that for sufficiently big n none p^nth root of α is a pth power in k_{cyc}. Applying lemma 3.9 (iii) to such a root shows DIM$_S$ a). □

At the end of this section, we consider the torsion group of an elliptic curve. Let E/k be an elliptic curve. The following result is quoted as the Assumption FIN for E and k_∞ in section 4. Recall that by lemma 3.9 the conditions DIM$_S$ b), c), c') are always satisfied in the false Tate curve case.

Lemma 3.12. *Let v be a prime of k above p. Assume E has good ordinary reduction at v. Then, for $k_\infty = k_{cyc}(\alpha^{p^{-\infty}})$, we have $E(k_{\infty,w})_{p^\infty}$ is finite for $w|v$. In particular, $E(k_\infty)_{p^\infty}$ is finite.*

Proof. Let $\hat{E}_v$ be the formal group law of E and $\tilde{E}_v$ be the reduction at v. Then we have

$$0 \to \hat{E}_v(\mathfrak{M}(k_{\infty,w}))_{p^\infty} \to E(k_{\infty,w})_{p^\infty} \to \tilde{E}_v(\kappa_{\infty,w})_{p^\infty} \to 0$$

where $\mathfrak{M}(k_{\infty,w})$ is the maximal ideal of $k_{\infty,w}$ and $\kappa_{\infty,w}$ is the residue field of $k_{\infty,w}$. Since $\kappa_{\infty,w}$ is a finite field, $\tilde{E}_v(\kappa_{\infty,w})_{p^\infty}$ is a finite group. So we show $\hat{E}_v(\mathfrak{M}(k_{\infty,w}))_{p^\infty}$ is finite. Since E has good ordinary reduction at v, $\hat{E}_v(\mathfrak{M}(\overline{k_v}))_{p^\infty}$ is isomorphic to $\mathbb{Q}_p/\mathbb{Z}_p$ where $\mathfrak{M}(\overline{k_v})$ is the maximal ideal of $\overline{k_v}$. Thus, the field $k_v(\hat{E}_{v,p^\infty})$ is abelian extension of k_v. By a theorem of Imai ([I]), $k_{cyc,u} \cap k_v(\hat{E}_{v,p^\infty})$ is a finite extension of k_v where $u|w$. Since the maximal abelian extension of k_v in $k_{\infty,w}$ is $k_{cyc,u}$, we have

$$k_{\infty,w} \cap k_v(\hat{E}_{v,p^\infty}) = k_{cyc,u} \cap k_v(\hat{E}_{v,p^\infty})$$

This means $\hat{E}_v(\mathfrak{M}(k_{\infty,w}))_{p^\infty}$ is finite. □

4. Euler Characteristics

In this section, we do not assume the Assumption BASE, i.e. k does not necessarily contain the p-th roots of unity. Put

$$K = k(\mu_p) \text{ and } K_{cyc} = k(\mu_p)_{cyc} = k(\mu_{p^\infty}).$$

Let k_∞ be a Galois extension of k unramified outside a finite set of primes of k such that $k_\infty \supset K_{cyc}$ and $H := \mathrm{Gal}(k_\infty/K_{cyc})$ is isomorphic to $\mathbb{Z}_p$. Assume further k_∞ satisfies DIM c).

For an elliptic curve E/k and k_∞, with good ordinary reduction at p, we consider the following.

Assumption FIN: $E(k_\infty)_{p^\infty}$ is a finite group.

When k_∞/k is a "false Tate curve" extension (see subsection 3.3), DIM c) and FIN are always satisfied (Lemma 3.11 and 3.12).

We denote $G = \mathrm{Gal}(k_\infty/k)$ and $\Gamma = G/H$. Note that G may not be a pro-p group.

4.1. G-Euler Characteristics.

For an discrete G-module M, we define its Euler characteristic by

$$\chi(G, M) := \prod_{i=0}^{2} (\sharp H^i(G, M))^{(-1)^i}$$

if this is defined. In this section, we calculate the Euler characteristics of Selmer groups. The formula as well as its proof is similar to that obtained in [CH] Theorem 1.1 for GL_2-case.

Let E be an elliptic curve defined over k which has good reduction at all primes above p.

We define the p-Birch-Swinnerton-Dyer constant as

$$\rho_p(E/k) := \frac{\sharp Ш(E/k)_{p^\infty}}{(\sharp E(k)_{p^\infty})^2 \prod_v |c_v|_p} \times \prod_{v|p} (\sharp \tilde{E}_v(\kappa_v)_{p^\infty})^2.$$

Here, $Ш(E/k)$ is the Tate-Shafarevich group of E over k, κ_v is the residue field of k at v and $\tilde{E}_v$ is the reduction of E over κ_v. We denote by c_v the local Tamagawa factor at v, $[E(k_v) : E_0(k_v)]$, where $E_0(k_v)$ is the subgroup of $E(k_v)$ consisting from all of the points which maps to smooth points by reduction modulo v. $|*|_p$ denotes the p-adic valuation normalized such that $|p|_p = \frac{1}{p}$. For any prime v of k, let $L_v(E, s)$ be the local L-factor of E at v. Let $P_0(k_\infty/k)$ be the set of all primes of k which are not lying above p and ramified in k_∞/K_{cyc}. We put

$$P_1(k_\infty/k, E) := \{v \in P_0(k_\infty/k) |\ E/K \text{ has split multiplicative reduction at any } w|v \text{ of } K = k(\mu_p)\},$$

$$P_2(k_\infty/k, E) := \{v \in P_0(k_\infty/k) |\ E/K \text{ has good reduction at any } w|v \text{ of } K \text{ and } E(K_w)_{p^\infty} \neq 0.\}$$

and $\mathfrak{M} = \mathfrak{M}(k_\infty/k, E) := P_1(k_\infty/k, E) \cup P_2(k_\infty/k, E)$. We prove the following:

THEOREM 4.1. *Under* DIM c) *and* FIN, *assume* (i) $p \geq 5$, (ii) *E has good ordinary reduction at all primes above p,* (iii) $\mathrm{Sel}_{p^\infty}(E/k)$ *is finite and* (iv) $X_f(k_\infty) := \mathrm{Sel}_{p^\infty}(E/k_\infty)^\vee$ *is $\Lambda(G_0)$-torsion where $G_0 = \mathrm{Gal}(k_\infty/K)$ and $K = k(\mu_p)$. Then $\chi(G, \mathrm{Sel}_{p^\infty}(E/k_\infty))$ is defined and equals*

$$\rho_p(E/k) \times \prod_{v \in \mathfrak{M}} |L_v(E, 1)|_p.$$

Note that condition (iv) is already a consequence of (i)-(iii), whenever G itself happens to be a pro-p-group since the strong Nakayama's lemma holds for G. In fact, we prove more. Let us consider the usual fundamental diagram.

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Sel}_{p^\infty}(E/k) & \longrightarrow & H^1(k_S/k, E_{p^\infty}) & \underset{\lambda_k}{\longrightarrow} & \bigoplus_{v \in S} J_v(k) \\ & \downarrow r & & \downarrow g & & \downarrow \oplus h_v \\ 0 \longrightarrow & \mathrm{Sel}_{p^\infty}(E/k_\infty)^G & \longrightarrow & H^1(k_S/k_\infty, E_{p^\infty})^G & \underset{\psi_\infty}{\longrightarrow} & \bigoplus_{v \in S} J_v(k_\infty)^G. \end{array} \tag{4.7}$$

Here, S is a finite set of primes of k containing $S_p \cup S_{\mathrm{bad}} \cup S_{\mathrm{ram}}$ where S_{ram} is the set of primes which is ramified in k_∞/k, k_S is the maximal unramified extension of k outside S. For any finite extension L of k, we put

$$J_v(L) := \bigoplus_{w|v} H^1(L_w, E(\overline{L_w}))_{p^\infty}$$

and for infinite extension M, put $J_v(M) := \varinjlim_L J_v(L)$ where L runs over all finite extensions of k contained in M. Note that $\mathrm{Ind}_G^{G_w} X_w(k_\infty)$ defined in §2 is the Pontryagin dual of $J_v(k_\infty)$.
We have the following and we get Theorem 4.1 as an immediate corollary of this.

THEOREM 4.2. *Assume the same hypothesis of Theorem 4.1. Then we have*

(i) $\sharp H^0(G, \mathrm{Sel}_{p^\infty}(E/k_\infty)) = \rho_p(E/k) \times \prod_{v \in \mathfrak{M}} |L_v(E, 1)|_p \times \sharp(\mathrm{Coker}(\psi_\infty))$,
(ii) $\sharp H^1(G, \mathrm{Sel}_{p^\infty}(E/k_\infty)) = \sharp(\mathrm{Coker}(\psi_\infty))$,
(iii) $H^i(G, \mathrm{Sel}_{p^\infty}(E/k_\infty)) = 0$ *for* $i \geq 2$.

We split the proof of Theorem 4.2 into some subsections.
Throughout this section, we assume the conditions of Theorem 4.1 except condition (iv) if not explicitly stated.

4.2. Global cohomology. First, we consider about the map g. We prove

Lemma 4.3.
$$\frac{\sharp \mathrm{Ker}(g)}{\sharp \mathrm{Coker}(g)} = \sharp E(k)_{p^\infty}$$

To prove this, we need the following lemma.

Lemma 4.4. *If a G-module M is finite, then $\chi(G, M)$ is defined and equals to 1.*

Proof. This is an immediate consequence of the Hochschild-Serre spectral sequence for
$$1 \to H \to G \to \Gamma \to 1$$
and the fact that the same statement of the Lemma holds if we replace G with Γ. □

Proof of Lemma 4.3.
By the Hochschild-Serre spectral sequence, we have
$$0 \to H^1(G, E(k_\infty)_{p^\infty}) \to H^1(k_S/k, E_{p^\infty}) \to H^1(k_S/k_\infty, E_{p^\infty})^G \to H^2(G, E(k_\infty)_{p^\infty}) \to H^2(k_S/k, E_{p^\infty}).$$
Since $\mathrm{Sel}_{p^\infty}(E/k)$ is finite, $H^2(k_S/k, E_{p^\infty}) = 0$ (see [CH] Lemma 4.3 or [CM]). Thus, we have that $\mathrm{Ker}(g) = H^1(G, E(k_\infty)_{p^\infty})$ and $\mathrm{Coker}(g) = H^2(G, E(k_\infty)_{p^\infty})$, which are finite. This prove the Lemma by Lemma 4.4 because of FIN. □

Next, we consider the global cohomology of k_∞. We first have the following. (See section 7.1 for a proof.)

Theorem 4.5. *Assume $X_f(k_\infty)$ is $\Lambda(G_0)$-torsion. Then we have*
(i) $H^2(k_S/k_\infty, E_{p^\infty}) = 0$ *and*
(ii) *The map* $H^1(k_S/k_\infty, E_{p^\infty}) \stackrel{\lambda_{k_\infty}}{\to} \bigoplus_{v \in S} J_v(k_\infty)$ *is surjective.*

As a Corollary, we have

Corollary 4.6. *If $X_f(k_\infty)$ is $\Lambda(G_0)$-torsion,*
$$H^i(G, H^1(k_S/k_\infty, E_{p^\infty})) = 0$$
for all $i \geq 1$ (and still for all $i \geq 2$ if $\mathrm{Sel}_{p^\infty}(E/k)$ is not assumed to be finite.)

Proof. By the above Theorem, $H^i(k_S/k_\infty, E_{p^\infty}) = 0$ for $i \geq 2$. So, we have the following by the Hochschild-Serre spectral sequence that
$$H^{i+1}(k_S/k, E_{p^\infty}) \to H^i(G, H^1(k_S/k_\infty, E_{p^\infty})) \to H^{i+2}(G, E_{p^\infty})$$
are exact for all $i \geq 1$. If $\mathrm{Sel}_{p^\infty}(E/k)$ is finite, $H^i(k_S/k, E_{p^\infty}) = 0$ for $i \geq 2$ (see [CH] Lemma 4.3 or [CM]). Since the p-cohomological dimension of G is 2, $H^i(G, E_{p^\infty}) = 0$ for $i \geq 3$. These proves the Corollary. □

4.3. Local cohomology. Next, we consider the cohomology of $J_v(k_\infty)$ and the kernel and cokernel of h_v.

Proposition 4.7. *For all $i \geq 1$, we have $H^i(G, J_v(k_\infty)) = 0$.*

Proof. By Shapiro's lemma,

$$H^i(G, J_v(k_\infty)) \cong H^i(G_w, H^1(k_{\infty,w}, E)_{p^\infty})$$

where $w|v$ and G_w is the decomposition group $\mathrm{Gal}(k_{\infty,w}/k_v)$ (see [CH] Lemma 2.8). Thus we show the latter is zero.

(i) *The case when v does not divide p.*

In this case, $H^1(k_{\infty,w}, E)_{p^\infty} \cong H^1(k_{\infty,w}, E_{p^\infty})$ (cf. [CH] §5.1 (59)). We also have $H^i(k_{\infty,w}, E_{p^\infty}) = 0$ for $i \geq 2$ because the p-cohomological dimension of $\mathrm{Gal}(\overline{k_v}/k_{\infty,w})$ is less than or equals 1. So we have by the Hochschild-Serre spectral sequence that

$$H^{i+1}(k_v, E_{p^\infty}) \to H^i(G, H^1(k_{\infty,w}, E_{p^\infty})) \to H^{i+2}(G_w, E(k_\infty)_{p^\infty})$$

are exact for all $i \geq 1$. It is also known $H^i(k_v, E_{p^\infty}) = 0$ for $i \geq 2$. Further, $H^i(G_w, E_{p^\infty}) = 0$ for $i \geq 3$ since the p-cohomological dimension of G_w is less than or equals 2. Thus we have the Lemma for $v \nmid p$.

(ii) *The case when v divides p.*

In this case, the proof is exactly same as that of [CH] Corollary 5.23 because $k_{\infty,w}$ is a deeply ramified extension. We have

$$H^1(k_{\infty,w}, E)_{p^\infty} \cong H^1(k_{\infty,w}, \tilde{E}_{v,p^\infty})$$

by [CG]. Then we get $H^i(G_w, H^1(k_{\infty,w}, \tilde{E}_{v,p^\infty})) = 0$ by the same argument using the Hochschild-Serre spectral sequence as (i) above because the p-cohomological dimension of $\mathrm{Gal}(\overline{k_v}/k_{\infty,w})$ is less than or equals 1 and $H^i(k_v, \tilde{E}_{v,p^\infty}) = 0$ for $i \geq 2$. □

Lemma 4.8. *Let v be a prime which does not divide p. If v is in $P_1(k_\infty/k, E) \cup P_2(k_\infty/k, E)$, then*

$$\frac{\sharp\mathrm{Ker}(h_v)}{\sharp\mathrm{Coker}(h_v)} = \left|\frac{c_v}{L_v(E,1)}\right|_p^{-1},$$

while otherwise, $\sharp\mathrm{Ker}(h_v)/\sharp\mathrm{Coker}(h_v) = |c_v|^{-1}$.

Proof. By Shapiro's lemma, the kernel and cokernel of h_v are isomorphic to those of the restriction map

$$H^1(k_v, E)_{p^\infty} \overset{\mathrm{res}_w}{\to} H^1(k_{\infty,w}, E)_{p^\infty}.$$

Since $v \nmid p$, E can be replaced by E_{p^∞}. So, $\mathrm{Ker}(h_v) \cong H^1(G_w, E(k_{\infty,w})_{p^\infty})$ and $\mathrm{Coker}(h_v) \cong H^2(G_w, E(k_{\infty,w})_{p^\infty})$.

First we consider the case v is not ramified in k_∞/k. Then, we have $k_{\infty,w} = K_{cyc,w}$. It is well known that $\sharp H^1(\mathrm{Gal}(K_{cyc,w}/k_v), E(K_{cyc,w})_{p^\infty}) = |c_v|_p^{-1}$ and $H^2(\mathrm{Gal}(K_{cyc,w}/k_v), E(K_{cyc,w})_{p^\infty}) = 0$.

Next consider the case where $E(K_w)_{p^\infty} = 0$ or the case where v has bad reduction which is not split multiplicative. In this case, $E(K_{cyc,w})_{p^\infty} = 0$ (cf.

[HM] Prop. 5.1), thus we have $E(k_{\infty,w})_{p^\infty}=0$. Thus $H^1(G_w,E(k_{\infty,w})_{p^\infty})$ and $\mathrm{H}^2(G_w,E(k_{\infty,w})_{p^\infty})$ are zero. Since we assume $p\geq 5$, $|c_v|_p=1$ in this case. Finally, consider the case $v\in P_1(k_\infty/k,E)\cup P_2(k_\infty/k,E)$. Then $k_{\infty,w}/K_w$ should be the maximal tame p-extension and therefore $k_{\infty,w}$ contains $k_v(E_{p^\infty})$. So we have $H^1(k_{\infty,w},E_{p^\infty})=0$ because there is no p-extension of $k_{\infty,w}$. Thus, $H^1(G_w,E(k_{\infty,w}))=H^1(k_v,E_{p^\infty})$ and $H^2(G_w,E(k_{\infty,w}))=0$. Therefore, Lemma follows from the fact that $\sharp H^1(k_v,E_{p^\infty})=|c_v/L_v(E,1)|_p^{-1}$ (cf. [CH] Lemma 5.6 or [CM]). □

Lemma 4.9. *Let v be a prime above p. Then*

$$\frac{\sharp\mathrm{Ker}(h_v)}{\sharp\mathrm{Coker}(h_v)}=(\sharp\tilde{E}_v(\kappa_v)_{p^\infty})^2.$$

Proof. By Shapiro's lemma,

$$\mathrm{Ker}(h_v)\cong H^1(G_w,E(k_{\infty,w}))_{p^\infty}\text{ and }\mathrm{Coker}(h_v)\cong H^2(G_w,E(k_{\infty,w}))_{p^\infty}.$$

Since $k_{\infty,w}$ is a deeply ramified extension, we have that

$$H^i(G_w,E(k_{\infty,w}))_{p^\infty}\cong H^i(G_w,\tilde{E}_v(\kappa_{\infty,w})_{p^\infty})$$

for $i\geq 2$ and

$$0\to H^1(k_v,\hat{E}_v(\mathfrak{M}(\overline{k_v})))_{p^\infty}\to H^1(G_w,E(k_{\infty,w}))_{p^\infty}\\ \to H^1(G_w,\tilde{E}_v(\kappa_{\infty,w})_{p^\infty})\to 0$$

is exact by the exactly same way as [C] Lemma 3.14. Here $\hat{E}_v$ is the formal group law for E, $\mathfrak{M}(\overline{k_v})$ is the maximal ideal of the integer ring of $\overline{k_v}$ and $\kappa_{\infty,w}$ is the residue field of $k_{\infty,w}$. It is known that

$$\sharp H^1(k_v,\hat{E}_v(\mathfrak{M}(\overline{k_v})))=\sharp\tilde{E}_v(\kappa_v)$$

(cf. [C] Lemma 3.13). Since $\tilde{E}_v(\kappa_{\infty,w})_{p^\infty}$ is finite by DIM c), we have $\chi(G_w,\tilde{E}_v(\kappa_{\infty,w})_{p^\infty})=1$ by the same way as Lemma 4.4. Thus we have

$$\sharp H^1(G_w,\tilde{E}_v(\kappa_{\infty,w})_{p^\infty})/\sharp H^2(G_w,\tilde{E}_v(\kappa_{\infty,w})_{p^\infty})=\sharp\tilde{E}_v(\kappa_v)_{p^\infty}.$$

Combining them, we have the Lemma. □

4.4. Proof of Theorem 4.2. Now we are ready to prove Theorem 4.2. To this aim let us assume conditions (i)-(iv). First, by Theorem 4.5,

$$0\to\mathrm{Sel}_{p^\infty}(E/k_\infty)\to H^1(k_S/k_\infty,E_{p^\infty})\overset{\lambda_{k_\infty}}{\to}\bigoplus_{v\in S}J_v(k_\infty)\to 0$$

is exact. Taking G-cohomology and by Lemma 4.6 and Proposition 4.7, we have

$$H^i(G,\mathrm{Sel}_{p^\infty}(E/k_\infty))=0$$

for $i \geq 2$. At the same time, we have that

$$0 \to \mathrm{Sel}_{p^\infty}(E/k_\infty)^G \to H^1(k_S/k_\infty, E_{p^\infty})^G \stackrel{\psi_\infty}{\to} \bigoplus_{v\, \epsilon\, S} J_v(k_\infty)^G \to H^1(G, \mathrm{Sel}_{p^\infty}(E/k_\infty)) \to 0$$

is exact, which means $\mathrm{Coker}\psi_\infty \cong H^1(G, \mathrm{Sel}_{p^\infty}(E/k_\infty))$.
Next, we calculate $\mathrm{Sel}_{p^\infty}(E/k_\infty)^G$. Consider the diagrams induced from the fundamental diagram (4.7),

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathrm{Sel}_{p^\infty}(E/k) & \longrightarrow & H^1(k_S/k, E_{p^\infty}) & \underset{\lambda_k}{\longrightarrow} & \mathrm{Im}\lambda_k & \longrightarrow 0 \\ & \downarrow r & & \downarrow g & & \downarrow \oplus h_v & \\ 0 \longrightarrow & \mathrm{Sel}_{p^\infty}(E/k_\infty)^G & \longrightarrow & H^1(k_S/k_\infty, E_{p^\infty})^G & \underset{\psi_\infty}{\longrightarrow} & \mathrm{Im}\psi_\infty & \longrightarrow 0, \end{array}$$

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathrm{Im}\lambda_k & \longrightarrow & \bigoplus_{v\, \epsilon\, S} J_v(k) & \longrightarrow & \mathrm{Coker}\lambda_k & \longrightarrow 0 \\ & \downarrow & & \downarrow \oplus h_v & & \downarrow & \\ 0 \longrightarrow & \mathrm{Im}\psi_\infty & \longrightarrow & \bigoplus_{v\, \epsilon\, S} J_v(k_\infty)^G & \longrightarrow & \mathrm{Coker}\psi_\infty & \longrightarrow 0. \end{array}$$

Since $\mathrm{Sel}_{p^\infty}(E/k)$ is finite, $\sharp\mathrm{Coker}\lambda_k = \sharp E(k)_{p^\infty}$ (cf. [CH] Lemma 2.7 or [CM]). The kernel and cokernel of $\oplus h_v$ are finite by Lemma 4.8 and 4.9. Therefore $\mathrm{Coker}\psi_\infty$ is finite by the latter diagram. By applying the Snake Lemma for the two diagrams, we have

$$\sharp\mathrm{Sel}_{p^\infty}(E/k_\infty)^G = \sharp\mathrm{Sel}_{p^\infty}(E/k) \times \frac{\sharp\mathrm{Coker}\psi_\infty}{\sharp\mathrm{Coker}\lambda_k} \times \prod_{v\, \epsilon\, S} \frac{\sharp\mathrm{Ker}h_v}{\sharp\mathrm{Coker}h_v} \times \frac{\sharp\mathrm{Coker}g}{\sharp\mathrm{Ker}g}.$$

Thus we have Theorem by combining Lemma 4.3, Lemma 4.8 and Lemma 4.9.

4.5. Truncated Euler Characteristics. The usual Euler characteristic at the beginning of this section is not defined for $\mathrm{Sel}_{p^\infty}(E/k_\infty)$ if $\mathrm{Sel}_{p^\infty}(E/k)$ is infinite, e.g. if $E(k)$ has a point of infinite order. To circumvent this problem (and since the higher cohomology groups $H^i(G, \mathrm{Sel}_{p^\infty}(E/k_\infty))$, $i \geq 2$, are conjecturally trivial), the truncated G-Euler characteristics was introduced by Coates-Schneider-Sujatha in the GL_2-case extending ideas of Schneider and Perrin-Riou in the cyclotomic situation. Similarly to Theorem 3.1 of [CSS2], we can calculate these modified Euler characteristics in our case.
For an G-module M, let

$$\phi_M : H^0(G, M) \to H^1(G, M)$$

be the composition of

$$H^0(G, M) \cong H^0(\Gamma, M^H) \stackrel{\psi_M}{\to} H^1(\Gamma, M^H) \stackrel{\mathrm{res}}{\to} H^1(G, M)$$

where ψ_M is the map induced from the natural map

$$H^0(\Gamma, M^H) \cong (M^H)^\Gamma \to (M^H)_\Gamma \cong H^1(\Gamma, M^H).$$

We define the truncated G-Euler characteristic of M as

$$\chi_t(G, M) := q(\phi_M)$$

where $q(\phi_M) := \sharp\mathrm{Ker}(\phi_M)/\sharp\mathrm{Cok}(\phi_M)$ and say that this is finite if both $\mathrm{Ker}(\phi_M)$ and $\mathrm{Cok}(\phi_M)$ are finite. Setting formally $H = 1$, e.g. $G = \Gamma$, in the above we obtain the definition of the modified Γ-Euler characteristic $\chi_t(\Gamma, N)$ of a discrete Γ-module N. Then we have

Theorem 4.10. *Assume that* (i) $p \geq 5$, (ii) *E has good ordinary reduction at all primes above p and* (iii) *$X_f(K_{cyc})$ is $\Lambda(\Gamma_0)$-torsion where $\Gamma_0 = \mathrm{Gal}(K_{cyc}/K)$. Then $\chi_t(G, \mathrm{Sel}_{p^\infty}(E/k_\infty))$ is finite if and only if $\chi_t(\Gamma, \mathrm{Sel}_{p^\infty}(E/K_{cyc}))$ is finite. Furthermore, if $\chi_t(\Gamma, \mathrm{Sel}_{p^\infty}(E/K_{cyc}))$ is finite, we have*

$$\chi_t(G, \mathrm{Sel}_{p^\infty}(E/k_\infty)) = \chi_t(\Gamma, \mathrm{Sel}_{p^\infty}(E/K_{cyc})) \times \prod_{\mathfrak{M}} |L_v(E, 1)|_p$$

where $\mathfrak{M}$ is defined in Theorem 4.1.

Remarks 4.11. As mentioned above we do not have to assume the finiteness of $\mathrm{Sel}_{p^\infty}(E/k)$ here. A formula for $\chi_t(\Gamma, \mathrm{Sel}_{p^\infty}(E/K_{cyc}))$ was obtained by Schneider [S] and Perrin-Riou [P2] involving p-adic heights and the constant $\rho_p(E/k)$. Thus, if we assume k contains μ_p ($k = K$), then we have another proof of Theorem 4.1. In fact, in this case, if we assume $\mathrm{Sel}_{p^\infty}(E/k)$ is finite then the assumption (iii) of Theorem 4.10 is true. Furthermore, we can prove $H^i(G, \mathrm{Sel}_{p^\infty}(E/k_\infty))$ is finite for $i = 0, 1$ and $H^2(G, \mathrm{Sel}_{p^\infty}(E/k_\infty)) = 0$. Thus we obtain the Theorem 4.1 as a corollary of Theorem 4.10 by using the formula for $\chi(\Gamma, \mathrm{Sel}_{p^\infty}(E/K_{cyc})) = \chi_t(\Gamma, \mathrm{Sel}_{p^\infty}(E/K_{cyc}))$.

Proof. The proof goes exactly similar to Theorem 3.1 of [CSS2]. Thus we give only a sketch. First, we see that

$$H^1(\Gamma, \mathrm{Sel}_{p^\infty}(E/k_\infty)^H) \xrightarrow{\sim} H^1(G, \mathrm{Sel}_{p^\infty}(E/k_\infty))$$

since $H^1(H, \mathrm{Sel}_{p^\infty}(E/k_\infty)) = 0$ by the assumption (iii) which is proved similarly as Lemma 2.5 of [CSS2]. Thus we have $\chi_t(G, \mathrm{Sel}_{p^\infty}(E/k_\infty)) = q(\psi)$ where

$$\psi : H^0(\Gamma, \mathrm{Sel}_{p^\infty}(E/k_\infty)^H) \to H^1(\Gamma, \mathrm{Sel}_{p^\infty}(E/k_\infty)^H).$$

Next, we define

$$\mathrm{Sel}'_{p^\infty}(E/K_{cyc}) := \mathrm{Ker}(H^1(k_S/K_{cyc}, E_{p^\infty}) \to \bigoplus_{S \setminus \mathfrak{M}} J_v(K_{cyc})).$$

Then we have

$$0 \to \mathrm{Sel}_{p^\infty}(E/K_{cyc}) \to \mathrm{Sel}'_{p^\infty}(E/K_{cyc}) \to \bigoplus_{\mathfrak{M}} J_v(K_{cyc}) \to 0$$

is exact by the assumption (iii). Thus,

$$\chi_t(\Gamma, \mathrm{Sel}'_{p^\infty}(E/K_{cyc})) = \chi_t(\Gamma, \mathrm{Sel}_{p^\infty}(E/K_{cyc})) \times \prod_{\mathfrak{M}} \chi_t(\Gamma, J_v(K_{cyc}))$$

and $\chi_t(\Gamma, J_v(K_{cyc})) = |L_v(E,1)|_p$ (cf. Lemma 3.4 of [CSS2]). Further, we can see the restriction map

$$\mathrm{res}: \mathrm{Sel}'_{p^\infty}(E/K_{cyc}) \to \mathrm{Sel}_{p^\infty}(E/k_\infty)^H$$

is defined and the kernel and cokernel of this map are finite (cf. Lemma 3.6 of [CSS2], see also Lemma 3.4 in section 3.)
Then, by the commutative diagram induced from the restriction

$$\begin{array}{ccc} H^0(\Gamma, \mathrm{Sel}_{p^\infty}(E/K_{cyc})) & \longrightarrow & H^0(\Gamma, \mathrm{Sel}_{p^\infty}(E/k_\infty)^H) \\ \psi' \downarrow & & \downarrow \psi \\ H^1(\Gamma, \mathrm{Sel}_{p^\infty}(E/K_{cyc})) & \longrightarrow & H^1(\Gamma, \mathrm{Sel}_{p^\infty}(E/k_\infty)^H) \end{array}$$

and Lemma 3.5 of [CSS2], we have $q(\psi) = q(\psi')(= \chi_t(\Gamma, \mathrm{Sel}'_{p^\infty}(E/K_{cyc})))$. Putting all together, we have the Theorem. □

4.6. A condition for triviality. Finally, we consider a question when the Selmer group $\mathrm{Sel}_{p^\infty}(E/k_\infty)$ is trivial. We assume here BASE,

$$k = K(= k(\mu_p)),\ G = G_0.$$

The following is an immediate corollary of Theorem 3.1.

Proposition 4.12. *We have*

$$\mathrm{Sel}_{p^\infty}(E/k_\infty) = 0 \textit{ if and only if } \chi(G, \mathrm{Sel}_{p^\infty}(E/k_\infty)) = 1.$$

Proof. Note that if $\mathrm{Sel}_{p^\infty}(E/k)$ is not finite then $\chi(G, \mathrm{Sel}_{p^\infty}(E/k_\infty))$ is not defined, since $\mathrm{Sel}_{p^\infty}(E/k_\infty)^G$ is not finite. Thus, we can see that $\chi(G, \mathrm{Sel}_{p^\infty}(E/k_\infty)) = 1$ if and only if both

(i) $\mathrm{Sel}_{p^\infty}(E/k)$ is finite and $\rho_p(E/k) = 1$.
(ii) $P_1(k_\infty/k, E) \cup P_2(k_\infty/k, E) = \varnothing$.

holds, since $\rho_p(E/k) \geq 1$ and $|L_v(E,1)|_p > 1$ if $v \in P_1 \cup P_2$. As is well known, (i) is equivalent to $\mathrm{Sel}_{p^\infty}(E/k_{cyc}) = 0$. Assume $\mathrm{Sel}_{p^\infty}(E/k_{cyc}) = 0$ and (ii). Then by the Theorem 3.1, $X_f(k_\infty)$ has rank 0 and is $\Lambda(H)$-torsionfree. Thus $X_f(k_\infty) = 0$. Assume $X_f(k_\infty) = 0$. Then $X_f(k_\infty)^H = 0$. By (3.4), we have $\mathrm{Sel}_{p^\infty}(E/k_{cyc}) = 0$ and (ii). □

Example 4.13. Let $E = X_1(11)$ defined by the equation $y^2 + y = x^3 - x$. Let $p = 5$, $k = \mathbb{Q}(\mu_5)$ and $k_\infty = \mathbb{Q}(\mu_{5^\infty}, \alpha^{5^{-\infty}})$ with $\alpha \in \mathbb{Q}^\times$. This satisfies DIM c) and FIN (subsection 3.3). Since $E(\mathbb{Q})_5 \cong \mathbb{Z}/5$, the condition (ii) in the proof of the Proposition 4.12 holds only when α is some power of ± 5. When $\alpha = (\pm 5)^n$, (i) and (ii) in the proof Proposition 4.12 hold. (For example, it is known that $\mathrm{Sel}_{p^\infty}(E/k_{cyc}) = 0$ by [CS]). Hence we have $\mathrm{Sel}_{p^\infty}(E/k_\infty) = 0$.
We see further structures of $X_f(k_\infty)$ for $\alpha = 11$ in §6.
Another example is $p = 7$ and the curve E defined by $y^2 + xy = x^3 - 141x + 657$ whose conductor is 294. This has good ordinary reduction at $p = 7$ over $k = \mathbb{Q}(\mu_7)$. For $k_\infty = \mathbb{Q}(\mu_{7^\infty}, \alpha^{7^{-\infty}})$ with $\alpha \in \mathbb{Q}^\times$, we see that $\mathrm{Sel}_{p^\infty}(E/k_\infty) = 0$ if and only if α is a power of ± 7 thanks to a result of Fisher ([F1], see also [CS]).

5. μ-invariants

In the GL_2-extension case, Coates and Sujatha (unpublished) and Howson [Ho2, §3] considered the behavior of the μ-invariant for Selmer groups of elliptic curves, hereby generalizing the formulas in the $\mathbb{Z}_p$ -case of Perrin-Riou (cyclotomic case) and Schneider (general case, also for abelian varieties). Under suitable assumptions, see below, analogous statements can be proven in our situation by almost literally the same proof as for [Ho2, thm 3.1, cor. 3.2]. To avoid redundancies in the literature we shall therefore just state the results with some comments and leave the detailed proof to the interested reader.

Assume that k contains μ_p and that k_∞ contains k_{cyc}. Since the Galois group $G = G(k_\infty/k) \cong \mathbb{Z}_p \rtimes \mathbb{Z}_p$ is without p-torsion, the Iwasawa algebras $\Lambda(G)$ and $\Omega(G) := F_p[\![G]\!]$ are both integral. Recall that the μ-invariant of a finitely generated $\Lambda(G)$-module M can be defined as

$$\mu(M) := \sum_{i\geq 0} \mathrm{rk}_{\Omega(G)} {}_{p^{i+1}}M/{}_{p^i}M$$

(cf. [V1]) but can be calculated via the relation

$$p^{\mu(M)} = \chi(G, M(p)),$$

where $M(p)$ denotes the $\mathbb{Z}_p$ -torsion submodule of M (see [Ho2, cor 8]).

Assume $\varphi : E_1 \to E_2$ is an isogeny of the elliptic curves E_1 and E_2 above k and denote by A the p-part of the group scheme $\ker\varphi$. *Throughout this subsection we assume that Assumption* SEQ_S *holds for* E_1 *or* E_2 *(and hence for both) and that Assumption* WL_S *holds for* E_1 *(and hence for* E_2*).*

The above isogeny induces a $\Lambda(G)$-homomorphism

$$\varphi_* : X_{f,2} \to X_{f,1}$$

of the corresponding Pontryagin duals $X_{f,i}$ of the Selmer groups of E_i, $i = 1, 2$.

Theorem 5.1. *Let $p \geq 5$. Then, under the above assumptions, the following holds*

$$\mu(\ker(\varphi_*))-\mu(\mathrm{coker}(\varphi_*))=\sum_{v|\infty}\log_p \#(A(k_v))-|k:\mathbb{Q}|\log_p \#A-\sum_{v|p}\log_p |\#\tilde{A}_v|_v,$$

where v denotes a place of k, $|-|_v$ its absolute value (normalized such that $|p|_v = p^{-[k_v:\mathbb{Q}_p]}$) and $\tilde{A}_v$ denotes the image of A under the reduction map of E_1 at v.

The theorem holds for more general pro-p Lie extensions without p-torsion as long as in addition to Assumption SEQ_S for E_1 or E_2 it holds that

$$H^2(k_S/k_\infty, E_{1,p^\infty}) \text{ is finite}$$

(The corresponding local condition, i.e. the finiteness of $\mathrm{H}^2(k_{\infty,w}, \overline{E}_{p^\infty})$ for all $w|v$, $v \in S_p \cup S_{bad} \cup S_{ram}$ where $\overline{E}$ denotes $\tilde{E}_v$ if $v|p$ and E otherwise, is always satisfied, see [CSS2, §2 (12),(13)]).

For the proof note also that the image of $E_{2,p^\infty}(k_\infty)$ and $\overline{E}_{2,p^\infty}$ in $H^1(k_S/k_\infty, A)$ and $\mathrm{H}^1(k_{\infty,w}, A)$ are always finite, because the cohomology

groups are annihilated by some power of p. Thus their Euler characteristic is 1. Furthermore, it is easy to see that the Euler characteristics $\chi(G, H^i(k_S/k_\infty, A))$ are well-defined for all $i \geq 0$.
By the additivity of the μ-invariant on short exact sequences of *torsion* modules it follows immediately (cf. [Ho2, cor 3.2])

Corollary 5.2. *Suppose, in addition to the assumptions of the theorem, that $X_{f,i}$ is a $\Lambda(G)$-torsion module for $i = 1$ or $i = 0$ (and hence for both). Then the difference between the μ-invariants of $X_{f,2}$ and $X_{f,1}$ is given by the following formula*

$$\mu(X_{f,2}) - \mu(X_{f,1}) = \sum_{v|\infty} \log_p \#(A(k_v)) - |k : \mathbb{Q}| \log_p \#A - \sum_{v|p} \log_p |\#\tilde{A}_v|_v,$$

where the notation is as in the theorem.

We conclude this section studying the relationship between the μ-invariants of the duals of the Selmer group over k_∞ on the one hand and over k_{cyc} on the other hand. In the GL_2-case this was investigated by Coates-Sujatha [CSS2, §2] and we will follow closely their arguments. We assume now that $p \geq 5$ and we keep the assumption BASE and that k_{cyc} is contained in k_∞. As before we set $H := G(k_\infty/k_{cyc})$ and $\Gamma := G(k_{cyc}/k)$. In order to distinguish between the two situations we shall write in the following $\mu_G(M)$ and $\mu_\Gamma(M)$ for the μ-invariant of a finitely generated $\Lambda(G)$- or $\Lambda(\Gamma)$-module M, respectively.

Theorem 5.3. *Let E be an elliptic curve defined over k with good ordinary reduction at S_p and assume that $X_f(k_{cyc})$ is a $\Lambda(\Gamma)$-torsion module. Then one always has $\mu_G(X_f(k_\infty))$ less than or equal to $\mu_\Gamma(X_f(k_{cyc}))$:*

$$\mu_G(X_f(k_\infty)) \leq \mu_\Gamma(X_f(k_{cyc})).$$

Remark 5.4. Assume that E is isogenous over k to an elliptic curve E' such that $\mu_\Gamma(X'_f(k_{cyc})) = 0$ where X'_f denotes the dual of Selmer of E'. Then

$$\mu_G(X_f(k_\infty)) = \mu_\Gamma(X_f(k_{cyc})).$$

Indeed, this follows immediately from the formulae for the change of the μ-invariant under isogeny over both k_∞ and k_{cyc}. More generally, the above equality holds if and only if the quotient $Z := X/T$ of $X := X_f(k_\infty)$ by its $\mathbb{Z}_p$ -torsion submodule $T := X_f(k_\infty)(p)$ is finitely generated over $\Lambda(H)$ (Indeed, we will see in the proof below, that equality is equivalent to the vanishing of $\mu_\Gamma(Z_H)$. Since Z_H is a $\Lambda(\Gamma)$-torsion module this in turn is equivalent to Z_H being a finitely generated $\mathbb{Z}_p$ -module. Now the claim follows from the Nakayama lemma).

Proof. We shall use the notation of the remark. By the analogue of [CSS2, lem. 2.5], we know that $H_1(H, X) = 0$. Since $\mathrm{cd}_p H = 1$, one immediately obtains that also $H_1(H, T) = 0$ and that $H_1(H, Z)$ has no p-torsion, because multiplication by p is injective on Z. But, again as $H_1(H, X) = 0$, we have that

$H_1(H, Z)$ injects into T_H, which is a $\mathbb{Z}_p$-torsion module. Thus we have shown that $H_1(H, Z)$ vanishes, too, and we have the exact sequence

$$0 \longrightarrow T_H \longrightarrow X_H \longrightarrow Z_H \longrightarrow 0$$

of $\Lambda(\Gamma)$-torsion modules. It is plain from this sequence that $\mu_\Gamma(T_H) \leq \mu_\Gamma(X_H)$ (with equality if and only if $\mu_\Gamma(Z_H)$ is zero).
Now we claim (i) that $\mu_\Gamma(T_H) = \mu_G(X)$ and (ii) that $\mu_\Gamma(X_H) = \mu_\Gamma(X_f(k_{cyc}))$. The latter claim is clear because it follows easily from the usual fundamental diagram 4.7 that the kernel and cokernel of the canonical map $X_H \to X_f(k_{cyc})$ are finitely generated over $\mathbb{Z}_p$. To prove (i), we use the fact that for a module which is annihilated by a power of p, the μ-invariant is given by the Euler characteristic (cf. [Ho2, cor. 1.8]). As $H_2(G, X) = 0$ (in theorem 4.2 we state this only under too restrictive assumptions, but use the validity of SEQ_S to derive this from the vanishing of $H_2(G, X_S)$ (corollary 4.6) and of $H_2(G, U_S)$ (proposition 4.7), which both hold in this generality) and as $\mathrm{cd}_p G = 2$, we see that $H_2(G, T) = 0$ and we obtain that

$$p^{\mu_G(X)} = p^{\mu(T)} = \frac{\#H_0(G, T)}{\#H_1(G, T)} = \frac{\#H_0(\Gamma, T_H)}{\#H_1(\Gamma, T_H)} = p^{\mu_\Gamma(T_H)}.$$

The last equality follows from the Hochschild-Serre spectral sequence using again the vanishing of $H_1(H, T)$. Thus the theorem follows. □

6. An example

In this section, we consider the following special example where $p = 5$ as a first case. Let $k = \mathbb{Q}(\mu_5)$ and k_{cyc} be the cyclotomic $\mathbb{Z}_5$-extension of k. Then, we put

$$k_\infty := k_{cyc}(\sqrt[5^\infty]{11}).$$

First, we have the following (cf. Lemma 3.9).

Lemma 6.1. (i) *k_∞ is unramified outside 5 and 11 over $\mathbb{Q}$.*
(ii) *The number of primes of k above 11 is four. They are not decomposed in k_∞/k. Further, they are totally ramified in k_∞/k_{cyc}.*
(iii) *There is a unique prime of k_∞ lying above 5, and it is totally ramified in the extension $k_\infty/\mathbb{Q}$.*

We consider the Selmer group over k_∞ of

$$E = X_1(11) : y^2 + y = x^3 - x^2,$$

the elliptic curve over $\mathbb{Q}$ of conductor 11. In this case, we can determine slightly more precise structure as a module over Iwasawa algebras.

Theorem 6.2. *Let $H = \mathrm{Gal}(k_\infty/k_{cyc})$. Then, the Pontryagin dual of the Selmer group $X_f(k_\infty) := \mathrm{Sel}_{p^\infty}(E/k_\infty)^\vee$ is free of rank four as a $\Lambda(H)$-module.*

It is shown that $\mathrm{Sel}_{p^\infty}(E/k_{cyc}) = 0$ in [CS]. Thus we have $X_f(k_\infty)$ is a submodule of $\Lambda(H)^{\oplus 4}$ whose cokernel is finite by Theorem 3.1 and Lemma 6.1. For $n \geq 1$, let H_n and F_n be as the same as subsection 3.1:

$$F_n := k_{cyc}(\sqrt[5^n]{11}) \text{ and } H_n := \mathrm{Gal}(k_\infty/F_n).$$

Here, we put $F_0 = k_{cyc}$ and $H_0 = H$. For the $\Lambda(H)$-freeness, it suffices to show that $\mathrm{Sel}_{p^\infty}(E/k_\infty)^{H_n}$ is cotorsion-free for any $n \geq 0$ by the structure theory of $\Lambda(H)$-modules. By (3.4) and Lemma 3.6, it is enough to show $\mathrm{Coker}(r'_n)$ is cotorsion-free. Taking $S = \{5, 11\}$ we have

$$H^1(H_n, E(k_\infty)_{5^\infty}) \to \bigoplus_{w|11, w|5} H^1(H_n, E(k_{\infty,w})_{5^\infty}) \to \mathrm{Coker}(r'_n) \to 0, \tag{6.8}$$

from (3.3). For $w|11$, $H^1(H_n, E(k_{\infty,w}))_{5^\infty} \cong \mathbb{Q}_p/\mathbb{Z}_p$ by Lemma 3.4, we have $\mathrm{Coker}(r'_n)$ is cotorsion-free if we show the following.

Lemma 6.3. *Let w be the (unique) prime of k_∞ above 5. Then,*

$$H^1(H_n, E(k_\infty)_{5^\infty}) \to H^1(H_n, E(k_{\infty,w}))_{5^\infty} \tag{6.9}$$

is an isomorphism.

To prove this, we have first

Lemma 6.4. $E(k_\infty)_{5^\infty} = E(\mathbb{Q})_{5^\infty} \cong \mathbb{Z}/5$.

Proof. The field adjoining all of 5-th division points of E is an extension of degree 5 over k. But it is well known that this is disjoint from $k(\sqrt[5]{11})$ and k_{cyc} over k. 5^2-th division points of E are defined over the field containing the maximal real subfield of $\mathbb{Q}(\mu_{11})$, which is not contained in k_∞. Therefore we have $E(k_\infty)_{5^\infty} = E(\mathbb{Q})_{5^\infty}$. □

By this Lemma, we have

$$H^1(H_n, E(k_\infty)_{5^\infty}) = \mathrm{Hom}(H_n, E(k_\infty)_{5^\infty}) \cong \mathbb{Z}/5. \tag{6.10}$$

Let w be the unique prime above 5. Let $\tilde{E}_5$ be the reduction of E modulo 5. Then it is well known that $\tilde{E}_5(\mathbb{F}_5) \cong \mathbb{Z}/5$. Since $k_\infty/\mathbb{Q}$ is totally ramified at 5 by Lemma 6.1, we have

$$\tilde{E}_5(\kappa_{\infty,w}) = \tilde{E}_5(\mathbb{F}_5) \cong \mathbb{Z}/5. \tag{6.11}$$

Further, we have the following.

Lemma 6.5. *The composition of the natural injection $E(k_\infty)_{5^\infty} \hookrightarrow E(k_{\infty,w})_{5^\infty}$ and the reduction map $E(k_{\infty,w})_{5^\infty} \to \tilde{E}_5(\kappa_{\infty,w})_{5^\infty}$ is an isomorphism.*

Proof. It is enough to show the same assertion replacing k_∞ by $\mathbb{Q}_5$ by Lemma 6.1 and (6.11). But this is well known (cf. [CS]). □

Now we can show Lemma 6.3. Since F_n is a deeply ramified extension, we have the following isomorphism by Coates-Greenberg:

$$H^1(H_n, E(k_{\infty,w}))_{5^\infty} \xrightarrow{\sim} H^1(H_n, \tilde{E}_5(\kappa_{\infty,w})_{5^\infty}).$$

By (6.11), $H^1(H_n, \tilde{E}_5(\kappa_{\infty,w})_{5^\infty}) = \mathrm{Hom}(H_n, \tilde{E}_5(\kappa_{\infty,w})_{5^\infty}) \cong \mathbb{Z}/5$. So,

$$H^1(H_n, E(k_\infty)_{5^\infty}) \to H^1(H_n, \tilde{E}_5(\kappa_\infty)_{5^\infty})$$

is an isomorphism by (6.10) and Lemma 6.5. □

The formula of corollary 5.2 enables us to calculate for $p = 5$ the μ-invariant of the elliptic curve $E_2 := X_0(11)$, given by the Weierstrass equation $y^2+y = x^3 - x^2 - 10x - 20$, see [Ho2, ex. in §3] for more details needed for this calculations. There is an isogeny $\varphi : E_1 \to E_2$ with $E_1 := X_1(11)$ and $A \cong \mathbb{Z}/5$. Since $\mu(X_{f,1}(k_\infty)) = 0$ by theorem 6.2, we obtain

$$\mu(X_{f,2}(k_\infty)) = \frac{1}{2}|k : \mathbb{Q}|,$$

where k is a finite extension of $\mathbb{Q}(\mu_5)$ inside $k_\infty = \mathbb{Q}(\mu_{p^\infty}, \sqrt[5^\infty]{11})$.
This result in turn can be used to calculate the μ-invariant of the Galois module

$$X_{cs}^S := G(L/k_\infty),$$

where L denotes the maximal unramified abelian p-extension of k_∞ in which all places lying above S are completely split. For further results on this module we refer the reader to [V3]. Let us now fix $k = \mathbb{Q}(\mu_5)$ and $E = X_0(11)$, i.e. $\mu(X_f) = 2$ by the above formula. Using the fact that $E_5 \cong \mu_5 \times \mathbb{Z}/5$ as $G_{\mathbb{Q}}$-module where $\mu_5 \cong \ker(E_5 \to \tilde{E}_5 \cong \mathbb{Z}/5)$ identifies with the kernel of the reduction map at 5, one easily obtains the following exact sequence of $\Lambda(G)$-modules

$$0 \to X_{cs}^S/5 \to X_f/5 \to X_S/5 \to 0,$$

where $X_S := H^1(G_S(k_\infty), E_{p^\infty})^\vee$ and $^\vee$ means taking the Pontryagin dual. Using the formula [Ho2, cor. 1.11] $\mathrm{rk}_\Omega M/pM = \mathrm{rk}_\Omega({}_pM) + \mathrm{rk}_\Lambda M$ where ${}_pM$ denotes the kernel of multiplication by p on a finitely generated Λ-module M, we conclude

$$\begin{aligned} 2 = \mu(X_f) \geq \mathrm{rk}_\Omega({}_5X_f) &= \mathrm{rk}_\Omega(X_f/5) \\ &= \mathrm{rk}_\Omega(X_S/5) + \mathrm{rk}_\Omega(X_{cs}^S/5) \\ &= \mathrm{rk}_\Lambda(X_S) + \mathrm{rk}_\Omega({}_5X_S) + \mathrm{rk}_\Omega({}_5X_{cs}^S) \\ &= 2 + \mathrm{rk}_\Omega({}_5X_S) + \mathrm{rk}_\Omega({}_5X_{cs}^S). \end{aligned}$$

Here we used that both X_f and X_{cs}^S are Λ-torsion modules and that $\mathrm{rk}_\Lambda(X_S) = 2$ by [OV2, thm 3.2]. Thus $\mathrm{rk}_\Omega({}_5X_S) = \mathrm{rk}_\Omega({}_5X_{cs}^S) = 0$ which implies

$$\mu(X_S) = \mu(X_{cs}^S) = 0$$

by [V1, rem 3.33]. Of course, the same calculation holds over the field $\mathbb{Q}(E_{5^\infty})$ thus showing the vanishing of $\mu(X_{nr}) = \mu(X_{cs}^S) = 0$ where X_{nr} denotes the Galois group of the p-Hilbert class field of $\mathbb{Q}(E_{5^\infty})$. We should point out that the modules X_{nr} and X_{cs}^S are probably pseudo-null, but that the vanishing of the μ-invariants is all we can show at the moment.

At the end of this section, we mention to the further structure of the Selmer group for $p = 5$, $E = X_1(11)$ and $\alpha = 11$. Let $\tilde{G} := \mathrm{Gal}(k_\infty/\mathbb{Q})$. Note that this is not a pro-p group.

Theorem 6.6. *The Pontryagin dual of the Selmer group $X_f(k_\infty)$ is cyclic over $\Lambda(\tilde{G})$.*

Proof. We see that (6.8) for $n = 0$ is an exact sequence of $\Lambda(\tilde{\Gamma})$-modules where $\tilde{\Gamma} = \mathrm{Gal}(k_{cyc}/\mathbb{Q})$. By Lemma 6.3,

$$\mathrm{Coker}(r_0) \cong \bigoplus_{u|11} H^1(H, E(k_{\infty,w})_{5^\infty}) \cong \mathrm{Coind}_\Gamma^{\tilde{\Gamma}}(H^1(H, E(k_{\infty,w})_{5^\infty})).$$

because the decomposition group of 11 in $\tilde{\Gamma}$ is $\Gamma = \mathrm{Gal}(k_{cyc}/k)$. Since we have $H^1(H, E(k_{\infty,w})_{5^\infty}) \cong \mathbb{Q}_p/\mathbb{Z}_p$ for $w|11$, its dual is cyclic over $\Lambda(\Gamma)$. (In fact, $H^1(H, E(k_{\infty,w})_{5^\infty}) \cong \mathbb{Q}_p/\mathbb{Z}_p(-1)$ as a Γ-module, but we omit the proof here.) Because $\mathrm{Sel}_{p^\infty}(E/k_\infty)^H \cong \mathrm{Coker}(r_0)$, $X_f(k_\infty)_H$ is isomorphic to $\Lambda(\tilde{\Gamma}) \otimes_{\Lambda(\Gamma)} H^1(H, E(k_{\infty,w})_{5^\infty})^\vee$, which is a cyclic $\Lambda(\tilde{\Gamma})$-module. Thus, to prove Theorem 6.6, we have only to see the following general Lemma which is an immediate consequence of Nakayama's lemma. □

Lemma 6.7. *Let $\tilde{G}$ be a profinite group which is not necessarily pro-p, and M a compact $\Lambda(G)$-module. Let H be a closed subgroup of $\tilde{G}$ which is a pro-p group. Then, if M_H is a cyclic $\Lambda(\tilde{G}/H)$-module we have M is cyclic over $\Lambda(\tilde{G})$.*

Finally, we propose an interesting question: what is the rank of $E(k_\infty)$? We know nothing about it so far. The only known result is $\mathrm{rank}(E(L)) = 0$ where $L = k(\mu_5, \sqrt[5]{11}) \subset k_\infty$ by Fisher ([F2]). See also Corollary 2.9.

7. Appendix

In this section, we collect some facts used in previous sections and prove them for the sake of completeness.

7.1. Surjectivity of the localization map. We see a relation between the Λ-torsionness of Selmer groups and the Assumptions WL_S and SEQ_S. We prove Theorem 4.5. The proofs are exactly the same as [P1] Lemma 4 and 5. Let F/k be a Galois extension with $G = \mathrm{Gal}(F/k)$. Let E be an elliptic curve defined over k. We analyze the localization map

$$\lambda_F : H^1(k_S/F, E_{p^\infty}) \to \bigoplus_{v\, \in\, S} J_v(F)$$

and $H^2(k_S/F, E_{p^\infty})$ where S is a set of primes of k containing $S_p \cup S_{\mathrm{bad}}$ and all the primes which are ramified in F/k.
First, we define the following module

$$\mathcal{R}_p(E/F) := \varprojlim_{n,M} \mathrm{Sel}_{p^n}(E/M).$$

Here, we denote

$$\mathrm{Sel}_{p^n}(E/M) := \mathrm{Ker}\left(H^1(k_S/M, E_{p^n}) \to \bigoplus_{v\, \in\, S} J_v(M) \right),$$

where M runs over all finite extensions of k contained in F and the limit is taken with respect to the corestrictions and the map induced by multiplication by p-maps, $E_{p^{n+1}} \to E_{p^n}$.

Theorem 7.1. *Assume that G is an infinite pro-p group. Further, assume $E(F)_{p^\infty}$ is finite. Then, there is an injection of $\Lambda(G)$-modules.*

$$\mathcal{R}_p(E/F) \hookrightarrow \mathrm{Hom}_{\Lambda(G)}(\mathrm{Sel}_{p^\infty}(E/F)^\vee, \Lambda(G)). \tag{7.12}$$

Here, $\mathrm{Hom}_{\Lambda(G)}(\mathrm{Sel}_{p^\infty}(E/F)^\vee, \Lambda(G))$ is considered as a left $\Lambda(G)$-module by its right action on $\Lambda(G)$ and the involution $g \to g^{-1}$.

Proof. For a finite subextension M of F/k, there is an exact sequence

$$0 \to E(M)_{p^\infty} \to \varprojlim_n \mathrm{Sel}_{p^n}(E/M) \to T_p(\mathrm{Sel}_{p^\infty}(E/M)) \to 0$$

where $T_p(*)$ is the Tate module of $*$. We note that

$$T_p(\mathrm{Sel}_{p^\infty}(E/M)) \cong \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Sel}_{p^\infty}(E/M)^\vee, \mathbb{Z}_p).$$

So we have the exact sequence by taking the inverse limit with respect to the corestrictions,

$$0 \to \varprojlim_M E(M)_{p^\infty} \to \mathcal{R}_p(E/F) \xrightarrow{\phi} \varprojlim_M \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Sel}_{p^\infty}(E/M)^\vee, \mathbb{Z}_p) \to 0$$

where M runs over all of finite Galois subextensions of F/k. By the assumption that $E(F)_{p^\infty}$ is finite, $\varprojlim_M E(M)_{p^\infty} = 0$ since G is infinite pro-p. So ϕ is an injection.

Next, we consider the restriction map

$$r_M : \mathrm{Sel}_{p^\infty}(E/M) \to \mathrm{Sel}_{p^\infty}(E/F)^{U_M}$$

with $U_M := \mathrm{Gal}(F/M)$. Then we have the following.

$$\begin{aligned} 0 \to \varprojlim_M \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Ker}(r_M)^\vee, \mathbb{Z}_p) \to \varprojlim_M \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Sel}_{p^\infty}(E/M)^\vee, \mathbb{Z}_p) \\ \xrightarrow{\psi} \varprojlim_M \mathrm{Hom}_{\mathbb{Z}_p}((\mathrm{Sel}_{p^\infty}(E/F)^\vee)_{U_M}, \mathbb{Z}_p). \end{aligned}$$

Here the inverse limits are taken w.r.t. the corestrictions for the first two terms. For the last, we take the limit w.r.t. the map induced from the map defined by

$$(\mathrm{Sel}_{p^\infty}(E/F)^\vee)_{U_M} \to (\mathrm{Sel}_{p^\infty}(E/F)^\vee)_{U_{M'}} : \ x \mapsto \sum_{\sigma \,\in\, U_M/U_{M'}} \sigma(x)$$

for $M' \supset M$. Since $\mathrm{Ker}(r_M)$ is contained in $H^1(U_M, E(F)_{p^\infty})$ and $E(F)_{p^\infty}$ is finite, $\mathrm{Ker}(r_M)$ is finite. So we have $\mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Ker}(r_M)^\vee, \mathbb{Z}_p) = 0$ and ψ is an injection.

Finally we see that

$$\mathrm{Hom}_{\mathbb{Z}_p}((\mathrm{Sel}_{p^\infty}(E/F)^\vee)_{U_M}, \mathbb{Z}_p) \cong \mathrm{Hom}_{\Lambda(G)}(\mathrm{Sel}_{p^\infty}(E/F)^\vee, \mathbb{Z}_p[G/U_M])$$

by the map

$$f \mapsto \left(x \in \mathrm{Sel}_{p^\infty}(E/F)^\vee \mapsto \sum_{\sigma \in G/U_M} f(\sigma^{-1}x)\sigma \in \mathbb{Z}_p[G/U_M] \right).$$

Thus we have the isomorphism

$$\varprojlim_M \mathrm{Hom}_{\mathbb{Z}_p}((\mathrm{Sel}_{p^\infty}(E/F)^\vee)_{U_M}, \mathbb{Z}_p) \cong \varprojlim_M \mathrm{Hom}_{\Lambda(G)}(\mathrm{Sel}_{p^\infty}(E/F)^\vee, \mathbb{Z}_p[G/U_M])$$

where the inverse limit of the right hand side is taken w.r.t the natural surjection $\mathbb{Z}_p[G/U_{M'}] \to \mathbb{Z}_p[G/U_M]$ for $M' \supset M$. Therefore,

$$\varprojlim_M \mathrm{Hom}_{\Lambda(G)}(\mathrm{Sel}_{p^\infty}(E/F)^\vee, \mathbb{Z}_p[G/U_M]) \cong \mathrm{Hom}_{\Lambda(G)}(\mathrm{Sel}_{p^\infty}(E/F)^\vee, \Lambda(G))$$

and we see that $\mathcal{R}_p(E/F)$ maps to this module injectively by the map $\psi \circ \phi$. □

As a consequence of this Theorem, we have the following (for odd p).

Theorem 7.2. *Assume G is a pro-p, p-adic Lie group with no p-torsion and $E(F)_{p^\infty}$ is finite. If $\mathrm{Sel}_{p^\infty}(E/F)^\vee$ is $\Lambda(G)$-torsion, then we have*

(i) *$H^2(k_S/F, E_{p^\infty}) = 0$ and*

(ii) *The map $H^1(k_S/F, E_{p^\infty}) \xrightarrow{\lambda_F} \bigoplus_{v \in S} J_v(F)$ is surjective.*

Proof. By the assumption that $\mathrm{Sel}_{p^\infty}(E/F)^\vee$ is $\Lambda(G)$-torsion, we have

$$\mathrm{Hom}_{\Lambda(G)}(\mathrm{Sel}_{p^\infty}(E/F)^\vee, \Lambda(G)) = 0.$$

Thus we have $\mathcal{R}_p(E/F) = 0$ by Theorem 7.1. This proves the Theorem because of the exact sequence

$$0 \to \mathrm{Sel}_{p^\infty}(E/F) \to H^1(k_S/F, E_{p^\infty}) \xrightarrow{\lambda_F} \bigoplus_{v \in S} J_v(F)$$
$$\to \mathcal{R}_p(E/F)^\vee \to H^2(k_S/F, E_{p^\infty}) \to 0.$$

by the Poitou-Tate global duality. □

7.2. Comparison of the Λ-ranks. Let $G \cong H \rtimes \Gamma$ where $H \cong \Gamma \cong \mathbb{Z}_p$. For any $\Lambda(G)$-module M, the H-coinvariants M_H have a structure as a $\Lambda(\Gamma)$-module.

Lemma 7.3. *Let M be a finitely generated $\Lambda(G)$-module. Then,*

$$\mathrm{rank}_{\Lambda(G)} M \leq \mathrm{rank}_{\Lambda(\Gamma)}(M_H).$$

Proof. For these G and H, the following fact is proved in the proof of [BH, last Theorem]: A finitely generated $\Lambda(G)$-module M is $\Lambda(G)$-torsion if M_H is $\Lambda(\Gamma)$-torsion (This fact fails in the GL_2-case in general.) It is easy to see that it is enough to show the Lemma when M is $\Lambda(G)$-torsion free. We use an induction on $n = \mathrm{rank}_{\Lambda(G)} M$. Assume $n = 1$. Then the above fact shows $\mathrm{rank}_{\Lambda(\Gamma)}(M_H) \geq 1$. If $n \geq 2$, then there exists an exact sequence $0 \to N \to M \to L \to 0$ where N and L are torsionfree $\Lambda(G)$-modules with $\mathrm{rank}_{\Lambda(G)} N =$

$n-1$ and $\operatorname{rank}_{\Lambda(G)} L = 1$. Since $L^H = 0$, the sequence $0 \to N_H \to M_H \to L_H \to 0$ is exact. Thus we have the Lemma by induction. □

7.3. Euler-Poincaré formula for Λ-ranks. For the convenience of the reader we include here the well-known determination of the alternating sum of the Λ-ranks of $\mathrm{H}^i(G_S(k_\infty), A)^\vee$ using Tate's global Euler-Poincaré characteristic formula (see also [OV2, thm. 3.2]).
For that purpose let p be any prime, k be a number field (totally imaginary, if $p = 2$), S a finite set of places of k containing S_p and S_∞, k_∞ a non-trivial Galois extension of k contained in k_S such that $G = G(k_\infty/k)$ is a pro-p p-adic Lie group without torsion element. As usual we write $r_1(k)$ and $r_2(k)$ for the number of real and complex places of k, respectively.
Furthermore, we denote by $A \cong (\mathbb{Q}_p/\mathbb{Z}_p)^d$ a discrete p-divisible p-primary $G_S(k)$-module of $\mathbb{Z}_p$-corank d. Then the cohomology groups $\mathrm{H}^i(G_S(k_\infty), A)^\vee$ are finitely generated Λ-modules, where $\Lambda = \Lambda(G)$ denotes the Iwasawa algebra of G. Their ranks are related as follows

Proposition 7.4.

$$\mathrm{rk}_\Lambda \mathrm{H}^1(G_S(k_\infty), A)^\vee - \mathrm{rk}_\Lambda \mathrm{H}^2(G_S(k_\infty), A)^\vee = (r_1(k)+r_2(k))d - \sum_{v \text{ real}} \dim_{\mathbb{F}_p} ({}_pA)^+,$$

where $(-)^+$ denotes the invariant part with respect to the complex conjugation and ${}_pA$ is the kernel of multiplication by p.

Note that $\mathrm{rk}_\Lambda \mathrm{H}^0(G_S(k_\infty), A)^\vee = 0$ because the dual of $A(k_\infty) \subseteq A$ is finitely generated over $\mathbb{Z}_p$.

Proof. Following [Ho2, thm. 1.1] the rank of any finitely generated Λ-module M can be calculated via its homology groups as

$$\mathrm{rk}_\Lambda M = \sum_{j\geq 0} (-1)^j \, \mathrm{rk}_{\mathbb{Z}_p} \mathrm{H}_i(G, M).$$

Using the Hochschild-Serre spectral sequence, the well known behaviour of Euler-characteristics with spectral sequences and the fact that in our situation $\mathrm{cd}_p G_S(k_\infty) \leq \mathrm{cd}_p G_S(k) \leq 2$, we obtain immediately that the term in the proposition of the left hand side is equal to

$$\begin{aligned}
\sum_{i\geq 0} (-1)^{i+1} \mathrm{rk}_\Lambda \mathrm{H}^i(G_S(k_\infty), A)^\vee &= \sum_{i,j\geq 0} (-1)^{i+j+1} \mathrm{rk}_{\mathbb{Z}_p} \mathrm{H}^j(G, \mathrm{H}^i(G_S(k_\infty), A))^\vee \\
&= \sum_{n\geq 0} (-1)^{n+1} \mathrm{rk}_{\mathbb{Z}_p} \mathrm{H}^n(G_S(k), A)^\vee \\
&= \sum_{n\geq 0}^{2} (-1)^{n+1} \dim_{\mathbb{F}_p} \mathrm{H}^n(G_S(k), {}_pA) \\
&= (r_1(k) + r_2(k))d - \sum_{v \text{ real}} \dim_{\mathbb{F}_p} ({}_pA)^+.
\end{aligned}$$

For the last equality we used Tate's global Euler-Poincaré characteristic formula, see e.g. [NSW, 8.6.14]. □

References

[BH] P.N. Balister and S. Howson, *Note on Nakayama's Lemma for Compact Λ-modules*, Asian J. Math. 1 (1997), no. 2, 224–229. 7.2

[B] B. J. Birch, *Cyclotomic fields and Kummer extensions,* Algebraic Number Theory (Proc. Instructional Conf., Brighton,1965), Thompson, Washington, D.C., 1967, pp. 85–93. 3.10, 3.3

[C] J. Coates, *Fragments of the GL_2 Iwasawa Theory of Elliptic curves without complex multiplication,* LNM 1716, Springer (1999), 1–50. 4.3

[CG] J. Coates and R. Greenberg *Kummer theory for abelian varieties over local fields,* Invent. Math. 124 (1996), no. 1-3, 129–174. 2, 3.1, 4.3

[CH] J. Coates and S. Howson, *Euler characteristics and elliptic curves II* J. math. Soc. Japan 53 (2001), 175–235. 1, 3.1, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4

[CM] J. Coates and G. McConnell, *Iwasawa theory of modular elliptic curves of analytic rank at most* 1 J. London. Math. Soc. 50 (1994), 243–264. 4.2, 4.2, 4.3, 4.4

[CSS1] J. Coates, P. Schneider and R. Sujatha, *Modules over Iwasawa algebras*, J. Inst. Math. Jussieu 2 (2003), 73–108. 1, 3.2

[CSS2] ———, *Links between cyclotomic and GL_2 Iwasawa theory*, this volume. 1, 2, 2, 4.5, 4.5, 5, 5, 5

[CS] J. Coates and R. Sujatha, *Galois cohomology of elliptic curves*, TIFR-AMS Lecture Notes, (2000). 4.13, 6, 6

[DSMS] J.D. Dixon, M.P.F. du Sautoy, A. Mann, and D. Segal, *Analytic pro-p groups*, 1st ed., London Mathematical Society Lecture Note, vol. 157, Cambridge University Press, 1991. 2

[F1] T. Fisher, *On* 5 *and* 7 *descents for elliptic curves*, Cambridge PhD Thesis (2000). 4.13

[F2] ———, *Descent calculation for the elliptic curves of conductor* 11, Proc. London Math. Soc. 86 (2003), 583–606. 6

[HM] Y. Hachimori and K. Matsuno: *An analogue of Kida's formula for the Selmer groups of elliptic curves*, J. Algebraic Geom. 8 (1999), 581–601. 3.1, 3.1, 3.1, 3.1, 4.3

[Ha1] M. Harris, *p-adic representations arising from descent on Abelian varieties*, Compos. Math. 39 (1979), 177-245. 2

[Ha2] ———, *The annihilators of p-adic induced modules*, J. Algebra 67 (1980), no. 1, 68–71. 1

[Ha3] ———, *Correction to p-adic representations arising from descent on Abelian varieties.*, Compos. Math. 121 (2000), no. 1, 105–108. 2

[Ho1] S. Howson, *Iwasawa theory of Elliptic Curves for p-adic Lie extensions*, Ph.D. thesis, University of Cambridge, July 1998. 2

[Ho2] ———, *Euler Characteristics as Invariants of Iwasawa Modules*, Proc. London Math. Soc. 85 (2002), 634–658. 5, 5, 5, 6, 7.3

[I] H. Imai, *A remark on the rational points of abelian varieties with values in cyclotomic $\mathbb{Z}_p$-extensions* , Proc. Japan. Acad. 51 (1975), 12–16. 3.1, 3.3

[K] K. Kato, *p-adic Hodge theory and values of zeta functions of modular forms*, to appear. 1, 2.2

[L] M. Lazard, *Groupes analytiques p-adiques*, Publ. Math. I.H.E.S. 26 (1965), 389–603.

[M] K. Matsuno, *Finite Λ-submodules of Selmer groups of abelian varieties over cyclotomic $\mathbb{Z}_p$-extensions* , J. Number Theory 99 (2003), 415–443. 3.1, 3.5

[NSW] J. Neukirch, A. Schmidt, and K. Wingberg, *Cohomology of number fields*, Grundlehren der mathematischen Wissenschaften, vol. 323, Springer, 2000. 2.2, 7.3

[OV1] Y. Ochi and O. Venjakob, *On the structure of Selmer groups over p-adic Lie extensions*, J. Algebraic Geometry 11 (2002), 547-580. 2, (i), 2.7, 2

[OV2] ———, *On the ranks of Iwasawa modules over p-adic Lie extensions*, Math. Proc. Camb. Phil. Soc. 135 (2003) 25–43. 2, 6, 7.3

[P1] B. Perrin-Riou, *Fonctions L p-adiques, theorie d'Iwasawa et points de Heegner* , Bull. Soc. Math. France 115 (1987), 399–456. 7.1

[P2] ———, *Theorie d'Iwasawa et hauteurs p-adiques (cas des varietes abeliennes)*, Progr. Math. 108, Birkhauser (1993), 203–220. 4.11

[P3] ———, *p-adic L-functions and p-adic representations*, American Mathematical Society, Providence, RI, 2000. 2.2

[S] P. Schneider, *p-adic height pairings II.* Invent. Math. 79 (1985), 329–374. 4.11

[V1] O. Venjakob, *On the structure theory of the Iwasawa algebra of a p-adic Lie group*, J. Eur. Math. Soc. 4 (2002), 271-311. 2, 2, 2, 5, 6

[V2] ———, *Iwasawa Theory of p-adic Lie Extensions*, to appear in Compositio Math. 2, 2, 2.5

[V3] ———, *A noncommutative Weierstrass preparation theorem and applications to Iwasawa theory* , J. Reine Angew. Math. 559 (2003) 153–191. 1, 3.2, 3.2, 3.3, 6

Yoshitaka Hachimori
Department of Mathematics
Faculty of Science
Gakushuin University
1-5-1, Mejiro, Toshima-ku
Tokyo 171-8588, Japan
yhachi@math.gakushuin.ac. jp

Otmar Venjakob
Universität Heidelberg
Mathematisches Institut
Im Neuenheimer Feld 288
69120 Heidelberg, Germany
otmar@mathi.uni-heidelberg.de
http://www.mathi.uni-heidelberg.de/~otmar/

Kato Homology of Arithmetic Schemes and Higher Class Field Theory over Local Fields

Dedicated to Kazuya Kato
on the occasion of his 50th birthday

Uwe Jannsen and Shuji Saito

Received: November 19, 2002
Revised: November 10, 2003

Abstract. For arithmetical schemes X, K. Kato introduced certain complexes $C^{r,s}(X)$ of Gersten-Bloch-Ogus type whose components involve Galois cohomology groups of all the residue fields of X. For specific (r,s), he stated some conjectures on their homology generalizing the fundamental isomorphisms and exact sequences for Brauer groups of local and global fields. We prove some of these conjectures in small degrees and give applications to the class field theory of smooth projecive varieties over local fields, and finiteness questions for some motivic cohomology groups over local and global fields.

2000 Mathematics Subject Classification: 11G25, 11G45, 14F42
Keywords and Phrases: Kato homology, Bloch-Ogus theory, niveau spectral sequence, arithmetic homology, higher class field theory

1. Introduction

The following two facts are fundamental in the theory of global and local fields. Let k be a global field, namely either a finite extension of $\mathbb{Q}$ or a function field in one variable over a finite field. Let $\mathbb{P}$ be the set of all places of k, and denote by k_v the completion of k at $v \in \mathbb{P}$. For a field L let $Br(L)$ be its Brauer group, and identify the Galois cohomology group $H^1(L, \mathbb{Q}/\mathbb{Z})$ with the group of the continuous characters on the absolute Galois group of L with values in $\mathbb{Q}/\mathbb{Z}$.

(1-1) For a finite place v, with residue field F_v, there are natural isomorphisms

$$Br(k_v) \xrightarrow{\cong} H^1(F_v, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\cong} \mathbb{Q}/\mathbb{Z},$$

where the first map is the residue map and the second is the evaluation of characters at the Frobenius element. For an archimedean place v there is an injection

$$Br(k_v) \xrightarrow{\cong} H^1(k_v, \mathbb{Q}/\mathbb{Z}) \hookrightarrow \mathbb{Q}/\mathbb{Z}.$$

(1-2) There is an exact sequence

$$0 \longrightarrow Br(k) \xrightarrow{\alpha} \bigoplus_{v \in \mathbb{P}} Br(k_v) \xrightarrow{\beta} \mathbb{Q}/\mathbb{Z} \longrightarrow 0,$$

where α is induced by the restrictions and β is the sum of the maps in (1-1).

In [K1] Kazuya Kato proposed a fascinating framework of conjectures that generalizes the stated facts to higher dimensional arithmetic schemes. In order to review these conjectures, we introduce some notations. For a field L and an integer $n > 0$ define the following Galois cohomology groups: If n is invertible in L, let $H^i(L, \mathbb{Z}/n\mathbb{Z}(j)) = H^i(L, \mu_n^{\otimes j})$ where μ_n is the Galois module of n-th roots of unity. If n is not invertible in L and L is of characteristic $p > 0$, let

$$H^i(L, \mathbb{Z}/n\mathbb{Z}(j)) = H^i(L, \mathbb{Z}/m\mathbb{Z}(j)) \oplus H^{i-j}(L, W_r\Omega^i_{L,log})$$

where $n = mp^r$ with $(p, m) = 1$. Here $W_r\Omega^i_{L,log}$ is the logarithmic part of the de Rham-Witt sheaf $W_r\Omega^i_L$ [Il, I 5.7]. Then one has a canonical identification $H^2(L, \mathbb{Z}/n\mathbb{Z}(1)) = Br(L)[n]$ where $[n]$ denotes the n-torsion part.
For an excellent scheme X and integers $n, r, s > 0$, and under certain assumptions (which are always satisfied in the cases we consider), Kato defined a homological complex $C^{r,s}(X, \mathbb{Z}/n\mathbb{Z})$ of Bloch-Ogus type (cf. [K1], §1):

$$\cdots \bigoplus_{x \in X_i} H^{r+i}(k(x), \mathbb{Z}/n\mathbb{Z}(s+i)) \to \bigoplus_{x \in X_{i-1}} H^{r+i-1}(k(x), \mathbb{Z}/n\mathbb{Z}(s+i-1)) \to \cdots$$
$$\cdots \to \bigoplus_{x \in X_1} H^{r+1}(k(x), \mathbb{Z}/n\mathbb{Z}(s+1)) \to \bigoplus_{x \in X_0} H^{r}(k(x), \mathbb{Z}/n\mathbb{Z}(s)).$$

Here $X_i = \{x \in X | \dim \overline{\{x\}} = i\}$, $k(x)$ denotes the residue field of x, and the term $\bigoplus_{x \in X_i}$ is placed in degree i. The differentials are certain residue maps generalizing the maps

$$Br(k_v)[n] = H^2(k_v, \mathbb{Z}/n\mathbb{Z}(1)) \longrightarrow H^1(F_v, \mathbb{Z}/n\mathbb{Z})$$

alluded to in (1-1). More precisely, they rely on the fact that one has canonical residue maps $H^i(K, \mathbb{Z}/n(j)) \to H^{i-1}(F, \mathbb{Z}/n(j-1))$ for a discrete valuation ring with fraction field K and residue field F.

Definition 1.1 *We define the Kato homology of X with coefficient in $\mathbb{Z}/n\mathbb{Z}$ as*

$$H_i^{r,s}(X, \mathbb{Z}/n\mathbb{Z}) = H_i(C^{r,s}(X, \mathbb{Z}/n\mathbb{Z})).$$

Note that $H_i^{r,s}(X, \mathbb{Z}/n\mathbb{Z}) = 0$ for $i \notin [0, d]$, $d = \dim X$. Kato's conjectures concern the following special values of (r, s).

Definition 1.2 *If X is of finite type over $\mathbb{Z}$, we put*

$$H_i^K(X, \mathbb{Z}/n\mathbb{Z}) = H_i^{1,0}(X, \mathbb{Z}/n\mathbb{Z}).$$

If X is of finite type over a global field or its completion at a place, we put

$$H_i^K(X, \mathbb{Z}/n\mathbb{Z}) = H_i^{2,1}(X, \mathbb{Z}/n\mathbb{Z}).$$

For a prime ℓ we define the Kato homology groups of X with coefficient in $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ as the direct limit of those with coefficient in $\mathbb{Z}/\ell^\nu\mathbb{Z}$ for $\nu > 0$.

The first conjecture of Kato is a generalization of (1-2) ([K1], 0.4).

Conjecture A *Let X be a smooth connected projective variety over a global field k. For $v \in \mathbb{P}$, let $X_v = X \times_k k_v$. Then the restriction maps induce isomorphisms*

$$H_i^K(X, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\cong} \bigoplus_{v \in \mathbb{P}} H_i^K(X_v, \mathbb{Z}/n\mathbb{Z}) \quad \text{for } i > 0,$$

and an exact sequence

$$0 \to H_0^K(X, \mathbb{Z}/n\mathbb{Z}) \to \bigoplus_{v \in \mathbb{P}} H_0^K(X_v, \mathbb{Z}/n\mathbb{Z}) \to \mathbb{Z}/n\mathbb{Z} \to 0.$$

If $\dim(X) = 0$, we may assume $X = \mathrm{Spec}(k)$. Then $H_i(X, \mathbb{Z}/n\mathbb{Z}) = H_i(X_v, \mathbb{Z}/n\mathbb{Z}) = 0$ for $i > 0$, and $H_0(X, \mathbb{Z}/n\mathbb{Z}) = Br(k)[n]$ and $H_0(X_v, \mathbb{Z}/n\mathbb{Z}) = Br(k_v)[n]$. Thus, in this case conjecture A is equivalent to (1-2). In case $\dim(X) = 1$, conjecture A was proved by Kato [K1]. The following is shown in [J4].

Theorem 1.3 *Conjecture A holds if* $\mathrm{ch}(k) = 0$ *and if one replaces the coefficients $\mathbb{Z}/n\mathbb{Z}$ with $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ for any prime ℓ.*

The main objective of this paper is to study the generalization of (1-1) to the higher dimensional case. Let A be a henselian discrete valuation ring with finite residue field F of characteristic p. Let K be the quotient field of A. Let $S = \mathrm{Spec}(A)$ and assume given the diagram

$$\text{(1-3)} \qquad \begin{array}{ccccc} X_\eta & \xrightarrow{j_X} & X & \xleftarrow{i_X} & X_s \\ \downarrow f_\eta & & \downarrow f & & \downarrow f_s \\ \eta & \xrightarrow{j} & S & \xleftarrow{i} & s \end{array}$$

in which s and η are the closed and generic point of S, respectively, the squares are cartesian, and f is flat of finite type. Then Kato defined a canonical *residue map*

$$\Delta_{X,n}^i \;:\; H_i^K(X_\eta, \mathbb{Z}/n\mathbb{Z}) \to H_i^K(X_s, \mathbb{Z}/n\mathbb{Z}),$$

and stated the following second conjecture ([K1], 5.1), which he proved for $\dim X_\eta = 1$.

Conjecture B *If f is proper and X is regular, $\Delta_{X,n}^i$ is an isomorphism for all $n > 0$ and all $i \geq 0$.*

If $X = S$, then $\Delta^0_{X,n}$ is just the map $H^2(K, \mathbb{Z}/n(1)) \to H^1(F, \mathbb{Z}/n)$ in (1-1). In general, conjecture B would allow to compute the Kato homology of X_η by that of the special fiber X_s. Our investigations are also strongly related to Kato's third conjecture ([K1], 0.3 and 0.5):

Conjecture C *Let $\mathcal{X}$ be a connected regular projective scheme of finite type over $\mathbb{Z}$. Then*

$$\widetilde{H}^K_i(\mathcal{X}, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\cong} \begin{cases} 0 & \text{if } i \neq 0, \\ \mathbb{Z}/n\mathbb{Z} & \text{if } i = 0. \end{cases}$$

Here the modified Kato homology $\widetilde{H}^K_i(\mathcal{X}, \mathbb{Z}/n\mathbb{Z})$ is defined as the homology of the modified Kato complex

$$\widetilde{C}^{1,0}(\mathcal{X}, \mathbb{Z}/n\mathbb{Z}) := Cone(\ C^{1,0}(\mathcal{X}, \mathbb{Z}/n\mathbb{Z})[1] \to C^{2,1}(\mathcal{X} \times_{\mathbb{Z}} \mathbb{R}, \mathbb{Z}/n\mathbb{Z})\ ).$$

The map $\widetilde{H}^K_0(\mathcal{X}, \mathbb{Z}/n) \to \mathbb{Z}/n\mathbb{Z}$ is induced by the maps $H^1(k(x), \mathbb{Z}/n\mathbb{Z}) \to \mathbb{Z}/n\mathbb{Z}$ for $x \in \mathcal{X}_0$ given by the evaluation of characters at the Frobenius (note that $k(x)$ is a finite field for $x \in \mathcal{X}_0$), together with the maps $H^2(k(y), \mathbb{Z}/n\mathbb{Z}(1)) = Br(k(y))[n] \hookrightarrow \mathbb{Z}/n\mathbb{Z}$ for $y \in (\mathcal{X} \times_{\mathbb{Z}} \mathbb{R})_0$. The canonical map $\widetilde{H}^K_i(\mathcal{X}, \mathbb{Z}/n\mathbb{Z}) \to H^K_i(\mathcal{X}, \mathbb{Z}/n\mathbb{Z})$ is an isomorphism if $\mathcal{X}(\mathbb{R})$ is empty or if n is odd.

Conjecture C in case $\dim(\mathcal{X}) = 1$ is equivalent to (the n-torsion part of) the classical exact sequence (1-2) for $k = k(\mathcal{X})$, the function field of $\mathcal{X}$. In case $\dim(\mathcal{X}) = 2$ conjecture C is proved in [K1] and [CTSS], as a consequence of the class field theory of $\mathcal{X}$. The other known results concern the case that $\mathcal{X} = Y$ is a smooth projective variety over a finite field F: In [Sa4] it is shown that $H^K_3(Y, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = 0$ if $\ell \neq \mathrm{ch}(F)$ and $\dim(Y) = 3$. This is generalized in [CT] and [Sw] where the $\mathbb{Q}_\ell/\mathbb{Z}_\ell$-coefficient version of Conjecture C in degrees $i \leq 3$ is proved for all primes ℓ and for Y of arbitrary dimension over F.

As we have seen, conjecture C can be regarded as another generalization of (1-2). In fact, conjectures A, B, and C are not unrelated: If $\mathcal{X}$ is flat over $\mathbb{Z}$, it is geometrically connected over $\mathcal{O}_k$, the ring of integers in some number field k. Then the generic fiber $X = \mathcal{X}_k$ is smooth, and we get a commutative diagram with exact rows

$$\text{(1-4)} \qquad \begin{array}{ccccccccc} 0 & \to & \oplus_v C^{1,0}(Y_v) & \to & \oplus_v C^{1,0}(\mathcal{X}_v) & \to & \oplus_v C^{2,1}(X_{k_v})[-1] & \to & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \to & \oplus_v C^{1,0}(Y_v) & \to & C^{1,0}(\mathcal{X}) & \to & C^{2,1}(X)[-1] & \to & 0. \end{array}$$

Here $\mathcal{X}_v = \mathcal{X} \times_{\mathcal{O}_k} \mathcal{O}_v$ for the ring of integers $\mathcal{O}_v$ in k_v, and $Y_v = \mathcal{X} \times_{\mathcal{O}_k} F_v$ is the fiber over v, if v is finite. If v is infinite, we let $Y_v = \emptyset$ and $C^{1,0}(\mathcal{X}_v, \mathbb{Z}/n\mathbb{Z}) = C^{2,1}(X_v, \mathbb{Z}/n\mathbb{Z})[-1]$. Thus conjecture B for $\mathcal{X}_v$ means that $C^{1,0}(\mathcal{X}_v)$ is acyclic for finite v, and any two of the conjectures imply the third one in this case.

On the other hand, conjecture C for a smooth projective variety over a finite field allows to compute the Kato homology of X_s in (1-3), at least in the case of semi-stable reduction: Assume that X is proper over S in (1-3), and that the reduced special fiber $Y = (X_s)_{red}$ is a strict normal crossings variety. In §3 we construct a *configuration map*

$$\gamma^i_{X_s,n} \ : \ H^K_i(X_s, \mathbb{Z}/n\mathbb{Z}) \to H_i(\Gamma_{X_s}, \mathbb{Z}/n\mathbb{Z}).$$

Here Γ_{X_s}, the *configuration (or dual) complex* of X_s, is a simplicial complex whose $(r-1)$-simplices $(r \geq 1)$ are the connected components of

$$Y^{[r]} = \coprod_{1 \leq j_1 < \cdots < j_r \leq N} Y_{j_1} \cap \cdots \cap Y_{j_r},$$

where $Y_1, \ldots, Y_N$ are the irreducible components of Y. This complex has been studied very often in the literature for a curve X/S, in which case Γ_{X_s} is a graph. In case $X = \mathrm{Spec}(\mathcal{O}_K)$, $\gamma^0_{X_s,n}$ is nothing but the map $H^1(F, \mathbb{Z}/n\mathbb{Z}) \to \mathbb{Z}/n\mathbb{Z}$ in (1-1). For a prime ℓ, let

$$\gamma^i_{X_s,\ell^\infty} \ : \ H^K_i(X_s, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to H_i(\Gamma_{X_s}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$$

be the inductive limit of $\gamma^i_{X_s,\ell^\nu}$ for $\nu > 0$. Then we show in 3.9:

Theorem 1.4 *The map $\gamma^j_{X_s,n}$ is an isomorphism if Conjecture C is true in degree i for all $i \leq j$ and for any connected component of $Y^{[r]}$, for all $r \geq 1$. The analogous fact holds with $\mathbb{Q}_\ell/\mathbb{Z}_\ell$-coefficients. In particular, $\gamma^j_{X_s,n}$ is an isomorphism for $j = 0, 1, 2$ and all $n > 0$, and $\gamma^3_{X_s,\ell^\infty}$ is an isomorphism for all primes ℓ.*

Our main results on Conjecture B now are as follows.

Theorem 1.5 *Let n be invertible in K, and assume that X is proper over S.*

(1) *If X_η is connected, one has isomorphisms*

$$H^K_0(X_\eta, \mathbb{Z}/n\mathbb{Z}) \xrightarrow[\sim]{\Delta^0_{X,n}} H^K_0(X_s, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}/n\mathbb{Z}.$$

(2) *If X is regular, $\Delta^1_{X,n}$ is an isomorphism.*

In the proof of Theorem 1.5, given in §5, an important role is played by the class field theory for varieties over local fields developed in [Bl], [Sa1] and [KS1].

In §6 we propose a strategy to show the $\mathbb{Q}_\ell/\mathbb{Z}_\ell$-coefficient version of Conjecture B in degrees ≥ 2 (cf. Proposition 6.4 and the remark at the end of §6) and then show the following result. Fix a prime ℓ different from $\mathrm{ch}(K)$. Passing to the inductive limit, the maps Δ^i_{X,ℓ^ν} induce

$$\Delta^i_{X,\ell^\infty} \ : \ H^K_i(X_\eta, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to H^K_i(X_s, \mathbb{Q}_\ell/\mathbb{Z}_\ell).$$

Theorem 1.6 *Let X be regular, projective over S, and with strict semistable reduction. Then Δ^2_{X,ℓ^∞} is an isomorphism and Δ^3_{X,ℓ^∞} is surjective.*

We note that the combination of Theorems 1.4, 1.5 and 1.6 gives a simple description of $H_i^K(X_\eta, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ for $i \leq 2$, in terms of the configuration complex of X_s.

The method of proof for 1.6 is as follows. In [K1] Kato defined the complexes $C^{r,s}(X, \mathbb{Z}/n)$ and the residue map $\Delta^i_{X,n}$ by using his computations with symbols in the Galois cohomology of discrete valuation fields of mixed characteristic [BK]. To handle these objects more globally and to obtain some compatibilities, we give an alternative definition in terms of a suitable *étale homology theory* , in particular for schemes over discrete valuation rings, in §2.

We will have to use the fact that the complexes defined here, following the method of Bloch and Ogus [BO], agree with the Kato complexes, as defined in [K1], because our constructions rely on the Bloch-Ogus method, while we have to use several results in the literature stated for Kato's definition (although even there the agreement is sometimes used implicitely). For the proof that the complexes agree (up to some signs) we refer the reader to [JSS].

Given this setting, the residue map $\Delta^i_{X,n}$ is then studied in §4 by a square

$$\begin{array}{ccc} H^{et}_{a-2}(X_\eta, \mathbb{Z}/n\mathbb{Z}(-1)) & \xrightarrow{\epsilon_{X_\eta}} & H^K_a(X_\eta, \mathbb{Z}/n\mathbb{Z}) \\ \downarrow \Delta^{et}_X & & \downarrow \Delta^K_X \\ H^{et}_{a-1}(X_s, \mathbb{Z}/n\mathbb{Z}(0)) & \xrightarrow{\epsilon_{X_s}} & H^K_a(X_s, \mathbb{Z}/n\mathbb{Z}), \end{array} \tag{1-5}$$

in which the groups on the left are étale homology groups, and the maps ϵ are constructed by the theory in §2. The shifts of degrees by -2 and -1 correspond to the fact that the cohomological dimensions of K and F are 2 and 1, respectively. If X_η is smooth of pure dimension d, then $H^{et}_{a-2}(X_\eta, \mathbb{Z}/n\mathbb{Z}(-1)) \cong H^{2d-a+2}_{et}(X_\eta, \mathbb{Z}/n\mathbb{Z}(d+1))$, similarly for X_s. But X_s will not in general be smooth, and then étale cohomology does not work. The strategy is to show that Δ^{et}_X and ϵ_{X_s} are bijective and that ϵ_{X_η} is surjective, at least if the coefficients are replaced by $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ (to use weight arguments), and if X is replaced by a suitable "good open" U (to get some vanishing in cohomology).

The proof of the p-part, i.e., for $\mathbb{Q}_p/\mathbb{Z}_p$ with $p = \mathrm{ch}(F)$, depends on two results not published yet. One is the purity for logarithmic de Rham-Witt sheaves stated in formula (4-2) (taken from [JSS]) and in Proposition 4.12 (due to K. Sato [Sat3]). The other is a calculation for p-adic vanishing cycles sheaves, or rather its consequence as stated in Lemma 4.22. It just needs the assumption $p \geq \dim(X_\eta)$ and will be contained in [JS]. If we only use the results from [BK], [H] and [Ts2], we need the condition $p \geq \dim(X_\eta) + 3$, and have to assume $p \geq 5$ in Theorem 1.6.

Combining Theorems 1.5 and 1.6 with Theorem 1.3 one obtains the following result concerning conjecture C (cf. (1-4)).

Theorem 1.7 *Let k be a number field with ring of integers $\mathcal{O}_k$. Let $f : \mathcal{X} \to S$ be a regular proper flat geometrically connected scheme over $S := \mathrm{Spec}(\mathcal{O}_k)$. Assume* *that $\mathcal{X}$ has strict semistable reduction around every closed fiber of f. Then we have*

$$\widetilde{H}_i^K(\mathcal{X}, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\cong} \begin{cases} 0 & \text{if } 1 \le i \le 3, \\ \mathbb{Q}/\mathbb{Z} & \text{if } i = 0. \end{cases}$$

We give an application of the above results to the class field theory of surfaces over local fields. Let K be a non-archimedean local field as in (1-3), and let V be a proper variety over $\eta = \mathrm{Spec}(K)$. Then we have the *reciprocity map for V*

$$\rho_V \ : \ SK_1(V) \to \pi_1^{ab}(V)$$

introduced in the works [Bl], [Sa1] and [KS1]. Here $\pi_1^{ab}(V)$ is the abelian algebraic fundamental group of V and

$$SK_1(V) = \mathrm{Coker}(\bigoplus_{x \in V_1} K_2(y) \xrightarrow{\partial} \bigoplus_{x \in V_0} K_1(x))$$

where $K_q(x)$ denotes the q-th algebraic K-group of $k(x)$, and ∂ is induced by tame symbols. The definition of ρ_V will be recalled in §5. For an integer $n > 0$ prime to $\mathrm{ch}(K)$ let

$$\rho_{V,n} \ : \ SK_1(V)/n \to \pi_1^{ab}(V)/n$$

denote the induced map. There exists the fundamental exact sequence (cf. §5)

(1-6) $$H_2^K(V, \mathbb{Z}/n\mathbb{Z}) \to SK_1(V)/n \xrightarrow{\rho_{V,n}} \pi_1^{ab}(V)/n \to H_1^K(V, \mathbb{Z}/n\mathbb{Z}) \to 0.$$

Combined with 1.5 (2) and 1.4 it describes the cokernel of $\rho_{V,n}$ - which is the quotient $\pi_1^{ab}(V)^{c.d.}$ of the abelianized fundamental group classifying the covers in which every point of V splits completely - in terms of the first configuration homology of the reduction in the case of semi-stable reduction. This generalizes the results for curves in [Sa1]. Moreover, (1-6) immediately implies that $\rho_{V,n}$ is injective if $\dim(V) = 1$, which was proved in [Sa1] assuming furthermore that V is smooth. In general $\mathrm{Ker}(\rho_{V,n})$ is controlled by the Kato homology $H_2^K(V, \mathbb{Z}/n\mathbb{Z})$. Sato [Sat2] constructed an example of a proper smooth surface V over K for which $\rho_{V,n}$ is not injective, which implies that the first map in the above sequence is not trivial in general. The following conjecture plays an important role in controlling $\mathrm{Ker}(\rho_{V,n})$. Let L be a field, and let ℓ be a prime different from $\mathrm{ch}(L)$.

Conjecture $BK_q(L, \ell)$: The group $H^q(L, \mathbb{Q}_\ell/\mathbb{Z}_\ell(q))$ is divisible.

This conjecture is a direct consequence of the Bloch-Kato conjecture asserting the surjectivity of the symbol map $K_q^M(L) \to H^q(L, \mathbb{Z}/\ell\mathbb{Z}(q))$ from Milnor K-theory to Galois cohomology. The above form is weaker if restricted to particular fields L, but known to be equivalent if stated for all fields. By Kummer theory, $BK_1(L, \ell)$ holds for any L and any ℓ. The celebrated work of [MS] shows that $BK_2(L, \ell)$ holds for any L and any ℓ. Voevodsky [V] proved $BK_q(L, 2)$ for any L and any q.

Quite generally, the validity of this conjecture would allow to extend results from $\mathbb{Q}_\ell/\mathbb{Z}_\ell$-coefficients to arbitrary coefficients, by the following result (cf. Lemma 7.3; for extending 1.3 and 1.7 one would need $BK_q(L, \ell)$ over number fields):

Lemma *Let V be of finite type over K, and let ℓ be a prime. Assume that either $\ell = \mathrm{ch}(K)$, or that $BK_{i+1}(K(x), \ell)$ holds for all $x \in V_i$ and $BK_i(K(x), \ell)$ holds for all $x \in V_{i-1}$. Then we have an exact sequence*

$$0 \to H^K_{i+1}(V, \mathbb{Q}_\ell/\mathbb{Z}_\ell)/\ell^\nu \to H^K_i(V, \mathbb{Z}/\ell^\nu) \to H^K_i(V, \mathbb{Q}_\ell/\mathbb{Z}_\ell)[\ell^\nu] \to 0.$$

In §7 we combine this observation with considerations about norm maps, to obtain the following results on surfaces. Let P be a set (either finite or infinite) of rational primes different from $\mathrm{ch}(K)$. Call an abelian group P-divisible if it is ℓ-divisible for all $\ell \in P$.

Theorem 1.8 *Let V be an irreducible, proper and smooth surface over K. Assume $BK_3(K(V), \ell)$ for all $\ell \in P$, where $K(V)$ is the function field of V.*

(1) *Then $\mathrm{Ker}(\rho_V)$ is the direct sum of a finite group and an P-divisible group.*

(2) *Assume further that there exists a finite extension K'/K and an alteration (=surjective, proper, generically finite morphism) $f : W \to V \times_K K'$ such that W has a semistable model X' over A', the ring of integers in K', with $H_2(\Gamma_{X'_{s'}}, \mathbb{Q}) = 0$ for its special fibre $X'_{s'}$. Then $\mathrm{Ker}(\rho_V)$ is P-divisible.*

(3) *If V has good reduction, then the reciprocity map induces isomorphisms*

$$\rho_{V,\ell^\nu} : SK_1(V)/\ell^\nu \xrightarrow{\cong} \pi_1^{ab}(V)/\ell^\nu$$

for all $\ell \in P$ and all $\nu > 0$. In particular, $\mathrm{Ker}(\rho_V)$ is ℓ-divisible.

Theorem 1.9 *Assume that V is an irreducible variety of dimension 2 over a local field K. Assume $BK_3(K(V), \ell)$ for all $\ell \in P$. If V is not proper (resp. proper), $SK_1(V)$ (resp. $\mathrm{Ker}(N_{V/K})$) is the direct sum of a finite group and a P-divisible group. Here $N_{V/K} : SK_1(V) \to K^*$ is the norm map introduced in §6.*

We remark that Theorem 1.8 generalizes [Sa1] where the kernel of the reciprocity map for curves over local fields is shown to be divisible under no assumption. Another remark is that Szamuely [Sz] has studied the reciprocity map for varieties over local fields and its kernel. His results require stronger assumptions than ours while it affirms that the kernel is uniquely divisible. We note however that Sato's example in [Sat2] also implies that the finite group in Theorem 1.8 is non-trivial in general.

The authors dedicate this paper to K. Kato, whose work and ideas have had a great influence on their own research and many areas of research in arithmetic in general. It is also a pleasure to acknowledge valuable help they received from T. Tsuji and K. Sato via discussions and contributions. The first author gratefully acknowledges the hospitality of the Research Institute for Mathematical Sciences at Kyoto and his kind host, Y. Ihara, during 6 months in 1998/1999, which allowed to write a major part of the paper. For the final write-up he profited from the nice working atmosphere at the Isaac Newton Institute for Mathematical Sciences at Cambridge. Finally we thank the referee for some helpful comments.

2. Kato complexes and Bloch-Ogus theory

It is well-known, although not made precise in the literature, that for a smooth variety over a field, one may construct the Kato complexes via the niveau spectral sequence for étale cohomology constructed by Bloch and Ogus [BO]. In this paper we will however need the Kato complexes for singular varieties and for schemes over discrete valuation rings, again not smooth. It was a crucial observation for us that for these one gets similar results by using étale *homology* (whose definition is somewhat subtle for p-coefficients with p not invertible on the scheme). This fits also well with the required functorial behavior of the Kato complexes, which is of 'homological' nature: covariant for proper morphisms, and contravariant for open immersions.
In several instances one could use still use étale *cohomology*, by embedding the schemes into a smooth ambient scheme and taking étale cohomology *with supports* (cf. 2.2 (b) and 2.3 (f)). But then the covariance for arbitrary proper morphisms became rather unnatural, and there were always annoying degree shifts in relation to the Kato homology. Therefore we invite the readers to follow our homological approach.

The following definition formalizes the properties of a homology of Borel-Moore type. It is useful for dealing with étale and Kato homology together, and for separating structural compatibilities from explicit calculations.

A. General results Let $\mathcal{C}$ be a category of noetherian schemes such that for any object X in $\mathcal{C}$, every closed immersion $i : Y \hookrightarrow X$ and every open immersion $j : V \hookrightarrow X$ is (a morphism) in $\mathcal{C}$.

Definition 2.1 (a) Let $\mathcal{C}_*$ be the category with the same objects as $\mathcal{C}$, but where morphisms are just the proper maps in $\mathcal{C}$. A homology theory on $\mathcal{C}$ is a sequence of covariant functors

$$H_a(-): \ \mathcal{C}_* \to (\text{abelian groups}) \qquad (a \in \mathbb{Z})$$

satisfying the following conditions:

(i) For each open immersion $j : V \hookrightarrow X$ in $\mathcal{C}$, there is a map $j^* : H_a(X) \to H_a(V)$, associated to j in a functorial way.

(ii) If $i : Y \hookrightarrow X$ is a closed immersion in X, with open complement $j : V \hookrightarrow X$, there is a long exact sequence (called localization sequence)

$$\cdots \xrightarrow{\delta} H_a(Y) \xrightarrow{i_*} H_a(X) \xrightarrow{j^*} H_a(V) \xrightarrow{\delta} H_{a-1}(Y) \longrightarrow \ldots .$$

(The maps δ are called the connecting morphisms.) This sequence is functorial with respect to proper maps or open immersions, in an obvious way.

(b) A morphism between homology theories H and H' is a morphism $\phi : H \to H'$ of functors on $\mathcal{C}_*$, which is compatible with the long exact sequences from (ii).

Before we go on, we note the two examples we need.

Examples 2.2 (a) This is the basic example. Let S be a noetherian scheme, and let $\mathcal{C} = Sch_{sft}/S$ be the category of schemes which are separated and of finite type over S. Let $\Lambda = \Lambda_S \in D^b(S_{\acute{e}t})$ be a bounded complex of étale sheaves on S. Then one gets a homology theory $H = H^\Lambda$ on $\mathcal{C}$ by defining

$$H_a^\Lambda(X) := H_a(X/S;\Lambda) := H^{-a}(X_{\acute{e}t}, Rf^!\Lambda)$$

for a scheme $f : X \to S$ in Sch_{sft}/S (which may be called the étale homology of X over (or relative) S with values in Λ). Here $Rf^!$ is the right adjoint of $Rf_!$ defined in [SGA 4.3, XVIII, 3.1.4]. For a proper morphism $g : Y \to X$ between schemes $f_Y : Y \to S$ and $f_X : X \to S$ in Sch_{sft}/S, the trace (=adjunction) morphism $tr : g_*Rg^! \to id$ induces a morphism

$$R(f_Y)_*Rf_Y^!\Lambda = R(f_X)_*Rg_*Rg^!Rf_X^!\Lambda \xrightarrow{tr} R(f_X)_*Rf_X^!\Lambda$$

which gives the covariant functoriality, and the contravariant functoriality for open immersions is given by restriction. The long exact localization sequence 1.1 (ii) comes from the exact triangle

$$i_*Rf_Y^!\Lambda = i_*Ri^!Rf_X^!\Lambda \to Rf_X^!\Lambda \to Rj_*j^*Rf_X^!\Lambda = Rj_*Rf_V^!\Lambda \to .$$

(b) Sometimes (but not always) it suffices to consider the following more down to earth version (avoiding the use of homology and Grothendieck-Verdier duality). Let X be a fixed noetherian scheme, and let $\mathcal{C} = Sub(X)$ be the category of subschemes of X, regarded as schemes over X (Note that this implies that there is at most one morphism between two objects). Let $\Lambda = \Lambda_X$ be an étale sheaf (resp. a bounded below complex of étale sheaves on X). Then one gets a homology theory $H = H^{\Lambda_X}$ on $Sub(X)$ by defining

$$H_a^\Lambda(Z) := H_a(Z/X;\Lambda) := H_Z^{-a}(U_{\acute{e}t}, \Lambda|U),$$

as the étale cohomology (resp. hypercohomology) with supports in Z, where U is any open subscheme of X containing Z as a closed subscheme. For the proper morphisms in $Sub(X)$, which are the inclusions $Z' \hookrightarrow Z$, the covariantly associated maps are the canonical maps $H_{Z'}^{-a}(U_{\acute{e}t}, \Lambda|U) \to H_Z^{-a}(U_{\acute{e}t}, \Lambda|U)$. The contravariant functoriality for open subschemes is given by the obvious restriction maps. We may extend everything to the equivalent category $Im(X)$ of immersions $S \hookrightarrow X$, regarded as schemes over X, and we will identify $Sub(X)$ and $Im(X)$.

Remarks 2.3 (a) For any homology theory H and any integer N, we get a shifted homology theory $H[N]$ defined by setting $H[N]_a(Z) = H_{a+N}(Z)$ and multiplying the connecting morphisms by $(-1)^N$.
(b) If H is a homology theory on $\mathcal{C}$, then for any scheme X in $\mathcal{C}$ the restriction of H to the subcategory $\mathcal{C}/X$ of schemes over X is again a homology theory.

(c) Let H be a homology theory on $\mathcal{C}/X$), and let $Z \hookrightarrow X$ be an immersion. Then the groups

$$H_a^{(Z)}(T) := H_a(T \times_X Z)$$

again define a homology theory on $\mathcal{C}/X$. For an open immersion $j : U \hookrightarrow X$ (resp. closed immersion $i : Z \hookrightarrow X$) one has an obvious morphism of homology theories $j^* : H \to H^{(U)}$ (resp. $i_* : H^{(Y)} \to H$).
(d) In the situation of 2.2 (a), let $X \in Ob(Sch_{sft}/S)$. Then by functoriality of $f \rightsquigarrow Rf^!$ the restriction of H^{Λ_S} to $\mathcal{C}/X = Sch_{sft}/X$ can be identified with H^{Λ_X} for $\Lambda_X = Rf_X^! \Lambda_S$.
(e) Since for a subscheme $ji : Z \overset{i}{\hookrightarrow} U \overset{j}{\hookrightarrow} X$, with i closed and j open immersion, we have

$$H_Z^{-a}(U_{\acute{e}t}, \Lambda_X|U) = H^{-a}(Z_{\acute{e}t}, Ri^!(j^*\Lambda_X) = H^{-a}(Z_{\acute{e}t}, R(ji)^!\Lambda_X)$$

the notation $H_a(Z/X; \Lambda_X)$ has the same meaning in 2.2 (b) as in 2.2 (a), and the restriction of H^{Λ_S} to $Sub(X)$ coincides with H^{Λ_X} from 2.2 (b).
(f) If moreover $f_X : X \to S$ is smooth of pure dimension d and $\Lambda_S = \mathbb{Z}/n(b)$ for integers n and b with n invertible on S, then by purity we have $Rf_X^! \mathbb{Z}/n(b) = \mathbb{Z}/n(b+d)[2d]$, so that H^{Λ_S} restricted to $Sub(X)$ is $H^{\Lambda_X}[2d]$ for $\Lambda_X = \mathbb{Z}/n(b+d)$.
(g) In the situation of 2.2 (a), any morphism $\psi : \Lambda_S \to \Lambda'_S$ in $D^b(S_{\acute{e}t})$ induces a morphism between the associated homology theories. Similarly for 2.2 (b) and a morphism $\psi : \Lambda_X \to \Lambda'_X$ of (complexes of) sheaves on X.

The axioms in 2.1 already imply the following property, which is known for example 2.2.
Let $Y, Z \subset X$ be a closed subschemes with open complement $U, V \subset X$, respectively. Then we get an infinite diagram of localization sequences

$$\begin{array}{ccccccccccc}
\dots & H_{a-1}(Y\cap Z) & \to & H_{a-1}(Z) & \to & H_{a-1}(U\cap Z) & \overset{\delta}{\to} & H_{a-2}(Y\cap Z) & \dots \\
 & \uparrow\delta & & \uparrow\delta & & \uparrow\delta & (-) & \uparrow\delta & \\
\dots & H_a(Y\cap V) & \to & H_a(V) & \to & H_a(U\cap V) & \overset{\delta}{\to} & H_{a-1}(Y\cap V) & \dots \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\dots & H_a(Y) & \to & H_a(X) & \to & H_a(U) & \overset{\delta}{\to} & H_{a-1}(Y) & \dots \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\dots & H_a(Y\cap Z) & \to & H_a(Z) & \to & H_a(U\cap Z) & \overset{\delta}{\to} & H_{a-1}(Y\cap Z) & \dots
\end{array}$$

Lemma 2.4 *The above diagram is commutative, except for the squares marked* (−)*, which anticommute.*

Proof For all squares except for the one with the four δ's, the commutativity follows from the functoriality in 2.1 (ii), so it only remains to consider that square marked (-). Since $(Y \cup Z) \setminus (Y \cap Z)$ is the disjoint union of $Y \setminus Z = Y \cap V$ and $Z \setminus Y = U \cap Z$, from 2.1 (ii) we have an isomorphism

$$H_{a-1}((Y \cup Z) \setminus (Y \cap Z)) \cong H_{a-1}(U \cap Z) \oplus H_{a-1}(Y \cap V)$$

and a commutative diagram from the respective localization sequences

$$\begin{array}{ccccc} H_{a-1}(X) & \to & H_{a-1}(X \setminus (Y \cap Z)) & & \\ \uparrow & & \uparrow & & \\ H_{a-1}(Y \cup Z) & \to & H_{a-1}(U \cap Z) \oplus H_{a-1}(Y \cap V) & \stackrel{\delta+\delta}{\to} & H_{a-2}(Y \cap Z) \\ \uparrow & & \uparrow (\delta, \delta) & & \\ H_a(X \setminus (Y \cup Z)) & = & H_a(U \cap V) & & . \end{array}$$

As indicated, the connecting morphisms are given by the product $\alpha = (\delta, \delta)$ and the sum $\beta = \delta + \delta$, respectively, of the connecting morphisms from the square marked (-), as one can see by applying the functoriality 2.1 (ii). Now the diagram implies that the composition $\beta \circ \alpha$ is zero, hence the claim.

Corollary 2.5 *The maps* $\delta : H_a(T \times_X U) \to H_{a-1}(T \times_X Y)$, *for* $T \in \mathcal{C}/X$, *define a morphism of homology theories* $\delta : H^{(U)}[1] \to H^{(Y)}$.

We shall also need the following Mayer-Vietoris property.

Lemma 2.6 *Let* $X = X_1 \cup X_2$ *be the union of two closed subschemes* $i_\nu : X_\nu \hookrightarrow X$, *and let* $k_\nu : X_1 \cap X_2 \hookrightarrow X_\nu$ *be the closed immersions of the (scheme-theoretic) intersection. Then there is a long exact Mayer-Vietoris sequence*

$$\to H_a(X_1 \cap X_2) \stackrel{(k_{1*}, -k_{2*})}{\longrightarrow} H_a(X_1) \oplus H_a(X_2) \stackrel{i_{1*}+i_{2*}}{\longrightarrow} H_a(X) \stackrel{\delta}{\to} H_{a-1}(X_1 \cap X_2) \to .$$

This sequence is functorial with respect to proper maps, localization sequences and morphisms of homology theories, in the obvious way.

Proof The exact sequence is induced in a standard way (via the snake lemma) from the commutative ladder of localization sequences

$$\begin{array}{cccccccccc} .. & H_a(X_2) & \stackrel{i_{2*}}{\to} & H_a(X) & \to & H_a(X \setminus X_2) & \to & H_{a-1}(X_2) & .. \\ & \uparrow k_{2*} & & \uparrow i_{1*} & & \| & & \uparrow & \\ .. & H_a(X_1 \cap X_2) & \stackrel{k_{1*}}{\to} & H_a(X_1) & \to & H_a(X_1 \setminus X_1 \cap X_2) & \to & H_{a-1}(X_1 \cap X_2) & .. \end{array}$$

The functorialities are clear from the functoriality of this diagram.

Now we come to the main object of this chapter. As in [BO] one proves the existence of the following *niveau spectral sequence*, by using the niveau filtration on the homology and the method of exact couples.

Proposition 2.7 *If H is a homology theory on $\mathcal{C}$, then, for every $X \in Ob(\mathcal{C})$, there is a spectral sequence of homological type*

$$E^1_{r,q}(X) = \bigoplus_{x \in X_r} H_{r+q}(x) \quad \Rightarrow \quad H_{r+q}(X).$$

Here $X_r = \{x \in X \mid \dim x = r\}$ and

$$H_a(x) = \lim_{\rightarrow} H_a(V)$$

for $x \in X$, where the limit is over all open non-empty subschemes $V \subseteq \overline{\{x\}}$. This spectral sequence is covariant with respect to proper morphisms in $\mathcal{C}$ and contravariant with respect to open immersions.

Remarks 2.8 (a) Since we shall partially need it, we briefly recall the construction of this spectral sequence. As in [BO], for any scheme $T \in \mathcal{C}$ let $\mathcal{Z}_r = \mathcal{Z}_r(T)$ be the set of closed subsets $Z \subset T$ of dimension $\leq r$, ordered by inclusion, and let $\mathcal{Z}_r/\mathcal{Z}_{r-1}(T)$ be the set of pairs $(Z, Z') \in \mathcal{Z}_r \times \mathcal{Z}_{r-1}$ with $Z' \subset Z$, again ordered by inclusion. For every $(Z, Z') \in \mathcal{Z}_r/\mathcal{Z}_{r-1}(X)$, one then has an exact localization sequence

$$\ldots \to H_a(Z') \to H_a(Z) \to H_a(Z \setminus Z') \xrightarrow{\delta} H_{a-1}(Z') \to \ldots,$$

and the limit of these, taken over $\mathcal{Z}_r/\mathcal{Z}_{r-1}(X)$, defines an exact sequence denoted

$$\ldots H_a(\mathcal{Z}_{r-1}(X)) \to H_a(\mathcal{Z}_r(X)) \to H_a(\mathcal{Z}_r/\mathcal{Z}_{r-1}(X)) \xrightarrow{\delta} H_{a-1}(\mathcal{Z}_{r-1}(X)) \ldots.$$

The collection of these sequences for all r, together with the fact that one has $H_*(\mathcal{Z}_r(X)) = 0$ for $r < 0$ and $H_*(\mathcal{Z}_r(X)) = H_*(X)$ for $r \geq \dim X$, gives the spectral sequence in a standard way, e.g., by exact couples. Here

$$E^1_{r,q}(X) = H_{r+q}(\mathcal{Z}_r/\mathcal{Z}_{r-1}(X)) = \bigoplus_{x \in X_r} H_{r+q}(x).$$

The differentials are easily described, e.g., in the same way as in [J3] for a filtered complex (by renumbering from cohomology to homology). In particular, the E^1-differentials are the compositions

$$H_{r+q}(\mathcal{Z}_r/\mathcal{Z}_{r-1}(X)) \xrightarrow{\delta} H_{r+q-1}(\mathcal{Z}_{r-1}(X)) \to H_{r+q-1}(\mathcal{Z}_{r-1}/\mathcal{Z}_{r-2}(X)).$$

Moreover the 'edge isomorphisms' $E^\infty_{r,q} \cong E^{r,q}_{r+q}$ are induced by

$$H_{r+q}(\mathcal{Z}_r/\mathcal{Z}_{r-1}(X) \leftarrow H_{r+q}(\mathcal{Z}_r(X)) \to H_{r+q}(\mathcal{Z}_\infty(X)) = H_{r+q}(X)$$

(b) This shows that the differential

$$d^1_{r,q} : \bigoplus_{x \in X_r} H_{r+q}(x) \quad \to \quad \bigoplus_{x \in X_{r-1}} H_{r+q-1}(x)$$

has the following description. For $x \in X_r$ and $y \in X_{r-1}$ define

$$\delta_X^{loc}\{x,y\} := \delta_{X,a}^{loc}\{x,y\} : H_a(x) \to H_{a+1}(y)$$

as the map induced by the connecting maps $H_a(V \setminus \overline{\{y\}}) \xrightarrow{\delta} H_{a-1}(V \cap \overline{\{y\}})$ from 2.1 (ii), for all open $V \subset \overline{\{x\}}$. Then the components of $d^1_{r,q}$ are the $\delta^{loc}_{X,r+q}\{x,y\}$. Note that $\delta_X^{loc}\{x,y\} = 0$ if y is not contained in $\overline{\{x\}}$.
(c) Every morphism $\phi : H \to H'$ between homology theories induces a morphism between the associated niveau spectral sequences.

We note some general results for fields and discrete valuation rings.

Proposition 2.9 *Let $S = Spec(F)$ for a field F, let X be separated and of finite type over F, and let H be a homology theory on $Sub(X)$. If $i : Y \hookrightarrow X$ is a closed subscheme and $j : U = X \setminus Y \hookrightarrow X$ is the open complement, then the following holds.*
(a) For all r, q the sequence

$$0 \to E^1_{r,q}(Y) \xrightarrow{i_*} E^1_{r,q}(X) \xrightarrow{j^*} E^1_{r,q}(U) \to 0$$

is exact.
(b) The connecting morphisms $\delta : H_a(Z \cap U) \to H_{a-1}(Z \cap Y)$, for $T \in Sub(X)$, induce a morphism of spectral sequences

$$\delta : E^1_{r,q}(U)^{(-)} \longrightarrow E^1_{r-1,q}(Y),$$

where the superscript $^{(-)}$ means that all differentials in the original spectral sequence (but not the edge isomorphisms $E^\infty_{r,q} \cong E^{r,q}_{r+q}$) are multiplied by -1.

Proof (a): One has always $X_r \cap Y = Y_r$, and since X is of finite type over a field, we also have $X_r \cap U = U_r$.
(b): This morphism is induced by the morphism of homology theories $\delta : H^{(U)}[1] \to H^{(Y)}$ and the construction of the spectral sequences, noting the following. For a closed subset $Z \subset U$ let $\overline{Z}$ be the closure in X and $\delta(Z) = \overline{Z} \cap Y$. For $(Z, Z') \in \mathcal{Z}_r/\mathcal{Z}_{r-1}(U)$ one then has $(\delta(Z), \delta(Z')) \in \mathcal{Z}_{r-1}/\mathcal{Z}_{r-2}(Y)$, and a commutative diagram via localization sequences

$$\begin{array}{ccccccccc}
.. & H_a(\overline{Z'}) & \to & H_a(\overline{Z}) & \to & H_a(\overline{Z} \setminus \overline{Z'}) & \xrightarrow{\delta} & H_{a-1}(\overline{Z'}) & .. \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
.. & H_a(\delta(Z')) & \to & H_a(\delta(Z)) & \to & H_a(\delta(Z) \setminus \delta(Z')) & \xrightarrow{\delta} & H_{a-1}(\delta(Z')) & .. \\
 & \uparrow \delta & & \uparrow \delta & & \uparrow \delta & & \uparrow \delta & \\
.. & H_{a+1}(Z') & \to & H_{a+1}(Z) & \to & H_{a+1}(Z \setminus Z') & \xrightarrow{-\delta} & H_a(Z') & ..
\end{array}$$

This shows that one gets a map of the exact couples defining the spectral sequences and hence of the spectral sequences themselves, with the claimed shift and change of signs. Note that every differential in the spectral sequence involves a connecting morphism once, whereas the edge isomorphisms do not involve any connecting morphism; this gives the signs in $E^{(-)}$.

Corollary 2.10 *Let $\mathcal{C}$ be a subcategory of $Sch_{sft}/Spec(F)$. For every fixed q, the family of functors $(E^2_{r,q})_{r\in\mathbb{Z}}$ defines a homology theory on $\mathcal{C}$.*

Proof The functoriality for proper morphisms and immersions comes from that of the spectral sequence noted in 2.8 (c). Moreover, in the situation of 2.9, we get an exact sequence of complexes

$$0 \to E^1_{\bullet,q}(Y) \to E^1_{\bullet,q}(X) \to E^1_{\bullet,q}(U) \to 0,$$

whose associated long exact cohomology sequence is the needed long exact sequence

$$\ldots \to E^2_{r,q}(Y) \to E^2_{r,q}(X) \to E^2_{r,q}(U) \xrightarrow{\delta} E^2_{r-1,q}(Y) \to \ldots .$$

Its functoriality for proper morphisms and open immersions comes from the functoriality of the mentioned exact sequence of complexes.

Remark 2.11 By the construction in 2.9 (b), the components of the maps δ on E^1-level,

$$\delta : E^1_{r,q}(U) = \bigoplus_{x\in U_r} H_{r+q}(x) \to \bigoplus_{y\in Y_{r-1}} H_{r+q}(y) = E^1_{r-1,q}(Y)$$

are the maps $\delta^{loc}_X\{x,y\}$. This also shows that the associated maps on the E^2-level coincide with the connecting morphisms in 2.10.

We now turn to discrete valuation rings.

Proposition 2.12 *Let $S = Spec(A)$ for a discrete valuation ring A, let X be separated of finite type over S, and let H be a homology theory on $Sub(X)$. Let η and s be the generic and closed point of S, respectively, and write $Z_\eta = Z\times_S \eta$ and $Z_s = Z\times_S s$ for any $Z \in Ob(Sub(X))$.*
(a) The connecting morphisms $\delta : H_a(Z_\eta) \to H_{a-1}(Z_s)$ induce a morphism of spectral sequences

$$\Delta_X : E^1_{r,q}(X_\eta)^{(-)} \ \to\ E^1_{r,q-1}(X_s),$$

where the superscript ${}^{(-)}$ has the same meaning as in 2.9. This morphism is functorial with respect to closed and open immersions, so that one gets commutative diagrams

$$\begin{array}{ccccccccc} 0 & \to & E^1_{r,q}(Y_\eta)^{(-)} & \to & E^1_{r,q}(X_\eta)^{(-)} & \to & E^1_{r,q}(U_\eta)^{(-)} & \to & 0 \\ & & \downarrow \Delta_Y & & \downarrow \Delta_X & & \downarrow \Delta_U & & \\ 0 & \to & E^1_{r,q}(Y_s) & \to & E^1_{r,q}(X_s) & \to & E^1_{r,q}(U_s) & \to & 0 \end{array}$$

for every closed subscheme Y in X, with open complement U.
(b) If X is proper over S, the open immersion $j : X_\eta \to X$ induces a morphism of spectral sequences

$$j^* : E^1_{r,q}(X) \ \to\ E^1_{r-1,q}(X_\eta)$$

such that

$$0 \to E^1_{r,q}(X_s) \xrightarrow{i_*} E^1_{r,q}(X) \xrightarrow{j^*} E^1_{r-1,q}(X_\eta) \to 0$$

is exact for all r and q, where $i : X_s \hookrightarrow X$ is the closed immersion of the special fiber X_s into X.

Proof. (a): As in 2.9 (b), this morphism is induced by the morphism of homology theories $\delta : H^{(X_\eta)}[1] \to H^{(X_s)}$ and the construction of the spectral sequences, noting the following in the present case: For $Z \in \mathcal{Z}_r(X_\eta)$ one now has $\delta(Z) = \overline{Z} \cap X_s \in \mathcal{Z}_r(X_s)$, where $\overline{Z}$ denotes the closure of Z in X.
(b): If $X \to S$ is proper, then $X_r \cap X_\eta = (X_\eta)_{r-1}$.

Remarks 2.13 (a) Proposition 2.12 (b) will in general be false if X is not proper over S, because $X_r \cap X_\eta$ will in general be different from $(X_\eta)_{r-1}$ (e.g., for $X = \operatorname{Spec}(K)$).
(b) By definition of Δ_X, the components of the map on E^1-level,

$$\Delta_X : \bigoplus_{x \in (X_\eta)_r} H_{r+q}(x) \longrightarrow \bigoplus_{x \in (X_s)_r} H_{r+q+1}(x)$$

are the maps $\delta^{loc}_X\{x, y\}$.

We study now two important special cases of Example 2.2 (a) (resp.(b)).

B. Étale homology over fields Let $S = \operatorname{Spec}(F)$ for a field F, and fix integers n and b. We consider two cases.

(i) n is invertible in F, and b is arbitrary.

(ii) F is a perfect field of characteristic $p > 0$, and $n = p^m$ for a positive integer m. Then we only consider the case $b = 0$.

We consider the homology theory

$$H_a(X/F, \mathbb{Z}/n(b)) := H_a(X/S; \mathbb{Z}/n(-b)) = H^{-a}(X_{\acute{e}t}, Rf^!\mathbb{Z}/n(-b))$$

(for $f : X \to S$) of 2.2 (a) on Sch_{sft}/S associated to the following complex of étale sheaves $\mathbb{Z}/n(-b)$ on S. In case (i) we take the usual $(-b)$-fold Tate twist of the constant sheaf $\mathbb{Z}/n$ and get the homology theory considered by Bloch and Ogus in [BO]. In case (ii) we define the complex of étale sheaves

$$\mathbb{Z}/p^m(i) := \mathbb{Z}/p^m(i)_T := W_m\Omega^i_{T,log}[-i],$$

for every T of finite type over F and every non-negative integer i, so that

$$H_a(X/F, \mathbb{Z}/n(b)) = H^{-a+b}(X, Rf^!W_m\Omega^{-b}_{F,log}).$$

Here $W_m\Omega^i_{T,log}$ is the logarithmic de Rham-Witt sheaf defined in [Il]. Note that $\mathbb{Z}/n(0)$ is just the constant sheaf $\mathbb{Z}/n$, and that $W_m\Omega^i_{F,log}$ is not defined for $i < 0$ and 0 for $i > 0$ (That is why we just consider $b = 0$ in case (ii)).

The niveau spectral sequence 2.7 associated to our étale homology is

$$E^1_{r,q}(X/F, \mathbb{Z}/n(b)) = \bigoplus_{x\in X_r} H_{r+q}(x/F, \mathbb{Z}/n(b)) \Rightarrow H_{r+q}(X/F, \mathbb{Z}/n(b)).$$

Theorem 2.14 *Let X be separated and of finite type over F.*
(a) There are canonical isomorphisms

$$H_a(x/F, \mathbb{Z}/n(b)) \cong H^{2r-a}(k(x), \mathbb{Z}/n(r-b)) \quad \textit{for} \;\; x \in X_r.$$

(b) If the cohomological ℓ-dimension $cd_\ell(F) \le c$ for all primes ℓ dividing n, then one has $E^1_{r,q}(X/F, \mathbb{Z}/n(b)) = 0$ for all $q < -c$, and, in particular, canonical edge morphisms

$$\epsilon(X/F) : H_{a-c}(X/F, \mathbb{Z}/n(b)) \longrightarrow E^2_{a,-c}(X/F, \mathbb{Z}/n(b)).$$

(c) If X is smooth of pure dimension d over F, then there are canonical isomorphisms

$$H_a(X/F, \mathbb{Z}/n(b)) \cong H^{2d-a}(X_{\acute{e}t}, \mathbb{Z}/n(d-b)).$$

Proof (c): If $f : X \to \mathrm{Spec}(F)$ is smooth of pure dimension d, then one has a canonical isomorphism of sheaves

$$\alpha_X : Rf^!\mathbb{Z}/n(-b)_S \cong \mathbb{Z}/n(d-b)[2d], \tag{2-1}$$

and (c) follows by taking the cohomology. In case (i) the isomorphism α_X is the Poincaré duality proved in [SGA 4.3, XVIII, 3.2.5]. In case (ii) it amounts to a purity isomorphism $Rf^!\mathbb{Z}/p^m \cong W_m\Omega^d_{X,log}[d]$ which is proved in [JSS].
Independently of [JSS] we note the following. In the case of a finite field F (which suffices for the later applications) we may deduce (c) in case (ii) from results of Moser [Mo] as follows. By [Mo] we have a canonical isomorphism of finite groups

$$Ext^i(\mathcal{F}, W_m\Omega^d_{X,log}) \cong H^{d+1-i}_c(X, \mathcal{F})^\vee,$$

for any constructible $\mathbb{Z}/p^m$-sheaf $\mathcal{F}$ on X. Here $M^\vee = Hom(M, \mathbb{Z}/p^m)$ for a $\mathbb{Z}/p^m$-module M. Applying this to $\mathcal{F} = \mathbb{Z}/p^m$, we get an isomorphism

$$H^i(X, W_m\Omega^d_{X,log}) \cong H^{d+1-i}_c(X, \mathbb{Z}/p^m)^\vee. \tag{2-2}$$

On the other hand, by combining Artin-Verdier duality [SGA 4.3, XVIII, 3.1.4]) and duality for Galois cohomology over F (cf. also 5.3 (2) below), one gets a canonical isomorphism of finite groups

$$H_j(X, \mathbb{Z}/p^m(0))) = H^{-j}(X, Rf^!\mathbb{Z}/p^m) \cong H^{1+j}_c(X, \mathbb{Z}/p^m)^\vee. \tag{2-3}$$

Putting together (2-2) and (2-3) we obtain (c):

$$H^{2d-a}(X,\mathbb{Z}/p^m(d)) \stackrel{def}{=} H^{d-a}(X,W_m\Omega^d_{X,log}) \cong H_a(X,\mathbb{Z}/p^m(0))).$$

(a): By topological invariance of étale cohomology we may assume that F is perfect also in case (i). Then every point $x \in X_r$ has an open neighbourhood $V \subset \overline{\{x\}}$ which is smooth of dimension r over F. Thus (a) follows from (c) and the compatibility of étale cohomology with limits.
(b): If $x \in X_r$, then $k(x)$ is of transcendence degree r over F, and hence $cd_\ell(F) \le c$ implies $cd_\ell(k(x)) \le c + r$. Hence in case (i) $H_{r+q}(x/F,\mathbb{Z}/n(b)) = H^{r-q}(k(x),\mathbb{Z}/n(r-b)) = 0$ for $r - q > c + r$, i.e., $q < -c$. In case (ii), since $cd_p(L) \le 1$ for every field of characteristic $p > 0$, we have $H_{r+q}(x/F,\mathbb{Z}/p^m(0)) = H^{-q}(k(x),W_m\Omega^r_{log}) = 0$ for $-q > 1$, which shows the claim unless $cd_p(F) = 0$. In this case we may assume that F is algebraically closed, by a usual norm argument, because every algebraic extension of F has degree prime to p [Se, I 3.3 Cor. 2]. Then $H^i(k(x),W_m\Omega^r_{log}) = 0$ for $i > 0$ by a result of Suwa ([Sw, Lem. 2.1], cf. the proof of Theorem 3.5 (a) below), because $k(x)$ is the limit of smooth affine F-algebras by perfectness of F.

We shall need the following result from [JSS].

Lemma 2.15 *Via the isomorphisms 2.14 (a), the homological complex*

$E^1_{\bullet,q}(X/F,\mathbb{Z}/n(b))$:

$$\dots \bigoplus_{x\in X_r} H_{r+q}(x/F,\mathbb{Z}/n(b)) \to \bigoplus_{x\in X_{r-1}} H_{r+q-1}(x/F,\mathbb{Z}/n(b)) \dots$$

$$\dots \to \bigoplus_{x\in X_0} H_q(x/F,\mathbb{Z}/n(b))$$

(with the last term placed in degree zero) coincides with the Kato complex

$C_n^{-q,-b}(X)$:

$$\dots \bigoplus_{x\in X_r} H^{r-q}(k(x),\mathbb{Z}/n(r-b)) \to \bigoplus_{x\in X_{r-1}} H^{r-q-1}(k(x),\mathbb{Z}/n(r-b-1)) \dots$$

$$\dots \to \bigoplus_{x\in X_0} H^{-q}(k(x),\mathbb{Z}/n(-b))$$

up to signs.

We also note the following functoriality.

Lemma 2.16 *The edge morphisms ϵ from 2.14 (b) define a morphism of homology theories on Sch_{sft}/F*

$$\epsilon : H_{\bullet-c}(-/F,\mathbb{Z}/n(b)) \longrightarrow E^2_{\bullet,-c}(-/F,\mathbb{Z}/n(b)) \stackrel{2.15}{=} H_\bullet(C_n^{-c,-b}(-))$$

Proof Note that the target is a homology theory by 2.10. The functoriality for proper morphisms and open immersions is clear from the functoriality of the niveau spectral sequence. The compatibility with the connecting morphisms of localization sequences follows from 2.8 (b) and remark 2.11.

C. Étale homology over discrete valuation rings Let $S = \operatorname{Spec} A$ for a discrete valuation ring A with residue field F and fraction field K. Let $j : \eta = \operatorname{Spec}(K) \hookrightarrow S$ be the open immersion of the generic point, and let $i : s = Spec(F) \hookrightarrow S$ be the closed immersion of the special point. Let n and b be integers. We consider two cases:

(i) n is invertible on S and b is arbitrary.

(ii) K is a field of characteristic 0, F is a perfect field of characteristic $p > 0$, $n = p^m$ for a positive integer m, and $b = -1$.

We consider the homology theory

$$H_a(X/S, \mathbb{Z}/n(b)) := H_a(X/S; \mathbb{Z}/n(-b)_S) = H^{-a}(X_{\acute{e}t}, Rf^!\mathbb{Z}/n(-b)_S)$$

(for $f : X \to S$) of 2.2 (a) on Sch_{sft}/S associated to the complex $\mathbb{Z}/n(-b)_S \in D^b(S_{\acute{e}t})$ defined below. The associated niveau spectral sequence is

$$E^1_{p,q}(X/S, \mathbb{Z}/n(b)) = \bigoplus_{x \in X_p} H_{p+q}(x/S, \mathbb{Z}/n(b)) \Rightarrow H_{p+q}(X/S, \mathbb{Z}/n(b)).$$

In case (i), $\mathbb{Z}/n(-b)_S$ is the usual Tate twist of the constant sheaf $\mathbb{Z}/n$ on S. In case (ii) it is the complex of étale sheaves on S

$$\mathbb{Z}/n(1)_S := Cone(Rj_*(\mathbb{Z}/n(1))_\eta \xrightarrow{\sigma} i_*(\mathbb{Z}/n)_s[-1])[-1]$$

considered in [JSS].

For the convenience of the reader, we add some explanation. By definition, $(\mathbb{Z}/n)_s$ is the constant sheaf with value $\mathbb{Z}/n$ on s, and $(\mathbb{Z}/n(1))_\eta$ is the locally constant sheaf $\mathbb{Z}/n(1) = \mu_n$ of n-th roots of unity on η. Note that n is invertible on η. The complex $Rj_*(\mathbb{Z}/n(1))_\eta$ is concentrated in degrees 0 and 1: Pulling back by j^* one gets $(\mathbb{Z}/n(1))_\eta$, concentrated in degree zero, and pulling back by i^* the stalk of the i-th cohomology sheaf is $H^i(K_{sh}, \mu_n)$, where K_{sh} is the strict Henselization of K. Since K_{sh} has cohomological dimension at most 1, the claim follows. Given this, and adjunction for i, the morphism σ is determined by a map $i^*R^1j_*(\mathbb{Z}/n(1))_\eta \to (\mathbb{Z}/n)_s$. Since sheaves on s are determined by their stalks as Galois modules, it suffices to describe the map on stalks

$$H^1(K_{sh}, \mu_n) = K_{sh}^\times/(K_{sh}^\times)^n \longrightarrow \mathbb{Z}/n$$

which we take to be the map induced by the normalized valuation.
We remark that $(\mathbb{Z}/n(1))_S$ is well-defined up to unique isomorphism, although forming a cone is not in general a well-defined operation in the derived category. But in our case, the source A of σ is concentrated in degrees 0 and 1, and the target B is concentrated in degree 1, so that $Hom(A[1], B) = 0$ in the derived category, and we can apply [BBD, 1.1.10].

Let X be separated of finite type over S, and use the notations s, η, X_s and X_η from Proposition 2.12.

Lemma 2.17 *There are isomorphisms of spectral sequences*

$$\begin{aligned} E^1_{r,q}(X_\eta/S,\mathbb{Z}/n(b)) &\cong E^1_{r,q}(X_\eta/\eta,\mathbb{Z}/n(b)) \\ E^1_{r,q}(X_s/S,\mathbb{Z}/n(b)) &\cong E^1_{r,q+2}(X_s/s,\mathbb{Z}/n(b+1)). \end{aligned}$$

Proof. One has canonical isomorphisms

$$\begin{aligned} j^*\mathbb{Z}/n(-b)_S &\cong \mathbb{Z}/n(-b)_\eta \\ Ri^!\mathbb{Z}/n(-b)_S &\cong \mathbb{Z}/n(-b-1)_s[-2]. \end{aligned}$$

This is clear for j^*. For $i^!$ it is the purity for discrete valuation rings [SGA 5, I,5.1] in case (i), and follows from the definition of $\mathbb{Z}/n(1)_S$ in case (ii). Thus the claim follows from remarks 2.3 (d) and (g), which imply isomorphisms of homology theories on Sch_{sft}/η and Sch_{sft}/s, respectively.

$$\begin{aligned} H_a(X_\eta/S;\mathbb{Z}/n(-b)) &\cong H_a(X_\eta/\eta;\mathbb{Z}/n(-b)) \\ H_a(X_s/S;\mathbb{Z}/n(-b)) &\cong H_{a+2}(X_s/s;\mathbb{Z}/n(-b-1)). \end{aligned}$$

Definition 2.18 Define the *residue morphism*

$$\Delta_X : C_n^{-a,-b}(X_\eta)^{(-)} \to C_n^{-a-1,-b-1}(X_s)$$

between the Kato complexes by the commutative diagram

$$\begin{array}{ccc} C_n^{-a,-b}(X_\eta)^{(-)} & \xrightarrow{\Delta_X} & C_n^{-a-1,-b-1}(X_s) \\ \|\wr\ 2.15 & & \|\wr\ 2.15 \\ E^1_{\bullet,a}(X_\eta/\eta,\mathbb{Z}/n(b))^{(-)} & & E^1_{\bullet,a+1}(X_s/s,\mathbb{Z}/n(b+1)) \\ \|\wr\ 2.17 & & \|\wr\ 2.17 \\ E^1_{\bullet,a}(X_\eta/S,\mathbb{Z}/n(b))^{(-)} & \underset{2.12(a)}{\xrightarrow{\Delta_X}} & E^1_{\bullet,a-1}(X_s/S,\mathbb{Z}/n(b)) \end{array}$$

Remark 2.19 By 2.12 (a), the residue map is compatible with restrictions for open immersions and push-forwards for closed immersion. Thus, if Y is closed in X, with open complement $U = X \setminus Y$, we get a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & C_n^{-a,-b}(Y_\eta) & \to & C_n^{-a,-b}(X_\eta) & \to & C_n^{-a,-b}(U_\eta) & \to & 0 \\ & & \downarrow \Delta_Y & & \downarrow \Delta_X & & \downarrow \Delta_U & & \\ 0 & \to & C^{-a-1,-b-1}(Y_s) & \to & C_n^{-a-1,-b-1}(X_s) & \to & C^{-a-1,-b-1}(U_s) & \to & 0 \end{array}$$

In view of 2.13 (b), the following is proved in [JSS].

Lemma 2.20 *For $x \in (X_\eta)_r$ and $y \in (X_s)_r$ the component*

$$\Delta_X\{x,y\} : H^{r+a+1}(k(x), \mathbb{Z}/n(r-b+1)) \to H^{r+a}(k(y), \mathbb{Z}/n(r-b))$$

of Δ_X coincides with the residue map $\delta_X^{Kato}\{x,y\}$ used by Kato in the complex $C^{a,-b}(X)$.

This gives the relationship between étale homology and the Kato complexes also in the case of a discrete valuation ring:

Corollary 2.21 *If X is proper over S, then the following holds.*
(a) The residue map Δ_X from 2.18 coincides with the map considered by Kato in Conjecture B (cf. the introduction).
(b) The homological complex

$E^1_{\bullet,q}(X/S, \mathbb{Z}/n(b))$:

$$\ldots \to \bigoplus_{x\in X_r} H_{r+q}(x/S, \mathbb{Z}/n(b)) \to \bigoplus_{x \in X_{r-1}} H_{r+q-1}(x/S, \mathbb{Z}/n(b)) \to \ldots$$

$$\ldots \to \bigoplus_{x\in X_0} H_q(x/S, \mathbb{Z}/n(b))$$

(with the last term placed in degree zero) coincides with the Kato complex

$C_n^{-q-2,-b-1}$:

$$\ldots \bigoplus_{x\in X_r} H^{r-q-2}(k(x), \mathbb{Z}/n(r-b-1)) \to \bigoplus_{x\in X_{r-1}} H^{r-q-3}(k(x), \mathbb{Z}/n(r-b-2)) \ldots$$

$$\ldots \to \bigoplus_{x\in X_0} H^{-q-2}(k(x), \mathbb{Z}/n(-b-1)).$$

Proof. Since $X_r \cap X_s = (X_s)_r$ and $X_r \cap X_\eta = (X_\eta)_{r-1}$, we have

$$H_a(x/S, \mathbb{Z}/n(b)) = H^{2r-a-2}(k(x), \mathbb{Z}/n(r-b-1)) \quad \text{for all} \quad x \in X_r.$$

Hence the components agree in (b). It then follows from 2.13 (b), 2.15 and Kato's definitions that (a) and (b) are equivalent, and that (a) holds by lemma 2.20.

3. Kato complexes and étale homology over finite fields

3.1 In this section, F is a finite field of characteristic $p > 0$, and $n > 0$ is an integer. Let

$$Y \xrightarrow{f} \mathrm{Spec}(F) \quad \rightsquigarrow \quad H_a^{et}(Y/F, \mathbb{Z}/n(0)) := H^{-a}(Y_{\acute{e}t}, Rf^!\mathbb{Z}/n(0))$$

be the étale homology with coefficients $\mathbb{Z}/n(0)$ over F, and let

$$E^1_{r,q}(Y/F, \mathbb{Z}/n(0)) \quad \Rightarrow \quad H^{et}_{r+q}(Y/F, \mathbb{Z}/n(0)).$$

be the associated niveau spectral sequence (cf. 2.A). Since F has cohomological dimension 1, theorems 2.14 (b) and theorem 2.21 give a canonical edge morphism

$$\epsilon_Y : H^{et}_{a-1}(Y/F, \mathbb{Z}/n(0)) \to E^2_{a,-1}(Y/F, \mathbb{Z}/n(0)) \ = \ H_a(C^{1,0}(Y, \mathbb{Z}/n)) \ = \ H^K_a(Y, \mathbb{Z}/n)$$

from étale to Kato homology which we want to study more closely for certain varieties.

For the étale homology we use the Hochschild-Serre spectral sequence, which in our case just becomes the collection of short exact sequences

$$0 \to H^{et}_{a+1}(\overline{Y}/\overline{F}, \mathbb{Z}/n(0))_\Gamma \xrightarrow{\alpha} H^{et}_a(Y/F, \mathbb{Z}/n(0)) \xrightarrow{\beta} H^{et}_a(\overline{Y}/\overline{F}, \mathbb{Z}/n(0))^\Gamma \to 0,$$

where $\Gamma = Gal(\overline{F}/F)$, for an algebraic closure $\overline{F}$ of F, is the absolute Galois group of F, and $\overline{Y} = Y \times_F \overline{F}$. Here we have used that the cohomological dimension of Γ is 1 , and that one has a canonical isomorphisms $H^1(\Gamma, M) \cong M_\Gamma$ for any Γ-module M. If we pass to the inductive limit, we get versions with coefficients $\mathbb{Q}_\ell/\mathbb{Z}_\ell(0)$ or $\mathbb{Q}/\mathbb{Z}(0)$ which we will treat as well. In the following, we shall suppress F and $\overline{F}$ in the notations.

For the Kato homology the following conjecture by Kato, which is a special case of conjecture C in the introduction, will play an important role. Let ℓ be a prime and let ν be a natural number or ∞. If $\nu = \infty$, we define $\mathbb{Z}/\ell^\infty := \mathbb{Q}_\ell/\mathbb{Z}_\ell$.

Conjecture $K(F, \mathbb{Z}/\ell^\nu)$ If X is a connected smooth projective variety over F, then

$$H^K_i(X, \mathbb{Z}/\ell^\nu) \xrightarrow{\cong} \begin{cases} 0 & \text{if } i \neq 0, \\ \mathbb{Z}/\ell^\nu & \text{if } i = 0. \end{cases}$$

3.2 More precisely, we want to study ϵ_Y for strict normal crossings varieties, i.e., reduced separated varieties Y with smooth irreducible components $Y_1, \ldots, Y_N$ intersecting transversally. Let

$$Y_{i_1,\ldots,i_s} := Y_{i_1} \times_Y \ldots \times_Y Y_{i_s}$$

be the scheme-theoretic intersection of $Y_{i_1}, \ldots, Y_{i_s}$, and write

$$Y^{[s]} := \coprod_{1 \le i_1 < \cdots < i_s \le N} Y_{i_1,\ldots,i_s}$$

for the disjoint union of the s-fold intersections of the Y_i, for $s > 0$. By assumption, all $Y^{[s]}$ are smooth. Denote by

$$i^{[s]} : Y^{[s]} \longrightarrow Y$$

the canonical morphism induced by the immersions $Y_{i_1,\ldots,i_s} \hookrightarrow Y$, and let

$$\delta_\nu : Y^{[s]} \longrightarrow Y^{[s-1]} \qquad (\nu = 1, \ldots, s)$$

be the morphism induced by the closed immersions

$$Y_{i_1,\ldots,i_s} \hookrightarrow Y_{i_1,\ldots,\hat{i_\nu},\ldots,i_s}.$$

Definition 3.3 A *good divisor* on Y is a reduced closed subscheme $Z \subset Y$ of pure codimension 1 such that

(a) Z intersects all subschemes $Y_{i_1,\ldots,i_s}$ transversally.

(b) $U = Y \setminus Z$ is affine.

Lemma 3.4 *Let X be a smooth proper variety over a field L, and let D be a smooth divisor on X such that $X \setminus D$ is affine. If X is connected of dimension > 1, then D is connected.*

Proof We may assume that X is geometrically connected over L, and, by base change, that L is algebraically closed. Let $d = \dim X$, and let ℓ be a prime invertible in L. Then we get an exact Gysin sequence

$$0 = H^{2d-1}_{et}(U, \mathbb{Z}/\ell\mathbb{Z}) \to H^{2d-2}_{et}(D, \mathbb{Z}/\ell\mathbb{Z}) \to H^{2d}_{et}(X, \mathbb{Z}/\ell\mathbb{Z}) \to H^{2d}_{et}(U, \mathbb{Z}/\ell\mathbb{Z}) = 0$$

where the vanishing comes from weak Lefschetz (note that $2d - 1 > d = \dim U$). Since $\dim_{\mathbb{Z}/\ell\mathbb{Z}} H^{2d}_{et}(X, \mathbb{Z}/\ell\mathbb{Z})$ is the number of proper connected components of a purely d-dimensional smooth variety X, the claim follows.

If Z is a good divisor, then it is again a strict normal crossing variety. By the lemma, the intersections $Y_i \cap Z$ are the connected components of Z, unless $\dim(Y_i) = 1$. If Y is projective, then a good divisor always exists over any infinite field extension of F by the Bertini theorem. The main result of this section is:

Theorem 3.5 *Let Y be a proper strict normal crossings variety of pure dimension d over F, and let ℓ be a prime number. Let $Z \subset Y$ be a good divisor, and let $U = Y \setminus Z$ be the open complement.*
(a) One has $H^{et}_{a-1}(U, \mathbb{Z}/\ell^\nu(0)) = 0 = H^{et}_a(\overline{U}, \mathbb{Z}/\ell^\nu)$ for $a < d$ and all $\nu \geq 1$.
(b) If the Kato conjecture $K(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ holds in degrees $\leq m$, then the map

$$\epsilon_U : H^{et}_{a-1}(U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(0)) \longrightarrow H_a(C^{1,0}(U, \mathbb{Q}_\ell/\mathbb{Z}_\ell)) = H^K_a(U, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$$

is an isomorphism for all $a \leq min(m, d)$.

Proof We prove the theorem by double induction on the dimension and the number N of components of Y. If dim $Y = 0$, then the claim is trivially true. Now let Y be of pure dimension d, and assume the statements are true in smaller dimension.
(1) If $N = 1$, i.e., Y has only one component, then Y is smooth and proper. Let $Z \subset Y$ be a good divisor. Then $U = Y \setminus Z$ is affine. For $\ell \neq p$ one then has $H_a^{et}(\overline{U}, \mathbb{Z}/\ell^\nu(0)) = H_{et}^{2d-a}(\overline{U}, \mathbb{Z}/\ell^\nu(d)) = 0$ for $a < d$ by weak Lefschetz. For $\ell = p$ one has $H_a^{et}(\overline{U}, \mathbb{Z}/p^r(0)) = H_{et}^{2d-a}(\overline{U}, \mathbb{Z}/p^r(d)) = H^{d-a}(\overline{U}, W_r\Omega^d_{U,log}) = 0$ for $d - a > 0$, i.e., $a < d$ as well, by 2.14 (c) and the weak Lefschetz theorem proved in [Sw, Lemma 2.1]. By the exact sequence

$$0 \to H_a^{et}(\overline{U}, \mathbb{Z}/\ell^\nu(0))_\Gamma \to H_{a-1}^{et}(U, \mathbb{Z}/\ell^\nu(0)) \to H_{a-1}^{et}(\overline{U}, \mathbb{Z}/\ell^\nu(0))^\Gamma \to 0$$

we see that $H_{a-1}^{et}(U, \mathbb{Z}/\ell^\nu(0)) = 0$ for $a < d$ as claimed in (a).
For (b) we may assume that Y is geometrically connected. First let dim $Y = 1$. In this case $H_1^{et}(\overline{Y}, \mathbb{Q}/\mathbb{Z}(0)) = H^1_{et}(\overline{Y}, \mathbb{Q}/\mathbb{Z}(1)) = \mathrm{Tor}(Pic(\overline{Y}))$ is the torsion of the Jacobian of $\overline{Y}$, hence $H_1^{et}(\overline{Y}, \mathbb{Q}/\mathbb{Z}(0))_\Gamma = 0$, by Weil's theorem. The map

$$H_0^{et}(Z, \mathbb{Q}/\mathbb{Z}(0)) \to H_0^{et}(Y, \mathbb{Q}/\mathbb{Z}(0))$$

is therefore identified with the Gysin map

$$H^0_{et}(\overline{Z}, \mathbb{Q}/\mathbb{Z}(0))^\Gamma \to H^2_{et}(\overline{Y}, \mathbb{Q}/\mathbb{Z}(1))^\Gamma \cong \mathbb{Q}/\mathbb{Z},$$

which is surjective: It has a left inverse up to isogeny, and the target is divisible. Hence the upper row in the following commutative diagram of localization sequences is exact

$$\begin{array}{ccccccccc} 0 & \to & H_0^{et}(U, \mathbb{Q}/\mathbb{Z}(0)) & \to & H_{-1}^{et}(Z, \mathbb{Q}/\mathbb{Z}(0)) & \to & H_{-1}^{et}(Y, \mathbb{Q}/\mathbb{Z}(0)) & \to & 0 \\ & & \downarrow \epsilon_U & & \downarrow \epsilon_Z & & \downarrow \epsilon_Y & & \\ 0 & \to & H_1^K(U, \mathbb{Q}/\mathbb{Z}) & \to & H_0^K(Z, \mathbb{Q}/\mathbb{Z}) & \to & H_0^K(Y, \mathbb{Q}/\mathbb{Z}) & \to & 0. \end{array}$$

Note that $H_{-1}^{et}(U, \mathbb{Q}/\mathbb{Z}(0)) = 0$ by the first step. Since the Kato conjecture is known for Z and Y (cf. the introduction), one has $H_1^K(Y, \mathbb{Q}/\mathbb{Z}) = 0$, and an isomorphism $H_0^K(Y, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}$ via the trace map to Spec(F). Therefore ϵ_Z and ϵ_Y are isomorphisms, and we conclude that the bottom sequence is exact (recall that Kato homology gives a homology theory in the sense of 2.1), ϵ_U is an isomorphism, and $H_0^K(U, \mathbb{Q}/\mathbb{Z}) = 0$. This settles the case dim $Y = 1$.
Now assume dim $Y > 1$. The long exact localization sequence for the Kato homology,

$$\dots H_a^K(Y, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to H_a^K(U, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to H_{a-1}^K(Z, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to \dots$$

$$\to H_1^K(U, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to H_0^K(Z, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \xrightarrow{\beta} H_0^K(Y, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to H_0^K(U, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to 0$$

then shows that $H_a^K(U, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = 0$ for all $0 \leq a \leq m$ if the Kato conjecture is true in dimensions $\leq m$ for Y and Z. Note that β is an isomorphism since Z is connected for dim $Y > 1$ by lemma 3.4. So it remains to show that

$H^{et}_{d-1}(U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(0)) = 0$ in the considered case as well. The above weak Lefschetz results imply that

$$H^{et}_d(\overline{U}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(0))_\Gamma \cong H^{et}_{d-1}(U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(0))$$

is divisible, and a quotient of $H^{et}_d(\overline{U}, \mathbb{Q}_\ell(0))_\Gamma$. But the exact sequence

$$H^{et}_d(\overline{Y}, \mathbb{Q}_\ell(0)) \to H^{et}_d(\overline{U}, \mathbb{Q}_\ell(0)) \to H^{et}_{d-1}(\overline{Z}, \mathbb{Q}_\ell(0))$$

shows that the middle group is mixed of weights $-d$ and $-d+1$, since $H^{et}_a(\overline{X}, \mathbb{Q}_\ell(b)) \cong H^{2d-a}_{et}(\overline{X}, \mathbb{Q}_\ell(d-b))$ is pure of weight $b-a$ for a smooth proper variety X by Deligne's proof of the Weil conjecture (for the case $\ell = p$ one uses results of Katz-Messing cf. [G-S]). Hence $H^{et}_d(\overline{U}, \mathbb{Q}_\ell(0))_\Gamma = 0$ for $d > 1$.
(2) Finally we carry out the induction step for the induction on N, the number of components of Y. Let

$$Y' = \bigcup_{i=1}^{N-1} Y_i.$$

Then $Y' \cap Y_N$ is a strict normal crossings variety, and $Z \cap Y'$, $Z \cap Y_N$ and $Z \cap Y' \cap Y_N$ are good divisors on Y', Y_N and $Y' \cap Y_N$, respectively. Write $\overset{\circ}{Y} = U = Y \setminus Z$, $\overset{\circ}{Y}{}' = Y' \setminus Z$ and $\overset{\circ}{Y}_N = Y_N \setminus Z$, and note that $\overset{\circ}{Y}{}' \cap \overset{\circ}{Y}_N = (Y' \cap Y_N) \setminus Z$. By 2.6 and 2.16 we get a commutative diagram of Mayer-Vietoris sequences

$$\begin{array}{ccccccc}
\ldots H^{et}_{a-1}(\overset{\circ}{Y}{}' \cap \overset{\circ}{Y}_N) & \to & H^{et}_{a-1}(\overset{\circ}{Y}{}') \oplus H^{et}_{a-1}(\overset{\circ}{Y}_N) & \to & H^{et}_{a-1}(\overset{\circ}{Y}) & \to & H^{et}_{a-2}(\overset{\circ}{Y}{}' \cap \overset{\circ}{Y}_N) \ldots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\ldots H^K_a(\overset{\circ}{Y}{}' \cap \overset{\circ}{Y}_N) & \to & H^K_a(\overset{\circ}{Y}{}') \oplus H^K_a(\overset{\circ}{Y}_N) & \to & H^K_a(\overset{\circ}{Y}) & \to & H^K_{a-1}(\overset{\circ}{Y}{}' \cap \overset{\circ}{Y}_N) \ldots
\end{array}$$

by taking the maps ϵ as vertical maps. Here we abbreviated $H^{et}_a(-)$ for $H^{et}_a(-, \ell^\nu(0))$ and $H^K_a(-)$ for $H^K_a(-, \ell^\nu)$, respectively, where $\nu \in \mathbb{N} \cup \{\infty\}$ is fixed. By induction on N, (a) holds for $\overset{\circ}{Y}{}'$, $\overset{\circ}{Y}_N$ and $\overset{\circ}{Y}{}' \cap \overset{\circ}{Y}_N$, hence also for $\overset{\circ}{Y} = U$ by the upper row. Now let $\nu = \infty$. By induction the vertical maps are then isomorphisms for $\overset{\circ}{Y}{}'$ and $\overset{\circ}{Y}_N$ if $a \le \min(d, m)$, and for $\overset{\circ}{Y}{}' \cap \overset{\circ}{Y}_N$ if $a \le \min(d-1, m)$, so we conclude by the 5-lemma. In fact, in case $a = d \le m$ note that $H^K_d(\overset{\circ}{Y}{}' \cap \overset{\circ}{Y}_N) = 0$. So even if $H^{\acute{e}t}_{d-1}(\overset{\circ}{Y}{}' \cap \overset{\circ}{Y}_N)$ may be non-zero (because $d > d-1 = \dim Y' \cap Y_N$), the 5-lemma in its stronger form applies.

We now come to the proof of theorem 1.4 in the introduction. We use the following spectral sequence.

Proposition 3.6 *Let Y be a noetherian scheme, let $\mathcal{Y} = (Y_1, \ldots, Y_N)$ be an ordered tuple of closed subschemes with $Y = \bigcup Y_i$, and let the notations be as in 3.2 (although we do not assume any normal crossing condition). Let n, r and b be integers such that the Kato complex $C^{r,b}(Y, \mathbb{Z}/n)$ is defined.*
(a) There is a spectral sequence of homological type

$$E^1_{s,t}(\mathcal{Y}, \mathbb{Z}/n) = H_t(C^{r,b}(Y^{[s+1]}, \mathbb{Z}/n)) \;\Rightarrow\; H_{s+t}(C^{r,b}(Y, \mathbb{Z}/n)),$$

in which the E^1–differential is $d^1_{s,t} = \sum_{\nu=1}^{s+1}(-1)^{\nu+1}(\delta_\nu)_*$, with $(\delta_\nu)_*$ being the homomorphism induced by the map δ_ν from 3.2.
(b) One has $E^1_{s,t}(\mathcal{Y},\mathbb{Z}/n) = 0$ for all $t<0$ and hence canonical edge morphisms

$$e_a^{\mathcal{Y},n}:\ H_a(C^{r,b}(Y,\mathbb{Z}/n)) \to E^2_{a,0}(\mathcal{Y},\mathbb{Z}/n) = H_a(H_0(C^{r,b}(Y^{[\bullet+1]},\mathbb{Z}/n))).$$

(c) Let Y be of finite type over the finite field F. Define the following complex

$$C(\mathcal{Y},\mathbb{Z}/n):\qquad \ldots \to (\mathbb{Z}/n)^{\pi_0(Y^{[s+1]})} \xrightarrow{d_s} (\mathbb{Z}/n)^{\pi_0(Y^{[s]})} \to \ \ldots\ \to (\mathbb{Z}/n)^{\pi_0(Y^{[1]})}.$$

Here $\pi_0(Z)$ is the set of connected components of a scheme Z, the last term of the complex is placed in degree 0, and the differential d_s is $\sum_{\nu=1}^{s+1}(-1)^{\nu+1}(\delta_\nu)_*$, where $(\delta_\nu)_*$ is the obvious homomorphism induced by the map δ_ν from 3.2. If Y is proper, then there is a canonical homomorphism

$$tr:\ E^2_{a,0}(\mathcal{Y}) \longrightarrow H_a(\mathcal{Y},\mathbb{Z}/n) := H_a(C(\mathcal{Y},\mathbb{Z}/n)).$$

If Y is a proper strict normal crossing variety, and $\mathcal{Y} = (Y_1,\ldots,Y_N)$ consists of the irreducible components of Y, then this map is an isomorphism.

Proof (a): Write C for $C^{r,b}$. There is an exact sequence of complexes

$$\ldots \to C(Y^{[s+1]},\mathbb{Z}/n) \xrightarrow{d_s} C(Y^{[s]},\mathbb{Z}/n) \to\ \ldots\ \to C(Y^{[1]},\mathbb{Z}/n) \xrightarrow{\pi_*} C(Y,\mathbb{Z}/n) \to 0,$$

where $d_s = \sum_{\nu=1}^{s+1}(-1)^{\nu+1}(\delta_\nu)_*$, and π_* is induced by the covering map $\pi: Y^{[1]} \to Y$. The exactness is standard. A simple proof can be given by using induction on the number of components and the exact sequence of complexes

$$0 \to C(Y',\mathbb{Z}/n\mathbb{Z}) \to C(Y,\mathbb{Z}/n\mathbb{Z}) \to C(Y_N \setminus Y',\mathbb{Z}/n\mathbb{Z}) \to 0,$$

for Y, Y' and Y_N as in the proof of theorem 3.5.
Then the spectral sequence is induced by the naive filtration of the above sequence of complexes, i.e.,by the second filtration of the double complex $C_\bullet(Y^{[\bullet]},\mathbb{Z}/n\mathbb{Z})$, whose associated total complex is quasiisomorphic to $C(Y,\mathbb{Z}/n\mathbb{Z})$.
(b): The first claim is clear, since the Kato complex $C(-,\mathbb{Z}/n\mathbb{Z})$ is zero in negative degrees, and the second claim follows from this.
(c): If Y is proper, then the covariant functoriality gives a trace map

$$tr: C^{1,0}(Y,\mathbb{Z}/n\mathbb{Z}) \to C^{1,0}(\mathrm{Spec}(F),\mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z},$$

inducing a map $tr: H_0(C^{1,0}(Y,\mathbb{Z}/n\mathbb{Z})) \to \mathbb{Z}/n\mathbb{Z}$. This is functorial in Y, and by restricting to the connected components we get a morphism of complexes

$$E^1_{\bullet,0}(\mathcal{Y},\mathbb{Z}/n) = H_0(C^{1,0}(Y^{[\bullet+1]},\mathbb{Z}/n)) \xrightarrow{tr} (\mathbb{Z}/n)^{\pi_0(Y^{[\bullet+1]})} = C(\mathcal{Y},\mathbb{Z}/n)$$

giving the wanted map tr. It is an isomorphism for strict normal crossings varieties, since $tr: H_0(C^{1,0}(X,\mathbb{Z}/n\mathbb{Z})) \to \mathbb{Z}/n\mathbb{Z}$ is an isomorphism for smooth proper connected X (the known case $i=0$ of Kato's conjecture $K(F,\mathbb{Z}/n)$, cf. [CT], [Sw], or 5.2 below).

As before, we also have versions with coefficients in $\mathbb{Z}/\ell^\infty := \mathbb{Q}_\ell/\mathbb{Z}_\ell$ for a prime number ℓ of the complexes and statements above.

Definition 3.7 Let Y be a proper strict normal crossings variety over F. If $Y_1, \ldots, Y_N$ are the irreducible components of Y, and $\mathcal{Y} = (Y_1, \ldots, Y_N)$, then we define $C(\Gamma_Y, \mathbb{Z}/n\mathbb{Z}) := C(\mathcal{Y}, \mathbb{Z}/n\mathbb{Z})$ and $H_a(\Gamma_Y, \mathbb{Z}/n\mathbb{Z}) := H_a(\mathcal{Y}, \mathbb{Z}/n\mathbb{Z})$ and call it the *configuration chain complex* and *configuration homology* of Y, respectively.

Remark 3.8 If Y is a (proper) strict normal crossings variety, let us define its *configuration (or dual) complex* Γ_Y as the simplicial complex, whose $s-$simplices correspond to the conncted components of $Y^{[s+1]}$, with the face maps given by the δ_ν. Then the group $H_a(\Gamma_Y, \mathbb{Z}/n\mathbb{Z})$, as defined above, does in fact compute the a-th homology of Γ_Y with coefficients in $\mathbb{Z}/n\mathbb{Z}$. If Y has dimension d, then Γ_Y has at most dimension d, and $C_a(\Gamma_Y, \mathbb{Z}/n\mathbb{Z}) = 0$ for $a > d$. If Y is a curve, then Γ_Y is a graph and is also called the intersection graph of Y.

Theorem 3.9 *Let Y be a proper strict normal crossings variety of pure dimension d over F, let ℓ be a prime number, and let ν be a natural number or ∞. Then there is a canonical map*

$$\gamma = \gamma_a^{Y,\ell^\nu} : H_a(Y, \mathbb{Z}/\ell^\nu) \longrightarrow H_a(\Gamma_Y, \mathbb{Z}/\ell^\nu),$$

where we define $\mathbb{Z}/\ell^\infty := \mathbb{Q}_\ell/\mathbb{Z}_\ell$. This map is an isomorphism for all $a \leq m$ if the Kato conjecture $K(F, \mathbb{Z}/\ell^\nu)$ is true in degrees $\leq m$.

Proof We define γ_a^{Y,ℓ^ν} as the composition

$$\gamma_a^{Y,\ell^\nu} : \ H_a(C^{1,0}(Y, \mathbb{Z}/\ell^\nu)) \stackrel{e_a^{\mathcal{Y},\ell^\nu}}{\longrightarrow} E^2_{a,0}(\mathcal{Y}, \mathbb{Z}/n) \stackrel{tr}{\longrightarrow} H_a(C(\Gamma_Y, \mathbb{Z}/\ell^\nu)),$$

where $\mathcal{Y} = (Y_1, \ldots, Y_N)$. By 3.6 (b) , tr is an isomorphism. If the Kato conjecture $K(F, \mathbb{Z}/\ell^\nu)$ holds in degrees $\leq m$, then one has $E^1_{p,q}(\mathcal{Y}, \mathbb{Z}/n) = 0$ for all $q = 1, \ldots, m$, so that $e_a^{\mathcal{Y},\ell^\nu}$ is an isomorphism in degrees $a \leq m$.

Remark 3.10 As the referee points out, we have morphisms of complexes

$$C^{1,0}(Y, \mathbb{Z}/\ell^\nu) \stackrel{quis}{\leftarrow} totC^{1,0}_\bullet(Y^{[\bullet]}, \mathbb{Z}/\ell^\nu) \ \rightarrow \ E^1_{\bullet,0}(\mathcal{Y}, \mathbb{Z}/\ell^\nu) \stackrel{tr}{\rightarrow} C(\Gamma_Y, \mathbb{Z}/\ell^\nu)$$

which induce γ and are quasi-isomorphisms if $K(F, \mathbb{Z}/\ell^\nu)$ holds.

4. The residue map in étale homology

We consider the same situation and notations as in 2.C: Hence we have $S = \mathrm{Spec}(A)$ for a discrete valuation ring A, with generic point $\eta = \mathrm{Spec}(K)$ and special point $s = \mathrm{Spec}(F)$. But in this section we assume that A is henselian.

4.1 Let $f : X \to S$ be a scheme which is separated and flat of finite type over S. We get a diagram with cartesian squares

$$\begin{array}{ccccc} X_\eta & \xrightarrow{j_X} & X & \xleftarrow{i_X} & X_s \\ \downarrow f_\eta & & \downarrow f & & \downarrow f_s \\ \eta & \xrightarrow{j} & S & \xleftarrow{i} & s \end{array}$$

The connecting morphisms $\delta : H_a(X_\eta/S, \mathbb{Z}/n(-1)) \to H_{a-1}(X_s/S, \mathbb{Z}/n(-1))$ in étale homology over S together with lemma 2.17 give residue maps in étale homology

$$\Delta_X : \ H_a(X_\eta, \mathbb{Z}/n(-1)) \longrightarrow H_{a+1}(X_s, \mathbb{Z}/n(0)),$$

which we want to study under suitable conditions. We shall also consider the versions with $\mathbb{Z}/n$ replaced by $\mathbb{Q}/\mathbb{Z}$ or $\mathbb{Q}_\ell/\mathbb{Z}_\ell$. Here we omit K and F in the notation for the homology of schemes over K and F, respectively, as in section 3. For a fixed algebraic closure $\overline{F}$ of F, let $\Gamma = \mathrm{Gal}(\overline{F}/F)$, $\overline{s} = \mathrm{Spec}(\overline{F})$, and $X_{\overline{s}} = X_s \times_s \overline{s}$. Similarly, let $\overline{K}$ be a fixed separable closure of K and write $\overline{\eta} = \mathrm{Spec}(\overline{K})$ and $X_{\overline{\eta}} = X_\eta \times_\eta \overline{\eta}$. We shall also consider the homology groups of $X_{\overline{s}}/\overline{F}$ and $X_{\overline{\eta}}/\overline{K}$, and omit the fields $\overline{F}$ and $\overline{K}$ in the notations as well.

Let $f : X \longrightarrow S$ be regular of pure relative dimension $d \geq 1$ and with strict semi-stable reduction. Hence X_s is a strict normal crossings variety over F.

Definition 4.2 By a *good divisor* on X we mean a divisor $Z \hookrightarrow X$ which is flat over S and for which Z_s is a good divisor in X_s in the sense of 3.3.

Proposition 4.3 *(a) If Z is a good divisor on X, then Z is regular and has strict semi-stable reduction, and is of pure relative dimension $d-1$.*

(b) If X is projective over S and F has infinitely many elements, then there is always a good divisor.

Proof (cf. also [JS]) Let $Y_1, \ldots, Y_M$ be the irreducible components of the special fibre $Y := X_s$. Then all r-fold intersections

$$Y_{i_1,\ldots,i_r} := Y_{i_1} \cap Y_{i_2} \cap \ldots Y_{i_r}$$

$(1 \leq i_1 < i_2 < \ldots < i_r \leq M)$ are smooth. If $X \hookrightarrow \mathbb{P}^N_S$ is a projective embedding and F is infinite, then by the Bertini theorem there is a hyperplane $H_0 \subset \mathbb{P}^N_F = \mathbb{P}^N_S \times_S s$, defined over F, intersecting all (irreducible components of all) varieties

$Y_{i_1,\dots,i_r}$ transversally. Let $H \subset \mathbb{P}^N_S$ be any hyperplane lifting H_0. Then the scheme-theoretic intersection $Z = H \cdot X = H \times_{\mathbb{P}^N_S} X$ is flat over S (the generic fibre is non-empty), and is a good divisor, by definition. It thus suffices to show (a).
We may assume that F is algebraically closed. Let x be a closed point of X_s which is contained in Z. The completion $\hat{\mathcal{O}}_{X,x}$ of the local ring $\mathcal{O}_{X,x}$ is isomorphic to

$$B = A[[x_1,\dots,x_{d+1}]]/\langle x_1\dots x_r - \pi\rangle \qquad (1 \le r \le d+1)$$

where π is a prime element in A. Since X is regular, Z is defined by one local equation at x. Let f be the image of the local equation in B, and let $\mathfrak{n} \subseteq B$ be the maximal ideal. Then $f \in \mathfrak{n}$, and the elements $x_1,\dots,x_r$ are the images of the local equations for $Y_{i_1},\dots,Y_{i_r}$ for suitable $1 \le i_1 < \dots < i_r \le M$. Thus the trace of $Y_{i_1,\dots,i_r}$ in $\hat{\mathcal{O}}_{X,x} \cong B$ corresponds to the quotient

$$B' = B/\langle x_1,\dots,x_r\rangle \cong F[[x_{r+1},\dots,x_{d+1}]] \quad .$$

Since $x \in H$ and, by assumption, H does not intersect the zero-dimensional varieties $Y_{i_1,\dots,i_{d+1}}$, we may assume $r < d+1$. Then H intersects $Y_{i_1,\dots,i_r}$ transversally at x if and only if the image of f in B' lies in $\mathfrak{n}' - (\mathfrak{n}')^2$, where $\mathfrak{n}'$ is the maximal ideal of B'. Since $\mathfrak{n}'/(\mathfrak{n}')^2 \cong \mathfrak{n}/(\mathfrak{n}^2 + \langle x_1,\dots,x_r\rangle)$ we see that f has non-zero image in $\mathfrak{n}/(\mathfrak{n}^2 + \langle x_1,\dots,x_r\rangle)$.
Now the elements $x_i \bmod \mathfrak{n}^2$ $(i = 1,\dots,d+1)$ form an F-basis of $\mathfrak{n}/\mathfrak{n}^2$. Hence we have

$$f \equiv \sum_{i=1}^{d+1} a_i x_i \mod \mathfrak{n}^2$$

with elements $a_i \in A$ which are determined modulo $\langle\pi\rangle$. By our condition on f, a_i must be a unit for one i with $i > r$, and by possibly renumbering and multiplying f by a unit we may assume $i = d+1$, and $a_{d+1} = 1$. But then

$$B/\langle f\rangle \cong A[[x_1,\dots,x_d]]/\langle x_1\dots x_r - \pi\rangle$$

which proves the claim. Note that the irreducible components of $(H\cdot X)_s = H_s \cdot X_s$ are the connected components of the smooth varieties $H_s \cdot Y_i$.

Good divisors Z and "good opens" $U = X \setminus Z$ are useful because of the following:

Theorem 4.4 *Assume that F is a finite field of characteristic p, and that X is proper of pure relative dimension d and has strict semi-stable reduction. Let $Z \hookrightarrow X$ be a good divisor, and let $U = X \setminus Z$ be the open complement.*
(a) One has $H_{a-2}(U_\eta, \mathbb{Z}/n(-1)) = H_{a-1}(U_s, \mathbb{Z}/n(0)) = 0$ if $a < d$.
(b) The map

$$\Delta_U :\ H_{a-2}\big(U_\eta, \mathbb{Z}/n(-1)\big) \to H_{a-1}\big(U_s, \mathbb{Z}/n(0)\big)$$

is an isomorphism for all n prime to p and all $a \le d$.
(c) Assume $\mathrm{ch}(K) = 0$*. The map*

$$\Delta_U :\ H_{d-2}(U_\eta, \mathbb{Q}_p/\mathbb{Z}_p(-1)) \to H_{d-1}(U_s, \mathbb{Q}_p/\mathbb{Z}_p(0))$$

is a surjective isogeny, and it is an isomorphism if $p \ge d$ or if X is smooth over S.

The rest of this section is devoted to the proof of this theorem. Part (a) for U_s was proved in 3.5 (a), and it follows directly from weak Lefschetz for U_η: Since U_η is smooth of dimension d, we have $H_{a-2}(U_\eta, \mathbb{Z}/n(o)) \cong H^{2d-a+2}(U_\eta, \mathbb{Z}/n(d))$, which is zero for $a < d$ by weak Lefschetz, since K has cohomological dimension at most 2. Now we turn to parts (b) and (c).

4.5 From the localization sequence together with Lemma 2.17 we get a long exact sequence

$$\cdots \to H_a(X/S, \mathbb{Z}/n(-1)) \to H_a(X_\eta, \mathbb{Z}/n(-1)) \stackrel{\Delta_X}{\to}$$

$$H_{a+1}\big(X_s, \mathbb{Z}/n(0)\big) \to H_{a-1}(X/S, \mathbb{Z}/n(-1)) \to \dots .$$

This sequence shows that kernel and cokernel of the residue map Δ_X are controlled by the homology groups $H_\bullet(X/S, \mathbb{Z}/n(-1)) = H^{-\bullet}(X,\, Rf^!\, \mathbb{Z}/n(1)_S\,)$. We study these cohomology groups in the following. First we consider the prime-to-p case, where $\mathbb{Z}/n(1)_S$ is the usual sheaf $\mathbb{Z}/n(1) = \mu_n$, the first Tate twist of the constant sheaf $\mathbb{Z}/n$.

Lemma 4.6 *Let n be prime to p, and let b be any integer. There are canonical isomorphisms*

(a) $$Rf^!\mathbb{Z}/n(b) \stackrel{\sim}{\longrightarrow} \mathbb{Z}/n(b+d)[2d]$$

(b) $$H_a(X/S, \mathbb{Z}/n(b)) \stackrel{\sim}{\longrightarrow} H^{2d-a}\big(X, \mathbb{Z}/n(d-b)\big)$$

Proof (b) follows from (a), and (a) is a special case of Grothendieck's purity conjecture for excellent schemes, a proof of which has been announced by Gabber and written down by Fujiwara [Fu].

Lemma 4.7 *For X and U as in 4.4, and n prime to p, one has*
(a) $H^i(X, \mathbb{Z}/n(j)) \cong H^i(X_s, \mathbb{Z}/n(j))$ for all $i, j \in \mathbb{Z}$,
(b) $H^i(U, \mathbb{Z}/n(j)) \cong H^i(U_s, \mathbb{Z}/n(j))$ for all $i, j \in \mathbb{Z}$.

Proof Part (a) follows from the proper base change theorem. For (b) let $g : U \to S$ be the structural morphism, and let $g_s : U_s \to s$ be the base change to $s = \text{Spec } F$. Then it follows from [R-Z, 2.18 and 2.19] that the base change morphism

$$i^* Rg_* \mathbb{Z}/n(b) \to Rg_{s*}\mathbb{Z}/n(b),$$

is an isomorphism, too, for the complement $U = X \setminus Z$ of a good divisor Z, and we obtain (b).

With this, we can now prove Theorem 4.4 (b): Recall that, by assumption, U_s has dimension d and is affine. By weak Lefschetz and the Hochschild-Serre exact sequences

$$0 \to H^{i-1}(U_{\overline{s}}, \mathbb{Z}/n(j))_\Gamma \to H^i(U_s, \mathbb{Z}/n(j)) \to H^i(U_{\overline{s}}, \mathbb{Z}/n(j))^\Gamma \to 0$$

we conclude that $H^i(U_s, \mathbb{Z}/n(j)) = 0$ for $i > d+1$. On the other hand, by 4.5 and 4.6 (b) we have a long exact sequence

$$\begin{array}{llllll} \to & H^{2d-a+2}(U_s, \mathbb{Z}/n(-1)) & \to & H_{a-2}(U_\eta, \mathbb{Z}/n(-1)) & \stackrel{\Delta_U}{\to} & H_{a-1}(U_s, \mathbb{Z}/n(0)) \\ \to & H^{2d-a+3}(U_s, \mathbb{Z}/n(-1)) & \to & & & . \end{array}$$

This evidently shows the claimed bijectivity of Δ_U for all $a \le d$.

4.8 Now we prove Theorem 4.4 (c). We first fix some notation. Let A denote a henselian discrete valuation ring with perfect (not necessarily finite) residue field F of characteristic $p > 0$ and with quotient field K of characteristic 0. Let $S = \mathrm{Spec}(A)$ and assume given a diagram as in 4.1:

$$\begin{array}{ccccc} X_\eta & \stackrel{j_X}{\longrightarrow} & X & \stackrel{i_X}{\longleftarrow} & X_s \\ \downarrow f_\eta & & \downarrow f & & \downarrow f_s \\ \eta & \stackrel{j}{\longrightarrow} & S & \stackrel{i}{\longleftarrow} & s \end{array}$$

Assume that X is proper and generically smooth with strict semistable reduction of relative dimension d over S. Let L (resp. M_X) be the log structure on S (resp. X) associated to $s \hookrightarrow S$ (resp. $X_s \hookrightarrow X$) and let L_s (resp. M_{X_s}) be its inverse image on s (resp. X_s). We have the Cartesian square of morphisms of log-schemes

$$\begin{array}{ccc} (X_s, M_{X_s}) & \to & (X, M_X) \\ \downarrow f_s & & \downarrow f \\ (s, L_s) & \to & (S, L) \end{array}$$

where f and f_s is log smooth. Assume now given $Z \subset X$, a good divisor in the sense of Definition 4.2 and let $U = X - Z$. We have the diagram of immersions

$$\begin{array}{ccccc} U_s & \stackrel{i_U}{\longrightarrow} & U & \stackrel{j_U}{\longleftarrow} & U_\eta \\ \downarrow \phi & & \downarrow \phi & & \downarrow \phi \\ X_s & \stackrel{i_X}{\longrightarrow} & X & \stackrel{j_X}{\longleftarrow} & X_\eta \\ \uparrow \tau & & \uparrow \tau & & \uparrow \tau \\ Z_s & \stackrel{i_Z}{\longrightarrow} & Z & \stackrel{j_Z}{\longleftarrow} & Z_\eta \end{array}$$

Let M_U (resp. M_Z) be the log structure on X (resp. Z) associated to $X_s \cup Z \subset X$ (resp. $Z_s \hookrightarrow Z$) and let M_{U_s} (resp. M_{Z_s}) be its inverse image on X_s (resp. Z_s). By definition we have $M_X = \mathcal{O}_X \cap (j_X)_* \mathcal{O}^*_{U_\eta}$, $M_U = \mathcal{O}_X \cap (\phi j_U)_* \mathcal{O}^*_{U_\eta}$, and $M_Z = \mathcal{O}_Z \cap (j_Z)_* \mathcal{O}^*_{Z_\eta}$. Let

$$W_n\omega^\cdot_{X_s}, \quad W_n\omega^\cdot_{X_s}(\log Z), \quad W_n\omega^\cdot_{Z_s}$$

denote the de Rham-Witt complexes associated to the log smooth schemes (X_s, M_{X_s}), (X_s, M_{U_s}), and (Z_s, M_{Z_s}) over (s, L_s) respectively (cf. [HK]). For $n = 1$ we use the simplified notation $\omega^{\cdot}_{X_s}$, $\omega^{\cdot}_{X_s}(\log Z)$, and $\omega^{\cdot}_{Z_s}$ for the de Rham-Witt complexes. They coincide with the complexes of logarithmic differentials defined in [K3], (1.9). We have the map of sheaves on $(X_s)_{et}$ (cf. [HK],(4.9)).

$$dlog \;:\; (M_{U_s}^{gp})^{\otimes q} \to W_n\omega^q_{X_s}(\log Z);\; a_1 \otimes \cdots \otimes a_q \to dlog(a_1) \wedge \cdots \wedge dlog(a_q).$$

We define the etale subsheaf $W_n\omega^q_{X_s}(\log Z)_{log} \subset W_n\omega^q_{X_s}(\log Z)$ to be the image of the above map. In a similar way we define the subsheaves

$$W_n\omega^q_{X_s,log} \subset W_n\omega^q_{X_s} \quad \text{and} \quad W_n\omega^q_{Z_s,log} \subset W_n\omega^q_{Z_s}.$$

Lemma 4.9 *The sheaf $W_n\omega^q_{X_s}(\log Z)_{log}$ is flat over $\mathbb{Z}/p^n\mathbb{Z}$ and we have an exact sequence*

$$0 \to W_n\omega^q_{X_s}(\log Z)_{log} \xrightarrow{p^m} W_{m+n}\omega^q_{X_s}(\log Z)_{log} \to W_m\omega^q_{X_s}(\log Z)_{log} \to 0.$$

The analogous facts hold for $W_n\omega^q_{X_s,log}$ and $W_n\omega^q_{Z_s,log}$.

Proof This follows from [H], (2.6) and [Lo] (1.5.4).

By [H], (1.6.1) and [Ts3], (3.1.5) the sheaf $R^m(\phi j_U)_*\mathbb{Z}/p^n\mathbb{Z}(m)$ is generated by symbols $\{x_1, \dots, x_m\} = \{x_1\} \cup \dots \cup \{x_m\}$ with local sections $x_1, \dots, x_m$ of M_U^{gp}, i.e., by the cup products of the images $\{x_1\}, \dots, \{x_m\}$ of these elements under the Kummer map $M_U^{gp} \to R^1(\phi j_U)_*\mathbb{Z}/p^n\mathbb{Z}(1)$. There exists a natural morphism

$$\text{(4-1)} \qquad \delta_m^{sym} \;:\; i_X^* R^m(\phi j_U)_*\mathbb{Z}/p^n\mathbb{Z}(m) \longrightarrow W_n\omega^m_{X_s}(\log Z)_{log}$$

sending $\{x_1 \dots x_m\}$ to $dlog(x_1) \wedge \dots \wedge dlog(x_m)$. In case $m = d+1$, $W_n\omega^{d+1}_{X_s,log} = 0$ and $R^{d+1}(\phi j_U)_*\mathbb{Z}/p^n\mathbb{Z}(d+1)$ is generated by symbols $\{\pi_K, x_1, \dots, x_d\}$ with $x_1, \dots, x_d$ as above and π_K, a prime element of K. Then there exists the natural morphism

$$\delta_{tame} :\; i_X^* R^{d+1}(\phi j_U)_*\mathbb{Z}/p^n\mathbb{Z}(d+1) \longrightarrow W_n\omega^d_{X_s}(\log Z)_{log}$$

which maps $\{\pi_K, x_1, \dots, x_d\}$ to $dlog(x_1) \wedge \dots \wedge dlog(x_d)$.

Lemma 4.10 $R^q(\phi j_U)_*\mathbb{Z}/p^n\mathbb{Z}(d+1) = 0$ *for* $q > d+1$.

Proof Since ϕj_U is affine, the stalk of $R^q(\phi j_U)_*\mathbb{Z}/p^n\mathbb{Z}(d+1)$ at a geometric point $\overline{x}$ over $x \in X_s$ is a limit of groups $H^q(V, \mathbb{Z}/p^n\mathbb{Z}(d+1))$ where V is an affine variety of dimension d over K^{ur}, the maximal unramified extension of K. The lemma now follows from the fact that $cd(K^{ur}) = 1$ and $cd(V \times \overline{K}) = d$, cf. [SGA4], XIV 3.1.

By the above lemma δ_{tame} induces a morphism in $D^b(X_{et})$

$$\delta_{tame}: \; R(\phi j_U)_*\mathbb{Z}/p^n\mathbb{Z}(d+1) \to (i_X)_* W_n\omega^d_{X_s}(\log Z)_{log}[-d-1]$$

Proposition 4.11 *Let $g: U \to S$, $g_\eta : U_\eta \to \eta$, and $g_s : U_s \to s$ be the structural morphisms. There is a canonical isomorphism in $D^b(X_{s,\acute{e}t})$,*

$$\alpha_{X,Z}: \; R\phi_* Rg_s^! \mathbb{Z}/p^n\mathbb{Z}(0)_s \cong W_n\omega^d_{X_s}(\log Z)_{log}[d]$$

which fits into the following commutative diagram

$$\begin{array}{ccc} H^{2d-a}(U_\eta, \mathbb{Z}/p^n\mathbb{Z}(d+1)) & \stackrel{\delta_{tame}}{\longrightarrow} & H^{d-a-1}(X_s, W_n\omega^d_{X_s}(\log Z)_{log}) \\ \uparrow \cong \text{2.14 (c)} & & \uparrow \cong \alpha_{X,Z} \\ H^{-a}(U_\eta, Rg_\eta^! \mathbb{Z}/p^n\mathbb{Z}(1)_\eta) & \stackrel{\delta}{\longrightarrow} & H^{-a-1}(U_s, Rg_s^! \mathbb{Z}/p^n\mathbb{Z}(0)_s) \\ \| & & \| \\ H_a(U_\eta, \mathbb{Z}/p^n\mathbb{Z}(-1)) & \stackrel{\Delta_U}{\longrightarrow} & H_{a+1}(U_s, \mathbb{Z}/p^n\mathbb{Z}(0)) \end{array}$$

where δ is induced by the map $\sigma : Rj_*\mathbb{Z}/p^n\mathbb{Z}(1)_\eta \to Ri_*\mathbb{Z}/p^n\mathbb{Z}(0)_s[-1]$ in §2 C by base change for $g^!$.

Proof The commutativity of the lower diagram is a direct consequence of the definition of Δ_U. In [JSS] it is proved that there is a canonical isomorphism,

$$\alpha_{U_s} : Rg_s^! \mathbb{Z}/p^n\mathbb{Z}(0)_s \cong W_n\omega^d_{U_s,log}[d] \tag{4-2}$$

which fits into the following commutative diagram, in which α_{U_η} is the purity isomorphism for the smooth morphism g_η (cf. (2-1))

$$\begin{array}{ccc} R(j_U)_*\mathbb{Z}/p^n\mathbb{Z}(d+1)[2d] & \stackrel{\delta_{tame}}{\longrightarrow} & (i_U)_* W_n\omega^d_{U_s,log}[d-1] \\ \uparrow \cong \alpha_{U_\eta} & & \uparrow \cong \alpha_{U_s} \\ R(j_U)_* Rg_\eta^! \mathbb{Z}/p^n\mathbb{Z}(1)_\eta & \stackrel{\delta}{\longrightarrow} & (i_U)_* Rg_s^! \mathbb{Z}/p^n\mathbb{Z}(0)_s[-1]. \end{array}$$

Applying $R\phi_*$ to it, and noting $\phi j_U = j_X \phi$, we get the commutative diagram

$$\begin{array}{ccc} R(\phi j_U)_*\mathbb{Z}/p^n\mathbb{Z}(d+1)[2d] & \stackrel{\delta_{tame}}{\longrightarrow} & (i_X)_* W_n\omega^d_{X_s}(\log Z)_{log}[d-1] \\ \| & & \downarrow \gamma \\ R(\phi j_U)_*\mathbb{Z}/p^n\mathbb{Z}(d+1)[2d] & \stackrel{\delta_{tame}}{\longrightarrow} & (i_X)_* R\phi_* W_n\omega^d_{U_s,log}[d-1] \\ \uparrow \cong & & \uparrow \cong \\ R(\phi j_U)_* Rg_\eta^! \mathbb{Z}/p^n\mathbb{Z}(1)_\eta & \stackrel{\delta}{\longrightarrow} & (i_X)_* R\phi_* Rg_s^! \mathbb{Z}/p^n\mathbb{Z}(0)_s[-1] \end{array}$$

where $W_n\omega^d_{U_s,log} = \phi^* W_n\omega^d_{X_s}(\log Z)_{log}$ and γ is the adjunction map. Thus Proposition 4.11 follows from the following result shown in [Sat3].

Proposition 4.12 *The adjunction induces an isomorphism*

$$W_n\omega^d_{X_s}(\log Z)_{log} \stackrel{\cong}{\longrightarrow} R\phi_* W_n\omega^d_{U_s,log}.$$

In view of Proposition 4.11, Theorem 4.4 (c) is now implied by the following result.

Theorem 4.13 *Assume that F is finite. Let*

$$\delta^{\infty}_{tame} \ : \ H^{d+2}(U_\eta, \mathbb{Q}_p/\mathbb{Z}_p(d+1)) \to H^1(X_s, W_\infty\omega^d_{X_s}(\log Z)_{log})$$

be obtained from the maps δ_{tame} in Proposition 4.11 by passing to the inductive limit over n, where $W_\infty\omega^q_{X_s}(\log Z)_{log} = \varinjlim_n W_n\omega^q_{X_s}(\log Z)_{log}$ (with transition maps denoted p^m in 4.9). Then δ^{∞}_{tame} is a surjective isogeny and it is an isomorphism if $p \geq d$ or if X is smooth over S.

In order to show this theorem, we want to compare the map δ_{tame} in Proposition 4.11 with another one, to be able to quote some results of Sato [Sat1] and Tsuji [Ts3]. Let $\overline{K}$ be a fixed separable closure of K and recall the notation in 4.1. Let $\overline{A}$ be the integral closure of A in $\overline{K}$ and put $\overline{S} = \mathrm{Spec}(\overline{A})$ and $\overline{\eta} = \mathrm{Spec}(\overline{K})$. By base change, we obtain the diagram

$$\begin{array}{ccccc} U_{\overline{s}} & \xrightarrow{\overline{i}_U} & \overline{U} & \xleftarrow{\overline{j}_U} & U_{\overline{\eta}} \\ \downarrow \phi & & \downarrow \phi & & \downarrow \phi \\ X_{\overline{s}} & \xrightarrow{\overline{i}_X} & \overline{X} & \xleftarrow{\overline{j}_X} & X_{\overline{\eta}} \\ \uparrow \tau & & \uparrow \tau & & \uparrow \tau \\ Z_{\overline{s}} & \xrightarrow{\overline{i}_Z} & \overline{Z} & \xleftarrow{\overline{j}_Z} & Z_{\overline{\eta}} \end{array}$$

where $\overline{X} = X \times_S \overline{S}$ and so on. Passing to the limit over the base changes by all finite extensions of K contained in $\overline{K}$, the maps δ^{sym}_m from (4-1) induce a morphism

$$\text{(4-3)} \qquad \overline{\delta}^{sym}_m : \overline{i}^*_X R^m(\phi\overline{j}_U)_*\mathbb{Z}/p^n\mathbb{Z}(m) \to W_n\omega^m_{X_{\overline{s}}}(\log Z)_{log}.$$

of etale sheaves on $X_{\overline{s}}$. It induces the map

$$\text{(4-4)} \qquad \overline{\delta}^{sym}_d : H^d_{et}(U_{\overline{\eta}}, \mathbb{Z}/p^n\mathbb{Z}(d)) \to H^0(X_{\overline{s}}, W_n\omega^d_{X_{\overline{s}}}(\log Z)_{log}).$$

Lemma 4.14 *Assume that F is finite. Let $G = \mathrm{Gal}(\overline{K}/K)$ (resp. $\Gamma = \mathrm{Gal}(\overline{F}/F)$) be the absolute Galois group of K (resp. F). Then the following diagram is commutative*

$$\begin{array}{ccc} H^d(U_{\overline{\eta}}, \mathbb{Z}/p^n\mathbb{Z}(d))_G & \xrightarrow{\overline{\delta}^{sym}_d} & H^0(X_{\overline{s}}, W_n\omega^d_{X_{\overline{s}}}(\log Z)_{log})_\Gamma \\ \downarrow \phi_K & & \downarrow \phi_F \\ H^2(K, H^d(U_{\overline{\eta}}, \mathbb{Z}/p^n\mathbb{Z}(d+1))) & & H^1(F, H^0(X_{\overline{s}}, W_n\omega^d_{X_{\overline{s}}}(\log Z)_{log})) \\ \downarrow \cong (a) & & \downarrow \cong (b) \\ H^{d+2}(U_\eta, \mathbb{Z}/p^n\mathbb{Z}(d+1)) & \xrightarrow{\delta_{tame}} & H^1(X_s, W_n\omega^d_{X_s}(\log Z)_{log}) \end{array}$$

where ϕ_K (resp. ϕ_F) are the isomorphisms coming from the duality theorems for Galois cohomology of K (resp. F), and where the isomorphisms (a) and (b) come from the Hochschild-Serre spectral sequences.

Proof (We are indebted to every reader who finds a simpler proof.) The maps (a) and (b) are isomorphisms since $H^r(U_{\overline{\eta}}, \mathbb{Z}/p^n\mathbb{Z}(d+1)) = 0$ for $r > d$ by [SGA4], XIV 3.1 and $H^s(X_{\overline{s}}, W_n\omega^d_{X_{\overline{s}}}(\log Z)_{log}) = H_{d-s}(U_{\overline{s}}, \mathbb{Z}/p^n\mathbb{Z}(0)) = 0$ for $s > 0$ by Proposition 4.11 and Theorem 3.5. Recall that local duality (resp. duality for Γ) gives isomorphisms

$$\phi_K : M_G \xrightarrow{\sim} H^2(K, M(1)) \qquad (\text{resp.} \quad \phi_F : N_\Gamma \xrightarrow{\sim} H^1(F, N)\ )$$

for any discrete torsion G-module M (resp. Γ-module N). For the proof of the commutativity we first note that we may pass to any finite extension K'/K. In fact, the diagram obtained by base change to $S' = \mathrm{Spec}(\mathcal{O}_{K'}) = \{\eta', s'\}$ is related to the previous one by the corestrictions from K' to K (resp. $F' = k(s')$ to F) and the trace maps from $U'_{\eta'}$ to U_η (resp. $X'_{s'}$ to X_s). This is clear for the top and the vertical maps of the diagram. For the bottom, we may translate back to homology via 4.11, and then have to show that the following diagram commutes:

$$\begin{array}{ccc} H_a(U'_{\eta'}, \mathbb{Z}/p^n\mathbb{Z}(-1)) & \xrightarrow{\Delta'_U} & H_{a+1}(U'_{s'}, \mathbb{Z}/p^n\mathbb{Z}(0)) \\ \downarrow tr_{K'/K} & & \downarrow tr_{F'/F} \\ H_a(U_\eta, \mathbb{Z}/p^n\mathbb{Z}(-1)) & \xrightarrow{\Delta_U} & H_{a+1}(U_s, \mathbb{Z}/p^n\mathbb{Z}(0)) \end{array} .$$

This commutativity follows from the covariance of étale homology over S (and its compatibility with localization sequences) once we show that the étale homology of U' over S' (which is considered in the first line) coincides with the étale homology of U' over S. This holds, however, because $Rg^!(\mathbb{Z}/p^n(1))_S = (\mathbb{Z}/p^n(1))_{S'}$ for $g : S' \to S$ and the complexes defined in §2 C , as one easily sees (a detailed proof can be found in [JSS]).

Thus it suffices to consider elements which are in the image of the map $H^d(U_\eta, , \mathbb{Z}/p^n\mathbb{Z}(d)) \to H^d(U_{\overline{\eta}}, , \mathbb{Z}/p^n\mathbb{Z}(d))^G \to H^d(U_{\overline{\eta}}, \mathbb{Z}/p^n\mathbb{Z}(d))_G$. Let

$$tr : H^2(K, \mathbb{Z}/p^n\mathbb{Z}(1)) \underset{\sim}{\longrightarrow} H^1(F, \mathbb{Z}/p^n\mathbb{Z}) \underset{\sim}{\xrightarrow{tr}} \mathbb{Z}/p^n\mathbb{Z}$$

be the canonical (trace) isomorphisms of (1-1) in the introduction, and let

$$\chi_K \in H^2(K, \mathbb{Z}/p^n\mathbb{Z}(1)) \quad \text{and} \quad \chi_F \in H^1(F, \mathbb{Z}/p^n\mathbb{Z})$$

be the inverse images of $1 \in \mathbb{Z}/p^n\mathbb{Z}$ under the isomorphisms. Then the composition

$$H^d(U_{\overline{\eta}}, \mathbb{Z}/p^n\mathbb{Z}(d))^G \longrightarrow H^d(U_{\overline{\eta}}, \mathbb{Z}/p^n\mathbb{Z}(d))_G \xrightarrow{\phi_K} H^2(K, H^d(U_{\overline{\eta}}, \mathbb{Z}/p^n\mathbb{Z}(d+1)))$$

is the cup product with χ_K. Now, by the compatibility of cup product with Hochschild-Serre spectral sequences we have a commutative diagram of cup product pairings

$$\begin{array}{ccccc} H^2(U_\eta, 1) & \times & H^d(U_\eta, d) & \xrightarrow{\cup} & H^{d+2}(U_\eta, d+1) \\ \uparrow & & \downarrow & & \uparrow \\ H^1(\Gamma, H^1(U_\eta^{ur}, 1)) & \times & H^d(U_\eta^{ur}, d)^\Gamma & \xrightarrow{\cup} & H^1(\Gamma, H^{d+1}(U_\eta^{ur}, d+1)) \\ \uparrow & & \downarrow & & \uparrow \\ H^1(\Gamma, H^1(K^{ur}, 1)) & \times & H^d(U_{\overline{\eta}}, d)^G & \xrightarrow{\cup} & H^1(\Gamma, H^1(K^{ur}, H^d(U_{\overline{\eta}}, d+1))) \\ \|\wr & & \| & & \|\wr \\ H^2(K, 1) & \times & H^d(U_{\overline{\eta}}, d)^G & \xrightarrow{\cup} & H^2(K, H^d(U_{\overline{\eta}}, d+1)). \end{array} \tag{4-5}$$

Here we have omitted the coefficients $\mathbb{Z}/p^n$ (but indicated the Tate twists), K^{ur} is the maximal unramified extension of K, $\Gamma = \mathrm{Gal}(K^{ur}/K)$, and the vertical maps are restrictions or come from the obvious Hochschild-Serre spectral sequences. The middle diagram comes from the commutative diagram

$$\begin{array}{ccccc} H^1(U_\eta^{ur},1) & \times & H^d(U_\eta^{ur},d) & \xrightarrow{\cup} & H^{d+1}(U_\eta^{ur},d+1) \\ \uparrow & & \downarrow & & \uparrow \\ H^1(K^{ur},1) & \times & H^0(K^{ur},H^d(U_{\overline{\eta}},d)) & \xrightarrow{\cup} & H^1(K^{ur},H^d(U_{\overline{\eta}},d+1), \end{array}$$

where the bottom cup product is induced by the pairing $H^0 \times H^d \to H^{d+1}$ for $U_{\overline{\eta}}$. The right vertical composition in (4-5) is the isomorphism (a) in Lemma 4.14. Putting things together, we get the following diagram
(4-6)

$$\begin{array}{ccccc} H^d(U_\eta,d) & \xrightarrow{\alpha} & H^{d+1}(U_\eta,d+1) & \xrightarrow{\delta_{tame}} & H^0(X_s,W_n\omega^d) \\ \downarrow & & \downarrow & & \downarrow \\ H^d(U_\eta^{ur},\mathbb{Z}/p^n\mathbb{Z}(d))^\Gamma & \xrightarrow{\beta} & H^{d+1}(U_\eta^{ur},d+1)_\Gamma & \xrightarrow{\delta_{tame}} & H^0(X_{\overline{s}},W_n\omega^d)_\Gamma \\ \| & & \downarrow \phi_F & & \downarrow \phi_F \\ H^d(U_\eta^{ur},\mathbb{Z}/p^n\mathbb{Z}(d))^\Gamma & \xrightarrow{\gamma} & H^1(\Gamma,H^{d+1}(U_\eta^{ur},d+1)) & \xrightarrow{\delta_{tame}} & H^1(\Gamma,H^0(X_{\overline{s}},W_n\omega^d)) \\ \downarrow res\phi_K & & \downarrow & & \downarrow \\ H^2(K,H^d(U_{\overline{\eta}},d+1)) & \xrightarrow{(a)} & H^{d+2}(U_\eta,d+1) & \xrightarrow{\delta_{tame}} & H^1(X_s,W_n\omega^d), \end{array}$$

where $W_n\omega^d$ stands for $W_n\omega^d_{X_s}(logZ)_{log}$. Here γ is the cup products with the image $\tilde{\chi}_K$ of χ_K in $H^1(\Gamma,H^1(U_\eta^{ur},1))$, and $res\phi_K$ is the composition of $res : H^d(U_\eta^{ur},d)^\Gamma \to H^d(U_{\overline{\eta}},d)^G$ and ϕ_K. Hence the bottom of the diagram is commutative by (4-5). The diagram involving β and γ is commutative if β is cup product with the element x corresponding to $\tilde{\chi}_K$ under the isomorphism $\phi_F : H^1(U_\eta^{ur},1)_\Gamma \to H^1(\Gamma,H^1(U_\eta^{ur},1))$. Consider the commutative diagram

$$\begin{array}{ccccc} H^2(K,\mathbb{Z}/p^n\mathbb{Z}(1)) & \to & H^1(\Gamma,H^1(K^{ur},\mathbb{Z}/p^n\mathbb{Z}(1))) & \to & H^1(\Gamma,\mathbb{Z}/p^n\mathbb{Z}(1)) \\ & & \uparrow \phi_F & & \uparrow \phi_F \\ & & H^1(K^{ur},\mathbb{Z}/p^n\mathbb{Z}(1))_\Gamma & \xrightarrow{v} & \mathbb{Z}/p^n\mathbb{Z}. \end{array}$$

Here the left horizonal map comes from the Hochschild-Serre spectral sequence, and the two right horizontal maps are induced by the Kummer isomorphism $H^1(K^{ur},\mathbb{Z}/p^n\mathbb{Z}(1)) \cong (K^{ur})^\times/p^n$ and the normalized valuation $v : (K^{ur})^\times \to \mathbb{Z}$. The composition in the top row is the map considered above and maps χ_K to χ_F, and the right vertical map sends 1 to χ_F. This shows that the element x is the image $\{\pi_K\}$ of a uniformizing element π_K of K under the Kummer isomorphism and the map $H^1(K^{ur},\mathbb{Z}/p^n\mathbb{Z}(1)) \to H^1(U_\eta^{ur},\mathbb{Z}/p^n\mathbb{Z}(1)_\Gamma$. Note that π_K is also a uniformizing element of K^{ur}. Now we see that the diagram involving α and β is commutative, if α is the cup product with $\{\pi_K\}$, which now denotes the image of π_K under $K^\times/p^n \to H^1(K,\mathbb{Z}/p^n\mathbb{Z}(1)) \to H^1(U_\eta,\mathbb{Z}/p^n\mathbb{Z}(1))$.
Thus, diagram (4-6) is commutative with $\alpha = \{\pi_K\}\cup$. Now, in view of the observation in the beginning of the proof, we obtain the commutativity in 4.14 by observing that the composition

$$H^d(U_\eta,d) \xrightarrow{\{\pi_K\}\cup} H^{d+1}(U_\eta,d+1) \xrightarrow{\delta_{tame}} H^0(X_s,W_n\omega^d)$$

in the top of (4-6) is just δ_d^{sym}, since on sheaf level, δ_d^{sym} is the composition

$$R^d(\phi j_U)_*\mathbb{Z}/p^n\mathbb{Z}(d) \stackrel{\{\pi_K\}\cup}{\to} R^{d+1}(\phi j_U)_*\mathbb{Z}/p^n\mathbb{Z}(d+1) \stackrel{\delta_{tame}}{\to} (i_X)_* W_n\omega^d_{X_s}(logZ)_{log},$$

where now $\{\pi_K\}$ denotes the global section of $R^1(\phi j_U)_*\mathbb{Z}/p^n\mathbb{Z}(1)$ corresponding to π_K.

We start the proof of Theorem 4.13. First we note that $H^1(X_s, W_\infty\omega^d_{X_s}(\log Z)_{log})$ is of cofinite type, by [GS, 4.18] and Lemma 4.16 below. We then claim that it is divisible. Indeed, considering the long exact sequence arising from the sequence (cf. Lemma 4.9)

$$0 \to W_n\omega^d_{X_s}(\log Z)_{log} \to W_\infty\omega^d_{X_s}(\log Z)_{log} \xrightarrow{p^n} W_\infty\omega^d_{X_s}(\log Z)_{log} \to 0,$$

the claim follows from the fact that $H^2(X_s, W_n\omega^d_{X_s}(\log Z)_{log}) = 0$ by Proposition 4.11 and Theorem 3.5 (a). Thus the first assertion of Theorem 4.13 follows from Lemmma 4.14 and the following result.

Proposition 4.15 *Let*

$$\overline{\delta}_d^{sym} : \ H^d(U_{\overline{\eta}}, \mathbb{Z}_p(d))_I \to \varprojlim_n H^0(X_{\overline{s}}, W_n\omega^d_{X_{\overline{s}}}(\log Z)_{log})$$

be the the map induced by the maps $\overline{\delta}_d^{sym}$ from (4-4), where $I \subset G$ is the inertia subgroup. Then $\overline{\delta}_d^{sym}$ has torsion kernel and cokernel.

We need the following two lemmas.

Lemma 4.16 *For every integers $n, t > 0$ the sequence*

$$0 \to W_n\omega^t_{X_s,log} \to W_n\omega^t_{X_s}(\log Z)_{log} \xrightarrow{Res_Z} \tau_* W_n\omega^{t-1}_{Z_s,log} \to 0.$$

is exact where Res_Z is the residue along $Z \subset X$.

Proof By Lemma 4.9 it suffices to show the exactness of the sequence

$$0 \to \omega^t_{X_s,log} \to \omega^t_{X_s}(\log Z)_{log} \to \tau_*\omega^{t-1}_{Z_s,log} \to 0. \tag{4-7}$$

It follows immediately from the definition of log-differentials that the sequences

$$0 \to \omega^t_{X_s} \to \omega^t_{X_s}(\log Z) \to \tau_*\omega^{t-1}_{Z_s} \to 0,$$

is exact. Except for the surjectivity of the last map, the exactness of (4-7) then follows from the commutative exact diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 \to & & \omega^t_{X_s,log} & \to & \omega^t_{X_s}(\log Z)_{log} & \to & \tau_*\omega^{t-1}_{Z_s,log} \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 \to & & \omega^t_{X_s,d=0} & \to & \omega^t_{X_s}(\log Z)_{d=0} & \to & \tau_*\omega^{t-1}_{Z_s,d=0} \\
 & & \downarrow C-1 & & \downarrow C-1 & & \downarrow C-1 \\
0 \to & & \omega^t_{X_s} & \to & \omega^t_{X_s}(\log Z) & \to & \tau_*\omega^{t-1}_{Z_s}
\end{array}$$

where the vertical exact sequences follow from [Ts3], Th.A.3 and A.4. Finally the surjectivity onto $\tau_*\omega^{t-1}_{Z_s,log}$ is a consequence of the fact that $\omega^q_{Z_s,log}$ is generated by $dlog$-differentials. This completes the proof of Lemma 4.16.

Lemma 4.17 *For every integer $m > 0$ write*

$$M^m_n(X) = (i_X)^* R^m (j_X)_* \mathbb{Z}/p^n\mathbb{Z}(m),$$
$$M^m_n(U) = (i_X)^* R^m (\phi j_U)_* \mathbb{Z}/p^n\mathbb{Z}(m),$$
$$M^m_n(Z) = (i_Z)^* R^m (j_Z)_* \mathbb{Z}/p^n\mathbb{Z}(m).$$

Then we have a commutative diagram with exact horizontal sequences

$$\begin{array}{ccccccc} & M^m_n(X) & \to & M^m_n(U) & \xrightarrow{\beta} & M^{m-1}_n(Z) & \to 0 \\ & \downarrow & & \downarrow \delta^{sym}_m & & \downarrow & \\ 0 \to & W_n\omega^m_{X_s,log} & \to & W_n\omega^m_{X_s}(\log Z)_{log} & \xrightarrow{Res_Z} & \tau_* W_n\omega^{m-1}_{Z_s,log} & \to 0 \end{array}$$

where the vertical arrows are the morphism δ^{sym}_m defined for U in (4-1), and its variants for X and Z, respectively. The upper sequence is induced by the distinguished triangle

$$R(j_X)_*\mathbb{Z}/p^n\mathbb{Z}(m)_{X_\eta} \to R(\phi j_U)_*\mathbb{Z}/p^n\mathbb{Z}(m)_{U_\eta} \to R(\tau j_Z)_*\mathbb{Z}/p^n\mathbb{Z}(m-1)_{Z_\eta}[-1]$$

arising from the localization theory for the smooth pair $Z_\eta \hookrightarrow X_\eta$.

Proof We remark that the above sheaves the p-adic vanishing cycles are generated by local symbols. One can check $\beta(\{s, x_1, \ldots, x_{m-1}\}) = \{\overline{x}_1, \ldots, \overline{x}_{m-1}\}$, where s is a local equation of Z in X and x_i (resp. $\overline{x}_i$) with $1 \le i \le m-1$ are local sections of M^{gp}_X (resp. the image of x_i in M^{gp}_Z). This shows the surjectivity of β. The commutativity of the diagram is verified by using the description of the values of δ^{sym}_m on symbols. This completes the proof of Lemma 4.17.

We proceed with the proof of Proposition 4.15. Writing

$$\overline{M}^m_n(X) = (\bar{i}_X)^* R^m (\bar{j}_X)_* \mathbb{Z}/p^n\mathbb{Z}(m),$$
$$\overline{M}^m_n(U) = (\bar{i}_X)^* R^m (\phi \bar{j}_U)_* \mathbb{Z}/p^n\mathbb{Z}(m),$$
$$\overline{M}^m_n(Z) = (\bar{i}_Z)^* R^m (\bar{j}_Z)_* \mathbb{Z}/p^n\mathbb{Z}(m),$$

we have the exact sequence

$$\cdots \to \overline{M}^{m-1}_n(U)(1) \to \overline{M}^{m-2}_n(Z)(1) \to \overline{M}^m_n(X) \to \overline{M}^m_n(U) \to \overline{M}^{m-1}_n(Z) \to \cdots .$$

Hence the surjectivity of β in Lemma 4.17 implies the exactness of

$$0 \to \overline{M}^m_n(X) \to \overline{M}^m_n(U) \to \overline{M}^{m-1}_n(Z) \to 0.$$

Thus, taking the cohomology and passing to the limit over finite extensions of K, the diagram in Lemma 4.17, with $m = d$, gives rise to the commutative diagram

$$\begin{array}{ccc} & & 0 \\ & & \downarrow \\ H^d(X_{\overline{\eta}}, \mathbb{Z}_p(d))_I & \to & \varprojlim_n H^0(X_{\overline{s}}, W_n\omega^d_{X_{\overline{s}},log}) \\ \downarrow & & \downarrow \\ H^d(U_{\overline{\eta}}, \mathbb{Z}_p(d))_I & \xrightarrow{\overline{\delta}^{sym}_d} & \varprojlim_n H^0(X_{\overline{s}}, W_n\omega^d_{X_{\overline{s}}}(\log Z)_{log}) \\ \downarrow & & \downarrow \\ H^d(Z_{\overline{\eta}}, \mathbb{Z}_p(d))_I & \to & \varprojlim_n H^0(Z_{\overline{s}}, W_n\omega^d_{Z_{\overline{s}},log}) \\ \downarrow & & \downarrow \\ H^{d+1}(X_{\overline{\eta}}, \mathbb{Z}_p(d))_I & \to & \varprojlim_n H^1(X_{\overline{s}}, W_n\omega^d_{X_{\overline{s}},log}) \end{array}$$

where the second horizontal arrow is the map $\overline{\delta}^{sym}_d$ for U from Proposition 4.14, and the first one is its analogue for X, induced by the analogue of (4-3)

$$\overline{\delta}^{sym}_{X,d} : \overline{i}^*_X R^d \overline{j}_{X*}\mathbb{Z}/p^n\mathbb{Z}(d) \to W_n\omega^d_{X_{\overline{s}},log}.$$

The fourth horizontal arrow is induced by $\overline{\delta}^{sym}_{X,d}$ as well, by noting the fact that $R^q\overline{j}_{X*}\mathbb{Z}/p^n\mathbb{Z}(d) = 0$ for $q > d$, which can be shown by the same argument as in Lemma 4.10. By similar reasoning, the third horizontal map is induced by the corresponding morphism $\overline{\delta}^{sym}_{Z,d-1}$ for Z. The left vertical sequence comes from the localization exact sequence

$$H^d(X_{\overline{\eta}}, \mathbb{Z}_p(d)) \to H^d(U_{\overline{\eta}}, \mathbb{Z}_p(d)) \to H^{d-1}(Z_{\overline{\eta}}, \mathbb{Z}_p(d-1)) \xrightarrow{\tau_*} H^{d+1}(X_{\overline{\eta}}, \mathbb{Z}_p(d))$$

via taking coinvariants under I, and it remains exact modulo torsion, since τ_* is split surjective modulo torsion by the hard Lefschetz theorem. Using the semistable comparison theorem on the comparison of p-adic étale cohomology and log-crystalline cohomology for proper semistable families, one can show (cf. [Sat1], Lemma 3.3 and [Ts3], (3.1.12) and (3.2.7)) that the first horizontal arrow and the last two ones are isomorphisms modulo torsion. Hence the second arrow is an isomorphism modulo torsion, too. This completes the proof of Proposition 4.15.

Next we show the second claim in Theorem 4.13, i.e., that δ^{∞}_{tame} is an isomorphism provided $p \geq d$. Denote by $\mathbb{Z}/p^n\mathbb{Z}(d+1)_{(X,Z)}$ the mapping fiber of

$$\delta_{tame} : R(\phi j_U)_*\mathbb{Z}/p^n\mathbb{Z}(d+1) \to (i_X)_* W_n\omega^d_{X_s}(\log Z)_{log}[-d-1]$$

and let $\mathbb{Q}_p/\mathbb{Z}_p(d+1)_{(X,Z)} = \varinjlim_n \mathbb{Z}/p^n\mathbb{Z}(d+1)_{(X,Z)}$. By definition we have the exact sequence

$$H^{d+2}(X, \mathbb{Q}_p/\mathbb{Z}_p(d+1)_{(X,Z)}) \to H^{d+2}(U_\eta, \mathbb{Q}_p/\mathbb{Z}_p(d+1)) \xrightarrow{\delta^{\infty}_{tame}} H^1(X_s, W_\infty\omega^d_{X_s}(\log Z)_{log}).$$

We know already that $\mathrm{Ker}(\delta^{\infty}_{tame})$ is finite, so its vanishing follows once we show that $H^{d+2}(X, \mathbb{Q}_p/\mathbb{Z}_p(d+1)_{(X,Z)})$ is divisible. In view of the distinguished triangle

$$\mathbb{Z}/p\mathbb{Z}(d+1)_{(X,Z)} \to \mathbb{Q}_p/\mathbb{Z}_p(d+1)_{(X,Z)} \xrightarrow{p} \mathbb{Q}_p/\mathbb{Z}_p(d+1)_{(X,Z)} \to,$$

which follows from Lemma 4.9, it suffices to show $H^{d+3}(X, \mathbb{Z}/p\mathbb{Z}(d+1)_{(X,Z)}) = 0$. Thus the claim follows from the following.

Theorem 4.18 *Assuming $p \geq d$, $H^q(X, \mathbb{Z}/p\mathbb{Z}(d+1)_{(X,Z)}) = 0$ for $q \geq d+3$.*

We remark that F is not assumed to be finite in Theorem 4.18. We need the following three lemmas.

Lemma 4.19 *Let the assumption be as above. Then there exists a trace map*

$$H^d(X_s, \omega^d_{X_s}) \xrightarrow{\cong} F.$$

For a locally free $\mathcal{O}_{X_s}$-module $\mathcal{M}$ the natural pairing

$$H^j(X_s, \omega^i_{X_s}(\log Z) \otimes \mathcal{M}) \times H^{d-j}(X_s, \mathcal{H}om_{\mathcal{O}_{X_s}}(\mathcal{M} \otimes \mathcal{O}_X(Z), \omega^{d-i}_{X_s}(\log Z))) \to H^d(X_s, \omega^d_{X_s}) \xrightarrow{\cong} F$$

is a perfect pairing of finite-dimensional F-vector spaces.

Proof By the isomorphism just after [Ts1], Th. 2.21, $\omega^d_{X_s}$ placed at degree d is the dualizing complex for X_s. The assertion follows from the isomorphisms

$$(\omega^i_{X_s}(\log Z))^\vee \otimes \omega^d_{X_s}(\log Z) \cong \omega^{d-i}_{X_s}(\log Z), \quad \omega^d_{X_s}(\log Z) = \omega^d_{X_s} \otimes \mathcal{O}(Z).$$

Lemma 4.20 *For any ample line bundle $\mathcal{L}$ on X_s, we have*

$$H^j(X_s, \omega^i_{X_s}(\log Z) \otimes \mathcal{L}^{-1}) = 0 \quad \text{for } i+j < \min\{d, p\}.$$

Proof The assertion follows from [K3], Th. 4.12 by the same argument as the proof of [DI], Cor. 2.8.

Lemma 4.21 *$H^j(X_s, \omega^i_{X_s}(\log Z)) = 0$ for $i+j > \max\{d, 2d-p\}$.*

Proof By Lemma 4.19 it suffices to show $H^j(X_s, \omega^i_{X_s}(\log Z) \otimes \mathcal{O}_X(Z)^{-1}) = 0$ for $i+j < d$, which follows from Lemma 4.20 since $\mathcal{O}_X(Z)$ is ample.

We start the proof of Theorem 4.18. Write ω^t for $\omega^t_{X_s}(\log Z)_{log}$, as well as

$$B^t = \mathrm{Im}(d : \omega^{t-1} \to \omega^t), \quad \text{and} \quad Z^t = \mathrm{Ker}(d : \omega^t \to \omega^{t+1}).$$

We may assume that K contains the p-th roots of unity, and can omit all Tate twists. Recall that $\mathbb{Z}/p\mathbb{Z}(d+1)_{(X,Z)}$ is concentrated in degree $[0, d+1]$.

Lemma 4.22 *Assume* $p \geq d$. *For every* $t \leq d+1$, *the homology sheaf* $\mathcal{H}^t(i_X^*\mathbb{Z}/p\mathbb{Z}(d+1)_{(X,Z)})$ *has a finite filtration whose subquotients are*

$$\omega_{log}^t, \quad \omega_{log}^{t-1}, \quad B^t, \quad B^{t-1} \quad \text{and} \quad \omega^{t-1}/B^{t-1}.$$

Proof By the Bloch-Kato-Hyodo theorems for the sheaf of the p-adic vanishing cycles $(i_X)^*R^t(\phi j_U)_*\mathbb{Z}/p\mathbb{Z}(t)$, as proved in [BK], [H] and, in particular, [Ts2 Proposition A.15], this holds under the condition $p \geq d+3$, which comes from the use of the syntomic complexes. For the result under the weaker assumption $p \geq d$, which uses the special structure of the good open U, we refer the reader to [JS].

By this result, it suffices to show the following.

Lemma 4.23 *Assume* $p \geq d$. *We have* $H^s(X_s, Q) = 0$ *for* $s+t \geq d+3$ *and for each of the above subquotients* Q.

Proof Write $Y = X_s$. By Lemma 4.21 and the assumption $p \geq d$ we have

(1) $H^s(Y, \omega^t) = 0$ for $s+t > d$ so that $H^s(Y, \omega^{t-1}) = 0$ for $s+t > d+1$.

Via the Cartier isomorphism (cf. [K3], Th. 4.12 and [Ts2], Th. A.3) we get

(2) $H^s(Y, Z^t/B^t) = 0$ for $s+t > d$.

Now we use descending induction on t to show that

(a_t) $H^s(Y, \omega^t/B^t) = 0$ for $s+t > d$,

(b_t) $H^s(Y, B^{t+1}) = 0$ for $s+t > d$,

(c_t) $H^s(Y, Z^t) = 0$ for $s+t > d+1$.

We start with the case $t = d$ where (a_d) (resp. (c_d)) follows from (2) (resp. (1)) by noting $Z^d = \omega^d$, and (b_d) follows from $B^{d+1} = 0$. Now assume that we have shown (a_t), (b_t), (c_t) for some $t \leq d$. For the induction step we use the exact sequence

$$H^{s-1}(Y, Z^t/B^t) \to H^s(Y, B^t) \to H^s(Y, Z^t).$$

We have $H^{s-1}(Y, Z^t/B^t) = 0$ if $s-1+t > d$ by (2) and $H^s(Y, Z^t) = 0$ if $s+t > d+1$ by (c_t) so that $H^s(Y, B^t) = 0$ if $s+t > d+1$, which implies (b_{t-1}). Next we look at the exact sequence

$$H^{s-1}(Y, B^t) \to H^s(Y, Z^{t-1}) \to H^s(Y, \omega^{t-1})$$

associated to the exact sequence $0 \to Z^{t-1} \to \omega^{t-1} \xrightarrow{d} B^t \to 0$. We have $H^{s-1}(Y, B^t) = 0$ if $s-1+t-1 > d$ by (b_{t-1}) and $H^s(Y, \omega^{t-1}) = 0$ if $s+t-1 > d$ by (1) so that $H^s(Y, Z^{t-1}) = 0$ if $s+t > d+2$, which implies (c_{t-1}). Now consider the exact sequence

$$H^s(Y, Z^{t-1}/B^{t-1}) \to H^s(Y, \omega^{t-1}/B^{t-1}) \to H^s(Y, \omega^{t-1}/Z^{t-1}).$$

We have $H^s(Y, Z^{t-1}/B^{t-1}) = 0$ if $s+t-1 > d$ by (2) and $H^s(Y, \omega^{t-1}/Z^{t-1}) \cong H^s(Y, B^t) = 0$ if $s+t-1 > d$ by (b_{t-1}) so that $H^s(Y, \omega^{t-1}/B^{t-1}) = 0$ if $s+t > d+1$, which implies (a_{t-1}). This completes the proof of (a_t), (b_t), (c_t) for $\forall t \leq d$. Note that (b_t) implies $H^s(Y, B^{t-1}) = 0$ for $s+t > d+2$ and (a_t) implies $H^s(Y, \omega^{t-1}/B^{t-1}) = 0$ for $s+t > d+1$.

Finally we look at the exact sequence

$$H^{s-1}(Y, \omega^t/B^t) \to H^s(Y, \omega^t_{log}) \to H^s(Y, \omega^t)$$

associated to the exact sequence

$$0 \to \omega^t_{log} \to \omega^t \xrightarrow{1-C^{-1}} \omega^t/B^t \to 0.$$

We have $H^{s-1}(Y, \omega^t/B^t) = 0$ if $s-1+t > d$ by (a_t) and $H^s(Y, \omega^t) = 0$ if $s+t > d$ by (1) so that $H^s(Y, \omega^t_{log}) = 0$ for $s+t > d+1$ and hence $H^s(Y, \omega^{t-1}_{log}) = 0$ for $s+t > d+2$ by the exact sequence. This completes the proof of Lemma 4.23 and hence that of Theorem 4.18.

It remains to show the last claim of Theorem 4.13, i.e., that δ^{∞}_{tame} is an isomorphism if X is smooth over S. It suffices to show $H^{d+2}(U_\eta, \mathbb{Q}_p/\mathbb{Z}_p(d+1)) = 0$ assuming $d \geq 2$. With $G = \mathrm{Gal}(\overline{K}/K)$, we have isomorphisms

$$H^{d+2}(U_\eta, \mathbb{Q}_p/\mathbb{Z}_p(d+1)) \simeq H^2(K, H^d(U_{\overline{\eta}}, \mathbb{Q}_p/\mathbb{Z}_p(d+1))) \simeq \mathrm{Hom}(H^d_c(U_{\overline{\eta}}, \mathbb{Z}_p)^G, \mathbb{Q}_p/\mathbb{Z}_p)$$

by local duality (cf. Lemma 4.14) and Poincaré duality. By the weak Lefschetz theorem $H^d_c(U_{\overline{\eta}}, \mathbb{Z}_p)$ is torsion free. Therefore it suffices to show $H^d_c(U_{\overline{\eta}}, \mathbb{Q}_p)^G = 0$. Noting the exact sequence

$$H^{d-1}(Z_{\overline{\eta}}, \mathbb{Q}_p) \to H^d_c(U_{\overline{\eta}}, \mathbb{Q}_p) \to H^d(X_{\overline{\eta}}, \mathbb{Q}_p)$$

and that Z is smooth over S by the assumption, this vanishing follows from:

Lemma 4.24 *If X is proper and smooth over S, then for any $i > 0$ and any G-subquotient V of $H^i(X_{\overline{\eta}}, \mathbb{Q}_p)$ one has $V^G = 0$.*

Proof By the B_{cris}-comparison isomorphism ([FM], [Fa])

$$H^i(X_{\overline{\eta}}, \mathbb{Q}_p) \otimes B_{cris} \simeq H^i_{cris}(X_s/W(F)) \otimes B_{cris},$$

the claim follows from the Weil conjecture for the crystalline cohomology ([KM]). In fact, $V^G \cong D_{cris}(V)^{F=1}$, where $D_{cris}(V)$ is a subquotient of $H^i_{cris}(X_s/W(F)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and F is induced by the crystalline Frobenius.

5. Δ^0_X, Δ^1_X, and class field theory

Let A be a henselian discrete valuation ring with finite residue field F of characteristic p. Let K be the quotient field of A. Let $S = \mathrm{Spec}(A)$ and consider a diagram like (1-3) in the introduction

$$\begin{array}{ccccc} X_\eta & \xrightarrow{j_X} & X & \xleftarrow{i_X} & X_s \\ \downarrow f_\eta & & \downarrow f & & \downarrow f_s \\ \eta & \xrightarrow{j} & S & \xleftarrow{i} & s \end{array}$$

In this section we prove Theorem 1.5, by using the relation between the Kato complexes of X_η and X_s and the class field theory developed in [Bl], [Sa1] and [KS1].

Definition 5.1 *For a scheme V of finite type over a field we put*

$$SK_1(V) = \mathrm{Coker}(\bigoplus_{x\in V_1} K_2(x) \xrightarrow{\partial} \bigoplus_{x\in V_0} K_1(x)),$$

$$CH_0(V) = \mathrm{Coker}(\bigoplus_{x\in V_1} K_1(x) \xrightarrow{\partial} \bigoplus_{x\in V_0} K_0(x)),$$

where K_* denotes algebraic K-groups and boundary maps are induced by localization theory for algebraic K-theory.

The components of the differential ∂ for $SK_1(V)$ (resp. $CH_0(V)$) are given by tame symbols (resp. valuations). By definition $CH_0(V)$ is the Chow group of zero-cycles on V. In case V is smooth of pure dimension d, $SK_1(V)$ coincides with Bloch's higher Chow group $CH^{d+1}(V,1)$ by [La], Lem. 2.8.

Note that $\mathrm{cd}(K) = 2$ and $\mathrm{cd}(F) = 1$ (cf. [Se], Ch.II, §2 and §4). By Proposition 2.12 (a) and Lemma 2.15 we have the commutative diagram

(5-1)
$$\begin{array}{ccc} H^{et}_{i-2}(X_\eta, \mathbb{Z}/n\mathbb{Z}(-1)) & \xrightarrow{\epsilon^i_{X_\eta}} & H^K_i(X_\eta, \mathbb{Z}/n\mathbb{Z}) \\ \downarrow \Delta_X & & \downarrow \Delta_X \\ H^{et}_{i-1}(X_s, \mathbb{Z}/n\mathbb{Z}(0)) & \xrightarrow{\epsilon^i_{X_s}} & H^K_i(X_s, \mathbb{Z}/n\mathbb{Z}), \end{array}$$

where the vertical maps are the residue maps (2.12 (a), 2.18) and the horizontal maps are edge homomorphisms (2.14 (b)) of the following spectral sequences (note 2.14 (a))

(5-2)
$$E^1_{p,q}(X_\eta) = \bigoplus_{x\in (X_\eta)_p} H^{p-q}(x, \mathbb{Z}/n\mathbb{Z}(p+1)) \Rightarrow H^{et}_{p+q}(X_\eta, \mathbb{Z}/n\mathbb{Z}(-1)),$$

(5-3)
$$E^1_{p,q}(X_s) = \bigoplus_{x\in (X_s)_p} H^{p-q}(x, \mathbb{Z}/n\mathbb{Z}(p)) \Rightarrow H^{et}_{p+q}(X_s, \mathbb{Z}/n\mathbb{Z}(0)).$$

Lemma 5.2 *The maps $\epsilon^0_{X_\eta}$ and $\epsilon^0_{X_s}$ are isomorphims and we have the commutative diagram with exact horizontal sequences*

$$\begin{array}{ccccccccc} 0 \to & \mathrm{Cok}(\epsilon^2_{X_\eta}) & \to & SK_1(X_\eta)/n & \xrightarrow{\alpha_{X_\eta}} & H^{et}_{-1}(X_\eta, \mathbb{Z}/n(-1)) & \xrightarrow{\epsilon^1_{X_\eta}} & H^K_1(X_\eta, \mathbb{Z}/n) & \to 0 \\ & \downarrow \Delta_X & & \downarrow \partial_X & & \downarrow \Delta_X & & \downarrow \Delta_X & \\ 0 \to & \mathrm{Cok}(\epsilon^2_{X_s}) & \to & CH_0(X_s)/n & \xrightarrow{\alpha_{X_s}} & H^{et}_0(X_s, \mathbb{Z}/n(0)) & \xrightarrow{\epsilon^1_{X_s}} & H^K_1(X_s, \mathbb{Z}/n) & \to 0 \end{array}$$

Here ∂_X comes from the localization theory for algebraic K-theory on X.

Proof Lemma 2.14 implies that $E^1_{p,q}(X_\eta) = 0$ unless $p \geq q$ and $q \geq -2$ and $E^1_{p,q}(X_s) = 0$ unless $p \geq q$ and $q \geq -1$ and that

$$E^2_{p,-2}(X_\eta) = H^K_p(X_\eta, \mathbb{Z}/n\mathbb{Z}) \quad \text{and} \quad E^1_{p,-1}(X_s) = H^K_p(X_s, \mathbb{Z}/n\mathbb{Z}),$$
$$E^2_{0,-1}(X_\eta) \cong SK_1(X_\eta)/n \quad \text{and} \quad E^2_{0,0}(X_s) \cong CH_0(X_s)/n.$$

Here the isomorphisms in the second row follows from the following commutative diagrams established in [K1], Lem. 1.4

$$\begin{array}{ccc} \bigoplus_{x\in(X_\eta)_1} K_2(x)/n & \to & \bigoplus_{x\in(X_\eta)_0} K_1(x)/n \\ \downarrow \wr & & \downarrow \wr \\ \bigoplus_{x\in(X_\eta)_1} H^2(x, \mathbb{Z}/n\mathbb{Z}(2)) & \to & \bigoplus_{x\in(X_\eta)_0} H^1(x, \mathbb{Z}/n\mathbb{Z}(1)) \end{array}$$

$$\begin{array}{ccc} \bigoplus_{x\in(X_s)_1} K_1(x)/n & \to & \bigoplus_{x\in(X_s)_0} K_0(x)/n \\ \downarrow \wr & & \downarrow \wr \\ \bigoplus_{x\in(X_s)_1} H^1(x, \mathbb{Z}/n\mathbb{Z}(1)) & \to & \bigoplus_{x\in(X_s)_0} H^1(x, \mathbb{Z}/n\mathbb{Z}) \end{array}$$

where the top side maps are the boundary maps of the Gersten complex for algebraic K-theory and the bottom side maps are the d_1-differentials of the spectral sequences (5-2) and (5-3). The vertical maps are the Galois symbol maps and they are isomorphisms by Kummer theory and [MS]. The proposition follows easily from these facts.

Let

$$tr_s : H^1(s, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}/n\mathbb{Z} \quad \text{and} \quad tr_\eta : H^2(\eta, \mathbb{Z}/n\mathbb{Z}(1)) \xrightarrow{\cong} \mathbb{Z}/n\mathbb{Z}$$

be the evaluation at the Frobenius substitution of the finite field F, and the composite of tr_s and the residue map $H^2(\eta, \mathbb{Z}/n\mathbb{Z}(1)) \to H^1(s, \mathbb{Z}/n\mathbb{Z})$, respectively. For a scheme Z denote by $D^b_c(Z, \mathbb{Z}/n\mathbb{Z})$ the derived category of complexes of étale sheaves of $\mathbb{Z}/n\mathbb{Z}$-modules on Z with bounded constructible cohomology sheaves.

Lemma 5.3 *Assume that f is proper.*
(1) For any $K \in D^b_c(X_\eta, \mathbb{Z}/n\mathbb{Z})$ the pairing

$$H^i(X_\eta, D_{X_\eta}(K)) \times H^{2-i}(X_\eta, K) \to H^2(X_\eta, Rf^!_\eta \mathbb{Z}/n(1)) \xrightarrow{tr_{X_\eta}} H^2(\eta, \mathbb{Z}/n(1)) \xrightarrow{tr_\eta} \mathbb{Z}/n\mathbb{Z}$$

is a perfect pairing of finite groups. Here $D_{X_\eta}(K) = R\mathcal{H}om(K, Rf^!_\eta \mathbb{Z}/n\mathbb{Z}(1))$ and tr_{X_η} is induced by the trace morphisms $Rf_{\eta *}Rf^!_\eta \mathbb{Z}/n\mathbb{Z}(1) \to \mathbb{Z}/n\mathbb{Z}(1)$.
(2) For any $K \in D^b_c(X_s, \mathbb{Z}/n\mathbb{Z})$ the pairing

$$H^i(X_s, D_{X_s}(K)) \times H^{1-i}(X_s, K) \to H^1(X_s, Rf^!_s \mathbb{Z}/n\mathbb{Z}) \xrightarrow{tr_{X_s}} H^1(s, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{tr_s} \mathbb{Z}/n\mathbb{Z}$$

is a perfect pairing of finite groups. Here $D_{X_s}(K) = R\mathcal{H}om(K, Rf^!_s \mathbb{Z}/n\mathbb{Z})$ and tr_{X_s} is induced by the trace morphisms $Rf_{s*}Rf^!_s \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$.

Proof This follows immediately from the Artin-Verdier duality for f_η and f_s together with the duality theorems for Galois cohomology of K and F (cf. [Sa3] and [CTSS]).

Proof of Theorem 1.5 (1) Assume that f is proper and X_η is connected. Then the bijectivity of $\Delta_X^0 = \Delta_X^{K,0}$ immediately follows from the following commutative diagram deduced from Lemmas 5.2 and 5.3

$$\begin{array}{ccccc} H_0^K(X_\eta, \mathbb{Z}/n\mathbb{Z}) & \underset{\sim}{\overset{\epsilon^0_{X_\eta}}{\longleftarrow}} & H^2(X_\eta, Rf_\eta^! \mathbb{Z}/n\mathbb{Z}(1)) & \underset{\sim}{\overset{tr_{X_\eta}}{\longrightarrow}} & H^2(\eta, \mathbb{Z}/n\mathbb{Z}(1)) \\ \downarrow \Delta_X^{K,0} & & \downarrow \Delta_X^{et,-2} & & \downarrow\wr\ \Delta_S^{et,-2} \\ H_0^K(X_s, \mathbb{Z}/n\mathbb{Z}) & \underset{\sim}{\overset{\epsilon^0_{X_s}}{\longleftarrow}} & H^1(X_s, Rf_s^! \mathbb{Z}/n\mathbb{Z}) & \underset{\sim}{\overset{tr_{X_s}}{\longrightarrow}} & H^1(s, \mathbb{Z}/n\mathbb{Z}) \end{array}$$

Proof of Theorem 1.5 (2) We need to recall the class field theory of X_η and X_s developed in [Bl], [KS1] and [Sa1]. For a scheme Z we let

$$\pi_1^{ab}(Z) = \mathrm{Hom}(H^1(Z_{et}, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z})$$

be the abelian algebraic fundamental group of Z. Let V be a proper scheme over K. For each $x \in V_0$ we have the map

$$K(x)^* \to \mathrm{Gal}(K(x)^{ab}/K(x)) = \pi_1^{ab}(x) \to \pi_1^{ab}(V),$$

where the first map is the reciprocity map for $K(x)$ which is a henselian discrete valuation field with finite residue field. The second map comes from the covariant functoriality of π_1^{ab}. Taking the sum of β_x over $x \in V_0$, we get the map

$$\tilde{\rho}_V \ : \ \bigoplus_{x \in V_0} K(x)^* \to \pi_1^{ab}(V).$$

Now the reciprocity law proved in [Sa1] and [Sa2] implies that $\tilde{\rho}_V$ factors through

$$\rho_V \ : \ SK_1(V) \to \pi_1^{ab}(V).$$

If Y is a proper scheme over a finite field F, we have the reciprocity map

$$\rho_Y \ : \ CH_0(Y) \to \pi_1^{ab}(Y)$$

defined in a similar way by using the following map for each $x \in Y_0$

$$\mathbb{Z} \to \mathrm{Gal}(F(x)^{ab}/F(x)) = \pi_1^{ab}(x) \to \pi_1^{ab}(Y),$$

where the first map sends $1 \in \mathbb{Z}$ to the Frobenius substitution over $F(x)$.

Now we return to the situation in Lemma 5.2. By 5.3 we have the isomorphism

$$H_{-1}^{et}(X_\eta, \mathbb{Z}/n\mathbb{Z}(-1)) \xrightarrow{\cong} \pi_1^{ab}(X_\eta)/n \quad (\text{resp. } H_0^{et}(X_s, \mathbb{Z}/n\mathbb{Z}(0)) \xrightarrow{\cong} \pi_1^{ab}(X_s)/n),$$

and we claim that the composite map of the isomorphism and the map α_{X_η} (resp. α_{X_s}) coincides with ρ_{X_η} (resp. ρ_{X_s}). Indeed, for $x \in (X_\eta)_0$ let $i_x : x \to X_\eta$ be the inclusion and put $\pi_x = f_\eta i_x : x \to \eta$. Consider the following composite map β_x

$$H^1(x, \mathbb{Z}/n(1)) \cong H^1(x, Ri_x^! Rf_\eta^! \mathbb{Z}/n(1)) \to H^1(X_\eta, Rf_\eta^! \mathbb{Z}/n(1)) = H_{-1}^{et}(X_\eta, \mathbb{Z}/n{-1}),$$

where the isomorphism comes from $Ri_x^! Rf_\eta^! \mathbb{Z}/n\mathbb{Z}(1) = R\pi_x^! \mathbb{Z}/n\mathbb{Z}(1) = \pi_x^* \mathbb{Z}/n\mathbb{Z}(1)$. By definition α_{X_η} is induced by the sum over $x \in (X_\eta)_0$ of the composite of β_x and the map $K(x)^*/n \xrightarrow{\cong} H^1(x, \mathbb{Z}/n\mathbb{Z}(1))$. Via the duality in Lemma 5.3 (1) and the local duality for Galois cohomology of $K(x)$, β_x is identified with the dual of $H^1(X_\eta, \mathbb{Z}/n\mathbb{Z}) \to H^1(x, \mathbb{Z}/n\mathbb{Z})$, the restriction map via i_x. This proves the desired assertion for ρ_{X_η}. The assertion for ρ_{X_s} is shown by the same argument. Thus we get the commutative diagram with exact rows

$$\begin{array}{ccccc} SK_1(X_\eta)/n & \xrightarrow{\rho_{X_\eta}/n} & \pi_1^{ab}(X_\eta)/n & \to & H_1^K(X_\eta, \mathbb{Z}/n\mathbb{Z}) \to 0 \\ \downarrow \partial_X & & \downarrow \delta_X & & \downarrow \Delta_X \\ CH_0(X_s)/n & \xrightarrow{\rho_{X_s}/n} & \pi_1^{ab}(X_s)/n & \to & H_1^K(X_s, \mathbb{Z}/n\mathbb{Z}) \to 0 \end{array}$$

where δ_X is the dual of $H^1(X_s, \mathbb{Z}/n\mathbb{Z}) \cong H^1(X, \mathbb{Z}/n\mathbb{Z}) \to H^1(X_\eta, \mathbb{Z}/n\mathbb{Z})$, and hence is the specialization map on fundamental groups. By definition of the reciprocity map, the cokernel of ρ_{X_η} is the quotient $\pi_1^{ab}(X_\eta)^{c.d.}$ classifying the abelian coverings in which every closed point of X_η is completely decomposed. Similarly $\mathrm{Coker}(\rho_{X_s}) = \pi_1^{ab}(X_s)^{c.d.}$, where the latter classifies the completely decomposed abelian coverings of X_s. Therefore Theorem 1.5 (2) follows from the next lemma.

Lemma 5.4 *If $f : X \to S$ is proper, with X regular, then the specialization map $\delta_X : \pi_1(X_\eta) \to \pi_1(X_s)$ (where we omitted suitable base points) is surjective and induces an isomorphism*

$$\pi_1^{ab}(X_\eta)^{c.d.} \xrightarrow{\sim} \pi_1^{ab}(X_s)^{c.d.}.$$

Proof The map δ_X factorizes as $\pi_1(X_\eta) \twoheadrightarrow \pi_1(X) \xleftarrow{\sim} \pi_1(X_s)$, in which the first map is surjective because X is normal [SGA 1] V 8.2, and the second map is an isomorphism because X is proper [Ar] (3.1), (3.4). The second claim of the lemma then follows from [Sa2], Proposition 3.12, which characterizes the image of $H^1(X_s, \mathbb{Q}/\mathbb{Z}) \to H^1(X_\eta, \mathbb{Q}/\mathbb{Z})$ as consisting of those characters on $\pi_1^{ab}(X_\eta)$ whose associated character on $SK_1(X_\eta)$ factors through $\partial_X : SK_1(X_\eta) \to CH_0(X_s)$.

6. Δ_X^2, Δ_X^3, and finiteness results for Kato homology

Let the notations be as in the beginning of the previous section. Recall that K is the quotient field of A, a henselian discrete valuation ring with finite residue field F of characteristic p. In this section we prove a crucial result that allows to control the second Kato homology of varieties over K by étale homology (Theorem 6.1. It enables us complete the proof of Theorem 1.6 and to deduce finiteness results for Kato homology over local and global fields. We also present a strategy to show Conjecture B in general (cf. Proposition 6.4). In the whole section, ℓ denotes a prime different from $\mathrm{ch}(K)$. Let V be a separated scheme of finite type over K, and let

$$\epsilon_V^i \;:\; H_{i-2}^{et}(V, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-1)) \to H_i^K(V, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$$

be the map considered in (5-1) (for $V = X_\eta$). Recall that for a proper scheme V over K, we have the norm map

$$N_{V/K} \ : \ SK_1(V) \to K^*$$

induced by the sum of the norm maps $K(x)^* \to K^*$ for $x \in V_0$ (cf. [Sa1]).

Theorem 6.1 *(1) If V is affine, then ϵ^2_V is surjective.*
(2) If V is proper and geometrically irreducible over K, then $\mathrm{Coker}(\epsilon^2_V)$ *is finite, and vanishes if $\ell \not| \, |\mathrm{Coker}(N_{V/K})|$.*

For the proof of Theorem 6.1, we need some preliminaries, in particular the following generalization of $N_{V/K}$. If $f : V \to M$ is a proper morphism with V and M of finite type over L, there is a norm map

$$N_{V/M} \ : \ SK_1(V) \to SK_1(M)$$

induced by the norm maps $K(x)^* \to K(f(x))^*$ for $x \in V_0$. For $M = \mathrm{Spec}(K)$ we have $N_{V/M} = N_{V/K}$.

Lemma 6.2 *(1) For a proper non-empty scheme V over K,* $\mathrm{Coker}(N_{V/K})$ *is finite.*
(2) For a proper surjective morphism $f : V \to M$ with V and M of finite type over K, $\mathrm{Coker}(N_{V/M})$ *is of finite exponent.*

Proof The first assertion is clear, since $N_{L/K}(L^*) \subset K^*$ is of finite index for any finite extension L/K. As for the second we may clearly assume that V and M are irreducible. If f is finite and flat, there is a map $f^* : SK_1(M) \to SK_1(V)$ induced by the natural inclusions $K(y)^* \to \oplus K(x)^*$ for $y \in M_0$ where the sum ranges over all $x \in V_0$ such that $f(x) = y$. We have $N_{V/M}f^* = [K(V) : K(M)]$, from which the assertion follows. In general, we proceed by induction on $\dim(M)$. If $\dim(M) = 0$, the claim follows from the first assertion of the lemma. By induction on $\dim(M)$ we may then replace V/M by $f^{-1}(U)/U$ for any non-empty open subset $U \subset M$ in view of the commutative diagram

$$\begin{array}{ccccccc} SK_1(f^{-1}(Z)) & \to & SK_1(V) & \to & SK_1(f^{-1}(U)) & \to 0 \\ \downarrow & & \downarrow & & \downarrow & \\ SK_1(Z) & \to & SK_1(M) & \to & SK_1(U) & \to 0 \end{array}$$

where $Z = M \setminus U$ and the vertical maps are the norm maps. However, replacing M with some non-empty open subset, we may assume that there exists a finite flat morphism $f' : N \to M$ and a proper morphism $g : N \to V$ such that $f \circ g = f'$. We have $N_{V/M} \cdot N_{N/V} = N_{N/M}$. Thus the desired assertion follows from the finite flat case. This completes the proof of Lemma 6.2.

Proposition 6.3 *(1) If V is affine, then $SK_1(V) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell = 0$.*
(2) If V is irreducible and not proper over K, then $SK_1(V) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell = 0$.
(3) If V is geometrically irreducible and proper over K, $\mathrm{Ker}(N_{V/K}) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell = 0$.

Proof If V is a proper smooth curve, the claim follows from the class field theory for curves over local fields [Sa1], Th. 4.1 and Th. 5.1. In what follows we reduce to this crucial case. First we note that claim (2) follows from claim (3). Indeed, for irreducible non-proper U there is an open immersion $U \subset V$ such that V is irreducible and proper over K with $Z := V \setminus U$ non-empty [N]. By possibly enlarging K, we may assume that V is geometrically irreducible over K. In view of the exact sequence

$$SK_1(Z) \to SK_1(V) \to SK_1(U) \to 0, \tag{6-1}$$

it suffices to show the surjectivity of $SK_1(Z) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \to SK_1(V) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell$. We have the commutative diagram

(6-2)

$$\begin{array}{cccccccc}
0 \to & \mathrm{Ker}(N_{Z/K}) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell & \to & SK_1(Z) \otimes \mathbb{Q}/\mathbb{Z} & \overset{N_{Z/K}}{\to} & K^* \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell & \to 0 \\
& \downarrow \alpha & & \downarrow \beta & & \| & \\
0 \to & \mathrm{Ker}(N_{V/K}) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell & \to & SK_1(V) \otimes \mathbb{Q}/\mathbb{Z} & \overset{N_{V/K}}{\to} & K^* \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell & \to 0 .
\end{array}$$

Since $\mathrm{Coker}(N_{Z/K})$ and the torsion part of K^* are finite, the horizontal sequences are exact up to finite groups. By the assumption we have $\mathrm{Ker}(N_{V/K}) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell = 0$ so that β is surjective up to finite groups, hence surjective.

In particular, we see that claim (1) holds for smooth affine curves over K. By 6.2 (2) it also holds for an arbitrary irreducible affine curve V over K, since there is a finite surjective morphism $C \to V$ with C affine and smooth. Now let V be arbitrary affine. Since every closed point of V lies on some irreducible curve $Z \hookrightarrow V$, the natural maps $SK_1(Z) \to SK_1(V)$ induce a surjection $\bigoplus_{Z \subset V} SK_1(Z) \to SK_1(V)$, where Z ranges over all irreducible closed subschemes of V with $\dim(Z) = 1$. As these are necessarily affine, claim (1) follows for V.

As for claim (3), let V is proper and geometrically irreducible over K. By Chow's lemma there is a proper birational morphism $Z \to V$ with Z projective, and we have diagram (6-2), exact up to finite groups, also for this morphism. By Lemma 6.2 (2) the map β then is surjective, hence α is surjective as well. Thus we may assume that V is projective. We proceed by induction on $\dim(V)$. In case $\dim(V) = 1$ we may replace V with its normalization (by Lemma 6.2), and we may pass to any inseparable extension L of K, because the norm induces an isomorphism $L^* \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \to K^* \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell$. We thus reduce to the treated case of proper smooth curves. Assume $\dim(V) > 1$. Then, by Bertini's theorem, there is a hyperplane section $Z \subset V$ which is defined over K and geometrically irreducible over K. Then $U = V \setminus Z$ is affine, and we conclude from claim (1) that $SK_1(U) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell = 0$. Now we consider (6-1) and (6-2) for this triple (V, Z, U). From (6-1) we get that the map β in (6-2) is surjective, and hence so is α. By induction on the dimension we may assume that $\mathrm{Ker}(N_{Z/K}) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell = 0$, and hence we also get $\mathrm{Ker}(N_{V/K}) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell = 0$ as wanted.

Now we show Theorem 6.1. In case V is affine, it follows immediately from 6.3 (1) and the exact sequence (cf. Lemma 5.2)

$$H_0^{et}(V, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-1)) \xrightarrow{\epsilon_V^2} H_2^K(V, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to SK_1(V) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \xrightarrow{\alpha_V} H_{-1}^{et}(V, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-1)).$$

In case $f : V \to \mathrm{Spec}(K)$ is proper, we use the commutative diagram

$$\begin{array}{ccccc} SK_1(V)\otimes\mathbb{Q}_\ell/\mathbb{Z}_\ell & \xrightarrow{\alpha_V} & H^{et}_{-1}(V,\mathbb{Q}_\ell/\mathbb{Z}_\ell(-1)) & = & H^1(V,Rf^!\mathbb{Q}_\ell/\mathbb{Z}_\ell(1))) \\ \downarrow\mu & & \downarrow f_* & & \downarrow tr_{V/K} \\ K^*\otimes\mathbb{Q}_\ell/\mathbb{Z}_\ell & \xrightarrow{\cong} & H^{et}_{-1}(\mathrm{Spec}(K),\mathbb{Q}_\ell/\mathbb{Z}_\ell(-1)) & = & H^1(K,\mathbb{Q}_\ell/\mathbb{Z}_\ell(1)) \end{array}$$

where μ is induced by $N_{V/K}$. By the above two diagrams we have $\mathrm{Coker}(\epsilon^2_V) = \mathrm{Ker}(\alpha_V) \subset \mathrm{Ker}(\mu)$. But by 6.2 (1) and 6.3 (3), $\mathrm{Ker}(\mu)$ is finite and vanishes if $\ell \not| \, |\mathrm{Coker}(N_{V/K})|$. This completes the proof of Theorem 6.1.

Now we turn to the proof of Theorem 1.6. In what follows we assume that X is projective and generically smooth with strict semistable reduction over $S = \mathrm{Spec}(A)$. Assume also that we are given $Z \subset X$, a good divisor in the sense of Definition 4.2. Write $U = X \setminus Z$. We fix a prime ℓ different from $\mathrm{ch}(K)$ and consider the residue maps

$$\Delta^i_X : H^K_i(X_\eta,\mathbb{Q}_\ell/\mathbb{Z}_\ell) \to H^K_i(X_s,\mathbb{Q}_\ell/\mathbb{Z}_\ell)$$
$$\Delta^i_Z : H^K_i(Z_\eta,\mathbb{Q}_\ell/\mathbb{Z}_\ell) \to H^K_i(Z_s,\mathbb{Q}_\ell/\mathbb{Z}_\ell).$$

Our strategy is to use induction on $\dim(X)$. Let

$$\epsilon^i_{U_\eta} \; : \; H^{et}_{i-2}(U_\eta,\mathbb{Q}_\ell/\mathbb{Z}_\ell(-1)) \to H^K_i(U_\eta,\mathbb{Q}_\ell/\mathbb{Z}_\ell)$$

be the map considered in (5-1).

Proposition 6.4 *Fix an integer $q \geq 2$. Assume the following conditions hold.*

(1) The Kato conjecture $K(F,\mathbb{Q}_\ell/\mathbb{Z}_\ell)$ (cf. 3.1) is true in degree $\leq q+1$.

(2) $\epsilon^q_{U_\eta}$ is surjective.

(3) Δ^{q-1}_Z and Δ^q_Z are isomorphisms and Δ^{q+1}_Z is surjective.

(4) One of the following conditions is satisfied:

- *(a) X is smooth over S.*
- *(b) $\ell \neq p := \mathrm{ch}(F)$.*
- *(c) $\ell = p \geq q$.*
- *(d) $q < d$.*

Then Δ^q_X is an isomorphism and Δ^{q+1}_X is surjective.

Proof For $i = q, q+1$ we look at the following commutative diagram of Kato homology groups obtained from 2.17 and 2.18, with the coefficients $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ omitted in the notation.

$$\begin{array}{ccccccccc} H^K_{i+1}(U_\eta) & \to & H^K_i(Z_\eta) & \to & H^K_i(X_\eta) & \to & H^K_i(U_\eta) & \to & H^K_{i-1}(Z_\eta) \\ \downarrow\Delta^{i+1}_U & & \downarrow\Delta^i_Z & & \downarrow\Delta^i_X & & \downarrow\Delta^i_U & & \downarrow\Delta^{i-1}_Z \\ H^K_{i+1}(U_s) & \to & H^K_i(Z_s) & \to & H^K_i(X_s) & \to & H^K_i(U_s) & \to & H^K_{i-1}(Z_s) \end{array}$$

Since $H^K_i(X_\eta) = H^K_i(X_s) = 0$ for $i > d := \dim(X_\eta)$, we may assume $q \leq d$. The proposition follows from the diagram by using the following result.

Lemma 6.5 *Assume $q \le d$ and the conditions 6.4 (1) and (2) for q.*

(1) Δ_U^i *is surjective for $i \le q+1$.*

(2) $H_q^K(U_\eta, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = H_q^K(U_s, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = 0$ *if $q < d$.*

(3) Δ_U^q *is an isomorphism if one of the conditions in 6.4 (4) is satisfied.*

Proof Consider the commutative diagram

$$\begin{array}{ccc} H^{et}_{i-2}(U_\eta, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-1)) & \xrightarrow{\epsilon^i_{U_\eta}} & H_i^K(U_\eta, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \\ \downarrow \Delta_U^{i,et} & & \downarrow \Delta_U^i \\ H^{et}_{i-1}(U_s, \mathbb{Q}_\ell/\mathbb{Z}_\ell(0)) & \xrightarrow{\epsilon^i_{U_s}} & H_i^K(U_s, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \end{array}$$

where the vertical maps are the respective residue maps. The left residue map $\Delta^{i,et}$ is surjective if $i \le d$ by Theorem 4.4, and by Theorem 3.5, condition 6.4 (1) implies that $\epsilon^i_{U_s}$ is an isomorphism for $i \le \min(q+1, d)$. This proves the first assertion. For the second assertion note that $H^{et}_{q-2}(U_\eta, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-1)) = H^{et}_{q-1}(U_s, \mathbb{Q}_\ell/\mathbb{Z}_\ell(0)) = 0$ if $q < d$ by Theorem 3.5 and Theorem 4.4. Thus the claim follows since $\epsilon^q_{U_s}$ and $\epsilon^q_{U_\eta}$ are surjective by the assumptions 6.4 (1) and (2). For (3) we may assume $q = d$ by (2). Then the assertion follows since, by Theorem 4.4, the residue map $\Delta_U^{i,et}$ is an isomorphism for $i = d(= q)$ if one of the conditions 6.4 (4)(a),(b),(c) is satisfied.

We can now prove Theorem 1.6. We proceed by induction on $\dim(X)$. First we claim that we may assume the existence of a good divisor $Z \subset X$. Indeed, by Proposition 4.3 such a divisor exists after replacing F with a finite extension of degree prime to ℓ and K with the corresponding unramified extension. Then the claim follows for the original F and K by a standard norm argument. Given a good divisor, Theorem 1.6 (1) follows from Proposition 6.4 with $q = 2$: Condition (1) is satisfied since the Kato conjecture $K(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ is known in degrees ≤ 3, as recalled in the introduction. Condition (2) is satisfied by Lemma 6.1 (1). Condition (3) is satisfied by the induction hypothesis and Theorem 1.5. Condition (4) is satisfied since every prime is not less than 2. This completes the proof of Theorem 1.6.

Remark 6.6 Proposition 6.4 tells that, assuming that the Kato conjecture $K(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ holds, an essential obstacle against showing Conjecture B in degrees > 2 is the surjectivity of $\epsilon^i_{U_\eta}$ for $i > 2$. In case $i = 2$ the class field theory for curves over local fields [Sa1] plays a crucial role to prove it (cf. the proof of Lemma 6.1). In case $i > 2$ we do not have any effective means to approach this problem at present.

We close this section with the following applications of Theorems 6.1 and 1.6.

Corollary 6.7 *Let V be a scheme of finite type over K.*
(1) $H_i^K(V, \mathbb{Z}/n\mathbb{Z})$ is finite for $i = 0, 1$ and for all $n > 0$ invertible in K.
(2) $H_2^K(V, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ is of cofinite type.

Proof There is an affine open subscheme $U \hookrightarrow V$ with complement $Z = V \setminus U$ such that $\dim(Z) < \dim(V)$. By induction on $\dim(V)$ and by the exact sequences

$$H_i^K(Z, \mathbb{Z}/n\mathbb{Z}) \to H_i^K(V, \mathbb{Z}/n\mathbb{Z}) \to H_i^K(U, \mathbb{Z}/n\mathbb{Z})$$

we may thus assume that V is affine. By the following Lemma, the claim then follows from Lemma 5.2 for H_0^K and H_1^K, and from Theorem 6.1 (1) for H_2^K.

Lemma 6.8 *If V separated, and n invertible in K, $H_i^{et}(V, \mathbb{Z}/n\mathbb{Z}(j))$ is finite for for all $i, j \in \mathbb{Z}$. In particular, $H_i^{et}(V, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$ is of cofinite type for all $i, j \in \mathbb{Z}$.*

Proof It suffices to show the first claim; the second then follows via the Kummer sequence $0 \to \mathbb{Z}/\ell\mathbb{Z}(-j) \to \mathbb{Q}_\ell/\mathbb{Z}_\ell(-j) \xrightarrow{\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell(-j) \to 0$. For an open immersion $V \hookrightarrow V'$ with complement $Z = V' \setminus V$, we have the exact sequence

$$H_i^{et}(Z, \mathbb{Z}/n\mathbb{Z}(j)) \to H_i^{et}(V', \mathbb{Z}/n\mathbb{Z}(j)) \to H_i^{et}(V, \mathbb{Z}/n\mathbb{Z}(j)) \to H_{i-1}^{et}(Z, \mathbb{Z}/n\mathbb{Z}(j)).$$

By induction on $\dim(V)$ and embedding V into a proper K-scheme [N], we may thus assume that $g : V \to \mathrm{Spec}(K)$ is proper. By Lemma 5.3 we are reduced to show the finiteness of $H^a(V, \mathbb{Z}/n\mathbb{Z}(b)) = H^a(K, Rg_*\mathbb{Z}/n\mathbb{Z}(b))$ for proper g, which is a consequence of [SGA4], XIV Th.1.1 and [Se], Ch.II §5.2.

Corollary 6.9 *If X is smooth and projective over S, then $H_2^K(X_\eta, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = 0$.*

Proof This is an immediate consequence of Theorems 1.6 and 1.4.

Corollary 6.10 *For a projective smooth variety Z over a number field, $H_2^K(Z, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ is of cofinite type.*

Proof This follows from Theorem 1.3 and Corollaries 6.7 and 6.9 by noting the following: Let the notation be as in Theorem 1.3. For an imaginary place v, one has $H_i^K(Z_v, \mathbb{Q}/\mathbb{Z}) = 0$ because $cd(k(x)) = p$ for $x \in (Z_v)_p$. For a real place v, $H_i^K(Z_v, \mathbb{Q}/\mathbb{Z})$ is a 2-torsion finite group by results of [Sch] 19.5.1 and 17.7.

7. An application: The kernel of the reciprocity map

Let the notations be as in the beginning of §5. Recall that K is the quotient field of A, a henselian discrete valuation ring with finite residue field F of characteristic p. We assume that f_η is proper. Let $k(X_\eta)$ denote the function field of X_η. In this section we study the kernel of the reciprocity map

$$\rho_{X_\eta} \; : \; SK_1(X_\eta) \to \pi_1^{ab}(X_\eta)$$

and prove Theorems 1.8 and 1.9. For an integer $n > 0$ prime to $\mathrm{ch}(K)$ let

$$\rho_{X_\eta, n} \; : \; SK_1(X_\eta)/n \to \pi_1^{ab}(X_\eta)/n$$

be the induced map. By Lemmas 5.2 and 5.3 we have an exact sequence

$$\text{(7-1)} \qquad \mathrm{Coker}(\epsilon^2_{X_\eta,n}) \to SK_1(X_\eta)/n \xrightarrow{\rho_{X_\eta,n}} \pi_1^{ab}(X_\eta)/n \to H_1^K(X_\eta, \mathbb{Z}/n\mathbb{Z}) \to 0$$

where $\epsilon^i_{X_\eta,n} : H^{et}_{i-2}(X_\eta, \mathbb{Z}/n\mathbb{Z}(-1)) \to H_i^K(X_\eta, \mathbb{Z}/n\mathbb{Z})$ is as in (5-1). We need the following theorem.

Theorem 7.1 *Let V be a smooth proper geometrically irreducible variety over K. Let $\pi_1^{ab}(V)^{geo}$ be the kernel of the natural surjection $\pi_1^{ab}(V) \to \mathrm{Gal}(K^{ab}/K)$. There is an exact sequence*

$$0 \to T \to \pi_1^{ab}(V)^{geo} \to \hat{\mathbb{Z}}^r \to 0,$$

where T is finite and r is the F-rank of the special fiber of the Néron model of the Albanese variety of V.

For the pro-ℓ-part, with $\ell \neq p := \mathrm{ch}(F)$, this is essentially due to Grothendieck. The result for the pro-p-part is due to T. Yoshida [Y].

Theorem 7.2 *For V as in Theorem 7.1, the image of $\mathrm{Ker}(N_{V/K})$ under $\rho_V : SK_1(V) \to \pi_1^{ab}(V)$ is finite.*

Proof By Theorem 7.1 it suffices to show that the image is torsion. By a similar argument as in the proof of Proposition 6.3 (1) we may reduce to the case that V is a proper smooth curve. Then the assertion follows from [Sa1], Th. 4.1, together with [Y], Th. 5.1 (for the p-part if $\mathrm{ch}(K) = p$).

Lemma 7.3 *Let V be of finite type over K. For a prime ℓ we have an exact sequence*

$$0 \to H^K_{i+1}(V, \mathbb{Q}_\ell/\mathbb{Z}_\ell)/\ell^\nu \to H_i^K(V, \mathbb{Z}/\ell^\nu\mathbb{Z}) \to H_i^K(V, \mathbb{Q}_\ell/\mathbb{Z}_\ell)[\ell^\nu] \to 0$$

either if $\ell = \mathrm{ch}(K)$, or if $\ell \neq \mathrm{ch}(K)$, $BK_{i+1}(K(x), \ell)$ (cf. the introduction) holds for all $x \in V_i$ and $BK_i(K(x), \ell)$ holds for all $x \in V_{i-1}$.

Proof (cf. [CT], §2) We use the Kummer sequences of étale sheaves

$$\begin{array}{ll} 0 \to \mathbb{Z}/\ell^\nu\mathbb{Z}(r) \to \mathbb{Q}_\ell/\mathbb{Z}_\ell(r) \xrightarrow{\ell^\nu} \mathbb{Q}_\ell/\mathbb{Z}_\ell(r) \to 0 & (\text{for } \ell \neq \mathrm{ch}(K)), \\ 0 \to W_\nu\Omega^r_{log} \to W_\infty\Omega^r_{log} \xrightarrow{\ell^\nu} W_\infty\Omega^r_{log} \to 0 & (\text{for } \ell = \mathrm{ch}(K)). \end{array}$$

For $x \in V_j$ they give rise to the exact sequence

$$\begin{aligned} H^{j+1}(K(x), \mathbb{Q}_\ell/\mathbb{Z}_\ell(j+1)) \xrightarrow{\ell^\nu} H^{j+1}(K(x), \mathbb{Q}_\ell/\mathbb{Z}_\ell(j+1)) \to H^{j+2}(K(x), \mathbb{Z}/\ell^\nu\mathbb{Z}(j+1)) \\ \xrightarrow{\iota} H^{j+2}(K(x), \mathbb{Q}_\ell/\mathbb{Z}_\ell(j+1)) \xrightarrow{\ell^\nu} H^{j+2}(K(x), \mathbb{Q}_\ell/\mathbb{Z}_\ell(j+1)) \to 0, \end{aligned}$$

(where we let $H^s(K(x),\mathbb{Q}_\ell/\mathbb{Z}_\ell(r)) := H^{s-r}(K(x),W_\infty\Omega^r_{log})$, cf. 2 B and the introduction). The surjectivity of the last map follows from $H^{j+3}(K(x),\mathbb{Z}/\ell^\nu\mathbb{Z}(j+1)) = 0$, which follows from [Se], Ch.II §4 and §6 for $\ell \neq \mathrm{ch}(K)$. For $\ell = p = \mathrm{ch}(K)$, the vanishing of $H^2(K(x),W_\nu\Omega^{j+1})$ follows from $cd_p(K(x)) \leq 1$, cf. the proof of 2.14 (c). In case $\ell \neq \mathrm{ch}(K)$ the assumption $BK_{j+1}(K(x),\ell)$ says that $H^{j+1}(K(x),\mathbb{Q}_\ell/\mathbb{Z}_\ell(j+1))$ is divisible so that ι is injective. In case $\ell = \mathrm{ch}(K)$ we get the same conclusion by [BK], Th. 2.1. The desired assertion now follows from the sequence of Kato complexes

$$0 \to C^{2,1}(V,\mathbb{Z}/\ell^\nu\mathbb{Z}) \to C^{2,1}(V,\mathbb{Q}_\ell/\mathbb{Z}_\ell) \xrightarrow{\ell^\nu} C^{2,1}(V,\mathbb{Q}_\ell/\mathbb{Z}_\ell) \to 0$$

and the exactness properties derived from the above exact sequences for $j = i-1, i, i+1$.

Lemma 7.4 *Let V and W be irreducible and smooth of dimension d over a field L and let $f : W \to V$ be proper and generically finite of degree N. For any integers $r, s,$, and any integer n invertible in L, the cokernel of the map*

$$f_* \ : \ H^{r,s}_d(W,\mathbb{Z}/n\mathbb{Z}) \to H^{r,s}_d(V,\mathbb{Z}/n\mathbb{Z})$$

induced by f in the Kato homology is annihilated by N.

Proof We have the commutative diagram

$$\begin{array}{ccc} H^{r,s}_d(W,\mathbb{Z}/n\mathbb{Z}) & \stackrel{\hookrightarrow}{\longrightarrow} & H^{d+r}(L(W),\mathbb{Z}/n\mathbb{Z}(d+s)) \\ \downarrow f_* & & \downarrow Cor_{L(W)/L(V)} \\ H^{r,s}_d(V,\mathbb{Z}/n\mathbb{Z}) & \stackrel{\hookrightarrow}{\longrightarrow} & H^{d+r}(L(V),\mathbb{Z}/n\mathbb{Z}(d+s)) \end{array}$$

where $Cor_{L(W)/L(V)}$ is the corestriction map for Galois cohomology. We claim that the restriction map

$$Res_{L(W)/L(V)} \ : \ H^{d+r}(L(V),\mathbb{Z}/n\mathbb{Z}(d+s)) \to H^{d+r}(L(W),\mathbb{Z}/n\mathbb{Z}(d+s))$$

induces $f^* : H^{r,s}_d(V,\mathbb{Z}/n\mathbb{Z}) \to H^{r,s}_d(W,\mathbb{Z}/n\mathbb{Z})$, which proves the lemma since the composite map $Cor_{L(W)/L(V)} \circ Res_{L(W)/L(V)}$ is the multiplication by N. If f is flat, the assertion follows from the contravariant functoriality of the Kato complexes for flat morphisms. In general it follows from the canonical isomorphism

$$H^{r,s}_d(W,\mathbb{Z}/n\mathbb{Z}) \cong H^0(W_{Zar},\mathcal{H}^{d+r}(\mathbb{Z}/n\mathbb{Z}(d+s)))$$

and the similar isomorphism for V following from [BO] by the assumption of smoothness. Here $\mathcal{H}^{d+r}(\mathbb{Z}/n\mathbb{Z}(d+s))$ is the Zariski sheaf associated to $U \to H^{d+r}(U_{et},\mathbb{Z}/n\mathbb{Z}(d+s))$.

Lemma 7.5 *Assume that X_η is irreducible and proper of dimension 2. If $\ell \neq \mathrm{ch}(K)$ is a prime such that $BK_3(K(X_\eta),\ell)$ holds, then $|\mathrm{Ker}(\rho_{X_\eta,\ell^\nu})|$ is bounded with respect to ν.*

Proof Lemma 5.2 induces the following commutative diagram
(7-2)

$$\begin{array}{ccccc} H_0^{et}(X_\eta,\mathbb{Z}/\ell^\nu\mathbb{Z}(-1)) & \to & H_2^K(X_\eta,\mathbb{Z}/\ell^\nu\mathbb{Z}) & \to & \mathrm{Ker}(\rho_{X_\eta,\ell^\nu}) \to 0 \\ \downarrow\alpha & & \downarrow\gamma & & \\ H_0^{et}(X_\eta,\mathbb{Q}_\ell/\mathbb{Z}_\ell(-1))[\ell^\nu] & \xrightarrow{\epsilon[\ell^\nu]} & H_2^K(X_\eta,\mathbb{Q}_\ell/\mathbb{Z}_\ell)[\ell^\nu] & & \\ \downarrow\hookrightarrow & & \downarrow\hookrightarrow & & \\ H_0^{et}(X_\eta,\mathbb{Q}_\ell/\mathbb{Z}_\ell(-1)) & \xrightarrow{\epsilon} & H_2^K(X_\eta,\mathbb{Q}_\ell/\mathbb{Z}_\ell) & & \end{array}$$

where the upper horizontal sequence is exact, α is surjective, γ is an isomorphism by Lemma 7.3, and $\mathrm{Coker}(\epsilon)$ is finite by Theorem 6.1. Since $H_0^{et}(X_\eta,\mathbb{Q}_\ell/\mathbb{Z}_\ell(-1))$ is of cofinite type by Lemma 6.8, this implies that $\mathrm{Ker}(\rho_{X_\eta,\ell^\nu})$ is finite and that

$$\begin{aligned} |\mathrm{Ker}(\rho_{X_\eta,\ell^\nu})| = |\mathrm{Coker}(\epsilon[\ell^\nu])| &\leq |\mathrm{Coker}(\epsilon)|\cdot|\mathrm{Ker}(\epsilon)/\ell^\nu| \\ &\leq |\mathrm{Coker}(\epsilon)|\cdot|\mathrm{Ker}(\epsilon)/\mathrm{Div}(\mathrm{Ker}(\epsilon))|, \end{aligned}$$

where $\mathrm{Div}(A)$ denotes the maximal divisible subgroup of an abelian group A. This proves Lemma 7.5.

Lemma 7.6 *With assumption be as Theorem 1.8 (1), let I_P be the set of all integers whose prime divisors belong to P. Then $\mathrm{Ker}(\rho_{X_\eta,n})$ is bounded with respect to $n \in I_P$.*

Proof By Lemma 7.5 it suffices to show $\mathrm{Ker}(\rho_{X_\eta,\ell^\nu}) = 0$ for almost all $\ell \in P$, $\ell \neq \mathrm{ch}(F)$. We claim that we may assume that X is projective over S having semistable reduction over S. Indeed, by [dJ] there exists a finite morphism $S' \to S$ and an alteration $\widetilde{X} \to X$ with $\widetilde{X}/S'$ satisfying the condition. Noting that (7-1) is covariantly functorial for proper morphisms, we get the commutative diagram

$$\begin{array}{ccccc} H_2^K(\widetilde{X}_\eta,\mathbb{Z}/\ell^\nu\mathbb{Z}) & \to & \mathrm{Ker}(\rho_{\widetilde{X}_\eta,\ell^\nu}) & \to 0 \\ \downarrow & & \downarrow & \\ H_2^K(X_\eta,\mathbb{Z}/\ell^\nu\mathbb{Z}) & \to & \mathrm{Ker}(\rho_{X_\eta,\ell^\nu}) & \to 0 \end{array}$$

Hence the claim follows from Lemma 7.4. Then by Proposition 4.3 and a finite étale base change we may furthermore assume that there exists a very good divisor $Z \subset X$ in the sense of Definition 3.3. Let $U = X \setminus Z$.

Claim 1 *If $\ell \in P$, then $\mathrm{Ker}(\rho_{X_\eta,\ell^\nu}) \subset \mathrm{Im}(SK_1(Z_\eta))$ for all ν.*

Proof We have the commutative diagram

$$\begin{array}{ccccccc} SK_1(Z_\eta)/\ell^\nu & \to & SK_1(X_\eta)/\ell^\nu & \to & SK_1(U_\eta)/\ell^\nu & \to 0 \\ & & \downarrow\alpha_{X_\eta} & & \downarrow\alpha_{U_\eta} & \\ & & H_{-1}^{et}(X_\eta,\mathbb{Z}/\ell^\nu\mathbb{Z}(-1)) & \to & H_{-1}^{et}(U_\eta,\mathbb{Z}/\ell^\nu\mathbb{Z}(-1)) & \end{array}$$

in which α_{X_η} can be identified with ρ_{X_η,ℓ^ν} (cf. 5.2 and 5.3). Thus it suffices to show that α_{U_η} is an injection. By Lemma 5.2 we have the commutative diagram

$$\begin{array}{ccccc} H_0^{et}(U_\eta,\mathbb{Z}/\ell^\nu\mathbb{Z}(-1)) & \xrightarrow{\epsilon_{U_\eta}} & H_2^K(U_\eta,\mathbb{Z}/\ell^\nu\mathbb{Z}) & \to & \mathrm{Ker}(\alpha_{U_\eta})\to 0 \\ \downarrow & & \downarrow & & \\ H_0^{et}(U_\eta,\mathbb{Q}_\ell/\mathbb{Z}_\ell(-1))[\ell^\nu] & \xrightarrow{\cong} & H_2^K(U_\eta,\mathbb{Q}_\ell/\mathbb{Z}_\ell)[\ell^\nu] & & \end{array}$$

where the upper horizontal sequence is exact and the left vertical map is surjective. The right vertical arrow is an isomorphism by Lemma 7.3 and the lower horizontal map is an isomorphism by the proof of Theorem 1.6 (note that $\ell\neq \mathrm{ch}(F)$). This proves the desired assertion.

Claim 2 *If* $\ell \not| \,|\mathrm{Coker}(N_{Z_\eta/K})|$, *then* $\mathrm{Ker}(\rho_{X_\eta,\ell^\nu})\subset \mathrm{Im}(\mathrm{Ker}(N_{Z_\eta/K}))$.

Proof We have the commutative diagram

$$\begin{array}{ccccccc} 0\to & \mathrm{Ker}(N_{Z_\eta/K})/\ell^\nu & \to & SK_1(Z_\eta)/\ell^\nu & \to & K^*/\ell^\nu & \to 0 \\ & \downarrow & & \downarrow & & \| & \\ 0\to & \mathrm{Ker}(N_{X_\eta/K})/\ell^\nu & \to & SK_1(X_\eta)/\ell^\nu & \to & K^*/\ell^\nu & \to 0 \\ & & & \downarrow \rho_{X_\eta,\ell^\nu} & & \downarrow \rho_{\eta,\ell^\nu} & \\ & & & \pi_1^{ab}(X_\eta)/\ell^\nu & \to & \pi_1^{ab}(\eta)/\ell^\nu & \end{array}$$

where ρ_{η,ℓ^ν} is an isomorphism by local class field theory. Under the assumption of the claim, the horizontal sequences are exact. Now the claim follows easily from Claim 1 by simple diagram chasing.

To finish the proof of Lemma 7.6 it suffices to show $\mathrm{Ker}(N_{Z_\eta/K})$ is ℓ-divisible for almost all ℓ. Since $\dim(Z_\eta)=1$, this follows from the class field theory for curves over local fields. Indeed, the kernel of

$$\mathrm{Ker}(N_{Z_\eta/K})\hookrightarrow SK_1(Z_\eta)\xrightarrow{\rho_V}\pi_1^{ab}(Z_\eta)$$

is ℓ-divisible by [Sa1], Th.5.1 so that the assertion follows from Theorem 7.2.

Proof of Theorem 1.8(1): Consider the commutative diagram

$$\begin{array}{ccccccc} 0\to & \mathrm{Ker}(\rho_{X_\eta}) & \to & SK_1(X_\eta) & \to & \pi_1^{ab}(X_\eta) \\ & \downarrow & & \downarrow \pi_1 & & \downarrow \pi_2 \\ 0\to & \varprojlim_{n\in I_P}\mathrm{Ker}(\rho_{X_\eta,n}) & \to & \varprojlim_{n\in I_P} SK_1(X_\eta)/n & \to & \varprojlim_{n\in I_P}\pi_1^{ab}(X_\eta)/n \end{array}$$

Theorem 7.1 implies that

$$\varprojlim_{n\in I_P}\pi_1^{ab}(X_\eta)/n=\prod_{\ell\in P}\pi_1^{ab}(X_\eta)(\ell)\quad\text{and}\quad \mathrm{Ker}(\pi_2)=\prod_{\ell\notin P}\pi_1^{ab}(X_\eta)(\ell).$$

and that the torsion part of the former is finite and that the latter is P-torsion free, namely $nx=0$ with $n\in I_P$ and $x\in\mathrm{Ker}(\pi_2)$ implies $x=0$. Lemma 7.6 implies $\varprojlim_{n\in I_P}\mathrm{Ker}(\rho_{X_\eta,n})$ is finite so that $\mathrm{Ker}(\pi_1)$ is P-divisible by Lemma 7.7 below. Hence the diagram implies that $\mathrm{Ker}(\rho_{X_\eta})$ is an extension of a finite group by a P-divisible group. Now Theorem 1.8(1) follows from Lemma 7.8 below.

Lemma 7.7 *Assume given an abelian group A, a projective system of abelian groups $\{B_n\}_{n\in I_P}$ and a projective system $\{A/nA \xrightarrow{\varphi_n} B_n\}_{n\in I_P}$ of homomorphisms. Write*

$$\hat{\varphi} := \varprojlim_{n\in I_P} \varphi_n \;:\; \hat{A} \to \hat{B}, \quad (\hat{A} = \varprojlim_{n\in I_P} A/nA,\ \hat{B} := \varprojlim_{n\in I_P} B_n)$$

Assume that there exists $0 \neq N \in I_P$ such that $N\cdot\hat{B}_{tor} = 0$ and that $N\cdot\mathrm{Ker}(\hat{\varphi}) = 0$. Then $D := \cap_{n\in I_P} nA$ is the maximal P-divisible subgroup of A.

Proof It is evident that D contains the maximal P-divisible subgroup of A. Thus it suffices to show that D is P-divisible. Let $\pi : A \to \hat{A}$ be the natural map and put $D' = \mathrm{Ker}(\hat{\varphi}\circ\pi)$. By the assumption $D = \mathrm{Ker}(\pi) \subset D'$ and $N \cdot D' \subset D$. Take $x \in D$. For any $n \in I_P$ there exists $y \in A$ such that $x = nN^2y$. Then we have $0 = \hat{\varphi}(\pi(x)) = nN^2\hat{\varphi}(\pi(y))$ so that $\hat{\varphi}(\pi(y)) \in \hat{B}_{tor}$. By the assumption this implies $N\hat{\varphi}(\pi(y)) = \hat{\varphi}(\pi(Ny)) = 0$. Hence $z := Ny \in D'$ so that $x = n(Nz) \in n\cdot D$. This completes the proof of Lemma 7.7.

Lemma 7.8 *Assume given $0 \to D \to A \xrightarrow{\pi} T \to 0$, an exact sequence of abelian groups, where T is torsion and D is P-divisible. Write $T = T(P)\oplus T'$ where $T(P)$ is P-torsion and T' has no P-torsion element. Put $D' = \pi^{-1}(T')$. Then D' is P-divisible and we have $A \cong D' \oplus T(P)$.*

Proof We have the exact sequence $0 \to D \to D' \to T' \to 0$. By definition T' is P-divisible and hence D' is P-divisible. We show $A \cong D' \oplus T(P)$. It suffices to construct a map $s_\ell : T\{\ell\} \to A$ for each $\ell \in P$ such that $\pi \circ s_\ell$ is the identity, where $M\{\ell\}$ denotes the ℓ-primary torsion part of an abelian group M. We have an exact Tor-sequence

$$0 \to D'\{\ell\} \to A\{\ell\} \to T\{\ell\} \to 0 = D' \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell$$

which can be viewed as an exact sequence of $\mathbb{Z}_\ell$-modules. Since $D'\{\ell\}$ is ℓ-divisible, it is an injective $\mathbb{Z}_\ell$-module (cf. [HS], Th. 7.1). Thus the above sequence splits and we get the desired map s_ℓ. This completes the proof of Lemma 7.8.

Proof of Theorem 1.9: If V is smooth and proper, the claim follows from Theorem 1.8 (1) and Theorem 7.2. If V is proper, there exists an alteration $\widetilde{V} \to V$ with $\widetilde{V}$ smooth and proper [dJ], and we get a commutative diagram

$$\begin{array}{ccc} SK_1(\widetilde{V}) & \overset{N_{\widetilde{V}/V}}{\to} & SK_1(V) \\ \downarrow N_{\widetilde{V}/K} & & \downarrow N_{V/K} \\ K^* & = & K^* \end{array}$$

in which $N_{\widetilde{V}/V}$ has finite cokernel by Lemma 6.2. Hence the claim for V follows from that for $\widetilde{V}$. If V is not proper, take an open immersion $V \hookrightarrow W$ with W

proper over K, and let $Z = W \setminus V$. Then we have a commutative diagram with exact top row

$$\begin{array}{ccccc} SK_1(Z) & \to & SK_1(W) & \to & SK_1(V) \to 0 \\ \downarrow N_{Z/K} & & \downarrow N_{W/K} & & \\ K^* & = & K^* & & \end{array}$$

It gives a map $\mathrm{Ker}(N_{W/K}) \to SK_1(V)$ with finite cokernel, which shows that it suffices to consider W, i.e. the case that V is proper. This completes the proof of Theorem 1.9.

Proof of Theorem 1.8 (2) and (3): Lemmas 5.2 and 5.3 induce the exact sequence for $n \in I_P$

$$H_2^K(X_\eta, \mathbb{Q}/\mathbb{Z})[n] \to SK_1(X_\eta)/n \xrightarrow{\rho_{X_\eta,n}} \pi_1^{ab}(X_\eta)/n \tag{7-3}$$

where we used that $H_2^K(X_\eta, \mathbb{Z}/n\mathbb{Z}) \cong H_2^K(X_\eta, \mathbb{Q}/\mathbb{Z})[n]$ by Lemma 7.3. The assumption $H_2(\Gamma_{\widetilde{X}_s}, \mathbb{Q}) = 0$ implies that $H_2(\Gamma_{\widetilde{X}_s}, \mathbb{Q}/\mathbb{Z})$ is finite. On the other hand, we have maps

$$H_2^K(\widetilde{X}_\eta, \mathbb{Q}/\mathbb{Z}') \xrightarrow{\Delta^2_{\widetilde{X}}} H_2^K(\widetilde{X}_s, \mathbb{Q}/\mathbb{Z}') \xrightarrow{\gamma_{\widetilde{X}_s}} H_2^K(\Gamma_{\widetilde{X}_s}, \mathbb{Q}/\mathbb{Z}')\ , \quad \mathbb{Q}/\mathbb{Z}' = \bigoplus_{\ell \neq \mathrm{ch}(K)} \mathbb{Q}_\ell/\mathbb{Z}_\ell,$$

where $\Delta^2_{\widetilde{X}}$ is an isomorphism by Theorem 1.6 and $\gamma_{\widetilde{X}_s}$ is an isomorphism by Theorem 1.4. Hence $H_2^K(\Gamma_{\widetilde{X}_s}, \mathbb{Q}/\mathbb{Z}')$ is finite, and by Lemma 7.4, $H_2^K(X_\eta, \mathbb{Q}/\mathbb{Z}')$ is finite as well. Thus, passing to the limit, the sequences (7-3) induces an injection

$$\varprojlim_{n \in I_P} SK_1(X_\eta)/n \hookrightarrow \varprojlim_{n \in I_P} \pi_1^{ab}(X_\eta)/n,$$

because $\varprojlim A[n] = 0 = \varprojlim^1 A[n]$ for any finite abelian group A. Now Theorem 1.8 (2) follows from Lemma 7.7 by the same argument as in the proof of Theorem 1.8(1). Finally Theorem 1.8 (3) follows from 1.8 (2) together with Theorems 1.4 and 1.5, because in the case of good reduction the complex Γ_{X_s} is contractible.

References

[Ar] M. Artin, *Algebraic approximation of structures over complete local rings*, Publ. Math. IHES 36 (1969), 23–58.

[BBD] A. Beilinson, J. Bernstein, and P. Deligne, *Faisceaux pervers*, Astérisque 100 (1982), 3–171.

[Bl] S. Bloch, *Higher Algebraic K-theory and class field theory for arithmetic surfaces*, Ann. of Math. 114 (1981), 229–265.

[BK] S. Bloch and K. Kato, *p-adic etale cohomology*, Publ. Math. IHES 63 (1986), 107–152.

[BO] S. Bloch and A. Ogus, *Gersten's conjecture and the homology of schemes*, Ann. Ec. Norm. Sup. 4 serie 7 (1974), 181–202.

[CT] J.-L. Colliot-Thélène, *On the reciprocity sequence in the higher class field theory of function fields*, Algebraic K-Theory and Algebraic Topology (Lake Louise, AB, 1991), (J.F. Jardine and V.P. Snaith, ed), 35–55, Kluwer Academic Publishers, 1993.

[CTSS] J.-L. Colliot-Thélène, J.-J. Sansuc and C. Soulé, *Torsion dans le groupe de Chow de codimension deux*, Duke Math. J. 50 (1983), 763–801.

[De] P. Deligne, *La conjecture de Weil II*, Publ. Math. IHES 52 (1981), 313–428.

[DI] P. Deligne and L. Illusie, *Relèvements modulo p^2 et décomposition du complexe de de Rham* Invent. Math. 89 (1987), 247–270.

[dJ] de Jong, A.J., *Smoothness, semi-stability and alterations*, Publ. Math. IHES 83 (1996), 51–93.

[Fa] G. Faltings, *Crystalline cohomology and p-adic repesentations*, in: Algebraic Analysis, Geometry and Number Theory, The John Hopkins University Press, Baltimore (1989), 25–80

[FM] J.-M. Fontaine and W. Messing, *p-adic periods and p-adic étale cohomology*, Contemporary Math. 67 (1987), 179–207

[Fu] K. Fujiwara, *A proof of the absolute purity conjecture (after Gabber)*, Algebraic Geometry (Azumino, JAPAN, 2001), Advanced Studies in Pure Math. 36 (2002), 153–184

[GS] M. Gros and N. Suwa, *Application d'Abel-Jacobi p-adique et cycles algébriques* Duke Math. J. 57 (1998), 579–613

[HS] P.J. Hilton and U. Stammbach, *A course in Homological Algebra*, GTM, 4, Springer-Verlag, 1970.

[H] O. Hyodo, *A note on p-adic etale cohomology in the semi-stable reduction case*, Invent. Math. 91 (1988), 543–557.

[HK] O. Hyodo and K. Kato, *Semi-stable reduction and crystalline cohomology with logarithmic poles*, in: Périodes p-adiques, Astérisque 223 (1994), 221-268.

[Il] L. Illusie, *Complexe de De Rham-Witt et cohomologie cristalline*, Ann. Scient. ENS 12 (1979), 501–661.

[J1] U. Jannsen, *Principe de Hasse cohomologique*, Progress in Mathematics 102, Séminaire de théorie des nombres de Paris (1989-90), (Sinnou David, Editor), 121–138, Birkhäuser, Boston/Basel/ Berlin, 1992.

[J2] U. Jannsen, *Letter from Jannsen to Gross on higher Abel-Jacobi maps*, The arithmetic and geometry of algebraic cycles (Banff, AB, 1998), 261–275, Kluwer Academic Publishers, 2000.

[J3] U. Jannsen, *Hasse principles for higher dimensional fields*, in preparation (see www.mathematik.uni-regensburg.de/Jannsen).

[JS] U. Jannsen and S. Saito, *Bertini and Lefschetz theorems for schemes over discrete valuation rings*, in preparation (see www.mathematik.uni-regensburg.de/Jannsen).

[JSS] U. Jannsen, S. Saito and K. Sato, *Etale duality for constructible sheaves on arithmetic schemes*, in preparation (see www.mathematik.uni-regensburg.de/Jannsen).

[K1] K. Kato, *A Hasse principle for two dimensional global fields*, J. für die reine und angew. Math. 366 (1986), 142–183.

[K2] K. Kato, *Semi-stable reduction and p-adic etale cohomology*, in: Périodes p-adiques, Astèrisque 223 (1994), 269–293.

[K3] K. Kato, *Logarithmic structures of Fontaine-Illusie*, in: Algebraic Analysis, Geometry and Number Theory, 191–224, The Johns Hopkins Univ. Press (1989).

[KM] N. Katz and W. Messing, *Some consequences of the Riemann hypothesis for varieties over finite fields*, Invent. Math. 23 (1974), 73–77

[KS1] K. Kato and S. Saito, *Unramified class field theory of arithmetic surfaces*, Ann. of Math. 118 (1985), 241–275.

[KS2] K. Kato and S. Saito, *Global class field theory of arithmetic schemes*, Contemporary Math. 55, I (1986), 255–331.

[La] S. Landsburg, *Relative Chow groups*, Illinois J. of Math. 35 (1991), 618–641.

[Lo] P. Lorenzon, *Logarithmic Hodge-Witt forms and Hyodo-Kato cohomology*, J. Algebra 249, (2002), 247–265.

[MS] A.S. Merkurjev and A. A. Suslin, *K-cohomology of Severi-Brauer Varieties and the norm residue homomorphism*, Math. USSR Izvestiya 21 (1983), 307–340.

[Mo] T. Moser, *A duality theorem for étale p-torsion sheaves on complete varieties over a finite field*, Compositio Math. 117 (1999), 123–152.

[N] M. Nagata, *A generalization of the imbedding problem of an abstract variety in a complete variety* J. Math. Kyoto Univ. 3 (1963), 89–102

[RZ] M. Rapoport and T. Zink, *Über die lokale Zetafunktion von Shimuravarietäten. Monodoromiefiltration und verschwindende Zyklen in ungleicher Charakteristik*, Invent. Math. 68 (1982), 21–101.

[Sa1] S. Saito, *Class field theory for curves over local fields*, J. Number Theory 21 (1985), 44–80.

[Sa2] S. Saito, *Unramified class field theory of arithmetical schemes*, Ann. of Math. 121 (1985), 251–281.

[Sa3] S. Saito, *A global duality theorem for varieties over global fields*, Algebraic K-theory: connections with geometry and topology (Lake Louise, AB, 1997), (J.F. Jardine and V.P. Snaith, ed), 425–444, Kluwer Academic Publishers, 1998.

[Sa4] S. Saito, *Cohomological Hasse principle for a threefold over a finite field*, Algebraic K-Theory and Algebraic Topology (Lake Louise, AB, 1991), (P.G. Goerss and J.F. Jardine, ed), 229–241, Kluwer Academic Publishers, 1993.

[Sat1] K. Sato, *Injectivity of the torsion cycle map of codimension two of varieties over p-adic fields with semi-stable reduction*, J. reine angew. Math. 501 (1998), 221–235.

[Sat2] K. Sato, *Non-divisible cycles on a $K3$ surface over p-adic fields*, preprint, TECH-MATH No. 12 (1999), Tokyo Inst. of Technology.

[Sat3] K. Sato, *Purity for a logarithmic Hodge-Witt sheaf on semi-stable varieties*, preprint.

[Sch] C. Scheiderer, *Real and Étale Cohomology*, Lecture Notes in Math. 1588, Springer-Verlag, 1991.

[Se] J.-P. Serre, *Cohomologie Galoisienne*, Lecture Notes in Math. 5, Springer-Verlag, 1965.

[Sw] N. Suwa, *A note on Gersten's conjecture for logarithmic Hodge-Witt sheaves*, K-theory 9 (1995), 245–271.

[Sz] T. Szamuely, *Sur la théorie des corps de classes pour les variétés sur les corps p-adiques*, J. reine angew. Math. 525 (2000), 183–212.

[Ts1] T. Tsuji, *Poincaré duality for logarithmic crystalline cohomology*, Compositio Math. 118 (1999), 11–41.

[Ts2] T. Tsuji, *On p-adic nearby cycles of log smooth families*, Bull. Soc. math. France 128 (2000), 529–575.

[Ts3] T. Tsuji, *The maximal unramified quotients of integral p-adic étale cohomology groups*, preprint.

[Y] T. Yoshida, *Finiteness theorems in the class field theory of varieties over local fields*, To appear in J. Number Theory.

[V] V. Voevodsky, *On 2-torsion in motivic cohomology*, preprint.

[SGA1] A. Grothendieck et al., *Revêtements tales et Groupe Fondamental* Lect. Notes in Math. 224, Springer, berlin 1971.

[SGA 4.3] M. Artin, A. Grothendieck, J.L. Verdier et al., *Théorie des Topos et Cohomologie Etale des Schemas, Tome 3*, Lect. Notes in Math. 305, Springer, Berlin 1973.

[SGA 5] A. Grothendieck et al., *Cohomologie l-adique et Fonctions L*, Lect. Notes in Math. 589, Springer, Berlin 1977.

Uwe Jannsen
Fakultät für Mathematik
Universität Regensburg
Universitätsstr. 31
93040 Regensburg
GERMANY
uwe.jannsen@mathematik.uni-regensburg.de

Shuji Saito
Graduate School of Mathematics
Nagoya University
Chikusa-ku
NAGOYA 464-8602
JAPAN
sshuji@msb.biglobe.ne.jp

On the Structure of Ideal Class Groups of CM-Fields

dedicated to Professor K. Kato on his 50th birthday

Masato Kurihara

Received: November 30, 2002
Revised: June 30, 2003

Abstract. For a CM-field K which is abelian over a totally real number field k and a prime number p, we show that the structure of the χ-component A_K^χ of the p-component of the class group of K is determined by Stickelberger elements (zeta values) (of fields containing K) for an odd character χ of $\mathrm{Gal}(K/k)$ satisfying certain conditions. This is a generalization of a theorem of Kolyvagin and Rubin. We define higher Stickelberger ideals using Stickelberger elements, and show that they are equal to the higher Fitting ideals. We also construct and study an Euler system of Gauss sum type for such fields. In the appendix, we determine the initial Fitting ideal of the non-Teichmüller component of the ideal class group of the cyclotomic $\mathbf{Z}_p$-extension of a general CM-field which is abelian over k.

0 Introduction

It is well-known that the cyclotomic units give a typical example of Euler systems. Euler systems of this type were systematically investigated by Kato [8], Perrin-Riou [14], and in the book by Rubin [18]. In this paper, we propose to study Euler systems of Gauss sum type which are not Euler systems in the sense of [18]. We construct an Euler system in the multiplicative groups of CM-fields, which is a generalization of the Euler system of Gauss sums, and generalize a structure theorem of Kolyvagin and Rubin for the minus class groups of imaginary abelian fields to general CM-fields.

The aim of this paper is to prove the structure theorem (Theorem 0.1 below), and we do not pursue general results on the Euler systems of Gauss sum type in this paper. One of very deep and remarkable works of Kato is his construction of the Euler system (which lies in $H^1(T)$) for a $\mathbf{Z}_p$-representation T associated to a modular form. We remark that we do not have an Euler system of Gauss sum type in $H^1(T)$, but fixing $n > 0$ we can find an Euler system of Gauss sum type in $H^1(T/p^n)$, which will be studied in our forthcoming paper.

We will describe our main result. Let k be a totally real number field, and K be a CM-field containing k such that K/k is finite and abelian. We consider an odd prime number p and the p-primary component $A_K = Cl_K \otimes \mathbf{Z}_p$ of the ideal class group of K. Suppose that p does not divide $[K : k]$. Then, A_K is decomposed into $A_K = \bigoplus_\chi A_K^\chi$ where A_K^χ is the χ-component which is an O_χ-module (where $O_\chi = \mathbf{Z}_p[\mathrm{Image}\,\chi]$, for the precise definition, see 1.1), and χ ranges over $\mathbf{Q}_p$-conjugacy classes of $\overline{\mathbf{Q}_p}^\times$-valued characters of $\mathrm{Gal}(K/k)$ (see also 1.1).

For $k = \mathbf{Q}$ and $K = \mathbf{Q}(\mu_p)$ (the cyclotomic field of p-th roots of unity), Rubin in [17] described the detail of Kolyvagin's method ([10] Theorem 7), and determined the structure of $A^\chi_{\mathbf{Q}(\mu_p)}$ as a $\mathbf{Z}_p$-module for an odd χ, by using the Euler system of Gauss sums (Rubin [17] Theorem 4.4). We generalize this result to arbitrary CM-fields.

In our previous paper [11], we proposed a new definition of the Stickelberger ideal. In this paper, for certain CM-fields, we define higher Stickelberger ideals which correspond to higher Fitting ideals. In §3, using the Stickelberger elements of fields containing K, we define the higher Stickelberger ideals $\Theta_{i,K} \subset \mathbf{Z}_p[\mathrm{Gal}(K/k)]$ for $i \geq 0$ (cf. 3.2). Our definition is different from Rubin's. (Rubin defined the higher Stickelberger ideal using the argument of Euler systems. We do not use the argument of Euler systems to define our $\Theta_{i,K}$.) We remark that our $\Theta_{i,K}$ is numerically computable, since the Stickelberger elements are numerically computable. We consider the χ-component $\Theta^\chi_{i,K}$.

We study the structure of the χ-component A_K^χ as an O_χ-module. We note that p is a prime element of O_χ because the order of $\mathrm{Image}\,\chi$ is prime to p.

Theorem 0.1. *We assume that the Iwasawa μ-invariant of K is zero (cf. Proposition 2.1), and χ is an odd character of* $\mathrm{Gal}(K/k)$ *such that $\chi \neq \omega$ (where ω is the Teichmüller character giving the action on μ_p), and that $\chi(\mathfrak{p}) \neq 1$ for every prime $\mathfrak{p}$ of k above p. Suppose that*

$$A_K^\chi \simeq O_\chi/(p^{n_1}) \oplus \ldots \oplus O_\chi/(p^{n_r})$$

with $0 < n_1 \leq \ldots \leq n_r$. Then, for any i with $0 \leq i < r$, we have

$$(p^{n_1+\ldots+n_{r-i}}) = \Theta^\chi_{i,K}$$

and $\Theta^\chi_{i,K} = (1)$ for $i \geq r$. Namely,

$$A_K^\chi \simeq \bigoplus_{i \geq 0} \Theta^\chi_{i,K}/\Theta^\chi_{i+1,K}.$$

In the case $K = \mathbf{Q}(\mu_p)$ and $k = \mathbf{Q}$, Theorem 0.1 is equivalent to Theorem 4.4 in Rubin [17].

This theorem says that the structure of A_K^χ as an O_χ-module is determined by the Stickelberger elements. Since the Stickelberger elements are defined from the partial zeta functions, we may view our theorem as a manifestation of a very general phenomena in number theory that zeta functions give us information on various important arithmetic objects.

In general, for a commutative ring R and an R-module M such that

$$R^m \xrightarrow{f} R^r \longrightarrow M \longrightarrow 0$$

is an exact sequence of R-modules, the i-th Fitting ideal of M is defined to be the ideal of R generated by all $(r-i)\times(r-i)$ minors of the matrix corresponding to f for i with $0 \leq i < r$. If $i \geq r$, it is defined to be R. (For more details, see Northcott [13]). Using this terminology, Theorem 0.1 can be simply stated as

$$\mathrm{Fitt}_{i,O_\chi}(A_K^\chi) = \Theta_{i,K}^\chi$$

for all $i \geq 0$.

The proof of Theorem 0.1 is divided into two parts. We first prove the inclusion $\mathrm{Fitt}_{i,O_\chi}(A_K^\chi) \supset \Theta_{i,K}^\chi$. To do this, we need to consider a general CM-field which contains K. Suppose that F is a CM-field containing K such that F/k is abelian, and F/K is a p-extension. Put $R_F = \mathbf{Z}_p[\mathrm{Gal}(F/k)]$. For a character χ satisfying the conditions in Theorem 0.1, we consider $R_F^\chi = O_\chi[\mathrm{Gal}(F/K)]$ and $A_F^\chi = A_F \otimes_{R_F} R_F^\chi$ where $\mathrm{Gal}(K/k)$ acts on O_χ via χ. For the χ-component $\theta_F^\chi \in R_F^\chi$ of the Stickelberger element of F (cf. 1.2), we do not know whether $\theta_F^\chi \in \mathrm{Fitt}_{0,R_F^\chi}(A_F^\chi)$ always holds or not (cf. Popescu [15] for function fields). But we will show in Corollary 2.4 that the dual version of this statement holds, namely

$$\iota(\theta_F^\chi) \in \mathrm{Fitt}_{0,R_F^{\chi^{-1}}}((A_F^\chi)^\vee)$$

where $\iota : R_F \longrightarrow R_F$ is the map induced by $\sigma \mapsto \sigma^{-1}$ for $\sigma \in \mathrm{Gal}(F/k)$, and $(A_F^\chi)^\vee$ is the Pontrjagin dual of A_F^χ. We can also determine the right hand side $\mathrm{Fitt}_{0,R_F^{\chi^{-1}}}((A_F^\chi)^\vee)$. In the Appendix, for the cyclotomic $\mathbf{Z}_p$-extension F_∞/F, we determine the initial Fitting ideal of (the Pontrjagin dual of) the non-ω component of the p-primary component of the ideal class group of F_∞ as a $\Lambda_F = \mathbf{Z}_p[[\mathrm{Gal}(F_\infty/k)]]$-module (we determine $\mathrm{Fitt}_{0,\Lambda_F}((A_{F_\infty})^\vee)$ except ω-component, see Theorem A.5). But for the proof of Theorem 0.1, we only need Corollary 2.4 which can be proved more simply than Theorem A.5, so we postpone Theorem A.5 and its proof until the Appendix. Concerning the Iwasawa module $X_{F_\infty} = \varprojlim A_{F_n}$ where F_n is the n-th layer of F_∞/F, we computed in [11] the initial Fitting ideal under certain hypotheses, for example, if $F/\mathbf{Q}$ is abelian. Greither in his recent paper [4] computed the initial Fitting ideal of X_{F_∞} more generally.

In our previous paper [11] §8, we showed that information on the initial Fitting ideal of the class group of F yields information on the higher Fitting ideals of the class group of K. Using this method, we will show $\mathrm{Fitt}_{i,O_\chi}(A_K^\chi) \supset \Theta_{i,K}^\chi$ in Proposition 3.2.

In order to prove the other inclusion, we will use the argument of Euler systems. By Corollary 2.4 which was mentioned above, we obtain

$$\theta_F^\chi A_F^\chi = 0.$$

(We remark that this has been obtained recently also in Greither [4] Corollary 2.7.) Using this property, we show that for any finite prime ρ of F there is an element $g_{F,\rho}^\chi \in (F^\times \otimes \mathbf{Z}_p)^\chi$ such that $\mathrm{div}(g_{F,\rho}^\chi) = \theta_F^\chi[\rho]^\chi$ in the divisor group where $[\rho]$ is the divisor corresponding to ρ (for the precise relation, see §4). These $g_{F,\rho}^\chi$'s become an Euler system of Gauss sum type (see §4). For the Euler system of Gauss sums, a crucial property is Theorem 2.4 in Rubin [18] which is a property on the image in finite fields, and which was proved by Kolyvagin, based on the explicit form of Gauss sums. But we do not know the explicit form of our $g_{F,\rho}^\chi$, so we prove, by a completely different method, the corresponding property (Proposition 4.7) which is a key proposition in §4.

It is possible to generalize Theorem 0.1 to characters of order divisible by p satisfying some conditions. We hope to come back to this point in our forthcoming paper.

I would like to express my sincere gratitude to K. Kato for introducing me to the world of arithmetic when I was a student in the 1980's. It is my great pleasure to dedicate this paper to Kato on the occasion of his 50th birthday. I would like to thank C. Popescu heartily for a valuable discussion on Euler systems. I obtained the idea of studying the elements $g_{F,\rho}^\chi$ from him. I would also like to thank the referee for his careful reading of this manuscript, and for his pointing out an error in the first version of this paper. I heartily thank C. Greither for sending me his recent preprint [4].

Notation

Throughout this paper, p denotes a fixed odd prime number. We denote by $\mathrm{ord}_p : \mathbf{Q}^\times \longrightarrow \mathbf{Z}$ the normalized discrete valuation at p. For a positive integer n, μ_n denotes the group of all n-th roots of unity. For a number field F, O_F denotes the ring of integers. For a group G and a G-module M, M^G denotes the G-invariant part of M (the maximal subgroup of M on which G acts trivially), and M_G denotes the G-coinvariant of M (the maximal quotient of M on which G acts trivially). For a commutative ring R, $R^\times$ denotes the unit group.

1 Preliminaries

1.1. Let $\mathcal{G}$ be a profinite abelian group such that $\mathcal{G} = \Delta \times \mathcal{G}'$ where $\#\Delta$ is finite and prime to p, and $\mathcal{G}'$ is a pro-p group. We consider the completed group ring $\mathbf{Z}_p[[\mathcal{G}]]$ which is decomposed into

$$\mathbf{Z}_p[[\mathcal{G}]] = \mathbf{Z}_p[\Delta][[\mathcal{G}']] \simeq \bigoplus_{\chi} O_\chi[[\mathcal{G}']]$$

where χ ranges over all representatives of $\mathbf{Q}_p$-conjugacy classes of characters of Δ (a $\overline{\mathbf{Q}_p}^{\times}$-valued character χ is said to be $\mathbf{Q}_p$-conjugate to χ' if $\sigma\chi = \chi'$ for some $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$), and O_χ is $\mathbf{Z}_p[\mathrm{Image}\,\chi]$ as a $\mathbf{Z}_p$-module, and Δ acts on it via χ ($\sigma x = \chi(\sigma)x$ for $\sigma \in \Delta$ and $x \in O_\chi$). Hence, any $\mathbf{Z}_p[[\mathcal{G}]]$-module M is decomposed into $M \simeq \bigoplus_\chi M^\chi$ where

$$M^\chi \simeq M \otimes_{\mathbf{Z}_p[\Delta]} O_\chi \simeq M \otimes_{\mathbf{Z}_p[[\mathcal{G}]]} O_\chi[[\mathcal{G}']].$$

In particular, M^χ is an $O_\chi[[\mathcal{G}']]$-module. For an element x of M, the χ-component of x is denoted by $x^\chi \in M^\chi$.
Let 1_Δ be the trivial character $\sigma \mapsto 1$ of Δ. We denote by M^1 the trivial character component, and define M^* to be the component obtained from M by removing M^1, namely

$$M = M^1 \oplus M^*.$$

Suppose further that $\mathcal{G}' = G \times \mathcal{G}''$ where G is a finite p-group. Let ψ be a character of G. We regard $\chi\psi$ as a character of $\mathcal{G}_0 = \Delta \times G$, and define $M^{\chi\psi}$ by $M^{\chi\psi} = M \otimes_{\mathbf{Z}_p[\mathcal{G}_0]} O_{\chi\psi}$ where $O_{\chi\psi}$ is $O_{\chi\psi} = \mathbf{Z}_p[\mathrm{Image}\,\chi\psi]$ on which $\mathcal{G}_0$ acts via $\chi\psi$. By definition, if $\chi \neq 1_\Delta$, we have $M^{\chi\psi} \simeq (M^*)^{\chi\psi}$.

Let k be a totally real number field and F be a CM-field such that F/k is finite and abelian, and $\mu_p \subset F$. We denote by F_∞/F the cyclotomic $\mathbf{Z}_p$-extension, and put $\mathcal{G} = \mathrm{Gal}(F_\infty/k)$. We write $\mathcal{G} = \Delta \times \mathcal{G}'$ as above. A $\mathbf{Z}_p[[\mathcal{G}]]$-module M is decomposed into $M = M^+ \oplus M^-$ with respect to the action of the complex conjugation where $M^\pm$ is the $\pm$-eigenspace. By definition, $M^- = \bigoplus_{\chi:\mathrm{odd}} M^\chi$ where χ ranges over all odd characters of Δ. We consider the Teichmüller character ω giving the action of Δ on μ_p, and define $M^\sim$ to be the component obtained from M^- by removing M^ω, namely

$$M^- = M^\sim \oplus M^\omega.$$

For an element x of M, we write $x^\sim$ the component of x in $M^\sim$.

1.2. Let k, F, F_∞ be as in 1.1, and S be a finite set of finite primes of k containing all the primes which ramify in F/k. We define in the usual way the partial zeta function for $\sigma \in \mathrm{Gal}(F/k)$ by

$$\zeta_S(s,\sigma) = \sum_{\substack{(\mathfrak{a},F/k)=\sigma \\ \mathfrak{a} \text{ is prime to } S}} N(\mathfrak{a})^{-s}$$

for $\mathrm{Re}(s) > 1$ where $N(\mathfrak{a})$ is the norm of $\mathfrak{a}$, and $\mathfrak{a}$ runs over all integral ideals of k, coprime to the primes in S such that the Artin symbol $(\mathfrak{a}, F/k)$ is equal to σ. The partial zeta functions are meromorphically continued to the whole complex plane, and holomorphic everywhere except for $s = 1$. We define

$$\theta_{F,S} = \sum_{\sigma \in \mathrm{Gal}(F/k)} \zeta_S(0, \sigma)\sigma^{-1}$$

which is an element of $\mathbf{Q}[\mathrm{Gal}(F/k)]$ (cf. Siegel [21]). Suppose that S_F is the set of ramifying primes of k in F/k. We simply write θ_F for θ_{F,S_F}. We know by Deligne and Ribet the non ω-component $(\theta_{F,S})^{\sim} \in \mathbf{Q}_p[\mathrm{Gal}(F/k)]^{\sim}$ is in $\mathbf{Z}_p[\mathrm{Gal}(F/k)]^{\sim}$. In particular, for a character χ of Δ with $\chi \neq \omega$, we have $(\theta_{F,S})^{\chi} \in \mathbf{Z}_p[\mathrm{Gal}(F/k)]^{\chi}$.
Suppose that S contains all primes above p. Let F_n denote the n-th layer of the cyclotomic $\mathbf{Z}_p$-extension F_∞/F, and consider $(\theta_{F_n,S})^{\sim} \in \mathbf{Z}_p[\mathrm{Gal}(F_n/k)]^{\sim}$. These $\theta^{\sim}_{F_n,S}$'s become a projective system with respect to the canonical restriction maps, and we define

$$\theta^{\sim}_{F_\infty,S} \in \mathbf{Z}_p[[\mathrm{Gal}(F_\infty/k)]]^{\sim}$$

to be their projective limit. This is essentially (the non ω-part of) the p-adic L-function of Deligne and Ribet [1].

2 Initial Fitting ideals

Let k, F, F_∞ be as in §1. We denote by k_∞/k the cyclotomic $\mathbf{Z}_p$-extension, and assume that $F \cap k_\infty = k$. Our aim in this section is to prove Proposition 2.1 and Corollary 2.4 below.

2.1. Let S be a finite set of finite primes of k containing ramifying primes in F_∞/k. We denote by F^+ the maximal real subfield of F. Put $\Lambda_F = \mathbf{Z}_p[[\mathrm{Gal}(F_\infty/k)]]$ and $\Lambda_{F^+} = \mathbf{Z}_p[[\mathrm{Gal}(F^+_\infty/k)]]$ which is naturally isomorphic to the plus part Λ_F^+ of Λ_F. We denote by $\mathcal{M}_{\infty,S}$ the maximal abelian pro-p extension of F^+_∞ which is unramified outside S, and by $\mathcal{X}_{F^+_\infty,S}$ the Galois group of $\mathcal{M}_{\infty,S}/F^+_\infty$. We study $\mathcal{X}_{F^+_\infty,S}$ which is a torsion Λ_{F^+}-module.
We consider a ring homomorphism $\tau^{-1}\iota : \Lambda_F \longrightarrow \Lambda_F$ which is defined by $\sigma \mapsto \kappa(\sigma)\sigma^{-1}$ for $\sigma \in \mathrm{Gal}(F_\infty/k)$ where $\kappa : \mathrm{Gal}(F_\infty/k) \longrightarrow \mathbf{Z}_p^\times$ is the cyclotomic character giving the action of $\mathrm{Gal}(F_\infty/k)$ on μ_{p^∞}. Let $(\Lambda_F)^{\sim}$ and $(\Lambda_F^+)^* = (\Lambda_{F^+})^*$ be as in §1.1. Then, $\tau^{-1}\iota$ induces

$$\tau^{-1}\iota : (\Lambda_F)^{\sim} \longrightarrow (\Lambda_{F^+})^*.$$

Let $\theta^{\sim}_{F_\infty,S} \in (\Lambda_F)^{\sim}$ be the Stickelberger element defined in 1.2.

Proposition 2.1. *Assume that the the Iwasawa μ-invariant of F is zero, namely $\mathcal{X}_{F^+_\infty,S}$ is a finitely generated $\mathbf{Z}_p$-module. Then,* $\mathrm{Fitt}_{0,\Lambda_{F^+}}((\mathcal{X}_{F^+_\infty,S})^*)$ *is generated by $\tau^{-1}\iota(\theta^{\sim}_{F_\infty,S})$ except the trivial character component, namely*

$$\mathrm{Fitt}_{0,\Lambda_{F^+}}((\mathcal{X}_{F^+_\infty,S})^*)^* = (\tau^{-1}\iota(\theta^{\sim}_{F_\infty,S})).$$

Proof. We use the method in [11]. In fact, the proof of this proposition is much easier than that of Theorem 0.9 in [11].
We decompose $\mathcal{G} = \mathrm{Gal}(F_\infty/k)$ as in 1.1 ($\mathcal{G} = \Delta \times \mathcal{G}'$). Suppose that $\mathbf{c}$ is the complex conjugation in Δ and put $\Delta^+ = \Delta/ < \mathbf{c} >$, and $\mathcal{G}_0 = \mathrm{Gal}(F^+/k)$. Then, we can write $\mathcal{G}_0 = \Delta^+ \times G$ where G is a p-group. For a character χ of Δ^+ with $\chi \neq 1_{\Delta^+}$, and a character ψ of G, we regard $\chi\psi$ as a character of $\mathcal{G}_0$. We consider $(\mathcal{X}_{F^+_\infty,S})^{\chi\psi}$ which is an $O_{\chi\psi}[[\mathrm{Gal}(F_\infty/F)]]$-module (cf. 1.1). Our assumption of the vanishing of the μ-invariant implies that $(\mathcal{X}_{F^+_\infty,S})^{\chi\psi}$ is a finitely generated $O_{\chi\psi}$-module. We will first show that $(\mathcal{X}_{F^+_\infty,S})^{\chi\psi}$ is a free $O_{\chi\psi}$-module.
Let $H \subset G$ be the kernel of ψ, and M be the subfield of F corresponding to H, namely $\mathrm{Gal}(F/M) = H$. We denote by M_∞ the cyclotomic $\mathbf{Z}_p$-extension of M and regard H as the Galois group of F_∞/M_∞. We will see that the H-coinvariant $((\mathcal{X}_{F^+_\infty,S})^\chi)_H$ is naturally isomorphic to $(\mathcal{X}_{M^+_\infty,S})^\chi$. In fact, by taking the dual, it is enough to show that the natural map $H^1_{et}(O_{M^+_\infty}[1/S], \mathbf{Q}_p/\mathbf{Z}_p)^{\chi^{-1}} \longrightarrow (H^1_{et}(O_{F^+_\infty}[1/S], \mathbf{Q}_p/\mathbf{Z}_p)^{\chi^{-1}})^H$ of etale cohomology groups is bijective where $O_{M^+_\infty}[1/S]$ (resp. $O_{F^+_\infty}[1/S]$) is the ring of S-integers in M^+_∞ (resp. F^+_∞). This follows from the Hochschild-Serre spectral sequence and $H^1(H, \mathbf{Q}_p/\mathbf{Z}_p)^{\chi^{-1}} = H^2(H, \mathbf{Q}_p/\mathbf{Z}_p)^{\chi^{-1}} = 0$. Hence, regarding $\chi\psi$ as a character of $\mathrm{Gal}(M^+/k)$, we have

$$(\mathcal{X}_{F^+_\infty,S})^{\chi\psi} = (\mathcal{X}_{M^+_\infty,S})^{\chi\psi}.$$

We note that $(\mathcal{X}_{M^+_\infty,S})^\chi$ does not have a nontrivial finite $O_\chi[[\mathcal{G}']]$-submodule (Theorem 18 in Iwasawa [5]), so is free over O_χ by our assumption of the μ-invariant. We will use the same method as Lemma 5.5 in [11] to prove that $(\mathcal{X}_{M^+_\infty,S})^{\chi\psi}$ is free over $O_{\chi\psi}$. We may assume $\psi \neq 1_G$, so p divides the order of $\mathrm{Gal}(M^+/k)$. Let C be the subgroup of $\mathrm{Gal}(M^+/k)$ of order p, M' the subfield such that $\mathrm{Gal}(M^+/M') = C$, and put $N_C = \Sigma_{\sigma\in C}\sigma$. We have an isomorphism $(\mathcal{X}_{M^+_\infty,S})^{\chi\psi} \simeq (\mathcal{X}_{M^+_\infty,S})^\chi/(N_C)$. Let σ_0 be a generator of C. In order to prove that $(\mathcal{X}_{M^+_\infty,S})^{\chi\psi}$ is free over $O_{\chi\psi}$, it is enough to show that the map

$$\sigma_0 - 1 : (\mathcal{X}_{M^+_\infty,S})^\chi/(N_C) \longrightarrow (\mathcal{X}_{M^+_\infty,S})^\chi$$

is injective. Hence, it suffices to show $((\mathcal{X}_{M^+_\infty,S})^\chi)^C = N_C((\mathcal{X}_{M^+_\infty,S})^\chi)$, hence to show $\hat{H}^0(C, (\mathcal{X}_{M^+_\infty,S})^\chi) = 0$. Taking the dual, it is enough to show $H^1(C, H^1_{et}(O_{M^+_\infty}[1/S], \mathbf{Q}_p/\mathbf{Z}_p)^{\chi^{-1}}) = 0$. This follows from the Hochschild-Serre spectral sequence and $H^2_{et}(O_{M'_\infty}[1/S], \mathbf{Q}_p/\mathbf{Z}_p) = 0$ (which is a famous

property called the weak Leopoldt conjecture and which follows immediately from the vanishing of the p-component of the Brauer group of M'_∞).
Thus, $(\mathcal{X}_{F^+_\infty,S})^{\chi\psi}$ is a free $O_{\chi\psi}$-module of finite rank. This shows that $\mathrm{Fitt}_{0,O_{\chi\psi}[[\mathrm{Gal}(F_\infty/F)]]}((\mathcal{X}_{F^+_\infty,S})^{\chi\psi})$ coincides with its characteristic ideal. By Wiles [25] and our assumption, the μ-invariant of $(\tau^{-1}\iota(\theta^{\sim}_{F_\infty,S}))^{\chi\psi}$ is also zero, and by the main conjecture proved by Wiles [25], we have

$$\mathrm{Fitt}_{0,O_{\chi\psi}[[\mathrm{Gal}(F_\infty/F)]]}((\mathcal{X}_{F^+_\infty,S})^{\chi\psi}) = (\tau^{-1}\iota(\theta^{\sim}_{F_\infty,S})^{\chi\psi}).$$

This holds for any χ and ψ with $\chi \neq 1_\Delta$. Hence, by Corollary 4.2 in [11], we obtain the conclusion of Proposition 2.1.

2.2. For any number field $\mathcal{F}$, we denote by $A_{\mathcal{F}}$ the p-primary component of the ideal class group of $\mathcal{F}$. Let F be as above. We define

$$A_{F_\infty} = \varinjlim A_{F_n}$$

where F_n is the n-th layer of F_∞/F. We denote by $(A_{F_\infty})^\vee$ the Pontrjagin dual of A_{F_∞}. Let S_p be the set of primes of k lying over p. By the orthogonal pairing in P.276 of Iwasawa [5] which is defined by the Kummer pairing, we have an isomorphism

$$(\mathcal{X}_{F^+_{\infty,S_p}})^* \simeq (A^{\sim}_{F_\infty})^\vee(1).$$

Let $\iota : \Lambda_F \longrightarrow \Lambda_F$ be the ring homomorphism induced by $\sigma \mapsto \sigma^{-1}$ for $\sigma \in \mathrm{Gal}(F_\infty/k)$. For a character χ of Δ, ι induces a ring homomorphism $\Lambda_F^\chi \longrightarrow \Lambda_F^{\chi^{-1}}$ which we also denote by ι. Since there is a natural surjective homomorphism $(\mathcal{X}_{F^+_{\infty,S}})^* \longrightarrow (\mathcal{X}_{F^+_{\infty,S_p}})^*$, Proposition 2.1 together with the above isomorphism implies

Corollary 2.2. *Let χ be an odd character of Δ such that $\chi \neq \omega$. Under the assumption of Proposition 2.1, we have*

$$\iota(\theta^\chi_{F_\infty,S}) \in \mathrm{Fitt}_{0,\Lambda_F^{\chi^{-1}}}((A^\chi_{F_\infty})^\vee).$$

Next, we consider a general CM-field F such that F/k is finite and abelian (Here, we do not assume $\mu_p \subset F$). Put $R_F = \mathbf{Z}_p[\mathrm{Gal}(F/k)]$. Let G be the p-primary component of $\mathrm{Gal}(F/k)$, and $\mathrm{Gal}(F/k) = \Delta \times G$. Suppose that χ is an odd character of Δ with $\chi \neq \omega$. We consider $R_F^\chi = O_\chi[G]$, and define $\iota : R_F \longrightarrow R_F$ and $\iota : R_F^\chi \longrightarrow R_F^{\chi^{-1}}$ similarly as above. If we assume that the Iwasawa μ-invariant of F vanishes, $(\mathcal{X}_{F(\mu_p)_\infty,S})^{\chi^{-1}\omega}$ is a finitely generated O_χ-module, so we can apply the proof of Proposition 2.1 to get $\iota(\theta^\chi_{F_\infty,S}) \in \mathrm{Fitt}_{0,\Lambda_F^{\chi^{-1}}}((A^\chi_{F_\infty})^\vee)$. Since $A_F^- \longrightarrow A^-_{F(\mu_p)_\infty}$ is injective ([24] Prop.13.26), $(A^\chi_{F(\mu_p)_\infty})^\vee \longrightarrow (A_F^\chi)^\vee$ is surjective. The image of $\iota(\theta^\chi_{F_\infty,S}) \in \Lambda_F^{\chi^{-1}}$ in $R_F^{\chi^{-1}}$ is $\iota(\theta^\chi_{F,S})$. Hence, we obtain

Corollary 2.3. *Assume that the Iwasawa μ-invariant of F is zero. Then, we have*

$$\iota(\theta^{\chi}_{F,S}) \in \mathrm{Fitt}_{0,R_F^{\chi^{-1}}}((A_F^{\chi})^{\vee}).$$

Let $S_{F(\mu_p)_\infty}$ (resp. S_F) be the set of ramifying primes in $F(\mu_p)_\infty/k$ (resp. F/k). Note that $S_{F(\mu_p)_\infty} \setminus S_F \subset S_p$ and

$$\theta_{F,S_{F(\mu_p)_\infty}} = (\Pi_{\mathfrak{p}\in S_{F(\mu_p)_\infty}\setminus S_F}(1-\varphi_{\mathfrak{p}}^{-1}))\theta_{F,S_F}$$

where $\varphi_{\mathfrak{p}}$ is the Frobenius of $\mathfrak{p}$ in $\mathrm{Gal}(F/k)$. If $\chi(\mathfrak{p}) \neq 1$ for all $\mathfrak{p}$ above p, $(1-\varphi_{\mathfrak{p}}^{-1})^{\chi}$ is a unit of $R_F^{\chi^{-1}}$ because the order of χ is prime to p. Therefore, we get

Corollary 2.4. *Assume that the Iwasawa μ-invariant of F is zero, and that $\chi(\mathfrak{p}) \neq 1$ for all $\mathfrak{p}$ above p. Then, we have*

$$\iota(\theta^{\chi}_{F}) \in \mathrm{Fitt}_{0,R_F^{\chi^{-1}}}((A_F^{\chi})^{\vee}).$$

3 Higher Stickelberger ideals

In this section, for a finite abelian extension K/k whose degree is prime to p, we will define the ideal $\Theta_{i,K} \subset \mathbf{Z}_p[\mathrm{Gal}(K/k)]$ for $i \geq 0$. We also prove the inclusion $\Theta^{\chi}_{i,K} \subset \mathrm{Fitt}_{i,O_\chi}(A_K^{\chi})$ for K and χ as in Theorem 0.1.

3.1. In this subsection, we assume that O is a discrete valuation ring with maximal ideal (p). We denote by ord_p the normalized discrete valuation of O, so $\mathrm{ord}_p(p) = 1$. For n, $r > 0$, we consider a ring

$$A_{n,r} = O[[S_1, ..., S_r]]/((1+S_1)^{p^n}-1, ..., (1+S_r)^{p^n}-1).$$

Suppose that f is an element of $A_{n,r}$ and write $f = \Sigma_{i_1,...i_r\geq 0} a_{i_1...i_r} S_1^{i_1}...S_r^{i_r}$ mod $\mathcal{I}$ where $\mathcal{I} = ((1+S_1)^{p^n}-1, ..., (1+S_r)^{p^n}-1)$. For positive integers i and s, we set $s' = \min\{x \in \mathbf{Z} : s < p^x\}$. Assume $s' \leq n$. If $0 < j < p^{s'}$, we have $\mathrm{ord}_p(\binom{p^n}{j}) = \mathrm{ord}_p(p^n!/(j!(p^n-j)!)) \geq n-s'+1$. Hence, for $i_1, ..., i_r \leq s < p^{s'}$, $a_{i_1,...,i_r}$ mod $p^{n-s'+1}$ is well-defined from $f \in A_{n,r}$. For positive integers i and s with $s' \leq n$, we define $I_{i,s}(f)$ to be the ideal of O which is generated by $p^{n-s'+1}$ and

$$\{a_{i_1,...,i_r} \ : \ 0 \leq i_1, ..., i_r \leq s \text{ and } i_1 + ... + i_r \leq i\}.$$

Since $a_{i_1,...,i_r}$ is well-defined mod $p^{n-s'+1}$, $I_{i,s}(f) \subset O$ is well-defined for any i and $s \in \mathbf{Z}_{>0}$ such that $n \geq s'$.

Lemma 3.1. *Let $\alpha : A_{n,r} \longrightarrow A_{n,r}$ be the homomorphism of O-algebras defined by $\alpha(S_k) = \Pi_{j=1}^r (1+S_j)^{a_{kj}} - 1$ for $1 \leq k,\ j \leq r$ such that $(a_{kj}) \in GL_n(\mathbf{Z}/p^n\mathbf{Z})$. Then, we have*

$$I_{i,s}(\alpha(f)) = I_{i,s}(f).$$

Proof. It is enough to show $I_{i,s}(f) \subset I_{i,s}(\alpha(f))$ because if we obtain this inclusion, the other inclusion is also obtained by applying it to α^{-1}. Further, since (a_{kj}) is a product of elementary matrices, it suffices to show the inclusion in the case that α corresponds to an elementary matrix, in which case, the inclusion can be easily checked.

In particular, let $\iota : A_{n,r} \longrightarrow A_{n,r}$ be the ring homomorphism defined by $\iota(S_k) = (1 + S_k)^{-1} - 1$ for $k = 1, ..., r$. Then, we have

$$I_{i,s}(\iota(f)) = I_{i,s}(f)$$

which we will use later.

3.2. Suppose that k is totally real, K is a CM-field, and K/k is abelian such that p does not divide $[K : k]$. Put $\Delta = \mathrm{Gal}(K/k)$. For $i \geq 0$, we will define the higher Stickelberger ideal $\Theta_{i,K} \subset \mathbf{Z}_p[\Delta]$. Since $\mathbf{Z}_p[\Delta] \simeq \bigoplus_\chi O_\chi$, it is enough to define $(\Theta_{i,K})^\chi$. We replace K by the subfield corresponding to the kernel of χ, and suppose the conductor of K/k is equal to that of χ.

For n, $r > 0$, let $\mathcal{S}_{K,n,r}$ denote the set of CM fields F such that $K \subset F$, F/k is abelian, and F/K is a p-extension satisfying $\mathrm{Gal}(F/K) \simeq (\mathbf{Z}/p^n)^{\oplus r}$. For $F \in \mathcal{S}_{K,n,r}$, we have an isomorphism

$$\mathbf{Z}_p[\mathrm{Gal}(F/k)]^\chi \simeq \mathbf{Z}_p[\Delta]^\chi[\mathrm{Gal}(F/K)] = O_\chi[\mathrm{Gal}(F/K)].$$

Fixing generators of $\mathrm{Gal}(F/K)$, we have an isomorphism between $O_\chi[\mathrm{Gal}(F/K)]$ and $A_{n,r}$ with $O = O_\chi$ in 3.1 (the fixed generators σ_1,...,σ_r correspond to $1 + S_1$,...,$1 + S_r$).

We first assume χ is odd and $\chi \neq \omega$. Then, θ_F^χ is in $\mathbf{Z}_p[\mathrm{Gal}(F/k)]^\chi = O_\chi[\mathrm{Gal}(F/K)]$ (cf. 1.2). Using the isomorphism between $O_\chi[\mathrm{Gal}(F/K)]$ and $A_{n,r}$, for i and s such that $n \geq s'$, we define the ideal $I_{i,s}(\theta_F^\chi)$ of O_χ (cf. 3.1). By Lemma 3.1, $I_{i,s}(\theta_F^\chi)$ does not depend on the choice of generators of $\mathrm{Gal}(F/K)$.

We define $(\Theta_{0,K})^\chi = (\theta_K^\chi)$. Suppose that $(\Theta_{0,K})^\chi = (p^m)$. If $m = 0$, we define $(\Theta_{i,K})^\chi = (1)$ for all $i \geq 0$. We assume $m > 0$. We define $\mathcal{S}_{K,n} = \bigcup_{r>0} \mathcal{S}_{K,n,r}$. We define $(\Theta_{i,s,K})^\chi$ to be the ideal generated by all $I_{i,s}(\theta_F^\chi)$'s where F ranges over all fields in $\mathcal{S}_{K,n}$ for all $n \geq m + s' - 1$ where $s' = \min\{x \in \mathbf{Z} : s < p^x\}$ as in 3.1, namely

$$(\Theta_{i,s,K})^\chi = \bigcup_{\substack{F \in \mathcal{S}_{K,n} \\ n \geq m+s'-1}} I_{i,s}(\theta_F^\chi).$$

We define $(\Theta_{i,K})^\chi$ by $(\Theta_{i,K})^\chi = \bigcup_{s>0} (\Theta_{i,s,K})^\chi$. For χ satisfying the condition of Theorem 0.1, we will see later in §5 that $(\Theta_{i,K})^\chi = (\Theta_{i,1,K})^\chi$.

For $F \in \mathcal{S}_{K,m}$ with $m > 0$, $I_{i,1}(\theta_F^\chi)$ contains p^m (note that $s' = 1$ when $s = 1$), so $p^m \in (\Theta_{i,K})^\chi$. Since $(\Theta_{0,K})^\chi = (p^m)$, $(\Theta_{0,K})^\chi$ is in $(\Theta_{i,K})^\chi$. It is also clear from definition that $(\Theta_{i,s,K})^\chi \subset (\Theta_{i+1,s,K})^\chi$ for $i > 0$ and $s > 0$. Hence, we have a sequence of ideals

$$(\Theta_{0,K})^\chi \subset (\Theta_{1,K})^\chi \subset (\Theta_{2,K})^\chi \subset ...$$

We do not use the ω-component in this paper, but for $\chi = \omega$, we define $(\Theta_{0,K})^\chi = (\theta_K \operatorname{Ann}_{\mathbf{Z}_p[\mathrm{Gal}(K/k)]}(\mu_{p^\infty}(K)))^\chi$. For $i > 0$, $(\Theta_{i,K})^\chi$ is defined similarly as above by using $x\theta_F^\chi$ instead of θ_F^χ where x ranges over elements of $\operatorname{Ann}_{\mathbf{Z}_p[\mathrm{Gal}(F/k)]}(\mu_{p^\infty}(F))^\chi$. For an even χ, we define $(\Theta_{i,K})^\chi = (0)$ for all $i \geq 0$.

Proposition 3.2. *Suppose that K and χ be as in Theorem 0.1. Then, for any $i \geq 0$, we have*

$$(\Theta_{i,K})^\chi \subset \mathrm{Fitt}_{i,O_\chi}(A_K^\chi).$$

Proof. At first, by Theorem 3 in Wiles [26] we know $\# A_K^\chi = \#(O_\chi/(\theta_K^\chi))$, hence $(\Theta_{0,K})^\chi = \mathrm{Fitt}_{0,O_\chi}(A_K^\chi)$. (In our case, this is a direct consequence of the main conjecture proved by Wiles [25].) We assume $i > 0$. By the definition of $(\Theta_{i,K})^\chi$, we have to show $I_{i,s}(\theta_F^\chi) \subset \mathrm{Fitt}_{i,O_\chi}(A_K^\chi)$ for $F \in \mathcal{S}_{K,n,r}$ where the notation is the same as above. By Lemma 3.1, $I_{i,s}(\theta_F^\chi) = I_{i,s}(\iota(\theta_F^\chi))$. Hence, it is enough to show

$$I_{i,s}(\iota(\theta_F^\chi)) \subset \mathrm{Fitt}_{i,O_\chi}(A_K^\chi).$$

We will prove this inclusion by the same method as Theorem 8.1 in [11]. We write $O = O_\chi = O_{\chi^{-1}}$, and $G = \mathrm{Gal}(F/K)$. As in 3.2, we fix an isomorphism $O[\mathrm{Gal}(F/K)] \simeq A_{n,r}$ by fixing generators of G. We consider $(A_F^\chi)^\vee = (A_F \otimes_{\mathbf{Z}_p[\Delta]} O)^\vee = (A_F \otimes_{\mathbf{Z}_p[\mathrm{Gal}(F/k)]} O[G])^\vee$ which is an $O[G]$-module. Since F/K is a p-extension, it is well-known that the vanishing of the Iwasawa μ-invariant of K implies the vanishing of the Iwasawa μ-invariant of F ([6] Theorem 3). By Corollary 2.4, we have

$$\iota(\theta_F^\chi) \in \mathrm{Fitt}_{0,O[G]}((A_F^\chi)^\vee).$$

Since $\chi \neq \omega$ and χ is odd, for a unit group $O_F^\times$, we have $(O_F^\times \otimes \mathbf{Z}_p)^\chi = \mu_{p^\infty}(F)^\chi = 0$, so $H^1(\mathrm{Gal}(F/K), O_F^\times)^\chi = H^1(\mathrm{Gal}(F/K), (O_F^\times \otimes \mathbf{Z}_p)^\chi) = 0$. This shows that the natural map $A_K^\chi \longrightarrow A_F^\chi$ is injective. Hence, regarding A_K^χ as an $O[G]$-module (G acting trivially on it), we have

$$\mathrm{Fitt}_{0,O[G]}((A_F^\chi)^\vee) \subset \mathrm{Fitt}_{0,O[G]}((A_K^\chi)^\vee),$$

and

$$\iota(\theta_F^\chi) \in \mathrm{Fitt}_{0,O[G]}((A_K^\chi)^\vee).$$

Hence, by the lemma below, we obtain

$$I_{i,s}(\iota(\theta_F^\chi)) \subset \mathrm{Fitt}_{i,O}(A_K^\chi).$$

This completes the proof of Proposition 3.2.

Lemma 3.3. *Put* $I_G = (S_1, ..., S_r)$. *Then,* $\mathrm{Fitt}_{0,O[G]}((A_K^\chi)^\vee)$ *is generated by* $\mathrm{Fitt}_{j,O}(A_K^\chi)(I_G)^j$ *for all* $j \geq 0$.

Proof. Put $M = (A_K^\chi)^\vee$. Since O is a discrete valuation ring, $M^\vee$ is isomorphic to M as an O-module. Hence, $\mathrm{Fitt}_{j,O}(M) = \mathrm{Fitt}_{j,O}(M^\vee) = \mathrm{Fitt}_{j,O}(A_K^\chi)$.
We take generators $e_1,...,e_m$ and relations $\Sigma_{k=1}^m a_{kl}e_k = 0$ ($a_{kl} \in O$, $l = 1, 2, ..., m$) of M as an O-module. Put $A = (a_{kl})$. We also consider a relation matrix of M as an $O[G]$-module. By definition, I_G annihilates M. Hence, the relation matrix of M as an $O[G]$-module is of the form

$$\left(\begin{array}{cccccccccc} S_1 & ... & S_r & ... & ... & 0 & ... & 0 & & \\ 0 & ... & 0 & ... & ... & 0 & ... & 0 & & \\ . & ... & . & ... & ... & . & ... & . & & A \\ . & ... & . & ... & ... & . & ... & . & & \\ 0 & ... & 0 & ... & ... & S_1 & ... & S_r & & \end{array}\right).$$

Therefore, $\mathrm{Fitt}_{0,O[G]}(M)$ is generated by $\mathrm{Fitt}_{j,O}(M)(I_G)^j$ for all $j \geq 0$.

4 Euler systems

Let K/k be a finite and abelian extension of degree prime to p. We also assume that K is a CM-field, and the Iwasawa μ-invariant of K is zero. We consider a CM field F such that F/k is finite and abelian, $F \supset K$, and F/K is a p-extension. Since the Iwasawa μ-invariant of F is also zero, by Corollary 2.4, we have $\iota(\theta_F^\sim)(A_F^\sim)^\vee = 0$. Hence, we have

$$\theta_F^\sim A_F^\sim = 0.$$

We denote by $O_F^\times$, Div_F, and A_F the unit group of F, the divisor group of F, and the p-primary component of the ideal class group of F. We write $[\rho]$ for the divisor corresponding to a finite prime ρ, and write an element of Div_F of the form $\Sigma a_i[\rho_i]$ with $a_i \in \mathbf{Z}$. If $(x) = \Pi\rho_i^{a_i}$ is the prime decomposition of $x \in F^\times$, we write $\mathrm{div}(x) = \Sigma a_i[\rho_i] \in \mathrm{Div}_F$. Consider an exact sequence $0 \longrightarrow O_F^\times \otimes \mathbf{Z}_p \longrightarrow F^\times \otimes \mathbf{Z}_p \xrightarrow{\mathrm{div}} \mathrm{Div}_F \otimes \mathbf{Z}_p \longrightarrow A_F \longrightarrow 0$. Since the functor $M \mapsto M^\sim$ is exact and $(O_F^\times \otimes \mathbf{Z}_p)^\sim = 0$,

$$0 \longrightarrow (F^\times \otimes \mathbf{Z}_p)^\sim \xrightarrow{\mathrm{div}} (\mathrm{Div}_F \otimes \mathbf{Z}_p)^\sim \longrightarrow A_F^\sim \longrightarrow 0$$

is exact. For any finite prime ρ_F of F, since the class of $\theta_F^\sim[\rho_F]^\sim$ in $A_F^\sim$ vanishes, there is a unique element g_{F,ρ_F} in $(F^\times \otimes \mathbf{Z}_p)^\sim$ such that

$$\mathrm{div}(g_{F,\rho_F}) = \theta_F^\sim[\rho_F]^\sim.$$

By this property, we have

Lemma 4.1. *Suppose that M is an intermediate field of F/K, and S_F (resp. S_M) denotes the set of ramifying primes of k in F/k (resp. M/k). Let ρ_M be a prime of M, ρ_F be a prime of F above ρ_M, and $f = [O_F/\rho_F : O_F/\rho_M]$. Then, we have*

$$N_{F/M}(g_{F,\rho_F}) = (\prod_{\lambda \in S_F \setminus S_M} (1-\varphi_\lambda^{-1}))^{\sim}(g_{M,\rho_M})^f$$

where $N_{F/M} : F^\times \longrightarrow M^\times$ is the norm map, and φ_λ is the Frobenius of λ in $\mathrm{Gal}(M/k)$.

Proof. In fact, we have

$$\mathrm{div}(N_{L/M}(g_{F,\rho_F})) = c_{F/M}(\theta_F)^{\sim}[N_{F/M}(\rho_F)]^{\sim}$$

where $c_{F/M} : \mathbf{Z}_p[\mathrm{Gal}(F/k)] \longrightarrow \mathbf{Z}_p[\mathrm{Gal}(M/k)]$ is the map induced by the restriction $\sigma \mapsto \sigma_{|M}$ and $N_{F/M}(\rho_F)$ is the norm of ρ_F. By a famous property of the Stickelberger elements (see Tate [23] p.86), we have

$$c_{F/M}(\theta_F^{\sim}) = ((\prod_{\lambda \in S_F \setminus S_M} (1-\varphi_\lambda^{-1}))\theta_M)^{\sim},$$

hence the right hand side of the first equation is equal to $((\prod_{\lambda \in S_F \setminus S_M}(1-\varphi_\lambda^{-1}))\theta_M)^{\sim} f[\rho_M]^{\sim}$. This is also equal to $\mathrm{div}((\prod_{\lambda \in S_F \setminus S_M}(1-\varphi_\lambda^{-1}))^{\sim}(g_{M,\rho_M})^f)$. Since div is injective, we get this lemma.

Remark 4.2. By the property $\theta_F^{\sim} A_F^{\sim} = 0$, we can also obtain an Euler system in some cohomology groups by the method of Rubin in [18] Chapter 3, section 3.4. But here, we consider the Euler system of these g_{F,ρ_F}'s, which is an analogue of the Euler system of Gauss sums. I obtained the idea of studying the elements g_{F,ρ_F} from C. Popescu through a discussion with him.

Let H_k be the Hilbert p-class field of k, namely the maximal abelian p-extension of k which is unramified everywhere. Suppose that the p-primary component A_k of the ideal class group of k is decomposed into $A_k = \mathbf{Z}/p^{a_1}\mathbf{Z} \oplus ... \oplus \mathbf{Z}/p^{a_s}\mathbf{Z}$. We take and fix a prime ideal $\mathfrak{q}_j$ which generates the j-th direct summand for each $j = 1, ..., s$. We take $\xi_j \in k^\times$ such that $\mathfrak{q}_j^{p^{a_j}} = (\xi_j)$ for each j. Let $\mathcal{U}$ denote the subgroup of $k^\times$ generated by the unit group $O_k^\times$ and ξ_1,...,ξ_s. For a positive integer $n > 0$, we define $\mathcal{P}_n$ to be the set of primes of k with degree 1 which are prime to $p\mathfrak{q}_1 \cdot ... \cdot \mathfrak{q}_s$, and which split completely in $KH_k(\mu_{p^n}, \mathcal{U}^{1/p^n})$.

Lemma 4.3. *Suppose $\lambda \in \mathcal{P}_n$. Then, there exists a unique cyclic extension $k_n(\lambda)/k$ of degree p^n, which is unramified outside λ, and in which λ is totally ramified.*

Proof. We prove this lemma by class field theory. Let $\mathcal{C}_k$ (resp. Cl_k) be the idele class group (resp. the ideal class group) of k. For a prime v, we denote by k_v the completion of k at v, and define U_{k_v} to be the unit group of the ring of integers of k_v for a finite prime v, and $U_{k_v} = k_v$ for an infinite prime v. We denote by $U^1_{k_v}$ the group of principal units for a finite prime v. We define $\mathcal{C}_{k,\lambda,n}$ which is a quotient of $\mathcal{C}_k \otimes \mathbf{Z}_p$ by

$$\mathcal{C}_{k,\lambda,n} = ((k_\lambda^\times/U^1_{k_\lambda}) \otimes \mathbf{Z}/p^n\mathbf{Z} \oplus \bigoplus_{v\neq\lambda}(k_v^\times/U_{k_v}) \otimes \mathbf{Z}_p)/(\text{the image of } k^\times)$$

where v ranges over all primes except λ. Since λ splits in H_k, the class of λ in $Cl_k \otimes \mathbf{Z}_p = A_k$ is trivial. Hence, the natural map

$$\bigoplus_v k_v^\times \otimes \mathbf{Z}_p \longrightarrow \bigoplus_v (k_v^\times/U_{k_v}) \otimes \mathbf{Z}_p \longrightarrow (\bigoplus_v \mathbf{Z}_p)/(\text{the image of } k^\times) = A_k$$

(v ranges over all primes) induces $\mathcal{C}_{k,\lambda,n} \longrightarrow A_k$, and we have an exact sequence

$$(U_{k_\lambda}/U^1_{k_\lambda}) \otimes \mathbf{Z}/p^n\mathbf{Z} \stackrel{a}{\longrightarrow} \mathcal{C}_{k,\lambda,n} \stackrel{b}{\longrightarrow} A_k \longrightarrow 0.$$

Let $\kappa(\lambda)$ denote the residue field of λ. Since λ splits in $k(\mu_{p^n})$, $(U_{k_\lambda}/U^1_{k_\lambda}) \otimes \mathbf{Z}/p^n\mathbf{Z} = \kappa(\lambda)^\times \otimes \mathbf{Z}/p^n\mathbf{Z}$ is cyclic of order p^n. Since λ splits in $k(\mu_{p^n}, (O_k^\times)^{1/p^n})$, $O_k^\times$ is in $(U_{k_\lambda})^{p^n}$ and a is injective (Rubin [18] Lemma 4.1.2 (i)). Next, we will show that the exact sequence

$$0 \longrightarrow (U_{k_\lambda}/U^1_{k_\lambda}) \otimes \mathbf{Z}/p^n\mathbf{Z} \stackrel{a}{\longrightarrow} \mathcal{C}_{k,\lambda,n} \stackrel{b}{\longrightarrow} A_k \longrightarrow 0$$

splits. Let $\mathfrak{q}_j$, a_j, ξ_j be as above. Suppose that $\pi_{\mathfrak{q}_j}$ is a uniformizer of $k_{\mathfrak{q}_j}$. We denote by $\Pi_{\mathfrak{q}_j}$ the idele whose $\mathfrak{q}_j$-component is $\pi_{\mathfrak{q}_j}$ and whose v-component is 1 for every prime v except for $\mathfrak{q}_j$ (the λ-component is also 1). Since λ splits in $k(\xi_j^{1/p^n})$, we have $\xi_j \in (U_{k_\lambda})^{p^n}$. Hence, the class of $\xi_j \in k^\times$ in $(k_\lambda^\times/U^1_{k_\lambda}) \otimes \mathbf{Z}/p^n\mathbf{Z} \oplus \bigoplus_{v\neq\lambda}(k_v^\times/U_{k_v}) \otimes \mathbf{Z}_p$ coincides with $(\Pi_{\mathfrak{q}_j})^{p^{a_j}}$. This shows that the class $[\Pi_{\mathfrak{q}_j}]_{\mathcal{C}_{k,\lambda,n}}$ of $\Pi_{\mathfrak{q}_j}$ in $\mathcal{C}_{k,\lambda,n}$ has order p^{a_j} because $b([\Pi_{\mathfrak{q}_j}]_{\mathcal{C}_{k,\lambda,n}}) = [\mathfrak{q}_j]_{A_k}$ where $[\mathfrak{q}_j]_{A_k}$ is the class of $\mathfrak{q}_j$ in A_k. We define a homomorphism $b' : A_k \longrightarrow \mathcal{C}_{k,\lambda,n}$ by $[\mathfrak{q}_j]_{A_k} \mapsto [\Pi_{\mathfrak{q}_j}]_{\mathcal{C}_{k,\lambda,n}}$ for all $j = 1, ..., s$. Clearly, b' is a section of b, hence the above exact sequence splits. By class field theory, this implies that there is a cyclic extension $k_n(\lambda)/k$ of degree p^n, which is linearly disjoint with k_H/k. From the construction, we know that λ is totally ramified in $k_n(\lambda)$, and $k_n(\lambda)/k$ is unramified outside λ. It is also clear that $k_n(\lambda)$ is unique by class field theory.

As usual, we consider Kolyvagin's derivative operator. Put $G_\lambda = \mathrm{Gal}(k_n(\lambda)/k)$, and fix a generator σ_λ of G_λ for $\lambda \in \mathcal{P}_n$. We define $N_\lambda = \Sigma_{i=0}^{p^n-1}\sigma_\lambda^i \in \mathbf{Z}[G_\lambda]$ and $D_\lambda = \Sigma_{i=0}^{p^n-1} i\sigma_\lambda^i \in \mathbf{Z}[G_\lambda]$. For a squarefree product $\mathfrak{a} = \lambda_1 \cdot ... \cdot \lambda_r$ with $\lambda_i \in \mathcal{P}_n$, we define $k_n(\mathfrak{a})$ to be the compositum $k_n(\lambda_1)...k_n(\lambda_r)$, and $K_{n,(\mathfrak{a})} = Kk_n(\mathfrak{a})$. We simply write $K_{(\mathfrak{a})}$ for $K_{n,(\mathfrak{a})}$ if no confusion arises. For $\mathfrak{a} = \lambda_1 \cdot ... \cdot \lambda_r$, we also define $N_\mathfrak{a} = \prod_{i=1}^r N_{\lambda_i}$ and $D_\mathfrak{a} = \prod_{i=1}^r D_{\lambda_i} \in \mathbf{Z}[\mathrm{Gal}(k_n(\mathfrak{a})/k)] = \mathbf{Z}[\mathrm{Gal}(K_{(\mathfrak{a})}/K)]$. For a finite prime ρ of k which splits completely in $K_{(\mathfrak{a})}$, we take a prime $\rho_{K_{(\mathfrak{a})}}$ of $K_{(\mathfrak{a})}$. By the standard method of Euler systems

(cf. Lemmas 2.1 and 2.2 in Rubin [17], or Lemma 4.4.2 (i) in Rubin [18]), we know that there is a unique $\kappa_{\mathfrak{a},\rho_{K_{(\mathfrak{a})}}} \in (K^\times \otimes \mathbf{Z}/p^n)^\sim$ whose image in $(K^\times_{(\mathfrak{a})} \otimes \mathbf{Z}/p^n)^\sim$ coincides with $D_\mathfrak{a}(g_{K_{(\mathfrak{a})},\rho_{K_{(\mathfrak{a})}}})$. We also have an element $\delta_\mathfrak{a} \in \mathbf{Z}/p^n[\mathrm{Gal}(K/k)]^\sim$ such that $D_\mathfrak{a}\theta^\sim_{K_{(\mathfrak{a})}} \equiv \nu_{K_{(\mathfrak{a})}/K}(\delta_\mathfrak{a}) \pmod{p^n}$ where $\nu_{K_{(\mathfrak{a})}/K} : \mathbf{Z}_p[\mathrm{Gal}(K/k)]^\sim \longrightarrow \mathbf{Z}_p[\mathrm{Gal}(K_{(\mathfrak{a})}/k)]^\sim$ is the map induced by $\sigma \mapsto \sum_{\tau_{|K}=\sigma} \tau$ for $\sigma \in \mathrm{Gal}(K/k)$. This $\delta_\mathfrak{a}$ is also determined uniquely by this property. We sometimes write $\kappa_\mathfrak{a}$ for $\kappa_{\mathfrak{a},\rho_{K_{(\mathfrak{a})}}}$ if no confusion arises.
We take an odd character χ of $\mathrm{Gal}(K/k)$ such that $\chi \neq \omega$, and consider the χ-component $\kappa^\chi_\mathfrak{a} \in (K^\times \otimes \mathbf{Z}/p^n)^\chi$, $\delta^\chi_\mathfrak{a} \in \mathbf{Z}/p^n[\mathrm{Gal}(K/k)]^\chi = O_\chi$, ...etc.

LEMMA 4.4. *Put* $S_i = \sigma_{\lambda_i} - 1 \in O_\chi[\mathrm{Gal}(K_{(\mathfrak{a})}/K)]$. *Then, we have*

$$\theta^\chi_{K_{(\mathfrak{a})}} \equiv (-1)^r \delta^\chi_\mathfrak{a} S_1 \cdot ... \cdot S_r \pmod{(p^n, S_1^2, ..., S_r^2)}.$$

Proof. We first prove $\theta^\chi_{K_{(\mathfrak{a})}} \equiv aS_1 \cdot ... \cdot S_r \mod (S_1^2, ..., S_r^2)$ for some $a \in O_\chi$ by induction on r. For any subfields M_1 and M_2 such that $K \subset M_1 \subset M_2 \subset K_{(\mathfrak{a})}$, we denote by $c_{M_2/M_1} : O_\chi[\mathrm{Gal}(M_2/K)] \longrightarrow O_\chi[\mathrm{Gal}(M_1/K)]$ the map induced by the restriction $\sigma \mapsto \sigma_{|M_1}$. Since $c_{K_{(\lambda_1)}/K}(\theta^\chi_{K_{(\lambda_1)}}) = ((1-\varphi^{-1}_{\lambda_1})\theta_K)^\chi$ (cf. Tate [23] p.86) and λ_1 splits completely in K, we have $c_{K_{(\lambda_1)}/K}(\theta^\chi_{K_{(\lambda_1)}}) = 0$. Hence, $S_1 = \sigma_{\lambda_1} - 1$ divides $\theta^\chi_{K_{(\lambda_1)}}$. So the first assertion was verified for $r = 1$.
Let $\mathfrak{a}_i = \mathfrak{a}/\lambda_i$ for i with $1 \leq i \leq r$. Then, we have $c_{K_{(\mathfrak{a})}/K_{(\mathfrak{a}_i)}}(\theta^\chi_{K_{(\mathfrak{a})}}) = ((1 - \varphi^{-1}_{\lambda_i})\theta_{K_{(\mathfrak{a}_i)}})^\chi$. Since λ_i splits completely in K, φ_{λ_i} is in $\mathrm{Gal}(K_{(\mathfrak{a}_i)}/K)$. Hence, $1 - \varphi^{-1}_{\lambda_i}$ is in the ideal $I_{\mathrm{Gal}(K_{(\mathfrak{a}_i)}/K)} = (S_1, ..., S_{i-1}, S_{i+1}, ..., S_r)$. This implies that $c_{K_{(\mathfrak{a})}/K_{(\mathfrak{a}_i)}}(\theta^\chi_{K_{(\mathfrak{a})}})$ is in the ideal $(S_1^2, ..., S_{i-1}^2, S_{i+1}^2, ..., S_r^2)$ by the hypothesis of the induction. This holds for all i, so $\theta^\chi_{K_{(\mathfrak{a})}}$ can be written as $\theta^\chi_{K_{(\mathfrak{a})}} = \alpha + \beta$ where α is divisible by all S_i for $i = 1, ..., r$, and β is in $(S_1^2, ..., S_r^2)$. Therefore, $\theta^\chi_{K_{(\mathfrak{a})}} \equiv aS_1 \cdot ... \cdot S_r \mod (S_1^2, ..., S_r^2)$ for some $a \in O_\chi$.
Next, we determine $a \mod p^n$. Note that $S_i D_{\lambda_i} \equiv -N_{\lambda_i} \pmod{p^n}$. Hence, $S_i^2 D_{\lambda_i} \equiv 0 \pmod{p^n}$. Thus, we have

$$D_\mathfrak{a}(\theta^\chi_{K_{(\mathfrak{a})}}) \equiv D_\mathfrak{a}(aS_1 \cdot ... \cdot S_r) \equiv (-1)^r N_\mathfrak{a}(a) \pmod{p^n}.$$

Hence, $N_\mathfrak{a}((-1)^r a) = \nu_{K_{(\mathfrak{a})}/K}((-1)^r a) \equiv \nu_{K_{(\mathfrak{a})}/K}(\delta^\chi_\mathfrak{a}) \pmod{p^n}$, which implies $\delta^\chi_\mathfrak{a} \equiv (-1)^r a \pmod{p^n}$ because $\nu_{K_{(\mathfrak{a})}/K}$ mod p^n is injective. This completes the proof of Lemma 4.4.

We put $G = \mathrm{Gal}(K_{(\mathfrak{a})}/K)$. As in §3, we have an isomorphism $O_\chi[G] \simeq A_{n,r}$ by the correspondence $\sigma_{\lambda_j} \leftrightarrow 1 + S_j$ where $A_{n,r}$ is the ring in 3.1 with $O = O_\chi$. For i, $s > 0$ and $\theta^\chi_{K_{(\mathfrak{a})}} \in O_\chi[G]$, we have an ideal $I_{i,s}(\theta^\chi_{K_{(\mathfrak{a})}})$ of O_χ as in 3.2. By the definition of $I_{i,s}(\theta^\chi_{K_{(\mathfrak{a})}})$ and Lemma 4.4, we know that $I_{r,1}(\theta^\chi_{K_{(\mathfrak{a})}})$ is generated by $\delta^\chi_\mathfrak{a}$ and p^n. Thus, we get

Corollary 4.5.
$$I_{r,1}(\theta^{\chi}_{K_{(\mathfrak{a})}}) = (\delta^{\chi}_{\mathfrak{a}}, p^n).$$

For a prime λ of k, we define the subgroup Div^{λ}_K of $\mathrm{Div}_K \otimes \mathbf{Z}_p$ by $\mathrm{Div}^{\lambda}_K = \bigoplus_{\lambda_K|\lambda} \mathbf{Z}_p[\lambda_K]$ where λ_K ranges over all primes of K above λ. We fix a prime λ_K, then $\mathrm{Div}^{\lambda}_K = \mathbf{Z}_p[\mathrm{Gal}(K/k)/D_{\lambda_K}][\lambda_K]$ where D_{λ_K} is the decomposition group of λ_K in $\mathrm{Gal}(K/k)$. Let $\mathrm{div}_{\lambda} : (K^{\times} \otimes \mathbf{Z}_p)^{\chi} \longrightarrow (\mathrm{Div}^{\lambda}_K)^{\chi}$ be the map induced by the composite of $\mathrm{div} : K^{\times} \otimes \mathbf{Z}_p \longrightarrow \mathrm{Div}_K \otimes \mathbf{Z}_p$ and the projection $\mathrm{Div}_K \otimes \mathbf{Z}_p \longrightarrow \mathrm{Div}^{\lambda}_K$. The following lemma is immediate from the defining properties of $\kappa_{\mathfrak{a},\rho_{K_{(\mathfrak{a})}}}$ and $\delta_{\mathfrak{a}}$, which we stated above.

Lemma 4.6. *Assume that ρ is a finite prime of k which splits completely in $K_{(\mathfrak{a})}$. We take a prime $\rho_{K_{(\mathfrak{a})}}$ of $K_{(\mathfrak{a})}$ and a prime ρ_K of K such that $\rho_{K_{(\mathfrak{a})}} \mid \rho_K \mid \rho$.*
(i) $\mathrm{div}_{\rho}(\kappa^{\chi}_{\mathfrak{a},\rho_{K_{(\mathfrak{a})}}}) \equiv (\delta_{\mathfrak{a}}[\rho_K])^{\chi} \pmod{p^n}$.
(ii) *If λ is prime to $\mathfrak{a}\rho$, we have* $\mathrm{div}_{\lambda}(\kappa^{\chi}_{\mathfrak{a},\rho_{K_{(\mathfrak{a})}}}) \equiv 0 \pmod{p^n}$.

We next proceed to an important property of $\kappa^{\chi}_{\mathfrak{a},\rho_{K_{(\mathfrak{a})}}}$. Suppose that λ is a prime in $\mathcal{P}_n$ with $(\lambda, \mathfrak{a}) = 1$ and ρ is a prime with $(\rho, \mathfrak{a}\lambda) = 1$. We assume both ρ and λ split completely in $K_{(\mathfrak{a})}$. Put $W = \mathrm{Ker}(\mathrm{div}_{\lambda} : (K^{\times} \otimes \mathbf{Z}_p)^{\chi} \longrightarrow (\mathrm{Div}^{\lambda}_K)^{\chi})$, and $R^{\lambda}_K = \bigoplus_{\lambda_K|\lambda} \kappa(\lambda_K)^{\times}$ where $\kappa(\lambda_K)$ is the residue field of λ_K ($\kappa(\lambda_K)$ coincides with the residue field $\kappa(\lambda) = O_k/\lambda$ of λ because λ splits in K) and λ_K ranges over all primes of K above λ. We consider a natural map
$$\ell_{\lambda} : W/W^{p^n} \longrightarrow (R^{\lambda}_K/(R^{\lambda}_K)^{p^n})^{\chi}$$
induced by $x \mapsto (x \bmod \lambda_K)$. Note that $N(\lambda) \equiv 1 \pmod{p^n}$ because $\lambda \in \mathcal{P}_n$. So, $R^{\lambda}_K/(R^{\lambda}_K)^{p^n}$ is a free $\mathbf{Z}/p^n\mathbf{Z}[\mathrm{Gal}(K/k)]$-module of rank 1. We take a basis $u \in (R^{\lambda}_K/(R^{\lambda}_K)^{p^n})^{\chi}$, and define $\ell_{\lambda,u} : W/W^{p^n} \longrightarrow (\mathbf{Z}/p^n\mathbf{Z}[\mathrm{Gal}(K/k)])^{\chi} \simeq O_{\chi}/(p^n)$ to be the composite of ℓ_{λ} and $u \mapsto 1$. By Lemma 4.6 (ii), $\kappa^{\chi}_{\mathfrak{a},\rho_{K_{(\mathfrak{a})}}}$ is in W/W^{p^n} (note that $W/W^{p^n} \subset (K^{\times} \otimes \mathbf{Z}/p^n\mathbf{Z})^{\chi}$). We are interested in $\ell_{\lambda,u}(\kappa^{\chi}_{\mathfrak{a},\rho_{K_{(\mathfrak{a})}}})$. We take a prime $\rho_{K_{(\mathfrak{a})}}$ (resp. $\lambda_{K_{(\mathfrak{a})}}$) of $K_{(\mathfrak{a})}$ and a prime ρ_K (resp. λ_K) of K such that $\rho_{K_{(\mathfrak{a})}} \mid \rho_K \mid \rho$ (resp. $\lambda_{K_{(\mathfrak{a})}} \mid \lambda_K \mid \lambda$).

Proposition 4.7. *We assume that $\chi(\mathfrak{p}) \neq 1$ for any prime $\mathfrak{p}$ of k above p, and that $[\rho_K]$ and $[\lambda_K]$ yield the same class in A^{χ}_K. Then, there is an element $x \in W/W^{p^n}$ satisfying the following properties.*
(i) *For any prime λ' of k such that $(\lambda', \mathfrak{a}) = 1$, we have*
$$\mathrm{div}_{\lambda'}(\kappa^{\chi}_{\mathfrak{a},\rho_{K_{(\mathfrak{a})}}}/x) \equiv 0 \pmod{p^n}.$$
(ii) *Choosing u suitably, we have*
$$\frac{1 - N(\lambda)^{-1}}{p^n} \ell_{\lambda,u}(x) \equiv \delta^{\chi}_{\mathfrak{a}\lambda} \pmod{(\delta^{\chi}_{\mathfrak{a}}, p^n)}$$
where $N(\lambda) = \#\kappa(\lambda) = \#(O_k/\lambda)$.
In particular, in the case $\mathfrak{a} = (1)$ we can take $x = g^{\chi}_{K,\rho_K} \bmod W^{p^n}$.

This proposition corresponds to Theorem 2.4 in Rubin [17], which was proved by using some extra property of the Gauss sums. For our g_{F,ρ_F} we do not have the property corresponding to Lemma 2.5 in [17], so we have to give here a proof in which we use only the definition of g_{F,ρ_F}, namely $\mathrm{div}(g^{\chi}_{F,\rho_F}) = (\theta_F[\rho_F])^{\chi}$.

Proof of Proposition 4.7. We denote by $\lambda_{K_{(\mathfrak{a}\lambda)}}$ the unique prime of $K_{(\mathfrak{a}\lambda)}$ above $\lambda_{K_{(\mathfrak{a})}}$. Put $N = \mathrm{ord}_p(N(\lambda)-1)+2n$. We take by Chebotarev density theorem a prime ρ' of k which splits completely in $K_{(\mathfrak{a}\lambda)}(\mu_{p^N})$ such that the class of $[\rho'_{K_{(\mathfrak{a}\lambda)}}]$ in $A^{\chi}_{K_{(\mathfrak{a}\lambda)}}$ for a prime $\rho'_{K_{(\mathfrak{a}\lambda)}}$ of $K_{(\mathfrak{a}\lambda)}$ over ρ' coincides with the class of $[\lambda_{K_{(\mathfrak{a}\lambda)}}]$. Let ρ'_K be the prime below $\rho'_{K_{(\mathfrak{a}\lambda)}}$. Then, the class of $[\lambda_K]$, the class of $[\rho'_K]$, and the class of $[\rho_K]$ in A^{χ}_K all coincide. Hence, there is an element $a \in W$ such that $\mathrm{div}(a) = [\rho_K]-[\rho'_K]$. Define $x \in W/W^{p^n}$ by $x = \kappa^{\chi}_{\mathfrak{a},\rho'_{K_{(\mathfrak{a})}}} \cdot a^{\delta^{\chi}_{\mathfrak{a}}}$. By Lemma 4.6 (ii), $\mathrm{div}_{\lambda'}(\kappa^{\chi}_{\mathfrak{a},\rho_{K_{(\mathfrak{a})}}}/x) \equiv 0 \pmod{p^n}$ for a prime λ' such that $(\lambda', \mathfrak{a}\rho\rho') = 1$. By Lemma 4.6 (i), the same is true for $\lambda' = \rho$ and ρ'. Thus, we get the first assertion. In the case $\mathfrak{a} = (1)$, we take $y = g^{\chi}_{K,\rho'_K} a^{\theta^{\chi}_K}$. Then, $\mathrm{div}(y) = \mathrm{div}(g^{\chi}_{K,\rho_K})$, so $y = g^{\chi}_{K,\rho_K}$, and we have $g^{\chi}_{K,\rho_K} \bmod W^{p^n} = y \bmod W^{p^n} = x$.

In order to show the second assertion, it is enough to prove

$$\frac{1-N(\lambda)^{-1}}{p^n}\ell_{\lambda,u}(\kappa^{\chi}_{\mathfrak{a},\rho'_{K_{(\mathfrak{a})}}}) \equiv \delta^{\chi}_{\mathfrak{a}\lambda} \pmod{p^n} \tag{1}$$

for some u. Set $\mathrm{Div}^{\lambda}_{K_{(\mathfrak{a}\lambda)}} = \bigoplus_{v|\lambda}\mathbf{Z}_p[v]$ and $R^{\lambda}_{K_{(\mathfrak{a}\lambda)}} = \bigoplus_{v|\lambda}\kappa(v)^{\times} = \bigoplus_{v|\lambda}(O_{K_{(\mathfrak{a}\lambda)}}/v)^{\times}$ where v ranges over all primes of $K_{(\mathfrak{a}\lambda)}$ above λ. Since the primes of $K_{(\mathfrak{a})}$ above λ are totally ramified in $K_{(\mathfrak{a}\lambda)}$, $(\mathrm{Div}^{\lambda}_{K_{(\mathfrak{a}\lambda)}})^{\chi}$ is isomorphic to $O_{\chi}[\mathrm{Gal}(K_{(\mathfrak{a})}/K)]$ and $(R^{\lambda}_{K_{(\mathfrak{a}\lambda)}}/(R^{\lambda}_{K_{(\mathfrak{a}\lambda)}})^{p^n})^{\chi}$ is isomorphic to $O_{\chi}/(p^n)[\mathrm{Gal}(K_{(\mathfrak{a})}/K)]$. We consider $W_{K_{(\mathfrak{a}\lambda)}} = \mathrm{Ker}(\mathrm{div}_{\lambda} : (K^{\times}_{(\mathfrak{a}\lambda)} \otimes \mathbf{Z}_p)^{\chi} \longrightarrow (\mathrm{Div}^{\lambda}_{K_{(\mathfrak{a}\lambda)}})^{\chi})$ and a natural map

$$\ell_{\lambda,K_{(\mathfrak{a}\lambda)}} : W_{K_{(\mathfrak{a}\lambda)}}/W^{p^n}_{K_{(\mathfrak{a}\lambda)}} \longrightarrow (R^{\lambda}_{K_{(\mathfrak{a}\lambda)}}/(R^{\lambda}_{K_{(\mathfrak{a}\lambda)}})^{p^n})^{\chi}.$$

We take $b \in (K^{\times}_{(\mathfrak{a}\lambda)} \otimes \mathbf{Z}_p)^{\chi}$ such that $\mathrm{div}(b) = [\lambda_{K_{(\mathfrak{a}\lambda)}}] - [\rho'_{K_{(\mathfrak{a}\lambda)}}]$. Then, $b' = \ell_{\lambda,K_{(\mathfrak{a}\lambda)}}(b^{\sigma_{\lambda}-1})$ is a generator of $(R^{\lambda}_{K_{(\mathfrak{a}\lambda)}}/(R^{\lambda}_{K_{(\mathfrak{a}\lambda)}})^{p^n})^{\chi}$ as an $O_{\chi}/(p^n)[\mathrm{Gal}(K_{(\mathfrak{a})}/K)]$-module ([19] Chap.4 Prop.7 Cor.1). Using this b', we identify $(R^{\lambda}_{K_{(\mathfrak{a}\lambda)}}/(R^{\lambda}_{K_{(\mathfrak{a}\lambda)}})^{p^n})^{\chi}$ with $O_{\chi}/(p^n)[\mathrm{Gal}(K_{(\mathfrak{a})}/K)]$, and define

$$\ell_{\lambda,K_{(\mathfrak{a}\lambda)},b'} : W_{K_{(\mathfrak{a}\lambda)}}/W^{p^n}_{K_{(\mathfrak{a}\lambda)}} \longrightarrow O_{\chi}/p^n[\mathrm{Gal}(K_{(\mathfrak{a})}/K)].$$

Since λ splits completely in $K_{(\mathfrak{a})}$, $c_{K_{(\mathfrak{a}\lambda)}/K_{(\mathfrak{a})}}(\theta^{\chi}_{K_{(\mathfrak{a}\lambda)}}) = 0$ by the formula in the proof of Lemma 4.1. Hence, $\sigma_{\lambda}-1$ divides $\theta^{\chi}_{K_{(\mathfrak{a}\lambda)}}$. Since $(\sigma_{\lambda}-1)[\lambda_{K_{(\mathfrak{a}\lambda)}}] = 0$, we have $\theta^{\chi}_{K_{(\mathfrak{a}\lambda)}}[\lambda_{K_{(\mathfrak{a}\lambda)}}]^{\chi} = 0$. So, $\mathrm{div}(g^{\chi}_{K_{(\mathfrak{a}\lambda)},\rho'_{K_{(\mathfrak{a}\lambda)}}}) = \mathrm{div}((b^{-\theta_{K_{(\mathfrak{a}\lambda)}}})^{\chi}) =$

$\theta^{\chi}_{K_{(\mathfrak{a}\lambda)}}[\rho'_{K_{(\mathfrak{a}\lambda)}}]^{\chi}$. The injectivity of div implies that $g^{\chi}_{K_{(\mathfrak{a}\lambda)},\rho'_{K_{(\mathfrak{a}\lambda)}}} = (b^{-\theta_{K_{(\mathfrak{a}\lambda)}}})^{\chi}$. Further, by Lemma 4.4, we can write

$$\theta^{\chi}_{K_{(\mathfrak{a}\lambda)}} \equiv (-1)^{r+1}\delta^{\chi}_{\mathfrak{a}\lambda}S_1 \cdot ... \cdot S_r(\sigma_\lambda - 1) + \beta \pmod{p^n}$$

where $\beta \in (S_1^2, ..., S_r^2, (\sigma_\lambda - 1)^2)$. Since $\sigma_\lambda - 1$ divides $\theta^{\chi}_{K_{(\mathfrak{a}\lambda)}}$, $\sigma_\lambda - 1$ also divides β. We write $\beta = (\sigma_\lambda - 1)\beta'$. So $\theta^{\chi}_{K_{(\mathfrak{a}\lambda)}} \equiv (\sigma_\lambda - 1)((-1)^{r+1}\delta^{\chi}_{\mathfrak{a}\lambda}S_1 \cdot ... \cdot S_r + \beta')$ (mod p^n). Then,

$$\begin{aligned}
\ell_{\lambda,K_{(\mathfrak{a}\lambda)},b'}(g^{\chi}_{K_{(\mathfrak{a}\lambda)},\rho'_{K_{(\mathfrak{a}\lambda)}}}) &= \ell_{\lambda,K_{(\mathfrak{a}\lambda)},b'}((b^{-\theta_{K_{(\mathfrak{a}\lambda)}}})^{\chi}) \\
&= -c_{K_{(\mathfrak{a}\lambda)}/K_{(\mathfrak{a})}}((-1)^{r+1}\delta^{\chi}_{\mathfrak{a}\lambda}S_1 \cdot ... \cdot S_r + \beta') \\
&= (-1)^r\delta^{\chi}_{\mathfrak{a}\lambda}S_1 \cdot ... \cdot S_r - c_{K_{(\mathfrak{a}\lambda)}/K_{(\mathfrak{a})}}(\beta').
\end{aligned}$$

Since $c_{K_{(\mathfrak{a}\lambda)}/K_{(\mathfrak{a})}}(\beta') \in (S_1^2, ..., S_r^2)$, using $S_iD_{\lambda_i} \equiv -N_{\lambda_i} \pmod{p^n}$ and $S_i^2D_{\lambda_i} \equiv 0 \pmod{p^n}$, we have

$$\begin{aligned}
\ell_{\lambda,K_{(\mathfrak{a}\lambda)},b'}((g^{\chi}_{K_{(\mathfrak{a}\lambda)},\rho'_{K_{(\mathfrak{a}\lambda)}}})^{D_{\mathfrak{a}}}) &= D_{\mathfrak{a}}((-1)^r\delta^{\chi}_{\mathfrak{a}\lambda}S_1 \cdot ... \cdot S_r - c_{K_{(\mathfrak{a}\lambda)}/K_{(\mathfrak{a})}}(\beta')) \\
&= N_{\mathfrak{a}}\delta^{\chi}_{\mathfrak{a}\lambda} \\
&= \nu_{K_{(\mathfrak{a})}/K}(\delta^{\chi}_{\mathfrak{a}\lambda}).
\end{aligned}$$

We similarly define $W_{K_{(\mathfrak{a})}} = \mathrm{Ker}(\mathrm{div}_\lambda \text{ for } K_{(\mathfrak{a})}) \subset (K^{\times}_{(\mathfrak{a})} \otimes \mathbf{Z}_p)^{\chi}$. Recall that $W = \mathrm{Ker}(\mathrm{div}_\lambda \text{ for } K) \subset (K^{\times} \otimes \mathbf{Z}_p)^{\chi}$. Let ℓ_λ (resp. $\ell_{\lambda,K_{(\mathfrak{a})}}$, $\ell_{\lambda,K_{(\mathfrak{a}\lambda)}}$) be the natural map $W_K/W_K^{p^n} \longrightarrow (R_K^\lambda/(R_K^\lambda)^{p^n})^{\chi}$ (resp. $W_{K_{(\mathfrak{a})}}/W_{K_{(\mathfrak{a})}}^{p^n} \longrightarrow (R^\lambda_{K_{(\mathfrak{a})}}/(R^\lambda_{K_{(\mathfrak{a})}})^{p^n})^{\chi}$, $W_{K_{(\mathfrak{a}\lambda)}}/W_{K_{(\mathfrak{a}\lambda)}}^{p^n} \longrightarrow (R^\lambda_{K_{(\mathfrak{a}\lambda)}}/(R^\lambda_{K_{(\mathfrak{a}\lambda)}})^{p^n})^{\chi}$). We have a commutative diagram

$$\begin{array}{ccccc}
W_K/W_K^{p^n} & \longrightarrow & W_{K_{(\mathfrak{a})}}/W_{K_{(\mathfrak{a})}}^{p^n} & \longrightarrow & W_{K_{(\mathfrak{a}\lambda)}}/W_{K_{(\mathfrak{a}\lambda)}}^{p^n} \\
\big\downarrow{\scriptstyle \ell_\lambda} & & \big\downarrow{\scriptstyle \ell_{\lambda,K_{(\mathfrak{a})}}} & & \big\downarrow{\scriptstyle \ell_{\lambda,K_{(\mathfrak{a}\lambda)}}} \\
(R_K^\lambda/(R_K^\lambda)^{p^n})^{\chi} & \longrightarrow & (R^\lambda_{K_{(\mathfrak{a})}}/(R^\lambda_{K_{(\mathfrak{a})}})^{p^n})^{\chi} & \longrightarrow & (R^\lambda_{K_{(\mathfrak{a}\lambda)}}/(R^\lambda_{K_{(\mathfrak{a}\lambda)}})^{p^n})^{\chi}
\end{array}$$

where the horizontal arrows are the natural maps. We take a generator u' of $(R^\lambda_{K_{(\mathfrak{a})}}/(R^\lambda_{K_{(\mathfrak{a})}})^{p^n})^{\chi}$ as an $O_\chi/(p^n)[\mathrm{Gal}(K_{(\mathfrak{a})}/K)]$-module, and a generator u'' of $(R_K^\lambda/(R_K^\lambda)^{p^n})^{\chi}$ as an $O_\chi/(p^n)$-module such that the diagram

$$\begin{array}{ccccc}
W_K/W_K^{p^n} & \longrightarrow & W_{K_{(\mathfrak{a})}}/W_{K_{(\mathfrak{a})}}^{p^n} & \longrightarrow & W_{K_{(\mathfrak{a}\lambda)}}/W_{K_{(\mathfrak{a}\lambda)}}^{p^n} \\
\big\downarrow{\scriptstyle \ell_{\lambda,u''}} & & \big\downarrow{\scriptstyle \ell_{\lambda,K_{(\mathfrak{a})},u'}} & & \big\downarrow{\scriptstyle \ell_{\lambda,K_{(\mathfrak{a}\lambda)},b'}} \\
O_\chi/(p^n) & \xrightarrow{\nu_{K_{(\mathfrak{a})}/K}} & O_\chi/(p^n)[\mathrm{Gal}(K_{(\mathfrak{a})}/K)] & \xrightarrow{id} & O_\chi/(p^n)[\mathrm{Gal}(K_{(\mathfrak{a})}/K)]
\end{array}$$

commutes where $\nu_{K_{(\mathfrak{a})}/K}$ is the norm map defined before Lemma 4.4, and id is the identity map.

Using the above computation of $\ell_{\lambda,K_{(\mathfrak{a}\lambda)},b'}((g^{\chi}_{K_{(\mathfrak{a}\lambda)},\rho'_{K_{(\mathfrak{a}\lambda)}}})^{D_{\mathfrak{a}}})$, if we get

$$\frac{1-N(\lambda)^{-1}}{p^n}\ell_{\lambda,K_{(\mathfrak{a})}}(g^{\chi}_{K_{(\mathfrak{a})},\rho'_{K_{(\mathfrak{a})}}}) = \ell_{\lambda,K_{(\mathfrak{a}\lambda)}}(g^{\chi}_{K_{(\mathfrak{a}\lambda)},\rho'_{K_{(\mathfrak{a}\lambda)}}}), \qquad (2)$$

we obtain (1) from the above commutative diagram.
The relation (2) is sometimes called the "congruence condition", and can be proved by the method of Rubin [18] Corollary 4.8.1 and Kato [8] Prop.1.1. Put $L = K_{(\mathfrak{a})}(\mu_{p^N})$ and $L_{(\lambda)} = K_{(\mathfrak{a}\lambda)}(\mu_{p^N})$ (Recall that N was chosen in the beginning of the proof). We take a prime $\rho'_{L_{(\lambda)}}$ of $L_{(\lambda)}$ above $\rho'_{K_{(\mathfrak{a}\lambda)}}$, and denote by ρ'_L the prime of L below $\rho'_{L_{(\lambda)}}$. We define $\ell_{\lambda,L} : W_L/W_L^{p^N} \longrightarrow (R^{\lambda}_L/(R^{\lambda}_L)^{p^N})^{\chi}$, and $\ell_{\lambda,L_{(\lambda)}} : W_{L_{(\lambda)}}/W_{L_{(\lambda)}}^{p^N} \longrightarrow (R^{\lambda}_{L_{(\lambda)}}/(R^{\lambda}_{L_{(\lambda)}})^{p^N})^{\chi}$ similarly. We identify $(R^{\lambda}_L/(R^{\lambda}_L)^{p^N})^{\chi}$ with $(R^{\lambda}_{L_{(\lambda)}}/(R^{\lambda}_{L_{(\lambda)}})^{p^N})^{\chi}$ by the map induced by the inclusion. Then, the norm map induces the multiplication by p^n. Since $N_{L_{(\lambda)}/L}(g^{\chi}_{L_{(\lambda)},\rho'_{L_{(\lambda)}}}) = (1-\varphi_{\lambda}^{-1})g^{\chi}_{L,\rho'_L}$, we have $p^n\ell_{\lambda,L_{(\lambda)}}(g^{\chi}_{L_{(\lambda)},\rho'_{L_{(\lambda)}}}) = (1 - N(\lambda)^{-1})\ell_{\lambda,L}(g^{\chi}_{L,\rho'_L})$. Hence,

$$\ell_{\lambda,L_{(\lambda)}}(g^{\chi}_{L_{(\lambda)},\rho'_{L_{(\lambda)}}}) \equiv p^{-n}(1-N(\lambda)^{-1})\ell_{\lambda,L}(g^{\chi}_{L,\rho'_L}) \pmod{p^{N-n}}.$$

Let $\mathcal{S}$ be the set of primes of k ramifying in $L_{(\lambda)}$ and not ramifying in $K_{(\mathfrak{a}\lambda)}$. Note that if $\mathfrak{p} \in \mathcal{S}$, $\mathfrak{p}$ is a prime above p. By Lemma 4.1 we have $N_{L_{(\lambda)}/K_{(\mathfrak{a}\lambda)}}(g^{\chi}_{L_{(\lambda)},\rho'_{L_{(\lambda)}}}) = \epsilon_{K_{(\mathfrak{a}\lambda)}}g^{\chi}_{K_{(\mathfrak{a}\lambda)},\rho'_{K_{(\mathfrak{a}\lambda)}}}$ and $N_{L/K_{(\mathfrak{a})}}(g^{\chi}_{L,\rho'_L}) = \epsilon_{K_{(\mathfrak{a})}}g^{\chi}_{K_{(\mathfrak{a})},\rho'_{K_{(\mathfrak{a})}}}$ where $\epsilon_{K_{(\mathfrak{a}\lambda)}} = (\Pi_{\mathfrak{p}\in\mathcal{S}}(1-\varphi_{\mathfrak{p}}^{-1}))^{\chi} \in O_{\chi}[\mathrm{Gal}(K_{(\mathfrak{a}\lambda)}/K)]$ and $\epsilon_{K_{(\mathfrak{a})}} = c_{K_{(\mathfrak{a}\lambda)}/K_{(\mathfrak{a})}}(\epsilon_{K_{(\mathfrak{a}\lambda)}})$ ($c_{K_{(\mathfrak{a}\lambda)}/K_{(\mathfrak{a})}}$ is the restriction map). Since we assumed $\chi(\mathfrak{p}) \neq 1$ for all $\mathfrak{p}$ above p, $\epsilon_{K_{(\mathfrak{a}\lambda)}}$ is a unit of $O_{\chi}[\mathrm{Gal}(K_{(\mathfrak{a}\lambda)}/K)]$. Hence, we obtain (2) by taking the norms $N_{L_{(\lambda)}/K_{(\mathfrak{a}\lambda)}}$ of both sides of the above formula. This completes the proof of Proposition 4.7.

5 The other inclusion

In this section, for K and χ in Theorem 0.1 and $i \geq 0$, we will prove $\mathrm{Fitt}_{i,O_{\chi}}(A^{\chi}_K) \subset (\Theta_{i,K})^{\chi}$ to complete the proof of Theorem 0.1. More precisely, we will show $\mathrm{Fitt}_{i,O_{\chi}}(A^{\chi}_K) \subset (\Theta_{i,1,K})^{\chi}$.
As in Theorem 0.1, suppose that

$$A^{\chi}_K \simeq O_{\chi}/(p^{n_1}) \oplus ... \oplus O_{\chi}/(p^{n_r})$$

with $0 < n_1 \leq ... \leq n_r$. We take generators $\mathbf{c}_1$,...,$\mathbf{c}_r$ corresponding to the above isomorphism ($\mathbf{c}_j$ generates the j-th direct summand). Let $\mathcal{P}_n$ be as in §4. We define

$$\mathcal{Q}_j = \{\lambda \in \mathcal{P}_n : \text{ there is a prime } \lambda_K \text{ of } K \text{ above } \lambda \text{ such that the class of } \lambda_K \text{ in } A^{\chi}_K \text{ is } \mathbf{c}_j\},$$

and $\mathcal{Q} = \bigcup_{1 \le j \le r} \mathcal{Q}_j$. We consider an exact sequence

$$0 \longrightarrow (K^\times \otimes \mathbf{Z}_p)^\chi \xrightarrow{\text{div}} (\text{Div}_K \otimes \mathbf{Z}_p)^\chi \longrightarrow A_K^\chi \longrightarrow 0.$$

For $\lambda \in \mathcal{Q}$, we have $(\mathbf{Z}_p[\text{Gal}(K/k)][\lambda_K])^\chi = O_\chi[\lambda_K]^\chi$. We define $M_{\mathcal{Q}}$ to be the inverse image of $\bigoplus_{\lambda \in \mathcal{Q}} O_\chi[\lambda_K]^\chi$ by div : $(K^\times \otimes \mathbf{Z}_p)^\chi \longrightarrow (\text{Div}_K \otimes \mathbf{Z}_p)^\chi$. On the other hand, as an abstract O_χ-module, A_K^χ fits into an exact sequence

$$0 \longrightarrow \bigoplus_{j=1}^{r} O_\chi e_j \xrightarrow{f} \bigoplus_{j=1}^{r} O_\chi e'_j \xrightarrow{g} A_K^\chi \longrightarrow 0$$

where (e_j) and (e'_j) are bases of free O_χ-modules of rank r, f is the map $e_j \mapsto p^{n_j} e'_j$, and g is induced by $e'_j \mapsto \mathbf{c}_j$. We define $\beta : \bigoplus_{\lambda \in \mathcal{Q}} O_\chi[\lambda_K]^\chi \longrightarrow \bigoplus_{j=1}^r O_\chi e'_j$ by $[\lambda_K]^\chi \mapsto e'_j$ for all $\lambda \in \mathcal{Q}_j$ and $j = 1, ..., r$. Then, β induces $\alpha : M_{\mathcal{Q}} \longrightarrow \bigoplus_{j=1}^r O_\chi e_j$, and we have a commutative diagram of O_χ-modules

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_{\mathcal{Q}} & \xrightarrow{\text{div}} & \bigoplus_{\lambda \in \mathcal{Q}} O_\chi[\lambda_K]^\chi & \longrightarrow & A_K^\chi & \longrightarrow & 0 \\
 & & \downarrow \alpha & & \downarrow \beta & & \| & & \\
0 & \longrightarrow & \bigoplus_{j=1}^r O_\chi e_j & \xrightarrow{f} & \bigoplus_{j=1}^r O_\chi e'_j & \xrightarrow{g} & A_K^\chi & \longrightarrow & 0.
\end{array}$$

Put $m = \text{length}_{O_\chi}(A_K^\chi)$. We take $n > 0$ such that $n \ge 2m$ and $\mu_{p^{n+1}} \not\subset K$. We use the same notation as in Proposition 4.7. Especially, we consider

$$\ell_\lambda : W/W^{p^n} \longrightarrow (R_K^\lambda/(R_K^\lambda)^{p^n})^\chi \simeq O_\chi/(p^n)$$

for $\lambda \in \mathcal{Q}$.

Lemma 5.1. *Suppose that $\mathfrak{a}$, λ, ρ,...etc satisfy the hypotheses of Proposition 4.7. We further assume that the primes dividing $\mathfrak{a}\lambda\rho$ are all in $\mathcal{Q}$. Then, there exists $\tilde{\kappa}^\chi_{\mathfrak{a},\rho_{K(\mathfrak{a})}} \in M_{\mathcal{Q}}$ satisfying the following properties.*
(i) *For any prime λ' such that $(\lambda', \mathfrak{a}\rho) = 1$, we have*

$$\text{div}_{\lambda'}(\tilde{\kappa}^\chi_{\mathfrak{a},\rho_{K(\mathfrak{a})}}) = 0.$$

(ii)

$$\text{div}_\rho(\tilde{\kappa}^\chi_{\mathfrak{a},\rho_{K(\mathfrak{a})}}) \equiv (\delta_{\mathfrak{a}}[\rho_K])^\chi \pmod{p^n}.$$

(iii)

$$\frac{1 - N(\lambda)^{-1}}{p^n} \ell_\lambda(\tilde{\kappa}^\chi_{\mathfrak{a},\rho_{K(\mathfrak{a})}}) \equiv u \delta^\chi_{\mathfrak{a}\lambda} \pmod{(\delta^\chi_{\mathfrak{a}}, p^m)}$$

for some $u \in O_\chi^\times$.

Proof. Let x be an element in W/W^{p^n}, which satisfies the conditions in Proposition 4.7, and take a lifting $y \in W$ of x. By Proposition 4.7 (i) and Lemma 4.6, we can write $\text{div}(y) = \mathcal{A} + p^n \mathcal{B}$ where $\mathcal{A}$ is a divisor whose support is contained in the primes dividing $\mathfrak{a}\rho$. Since the class of $p^m\mathcal{B}$ in A_K^χ is zero, we can take $z \in (K^\times \otimes \mathbf{Z}_p)^\chi$ such that $\mathrm{div}(z) = p^m\mathcal{B}$. Put $\tilde{\kappa}^\chi_{\mathfrak{a},\rho_{K_{(\mathfrak{a})}}} = y/z^{p^{n-m}}$. Then, $\tilde{\kappa}^\chi_{\mathfrak{a},\rho_{K_{(\mathfrak{a})}}}$ is in $M_\mathcal{Q}$, and satisfies the above properties (i), (ii), and (iii) by Proposition 4.7 and Lemma 4.6.

We will prove Theorem 0.1. First of all, as we saw in Proposition 3.2, $\mathrm{Fitt}_{0,O_\chi}(A_K^\chi) = \Theta^\chi_{0,K}$. Recall that we put $m = \mathrm{length}_{O_\chi}(A_K^\chi)$, so $\mathrm{Fitt}_{0,O_\chi}(A_K^\chi) = (p^m)$. Next, we consider the commutative diagram before Lemma 5.1. We denote by $\alpha_j : M_\mathcal{Q} \longrightarrow O_\chi e_j \simeq O_\chi$ the composite of α and the j-th projection. We take $\rho_r \in \mathcal{Q}_r$ and a prime $\rho_{r,K}$ of K above ρ_r. We consider $g^\chi_{K,\rho_{r,K}} \in M_\mathcal{Q}$. We choose $\lambda_r \in \mathcal{Q}_r$ such that $\lambda_r \neq \rho_r$, $\mathrm{ord}_p(N(\lambda_r) - 1) = n$, the class of $\lambda_{r,K}$ in A_K^χ coincides with the class of $\rho_{r,K}$, and $\alpha_r(g^\chi_{K,\rho_{r,K}}) \bmod p^n = u'\ell_{\lambda_r}(g^\chi_{K,\rho_{r,K}})$ for some $u' \in O_\chi^\times$. This is possible by Chebotarev density theorem (Theorem 3.1 in Rubin [17], cf. also [16]). By the commutative diagram before Lemma 5.1 and $\mathrm{div}(g^\chi_{K,\rho_{r,K}}) = \theta_K^\chi[\rho_{r,K}]^\chi$, we have

$$\mathrm{ord}_p(\alpha_r(g^\chi_{K,\rho_{r,K}})) + n_r = \mathrm{ord}_p(\theta_K^\chi) = m. \tag{3}$$

On the other hand, by Proposition 4.7, we have $\ell_{\lambda_r}(g^\chi_{K,\rho_{r,K}}) = u\delta^\chi_{\lambda_r} \bmod (p^m)$ for some $u \in O_\chi^\times$. Hence, $\alpha_r(g^\chi_{K,\rho_{r,K}}) \equiv u'\ell_{\lambda_r}(g^\chi_{K,\rho_{r,K}}) \equiv uu'\delta^\chi_{\lambda_r} \bmod (p^m)$. From (3), $\delta^\chi_{\lambda_r} \bmod p^m \neq 0$, hence, $\mathrm{ord}_p(\alpha_r(g^\chi_{K,\rho_{r,K}})) = \mathrm{ord}_p(\delta^\chi_{\lambda_r})$ (for a nonzero element x in O_χ/p^m, $\mathrm{ord}_p(x)$ is defined to be $\mathrm{ord}_p(\tilde{x})$ where $\tilde{x}$ is a lifting of x to O_χ). Therefore, we have

$$\mathrm{ord}_p(\delta^\chi_{\lambda_r}) = m - n_r.$$

Hence, $\mathrm{Fitt}_{1,O_\chi}(A_K^\chi) = (p^{m-n_r})$ is generated by $I_{1,1}(\theta^\chi_{K_{(\lambda_r)}})$ by Corollary 4.5. Thus, $\mathrm{Fitt}_{1,O_\chi}(A_K^\chi) \subset (\Theta_{1,1,K})^\chi \subset (\Theta_{1,K})^\chi$.

For any $i > 1$, we prove $\mathrm{Fitt}_{i,O_\chi}(A_K^\chi) \subset \Theta^\chi_{i,K}$ by the same method. We will show that we can take $\lambda_r \in \mathcal{Q}_r$, $\lambda_{r-1} \in \mathcal{Q}_{r-1}$,...inductively such that $\delta^\chi_{\mathfrak{a}_i}$ generates $\mathrm{Fitt}_{i,O_\chi}(A_K^\chi)$ where $\mathfrak{a}_i = \lambda_r \cdot ... \cdot \lambda_{r-i+1}$. For i such that $1 < i \leq r$, suppose that λ_r,...,λ_{r-i+2} were defined. We first take $\rho_{r-i+1} \in \mathcal{Q}_{r-i+1}$, which splits completely in $K_{(\mathfrak{a}_{i-1})}$. We consider $\kappa = \tilde{\kappa}^\chi_{\mathfrak{a}_{i-1},\rho_{r-i+1,K_{(\mathfrak{a}_{i-1})}}} \in M_\mathcal{Q}$ where we used the notation in Lemma 5.1. We choose $\lambda_{r-i+1} \in \mathcal{Q}_{r-i+1}$ such that $\lambda_{r-i+1} \neq \rho_{r-i+1}$, $\mathrm{ord}_p(N(\lambda_{r-i+1})-1) = n$, λ_{r-i+1} splits completely in $K_{(\mathfrak{a}_{i-1})}$, the class of $\lambda_{r-i+1,K}$ in A_K^χ coincides with the class of $\rho_{r-i+1,K}$ in A_K^χ, and $\alpha_{r-i+1}(\kappa) \bmod p^n = u'\ell_{\lambda_{r-i+1}}(\kappa)$ for some $u' \in O_\chi^\times$. This is also possible by Chebotarev density theorem (Theorem 3.1 in Rubin [17], cf. also [16]). By Lemma 5.1 (ii), $\mathrm{div}_{\rho_{r-i+1}}(\kappa) \equiv \delta^\chi_{\mathfrak{a}_{i-1}}[\rho_{r-i+1}]^\chi \pmod{p^n}$. Hence, from the commutative diagram before Lemma 5.1, we obtain

$$\mathrm{ord}_p(\alpha_{r-i+1}(\kappa)) + n_{r-i+1} = \mathrm{ord}_p(\delta^\chi_{\mathfrak{a}_{i-1}}).$$

By the hypothesis of the induction, we have $\mathrm{ord}_p(\delta^\chi_{\mathfrak{a}_{i-1}}) = n_1 + ... + n_{r-i+1}$. It follows that

$$\mathrm{ord}_p(\alpha_{r-i+1}(\kappa)) = n_1 + ... + n_{r-i}.$$

On the other hand, by Lemma 5.1 (iii), we have

$$\begin{aligned}\mathrm{ord}_p(\alpha_{r-i+1}(\kappa)) &= \mathrm{ord}_p(\ell_{\lambda_{r-i+1}}(\kappa)) = \mathrm{ord}_p(\delta^{\chi}_{\mathfrak{a}_{i-1}\lambda_{r-i+1}}) \\ &= \mathrm{ord}_p(\delta^{\chi}_{\mathfrak{a}_i}).\end{aligned}$$

Therefore,

$$\mathrm{ord}_p(\delta^{\chi}_{\mathfrak{a}_i}) = n_1 + ... + n_{r-i}.$$

This shows that $\delta^{\chi}_{\mathfrak{a}_i}$ generates $\mathrm{Fitt}_{i,O_\chi}(A^{\chi}_K) = (p^{n_1+\cdots+n_{r-i}})$. Hence, by Corollary 4.5 we obtain

$$I_{i,1}(\theta^{\chi}_{K_{(\mathfrak{a}_i)}}) = (\delta^{\chi}_{\mathfrak{a}_i}) = \mathrm{Fitt}_{i,O_\chi}(A^{\chi}_K).$$

Thus, we have $\mathrm{Fitt}_{i,O_\chi}(A^{\chi}_K) \subset (\Theta_{i,1,K})^{\chi} \subset (\Theta_{i,K})^{\chi}$.
Note that for $i = r$, we have got $\Theta^{\chi}_{r,K} = (1)$. Hence, $\Theta^{\chi}_{i,K} = (1)$ for all $i \geq r$, and we have $\mathrm{Fitt}_{i,O_\chi}(A^{\chi}_K) = \Theta^{\chi}_{i,K}$ for all $i \geq 0$. This completes the proof of Theorem 0.1.

A appendix

In this appendix, we determine the initial Fitting ideal of the Pontrjagin dual $(A^{\sim}_{F_\infty})^{\vee}$ (cf. §2) of the non-ω component of the p-primary component of the ideal class group as a $\mathbf{Z}_p[[\mathrm{Gal}(F_\infty/k)]]$-module for the cyclotomic $\mathbf{Z}_p$-extension F_∞ of a CM-field F such that F/k is finite and abelian, under the assumption that the Leopoldt conjecture holds for k and the μ-invariant of F vanishes. Our aim is to prove Theorem A.5. For the initial Fitting ideal of the Iwasawa module $X_{F_\infty} = \varprojlim A_{F_n}$ of F_∞, see [11] and Greither's results [3], [4].
Suppose that λ_1,...,λ_r are all finite primes of k, which are prime to p and ramifying in F_∞/k. We denote by P_{λ_i} the p-Sylow subgroup of the inertia subgroup of λ_i in $\mathrm{Gal}(F_\infty/k)$. We first assume that

$$(*) \qquad P_{\lambda_1} \times ... \times P_{\lambda_r} \subset \mathrm{Gal}(F_\infty/k).$$

(Compare this condition with the condition (A_p) in [11] §3.) We define a set $\mathcal{H}$ of certain subgroups of $\mathrm{Gal}(F_\infty/k)$ by

$$\mathcal{H} = \{H_1 \times ... \times H_r \mid H_i \text{ is a subgroup of } P_{\lambda_i} \text{ for all } i \text{ such that } 1 \leq i \leq r\}.$$

We also define

$$\mathcal{M} = \{M_\infty \mid k \subset M_\infty \subset F_\infty, \; M \text{ is the fixed field of some } H \in \mathcal{H}\}.$$

For an intermediate field M_∞ of F_∞/k, we denote by

$$\nu_{F_\infty/M_\infty} : \mathbf{Z}_p[[\mathrm{Gal}(M_\infty/k)]]^{\sim} \longrightarrow \mathbf{Z}_p[[\mathrm{Gal}(F_\infty/k)]]^{\sim}$$

the map induced by $\sigma \mapsto \Sigma_{\tau_{|M_\infty}=\sigma}\tau$ for $\sigma \in \mathrm{Gal}(M_\infty/k)$. We define $\Theta^{\sim}_{F_\infty/k}$ to be the $\mathbf{Z}_p[[\mathrm{Gal}(F_\infty/k)]]^{\sim}$-module generated by $\nu_{F_\infty/M_\infty}(\theta^{\sim}_{M_\infty})$ for all $M_\infty \in \mathcal{M}_{F_\infty/k}$.
Put $\Lambda_F = \mathbf{Z}_p[[\mathrm{Gal}(F_\infty/k)]]$. Let $\iota : \Lambda_F \longrightarrow \Lambda_F$ be the map defined by $\sigma \mapsto \sigma^{-1}$ for all $\sigma \in \mathrm{Gal}(F_\infty/k)$. For a Λ_F-module M, we denote by $M^{\approx}$ to be the component obtained from M^- by removing $M^{\omega^{-1}}$, namely $M^- = M^{\approx} \oplus M^{\omega^{-1}}$ if $\mu_p \subset F$, and $M^- = M^{\approx}$ otherwise (cf. 1.1). The map ι induces $M^{\sim} \stackrel{\iota}{\longrightarrow} M^{\approx}$ which is bijective.

PROPOSITION A.1. *We assume that the μ-invariant of F vanishes. Under the assumption of* $(*)$, *we have*

$$\mathrm{Fitt}_{0,\Lambda_F}((A^{\sim}_{F_\infty})^{\vee})^{\approx} = \iota(\Theta^{\sim}_{F_\infty/k}).$$

Proof. This can be proved by the same method as the proof of Theorem 0.9 in [11] by using a slight modification of Lemma 4.1 in [11]. In fact, instead of Corollary 5.3 in [11], we can use

LEMMA A.2. *Let L/K be a finite abelian p-extension of CM-fields. Suppose that P is a set of primes of K_∞ which are ramified in L_∞ and prime to p. For $v \in P$, e_v denotes the ramification index of v in L_∞/K_∞. Then, we have an exact sequence*

$$0 \longrightarrow A^{\sim}_{K_\infty} \longrightarrow (A^{\sim}_{L_\infty})^{\mathrm{Gal}(L_\infty/K_\infty)} \longrightarrow (\bigoplus_{v\in P} \mathbf{Z}/e_v\mathbf{Z})^{\sim} \longrightarrow 0$$

Proof of Lemma A.2. It is enough to prove $\hat{H}^0(L_\infty/K_\infty, A^{\sim}_{L_\infty}) = (\bigoplus_{v\in P} \mathbf{Z}/e_v\mathbf{Z})^{\sim}$. Let P'_n be the set of primes of K_n ramifying in L_n. Then, by Lemma 5.1 (ii) in [11], we have $\hat{H}^0(L_n/K_n, A^{\sim}_{L_n}) = (\bigoplus_{v\in P'_n} H^1(L_{n,w}/K_{n,v}, O^{\times}_{L_{n,w}}))^{\sim} = (\bigoplus_{v\in P'_n} \mathbf{Z}/e_v\mathbf{Z})^{\sim}$ where w is a prime of L_n above v. If v is a prime above p, it is totally ramified in K_∞ for sufficiently large n, hence we have $\varinjlim(\bigoplus_{v\in P'_n, v|p} \mathbf{Z}/e_v\mathbf{Z})^{\sim} = 0$. On the other hand, $\varinjlim(\bigoplus_{v\in P'_n, v\nmid p} \mathbf{Z}/e_v\mathbf{Z})^{\sim} = (\bigoplus_{v\in P} \mathbf{Z}/e_v\mathbf{Z})^{\sim}$. Thus, we get $\hat{H}^0(L_\infty/K_\infty, A^{\sim}_{L_\infty}) = (\bigoplus_{v\in P} \mathbf{Z}/e_v\mathbf{Z})^{\sim}$.

Next, we consider a general CM-field F with F/k finite and abelian. We assume that the Leopodt conjecture holds for k.

LEMMA A.3. (Iwasawa) *Let λ be a prime of k not lying above p. Suppose that $k(\lambda p^\infty)$ is the maximal abelian pro-p extension of k, unramified outside $p\lambda$. Then the ramification index of λ in $k(\lambda p^\infty)$ is p^{n_λ} where $n_\lambda = \mathrm{ord}_p(N(\lambda)-1)$.*

In fact, Iwasawa proved that the Leopoldt conjecture implies the existence of "λ-field" ($\mathfrak{q}$-field) in his terminology ([5] Theorem 1). This means that the ramification index of λ is p^{n_λ}.

LEMMA A.4. *Let F/k be a finite abelian extension such that F is a CM-field. Then, there is an abelian extension F'/k satisfying the following properties.*
(i) *$F'_\infty \supset F_\infty$, and the extension F'_∞/F_∞ is a finite abelian p-extension which is unramified outside p.*
(ii) *F'_∞ satisfies the condition $(*)$.*

Proof. This follows from Lemme 2.2 (ii) in Gras [2], but we will give here a proof. Suppose that $\lambda_1,...,\lambda_r$ are all finite primes ramifying in F_∞/k, and prime to p. We denote by $e_{\lambda_i}^{(p)}$ the p-component of the ramification index of λ_i in F_∞. By class field theory, $e_{\lambda_i}^{(p)} \leq p^{n_{\lambda_i}}$. We take a subfield k_i of $k(\lambda_i p^\infty)$ such that k_i/k is a p-extension, and the ramification index of k_i/k is $e_{\lambda_i}^{(p)}$. This is possible by Lemma A.3. Take F' such that $F'_\infty = F_\infty k_1...k_r$. It is clear that F' satisfies the condition (i). Since $k_1...k_r \subset F'_\infty$, F' satisfies the condition $(*)$.

We define $\iota(\Theta^{\sim}_{F_\infty/k})$ by $\iota(\Theta^{\sim}_{F_\infty/k}) = c_{F'_\infty/F_\infty}(\iota(\Theta^{\sim}_{F'_\infty/k}))$ where $c_{F'_\infty/F_\infty} : \Lambda_{F'} \longrightarrow \Lambda_F$ is the restriction map. This $\iota(\Theta^{\sim}_{F_\infty/k})$ does not depend on the choice of F'. In fact, we have

THEOREM A.5. *We assume the Leopoldt conjecture for k and the vanishing of the μ-invariant of F. Then, we have*

$$\mathrm{Fitt}_{0,\Lambda_F}((A^{\sim}_{F_\infty})^\vee)^\approx = \iota(\Theta^{\sim}_{F_\infty/k}).$$

Proof. We take F' as in Lemma A.4. By Proposition A.1, Theorem A.5 holds for F'_∞. Since F'_∞/F_∞ is unramified outside p, by Lemma A.2 the natural map $A^{\sim}_{F_\infty} \longrightarrow (A^{\sim}_{F'_\infty})^{\mathrm{Gal}(F'_\infty/F_\infty)}$ is an isomorphism. Hence, we get

$$\begin{aligned}\mathrm{Fitt}_{0,\Lambda_F}((A^{\sim}_{F_\infty})^\vee)^\approx &= c_{F'_\infty/F_\infty}(\mathrm{Fitt}_{0,\Lambda_{F'}}((A^{\sim}_{F'_\infty})^\vee)^\approx) = c_{F'_\infty/F_\infty}(\iota(\Theta^{\sim}_{F'_\infty/k})) \\ &= \iota(\Theta^{\sim}_{F_\infty/k}).\end{aligned}$$

This completes the proof of Theorem A.5.

REFERENCES

[1] Deligne, P. and Ribet, K., Values of abelian L-functions at negative integers over totally real fields, Invent. math. 59 (1980), 227-286.
[2] Gras, G., Théorie des genres analytique des fonctions L p-adiques des corps totalement réels, Invent. math. 86 (1986), 1-17.
[3] Greither C., Some cases of Brumer's conjecture for abelian CM extensions of totally real fields, Math. Zeitschrift 233 (2000), 515-534.
[4] Greither, C., Computing Fitting ideals of Iwasawa modules, preprint.
[5] Iwasawa, K., On $\mathbf{Z}_l$-extensions of algebraic number fields, Ann. of Math. 98 (1973), 246-326.
[6] Iwasawa, K., On the μ-invariants of $\mathbf{Z}_l$-extensions, *Number Theory, Algebraic Geometry and Commutative Algebra, in honor of Yasuo Akizuki*, Kinokuniya, Tokyo 1973, 1-11.

[7] Iwasawa, K., A simple remark on Leopoldt's conjecture, Kenkichi Iwasawa Collected Papers, Vol II, Springer-Verlag Tokyo 2001, 862-870.
[8] Kato, K., Euler systems, Iwasawa theory, and Selmer groups, Kodai Math. J. 22 (1999), 313-372.
[9] Kato, K., p-adic Hodge theory and values of zeta functions of modular forms, preprint
[10] Kolyvagin, V.A., Euler systems, The Grothendieck Festschrift Vol II (1990), 435-483.
[11] Kurihara, M., Iwasawa theory and Fitting ideals, J. reine angew. Math. 561 (2003), 39-86.
[12] Mazur, B. and Wiles, A., Class fields of abelian extensions of Q, Invent. math. 76 (1984), 179-330.
[13] Northcott, D. G., *Finite free resolutions*, Cambridge Univ. Press, Cambridge New York 1976.
[14] Perrin-Riou, B., Systèmes d'Euler p-adiques et théorie d'Iwasawa, Ann. Inst. Fourier 48 (1998) 1231-1307.
[15] Popescu, C.D., Stark's question and a strong form of Brumer's conjecture, preprint.
[16] Rubin, K., The main conjecture, Appendix to *Cyclotomic Fields I and II* by S. Lang, Graduate Texts in Math. 121, Springer-Verlag 1990, 397-419.
[17] Rubin, K., Kolyvagin's system of Gauss sums, *Arithmetic Algebraic Geometry*, G. van der Geer et al eds, Progress in Math 89, Birkhäuser 1991, 309-324.
[18] Rubin, K., *Euler systems*, Annals of Math. Studies 147, Princeton Univ. Press 2000.
[19] Serre, J.-P., *Corps Locaux*, Hermann, Paris 1968 (troisième édition).
[20] Serre, J.-P., Sur le résidu de la fonction zêta p-adique, Comptes Rendus Acad. Sc. Paris, t.287 (1978), Série A, 183-188.
[21] Siegel, C. R., Über die Fourierschen Koeffizienten von Modulformen, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II 1970, 15-56.
[22] Solomon, D., On the classgroups of imaginary abelian fields, Ann. Inst. Fourier, Grenoble 40-3 (1990), 467-492.
[23] Tate J., *Les conjectures de Stark sur les Fonctions L d'Artin en* $s = 0$, Progress in Math. 47, Birkhäuser 1984.
[24] Washington, L., *Introduction to cyclotomic fields*, Graduate Texts in Math. 83, Springer-Verlag 1982.
[25] Wiles, A., The Iwasawa conjecture for totally real fields, Ann. of Math. 131 (1990), 493-540.
[26] Wiles, A., On a conjecture of Brumer, Ann. of Math. 131 (1990), 555-565.

Masato Kurihara
Department of Mathematics
Tokyo Metropolitan University
Hachioji, Tokyo, 192-0397
Japan
kurihara@math.metro-u.ac.jp

Absolute Derivations and Zeta Functions

Dedicated to Professor Kazuya Kato's 50th birthday

Nobushige Kurokawa, Hiroyuki Ochiai, and Masato Wakayama

Received: September 24, 2002
Revised: December 17, 2002

Abstract. Just as the function ring case we expect the existence of the coefficient field for the integer ring. Using the notion of one element field in place of such a coefficient field, we calculate absolute derivations of arithmetic rings. Notable examples are the matrix rings over the integer ring, where we obtain some absolute rigidity. Knitting up prime numbers via absolute derivations we speculate the arithmetic landscape. Our result is only a trial to a proper foundation of arithmetic.

2000 Mathematics Subject Classification: 11R27, 11R42, 14G10
Keywords and Phrases: absolute derivations, absolute mathematics, absolute schemes

Kronecker and many excellent arithmeticians attempted to study the arithmetic geometry by looking at the intimate analogy between $\mathbf{Z}$ and $\mathbf{F}_p[T]$. Although these two objects are similar in some respects, there exists a quite clear difference: the non-existence (or "invisibility") of the constant (coefficient) field of $\mathbf{Z}$. Zeta functions suggest to compare

$$\hat{\zeta}_{\mathbf{Z}}(s) = \frac{\det(R - (s - \frac{1}{2}))}{s(s-1)}$$

and

$$\hat{\zeta}_{\mathbf{F}_p[T]}(s) = \frac{1}{(1-p^{-s})(1-p^{-(s-1)})},$$

where $\hat{\zeta}$ denotes the "completed zeta function"; in the latter case we know good cohomologies with $\dim H^0(\mathbf{F}_p[T]) = 1$, $\dim H^1(\mathbf{F}_p[T]) = 0$, $\dim H^2(\mathbf{F}_p[T]) = 1$

and $H^i(\mathbf{F}_p[T]) = 0$ for $i > 2$. Up to now, we have not come across a cohomology theory such as $\dim H^0(\mathbf{Z}) = 1$, $\dim H^1(\mathbf{Z}) = \infty$ with a skew-hermitian operator $R : H^1(\mathbf{Z}) \to H^1(\mathbf{Z})$, $\dim H^2(\mathbf{Z}) = 1$ and $H^i(\mathbf{Z}) = 0$ for $i > 2$. Yet we can try to figure out the nature of the "constant field" $\mathbf{F}_1$ of $\mathbf{Z}$ (Manin [9], Deninger [1], Kurokawa [6]). As a first little step we calculate $\mathbf{F}_1$-derivations (in other words, "absolute derivations") of $\mathbf{Z}$ and allied objects here.

The authors thank Professor Kazuya Kato for his patient listening to our primitive tales in old days.
[One of our friends indicates the appearance of KAZUYA by looking at the leading alphabets of sentences in the introduction: the readers are welcome to find such an accidental coincidence.]

1 Absolute derivations

We define an absolute derivation of a ring R as a map $D : R \to R$ satisfying the condition (Leibniz rule)

$$D(ab) = D(a)b + aD(b) \quad \text{for all } a, b \in R.$$

If an absolute derivation D satisfies moreover the additivity property

$$D(a+b) = D(a) + D(b),$$

it is called a derivation of R. Here the word "absolute" indicates objects over "the one element field $\mathbf{F}_1$". Elements of the absolute mathematics are briefly described in § 2 below. We denote by $\mathrm{Der}_{\mathbf{F}_1}(R)$ the set of all absolute derivations of R, and by $\mathrm{Der}_{\mathbf{Z}}(R)$ the set of all derivations of R.

1.1

We first determine the absolute derivations of the most simple but the fundamental case $R = \mathbf{Z}$. For each prime p, define a map $\frac{\partial}{\partial p} : \mathbf{Z} \to \mathbf{Z}$ by

$$\frac{\partial}{\partial p}(x) = \frac{x}{p} \cdot \mathrm{ord}_p(x).$$

Here $\mathrm{ord}_p(x)$ denotes the p-order of $x \in \mathbf{Z}$, that is, the integer ℓ such that x is divisible by p^ℓ but is not divisible by $p^{\ell+1}$. Namely we have

$$\frac{\partial}{\partial p}(x) = \begin{cases} 0 & \text{if } p \nmid x \\ \ell p^{\ell-1} \cdot m & \text{if } x = p^\ell \cdot m \ (\ell \geq 1, p \nmid m) \end{cases}$$

and put $\frac{\partial}{\partial p}(0) = 0$. It is easy to see that $\frac{\partial}{\partial p}$ satisfies the Leibniz rule;

$$\frac{\partial}{\partial p}(xy) = \frac{\partial}{\partial p}(x)y + x\frac{\partial}{\partial p}(y),$$

whence $\frac{\partial}{\partial p} \in \mathrm{Der}_{\mathbf{F}_1}(\mathbf{Z})$. In fact, since $\mathrm{ord}_p(xy) = \mathrm{ord}_p(x) + \mathrm{ord}_p(y)$, we see that

$$\begin{aligned}\frac{\partial}{\partial p}(xy) &= \frac{xy}{p}\cdot \mathrm{ord}_p(xy)\\ &= \frac{xy}{p}\left(\mathrm{ord}_p(x)+\mathrm{ord}_p(y)\right)\\ &= \left(\frac{x}{p}\cdot \mathrm{ord}_p(x)\right) y + x\left(\frac{y}{p}\cdot \mathrm{ord}_p(y)\right)\\ &= \frac{\partial}{\partial p}(x)y + x\frac{\partial}{\partial p}(y).\end{aligned}$$

The following theorem shows these $\frac{\partial}{\partial p}$'s ($p$: prime numbers) span the set of the absolute derivations of $\mathbf{Z}$.

Theorem 1 *We have the following direct product decomposition:*

$$\mathrm{Der}_{\mathbf{F}_1}(\mathbf{Z}) = \widehat{\bigoplus_{p:\mathrm{prime}}} \mathbf{Z}\frac{\partial}{\partial p} := \left\{ \sum_p c_p \frac{\partial}{\partial p}\ ;\ c_p \in \mathbf{Z} \right\} \subset \mathrm{End}_{\mathbf{F}_1}(\mathbf{Z}),$$

where $\mathrm{End}_{\mathbf{F}_1}(\mathbf{Z}) = \mathrm{Map}(\mathbf{Z},\mathbf{Z})$.

Proof: Note first that the infinite sum $\sum_p c_p \frac{\partial}{\partial p}(x) \in \widehat{\bigoplus_p} \mathbf{Z}\frac{\partial}{\partial p}$ is well-defined since for each $x \in \mathbf{Z}$, $\frac{\partial}{\partial p}(x) = 0$ except the finite number of p. It is also easy to see that such an expression is unique. The fact that the sum of absolute derivations is also an absolute derivation, shows clearly that

$$\mathrm{Der}_{\mathbf{F}_1}(\mathbf{Z}) \supset \widehat{\bigoplus_p}\mathbf{Z}\frac{\partial}{\partial p}.$$

It is therefore enough to prove that any $D \in \mathrm{Der}_{\mathbf{F}_1}(\mathbf{Z})$ can be written as

$$D = \sum_p D(p)\frac{\partial}{\partial p}.$$

In order to see this we show that the absolute derivation D is completely determined by its values $D(p)$ on prime numbers $p = 2, 3, 5, \ldots$. By successive use of the Leibniz rule it is obvious to see that

$$D(p^\ell) = \ell p^{\ell-1} D(p).$$

Remark also that $D(0) = D(1) = D(-1) = 0$. Actually, the Leibniz rule shows

$$\begin{gathered} D(0) = D(0\cdot 0) = D(0)\cdot 0 + 0\cdot D(0) = 0,\\ D(1) = D(1\cdot 1) = D(1)\cdot 1 + 1\cdot D(1) = 2D(1),\end{gathered}$$

whence it follows that $D(1) = 0$. Further,

$$0 = D(1) = D((-1)\cdot(-1)) = D(-1)(-1) + (-1)D(-1) = -2D(-1).$$

Since any non-zero $x \in \mathbf{Z}$ can be written as $x = \pm p_1^{i_1} p_2^{i_2} \cdots p_\ell^{i_\ell}$ by primes p_j, using the Leibniz rule again the assertion follows, that is, D is completely determined once the values $D(p)$ are given. To confirm the relation $D = \sum_p D(p)\frac{\partial}{\partial p}$ holds it suffices to show $D(q) = \left(\sum_p D(p)\frac{\partial}{\partial p}\right)(q)$ for each prime q. Since

$$\frac{\partial}{\partial p}(q) = \begin{cases} 1 & (p = q) \\ 0 & (p \neq q), \end{cases}$$

it is actually immediate to have

$$\Big(\sum_p D(p)\frac{\partial}{\partial p}\Big)(q) = \sum_p D(p)\frac{\partial}{\partial p}(q) = D(q).$$

This completes the proof. □

Remark 1 *It is easy to see that* $\mathrm{Der}_{\mathbf{Z}}(\mathbf{Z}) = 0$*. In fact, since* $D(0) = D(1) = D(-1) = 0$ *as above, the additivity asserts* $D(m) = 0$ *for* $m \in \mathbf{Z}$*.*

Remark 2 *We have for primes* p, q

$$\left[\frac{\partial}{\partial p}, q\right] = \delta_{pq},$$

where q *in the left-hand side is regarded as a multiplication operator, since*

$$\begin{aligned} \left[\frac{\partial}{\partial p}, q\right](x) &= \frac{\partial}{\partial p}(qx) - q\frac{\partial}{\partial p}(x) \\ &= \Big(\frac{\partial}{\partial p}(q)x + q\frac{\partial}{\partial p}(x)\Big) - q\frac{\partial}{\partial p}(x) = \frac{\partial}{\partial p}(q)x = \delta_{pq}\cdot x. \end{aligned}$$

1.2

We have the similar statement for $\mathbf{Z}[i]$ and $\mathbf{F}_p[T]$.
Let $\mathbf{Z}[i] = \{m + ni;\ \ m, n \in \mathbf{Z}\}$ $(i = \sqrt{-1})$ be the ring of Gaussian integers. Let $\{\pi\}$ be a complete set of representatives of the prime elements in $\mathbf{Z}[i]$. Define the map $\frac{\partial}{\partial \pi} \in \mathrm{End}_{\mathbf{F}_1}(\mathbf{Z}[i])$ by

$$\frac{\partial}{\partial \pi}(x) = \frac{x}{\pi}\cdot \mathrm{ord}_\pi(x).$$

Then similar to the theorem above we obtain the following:

Theorem 2 *We have*

$$\mathrm{Der}_{\mathbf{F}_1}(\mathbf{Z}[i]) = \widehat{\bigoplus_{\pi}}\mathbf{Z}[i]\frac{\partial}{\partial \pi} \subset \mathrm{End}_{\mathbf{F}_1}(\mathbf{Z}[i]).$$

Moreover, for prime elements π, π' *we have*

$$\left[\frac{\partial}{\partial \pi}, \pi'\right] = \delta_{\pi\pi'}.$$

Now we consider the case $R = \mathbf{F}_p[T]$, the polynomial ring over the finite field $\mathbf{F}_p$. Any $f \in \mathbf{F}_p[T]$ can be factorized uniquely as

$$f = c \cdot h_1^{e_1} \cdot h_2^{e_2} \cdots h_\ell^{e_\ell},$$

where $c \in \mathbf{F}_p$, $e_i \in \mathbf{Z}_{\geq 0}$ and $h_1, h_2, \ldots, h_\ell$ are monic irreducible polynomials, that is, the prime elements in $\mathbf{F}_p[T]$. We define the order $\mathrm{ord}_h(f)$ in an obvious way; $\mathrm{ord}_{h_i}(f) = e_i$. Quite similarly as Theorem 1 we have the following theorem.

Theorem 3 *For a monic irreducible polynomial $h \in \mathbf{F}_p[T]$, define a map $\frac{\partial}{\partial h}$: $\mathbf{F}_p[T] \to \mathbf{F}_p[T]$ by*

$$\frac{\partial}{\partial h} f(T) = \frac{f(T)}{h(T)} \cdot \mathrm{ord}_h(f).$$

Then we have

$$\mathrm{Der}_{\mathbf{F}_1}(\mathbf{F}_p[T]) = \widehat{\bigoplus_{h:\text{monic irred.}}} \mathbf{F}_p[T] \frac{\partial}{\partial h}.$$

Remark 3 *We notice that $\frac{\partial}{\partial h}$ is different from the usual derivation. For example, $\frac{\partial}{\partial T}(T^2) = 2T$ and $\frac{\partial}{\partial T}(T^2 + 1) = 0$ here.*

1.3

For some general unique factorization domains, we have the following result:

Theorem 4 *Let R be a commutative unique factorization domain whose unit group $R^\times$ is a finitely generated abelian group. Fix a set of representative $P_0(R)$ of irreducible elements of $R \setminus (R^\times \cup \{0\})$ modulo $R^\times$, and a set of generators $P_1(R)$ of $R^\times$ modulo $R^\times_{\mathrm{tor}}$, where $R^\times_{\mathrm{tor}}$ is the subgroup of torsion elements. Put $P(R) = P_0(R) \cup P_1(R)$. Each element $a \in R \setminus \{0\}$ can be uniquely written as*

$$a = u \prod_{\pi \in P(R)} \pi^{m(\pi)},$$

where $m(\pi) \in \mathbf{Z}_{\geq 0}$ if $\pi \in P_0(R)$, $m(\pi) \in \mathbf{Z}$ if $\pi \in P_1(R)$, and $u \in R^\times_{\mathrm{tor}}$. We define

$$\frac{\partial}{\partial \pi}(a) = m(\pi) \frac{a}{\pi}$$

and

$$\frac{\partial}{\partial \pi}(0) = 0.$$

Then

$$\mathrm{Der}_{\mathbf{F}_1}(R) = \widehat{\bigoplus_{\pi \in P(R)}} R \frac{\partial}{\partial \pi}.$$

Proof. Each $a \in R \setminus \{0\}$ is uniquely written as

$$a = u' \prod_{\pi \in P_0(R)} \pi^{m(\pi)}$$

with $u' \in R^\times$. Since $R^\times / R^\times_{\mathrm{tor}}$ is a free abelian group, we can write

$$u' = u \prod_{\pi \in P_1(R)} \pi^{m(\pi)}$$

with $u \in R^\times_{\mathrm{tor}}$ uniquely, where $P_1(R)$ is a finite set. It is clear that $\frac{\partial}{\partial \pi} \in \mathrm{Der}_{\mathbf{F}_1}(R)$ by definition.

Now, take any $X \in \mathrm{Der}_{\mathbf{F}_1}(R)$. We show

$$X = \sum_{\pi \in P(R)} X(\pi) \frac{\partial}{\partial \pi}.$$

Take an $a \in R \setminus \{0\}$. Express it as

$$a = u \prod_{\pi \in P(R)} \pi^{m(\pi)}$$

with $u \in R^\times_{\mathrm{tor}}$. Then, using $X(u) = 0$ we have

$$X(a) = \sum_{\pi \in P(R)} m(\pi) \frac{a}{\pi} X(\pi) = \sum_{\pi \in P(R)} X(\pi) \frac{\partial}{\partial \pi}(a).$$

Hence,

$$X = \sum_{\pi \in P(R)} X(\pi) \frac{\partial}{\partial \pi}. \quad \square$$

Example 1 (1) *If* $R = \mathbf{Z}[\sqrt{2}]$, *then* $P_1(R) = \{1 + \sqrt{2}\}$.

(2) *If* $R = \mathbf{Z}[T_1, \ldots, T_n]$, *then* $P_1(R) = \emptyset$.

1.4

We note on a special subset of $\mathrm{Der}_{\mathbf{F}_1}(R)$ for $R = \mathbf{Z}$ and $\mathbf{Z}[i]$.

Theorem 5 *Let* p *be a prime. Then* $\mathfrak{g}_p = \mathbf{Z}\frac{\partial}{\partial p}$ *is closed under the Lie bracket defined by the commutator* $[\cdot, \cdot]$ *of* $\mathrm{End}_{\mathbf{F}_1}(\mathbf{Z})$. *Similarly, for a prime element* π *in* $\mathbf{Z}[i]$, *the subset* $\mathfrak{g}_\pi = \mathbf{Z}[i]\frac{\partial}{\partial \pi}$ *of* $\mathrm{Der}_{\mathbf{F}_1}(\mathbf{Z}[i])$ *is closed under the commutator of* $\mathrm{End}_{\mathbf{F}_1}(\mathbf{Z}[i])$.

Proof: The first assertion is easily confirmed by using the relations

$$[p^\ell \frac{\partial}{\partial p}, p^m \frac{\partial}{\partial p}] \quad = \quad (m-\ell)p^{\ell+m-1}\frac{\partial}{\partial p}.$$

The assertion for $\mathbf{Z}[i]$ can be proved similarly. □

Remark 4 *Define $H = p\partial,\ E = -p^2\partial,\ F = \partial$ with $\partial = \frac{\partial}{\partial p}$. Then the formula above implies the following commutation relations*

$$[H, E] = E, \quad [H, F] = -F, \quad [E, F] = 2H.$$

These commutation relations coincide with those of standard generators $\{H,\ E,\ F\}$ of the simple Lie algebra $\mathfrak{sl}_2$. Hence, the operator $C := 2H^2 + EF + FE$ can be considered as an "absolute" Casimir operator (cf. [7]). This element C commutes with each H, E, F, and moreover it is not hard to see that C vanishes under the map $\mathfrak{g}_p \to \mathrm{End}_{\mathbf{F}_1}(\mathbf{Z})$. It would be interesting to study the "absolute Virasoro algebra" extending $\mathfrak{g}_p$ with its physical implications.

1.5

Let R be a (non-commutative) ring. For an element $a \in R$, we define an inner derivation D_a by $D_a(b) = ab - ba$. It is easy to see that $D_a \in \mathrm{Der}_{\mathbf{Z}}(R)$. We denote the set of all inner derivations by $\mathrm{InnDer}_{\mathbf{Z}}(R)$. We have

$$\mathrm{InnDer}_{\mathbf{Z}}(R) \subset \mathrm{Der}_{\mathbf{Z}}(R) \subset \mathrm{Der}_{\mathbf{F}_1}(R).$$

The main result of this subsection is the following:

Theorem 6 $\mathrm{Der}_{\mathbf{F}_1}(M_2(\mathbf{Z})) = \mathrm{InnDer}_{\mathbf{Z}}(M_2(\mathbf{Z}))$.

Let D be an absolute derivation of $M_2(\mathbf{Z})$.

Lemma 1 $D(0) = D(I_2) = D(-I_2) = 0$.

Proof: Actually, $D(0) = D(00) = 0$, $D(I_2) = D(I_2 I_2) = 2D(I_2)$, $0 = D((-I_2)(-I_2)) = -2I_2 D(-I_2)$. □

Let $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $N' = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. These elements satisfy the following relations $HN = N$, $NH = -N$, $H^2 = I_2$, $HN' = -N'$, $N'H = N'$, $NN' = E_{11}$, $N'N = E_{22}$.

Lemma 2 *(i) There exist $a, b \in \mathbf{Z}$ such that $D(N) = aH + bN$.*

(ii) There exist $a', b' \in \mathbf{Z}$ such that $D(N') = a'H + b'N'$.

Proof: $0 = D(0) = D(N^2) = D(N)N + ND(N)$. Then (i) follows easily. The argument for N' is the same. □

Lemma 3 *$D(H) = -2a'N - 2aN'$.*

Proof: $D(N) = D(HN) = HD(N) + D(H)N$. Then $D(H)N = a(H - I_2)$. Also,

$$-D(N) = D(-N) = D(NH) = D(N)H + ND(H).$$

Then $ND(H) = -a(H + I_2)$. By these two conditions, the diagonal entries of $D(H)$ are both zero, and the $(2,1)$-entry of $D(H)$ is $-2a$.
By $D(HN') = D(-N')$, we have $D(H)N' = -a'(I + H)$. Then the $(1,2)$-entry of $D(H)$ is $-2a'$. □

Lemma 4 *There exists $q \in \mathbf{Z}$ such that for all $c \in \mathbf{Z}$*

$$\begin{aligned} D(H + cN) &= D(H) + c(aH + qN), \\ D(I_2 + cN) &= c(aH + qN). \end{aligned}$$

Proof: Let $A_c := H + cN$ for $c \in \mathbf{Z}$. Since

$$D(N) = D(A_c N) = A_c D(N) + D(A_c)N$$

and (i) of Lemma 2, we have $D(A_c)N = acN + a(H - I_2)$. Also,

$$-D(N) = D(-N) = D(NA_c) = D(N)A_c + ND(A_c).$$

Then $ND(A_c) = -a(H+I_2) - acN$. By these two conditions, we have $D(A_c) = q_c N + acH - 2aN'$. By setting $c = 0$, we have $D(H) = q_0 N - 2aN'$. Then $q_0 = -2a'$.
Notice

$$\begin{aligned} D(I_2 + (c' - c)N) &= D(A_c A_{c'}) \\ &= A_c D(A_{c'}) + D(A_c)A_{c'} \\ &= (c' - c)aH + (q_{c'} - q_c)N. \end{aligned}$$

Then $q_{c'+c} - q_c = q_{c'} - q_0$. This means that $\mathbf{Z} \ni c \mapsto q_c - q_0 \in \mathbf{Z}$ is an additive map. We set $q = q_1 - q_0$, then $q_c = cq + q_0$. This proves the lemma. □

Lemma 5 *We have*

$$\begin{aligned} D(H + cN') &= D(H) + c(a'H - qN'), \\ D(I_2 + cN') &= c(a'H - qN'). \end{aligned}$$

Proof: By a similar argument, there exists $q' \in \mathbf{Z}$, independent of c, such that

$$\begin{aligned} D(H+cN') &= D(H)+c(a'H+q'N'), \\ D(I_2+cN') &= c(a'H+q'N') \end{aligned}$$

for all $c \in \mathbf{Z}$. In fact, let $A'_c := H + cN'$ for $c \in \mathbf{Z}$. Since

$$D(-N') = D(A'_c N') = A'_c D(N') + D(A'_c)N'$$

and (ii) of Lemma 2, we have $D(A'_c)N' = -a'cN' - a'(H+I_2)$. Also,

$$D(N') = D(N') = D(N'A'_c) = D(N')A'_c + N'D(A'_c).$$

Then $N'D(A'_c) = a'(H-I_2)+a'cN'$. By these two conditions, we have $D(A'_c) = q'_c N' + a'cH - 2a'N$. By setting $c = 0$, we have $D(H) = q'_0 N' - 2a'N$. Then $q'_0 = -2a$. The remaining is similar to the previous lemma.
Now we put $B = (I_2 + N)(I_2 - N')$. Then, $\det(B) = 1$ and $\mathrm{tr}(B) = 1$ show that $B^3 = -I_2$. Hence

$$0 = D(B^3) = D(B)B^2 + BD(B)B + B^2 D(B).$$

By multiplying $-B$ from the left, we have

$$BD(B)B^{-1} + B^{-1}D(B)B + D(B) = 0.$$

Taking the trace, $3\mathrm{tr}(D(B)) = 0$. On the other hand, calculate as

$$\begin{aligned} D(B) &= D(I_2+N)(I_2-N') + (I_2+N)D(I_2-N') \\ &= (aH+qN)(I-N') - (I+N)(a'H+q'N'), \end{aligned}$$

then $\mathrm{tr}(D(B)) = -q' - q$. Thus $q' = -q$. □

Proof of Theorem 6: We use the notation above, especially, $a, a', q \in \mathbf{Z}$. Let $Y = a'N - aN' + qE_{11} \in M_2(\mathbf{Z})$. We note that

$$\begin{aligned} &[Y,H] = -2a'N - 2aN', \\ &[Y,N] = aH + qN, \\ &[Y,N'] = a'H - qN'. \end{aligned}$$

We consider the corresponding inner derivation $D_Y \in \mathrm{InnDer}_{\mathbf{Z}}(M_2(\mathbf{Z}))$. Then $D(H) = D_Y(H)$, $D(I_2+cN) = D_Y(I_2+cN)$ and $D(I_2+cN') = D_Y(I_2+cN')$ for all $c \in \mathbf{Z}$. Recall the fact that the group $GL(2,\mathbf{Z})$ is generated by $\{H, I_2 + cN, I_2+cN' \mid c \in \mathbf{Z}\}$. Then we know that $D(A) = D_Y(A)$ for all $A \in GL(2,\mathbf{Z})$. We put $Z := D - D_Y \in \mathrm{Der}_{\mathbf{F}_1}(M_2(\mathbf{Z}))$. Then we have proved that $Z(A) = 0$ for all $A \in GL(2,\mathbf{Z})$, $Z(N) = bN$ for some $b \in \mathbf{Z}$. We set $K = N + N' \in GL(2,\mathbf{Z})$. Then $NKN = N$ implies that

$$Z(N)KN + NKZ(N) = Z(N).$$

This shows $bN = 0$ and so $b = 0$.
Now we take a non-zero integer $\lambda \in \mathbf{Z}$. Consider a matrix $A \in GL(2, \mathbf{Z})$ with $(2, 1)$-entry of A is divisible by λ. We define $A' \in GL(2, \mathbf{Z})$ by

$$(E_{11} + \lambda E_{22})A = A'(E_{11} + \lambda E_{22}).$$

Then

$$\begin{aligned} Z(E_{11} + \lambda E_{22})A &= A'Z(E_{11} + \lambda E_{22}), \\ (E_{11} + \lambda E_{22})^{-1}Z(E_{11} + \lambda E_{22})A &= A(E_{11} + \lambda E_{22})^{-1}Z(E_{11} + \lambda E_{22}) \end{aligned}$$

for all A as above. Such an A is so many, this equality holds for all $A \in M_2(\mathbf{Z})$. This means that $(E_{11} + \lambda E_{22})^{-1}Z(E_{11} + \lambda E_{22})$ is a scalar matrix since it commutes with all the right multiplication. Therefore, there exists $\tau(\lambda) \in \mathbf{Z}$ such that

$$Z(E_{11} + \lambda E_{22}) = \tau(\lambda)(E_{11} + \lambda E_{22}).$$

On the other hand, we have $(E_{11} + \lambda E_{22})N = N$. Since $Z(N) = 0$, we have $Z(E_{11} + \lambda E_{22})N = 0$. This shows that $\tau(\lambda) = 0$. We have proved $Z(E_{11} + \lambda E_{22}) = 0$ for all non-zero $\lambda \in \mathbf{Z}$. By considering $K(E_{11} + \lambda E_{22})K$, we know that $Z(\lambda E_{11} + E_{22}) = 0$. This proves that $Z(A) = 0$ for all $A \in M_2(\mathbf{Z})$ with $\det(A) \neq 0$.
For any matrix C in $M_2(\mathbf{Z})$ of rank one, there exist $A, A' \in M_2(\mathbf{Z})$ with $\det A \neq 0$, $\det A' \neq 0$ such that $C = ANA'$. This shows that $Z(C) = 0$. Finally we have $Z(0) = 0$. □

This result would suggest a kind of rigidity or semi-simplicity of $M_2(\mathbf{Z})$ as an absolute algebra, but it is not quite sure, at this moment, that such a notion can be formulated rigorously. We also remark that some argument can be extended to other non-commutative algebras. For example, the following result is proved in [10]: Let $R \ni 1$ be a ring contained in the algebraic closure $\bar{\mathbf{Q}}$. Then for each $n \geq 2$,

$$\mathrm{Der}_{\mathbf{F}_1}(M_n(R)) = \mathrm{InnDer}_{\mathbf{Z}}(M_n(R)) = \{D_a \mid a \in M_n(R)\}.$$

1.6 Absolute Hochschild cohomology

Theorem 6 can be stated as

$$H^1_{\mathbf{F}_1}(M_2(\mathbf{Z}), M_2(\mathbf{Z})) = 0,$$

where the left-hand side indicates the absolute Hochschild cohomology in the following sense. Let R be a ring and M be an R-bimodule. Let $C^0_{\mathbf{F}_1} = M$, $C^1_{\mathbf{F}_1} = Map(R, M)$, $C^2_{\mathbf{F}_1} = Map(R \times R, M)$, $C^n_{\mathbf{F}_1} = Map(R^n, M)$. We define a derivation $\delta^n : C^n_{\mathbf{F}_1} \to C^{n+1}_{\mathbf{F}_1}$ by

$$\begin{aligned}(\delta^0 m)(a) &= am - ma,\\ (\delta^1 f)(a_1, a_2) &= a_1 f(a_2) - f(a_1 a_2) + f(a_1)a_2,\\ (\delta^n f)(a_1, \ldots, a_{n+1}) &= a_1 f(a_2, \ldots, a_{n+1}) + \sum_{i=1}^{n} (-1)^i f(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+1})\\ &\quad +(-1)^{n+1} f(a_1, \ldots, a_n)a_{n+1}.\end{aligned}$$

Then $(\delta^*, C^*_{\mathbf{F}_1})$ is a chain complex of abelian groups. We call the cohomology groups $H^n_{\mathbf{F}_1}(R, M) = \mathrm{Ker}\delta^n/\mathrm{Im}\delta^{n-1}$ of this chain complex by the absolute Hochschild cohomology (cf. [8]). In particular,

$$H^0_{\mathbf{F}_1}(R, R) = \mathrm{Ker}\delta^0 = \{b \in R \mid ba = ab \text{ for all } a \in R\},$$

the center of R, and

$$H^1_{\mathbf{F}_1}(R, R) = \mathrm{Der}_{\mathbf{F}_1}(R)/\mathrm{InnDer}_{\mathbf{Z}}(R).$$

For example, Theorem 1 says

$$H^1_{\mathbf{F}_1}(\mathbf{Z}, \mathbf{Z}) = \widehat{\bigoplus_{p:\text{prime}}} \mathbf{Z}\frac{\partial}{\partial p}.$$

2 Absolute mathematics

We explain the background material of absolute mathematics, i.e., the mathematics over $\mathbf{F}_1$. As noted in Manin [9] the first appearance of $\mathbf{F}_1$ seems to be in $GL_n(\mathbf{F}_1) = S_n$ where S_n is the symmetric group of order n. This might be a folklore, but a much precise reference was supplied by Soulé [11] citing a paper [12] by Tits. There it seems that Tits conjectured $G(\mathbf{F}_1) = W(G)$ for each algebraic group G, where $W(G)$ is the Weyl group; in the case $G = GL_n$ we get $GL_n(\mathbf{F}_1) = W(GL_n) = S_n$ again.

We consider that $GL_n(\mathbf{F}_1) = S_n$ suggests to identify the category $\mathbf{Mod}(\mathbf{F}_1)$ of $\mathbf{F}_1$-modules with the category $\mathbf{Set}$ of sets. Let denote the free $\mathbf{F}_1$-module over a set X by $\mathbf{F}_1^{(X)}$. Then the more precise expectation is as follows: for objects X and Y of $\mathbf{Set}$, the corresponding objects of $\mathbf{Mod}(\mathbf{F}_1)$ are $\mathbf{F}_1^{(X)}$ and $\mathbf{F}_1^{(Y)}$ respectively with the corresponding morphisms

$$\mathrm{Hom}_{\mathbf{Set}}(X, Y) \cong \mathrm{Hom}_{\mathbf{Mod}(\mathbf{F}_1)}(\mathbf{F}_1^{(X)}, \mathbf{F}_1^{(Y)}).$$

Hence, especially for $X = \{1, 2, \ldots, n\}$ it would hold that

$$M_n(\mathbf{F}_1) = \mathrm{End}_{\mathbf{Mod}(\mathbf{F}_1)}(\mathbf{F}_1^n) = \mathrm{End}_{\mathbf{Set}}(\{1, 2, \ldots, n\})$$

and

$$GL_n(\mathbf{F}_1) = \mathrm{Aut}_{\mathbf{Mod}(\mathbf{F}_1)}(\mathbf{F}_1^n) = \mathrm{Aut}_{\mathbf{Set}}(\{1, 2, \ldots, n\}) = S_n,$$

which give $\#M_n(\mathbf{F}_1) = n^n$ and $\#GL_n(\mathbf{F}_1) = n!$. For example

$$M_2(\mathbf{F}_1) = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\}$$

and

$$GL_2(\mathbf{F}_1) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} = S_2.$$

(Here we may omit 0's respecting $\mathbf{F}_1 = \{1\}$.)
Furthermore we identify the category $\mathbf{Alg}(\mathbf{F}_1)$ of $\mathbf{F}_1$-algebras with the category **Monoid** of monoids according to the following picture:

$$\left(\begin{array}{ccc} & \mathbf{Alg}(\mathbf{Z}) & \\ \swarrow & & \searrow \\ \mathbf{Mod}(\mathbf{Z}) & & \mathbf{Alg}(\mathbf{F}_1) \\ \searrow & & \swarrow \\ & \mathbf{Mod}(\mathbf{F}_1) & \end{array} \right) = \left(\begin{array}{ccc} & \mathbf{Ring} & \\ \swarrow & & \searrow \\ \mathbf{Ab} & & \mathbf{Monoid} \\ \searrow & & \swarrow \\ & \mathbf{Set} & \end{array} \right)$$

under the forgetful functors

$$\begin{array}{ccc} & (R, \times, +) & \\ \swarrow & & \searrow \\ (R, +) & & (R, \times) \\ \searrow & & \swarrow \\ & R. & \end{array}$$

Then, the absolute derivations $\mathrm{Der}_{\mathbf{F}_1}(R)$ of a ring R studied in §1 would be understood by looking at the multiplicative monoid structure of the ring R. (Actually we do not forget completely the additive structure.) Now we state a problem to which the absolute mathematics may be applicable. Let X be a (projective smooth) scheme of finite type over $\mathrm{Spec}(\mathbf{Z})$. The Hasse zeta function is defined as

$$\zeta_X(s) = \prod_{x \in X_0} (1 - N(x)^{-s})^{-1}$$

where x runs over the set X_0 of closed points (0-dimensional points) of X and $N(x)$ is the cardinality of the residue field at x. It is expected that there exists the so-called gamma factor $\Gamma_X(s)$ and that the completed zeta function $\hat{\zeta}_X(s) = \zeta_X(s)\Gamma_X(s)$ is meromorphic in $s \in \mathbf{C}$ with the functional equation $s \leftrightarrow \dim(X) - s$. For our purpose it is convenient to formulate the following conjecture (cf. Kurokawa [4], Deninger [2]): There would exist cohomologies $H^m(X)$ for $m = 0, 1, \ldots, 2\dim(X)$ with skew-hermitian operators $R^m : H^m(X) \to H^m(X)$ satisfying

$$\hat{\zeta}_X(s) = \prod_{m=0}^{2\dim(X)} \det\Big(R^m - (s - \frac{m}{2})\Big)^{(-1)^{m+1}}.$$

We refer to Deninger [1], [2] for various investigations. For example, in the case $X = \mathrm{Spec}(\mathbf{Z})$, we expect

$$\hat{\zeta}(s) = \zeta(s)\pi^{-s/2}\Gamma(\frac{s}{2}) = \frac{\det(R^1 - (s - \frac{1}{2}))}{s(s-1)},$$

where we consider that $H^0(\mathrm{Spec}(\mathbf{Z}))$ and $H^2(\mathrm{Spec}(\mathbf{Z}))$ are one dimensional with the trivial R^0 and R^2. Of course $H^1(\mathrm{Spec}(\mathbf{Z}))$ should be infinite dimensional.

Problem Can such a cohomology $H^m(X)$ and an operator $R^m : H^m(X) \to H^m(X)$ be constructed via the absolute mathematics ? Is $H^m(X)$ interpreted as a cohomology of the associated absolute scheme $X^{\mathrm{abs}} \to \mathrm{Spec}(\mathbf{F}_1)$?

Some trials will be made in §3 and §4 below.

3 Zeta functions for absolute derivations

3.1

For a map $X : \mathbf{Z} \to \mathbf{C}$, we define the zeta function attached to X by the Dirichlet series:

$$\zeta(s, X) := \sum_{n=1}^{\infty} \frac{X(n)}{n^s}.$$

Lemma 6

$$\zeta(s, \frac{\partial}{\partial p}) = \frac{\zeta(s-1)}{p^s - p}.$$

Proof: We start with

$$\frac{\partial}{\partial p}(n) = \frac{n \cdot \mathrm{ord}_p(n)}{p}.$$

Then

$$\zeta(s, \frac{\partial}{\partial p}) = \frac{1}{p}\sum_{n=1}^{\infty} \mathrm{ord}_p(n) n^{-(s-1)}.$$

We express $n = p^k m$ with $(p, m) = 1$, and $k \geq 0$.

$$\begin{aligned} \zeta(s, \frac{\partial}{\partial p}) &= \frac{1}{p} \sum_{m:(p,m)=1} m^{-(s-1)} \sum_{k=0}^{\infty} k p^{-k(s-1)} \\ &= \frac{1}{p} \times \zeta(s-1)(1 - p^{-(s-1)}) \times \frac{p^{-(s-1)}}{(1 - p^{-(s-1)})^2} \\ &= \zeta(s-1) \times \frac{1}{p^s - p}. \end{aligned}$$

□

Theorem 7 *For an $X \in \mathrm{Der}_{\mathbf{F}_1}(\mathbf{Z})$ of finite type, i.e., $X(p) = 0$ for all but finitely many primes p,*

$$\zeta(s, X) := \sum_{n=1}^{\infty} \frac{X(n)}{n^s} = \zeta(s-1) \sum_{p:\text{primes}} \frac{X(p)}{p^s - p}.$$

This can be extended to a meromorphic function on the whole complex plane. The special value at $s = 0$ is given by

$$\zeta(0, X) = \frac{1}{12} \sum_{p} \frac{X(p)}{p-1}.$$

Proof: It follows directly from Lemma 6 and $X = \sum_p X(p) \frac{\partial}{\partial p}$. □

3.2 Quantum noncommutativity

We introduce the noncommutativity of primes as "$\zeta(0, [\frac{\partial}{\partial p}, \frac{\partial}{\partial q}])$".
First we give a rather explicit formula of the zeta of the commutator of absolute derivations.

Lemma 7

$$\zeta(s, [\frac{\partial}{\partial p}, \frac{\partial}{\partial q}]) = \frac{1}{pq} \zeta(s-1) \left((1 - q^{-(s-1)}) \sum_{k=1}^{\infty} p^k \frac{q^{-p^k(s-1)}}{(1 - q^{-p^k(s-1)})^2} \right.$$
$$\left. - (1 - p^{-(s-1)}) \sum_{k=1}^{\infty} q^k \frac{p^{-q^k(s-1)}}{(1 - p^{-q^k(s-1)})^2} \right).$$

Proof: We start with

$$\frac{\partial}{\partial p}(\frac{\partial}{\partial q}(n)) = \frac{n}{pq} \left(\mathrm{ord}_q(n) \mathrm{ord}_p(n) + \mathrm{ord}_q(n) \mathrm{ord}_p(\mathrm{ord}_q(n)) \right).$$

Then

$$[\frac{\partial}{\partial p}, \frac{\partial}{\partial q}](n) = \frac{n}{pq} \left(\mathrm{ord}_q(n) \mathrm{ord}_p(\mathrm{ord}_q(n)) - \mathrm{ord}_p(n) \mathrm{ord}_q(\mathrm{ord}_p(n)) \right).$$

We set $n = q^k m$ with $(q, m) = 1$, and $k \geq 0$. Then

$$\begin{aligned} \zeta(s, [\frac{\partial}{\partial p}, \frac{\partial}{\partial q}]) &= \frac{1}{pq} \left(\sum_{n=1} \mathrm{ord}_q(n) \mathrm{ord}_p(\mathrm{ord}_q(n)) n^{-(s-1)} - \cdots \right) \\ &= \frac{1}{pq} \left(\sum_{m:(q,m)=1} m^{-(s-1)} \sum_{k=1}^{\infty} k \, \mathrm{ord}_p(k) q^{-k(s-1)} - \cdots \right) \\ &= \frac{1}{pq} \left((1 - q^{-(s-1)}) \zeta(s-1) \sum_{k=1}^{\infty} k \, \mathrm{ord}_p(k) q^{-k(s-1)} - \cdots \right) \end{aligned}$$

We set $k = p^l m'$ with $(p, m') = 1$ and $l \geq 0$. Then

$$\begin{aligned}
\sum_{k=1}^{\infty} k \operatorname{ord}_p(k) q^{-k(s-1)} &= \sum_{l=1}^{\infty} \sum_{m':(p,m')=1} p^l m' l q^{-p^l m'(s-1))} \\
&= \sum_{l=1}^{\infty} l p^l \left(\sum_{m'=1}^{\infty} m' q^{-p^l m'(s-1)} - \sum_{m'=1}^{\infty} (pm') q^{-p^{l+1} m'(s-1)} \right) \\
&= \sum_{l=1}^{\infty} l p^l \left(\frac{q^{-p^l(s-1)}}{(1 - q^{-p^l(s-1)})^2} - p \frac{q^{-p^{l+1}(s-1)}}{(1 - q^{-p^{l+1}(s-1)})^2} \right) \\
&= \sum_{l=1}^{\infty} l p^l \frac{q^{-p^l(s-1)}}{(1 - q^{-p^l(s-1)})^2} - \sum_{l=0}^{\infty} l p^{l+1} \frac{q^{-p^{l+1}(s-1)}}{(1 - q^{-p^{l+1}(s-1)})^2} \\
&= \sum_{l=1}^{\infty} p^l \frac{q^{-p^l(s-1)}}{(1 - q^{-p^l(s-1)})^2}.
\end{aligned}$$

This proves the lemma. □

Remark 5 *Notice a partial functional equation under $s \leftrightarrow 2 - s$.*

Remark 6 *Let $\phi_p(z) = \sum_{n=1}^{\infty} \operatorname{ord}_p(n) z^n$. Then the Mellin transform of ϕ_p is $p\zeta(s+1, \frac{\partial}{\partial p})$, and the series in the lemma above is obtained as*

$$\begin{aligned}
\sum_{l=1}^{\infty} p^l \frac{q^{-p^l(s-1)}}{(1 - q^{-p^l(s-1)})^2} &= \sum_{l=1}^{\infty} p^l \left. \frac{z^{p^l}}{(1 - z^{p^l})^2} \right|_{z=q^{-(s-1)}} \\
&= z \frac{\partial}{\partial z} \sum_{l=1}^{\infty} \left. \frac{z^{p^l}}{1 - z^{p^l}} \right|_{z=q^{-(s-1)}} \\
&= z \phi_p'(z) \big|_{z=q^{-(s-1)}}.
\end{aligned}$$

Now we give a rigorous definition of the quantum noncommutativity motivated by the lemma above.

Definition The quantum noncommutativity, abbreviated as QNC, of p and q is defined by

$$\operatorname{QNC}(p, q) = \frac{1}{12pq} \left((q-1) \sum_{k=1}^{\infty} p^k \frac{q^{-p^k}}{(1 - q^{-p^k})^2} - (p-1) \sum_{k=1}^{\infty} q^k \frac{p^{-q^k}}{(1 - p^{-q^k})^2} \right).$$

Numerically,

$$\begin{aligned}\mathrm{QNC}(2,3) &= 0.00220482..,\\ \mathrm{QNC}(2,5) &= 0.00172077..,\\ \mathrm{QNC}(2,7) &= 0.00124803..,\\ \mathrm{QNC}(3,5) &= 0.00031155...\end{aligned}$$

Note that $\mathrm{QNC}(p,p)=0$ and it seems that $\mathrm{QNC}(p,q)>0$ for $p<q$.
We remark that $\mathrm{QNC}(p,q)$ has a neat expression using the Jackson integral (q-integral). We recall the following standard notions in q-analysis [3]. For an appropriate function $f(x)$, we define the Jackson integral

$$\int_1^\infty f(x)d_qx := \sum_{k=1}^\infty (q^k - q^{k-1}) f(q^k)$$

with a base q. For a real number x, we define the corresponding q-number

$$[x]_q := \frac{q^{x/2}-q^{-x/2}}{q^{1/2}-q^{-1/2}}.$$

Then the quantum noncommutativity is expressed as

$$\begin{aligned}\mathrm{QNC}(p,q) &= \frac{1}{12pq}\left((q-1)\sum_{k=1}^\infty p^k \frac{q^{-p^k}}{(1-q^{-p^k})^2} - (p-1)\sum_{k=1}^\infty q^k \frac{p^{-q^k}}{(1-p^{-q^k})^2}\right)\\ &= \frac{1}{12pq}\left(\frac{(q-1)}{(q^{1/2}-q^{-1/2})^2}\sum_{k=1}^\infty p^k\frac{1}{[p^k]_q^2} - \frac{(p-1)}{(p^{1/2}-p^{-1/2})^2}\sum_{k=1}^\infty q^k \frac{1}{[q^k]_p^2}\right)\\ &= \frac{1}{12(p-1)(q-1)}\left(\int_1^\infty \frac{d_px}{[x]_q^2} - \int_1^\infty \frac{d_qx}{[x]_p^2}\right).\end{aligned}$$

Question (A) Let $R=(r_{pq})_{p,q:\mathrm{primes}}$ with $r_{pq} =$"$\zeta(0,[\frac{\partial}{\partial p},\frac{\partial}{\partial q}])$"$=\mathrm{QNC}(p,q)$. Then does it hold that

$$\hat{\zeta}_{\mathbf{Z}}(s) = \frac{\det(1-R(s-\frac{1}{2}))}{s(s-1)}?$$

Here we may recall that

$$H^1_{\mathbf{F}_1}(\mathbf{Z},\mathbf{Z})_{\mathbf{C}} = \widehat{\oplus}_p \mathbf{C}\frac{\partial}{\partial p}.$$

(B) For a sheaf (automorphic or Galois) ρ of $\mathbf{Z}$, let $R_\rho = (r_{pq}(\rho))$ with

$$r_{pq}(\rho) = \left(\frac{\rho(p)+\rho(q)^*}{2}\right) r_{pq},$$

where $\rho(q)^*$ is the adjoint of $\rho(p)$. Then does it hold that

$$\hat{L}(s,\rho) = \frac{\det(1-R_\rho(s-\frac{1}{2}))}{s^{m(\rho)}(s-1)^{m(\rho)}}$$

where $m(\rho)$ is the multiplicity of the trivial representation in ρ ?

REMARK 7 *Let P be a set of "generalized primes" with the zeta function*

$$\zeta_P(s) = \prod_p (1 - N(p)^{-s})^{-1},$$

whose conjectual functional equation is $s \mapsto d(P) - s$. Let $R : V \to V$ be a linear operator on a complex vector space V. Assume that there is a basis $\{v_p \mid p \in P\}$ of V indexed by P. Let $\{e_\mu \mid \mu \in \mathrm{Spect}(R)\}$ be an R-eigen basis of V with $Re_\mu = \mu e_\mu$. Thus

$$\oplus_\mu \mathbf{C} e_\mu = V = \oplus_p \mathbf{C} v_p.$$

Take a suitable test function W such that $W(R)$ has a trace. Then we have (under a suitable condition) a so-called "trace formula"

$$\sum_\mu W(\mu) = \sum_p M(p)$$

where $M(p) = M(p,p)$ is given by

$$W(R) v_p = \sum_q M(q,p) v_q.$$

When

$$W(\mu) = \log(\mu - (s - \frac{d(P)}{2}))$$

and

$$M(p) = \log((1 - N(p)^{-s})^{-1})$$

we would obtain the determinant expression

$$\begin{aligned} \zeta_P(s) &= \prod_p (1 - N(p)^{-s})^{-1} \\ &= \prod_\mu (\mu - (s - \frac{d(P)}{2})) \\ &= \mathrm{Det}(R - (s - \frac{d(P)}{2})). \end{aligned}$$

By this way we would have the analytic continuation of the zeta functions and the L-functions.

4 TOWARDS ABSOLUTE SCHEMES

We try to set up the first step to $\mathbf{F}_1$-schemes. Recall that the usual scheme is coming from the affine scheme $\mathrm{Spec}_{\mathbf{Z}}(A) = \mathrm{Spec}(A)$ for a (commutative) ring A, where $\mathrm{Spec}_{\mathbf{Z}}(A)$ is the set of the prime ideals of A with the Zariski (Stone-Jacobson-Gelfand) topology. Since we consider an $\mathbf{F}_1$-algebra as a monoid, we

define Spec(M) for a monoid M. A typical example is the case $M = (A, \times)$ for a ring A.
Generalizing a bit, let X be an algebraic system having an associative multiplication with the identity element 1. We assume that X has the zero element 0 (satisfying $0 \cdot x = x \cdot 0 = 0$ for all $x \in X$). An equivalence relation α on X is called a congruence if α is compatible with the algebraic operations on X. For example, if $x \equiv x'$ and $y \equiv y' \mod \alpha$, then $xy \equiv x'y' \mod \alpha$. In other words, such an α is associated to the residue (quotient) algebraic system X/α. We denote by Cong(X) the set of all congruences on X.

Example 2 *A congruence on a (not necessary commutative) ring corresponds to a two-sided ideal.*

Example 3 *A congruence on a group corresponds to a normal subgroup.*

We say that a congruence α on X is a prime congruence if X/α is integral in the sense that X/α has no (non-zero) zero divisors: if $x \not\equiv 0$, $y \not\equiv 0 \mod \alpha$, then $xy \not\equiv 0 \mod \alpha$. We denote by Spec($X$) the set of the prime congruences on X with the following topology: the closed subsets are

$$V(\beta) = \{\alpha \leq \beta \mid \alpha \text{ is a prime congruence}\}$$

for $\beta \in \mathrm{Cong}(X)$. Here $\alpha \leq \beta$ means that $x \equiv 0 \mod \beta$ implies $x \equiv 0 \mod \alpha$. As in the usual case, it is checked that such $V(\beta)$'s satisfy the needed conditions for closed sets by

$$V(\mathrm{id}) = X, \quad V(\mathrm{triv}) = \emptyset,$$

$$V(\beta_1) \cup \cdots \cup V(\beta_n) = V(\beta_1 \wedge \cdots \wedge \beta_n),$$
$$\bigcap_{\lambda \in \Lambda} V(\beta_\lambda) = V(\sum_\lambda \beta_\lambda),$$

where $X/\mathrm{triv} = \{1\}$, $X/\mathrm{id} = X$, and $\sum_\lambda \beta_\lambda$ denotes the congruence generated by β_λ's.
For a ring A, we define

$$(\mathrm{Spec}_{\mathbf{Z}}(A))^{\mathrm{abs}} = \mathrm{Spec}_{\mathbf{F}_1}(A) = \mathrm{Spec}((A, \times)).$$

[Notice that Spec($(A, \times)$) is not the set of usual "ideals" of $(A, \times)$.] Then we have the natural map

$$\mathrm{Spec}_{\mathbf{Z}}(A) \longrightarrow \mathrm{Spec}_{\mathbf{F}_1}(A)$$

since each congruence on the ring A induces a congruence on the multiplicative monoid $(A, \times)$. The "local-global" map

$$A \longrightarrow \prod_{\alpha \in \mathrm{Spec}_{\mathbf{F}_1}(A)} (A/\alpha)$$

refines the usual local global map

$$A \longrightarrow \prod_{\gamma \in \mathrm{Spec}_{\mathbf{Z}}(A)} (A/\gamma).$$

For example, $\mathrm{Spec}_{\mathbf{F}_1}(\mathbf{Z})$ and $\prod_{\alpha \in \mathrm{Spec}_{\mathbf{F}_1}(\mathbf{Z})} (\mathbf{Z}/\alpha)$ are both big and unusual.

Remark 8 *The absolute fundamental group $\pi_1(\mathrm{Spec}_{\mathbf{F}_1}(\mathbf{Z}))$ would be interesting from the view point of the Langlands conjecture since for each unramified automorphic representation π of $GL_n(\mathbf{A})$ ($\mathbf{A}$ being the adele ring of $\mathbf{Q}$) we may have an n-dimensional representation (local system)*

$$\rho : \pi_1(\mathrm{Spec}_{\mathbf{F}_1}(\mathbf{Z})) \to GL_n(\mathbf{C})$$

satisfying

$$L(s,\pi) = L(s,\rho).$$

Now we notice on the $\mathbf{F}_1$-tensor product. We define the $\mathbf{F}_1$-tensor product of rings A and B as

$$A \otimes_{\mathbf{F}_1} B = (A,\times) * (B,\times)$$

the free product of monoids under the identification

$$1_A = 1_B = 1 \text{ and } 0_A = 0_B = 0.$$

Theorem 8

$$\mathrm{Spec}_{\mathbf{F}_1}(A \otimes_{\mathbf{F}_1} B) = \mathrm{Spec}_{\mathbf{F}_1}(A) \times \mathrm{Spec}_{\mathbf{F}_1}(B).$$

Proof: We show

$$\mathrm{Spec}(M * N) \cong \mathrm{Spec}(M) \times \mathrm{Spec}(N)$$

by sending $\alpha * \beta$ to (α, β), where M and N are multiplicative monoids having 1 and 0. Here $\alpha * \beta$ is the natural congruence on $M * N$ coming from α and β. Since it is easy to see that $\alpha * \beta$ is a prime congruence on $M * N$ if α and β are prime congruences on M and N respectively by

$$(M * N)/(\alpha * \beta) \cong (M/\alpha) * (N/\beta),$$

it is sufficient to show that the map is surjective. Let γ be a prime congruence on $M * N$. Then we obtain a congruence α on M and a congruence β on N by restricting γ to M and N respectively, and it holds that $\gamma = \alpha * \beta$. Since

$$(M * N)/\gamma \cong (M/\alpha) * (N/\beta),$$

we know that α (resp. β) is a prime congruence on M (resp. N). □

Thus we have

$$\mathrm{Spec}_{\mathbf{F}_1}(\mathbf{Z} \otimes_{\mathbf{F}_1} \mathbf{Z}) = \mathrm{Spec}_{\mathbf{F}_1}(\mathbf{Z}) \times \mathrm{Spec}_{\mathbf{F}_1}(\mathbf{Z})$$

where $\mathbf{Z} \otimes_{\mathbf{F}_1} \mathbf{Z} = \mathbf{Z} * \mathbf{Z}$, as expected in [4]–[7] and [9].

References

[1] C. Deninger: Local L-factors of motives and regularized determinants. Invent. Math. 107 (1992), no. 1, 135–150.

[2] C. Deninger: Some analogies between number theory and dynamical systems on foliated spaces, Proceedings of the ICM, Vol. I (Berlin, 1998) Doc. Math. 1998, Extra Vol. I, 163–186.

[3] V. Kac and P. Cheung: Quantum Calculus, Springer-Verlag, Berlin, 2002.

[4] N. Kurokawa: On some Euler products. I, Proc. Japan Acad. Ser. A Math. Sci. 60(1984), no. 9, 335–338.

[5] N. Kurokawa: Multiple zeta functions: an example, Adv. Stud. Pure Math. 21(1992), 219–226.

[6] N. Kurokawa: On $\mathbf{F}_1$. Proceedings of the Symposium on Algeraic Geometry at Kinosaki in 1994 Oct., pp38–46.

[7] N. Kurokawa and M. Wakayama: Absolute Casimir Elements (in Japanese), Iwanami, Tokyo, 2002.

[8] J. L. Loday: Cyclic Homology, second ed., Springer-Verlag, Berlin, 1998.

[9] Yu. I. Manin: Lectures on zeta functions and motives (according to Deninger and Kurokawa). Astérisque 228 (1995) 121–163.

[10] H. Ochiai: Monoidal absolute derivations. A talk at the Conference "Zetas and Trace Formulas" in Okinawa 2002.

[11] C. Soulé: On the field with one element, Max-Planck-Institut für Mathematik, Arbeitstagung 1999, 5pages; IHES prépublications M/55, 1999.

[12] J. Tits: Sur les analogues algébriques des groupes semi-simple complexes, Colloque d'algèbre supérieure, Bruxelles, 1956, 261–289, Centre Belge de Recherches Mathématiques, Gauthier-Villars, 1957.

Nobushige Kurokawa
Department of Mathematics
Tokyo Institute of Technology
Oh-okayama Meguro-ku, Tokyo
152-8551 Japan
kurokawa@math.titech.ac.jp

Hiroyuki Ochiai
Department of Mathematics
Tokyo Institute of Technology
Current address:
Department of Mathematics,
Nagoya University
Chikusa, Nagoya 464-8602, Japan
ochiai@math.nagoya-u.ac.jp

Masato Wakayama
Faculty of Mathematics
Kyushu University
Hakozaki Higashi-ku, Fukuoka
812-8581 Japan
wakayama@math.kyushu.u-ac.jp

Studying the Growth of Mordell-Weil

To Kazuya Kato, with admiration

Barry Mazur and Karl Rubin

Received: December 1, 2002
Revised: February 16, 2003

Abstract. We study the growth of the Mordell-Weil groups $E(K_n)$ of an elliptic curve E as K_n runs through the intermediate fields of a $\mathbf{Z}_p$-extension. We describe those $\mathbf{Z}_p$-extensions K_∞/K where we expect the ranks to grow to infinity. In the cases where we know or expect the rank to grow, we discuss where we expect to find the submodule of universal norms.

2000 Mathematics Subject Classification: Primary 11G05, 11G40; Secondary 11R23, 14G05

1 Introduction

Fix an elliptic curve E over $\mathbf{Q}$ of conductor N, and an odd prime number p. Let K be a number field and K_∞/K a $\mathbf{Z}_p$-extension. For $n \geq 0$ denote by K_n/K the intermediate extension in K_∞/K of degree p^n. What is the rate of growth of the Mordell-Weil groups $E(K_n)$ as n tends to ∞? The shape of these asymptotics for general number fields K and prime numbers p of good ordinary reduction is given by the following result, conditional on the Shafarevich-Tate conjecture (more specifically, on the finiteness of p-primary components of Shafarevich-Tate groups).

Proposition 1.1. *If the Shafarevich-Tate conjecture is true, and E has good, ordinary reduction at p, then there is a non-negative integer $r = r(E, K_\infty/K)$ such that*

$$\mathrm{rank}_{\mathbf{Z}}(E(K_n)) = rp^n + O(1)$$

where the bound $O(1)$ depends on E and K_∞/K, but not on n.

We will review the proof of this proposition in §3 below, Proposition 3.2. For convenience, if the above asymptotics hold, we will say that $(E, K_\infty/K)$ has *growth number* $r = r(E, K_\infty/K)$. In particular, if E does not have complex multiplication by a subfield of K, then $(E, K_\infty/K)$ has growth number zero if and only if $E(K_\infty)$ is finitely generated. When $r(E, K_\infty/K) > 0$ let us say that the Mordell-Weil rank of E has *positive growth* relative to K_∞/K. Recent work has significantly increased our knowledge of questions related to growth number, and has allowed us to compute these growth numbers in two specific instances.

Example 1.2. When K is a real abelian field there is only one $\mathbf{Z}_p$-extension K_∞/K (the cyclotomic $\mathbf{Z}_p$-extension) and Kato's celebrated work (cf. [K, Sc, Ru]) shows that $E(K_n)$ stabilizes for large n. Therefore $E(K_\infty)$ is a finitely generated group and the growth number vanishes. This had been conjectured in [M1], at least for good, ordinary, primes p.

Example 1.3. Suppose $K/\mathbf{Q}$ is a quadratic imaginary field and let K_∞/K denote the anti-cyclotomic $\mathbf{Z}_p$-extension. This is the unique $\mathbf{Z}_p$-extension of K such that $K_\infty/\mathbf{Q}$ is Galois with nonabelian Galois group. It was conjectured in [M2] (for the case of primes p of good, ordinary, reduction for E; but see, e.g., [V] for arbitrary p) that the growth number $r(E, K_\infty/K)$ is at most 2. Moreover, the conjectures in [M2] assert that

- the growth number $r = 2$ can only occur in the "exceptional case" where E has complex multiplication by K,
- if we are not in this "exceptional case" then the distinction between growth number 0 and 1 is determined by the root number in the functional equation satisfied by the relevant L-functions.

A good deal of this conjecture, as well as other information about the p-anti-cyclotomic arithmetic of elliptic curves over $\mathbf{Q}$, is now established, thanks to work of Gross & Zagier, Kolyvagin, Perrin-Riou, Bertolini & Darmon, Vatsal, Cornut, Nekovář, and Zhang.

The object of this note is to indicate a slightly wider context where "signs of functional equations" would lead one to conjecture positive growth of Mordell-Weil ranks, and to formulate some open problems suggested by the recent work regarding anti-cyclotomic extensions of quadratic imaginary fields.
Specifically, we will do two things. Let E be an elliptic curve over $\mathbf{Q}$. For a number field K, call a $\mathbf{Z}_p$-extension K_∞/K *new* over K if it is not the base extension of a $\mathbf{Z}_p$-extension of a proper subfield of K.

(i) Assuming standard conjectures, we will consider *new* $\mathbf{Z}_p$-extensions K_∞/K of number fields K that have the property that (in a sense to be made precise) the root number in the functional equation predicts positive growth for E relative to K_∞/K. These $\mathbf{Z}_p$-extensions have a very tight structure. As we shall see, in such a situation K has an automorphism of order 2, and if $K^+ \subset K$ is the fixed field of this automorphism then K_∞/K^+ is an infinite dihedral extension. To be sure, the "classical case" where K is a quadratic imaginary field is of this form, but–again as we shall see–there are quite a number of other "nonclassical" instances (in general K need not even be a CM field). An open problem is to prove positive growth in all (or even just *one*) of these "nonclassical" instances.

(ii) When K is a quadratic imaginary field we shall discuss a certain intriguing p-adic "line" in the $\mathbf{Q}_p$-vector space $E(K) \otimes \mathbf{Q}_p$ that can be constructed, under certain hypotheses, via universal norms from the anti-cyclotomic $\mathbf{Z}_p$-extension of K. This line is stable under the action of complex conjugation and we offer a conjecture, and some beginning numerical evidence, about its "sign." In the "nonclassical" instances alluded to above, where at present we cannot prove positive growth, if the growth number is 1 we can define a similar "sign" (see Remark 4.12).

We only consider here the growth of Mordell-Weil groups in $\mathbf{Z}_p$-power extensions of number fields, i.e., infinite extensions F/K where $\mathrm{Gal}(F/K)$ is an abelian p-adic Lie group. There are recent studies (see for example [CH]) by Coates, Howson, and others concerning the growth of Mordell-Weil groups in infinite extensions F/K where $\mathrm{Gal}(F/K)$ is a *nonabelian* p-adic Lie group.
We conclude this introduction with an example in which Proposition 1.1, along with the rest of this paper, does not apply.

EXAMPLE 1.4. Suppose E has complex multiplication by the quadratic imaginary field K, p is a prime of good, supersingular reduction, and K_∞/K is the anticyclotomic $\mathbf{Z}_p$-extension of K. Then standard conjectures imply (see the remark near the end of §1 of [G1])

$$\mathrm{rank}_{\mathbf{Z}}(E(K_n)) = \begin{cases} \frac{p^{2[n+1/2]}-1}{p+1} + O(1) & \text{if } W(E/\mathbf{Q}) = +1 \\ \frac{p^{2[n/2]+1}+1}{p+1} + O(1) & \text{if } W(E/\mathbf{Q}) = -1 \end{cases}$$

where $W(E/\mathbf{Q})$ is the root number in the functional equation of the L-function $L(E/\mathbf{Q}, s)$ (see §2.2) and $[\ \ ]$ is the greatest integer function.

2 WHERE CAN WE FIND MORDELL-WEIL CONTRIBUTION IN BULK?

Fix an odd prime p. Let E be an elliptic curve over $\mathbf{Q}$, $K/\mathbf{Q}$ a finite Galois extension of degree prime to p, and L the pro-p-abelian extension of K that is the compositum of all $\mathbf{Z}_p$-extensions of K in $\bar{\mathbf{Q}}$. Put $\Delta := \mathrm{Gal}(K/\mathbf{Q})$ and $\Gamma := \mathrm{Gal}(L/K) \cong \mathbf{Z}_p^d$, so we have $d \geq r_2(K) + 1$ with equality if Leopoldt's conjecture holds, where $r_2(K)$ is the number of complex archimedean places of K. The extension $L/\mathbf{Q}$ is Galois, and "conjugation" in $\mathcal{G} := \mathrm{Gal}(L/\mathbf{Q})$ induces a natural action of the group Δ on Γ. By the classical Schur-Zassenhaus

Theorem, $\mathcal{G}$ is noncanonically isomorphic to the semi-direct product of Δ and Γ, the semi-direct product being formed via this conjugation action.

Example 2.1 (Mordell-Weil growth when K is totally real). Assume that Leopoldt's conjecture holds. If K is (totally) real, it follows that $d = 1$ and the unique $\mathbf{Z}_p$-extension of K is the base change of the p-cyclotomic $\mathbf{Z}_p$-extension over $\mathbf{Q}$. The action of Δ on Γ is therefore trivial. In this case it has been conjectured that $E(L)$ is finitely generated (cf. [M1] when p is of good ordinary reduction for E) and this conjecture has been proved by Kato [K] in the case where $K/\mathbf{Q}$ is real abelian.

Example 2.2 (Mordell-Weil growth when K is totally complex). Assume that Leopoldt's conjecture holds, and that $K/\mathbf{Q}$ is (totally) complex. Consider the *$\mathbf{Q}_p$-vector space of $\mathbf{Z}_p$-extensions of K*. That is, form the vector space

$$V(K) := \mathrm{Hom}_{\mathrm{cont}}(\Gamma, \mathbf{Q}_p),$$

whose one-dimensional $\mathbf{Q}_p$-subspaces are in one-to-one correspondence with $\mathbf{Z}_p$-extensions of K. The vector space $V(K)$ inherits a natural $\mathbf{Q}_p$-linear representation of Δ. An exercise using class field theory gives us that the character of this Δ-representation is

$$\mathrm{Ind}_{\langle\sigma\rangle}^{\Delta}\chi \oplus \mathbf{1},$$

where $\mathbf{1}$ is the trivial character on Δ, $\sigma \in \Delta$ is a complex conjugation involution,

$$\chi : \langle\sigma\rangle \to \{\pm 1\} \subset \mathbf{C}^{\times}$$

is the nontrivial character on $\langle\sigma\rangle = \{1, \sigma\} \subset \Delta$, and $\mathrm{Ind}_{\langle\sigma\rangle}^{\Delta}$ refers to induction of characters from the subgroup $\langle\sigma\rangle$ to Δ. Here the "$\mathbf{1}$" corresponds to the cyclotomic $\mathbf{Z}_p$-extension and the "$\mathrm{Ind}_{\langle\sigma\rangle}^{\Delta}\chi$" cuts out a hyperplane in $V(K)$ that we call the *anti-cyclotomic hyperplane*

$$V(K)^{\mathrm{anti-cyc}} \subset V(K).$$

2.1 Representations in Mordell-Weil.

The "big" Mordell-Weil group $E(L)$ of all L-rational points of E is a union of finite-rank $\mathcal{G} = \mathrm{Gal}(L/\mathbf{Q})$-stable subgroups, so one way of trying to understanding the growth of Mordell-Weil is to ask, for finite dimensional irreducible complex characters τ of $\mathcal{G}$, whether or not the $\mathcal{G}$-representation space V of τ occurs in $E(F) \otimes \mathbf{C}$ for some intermediate Galois extension $F/\mathbf{Q}$. Mackey's criterion (cf. [Se] §8.2, Prop. 25) gives a description of all the irreducible characters on $\mathcal{G}$ in terms of induced characters. Here, in any event, is how to get quite a few irreducible characters of $\mathcal{G}$.
Define an action of Δ on the group of (continuous, hence finite order) characters of Γ as follows. For $\psi \in \mathrm{Hom}_{\mathrm{cont}}(\Gamma, \mathbf{C}^{\times})$ and $\delta \in \Delta$, define ψ^{δ} by the formula

$$\psi^{\delta}(\gamma) := \psi(\delta^{-1}(\gamma)) \quad \text{for } \gamma \in \Gamma.$$

For such a ψ let $H_\psi \subset \Delta$ be the stabilizer of ψ, let $K_\psi \subset K$ be the fixed field of H_ψ, and put $\mathcal{G}_\psi := \mathrm{Gal}(L/K_\psi) \subset \mathcal{G}$.

Lemma 2.3. *With notation as above, every $\psi : \Gamma \to \mathbf{C}^\times$ extends uniquely to a character $\psi_0 : \mathcal{G}_\psi \to \mathbf{C}^\times$ of p-power order (that is, the restriction of ψ_0 to Γ is ψ).*

Proof. Fix ψ. Since $|\Delta|$ is prime to p, $\mathcal{G}_\psi$ is (noncanonically) a semidirect product of H_ψ and Γ. More precisely, there is a subgroup $\tilde{H}_\psi \subset \mathcal{G}_\psi$ that projects isomorphically to H_ψ, and such that $\mathcal{G}_\psi = \Gamma \cdot \tilde{H}_\psi$. We define ψ_0 by $\psi_0(\gamma h) = \psi(\gamma)$ for $h \in \tilde{H}_\psi$ and $\gamma \in \Gamma$. Then ψ_0 is a homomorphism because $\psi^h = \psi$ for $h \in H_\psi$, ψ_0 clearly has p-power order, and the restriction of ψ_0 to Γ is ψ.

If ψ_0' is another such extension of ψ, then $\psi_0'\psi_0^{-1}$ is trivial on Γ, and hence is the inflation of a character of H_ψ of p-power order, which must be trivial. □

Definition 2.4. Suppose $\psi : \Gamma \to \mathbf{C}^\times$. Lemma 2.3 shows that ψ is the restriction to K of a character of a $\mathbf{Z}_p$-extension of K_ψ, and clearly K_ψ is minimal with this property. We will call K_ψ the *level* of ψ, and we say that ψ is *new of level* K_ψ. We will say that ψ is *generic* if ψ is new of level K, i.e., if H_ψ is trivial.

Let $\psi_0 : \mathcal{G}_\psi \to \mathbf{C}^\times$ be the extension of ψ given by Lemma 2.3, and define

$$\phi_\psi := \mathrm{Ind}_{\mathcal{G}_\psi}^{\mathcal{G}} \psi_0,$$

the induced character from $\mathcal{G}_\psi$ to $\mathcal{G}$. By [Se] §8.2, Proposition 25, ϕ_ψ is an irreducible character of $\mathcal{G}$. We will also say that ϕ_ψ has level K_ψ, and that ϕ_ψ is generic if ψ is generic.

Proposition 2.5. *Suppose $\psi : \Gamma \to \mathbf{C}^\times$, and let $\phi_\psi = \mathrm{Ind}_{\mathcal{G}_\psi}^{\mathcal{G}} \psi_0$ be the induced character given by Definition 2.4.*

Then ϕ_ψ is real-valued if and only if there is an element $\sigma \in \Delta$ such that $\psi^\sigma = \psi^{-1}$. Such a σ lies in the normalizer $N(H_\psi)$ of H_ψ in Δ, and if $\psi \neq \mathbf{1}$ then σ has order 2 in $N(H_\psi)/H_\psi$.

Proof. Let $\langle\ ,\ \rangle_G$ denote the usual pairing on characters of a profinite group G. Then using Frobenius reciprocity ([Se] Theorem 13, §7.2) for the second equivalence, and [Se] Proposition 22, §7.3, for the third and fourth,

$$\begin{aligned} \phi_\psi \text{ is real-valued} &\iff \langle \phi_\psi, \bar{\phi}_\psi \rangle_{\mathcal{G}} > 0 \\ &\iff \langle \mathrm{Res}_{\mathcal{G}_\psi}^{\mathcal{G}} \phi_\psi, \bar{\psi}_0 \rangle_{\mathcal{G}_\psi} > 0 \\ &\iff \langle \mathrm{Ind}_{H_\sigma}^{\mathcal{G}_\psi} (\psi^\sigma)_0, \bar{\psi}_0 \rangle_{\mathcal{G}_\psi} > 0 \text{ for some } \sigma \in \Delta \end{aligned}$$

where $H_\sigma = H_\psi \cap H_{\psi^\sigma}$ and $(\psi^\sigma)_0$ is the extension of ψ^σ to H_σ of Lemma 2.3

$$\iff \psi^\sigma = \bar{\psi} = \psi^{-1} \text{ for some } \sigma \in \Delta.$$

If $\psi^\sigma = \bar{\psi}$ then we have $\sigma H_\psi \sigma^{-1} = H_{\psi^\sigma} = H_\psi$, so $\sigma \in N(H_\psi)$, and $\psi^{\sigma^2} = \psi$ so $\sigma^2 \in H_\psi$. If further $\psi \neq \mathbf{1}$ then (since p is odd) $\psi^\sigma \neq \psi$, so $\sigma \notin H_\psi$. This completes the proof. □

COROLLARY 2.6. *Suppose $\psi \neq \mathbf{1}$. Then ϕ_ψ is real-valued if and only if there is a subfield $K_\psi^+ \subset K_\psi$ such that $[K_\psi : K_\psi^+] = 2$ and $L^{\ker(\psi_0)}$ is Galois over K_ψ^+ with dihedral Galois group. In particular, if ϕ_ψ is real-valued then $[\Delta : H_\psi]$ is even.*

Proof. This follows directly from Proposition 2.5. If ϕ_ψ is real-valued we take K_ψ^+ to be be the subfield of K_ψ fixed by the element σ of Proposition 2.5. Conversely, if there is such a field K_ψ^+, let σ be a lift to Δ of the nontrivial automorphism of K_ψ/K_ψ^+ and apply Proposition 2.5. □

2.2 MORDELL-WEIL AND ROOT NUMBERS

Let τ be the character of an irreducible finite dimensional complex $\mathcal{G}$-representation V. Appealing to a suitably general version of the conjecture of Birch and Swinnerton-Dyer, and assuming that the relevant L-function $L(E, \tau, s)$ has analytic continuation to the entire complex plane, we expect that the representation V occurs in $E(F) \otimes \mathbf{C}$ (for F the fixed field of the $\mathcal{G}$-representation V) if and only if the L-function $L(E, \tau, s)$ vanishes at $s = 1$. We also expect (but do not yet have, in general) a functional equation of the form

$$L(E, \tau, s) = W(E, \tau, s) L(E, \bar{\tau}, 2 - s),$$

where $W(E, \tau, s)$ is an explicit function involving the exponential function and the Γ-function. Even though the functional equation remains conjectural, there is an explicit definition of $W(E, \tau, s)$ (see for example [D, Ro]). If τ is real-valued, then the *root number* $W(E, \tau) := W(E, \tau, 1) = \pm 1$. Moreover, if $W(E, \tau, 1) = -1$ it would follow that $L(E, \tau, s)$ vanishes at $s = 1$ (to odd order). Then, by the Birch and Swinnerton-Dyer conjecture we would expect that the representation space V occurs in $E(L) \otimes \mathbf{C}$.

It is natural to ask if the really huge contribution to Mordell-Weil (if there is any) will come from this "expected" occurrence of representations in $E(L)$. This leads us to seek out real irreducible characters τ with root number $W(E, \tau) = -1$. Proposition 2.5 provides us with a substantial collection of real characters, and it remains to determine their root numbers.

EXAMPLE 2.7. Suppose first that K is quadratic imaginary. We also suppose, for simplicity, that the discriminant of the field K, the conductor N of the elliptic curve E, and the prime number p are pairwise relatively prime. The unique nontrivial element $\sigma \in \Delta$ is complex conjugation, and the nontrivial character $\chi : \langle \sigma \rangle = \Delta \to \mathbf{C}^\times$ corresponds in the usual way to the quadratic Dirichlet character χ_K attached to the field K.

Choose $\psi \in \mathrm{Hom}_{\mathrm{cont}}(\Gamma, \mathbf{C}^\times)^-$ (the subgroup of homomorphisms on which σ acts as -1), and put $\phi_\psi = \mathrm{Ind}_\Gamma^{\mathcal{G}} \psi$ as in Definition 2.4. Then ϕ_ψ is real and generic by Proposition 2.5. Viewing $\det \phi_\psi : \mathcal{G} \to \mathbf{C}^\times$ as a Dirichlet character in the usual way we have

$$W(E, \phi_\psi) = (\det \phi_\psi)(-N) \cdot W(E)^{\dim \phi_\psi}$$

where $W(E) = W(E, \mathbf{1})$ is the root number of E (see for example Proposition 10 of [Ro]). But $(\det \phi_\psi)$ is simply the inflation of χ, and $\dim \phi_\psi = 2$, so this identity simplifies to

$$W(E, \phi_\psi) = \chi_K(-N). \tag{1}$$

The surprising consequence of this formula, which we will see repeated even more generally, is that the root number does not depend upon much: it is independent of the choice of generic (real) character ψ. and the elliptic curve E only enters into the formula for the root number via its conductor.

This is the case that has seen extraordinary progress recently via the detailed study of the tower of Heegner points. We will return to this in §4 of this article.

We now return to the general case, where $K/\mathbf{Q}$ is Galois with group Δ, L is the maximal $\mathbf{Z}_p^d$-extension of K, and $\mathcal{G} = \mathrm{Gal}(L/\mathbf{Q})$. From now on we suppose that the discriminant $\mathrm{disc}(K)$ of the field K, the conductor N of the elliptic curve E, and the prime p are pairwise relatively prime.

If $H \subset \Delta$ let $\chi_{\Delta/H} : G_\mathbf{Q} \to \pm 1$ be the determinant of $\mathrm{Ind}_H^\Delta \mathbf{1}$, the permutation representation of Δ on the set of left cosets Δ/H. (If $H = \{1\}$ we will write simply χ_Δ.) We will view $\chi_{\Delta/H}$ (and every other one-dimensional character of $G_\mathbf{Q}$) as a Dirichlet character in the usual way.

THEOREM 2.8. *If $\psi \neq \mathbf{1}$ and ϕ_ψ is real-valued, then $W(E, \phi_\psi) = \chi_{\Delta/H_\psi}(-N)$. In particular characters of the same level have the same root number.*

Proof. We have $\phi_\psi = \mathrm{Ind}_{\mathcal{G}_\psi}^{\mathcal{G}} \psi_0$ from Definition 2.4. By Corollary 2.6, $\dim(\phi_\psi) = [\mathcal{G} : \mathcal{G}_\psi] = [\Delta : H]$ is even. Hence Proposition 10 of [Ro] ("a special case of a well-known formula") shows that

$$W(E, \phi_\psi) = (\det \phi_\psi)(-N) W(E)^{\dim(\phi_\psi)} = (\det \phi_\psi)(-N).$$

It remains to show that $\det \phi_\psi = \chi_{\Delta/H_\psi}$. We thank the referee for pointing out the following simple argument.

Let $\mathfrak{p}$ be a prime of $\bar{\mathbf{Q}}$ above p. Since ψ_0 has p-power order, $\psi_0 \equiv \mathbf{1} \pmod{\mathfrak{p}}$ and so

$$\det \phi_\psi = \det(\mathrm{Ind}_{\mathcal{G}_\psi}^{\mathcal{G}} \psi_0) \equiv \det(\mathrm{Ind}_{\mathcal{G}_\psi}^{\mathcal{G}} \mathbf{1}) = \det(\mathrm{Ind}_{H_\psi}^{\Delta} \mathbf{1}) = \chi_{\Delta/H_\psi} \pmod{\mathfrak{p}}.$$

Since p is odd and both $\det \phi_\psi$ and χ_{Δ/H_ψ} take only the values ± 1, it follows that $\det \phi_\psi = \chi_{\Delta/H_\psi}$. □

PROPOSITION 2.9. *The following are equivalent:*

(i) $\chi_\Delta \neq \mathbf{1}$,

(ii) Δ *has a nontrivial cyclic* 2*-Sylow subgroup,*

(iii) Δ *is the semi-direct product of a (normal) subgroup of odd order with a nontrivial cyclic* 2*-group.*

If these (equivalent) conditions hold then Δ *is solvable,* K *contains a unique quadratic subfield* $k/\mathbf{Q}$*, and* $\chi_\Delta = \chi_k$*, the quadratic character of* k*.*

Proof. We have

$$\chi_\Delta(\sigma) = \mathrm{sign}(\pi_\sigma)$$

where π_σ is the permutation of Δ given by left multiplication by σ. If $\sigma \in \Delta$ has order d, then the permutation π_σ is a product of $|\Delta|/d$ d-cycles, so

$$\chi_\Delta(\sigma) = \mathrm{sign}(\pi_\sigma) = (-1)^{(d-1)|\Delta|/d} = \begin{cases} -1 & \text{if } d \text{ is even and } |\Delta|/d \text{ is odd,} \\ 1 & \text{otherwise.} \end{cases}$$

Thus $\chi_\Delta(\sigma) = -1$ if and only if the cyclic subgroup generated by σ contains a nontrivial 2-Sylow subgroup of Δ, and so (i) is equivalent to (ii).
If (i) and (ii) hold, then $\ker(\chi)$ is a (normal) subgroup of index 2 in Δ, which also has a cyclic 2-Sylow subgroup. Proceeding by induction we get a filtration

$$\Delta = \Delta_0 \supset \Delta_1 \supset \cdots \supset \Delta_k$$

where $[\Delta_i : \Delta_{i+1}] = 2$ and $|\Delta_k|$ is odd.
We claim that Δ_k is normal in Δ. For if H is a conjugate of Δ_k, then $|H|$ is odd so we see by induction that for every $i \geq 1$ the map $H \to \Delta_{i-1}/\Delta_i$ is injective, i.e., $H \subset \Delta_i$. Thus $H \subset \Delta_k$, so Δ_k is normal. Therefore the Schur-Zassenhaus Theorem shows that Δ is the semidirect product of Δ_k with a 2-Sylow subgroup. This shows that (ii) implies (iii)[1], and it is immediate that (iii) implies (ii).
Suppose now that conditions (i)-(iii) hold. It follows from (iii) and the Feit-Thompson theorem that Δ is solvable. By (i), $\chi_\Delta = \chi_k$ for some quadratic field $k \subset K$, and by (ii), K contains at most one quadratic field. This completes the proof. □

Corollary 2.10. *Suppose that* ϕ_ψ *is real and generic. Then* $W(E, \phi) = -1$ *if and only if the equivalent conditions of Proposition 2.9 hold and* $\chi_k(-N) = -1$*.*

Proof. This is immediate from Theorem 2.8 and Proposition 2.9. □

Definition 2.11. Following Theorem 2.8, if $F \subset K$ we define $W(E/F)$ to be the common root number of all real characters ϕ_ψ of level F. (The proof of Theorem 2.8 shows that this common root number is also the root number in the functional equation of $L(E/F, s)$.)

[1] The fact that (ii) implies (iii) was proved by Frobenius in §6, p. 1039 of [F], by the same method we give here. Another proof was given by Burnside in [Bu], Corollary II of §244, p. 327. We thank Persi Diaconis for pointing us toward these references.

In particular if $F = K$, then $W(E/K) = \chi_k(-N)$ if Δ has nontrivial cyclic 2-Sylow subgroup (where k is the quadratic field inside K), and $W(E/K) = 1$ otherwise.

Suppose now that we are in such a situation where $W(E/K) = -1$. From the discussion above we expect significant growth of Mordell-Weil rank as we ascend the tower of intermediate finite extensions contained in $\mathbf{Z}_p^d$-extensions that correspond to the -1-eigenspace in $V(K)$ of some element of order 2 in Δ. We next work out the elementary features of two examples that are more general than the case of $K/\mathbf{Q}$ quadratic imaginary, and for which we do not, as yet, have any satisfactory theory of Mordell-Weil growth.

Example 2.12 ($K/\mathbf{Q}$ is a complex abelian Galois extension). In this case Leopoldt's conjecture is known to be true, there is a unique complex conjugation involution, and its -1-eigenspace is the entire anti-cyclotomic hyperplane.

Question 2.13. *Can one use towers of Heegner points in Shimura curves over totally real fields to account for (at least some of) the* expected *Mordell-Weil growth as one ascends the finite intermediate extensions of the anti-cyclotomic hyperplane?*

Example 2.14 ($K/\mathbf{Q}$ is a complex S_3-extension). To guard against complacency, we wish to mention this relatively simple case, where no analogue of the detailed results known in the quadratic imaginary case is currently available. Suppose $\Delta \cong S_3$, the symmetric group on three letters, and suppose that $W(E/K) = -1$. Let k be the quadratic field contained in K. Leopoldt's conjecture holds for K (for "easy" reasons: if Leopoldt's conjecture failed the natural homomorphism from global units to local units would be trivial, which it is not). We may decompose the four-dimensional Δ-representation space $V(K)$ into the sum of two Δ-stable planes

$$V(K) = V(K)_{\text{new}} \oplus V(k)$$

where the *new* plane $V(K)_{\text{new}} \subset V(K)^{\text{anti-cyc}}$ is of codimension one in the anti-cyclotomic hyperplane. There are three involutions $\sigma, \sigma', \sigma''$ in Δ, none of which act as scalars either on $V(K)_{\text{new}}$ or on $V(k)$. Let L_σ/K denote the unique $\mathbf{Z}_p^2$-extension of K on whose Galois group σ acts as -1, and similarly for σ' and σ''. We have, then, three $\mathbf{Z}_p^2$-extensions, L_σ/K, $L_{\sigma'}/K$ and $L_{\sigma''}/K$, sub-extensions of the anti-cyclotomic $\mathbf{Z}_p^3$-extension of K. Each of these $\mathbf{Z}_p^2$-extensions is the compositum of the "old" anti-cyclotomic $\mathbf{Z}_p$-extension of the quadratic subfield $k \subset K$ and the unique $\mathbf{Z}_p$-extension of K that corresponds to a line in $V(K)_{\text{new}}$ that is a -1-eigenspace for one of the three involutions in Δ.

Challenge 2.15. *As one ascends the intermediate fields of finite degree in each of these $\mathbf{Z}_p^2$-extensions we* expect *significant growth in the ranks of the corresponding Mordell-Weil groups. Find this Mordell-Weil contribution.*

Remark 2.16. Theorem 2.10 fails in the situation of Example 1.4 because in that case E has complex multiplication by K, so the conductor of E is not relatively prime to the discriminant of K.

3 Where is the support of the Selmer module?

Keep the notation of the previous section. In particular we continue to assume that the conductor of E, the discriminant of K, and the prime p are pairwise relatively prime.

3.1 Selmer modules

If F is an abelian extension of K we put $\Lambda_F = \mathbf{Z}_p[[\mathrm{Gal}(F/K)]]$; if $[F:K]$ is finite this is just the group ring $\mathbf{Z}_p[\mathrm{Gal}(F/K)]$, and if $\mathrm{Gal}(F/K) \cong \mathbf{Z}_p^d$ then Λ_F is noncanonically isomorphic to a power series ring $\mathbf{Z}_p[[T_1, \ldots, T_d]]$. If $F' \subset F$ then $\Lambda_{F'}$ is naturally a quotient of Λ_F.
For every number field F let $S_p(E/F) \subset H^1(F, E[p^\infty])$ be the classical Selmer group, which sits in the center of the exact sequence

$$0 \longrightarrow E(F) \otimes \mathbf{Q}_p/\mathbf{Z}_p \longrightarrow S_p(E/F) \longrightarrow Ш(E/F)[p^\infty] \longrightarrow 0 \qquad (2)$$

where $Ш(E/F)[p^\infty]$ is the p-part of the Shafarevich-Tate group of E over F. We extend this definition to possibly infinite algebraic extensions of $\mathbf{Q}$ by passing to a direct limit

$$S_p(E/F) := \varinjlim_{0} F' \subset F S_p(E/F')$$

where F' ranges through the finite extensions of $\mathbf{Q}$ in F. If F/K is Galois then $S_p(E/F)$ has a natural $\mathbf{Z}_p[[\mathrm{Gal}(F/K)]]$-module structure. Specifically, $S_p(E/L)$ is a Λ_L-module. We will refer to $S_p(E/L)$ as the *discrete Selmer module* attached to L/K. If $K_\infty \subset L$ is a $\mathbf{Z}_p$-extension of K we will write $K_n \subset K_\infty$ for the subfield of degree p^n over K, and we have the Λ_{K_∞}-module

$$S_p(E/K_\infty) = \varinjlim_{0} n S_p(E/K_n).$$

The following Control Theorem is a strengthening by Greenberg ([G2] Proposition 3.1) of a theorem of the first author [M1].

Theorem 3.1 (Control Theorem). *Suppose that K_∞ is a $\mathbf{Z}_p$-extension of K, and E has good ordinary reduction at p. Then the restriction maps $H^1(K_n, E[p^\infty]) \to H^1(K_\infty, E[p^\infty])$ induce maps*

$$S_p(E/K_n) \longrightarrow S_p(E/K_\infty)^{\mathrm{Gal}(K_\infty/K_n)}$$

whose kernels and cokernels are finite and bounded independently of n.

For every F we form the corresponding compact Selmer group

$$X(E/F) := \mathrm{Hom}(S_p(E/F), \mathbf{Q}_p/\mathbf{Z}_p).$$

Then $X(E/L)$ is a finitely generated Λ_L-module, and if K_∞ is a $\mathbf{Z}_p$-extension of K then $X(E/K_\infty)$ is finitely generated over Λ_{K_∞}.

Proposition 3.2. *Suppose K_∞ is a $\mathbf{Z}_p$-extension of K. If E has good ordinary reduction at p, and $Ш(E/K_n)[p^\infty]$ is finite for every n, then the growth number $r(E, K_\infty/K)$ defined in Proposition 1.1 is equal to the rank of the compact Selmer Λ_{K_∞}-module $X(E/K_\infty)$.*

Proof. Write $\Lambda_\infty = \Lambda_{K_\infty}$ and $\Lambda_n = \Lambda_{K_n} = \mathbf{Z}_p[\mathrm{Gal}(K_n/K)]$. The Control Theorem (Theorem 3.1) shows that there is a map with finite kernel and cokernel

$$X(E/K_\infty) \otimes_{\Lambda_\infty} \Lambda_n \to X(E/K_n).$$

If $Ш(E/K_n)[p^\infty]$ is finite then (2) shows that

$$\mathrm{rank}_{\mathbf{Z}}(E(K_n)) = \mathrm{rank}_{\mathbf{Z}_p}(X(E/K_n)) = \mathrm{rank}_{\mathbf{Z}_p}(X(E/K_\infty) \otimes_{\Lambda_\infty} \Lambda_n),$$

and the structure theory of Λ_{K_∞}-modules shows that

$$\mathrm{rank}_{\mathbf{Z}_p}(X(E/K_\infty) \otimes_{\Lambda_\infty} \Lambda_n) = p^n \mathrm{rank}_{\Lambda_\infty}(X(E/K_\infty)) + O(1). \qquad \square$$

3.2 The support of the Selmer module

Definition 3.3. Recall that $\Gamma = \mathrm{Gal}(L/K) \cong \mathbf{Z}_p^d$, so $\Lambda_L = \mathbf{Z}_p[[\Gamma]]$. The group Δ acts naturally and $\mathbf{Z}_p$-linearly on Γ.
If $F \subset K$ let L_F denote the compositum of all $\mathbf{Z}_p$-extensions of F. The group $\mathrm{Aut}(F)$ of automorphisms of F acts on $\Gamma_F := \mathrm{Gal}(L_F/F)$. If $\sigma \in \mathrm{Aut}(F)$ has order 2, we let $\Gamma^{\sigma,-}$ denote the maximal quotient of Γ on which σ acts as -1, and $L_F^{\sigma,-} \subset L_F$ the extension of F.
The *minus locus* in $\mathrm{Spec}\,\Lambda_L$ is the closed subscheme of $\mathrm{Spec}\,\Lambda_L$

$$\bigcup_{\substack{F \subset K \\ W(E/F) = -1}} \; \bigcup_{\substack{\sigma \in \mathrm{Aut}(F) \\ \sigma^2 = 1, \sigma \neq 1}} \mathrm{Spec}\,\Lambda_{KL_F^{\sigma,-}},$$

where we view $\mathrm{Spec}\,\Lambda_{KL_F^{\sigma,-}} \subset \mathrm{Spec}\,\Lambda_L$ via the projection map $\Lambda_L \twoheadrightarrow \Lambda_{KL_F^{\sigma,-}}$.

Using the root number considerations above, the following conjecture follows from a suitably general version of the Birch and Swinnerton-Dyer conjecture.

Conjecture 3.4. *The support of the compact Selmer module $X(E/L)$ in $\mathrm{Spec}\,\Lambda_L$ contains the minus locus.*

Example 3.5. For each quadratic imaginary field $F \subset K$ such that $\chi_F(-N) = -1$, the minus locus contains a $\mathbf{Z}_p$-line corresponding to the anticyclotomic $\mathbf{Z}_p$-extension of F, and Heegner points should provide the corresponding part of the Selmer group.
On the other hand, if $\chi_F(-N) = +1$ for every quadratic field F contained in K, then Theorem 2.8 shows that $W(E/F) = +1$ for *every* subfield F of K, and the minus locus is empty.
In Example 2.14, the minus locus consists of the union of the 3 $\mathbf{Z}_p$-planes $\operatorname{Spec}\Lambda_{L^\sigma}$, $\operatorname{Spec}\Lambda_{L^{\sigma'}}$, and $\operatorname{Spec}\Lambda_{L^{\sigma''}}$.

Let

$$\mathrm{MW}_p(E/L) := \mathrm{Hom}(E(L), \mathbf{Z}_p) = \mathrm{Hom}(E(L) \otimes \mathbf{Q}_p/\mathbf{Z}_p, \mathbf{Q}_p/\mathbf{Z}_p),$$

a quotient of $X(E/L)$. We will say that L/K has *unexpected large Mordell-Weil* if there exists a subscheme of Krull dimension greater than 1 in $\operatorname{Spec}\Lambda_L$ that is in the support of $\mathrm{MW}_p(E/L)$, but in the complement of the minus locus.

Question 3.6. *Are there Galois extensions $K/\mathbf{Q}$ such that L/K has unexpected large Mordell-Weil?*

We will call an abelian extension F/K a $\mathbf{Z}_p$-power extension of K if $\mathrm{Gal}(F/K) = \mathbf{Z}_p^k$ for some $k \geq 0$.

Question 3.7. *Are there $\mathbf{Z}_p$-power extensions $L_1, \ldots, L_k$ of K, and a finite subset $\Psi \subset \mathrm{Hom}(\Gamma, \mathbf{C}^\times)$, such that*

$$\{\psi : \Gamma \to \mathbf{C}^\times : \psi \textit{ occurs in } E(L) \otimes \mathbf{C}\} \;=\; \cup_i \mathrm{Hom}(\mathrm{Gal}(L_i/K), \mathbf{C}^\times) \;\cup\; \Psi?$$

Remark 3.8. Question 3.7 is motivated by work of Monsky, who was the first to study what the support of a Selmer module (in his case, an ideal class group module) over a $\mathbf{Z}_p^d$-extension should look like when $d \geq 2$. Iwasawa theory suggests that there should be an ideal $\mathcal{A} \subset \Lambda$ such that ψ occurs in $E(L) \otimes \mathbf{C}$ if and only if $\psi(\mathcal{A}) = 0$ (where we fix an embedding $\bar{\mathbf{Q}}_p \hookrightarrow \mathbf{C}$ so that we can view ψ as a character into $\bar{\mathbf{Q}}_p$). Monsky (Theorem 2.6 of [Mo]) showed that there is a set $\{L_1, \ldots, L_k\}$ of $\mathbf{Z}_p$-power extensions of K, and $\psi_i, \ldots, \psi_k \in \mathrm{Hom}(\Gamma, \mathbf{C}^\times)$ such that

$$\{\psi : \psi(\mathcal{A}) = 0\} \;=\; \cup_{i=1}^k \psi_i \cdot \mathrm{Hom}(\mathrm{Gal}(L_i/K), \mathbf{C}^\times).$$

Combining this with the root number calculation of Theorem 2.8 leads to Question 3.7. One can also ask the following variant of Question 3.7.

Question 3.9. *Is there a collection $L_1, \ldots, L_k$ of $\mathbf{Z}_p$-power extensions of K (not necessarily distinct) such that for every finite extension F of K in L we have*

$$\mathrm{rank}_{\mathbf{Z}}(E(F)) = \sum_{i=1}^{k} [F \cap L_i : K] + O(1)?$$

Here the bound $O(1)$ should depend only on E and K, not on F.

4 Where are the universal norms in Mordell-Weil?

Fix for this section a $\mathbf{Z}_p$-extension K_∞/K, and let $\Gamma := \mathrm{Gal}(K_\infty/K)$ and $\Lambda := \Lambda_{K_\infty} = \mathbf{Z}_p[[\Gamma]]$. As before, K_n will denote the extension of K of degree p^n in K_∞, and $\Lambda_n = \Lambda_{K_n} = \mathbf{Z}_p[\mathrm{Gal}(K_n/K)]$.

4.1 Universal norms

Definition 4.1. For $m \geq n$ consider the ($\mathbf{Z}_p$-linear) norm maps (or, perhaps, "trace maps" since one usually writes Mordell-Weil groups additively)

$$N_{m,n} : E(K_m) \otimes \mathbf{Z}_p \longrightarrow E(K_n) \otimes \mathbf{Z}_p.$$

Define

$$\mathcal{U}(E, K_\infty/K_n) := \bigcap_{m \geq n} N_{m,n}(E(K_m) \otimes \mathbf{Z}_p) \ \subset E(K_n) \otimes \mathbf{Z}_p.$$

Passing to the projective limits we define a $\mathrm{Gal}(K/\mathbf{Q})$-semi-linear Λ-module

$$\mathcal{U}(E/K_\infty) := \varprojlim_{0} n\, E(K_n) \otimes \mathbf{Z}_p = \varprojlim_{0} n\, \mathcal{U}(E, K_\infty/K_n),$$

and $\mathcal{U}(E, K_\infty/K_n)$ is the image of the projection $\mathcal{U}(E/K_\infty) \to E(K_n) \otimes \mathbf{Z}_p$.

Theorem 4.2. *Suppose E has good ordinary reduction at p, and $Ш(E/K_n)[p^\infty]$ is finite for every n. Let $r = \mathrm{rank}_\Lambda(X(E/K_\infty))$. Then*

(i) *$\mathcal{U}(E/K_\infty)$ has finite index in a free Λ-module of rank r,*

(ii) *$\mathcal{U}(E, K_\infty/K_n) \otimes \mathbf{Q}_p$ is a free $\Lambda_n \otimes \mathbf{Q}_p$-submodule of $E(K_n) \otimes \mathbf{Q}_p$ of rank r.*

Proof. Write simply X for the finitely generated Λ-module $X(E/K_\infty)$, and $X_n = X \otimes \Lambda_n$.
It follows from the Control Theorem (Theorem 3.1) that there is an injection with finite cokernel bounded independently of n

$$\mathrm{Hom}(X(E/K_n), \mathbf{Z}_p) \longrightarrow \mathrm{Hom}(X_n, \mathbf{Z}_p).$$

Further, it follows from (2) that if $Ш(E/K_n)[p^\infty]$ is finite then

$$\mathrm{Hom}(X(E/K_n), \mathbf{Z}_p) \cong E(K_n) \otimes \mathbf{Z}_p.$$

Taking inverse limits we get an injective map

$$\mathcal{U}(E/K_\infty) = \varprojlim (E(K_n) \otimes \mathbf{Z}_p) \longrightarrow \varprojlim(\mathrm{Hom}(X_n, \mathbf{Z}_p)) =: \mathcal{U}(X)$$

with finite cokernel, where $\mathcal{U}(X)$ is the abstract universal norm Λ-module studied in Appendix A.2. Since $\mathcal{U}(X)$ is free of rank r (Proposition A.20), this proves (i).

Consider the diagram

$$\begin{array}{ccccc}
\mathcal{U}(E/K_\infty)\otimes \mathbf{Q}_p & \longrightarrow & \mathcal{U}(E/K_\infty)\otimes \Lambda_n \otimes \mathbf{Q}_p & \longrightarrow & E(K_n)\otimes \mathbf{Q}_p \\
\downarrow \cong & & \downarrow \cong & & \downarrow \\
\mathcal{U}(X)\otimes \mathbf{Q}_p & \twoheadrightarrow & \mathcal{U}(X)\otimes \Lambda_n \otimes \mathbf{Q}_p & \hookrightarrow & \mathrm{Hom}(X_n, \mathbf{Q}_p)
\end{array}$$

where the horizontal maps are the natural projections. The lower right-hand map is injective by Proposition A.21, so the upper right-hand map is injective, which proves (ii). ☐

4.2 Anticyclotomic $\mathbf{Z}_p$-extensions of quadratic imaginary fields

From now on we assume that K is quadratic imaginary, and K_∞ is the anticyclotomic $\mathbf{Z}_p$-extension of K (the unique $\mathbf{Z}_p$-extension, Galois over $\mathbf{Q}$, on which the nontrivial element of $\mathrm{Gal}(K/\mathbf{Q})$ acts as -1). Let $\Lambda = \mathbf{Z}_p[[\mathrm{Gal}(K_\infty/K)]]$. We assume in addition that E has good ordinary reduction at p, that $Ш(E/K_n)[p^\infty]$ is finite for every n, and that every prime dividing the conductor N of E splits in K. The last assumption guarantees that $W(E/K) = \chi_K(-N) = -1$.

Theorem 4.3 (Vatsal [V], Cornut [Co]). *With assumptions as above,* $\mathrm{rank}_\Lambda(X(E/K_\infty)) = 1$.

Let $E(K)^\pm \subset E(K)$ denote the $+1$ and -1 eigenspaces for the action of $\Delta = \mathrm{Gal}(K/\mathbf{Q})$ on $E(K)$. Thus $E(K)^+ = E(\mathbf{Q})$, and $E(K)^- \cong E^{(K)}(\mathbf{Q})$ where $E^{(K)}$ is the quadratic twist of E by $K/\mathbf{Q}$. Also write $r_\pm = \mathrm{rank}_{\mathbf{Z}}(E(K)^\pm)$, and let $W(E/\mathbf{Q}) = \pm 1$ denote the root number in the functional equation of $L(E/\mathbf{Q}, s)$.

Corollary 4.4. *The universal norm subgroup $\mathcal{U}(E, K_\infty/K) \otimes \mathbf{Q}_p$ is a one-dimensional $\mathbf{Q}_p$-subspace of $E(K)\otimes\mathbf{Q}_p$, contained either in $E(K)^+ \otimes \mathbf{Q}_p$ or $E(K)^- \otimes \mathbf{Q}_p$.*

Proof. By Theorems 4.2(ii) and 4.3, $\dim_{\mathbf{Q}_p}(\mathcal{U}(E, K_\infty/K)\otimes\mathbf{Q}_p) = 1$. It is clear from the definition that $\mathcal{U}(E, K_\infty/K) \otimes \mathbf{Q}_p$ is stable under Δ, so it must lie in $E(K)^\pm \otimes \mathbf{Q}_p$. ☐

Definition 4.5. Following Corollary 4.4, if $\mathcal{U}(E, K_\infty/K)\otimes\mathbf{Q}_p \subset E(K)^+\otimes\mathbf{Q}_p$ (resp., $E(K)^-\otimes\mathbf{Q}_p$) we say that the *sign of the p-anti-cyclotomic norms* in $E(K)$ is $+1$ (resp., -1).
In the language of Appendix A.1, the sign of the p-anti-cyclotomic norms is $\mathrm{sign}(\mathcal{U}(E/K_\infty))$, the sign of the Δ-semi-linear Λ-module $\mathcal{U}(E/K_\infty)$.

Theorem 4.6 (Nekovář [N]). *Under the hypotheses above,*

$$(-1)^{r_\pm} = \pm W(E/\mathbf{Q}) \text{ and } \mathrm{rank}_{\mathbf{Z}}(E(K)) \text{ is odd.}$$

Proof. Using (1), the root numbers in the functional equations of $L(E/\mathbf{Q}, s)$, $L(E^{(K)}/\mathbf{Q}, s)$, and $L(E/K, s)$ are, respectively, $W(E/\mathbf{Q})$, $\chi_K(-N)W(E/\mathbf{Q})$, and $\chi_K(-N)$. By Theorem 2.10 and our assumption that all primes dividing N split in K (the "Heegner hypothesis"), $\chi_K(-N) = \chi_K(-1) = -1$. Thus the conclusions of the theorem follow from the Parity Conjecture (which predicts, for every number field F, that $\mathrm{rank}_{\mathbf{Z}}(E(F) \equiv \mathrm{ord}_{s=1} L(E, s) \pmod 2)$), which under our hypotheses was proved by Nekovář. □

Definition 4.7. By Theorem 4.6, $\mathrm{rank}_{\mathbf{Z}}(E(K)^+) \neq \mathrm{rank}_{\mathbf{Z}}(E(K)^-)$. We define

$$\epsilon(E/K) = \begin{cases} +1 & \text{if } \mathrm{rank}_{\mathbf{Z}}(E(K)^+) > \mathrm{rank}_{\mathbf{Z}}(E(K)^-), \\ -1 & \text{if } \mathrm{rank}_{\mathbf{Z}}(E(K)^-) > \mathrm{rank}_{\mathbf{Z}}(E(K)^+). \end{cases}$$

Conjecture 4.8 (Sign Conjecture). *We have*

$$\mathrm{sign}(\mathcal{U}(E/K_\infty)) = \epsilon(E/K).$$

In other words $\mathcal{U}(E, K_\infty/K) \otimes \mathbf{Q}_p$ *lies in the* larger *of* $E(K)^+ \otimes \mathbf{Q}_p$ *and* $E(K)^- \otimes \mathbf{Q}_p$, *and in particular* $\mathrm{sign}(\mathcal{U}(E/K_\infty))$ *is independent of* p.

Remark 4.9. Here is a brief heuristic argument that supports the Sign Conjecture. It is straightforward to prove that $\mathcal{U}(E, K_\infty/K)$ is contained in the nullspace of the (symmetric, bilinear) p-anti-cyclotomic height pairing (cf. [MT])

$$\langle\ ,\ \rangle_{\mathrm{anti-cycl}} : E(K) \otimes \mathbf{Z}_p \times E(K) \otimes \mathbf{Z}_p \longrightarrow \mathrm{Gal}(K_\infty/K) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p.$$

Moreover, this height pairing is compatible with the action of $\mathrm{Gal}(K/\mathbf{Q})$ in the sense that if τ is the nontrivial automorphism of K, then

$$\langle \tau(v), \tau(w) \rangle_{\mathrm{anti-cycl}} = -\langle v, w \rangle_{\mathrm{anti-cycl}}.$$

It follows that $E(K)^+ \otimes \mathbf{Z}_p$ and $E(K)^- \otimes \mathbf{Z}_p$ are each isotropic under the anti-cyclotomic height pairing, and the null-space of $\langle\ ,\ \rangle_{\mathrm{anti-cycl}}$ is the sum of the left and right nullspaces of the restriction of $\langle\ ,\ \rangle_{\mathrm{anti-cycl}}$ to

$$E(K)^+ \otimes \mathbf{Z}_p \times E(K)^- \otimes \mathbf{Z}_p \longrightarrow \mathrm{Gal}(K_\infty/K) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p. \tag{3}$$

It seems natural to conjecture that the pairing (3) is *as nondegenerate as possible.* (For a precise statement of this conjecture, and further discussion, see §3 of [BD1]). This "maximal nondegeneracy" would imply that (3) is either left or right nondegenerate, depending upon which of the two spaces $E(K)^-$ or $E(K)^+$ is larger, and this in turn would imply the Sign Conjecture.

4.3 Heegner points and the Sign Conjecture

We are grateful to Christophe Cornut who communicated to us Proposition 4.10 below and its proof.

Keep the assumptions of the previous section, and fix a modular parametrization $X_0(N) \twoheadrightarrow E$ and an embedding $K_\infty \hookrightarrow \mathbf{C}$. Let $\mathcal{H}_n \subset E(K_n)$ be the module of Heegner points as defined, for example, in [BD2] §2.5 or [M2] §19, and

$$\mathcal{H}_\infty = \varprojlim(\mathcal{H}_n \otimes \mathbf{Z}_p) \subset \mathcal{U}(E/K_\infty).$$

Then $\mathcal{H}_\infty$ is free of rank one over Λ, and stable under the action of $\Delta = \mathrm{Gal}(K/\mathbf{Q})$. In particular it is a Δ-semi-linear Λ-module in the sense of Appendix A.1, and $\mathrm{sign}(\mathcal{H}) \in \{1, -1\}$ describes the action of Δ on $\mathcal{H} \otimes_\Lambda \mathbf{Z}_p$ (Definition A.16).

Since $\mathcal{U}(E/K_\infty)$ is a torsion-free, rank-one Λ-modules (Theorem 4.2(i)), we can fix $\mathcal{L} \in \Lambda$ so that $\mathcal{L}\mathcal{U}(E/K_\infty) \subset \mathcal{H}_\infty$ and $[\mathcal{H}_\infty : \mathcal{L}\mathcal{U}(E/K_\infty)]$ is finite (take $\mathcal{L}$ to be a generator of the characteristic ideal of $\mathcal{U}(E/K_\infty)/\mathcal{H}_\infty$). Let I denote the augmentation ideal of Λ, the kernel of the natural map $\Lambda \twoheadrightarrow \mathbf{Z}_p$, and let $\mathrm{ord}_I(\mathcal{L})$ denote the largest power of I that contains $\mathcal{L}$. If we fix an isomorphism $\Lambda \cong \mathbf{Z}_p[[T]]$, then $\mathrm{ord}_I(\mathcal{L}) = \mathrm{ord}_{T=0}(\mathcal{L})$.

Proposition 4.10. (i) $\mathrm{sign}(\mathcal{H}_\infty) = -W(E/\mathbf{Q})$.

(ii) $\mathrm{sign}(\mathcal{U}(E/K_\infty)) = -W(E/\mathbf{Q})(-1)^{\mathrm{ord}_I(\mathcal{L})}$.

Proof. The Heegner module $\mathcal{H}_\infty$ has a generator h_∞ with the property that

$$\tau h_\infty = -W(E/\mathbf{Q})\sigma h_\infty$$

for some $\sigma \in \mathrm{Gal}(K_\infty/K)$, where τ is complex conjugation (see [Gr] §§1.4-1.5). Thus $\tau h_\infty \equiv -W(E/\mathbf{Q})h_\infty$ in $\mathcal{H}_\infty/I\mathcal{H}_\infty$, which proves (i). The second assertion then follows by Proposition A.17(ii). □

Conjecture 3.10 of Bertolini and Darmon in [BD1] asserts (in our notation) that

$$\mathrm{ord}_I(\mathcal{L}) = \max\{r_+, r_-\} - 1. \tag{4}$$

Corollary 4.11. *Conjecture 3.10 of Bertolini and Darmon [BD1] implies the Sign Conjecture.*

Proof. Let $\epsilon = \epsilon(E/K)$ as given by Definition 4.7, so $r_\epsilon = \max\{r_+, r_-\}$. Combining Proposition 4.10(ii), the conjecture (4), and Theorem 4.6 gives

$$\mathrm{sign}(\mathcal{U}(E/K_\infty)) = (-1)^{r_\epsilon} W(E/\mathbf{Q}) = \epsilon,$$

which is the Sign Conjecture. □

REMARK 4.12. Suppose now that K is arbitrary (possibly even non-Galois), and K_∞/K is a $\mathbf{Z}_p$-extension that is "new", in the sense that it is not the base-change of a $\mathbf{Z}_p$-extension of a proper subfield of K. Suppose further that the compact Selmer module $X(E/K_\infty)$ has rank 1 over $\Lambda = \mathbf{Z}_p[[\mathrm{Gal}(K_\infty/K)]]$. Then by Proposition 3.2 (assuming the Shafarevich-Tate conjecture) the growth number of $(E, K_\infty/K)$ is 1.

In the spirit of the questions from §3, we expect this can only happen if the characters ϕ_ψ are real-valued for every $\psi \in \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(K_\infty/K), \mathbf{C}^\times)$, and have root number $W(E, \phi_\psi) = W(E/K) = -1$. In this case Corollary 2.6 shows that K has an automorphism of order 2, with fixed field K^+, such that K_∞/K^+ is Galois with dihedral Galois group.

Thus in this setting, exactly as when K is quadratic imaginary, the universal norm module $\mathcal{U}(E, K_\infty)$ is a $\mathrm{Gal}(K/K^+)$-semi-linear Λ-module that is torsion-free of rank one over Λ, and we get a one-dimensional universal norm subspace $\mathcal{U}(E, K_\infty/K) \subset E(K) \otimes \mathbf{Q}_p$. Again $\mathcal{U}(E, K_\infty/K) \otimes \mathbf{Q}_p \subset E(K)^\pm \otimes \mathbf{Q}_p$, where $E(K)^\pm$ means the plus and minus spaces for the action of $\mathrm{Gal}(K/K^+)$, $\mathrm{rank}_{\mathbf{Z}}(E(K)^+)$ and $\mathrm{rank}_{\mathbf{Z}}(E(K)^-)$ (conjecturally) have opposite parity, and we can conjecture that $\mathcal{U}(E, K_\infty/K) \otimes \mathbf{Q}_p$ lies in the larger of $E(K)^+ \otimes \mathbf{Q}_p$ and $E(K)^- \otimes \mathbf{Q}_p$. As in Remark 4.9, such a conjecture would follow from the "maximal nondegeneracy" of the p-adic height attached to $(E, K_\infty/K)$.

4.4 COMPUTATIONAL EXAMPLES

The Sign Conjecture is trivially true if either $E(K)^+$ or $E(K)^-$ is finite. In the first nontrivial case, i.e., if $\min\{\mathrm{rank}(E(K)^\pm)\} = 1$, the Sign Conjecture would follow (using Remark 4.9) if the anti-cyclotomic height pairing is not identically zero.

Let us focus on this "first nontrivial case". Fix an elliptic curve E with $\mathrm{rank}_{\mathbf{Z}} E(\mathbf{Q}) = 2$, and let $R, S \in E(\mathbf{Q})$ generate a subgroup of finite index. If K is a quadratic field, let $E^{(K)}$ be the twist of E by the quadratic character attached to K. Fix a prime number p where E has good ordinary reduction. Let K run through the set $\mathcal{K}_E$ of all quadratic imaginary fields such that E and K satisfy our running hypotheses, and such that the rank of the Mordell-Weil group $E^{(K)}(\mathbf{Q})$ is 1; for each of these fields, choose a rational point $P_K \in E^{(K)}(\mathbf{Q}) \subset E(K)$ of infinite order. If the anti-cyclotomic height pairing is not identically zero, then the Sign Conjecture is true, so the p-anti-cyclotomic norm line for $K \in \mathcal{K}_E$ (call it ℓ_K) will lie in the $\mathbf{Q}_p$ vector space $E(\mathbf{Q}) \otimes \mathbf{Q}_p$. The *slope* of $\ell_K \subset E(\mathbf{Q}) \otimes \mathbf{Q}_p$ with respect to the basis $\{R, S\}$ is given by

$$\mathrm{slope}(\ell_K) = -\frac{h(S + P_K)}{h(R + P_K)}$$

where $h = h_{K_\infty/K}$ is the quadratic function $h(P) = \langle P, P \rangle_{\mathrm{anti-cycl}}$ on $E(K)$.

SOME NUMERICAL EXPERIMENTS. We are grateful to William Stein for providing us with the following data. Stein works with the prime $p = 3$ and performs two types of numerical experiment. The first experiment is to test whether or not the 3-anti-cyclotomic height pairing is not identically zero (and hence whether or not the Sign Conjecture is true) for optimal curves E of Mordell-Weil rank 2 of small conductor, and for the quadratic imaginary field K of smallest discriminant in $\mathcal{K}_E$. This Stein did for the elliptic curves labelled

389A, 433A,446D, 571B, 643A, 655A, 664A, 707A, 718B, 794A, 817A

in [Cr]. In all these instances, the height pairing is indeed not identically zero. We also expect that the lines ℓ_K (for $K \in \mathcal{K}_E$) generate the $\mathbf{Q}_p$-vector space $E(\mathbf{Q}) \otimes \mathbf{Q}_p$, and further that it is never the case that $\ell_K = \ell_{K'}$ for different fields K, K'. To test this, Stein computed the slopes of $\ell_K \pmod 9$ (and mod 27, when necessary) for certain selected curves E and various $K \in \mathcal{K}_E$. In particular

- for $E = 389\text{A}$ the first 6 $K \in \mathcal{K}_E$ have the property that the ℓ_K are distinct, with slopes (mod 9) equal to $4, 4, 4, 0, 7, 4$,
- for $E = 433\text{A}$ the first 4 $K \in \mathcal{K}_E$ have the property that the ℓ_K are distinct, with slopes (mod 9) equal to $2, 1, 6, \infty$,
- for $E = 571\text{B}$ the first 4 $K \in \mathcal{K}_E$ have the property that the ℓ_K are distinct, with slopes (mod 9) all equal to 6.

Remark 4.13. If the anti-cyclotomic height pairing is *not* identically zero, then computing it modulo some sufficiently high power of p will detect its nontriviality. Unfortunately, at present, we know of no recipe for an a priori upper bound on how high a power of p one must check, before being able to conclude with certainty that the height pairing is trivial. In other words, if the Sign Conjecture were to fail in some example, it seems difficult to formulate a numerical experiment that would prove this failure.

Appendix

A.1 Semi-linear Λ-modules

Suppose $\mathcal{G}$ is a profinite group with a normal subgroup of finite index $\Gamma \cong \mathbf{Z}_p^d$ for some natural number d. This is the situation discussed earlier in the paper, with $\mathcal{G} = \mathrm{Gal}(L/\mathbf{Q})$ and $\Gamma = \mathrm{Gal}(L/K)$. We let $\Delta = \mathcal{G}/\Gamma$ and we assume further that the order of Δ is prime to p.
Let $\Lambda = \mathbf{Z}_p[[\Gamma]]$. Then Δ acts on both Γ and Λ, and $\mathcal{G}$ is noncanonically isomorphic to the the semidirect product of Δ and Γ. We fix such a semidirect product decomposition.

Definition A.14. A $\mathbf{Z}_p[[\mathcal{G}]]$-module M is a Λ module with a *semi-linear* action of Δ in the following sense:

$$\delta(\lambda \cdot m) = \lambda^\delta \cdot \delta(m)$$

for all $\delta \in \Delta, \lambda \in \Lambda, m \in M$. We will call such a module a Δ-*semi-linear* Λ-*module*, or simply a semi-linear Λ-module if the group Δ and its action on Λ are understood.
Given a semi-linear Λ-module M define $M_\Gamma := M \otimes_\Lambda \mathbf{Z}_p$, and note that the semi-linear Δ-action on M induces a $\mathbf{Z}_p$-linear action of Δ on M_Γ. We can also produce semi-linear Λ-modules by "inverting" this procedure. Namely, if N is a $\mathbf{Z}_p[\Delta]$-module, then $M = N \otimes_{\mathbf{Z}_p} \Lambda$ (with the natural "diagonal" action of Δ) is a semi-linear Λ-module and $M_\Gamma = N$. We say that a semi-linear Λ-module M is *split* if $M \cong N \otimes_{\mathbf{Z}_p} \Lambda$ for some $\mathbf{Z}_p[\Delta]$-module N. Clearly a split semi-linear Λ-module M is determined, up to isomorphism, by the $\mathbf{Z}_p[\Delta]$-module M_Γ.

Lemma A.15. *If M is a semi-linear Λ-module that is free of finite rank over Λ, then M is split.*

Proof. For every subgroup H of finite index in Γ that is stable under Δ, the projection $M \otimes_\Lambda \mathbf{Z}_p[\Gamma/H] \twoheadrightarrow M_\Gamma$ of finite-type $\mathbf{Z}_p[\Delta]$-modules admits a section because $\mathbf{Z}_p[\Delta]$ is semisimple. It follows by a compactness argument that $M \twoheadrightarrow M_\Gamma$ admits a $\mathbf{Z}_p[\Delta]$-section $\iota : M_\Gamma \hookrightarrow M$, and it is straightforward to verify that $M = \iota(M_\Gamma) \otimes \Lambda$. □

Definition A.16. Suppose that M is a semi-linear Λ-module. If $M \otimes \mathbf{Q}_p$ is free of rank one over $\Lambda \otimes \mathbf{Q}_p$, we define the *sign* $\mathrm{sign}(M)$ to be the character

$$\chi_M : \Delta \to \mathrm{Aut}(M_\Gamma \otimes \mathbf{Q}_p) \cong \mathbf{Q}_p^\times.$$

By Lemma A.15, if M is free of rank one over Λ then $\mathrm{sign}(M)$ determines M up to isomorphism.
If Δ has order two (which is the case of most interest in this paper), and τ is the nontrivial element of Δ, we will also call $\chi_M(\tau) = \pm 1$ the sign of M.

Let I denote the augmentation ideal of Λ, the kernel of the natural map $\Lambda \twoheadrightarrow \mathbf{Z}_p$.

Proposition A.17. *Suppose that $\Gamma \cong \mathbf{Z}_p$, and let $\psi : \Delta \to \mathrm{Aut}(\Gamma) \cong \mathbf{Z}_p^\times$ be the character giving the action of Δ on Γ. Suppose $N \subset M$ are semi-linear Λ-modules, and $N \otimes \mathbf{Q}_p$, $M \otimes \mathbf{Q}_p$ are free of rank one over $\Lambda \otimes \mathbf{Q}_p$. Then there is an $\mathcal{L} \in \Lambda$ such that $N \otimes \mathbf{Q}_p = \mathcal{L} M \otimes \mathbf{Q}_p$, and for every such $\mathcal{L}$*

(i) $\mathrm{sign}(N) = \psi^{\lambda(\mathcal{L})}\mathrm{sign}(M)$, *where* $\lambda(\mathcal{L}) = \dim_{\mathbf{Q}_p}(M/N \otimes \mathbf{Q}_p)$ *is the λ-invariant of $\mathcal{L}$,*

(ii) $\mathrm{sign}(N) = \psi^r \mathrm{sign}(M)$, *where $r = \mathrm{ord}_I(\mathcal{L})$ is maximal such that $\mathcal{L} \in I^r$.*

Proof. The existence of such an $\mathcal{L} \in \Lambda$ is clear. Further, replacing M and N by their double duals $\mathrm{Hom}(\mathrm{Hom}(M, \Lambda), \Lambda)$ and $\mathrm{Hom}(\mathrm{Hom}(N, \Lambda), \Lambda)$, we may assume that M and N are both free of rank one over Λ and that $N = \mathcal{L}M$.
The natural isomorphism $\Gamma \xrightarrow{\sim} I/I^2$ by $\gamma \mapsto \gamma - 1$ is Δ-equivariant, so Δ acts on I/I^2 via ψ. Hence $\mathrm{sign}(IM)$, the character giving the action of Δ on $(IM)_\Gamma = (IM) \otimes_\Lambda \mathbf{Z}_p = M_\Gamma \otimes I/I^2$, is $\psi \cdot \mathrm{sign}(M)$ and by induction

$$\mathrm{sign}(I^r M) = \psi^r \mathrm{sign}(M) \tag{5}$$

for every $r \geq 0$.
Identify Λ with a power series ring $\mathbf{Z}_p[[T]]$, so that $I = T\mathbf{Z}_p[[T]]$. The Weierstrass Preparation Theorem shows that $\mathcal{L} = p^\mu T^r g(T)$, where g is a distinguished polynomial of degree $\lambda(\mathcal{L}) - r$ and $g(0) \neq 0$. Since $\mathrm{sign}(M) = \mathrm{sign}(pM)$, we may suppose that $\mu = 0$.
Write $\bar{M} = M/pM$ and $\bar{N} = N/pN$. Then we have $\bar{N} = I^{\lambda(\mathcal{L})}\bar{M}$. Using (5) we see that $\mathrm{sign}(N) \equiv \psi^{\lambda(\mathcal{L})}\mathrm{sign}(M) \pmod{p}$. Since $\mathrm{sign}(N)$ is a character of order prime to p, this proves (i).
To prove (ii), thanks to (5) it is enough to consider the case $r = 0$, i.e., $N = g(T)M$ with $g(0) \neq 0$. In that case

$$N_\Gamma = N \otimes_\Lambda \mathbf{Z}_p = g(0)(M \otimes_\Lambda \mathbf{Z}_p) = g(0)M_\Gamma$$

so $\mathrm{sign}(N) = \mathrm{sign}(M)$ as desired. □

A.2 Universal norms and growth of ranks in Iwasawa modules.

In this appendix Γ will denote a topological pro-p-group written multiplicatively and isomorphic to $\mathbf{Z}_p$, and Γ_n is the unique open subgroup in Γ of index p^n. If $\gamma \in \Gamma$ is a topological generator, then γ^{p^n} is a topological generator of Γ_n. Put $\Lambda := \mathbf{Z}_p[[\Gamma]]$ and for each $n \geq 0$, $\Lambda_n := \mathbf{Z}_p[\Gamma/\Gamma_n] = \mathbf{Z}_p[\gamma]/(\gamma^{p^n} - 1)$. We have $\Lambda_n = \Lambda/(\gamma^{p^n} - 1)\Lambda$, and $\Lambda = \varprojlim \Lambda_n$. IF $n \geq M$ there are natural Λ-module homomorphisms

- the canonical surjection $\pi_{n,m} : \Lambda_n \to \Lambda_m$,
- an injection $\nu_{n,m} : \Lambda_m \to \Lambda_n$, given by multiplication by

  $$\frac{\gamma^{p^n} - 1}{\gamma^{p^m} - 1} = \sum_{i=0}^{p^{n-m}-1} \gamma^{ip^m} \in \Lambda$$

  which takes cosets of $(\gamma^{p^m} - 1)\Lambda$ to cosets of $(\gamma^{p^n} - 1)\Lambda$,

and $\pi_{n,m} \circ \nu_{n,m} = p^{n-m}$.
Suppose M is a module of finite type over Λ. Put

$$M_n := M \otimes_\Lambda \Lambda_n = M/(\gamma^{p^n} - 1)M,$$

and let $\mathrm{rank}_{\mathbf{Z}_p}(M_n) = \dim_{\mathbf{Q}_p}(M_n \otimes_{\mathbf{Z}_p} \mathbf{Q}_p)$, the the $\mathbf{Z}_p$-rank of M_n. Then the structure theory of Λ-modules shows that

$$\mathrm{rank}_{\mathbf{Z}_p}(M_n) = rp^n + \epsilon_n$$

where the error term ϵ_n is non-negative, monotone non-decreasing with n, and bounded (i.e., there is an $n_0 \in \mathbf{Z}^+$ such that if $n \geq n_0$ then $\epsilon_n = \epsilon_{n_0}$).

Definition A.18. Put

$$M_n^* := \mathrm{Hom}_{\mathbf{Z}_p}(M_n, \mathbf{Z}_p).$$

If $n \geq m$ the map $1 \otimes \nu_{m,n} : M_m \to M_n$ induces a map $\nu_{m,n}^* : M_n^* \to M_m^*$. The *universal norm* Λ*-module attached to* M is the projective limit

$$\mathcal{U}(M) := \varprojlim_{0} n M_n^*$$

with respect to the maps $\nu_{n,m}^*$, with its natural Λ-module structure.

Lemma A.19. *There are canonical horizontal isomorphisms making the following diagram commute*

$$\begin{array}{ccc} M_n^* & \xrightarrow{\sim} & \mathrm{Hom}_{\Lambda_n}(M_n, \Lambda_n) \\ {\scriptstyle \nu_{m,n}^*}\big\downarrow & & \big\downarrow \\ M_m^* & \xrightarrow{\sim} & \mathrm{Hom}_{\Lambda_m}(M_m, \Lambda_m) \end{array}$$

where the right-hand map is induced by the isomorphism $M_m = M_n \otimes_\Lambda \Lambda_m$.

Proof. Define $p_1 : \Lambda_n \to \mathbf{Z}_p$ by $p_1(\sum_{i=0}^{p^n-1} a_i \gamma^i) = a_0$. It is straightforward to check that composition with p_1 induces a Λ_n-module isomorphism $\mathrm{Hom}_{\Lambda_n}(M_n, \Lambda_n) \xrightarrow{\sim} \mathrm{Hom}(M_n, \mathbf{Z}_p)$ (see for example [Br] Proposition VI.3.4), and that with these isomorphisms the diagram of the lemma commutes. □

The Λ-*dual* of M is $M^\bullet := \mathrm{Hom}_\Lambda(M, \Lambda)$, which is a free Λ-module (of finite rank). The Λ-rank of M is

$$\mathrm{rank}_\Lambda(M) := \dim_{\mathcal{K}}(M \otimes \mathcal{K})$$

where $\mathcal{K}$ is the field of fractions of Λ. Equivalently, the Λ-rank of M is the rank of the free module $M^\bullet$.

Proposition A.20. *Suppose* M *is a finitely generated* Λ*-module. Then* $\mathcal{U}(M) \cong M^\bullet$*, so in particular* $\mathcal{U}(M)$ *is free over* Λ *of rank* $\mathrm{rank}_\Lambda(M)$.

Proof. Clearly $M^\bullet \cong \varprojlim \mathrm{Hom}_{\Lambda_n}(M_n, \Lambda_n)$. Thus by Lemma A.19 we have an isomorphism

$$\mathcal{U}(M) \xrightarrow{\sim} M^\bullet$$

and the proposition follows. □

Proposition A.21. *Suppose* M *is a finitely generated* Λ*-module. Then the projection map* $\mathcal{U}(M) \to M_n^*$ *factors through an injective map* $\mathcal{U}(M) \otimes \Lambda_n \hookrightarrow M_n^*$*. In particular* $\cap_{m>n} \nu_{m,n}^*(M_m^*)$ *is a free* Λ_n*-submodule of* M_n^* *of rank equal to* $\mathrm{rank}_\Lambda(M)$.

Proof. Using the identification $\mathcal{U}(M) \cong M^\bullet$ of Proposition A.20, and the identification $M_n^* \cong \mathrm{Hom}_\Lambda(M_n, \Lambda_n) = \mathrm{Hom}_\Lambda(M, \Lambda_n)$ of Lemma A.19, the projection map $\mathcal{U}(M) \to M_n^*$ is identified with the natural composition

$$\mathrm{Hom}_\Lambda(M,\Lambda) \longrightarrow \mathrm{Hom}_\Lambda(M,\Lambda) \otimes \Lambda_n \xrightarrow{\beta_n} \mathrm{Hom}_\Lambda(M,\Lambda_n). \tag{6}$$

We need to show that the map β_n is injective.
Let $r = \mathrm{rank}_\Lambda(M)$. By Proposition A.20, $\mathrm{Hom}_\Lambda(M,\Lambda) \otimes \Lambda_n$ is a free $\mathbf{Z}_p$-module of rank rp^n, so to prove the injectivity of β_n it will suffice to show that the image of the composition (6) contains a $\mathbf{Z}_p$-module of rank rp^n.
Fix a free Λ-module N of rank r and a map $M \to N$ with finite cokernel (for example, we can take N to be the double dual $M^{\bullet\bullet}$, with the canonical map). We have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_\Lambda(M,\Lambda) & \longrightarrow & \mathrm{Hom}_\Lambda(M,\Lambda_n) \\
\uparrow & & \uparrow \\
\mathrm{Hom}_\Lambda(N,\Lambda) & \twoheadrightarrow & \mathrm{Hom}_\Lambda(N,\Lambda_n)
\end{array}$$

The vertical maps are injective because Λ and Λ_n have no p-torsion, and the bottom map is surjective because N is free. Since $\mathrm{Hom}_\Lambda(N, \Lambda_n)$ is free of rank rp^n over $\mathbf{Z}_p$, this proves that β_n is injective.
By definition the image of $\mathcal{U}(M) \otimes \Lambda_n \hookrightarrow M_n^*$ is $\cap_{m>n} \nu_{m,n}^*(M_m^*)$, and since $\mathcal{U}(M)$ is free of rank r over Λ, this module is free of rank r over Λ_n □

References

[BD1] M. Bertolini, H. Darmon, Derived p-adic heights, *Duke Math. J.* 117 (1995) 1517–1554.

[BD2] ______, Heegner points on Mumford-Tate curves, *Invent. math.* 126 (1996) 413–456.

[Br] K. Brown, Cohomology of groups, *Grad. Texts in Math.* 87, Springer, New York (1982).

[Bu] W. Burnside, Theory of groups of finite order (second edition), Cambridge University Press, Cambridge (1911).

[CH] J. Coates, S. Howson, Euler Characteristics and Elliptic Curves II, J. Math. Soc. Japan, 53 (2001) 175–235.

[Co] C. Cornut, Mazur's conjecture on higher Heegner points, *Invent. math.* 148 (2002) 495–523.

[Cr] J. Cremona, Algorithms for modular elliptic curves, Second edition. Cambridge: Cambridge University Press (1997).

[D] P. Deligne, Les constantes des équations fonctionelles des fonctions L. In: Modular Functions of One Variable II, *Lect. Notes in Math.* 349, Springer, New York (1973) 501–595.

[F] F. G. Frobenius, Über auflösbare Gruppen II, *Sitzungsbericht der Königlich Preußischen Akademie der Wissenshaften zu Berlin* (1895) 1027–1044.

[G1] R. Greenberg, On the Birch and Swinnerton-Dyer conjecture, *Invent. math.* 72 (1983) 241–265.

[G2] ———, Galois theory for the Selmer group of an abelian variety. To appear.

[Gr] B. Gross, Heegner points on $X_0(N)$. In: Modular forms (Durham, 1983), Horwood, Chichester (1984) 87–105.

[K] K. Kato, p-adic Hodge theory and values of zeta functions of modular forms. To appear.

[M1] B. Mazur, Rational points of abelian varieties with values in towers of number fields, *Invent. Math.* 18 (1972) 183–266.

[M2] ———, Modular curves and arithmetic, in: *Proceedings of the International Congress of Mathematicians (Warsaw, 1983)*, PWN, Warsaw (1984) 185–211.

[MT] B. Mazur, J. Tate, Canonical height pairings via biextensions. In: *Arithmetic and Geometry*, Progr. Math. 35, Birkhaüser, Boston (1983) 195–237.

[Mo] P. Monsky, On p-adic power series, *Math. Ann.* 255 (1981) 217–227.

[N] J. Nekovář, On the parity of ranks of Selmer groups. II, *C. R. Acad. Sci. Paris Sér. I Math.* 332 (2001) 99–104.

[Ro] D. Rohrlich, Galois theory, elliptic curves, and root numbers, *Compositio Math.* 100 (1996) 311–349.

[Ru] K. Rubin, Euler systems and modular elliptic curves, in: Galois representations in arithmetic algebraic geometry, A. J. Scholl and R. L. Taylor, eds., *London Math. Soc. Lect. Notes* 254 Cambridge Univ. Press, Cambridge (1998) 351–367.

[Sc] A. Scholl, An introduction to Kato's Euler systems, in: Galois representations in arithmetic algebraic geometry, A. J. Scholl and R. L. Taylor, eds., *London Math. Soc. Lect. Notes* 254 Cambridge Univ. Press, Cambridge (1998) 379–460.

[Se] J-P. Serre, Représentations linéaires des groupes finis. Hermann, Paris (1971).

[V] V. Vatsal, Uniform distribution of Heegner points, *Invent. math.* 148 (2002) 1–46.

Barry Mazur
Department of Mathematics
Harvard University
Cambridge MA 02138 USA
mazur@math.harvard.edu

Karl Rubin
Department of Mathematics
Stanford University
Stanford CA 94305 USA
rubin@math.stanford.edu

The Absolute Anabelian Geometry of Canonical Curves

Shinichi Mochizuki

Received: September 27, 2002
Revised: January 13, 2003

Abstract. In this paper, we continue our study of the issue of the extent to which a *hyperbolic curve over a finite extension of the field of p-adic numbers* is determined by the profinite group structure of its *étale fundamental group*. Our main results are that: (i) the theory of *correspondences* of the curve — in particular, its *arithmeticity* — is completely determined by its fundamental group; (ii) when the curve is a *canonical lifting* in the sense of *"p-adic Teichmüller theory"*, its *isomorphism class* is functorially determined by its fundamental group. Here, (i) is a consequence of a *"p-adic version of the Grothendieck Conjecture for algebraic curves"* proven by the author, while (ii) builds on a previous result to the effect that the *logarithmic special fiber* of the curve is functorially determined by its fundamental group.

2000 Mathematics Subject Classification: 14H25, 14H30
Keywords and Phrases: *hyperbolic curve, étale fundamental group, anabelian, correspondences, Grothendieck Conjecture, canonical lifting, p-adic Teichmüller theory*

Contents:

Introduction

Let X_K be a *hyperbolic curve* (cf. §0 below) over a *field K of characteristic* 0. Denote its *algebraic fundamental group* by Π_{X_K}. Thus, we have a *natural surjection*

$$\Pi_{X_K} \twoheadrightarrow G_K$$

of Π_{X_K} onto the *absolute Galois group* G_K of K. When K is a *finite extension of* $\mathbb{Q}$ *or* $\mathbb{Q}_p$, and one *holds* G_K *fixed*, then it is known (cf. [Tama], [Mzk6]) that one may *recover the curve* X_K *in a functorial fashion* from Π_{X_K}. This sort of result may be thought of as a *"relative result"* (i.e., over G_K).

In the present paper, we continue our study — begun in [Mzk7] — of "ABSOLUTE ANALOGUES" of such relative results. Since such absolute analogues are well understood in the case where K is a finite extension of $\mathbb{Q}$ (cf. the Introduction to [Mzk7]), we concentrate on the *p-adic case*. In the p-adic case, it is proven in [Mzk7] (cf. [Mzk7], Theorem 2.7) — by applying the work of [Tama] and the techniques of [Mzk5]— that (if X_K has *stable reduction*, then) the "LOGARITHMIC SPECIAL FIBER" OF X_K — i.e., the special fiber, equipped with its natural "log structure" (cf. [Kato]), of the "stable model" of X_K over the ring of integers $\mathcal{O}_K$ — *may be recovered solely from the abstract profinite group* Π_{X_K}. This result prompts the question (cf. [Mzk7], Remark 2.7.3):

What other information — e.g., the ISOMORPHISM CLASS OF X_K ITSELF *— can be recovered from the profinite group* Π_{X_K}*?*

In this present paper, we give *three partial answers* to this question (cf. [Mzk7], Remark 2.7.3), all of which revolve around the *central theme* that:

When X_K is, in some sense, "CANONICAL"*, there is a tendency for substantial information concerning X_K — e.g., its isomorphism class — to be recoverable from* Π_{X_K}.

Perhaps this "tendency" should not be surprising, in light of the fact that in some sense, a *"canonical"* curve is a curve which is *"rigid"*, i.e., has no moduli, hence should be "determined" by its special fiber (cf. Remark 3.6.3).

Our three partial answers are the following:

(a) The property that the *Jacobian* of the X_K be a *Serre-Tate canonical lifting* is determined by Π_{X_K} (Proposition 1.1).

(b) The theory of *correspondences* of X_K — in particular, whether or not X_K is *"arithmetic"* (cf. [Mzk3]) — is determined by Π_{X_K} (cf. Theorem 2.4, Corollary 2.5).

(c) The property that X_K be a *canonical lifting in the sense of the theory of [Mzk1] (cf. also [Mzk2])* is determined by Π_{X_K}; moreover, in this case, the *isomorphism class* of X_K is also determined by Π_{X_K} (cf. Theorem 3.6).

At a technical level, (a) is entirely elementary; (b) is a formal consequence of the *"p-adic version of the Grothendieck Conjecture" proven in [Mzk6], Theorem A*; and (c) is derived as a consequence of the theory of [Mzk1], together with [Mzk7], Theorem 2.7.

Finally, as a consequence of (c), we conclude (cf. Corollary 3.8) that the *set of points arising from curves over finite extensions of* $\mathbb{Q}_p$ *whose isomorphism classes are completely determined by* Π_{X_K} *forms a* Zariski dense *subset of the moduli stack over* $\mathbb{Q}_p$. This result (cf. Remark 3.6.2) constitutes the first *application* of the "p-adic Teichmüller theory" of [Mzk1], [Mzk2], to prove a *hitherto unknown result* that can be *stated* without using the terminology, concepts, or results of the theory of [Mzk1], [Mzk2]. Also, it shows that — unlike (a), (b) which only yield "useful information" concerning X_K in *"very rare cases"* — (c) may be applied to a *"much larger class of* X_K*"* (cf. Remarks 1.1.1, 2.5.1, 3.6.1).

Acknowledgements: I would like to thank *A. Tamagawa* for useful comments concerning earlier versions of this manuscript.

Section 0: Notations and Conventions

We will denote by $\mathbb{N}$ the set of *natural numbers*, by which we mean the set of integers $n \geq 0$. A *number field* is defined to be a finite extension of the field of rational numbers $\mathbb{Q}$.

Suppose that $g \geq 0$ is an *integer*. Then a *family of curves of genus* g

$$X \to S$$

is defined to be a smooth, proper, geometrically connected morphism $X \to S$ whose geometric fibers are curves of genus g.

Suppose that $g, r \geq 0$ are *integers* such that $2g-2+r > 0$. We shall denote the *moduli stack of* r*-pointed stable curves of genus* g (where we assume the points to be *unordered*) by $\overline{\mathcal{M}}_{g,r}$ (cf. [DM], [Knud] for an exposition of the theory of such curves; strictly speaking, [Knud] treats the finite étale covering of $\overline{\mathcal{M}}_{g,r}$ determined by *ordering* the marked points). The open substack $\mathcal{M}_{g,r} \subseteq \overline{\mathcal{M}}_{g,r}$ of smooth curves will be referred to as the *moduli stack of smooth* r*-pointed* *stable curves of genus* g or, alternatively, as the *moduli stack of hyperbolic curves of type* (g, r).

A *family of hyperbolic curves of type* (g, r)

$$X \to S$$

is defined to be a morphism which factors $X \hookrightarrow Y \to S$ as the composite of an open immersion $X \hookrightarrow Y$ onto the complement $Y \backslash D$ of a relative divisor $D \subseteq Y$ which is finite étale over S of relative degree r, and a family $Y \to S$ of curves of genus g. One checks easily that, if S is *normal*, then the pair (Y, D) is *unique up to canonical isomorphism*. (Indeed, when S is the spectrum of a field, this fact is well-known from the elementary theory of algebraic curves. Next, we consider an arbitrary *connected normal* S on which a prime l is *invertible* (which, by Zariski localization, we may assume without loss of generality). Denote by $S' \to S$ the finite étale covering parametrizing *orderings of the marked points* and *trivializations of the* l*-torsion points of the Jacobian of* Y. Note that $S' \to S$ is *independent* of the choice of (Y, D), since (by the normality of S) S' may be constructed as the *normalization* of S in the function field of S' (which is independent of the choice of (Y, D) since the restriction of (Y, D) to the generic point of S has already been shown to be unique). Thus, the uniqueness of (Y, D) follows by considering the classifying morphism (associated to (Y, D)) from S' to the finite étale covering of $(\mathcal{M}_{g,r})_{\mathbb{Z}[\frac{1}{l}]}$ parametrizing orderings of the marked points and trivializations of the l-torsion points of the Jacobian [since this covering is well-known to be a scheme, for l sufficiently large].)

We shall refer to Y (respectively, D; D; D) as the *compactification* (respectively, *divisor at infinity*; *divisor of cusps*; *divisor of marked points*) of X. A *family of hyperbolic curves* $X \to S$ is defined to be a morphism $X \to S$ such that the restriction of this morphism to each connected component of S is a *family of hyperbolic curves of type* (g, r) for some integers (g, r) as above.

Section 1: Serre-Tate Canonical Liftings

In this §, we observe (cf. Proposition 1.1 below) that the issue of whether or not the Jacobian of a p-adic hyperbolic curve is a *Serre-Tate canonical lifting* is completely determined by the abstract profinite group structure of its *arithmetic profinite group.*

Let p be a prime number. For $i = 1, 2$, let K_i be a finite extension of $\mathbb{Q}_p$, and $(X_i)_{K_i}$ a *proper hyperbolic curve* over K_i whose associated stable curve has stable reduction over $\mathcal{O}_{K_i}$. Denote the resulting *"stable model"* of $(X_i)_{K_i}$ over $\mathcal{O}_{K_i}$ by $(\mathcal{X}_i)_{\mathcal{O}_{K_i}}$.

Assume that we have chosen basepoints of the $(X_i)_{K_i}$ (which thus induce basepoints of the K_i) and suppose that we are given an *isomorphism of profinite groups* $\Pi_{(X_1)_{K_1}} \xrightarrow{\sim} \Pi_{(X_2)_{K_2}}$, which (by [Mzk7], Lemmas 1.1.4, 1.1.5) induces a *commutative diagram*:

$$\begin{array}{ccc} \Pi_{(X_1)_{K_1}} & \xrightarrow{\sim} & \Pi_{(X_2)_{K_2}} \\ \downarrow & & \downarrow \\ G_{K_1} & \xrightarrow{\sim} & G_{K_2} \end{array}$$

Proposition 1.1. (Group-Theoreticity of Serre-Tate Canonical Liftings) *The Jacobian of* $(X_1)_{K_1}$ *is Serre-Tate canonical if and only if the same is true of the Jacobian of* $(X_2)_{K_2}$.

Proof. Indeed, this follows from the fact that the Jacobian of $(X_i)_{K_i}$ is a Serre-Tate canonical lifting if and only if its p-adic Tate module *splits* (as a G_{K_i}-module) into a direct sum of an unramified G_{K_i}-module and the Cartier dual of an unramified G_{K_i}-module (cf. [Mess], Chapter V: proof of Theorem 3.3, Theorem 2.3.6; [Mess], Appendix: Corollary 2.3, Proposition 2.5). ○

Remark 1.1.1. As is shown in [DO] (cf. also [OS]), for $p > 2$, $g \geq 4$, the Serre-Tate canonical lifting of the Jacobian of a general proper curve of genus g in characteristic p is *not* a Jacobian. Thus, in some sense, one expects that:

There are not so many curves to which Proposition 1.1 may be applied.

From another point of view, if there exist infinitely many Jacobians of a given genus g over finite extensions of $\mathbb{Q}_p$ which are Serre-Tate canonical liftings, then one expects — cf. the *"André-Oort Conjecture"* ([Edix], Conjecture 1.3) — that every irreducible component of the Zariski closure of the resulting set of points in the moduli stack of principally polarized abelian varieties should be a "subvariety of Hodge type". Moreover, one expects that the intersection of such a subvariety with the Torelli locus (i.e., locus of Jacobians) in the moduli stack of principally polarized abelian varieties should typically be *"rather small"*. Thus, from this point of view as well, one expects that Proposition 1.1 should *not be applicable* to the "OVERWHELMING MAJORITY" of curves of genus $g \geq 2$.

Section 2: Arithmetic Hyperbolic Curves

In this §, we show (cf. Theorem 2.4 below) that the theory of *correspondences* (cf. [Mzk3]) of a p-adic hyperbolic curve is completely determined by the abstract profinite group structure of its *arithmetic profinite group*. We begin by reviewing and extending the *theory of [Mzk3]*, as it will be needed in the discussion of the present §.

Let X be a *normal connected algebraic stack* which is *generically "scheme-like"* (i.e., admits an open dense algebraic substack isomorphic to a scheme). Then we shall denote by

$$\mathrm{Loc}(X)$$

the *category* whose *objects* are (necessarily generically scheme-like) algebraic stacks Y that admit a finite étale morphism to X, and whose *morphisms* are finite étale morphisms of stacks $Y_1 \to Y_2$ (that do not necessarily lie over X!). Note that since these stacks are *generically scheme-like*, it makes sense to speak of the *(1-)category* of such objects (so long as our morphisms are finite étale), i.e., there is no need to work with 2-categories.

Given an object Y of $\mathrm{Loc}(X)$, let us denote by

$$\mathrm{Loc}(X)_Y$$

the *category* whose *objects* are morphisms $Z \to Y$ in $\mathrm{Loc}(X)$, and whose morphisms, from an object $Z_1 \to Y$ to an object $Z_2 \to Y$, are the morphisms $Z_1 \to Z_2$ over Y in $\mathrm{Loc}(X)$. Thus, by considering *maximal nontrivial decompositions* of the terminal object of $\mathrm{Loc}(X)_Y$ into a coproduct of nonempty objects of $\mathrm{Loc}(X)_Y$, we conclude that the *set of connected components of* Y may be recovered — *functorially!* — from the *category structure* of $\mathrm{Loc}(Y)$. Finally, let us observe that $\mathrm{Loc}(X)_Y$ may be identified with the category

$$\acute{\mathrm{E}}\mathrm{t}(Y)$$

of *finite étale coverings of* Y (and Y-morphisms).

We would also like to consider the *category*

$$\overline{\mathrm{Loc}}(X)$$

whose *objects* are generically scheme-like algebraic stacks which arise as *finite étale quotients* (in the sense of stacks!) of objects in $\mathrm{Loc}(X)$, and whose *morphisms* are finite étale morphisms of algebraic stacks. Note that $\overline{\mathrm{Loc}}(X)$ may be constructed *entirely category-theoretically* from $\mathrm{Loc}(X)$ by considering the *"category of objects of* $\mathrm{Loc}(X)$ *equipped with a (finite étale) equivalence relation"*. (We leave it to the reader to write out the routine details.)

Definition 2.1.

(i) X will be called *arithmetic* if $\overline{\mathrm{Loc}}(X)$ does not admit a terminal object.

(ii) X will be called a(n) *(absolute) core* if X is a terminal object in $\mathrm{Loc}(X)$.

(ii) X will be said to *admit a(n) (absolute) core* if there exists a terminal object Z in $\overline{\mathrm{Loc}}(X)$. In this case, $\overline{\mathrm{Loc}}(X) = \overline{\mathrm{Loc}}(Z)$, so we shall say that Z is a *core*.

Remark 2.1.1. Let k be a *field*. If X is a *geometrically normal, geometrically connected algebraic stack of finite type over* k, then we shall write

$$\mathrm{Loc}_k(X); \quad \overline{\mathrm{Loc}}_k(X)$$

for the categories obtained as above, except that we assume all the morphisms to be *k-morphisms*. Also, we shall say that X is *k-arithmetic*, or *arithmetic over* k (respectively, a *k-core*, or *core over* k), if $\overline{\mathrm{Loc}}_k(X)$ does not admit a terminal object (respectively, X is a terminal object in $\mathrm{Loc}_k(X)$). On the other hand, when k is *fixed*, and the entire discussion "takes place over k", then we shall often *omit* the "k-" from this terminology.

Remark 2.1.2. Thus, when $k = \mathbb{C}$, a hyperbolic curve X is k-arithmetic if and only if it is arithmetic in the sense of [Mzk3], §2. (Indeed, if X is *non-arithmetic* in the sense of [Mzk3], §2, then a terminal object in $\overline{\mathrm{Loc}}_k(X)$ — i.e., a *"(hyperbolic) core"* — is constructed in [Mzk3], §3, so X is non-k-arithmetic. Conversely, if X is *arithmetic* in the sense of [Mzk3], §2, then (cf. [Mzk3], Definition 2.1, Theorem 2.5) it corresponds to a *fuchsian group* $\Gamma \subseteq SL_2(\mathbb{R})/\{\pm 1\}$ which has *infinite index* in its commensurator $C_{SL_2(\mathbb{R})/\{\pm 1\}}(\Gamma)$ — a fact which precludes the existence of a *k-core*.) Moreover, issues over an *arbitrary algebraically closed* k *of characteristic zero* may always be *resolved over* $\mathbb{C}$, by Proposition 2.3, (ii), below.

Remark 2.1.3. If we *arbitrarily choose* a finite étale structure morphism to X for every object of $\mathrm{Loc}(X)$, then one verifies easily that every morphism of $\mathrm{Loc}(X)$ *factors* as the composite of an *isomorphism* (not necessarily over X!) with a *(finite étale) morphism over* X (i.e., relative to these arbtirary choices). A similar statement holds for $\mathrm{Loc}_k(X)$.

Definition 2.2. Let X be a smooth, geometrically connected, generically scheme-like algebraic stack of finite type over a field k of *characteristic zero.*

(i) We shall say that X is an *orbicurve* if it is of dimension 1.

(ii) We shall say that X is a *hyperbolic orbicurve* if it is an orbicurve which admits a compactification $X \hookrightarrow \overline{X}$ (necessarily unique!) by a *proper orbicurve* $\overline{X}$ over k such that if we denote the reduced divisor $\overline{X} \backslash X$ by $D \subseteq \overline{X}$, then

$\overline{X}$ is *scheme-like* near D, and, moreover, the line bundle $\omega_{\overline{X}/k}(D)$ on $\overline{X}$ has *positive degree.*

PROPOSITION 2.3. (INDEPENDENCE OF THE BASE FIELD)

(i) Let k^{sep} *be a separable closure of* k*;* X *a geometrically normal, geometrically connected algebraic stack of finite type over* k*. Then* X *is a* k*-core (respectively,* k*-arithmetic) if and only if* $X_{k^{\mathrm{sep}}} \stackrel{\mathrm{def}}{=} X \times_k k^{\mathrm{sep}}$ *is a* k^{sep}*-core (respectively,* k^{sep}*-arithmetic). Moreover, if* $X_{k^{\mathrm{sep}}}$ *admits a finite étale morphism* $X_{k^{\mathrm{sep}}} \to Z_{k^{\mathrm{sep}}}$ *to a* k^{sep}*-core* $Z_{k^{\mathrm{sep}}}$*, then* $Z_{k^{\mathrm{sep}}}$ *descends uniquely to a* k*-core* Z *of* X*.*

(ii) Suppose that k *is algebraically closed of characteristic* 0*, and that* X *is a* HYPERBOLIC ORBICURVE*. Next, let* k' *be an algebraically closed field containing* k*. Then the natural functors*

$$\mathrm{Loc}_k(X) \to \mathrm{Loc}_{k'}(X \otimes_k k'); \quad \overline{\mathrm{Loc}}_k(X) \to \overline{\mathrm{Loc}}_{k'}(X \otimes_k k')$$

(given by tensoring over k *with* k'*) are* EQUIVALENCES *of categories. In particular,* X *is a* k*-core (respectively,* k*-arithmetic) if and only if* $X \otimes_k k'$ *is a* k'*-core (respectively,* k'*-arithmetic).*

Proof. First, we observe that (i) is a formal consequence of the definitions. As for (ii), let us observe first that it suffices to verify the asserted *equivalences of categories.* These equivalences, in turn, are formal consequences of the following *two assertions* (cf. Remark 2.1.3):

(a) The natural functor $\acute{\mathrm{E}}\mathrm{t}(X) \to \acute{\mathrm{E}}\mathrm{t}(X \otimes_k k')$ is an equivalence of categories.

(b) If Y_1, Y_2 are finite étale over X, then

$$\mathrm{Isom}_k(Y_1, Y_2) \to \mathrm{Isom}_{k'}(Y_1 \otimes_k k', Y_2 \otimes_k k')$$

is bijective.

The proofs of these two assertions is an exercise in elementary algebraic geometry, involving the following well-known techniques:

(1) *descending* the necessary diagrams of finite étale morphisms over k' to a subfield $K \subseteq k'$ which is finitely generated over k;

(2) *extending* orbicurves over K to orbicurves over some k-variety V with function field K;

(3) *specializing* orbicurves over V to closed (i.e., k-valued) points v of V;

(4) *base-changing* orbicurves over V to formal completions $\widehat{V}_v$ of V at closed points v;

(5) *deforming* (log) étale morphisms of orbicurves over v to morphisms over the completions $\widehat{V}_v$;

(6) *algebrizing* such deformed morphisms (when the orbicurves involved are proper).

This "elementary exercise" is carried out (for assertion (a) above) in the case when X itself is *proper* in [SGA1], Exposé X, Theorem 3.8. When X is an arbitrary orbicurve as in the statement of (ii), the *same arguments* — centering around the *rigidity* of (log) étale morphisms under infinitesimal deformations — may be used, by considering *compactifications* $(\overline{X}, D)$ of X as in Definition 2.2, (ii), and replacing "étale" by *"étale away from D".* Note that we use the assumption that k is of characteristic zero here to ensure that *all ramification is tame.*

Finally, assertion (b) may be deduced by similar arguments — by applying, in (5) above, the *fact* (cf. Definition 2.2, (ii)) that, if $\overline{Y} \to \overline{X}$ is any finite morphism of orbicurves over k, then

$$H^0(\overline{Y}, \omega^{\vee}_{\overline{X}/k}(-D)|_{\overline{Y}}) = 0$$

(where "$\vee$" denotes the $\mathcal{O}_{\overline{X}}$*-dual*) in place of the *rigidity* of (log) étale morphisms used to prove assertion (a). ◯

Next, for $i = 1, 2$, let K_i be a *finite extension of* $\mathbb{Q}_p$ (where p is a prime number); let $(X_i)_{K_i}$ be a *hyperbolic curve* over K_i. Assume that we have chosen basepoints of the $(X_i)_{K_i}$, which thus induce basepoints/algebraic closures $\overline{K}_i$ of the K_i and determine *fundamental groups* $\Pi_{(X_i)_{K_i}} \stackrel{\text{def}}{=} \pi_1((X_i)_{K_i})$ and *Galois groups* $G_{K_i} \stackrel{\text{def}}{=} \text{Gal}(\overline{K}_i/K_i)$. Thus, for $i = 1, 2$, we have an *exact sequence*:

$$1 \to \Delta_{X_i} \to \Pi_{(X_i)_{K_i}} \to G_{K_i} \to 1$$

(where $\Delta_{X_i} \subseteq \Pi_{(X_i)_{K_i}}$ is defined so as to make the sequence exact). Here, we shall think of G_{K_i} as a *quotient* of $\Pi_{(X_i)_{K_i}}$ (i.e., not as an independent group to which $\Pi_{(X_i)_{K_i}}$ happens to surject). By [Mzk7], Lemmas 1.1.4, 1.1.5, this quotient is *characteristic*, i.e., it is *completely determined by the structure of* $\Pi_{(X_i)_{K_i}}$ *as a profinite group.*

THEOREM 2.4. (GROUP-THEORETICITY OF CORRESPONDENCES) *Any isomorphism* $\alpha : \Pi_{(X_1)_{K_1}} \xrightarrow{\sim} \Pi_{(X_2)_{K_2}}$ *induces equivalences of categories:*

$$\text{Loc}_{\overline{K}_1}((X_1)_{\overline{K}_1}) \xrightarrow{\sim} \text{Loc}_{\overline{K}_2}((X_2)_{\overline{K}_2}); \quad \overline{\text{Loc}}_{\overline{K}_1}((X_1)_{\overline{K}_1}) \xrightarrow{\sim} \overline{\text{Loc}}_{\overline{K}_2}((X_2)_{\overline{K}_2})$$

in a fashion that is FUNCTORIAL *in* α.

Proof. Since $\overline{\mathrm{Loc}}_{\overline{K}_i}((X_i)_{\overline{K}_i})$ may be *reconstructed "category-theoretically"* from $\mathrm{Loc}_{\overline{K}_i}((X_i)_{\overline{K}_i})$ (cf. the discussion at the beginning of the present §), in order to prove Theorem 2.4, it thus suffices to show that the isomorphism α induces an equivalence between the categories $\mathrm{Loc}_{\overline{K}_i}((X_i)_{\overline{K}_i})$.

Clearly, the class of *objects* of $\mathrm{Loc}_{\overline{K}_i}((X_i)_{\overline{K}_i})$ may be reconstructed as the class of objects of the category of finite sets with continuous Δ_{X_i}-action. To reconstruct the *morphisms*, it suffices (cf. Remark 2.1.3) to show that given any two *open subgroups* $H_1, J_1 \subseteq \Pi_{(X_1)_{K_1}}$ — which we may assume, without loss of generality, to *surject* onto G_{K_1} — and an isomorphism

$$H_1 \xrightarrow{\sim} J_1$$

that *arises "K_1-geometrically"* (i.e., from a K_1-scheme-theoretic isomorphism between the curves corresponding to H_1, J_1), it is necessarily the case that the corresponding isomorphism

$$H_2 \xrightarrow{\sim} J_2$$

between open subgroups $H_2, J_2 \subseteq \Pi_{(X_2)_{K_2}}$ *arises K_2-geometrically.*

But this *follows formally from the "p-adic version of the Grothendieck Conjecture" proven in [Mzk6], Theorem A*: Indeed, $H_1 \xrightarrow{\sim} J_1$ necessarily lies over an *inner automorphism* $\gamma_1 : G_{K_1} \xrightarrow{\sim} G_{K_1}$. In particular, $H_2 \xrightarrow{\sim} J_2$ lies over an isomorphism $\gamma_2 : G_{K_2} \xrightarrow{\sim} G_{K_2}$, which is obtained by conjugating γ_1 by *some fixed isomorphism* (not necessarily geometric!) arising from α between the *characteristic quotients* $G_{K_1} \xrightarrow{\sim} G_{K_2}$. Since the property of "being an inner automorphism" is *manifestly intrinsic*, we thus conclude that γ_2 is *also an inner automorphism.* This allows us to apply [Mzk6], Theorem A, which implies that $H_2 \xrightarrow{\sim} J_2$ arises K_2-geometrically, as desired. ○

COROLLARY 2.5. (CONSEQUENCES FOR CORES AND ARITHMETICITY) *Let*

$$\alpha : \Pi_{(X_1)_{K_1}} \xrightarrow{\sim} \Pi_{(X_2)_{K_2}}$$

be an isomorphism. Then:

(i) $(X_1)_{\overline{K}_1}$ *is* $\overline{K}_1$- ARITHMETIC *(respectively, a* $\overline{K}_1$- CORE*) if and only if* $(X_2)_{\overline{K}_2}$ *is* $\overline{K}_2$- ARITHMETIC *(respectively, a* $\overline{K}_2$- CORE*).*

(ii) Suppose that, for $i = 1, 2$, *we are given a finite étale morphism* $(X_i)_{K_i} \to (Z_i)_{K_i}$ *to a* K_i*-core* $(Z_i)_{K_i}$. *Then the isomorphism* α EXTENDS UNIQUELY *to an isomorphism* $\Pi_{(Z_1)_{K_1}} \xrightarrow{\sim} \Pi_{(Z_2)_{K_2}}$.

Proof. Assertion (i) is a formal consequence of Theorem 2.4 and Definition 2.1 (cf. also Remark 2.1.1). In light of Proposition 2.3, (i), assertion (ii) is a formal consequence of Theorem 2.4, at least over *some* corresponding finite Galois extensions K'_1, K'_2 of K_1, K_2. That the resulting extension $\Pi_{(Z_1)_{K'_1}} \xrightarrow{\sim} \Pi_{(Z_2)_{K'_2}}$ of α is *unique* is a formal consequence of the fact that every open subgroup of $\Pi_{(X_i)_{K_i}}$ has *trivial centralizer* in $\Pi_{(Z_i)_{K_i}}$ (cf. [Mzk7], Lemma 1.3.1, Corollary 1.3.3). Moreover, it follows formally from this triviality of centralizers that, by choosing corresponding normal open subgroups $H_i \subseteq \Pi_{(Z_i)_{K_i}}$ such that $H_i \subseteq \Pi_{X_{K'_i}}$, we may think of $\Pi_{(Z_i)_{K_i}}$ (and its various open subgroups) as *subgroups of* $\mathrm{Aut}(H_i)$, in a fashion which is *compatible* with α and its various (unique) extensions. Thus, since $\Pi_{(Z_i)_{K_i}}$ is *generated* by $\Pi_{(Z_i)_{K'_i}}$ and $\Pi_{(X_i)_{K_i}}$, we conclude that this extension $\Pi_{(Z_i)_{K'_1}} \xrightarrow{\sim} \Pi_{(Z_i)_{K'_2}}$ over the K'_i descends to some $\Pi_{(Z_1)_{K_1}} \xrightarrow{\sim} \Pi_{(Z_2)_{K_2}}$, as desired. ○

REMARK 2.5.1. Recall from the theory of [Mzk3] (cf. Remark 2.1.2; Proposition 2.3, (ii), of the discussion above) that $(X_i)_{\overline{K}_i}$ is *arithmetic* if and only if it admits a finite étale cover which is a finite étale cover of a *Shimura curve*, i.e., (equivalently) if there exists a Shimura curve in $\overline{\mathrm{Loc}}_{\overline{K}i}((X_i)_{\overline{K}_i})$. As is discussed in [Mzk3], Theorem 2.6, a theorem of Takeuchi states that for a given (g, r), there are only *finitely many* isomorphism classes of hyperbolic curves of type (g, r) (over a given algebraically closed field of characteristic zero) which are *arithmetic*. Moreover, a general hyperbolic curve of type (g, r) is not only non-arithmetic; it is, in fact, *equal to its own hyperbolic core* (cf. [Mzk3], Theorem 5.3). Thus, for general curves of a given type (g, r), the structure of the category $\overline{\mathrm{Loc}}((X_i)_{\overline{K}_i})$ is *not* sufficient to determine the isomorphism class of the curve. It is not clear to the author at the time of writing whether or not, in the case when $(X_i)_{K_i}$ is *arithmetic*, the structure of the category $\overline{\mathrm{Loc}}((X_i)_{\overline{K}_i})$ is sufficient to determine the isomorphism class of $(X_i)_{K_i}$. At any rate, just as was the case with Proposition 1.1 (cf. Remark 1.1.1), *Theorem 2.4 does not allow one to recover the isomorphism class of* $(X_i)_{K_i}$ *for "most"* $(X_i)_{K_i}$ — where here we take "most" to mean that (at least for (g, r) sufficiently large) the set of points determined by the curves for which it *is* possible to recover the isomorphism class of $(X_i)_{K_i}$ from the profinite group $(X_i)_{K_i}$ via the method in question *fails to be Zariski dense in the moduli stack of hyperbolic curves of type* (g, r) (cf. Corollary 3.8 below).

Finally, to give the reader a feel for the abstract theory — and, in particular, the state of affairs discussed in Remark 2.5.1 above — we consider the case of *punctured hemi-elliptic orbicurves*, in which the situation is understood somewhat explicitly:

DEFINITION 2.6. Let X be an *orbicurve* (cf. Definition 2.2, (i)) over a field of characteristic zero k.

(i) We shall say that X is a *hemi-elliptic orbicurve* if it is obtained by forming the quotient — in the *sense of stacks* — of an elliptic curve by the action of ± 1.

(ii) We shall say that X is a *punctured hemi-elliptic orbicurve* if it is obtained by forming the quotient — in the *sense of stacks* — of a once-punctured elliptic curve (i.e., the open subscheme given by the complement of the origin in an elliptic curve) by the action of ± 1.

Proposition 2.7. (Punctured Hemi-Elliptic Cores) *Let k be an algebraically closed field of characteristic 0; let X be a* punctured hemi-elliptic orbicurve *over k. Then if X is non-k-arithmetic, then X is a k-core. In particular, if X admits nontrivial automorphisms (over k), then X is k-arithmetic. Finally, there exist precisely 4 isomorphism classes of k-arithmetic X, which are described explicitly in [Take2], Theorem 4.1, (i).*

Proof. In the following discussion, we *omit* the "k-". Suppose that X is *non-arithmetic*. Write

$$Y \to X$$

for the unique *double (étale) covering by a punctured elliptic curve* Y, and

$$Y \to Z$$

for the unique morphism to the *core* (i.e., the terminal object in $\overline{\text{Loc}}_k(X) = \overline{\text{Loc}}_k(Y)$). Thus, the induced morphism

$$\overline{Y} = \overline{Y}^{\text{crs}} \to \overline{Z}^{\text{crs}}$$

on the *"coarse moduli spaces"* (cf. [FC], Chapter I, Theorem 4.10) associated to the *canonical compactifications* $\overline{Y}$, $\overline{Z}$ of the orbicurves Y, Z is a *finite ramified covering morphism* — whose degree we denote by d — from an elliptic curve $\overline{Y}$ to a copy of $\mathbb{P}^1_k \cong \overline{Z}^{\text{crs}}$. Note that since $\overline{Y}$ has only *one* "cusp" y_∞ (i.e. point $\in \overline{Y} \backslash Y$), and a point of $\overline{Y}$ is a cusp if and only if its image is a cusp in $\overline{Z}$, it follows that $\overline{Z}$ also has a *unique cusp* z_∞, and that y_∞ is the unique point of $\overline{Y}$ lying over z_∞. Moreover, because $\overline{Y} \to \overline{Z}^{\text{crs}}$ arises from a *finite étale* morphism $Y \to Z$, it follows that the ramification index of $\overline{Y} \to \overline{Z}^{\text{crs}}$ is the *same* at all points of $\overline{Y}$ lying over a given point of $\overline{Z}^{\text{crs}}$. Thus, applying the Riemann-Hurwitz formula yields:

$$0 = -2d + \sum_i \frac{d}{e_i}(e_i - 1)$$

where the e_i are the ramification indices over the points of $\overline{Z}^{\text{crs}}$ at which the covering morphism ramifies. Thus, we conclude that $2 = \sum_i \frac{1}{e_i}(e_i - 1)$. Since all of the e_i are *integers*, one verifies immediately that the only possibilities for the set of e_i's are the following:

$$(2,2,2,2);\ (2,3,6);\ (2,4,4);\ (3,3,3)$$

Note that it follows from the fact that y_∞ is the unique point of $\overline{Y}$ lying over z_∞ that d, as well as the ramification index at z_∞, is *necessarily equal to the largest* e_i. In the case of $(2,2,2,2)$, we thus conclude that $X = Z$, so X is a *core*, as asserted. In the other three cases, we conclude that Y is a *finite étale covering of the orbicurve determined by a "triangle group"* (cf. [Take1]) of one of the following types:

$$(2,3,\infty);\ (2,4,\infty);\ (3,3,\infty)$$

By [Take1], Theorem 3, (ii), such a triangle group is *arithmetic*, so X itself is *arithmetic*, thus contradicting our hypotheses. This completes the proof of the *first assertion* of Proposition 2.7.

The *second (respectively, third) assertion* of Proposition 2.7 is a formal consequence of the first assertion of Proposition 2.7 (respectively, [Take2], Theorem 4.1, (i)). ○

SECTION 3: HYPERBOLICALLY ORDINARY CANONICAL LIFTINGS

In this §, we would like to work over a *finite, unramified extension* K *of* $\mathbb{Q}_p$, where p is a *prime number* ≥ 5. We denote the ring of integers (respectively, residue field) of K by A (respectively, k). Since $A \cong W(k)$ (the ring of Witt vectors with coefficients in k), we have a natural *Frobenius morphism*

$$\Phi_A : A \to A$$

which lifts the Frobenius morphism $\Phi_k : k \to k$ on k. In the following discussion, the result of base-changing over A (respectively, A; A; $\mathbb{Z}_p$) with k (respectively, K; with A, via Φ_A; $\mathbb{Z}/p^n\mathbb{Z}$, for an integer $n \geq 1$) will be denoted by a *subscript* k (respectively, *subscript* K; *superscript* F; *subscript* $\mathbb{Z}/p^n\mathbb{Z}$).

Let

$$(X \to S \stackrel{\text{def}}{=} \text{Spec}(A),\ D \subseteq X)$$

be a *smooth, pointed curve of type* (g, r) (for which D is the divisor of marked points), where $2g-2+r > 0$ — cf. §0. In the following discussion, we would like to consider the extent to which (X, D) is a *canonical lifting* of (X_k, D_k), in the sense of [Mzk1], Chapter III, §3; Chapter IV. We refer also to the Introduction of [Mzk2] for a *survey of "p-adic Teichmüller theory"* (including the theory of [Mzk1]).

LEMMA 3.1. (CANONICALITY MODULO $\boldsymbol{p}^2$) *Suppose that*

$$Y_K \to X_K$$

is a finite ramified morphism of smooth, proper, geometrically connected curves over K *which is unramified away from* D_K. *Denote the reduced induced subscheme associated to the inverse image in* Y_K *of* D_K *by* $E_K \subseteq Y_K$. *Suppose further that the reduction*

$$Y_k \to X_k$$

modulo p *of the normalization* $Y \to X$ *of* X *in* Y_K *has the following form:*

(i) Y_k *is reduced;*

(ii) Y_k *is smooth over* k, *except for a total of precisely* $\frac{1}{2}(p-1)(2g-2+r)$ (≥ 2) *nodes. Moreover, the "order" of the deformation of each node determined by* Y *is equal to* 1 *(equivalently:* Y *is* REGULAR *at the nodes), and the special fiber* $E_k \subseteq Y_k$ *of the closure* $E \subseteq Y$ *of* E_K *in* Y *is a reduced divisor (equivalently: a divisor which is* ÉTALE *over* k*) at which* Y_k *is smooth.*

(iii) Y_k *has precisely two irreducible components* C_V, C_F. *Here, the morphism* $C_V \to X_k$ *(respectively,* $C_F \to X_k$*) is an isomorphism (respectively,* k*-isomorphic to the relative Frobenius morphism* $\Phi_{X_k/k} : X_k^F \to X_k$ *of* X_k*).*

(iv) (X_k, D_k) *admits a nilpotent ordinary indigenous bundle (cf. [Mzk1], Chapter II, Definitions 2.4, 3.1) whose supersingular divisor (cf. [Mzk1], Chapter II, Proposition 2.6, (3)) is equal to the image of the nodes of* Y_k *in* X_k.

Then (X, D) *is isomorphic modulo* p^2 *to the* CANONICAL LIFTING *(cf. [Mzk1], Chapter III, §3; Chapter IV) determined by the nilpotent indigenous bundle of (iv).*

REMARK 3.1.1. In the context of Lemma 3.1, we shall refer to the points of X_k which are the image of nodes of Y_k as *supersingular points* and to points which are not supersingular as *ordinary*. Moreover, the open subscheme of ordinary points will be denoted by

$$X_k^{\text{ord}} \subseteq X_k$$

and the corresponding *p-adic formal open subscheme* of $\widehat{X}$ (the p-adic completion of X) by $\widehat{X}^{\text{ord}}$. Also, we shall consider X (respectively, Y) to be equipped with the *log structure* (cf. [Kato] for an introduction to the theory of log structures) determined by the monoid of regular functions invertible on $X_K \backslash D_K$ (respectively, $Y_K \backslash E_K$) and denote the resulting *log scheme* by $X^{\log}$ (respectively, $Y^{\log}$). Thus, the morphism of schemes $Y \to X$ extends uniquely to a morphism of log schemes $Y^{\log} \to X^{\log}$.

Proof. First, let us observe that

$$(Y^{\log})^{\text{ord}}_k \cong \{(X^{\log})^{\text{ord}}_k\}^F \bigcup (X^{\log})^{\text{ord}}_k$$

where the isomorphism is the *unique* isomorphism lying over $X^{\log}_k$. Since $(X^{\log})^{\text{ord}}$, $(Y^{\log})^{\text{ord}}$ are *log smooth* over A, it follows that the inclusion $\{(X^{\log})^{\text{ord}}_k\}^F \hookrightarrow (Y^{\log})^{\text{ord}}_k$ lifts to a (not necessarily unique!) inclusion

$$\{(X^{\log})^{\text{ord}}_{\mathbb{Z}/p^2\mathbb{Z}}\}^F \hookrightarrow (Y^{\log})^{\text{ord}}_{\mathbb{Z}/p^2\mathbb{Z}}$$

whose *composite*

$$\Psi^{\log} : \{(X^{\log})^{\text{ord}}_{\mathbb{Z}/p^2\mathbb{Z}}\}^F \to (X^{\log})^{\text{ord}}_{\mathbb{Z}/p^2\mathbb{Z}}$$

with the natural morphism $(Y^{\log})^{\text{ord}}_{\mathbb{Z}/p^2\mathbb{Z}} \to (X^{\log})^{\text{ord}}_{\mathbb{Z}/p^2\mathbb{Z}}$ *is* nevertheless *independent of the choice of lifting of the inclusion.* Indeed, this is formal consequence of the fact that $\Psi^{\log}$ is a *lifting of the Frobenius morphism* on $(X^{\log})^{\text{ord}}_k$ (cf., e.g., the discussion of [Mzk1], Chapter II, the discussion preceding Proposition 1.2, as well as Remark 3.1.2 below).

Of course, $\Psi^{\log}$ might not be regular at the supersingular points, but we may estimate the *order of the poles of* $\Psi^{\log}$ at the supersingular points as follows: Since Y is assumed to be *regular*, it follows that the completion of $Y_{\mathbb{Z}/p^2\mathbb{Z}}$ at a supersingular point ν is given by the formal spectrum Spf of a complete local ring isomorphic to:

$$R_Y \stackrel{\text{def}}{=} (A/p^2 \cdot A)[[s,t]]/(st - p)$$

(where s, t are indeterminates). Thus, modulo p, this completion is a *node*, with the property that *precisely one* branch — i.e., irreducible component — of this node lies on C_F (respectively, C_V). (Indeed, this follows from the fact that both C_F and C_V are *smooth* over k.) Suppose that the irreducible component lying on C_F is defined locally (modulo p) by the equation $t = 0$. Thus, the ring of regular functions on the ordinary locus of C_F restricted to this formal neighborhood of a supersingular point is given by $k[[s]][s^{-1}]$. The connected component of $(Y^{\log})^{\mathrm{ord}}_{\mathbb{Z}/p^2\mathbb{Z}}$ determined by C_F may be thought of as the *open subscheme* "$s \neq 0$". Here, we recall that the parameter s in this discussion is uniquely determined *up to multiplication by an element of* $R_Y^{\times}$ — cf. [Mzk4], §3.7.

Next, let us write:

$$R'_X \stackrel{\mathrm{def}}{=} \mathrm{Im}(R_Y) \subseteq R_Y[s^{-1}]$$

for the image of R_Y in $R_Y[s^{-1}]$. Then since

$$t = s^{-1} \cdot p \in R_Y[s^{-1}] = (A/p^2 \cdot A)[[s]][s^{-1}]$$

it follows that $R'_X = (A/p^2 \cdot A)[[s]][s^{-1} \cdot p]$. In particular, if we think of

$$R_X \stackrel{\mathrm{def}}{=} (A/p^2 \cdot A)[[s]] \subseteq R'_X$$

— so $R_X[s^{-1}] = R'_X[s^{-1}] = R_Y[s^{-1}]$ — as a local *smooth lifting* of C_F at ν, we thus conclude that *arbitrary regular functions on Spf*(R_Y) *restrict to meromorphic functions on Spf*(R_X) *with poles of order* ≤ 1 *at* ν. Thus, since $\Psi^{\log}$ arises from an *everywhere regular* morphism $Y^{\log} \to X^{\log}$, we conclude that:

$\Psi^{\log}$ *has poles of order* ≤ 1 *at the supersingular points.*

But then the conclusion of Lemma 3.1 follows formally from [Mzk1], Chapter II, Proposition 2.6, (4); Chapter IV, Propositions 4.8, 4.10, Corollary 4.9. ○

Remark 3.1.2. The fact — cf. the end of the first paragraph of the proof of Lemma 3.1 — that the order of a pole of a Frobenius lifting is *independent* of the choice of smooth lifting of the domain of the Frobenius lifting may be understood more *explicitly* in terms of the coordinates used in the latter portion of the proof of Lemma 3.1 as follows: Any *"coordinate transformation"*

$$s \mapsto s + p \cdot g(s)$$

(where $g(s) \in k[[s]][s^{-1}]$) *fixes* — since we are working *modulo* p^2 — functions of the form $s^p + p \cdot f(s)$ (where $f(s) \in k[[s]][s^{-1}]$). This shows that the order of the pole of $f(s)$ does not depend on the choice of parameter s.

In the situation of Lemma 3.1, let us denote the *natural morphism of fundamental groups* (induced by $(\widehat{X}^{\log})^{\mathrm{ord}} \to X^{\log}$) by

$$\Pi_{(\widehat{X}^{\log})^{\mathrm{ord}}} \stackrel{\mathrm{def}}{=} \pi_1((\widehat{X}^{\log})^{\mathrm{ord}}_K) \ \to \ \Pi_{X^{\log}} \stackrel{\mathrm{def}}{=} \pi_1(X^{\log}_K) = \pi_1(X_K \backslash D_K)$$

(for some fixed choice of basepoints). Here, we observe that (by the main theorem of [Vala]) $(\widehat{X}^{\log})^{\mathrm{ord}}$ is *excellent*, so the *normalization* of $(\widehat{X}^{\log})^{\mathrm{ord}}$ in a finite étale covering of $(\widehat{X}^{\log})^{\mathrm{ord}}_K$ is *finite* over $(\widehat{X}^{\log})^{\mathrm{ord}}$. Thus, $(\widehat{X}^{\log})^{\mathrm{ord}}_K$ has a "well-behaved theory" of finite étale coverings which is *compatible with étale localization on* $(\widehat{X}^{\log})^{\mathrm{ord}}$. Also, before proceeding, we observe that $\Pi_{X^{\log}}$ fits into an exact sequence:

$$1 \to \Delta_{X^{\log}} \to \Pi_{X^{\log}} \to G_K \to 1$$

(where $\Delta_{X^{\log}}$ is defined so as to make the sequence exact).

Lemma 3.2. (The Ordinary Locus Modulo $\boldsymbol{p}^2$) *Let $V_{\mathbb{F}_p}$ be a 2-dimensional $\mathbb{F}_p$-vector space equipped with a continuous action of $\Pi_{X^{\log}}$ up to ± 1 — i.e., a representation*

$$\Pi_{X^{\log}} \to GL_2^{\pm}(V_{\mathbb{F}_p}) \stackrel{\mathrm{def}}{=} GL_2(V_{\mathbb{F}_p})/\{\pm 1\}$$

— such that:

(i) The determinant of $V_{\mathbb{F}_p}$ is isomorphic (as a $\Pi_{X^{\log}}$-module) to $\mathbb{F}_p(1)$.

(ii) There exists a finite log étale Galois covering (i.e., we assume tame ramification over D)

$$X^{\log}_{\pm} \to X^{\log}$$

such that the action of $\Pi_{X^{\log}}$ up to ± 1 on $V_{\mathbb{F}_p}$ lifts to a (usual) action (i.e., without sign ambiguities) of $\Pi_{X^{\log}_{\pm}} \subseteq \Pi_{X^{\log}}$ on $V_{\mathbb{F}_p}$. This action is uniquely determined up to tensor product with a character of $\Pi_{X^{\log}_{\pm}}$ of order 2.

(iii) The finite étale covering $Y^{\log}_K \to X^{\log}_K$ determined by the finite $\Pi_{X^{\log}}$-set of 1-dimensional $\mathbb{F}_p$-subspaces of $V_{\mathbb{F}_p}$ satisfies the hypotheses of Lemma 3.1.

(iv) Write

$$(Z^{\log}_{\pm})_K \to (X^{\log}_{\pm})_K$$

for the finite log étale covering (of degree $p^2 - 1$) corresponding to the nonzero portion of $V_{\mathbb{F}_p}$. Write

$$(Y^{\log}_{\pm})_K \to (X^{\log}_{\pm})_K$$

for the finite log étale covering of smooth curves which is the composite (i.e., normalization of the fiber product over $X^{\log}_K$) of the coverings $(Z^{\log}_{\pm})_K$, $Y^{\log}_K$

of $X_K^{\log}$. (Thus, $(Z_{\pm}^{\log})_K$ maps naturally to $(Y_{\pm}^{\log})_K$, hence also to $Y_K^{\log}$.) Let us refer to an irreducible component of the special fiber of a stable reduction of $(Y_{\pm}^{\log})_K$ (respectively, $(Z_{\pm}^{\log})_K$) over some finite extension of K that maps FINITELY *to the irreducible component "C_F" (cf. Lemma 3.1) as being* "OF C_F-TYPE" *and "associated to $(Y_{\pm}^{\log})_K$ (respectively, $(Z_{\pm}^{\log})_K$)". Then any irreducible component of C_F-type associated to $(Z_{\pm}^{\log})_K$ is* ÉTALE AND FREE OF NODES *over the* ORDINARY LOCUS *of any irreducible component of C_F-type associated to $(Y_{\pm}^{\log})_K$.*

Then (after possibly tensoring $V_{\mathbb{F}_p}$ with a character of $\Pi_{X_{\pm}^{\log}}$ of order 2) the étale local system $\mathcal{E}_{\mathbb{F}_p}^{\mathrm{ord}}$ on $(\widehat{X}_{\pm}^{\log})_K^{\mathrm{ord}}$ determined by the $\Pi_{(\widehat{X}_{\pm}^{\log})^{\mathrm{ord}}}$-module $V_{\mathbb{F}_p}$ arises from a (logarithmic) finite flat group scheme on $(\widehat{X}_{\pm}^{\log})^{\mathrm{ord}}$. Moreover, the $\Pi_{(\widehat{X}_{\pm}^{\log})^{\mathrm{ord}}}$-module $V_{\mathbb{F}_p}$ fits into an exact sequence:

$$0 \to (V_{\mathbb{F}_p}^{\mathrm{etl}})^{\vee}(1) \to V_{\mathbb{F}_p} \to V_{\mathbb{F}_p}^{\mathrm{etl}} \to 0$$

where $V_{\mathbb{F}_p}^{\mathrm{etl}}$ is a 1-dimensional $\mathbb{F}_p$-space "up to ± 1" whose $\Pi_{(\widehat{X}_{\pm}^{\log})^{\mathrm{ord}}}$-action arises from a finite étale local system on $(X_{\pm}^{\mathrm{ord}})_k \subseteq (X_{\pm})_k$, and the "$\vee$" denotes the $\mathbb{F}_p$-linear dual.

Proof. As was seen in the proof of Lemma 3.1, we have an *isomorphism*

$$(Y^{\log})_k^{\mathrm{ord}} \cong \{(X^{\log})_k^{\mathrm{ord}}\}^F \bigcup (X^{\log})_k^{\mathrm{ord}}$$

which thus determines a *decomposition of $(\widehat{Y}^{\log})^{\mathrm{ord}}$ into two connected components*. Moreover, the *second* connected component on the right-hand side corresponds to a *rank one quotient* $V_{\mathbb{F}_p} \twoheadrightarrow Q_{\mathbb{F}_p}$ which is *stabilized* by the action of $\Pi_{(\widehat{X}^{\log})^{\mathrm{ord}}}$, while the *first* connected component on the right-hand side parametrizes *splittings* of this quotient $V_{\mathbb{F}_p} \twoheadrightarrow Q_{\mathbb{F}_p}$. Here, we observe that $Q_{\mathbb{F}_p}^{\otimes 2}$ admits a *natural* $\Pi_{(\widehat{X}^{\log})^{\mathrm{ord}}}$*-action* (i.e., without sign ambiguities).

Now *any choice of isomorphism* between $\{(\widehat{X}^{\log})^{\mathrm{ord}}\}^F$ and the *first* connected component of $(\widehat{Y}^{\log})^{\mathrm{ord}}$ determines a *lifting of Frobenius*

$$\Phi^{\log} : \{(\widehat{X}^{\log})^{\mathrm{ord}}\}^F \to (\widehat{X}^{\log})^{\mathrm{ord}}$$

which is *ordinary* (by the conclusion of Lemma 3.1 — cf. [Mzk1], Chapter IV, Proposition 4.10). Thus, by the *general theory of ordinary Frobenius liftings*, $\Phi^{\log}$ determines, in particular, a *(logarithmic) finite flat group scheme* annihilated by p which is an *extension* of the trivial finite flat group scheme

"$\mathbb{F}_p$" by the finite flat group scheme determined by the Cartier dual of some étale local system of one-dimensional $\mathbb{F}_p$-vector spaces on X_k^{ord} (cf. [Mzk1], Chapter III, Definition 1.6); denote the $\Pi_{(\widehat{X}^{\log})^{\mathrm{ord}}}$-module corresponding to this étale local system by $\Omega_{\mathbb{F}_p}$. Moreover, it is a formal consequence of this general theory that $\Phi_K^{\log}$ is precisely the covering of $(\widehat{X}^{\log})_K^{\mathrm{ord}}$ determined by considering *splittings of this extension.* Thus, since the Galois closure of this covering has Galois group given by the *semi-direct product* of a cyclic group of order p with a cyclic group of order $p-1$, we conclude (by the elementary group theory of such a semi-direct product) that we have an *isomorphism of* $\Pi_{(\widehat{X}^{\log})^{\mathrm{ord}}}$*-modules*: $(Q_{\mathbb{F}_p})^{\otimes -2}(1) \cong \Omega_{\mathbb{F}_p}(1)$, i.e.,

$$(Q_{\mathbb{F}_p})^{\otimes -2} \cong \Omega_{\mathbb{F}_p}$$

We thus conclude that the local system $\mathcal{E}_{\mathbb{F}_p}^{\Phi^{\log}}$ on $(\widehat{X}^{\log})_K^{\mathrm{ord}}$ determined by this (logarithmic) finite flat group scheme arising from the general theory satisfies:

$$\mathcal{E}_{\mathbb{F}_p}^{\Phi^{\log}}|_{(\widehat{X}_{\pm}^{\log})_K^{\mathrm{ord}}} \cong \mathcal{E}_{\mathbb{F}_p}^{\mathrm{ord}} \otimes_{\mathbb{F}_p} Q_{\mathbb{F}_p}^{\vee}$$

Next, let us write χ_Q for the *character* (valued in $\mathbb{F}_p{}^{\times}$) of $\Pi_{(\widehat{X}_{\pm}^{\log})^{\mathrm{ord}}}$ corresponding to $Q_{\mathbb{F}_p}$. Now it follows formally from condition (iv) of the statement of Lemma 3.2 that the *finite étale covering of* $(\widehat{X}_{\pm}^{\log})_K^{\mathrm{ord}}$ *determined by* $\mathrm{Ker}(\chi_Q)$

$$W_Q \to (\widehat{X}_{\pm}^{\log})_K^{\mathrm{ord}}$$

is *dominated* by the composite of some *finite étale covering of* $\widehat{X}_{\pm}^{\mathrm{ord}}$ and a *"constant covering"* (i.e., a covering arising from a finite extension L of K). Thus, the only *ramification* that may occur in the covering W_Q arises from ramification of the "constant covering", i.e., the finite extension L/K. Moreover, (since $(Q_{\mathbb{F}_p})^{\otimes -2} \cong \Omega_{\mathbb{F}_p}$) the covering determined by $\mathrm{Ker}(\chi_Q^2)$ is *unramified*, so, in fact, we may take L to be the extension $K(p^{\frac{1}{2}})$. This implies that we may write

$$\chi_Q = \chi'_Q \cdot \chi''_Q$$

where the covering determined by the kernel of χ'_Q (respectively, χ''_Q) is finite étale over $\widehat{X}_{\pm}^{\mathrm{ord}}$ (respectively, the covering arising from base-change from K to L). On the other hand, since χ''_Q *extends naturally* to $\Pi_{X_{\pm}^{\log}}$, we may assume (without loss of generality — cf. condition (ii) of the statement of Lemma 3.2) that χ''_Q is *trivial*, hence that $Q_{\mathbb{F}_p}$ *arises from an étale local system on* $(X_{\pm}^{\mathrm{ord}})_k$.

But this — together with the isomorphism $\mathcal{E}^{\Phi^{\log}}_{\mathbb{F}_p}|_{(\widehat{X}^{\log}_{\pm})^{\mathrm{ord}}_K} \cong \mathcal{E}^{\mathrm{ord}}_{\mathbb{F}_p} \otimes_{\mathbb{F}_p} Q^{\vee}_{\mathbb{F}_p}$ — implies the conclusion of Lemma 3.2. ○

Lemma 3.3. (Global Logarithmic Finite Flat Group Scheme) *Let $V_{\mathbb{F}_p}$ be a 2-dimensional $\mathbb{F}_p$-vector space equipped with a continuous action of $\Pi_{X^{\log}}$ up to ± 1 which satisfies the hypotheses of Lemma 3.2. Then the étale local system $\mathcal{E}_{\mathbb{F}_p}$ on $(X^{\log}_{\pm})_K$ determined by the $\Pi_{X^{\log}_{\pm}}$-module $V_{\mathbb{F}_p}$ arises from a (logarithmic) finite flat group scheme on $X^{\log}_{\pm}$.*

Proof. Write

$$Z \to X_{\pm}$$

for the *normalization* of $X_{\pm}$ in the finite étale covering of $(X^{\log}_{\pm})_K$ determined by the local system $\mathcal{E}_{\mathbb{F}_p}$. Since $X_{\pm}$ is *regular of dimension* 2, it thus follows that Z is *finite and flat* (by the "Auslander-Buchsbaum formula" and "Serre's criterion for normality" — cf. [Mtmu], p. 114, p. 125) over $X_{\pm}$. Moreover, since Z_K is already equipped with a structure of (logarithmic) finite flat group scheme, which, by the conclusion of Lemma 3.2, extends naturally over the generic point of the special fiber of X — since it extends (by the proof of Lemma 3.2) to a *regular*, hence normal, (logarithmic) finite flat group scheme over the ordinary locus — it thus suffices to verify that this finite flat group scheme structure *extends (uniquely) over the supersingular points of* $X_{\pm}$. But this follows formally from [Mtmu], p. 124, Theorem 38 — i.e., the fact (applied to $X_{\pm}$, not Z!) that a meromorphic function on a normal scheme is regular if and only if it is regular at the primes of height 1 — and the fact that Z (hence also $Z \times_{X_{\pm}} Z$) is *finite and flat* over $X_{\pm}$. ○

Lemma 3.4. (The Associated Dieudonné Crystal Modulo $\boldsymbol{p}$) *Let $V_{\mathbb{F}_p}$ be a 2-dimensional $\mathbb{F}_p$-vector space equipped with a continuous action of $\Pi_{X^{\log}}$ up to ± 1 which satisfies the hypotheses of Lemma 3.2. Suppose, further, that the $\Pi_{X^{\log}}$-module (up to ± 1) $V_{\mathbb{F}_p}$ satisfies the following condition:*

$(\dagger_M)$ *The G_K-module*

$$M \stackrel{\mathrm{def}}{=} H^1(\Delta_{X^{\log}}, \mathrm{Ad}(V_{\mathbb{F}_p}))$$

(where $\mathrm{Ad}(V_{\mathbb{F}_p}) \subseteq \mathrm{End}(V_{\mathbb{F}_p})$ is the subspace of endomorphisms whose trace $= 0$) fits into an exact sequence:

$$0 \to \mathbb{G}^{-1}(M) \to M \to \mathbb{G}^2(M) \to 0$$

where $\mathbb{G}^2(M)$ (respectively, $\mathbb{G}^{-1}(M)$) is isomorphic to the result of tensoring an UNRAMIFIED *G_K-module whose dimension over $\mathbb{F}_p$ is equal to $3g - 3 + r$ with the Tate twist $\mathbb{F}_p(2)$ (respectively, $\mathbb{F}_p(-1)$).*

Then the "$\mathcal{MF}^{\nabla}$-object (up to ± 1)" (cf. [Falt], §2) determined by the $\Pi_{X^{\log}}$-module (up to ± 1) $V_{\mathbb{F}_p}$ (cf. Lemma 3.3) arises from the (unique) nilpotent ordinary indigenous bundle of Lemma 3.1, (iv).

Proof. First, we recall from the theory of [Falt], §2, that the conclusion of Lemma 3.3 implies that $V_{\mathbb{F}_p}$ arises from an *"$\mathcal{MF}^{\nabla}$-object (up to ± 1)"* on $X^{\log}$, as in the theory of [Falt], §2. Here, the reader uncomfortable with "$\mathcal{MF}^{\nabla}$-objects up to ± 1" may instead work with a usual $\mathcal{MF}^{\nabla}$-object over $X^{\log}_{\pm}$ equipped with an "action of $\mathrm{Gal}(X^{\log}_{\pm}/X^{\log})$ up to ± 1". Write $\mathcal{F}_k$ for the vector bundle on $(X_{\pm})_k$ underlying the $\mathcal{MF}^{\nabla}$-object on $X^{\log}_{\pm}$ determined by $\mathcal{E}_{\mathbb{F}_p}$.

Thus, $\mathcal{F}_k$ is a vector bundle of *rank* 2, whose Hodge filtration is given by a *subbundle* $F^1(\mathcal{F}_k) \subseteq \mathcal{F}_k$ *of rank* 1. Moreover, the *Kodaira-Spencer morphism* of this subbundle, as well as the *"Hasse invariant"*

$$\Phi^*_{X_{\pm}} F^1(\mathcal{F}_k)^{\vee} \hookrightarrow \mathcal{F}_k \twoheadrightarrow \mathcal{F}_k/F^1(\mathcal{F}_k) \cong F^1(\mathcal{F}_k)^{\vee}$$

(where $\Phi_{X_{\pm}}$ is the Frobenius morphism on $X_{\pm}$; and the injection is the morphism that arises from the Frobenius action on the $\mathcal{MF}^{\nabla}$-object in question), is *generically nonzero*. Indeed, these facts all follow immediately from our analysis of $\mathcal{E}_{\mathbb{F}_p}$ over the ordinary locus in the proof of Lemma 3.2.

Next, let us observe that *this Hasse invariant has at least one zero*. Indeed, if it were nonzero everywhere, it would follow formally from the general theory of $\mathcal{MF}^{\nabla}$-objects (cf. [Falt], §2) that the $\Pi_{X^{\log}}$-module (up to ± 1) $V_{\mathbb{F}_p}$ admits a $\Pi_{X^{\log}}$-*invariant subspace of* $\mathbb{F}_p$-*dimension* 1 — cf. the situation over the *ordinary locus* in the proof of Lemma 3.2, over which the Hasse invariant is, in fact, nonzero. On the other hand, this implies that $Y_K \to X_K$ admits a section, hence that Y is *not connected*. But this contradicts the fact (cf. the proof of Lemma 3.1) that the two irreducible components C_V, C_F of Y_k (cf. Lemma 3.1, (iii)) necessarily meet at the nodes of Y_k. (Here, we recall from Lemma 3.1, (ii), that there exists at least one node on Y_k.)

In particular, it follows from the fact that the Hasse invariant is generically nonzero, but not nonzero everywhere that *the degree of the line bundle $F^1(\mathcal{F})$ on $(X_{\pm})_k$ is positive.* Note that since $F^1(\mathcal{F})^{\otimes 2}$ descends naturally to a line bundle $\mathcal{L}$ on X_k, we thus obtain that

$$1 \leq \deg(\mathcal{L}) \leq 2g - 2 + r$$

(where the second inequality follows from the fact that the Kodaira-Spencer morphism in nonzero).

Now, we conclude — from the *p-adic Hodge theory* of [Falt], §2; [Falt], §5, Theorem 5.3 — that the condition ($\dagger_M$) implies various consequences concerning the *Hodge filtration on the first de Rham cohomology module* of Ad of the $\mathcal{MF}^\nabla$-object determined by $V_{\mathbb{F}_p}$, which may be summarized by the inequality:

$$h^1(X_k, \mathcal{L}^{-1}) = h^0(X_k, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_X) \geq 3g - 3 + r$$

(where "h^i" denotes the dimension over k of "H^i") — cf. [Mzk1], IV, Theorem 1.3 (and its proof), which, in essence, addresses the $\mathbb{Z}_p$ analogue of the present $\mathbb{F}_p$-vector space situation. (Note that here we make *essential use* of the hypothesis $p \geq 5$.) Thus, (cf. *loc. cit.*) we conclude that (since $\deg(\mathcal{L}) > 0$) the line bundle $\mathcal{L} \otimes_{\mathcal{O}_X} \omega_X$ on X_k is *nonspecial*, hence (by the above inequality) that:

$$\begin{aligned} \deg(\mathcal{L}) &= \deg(\mathcal{L} \otimes_{\mathcal{O}_X} \omega_X) - \deg(\omega_X) \\ &= h^0(X_k, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_X) + (g-1) - 2(g-1) \\ &\geq 3g - 3 + r - (g-1) = 2g - 2 + r \end{aligned}$$

Combining this with the inequalities of the preceding paragraph, we thus obtain that $\deg(\mathcal{L}) = 2g - 2 + r$, so the $\mathcal{MF}^\nabla$-object in question is an *indigenous bundle*, which is necessarily equal to the indigenous bundle of Lemma 3.1, (iv), since the supersingular locus of the former is contained in the supersingular locus of the latter (cf. [Mzk1], Chapter II, Proposition 2.6, (4)). ○

Lemma 3.5. (Canonical Deformations Modulo Higher Powers of $\boldsymbol{p}$) *Let $V_{\mathbb{F}_p}$ be a 2-dimensional $\mathbb{F}_p$-vector space equipped with a continuous action of $\Pi_{X^{\log}}$ up to ± 1 which satisfies the hypotheses of Lemmas 3.2, 3.4. Suppose that, for some $n \geq 1$:*

(i) (X, D) is isomorphic modulo p^n to a canonical lifting *(as in [Mzk1], Chapter III, §3; Chapter IV).*

(ii) $V_{\mathbb{F}_p}$ is the reduction modulo p of a rank 2 free $\mathbb{Z}/p^n\mathbb{Z}$-module $V_{\mathbb{Z}/p^n\mathbb{Z}}$ with continuous $\Pi_{X^{\log}}$-action up to ± 1.

Then (X, D) is isomorphic modulo p^{n+1} to a canonical lifting, *and the $\Pi_{X^{\log}}$-set $\mathbb{P}(V_{\mathbb{Z}/p^n\mathbb{Z}})$ (of free, rank one $\mathbb{Z}/p^n\mathbb{Z}$-module quotients of $V_{\mathbb{Z}/p^n\mathbb{Z}}$) is isomorphic to the projectivization of the* canonical representation *modulo p^n (cf. [Mzk1], Chapter IV, Theorem 1.1) associated to (X, D). Finally, if the determinant of $V_{\mathbb{Z}/p^n\mathbb{Z}}$ is isomorphic to $(\mathbb{Z}/p^n\mathbb{Z})(1)$, then the $\Pi_{X^{\log}}$-module (up to ± 1) $V_{\mathbb{Z}/p^n\mathbb{Z}}$ is isomorphic to the canonical representation modulo p^n.*

Proof. First, we observe that the case $n = 1$ is a formal consequence of Lemmas 3.1, 3.4. The case of general n is then, in essence, a *formal consequence of the theory of [Mzk1], Chapter V, §1* — cf. especially, Theorem 1.7, and the discussion following it. We review the details as follows:

The *space of deformations* (of the projectivization) of the $\Pi_{X^{\log}}$-module "up to ± 1" $V_{\mathbb{F}_p}$ may be thought of as a *formal scheme*

$$\mathcal{R}$$

which is noncanonically isomorphic to $\mathrm{Spf}(\mathbb{Z}_p[[t_1, \dots, t_{2(3g-3+r)}]])$ (where the t_i's are indeterminates) — cf. the discussion preceding [Mzk1], Chapter V, Lemma 1.5. Write $\mathcal{R}^{\mathrm{PD}}$ for the *p-adic completion of the PD-envelope of* $\mathcal{R}$ at the $\mathbb{F}_p$-valued point defined by "$V_{\mathbb{F}_p}$". Note that $\mathcal{R}$ and $\mathcal{R}^{\mathrm{PD}}$ are equipped with a *natural* G_K*-action.* Then according to the theory of *loc. cit.*, there is a G_K*-equivariant closed immersion of formal schemes* (cf. Remark 3.5.1 below)

$$\kappa^{\mathrm{PD}} : \mathrm{Spf}(\widehat{\mathcal{D}}^{\mathrm{Gal}}) \hookrightarrow \mathcal{R}^{\mathrm{PD}}$$

where $\widehat{\mathcal{D}}^{\mathrm{Gal}}$ is noncanonically isomorphic to the p-adic completion of the PD-envelope at the closed point of a power series ring $\mathbb{Z}_p[[t_1, \dots, t_{3g-3+r}]]$ equipped with a natural G_K-action.

Now it follows from the theory of *loc. cit.*, the induction hypothesis on n, and the assumptions (i), (ii) of Lemma 3.5, that $V_{\mathbb{Z}/p^{n-1}\mathbb{Z}} \stackrel{\mathrm{def}}{=} V_{\mathbb{Z}/p^n\mathbb{Z}} \otimes \mathbb{Z}/p^{n-1}\mathbb{Z}$ corresponds (at least projectively) to the *canonical representation modulo* p^{n-1}, hence determines a G_K*-invariant rational point*

$$\sigma_{n-1} \in \mathcal{R}^{\mathrm{PD}}(\mathbb{Z}/p^{n-1}\mathbb{Z})$$

that *lies in the image* $\mathrm{Im}(\kappa^{\mathrm{PD}})$ *of* κ^{PD}. Thus, the point

$$\sigma_n \in \mathcal{R}^{\mathrm{PD}}(\mathbb{Z}/p^n\mathbb{Z})$$

determined by $V_{\mathbb{Z}/p^n\mathbb{Z}}$ may be regarded as a G_K*-invariant deformation* of σ_{n-1}. Note that the set of deformations of σ_{n-1} naturally forms a *torsor* $\mathcal{T}$ *over the* $\mathbb{F}_p$*-vector space* M of Lemma 3.4. Since $\sigma_{n-1} \in \mathrm{Im}(\kappa^{\mathrm{PD}})$, this torsor is equipped with a natural G_K*-stable subspace*

$$\mathcal{T}'' \subseteq \mathcal{T}$$

(consisting of the deformations that lie $\in \mathrm{Im}(\kappa^{\mathrm{PD}})$) which is (by the theory of *loc. cit.*) a *torsor over* $\mathbb{G}^{-1}(M)$. In particular, this subspace determines a G_K*-invariant trivialization* τ' of the $\mathbb{G}^2(M)$-torsor

$$(\mathcal{T} \twoheadrightarrow)\ \mathcal{T}'$$

given by the "change of structure group $M \twoheadrightarrow \mathbb{G}^2(M)$".

Now let us observe that by ($\dagger_M$) and the fact that $p \geq 5$ — so the square of the cyclotomic character $G_K \to \mathbb{F}_p{}^\times$ is *nontrivial* — $\mathbb{G}^2(M)$ has no nontrivial Galois invariants, i.e.:

$$\mathbb{G}^2(M)^{G_K} = 0$$

Thus, we conclude that τ' is the *unique* G_K-invariant point of $\mathcal{T}'$, hence that the image in $\mathcal{T}'$ of the point $\tau_n \in \mathcal{T}$ determined by σ_n is necessarily equal to τ', i.e., $\tau_n \in \mathcal{T}''$ — or, in other words, $\sigma_n \in \mathrm{Im}(\kappa^{\mathrm{PD}})$.

On the other hand, if we interpret the *Galois-theoretic fact* that σ_{n-1} lifts to a *G_K-invariant $\sigma_n \in \mathrm{Im}(\kappa^{\mathrm{PD}})$* in terms of the *original finite étale coverings* — combinatorial information concerning which the Galois theory is intended to encode — then we obtain the following conclusion: The *A-valued point*

$$\alpha \in \mathcal{N}(A)$$

of the $\mathbb{Z}_p$-smooth p-adic formal scheme $\mathcal{N} \stackrel{\mathrm{def}}{=} \mathcal{N}^{\mathrm{ord}}_{g,r}$ (cf. [Mzk1], Chapter III, §2) determined by (X, D) and the nilpotent ordinary indigenous bundle modulo p of Lemma 3.4 not only lies — by assumption (i) of the statement of Lemma 3.5 — in the image of $\mathcal{N}(A)$ under the $(n-1)$-st iterate $\Phi^{n-1}_{\mathcal{N}}$ of the *canonical Frobenius lifting*

$$\Phi_{\mathcal{N}} : \mathcal{N} \to \mathcal{N}$$

on (cf. [Mzk1], Chapter III, Theorem 2.8) but also in the image of $\mathcal{N}(A)$ under the *n-th iterate $\Phi^n_{\mathcal{N}}$ of $\Phi_{\mathcal{N}}$*. (Indeed, the restriction of the covering $\Phi^{n-1}_{\mathcal{N}}$ (respectively, $\Phi^n_{\mathcal{N}}$) to α *admits a section*, determined by the G_K-invariant rational point σ_{n-1} (respectively, σ_n).) Thus, we conclude that $X^{\log}$ is *canonical modulo p^{n+1}*. Finally, since

$$\mathbb{G}^{-1}(M)^{G_K} = 0$$

we conclude that σ_n is the *unique G_K-invariant lifting* of σ_{n-1} to $\mathbb{Z}/p^n\mathbb{Z}$, hence that $V_{\mathbb{Z}/p^n\mathbb{Z}}$ corresponds (projectively) to the *canonical representation modulo p^n*, as desired. ○

Remark 3.5.1. In some sense, it is natural to think of $\mathrm{Im}(\kappa^{\mathrm{PD}})$ (cf. the proof of Lemma 3.5) as the *"crystalline locus"* in the space of "all" representations $\mathcal{R}^{\mathrm{PD}}$.

THEOREM 3.6. (GROUP-THEORETICITY OF CANONICAL LIFTINGS) *Let $p \geq 5$ be a prime number. For $i = 1, 2$, let K_i be an* UNRAMIFIED *finite extension of $\mathbb{Q}_p$, and (X_i, D_i) a* SMOOTH POINTED CURVE *of type (g_i, r_i) over $\mathcal{O}_{K_i}$, where $2g_i - 2 + r_i > 0$. Assume that we have chosen basepoints of the $(X_i)_{K_i} \backslash (D_i)_{K_i}$ (which thus induce basepoints of the K_i); denote the resulting fundamental group by $\Pi_{X_i^{\log}}$. Suppose that we have been given an* ISOMORPHISM OF PROFINITE GROUPS*:*

$$\Pi_{X_1^{\log}} \xrightarrow{\sim} \Pi_{X_2^{\log}}$$

Then:

(i) (X_1, D_1) is a CANONICAL LIFTING *(in the sense of [Mzk1], Chapter III, §3; Chapter IV) if and only if (X_2, D_2) is so.*

(ii) If at least one of the (X_i, D_i) is a canonical lifting, then the isomorphism on logarithmic special fibers of [Mzk7], Theorem 2.7, LIFTS *(uniquely) to an isomorphism $(X_1, D_1) \xrightarrow{\sim} (X_2, D_2)$ over $\mathcal{O}_{K_1} \cong W(k_1) \xrightarrow{\sim} W(k_2) \cong \mathcal{O}_{K_2}$.*

Proof. Let us verify (i). Since, by [Mzk7], Theorem 2.7 (and [Mzk7], Proposition 1.2.1, (vi)), the logarithmic special fibers of all stable reductions of all finite étale coverings of $(X_i)_{K_i} \backslash (D_i)_{K_i}$ are *"group-theoretic"*, it follows that the conditions (i), (ii), (iii), (iv) of Lemma 3.1, as well as the conditions (i), (ii), (iii), (iv) of Lemma 3.2, are all *group-theoretic* conditions which, moreover, (by the theory of [Mzk1], Chapter III, §3; Chapter IV) are satisfied by canonical liftings. (Here, relative to the assertion that canonical liftings satisfy Lemma 3.1, (ii), and Lemma 3.2, (iv), we remind the reader that:

(1) Since $p \geq 5$, the special fiber of the curve Y of Lemma 3.1 has ≥ 2 nodes (cf. Lemma 3.1, (ii)), which implies that the curve Y is *stable* (i.e., even without the marked points).

(2) The *smooth locus* of *any* model of a curve over a discrete valuation ring *necessarily* maps to the *stable model* (whenever it exists) of the curve — cf., e.g., [JO]— and, moreover, whenever this map is *quasi-finite*, necessarily embeds as an *open subscheme of the smooth locus* of the stable model.)

Moreover, (since the cyclotomic character and inertia subgroup are group-theoretic — cf. [Mzk7], Proposition 1.2.1, (ii), (vi) — it follows that) condition $(\dagger_M)$ of Lemma 3.4 is a *group-theoretic* condition which (by the theory of *loc. cit.*) is satisfied by canonical liftings. Thus, successive application of Lemma 3.5 for $n = 1, 2, \ldots$ implies assertion (i) of the statement of Theorem 3.6.

Next, let us observe that when one (hence both) of the (X_i, D_i) is a canonical lifting, it follows from Lemma 3.1 that the isomorphism of special fibers

$$(X_1, D_1)_{k_1} \xrightarrow{\sim} (X_2, D_2)_{k_2}$$

(lying over some isomorphism $k_1 \stackrel{\sim}{\to} k_2$) determined by [Mzk7], Theorem 2.7 is compatible with the nilpotent ordinary indigenous bundles on either side that give rise to the canonical liftings. On the other hand, by the theory of *loc. cit.*, the lifting of $(X_i, D_i)_{k_i}$ over $\mathcal{O}_{K_i} \cong W(k_i)$ is determined uniquely by the fact that this lifting is "canonical" (i.e., when it is indeed the case that it is canonical!). This completes the proof of assertion (ii) of the statement of Theorem 3.6. ○

Remark 3.6.1. Thus, (cf. Remarks 1.1.1, 2.2.1) unlike the situation with Proposition 1.1, Theorem 2.2:

Theorem 3.6 provides, for each hyperbolic (g, r), *"lots of examples"* — in particular, an example lifting a general curve of type (g, r) in characteristic $p \geq 5$ — of (X_K, D_K) which are *group-theoretically determined solely by the profinite group* $\pi_1(X_K \backslash D_K)$.

In fact, the exact same arguments of the present §3 show that the *analogue of Theorem 3.6 also holds for "Lubin-Tate canonical curves (for various tones* ϖ*)"* — i.e., the curves defined by considering *canonical points* (cf. [Mzk2], Chapter VIII, §1.1) associated to the *canonical Frobenius lifting* of [Mzk2], Chapter VIII, Theorem 3.1, in the case of a "VF-pattern Π of pure tone ϖ".

One feature of the Lubin-Tate case that *differs* (cf. [Mzk7], Proposition 1.2.1, (vi)) from the "classical ordinary" case of [Mzk1] is that it is not immediately clear that the *"Lubin-Tate character"*

$$\chi : G_K \to W(\mathbb{F}_q)^\times$$

(where $\mathbb{F}_q$ is a finite extension of $\mathbb{F}_p$) is *group-theoretic.* Note, however, that at least for the *"portion of* χ *modulo* p*"*

$$G_K \to \mathbb{F}_q{}^\times$$

group-theoreticity — at least after passing to a *finite unramified extension* of the original base field K (which does not present any problems, from the point of view of proving the Lubin-Tate analogue of Theorem 3.6) — is a consequence of the fact that the *field structure* on (the union of 0 and) the torsion in G_K^{ab} of order prime to p is group-theoretic (cf. [Mzk7], Lemma 2.6, Theorem 2.7). This much is *sufficient for the Lubin-Tate analogues* of Lemmas 3.1, 3.2, 3.3, 3.4, 3.5 (except for the last sentence of the statement of Lemma 3.5, which is, at any rate, not necessary to prove Theorem 3.6), and hence of Theorem 3.6. Finally, once one has proved that the curve in question is (Lubin-Tate) *canonical* and recovered its canonical representation $V_{\mathbb{Z}_p}$, at least *projectively*, then it follows, by considering the $W(\mathbb{F}_q)$-module analogue of the exact sequence of $\mathbb{F}_p$-vector spaces in the conclusion of Lemma 3.2, over the ordinary locus, that (even if one only knows $V_{\mathbb{Z}_p}$ "projectively") one may *recover the Lubin-Tate character* by forming

$$\text{Hom(third nonzero term of the exact sequence,} \\ \text{first nonzero term of the exact sequence)}$$

— at least *up to un "étale twist"*, i.e., up to multiplication by some *unramified* character $G_K \to W(\mathbb{F}_q)^\times$. (We leave the remaining routine technical details of the Lubin-Tate case to the enthusiastic reader.) At the time of writing, it is not clear to the author whether or not it is possible to eliminate this "étale twist". Since this étale twist corresponds to the *well-known dependence of the Lubin-Tate group on the choice of uniformizer*, this indeterminacy with respect to an étale twist may be thought of as being related to the fact that (at the present time) the author is unable to prove the *group-theoreticity of the natural uniformizer "p"* — cf. [Mzk7], Remark 2.7.2.

At any rate, the theory of the present §3 constitutes the case of "tone 0". Moreover, one checks easily — by considering the "FL-bundle" (as in [Mzk1], Chapter II, Proposition 1.2) determined by the lifting modulo p^2 — that canonical curves of distinct tones are never isomorphic. Thus, (at least when $r = 0$) the Lubin-Tate canonical curves give rise to (strictly) more examples of (proper) hyperbolic X_K which are group-theoretically determined solely by the profinite group $\pi_1(X_K)$. This prompts the following interesting question: Is this "list" complete? — i.e.:

Do there exist any other curves (X_K, D_K) that are group-theoretically determined solely by the profinite group $\pi_1(X_K \backslash D_K)$?

At the time of writing, the author does not even have a conjectural answer to this question.

Remark 3.6.2. One interesting aspect of the theory of the present §3 is that, to the knowledge of the author:

It constitutes the first *application* of the "p-adic Teichmüller theory" of [Mzk1], [Mzk2], to prove a *hitherto unknown result* (cf. Corollary 3.8 below) that lies *outside* — i.e., can be *stated* without using the terminology, concepts, or results of — the theory of [Mzk1], [Mzk2].

Moreover, not only does this constitute the first application of "p-adic Teichmüller theory" to prove a *new* result (cf. the proof of the irreducibility of the moduli stack via p-adic Teichmüller theory in [Mzk2], Chapter III, §2.5

— an application, albeit to an *old* result), it is interesting relative to the *original philosophical motivation* for this theory, involving the analogy to *uniformization theory/Teichmüller theory over the complex numbers*, which was to construct a p-adic theory that would allow one to prove a *"p-adic version of the Grothendieck Conjecture"* as in [Mzk6] — cf. [Mzk2], Introduction, §0.10, for more on these ideas.

REMARK 3.6.3. One interesting point of view is the following:

For a hyperbolic curve X_K over a finite extension K of $\mathbb{Q}_p$, consideration of the profinite group Π_{X_K} should be thought of as the *arithmetic analogue* of considering a hyperbolic curve (which is given *a priori*) over $\mathbb{F}_p[[t]][t^{-1}]$ — where t is an indeterminate which, perhaps, should be thought of as the *symbol "p"* — "IN THE ABSOLUTE", i.e., "stripped of its structure morphism to a *specific* copy of $\mathbb{F}_p[[t]][t^{-1}]$", or, alternatively, "when we allow the indeterminate t to *vary freely* (in $\mathbb{F}_p[[t]]$)".

Thus, from this point of view, it is *natural* to expect that the hyperbolic curves X_K most likely to be recoverable from the absolute datum Π_{X_K} are those which are "defined over some (fictitious) ABSOLUTE FIELD OF CONSTANTS inside K" — hence which have moduli that are *invariant* with respect to changes of variable $t \mapsto t + t^2 + \ldots$. In particular, since one expects *canonical curves* to be arithmetic analogues of "curves defined over the constant field" (cf. [Mzk2], Introduction, §2.3), it is perhaps not surprising that they should satisfy the property of Theorem 3.6.

Moreover, this point of view suggests that:

Perhaps it is natural to regard Theorem 3.6 as the proper analogue for hyperbolic canonical curves of the fact that Serre-Tate canonical liftings (of abelian varieties) are defined over number fields (cf. [Mzk2], Introduction, §2.1, Open Question (7)).

Indeed, one way of showing that Serre-Tate canonical liftings — or, indeed, arbitrary abelian varieties with lots of endomorphisms — are defined over number fields is by thinking of the algebraic extension $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ over which such abelian varieties are a priori defined *"in the absolute"*, i.e., as a *transcendental extension of* $\mathbb{Q}$ and considering what happens when one transports such abelian varieties via *arbitrary field automorphisms of* $\overline{\mathbb{Q}}_p$. Such field automorphisms are reminiscent of the *"changes of variable"* appearing in the approach to thinking about "recovering X_K from the absolute datum Π_{X_K}" described above.

REMARK 3.6.4. The *"rigidity"* of canonical curves — in the sense that they are determined by the existence of the *unique* G_K-invariant lifting "$V_{\mathbb{Z}_p}$" of the representation $V_{\mathbb{F}_p}$ (cf. Lemma 3.5 and its proof) — is reminiscent, at least at a technical level, of the theory of deformations of representations applied in *Wiles' famous proof of the "modularity conjecture"* (cf. [Wiles]). It would be interesting if this analogy could be pursued in more detail in the future.

REMARK 3.6.5. Just as the Serre-Tate canonical coordinates are used in [Mzk5], §9, to prove a *weak p-adic Grothendieck Conjecture-type result* for hyperbolic curves over p-adic fields whose Jacobians have ordinary reduction, the techniques of the present § may be applied — by using the *canonical coordinates* of [Mzk1]— to prove a *similar (but in some sense even weaker) p-adic Grothendieck Conjecture-type result* for hyperbolic curves over absolutely unramified p-adic fields which are isomorphic to *canonical curves (as in [Mzk1]) modulo* p^2. Thus, the true significance of the theory of the present § lies in its *wide applicability in the canonical lifting case* (cf. Corollary 3.8 below), a feature which differs substantially from the theory in the case of ordinary Jacobians (cf. Remark 1.1.1).

DEFINITION 3.7. Let Y_L be a hyperbolic curve over a finite extension L of $\mathbb{Q}_p$. Then we shall say that Y_L is *absolute* if for every other hyperbolic curve $Y'_{L'}$ over a finite extension L' of $\mathbb{Q}_p$,

$$\pi_1(Y_L) \cong \pi_1(Y'_{L'})$$

(as profinite groups) implies that $Y'_{L'}$ is isomorphic as a $\mathbb{Q}_p$-scheme to Y_L. Also, we shall refer to points in moduli stacks of hyperbolic curves over $\mathbb{Q}_p$ that are defined by absolute hyperbolic curves as *absolute.*

COROLLARY 3.8. (APPLICATION OF ***p***-ADIC TEICHMÜLLER THEORY)

(i) A general pointed smooth curve (X_k, D_k) *of type* (g, r)*, where* $2g-2+r > 0$*, over a finite field* k *of characteristic* $p \geq 5$ *may be lifted to a pointed smooth curve* (X_K, D_K) *over the quotient field* K *of the ring of Witt vectors* $A = W(k)$ *such that the hyperbolic curve* $X_K \backslash D_K$ *is* ABSOLUTE.

(ii) In particular, for each (g, r)*,* $p \geq 5$*, there exists a* ZARISKI DENSE *— hence (at least when* $3g-3+r \geq 1$*) infinite — set of* ABSOLUTE POINTS*, valued in absolutely unramified finite extensions of* $\mathbb{Q}_p$*, of the moduli stack of hyperbolic curves of type* (g, r) *over* $\mathbb{Q}_p$*.*

Proof. Assertion (i) follows formally from Theorem 3.6; [Mzk7], Lemmas 1.1.4, 1.1.5; and [Mzk7], Proposition 1.2.1, (v). Assertion (ii) follows formally from assertion (i) and the following elementary argument: The scheme-theoretic closure of the points of assertion (i) in the (compactified) moduli stack over $\mathbb{Z}_p$ forms a $\mathbb{Z}_p$- *flat* proper algebraic stack Z with the property that $Z \otimes \mathbb{F}_p$ is equal to the entire moduli stack over $\mathbb{F}_p$, hence is smooth of dimension $3g-3+r$. But this implies — by $\mathbb{Z}_p$-*flatness* — that $Z \otimes \mathbb{Q}_p$ is also of dimension $3g-3+r$, hence equal to the entire moduli stack over $\mathbb{Q}_p$, as desired. ◯

BIBLIOGRAPHY

Unpublished RIMS preprints are available as .ps files at:

`http://www.kurims.kyoto-u.ac.jp/~kenkyubu/paper/all.html`

[DM] P. Deligne and D. Mumford, The Irreducibility of the Moduli Space of Curves of Given Genus, *IHES Publ. Math.* 36 (1969), pp. 75-109.

[DO] B. Dwork and A. Ogus, Canonical liftings of Jacobians, *Compositio Math.* 58 (1986), pp. 111-131.

[Edix] B. Edixhoven, On the André-Oort conjecture for Hilbert modular surfaces, *Moduli of abelian varieties (Texel Island, 1999)*, *Progr. Math.* 195, Birkhäuser (2001), pp. 133-155.

[Falt] G. Faltings, Crystalline Cohomology and p-adic Galois Representations, *Proceedings of the First JAMI Conference*, Johns Hopkins University Press (1990), pp. 25-79.

[FC] G. Faltings and C.-L. Chai, *Degenerations of Abelian Varieties*, Springer-Verlag (1990).

[Groth] A. Grothendieck, Letter to G. Faltings (June 1983) in Lochak, L. Schneps, *Geometric Galois Actions; 1. Around Grothendieck's Esquisse d'un Programme*, *London Math. Soc. Lect. Note Ser.* 242, Cambridge Univ. Press (1997).

[JO] A. J. de Jong and F. Oort, On Extending Families of Curves, *Journal of Alg. Geom.* 6 (1997), pp. 545-562.

[Kato] K. Kato, Logarithmic Structures of Fontaine-Illusie, *Proceedings of the First JAMI Conference*, Johns Hopkins University Press (1990), pp. 191-224.

[Knud] F. F. Knudsen, The Projectivity of the Moduli Space of Stable Curves, II, *Math. Scand.* 52 (1983), 161-199.

[Mtmu] H. Matsumura, *Commutative Algebra (Second Edition)*, The Benjamin/Cummings Publishing Company (1980).

[Mess] W. Messing, *The Crystals Associated to Barsotti-Tate Groups; with Applications to Abelian Schemes*, *Lecture Notes in Mathematics* 264, Springer-Verlag (1972).

[Mzk1] S. Mochizuki, A Theory of Ordinary p-adic Curves, *Publ. of RIMS* 32 (1996), pp. 957-1151.

[Mzk2] S. Mochizuki, *Foundations of p-adic Teichmüller Theory*, AMS/IP Studies in Advanced Mathematics 11, American Mathematical Society/International Press (1999).

[Mzk3] S. Mochizuki, Correspondences on Hyperbolic Curves, *Journ. Pure Appl. Algebra* 131 (1998), pp. 227-244.

[Mzk4] S. Mochizuki, The Geometry of the Compactification of the Hurwitz Scheme, *Publ. of RIMS* 31 (1995), pp. 355-441.

[Mzk5] S. Mochizuki, The Profinite Grothendieck Conjecture for Closed Hyperbolic Curves over Number Fields, *J. Math. Sci., Univ. Tokyo* 3 (1996), pp. 571-627.

[Mzk6] S. Mochizuki, The Local Pro-p Anabelian Geometry of Curves, *Invent. Math.* 138 (1999), pp. 319-423.

[Mzk7] S. Mochizuki, *The Absolute Anabelian Geometry of Hyperbolic Curves*, RIMS Preprint No. 1363 (June 2002).

[NTM] H. Nakamura, A. Tamagawa, and S. Mochizuki, The Grothendieck Conjecture on the Fundamental Groups of Algebraic Curves, *Sugaku Expositions* 14 (2001), pp. 31-53.

[OS] F. Oort and T. Sekiguchi, The canonical lifting of an ordinary Jacobian variety need not be a Jacobian variety, *J. Math. Soc. Japan* 38 (1986), pp. 427-437.

[SGA1] *Revêtement étales et groupe fondamental*, Séminaire de Géometrie Algébrique du Bois Marie 1960-1961 (SGA1), dirigé par A. Grothendieck, augmenté de deux exposés de M. Raynaud, *Lecture Notes in Mathematics* 224, Springer-Verlag (1971).

[Take1] K. Takeuchi, Arithmetic Triangle Groups, *Journ. Math. Soc. Japan* 29 (1977), pp. 91-106.

[Take2] K. Takeuchi, Arithmetic Fuchsian Groups with Signature $(1; e)$, *Journ. Math. Soc. Japan* 35 (1983), pp. 381-407.

[Tama] A. Tamagawa, The Grothendieck Conjecture for Affine Curves, *Compositio Math.* 109 (1997), pp. 135-194.

[Vala] P. Valabrega, A Few Theorems on Completion of Excellent Rings, *Nagoya Math J.* 61 (1976), pp. 127-133.

[Wiles] A. Wiles, Modular elliptic curves and Fermat's last theorem, *Ann. of Math.* 141 (1995), pp. 443-551.

Shinichi Mochizuki
Research Institute
for Mathematical Sciences
Kyoto University
Kyoto 606-8502, Japan
motizuki@kurims.kyoto-
u.ac.jp

Refinement of Tate's Discriminant Bound and Non-Existence Theorems for Mod p Galois Representations

Dedicated to Professor Kazuya Kato
on the occasion of his fiftieth birthday

Hyunsuk Moon[1] and Yuichiro Taguchi

Received: November 30, 2002
Revised: February 4, 2003

Abstract. Non-existence is proved of certain continuous irreducible mod p representations of degree 2 of the absolute Galois group of the rational number field. This extends previously known results, the improvement based on a refinement of Tate's discriminant bound.

2000 Mathematics Subject Classification: 11F80, 11R29, 11R39
Keywords and Phrases: Mod p Galois Representation, Discriminant

Introduction. Let $G_{\mathbb{Q}}$ be the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of the rational number field $\mathbb{Q}$, and $\overline{\mathbb{F}}_p$ an algebraic closure of the prime field $\mathbb{F}_p$ of p elements. In this paper, we are motivated by Serre's conjecture [19] to prove that there exists no continuous irreducible representation $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ unramified outside p for $p \leq 31$ and with small Serre weight k. This extends the previous works by Tate [21], Serre [18], Brueggeman [1], Fontaine [5], Joshi [6] and Moon [11], [12]. Our main result is:

Theorem 1. *There exists no continuous irreducible representation $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ which is unramified outside p and of reduced Serre weight k (cf. Sect. 1) in the following cases marked with $\times$, and the same is true if we assume the Generalized Riemann Hypothesis (GRH) in the following cases marked with $\times_{\mathrm{R}}$:*

[1]The first author was supported by the JSPS Postdoctoral Fellowship for Foreign Researchers.

$k\backslash p$	2	3	5	7	11	13	17	19	23	29	31
2	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times_{\mathrm{R}}$	$\times_{\mathrm{R}}$	$\times_{\mathrm{R}}$
3	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	f	f	f
4		$\times$	$\times$	$\times$	$\times_{\mathrm{R}}$	$\times_{\mathrm{R}}$	$\times_{\mathrm{R}}$	$\times_{\mathrm{R}}$	$\times_{\mathrm{R}}$	$\mathrm{f_R}$	$\mathrm{f_R}$
5			$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	f	f	f
6			$\times_{\mathrm{R}}$	$\times_{\mathrm{R}}$	$\times_{\mathrm{R}}$	$\times_{\mathrm{R}}$	$\mathrm{f_R}$	$\mathrm{f_R}$	?	?	?
7				$\times$	$\times$	$\times$	$\times$	$\times$	f	f	$\mathrm{f_R}$
8				?	?	?	?	?	?	?	?
9					$\times_{\mathrm{R}}$	$\times_{\mathrm{R}}$	$\times_{\mathrm{R}}$	$\times_{\mathrm{R}}$	$\mathrm{f_R}$	$\mathrm{f_R}$	$\mathrm{f_R}$
10					?	?	?	?	?	?	?
11					$\times_{\mathrm{R}}$	$\times_{\mathrm{R}}$	$\times_{\mathrm{R}}$	$\times_{\mathrm{R}}$	$\mathrm{f_R}$	$\mathrm{f_R}$	$\mathrm{f_R}$
12					$\exists$	$\exists$	$\exists$	$\exists$	?	$\exists$	$\exists$
13						$\mathrm{f_R}$	$\times_{\mathrm{R}}$	$\times_{\mathrm{R}}$	$\mathrm{f_R}$	$\mathrm{f_R}$	$\mathrm{f_R}$
14						?	?	?	?	?	?
15							$\mathrm{f_R}$	$\mathrm{f_R}$	$\mathrm{f_R}$	$\mathrm{f_R}$	$\mathrm{f_R}$
16							$\exists$	$\exists$	$\exists$	$\exists$	?
17							?	?	?	$\mathrm{f_R}$	$\mathrm{f_R}$
18							$\exists$	$\exists$	$\exists$	$\exists$	$\exists$
19								?	?	?	?
20								$\exists$	$\exists$	$\exists$	$\exists$

In this table, an f *(resp.* $\mathrm{f_R}$*) means that, unconditionally (resp. under the GRH), there exist only finitely many* ρ *in that case, and an* $\exists$ *means that there does exist an irreducible representation in that case. A ? means that the non-existence/finiteness is unknown (at present) in that case.*

Note that the reduced Serre weight takes values $1 \leq k \leq p+1$; the table can be continued further down to $k = 32$ in an obvious manner (with many ?'s and some $\exists$'s). The case $k = 1$ of the Theorem is trivial since $k = 1$ means that ρ is unramified at p. In the above table, the cases $p = 2, 3, 5$ are proved respectively in [21], [18], [1]. The case where $p = 7$ and ρ is even (i.e. k is odd) is proved in [12]. For $k = 2$ and $p \leq 17$, Fontaine [5] proved the non-existence of certain types of finite flat group schemes (not just of two-dimensional Galois representations). Joshi [6] proved the non-existence of ρ for $p \leq 13$ and of Hodge-Tate weight $1, 2$ (instead of Serre weight $2, 3$; presumably, one has $k-1$ = the Hodge-Tate weight in the sense of Joshi if the Serre weight k satisfies $1 \leq k \leq p-1$). The representations marked with $\exists$ are provided by cusp forms (mod p) of weight 12, 16, 18, 20 and level 1 (cf. [16], §3.3–3.5).

As a corollary, it follows from this theorem that, under the GRH and for $3 \leq p \leq 31$, (i) any finite flat group scheme over $\mathbb{Z}$ of type (p, p) is the direct sum of two group schemes which are isomorphic to $\mathbb{Z}/p\mathbb{Z}$ or μ_p (cf. [19], Théorème 3); and (ii) any p-divisible group over $\mathbb{Z}$ of height 2 is the direct sum of two p-divisible groups which are either constant or multiplicative (cf. [5], Théorème 4 and its Corollaries).

Our strategy in the proof is basically the same as in the above cited works; to deduce contradiction by comparing two kinds of inequalities of the opposite direction for the discriminant of the field corresponding to the kernel of ρ — one from above (the Tate bound), and the other from below (the Odlyzko bound). The novelty in this paper is in the refinement of the Tate bound (Theorem 3), which gives the precise value of the discriminant in terms of the reduced Serre weight $\tilde{k}(\rho)$ of ρ. This is done in Section 1. In Section 2, we compare this with the Odlyzko bound ([14] and [15]) to prove the above Theorem. To deal with the case where ρ is odd and has solvable image, we use the fact that Serre's conjecture is true for such ρ if $p \geq 3$ ([7]).

Another interesting case to consider is where the representation ρ has Serre weight 1 (i.e. unramified at p) and non-trivial Artin conductor outside p. Although a mod p modular form in Katz' sense lifts to a classical one of the same weight in most cases if the weight is ≥ 2, this may not be the case for weight 1 forms (Lemma 1.9 of [4]). If this is the case and Serre's conjecture is true, then an odd and irreducible ρ of Serre weight 1 is put under a severe restraint on its image. Indeed, if ρ comes from a mod p eigenform f which lifts to a classical eigenform F of weight 1, then ρ has also to lift to an Artin representation $\rho_F : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C})$ associated to F ([2]). In particular, in such a case, an irreducible ρ cannot have image of order divisible by p (or equivalently, its projective image cannot contain a subgroup isomorphic to $\mathrm{PSL}_2(\mathbb{F}_p)$) if $p \geq 5$. Conversely, if there are no such representations $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ and if the Artin conjecture is true, then any mod p eigenform of weight 1 lifts, at least "outside the level", to a classical eigenform of weight 1. In this vein, we prove:

THEOREM 2. *Assume the GRH. Then for each prime $p \geq 5$, there exists a positive integer N_p such that there exists no continuous representation $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ with reduced Serre weight 1, $N(\rho) \leq N_p$ and projective image containing a subgroup isomorphic to $\mathrm{PSL}_2(\mathbb{F}_p)$. The N_p can be computed explicitly; for large enough p (say, $p \geq 1000003$), we can take $N_p = 44$, and for some small p, we can take $N_5 = 20$, $N_7 = 24$, $N_{11} = 29$, ..., $N_{31} = 34$, ...*

This is just a simple application of the Odlyzko bound. One can give also an unconditional version of this theorem. Theorem 2, together with some extensions of Theorem 1 to the case of non-trivial Artin conductors, is proved in Section 3.

In this paper, we follow the definitions, notations and conventions in [4] for, e.g., the Serre weight $k(\rho)$, the notion of mod p modular forms, and the formulation of Serre's conjecture. There are slight differences (cf. [4], §1) between these and those of Serre's original ones in [19].

It is our pleasure to dedicate this paper to Professor Kazuya Kato on the occasion of his fiftieth birthday. The second named author got interested in Serre's conjecture when he read the paper [19] as a graduate student under the direction of Professor Kato, and a decade later his continued interest was conveyed to the first named author.

1. REFINEMENT OF THE TATE BOUND. In this section, we refine Tate's discriminant bound [21] for the finite Galois extension $K/\mathbb{Q}_p$ corresponding to

the kernel of a continuous representation $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ of the absolute Galois group $G_{\mathbb{Q}_p}$ of the p-adic number field $\mathbb{Q}_p$. Namely, we give a formula which gives the valuation of the different $\mathcal{D}_{K/\mathbb{Q}_p}$ of $K/\mathbb{Q}_p$ in terms of the reduced Serre weight (defined below) of ρ.

Let $k(\rho)$ be the Serre weight of ρ, and χ the mod p cyclotomic character. Then by the definition of $k(\rho)$, there exists an integer α (mod $(p-1)$) such that $k(\chi^{-\alpha} \otimes \rho) \leq p+1$. It will be convenient for our purpose to define the *reduced Serre weight* $\tilde{k}(\rho)$ of ρ by

$$\tilde{k}(\rho) \; := \; k(\chi^{-\alpha} \otimes \rho)$$

with the α which minimizes the value of $k(\chi^{-\alpha} \otimes \rho)$. This α (mod $(p-1)$) is unique unless the restriction of ρ to an inertia group at p is the direct sum of two different powers of χ.

If ρ is tamely ramified, then we have $v_p(\mathcal{D}_{K/\mathbb{Q}_p}) < 1$, where v_p denotes the valuation of K normalized by $v_p(p) = 1$. So we assume from now on that ρ is wildly ramified. Let us recall the definition of the Serre weight $k(\rho)$ in this case. A wildly ramified representation ρ, restricted to an inertia group I_p at p, has the following form:

$$\rho|_{I_p} \sim \begin{pmatrix} \chi^\beta & * \\ 0 & \chi^\alpha \end{pmatrix} \qquad \text{with } * \neq 0, \tag{1.1}$$

where $\sim$ denotes the equivalence relation of representations of I_p. Take the integers α and β (uniquely) so that $0 \leq \alpha \leq p-2$ and $1 \leq \beta \leq p-1$. We set $a = \min(\alpha, \beta)$, $b = \max(\alpha, \beta)$, and define

$$k(\rho) \; := \; \begin{cases} 1 + pa + b + p - 1 & \text{if } \beta - \alpha = 1 \text{ and } \chi^{-\alpha} \otimes \rho \text{ is not finite,} \\ 1 + pa + b & \text{otherwise.} \end{cases}$$

Thus, if we write

$$\rho|_{I_p} \sim \chi^\alpha \begin{pmatrix} \chi^{k-1} & * \\ 0 & 1 \end{pmatrix}$$

with $2 \leq k \leq p$, then we have

$$\tilde{k}(\rho) \; = \; \begin{cases} p+1 & \text{if } k = 2 \text{ and } \chi^{-\alpha} \otimes \rho \text{ is not finite,} \\ k & \text{otherwise.} \end{cases}$$

We shall prove

THEOREM 3. *Suppose $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is wildly ramified, with α, β as in (1.1). Let $\tilde{k} = \tilde{k}(\rho)$ be the reduced Serre weight of ρ. Put $d := (\alpha, \beta, p-1) = (\alpha, \tilde{k}-1, p-1)$. Let p^m be the wild ramification index of $K/\mathbb{Q}_p$. Then we have*

$$v_p(\mathcal{D}_{K/\mathbb{Q}_p}) \; = \; \begin{cases} 1 + \frac{\tilde{k}-1}{p-1} - \frac{\tilde{k}-1+d}{(p-1)p^m} & \text{if } 2 \leq \tilde{k} \leq p, \\ 2 + \frac{1}{(p-1)p} - \frac{2}{(p-1)p^m} & \text{if } \tilde{k} = p+1. \end{cases}$$

Remarks. (1) The value of $v_p(\mathcal{D}_{K/\mathbb{Q}_p})$ is the largest in the last case, so we have in general

$$v_p(\mathcal{D}_{K/\mathbb{Q}_p}) \ \leq \ 2 + \frac{1}{(p-1)p} - \frac{2}{(p-1)p^m}.$$

This bound coincides with Tate's one ([21], Remark 1 on p. 155) if $m = 1$ or $p = 2$, and is smaller if $m > 1$ and $p > 2$.

(2) The case of $\tilde{k} = 2$ is comparable to (the $n = 1$ case of) the bound of Fontaine ([5], Théorème 1); the main term $1 + 1/(p-1)$ is the same. We have the correction term $-2/(p-1)p^m$.

(3) Suppose $2 < \tilde{k} \leq p$. If $d_0 := (\tilde{k}-1, p-1) \geq 2$, then the value of $d = (\alpha, \tilde{k}-1, p-1)$ may vary if ρ is twisted by a power of χ. The largest value d_0 is attained by $\chi^{-\alpha} \otimes \rho$. So the minimum value of $v_p(\mathcal{D}_{K/\mathbb{Q}_p})$, with $K/\mathbb{Q}_p$ corresponding to $\mathrm{Ker}(\chi^i \otimes \rho)$ for various i, is $1 + \frac{\tilde{k}-1}{p-1} - \frac{\tilde{k}-1+d_0}{(p-1)p^m}$.

Proof. Let $K_0/\mathbb{Q}_p$ (resp. $K_1/\mathbb{Q}_p$) be the maximal unramified (resp. maximal tamely ramified) subextension of $K/\mathbb{Q}_p$ (so K_1/K_0 is cut out by the representation $\chi^\alpha \oplus \chi^\beta$ and K/K_1 by the representation $\left(\begin{smallmatrix}1 & * \\ 0 & 1\end{smallmatrix}\right)$). Then K_1 is a subfield of $K_0(\zeta_p)$, where ζ_p is a primitive pth root of unity, and K_1/K_0 has degree (and ramification index) $e := (p-1)/d$. The extension K/K_1 has degree (and ramification index) p^m. Set $\Delta = \mathrm{Gal}(K_1/K_0)$ and $H = \mathrm{Gal}(K/K_1)$. Then Δ may be identified with a quotient of $\mathrm{Gal}(K_0(\zeta_p)/K_0) \simeq (\mathbb{Z}/p\mathbb{Z})^\times$. In fact, we have $\Delta \simeq ((\mathbb{Z}/p\mathbb{Z})^\times)^d \simeq \mathbb{Z}/e\mathbb{Z}$, and its character group $\widehat{\Delta}$ is generated by χ^d. The group Δ acts on the $\mathbb{F}_p$-module H by conjugation and, in view of (1.1), this action is via $\chi^{\beta-\alpha} = \chi^{\tilde{k}-1}$;

$$\begin{pmatrix}\chi^\beta(\sigma) & b(\sigma) \\ 0 & \chi^\alpha(\sigma)\end{pmatrix}\begin{pmatrix}1 & * \\ 0 & 1\end{pmatrix}\begin{pmatrix}\chi^\beta(\sigma) & b(\sigma) \\ 0 & \chi^\alpha(\sigma)\end{pmatrix}^{-1} = \begin{pmatrix}1 & \chi^{\beta-\alpha}(\sigma)* \\ 0 & 1\end{pmatrix}$$

for $\sigma \in I_p$. Thus we have $H = H(\chi^{\tilde{k}-1})$ if we denote by $\mathcal{H}(\chi^i)$ the χ^i-part (= the part on which $\sigma \in \Delta$ acts by multiplication by $\chi^i(\sigma)$) of any $\mathbb{F}_p[\Delta]$-module $\mathcal{H}$.

Now set $U = (1+\pi\mathcal{O})^\times/(1+\pi\mathcal{O})^p$, where π (resp. $\mathcal{O}$) is a uniformizer (resp. the integer ring) of K_1. Here and elsewhere, we denote by $(1+\pi\mathcal{O})^p$ the subgroup of pth powers in $(1+\pi\mathcal{O})^\times$. By local class field theory, we have the reciprocity map

$$r: \ U \twoheadrightarrow H.$$

The Galois group Δ acts naturally on U, so U decomposes as $U = \oplus_{i=1}^e U(\chi^{di})$. Since the map r is compatible with the actions of Δ on U and H, only $U(\chi^{\tilde{k}-1})$ is mapped onto H and the other parts go to 0;

$$r(U(\chi^{di})) \ = \ \begin{cases} 0 & \text{if } di \not\equiv \tilde{k}-1 \pmod{p-1}, \\ H & \text{if } di \equiv \tilde{k}-1 \pmod{p-1}. \end{cases} \tag{1.2}$$

Next we shall examine $U(\chi^i)$ more closely. Any element of U can be represented by an element $1 + u_1\pi + u_2\pi^2 + \ldots$ of $(1+\pi\mathcal{O})^\times$, where u_i are units of K_0.

Claim. For any $\sigma \in \Delta$, a unit u of K_0, and $i \geq 1$, one has

$$\sigma(1+u\pi^i) \equiv (1+u\pi^i)^{\chi^{di}(\sigma)} \pmod{\pi^{i+1}}.$$

Proof. By considering K_1 as a subfield of $K_0(\zeta_p)$, we may reduce this to the case of $K_1 = K_0(\zeta_p)$ and $d = 1$. Also the validity of the Claim is independent of the choice of a uniformizer π. So it is enough to show

$$\sigma(1+u\pi^i) \equiv (1+u\pi^i)^{\chi^{i}(\sigma)} \pmod{\pi^{i+1}}$$

assuming that $\pi = \zeta_p - 1$. Since $\sigma(\zeta_p) = \zeta_p^{\chi(\sigma)}$, we have $\sigma(\pi) \equiv \chi(\sigma)\pi \pmod{\pi^2}$, hence if u is a unit of K_0 then $\sigma(u\pi^i) \equiv \chi^i(\sigma)u\pi^i \pmod{\pi^{i+1}}$. This implies the above congruence. □

Let $U^{(i)}$ be the image of $(1+\pi^i\mathcal{O})^\times$ in U. Note that $(1+p\pi^2\mathcal{O})^\times \subset (1+\pi\mathcal{O})^p$ (i.e. $U^{(e+2)} = U^{(p+1)} = 0$) if $d = 1$, and $(1+p\pi\mathcal{O})^\times \subset (1+\pi\mathcal{O})^p$ (i.e. $U^{(e+1)} = 0$) if $d \geq 2$. By the above Claim, we have

$$(1.3) \qquad \begin{cases} U(\chi^{di}) \xrightarrow{\sim} U^{(i)}/U^{(i+1)} & \text{if } d \geq 2 \text{ or } 2 \leq i \leq e, \\ U(\chi) \xrightarrow{\sim} U^{(1)}/U^{(2)} \oplus U^{(p)} & \text{if } d = i = 1. \end{cases}$$

This shows that, if $d \geq 2$ or $\tilde{k} \neq 2, p+1$, then by (1.2) we have

$$(1.4) \qquad r(U^{(i)}) = \begin{cases} 0 & \text{if } i > \frac{\tilde{k}-1}{d}, \\ H & \text{if } i \leq \frac{\tilde{k}-1}{d}. \end{cases}$$

If $d = 1$ and $\tilde{k} = 2, p+1$, we claim that $r(U^{(p)}) = 0$ if and only if $\tilde{k} = 2$, so that (1.4) is valid also in this case. Indeed, it is proved in §2.8 of [19] that $\tilde{k} = 2$ (i.e. $(\chi^{-\alpha} \otimes \rho)|_{I_p}$ is finite) if and only if K/K_1 is "peu ramifiée", i.e., K is obtained by adjoining pth roots of *units* of K_1 (actually, this was his original definition of the Serre weight's being 2). Suppose $\tilde{k} = 2$ or $p+1$. By (1.3), a non-trivial cyclic subextension $K_1(\xi^{1/p})/K_1$ has conductor (π^2) or (π^{p+1}), and accordingly has different (π^2) or (π^{p+1}). But the different is easily seen to divide (p) if ξ is a unit. Thus K/K_1 is peu ramifiée if and only if $r(U^{(p)}) = 0$. To calculate the value of $v_p(\mathcal{D}_{K/\mathbb{Q}_p})$, we now distinguish the two cases, $2 \leq \tilde{k} \leq p$ and $\tilde{k} = p+1$.

Case $2 \leq \tilde{k} \leq p$: By (1.4), any non-trivial character $\psi \in \widehat{H} := \mathrm{Hom}(H, \mathbb{C}^\times)$ has conductor $(\pi^{(\tilde{k}-1)/d+1})$. By the Führerdiskriminantenproduktformel, we have

$$\begin{aligned} v_p(\mathcal{D}_{K/K_1}) &= \frac{1}{[K:K_1]} v_p(d_{K/K_1}) \\ &= \frac{p^m-1}{p^m}\left(\frac{\tilde{k}-1}{d}+1\right)v_p(\pi) = \left(\frac{\tilde{k}-1}{p-1}+\frac{1}{e}\right)\left(1-\frac{1}{p^m}\right). \end{aligned}$$

Combining this with the tame part

$$v_p(\mathcal{D}_{K_1/K_0}) = \frac{1}{[K_1:K_0]} v_p(d_{K_1/K_0}) = 1 - \frac{1}{e},$$

we have

$$v_p(\mathcal{D}_{K/\mathbb{Q}_p}) \;=\; 1+\frac{\tilde{k}-1}{p-1}-\frac{\tilde{k}-1+d}{(p-1)p^m}.$$

Case $\tilde{k}=p+1$: We have $d=1$ in this case, and (1.4) shows that non-trivial characters $\psi\in\widehat{H}$ have conductor either (π^2) or (π^{p+1}). In fact, exactly one pth of all the characters have conductor dividing (π^2) and the rest have conductor (π^{p+1}) (this is remarked in Remarque (2) in §2.4 of [19], and a similar fact had been noticed already in the proof of the Lemma in [21]). We reproduce here the proof given in [10], Lemma 3.5.4. This follows from the fact that the subgroup $(1+\pi\mathcal{O})^p$ has index p in $(1+\pi p\mathcal{O})^\times$. To show this, consider the commutative diagram

$$\begin{array}{ccc}
(1+\pi\mathcal{O})^p/(1+\pi^2p\mathcal{O})^\times & \xrightarrow{\subset} & (1+\pi p\mathcal{O})^\times/(1+\pi^2p\mathcal{O})^\times \\
\wr\downarrow & & \downarrow\wr \\
\wp(\mathbb{F}) & \xrightarrow[\subset]{} & \mathbb{F},
\end{array}$$

where $\mathbb{F}$ is (the additive group of) the residue field of K_1, $\wp(\mathbb{F})$ is the subgroup $\{x+(\pi^{p-1}/p)x^p;\ x\in\mathbb{F}\}$ of $\mathbb{F}$, and the right vertical arrow is the map $1+\pi px(\mathrm{mod}\ \pi^2p)\mapsto x(\mathrm{mod}\ \pi)$. This map induces the map $(1+\pi x)^p(\mathrm{mod}\ \pi^2p)\mapsto x+(\pi^{p-1}/p)x^p(\mathrm{mod}\ \pi)$ on the left-hand side. We claim that $\wp(\mathbb{F})$ has index p in $\mathbb{F}$. This is equivalent to that the map

$$\begin{array}{rccc}
\wp: & \mathbb{F} & \to & \mathbb{F} \\
 & x & \mapsto & x+ux^p,
\end{array}$$

where $u:=\pi^{p-1}/p$ (mod π), has kernel of dimension 1 over $\mathbb{F}_p$. The dimension depends only on the class of u in $\mathbb{F}^\times/(\mathbb{F}^\times)^{p-1}$, which is independent of the choice of a uniformizer π of K_1. Since $K_1=K_0(\zeta_p)=K_0((-p)^{1/(p-1)})$ now, we may take π so that $\pi^{p-1}/p=-1$, in which case the kernel has dimension 1. Now again by the Führerdiskriminantenproduktformel, we have

$$\begin{aligned}
v_p(\mathcal{D}_{K/K_1}) &= \frac{1}{[K:K_1]}v_p(d_{K/K_1}) \\
&= \frac{1}{p^m}((p^m-p^{m-1})(p+1)+(p^{m-1}-1)2)v_p(\pi) \\
&= 1+\frac{2}{p-1}-\frac{1}{p}-\frac{2}{(p-1)p^m}.
\end{aligned}$$

Combining this with the tame part

$$v_p(\mathcal{D}_{K_1/K_0}) \;=\; 1-\frac{1}{p-1},$$

we obtain

$$v_p(\mathcal{D}_{K/\mathbb{Q}_p}) \;=\; 2+\frac{1}{(p-1)p}-\frac{2}{(p-1)p^m}.$$

2. Proof of Theorem 1. In this section, we prove Theorem 1. As in [21], the proof splits into two cases, according as $G = \mathrm{Im}(\rho)$ is solvable or not. We assume $p \geq 5$ since the cases $p = 2$ and 3 are done respectively in [21] and [18] (cf. also [1] and [12] for the cases of $p = 5, 7$).

(1) Solvable case. Suppose G is solvable. To deal with the cases $p \leq 19$, we proceed as follows: According to [20], a maximal irreducible solvable subgroup $\mathbf{G}$ of $\mathrm{GL}_2(\overline{\mathbb{F}}_p)$ has the following structure: either

(i) Imprimitive case: $\mathbf{G}$ is isomorphic to the wreath product $\overline{\mathbb{F}}_p^\times \wr (\mathbb{Z}/2\mathbb{Z})$, or

(ii) Primitive case: one has exact sequences

$$\begin{aligned} 1 \to \mathbf{A} \to \quad \mathbf{G} \quad \to \overline{\mathbf{G}} \to 1, &\qquad \text{with } \overline{\mathbf{G}} \simeq \mathrm{SL}_2(\mathbb{F}_2) \simeq S_3, \\ 1 \to \overline{\mathbb{F}}_p^\times \to \quad \mathbf{A} \quad \to \overline{\mathbf{A}} \to 1, &\qquad \text{with } \overline{\mathbf{A}} \simeq \mathbb{F}_2^{\oplus 2}. \end{aligned}$$

Note that, in either case, a finite subgroup of $\mathbf{G}$ has order prime to p. So, when $p \leq 19$, we are done if we show the following lemma, since the pth cyclotomic field $\mathbb{Q}(\zeta_p)$ has class number 1 for $p \leq 19$.

Lemma 1. *If $\mathbb{Q}(\zeta_p)$ has class number 1, then there exists no non-abelian solvable extension of $\mathbb{Q}$ which is unramified outside p and of degree prime to p.*

Proof. It is enough to show that there exists no non-trivial abelian extension of $\mathbb{Q}(\zeta_p)$ which is unramified outside p and of degree prime to p. Let $\mathcal{O}_p$ be the p-adic completion of the integer ring of $\mathbb{Q}(\zeta_p)$. By class field theory (together with the assumption "class number 1"), the Galois group of the maximal such extension is isomorphic to the quotient of $\mathcal{O}_p^\times/(1 + (\zeta_p - 1)\mathcal{O}_p)^\times \simeq \mathbb{F}_p^\times$ by the image of the global units. This group is trivial since we have at least the cyclotomic units $(\zeta_p^i - 1)/(\zeta_p - 1) \equiv i \pmod{\zeta_p - 1}$, $1 \leq i \leq p - 1$. □

To deal with the odd cases with $p \geq 23$, we appeal to the solvable case of Serre's conjecture:

Theorem 4 (cf. [7]). *Let $p \geq 3$. Let $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be an odd and irreducible representation with solvable image. Then ρ is modular of the type predicted by Serre.*

Proof. If ρ is irreducible and $G := \mathrm{Im}(\rho)$ is solvable, then as we saw above, either G has order prime to p (if $p \geq 5$) or $p = 3$ and G is an extension of a subgroup of the symmetric group S_3 by a finite solvable group of order prime to 3. By Fong-Swan's theorem (Th. 38 of [17]), there is an odd and irreducible lifting $\hat{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O})$ of ρ to some ring $\mathcal{O}$ of algebraic integers. By Langlands-Tunnell ([8], [22]), $\hat{\rho}$, and hence ρ, is modular of weight 1. By the ε-conjecture ([4], Th. 1.12), ρ is modular of the type predicted by Serre. □

By this theorem, we can exclude the possibility of the existence of ρ with solvable image, unramified outside p, and with even Serre weight $k(\rho) \leq 10$.

(2) Non-solvable case. Suppose $G = \mathrm{Im}(\rho)$ is non-solvable. In this case, we compare the discriminant bound in Section 1 and the Odlyzko bound ([14], [15]) to deduce contradictions. We distinguish the two cases where ρ is odd

and even. If ρ is even, then the complex conjugation is mapped by ρ to $\pm\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, so the field $\overline{\mathbb{Q}}^{\mathrm{Ker}(\rho)}$ cut out by ρ is totally real or CM. Note that the Odlyzko bound is much better (i.e. gives larger values) for totally real fields. Let K be either $\overline{\mathbb{Q}}^{\mathrm{Ker}(\rho)}$ or its maximal real subfield according as ρ is odd or even. Let $n := [K : \mathbb{Q}]$ (so $n = |G|$ or $|G|/2$ according as ρ is odd or even), and let $d_K^{1/n}$ denote the root discriminant of K.

For the Odlyzko bound to work for our purpose, the degree $n = [K : \mathbb{Q}]$ has to be large to a certain extent. Set $G_1 := G \cap \mathrm{SL}_2(\overline{\mathbb{F}}_p)$. We have an exact sequence

$$1 \to G_1 \to G \to \det(G) \to 1.$$

Since $\det\rho = \chi^{k-1}$, we have $\det(G) = (\mathbb{F}_p^\times)^{k-1} \simeq \mathbb{Z}/e\mathbb{Z}$ if we put $e := (p-1)/(k-1, p-1)$. If G is non-solvable, so is the image $\overline{G}_1$ of G_1 in $\mathrm{PSL}_2(\overline{\mathbb{F}}_p)$, and hence we have $|\overline{G}_1| \geq 60$. Furthermore, Brueggeman makes a nice observation after the proof of Lemma 3.1 of [1] as follows: Since G_1 is non-solvable, it contains an element of order 2, which must be $-\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ as it is the only element of order 2 of $\mathrm{SL}_2(\overline{\mathbb{F}}_p)$ if $p \neq 2$. Thus we have $|G| \geq 120e$.

If ρ is at most tamely ramified, then we have $d_K^{1/n} < p$. On the other hand, if $n \geq 120e$, by the Odlyzko bound [14], we have $d_K^{1/n} > p$ in all the cases we need (assuming the GRH for $p = 23, 29, 31$). Thus we may assume ρ is wildly ramified.

If p^m divides the order of G (hence of G_1) and ρ is irreducible, then by §§251–253 of [3], the image $\overline{G}_1$ of G_1 in $\mathrm{PGL}_2(\overline{\mathbb{F}}_p)$ coincides with a conjugate of $\mathrm{PSL}_2(\mathbb{F}_{p^m})$. Thus we have $n = |G| \geq 2e \times |\mathrm{PSL}_2(\mathbb{F}_{p^m})| = e(p^{2m}-1)p^m$ if ρ is odd, and $n = |G|/2 \geq e \times |\mathrm{PSL}_2(\mathbb{F}_{p^m})| = e(p^{2m}-1)p^m/2$ if ρ is even. Let us denote these values by $n(p^m, k)$;

$$n(p^m, k) := \begin{cases} e(p^{2m}-1)p^m & \text{if } k \text{ is even},\\ e(p^{2m}-1)p^m/2 & \text{if } k \text{ is odd}. \end{cases}$$

To show the non-existence of a ρ, it is enough to show the non-existence of a twist $\chi^{-\alpha} \otimes \rho$ of it. So in what follows, we may assume that ρ has Serre weight $k \leq p+1$ (hence $d = (k-1, p-1)$ in the notation of Theorem 3) for our ρ; this minimizes the bound of Theorem 3 (see Remark (3) after Theorem 3).

We compare inequalities implied by Odlyzko and Tate bounds for each (p, k, m) to deduce contradictions proving the non-existence of ρ, the Odlyzko bound being calculated with $n \geq n(p^m, k)$ by using either [14] or [15] (Eqn. (10) (assuming the GRH) and (16) of *loc. cit.*). In general, under the GRH and for not too large n, the values from [14] are better, and otherwise we use [15]. In most cases, it is enough to compare the $n \geq n(p^1, k)$ case of the Odlyzko bound and the $m = \infty$ case of the Tate bound. Sometimes, however, it happens that we have to look at the cases $m = 1$ and $m \geq 2$ separately.

Also, to prove the finiteness of ρ's, we only need to have the contradictions for sufficiently large n, because if the degree n is bounded, by the Hermite-Minkowski theorem, there exist only finitely many extensions $K/\mathbb{Q}$ which are unramified outside a given finite set of primes and of degree $\leq n$. Thus we only need to compare the Tate bound with $m = \infty$ and the asymptotic Odlyzko bound, which says that, for sufficiently large $n = [K : \mathbb{Q}]$, one has

$$d_K^{1/n} > \begin{cases} 22.381 & \text{for any } K, \\ 60.839 & \text{for totally real } K, \\ 44.763 & \text{under GRH, for any } K, \\ 215.332 & \text{under GRH, for totally real } K. \end{cases}$$

The comparison for proving the finiteness is easily done, so in the following we focus on the proof of the non-existence. As typical cases, we present here only the proof of the cases of $p = 11$ and 23.

Case $p = 11$: For $k = 2, ..., 12$, we have respectively $n(11, k) = 13200$, 3300, 13200, 3300, 2640, 3300, 13200, 3300, 13200, 660, 13200. If $n \geq n(11, k)$, the Odlyzko bound implies

$$d_K^{1/n} > \begin{cases} 22.108 & \text{for } k = 2, 4, 8, 10, 12, \\ 58.598 & \text{for } k = 3, 5, 7, 9, \\ 21.592 & \text{for } k = 6, \\ 54.517 & \text{for } k = 11, \\ 34.768 & \text{under GRH, for } k = 2, 4, 8, 10, 12, \\ 122.112 & \text{under GRH, for } k = 3, 5, 7, 9, \\ 31.645 & \text{under GRH, for } k = 6, \\ 97.979 & \text{under GRH, for } k = 11. \end{cases} \tag{2.1}$$

On the other hand, the Tate bound ($m = \infty$) implies

$$d_K^{1/n} \leq \begin{cases} 13.981 & \text{if } k = 2, \\ 17.770 & \text{if } k = 3, \\ 22.585 & \text{if } k = 4, \\ 28.705 & \text{if } k = 5, \\ 36.483 & \text{if } k = 6, \\ 46.370 & \text{if } k = 7, \\ 58.935 & \text{if } k = 8, \\ 74.905 & \text{if } k = 9, \\ 95.203 & \text{if } k = 10, \\ 121 & \text{if } k = 11, \\ 123.667 & \text{if } k = 12. \end{cases} \tag{2.2}$$

Comparing (2.1) and (2.2), we obtain contradictions for $k = 2, 3, 5, 7$, and also for $k = 4, 9$ assuming the GRH. For $k = 6, 11$, we look at the cases $m = 1$ and

$m \geq 2$ separately. If $m = 1$, the Tate bound implies

$$d_K^{1/n} < \begin{cases} 29.338 & \text{if } k = 6, \\ 78.243 & \text{if } k = 11. \end{cases} \tag{2.3}$$

Comparing (2.1) and (2.3), we obtain contradictions for $k = 6, 11$ assuming the GRH. For $m = 2$ and $k = 6, 11$, we have $n(11^2, k) = 3542880, 885720$. If $n \geq n(11^2, k)$, the Odlyzko bound implies

$$d_K^{1/n} > \begin{cases} 40.458 & \text{under GRH, for } k = 6, \\ 168.971 & \text{under GRH, for } k = 11. \end{cases} \tag{2.4}$$

Comparing (2.2) and (2.4), we obtain contradictions for $k = 6, 11$ assuming the GRH.

Case $p = 23$: For $p = 23, 29, 31$, we rely on Theorem 4 in the solvable image case, so we can prove the non-existence at most in the odd case (i.e. when k is even). Let $p = 23$. We have $n(23, k) = 267168$ for $k = 2, 4, 6$. If n is greater than or equal to this value, the Odlyzko bound implies

$$d_K^{1/n} > 37.994 \qquad \text{under GRH.} \tag{2.5}$$

On the other hand, the Tate bound implies

$$d_K^{1/n} < \begin{cases} 26.524 & \text{if } k = 2, \\ 35.272 & \text{if } k = 4, \\ 46.905 & \text{if } k = 6. \end{cases} \tag{2.6}$$

Comparing (2.5) and (2.6), we obtain contradictions for $k = 2, 4$.

3. Representations with non-trivial Artin conductor.

In this section, we prove Theorem 2 and extend Theorem 1 to some other cases where the representations ρ have non-trivial Artin conductors outside p. We present in §3.1 (resp. §3.2) the cases where we can prove the non-existence (resp. finiteness) of $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$. We denote by $N(\rho)$ the Artin conductor of ρ outside p. In both cases, we use:

LEMMA 2. *Let $K/\mathbb{Q}$ be the extension which corresponds to the kernel of ρ, and $n = [K : \mathbb{Q}]$. Let d'_K be the prime-to-p part of the discriminant of K. Then if $|d'_K| > 1$, we have*

$$|d'_K|^{1/n} < N(\rho).$$

Proof. This is Lemma 3.2, (ii) of [13]. Note that, in the proof there, one has $i_{E/F} > 0$ if the extension E/F is ramified, whence the strict inequality in the above lemma. □

3.1. Non-existence.

We first prove Theorem 2. Let $K/\mathbb{Q}$ be the extension corresponding to the kernel of the representation ρ. If for example $p \geq 1000003$, then for $n \geq 2 \times |\mathrm{PSL}_2(\mathbb{F}_p)| \geq 4000036000104000096$, the Odlyzko bound implies, under the GRH, that the root discriminant of K is $> 44.17....$ Noticing

Lemma 2, we conclude that there is no ρ which is unramified at p, with $N(\rho) \leq 44$, and has projective image containing $\mathrm{PSL}_2(\mathbb{F}_p)$.

To extend Theorem 1, we consider as in Section 2 the solvable and non-solvable cases separately. We shall consider only the odd cases. In the solvable case, by Theorem 4, we only need to calculate the dimension of the $\mathbb{C}$-vector space $S_k(\Gamma_1(N))$ of cusp forms of weight k with respect to the congruence subgroup $\Gamma_1(N)$. This is done by using, e.g., Chapters 2 and 3 of [9]. If $N \geq 2$, the values of (N, k) for which $S_k(\Gamma_1(N)) = 0$ are:

$(N, k) = (2, 2), (2, 4), (2, 6), (2, \text{odd})$;

$(N, k) = (3, 2), (3, 3), (3, 4), (3, 5); (4, 2), (4, 3), (4, 4); (5, 2), (5, 3); (6, 2), (6, 3)$;

and

$(N, 2)$ for $N = 7, 8, 9, 10, 12$.

The non-solvable case is also done in a similar way to that in Section 2 by comparing various discriminant bounds, except that we take the Artin conductor into account. Combining with the solvable case, we obtain:

Theorem 5. *There exists no odd and irreducible representation* $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ *of reduced Serre weight* k *and Artin conductor* N *outside* p *in the following cases:*

Case $N = 2$*:* $(p, k) = (3, 2), (3, 3), (3, 4); (5, 2); (7, 2)$.

Case $N = 3$*:* $(p, k) = (2, 2), (2, 3)$.

Case $N = 4$*:* $(p, k) = (3, 2)$.

Case $N = 5$*:* $(p, k) = (2, 2)$.

Assuming the GRH, we obtain the non-existence of ρ*, besides the above cases, in the following cases:*

Case $N = 2$*:* $(p, k) = (5, 3); (7, 3); (11, 2); (13, 2)$.

Case $N = 3$*:* $(p, k) = (5, 2); (7, 2)$.

Case $N = 4$*:* $(p, k) = (3, 3)$.

Case $N = 5$*:* $(p, k) = (3, 2)$.

3.2. Finiteness. To prove the finiteness of the set of isomorphism classes of semisimple representations $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ with bounded Artin conductor $N(\rho)$, we only need to compare the lower bound of the discriminants by Odlyzko and the upper bound obtained as the product of the one in Theorem 3 with $m = \infty$ and the one in Lemma 2. Here we give only the results for odd representations under the assumption of the GRH. Other cases (even and/or unconditional) can be obtained similarly.

Theorem 6. *Assume the GRH. Then there exist only finitely many isomorphism classes of odd and semisimple representations* $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ *with reduced Serre weight* k *and Artin conductor* N *outside* p *in the following cases:*

(1) $k = 1$*, any* p*, and* $N \leq 44$.

(2) $p = 2$*:* *(*$k = 2$ *and* $N \leq 11$*), (*$k = 3$ *and* $N \leq 7$*).*

(3) $p = 3$*:* *(*$k = 2$ *and* $N \leq 8$*), (*$k = 3$ *and* $N \leq 4$*), (*$k = 4$ *and* $N \leq 4$*).*

(4) For other p *and* $k > 1$*;*
$N = 2$ *and* $(p, k) = (5, 2), (5, 4), (7, 2), (7, 4), (11, 2), (13, 2)$.
(Note that, when $N = 2$*, the representation* ρ *is odd if and only if* k *is even.)*
$N = 3$ *and* $(p, k) = (5, 2), (5, 3), (7, 2), (7, 3), (11, 2)$.
$N = 4$ *and* $(p, k) = (5, 2), (7, 2)$.

To keep the table compact, we classified the cases in an unsystematic manner. We hope to give a more convenient table on a suitable web site.

References

[1] S. Brueggeman, *The nonexistence of certain Galois extensions unramified outside 5*, J. Number Theory 75(1999), 47–52

[2] P. Deligne and J.-P. Serre, *Formes modulaires de poids* 1, Ann. Sci. Ecole Norm. Sup. 7(1974), 507–530

[3] L. E. Dickson, *Linear Groups*, Teubner, 1901, Leibzig

[4] B. Edixhoven, *Serre's conjectures*, in: "Modular Forms and Fermat's Last Theorem" (G. Cornell, J.H. Silverman, G. Stevens (eds.)), Springer-Verlag, 1997

[5] J.-M. Fontaine, *Il n'y a pas de variété abélienne sur* $\mathbb{Z}$, Invent. Math. 81(1985), 515–538

[6] K. Joshi, *Remarks on methods of Fontaine and Faltings*, Intern. Math. Res. Notices 22(1999), 1199–1209

[7] E. Kimura, H. Moon and Y. Taguchi, in preparation

[8] R. Langlands, *Base Change for* GL(2), Princeton Univ. Press

[9] T. Miyake, *Modular Forms*, Springer-Verlag, 1989

[10] H. Moon, *On the finiteness of mod p Galois representations*, Thesis, Tokyo Metropolitan University, 2000

[11] H. Moon, *Finiteness results on certain mod p Galois representations*, J. Number Theory 84(2000), 156–165

[12] H. Moon, *The non-existence of certain mod p Galois representations*, Bull. Korean Math. Soc. 40(2003), 537–544

[13] H. Moon and Y. Taguchi, *Mod p Galois representations of solvable image*, Proc. A.M.S. 129(2001), 2529–2534

[14] A. M. Odlyzko, *Discriminant bounds*, unpublished manuscript (1976), available at:
`http://www.dtc.umn.edu/~odlyzko/unpublished/index.html`

[15] G. Poitou, *Sur les petits discriminants*, Sém. Delange-Pisot-Poitou (Théorie des nombres), 18e année, 1976/77, n° 6

[16] J.-P. Serre, *Congruences et formes modulaires (d'aprés H.P.F. Swinnerton-Dyer)*, Sém. Bourbaki 1971/72, n° 416, in: Œuvres III, Springer-Verlag, 1986, pp. 74–88

[17] J.-P. Serre, *Représentations Linéaires des Groupes Finis* (5ème éd.), Hermann, 1998

[18] J.-P. Serre, Note $229._2$ on p. 710, Œuvres III, Springer-Verlag, 1986

[19] J.-P. Serre, *Sur les représentations modulaires de degré 2 de* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, Duke Math. J. 54(1987), 179–230

[20] D. A. Suprunenko, *Matrix Groups*, A.M.S., Providence, 1976

[21] J. Tate, *The non-existence of certain Galois extensions of* $\mathbb{Q}$ *unramified outside* 2, Contemp. Math. 174(1994), 153–156

[22] J. Tunnell, *Artin's conjecture for representations of octahedral type*, Bull. Amer. Math. Soc. 5(1981), 173–175

Hyunsuk Moon
Graduate School of Mathematics
Kyushu University 33
Fukuoka 812-8581
Japan
moon@math.kyushu-u.ac.jp

Yuichiro Taguchi
Graduate School of Mathematics
Kyushu University 33
Fukuoka 812-8581
Japan
taguchi@math.kyushu-u.ac.jp

On the Logarithmic Riemann-Hilbert Correspondence

Arthur Ogus

Received: October 15, 2002
Revised: May 23, 2003

Abstract. We construct a classification of coherent sheaves with an integrable log connection, or, more precisely, sheaves with an integrable connection on a smooth log analytic space X over $\mathbf{C}$. We do this in three contexts: sheaves and connections which are equivariant with respect to a torus action, germs of holomorphic connections, and finally global log analytic spaces. In each case, we construct an equivalence between the relevant category and a suitable combinatorial or topological category. In the equivariant case, the objects of the target category are graded modules endowed with a group action. We then show that every germ of a holomorphic connection has a canonical equivariant model. Global connections are classified by locally constant sheaves of modules over a (varying) sheaf of graded rings on the topological space X_{log}. Each of these equivalences is compatible with tensor product and cohomology.

Keywords and Phrases: De Rham cohomology, Log scheme

Contents

0 Introduction

Let $X/\mathbf{C}$ be a smooth proper scheme of finite type over the complex numbers and let X_{an} be its associated complex analytic space. The classical Riemann-Hilbert correspondence provides an equivalence between the category $L_{coh}(\mathbf{C}_X)$ of locally constant sheaves of finite dimensional $\mathbf{C}$-vector spaces V on X_{an} and the category $MIC_{coh}(X/\mathbf{C})$ of coherent sheaves (E, ∇) with integrable connection on $X/\mathbf{C}$. This correspondence is compatible with formation of tensor products and with cohomology: if an object V of $L_{coh}(\mathbf{C}_X)$ corresponds to an object (E, ∇) of $MIC_{coh}(X/\mathbf{C})$, there is a canonical isomorphism

$$H^i(X_{an}, V) \cong H^i(X, E \otimes \Omega^{\cdot}_{X/\mathbf{C}}), \tag{0.0.1}$$

where $E \otimes \Omega^{\cdot}_{X/\mathbf{C}}$ is the De Rham complex of (E, ∇).

When X is no longer assumed to be proper, such an equivalence and equation (0.0.1) still hold, provided one restricts to connections with regular singularities at infinity [3]. Among the many equivalent characterizations of this condition, perhaps the most precise is the existence of a smooth compactification $\overline{X}$ of X such that the complement $\overline{X} \setminus X$ is a divisor Y with simple normal crossings and such that (E, ∇) prolongs to a locally free sheaf $\overline{E}$ endowed with a connection with log poles $\nabla \colon \overline{E} \to \overline{E} \otimes \Omega^1_{\overline{X}/\mathbf{C}}(\log Y)$. In general there are many possible choices of $\overline{E}$, *some* of which have the property that the natural map

$$H^i(\overline{X}, \overline{E} \otimes \Omega^{\cdot}_{\overline{X}/\mathbf{C}}(\log Y)) \to H^i(X, E \otimes \Omega^{\cdot}_{X/\mathbf{C}}) \tag{0.0.2}$$

is an isomorphism.

In some situations, it is more natural to view the compactification data $(\overline{X}, \overline{E})$ as the fundamental object of study. To embody this point of view in the notation, let $(\underline{X}, Y)$ denote a pair consisting of a smooth scheme $\underline{X}$ over $\mathbf{C}$ together with a reduced divisor with strict normal crossings Y on $\underline{X}$, and let $X^* := \underline{X} \setminus Y$. Write $\mathcal{O}_X$ for $\mathcal{O}_{\underline{X}}$, and let M_X denote the sheaf of sections of $\mathcal{O}_X$ which become units on X^*. Then M_X is a (multiplicative) submonoid of $\mathcal{O}_X$ containing $\mathcal{O}_X^*$, and the natural map of sheaves of monoids $\alpha_X \colon M_X \to \mathcal{O}_X$ defines a "log structure" [6] on $\underline{X}$. The datum of $(\underline{X}, Y)$ is in fact equivalent to the datum of the "log scheme" $X := (\underline{X}, \alpha_X)$. The quotient monoid sheaf $\overline{M}_X := M_X/\mathcal{O}_X^*$ is exactly the sheaf of anti-effective divisors with support in Y. This sheaf is locally constant on a stratification of X and has finitely generated stalks, making it an essentially combinatorial object, which encodes in a convenient way much of the combinatorics of the geometry of $(\underline{X}, Y)$. For example, one can easily control the geometry of those closed subschemes of $\underline{X}$ which are defined by coherent sheaves of ideals K in the sheaf of monoids $\overline{M}_X$. Such a scheme $\underline{Z}$ inherits a log structure $\alpha_Z \colon M_Z \to \mathcal{O}_Z$ from that of X, and the sheaf of ideals K defines a sheaf of ideals K_Z in M_Z which is annihilated by α_Z. If one adds this extra datum to the package, one obtains an *idealized log scheme* (Z, α_Z, K_Z). Many of the techniques of logarithmic de Rham cohomology work as well for Z as they do for X, a phenomenon explained by the fact that (Z, α_Z, K_Z) is smooth over $\mathbf{C}$ in the category of idealized log schemes. Conversely, any fine saturated idealized log scheme X which is smooth over $\mathbf{C}$ (in the sense of Grothendieck's general notion of smoothness) is, locally in the étale topology, isomorphic to the idealized log scheme associated to the quotient monoid algebra $\mathbf{C}[P]$ by an ideal $K \subseteq P$, where P is a finitely generated, integral, and saturated monoid.

In [7], Kato and Nakayama construct, for any log scheme X of finite type over $\mathbf{C}$, a commutative diagram of ringed topological spaces

$$\begin{array}{ccc} X_{an}^* & \xrightarrow{j_{log}} & X_{log} \\ & \searrow^{j} & \downarrow{\scriptstyle \tau} \\ & & X_{an} \end{array}$$

The morphism τ is surjective and proper and can be regarded as a relative compactification of the open immersion j. We show in (3.1.2) that, if X is smooth, X_{an}^* and X_{log} have the same local homotopy type. Since τ is proper, it is much easier to work with than the open immersion j. The construction of X_{log} also works in the idealized case. Here X_{an}^* can be empty, hence useless, while its avatar X_{log} remains. These facts justify the use of the space X_{log} as a substitute for X_{an}^* as the habitat for log topology.

Let $X/\mathbf{C}$ be a smooth, fine, and saturated idealized log analytic space, let $\Omega^1_{X/\mathbf{C}}$ be the sheaf of log Kahler differentials, and let $MIC_{coh}(X/\mathbf{C})$ denote the category of coherent sheaves E on X equipped with an integrable (log) connection $\nabla \colon E \to E \otimes \Omega^1_{X/\mathbf{C}}$. One of the main results of [7] is a Riemann-Hilbert correspondence for a subcategory $MIC_{nilp}(X/\mathbf{C})$ of $MIC_{coh}(X/\mathbf{C})$. This consists of objects (E, ∇) which, locally on X, admit a filtration whose associated graded object "has no poles." (In the classical divisor with normal crossings case, such an object corresponds to the "canonical extension" of a connection with regular singular points and nilpotent residue map [3, II,5.2].) Kato and Nakayama establish a natural equivalence between $MIC_{nilp}(X/\mathbf{C})$

and a category $L_{unip}(X_{log})$ of locally constant sheaves of $\mathbf{C}$-modules on X_{log} with unipotent monodromy relative to τ. Note that if (E,∇) is an object of $MIC_{nilp}(X/\mathbf{C})$, then E is locally free, but that this is not true for a general (E,∇) in $MIC_{coh}(X/\mathbf{C})$.

Our goal in this paper is to classify the category $MIC_{coh}(X/\mathbf{C})$ of *all* coherent sheaves on X, with no restriction on E or its monodromy, in terms of suitable topological objects on X_{log}. These will be certain sheaves of $\mathbf{C}$-vector spaces plus some extra data to keep track of the choice of coherent extension. The extra data we need involve the exponents of the connection. These can be thought of in the following way. At a point x of X, one can associate to a module with connection (E,∇) its *residue* at x. This is a family of commuting endomorphisms of $E(x)$ parameterized by $T_{\overline{M},x} := \operatorname{Hom}(\overline{M}^{gp}_{X,x},\mathbf{C})$; it gives $E(x)$ the structure of a module over the symmetric algebra of $T_{\overline{M},x}$. The support of this module is then a finite subset of the maximal spectrum of $S^{\cdot}T_{\overline{M},x}$, which is just $\mathbf{C}\otimes\overline{M}^{gp}_{X,x}$. The *exponents* of the connection are the *negatives* of these eigenvalues; they are all zero for objects of $MIC_{nilp}(X/\mathbf{C})$. Let Λ denote the pullback of the sheaf $\mathbf{C}\otimes\overline{M}^{gp}_X$ to X_{log}, regarded as a sheaf of $\overline{M}_X$-sets induced from the *negative* of the usual inclusion $\overline{M}_X\to\mathbf{C}\otimes\overline{M}_X$. The map $\overline{M}_X\to\Lambda$ sending p to $-1\otimes p$ endows the pullback $\mathbf{C}^{log}_X$ of $\mathbf{C}[\overline{M}_X]/K_X$ to X_{log} with the structure of a Λ-graded algebra. It then makes sense to speak of sheaves of Λ-graded (or indexed) $\mathbf{C}^{log}_X$-modules. In §3, we describe a category $L_{coh}(\mathbf{C}^{log}_X)$ of "coherent" sheaves of Λ-graded $\mathbf{C}^{log}_X$-modules and prove the following theorem:

Theorem *Let $X/\mathbf{C}$ be a smooth fine, and saturated idealized log analytic space over the complex numbers. There is an equivalence of tensor categories:*

$$\mathcal{V}\colon MIC_{coh}(X/\mathbf{C}) \longrightarrow L_{coh}(\mathbf{C}^{log}_X)$$

compatible with pullback via morphisms $X'\to X$.

As in [7], the equivalence can be expressed with the aid of a sheaf of rings $\tilde{\mathcal{O}}^{log}_X$ on X_{log} which simultaneously possesses the structure of a Λ-graded $\mathbf{C}^{log}_X$-module and an exterior differential:

$$d\colon \tilde{\mathcal{O}}^{log}_X \to \tilde{\Omega}^{1,log}_{X/\mathbf{C}} := \tilde{\mathcal{O}}^{log}_X \otimes_{\tau^{-1}\mathcal{O}_X} \tau^{-1}\Omega^1_{X/\mathbf{C}},$$

whose kernel is exactly $\mathbf{C}^{log}_X$. If (E,∇) is an object of $MIC_{coh}(X/\mathbf{C})$, $\tilde{E} := \tilde{\mathcal{O}}^{log}_X \otimes_{\tau^{-1}\mathcal{O}_X} \tau^{-1}E$ inherits a "connection"

$$\tilde{\nabla}\colon \tilde{E}\to\tilde{E}\otimes_{\tilde{\mathcal{O}}^{log}_X}\tilde{\Omega}^{1,log}_{X/\mathbf{C}},$$

and $\mathcal{V}(E,\nabla)$ is the Λ-graded $\mathbf{C}^{log}_X$-module $\tilde{E}^{\tilde{\nabla}}$. Conversely, if V is an object of

$L_{coh}(\mathbf{C}_X^{log})$, then $\tilde{V} := \tilde{\mathcal{O}}_X^{log} \otimes_{\mathbf{C}_X^{log}} V$ inherits a graded connection

$$\tilde{\nabla} := d \otimes \mathrm{id} \colon \tilde{V} \to \tilde{V} \otimes_{\tilde{\mathcal{O}}_X^{log}} \tilde{\Omega}^{1,log}_{X/\mathbf{C}}.$$

Pushing forward by τ and taking the degree zero parts, one obtains an $\mathcal{O}_X$-module which we denote by $\tau_*^{\Lambda}\tilde{V}$ and which inherits a (logarithmic) connection ∇; this gives a quasi-inverse to the functor $(E,\nabla) \mapsto \mathcal{V}(E,\nabla)$.

The equivalence provided by the theorem is also compatible with cohomology. A Poincaré lemma asserts that the map:

$$V \to E \otimes \tilde{\Omega}^{\cdot,log}_{X/\mathbf{C}}$$

from V to the De Rham complex of $\tau^{-1}E \otimes \tilde{\mathcal{O}}_X^{log}$ is a quasi-isomorphism. An analogous topological calculation asserts that the map

$$E \otimes \Omega^{\cdot}_{X/\mathbf{C}} \to R\tau_*^{\Lambda}(\tau^{-1}E \otimes \tilde{\Omega}^{\cdot,log}_{X/\mathbf{C}})$$

is a quasi-isomorphism, where $R\tau_*^{\Lambda}$ means the degree zero part of $R\tau_*$. One can conclude that the natural maps

$$H^i(X, E \otimes \Omega^{\cdot}_{X/\mathbf{C}}) \to H^i(X_{log}, E \otimes \tilde{\Omega}^{\cdot,log}_{X/\mathbf{C},0}) \leftarrow \ H^i(X_{log}, V_0)$$

are isomorphisms. Note that in the middle and on the right, we take only the part of degree zero; this reflects the well-known fact that in general, logarithmic De Rham cohomology does not calculate the cohomology on the complement of the log divisor without further conditions on the exponents [3, II, 3.13]. The grading structure on the topological side obviates the unpleasant choice of a section of the map $\mathbf{C} \to \mathbf{C}/\mathbf{Z}$ which is sometimes made in the classical theory [3, 5.4]; it has the advantage of making our correspondence compatible with tensor products.

The question of classifying coherent sheaves with integrable logarithmic connection is nontrivial even locally. A partial treatment in the case of a divisor with normal crossings is due to Deligne and briefly explained in an appendix to [4]. The discussion there is limited to the case of torsion-free sheaves and is expressed in terms of $\mathbf{Z}^r$-filtered local systems $(V, F_\cdot)$ of $\mathbf{C}$-vector spaces. In our coordinate-free formalism, $\overline{M}_X^{gp}$ replaces $\mathbf{Z}^r$, and the filtered local system $(V, F_\cdot)$ is replaced by its graded Rees-module $\oplus_m F_m V$.

Because some readers may be primarily concerned with the local problem, and/or may not appreciate logarithmic geometry, we discuss the local Riemann-Hilbert correspondence, in which the logarithmic techniques reduce to toroidal methods which may be more familiar, before the global one. We shall in fact describe this correspondence in two ways: one in terms of certain normalized representations of a "logarithmic fundamental group," and one in terms of equivariant nilpotent Higgs modules. Then the proof of the global theorem stated above amounts to formulating and verifying enough compatibilities so that one can reduce to the local case.

The paper has three sections, dealing with the Riemann-Hilbert correspondence in the equivariant, local, and global settings, respectively. The first section discusses homogeneous connections on affine toric varieties. Essentially, these are modules with integrable connection which are equivariant with respect to the torus action. These are easy to classify, for example in terms of equivariant Higgs modules. Once this is done, it is quite easy to describe an equivariant Riemann-Hilbert correspondence for such modules. It takes some more care to arrange the correspondence in a way that will be compatible with the global formulation we need later. The next section is devoted to the local Riemann-Hilbert correspondence. The main point is to show that the category of analytic germs of connections at the vertex of an affine toric variety is equivalent to the category of coherent equivariant connections (and hence also to the category of equivariant Higgs modules). There are two key ingredients: the first is the study of connections on modules of finite length (using Jordan normal form) and, by passing to the limit, of formal germs. Ahmed Abbes has pointed out the similarity between this construction and the technique of "decompletion" used by Fontaine in an analogous p-adic situation [5]. Our second key ingredient is a convergence theorem which shows that the formal completion functor is an equivalence on germs. Since our analytic spaces are only log smooth and our sheaves are not necessarily locally free, such a theorem is not standard. Instead of trying a dévissage technique to reduce to the classical case, we prove convergence from scratch, using direct estimates of the growth of terms of formal power series indexed by a monoid. In the last section, we globalize the Riemann-Hilbert correspondence by defining $\tilde{\mathcal{O}}_X^{log}$ and showing that it agrees, in a suitable sense, with the equivariant constructions in the first section. To illustrate the power of our somewhat elaborate main theorem, we show how it immediately implies a logarithmic version (3.4.9) of Deligne's comparison theorem [3, II, 3.13]. Our version says that the map (0.0.2) is an isomorphism provided that, at each $x \in X$, the intersection of the set of exponents of E (viewed as a subset of $\overline{\mathbf{C}} \otimes M_{X,x}$) with $\overline{M}^{gp}_{X,x}$ lies in $\overline{M}_{X,x}$. (In fact our result is slightly stronger, as well more general, than Deligne's original version.) We also explain how it immediately implies the existence of a logarithmic version of the Kashiwara-Malgrange V-filtration and of Deligne's meromorphic to analytic comparison theorem.

Since this paper seems long enough in its current state, we have not touched upon several obvious problems, which we expect present varying degrees of difficulty. These include a notion of regular singular points for modules with connection on a log scheme, and especially the functoriality of the Riemann-Hilbert correspondence with respect to direct images. We leave completely untouched moduli problems of log connections, referring to work by N. Nitsure in [9] and [10] on this subject.

The proofs given in the admirably short [7] use a dévissage argument, along with resolution of toric singularities, to reduce to the classical case of a divisor with normal crossings and a reference to [3]. Our point of view is that the monoidal models rendered natural by the log point of view are so convenient that it is natural and easy to give direct proofs, including proofs of the basic convergence results in the analytic setting. Thus our treatment is logically independent of [7] and even [3]. (Of course, these sources were fundamental inspirations.)

I would like to thank Hélène Esnault for pointing out the existence of Deligne's classification in the appendix to [4] and for her encouragement in this attempt to follow up on it. Acknowledgments are also due to Toshirharu Matsubara and Maurizio Cailotto, whose preliminary manuscripts on log connections were very helpful. I would also like to thank Ahmed Abbes for the interest he has shown in this work and the hospitality he provided at the University of Paris (Epinay-Villetaneuse), where I was able to carry out some important rethinking of the presentation. I am especially grateful to the referee for his meticulous work which revealed over two hundred errors and/or ambiguities in the first version of this manuscript. Finally, it is a pleasure and honor to be able to dedicate this work to Kazuya Kato, whose work on the foundations and applications of log geometry has been such an inspiration.

1 An equivariant Riemann-Hilbert correspondence

1.1 Logarithmic and equivariant geometry

Smooth log schemes are locally modeled on affine monoid schemes, and the resulting toric geometry is a powerful tool in their analysis. We shall review the basic setup and techniques of affine monoid schemes (affine toric varieties) and then describe an equivariant Riemann-Hilbert correspondence for such schemes. This will be the main computational tool in our proof of the local and global correspondences in the next sections.

We start working over a commutative ring R, which later will become the field of complex numbers. All our monoids will be commutative unless otherwise stated. A monoid P is said to be *toric* if it is finitely generated, integral, and saturated and in addition P^{gp} is torsion free. If P is a monoid, we let $R[P]$ denote the monoid algebra of P over R, and write $e(p)$ or e_p for the element of $R[P]$ corresponding to an element p of P. If K is an ideal of P, we write $R[K]$ for the ideal of $R[P]$ generated by the elements of K and $R[P,K]$ for the quotient $R[P]/R[K]$. By an *idealized monoid* we mean a pair (P,K), where K is an ideal in a monoid P. Sometimes we simply write P for an idealized monoid (Q,K) and $R[P]$ for $R[Q,K]$.

We use the terminology of log geometry from, for example, [6]. Thus a *log scheme* is a scheme X, together with a sheaf of commutative monoids M_X on $X_{ét}$ and a morphism of sheaves of monoids α_X from M_X to the multiplicative monoid $\mathcal{O}_X$ which induces an isomorphism $\alpha_X^{-1}(\mathcal{O}_X^*) \to \mathcal{O}_X^*$. Then α induces an isomorphism from the sheaf of units M_X^* of M_X to $\mathcal{O}_X^*$; we denote by λ_X the inverse of this isomorphism and by $\overline{M}_X$ the quotient of M_X by $\mathcal{O}_X^*$. All our log schemes will be coherent, fine, and saturated; for the definitions and basic properties of these notions, we refer again to [6]. An *idealized log scheme* is a log scheme with a sheaf of ideals $K_X \subseteq M_X$ such that $\alpha_X(k) = 0$ for every local section k of K_X. A sheaf of ideals K_X of M_X is said to be *coherent* if it is locally generated by a finite number of sections, and we shall always assume this is the case. Morphisms of log schemes and idealized log schemes are defined in the obvious way. A morphism $f\colon X \to Y$ of fs idealized log schemes is *strict* if the induced map $f^{-1}\overline{M}_Y \to \overline{M}_X$ is an isomorphism, and it is *ideally strict* if the morphism $f^{-1}\overline{K}_Y \to \overline{K}_X$ is also an isomorphism.

If P is a monid, we let $\mathsf{A_P}$ denote the log scheme $\mathrm{Spec}(P \to R[P])$ and $\underline{\mathsf{A}}_\mathsf{P}$ its underlying scheme, *i.e.*, with trivial log structure. If A is an R-algebra, the set $\underline{\mathsf{A}}_\mathsf{P}(A)$ of A-valued points of $\underline{\mathsf{A}}_\mathsf{P}$ can be identified with the set of homomorphisms from the monoid P to the multiplicative monoid underlying A. This set has a natural monoid structure, and thus $\underline{\mathsf{A}}_\mathsf{P}$ can be viewed as a monoid object in the category of R-schemes. The canonical map $P \to P^{gp}$ induces a morphism $\mathsf{A}^*_P := \mathsf{A}_{\mathsf{P}^{\mathsf{gp}}} = \underline{\mathsf{A}}_{\mathsf{P}^{\mathsf{gp}}} \to \underline{\mathsf{A}}_\mathsf{P}$ which identifies A^*_P with the group scheme of units of $\underline{\mathsf{A}}_\mathsf{P}$. The natural morphism of log schemes $\mathsf{A_P} \to \underline{\mathsf{A}}_\mathsf{P}$ is injective on A-valued points, and its image coincides with the image of the map $\underline{\mathsf{A}}^*_\mathsf{P} \to \underline{\mathsf{A}}_\mathsf{P}$. If K is an ideal of P, the subscheme $\underline{\mathsf{A}}_{\mathsf{P},\mathsf{K}} := \mathrm{Spec}(R[P,K])$ it defines is invariant under the monoid action of $\underline{\mathsf{A}}_\mathsf{P}$ on itself, so that $\underline{\mathsf{A}}_{\mathsf{P},\mathsf{K}}$ defines an ideal of the monoid scheme $\underline{\mathsf{A}}_\mathsf{P}$. Also, K generates a (coherent) sheaf of ideals K_X in the sheaf of monoids M_X of $\mathsf{A_P}$, and the restrictions of M_X and K_X to $\underline{\mathsf{A}}_{\mathsf{P},\mathsf{K}}$ give it the structure of an idealized log scheme $\mathsf{A}_{\mathsf{P},\mathsf{K}}$. It can be shown that, using Grothendieck's definition of smoothness via ideally strict infinitesimal thickenings as in [6], the ideally smooth log schemes over $\mathrm{Spec}\, R$ are exactly those that are, locally in the étale topology, isomorphic to $\mathsf{A}_{\mathsf{P},\mathsf{K}}$ for some P and K. Note that these are the log schemes considered by Kato and Nakayama in [7].

Suppose from now on that P is toric. Then A^*_P is a torus with character group P^{gp}, and the evident map A^*_P to $\underline{\mathsf{A}}_\mathsf{P}$ is an open immersion. The complement F of a prime ideal $\mathfrak{p}$ of P is by definition a *face* of P. It is a submonoid of P, and there is a natural isomorphism of monoid algebras $R[F] \cong R[P]/\mathfrak{p}$, inducing an isomorphism $\underline{\mathsf{A}}_{\mathsf{P},\mathfrak{p}} \cong \underline{\mathsf{A}}_\mathsf{F}$. If k is an algebraically closed field, $\underline{\mathsf{A}}_{\mathsf{P},\mathfrak{p}}(k)$ is the closure of an orbit of the action of $\mathsf{A}^*_P(k)$ on $\underline{\mathsf{A}}_\mathsf{P}(k)$, and in this way the set of all faces of P parameterizes the set of orbits of $\underline{\mathsf{A}}_\mathsf{P}(k)$. In particular, the maximal ideal P^+ of P is the complement of the set of units P^* of P, and defines the minimal orbit of $\underline{\mathsf{A}}_\mathsf{P}$.

The map $P \to R$ sending every element of P^* to 1 and every element of P^+ to 0 is a homomorphism of monoids, and hence defines an R-valued point of $\underline{\mathsf{A}}_\mathsf{P}$, called the *vertex* of $\underline{\mathsf{A}}_\mathsf{P}$. The vertex belongs to $\underline{\mathsf{A}}_{\mathsf{P},\mathsf{K}}$ for every proper ideal K of P. By definition $\overline{P} := P/P^*$; and the surjection $P \to \overline{P}$ induces a strict closed immersion $\mathsf{A}_{\overline{\mathsf{P}}} \to \mathsf{A_P}$. The inclusion $P^* \to P$ defines a (log) smooth morphism $\mathsf{A_P} \to \mathsf{A}_{\mathsf{P}^*}$; note that $\mathsf{A}_{\mathsf{P}^*}$ is a torus and that $\mathsf{A}_{\overline{\mathsf{P}}}$ is the inverse image of its origin 1 under this map. Thus there is a Cartesian diagram:

$$\begin{array}{ccccc} v & \longrightarrow & \underline{\mathsf{A}}_{\overline{\mathsf{P}}} & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow \\ \underline{\mathsf{A}}_{\mathsf{P},\mathsf{P}^+} & \longrightarrow & \underline{\mathsf{A}}_{\mathsf{P}} & \longrightarrow & \mathsf{A}_{\mathsf{P}^*} \end{array}$$

The action of the torus A^*_P on $\underline{\mathsf{A}}_{\mathsf{P}}$ manifests itself algebraically in terms of a P^{gp}-grading on $R[P]$: $R[P]$ is a direct sum of R-modules $R[P] = \oplus\{A_p : p \in P^{gp}\}$, and the multiplication map sends $A_p \otimes A_q$ to A_{p+q}. Quasi-coherent sheaves on A_{P} which are equivariant with respect to the torus action correspond to P^{gp}-graded modules over $R[P]$.

More generally, if S is a P-set, there is a notion of an *S-graded $R[P]$-module.* This is an $R[P]$-module V together with a direct sum decomposition $V = \oplus\{V_s : s \in S\}$, such that for every $p \in P$, multiplication by $e_p \colon V \to V$ maps each V_s to V_{p+s}. For example, $R[S]$ is defined to be the free R-module generated by s in degree s, and if e_s is a basis in degree s and $p \in P$, $e_p e_s := e_{p+s}$. Morphisms of S-graded modules are required to preserve the grading. We denote by $Mod^S_R(P)$ the category of S-graded $R[P]$-modules and S-graded maps, and if K is an ideal of P, we denote by $Mod^S_R(P, K)$ the full subcategory consisting of those modules annihilated by K (*i.e.*, by the ideal of $R[P]$ generated by K). If the ring R is understood we may drop it from the notation.

Equivalently, one can work with *S-indexed R-modules.* Recall that the *transporter* of a P-set S is the category whose objects are the elements of S and whose morphisms from an object $s \in S$ to an object $s' \in S$ are the elements $p \in P$ such that $p+s = s'$, (with composition defined by the monoid law of P). Then an *S-indexed R-module* is by definition a functor F from the transporter of S to the category of R-modules. If F is an S-indexed R-module, then $\oplus\{F(s) : s \in S\}$ has a natural structure of an S-graded $R[P]$-module. This construction gives an isomorphism between the category of S-graded $R[P]$-modules and the category of S-indexed R-modules, and we shall not distinguish between these two notions in our notation. See also the discussion by Lorenzon [8].

If the action of P on S extends to a free action of P^{gp} on the localization of S by P we say that S is *potentially free.* If S is potentially free, then whenever s and s' are two elements of S and p is an element of P such that $s' = p + s$, then p is unique, and the transporter category of S becomes a pre-ordered set. In this case, an S-indexed module F for which all the transition maps are injective amounts to an S-filtered R-module, and the corresponding S-graded $R[P]$-module is torsion free.

In particular let $\phi \colon P \to Q$ be a morphism of monoids. Then Q inherits an action of P, and so it makes sense to speak of a *Q-graded $R[P]$-module.* The morphism ϕ also defines a morphism of monoid schemes $\underline{\mathsf{A}}_\phi \colon \underline{\mathsf{A}}_{\mathsf{Q}} \to \underline{\mathsf{A}}_{\mathsf{P}}$, and hence an action $\mu \colon \underline{\mathsf{A}}_{\mathsf{P}} \times \underline{\mathsf{A}}_{\mathsf{Q}} \to \underline{\mathsf{A}}_{\mathsf{P}}$ of $\underline{\mathsf{A}}_{\mathsf{Q}}$ on $\underline{\mathsf{A}}_{\mathsf{P}}$. A Q-grading on an

$R[P]$-module E then corresponds to an $\underline{\mathsf{A}}_{\mathsf{Q}}$-equivariant quasi-coherent sheaf $\tilde{E}$ on $\underline{\mathsf{A}}_{\mathsf{P}}$, *i.e.*, a quasi-coherent sheaf $\tilde{E}$ together with a linear map $\mu^*\tilde{E} \to pr_1^*\tilde{E}$ on $\underline{\mathsf{A}}_{\mathsf{P}} \times \underline{\mathsf{A}}_{\mathsf{Q}}$ satisfying a suitable cocycle condition. We shall be especially concerned with the case in which Q is a submonoid of $R \otimes P^{gp}$, or even all of $R \otimes P^{gp}$.

Remark 1.1.1 Let $\phi\colon P \to Q$ be a morphism of monoids, let S (resp. T) be a P-set (resp. a Q-set) and let $\psi\colon S \to T$ be a morphism of P-sets over ϕ. Then if E is an object of $Mod_R^S(P)$, the tensor product $R[Q] \otimes_{R[P]} E$ has a natural T-grading, uniquely determined by the fact that if $x \in E$ has degree s and $q \in Q$, then $e_q \otimes x$ has degree $q + \psi(s)$. This works because if $p \in P$, $(q + \phi(p)) + \psi(s) = q + \psi(p + s)$. We denote this T-graded $R[P]$-module by $\phi_\psi^*(E)$. If F is an object of $Mod_R^T(Q)$, then there is a natural map of $R[P]$-modules:

$$\phi_*^\psi(F) := \oplus_{s\in S} F_{\psi(s)} \to \phi_* F = \oplus_{t\in T} F_t.$$

Furthermore $\phi_*^\psi(F)$ is naturally S-graded, and the functor ϕ_*^ψ is right adjoint to the functor ϕ_ψ^*. For example, if P is the zero monoid and $T = Q^{gp}$, then the adjoint to the functor $Mod(R) \to Mod_R^{Q^{gp}}(Q)$ is the functor which takes a Q^{gp}-graded module to its component of degree zero. We denote this functor by π_*^Q.

Proposition 1.1.2 *Let P be an integral monoid, let S be a potentially free P-set, and view the orbit space S/P^* as a $\overline{P}$-set, so that the projection $\pi\colon S \to S/P^*$ is a morphism over the morphism $\pi\colon P \to \overline{P}$.*

1. *The base-change functor*

$$\pi_\pi^*\colon Mod_R^S(P) \to Mod_R^{S/P^*}(\overline{P})$$

is an equivalence of categories.

2. *If E is any object of $Mod_R^S(P)$, $\overline{E} := \pi_\pi^* E$, and $s \in S$ maps to $\overline{s} \in S/P^*$, then the natural map $E_s \to \overline{E}_{\overline{s}}$ is an isomorphism.*

Proof: Let I be kernel of the surjective map $R[P] \to R[\overline{P}]$. This is the ideal generated by the set of elements of the form $1 - e_u : u \in P^*$. If E is an object of $Mod_R^S(P)$, then $\overline{E} := \pi_\pi^* E \cong E/IE$. Since S is potentially free as a P-set, the action of the group P^* on S is free. Thus an element t of S/P^*, viewed as a subset of S, is a torsor under the action of P^*. Let $E_t := \oplus\{E_s : s \in t\}$. Then E_t has a natural action of $R[P^*]$ and $\overline{E}_t \cong E_t \otimes_{R[P^*]} R$, where $R[P^*] \to R$ is the map sending every element of P^* to 1_R. Let $t := \overline{s}$, and let J be the kernel of the augmentation map $R[P^*] \to R$. Since J and I have the same generators, $\overline{E}_t \cong E_t/JE_t$. For each $s' \in t$, there is a unique $u' \in P^*$ such that $s = u's'$, and multiplication by $e_{u'}$ defines an isomorphism $\iota_{s'}\colon E_{s'} \to E_s$. The sum of all these defines a morphism ι of R-modules $E_t \to E_s$. If $u \in P^*$ and $s' := us''$, then $u''u = u'$, and hence $\iota_{u''} \circ \cdot e_u = \iota_{u'}$. Thus ι factors through a morphism $\bar{\iota}$ of R-modules $E_t/JE_t \to E_s$. The inclusion $E_s \to E_t$ induces a section j: $\iota \circ j = \mathrm{id}$. Since the map $j\colon E_s \to E_t/JE_t$ is also evidently surjective, it is an isomorphism inverse to π_s. This proves (1.1.2.2), which implies that the functor π^*_π is fully faithful. One checks immediately that π^π_* is a quasi-inverse. □

With the notation of the proposition above, suppose that E is an S-graded $R[P]$-module. The map $\eta\colon R[P] \to R$ sending P to 1 can be thought of as a generic R-valued point of $\underline{\mathsf{A}}_{\mathsf{P}}$. Indeed, this map factors through $R[P^{gp}]$, and the above result shows that it induces an equivalence from the category of $S \otimes P^{gp}$-graded modules to the category of R-modules. Let E_η denote the R-module $\eta^* E$. For each $s \in S$, there is a map of R-modules

$$cosp_{s,\eta}\colon E_s \to E_\eta.$$

Corollary 1.1.3 *If E and $s \in S$ are as above, suppose that E is torsion free as an $R[P]$-module and also that it admits a set of homogeneous generators in degrees $t \leq s$ (i.e., for each generating degree t, there exists $p \in P$ with $s = p + t$). Then the cospecialization map $cosp_{s,\eta}$ is an isomorphism.*

Proof: Let $E' := E \otimes R[P^{gp}]$. Since E is torsion free, the map from E to E' is injective. The proposition shows that for any $s' \in S \otimes P^{gp}$, the map $E'_{s'} \to E_\eta$ is bijective. So it suffices to see that the map $E_s \to E_\eta$ is surjective. Any $x' \in E'_{s'}$ is a sum of elements of the form $e_q x_q$, where $q \in P^{gp}$ and $x_q \in E$ is a homogeneous generator of some degree $t \leq s$. Thus it suffices to show that if x' is equal to such an $e_q x_q$, then its image in E_η is in the image of E_s. Write $s = p + t$, with $p \in P$, so that $x' = e_q x_q = e_{q-p}(e_p x_q)$. Then $e_p x_q \in E_s$ has the same image in E_η as does x'. □

1.2 Equivariant differentials and connections

Let P be a toric monoid and let $X := \mathsf{A}_{\mathsf{P}}$; since X is affine, we may and shall identify quasi-coherent sheaves with $R[P]$-modules. We refer to [6] and [11] for the definitions and basic properties of the (log) differentials $\Omega^1_{X/R}$ and modules with connection on X/R. Recall in particular that $\Omega^1_{X/R}$ is the quasi-coherent sheaf on X corresponding to the $R[P]$-module

$$R[P] \otimes_{\mathbf{Z}} P^{gp} \cong R[P] \otimes_R \Omega_{P/R},$$

where $\Omega_{P/R} := R \otimes P^{gp}$. If $p \in P$, we sometimes denote by dp the class of $1 \otimes p^{gp}$ in $\Omega_{P/R}$ We write $\Omega^i_{P/R}$ for the ith exterior power of $\Omega_{P/R}$ and $T_{P/R}$ for its dual; we shall drop the subscripts if there seems to be no risk of confusion. An element p of P defines a global section $\beta(p)$ of M_X, and

$$dlog\, \beta(p) = dp = 1 \otimes p^{gp}$$

in $\Omega_{P/R} \subseteq \Omega^1_{X/R}$. Such an element p also defines a basis element e_p of $R[P]$, and $de_p = e_p dp \in \Omega^1_{X/R}$. The grading of $\Omega^1_{X/R}$ for which d is homogeneous of degree zero corresponds to the action of A^*_P on $\Omega^1_{X/R}$ induced by functoriality; under this action, $\Omega_{P/R} \subseteq \Omega^1_{X/R}$ is the set of invariant forms, *i.e.*, the component of degree zero. The dual $T_{P/R}$ of $\Omega_{P/R}$ can be thought of as the module of equivariant vector fields on $\mathsf{A_P}$. If E is an $R[P]$-module, a connection on the corresponding sheaf on X corresponds to a map

$$\nabla\colon E \to E \otimes_{R[P]} \Omega^1_{X/R} \cong E \otimes_R \Omega_{P/R},$$

and the Leibniz rule reduces to the requirement that

$$\nabla(e_p x) = e_p x \otimes dp + e_p \nabla(x).$$

for $p \in P$ and $x \in E$.

REMARK 1.2.1 If K is an ideal of P, let $\mathsf{A_{P,K}}$ be the idealized log subscheme of $\mathsf{A_P}$ defined by K. Then the structure sheaf of $\mathsf{A_{P,K}}$ corresponds to $R[P,K]$ and $\Omega^1_{X/R}$ to $R[P,K] \otimes_R \Omega^1_{P/R}$. Thus the category of modules with integrable connection on $\mathsf{A_{P,K}}/R$ can be identified with the full subcategory of modules with integrable connection on $\mathsf{A_P}/R$ annihilated by K. This remark reduces the local study of connections on idealized log schemes to the case in which the ideal is empty.

Suppose now that S is a P-set and (E, ∇) is an S-graded $R[P]$-module with an integrable log connection. The S-grading on E induces an S-grading on $\Omega^1_{P/R} \otimes E$; we say that ∇ is *homogeneous* if it preserves the grading. Thus for each $s \in S$ and $p \in P$, there is a commutative diagram

$$\begin{array}{ccc} E_s & \xrightarrow{\nabla + dp} & E_s \otimes_R \Omega_{P/R} \\ \Big\downarrow e_p & & \Big\downarrow e_p \\ E_{p+s} & \xrightarrow{\nabla} & E_{p+s} \otimes_R \Omega_{P/R} \end{array}$$

For example, the data of a homogeneous log connection on $R[S]$ amounts simply to a morphism of P-sets $d\colon S \to \Omega^1_{P/R}$. Note that such a morphism defines a pairing $\langle\ ,\ \rangle\colon T_{P/R} \times S \to R$.

DEFINITION 1.2.2 *Let (P,K) be an idealized monoid and R a ring. Then a set of* exponential data *for (P,K) over R is an abelian group Λ together with homomorphisms $P \to \Lambda$ and $\Lambda \to \Omega_{P/R}$ whose composition is the map $p \mapsto dp$. The data are said to be* rigid *if for every nonzero $\lambda \in \Lambda$, there exists a $t \in T_{P/R}$ such that $\langle t, \lambda \rangle \in R^*$.*

Typical examples are $\Lambda = P^{gp}$, $\Lambda = R \otimes P^{gp}$, and $\Lambda = k \otimes P^{gp}$, where k is a field contained in R. Rigidity implies that $\Lambda \to \Omega$ is injective, and is equivalent to this if R is a field. Note that if R is flat over $\mathbf{Z}$, the map $P^{gp} \to \Omega$ is also injective.

We sometimes just write Λ for the entire set of exponential data. Given such data, P acts on Λ and it makes sense to speak of a Λ-graded $R[P]$-module with homogeneous connection. For example, $R[P]$ can be viewed as a Λ-graded $R[P]$-module, where e_p is given degree $\delta(p)$ as in (1.1.1), and the connection d is Λ-graded. Because the homomorphism $\Lambda \to \Omega_{P/R}$ is also a map of P-sets, $R[\Lambda]$ also has such a structure. Associated to the map $P \to \Lambda$ is a map from the torus A_Λ to A_P and a consequent action of A_Λ on A_P. Then a Λ-graded $R[P]$-module with a homegeneous connection corresponds to a quasi-coherent sheaf with a connection on A_P which are together equivariant with respect to this action.

Definition 1.2.3 *Let (P, K) be an idealized toric monoid and let*

$$P \xrightarrow{\delta} \Lambda \xrightarrow{\epsilon} \Omega_{P/R}$$

be a set of exponential data for P/R.

1. *$MIC^\Lambda(P, K/R)$ is the category of Λ-graded $R[P]$-modules with homogeneous connection and morphisms preserving the connections and gradings.*
2. *An object (E, ∇) of $MIC^\Lambda(P, K/R)$ is said to be* normalized *if for every $t \in T_{P/R}$ and every $\lambda \in \Lambda$ the endomorphism of E_λ induced by $\nabla_t - \langle t, \lambda\rangle$ is locally nilpotent. The full subcategory of $MIC^\Lambda(P, K/R)$ consisting of the normalized (resp. of the normalized and finitely generated) objects is denoted by $MIC^\Lambda_*(P, K/R)$ (resp. $MIC^\Lambda_{coh}(P, K/R)$).*

Remark 1.2.4 Let $MIC(P, K/R)$ be the category of $R[P, K]$-modules with integrable log connection but no grading. If the exponential data are rigid, the functor $MIC^\Lambda_*(P, K/R) \to MIC(P, K/R)$ is fully faithful. To see this, note first that, since the category $MIC^\Lambda(P, K/R)$ has internal Hom's, it suffices to check that if (E, ∇) is an object of $MIC^\Lambda_*(P, K/R)$ and $e \in E$ is horizontal, then $e \in E_0$. In other words, we have to show that ∇ is injective on E_λ if $\lambda \neq 0$. Since the data are rigid, if $\lambda \neq 0$ there exists a $t \in T$ such that $\langle t, \lambda\rangle$ is a unit, and the action of ∇_t on E_λ can be written as $\langle t, \lambda\rangle$ plus a locally nilpotent endomorphism. It follows that ∇_t is an isomorphism on E_λ.

When the choice of Λ is clear or fixed in advance, we shall permit ourselves to drop it from the notation. We also sometimes use the same letter to denote an element of P or Λ and its image in Λ or $\Omega_{P/R}$. This is safe to do if the maps $P \to \Lambda$ and $\Lambda \to \Omega_{P/R}$ are injective.

Example 1.2.5 The differential $d\colon R[P, K] \to R[P, K] \otimes_R \Omega_{P/R}$ defines an object of $MIC^\Lambda_{coh}(P/R)$, for any Λ. More generally, choose $\lambda \in \Lambda$, and let

L^λ denote the free Λ-graded $R[P,K]$-module generated by a single element x_λ in degree λ, with the connection d such that $d(e_p x_\lambda) = e_p x_\lambda \otimes (dp + \epsilon(\lambda))$. If $t \in T_{P/R}$, then $d_t(e_p x_\lambda) = \langle t, p + \lambda\rangle$. Since $e_p x_\lambda$ has degree $\delta(p) + \lambda$ and $d_t - \langle t, dp + \epsilon(\lambda)\rangle = 0$ in this degree, L^λ belongs to $MIC^\Lambda_{coh}(P, K/R)$. For λ and λ' in Λ there is a homogeneous and horizontal isomorphism $L^\lambda \otimes L^{\lambda'} \to L^{\lambda+\lambda'}$ sending $x_\lambda \otimes x_{\lambda'}$ to $x_{\lambda+\lambda'}$, and in this way one finds a ring structure on the direct sum $\oplus\{L^\lambda : \lambda \in \Lambda\}$, compatible with the connection. This direct sum is in some sense a universal diagonal object of $MIC^\Lambda_*(P, K/R)$. The ring $\oplus_\lambda L^\lambda$ can be identified with the tensor product of the monoid algebras $R[P,K]$ and $R[\Lambda]$, or with the quotient of the monoid algebra $R[P \oplus \Lambda]$ of $P \oplus \Lambda$ by the ideal generated by K. We shall also denote it by $R[P,K,\Lambda]$. Note the unusual grading: the degree of $e_p x_\lambda$ is $\delta(p) + \lambda$. The ring $R[P,K,\Lambda]$ admits another Λ grading, in which $e_p x_\lambda$ has degree λ. In fact it is naturally $\Lambda \oplus \Lambda$ graded. For convenience, shall set $\Lambda' := \Lambda$ and say that $e_p x_{\lambda'}$ has Λ-*degree* $\delta(p) + \lambda'$ and Λ'-*degree* λ'. When we need to save space, we shall let P stand for the pair (P,K) and just write $R[P,\Lambda]$ instead of $R[P,K,\Lambda]$.

Example 1.2.6 One can also construct a universal nilpotent object as follows. Let $\Omega := \Omega_{P/R}$, and for each $n \in \mathbf{N}$, let $\Omega \to \Gamma^n(\Omega)$ denote the universal polynomial law of degree n [2, Appendix A] over R. Thus, $\Gamma^n(\Omega)$ is the R-linear dual of the nth symmetric power of $T_{P/R}$, and $\Gamma^\cdot(\Omega) := \oplus_n \Gamma^n(\Omega)$ is the divided power polynomial algebra on Ω. It has an exterior derivative d which maps $\Gamma^n(\Omega)$ to $\Gamma^{n-1}(\Omega) \otimes \Omega$, defined by

$$d\omega_1^{[I_1]} \cdots \omega_n^{[I_n]} := \sum_i \omega_1^{[I_1]} \cdots \omega_i^{[I_i - 1]} \cdots \omega_n^{[I_n]} \otimes \omega_i \tag{1.2.1}$$

Of course, if R is a $\mathbf{Q}$-algebra, $\Gamma^n(\Omega)$ can be identified with the nth symmetric power of Ω. Let $N(P,K) := R[P,K] \otimes_R \Gamma^\cdot(\Omega)$, graded so that $\Gamma^\cdot(\Omega)$ has degree zero, and let

$$\nabla \colon N(P,K) \to N(P,K) \otimes_R \Omega_{P/R} := d \otimes \mathrm{id} + \mathrm{id} \otimes d$$

be the extension of d satisfying the Leibniz rule with respect to $R[P]$. Then $N(P,K) \in MIC^\Lambda_*(P,K)$. Note that $N_*(P,K)$ has an exhaustive filtration $F_.$, where $F_n := \sum_{i \le n} R[P,K] \otimes \Gamma^i(\Omega)$, and the associated graded connection is constant.

1.3 Equivariant Higgs fields

Let X be a smooth scheme over R, let $\Omega_{X/R}$ be its sheaf of Kahler differentials, and let $T_{X/R}$ be the dual of $\Omega_{X/R}$. Recall [13] that a *Higgs field* on a sheaf F of $\mathcal{O}_X$-modules is an $\mathcal{O}_X$-linear map $\theta \colon F \to F \otimes \Omega^1_{X/R}$ such that the composite $F \to F \otimes \Omega^1_{X/R} \to F \otimes \Omega^2_{X/R}$ vanishes. Such a θ is equivalent to an action of the symmetric algebra $S^\cdot T_{X/R}$ on F, and hence defines a sheaf of $\mathcal{O}_{\mathbf{T}^*_{X/R}}$-modules, where $\mathbf{T}^*_{X/R} := \mathbf{V}T_{X/R} := \mathrm{Spec}_X\, S^{\cdot}T_{X/R}$ is the cotangent bundle of X/R. One can prolong a Higgs field θ to a complex

$$K^{\cdot}(F,\theta) := F \to F \otimes \Omega^1_{X/R} \to F \otimes \Omega^2_{X/R} \to \cdots$$

with $\mathcal{O}_X$-linear boundary maps induced by θ, called the *Higgs complex* of (F,θ). All these constructions make sense with $T_{X/R}$ replaced by any locally free sheaf T of $\mathcal{O}_X$-modules, and we call (F,θ) an *$\mathcal{O}_X$-T-module* or *T-Higgs-module* in the general case.

One can define internal tensor products and duals in the category of T-Higgs modules in the same way one does for modules with connection. For example, if θ and θ' are T-Higgs fields on F and F' respectively, then $\theta \otimes \mathrm{id} + \mathrm{id} \otimes \theta'$ is the Higgs field on $F \otimes F'$ used to define the internal tensor product. If ω is a section of the dual Ω of T, the *ω-twist* of a T-Higgs field θ is the T-Higgs field $\theta + \mathrm{id} \otimes \omega$. An R-T module (F,θ) is said to be *nilpotent* if θ_t defines a locally nilpotent endomorphism of F for every $t \in T$. This means that the corresponding sheaf on $\mathbf{V}T$ is supported on the zero section.

A *Jordan decomposition* of a T-Higgs module (E,θ) is a direct sum decomposition $E \cong \oplus E_\omega : \omega \in \Omega$ such that each E_ω is invariant under θ and is the ω-twist of a nilpotent T-Higgs module. For example, if R is an algebraically closed field and E is finitely generated, then E can be viewed as a module of finite length over $S^{\cdot}T$ and its support is a finite subset of the maximal spectrum of $S^{\cdot}T$, which can be canonically identified with Ω. Thus E admits a canonical Jordan decomposition $E \cong \oplus E_\omega$.

The following simple and well-known vanishing lemma will play a central role.

Lemma 1.3.1 *Let (F,θ) be a T-Higgs module and suppose there exists a $t \in T$ such that θ_t is an automorphism of F. Then the Higgs complex $K^{\cdot}(F,\theta)$ is homotopic to zero, hence acyclic.*

Proof: Interior multiplication by t defines a sequence of maps

$$\rho^i \colon F \otimes \Omega^i \to F \otimes \Omega^{i-1}.$$

One verifies easily that $\kappa := d\rho + \rho d$ is $\theta_t \otimes \mathrm{id}$. Thus θ_t induces the zero map on cohomology, and since θ_t is an isomorphism, the cohomology vanishes. □

We shall see that there is a simple relationship between equivariant Higgs fields and equivariant connections. In fact there are two constructions we shall use.

Definition 1.3.2 *Let P be an idealized toric monoid and $P \xrightarrow{d} \Lambda \xrightarrow{\epsilon} \Omega_{P/R}$ a set of exponential data for P.*

1. *$HIG^\Lambda(P/R)$ is the category of Λ-graded $R[P]$-$T_{P/R}$ modules. That is, the objects are pairs (E, θ), where E is a Λ-graded $R[P]$-module and*

$$\theta \colon E \to E \otimes_R \Omega_{P/R}$$

is a homogeneous map such that $\theta \wedge \theta = 0$, and the morphisms are the degree preserving maps compatible with θ.

2. *An object (E, θ) of $HIG^\Lambda(P/R)$ is* nilpotent *if for every $t \in T_{P/R}$, the endomorphism θ_t of E is locally nilpotent. The full subcategory of $HIG^\Lambda(P/R)$ consisting of nilpotent (resp., the nilpotent and finitely generated objects) is denoted by $HIG^\Lambda_*(P/R)$ (resp., $HIG^\Lambda_{coh}(P/R)$).*

Example 1.3.3 If $\lambda \in \Lambda$, let L^λ be the free Λ-graded $R[P]$-module generated in degree λ by x_λ. Then there is a unique $T_{P/R}$-Higgs field θ on L^λ such that $\theta(e_p x_\lambda) = e_p x_\lambda \otimes \epsilon(\lambda)$ for each $p \in P$. The isomorphism $L^\lambda \otimes L^{\lambda'} \to L^{\lambda+\lambda'}$ sending $x_\lambda \otimes x_{\lambda'}$ to $x_{\lambda+\lambda'}$ is compatible with the induced Higgs fields, so we get a Higgs field θ on $R[P, \Lambda] = \oplus L^\lambda$, compatible with the ring structure. Similarly there is a unique Higgs field on $N(P) = R[P] \otimes \Gamma^{\cdot}(\Omega)$ such that

$$\omega_1^{[I_1]} \cdots \omega_n^{[I_n]} \mapsto \sum_i \omega_1^{[I_1]} \cdots \omega_i^{[I_i - 1]} \cdots \omega_n^{[I_n]} \otimes \omega_i$$

for all I.

Let (E, ∇) be an object of $MIC^\Lambda(P/R)$. We can forget the $R[P]$-module structure of E and view it as an R-module. Since $T_{P/R}$ is a finitely generated free R-module, $\nabla \colon E \to E \otimes \Omega_{P/R}$ can be viewed as a $T_{P/R}$-Higgs field on E. If R is an algebraically closed field and E is finite dimensional over R, such fields are easy to analyze, using the Jordan decomposition. We can generalize this as follows.

Lemma 1.3.4 *Let $P \to \Lambda \to \Omega_{P/R}$ be a rigid set of exponential data for an idealized monoid P.*

1. *Let (E, ∇) be an object of $MIC(P, K/R)$. Suppose the corresponding $T_{P/R}$-Higgs module (E, ∇) admits a Jordan decomposition $E = \oplus E_\lambda$. Then this direct sum decomposition gives E the structure of a Λ-graded $R[P, K]$-module, and with this structure, $(E, \nabla) \in MIC^\Lambda_*(P, K/R)$. Thus, $MIC^\Lambda_*(P, K/R)$ is equivalent to the full subcategory of $MIC^\Lambda(P, K/R)$ whose corresponding $T_{P/R}$-Higgs modules admit a Jordan decomposition.*

2. *If $(E, \nabla) \in MIC^\Lambda_*(P, K/R)$, then its de Rham complex is acyclic except in degree zero.*

Proof: Let $\theta_\lambda := \nabla - \mathrm{id} \otimes \lambda$. The Leibniz rule implies that for each $p \in P$ and $t \in T$, $\theta_{t,\lambda+p} \circ e_p = e_p \circ \theta_{t,\lambda}$. It follows that $\theta^n_{t,\lambda+p} \circ e_p = e_p \circ \theta^n_{t,\lambda}$, for every $n \geq 0$. If $x \in E_\lambda$, then x is killed by some power of $\theta_{t,\lambda}$, and hence $e_p x$ is killed by some power of $\theta_{t,\lambda+p}$. For any λ', $\theta_{t,\lambda'} = \theta_{t,\lambda+p} + \langle t, \lambda' - p - \lambda\rangle$. If $\lambda' \neq p+\lambda$, we can choose t so that $\langle t, \lambda' - p - \lambda\rangle$ is a unit, and hence $\theta_{t,\lambda+p}$ acts injectively on $E_{\lambda'}$. It follows that the degree λ' piece of $e_p x$ is zero. In other words, e_p maps E_λ to $E_{p+\lambda}$. This shows that $\oplus E_\lambda$ gives E the structure of a Λ-graded $R[P]$-module. Evidently each E_λ is invariant under ∇, and killed by K, and with this grading, $(E, \nabla) \in MIC^\Lambda_*(P, K/R)$. We have already remarked in (1.2.4) that $MIC^\Lambda_*(P, K/R)$ is a full subcategory of $MIC(P, K/R)$. The $T_{P/R}$-Higgs module associated to every object of $MIC^\Lambda_*(P, K/R)$ admits a Jordan decomposition, by definition, and the above argument show that the converse is also true. This proves (1).

Let (E, ∇) be an object of $MIC^\Lambda_*(P, K/R)$. Its de Rham complex is Λ-graded, and its component in degree λ can be viewed as the Higgs complex associated to the linear map $\nabla \colon E_\lambda \to E_\lambda \otimes \Omega_{P/R}$. If $\lambda \neq 0$, then there exists a $t \in T_P$ such that $\langle t, \lambda\rangle$ is not zero, hence a unit. Since E is normalized, $\nabla_t - \langle t, \lambda\rangle$ is nilpotent, and hence ∇_t an isomorphism, in degree λ. By (1.3.1), this implies that the complex $E \otimes \Omega^{\cdot}_{P/R}$ is acyclic in degree λ and proves (2). $\square$

In general, suppose that E is an object of $MIC^\Lambda(P/R)$. Then the degree λ component of ∇ is a Higgs field on E_λ. Then

$$\theta_\lambda := \nabla - \mathrm{id}_{E_\lambda} \otimes \lambda \quad : \; E_\lambda \to E_\lambda \otimes_R \Omega^1_{P/R}$$

is another Higgs field, and evidently (E, ∇) is normalized if and only if this field is nilpotent for every $\lambda \in \Lambda$. Moreover, $\theta := \oplus_\lambda \theta_\lambda$ is $R[P]$-linear, and endows E with the structure of an equivariant $R[P]$-$T_{P/R}$-module. This Higgs module structure can be viewed as the difference between the given connection ∇ and the "trivial" connection coming from the action of Λ. This simple construction evidently gives a complete description of the category of (normalized) equivariant connections in terms of the category of (nilpotent) equivariant Higgs modules, and it will play a crucial role in our proof of the equivariant Riemann-Hilbert correspondence.

We shall see that the above correspondence can be expressed in terms of a suitable "integral transform." As it turns out, this integral transform introduces a sign. To keep things straight, we introduce the following notation. Let

$$P \xrightarrow{\delta} \Lambda \xrightarrow{\epsilon} \Omega_{P/R}$$

be a set of exponential data for a toric monoid P. Let $P' := -P \subseteq P^{gp}$, let $\Lambda' := \Lambda$, let $\epsilon' := \epsilon$, and let $\delta' \colon P' \to \Lambda'$ be the composite of the inclusion

$-P \to P^{gp}$ with $\delta^{gp} \colon P^{gp} \to \Lambda$. Thus we have a commutative diagram:

$$\begin{array}{ccccc} P & \xrightarrow{\delta} & \Lambda & \xrightarrow{\epsilon} & \Omega_{P/R} \\ \Big\downarrow{\scriptstyle -\mathrm{id}} & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle \mathrm{id}} \\ P' & \xrightarrow{-\delta'} & \Lambda' & \xrightarrow{\epsilon'} & \Omega_{P'/R}. \end{array}$$

(Note that the vertical arrow on the right is the map induced by the identity map $P^{gp} \to P^{gp} = (-P)^{gp}$ and is the negative of the map induced by functoriality from the isomorphism $P' \to P$.)

In the context of the above set-up, there is a completely trivial equivalence between the categories $Mod_R^{\Lambda}(P, K)$ and $Mod_R^{\Lambda'}(P', K')$, where $K' := -K$. Namely, if $(E, \nabla) \in Mod_R^{\Lambda}(P, K)$, then for each $\lambda' \in \Lambda' = \Lambda$, let $E'_{\lambda'} := E_{-\lambda'}$. If $p' \in P'$, $-p' \in P$, and one can define

$$\cdot e_{p'} \colon E'_{\lambda'} \to E'_{\lambda'+p'}$$

to be multiplication by $e_{-p'}$. This gives $\oplus E'_{\lambda'}$ the structure of a Λ'-graded $R[P', K']$-module, and it is evident that the functor $E \mapsto E'$ is an equivalence. This is too trivial to require a proof, but since it will be very useful in our following constructions, it is worth stating for further reference.

Proposition 1.3.5 *Let (P, K) be an idealized toric monoid endowed with exponential data $P \xrightarrow{\delta} \Lambda \xrightarrow{\epsilon} \Omega$ and let $P' \xrightarrow{\delta'} \Lambda' \xrightarrow{\epsilon'} \Omega$ be the corresponding exponential data for (P', K').*

1. *The functor $Mod_R^{\Lambda}(P, K) \to Mod_R^{\Lambda'}(P', K')$ described above is an equivalence of categories, compatible with tensor products and internal Hom.*

2. *If $(E, \nabla) \in MIC^{\Lambda}(P, K/R)$, let E' be the object of $Mod_R^{\Lambda'}(P', K')$ corresponding to E, and define $\theta' \colon E' \to E' \otimes_R \Omega$ by the following diagram:*

$$\begin{array}{ccc} E_\lambda & \xrightarrow{=} & E'_{-\lambda} \\ \Big\downarrow{\scriptstyle \nabla - \mathrm{id} \otimes \epsilon(\lambda)} & & \Big\downarrow{\scriptstyle \theta'} \\ E_\lambda \otimes \Omega & \xrightarrow{=} & E'_{-\lambda} \otimes \Omega \end{array}$$

Then θ' defines a Higgs field on E', and the corresponding functor $MIC^{\Lambda}(P, K/R) \to HIG^{\Lambda'}(P', K'/R)$ is an equivalence. Under this functor,an object (E, ∇) is normalized if and only if the corresponding (E', θ') is nilpotent.

□

The value of the above proposition will be enhanced by the fact that its functors can be realized geometrically, using the ring $R[P,\Lambda]$ described in (1.3.3) and (1.2.5). (Here P stands for an idealized monoid (P,K).)

We have morphisms of monoids:

$$\begin{array}{rcl} \phi\colon P \to P\oplus\Lambda & : & p \mapsto (p,0) \\ \eta\colon P\oplus\Lambda \to \Lambda & : & (p,\lambda) \mapsto \delta(p)+\lambda \\ \phi'\colon P' \to P\oplus\Lambda & : & p' \mapsto (-p',\delta'(p')) \\ \psi\colon P\oplus\Lambda \to \Lambda\oplus\Lambda & : & (p,\lambda) \mapsto (\delta(p)+\lambda,\lambda) \\ \pi\colon \Lambda \to \Lambda\oplus\Lambda & : & \lambda \mapsto (\lambda,0) \\ \pi'\colon \Lambda \to \Lambda\oplus\Lambda & : & \lambda \mapsto (0,\lambda) \end{array}$$

These fit into commutative diagrams:

$$\begin{array}{ccc} P & \xrightarrow{\phi} & P\oplus\Lambda \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\psi} \\ \Lambda & \xrightarrow{\pi} & \Lambda\oplus\Lambda \end{array} \qquad \begin{array}{ccc} P' & \xrightarrow{\phi'} & P\oplus\Lambda \\ {\scriptstyle\delta'}\downarrow & & \downarrow{\scriptstyle\psi} \\ \Lambda & \xrightarrow{\pi'} & \Lambda\oplus\Lambda \end{array}$$

Note that $\eta\circ\phi' = 0$ and that $\underline{\mathsf{A}}_\phi$ corresponds to the projection $q\colon \underline{\mathsf{A}}_{\mathsf{P}}\times\underline{\mathsf{A}}_{\Lambda} \to \underline{\mathsf{A}}_{\mathsf{P}}$. Let $q' := \underline{\mathsf{A}}_{\phi'}$. Then there is a commutative diagram:

$$\begin{array}{ccccc} \underline{\mathsf{A}}_{\Lambda} & \xrightarrow{\underline{\mathsf{A}}_\eta} & \underline{\mathsf{A}}_{\mathsf{P}}\times\underline{\mathsf{A}}_{\Lambda} & \xrightarrow{q'} & \underline{\mathsf{A}}_{\mathsf{P'}} \\ & {\scriptstyle\underline{\mathsf{A}}_\delta}\searrow & \downarrow{\scriptstyle q=pr_1} & & \\ & & \mathsf{A}_{\mathsf{P}} & & \end{array} \tag{1.3.1}$$

In this diagram, $\underline{\mathsf{A}}_\eta$ is a closed immersion, and identifies $\underline{\mathsf{A}}_{\Lambda}$ with ${q'}^{-1}(1_{\underline{\mathsf{A}}_{\mathsf{P'}}})$.

Recall from (1.2.5) that $R[P,\Lambda]$ is a $\Lambda\oplus\Lambda'$-graded ring, where $e_p x_{\lambda'}$ has degree $(\delta(p)+\lambda',\lambda')$. Thus, a Λ-Λ'-graded $R[P,\Lambda]$-module is an $R[P,\Lambda]$-module $\tilde{E}$ together with a direct sum decomposition into sub R-modules $\tilde{E} = \oplus\tilde{E}_{\lambda,\lambda'}$, such that multiplication by $e_p x_\mu$ maps $\tilde{E}_{\lambda,\lambda'}$ into $\tilde{E}_{\delta(p)+\lambda+\mu,\lambda'+\mu}$. The category of such objects (with bihomogeneous morphisms) will be denoted by $Mod_{\Lambda}^{\Lambda'}(P,\Lambda)$. The pair of morphisms (ϕ,π) induces a functor

$$q_\pi^*\colon Mod_R^{\Lambda}(P) \to Mod_{\Lambda}^{\Lambda'}(P,\Lambda) \quad : \quad E \mapsto E\otimes_{R[P]} R[P,\Lambda] \cong E\otimes_R R[\Lambda],$$

where $E\otimes_R R[\Lambda]$ is graded so that $e\otimes x_{\lambda'}$ has bidegree $(\lambda+\lambda',\lambda')$ if $e\in E$ has degree λ, as discussed in (1.1.1). Recall that its left adjoint, which we denote by q^π_* or $q^{\Lambda'}_*$, takes an object of $Mod^{\Lambda'}_\Lambda(P,\Lambda)$ to the Λ-graded $R[P]$-submodule consisting of the elements whose Λ'-degree is zero.

Recall that the connection ∇ on $R[P,\Lambda]$ sends $e_p x_\lambda$ to $e_p x_\lambda \otimes (dp + \epsilon(\lambda))$. In particular, $e_p x_{-p}$ is horizontal. This implies that, when $R[P,\Lambda]$ is regarded as an $R[P']$-module via q'^*, ∇ is $R[P']$-linear, and in fact defines an element of $HIG^{\Lambda'}(P'/R)$. More generally, if $(E,\nabla) \in MIC^\Lambda(P/R)$, the tensor product connection $\tilde{\nabla}$ on $q^*_\pi(E)$ is an equivariant Higgs field on the $R[P']$-module $q'_* q^*_\pi(E)$. On the other hand, if θ' is an equivariant Higgs field on a Λ'-graded $R[P']$-module E', the tensor product Higgs field $\tilde{\theta} := d \otimes \mathrm{id} + \mathrm{id} \otimes \theta'$ on $q'^*_{\pi'} E'$ is a connection over $R[P]$. Thus we have functors

$$\begin{aligned} q'^\Lambda_* q^* &: MIC^\Lambda(P/R) \to HIG^{\Lambda'}(P'/R) \\ q^{\Lambda'}_* q'^* &: HIG^{\Lambda'}(P'/R) \to MIC^\Lambda(P/R) \end{aligned} \tag{1.3.2}$$

Remark 1.3.6 Let $R[P',\Lambda']$ be the ring constructed from $P' \to \Lambda'$ the same way $R[P,\Lambda]$ was constructed from $R[P,\Lambda]$. Then $R[P',\Lambda']$ is a Λ'-Λ-graded R-algebra, where $e_{p'} x_{\lambda'}$ has degree $(p'+\lambda',\lambda')$. The isomorphism of monoids $P \oplus \Lambda \to P' \oplus \Lambda$ sending (p,λ) to $(-p, p+\lambda)$ induces an isomorphism of $R[\Lambda]$-algebras

$$\iota \colon R[P,\Lambda] \to R[P',\Lambda'] \quad : \quad e_p x_\lambda \mapsto e_{-p} x_{p+\lambda}.$$

It takes elements of degree (λ,λ') to elements of degree (λ',λ). Its inverse ι' is constructed from the data $P' \to \Lambda'$ just as ι was constructed from $P \to \Lambda$, and the map $q'^* \colon R[P'] \to R[P,\Lambda]$ is just the inclusion $R[P'] \to R[P',\Lambda']$ followed by ι'. Since Λ' is a group, Proposition (1.1.2) implies that q^*_π is an equivalence: $Mod^\Lambda_R(P) \to Mod^{\Lambda'}_\Lambda(P,\Lambda/R)$, with quasi-inverse $q^{\Lambda'}_*$. Of course, ι^* is also an equivalence, and hence so are the functors in (1.3.2).

Proposition 1.3.7 *The equivalence in Proposition (1.3.5) is given by the functors (1.3.2).*

Proof: For any $\lambda \in \Lambda$, x_λ is a unit of $R[P,\Lambda]$ and $\iota(x_\lambda) = x_\lambda \in R[P',\Lambda']$. Then multiplication by $x_{-\lambda}$ induces an isomorphism $q^*_\pi E \to \iota^* q^*_\pi E$ which takes E_λ to $E'_{-\lambda}$; this is the isomorphism in (1.3.5.2). If $e \in E_\lambda$,

$$\tilde{\nabla}(x_{-\lambda} e) = x_{-\lambda}\nabla(e) + (\nabla x_{-\lambda})e = x_{-\lambda}(\nabla e - \epsilon(\lambda)e) = x_{-\lambda}\theta'(e).$$

This proves the commutativity of the diagram in (1.3.5.2). □

1.4 Equivariant Riemann-Hilbert

Now let $R = \mathbf{C}$, and let $P \to \Lambda \to \Omega$ be a rigid set of exponential data. The universal cover of the analytic torus $\underline{\mathsf{A}}^{*an}_P$ is the exponential map

$$\exp \colon \mathbf{V}\Omega^{an} \to \underline{\mathsf{A}}^{*an}_P,$$

which we can describe as follows. Recall that $\mathbf{V}\Omega$ is the spectrum of the symmetric algebra $S^{\cdot}(\Omega)$, which is isomorphic to $\Gamma^{\cdot}(\Omega)$, since we are in characteristic zero. Thus the set of points of $\mathbf{V}\Omega^{an}$ is just $T := \mathrm{Hom}(P^{gp}, \mathbf{C})$, and an element ω of Ω defines a function on $\mathbf{V}\Omega^{an}$ whose value at $t \in T$ is $\langle t, \omega\rangle$. Then exp is the map taking an additive homomorphism $t\colon P^{gp} \to \mathbf{C}$ to the multiplicative homomorphism $\exp\circ t\colon P^{gp} \to \mathbf{C}^*$. The kernel of this map is the group $\mathrm{Hom}(P^{gp}, \mathbf{Z}(1))$, where $\mathbf{Z}(1)$ is the subgroup of $\mathbf{C}$ generated by $2\pi i$. Thus there is a canonical isomorphism:

$$\pi_1(P) := Hom(P^{gp}, \mathbf{Z}(1)) \cong \pi_1(\underline{\mathsf{A}}_P^{*an}) = Aut(\mathbf{V}\Omega^{an}/\,\underline{\mathsf{A}}_P^{*an}). \qquad (1.4.1)$$

We shall now introduce an "equivariant Riemann-Hilbert transform" which classifies objects of $MIC_*^{\Lambda}(P)$ in terms of suitably normalized graded representations of the fundamental group $\pi_1(P)$.

Definition 1.4.1 *Let (P, K) be an idealized toric monoid with a rigid set of exponential data $P \to \Lambda \to \Omega_{P/\mathbf{C}}$. Then $L^{\Lambda}(P, K)$ is the category of pairs (V, ρ), where V is a Λ-graded $\mathbf{C}[P, K]$-module and ρ is a homogeneous action of $\pi_1(P)$ on V. An object (V, ρ) of $L^{\Lambda}(P, K)$ is said to be* normalized *if for every $\gamma \in \pi_1(P)$ and $\lambda \in \Lambda$, the action of $\rho_\gamma - \exp\langle\gamma, \lambda\rangle$ on V_λ is locally nilpotent. The full subcategory of $L^{\Lambda}(P, K)$ consisting of the normalized objects (resp. the normalized and finitely generated objects) is denoted by $L_*^{\Lambda}(P, K)$ (resp. $L_{coh}^{\Lambda}(P, K)$).*

Note that the normalization condition in the definition above is compatible with multiplication by elements of $\mathbf{C}[P]$. More precisely, if $\lambda \in \Lambda$, $p \in P$, and $\gamma \in \pi_1(P)$, then multiplication by e_p takes V_λ to $V_{p+\lambda}$, and $\rho_\gamma \circ (\cdot e_p) = (\cdot e_p) \circ \rho_\gamma$. Moreover, $\langle p, \gamma\rangle \in \mathbf{Z}(1)$, so $\exp\langle\gamma, p + \lambda\rangle = \exp\langle\gamma, p\rangle$.

Remark 1.4.2 If P is a finitely generated abelian free group and $\Lambda = \mathbf{C} \otimes P$, the category $L_{coh}^{\Lambda}(P)$ can be simplified: it is equivalent to the category of finite dimensional $\mathbf{C}$-vector spaces equipped with an action of $\pi_1(P)$. More generally, let P be any idealized toric monoid, let Λ be a subgroup of $\mathbf{C} \otimes P^{gp}$ containing P^{gp} and let $\overline{\Lambda}$ be the image of Λ in $\mathbf{C} \otimes \overline{P}^{gp}$. Note that $\pi_1(\overline{P}) \subseteq \pi_1(P)$. Let $\overline{L}_{coh}^{\Lambda}(P)$ denote the category of finitely-generated $\overline{\Lambda}$-graded-$\mathbf{C}[\overline{P}]$-modules W equipped with an action ρ of $\pi_1(P)$ such that for each $\gamma \in \pi_1(\overline{P})$ and each $\overline{\lambda} \in \overline{\Lambda}$, the action of $\rho_\gamma e^{-\langle\gamma, \overline{\lambda}\rangle}$ on $W_{\overline{\lambda}}$ is unipotent. Then the evident functor (tensoring with $\mathbf{C}[\overline{P}]$) is an equivalence of categories:

$$L_{coh}^{\Lambda}(P) \to \overline{L}_{coh}^{\Lambda}(P).$$

Here is a sketch of why this is so. To see that it is fully faithful, let V be an object of $L_{coh}^{\Lambda}(P)$ and let $\overline{V} := V \otimes_{\mathbf{C}[P]} \mathbf{C}[\overline{P}]$; it is enough to prove that the natural map $V_0^{\pi_1} \to \overline{V}_{\overline{0}}^{\pi_1}$ is an isomorphism. Let $\Lambda^* := \Lambda \cap (\mathbf{C} \otimes P^*)$ and let $V_{\Lambda^*} := \oplus V_\lambda : \lambda \in \Lambda^*$. Then V_{Λ^*} is a Λ^*-graded $\mathbf{C}[P^*]$-module, and $\overline{V}_{\overline{0}}$ is the quotient of V_{Λ^*} by IV_{Λ^*}, where I is the kernel of the map $\mathbf{C}[P^*] \to \mathbf{C}$ sending every element of P^* to 1. Note that I is the $\mathbf{C}$-submodule of $\mathbf{C}[P^*]$ generated by the set of all $e_u - e_v : u, v \in P^*$. We have an exact sequence:

$$0 \to IV_{\Lambda^*} \to V_{\Lambda^*} \to \overline{V}_{\overline{0}} \to 0,$$

which remains exact if we restrict to the subspace on which the action of $\pi_1(\overline{P})$ is unipotent. The coherence of V implies that the unipotent part of V_{Λ^*} is exactly V_{P^*}, and $IV_{\Lambda^*} \cap V_{P^*} = IV_{P^*}$. Thus there is an exact sequence;

$$0 \to IV_{P^*} \to V_{P^*} \to \overline{V}_{\overline{0}}^{un} \to 0.$$

That is, $\overline{V}_{\overline{0}}^{un} := V_{P^*}/IV_{P^*} \cong V_{P^*} \otimes_{\mathbf{C}[P^*]} \mathbf{C}$. Then by (1.1.2), the natural map $V_0 \to \overline{V}_{\overline{0}}^{un}$ is an isomorphism, and it follows that $V_0^{\pi_1} \to \overline{V}_{\overline{0}}^{\pi_1}$ is an isomorphism, as desired.

For the essential surjectivity, let W be an object of $\overline{L}_{coh}^{\Lambda}(P)$. For each $\overline{\lambda} \in \overline{\Lambda}$, $W_{\overline{\lambda}}$ is a finite dimensional $\mathbf{C}[\pi_1(P)]$-module, and hence can be written as a direct sum of submodules $W_{\overline{\lambda},\chi}$, where χ ranges over the set S of homomorphisms $\pi_1(P) \to \mathbf{C}^*$. If $\lambda \in \Lambda$, let $e^{\lambda}\colon \pi_1 \to \mathbf{C}^*$ be the homomorphism taking $\gamma \in \pi_1(P)$ to $e^{\langle\gamma,\lambda\rangle}$. By hypothesis, if $W_{\overline{\lambda},\chi} \neq 0$, the restriction of χ to $\pi_1(\overline{P})$ is $e^{\overline{\lambda}}$. This implies that there exists a $\lambda \in \Lambda$ which maps to $\overline{\lambda}$ and such that $e^{\lambda} = \chi$, and the set of such λ is a torsor under P^*. For each $\lambda \in \Lambda$, let $V_\lambda := W_{\overline{\lambda},e^{\lambda}}$, and for $p \in P$, let multiplication by $e_p\colon V_\lambda \to V_{p+\lambda}$ be multiplication by $e_{\overline{p}}$. Then $\oplus V_\lambda$ is the desired object of $L_{coh}^{\Lambda}(P)$. $\square$

We can now define the equivariant Riemann-Hilbert correspondence:

$$\mathcal{V}\colon MIC_*^{\Lambda}(P,K) \to L_*^{\Lambda}(P',K').$$

Again we use the exponential data for P' deduced from the given exponential data for P. If (E,∇) is an object of $MIC_*^{\Lambda}(P,K)$, let V be its corresponding $\mathbf{C}[P',K']$-module, as described in (1.3.5). View $-\nabla$ as defining a Higgs field on the underlying $\mathbf{C}$-module of V, and let ρ be the corresponding action of $\pi_1(P)$:

$$\rho_\gamma := \exp(-\nabla_\gamma) \text{ for } \gamma \in \pi_1(P) \subseteq T.$$

Note that ρ_γ preserves the Λ-grading. It also commutes with the action of $\mathbf{C}[P']$ on V. To see this, recall that if $p \in P$, $\nabla_\gamma \circ \cdot e_{-p} = \cdot e_{-p} \circ (\nabla_\gamma - \cdot\langle\gamma,p\rangle)$, by the Leibniz rule. Hence

$$\rho_\gamma \circ \cdot e_{-p} = \exp(-\nabla_\gamma) \circ \cdot e_{-p} = \cdot e_{-p} \circ \exp(\nabla_{-\gamma} + \cdot\langle\gamma,p\rangle) = e_{-p} \circ \rho_\gamma \circ \cdot \exp\langle\gamma,p\rangle,$$

and $\exp\langle\gamma,p\rangle = 1$. Note also that if $\gamma \in \pi_1(P)$, $\nabla_\gamma - \langle\gamma,\lambda\rangle$ is locally nilpotent on E_λ. Hence $\exp(\nabla_\gamma)e^{-\langle\gamma,\lambda\rangle}$ is locally unipotent on E_λ and $\rho_\gamma e^{\langle\gamma,\lambda\rangle}$ is locally unipotent on $V_{-\lambda}$. Hence $\rho_\gamma - e^{-\langle\gamma,\lambda\rangle}$ is locally nilpotent on $V_{-\lambda}$, so $(V,\rho) \in L_*^{\Lambda}(P',K')$.

Proposition 1.4.3 *Let $P \to \Lambda \to \Omega_{P/\mathbf{C}}$ be a rigid set of exponential data for an idealized toric monoid (P, K), and let $P' \to \Lambda \to \Omega_{P/\mathbf{C}}$ be the corresponding exponential data for P'. The equivariant Riemann-Hilbert correspondence described above defines an equivalence of tensor categories*

$$\mathcal{V}\colon MIC_*^\Lambda(P, K) \to L_*^\Lambda(P', K').$$

If $(E, \nabla) \in MIC_*^\Lambda(P, K)$ and $(V, \rho) := \mathcal{V}(E, \nabla)$, then there is a canonical isomorphism

$$H^i_{DR}(E, \nabla) \cong H^i(\pi_1(P), V_0)$$

for all i. Moreover, if $\lambda \in \Lambda \setminus P^{gp}$, then $H^i(\pi_1(P), V_\lambda) = 0$ for all i.

Proof: It follows immediately from the construction that $\mathcal{V}$ is compatible with tensor product and duality, hence with internal Hom. To prove that it is fully faithful, it suffices to prove that if (E, ∇) is an object of $MIC_*^\Lambda(P)$ and $V = \mathcal{V}(E, \nabla)$, the map $E_0^\nabla \to V_0^{\pi_1}$ is an isomorphism. For each $\gamma \in \pi_1$, ∇_γ defines a nilpotent endomorphism of E_0, and it will suffice to prove that if $e \in E_0$, $\nabla_\gamma(e) = 0$ if and only if $\rho_\gamma(e) = e$. This follows from the formulas:

$$\begin{aligned} \rho_\gamma &= \mathrm{id} - \nabla_\gamma + \frac{\nabla_\gamma^2}{2!} - \cdots \\ -\nabla_\gamma &= (\rho_\gamma - 1) - \frac{(\rho_\gamma - 1)^2}{2} + \cdots \end{aligned}$$

More generally, one has the following result, which implies the statement about cohomology.

Lemma 1.4.4 *Let (E, θ) be a nilpotent $T_{P/\mathbf{C}}$-Higgs module and let $V := E$ with the action of $\pi := \pi_1(P)$ defined by $\rho_\gamma := \exp(-\theta_\gamma)$ for $\gamma \in \pi_1(P)$. Then there are natural isomorphisms:*

$$H^i_{HIG}(E, \theta) \cong H^i(\pi, (V, \rho)).$$

for all i.

Proof: The category of representations of π is equivalent to the category of $\mathbf{Z}[\pi]$-modules, and if M is such a module, $H^i(\pi, M) \cong \mathrm{Ext}^i_{\mathbf{Z}[\pi]}(\mathbf{Z}, M)$, where $\mathbf{Z}$ is the trivial module. Let $P^\cdot$ be a finitely generated and projective resolution of $\mathbf{Z}$ over $\mathbf{Z}[\pi]$. As a sequence of $\mathbf{Z}$-modules, $P^\cdot$ is split, and hence it remains exact when tensored over $\mathbf{Z}$ with any ring R. It follow that, if V is an R-module, $\mathrm{Ext}^i_{R[\pi]}(R, V) \cong \mathrm{Ext}^i_{\mathbf{Z}[\pi]}(\mathbf{Z}, V)$ for every i. Applying this with $R = \mathbf{C}$, we see that $H^i(\pi, V) \cong \mathrm{Ext}^i_{\mathbf{C}[\pi]}(\mathbf{C}, V)$ for all i. If the action of π on V is unipotent, then V is in fact a module for the formal completion $\hat{\mathbf{C}}[\pi]$ of $\mathbf{C}[\pi]$ at the vertex. Since this completion is flat over $\mathbf{C}[\pi]$, it follows that the natural map

$$\mathrm{Ext}^i_{\mathbf{C}[\pi]}(\mathbf{C}, V) \to \mathrm{Ext}^i_{\hat{\mathbf{C}}[\pi]}(\mathbf{C}, V)$$

is an isomorphism.

Let $Y := \mathsf{A}_\pi = \operatorname{Spec} \mathbf{C}[\pi]$, let $T := \mathbf{C} \otimes \pi$, and suppose that E and V are as in the lemma. The exponential map induces an isomorphism of formal schemes $\hat{\mathbf{V}}T \to \hat{Y}$, where $\hat{\mathbf{V}}T$ is the formal completion of $\mathbf{V}T$ along the zero section and $\hat{Y}$ is the formal completion of Y at the vertex. Under this isomorphism, if $\gamma \in \pi$, $\exp^* \gamma = \mathrm{id} + \gamma + \gamma^2/2! + \cdots$. The Higgs module (E, θ) can be thought of as quasi-coherent sheaf on $\mathbf{V}T$. Since E is nilpotent, it is supported on the zero section, and, up to a sign, $V \cong \exp_* E$. By [1], the Higgs cohomology of E is $\operatorname{Ext}^i_{S^\cdot T}(\mathbf{C}, E)$, where $\mathbf{C}$ corresponds to the zero section of $\mathbf{V}T$. As before, this Ext remains the same when computed on the formal completion. Thus

$$H^i_{HIG}(E, \theta) \cong \operatorname{Ext}^i_{S^\cdot T}(\mathbf{C}, E) \cong \operatorname{Ext}^i_{\hat{S}^\cdot T}(\mathbf{C}, E) \cong \operatorname{Ext}^i_{\hat{\mathbf{C}}[\pi]}(\mathbf{C}, V) \cong H^i(\pi, V).$$

□

To prove that $\mathcal{V}$ is essentially surjective, let (V, ρ) be an object of $L_*(P', K')$, and for each $\lambda \in \Lambda$ let $E_\lambda := V_{-\lambda}$, so that $\oplus E_\lambda$ is a Λ-graded $\mathbf{C}[P]$-module. For $\gamma \in \pi_1$, $\rho_\gamma e^{\langle \gamma, \lambda \rangle}$ induces a unipotent automorphism u_γ of E_λ, and hence $\log u_\gamma := (u_\gamma - 1) - \frac{(u_\gamma - 1)^2}{2} + \cdots$ is well defined and nilpotent. Let $\nabla_\gamma := -\log u_\gamma + \langle \gamma, \lambda \rangle$. Then $\exp(-\nabla_\gamma) = \rho_\gamma$. Furthermore, $\nabla_{\gamma_1 + \gamma_2} = \nabla_{\gamma_1} + \nabla_{\gamma_2}$, and $\nabla_\gamma \circ e_p = e_p \nabla_\gamma + \langle \gamma, dp \rangle$. Thus $(E, \nabla) \in MIC^\Lambda_*(P, K)$ and $\mathcal{V}(E, \nabla) = (V, \rho)$, so that $\mathcal{V}$ is essentially surjective. □

Remark 1.4.5 If $(E, \nabla) \in MIC^\Lambda_*(P, K)$, then its cohomology vanishes except in degree zero. This is not true for objects of $L^\Lambda_*(P', K')$, and this is why we have to specify taking the degree zero part in the isomorphism on cohomology. On the other hand, if $\lambda \in \Lambda \setminus P^{gp}$, then the support of V_λ (regarded as a sheaf on A_π) does not meet the vertex, so its cohomology is zero.

There is an evident functor $L^\Lambda_{coh}(P) \to L^\Lambda_{coh}(P^{gp})$. Recall from (1.4.2) that in the latter category, the grading is superfluous, and that the functor can be viewed as the functor which takes V to $V \otimes_{\mathbf{C}[P]} \mathbf{C}$ via the map $\mathbf{C}[P] \to \mathbf{C}$ sending P to 1. This corresponds to evaluating a "generic point" and so we denote the corresponding module by V_η. There is a cospecialization map $V \to V_\eta$ and hence a map on cohomology.

Corollary 1.4.6 *Let V be a torsion free object of $L^\Lambda_{coh}(P)$ and let $D \subseteq \Lambda$ be the set of the degrees of a minimal set of homogeneous generators for V. Suppose that $D \cap P^{gp} \subseteq -P$. Then the natural map*

$$H^i(\pi_1(P), V_0) \to H^i(\pi_1(P), V_\eta)$$

is an isomorphism.

Proof: Let $V' := \sum \{V_\lambda : \lambda \in P^{gp}\}$. Remark (1.4.5) shows that the natural map $H^i(\pi_1(P), V') \to H^i(\pi_1(P), V)$ is an isomorphism, and the same is true for V_η. Thus we may as well assume that $V' = V$. But then Corollary (1.1.3) shows that the hypothesis on the degrees of the generators implies that the natural map $V_0 \to V_\eta$ is an isomorphism. □

As stated, Proposition (1.4.3) is too artificial to be of much value. We shall show that in fact it can be formulated in a more geometric manner which we can then use in our proof of the global Riemann-Hilbert correspondence.

Tensoring together the fundamental examples $\mathbf{C}[P,\Lambda]$ (1.2.5) and $N(P)$ (1.2.6), we obtain the $\mathbf{C}[P]$-algebra

$$J(P,\Lambda) := \mathbf{C}[P,\Lambda] \otimes_{\mathbf{C}} \Gamma^{\cdot}(\Omega) \cong \mathbf{C}[P,\Lambda] \otimes_{\mathbf{C}[P]} N(P).$$

It has a connection ∇ and a Higgs field θ as explained in Example (1.3.3).

The connection ∇ is in some sense the universal connection in Jordan normal form. Indeed, we shall see that $J(P,\Lambda)$ can be viewed as a ring of multivalued functions which is large enough to solve all the differential equations coming from objects of $MIC^{\Lambda}_{*}(P/R)$. This fact is the main computational tool underlying the equivariant Riemann-Hilbert correspondence. First let us attempt to explain its geometric meaning.

The map $\delta\colon P \to \Lambda$ induces a map $\underline{\mathsf{A}}_{\Lambda} \to \underline{\mathsf{A}}_{\mathsf{P}}$. Let us write α for the canonical map from the analytic space $X^{an} \to X$.

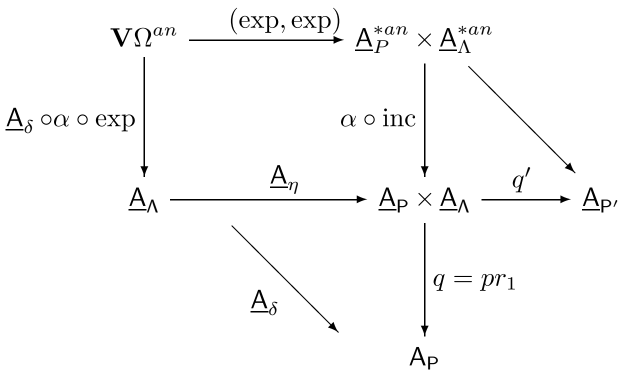
The rings of functions $\mathbf{C}[P]$ and $\mathbf{C}[\Lambda]$ on $\underline{\mathsf{A}}_{\mathsf{P}}$ and $\underline{\mathsf{A}}_{\Lambda}$ map to the ring of analytic functions on $\mathbf{V}\Omega^{an}$. For example, if $p \in P$ and $t \in T$,

$$\exp^*(e_p)(t) = \exp\langle t, dp\rangle.$$

Thus, the function associated to p is the logarithm of the function associated to e_p. Similarly, if $\lambda \in \Lambda$, then

$$\exp^*(x_\lambda)(t) := \exp\langle t, \delta\lambda\rangle,$$

so that we have maps exp from $\mathbf{V}\Omega^{an}$ to $\underline{\mathsf{A}}^{*an}_{P}$ and to $\underline{\mathsf{A}}^{*an}_{\Lambda}$. There is a commutative diagram (see (1.3.1))



Thus we obtain a map from $J(P,\Lambda)$ to the ring of analytic functions on $\mathbf{V}\Omega^{an}$. The group $\pi_1(P)$ acts on the ring of analytic functions on $\mathbf{V}\Omega^{an}$ by transport of structure, and preserves the subalgebra $\Gamma^{\cdot}(\Omega)$ of algebraic functions on $\mathbf{V}\Omega^{an}$ as well as the subring $\mathbf{C}[P,\Lambda]$. Let us make this explicit.

Lemma 1.4.7 *If $\gamma \in \pi_1(P)$, let ρ_γ act on $J(P,\Lambda)$ by*

$$\rho_\gamma := \exp(\theta_\gamma) = \exp \nabla_\gamma := e^{\nabla_\gamma} := \mathrm{id} + \frac{\nabla_\gamma}{1!} + \frac{\nabla_\gamma^2}{2!} + \cdots .$$

Then this action is compatible with the action on $\mathbf{V}\Omega$ via the exponential map and the diagram above.

Proof: The action of $\pi_1(P)$ on $\mathbf{V}\Omega^{an} = T$ is via translation: $\rho_\gamma(t) = t + \gamma$ if $\gamma \in \pi_1(P)$ and $t \in T := \mathrm{Hom}(P^{gp}, \mathbf{C})$. The induced action on the analytic functions on $\mathbf{V}\Omega^{an}$ is then by transport of structure, and in particular is by ring automorphisms. On the other hand, if $\gamma \in \pi_1(P)$ and $f_i \in J(P,\Lambda)$, then $\nabla_\gamma(f_1 + f_2) = \nabla_\gamma(f_1) + \nabla_\gamma(f_2)$, and $\nabla_\gamma(f_1 f_2) = \nabla_\gamma(f_1) f_2 + \nabla_\gamma(f_2) f_1$. It follows that $\exp(\nabla_\gamma)$ is also a ring automorphism of $J(P,\Lambda)$. Thus it suffices to check the compatibility of $\exp \nabla$ and ρ on a set of generators of the algebra $J(P,\Lambda)$. In particular, it suffices to check it for $\omega \in \Omega \subseteq \Gamma^{\cdot}(\Omega)$, $x_\lambda \in J(P,\Lambda)$, and $e_p \in \mathbf{C}[P]$. First of all, ∇_γ maps Ω to $\mathbf{C}$ and is zero on $\mathbf{C}$, and hence

$$e^{\nabla_\gamma}(\omega) = \omega + \nabla_\gamma(\omega) = \omega + \langle \gamma, \omega \rangle.$$

Thus

$$\begin{aligned} \langle t, e^{\nabla_\gamma}(\omega) \rangle &= \langle t, \omega \rangle + \langle \gamma, \omega \rangle \\ &= \langle t + \gamma, \omega \rangle \\ &= \langle \rho_\gamma(t), \omega \rangle \\ &= \langle t, \rho_\gamma(\omega) \rangle \end{aligned}$$

On the other hand, if $\lambda \in \Lambda$, $\nabla_\gamma(x_\lambda) = \langle \gamma, \lambda \rangle x_\lambda$, so $e^{\nabla_\gamma}(x_\lambda) = e^{\langle \gamma, \lambda \rangle} x_\lambda$. Pulling back to $\mathbf{V}\Omega^{an}$ and evaluating at t, we get

$$\begin{aligned} \langle t, e^{\nabla_\gamma}(x_\lambda) \rangle &= e^{\langle \gamma, \lambda \rangle} \langle t, x_\lambda \rangle \\ &= e^{\langle \gamma, \lambda \rangle} e^{\langle t, \lambda \rangle} \\ &= e^{\langle t + \gamma, \lambda \rangle} \\ &= \exp^*(x_\lambda)(t + \gamma) \\ &= \exp^*(x_\lambda)(\rho_\gamma t) \\ &= \exp^*(\rho_\gamma(x_\lambda))(t) \end{aligned}$$

Finally, if $p \in P$, $\rho_\gamma(e_p) = e_p$, and since $\nabla_\gamma e_p = \langle \gamma, p \rangle e_p$ and $\langle \gamma, p \rangle e_p \in \mathbf{Z}(1)$, e_p is also fixed by $\exp(\tilde{\nabla}_\gamma)$. This proves the compatibility of ρ with ∇. On the other hand, $\theta_\gamma(e_p x_\lambda \omega) = \nabla_\gamma(x_\lambda \omega) - \langle \gamma, p \rangle$, and $\langle \gamma, p \rangle \in \mathbf{Z}(1)$. Hence $\exp(\theta_\gamma) = \exp(\nabla_\gamma)$, and so ρ is also compatible with θ. □

Regarded as a $\mathbf{C}[P]$-module via the map q_*, $(J(P,\Lambda), d)$ is an object of $MIC_*^\Lambda(P)$. Regarded as a $\mathbf{C}[P']$-module via the map q'_*, $(J(P,\Lambda), \rho)$ is an object of $L^\Lambda(P')$, where $\rho := \exp(\nabla)$, since ∇ (hence ρ) is $\mathbf{C}[P']$-linear over q'_*. Let us check that it is normalized. Every element of degree λ' of $q'_*(J(P,\Lambda))$ can be written as a sum of elements of the form $e_p w x_{\lambda'}$ with $p \in P$, $w \in \Gamma^\cdot(\Omega_{P/\mathbf{C}})$, and $\lambda' \in \Lambda$, and

$$\rho_\gamma(e_p w x_{\lambda'}) = e^{\langle \gamma, p+\lambda'\rangle}(\exp \nabla_\gamma)(w) = e^{\langle\gamma,\lambda'\rangle}(\exp\nabla_\gamma)w.$$

Since $\exp\nabla_\gamma$ is locally unipotent on $\Gamma^\cdot(\Omega_{P/\mathbf{C}})$, $e^{\langle\gamma,\lambda'\rangle}\exp(\nabla_\gamma) - e^{\langle\gamma,\lambda'\rangle}$ is locally nilpotent. Note also that ρ_γ commutes with the connection ∇.

Now we can give description of the equivariant Riemann-Hilbert correspondence as an integral transform. If (E,∇) is an object of $MIC^\Lambda(P/\mathbf{C})$, let J^*E be $\tilde E := E \otimes_{\mathbf{C}[P]} J(P,\Lambda)$ with the Λ-Λ'-grading and connection $\tilde\nabla$ as described in the discussion preceding (1.3.2), and with the action $\tilde\rho$ of $\pi_1(P)$ defined by $\mathrm{id}_E \otimes \rho$. If (V,ρ) is an object of $L_*^{\Lambda'}(P')$, let $J'^*(V)$ be $\tilde V := V \otimes_{R[P']} J(P,\Lambda)$, with the Λ-Λ'-grading as above, with $\tilde\nabla := \mathrm{id}\otimes d$, and with $\tilde\rho$ the tensor product action. In both cases, we end up with a $J(P,\Lambda)$-module endowed with a Λ-Λ'-grading, a connection, and an action of $\pi_1(P)$. Let q'^{∇}_* be the functor which takes such an object to its horizontal sections, regarded as a Λ'-graded $\mathbf{C}[P']$-module with an action of $\pi_1(P)$. Also, let q_*^{Λ',π_1} denote the part of Λ'-degree zero which is fixed by $\tilde\rho$, regarded a Λ-graded $\mathbf{C}[P]$-module with connection.

Theorem 1.4.8 *Let $P \to \Lambda \to \Omega_{P/\mathbf{C}}$ be a rigid set of exponential data for an idealized toric monoid and let $P' \to \Lambda \to \Omega_{P/\mathbf{C}}$ be the corresponding exponential data for P'.*

1. *The functors*
$$\mathcal{V} := q'^{\nabla}_* J^* \colon MIC_*^\Lambda(P) \to L_*^\Lambda(P')$$
and
$$\mathcal{E} := q_*^{\Lambda',\pi_1} J'^* \colon L_*^\Lambda(P') \to MIC_*^\Lambda(P)$$
are the functors in the equivariant Riemann-Hilbert correspondence (1.4.3).

2. *If $(E,\nabla) \in MIC_*^\Lambda(P)$, let $(\tilde E, \tilde\nabla, \tilde\rho) := J^*(E)$. Then in the category $L_*^\Lambda(P')$,*
$$H^i_{DR}(\tilde E, \tilde\nabla, \rho) = \begin{cases} 0 & \text{if } i > 0 \\ \mathcal{V}(E,\nabla) & \text{if } i = 0. \end{cases}$$
Furthermore, the natural map $\mathcal{V}(E,\nabla)\otimes_{\mathbf{C}[P']} J(P,\Lambda) \to \tilde E$ is an isomorphism.

3. *If $(V,\rho) \in L_*^{\Lambda'}(P')$, let $(\tilde V, \tilde\nabla, \tilde\rho) := J^*(V,\rho)$. Then in the category $MIC_*^\Lambda(P)$,*
$$H^i(\pi_1(P), (\tilde V, \tilde\nabla, \rho))_{\Lambda'=0} = \begin{cases} 0 & \text{if } i > 0 \\ \mathcal{E}(V,\rho) & \text{if } i = 0. \end{cases}$$

Furthermore, the natural map $\mathcal{E}(V,\rho)\otimes_{\mathbf{C}[P]} J(P,\Lambda)\to\tilde{V}$ *is an isomorphism.*

We give the proof in the next section, where we deduce it from a more abstract construction which we call, for want of a better name, the "Jordan transform."

1.5 The Jordan transform

Most of the real work in this section makes sense over an arbitrary $\mathbf{Q}$-algebra R, so we temporarily revert to this generality. To simplify the notation, we let P be an idealized toric monoid (previously denoted (P,K)), and we let $P\to\Lambda\to\Omega$ be a rigid set of exponential data. We have seen in (1.2.5) and (1.3.3) that $R[P,\Lambda]$ carries a connection ∇ and a Higgs field θ relative to $R[P]$. Note that this is *not* the Higgs field θ' constructed from ∇ as in (1.3.5). To emphasize the symmetric nature of the constructions, we now write ∇' for θ. Indeed, ∇ is a Higgs field relative to $R[P']\subseteq R[P,\Lambda]$, and ∇' is a connection relative to $R[P']$. Note that ∇' and ∇ commute.

Let us summarize the structures $J(P,\Lambda):=R[P,\Lambda]\otimes_R\Gamma^{\cdot}(\Omega)$ carries.

1. It has a Λ-grading, where $e_px_\lambda\omega^{[i]}$ has degree $p+\lambda$, and there is a Λ-graded homomorphism
$$q\colon R[P]\to J(P,\Lambda)\quad:\quad e_p\mapsto e_px_0.$$

2. It has a second Λ-grading, (called the Λ'-grading) where $e_px_\lambda\omega^{[i]}$ has Λ'-degree λ, and a Λ'-graded homomorphism
$$q'\quad:\quad R[P']\to J(P,\Lambda)\quad:\quad e'_{p'}\mapsto e_{-p'}x_{p'}.$$

3. There is a map $\nabla\colon J(P,\Lambda)\to J(P,\Lambda)\otimes_R\Omega_{P/R}$ such that
$$\nabla\colon e_px_\lambda\omega^{[i]}\mapsto e_px_\lambda\omega^{[i]}\otimes(p+\lambda)+e_px_\lambda\omega^{[i-1]}\otimes\omega.$$
Then $q_*(J(P,\Lambda),\nabla)\in MIC^{\Lambda}_*(P/R)$, and $q'_*(J(P,\Lambda),\nabla)\in HIG^{\Lambda'}(P'/R)$.

4. There is a map $\nabla'\colon J(P,\Lambda)\to J(P,\Lambda)\otimes_R\Omega_{P/R}$ such that
$$\nabla'\colon e_px_\lambda\omega^{[i]}\mapsto e_px_\lambda\omega^{[i]}\otimes\lambda+e_px_\lambda\omega^{[i-1]}\otimes\omega.$$
Then $q_*(J(P,\Lambda),\nabla')\in HIG^{\Lambda}(P/R)$, and $q'_*(J(P,\Lambda),\nabla')\in MIC^{\Lambda'}_*(P'/R)$.

Note also that the set of elements of degree zero with respect to the Λ'-grading is just $R[P]\otimes\Gamma^{\cdot}(\Omega)$. Similarly, the set of elements of degree zero with respect to the Λ-grading is $R[P']\otimes\Gamma^{\cdot}(\Omega)$.

Let $MH^{\Lambda}_{\Lambda'}(P/R)$ denote the category of Λ-Λ'-graded $J(P,\Lambda)$-modules equipped with structures parallel to those of $J(P,\Lambda)$. In particular, an object $\tilde{E}$ of $MH^{\Lambda}_{\Lambda'}(P/R)$ is equipped with two commuting homogeneous maps:

$$\tilde{\nabla}, \tilde{\nabla}' \colon \tilde{E} \to \tilde{E} \otimes_R \Omega_{P/R}$$

where $\tilde{\nabla}$ is a homogeneous connection relative to $R[P]$ and a homogeneous Higgs structure relative to $R[P']$, and $\tilde{\nabla}'$ is a Higgs structure relative to $R[P]$ and a connection relative to $R[P']$.

Consider then the following functors:

1. If $(E,\nabla) \in MIC^{\Lambda}(P/R)$, let $J^*(E) := E \otimes_{R[P]} J(P,\Lambda)$, with the tensor product gradings, in which E is viewed as having Λ'-degree zero, and let $\tilde{\nabla} := \nabla \otimes \mathrm{id} + \mathrm{id} \otimes \nabla$ and $\tilde{\nabla}' := \mathrm{id}_E \otimes \nabla'$. Then $(J^*(E), \tilde{\nabla}, \tilde{\nabla}') \in MH^{\Lambda}_{\Lambda'}(P/R)$.

2. If $(E',\nabla') \in MIC^{\Lambda}(P'/R)$, let $J'^*(E') := E' \otimes_{R[P']} J(P,\Lambda)$ with the tensor product gradings, in which E' is viewed as having Λ'-degree zero, and let $\tilde{\nabla}' := \nabla' \otimes \mathrm{id} + \mathrm{id} \otimes \nabla'$, and $\tilde{\nabla} := \mathrm{id}_{E'} \otimes \nabla$. Then $(J'^*(E'), \tilde{\nabla}, \tilde{\nabla}') \in MH^{\Lambda}_{\Lambda'}(P/R)$.

3. If $(\tilde{E}, \tilde{\nabla}, \tilde{\nabla}') \in MH^{\Lambda}_{\Lambda'}(P/R)$, let $E := q^{\nabla'}_*(\tilde{E})$ (resp., $q^{\Lambda'}_*(\tilde{E})$) denote the elements which are killed by $\tilde{\nabla}'$ (resp., and of Λ'-degree zero.) Then E is a Λ-graded $R[P]$-module with a connection ∇ induced by $\tilde{\nabla}$, and $(E,\nabla) \in MIC^{\Lambda}(P/R)$.

4. If $(\tilde{E}, \tilde{\nabla}, \tilde{\nabla}') \in MH^{\Lambda}_{\Lambda'}(P/R)$, let $E' := q'^{\nabla}_*(\tilde{E})$ (resp., $E' := q'^{\Lambda}_*(\tilde{E})$) denote the elements which are killed by $\tilde{\nabla}$ (resp., and of Λ-degree zero.) Then E' is a Λ'-graded $R[P']$-module, with a connection ∇' induced by $\tilde{\nabla}'$, and $(E',\nabla') \in MIC^{\Lambda'}(P'/R)$.

Theorem 1.5.1 *Let $P \xrightarrow{\delta} \Lambda \xrightarrow{\epsilon} \Omega_{P/R}$ be a rigid set of exponential data for a toric idealized monoid. Then the functor $q'^{\nabla}_* J^*$ described above defines an equivalence of categories*

$$MIC^{\Lambda}_*(P/R) \to MIC^{\Lambda'}_*(P'/R).$$

This functor is compatible with tensor products and formation of cohomology, and has as quasi-inverse the functor $q^{\nabla'}_* J'^*$. Moreover:

1. *If $(E,\nabla) \in MIC^{\Lambda}_*(P/R)$ corresponds to $(E',\nabla') \in MIC^{\Lambda'}_*(P'/R)$, then for each λ there is a commutative diagram:*

$$\begin{array}{ccc} E_\lambda & \xrightarrow{\cong} & E'_{-\lambda} \\ {\scriptstyle -\nabla}\downarrow & & \downarrow{\scriptstyle \nabla'} \\ E_\lambda \otimes \Omega_{P/R} & \xrightarrow{\cong} & E'_{-\lambda} \otimes \Omega_{P/R} \end{array}$$

2. *If* $(E,\nabla) \in MIC^{\Lambda}_{*}(P/R)$, *then*

$$H^i_{DR}(J^*(E),\tilde{\nabla}) = \begin{cases} E' := q'^{\nabla}_{*} J^*(E) & \text{if } i = 0 \\ 0 & \text{if } i > 0. \end{cases}$$

$$H^i_{HIG}(J^*(E),\tilde{\nabla}') = \begin{cases} E & \text{if } i = 0 \\ 0 & \text{if } i > 0. \end{cases}$$

Furthermore, the natural map $E' \otimes_{R[P']} J(P,\Lambda) \to J^*(E)$ *is an isomorphism.*

3. *If* $(E',\nabla') \in MIC^{\Lambda'}_{*}(P'/R)$, *then*

$$H^i_{HIG}(J'^*(E'),\tilde{\nabla}') = \begin{cases} E := q^{\nabla'}_{*} J'^*(E') & \text{if } i = 0 \\ 0 & \text{if } i > 0. \end{cases}$$

$$H^i_{DR}(J'^*(E'),\tilde{\nabla}) = \begin{cases} E' & \text{if } i = 0 \\ 0 & \text{if } i > 0. \end{cases}$$

Furthermore, the natural map $E \otimes_{R[P]} J(P,\Lambda) \to J'^*(E')$ *is an isomorphism.*

We begin with some preliminary lemmas.

Lemma 1.5.2 *Let* (E,∇) *be an object of* $MIC^{\Lambda}_{*}(P/R)$.

1. *Let* $K^{\cdot}$ *be the De Rham complex of* (E,∇) *and let* $K^{\cdot}_{\Lambda=0}$ *be its degree zero part (with respect to the* Λ*-grading). Then the map* $K^{\cdot}_{\Lambda=0} \to K^{\cdot}$ *is a quasi-isomorphism.*

2. *Let* $\tilde{K}^{\cdot}$ *be the Higgs complex of* $(J^*(E),\tilde{\nabla}')$, *and let* $\tilde{K}^{\cdot}_{\Lambda'=0}$ *be its degree zero part with respect to the* Λ'*-grading. Then the map* $\tilde{K}^{\cdot}_{\Lambda'=0} \to \tilde{K}^{\cdot}$ *is a quasi-isomorphism.*

Proof: The first statement is an immediate consequence of (1.3.4.2). Let

$$E'' := E \otimes_R \Gamma^{\cdot}(\Omega) \subseteq \tilde{E} := J^*(E,\nabla) \cong E \otimes_R \Gamma^{\cdot}(\Omega) \otimes R[\Lambda] \cong E'' \otimes_{R[P]} R[P,\Lambda].$$

Then $E'' = \tilde{E}_{\Lambda'=0}$, and the action of $\tilde{\nabla}'$ on E'' is nilpotent. For $\lambda' \in \Lambda$, the action of $\tilde{\nabla}'$ on the degree λ'-component of $\tilde{E}$ is $\nabla'_{E''} + \mathrm{id} \otimes \lambda'$. By (1.3.1), its Higgs complex is then acyclic if $\lambda' \neq 0$. This proves (2). □

Lemma 1.5.3 *Let* T *be a free* R*-module with basis* $(t_1, \ldots t_n)$, *and let* Ω *be the dual of* T, *with dual basis* $(\omega_1, \ldots \omega_n)$. *If* (V,θ) *is a locally nilpotent* T*-Higgs module, let*

$$E'' := V \otimes_R \Gamma^{\cdot}(\Omega), \quad \text{and} \quad \nabla'' := \theta \otimes \mathrm{id} + \mathrm{id} \otimes d.$$

Let $\partial_i := \nabla''_{t_i}$ *and* $h\colon E'' \to E''$ *be* $\sum_I (-1)^{|I|} \omega^{[I]} \partial^I$, *where the sum is taken over all multi-indices* $I = (I_1, \ldots I_n)$ *with* $I_i \in \mathbf{N}$.

1. *h is independent of the bases, and defines a projection operator with image $E''^{\nabla''}$.*

2. *$E''^{\nabla''}$ is invariant under* $\mathrm{id} \otimes d$, *and h induces an isomorphism $h' \colon V \to E''^{\nabla''}$ fitting into a commutative diagram:*

$$\begin{array}{ccccc}
V & \xrightarrow{h'} & E''^{\nabla''} & \longrightarrow & V \otimes \Gamma^{\cdot}(\Omega) \\
\downarrow{\scriptstyle -\theta} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id} \otimes d} \\
V \otimes \Omega & \xrightarrow{h' \otimes \mathrm{id}} & E''^{\nabla''} \otimes \Omega & \longrightarrow & V \otimes \Gamma^{\cdot}(\Omega) \otimes \Omega
\end{array}$$

3. *The natural map $\Gamma^{\cdot}(\Omega) \otimes E''^{\nabla''} \to E''$ is an isomorphism, with inverse $\sum \omega^{[I]} \otimes h\partial^I$.*

4. *The De Rham cohomology $H^i_{DR}(E'')$ of E'' vanishes if $i > 0$.*

Proof: Most of this lemma is more or less standard, at least if one replaces the polynomial ring $\Gamma^{\cdot}(\Omega)$ by its formal completion at the origin. Notice first that for any $n > 0$, $\sum\{\omega^{[I]} \otimes t^I : |I| = n\}$ is the matrix for the canonical pairing between $\Gamma^n(\Omega)$ and $Sym^n(T)$. It follows that h (the Kasimir operator) is independent of the basis. The local nilpotence of the operators ∂_i implies that the operator h is well-defined, and the fact that it is a projection with image $E''^{\nabla''}$ is an immediate calculation. It is apparent from the definition that h' is injective. To see that it is surjective, write an arbitrary $e'' \in E''^{\nabla}$ as a sum $e'' = \sum \omega^{[I]} \otimes v_I$ with $v_I \in V$. Then e'' and $h'(v_0)$ are two elements of $E''^{\nabla''}$ which agree modulo the ideal $\Gamma^+(\Omega)$ of $\Gamma^{\cdot}(\Omega)$. It follows from the well-known complete version of this lemma that they agree in the formal completion at this ideal, and hence that they agree. This shows that h' is also surjective. Note that $\theta \circ h' = (h' \otimes \mathrm{id}) \circ \theta$. If $v \in V$, $\nabla'' h'(v) = 0$, and since $\nabla'' = \mathrm{id} \otimes d + \theta \otimes \mathrm{id}$,

$$\begin{aligned}
(\mathrm{id} \otimes d) \circ h'(v) &= -(\theta \otimes \mathrm{id}) \circ h'(v) \\
&= -(h' \otimes \mathrm{id}) \circ \theta(v)
\end{aligned}$$

This proves that the diagram in (2) commutes. Statement (3) is a straightforward calculation, and (4) then follows, since (3) reduces the computation of De Rham cohomology to the case of the trivial connection, which of course vanishes, by the Poincaré lemma in crystalline cohomology. □

Proof of Theorem (1.5.1) Let (E, ∇) be an object of $MIC^{\Lambda}_{*}(P/R)$ and let $(\tilde{E}, \tilde{\nabla}, \tilde{\nabla}')$ be $J^*(E, \nabla)$. Since (E, ∇) and $(J(P, \Lambda), \nabla)$ are normalized, so is $(\tilde{E}, \tilde{\nabla})$. We have

$$\tilde{E} := E \otimes_{R[P]} J(P, \Lambda) \cong E \otimes_R R[\Lambda] \otimes_R \Gamma^{\cdot}(\Omega).$$

Let $(V,\theta) := (E\otimes_R R[\Lambda],\tilde{\nabla})_{\Lambda=0}$ and let $E'' := \tilde{E}_\Lambda$ be the part of $\tilde{E}$ of Λ-degree zero. Thus

$$E'' := \tilde{E}_{\Lambda=0} \cong V\otimes_R \Gamma^{\cdot}(\Omega),$$

and (V,θ) is just the Higgs transform (1.3.7) of E. Since θ is nilpotent, (1.5.3) applies. Assembling the diagrams (1.3.5.2) and (1.5.3.2), we obtain a commutative diagram:

$$\begin{array}{ccccc}
E_\lambda & \xrightarrow{\cong} & V_{0,-\lambda} & \xrightarrow{\cong} & E''^{\nabla''}_{-\lambda} \\
\Big\downarrow{\scriptstyle \nabla - \mathrm{id}\otimes\lambda} & & \Big\downarrow{\scriptstyle \theta} & & \Big\downarrow{\scriptstyle -\mathrm{id}\otimes\nabla_{\Gamma^{\cdot}(\Omega)}} \\
E_\lambda\otimes\Omega & \longrightarrow & V_{0,-\lambda}\otimes\Omega & \xrightarrow{\cong} & E''^{\nabla''}_{-\lambda}\otimes\Omega.
\end{array}$$

Now $E''_{-\lambda} \subseteq \tilde{E}_{0,-\lambda}$, and by definition

$$\tilde{\nabla}' := \mathrm{id}_E\otimes\nabla' = \mathrm{id}\otimes\nabla_{\Gamma^{\cdot}(\Omega)} + \mathrm{id}\otimes(-\lambda)$$

in these degrees. The diagram shows that the map $\nabla - \mathrm{id}\otimes\lambda\colon E_\lambda \to E_\lambda\otimes\Omega$ corresponds to the map $-\mathrm{id}\otimes\nabla'_{\Gamma^{\cdot}\Omega} = -\nabla' - \mathrm{id}\otimes\lambda$. Thus ∇ corresponds to $-\nabla'$, and we get the commutative diagram in (1). This diagram implies that $q'^{\nabla}_{*} J^*E$ belongs to $MIC^{\Lambda'}_{*}(P'/R)$.

It follows from (1.5.2) that the map from the de Rham complex $K''^{\cdot}$ of E'' to $\tilde{K}^{\cdot}$ is a quasi-isomorphism. Lemma (1.5.3) implies that $H^i_{DR}(E'') = 0$ if $i > 0$, and since $K''^{\cdot} \to \tilde{K}^{\cdot}$ is a quasi-isomorphism, the same is true of $H^i_{DR}(\tilde{K}^{\cdot})$. Lemma (1.5.3) also implies that the natural map $E''^{\nabla''}\otimes_R\Gamma^{\cdot}(\Omega) \to E''$ is an isomorphism. Now $E''^{\nabla''}$ is in fact an $R[P']$-module, and this isomorphism can be rewritten as an isomorphism

$$E''^{\nabla''}\otimes_{R[P']}\otimes R[P']\otimes_R\Gamma^{\cdot}(\Omega) \to E''.$$

Tensoring with $R[\Lambda]$ and using the fact that the map $E''^{\nabla''} \to \tilde{E}^{\tilde{\nabla}}$ is an isomorphism, we see that the map

$$\tilde{E}^{\tilde{\nabla}}\otimes_{R[P']}\otimes R[P',\Lambda]\otimes_R\Gamma^{\cdot}(\Omega) \to E''\otimes_R R[\Lambda]$$

is an isomorphism. But by Proposition (1.3.7), the natural map

$$E''\otimes_R R[\Lambda] \to \tilde{E}$$

is an isomorphism. Hence the map

$$\tilde{E}^{\tilde{\nabla}}\otimes_{R[P']}\otimes J(P,\Lambda) \to \tilde{E}$$

is an isomorphism, proving the last statement of (2). The calculation of the Higgs cohomology of $(\tilde{E},\tilde{\nabla}')$ is done in the same way as the de Rham cohomology. This completes the proof of (2), and (3) follows by symmetry.

Now suppose that $(E,\nabla) \in MIC_*^{\Lambda}(P/R)$ and let $(E',\nabla') := {q'}_*^{\nabla} J^*(E,\nabla)$. As we have seen, $(E',\nabla') \in MIC_*^{\Lambda'}(P'/R)$. By the last part of (2),

$$J(P,\Lambda)\otimes E' \cong \tilde{E},$$

and so $q_*{}^{\nabla'}(J(P,\Lambda)\otimes E') \cong q_*{}^{\nabla'}\tilde{E} \cong E$. This implies that the composite $MIC_*^{\Lambda}(P/R) \to MIC_*^{\Lambda'}(P'/R) \to MIC_*^{\Lambda}(P/R)$ is isomorphic to the identity. A similar argument works starting with $MIC_*^{\Lambda'}(P'/R)$. This completes the proof of the theorem. □

Proof of (1.4.8) Let (E,∇) be an object of $MIC_*^{\Lambda}(P,\mathbf{C})$ and let $(V,\rho) := \mathcal{V}(E,\nabla)$. By construction, V is the $\mathbf{C}[P']$-module $q'^{\nabla}_* J^*(E,\nabla)$ of (1.5.1), and ρ is the map induced by $\tilde{\rho} := \mathrm{id}_E \otimes \rho_J$. Here ρ_J is the action of $\pi_1(P)$ on $J(P,\Lambda)$, which by (1.4.7) is $\mathrm{id}_E \otimes \exp\tilde{\nabla}' = \mathrm{id}_E \otimes \exp\theta$. The isomorphism $E \to V$ of (1) of (1.5.1) takes ∇' to $-\nabla$, and so the action ρ of (1.4.8) agrees with the action defined in (1.4.3). This proves (1) of (1.4.8), and (2) follows directly from (1.5.1.2). Conversely, let (V,ρ) be an object of $L_*^{\Lambda'}(P')$, and $\tilde{V} := V \otimes J(P,\Lambda)$. Then the action of π_1 on $\tilde{V}_{\Lambda'=0}$ is unipotent. Its logarithm is the nilpotent Higgs structure $\theta = -\nabla$, and so by (1.4.4), $q_*^{\Lambda',\pi_1}(\tilde{V}) = \tilde{V}^{\nabla} = E$. By (1.4.4), the Higgs cohomology of $\tilde{V}$ is the same as the group cohomology, and so (1.4.8.3) follows from (1.5.1.3). □

Remark 1.5.4 A morphism of toric monoids $P \to Q$ induces a map $\Omega_{P/R} \to \Omega_{Q/R}$. A compatible morphism of exponential data is a commutative diagram

$$\begin{array}{ccccc} P & \longrightarrow & \Lambda_P & \longrightarrow & \Omega_{P/R} \\ \downarrow & & \downarrow & & \downarrow \\ Q & \longrightarrow & \Lambda_Q & \longrightarrow & \Omega_{Q/R}. \end{array}$$

For example, if $\Lambda_P = P^{gp}$ or $k \otimes P^{gp}$ or $R \otimes P^{gp}$, there is an evident choice of $\Lambda_P \to \Lambda_Q$. Associated with such data are morphisms $R[P,\Lambda_P] \to R[Q,\Lambda_Q]$ and $J(P,\Lambda_P) \to J(Q,\Lambda_Q)$ and concomitant functors (with the subscripts on the Λ's omitted from the notation):

$$\begin{array}{rcl} MIC_*^{\Lambda}(P/R) & \to & MIC_*^{\Lambda}(Q/R) \\ HIG_*^{\Lambda}(P/R) & \to & HIG_*^{\Lambda}(Q/R) \\ MH_{\Lambda'}^{\Lambda}(P/R) & \to & MH_{\Lambda'}^{\Lambda}(Q/R) \end{array}$$

and, when $R = \mathbf{C}$,

$$L_*^{\Lambda}(P) \to L^{\Lambda}(Q).$$

It is easy to verify that the functors in (1.5.1) and (1.4.8) are compatible with these base change functors.

2 Formal and holomorphic germs

2.1 Exponents and the logarithmic inertia group

Let X be a smooth, fine, and saturated idealized log analytic space. If x is a point of X, let

$$\begin{array}{rcl} I_x : & = & \mathrm{Hom}(\overline{M}^{gp}_{X,x}, \mathbf{Z}(1)) \\ \Omega_{\overline{M}_x/\mathbf{C}} & := & \mathbf{C} \otimes \overline{M}^{gp}_{X,x} \\ T_{\overline{M}_x/\mathbf{C}} & := & Hom(\overline{M}^{gp}_{X,x}, \mathbf{C}) \cong \mathbf{C} \otimes I_x. \end{array}$$

The group I_x is called the *logarithmic inertia group at x*. It is the fundamental group of the torus $\mathsf{A}^*_{\overline{M}_{X,x}}$, and $T_{\overline{M}_x/\mathbf{C}}$ is the space of invariant vector fields on $\mathsf{A}^*_{\overline{M}_{X,x}}$.

It follows as in [12, 1.3.1] that there is a natural surjective map

$$\Omega^1_{X/\mathbf{C}}(x) \to \mathbf{C} \otimes \overline{M}^{gp}_{X,x}.$$

If (E, ∇) is a coherent sheaf with integrable connection on X, let $E(x) := E_x/m_x E_x$ be its fiber at x. Then there is a unique linear map ρ_x such that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\nabla} & E \otimes \Omega^1_{X/\mathbf{C}} \\ \downarrow & & \downarrow \\ E(x) & \xrightarrow{\rho_x} & E(x) \otimes \Omega_{\overline{M}_x/\mathbf{C}} \end{array}$$

It follows from the integrability of ∇ that the endomorphisms of $E(x)$ defined by evaluating ρ_x at any two elements of $T_{\overline{M}_x/\mathbf{C}}$ commute. Thus ρ_x defines a $T_{\overline{M}_x/\mathbf{C}}$-Higgs field on $E(x)$, and $E(x)$ becomes a module over the symmetric algebra $S^{\cdot}T_{\overline{M}_x/\mathbf{C}}$. Since $E(x)$ is finite dimensional over $\mathbf{C}$, it is supported at a finite set of maximal ideals of this algebra, *i.e.*, at a finite set of elements of $\Omega_{\overline{M}_x}$.

Definition 2.1.1 *Let (E, ∇) be a coherent sheaf with integrable connection on X and let x be a point of X. Then the* residue of (E, ∇) *at x is the map ρ_x in the diagram above, and the* exponents of (E, ∇) at x *are the* negatives *of the elements in $\Omega_{\overline{M}_x/\mathbf{C}} = \mathbf{C} \otimes \overline{M}^{gp}_{X,x}$ lying in the support of the $\mathbf{C}$-$T_{\overline{M}_x}$ module defined by ρ_x.*

To understand the choice of the sign in the definition of exponents, consider the connection on the structure sheaf of the logarithmic affine line with $\nabla(1) :=$

$\lambda \otimes dt/t$, where $\lambda \in \mathbf{C}$. Then the corresponding $\mathbf{C}$-$T_{\overline{M}_x}$-module, has support at λ. On the other hand, the horizontal sections of the connection are the constant multiples of $t^{-\lambda}$, so it is $-\lambda$ which appears as an "exponent." Note that formation of the residue is compatible with tensor products. In particular, the set of exponents of the tensor product of two connections (E_1, ∇_1) and (E_2, ∇_2) is the set of sums $\lambda_1 + \lambda_2$, with λ_i an exponent of E_i.

Our main local theorem gives an equivalence between the category of analytic germs of log connections and the category of normalized homogeneous connections considered in §1. Fix a point x of X and let $\overline{M}_{X,x} \to \Lambda \to \Omega_{\overline{M}_x/\mathbf{C}}$ be a rigid set of exponential data for $\overline{M}_{X,x}$. Let $MIC^{\Lambda}_{coh}(X_x)$ denote the category of germs of coherent sheaves with integrable connection all of whose exponents lie in Λ. This category is closed under extensions, tensor products, and duals (because Λ is a group). If $P \to \Lambda \to \mathbf{C} \otimes P^{gp}$ is a rigid set of exponential data for a toric monoid P, then the image $\overline{\Lambda}$ of Λ in $\mathbf{C} \otimes \overline{P}^{gp}$ defines a set of exponential data for $\overline{P}$, and we sometimes write $MIC^{\Lambda}_{coh}(X_x)$ for $MIC^{\overline{\Lambda}}_{coh}(X_x)$.

Theorem 2.1.2 *Let P be an idealized toric monoid with rigid exponential data Λ, let X be the log analytic space associated to A_P, and let $\hat{X}_v$ be the formal completion of X at its vertex v. Use $\overline{X}$ and similar notation for $\mathsf{A}_{\overline{\mathsf{P}}} \subseteq \mathsf{A}_\mathsf{P}$, where $\overline{P} := P/P^*$. Then the evident functors form a 2-commutative diagram:*

$$\begin{array}{ccccc}
 & \swarrow & MIC^{\Lambda}_{coh}(P/\mathbf{C}) \;\downarrow & \searrow & \\
MIC^{\overline{\Lambda}}_{coh}(\overline{P}/\mathbf{C}) & \xrightarrow{\overline{A}} & MIC^{\overline{\Lambda}}_{coh}(\overline{X}_v) & \xleftarrow{B_{an}} & MIC^{\Lambda}_{coh}(X_v/\mathbf{C}) \\
 & \searrow^{C} & \downarrow \overline{D} & & \downarrow D \\
 & & MIC^{\overline{\Lambda}}_{coh}(\hat{\overline{X}}_v/\mathbf{C}) & \xleftarrow{\hat{B}} & MIC^{\Lambda}_{coh}(\hat{X}_v/\mathbf{C}),
\end{array}$$

in which all the labeled arrows are equivalences of tensor categories, compatible with De Rham cohomology.

The proof will occupy the rest of this section.

Remark 2.1.3 Let (E, ∇) be an object of $MIC^{\Lambda}_{coh}(X_v/\mathbf{C})$ and let (E', ∇) be the corresponding object of $MIC^{\overline{\Lambda}}_{coh}(\overline{P}/\mathbf{C})$. Then (E, ∇) and (E', ∇) have the same restriction to $\hat{\overline{X}}$, and in particular they have the same residue and exponents. That is, the residue ρ of E can be identified with the endomorphism of E'/P^+E' induced by ∇. Since ∇ is normalized, $\{\lambda : (E'/P^+E')_\lambda \neq 0\}$ is the same as the support of the $T_{\overline{P}/\mathbf{C}}$-Higgs module defined by ρ. Note that this set is just the set of degrees of any minimal set of generators for E'/P^+E'. Let (V, ρ) be the equivariant Riemann-Hilbert transform (1.4.3) of (E', ∇). Since the degrees of V are the negative of the degrees of E', it follows that the set of exponents of (E, ∇) is exactly the set of minimal degrees of V.

2.2 Formal germs

We begin with the functor C; without loss of generality we may and shall assume that $P = \overline{P}$. Then v corresponds to the maximal ideal of $\mathbf{C}[P]$ generated by P^+, and the completion of $\mathbf{C}[P]$ at this ideal can be identified with the formal power series ring $\mathbf{C}[[P]]$. This is the set of functions $a \colon P \to \mathbf{C}$, where for $a, b \in \mathbf{C}[[P]]$, $(a+b)_p := a_p + b_p$ and $(ab)_r := \sum\{a_p b_q : p + q = r\}$. To see that the sum is finite, choose a local homomorphism $\phi \colon P \to \mathbf{N}$, and observe that each $\{p \in P : \phi(p) \leq n\}$ is finite. In fact, this set is the complement of an ideal K_n of P, and the set of such ideals $\{K_n : n \in \mathbf{N}\}$ is cofinal with the set of powers of P^+. If S is a free P set and V is a finitely generated S-graded $\mathbf{C}[P]$-module, the P^+-adic completion $\hat{V}$ of V can be identified with the product $\prod\{V_s : s \in S\}$. The action of P on S defines a partial ordering on S: $s \leq t$ if there exists $p \in P$ with $p + s = t$; such a p is unique if it exists, and we write $t - s$ for this p. Then if $a \in \mathbf{C}[[P]]$ and $v \in \prod V_s$, $(av)_t := \sum a_{t-s} v_s$. The P-set $\Lambda \subseteq \mathbf{C} \otimes P^{gp}$ is only potentially free, but if V is a finitely generated Λ-graded $\mathbf{C}[P]$-module, there exists a finitely generated free P-subset S of Λ such that $V_\lambda = 0$ for $\lambda \notin S$, and we can identify $\hat{V}$ with $\prod\{V_s : s \in S\} \cong \prod\{V_\lambda : \lambda \in \Lambda\}$.

It is now easy to see that the functor C is compatible with cohomology, *i.e.*, that if (E, ∇) is an object of $MIC^{\Lambda}_{coh}(P/\mathbf{C})$, the natural map

$$(E \otimes \Omega^{\cdot}_{X/\mathbf{C}})_0 \to \hat{E} \otimes \Omega^{\cdot}_{X/\mathbf{C}}$$

from the degree zero part of its de Rham complex to its completion is a quasi-isomorphism. Indeed, $\Omega^1_{X/\mathbf{C},v} \cong \mathcal{O}_{X,v} \otimes_{\mathbf{C}} \Omega_{P/\mathbf{C}}$, and $\hat{E} \otimes \Omega^{\cdot}_{X/\mathbf{C}}$ can be identified with the product: $\prod_\lambda (E \otimes \Omega^{\cdot}_{P/\mathbf{C}})_\lambda$. For each λ, the degree λ part of the complex $E \otimes \Omega^{\cdot}_{P/\mathbf{C}}$ can be identified with the Higgs complex of the $T_{P/\mathbf{C}}$-Higgs module (E, ∇_λ). Since (E, ∇) is normalized (1.2.3), this complex is acyclic whenever $\lambda \neq 0$, by (1.5.2). Since infinite products in the category of vector spaces commute with cohomology, the cohomology of the product identifies with the cohomology of the degree zero part of $E \otimes \Omega^{\cdot}_{X/\mathbf{C}}$, as required. Since the functor C is compatible with the formation of internal Hom's, it follows that it is also fully faithful.

It remains to prove that C is essentially surjective. Let (E, ∇) be an object of $MIC^{\Lambda}_{coh}(\hat{X}_v/\mathbf{C})$. The connection

$$\nabla \colon E \to E \otimes_{\mathbf{C}[P]} \Omega^1_{X/\mathbf{C}} \cong E \otimes_{\mathbf{C}} \Omega_{P/\mathbf{C}}$$

can be regarded as a $\mathbf{C}$-$T_{P/\mathbf{C}}$-module structure on E, which is easy to analyze if E is finite dimensional over $\mathbf{C}$. Indeed, such an E admits a Jordan decomposition

$$(E, \nabla) \cong \oplus\{(E_\lambda, \nabla_\lambda) : \lambda \in \Omega\},$$

where each $(E_\lambda, \nabla_\lambda)$ has support in λ, and Lemma (1.3.4) applies. In fact, $E_\lambda = 0$ unless $\lambda \in \Lambda$ by (2.2.1) below. Thus $E \cong \oplus_\lambda (E_\lambda, \nabla_\lambda)$ is an object of $MIC^\Lambda_{coh}(P/\mathbf{C})$ and it is evident that its formal completion at v is (E, ∇). This shows that any such (E, ∇) is in the essential image of C.

For the general case, we use a limit argument and the following lemma.

Lemma 2.2.1 *Let (E, ∇) be an object of $MIC^\Lambda_{coh}(\hat{X}_v/\mathbf{C})$ such that E is finite dimensional over $\mathbf{C}$. Then the support of (E, ∇) as a $T_{P/\mathbf{C}}$-Higgs module is contained in the P-subset S of $\mathbf{C} \otimes P^{gp}$ generated by the support of $(E(v), \nabla)$, and in particular is contained in Λ. If K is an ideal of P, then the support of KE is contained in the K-translate of the support of E.*

Proof: If K is any ideal of P, then the ideal $\mathbf{C}[K]$ of $\mathbf{C}[P]$ it generates is invariant under ∇ and defines an element of $MIC^\Lambda_{coh}(P/\mathbf{C})$. Since $\nabla e_k = e_k \otimes dk$, the support of the corresponding Higgs module is the image of K in Λ. Since there is a surjective map $\mathbf{C}[K] \otimes E \to KE$, the support of KE is contained in the support of $\mathbf{C}[K] \otimes E$, which is the K-translate of the support of E. This proves the second statement. Since E has finite length, it is annihilated by P^{+n} for some $n \in \mathbf{Z}^+$, and we prove the first statement by induction on n. If $n = 1$, $E \cong E/P^+E = E(v)$ and the result is trivial. In the general case, note that P^+E is invariant under the connection and annihilated by ${P^+}^{n-1}$, so the induction hypothesis implies that the support of P^+E is contained in the P-subset of $\mathbf{C} \otimes P^{gp}$ generated by the support of $P^+E/P^{+2}E$. As we have just seen, this is contained in $P^+ + S \subseteq S$. Then the exact sequence $0 \to P^+E \to E \to E/P^+E \to 0$ shows that the support of E is contained in S as well. □

Now let (E, ∇) be any object of $MIC^\Lambda_{coh}(\hat{X}_v/\mathbf{C})$. Choose a local homomorphism $\phi \colon P \to \mathbf{N}$. Then ϕ extends uniquely to a $\mathbf{C}$-linear map $\mathbf{C} \otimes P^{gp} \to \mathbf{C}$ which we also denote by ϕ. Let $K^n := \{p \in P : \phi(p) \geq n\}$, and let $E_n := E/K^nE$. If $n' \geq n$ there is an exact sequence of modules with connection

$$0 \to K^nE/K^{n'}E \to E_{n'} \to E_n \to 0.$$

Each of these terms is finite dimensional over $\mathbf{C}$, and the $\mathbf{C}$-$T_{P/\mathbf{C}}$-module it defines has support in Λ. For every λ, the corresponding sequence:

$$0 \to (K^nE/K^{n'}E)_\lambda \to E_{n',\lambda} \to E_{n,\lambda} \to 0$$

is again exact. Let S be the support of E/P^+E and choose $m \in \mathbf{Z}$ so that $m < Re(\phi(s))$ for all $s \in S$. Suppose $(K^nE/K^{n'}E)_\lambda \neq 0$. Then by lemma (2.2.1), λ can be written as $p + s$ with $p \in K^n$ and $s \in S$, and

$$Re(\phi(\lambda)) = \phi(p) + Re(\phi(s)) > n + m.$$

Thus if $n \geq Re(\phi(\lambda)) - m$, $(K^nE/K^{n'}E)_\lambda$ vanishes and the map $E_{n',\lambda} \to E_{n,\lambda}$ is an isomorphism. Let E_λ be the inverse limit, *i.e.*, the stable value of $E_{n,\lambda}$ for

n large. Then ∇ maps E_λ to E_λ, and $\oplus(E_\lambda, \nabla_\lambda)$ is an object of $MIC_{coh}(P/\mathbf{C})$, whose completion at the vertex is (E, ∇).

This completes the proof that C is essentially surjective, and it follows from the diagram that the same is true of $\overline{D}$.

The fact the arrow $\hat{B}$ is an equivalence follows from the following slightly stronger result, which is a consequence of the fact there is no log structure in the transverse direction.

Lemma 2.2.2 *Let $X^\vee$ denote the formal completion of X along $\overline{X}$. Then the natural functor*

$$MIC^{\Lambda}_{coh}(X^\vee/\mathbf{C}) \to MIC^{\Lambda}_{coh}(\overline{X}/\mathbf{C})$$

is an equivalence, compatible with cohomology.

Proof: Since $\overline{X}/\mathbf{C}$ is smooth, the category $MIC^{\Lambda}_{coh}(\overline{X}/\mathbf{C})$ is equivalent to a full subcategory of the category of coherent crystals on $\overline{X}/\mathbf{C}$, and the same holds for $X/\mathbf{C}$ [6, 6.2]. Since $\overline{X} \to X$ is a strict closed immersion, the fact that the above functor is an equivalence follows formally from the properties of crystals: $X^\vee$ is a limit of strict infinitesimal thickenings of $\overline{X}$, and hence a crystal on $\overline{X}$ has a natural value on $X^\vee$, and in fact also on any strict infinitesimal thickening of $X^\vee$. To check the result on De Rham cohomology, one can work locally, using the fact that $X^\vee$ looks locally like $\overline{X} \times \operatorname{Spf} \mathbf{C}[[t_1, \ldots t_n]]$, and argue as in the classical case. □

Since P is saturated, $\overline{P}^{gp}$ is a finitely generated free abelian group, and so the exact sequence $0 \to P^* \to P^{gp} \to \overline{P}^{gp} \to 0$ splits. Any splitting $\overline{P}^{gp} \to P^{gp}$ automatically maps $\overline{P}$ to P and induces a section of the map $\overline{X} \to X$. This implies that the functor $MIC^{\Lambda}_{coh}(P/\mathbf{C}) \to MIC^{\Lambda}_{coh}(\overline{P}/\mathbf{C})$ is essentially surjective. Since $\hat{B}$ is an equivalence, it follows from the diagram that D is also essentially surjective.

2.3 Convergent germs

Our first task is to establish a convenient description of the ring of germs of analytic functions at the vertex of $\underline{\mathsf{A}}_{\mathsf{P}}$ as a subring $\mathbf{C}\{P\}$ of $\mathbf{C}[[P]]$.

Proposition 2.3.1 *Let P be a fine sharp monoid, let v be the vertex of A^{*an}_P, and let T be the (necessarily finite) set of irreducible elements of P.*

1. *For $\delta \in \mathbf{R}^+$, let*

$$U_\delta := \{x \in \underline{\mathsf{A}}_{\mathsf{P}}(\mathbf{C}) : |x(t)| < \delta \text{ for all } t \in T\}.$$

Then $\{U_\delta : \delta \in \mathbf{R}^+\}$ forms a basis for the system of neighborhoods of v in $\mathsf{A}^{*an}_P(\mathbf{C})$ in the usual complex topology.

2. *If ϕ is a local homomorphism $P \to \mathbf{N}$ and $\alpha := \sum_p a_p e_p \in \mathbf{C}[[P]]$, then α converges in some neighborhood of v if and only if the set $\{\frac{\log|a_p|}{\phi(p)} : p \in P^+\}$ is bounded above.*

Proof: First suppose that $P = \mathbf{N}^n$. Then $X = \mathbf{C}^n$, v is the origin, U_δ is the polydisc about v of radius δ, and (1) is clear. If P is any fine sharp monoid, then T is finite and generates P as a monoid, and hence a bijection $\{1 \ldots n\} \to T$ induces a surjective homomorphism $\mathbf{N}^n \to P$ and a closed immersion $\underline{\mathsf{A}}_\mathsf{P} \to \mathbf{A}^n$. With respect to this closed immersion, U_δ is just the intersection of $\underline{\mathsf{A}}_\mathsf{P}(\mathbf{C})$ with the polydisc of radius δ about v. This proves (1) in general.

Suppose that $\alpha = \sum_p a_p e_p$, $c \in \mathbf{R}$, and $c \geq \phi(p)^{-1} \log|a_p|$ for every $p \in P^+$. Choose $\epsilon > 0$, let $\lambda_t := -(c+\epsilon)\phi(t)$ for each $t \in T$, and choose a positive number δ such that $\delta < e^{\lambda_t}$ for all t. Then U_δ is an open neighborhood of v in X, and if $x \in U_\delta$, $\log|x(t)| < \lambda_t$ for all t. Any $p \in P$ can be written $p = \sum n_t t$. It follows that for $x \in U_\delta$,

$$\begin{aligned}
\log|a_p x(p)| &= \log|a_p| + \log|x(p)| \\
&\leq c\phi(p) + \log|x(p)| \\
&\leq \sum_t n_t(c\phi(t) + \log|x(t)|) \\
&\leq \sum_i n_t(c\phi(t) + \lambda_t) \\
&\leq \sum_t n_t(-\epsilon\phi(t)) \\
&\leq -\epsilon\phi(p)
\end{aligned}$$

Thus $|a_p x(p)| \leq r^{\phi(p)}$, where $r := e^{-\epsilon} < 1$. As is well known, $\{p : \phi(p) = i\}$ has cardinality less than Ci^m for some C and m, so the set of partial sums of the series $\sum_p |a_p x(p)|$ is bounded by the set of partial sums of the series $\sum_i Ci^m r^i$. Since this latter series converges, so does the former.

Suppose on the other hand that $\{\phi(p)^{-1} \log|a_p| : p \in P^+\}$ is unbounded. For $c \in \mathbf{R}^+$, define $x_c : P \to \mathbf{C}$ by $x_c(p) := c^{-\phi(p)}$. Then $x_c \in \underline{\mathsf{A}}_\mathsf{P}(\mathbf{C})$, and if $\delta > 0$ and c is chosen large enough so that $\log c > (\phi(t))^{-1}(-\log\delta)$ for all $t \in T$, then $x_c \in U_\delta$. For every such c, there are infinitely many $p \in P^+$ such that $|a_p| > (c+1)^{\phi(p)}$. For any such p,

$$|a_p x_c(p)| \geq (1+c)^{\phi(p)} c^{-\phi(p)} = (1/c+1)^{\phi(p)} \geq 1,$$

so the series $\sum_p a_p x_c(p)$ cannot converge. □

Our next task is an existence and uniqueness result for formal and convergent solutions to certain differential equations. Recall that if $X = \mathsf{A}_\mathsf{P}$, a homomorphism $P \to \mathbf{N}$ defines an invariant vector field on X.

Proposition 2.3.2 *Let P be a sharp toric monoid, let $X := \mathsf{A}_{\mathsf{P}}$, let v be its vertex, and let (E,∇) be the germ of a coherent sheaf with integrable log connection on X^{an} at v. Suppose that $\phi\colon P \to \mathbf{N}$ is a local homomorphism such that $\phi(\lambda) < 0$ for every exponent λ of E at v.* [1] *Then ∇_ϕ acts bijectively on E and on $\hat{E}$.*

Let us first discuss the formal case. It suffices to prove that for each $n \in \mathbf{N}$, ∇_ϕ induces an automorphism of $E_n := E/K^nE$, where $K^n := \{p : \phi(p) \geq n\}$. Each E_n is finite dimensional over $\mathbf{C}$ and ∇ can be viewed as a $T_{P/\mathbf{C}}$-Higgs field on E_n. The support of E_n as a $T_{P/\mathbf{C}}$-Higgs module is a finite subset of $\Omega_{P/\mathbf{C}}$. By (2.2.1), its support is contained in the sub P-subset S of Λ generated by the support of the $T_{P/\mathbf{C}}$-Higgs module E/P^+E, *i.e.*, by the negative of the set of exponents. Thus $\phi(s) > 0$ for every $s \in S$, and hence ∇_ϕ is an automorphism of E_n.

To deal with convergence we must be more explicit. We have a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\ \nabla_\phi\ } & E \\ \downarrow & & \downarrow \\ \hat{E} & \xrightarrow{\ \hat{\nabla}_\phi\ } & \hat{E} \end{array}$$

It follows that $\nabla_\phi\colon E \to E$ is injective, and it remains to prove that it is surjective.

Let $(v_1,\dots v_n)$ be a subset of E whose reduction modulo P^+E forms a basis for E/P^+E, and let $V \subseteq E$ be its $\mathbf{C}$-linear span. Then V generates E as a module over the ring $\mathcal{O} := \mathcal{O}_{X^{an}_v}$. For each i, $\nabla_\phi(v_i) \in E$, and hence can be written (not necessarily uniquely) as a sum: $\sum a_{ij}v_j$, with $a_{ij} \in \mathcal{O}$. Let A denote the $n\times n$ matrix $\big(a_{ij}\big)$, and write A as a formal sum $\sum\{A_q e_q : q \in P\}$, where A_q is an $n \times n$ matrix in $\mathbf{C}$. For any $v \in V$,

$$\nabla_\phi(v) = \sum_q A_q(v)e_q$$

In particular, A_0 is the matrix of the endomorphism induced by ∇_ϕ on E/P^+E. The eigenvalues of this endomorphism are among those complex numbers of the form $\phi(s)$ for s in the support of $(E/P^+E,\nabla)$. By hypothesis, $\phi(p)+\phi(s) \neq 0$, for every $p \in P$ and s in this support. It follows that $A_0 + \phi(p)$ is invertible for every $p \in P$.

[1] One can show using a Baire category argument that a ϕ as in (2.3.2) exists if and only if the set of exponents does not meet P.

Any element v of $\hat{E}$ can be written as a formal sum $v = \sum v_q e_q$, with $v_q \in V$. Then:

$$\begin{aligned}
\nabla_\phi(v) &= \sum_q \Big(\nabla_\phi(v_q) e_q + v_q \langle \phi, de_q \rangle \Big) \\
&= \sum_q \sum_{q'} A_{q'}(v_q) e_{q'} e_q + \sum_q \phi(q) v_q e_q \\
&= \sum_p \Big(\sum_{q+q'=p} A_{q'}(v_q) \Big) e_p + \sum_p v_p \phi(p) e_p \\
&= \sum_p w_p e_p,
\end{aligned}$$

where

$$w_p := A_0(v_p) + \phi(p) v_p + \sum_{q<p} A_{p-q}(v_q).$$

Recall that $A_0 + \phi(p)$ is invertible; let B_p be its inverse. Then the above equation becomes:

$$B_p(w_p) = v_p + B_p \sum_{q<p} A_{p-q}(v_q).$$

In other words, if $w = \sum w_p e_p$, then the coefficients of $v = \hat{\nabla}_\phi^{-1}(w)$ are given recursively by the formula:

$$v_p = B_p(w_p) - B_p \sum_{q<p} A_{p-q}(v_q). \tag{2.3.1}$$

Note that the sum is finite since there are only finitely many q with $q < p$. We have to prove that if the series $\sum w_p e_p$ converges, so does the series $\sum v_p e_p$.

Since ϕ is local, there are only finitely many p with $\phi(p) \leq 2||A_0||$, and we can find a constant $M \geq 2$ such that $||B_p|| \leq M\phi(p)^{-1}$ for all these p. Let $\psi := \phi/M$. We claim that $||B_p|| \leq \psi(p)^{-1}$ for all $p \in P$. This is true by our choice of M if $\phi(p) \leq 2||A_0||$. If on the other hand $\phi(p) > 2||A_0||$, then

$$\begin{aligned}
||B_p|| &= ||\,(\phi(p) + A_0)^{-1}\,|| \\
&= \phi(p)^{-1} ||1 - \phi(p)^{-1} A_0 + \phi(p)^{-2} A_0^2 - \cdots|| \\
&\leq \phi(p)^{-1}(1 + 1/2 + 1/4 + \cdots) \\
&\leq 2\phi(p)^{-1} \\
&\leq \psi(p)^{-1}.
\end{aligned}$$

Since A and w are convergent there exists a positive real number s such that $||A_p||$ and $||w_p||$ are less than $s^{\psi(p)}$ for all p. Moreover, since ∇_ϕ is $\mathbf{C}$-linear, we may without loss of generality assume that $||w_0|| \leq ||B_0||^{-1}$, so that $||v_0|| \leq 1$. Let $y_p := ||v_p s^{-\psi(p)}||$ for $p \in P$. It will suffice to show that there exists a t such that $y_p \leq t^{\psi(p)}$ for all p.

By the formula (2.3.1),

$$y_p \leq ||B_p||||w_p||s^{-\psi(p)} + ||B_p|| \sum_{q<p} ||A_{p-q}||s^{-\psi(p-q)}||v_q||s^{-\psi(q)}$$

Hence

$$y_p \leq \frac{1}{\psi(p)} + \frac{1}{\psi(p)} \sum_{q<p} y_q. \tag{2.3.2}$$

Let ϵ be the minimum of $\psi(P^+)$, and choose c so that $c\epsilon > 2$. Then let $a_0 := 1$ and for $p \in P^+$ define a_p inductively by setting

$$a_p := c \sum_{q<p} a_q (1 - \frac{\psi(q)}{\psi(p)}).$$

If $q < p$, $\psi(p) - \psi(q) \geq \epsilon$. Hence if p is any element of P^+,

$$\begin{aligned} a_p &= \sum_{q<p} c a_q (\frac{\psi(p) - \psi(q)}{\psi(p)}) \\ &\geq \sum_{q<p} \frac{c\epsilon a_q}{\psi(p)} \\ &\geq \sum_{q<p} \frac{2a_q}{\psi(p)} \geq \sum_{q<p} \frac{a_q}{\psi(p)} + \sum_{q<p} \frac{a_q}{\psi(p)} \\ &\geq \frac{1}{\psi(p)} + \frac{1}{\psi(p)} \sum_{q<p} a_q \end{aligned}$$

Note that $y_0 = ||v_0|| \leq 1 = a_0$. Then it follows by induction on p from the previous inequality and (2.3.2) that $y_p \leq a_p$ for all p. Thus it suffices to prove that there exists a t such that $a_p \leq t^{\phi(p)}$ for all ϕ, *i.e.*, that the series $\sum a_p e_p$ in fact lies in $\mathbf{R}\{P\}$. This will follow from the following lemma.

LEMMA 2.3.3 *Let P be a fine sharp monoid, let $\phi \colon P \to (\mathbf{R}^{\geq}, +)$ be a local homomorphism, and let c be any positive real number. Define $a \colon P \to \mathbf{R}$ inductively setting $a(0) = 1$, and, if $p \in P^+$,*

$$a_p = c \sum_{q<p} a_q \Big(1 - \frac{\phi(q)}{\phi(p)}\Big)$$

Then $\sum a_p e_p$ belongs to the ring $\mathbf{R}\{P\}$ of germs of convergent elements of $\mathbf{R}[[P]]$, and is in fact independent of ϕ.

Proof: Let

$$f := \sum_{q \in P^+} e_q \in \mathbf{C}[[P]],$$

Evidently $f(x)$ converges for all x in $U_1 = \{x : q(x) < 1 \text{ for all } q \in P^+\}$, hence so does $g := \exp cf$. Write

$$g := \sum_{p \in P} b_p e_p.$$

Then

$$\begin{aligned} dg &= cgdf \\ \sum_{p \in P^+} b_p e_p dp &= c\Big(\sum_{q \in P} b_q e_q\Big)\Big(\sum_{q' \in P^+} e_{q'} dq'\Big) \\ &= \sum_{q \in P, q' \in P^+} c b_q e_{q'+q} dq' \\ &= \sum_{p \in P^+} \Big(\sum_{q<p} c b_q d(p-q)\Big) e_p \end{aligned}$$

Thus

$$\begin{aligned} b_p dp &= \sum_{q<p} c b_q (dp - dq) \\ b_p \phi(p) &= \sum_{q<p} c b_q (\phi(p) - \phi(q)) \end{aligned}$$

Hence $a_p = b_p$, and therefore $\sum a_p e_p$ lies in $\mathbf{R}\{P\}$. □

We next show that the functors D and $\overline{D}$ are compatible with cohomology. Since this implies that they are fully faithful, this will complete the proof of the theorem. Since $\overline{D}$ is a special case of D, the following result suffices.

Proposition 2.3.4 *If (E, ∇) is an object of $MIC_{coh}(X_v/\mathbf{C})$, then the natural map*

$$E \otimes \Omega^{\cdot}_{X/\mathbf{C}} \to \hat{E} \otimes \Omega^{\cdot}_{X/\mathbf{C}}$$

is a quasi-isomorphism.

Proof: Since P is a toric monoid, its unit group is a finitely generated free group, say of rank r, and there is an isomorphism $P \cong \overline{P} \oplus \mathbf{Z}^r$. The vertex v of X is the point sending every element of P^* to 1 and every element of $\overline{P}^+$ to 0. Let $Q := \mathbf{N}^r \oplus \overline{P}$, let $X'' := \operatorname{Spec}(\overline{P} \to \mathbf{C}[Q])$, and let v'' be the point of X'' sending Q^+ to zero. Finally, let $X' := \mathsf{A}_\mathbf{Q}$, and let $f \colon X' \to X''$ be the map which is the identity on underlying analytic spaces and the inclusion on log structures. Thus

$$\begin{aligned} X &\cong \operatorname{Spec}(\overline{P} \to \mathbf{C}[\overline{P}][t_1, t_1^{-1}, \ldots t_r, t_r^{-1}]) \cong \overline{X} \times \mathbf{G}_m^r \\ X' &\cong \operatorname{Spec}(\overline{P} \oplus \mathbf{N}^r \to \mathbf{C}[\overline{P}][x_1, \ldots x_r]) \cong \overline{X} \times \mathsf{A}_{\mathbf{N}^r} \\ X'' &\cong \operatorname{Spec}(\overline{P} \to \mathbf{C}[\overline{P}][x_1, \ldots x_r]) \cong \overline{X} \times \underline{\mathsf{A}}_{\mathbf{N}^r} \end{aligned}$$

The homomorphism sending x_i to $t_i - 1$ and which is the identity on $\overline{P}$ defines a strict open immersion of log schemes $X \to X''$ sending v to v''. Replacing X' by a neighborhood of the vertex v' of X', we find a map $X' \to X$ which is an isomorphism on underlying analytic spaces and which sends v' to v. Thus we may and shall identify the stalk of E at v with the stalk of its pullback to X' at v'. (In other words, we have added some log structure to X to get X'.)

Lemma 2.3.5 *For each i, the stalk at v of natural map*

$$E \otimes \Omega^i_{X/\mathbf{C}} \to E \otimes \Omega^i_{X'/\mathbf{C}}$$

is injective. Furthermore, as submodules of $\hat{E} \otimes \Omega^i_{X'/\mathbf{C}}$,

$$E \otimes \Omega^i_{X/\mathbf{C}} = (\hat{E} \otimes \Omega^i_{X/\mathbf{C}}) \cap (E \otimes \Omega^i_{X'/\mathbf{C}})$$

at v.

Proof: We can check the injectivity statement after passing to formal completions. Recall from (2.2.2) that, since E is a crystal on X, there is a coherent sheaf $\overline{E}$ on $\overline{X}$ such that $\hat{E} \cong \pi^* \hat{\overline{E}}$, where $\pi\colon X \to \overline{X}$ is the map induced by our chosen splitting of $P \to \overline{P}$. Let $Y := \mathsf{A}_{\mathbf{N}^r}$, so that $X' \cong \overline{X} \times Y$ and $X \cong \overline{X} \times \underline{Y}$ near v. Then $\Omega^1_{X/\mathbf{C}} \cong \Omega^1_{\overline{X}/\mathbf{C}} \oplus \Omega^1_{\underline{Y}/\mathbf{C}}$ and $\Omega^1_{X'/\mathbf{C}} \cong \Omega^1_{\overline{X}/\mathbf{C}} \oplus \Omega^1_{Y/\mathbf{C}}$; furthermore all these sheaves are free at v. It follows that $\hat{\overline{E}}$ and the cokernel of the map $\Omega^i_{X/\mathbf{C}} \to \Omega^i_{X'/\mathbf{C}}$ are tor-independent. This proves the injectivity. Note that $\Omega^i_{X/\mathbf{C}}$ and $\Omega^i_{X'/\mathbf{C}}$ are free, and $x\Omega^i_{X'/\mathbf{C}} \subseteq \Omega^i_{X/\mathbf{C}}$, where $x := x_1 \cdots x_r$. If $e \in \hat{E}$ and $xe \in E$, then it is clear from (2.3.1) that $e \in E$. Since X' and X have the same underlying analytic structure, it follows that $(\hat{E} \otimes \Omega^i_{X/\mathbf{C}}) \cap (E \otimes \Omega^i_{X'/\mathbf{C}}) = E \otimes \Omega^i_{X/\mathbf{C}}$. ☐

Choose a local homomorphism $\phi\colon Q \to \mathbf{N}$ and for $n \in \mathbf{N}$ let $K_n := \{q \in Q : \phi(q) \geq n\}$. Let $E^i := E_v \otimes \Omega^i_{X/\mathbf{C}}$ and let $E^{i\prime}_v := E_v \otimes \Omega^i_{X'/\mathbf{C}}$. Then $E^{\cdot\prime}$ is a complex, containing subcomplexes $E^{\cdot}$ and $K_n E^{\cdot\prime}$ for each n. There is a commutative diagram of exact sequences of complexes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E^{\cdot} \cap K_n E^{\cdot\prime} & \longrightarrow & E^{\cdot} & \longrightarrow & E^{\cdot}_n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \hat{E}^{\cdot} \cap K_n \hat{E}^{\cdot\prime} & \longrightarrow & \hat{E}^{\cdot} & \longrightarrow & \hat{E}^{\cdot}_n & \longrightarrow & 0
\end{array}$$

In this diagram, the quotient $E^{\cdot}_n$ is contained in $E^{\cdot\prime}_n := E^{\cdot\prime}/K_n E^{\cdot\prime}$ and annihilated by a power of the maximal ideal at v. Thus, the arrow on the right is an isomorphism of complexes. Our goal is to prove that the central arrow is a quasi-isomorphism, and so it will suffice to prove that the arrow on the left is a quasi-isomorphism. In particular, the following lemma suffices.

Lemma 2.3.6 *With the above notation, suppose that $n > \phi(\lambda)$ for every exponent λ of $E(v')$ on X'. Then $E^{\cdot} \cap K_n E^{\cdot\prime}$ and $\hat{E}^{\cdot} \cap K_n \hat{E}^{\cdot\prime}$ are acyclic.*

Proof: The homomorphism $\phi\colon Q \to \mathbf{N}$ induces a homomorphism $\Omega_{Q/\mathbf{C}} \to \mathbf{C}$, which can be regarded as an equivariant vector field on X'. It also induces for each i a homomorphism $\Omega^i_{Q/\mathbf{C}} \to \Omega^{i-1}_{Q/\mathbf{C}}$, by interior multiplication. These maps extend to $\mathbf{C}[Q]$-linear maps $E \otimes \Omega^i_{X'/\mathbf{C}} \to E \otimes \Omega^{i-1}_{X'/\mathbf{C}}$ sending $E \otimes \Omega^i_{X/\mathbf{C}}$ to $E \otimes \Omega^{i-1}_{X/\mathbf{C}}$ and $K_n E \otimes \Omega^i_{X'/\mathbf{C}}$ to $K_n E \otimes \Omega^{i-1}_{X'/\mathbf{C}}$. Let $\kappa := d\rho + \rho d$, *i.e.*, the Lie derivative with respect to ϕ. Then κ defines a morphism of complexes $E^{\cdot\prime} \to E^{\cdot\prime}$ which preserves the subcomplexes $E^{\cdot}$ and $K_n E^{\cdot\prime}$ and (hence) passes to the completions. By construction, κ is homotopic to zero. So to prove the complexes are acyclic, it suffices to prove that κ is an isomorphism on each of them.

Note that if $q \in Q$,

$$\kappa(e_q) = d\rho(e_q) + \rho d(e_q) = \rho(e_q dq) = \phi(q) e_q.$$

Furthermore, κ is a derivation, *i.e.*,

$$\kappa(\eta \wedge \omega) = \kappa(\eta) \wedge \omega + \eta \wedge \kappa(\omega)$$

if $\eta \in E \otimes \Omega^{\cdot}_{X/\mathbf{C}}$ and $\omega \in \Omega^i_{X/\mathbf{C}}$. If ω is equivariant, *i.e.*, if it lies in $\Omega^i_{Q/\mathbf{C}}$, $\kappa(\omega) := d\rho\omega + \rho d\omega = 0$. Thus if $e \in E$ and $\omega \in \Omega^i_{Q/\mathbf{C}}$, $\kappa(e \otimes \omega) = \nabla_\phi(e) \otimes \omega$. In other words, viewed as a map

$$\kappa\colon E \otimes_{\mathbf{C}} \Omega^i_{Q/\mathbf{C}} \to E \otimes_{\mathbf{C}} \Omega^i_{Q/\mathbf{C}},$$

$\kappa := \nabla_\phi \otimes \mathrm{id}$. Lemma (2.3.2) implies that ∇_ϕ acts bijectively on E and $\hat{E}$, and hence κ induces an automorphism of $E \otimes \Omega^{\cdot}_{X'/\mathbf{C}}$ and of $\hat{E} \otimes \Omega^{\cdot}_{X'/\mathbf{C}}$.

Since $K_n E$ and its quotients also satisfy the hypothesis of (2.3.2), κ also induces automorphisms of $K_n E^{\cdot\prime}$ and of $K_n E^{\cdot\prime}/K_{n'} E^{\cdot\prime}$ whenever $n' \geq n$. The image of $E^{\cdot} \cap K_n E^{\cdot\prime}$ in $K_n E^{\cdot\prime}/K_{n'} E^{\cdot\prime}$ is a finite dimensional subspace invariant under κ, and hence κ also acts as an automorphism of this subspace. Taking the limit over n', we see that κ induces an automorphism of $\hat{E}^{\cdot} \cap K_n \hat{E}^{\cdot\prime}$. It follows that κ is injective on $E^{\cdot} \cap K_n E^{\cdot\prime}$. If $e \in E^{\cdot} \cap K_n E^{\cdot\prime}$, there is a unique $\hat{e} \in \hat{E}^{\cdot} \cap K_n \hat{E}^{\cdot\prime}$ such that $\kappa(\hat{e}) = e$. But $e \in K_n E^{\cdot\prime}$ and so there is a unique $f \in K_n E^{\cdot\prime}$ such that $\kappa(f) = e$. Thus

$$\hat{e} = f \in \hat{E}^{\cdot} \cap E^{\cdot\prime} \cap K_n \hat{E}^{\cdot\prime} = E^{\cdot} \cap K_n E^{\cdot\prime}.$$

This proves that κ is an isomorphism of $E^{\cdot} \cap K_n E^{\cdot\prime}$ and completes the proof of the lemma. □

□

3 X_{log} and the global Riemann-Hilbert correspondence

3.1 X_{log} and its universal covering

If X is an idealized log scheme of finite type over $\mathbf{C}$, let X_{an} or X^{an} denote the corresponding log analytic space. Let us say that an idealized log analytic space X is *ideally log smooth* if it admits a covering by open subsets each of which is isomorphic to an open subset of $\underline{\mathsf{A}}^{an}_{P,K}$ for some fine idealized monoid (P, K). For such spaces, the sheaf of ideals K can be recovered from the log structure as the inverse image in M_X of 0. We let $\mathbf{S^1}$ be the unit circle, *i.e.*, $\{z \in \mathbf{C} : |z| = 1\}$ and $\mathbf{R}^{\geq}$ the multiplicative monoid of nonnegative real numbers. Thus the multiplication map $\mathbf{R}^{\geq} \times \mathbf{S^1} \to \mathbf{C}$ defines a log structure on $\mathbf{C}$, and we let ξ_{log} denote the corresponding log scheme $\mathrm{Spec}(\mathbf{R}^{\geq} \times \mathbf{S^1} \to \mathbf{C})$. Note that this log scheme is not integral or even coherent.

Kato and Nakayama have constructed in [7] a commutative diagram of ringed spaces:

$$\begin{array}{ccc} X^*_{an} & \xrightarrow{j_{log}} & X_{log} \\ & \searrow^{j} & \downarrow \tau \\ & & X_{an} \end{array}$$

We refer to [7] for the definition, but recall that the set underlying X_{log} is the set of $\mathbf{C}$-morphisms of log schemes $\xi_{log} \to X$ and that τ is the obvious map which forgets the log structure. This map is proper, and the fiber over a point x is a torsor under the group $\mathrm{Hom}(\overline{M}_{X,x}, \mathbf{S^1})$. Since X is saturated, this space is (noncanonically) isomorphic to $(\mathbf{S^1})^{r(x)}$, where $r(x)$ is the rank of $\overline{M}^{gp}_{X,x}$. The fundamental group I_x of the fiber $\tau^{-1}(x)$ (the *logarithmic inertia group at* x) can be canonically identified with $\mathrm{Hom}(\overline{M}^{gp}_{X,x}, \mathbf{Z}(1))$. Since this group is abelian and the fiber is connected, the choice of base point can be ignored.

When $X = \mathsf{A}_{\mathsf{P},\mathsf{K}}$, the space X_{log} has a convenient explicit description. If P is a monoid, let $C(P)$ denote the set of morphisms of monoids $\rho\colon P \to \mathbf{R}^{\geq}$, with the structure of topological monoid inherited from that of $\mathbf{R}^{\geq}$. If K is an ideal in P, let $C(P, K)$ be the set of those $\rho \in C(P)$ sending K to 0. This is a closed submonoid of $C(P)$, and in fact is an ideal in the monoid $C(P)$. Let $S(P)$ denote the set of morphisms of monoids $\sigma\colon P \to \mathbf{S^1}$, or, equivalently, $P^{gp} \to \mathbf{S^1}$, with its structure of topological group. If P is toric, P^{gp} is a finitely generated free abelian group, so $S(P)$ is a torus. Then if $X = \mathsf{A}_{\mathsf{P},\mathsf{K}}$, there is a canonical isomorphism $X_{log} \cong C(P, K) \times S(P)$. When K is a proper ideal, the map $c_0\colon P \to \mathbf{R}^{\geq}$ sending P^* to 1 and P^+ to 0 is a point of $C(P, K)$, and the pair $(c_0, 1)$ is a point of X_{log} lying over the vertex of X, which we call the vertex of X_{log}.

It will be useful for us to work with an explicit universal cover of X_{log} when $X = \mathsf{A}_{\mathsf{P},\mathsf{K}}$. Let $\mathbf{R}(1) \subseteq \mathbf{C}$ denote the set of purely imaginary numbers, which forms a topological group under addition, and let $Y(P)$ denote the set of homomorphisms of abelian groups from P^{gp} to $\mathbf{R}(1)$. Finally, let $\tilde{\mathsf{A}}^{log}_{P,K} := C(P,K) \times Y(P)$, with its natural structure of a topological monoid. If K is a proper ideal call $\tilde{v} := (c_0, 0)$ the vertex of $\tilde{\mathsf{A}}^{log}_{P,K}$.

Proposition 3.1.1 *Let K be a proper ideal in a toric monoid P and let $X := \mathsf{A}_{\mathsf{P},\mathsf{K}}$ and $\tilde{X}^{log} := \tilde{\mathsf{A}}^{log}_{P,K}$. Then the map*

$$\zeta\colon \tilde{X}^{log} = C(P,K) \times Y(P) \to C(P,K) \times S(P) = X_{log} : (\rho, y) \mapsto (\rho, \exp \circ y)$$

is a universal covering sending the vertex of $\tilde{X}^{log}$ to the vertex of X_{log}, with covering group canonically isomorphic to $\pi_1(P) := \mathrm{Hom}(P^{gp}, \mathbf{Z}(1))$. When P is a group, there is a natural isomorphism $\tilde{X}^{log} \cong \mathbf{V}\Omega^{an}_P$, under which the covering map ζ corresponds to the covering map exp *defined at the beginning of section (1.4).*

Proof: It is clear that id $\times$ exp is a covering map taking the vertex to the vertex. The exact sequence $0 \to \mathbf{Z}(1) \to \mathbf{R}(1) \to \mathbf{S^1} \to 0$ induces an exact sequence

$$0 \to \pi_1(P) \to Y(P) \to S(P) \to 0,$$

and so the covering group of ζ is canonically isomorphic to $\pi_1(P)$. To finish the proof, it will suffice to show that $\tilde{\mathsf{A}}^{log}_{P,K}$ is contractible. Choose a local homomorphism $\delta\colon P \to \mathbf{N}$. Then for any $t \in [0,1]$, t^δ defines a homomorphism $P \to \mathbf{R}^{\geq}$ and so is a point of $C(P)$. (Here we are using the convention that $0^0 = 1$.) Consider the continuous map

$$\tilde{\mathsf{A}}^{log}_{P,K} \times I \to \tilde{\mathsf{A}}^{log}_{P,K} : (x,t) \mapsto x_t \tag{3.1.1}$$

sending $(x,t) := ((\rho, y), t)$ to $x_t := (\rho^t t^\delta, ty)$. When $t = 1$, the map $x \mapsto x_t$ is the identity, and when $t = 0$, it is the constant map to the vertex, since $0^{\delta(p)}$ is 1 if $p \in P^*$ and 0 otherwise. If P is a group, then each element $c\colon P \to \mathbf{R}^{\geq}$ of $C(P)$ factors through $\mathbf{R}^+$, and so we can define $\tilde{c}\colon P \to \mathbf{R}$ to be $\log \circ c$. Then $C(P)$ can be identified with $\mathrm{Hom}(P, \mathbf{R})$ and $\tilde{X}^{log}$ with $\mathrm{Hom}(P, \mathbf{R}) \times \mathrm{Hom}(P, \mathbf{R}(1)) \cong \mathrm{Hom}(P, \mathbf{C}) = \mathbf{V}\Omega^{an}_P$. With this identification, ζ corresponds to exp. ☐

The complement $\mathfrak{p}$ of each face F of P not meeting K is a prime ideal of P containing K and defines a closed log subscheme of $\mathsf{A}_{\mathsf{P},\mathsf{K}}$ whose underlying scheme is isomorphic to $\underline{\mathsf{A}}_{\mathsf{F}}$. Let $X_{\mathfrak{p}}$ or X_F denote this log scheme; in fact $X_F \cong \mathrm{Spec}(P \to \mathbf{C}[F])$, where $P \to \mathbf{C}[F]$ is the obvious one on F and kills $\mathfrak{p}$. If x is a point of the dense open subset $\underline{X}^*_F = \mathsf{A}^*_{\mathsf{F}}$ of $\underline{X}_F$, the map $P \to \overline{M}_{X,x}$ induces an isomorphism $P/F \to \overline{M}_{X,x}$. Thus the family of faces F not meeting K defines a canonical stratification of $\mathsf{A}_{\mathsf{P},\mathsf{K}}$ on which the log structure is constant. We call this stratification, as well as the stratification it induces by pullback to X_{log} and its universal cover, the *canonical log stratification.* For each F, $\tau^{-1}(X_F^*)$ is the set of $(\rho, \sigma) \in C(P, K) \times S(P)$ such that $\rho^{-1}(\mathbf{R}^+) = F$. This space is homotopy equivalent to all of X_{log}, and the fiber over a point x is isomorphic to $S(P/F)$. In particular, $I_x \cong \mathrm{Hom}(P/F, \mathbf{Z}(1))$, and there is an exact sequence:

$$1 \to I_x \to \pi_1(X_F^{log}) \to \pi_1(X_{F,an}^*) \to 1.$$

Note that these strata are preserved by the contraction (3.1.1) above. That is, if $x \in \tilde{X}_F^{log*}$, then $x_t \in \tilde{X}_F^{log*}$ for all $t > 0$.

The following result may help explain the geometric significance of the construction of $\tau_X \colon X_{log} \to X_{an}$: it can be regarded as a compactification of the inclusion $X_{an}^* \to X_{an}$ which doesn't change its local homotopy type.

Theorem 3.1.2 *Suppose that $X/\mathbf{C}$ is a fine, smooth, and saturated log scheme (so $K_X = \emptyset$.) Then the map $j_{log} \colon X_{an}^* \to X_{log}$ is aspheric. That is, any point z of X_{log} has a basis of neighborhoods U such that $j_{log}^{-1}(U)$ is contractible. Consequently:*

1. *There are natural isomorphisms $\mathbf{Z} \cong Rj_{log*}\mathbf{Z}$, and $R\tau_*\mathbf{Z} \cong Rj_*\mathbf{Z}$.*
2. *If V is a locally constant abelian sheaf on X_{an}^*, then $j_{log*}V$ is locally constant on X_{log} and $R^i j_{log*}V = 0$ for $i > 0$.*

Proof: This question is local in a neighborhood of z in X_{log}, and hence also in a neighborhood of its image x in X. Since $X/\mathbf{C}$ is smooth, by [6, 3.5] there exists a toric monoid P and a strict étale map $f \colon X \to \mathsf{A_P}$. Thus the theorem follows from the following lemma. □

Lemma 3.1.3 *Let K be a proper ideal in a toric monoid P, let $X := \mathsf{A_{P,K}}$ and let z be a point of X_{log} lying over a point x of X. Then z has a cofinal system of open neighborhoods U such that for each face F of P such that $x \in X_F$, the intersection of U with the stratum $\tau^{-1}(X_F^*)$ is contractible.*

Proof: If $z = (\rho, \sigma)$, then $G := \rho^{-1}(\mathbf{R}^+)$ is the face of P corresponding to the log stratum containing x. Then $x \in X_G^* \subseteq X_F$, $G \subseteq F$, and X_G^* and X_F^* are contained in X_{P_G}, where P_G is the localization of P by G. Thus without loss of generality we may replace P by P_G. Then x lies in the minimal orbit X_{P^*}. Since this orbit and its inverse image in X_{log} are homogeneous, we may as well assume that z is the vertex v of X_{log}.

Fix a splitting of $P \to \overline{P}$ and choose finite sets of generators S^+ for $\overline{P}$ and S^* for P^*. For each $\epsilon > 0$, let $C_\epsilon(P, K)$ be the set of $\rho \in C(P, K)$ such that $\rho(s) < \epsilon$ for $s \in S^+$ and $|\rho(s) - 1| < \epsilon$ for $s \in S^*$. Similarly, let $S_\epsilon(P)$ denote the set of $\sigma \in S(P)$ such that $|\sigma(s) - 1| < \epsilon$ for all $s \in S$, and let $U_\epsilon := C_\epsilon(P, K) \times S_\epsilon(P)$. Then the family of these U_ϵ for $\epsilon > 0$ is a basis for the set of neighborhoods of v. If F is a face of P not meeting K, the inverse image of X_F in X_{log} can be identified with $C(F) \times S(P)$. Since F is a face of P, $P^* = F^*$, the splitting $P \cong P^* \oplus \overline{P}$ induces a splitting $F \cong F^* \oplus \overline{F}$, and $S^+ \cap \overline{F}$ is a set of generators for $\overline{F}$. Then the intersection of $\tau^{-1}(X_F)$ with U_ϵ becomes

$C_\epsilon(\overline{F})\times C_\epsilon(F^*)\times S_\epsilon(P)$, where $C_\epsilon(\overline{F})$ is the set of $\rho\colon \overline{F}\to \mathbf{R}^{\geq}$ such that $\rho(s)<\epsilon$ for all $s\in \overline{F}\cap S^+$ and $C_\epsilon(F^*)$ is the set of homomorphisms $F^*\to \mathbf{R}^{\geq}$ such that $|\rho(s)-1|<\epsilon$ for $s\in S^*$. Then $\tau^{-1}(X^*_F)\cap U_\epsilon$ is $C^*_\epsilon(\overline{F})\times C_\epsilon(F^*)\times S_\epsilon(P)$, where $C^*_\epsilon(\overline{F})$ is the set of $\rho\in C_\epsilon(\overline{F})$ which factor through $\overline{F}^{gp}$. Thus $C^*_\epsilon(\overline{F})$ is contained in the set $C^*(\overline{F})$ of homomorphisms $\overline{F}^{gp}\to \mathbf{R}^+$. Choosing a basis $(f_1,\dots f_n)$ for the finitely generated free abelian group $\overline{F}^{gp}$ and using the topological isomorphism $\log\colon \mathbf{R}^+\to\mathbf{R}$, we may identify $C^*(\overline{F})$ with the Euclidean space $E:=\mathbf{R}^n$. Then each $s\in S^+\cap\overline{F}$ can be written as a linear combination of the elements f_i and defines an element $\hat{s}$ in the dual of E. For each s, the set $E_s:=\{e\in E:\hat{s}(e)<\log\epsilon\}$ is convex. Thus $C^*_\epsilon(\overline{F})$ becomes identified with the intersection of the convex subsets $E_s : s\in S^+\cap\overline{F}$, which is therefore convex, hence contractible. Since $C_\epsilon(F^*)$ and $S_\epsilon(P)$ are evidently also contractible if $\epsilon<1$, the same is true of $U_\epsilon\cap\tau^{-1}(X^*_F)$. □

3.2 $\mathbf{C}^{log}_X$ and logarithmic local systems

We shall use the space X_{log} to globalize the local classification (2.1.2) of log connections, as explained in the introduction. Our first task is to give a more precise formulation of the global Riemann-Hilbert correspondence which takes into account the fact that the sheaf $\overline{M}_X$ is not constant. This will require the notion of cospecialization for certain constructible sheaves.

We begin by recalling the simple case of sheaves on intervals. Let $I=[0,1]$ be the closed unit interval and let F be a sheaf on I. If F is constant, then for any connected open subset U of I, the restriction map $F(I)\to F(U)$ is an isomorphism. Hence for any $a,b\in I$, the maps $F(I)\to F_a$ and $F(I)\to F_b$ are isomorphisms, and so there is a canonical isomorphism $F_a\to F_b$. More generally, suppose only that the restriction of F to $(0,1]$ is constant. Then if $a>0$, the restriction mapping $F((0,1])\to F((0,a))$ is bijective. Since F is a sheaf, the sequence

$$F(I)\longrightarrow F((0,1])\times F([0,a))\rightrightarrows F((0,a))$$

is exact, and it follows that the map $F(I)\to F([0,a))$ is an isomorphism. Since this is true for all $a>0$, the map $F(I)\to F_0$ is also an isomorphism. Hence there is a natural map

$$cosp_{0,b}: F_0\xrightarrow{\rho^{-1}_{I,0}} F(I)\xrightarrow{\rho_{I,1}} F_b \tag{3.2.1}$$

for any $b\in[0,1]$. Even more generally, suppose F is a sheaf on $[0,1]$ and that for some $c\in(0,1)$ the restrictions of F to $(0,c]$ and to $[c,1]$ are locally constant. Then they are constant, and since $\{(0,c],[c,1]\}$ is a locally finite closed cover of $(0,1]$, it follows that the restriction of F to $(0,1]$ is also constant. Hence for any $b \in [c, 1]$ there is a commutative diagram

$$\begin{array}{ccc} F_0 & \xrightarrow{cosp_{0,c}} & F_c \\ & \underset{cosp_{0,b}}{\searrow} & \downarrow{\scriptstyle cosp_{c,b}} \\ & & F_b \end{array}$$

Now let x and y be points in a topological space X and let F be a sheaf on X. By an *F-path (resp. strict F-path) from x to y* we shall mean a continuous function $\gamma\colon I \to X$ such that $\gamma(0) = x$, $\gamma(1) = y$, and such that the restriction of $\gamma^{-1}F$ to $(0, 1]$ (resp. and such that $\gamma^{-1}(F)$) is locally constant. Then the above construction defines a canonical cospecialization map (resp. isomorphism)

$$\gamma^*_{x,y}\colon F_x \to F_y.$$

If γ is an F-path from x to y and γ' is a strict F-path from y to z, then the concatenation $\gamma\gamma'$ is an F-path from x to z, and $\gamma^*_{x,z} = {\gamma'}^*_{y,z} \circ \gamma^*_{x,y}$. If X is a log scheme and x and y are points of X or X_{log}, we shall simply say *log path* instead of $\overline{M}_X$-path.

We shall need a toric version of this cospecialization construction. Let K be a proper ideal in a toric monoid P, let $X := \mathsf{A}_{\mathsf{P},\mathsf{K}}$, and let $\zeta\colon \tilde{X}^{log} \to X_{log}$ be the universal cover constructed in Proposition (3.1.1). For each face F of P not meeting K, let X^*_F denote the corresponding (locally closed) log stratum of X, and let $j_F\colon X^*_F \to X$ denote the inclusion. Thus j_F factors through the closure X_F of X^*_F in X. We denote by $\tilde{X}^*_F$ the inverse image of X^*_F in $\tilde{X}^{log}$, with similar notation for X^{log}_F and $\tilde{j}_F$. We say that a sheaf W on X_{log} or $\tilde{X}^{log}$ is *log constructible* if its restriction to each log stratum is locally constant.

Let W be a log constructible sheaf on $\tilde{X}^{log}$. Since the log strata are simply connected, the restriction of W to each log stratum $\tilde{X}^*_F$ is constant, and we let $W_F := W(\tilde{X}^*_F)$. Lemma (3.1.3) implies that each point of $\tilde{X}_F$ admits a neighborhood basis of open sets whose intersection with $\tilde{X}^*_F$ is connected (even contractible). It follows that $\tilde{j}_{F*}\tilde{j}^*_F W$ is canonically isomorphic to the constant sheaf W_F on $\tilde{X}^{log}_F$. If G is a face of F, the canonical map $\tilde{j}^*_G W \to \tilde{j}^*_G \tilde{j}_{F*}\tilde{j}^*_F W$ induces a map

$$cosp_{G,F}\colon W_G \to W_F.$$

Furthermore, there is a commutative diagram

$$\begin{array}{ccc} W & \longrightarrow & \tilde{j}_{F*}\tilde{j}^*_F W \\ \downarrow & & \downarrow \\ \tilde{j}_{G*}\tilde{j}^*_G W & \longrightarrow & \tilde{j}_{G*}\tilde{j}^*_G\tilde{j}_{F*}\tilde{j}^*_F W. \end{array}$$

Since $\tilde{j}_{F*}\tilde{j}_F^*W$ is the constant sheaf with value W_F, the same argument as before shows that the vertical arrow on the right is an isomorphism. If H is a face of G, we can pull back this diagram to $\tilde{X}_H^*$ and take global sections to obtain a commutative diagram

$$\begin{array}{ccc} W_H & & \\ {\scriptstyle cosp_{H,G}}\downarrow & \searrow{\scriptstyle cosp_{H,F}} & \\ W_G & \xrightarrow{cosp_{G,F}} & W_F. \end{array}$$

It is well-known and easy to check that W is determined completely by the family of sets W_F and cospecialization maps. Indeed, the functor which takes a log constructible sheaf on $\tilde{X}^{log}$ to the corresponding family of sets and maps is easily seen to be an equivalence. We should perhaps remark that it is not difficult to show that if $X := \mathsf{A}_{\mathsf{P},\mathsf{K}}$ and W is a log constructible sheaf on $\tilde{X}^{log}$, then the natural map from $W(\tilde{X}^{log})$ to the stalk of W at the vertex is an isomorphism. This fact can be used to give another interpretation of the cospecialization maps.

Let V be a log constructible sheaf on X_{log} and let $\tilde{V}$ be its pullback to $\tilde{X}^{log}$. Then each $\tilde{V}_F$ is equipped with a natural action of $\pi_1(P)$, and the cospecialization maps are compatible with this action. In this way one obtains an equivalence between the category of log constructible sheaves on X_{log} and the category of families of $\pi_1(P)$-sets and compatible cospecialization maps. Let x and y be points of X_{log} and choose points $\tilde{x}$ and $\tilde{y}$ of $\tilde{X}$ lying over them. Let $F(x)$ (resp. $F(y)$) be the face of P corresponding to the log stratum containing x (resp. y). Then if $F(x) \subseteq F(y)$,

$$cosp_{\tilde{x},\tilde{y}} \colon V_x \longrightarrow V_y$$

is by definition the map such that the diagram

$$\begin{array}{ccc} \tilde{V}_{F(x)} & \xrightarrow{cosp_{F(x),F(y)}} & \tilde{V}_{F(y)} \\ \downarrow & & \downarrow \\ V_x & \xrightarrow{cosp_{\tilde{x},\tilde{y}}} & V_y \end{array}$$

commutes. Here the left vertical arrow is the composite $\tilde{V}_{F(x)} \cong \tilde{V}(\tilde{X}_F^*) \cong \tilde{V}_{\tilde{x}} \cong V_x$, and the right one is defined similarly. Note that the map $cosp_{\tilde{x},\tilde{y}}$ depends on the choices of $\tilde{x}$ and $\tilde{y}$ lying over x and y, and that $cosp_{\tilde{x},\tilde{y}}$ is an isomorphism if x and y lie in the same log stratum. Note also that if $\tilde{z}$ is a point of $\tilde{X}$ such that $F(y) \subseteq F(z)$, then there is a commutative diagram

$$\begin{array}{ccc} V_x & & \\ {\scriptstyle cosp_{\tilde{x},\tilde{y}}}\downarrow & \searrow^{cosp_{\tilde{x},\tilde{z}}} & \\ V_y & \xrightarrow{cosp_{\tilde{y},\tilde{z}}} & V_z \end{array}$$

Remark 3.2.1 Let V be a log constructible sheaf on $X_{log} := \mathsf{A}_{P,K}^{log}$ and let γ be a continuous map from the unit interval I to X_{log} such that the image of $(0, 1]$ is contained in a single log stratum. Choose a point $\tilde{x}$ of $\tilde{X}$ lying over $x := \gamma(0)$, let $\tilde{\gamma}\colon I \to \tilde{X}$ be the lift of γ such that $\tilde{\gamma}(0) = \tilde{x}$, and let $\tilde{y} := \tilde{\gamma}(1)$. Then the following diagram commutes:

$$\begin{array}{ccc} \gamma^{-1}V_0 & \xrightarrow{cosp_{0,1}} & \gamma^{-1}V_1 \\ \downarrow & & \downarrow \\ V_{\tilde{x}} & \xrightarrow{cosp_{\tilde{x},\tilde{y}}} & V_{\tilde{y}} \end{array}$$

The notions of log stratification and log constructibility make sense more generally, at least locally. Let X be an ideally smooth fs log scheme over $\mathbf{C}$, let x be a $\mathbf{C}$-valued point of X, and consider the map $\alpha_{X,x}\colon M_{X,\overline{x}} \to \mathcal{O}_{X,\overline{x}}$. (We are temporarily writing $\overline{x}$ to remind ourselves that we are taking the stalks in the étale topology.) A point of the spectrum $\mathcal{O}_{X,\overline{x}}$ is said to be a *log branch at* x if it is the inverse image of a prime ideal in the monoid $M_{X,\overline{x}}$. Since X is, locally in some étale neighborhood of x, isomorphic to $\mathsf{A}_{\mathsf{P},\mathsf{K}}$ for some fs monoid P and some ideal K in P, there is a bijection between the set of log branches at x and the set of prime ideals of $M_{X,\overline{x}}$ containing $K_{X,x}$. Each log branch at x defines an irreducible and unibranch closed subscheme Z in some étale neighborhood of x, and the restriction of $\overline{M}_X$ and $\overline{K}_X$ to a dense open subset Z^o of Z is constant. We shall call a maximal such Z^o a *log stratum at* x. We use the same terminology for the inverse images of these sets in X_{log}.

Corollary 3.2.2 *Let X be an ideally smooth fs log scheme and let x be a point of X_{log}. Then x has an étale neighborhood U such that for every point y of U, there exists a log path from x to y.*

Proof: Without loss of generality we may assume that $X = \mathsf{A}_{\mathsf{P},\mathsf{K}}$, where P is a fine monoid and K is an ideal of P, and that x is the vertex of X_{log}. We may work in $\tilde{X}_{log}$ instead of X_{log}. Then the result follows from (3.1.1) and the discussion which follows it. □

Definition 3.2.3 *Let X be an idealized log scheme and let V be a sheaf on X_{log}. Then V is said to be* log constructible *if for each $x \in X_{log}$, V is locally constant on the log strata at x.*

Observe that this condition is local in the analytic topology of X. That is, if V is a sheaf on X_{log} and X_{an} admits a cover by open sets U such that the restriction of V to each $\tau^{-1}(U)$ is log constructible, then V is log constructible. The sheaves $\overline{M}_X$ and $\overline{K}_X$ are always log constructible. Let V be a log constructible sheaf on X_{log}, let x and y be points of X_{log}, and let γ be a log path from x to y. Then X admits an étale open cover $\{U_\lambda\}$ which admits charts as above, and the restriction of γ to each $\gamma^{-1}(U_\lambda) \cap (0,1]$ factors through a log stratum. It follows that the restriction of $\gamma^{-1}V$ to $(0,1]$ is locally constant, so γ defines a cospecialization map

$$\gamma^*_{x,y} \colon V_x \to V_y.$$

In particular, V is locally constant on the fiber $\tau^{-1}(x)$ of each point x of X, and hence if $z \in \tau^{-1}(x)$, V_z has a natural action ρ of the fundamental group I_x of $\tau^{-1}(x)$.

By a *sheaf of exponential data* for X we mean a log constructible sheaf of subgroups $\Lambda \subseteq \mathbf{C} \otimes \overline{M}^{gp}_X$ containing $\overline{M}^{gp}_X$. In practice, it will suffice to take $\Lambda := \mathbf{C} \otimes \overline{M}^{gp}_X$, but for some purposes it might be preferable to use $\mathbf{Q} \otimes \overline{M}^{gp}_X$ or $\overline{M}^{gp}_X$. We also write Λ for $\tau^{-1}\Lambda$ to simplify the notation. Let $\mathbf{C}^{log}_X$ denote the pullback to X_{log} of the quotient of the sheaf of monoid algebras $\mathbf{C}[-\overline{M}_X]$ by the ideal generated by $-\overline{K}_X$. This sheaf is also log constructible. The inclusion $-\overline{M}_X \to \Lambda$ defines an action of $-\overline{M}_X$ on Λ, so that one has a notion of a sheaf of Λ-graded $\mathbf{C}^{log}_X$-modules.

Definition 3.2.4 *Let $L^\Lambda_{coh}(\mathbf{C}^{log}_X)$ denote the category of Λ-graded sheaves V of $\mathbf{C}^{log}_X$-modules on X_{log} satisfying the following conditions:*

1. *V is log constructible.*

2. *For each $z \in X_{log}$, the stalk V_z of V at z is finitely generated over $\mathbf{C}^{log}_{X,z}$.*

3. *If x and y are points of X_{log} and γ is any log path from x to y, then the cospecialization map*

$$\gamma^*_{x,y} \colon V_x \otimes_{\mathbf{C}^{log}_{X,x}} \mathbf{C}^{log}_{X,y} \to V_y$$

is an isomorphism.

4. *If $z \in X_{log}$, $\gamma \in I_{\tau(z)}$, and $\lambda \in \Lambda_z$, then $\exp\langle\gamma,\lambda\rangle$ is the only eigenvalue of the action of ρ_γ on $V_{\lambda,z}$, i.e., $\rho_\gamma - \exp\langle\gamma,\lambda\rangle \colon V_{z,\lambda} \to V_{z,\lambda}$ is nilpotent.*

We shall say that a sheaf of Λ-graded $\mathbf{C}^{log}_X$-modules is *coherent* if it satisfies the above conditions. These perhaps need some explanation. Let $x = \tau(z)$, and note that in (4) of the above definition, $\gamma \in I_x = \mathrm{Hom}(\overline{M}^{gp}_{X,x}, \mathbf{Z}(1))$ and $\lambda \in \mathbf{C} \otimes \overline{M}^{gp}_{X,x}$, so that $\langle \gamma, \lambda \rangle \in \mathbf{C}$ makes sense. Moreover, if $m \in -\overline{M}_{X,x}$, $\exp\langle \gamma, m \rangle = 1$, so (4) is compatible with multiplication by elements of $\mathbf{C}^{log}_X$. Note also that (4) implies that the action of I_x on $V_{z,\lambda}$ is unipotent if $\lambda \in \overline{M}^{gp}_{X,x}$, and that (2) implies that each graded piece $V_{z,\lambda}$ is a finite dimensional $\mathbf{C}$-vector space. Using the compatibility of cospecialization with concatenation of log paths, one can easily check that condition (3), like the others, is local on X_{an}. Thus the category $L^{\Lambda}_{coh}(\mathbf{C}^{log}_X)$ is of local nature on X_{an}. Suppose V is log constructible, that X admits a toric chart above, and that γ is a log path from x to y for which (3) holds. Then it follows from the toric interpretation of the cospecialization map that if γ' is a log path from x to any point y' in the log stratum of y, ${\gamma'}^*_{x,y'}$ is also an isomorphism. Furthermore if a morphism $V \to V'$ in $L^{\Lambda}_{coh}(\mathbf{C}^{log}_X)$ induces an isomorphism on the stalks at some point z of X_{log}, then one sees easily from (3) and Corollary (3.2.2) that it induces an isomorphism in some neighborhood of z. Thus objects in $L^{\Lambda}_{coh}(\mathbf{C}^{log}_X)$ can be thought of as analogs of locally constant sheaves—of course, when the log structure is trivial, they are indeed locally constant.

Let us describe the category $L^{\Lambda}_{coh}(\mathbf{C}^{log}_X)$ explicitly when $X = \mathsf{A_P}$ for a toric monoid P endowed with a rigid set of exponential data $\Lambda \subseteq \mathbf{C} \otimes P^{gp}$. For each face F of P, the image Λ_F of Λ in $(\mathbf{C} \otimes P^{gp})/(\mathbf{C} \otimes F^{gp})$ defines a set of exponential data for P/F, and an inclusion $F \subseteq G$ induces a cospecialization map $\Lambda_F \to \Lambda_G$. We thus obtain a sheaf of exponential data on $\mathsf{A_P}$, also denoted by Λ. Let M be a $\overline{\Lambda}$-graded $\mathbf{C}[\overline{P}]$-module endowed with an action of $\pi_1(P)$ which satisfies the coherence conditions of (1.4.1), *i.e.*, an object of $\overline{L}^{\Lambda}_{coh}(P)$ (see (1.4.2)). For each face F of P, let $M_F := \mathbf{C}[P/F] \otimes_{\mathbf{C}[\overline{P}]} M$, with its natural structure of a Λ_F-grading and action of $\pi_1(P)$, and recall from (1.4.2) that $M \mapsto M_F$ is an equivalence when $F = P^*$. Since this construction is compatible with further dividing by faces, the family $\{M_F : F\}$ defines an object of $L^{\Lambda}_{coh}(\mathbf{C}^{log}_X)$. Conversely, if V is an object of $L^{\Lambda}_{coh}(\mathbf{C}^{log}_X)$, the restriction of V to $X^{log}_{P^*}$ is locally constant, and the evident maps

$$\Gamma(\tilde{X}^{log}_{P^*}, \tilde{V}) \longrightarrow \tilde{V}_{\tilde{v}} \longleftarrow V_v$$

are isomorphisms, where $v \in X_{log}$ and $\tilde{v} \in \tilde{X}^{log}$ are the vertices. The automorphism group of the covering $\tilde{X}^{log}_{P^*} \to X^{log}_{P^*}$ is $\pi_1(P)$, and it acts naturally on $\Gamma(\tilde{X}^{log}_{P^*}, \tilde{V}) \cong V_v$. Thus $\Gamma(\tilde{X}^{log}_{P^*}, \tilde{V})$ is an object of $\overline{L}^{\Lambda}_{coh}(P)$. This establishes the following equivalence, and the compatibilities which go along with it should be clear.

Proposition 3.2.5 *Let $X := \mathsf{A_P}$, where P is a toric monoid with rigid exponential data $\Lambda \subseteq \mathbf{C} \otimes P^{gp}$.*

1. *The functor*

$$V \mapsto \Gamma(\tilde{X}^{log}_{P^*}, \tilde{V}) \cong \tilde{V}_{\tilde{v}}$$

is an equivalence from the category $L^{\Lambda}_{coh}(\mathbf{C}^{log}_X)$ *to the category* $\overline{L}^{\Lambda}_{coh}(P)$. *A quasi-inverse is the functor taking an object* M *of* $\overline{L}^{\Lambda}_{coh}(P)$ *to the sheaf corresponding to the family* $\{M_F := \mathbf{C}[P/F] \otimes M\}$ *described above.*

2. *If* $(\phi, \psi)\colon (P, \Lambda_P) \to (Q, \Lambda_Q)$ *is a homomorphism of toric monoids and exponential data and* $Y := \mathsf{A_Q}$, *the diagram of functors*

$$\begin{array}{ccc} \overline{L}^{\Lambda}_{coh}(P) & \xrightarrow{\phi^*_\psi} & \overline{L}^{\Lambda}_{coh}(Q) \\ \downarrow & & \downarrow \\ L^{\Lambda}_{coh}(\mathbf{C}^{log}_X) & \xrightarrow{f^*_{log}} & L^{\Lambda}_{coh}(\mathbf{C}^{log}_Y) \end{array}$$

is 2*-commutative.*

3. *If* V *is an object of* $L^{\Lambda}_{coh}(\mathbf{C}^{log}_X)$ *and* F *is a face of* P, *then the cospecialization map* $cosp_{P^*,F}\colon \tilde{V}_{P^*} \to \tilde{V}_F$ *identifies* $\tilde{V}_F$ *with the tensor product* $\tilde{V}_P \otimes_{\mathbf{C}[\overline{P}]} \mathbf{C}[P/F]$.

□

3.3 The ring $\tilde{\mathcal{O}}^{log}_X$

To globalize the constructions of (1.4.8) and (2.1.2), we shall construct a sheaf of rings $\tilde{\mathcal{O}}^{log}_X$ on X_{log}, combining the constructions in [7] and [8]. Let us begin by reviewing the first of these.

If Y and X are topological spaces, let Y_X denote the sheaf which to every open set U of X assigns the set of continuous functions $U \to Y$. Recall from [7] that there is a commutative diagram with exact rows, in which the squares on the right are Cartesian:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{Z}(1) & \longrightarrow & \tau^{-1}(\mathcal{O}_X) & \xrightarrow{\exp} & \tau^{-1}\mathcal{O}^*_X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \lambda} & & \\ 0 & \longrightarrow & \mathbf{Z}(1) & \longrightarrow & \mathcal{L} & \xrightarrow{\pi} & \tau^{-1}M^{gp}_X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \tilde{h}} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & \mathbf{Z}(1) & \longrightarrow & \mathbf{R}(1)_{X_{log}} & \xrightarrow{\exp} & \mathbf{S}^1_{X_{log}} & \longrightarrow & 0 \end{array} \qquad (3.3.1)$$

To understand this diagram, recall that a point of X_{log} lying over a point x of X is a homomorphism of monoids $\sigma\colon M_{X,x} \to \mathbf{S}^1$ such that

$$\sigma(m)|\alpha_X(m)(x)| = \alpha_X(m)(x)$$

for every $m \in M_{X,x}$. If m is a local section of M_X, then the map sending such a σ to $\sigma(m_x)$ is a continuous function of σ, and so defines a section $h(m)$ of $\mathbf{S}^1_{X_{log}}$. This defines the homomorphism h in the diagram, and by definition, $\mathcal{L}$ is the fiber product of $\tau^{-1}M_X^{gp}$ and $\mathbf{R}(1)_{X_{log}}$ over $\mathbf{S}^1_{X_{log}}$. This defines the bottom two rows of the diagram, and the top row is just the pullback of the exponential exact sequence on X_{an}. The map $h \circ \lambda$ is arg, (*i.e.*, the map $u \mapsto |u|^{-1}u$). Let $Im\colon \tau^{-1}\mathcal{O}_X \to \mathbf{R}(1)_X$ be the map taking a function to its imaginary part. Then $\exp \circ Im = \arg \circ \exp$, and so there is a unique map ϵ with $\tilde{h}\epsilon = Im$ making the diagram commute. Since the big right rectangle is Cartesian, so is the upper square. Chasing the diagram shows that there is an exact sequence

$$0 \to \tau^{-1}\mathcal{O}_X \xrightarrow{\epsilon} \mathcal{L} \to \tau^{-1}\overline{M}_X^{gp} \longrightarrow 0. \qquad (3.3.2)$$

By definition, $\mathcal{O}_X^{log}$ is the universal $\tau^{-1}\mathcal{O}_X$-algebra equipped with a map $\mathcal{L} \to \mathcal{O}_X^{log}$ such that the diagram

$$\begin{array}{ccc} \tau^{-1}\mathcal{O}_X & \xrightarrow{\epsilon} & \mathcal{L} \\ & \searrow & \downarrow \\ & & \mathcal{O}_X^{log} \end{array}$$

commutes.

The ring $\mathcal{O}_X^{log}$ is adequate to deal with connections whose exponents vanish. In order to deal with the general case we adopt a construction of Lorenzon [8]. Recall that the exact sequence

$$0 \to \mathcal{O}_X^* \to M_X^{gp} \to \overline{M}_X^{gp} \to 0$$

defines a family of $\mathcal{O}_X^*$-torsors, hence invertible sheaves, indexed by $\overline{M}_X^{gp}$. If a and b are local sections of $\overline{M}_X^{gp}$, there is a map of the corresponding invertible sheaves $L_a \otimes L_b \to L_{a+b}$, and one obtains using these maps an $\overline{M}_X^{gp}$-indexed or graded $\mathcal{O}_X$-algebra $\mathcal{A}_X^{gp} := \oplus L_a$. The ring $\mathcal{O}_X^{log} \otimes \mathcal{A}_X^{gp}$ is sufficient to classify objects of $MIC^{\Lambda}_{coh}(X/\mathbf{C})$ when $\Lambda = \overline{M}_X^{gp}$, and the corresponding local systems have unipotent logarithmic monodromy. For the general case, we need to enlarge $\mathcal{A}_X^{gp}$ even more.

Consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{C} \otimes \tau^{-1}\mathcal{O}_X & \longrightarrow & \mathbf{C} \otimes \mathcal{L} & \longrightarrow & \mathbf{C} \otimes \tau^{-1}\overline{M}_X^{gp} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \tilde{\mu}} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & \mathcal{O}_X^* & \longrightarrow & \tilde{M}_X & \xrightarrow{\tilde{\pi}} & \mathbf{C} \otimes \tau^{-1}\overline{M}_X^{gp} & \longrightarrow & 0. \end{array} \qquad (3.3.3)$$

Here the top row is obtained by tensoring the sequence (3.3.2) with $\mathbf{C}$, and the bottom row is just the pushout by the map μ sending $a \otimes f$ to $\exp(af)$. Finally let

$$0 \longrightarrow \mathcal{O}_X^* \longrightarrow M_X^\Lambda \longrightarrow \Lambda \longrightarrow 0 \tag{3.3.4}$$

be the pullback of the bottom row of the above diagram by the map $\Lambda \subseteq \mathbf{C} \otimes \tau^{-1} M_X^{gp}$. If it seems to be unnecessary to specify the exponential data we write M_X^{log} instead of M_X^Λ. It follows from the exactness of the middle row of diagram (3.3.1) that there is an injection $\tau^{-1} M_X^{gp} \to M_X^\Lambda$ which agrees with $\tilde{\mu}$ when composed with π.

Let $\mathcal{A}_X^\Lambda$ (or $\mathcal{A}_X^{log}$) denote the Λ-graded $\mathcal{O}_X$-algebra corresponding to the exact sequence (3.3.4).

Proposition 3.3.1 *Let $d_\mathcal{L} \colon \mathcal{L} \to \tau^{-1}\Omega^1_{X/\mathbf{C}}$ denote the composition of the map $\pi \colon \mathcal{L} \to \tau^{-1} M_X^{gp}$ with $dlog \colon \tau^{-1} M_X^{gp} \to \tau^{-1}\Omega^1_{X/\mathbf{C}}$.*

1. *If f is a section of $\tau^{-1}(\mathcal{O}_X)$, then $d_\mathcal{L}\epsilon(f) = df$ in $\tau^{-1}(\Omega^1_{X/\mathbf{C}})$.*

2. *There is a unique homomorphism $\tilde{M}_X \xrightarrow{dlog} \tau^{-1}\Omega^1_{X/\mathbf{C}}$ such that*

$$dlog\tilde{\mu}(a \otimes \ell) = a d_\mathcal{L}(\ell)$$

for every section ℓ of $\mathcal{L}$ and every $a \in \mathbf{C}$.

3. *There is a unique additive and homogeneous homomorphism*

$$\nabla \colon \mathcal{A}_X^{log} \to \mathcal{A}_X^{log} \otimes \tau^{-1}\Omega^1_{X/\mathbf{C}},$$

satisfying the Leibniz rule with respect to $\tau^{-1}\mathcal{O}_X$ and such that if x_m is the section of $\mathcal{A}_X^{log}$ corresponding to a section m of M_X^{log},

$$\nabla x_m = x_m \otimes dlog(m).$$

4. *There is a natural map of Λ-graded rings $\iota \colon \mathbf{C}_X^{log} \to \mathcal{A}_X^{log}$, whose image is annihilated by d.*

Proof: By definition, $d_\mathcal{L}\epsilon(f) = dlog\pi\epsilon(f) = dlog\lambda exp(f) = df$, as asserted in (1). Let $\eta \colon \mathbf{C} \otimes \tau^{-1}\Omega^1_{X/\mathbf{C}} \to \tau^{-1}\Omega^1_{X/\mathbf{C}}$ be multiplication. If $a_i \in \mathbf{C}$ and $f_i \in \tau^{-1}(\mathcal{O}_X)$ for $i = 1 \ldots n$, it follows that

$$\eta \circ (\mathrm{id} \otimes d_\mathcal{L}) \circ (\mathrm{id} \otimes \epsilon)(\sum a_i \otimes f_i) = \sum a_i df_i = d \sum a_i f_i.$$

In particular, this is zero if $\sum a_i f_i$ is locally constant. The kernel of the map $\mu \colon \mathbf{C} \otimes \mathcal{O}_X \to \mathcal{O}_X^*$ is generated by the set of sums $\sum a_i \otimes f_i$ such that $\sum a_i f_i \in \mathbf{Z}(1)$, and in particular any such sum is killed by $\eta \circ (\mathrm{id} \otimes d_\mathcal{L}) \circ (\mathrm{id} \otimes \epsilon)$. Since $\tilde{M}_X$ is the quotient of $\mathbf{C} \otimes \mathcal{L}$ by the image of this kernel, there exists a unique $dlog$ as in (2).

To verify the existence of ∇, suppose that $\lambda \in \Lambda$, and let $\mathcal{A}^{log}_{X,\lambda}$ be the degree λ part of $\mathcal{A}^{log}_X$. This is an invertible $\tau^{-1}\mathcal{O}_X$-module, and if $m \in M^{log}_X$ maps to λ, there is a corresponding basis x_m of $\mathcal{A}^{log}_{X,\lambda}$. Then there is a unique

$$\nabla\colon \mathcal{A}^{log}_{X,\lambda} \to \mathcal{A}^{log}_{X,\lambda} \otimes \tau^{-1}\Omega^1_{X/\mathbf{C}}$$

satisfying the Leibniz rule such that $\nabla x_m = x_m \otimes dlog(m)$. We must verify that ∇ is independent of the choice of m. If m' is another section of M^{log}_X mapping to λ, then $m = um'$ for some $u \in \mathcal{O}^*_X$. Let ∇' be defined using m' in place of m. Then $x_m = ux_{m'}$ and $dlog(m) = dlog u + dlog m'$. Hence

$$\begin{aligned}
\nabla' x_m &= \nabla'(ux_{m'}) \\
&= du \otimes x_{m'} + u\nabla' x_{m'} \\
&= u^{-1}du \otimes x_m + udlog(m') \otimes x_{m'} \\
&= dlog u \otimes x_m + dlog(m') \otimes x_m \\
&= dlog m \otimes x_m \\
&= \nabla(x_m),
\end{aligned}$$

as required.

Let us continue to write the monoid law of M_X multiplicatively and that of $\overline{M}_X$ additively. A section m of M^{-1}_X defines an element m^{-1} of M_X; let $\iota(m) := \alpha(m^{-1})x_m \in \mathcal{A}^{log}_X$. If $u \in \mathcal{O}^*_X$ and $m' = um$, then

$$\alpha(m'^{-1})x_{m'} = u^{-1}\alpha(m^{-1})ux_m = \iota(m).$$

Thus $\iota(m)$ depends only on the image $\overline{m}$ of m in $\overline{M}^{-1}_X$, and we write $\iota(\overline{m})$ instead of $\iota(m)$. Then ι defines a homomorphism of graded rings $\mathbf{C}[-\overline{M}_X] \to \mathcal{A}^{log}_X$ sending $e_{\overline{m}}$ to $\iota(\overline{m})$. Since $\alpha(m^{-1})$ vanishes if and only if $m \in K_X$, ι factors through an injective homomorphism $\mathbf{C}^{log}_X := \mathbf{C}[-\overline{M}_X]/\mathbf{C}[-\overline{K}_X]$, which we also denote by ι. Furthermore,

$$\begin{aligned}
d\iota(m) &= \nabla(\alpha(m^{-1})x_m) \\
&= d\alpha(m^{-1})x_m + \alpha(m^{-1})\nabla x_m \\
&= \alpha(m^{-1})dlog(m^{-1})x_m + \alpha(m^{-1})dlog(m)x_m \\
&= -\alpha(m^{-1}dlog(m)x_m + \alpha(m^{-1})dlog(m)x_m \\
&= 0
\end{aligned}$$

□

Definition 3.3.2 *Let $X/\mathbf{C}$ be a fine saturated idealized log scheme with a sheaf of exponential data $\Lambda \subseteq \mathbf{C} \otimes \overline{M}^{gp}_X$. Then $\tilde{\mathcal{O}}^{log}_X$ is the Λ-graded $\tau^{-1}\mathcal{O}_X$-algebra $\mathcal{A}^{log}_X \otimes_{\tau^{-1}\mathcal{O}_X} \mathcal{O}^{log}_X$, and*

$$d\colon \tilde{\mathcal{O}}^{log}_X \to \tilde{\Omega}^{1,log}_{X/\mathbf{C}} := \tilde{\mathcal{O}}^{log}_X \otimes_{\tau^{-1}\mathcal{O}_X} \tau^{-1}\Omega^1_{X/\mathbf{C}}$$

is the map defined by the usual rule for the tensor product connection, using the connections defined above on $\mathcal{A}_X^{log}$ and $\mathcal{O}_X^{log}$.

Remark 3.3.3 Let $f\colon X \to Y$ be a morphism of fs idealized log schemes which is compatible, in the obvious sense, with sheaves of exponential data Λ_X and Λ_Y. Then there is a commutative diagram of ringed spaces

$$\begin{array}{ccc} X_{log} & \xrightarrow{f_{log}} & Y_{log} \\ \tau\downarrow & & \downarrow\tau \\ X & \xrightarrow{f} & Y \end{array}$$

which is Cartesian if f is strict [7]. It is straightforward to verify that f induces a map

$$\tau^{-1}\mathcal{O}_X \otimes_{(f\tau)^{-1}\mathcal{O}_Y} f_{log}^{-1}A \to A_X,$$

where A is $\mathcal{O}^{log}$, $\mathcal{A}^{log}$, or $\tilde{\mathcal{O}}^{log}$, compatible with the connections. If f is strict and the map $f^{-1}\Lambda_Y \to \Lambda_X$ is an isomorphism, then the above map is also an isomorphism.

We shall need an explicit description of the ring $\tilde{\mathcal{O}}_X^{log}$ when X is the log scheme associated to a toric monoid P. Let

$$\zeta\colon \tilde{X}^{log} := C(P) \times Y(P) \to X_{log}$$

be the universal covering constructed in (3.1.1). If $p \in P$ and $\tilde{x} = (\rho, y) \in \tilde{X}^{log}$, let $\hat{p}(\tilde{x}) := y(p) \in \mathbf{R}(1)$. Then $\hat{p}$ is a continuous function from $\tilde{X}^{log}$ to $\mathbf{R}(1)$, *i.e.*, a global section of $\mathbf{R}(1)_{\tilde{X}^{log}}$. The element p also defines a global section $\beta(p)$ of M_X, and in the diagram (3.3.1) pulled back to $\tilde{X}^{log}$, $h\beta(p) = \exp\hat{p}$. Thus $\tilde{\beta}(p) := (\hat{p}, \beta(p))$ is a global section of $\zeta^{-1}\mathcal{L}$, and $\tilde{\beta}$ is a map $P \to \zeta^{-1}\mathcal{L}$. We shall abuse notation and write $\mathcal{O}_{\tilde{X}}$ for the sheaf $\zeta^{-1}\tau^{-1}\mathcal{O}_{X_{an}}$.

Lemma 3.3.4 *Let $X := \mathsf{A}_\mathsf{P}$ and let $\zeta\colon \tilde{X}^{log} \to X_{log}$ be the universal covering.*

1. *The map $\tilde{\beta}$ described above fits into a cocartesian diagram:*

$$\begin{array}{ccc} \beta^{-1}(\mathcal{O}_X^*) & \xrightarrow{\tilde{\lambda}} & \mathcal{O}_{\tilde{X}} \\ \downarrow & & \downarrow\epsilon \\ P^{gp} & \xrightarrow{\tilde{\beta}} & \zeta^{-1}\mathcal{L}. \end{array}$$

2. *Let ρ be the action of $\pi_1(P) = Aut(\tilde{X}^{log}/X_{log})$ on $\mathcal{O}_{\tilde{X}}$ and $\zeta^{-1}\mathcal{L}$ by transport of structure. Then for each $p \in P^{gp}$, $\rho_\gamma(\tilde{\beta}(p)) = \tilde{\beta}(p) + \langle\gamma, p\rangle$. In particular, if z is a point in X_{log} then the action of $I_{\tau(x)}$ on $\mathcal{L}_z$ is given by*

$$\rho_\gamma(\ell) = \ell + \langle\gamma, \pi\ell\rangle$$

for any $\ell \in \mathcal{L}_z$ and $\gamma \in I_{\tau(x)}$.

3. *Let $\tilde{I}$ be the sheaf of ideals in the algebra $\mathcal{O}_{\tilde{X}} \otimes S^{\cdot}(P^{gp})$ generated by all elements of the form $\tilde{\lambda}(p) \otimes 1 - 1 \otimes p$ for p a local section of $\beta^{-1}\mathcal{O}_X^*$. Then the map $\tilde{\beta}$ induces an isomorphism of $\mathcal{O}_{\tilde{X}}$ -algebras*

$$(\mathcal{O}_{\tilde{X}} \otimes S^{\cdot}(P^{gp}))/\tilde{I} \to \zeta^{-1}\mathcal{O}_X^{log}.$$

Proof: In the diagram, $\beta^{-1}(\mathcal{O}_X^*)$ means the subsheaf of the constant sheaf P consisting of those elements of P which become units in M_X, open set by open set. Let p be a section of this sheaf on some open set $U \subseteq X_{an}$ and let $m := \beta(p) \in M_X^*(U)$ and $u := \alpha_X(m) \in \mathcal{O}_X^*(U)$. Then $\log|u| \in \mathbf{R}_X(U)$, $\hat{p} \in \mathbf{R}(1)(\tau\zeta)^{-1}(U)$, and $\tilde{\lambda}(p) := (\log|u|, \hat{p})$ is a section of $\zeta^{-1}(\mathcal{O}_X)$ such that $\exp\tilde{\lambda}(p) = u$. Then the diagram in (1) commutes. The fact that it is cocartesian follows from the exact sequence (3.3.2).

Recall that the action of $\pi_1(P)$ on $\tilde{X}^{log}$ is the action induced by translation and its inclusion as a subgroup. Thus if f is a function on $\tilde{X}^{log}$, $\rho_\gamma(f(\tilde{x})) = f(\tilde{x}) + f(\gamma\tilde{x})$ for each $\tilde{x} \in \tilde{X}^{log}$. Hence $\gamma^*(\hat{p}) = \hat{p} + \langle\gamma, p\rangle$ and $\gamma^*(\tilde{\beta}(p)) = \tilde{\beta}(p) + \langle\gamma, p\rangle$, and if $q \in \beta^{-1}(\mathcal{O}_X^*)$, $\rho_\gamma(\tilde{\lambda}(q)) = \tilde{\lambda}(q) + \langle\gamma, q\rangle$. This proves the formula for the action of ρ on $\zeta^{-1}\mathcal{L}$. Note that if $\gamma \in I_x$, then $\gamma^*(\tilde{\beta}(p)) - \tilde{\beta}(p)$ depends only on the image $\overline{\beta}(p)$ of p in $\overline{M}_X$. Let $\ell := \tilde{\beta}(p)$, and note that $\overline{\beta}(p) = \pi(\ell)$. This proves the formula for the action of I_x on $\mathcal{L}_z$, since the map $P \to \overline{M}_X^{gp} \cong \mathcal{L}_z/\mathcal{O}_{X,x}$ is surjective.

The map $\tilde{\beta}$ followed by the inclusion is a homomorphism $P^{gp} \to \zeta^{-1}\mathcal{O}_X^{log}$, and by the universal property of the symmetric algebra, this map extends uniquely to a homomorphism of algebras $\mathcal{O}_{\tilde{X}} \otimes S^{\cdot}(P^{gp}) \to \zeta^{-1}\mathcal{O}_X^{log}$. For any local section q of $\beta^{-1}\mathcal{O}_X^*$, the commutativity of the square in (1) and the triangle (3.3) imply that $1 \otimes p$ and $\tilde{\lambda}(p) \otimes 1$ have the same image in $\zeta^{-1}\mathcal{O}_X^{log}$, so that this homomorphism annihilates $\tilde{I}$. On the other hand, the map

$$\mathcal{O}_{\tilde{X}} \oplus P^{gp} \to (\mathcal{O}_{\tilde{X}} \otimes S^{\cdot}(P^{gp}))/\tilde{I}$$

sending (f, p) to $f \otimes 1 + 1 \otimes p$ factors through $\zeta^{-1}\mathcal{L}$, since the square in (1) is cocartesian and since for any $q \in \beta^{-1}\mathcal{O}_X^*$ the elements $1 \otimes [q]$ and $\tilde{\lambda}(q) \otimes 1$ have the same image in $(\mathcal{O}_{\tilde{X}} \otimes S^{\cdot}(P^{gp}))/\tilde{I}$. By the universal property of $\mathcal{O}_X^{log}$, these maps extend uniquely to a map $\zeta^{-1}\mathcal{O}_X^{log} \to (\mathcal{O}_{\tilde{X}} \otimes S^{\cdot}(P^{gp}))/\tilde{I}$, which is the inverse to the map in (3). ☐

Proposition 3.3.5 *Let P be a toric monoid with exponential data $\Lambda \subseteq \mathbf{C} \otimes P^{gp}$ and a proper ideal $K \subseteq P$, and let $X := \mathsf{A}_{\mathsf{P},\mathsf{K}}$.*

1. *Then there are natural maps, compatible with the connections and gradings, and actions of $\pi_1(P)$:*

$$\begin{array}{rcl} \Gamma(\tilde{X}^{log}, \mathcal{O}_{\tilde{X}}) \oplus P^{gp} & \to & \Gamma(\tilde{X}^{log}, \zeta^{-1}\mathcal{L}) \\ \Gamma(\tilde{X}^{log}, \mathcal{O}^*_{\tilde{X}}) \oplus \Lambda & \to & \Gamma(\tilde{X}^{log}, \zeta^{-1}\tilde{M}^{\Lambda}_X) \\ \Gamma(\tilde{X}^{log}, \mathcal{O}_{\tilde{X}}) \otimes \mathbf{Z}[\Lambda] & \to & \Gamma(\tilde{X}^{log}, \zeta^{-1}\mathcal{A}^{log}_X) \\ \Gamma(\tilde{X}^{log}, \mathcal{O}_{\tilde{X}}) \otimes \Gamma^{\cdot}[P^{gp}] & \to & \Gamma(\tilde{X}^{log}, \zeta^{-1}\mathcal{O}^{log}_X) \\ \Gamma(\tilde{X}^{log}, \mathcal{O}_{\tilde{X}}) \otimes_{\mathbf{C}[P]} J(P, \Lambda) & \to & \Gamma(\tilde{X}^{log}, \zeta^{-1}\tilde{\mathcal{O}}^{log}_X). \end{array}$$

2. *Suppose that P is sharp and let z (resp. v) be the vertex of X_{log} (resp. X). Then these maps induce isomorphisms on stalks:*

$$\begin{array}{rcl} \mathcal{O}_{X,v} \oplus P^{gp} & \to & \mathcal{L}_z \\ \mathcal{O}^*_{X,v} \oplus \Lambda & \to & \tilde{M}^{\Lambda}_{X,z} \\ \mathcal{O}_{X,v} \otimes \mathbf{Z}[\Lambda] & \to & \mathcal{A}^{log}_{X,z} \\ \mathcal{O}_{X,v} \otimes \Gamma^{\cdot}[P^{gp}] & \to & \mathcal{O}^{log}_{X,z} \\ \mathcal{O}_{X,v} \otimes_{\mathbf{C}[P]} J(P, \Lambda) & \to & \tilde{\mathcal{O}}^{log}_{X,z} \end{array}$$

These isomorphisms are compatible with the connections, gradings, and actions of $I_v = \pi_1(P)$.

Proof: We have already constructed the first of the maps in statement (1), and the construction of the remaining maps is then straightforward. Let γ be an element of I_x, let p be an element of P^{gp} and let a be an element of $\mathbf{C}$. Then $\tilde{\mu}(a \otimes \tilde{\beta}(p))$ is a global section of $\zeta^{-1}\tilde{M}_X$, and

$$\begin{array}{rcl} \rho_\gamma \tilde{\mu}(a \otimes \tilde{\beta}(p)) & = & \tilde{\mu}(a \otimes \rho_\gamma \tilde{\beta}(p)) \\ & = & \tilde{\mu}(a \otimes \tilde{\beta}(p) + a \otimes \langle \gamma, p \rangle) \\ & = & \tilde{\mu}(a \otimes \tilde{\beta}(p)) \mu(a \otimes \langle \gamma, p \rangle) \\ & = & \tilde{\mu}(a \otimes \tilde{\beta}(p)) \exp(a\langle \gamma, p \rangle) \\ & = & \tilde{\mu}(a \otimes \tilde{\beta}(p)) \exp\langle \gamma, a \otimes p \rangle \end{array}$$

It follows that if λ is any element of Λ and $\tilde{m}$ is its image in $\Gamma(\tilde{X}^{log}, \zeta^{-1}\tilde{M}_X)$, then

$$\rho_\gamma(\tilde{m}) = \tilde{m} \exp\langle \gamma, \lambda \rangle$$

This shows that the second arrow is compatible with the actions of $\pi_1(P)$. Let $x_{\tilde{m}}$ be the basis element of $\mathcal{A}_{X,\lambda}$ corresponding to $\tilde{m}$ and let $u := \exp\langle \gamma, \lambda \rangle$. Then

$$\rho_\gamma(x_{\tilde{m}}) = x_{\rho_\gamma(\tilde{m})} = x_{u\tilde{m}} = u x_{\tilde{m}}.$$

This shows that the third arrow is compatible with the actions of $\pi_1(P)$. Furthermore, from the definition in (3.3.2), $\nabla x_{\tilde m} = x_{\tilde m} \otimes dlog \tilde m$, so it is also compatible with the connections. The sheaf $\mathcal{O}_X^{log}$ is generated over $\mathcal{O}_X$ by $\mathcal{L}$, on which we have already calculated the action of I_x, and it follows that its action on all of $\mathcal{O}_X^{log}$ is as described. The same argument works with the connections. The statement for $\tilde{\mathcal{O}}_X^{log}$ follows, as does part (2) of the proposition. □

3.4 Logarithmic Riemann-Hilbert

We can at last give the precise statement of the logarithmic Riemann-Hilbert correspondence.

Definition 3.4.1 *Let $X/\mathbf{C}$ be an fs smooth idealized log scheme and let $\Lambda \subseteq \mathbf{C} \otimes \overline{M}_X^{gp}$ be a set of exponential data for X. Let $MIC_{coh}^{\Lambda}(X_{an}/\mathbf{C})$ be the category of coherent sheaves of $\mathcal{O}_X$-modules on X_{an} equipped with an integrable logarithmic connection all of whose exponents lie in Λ.*

1. *If (E, ∇) is an object of $MIC_{coh}^{\Lambda}(X_{an})$, let*

$$\tilde E := \tilde{\mathcal{O}}_X^{log} \otimes_{\tau^{-1}\mathcal{O}_X} \tau^{-1} E,$$

   with the induced connection $\tilde\nabla \colon \tilde E \to \tilde E \otimes \tilde\Omega^{1,log}_{X/\mathbf{C}}$, and let $\mathcal{V}(E,\nabla)$ be the sheaf of Λ-graded $\mathbf{C}_X^{log}$-modules $\tilde E^{\tilde\nabla}$.

2. *If V is an object of $L_{coh}^{\Lambda}(\mathbf{C}_X^{log})$, let $\tilde V := \tilde{\mathcal{O}}_X^{log} \otimes_{\mathbf{C}_X^{log}} V$, endowed with the connection $\tilde\nabla := d \otimes id$ and the tensor product Λ-grading, and let $(\mathcal{E}(V), \nabla) := \tau_*^{\Lambda}(\tilde V, \nabla)$, where the superscript Λ means the degree zero part with respect to the Λ-grading.*

Since the connection on $\tilde E$ is $\mathbf{C}_X^{log}$-linear and homogeneous, $\mathcal{V}(E,\nabla)$ is a sheaf of Λ-graded $\mathbf{C}_X^{log}$-modules. Thus the definition (1) above makes sense.

Theorem 3.4.2 *Let the notation be as in (3.4.1).*

1. *The functor $\mathcal{V}$ above is an equivalence of tensor categories*

$$MIC_{coh}^{\Lambda}(X_{an}) \to L_{coh}^{\Lambda}(\mathbf{C}_X^{log}),$$

   with quasi-inverse $\mathcal{E}$.

2. *If $f \colon X \to Y$ is a morphism of smooth idealized fs log schemes and (E,∇) is an object of $MIC_{coh}^{\Lambda}(Y)$, then there is a natural isomorphism in $L_{coh}^{\Lambda}(\mathbf{C}_X^{log})$:*

$$f_{log}^* \mathcal{V}(E,\nabla) \cong \mathcal{V}(f^*E, \nabla).$$

3. *Let (E,∇) be an object of $MIC_{coh}^{\Lambda}(X_{an})$ and let $V := \mathcal{V}(E,\nabla)$.*

(a) The natural map

$$\tilde{\mathcal{O}}_X^{log} \otimes_{\mathbf{C}_X^{log}} V \to \tilde{E} := \tilde{\mathcal{O}}_X^{log} \otimes_{\tau^{-1}\mathcal{O}_X} \tau^{-1}E$$

is an isomorphism, compatible with the Λ-gradings and connections.

(b) The natural map

$$\mathcal{V}(E,\nabla) \to \tilde{E} \otimes_{\tilde{\mathcal{O}}_X^{log}} \tilde{\Omega}^{\cdot,log}_{X/\mathbf{C}}$$

of complexes of abelian sheaves on X_{log} is a quasi-isomorphism.

(c) The natural map

$$E \otimes \Omega^{\cdot}_{X/\mathbf{C}} \to R\tau_*^{\Lambda}(\tilde{E} \otimes \tilde{\Omega}^{\cdot,log}_{X/\mathbf{C}})$$

is a quasi-isomorphism, where the superscript Λ means the degree zero part.

Proof: We will reduce the proof of the above global theorem to the local versions proved in the previous sections. Suppose that K is a proper ideal in a toric monoid P and let $X := \mathsf{A}_{\mathsf{P},\mathsf{K}}$. Let E be an object of $MIC^{\Lambda}_{coh}(P, K/\mathbf{C})$, and let E_{an} be the corresponding object of $MIC(X_{an}/\mathbf{C})$. Then $\mathcal{V}(E_{an})$ is a sheaf of graded $\mathbf{C}_X^{log}$-modules on X_{log}. Its stalk at the vertex z is a Λ-graded $\mathbf{C}[-P]$-module. Since all the sheaves involved in the construction of $\mathcal{V}(E_{an})$ are locally constant on the fibers of τ, it also is locally constant on the fibers. On the other hand, the equivariant Riemann-Hilbert transform V of E is an object of $L^{\Lambda}(-P/\mathbf{C})$. Thus it is a Λ-graded $\mathbf{C}[-P]$-module, endowed with an action of $\pi_1(P)$. Recall that in (3.3.5) we constructed a map $J(P,\Lambda) \to \Gamma(\tilde{X}^{log}, \zeta^{-1}\tilde{\mathcal{O}}_X^{log})$. Tensoring with E, and observing that the resulting map is compatible with connections, we find a commutative diagram

$$\begin{array}{ccc} V & \longrightarrow & \Gamma(\tilde{X}^{log}, \zeta^{-1}\mathcal{V}(E_{an})) \\ \downarrow & & \downarrow \\ E \otimes J(P,\Lambda) & \longrightarrow & \Gamma(\tilde{X}^{log}, \zeta^{-1}(E_{an} \otimes \tilde{\mathcal{O}}_X^{log})). \end{array}$$

Lemma 3.4.3 *Let $X := \mathsf{A}_{\mathsf{P},\mathsf{K}}$, let E be an object of $MIC^{\Lambda}_{coh}(P,K)$ and let $V \in L^{\Lambda}(-P,-K)$ be its equivariant Riemann-Hilbert transform (1.4.8). Let z be the vertex of X_{log}.*

1. *The map $V \to \Gamma(\tilde{X}^{log}, \zeta^{-1}\mathcal{V}(E_{an}))$ constructed above induces an isomorphism*

$$\overline{V} := V \otimes_{\mathbf{C}[-P]} \mathbf{C}[-\overline{P}] \cong \mathcal{V}(E_{an},\nabla)_z$$

in the category of $\overline{\Lambda}$-graded $\mathbf{C}[-\overline{P}]$-modules, compatible with the actions of $I_v \subseteq \pi_1(P)$.

2. The natural map $\mathcal{V}(E_{an},\nabla)_z \otimes_{\mathbf{C}_X^{log}} \tilde{\mathcal{O}}^{log}_{X,z} \to \tilde{E}_{an,z}$ is an isomorphism.

3. The natural map $\mathcal{V}(E_{an},\nabla)_z \to \tilde{E}_{an,z} \otimes \tilde{\Omega}^{\cdot}_{X/\mathbf{C}}$ is a quasi-isomorphism.

Proof: Let $\overline{X} := \mathsf{A}_{\overline{\mathsf{P}},\mathsf{K}}$ and let $(\overline{E},\nabla)$ be the image of (E,∇) in $MIC^{\overline{\Lambda}}_{coh}(\overline{P}/\mathbf{C})$. Because the functor B_{an} of (2.1.2) is fully faithful, the map $E^{\nabla}_{an,v} \to \overline{E}^{\nabla}_{an,v}$ is an isomorphism. Now $\tilde{E}_z := E \otimes \tilde{\mathcal{O}}^{log}_{X,z}$ is a direct limit of finitely generated modules with integrable connection. Applying the same remark to each of these and passing to the limit, we see that the map $\mathcal{V}(E_{an},\nabla)_z \to \mathcal{V}(\overline{E}_{an},\nabla)_z$ is also an isomorphism. This reduces the proof of the first and third statements to the case in which P is sharp. The second statement will also follow from the sharp case. Indeed, a section of $P \to \overline{P}$ induces a strict morphism $f\colon X \to \overline{X}$, so $f^*\tilde{\mathcal{O}}^{log}_{\overline{X}} \cong \tilde{\mathcal{O}}^{log}_X$. Thus for the remainder of the proof we may and shall assume that P is sharp.

Because the functor $\overline{A}$ of (2.1.2) is an equivalence, the map

$$E^{\nabla} \cong E^{\nabla}_{an,v} \tag{3.4.1}$$

is an isomorphism. Now $J(P,\Lambda)$ is a direct limit of objects J_a of $MIC^{\Lambda}_{coh}(P)$. For each a, $E \otimes J_a$ is an object of $MIC^{\Lambda}_{coh}(P)$, and so applying (3.4.1) to each of these produces an isomorphism

$$(E \otimes J_a)^{\nabla} \cong (E_{an,v} \otimes J_a)^{\nabla} \cong (E \otimes \mathcal{O}_{X_{an,v}} \otimes J_a)^{\nabla}.$$

Passing to the limit, we see by the last statement of (3.3.5) that the map

$$\big(E \otimes_{\mathbf{C}[P]} J(P,\Lambda)\big)^{\nabla} \to \big(E \otimes_{\mathbf{C}[P]} \mathcal{O}_{X_{an,v}} \otimes J(P,\Lambda)\big)^{\nabla} \to (E \otimes_{\mathbf{C}[P]} \tilde{\mathcal{O}}_{X^{log},v})^{\nabla}$$

is an isomorphism. By (1.4.8), the left hand side of this equation is the equivariant Riemann-Hilbert transform of E, which is in fact V, and the right side is by definition the stalk of $\mathcal{V}(E)$ at z. This proves (1). Recall from (1.4.8.2) that the natural map $V \otimes_{\mathbf{C}[-P]} J(P,\Lambda) \to E \otimes_{\mathbf{C}[P]} J(P,\Lambda)$ is an isomorphism. Statement (2) follows from this, after tensoring with $\mathcal{O}_{X_{an}}$. To prove (3), it will now suffice to show that $H^i(\tilde{E} \otimes \tilde{\Omega}^{\cdot}_X) = 0$ if $i > 0$. The same direct limit argument and Theorem (2.1.2) reduce this to the analogous computation in the category $MIC^{\Lambda}_{coh}(P/\mathbf{C})$, where it is a consequence of (1.4.8.2). □

We can now prove (2) of the theorem. Let (E,∇) be an object of $MIC^{\Lambda}_{coh}(Y_{an})$, and let $\tau^*_Y E := \tau^{-1}E \otimes \tilde{\mathcal{O}}^{log}_Y$, with a similar notation for X. As we have seen in (3.3.5), there is a natural map $f^{-1}_{log}\tilde{\mathcal{O}}^{log}_Y \to \tilde{\mathcal{O}}^{log}_X$, compatible with the exterior derivative and hence a natural and horizontal isomorphism

$$f^*_{log}\tau^*_Y E := f^{-1}_{log}\tau^*_Y E \otimes \tilde{\mathcal{O}}^{log}_X \cong \tau^*_X f^* E \tag{3.4.2}$$

Thus there is a natural map

$$\mathcal{V}(E) := (\tau^*_Y E)^{\nabla} \to f_{log*}(\tau^*_X f^* E)^{\nabla} = f_{log*}\mathcal{V}(f^*E).$$

By adjunction, we get a map

$$f^*_{log}\mathcal{V}(E) := f^{-1}_{log}\mathcal{V}(E) \otimes_{f^{-1}\mathbf{C}^{log}_Y} \mathbf{C}^{log}_X \to \mathcal{V}(f^*E),$$

which we are claiming is an isomorphism of $\mathbf{C}^{log}_X$-modules. It is clear from the local description (3.3.5) of $\tilde{\mathcal{O}}^{log}_X$ that it is faithfully flat over $\mathbf{C}^{log}_X$, so it suffices to prove that the map is an isomorphism after tensoring with $\tilde{\mathcal{O}}^{log}_X$. There is a commutative diagram

$$\begin{array}{ccc} f^*_{log}\mathcal{V}(E) \otimes_{\mathbf{C}^{log}_X} \tilde{\mathcal{O}}^{log}_X & \longrightarrow & \mathcal{V}(f^*E) \otimes_{\mathbf{C}^{log}_X} \tilde{\mathcal{O}}^{log}_X \\ \downarrow & & \downarrow \\ f^*_{log}\tau^*_Y E & \longrightarrow & \tau^*_X f^* E \end{array}$$

The lower horizontal map is the isomorphism (3.4.2) we started with, and we have already seen in (3.4.3) that the vertical arrows are isomorphisms. This implies that the arrow in (2) of the theorem is an isomorphism.

Lemma 3.4.4 *For any X as in the theorem and any $E \in MIC^{\Lambda}_{coh}(X_{an}/\mathbf{C})$, $\mathcal{V}(E)$ is log constructible (3.2.3).*

Proof: This can be verified locally in an analytic neighborhood of an arbitrary point x of X. Since $X/\mathbf{C}$ is fs and ideally log smooth, there exist a toric monoid P, an ideal K of P, and a strict étale map $X \to \mathsf{A}_{\mathsf{P},\mathsf{K}}$ sending x to the vertex. In the analytic topology, this map is locally an isomorphism, so we may and shall assume that $X = \mathsf{A}_{\mathsf{P},\mathsf{K}}$. By (2.1.2), there is a neighborhood of the vertex on which (E, ∇) is isomorphic to the analytification of an object (M, ∇) of $MIC^{\Lambda}_{coh}(P, K)$, so we may as well assume that (E, ∇) *is* this analytification. We may also assume that $\Lambda = \mathbf{C} \otimes P^{gp}$. A splitting of $P \to \overline{P}$ induces a map $X \to \overline{X}$, and as we observed in (2.2.2), (E, ∇) is isomorphic to the pullback of some $(\overline{E}, \nabla)$ on $\overline{X}$, in some neighborhood U of v. By part (2) of the theorem, formation of $\mathcal{V}$ is compatible with pullback, and it follows that $\mathcal{V}(E, \nabla)$ is also pulled back from $\overline{X}$. Hence it is constant on $U \cap \mathsf{A}_{\mathsf{P}^*}$. But $\mathsf{A}_{\mathsf{P}^*}$ is the log stratum containing v. Since the same argument works in a neighborhood of every point, $\mathcal{V}(E, \nabla)$ is locally constant on the canonical stratification of X_{log}. □

Lemma 3.4.5 *The functor $\mathcal{V}$ of Theorem (3.4.2) maps $MIC^{\Lambda}_{coh}(X_{an})$ into $L^{\Lambda}_{coh}(\mathbf{C}^{log}_X)$. In fact, suppose $X = \mathsf{A}_{\mathsf{P},\mathsf{K}}$, with P sharp, E is an object of $MIC^{\Lambda}(P, K)$ and V is its equivariant Riemann-Hilbert transform (1.4.8). Then the sheaf $\mathcal{V}(E_{an})$ is isomorphic to the object of $L^{\Lambda}(\mathbf{C}^{log}_X)$ corresponding to V via the equivalence in (3.2.5).*

Proof: Let be E an object of $MIC^{\Lambda}_{coh}(P, K)$. We have seen in (3.4.4) that $\mathcal{V}(E_{an})$ is log constructible. To prove that it lies in $L(\mathbf{C}^{log}_X)$ is a local question on X_{an}, so we may assume that $X = \mathsf{A}_{\mathsf{P},\mathsf{K}}$ and work in a neighborhood of the vertex. By (2.1.2), we may also assume that P is sharp. We claim that if $\tilde{x}$ is the vertex of $\tilde{X}^{log}$ and $\tilde{y}$ is any point of $\tilde{X}^{log}$, then

$$cosp^*_{\tilde{x},\tilde{y}} \colon \mathbf{C}^{log}_{X,y} \otimes \mathcal{V}(E_{an})_x \to \mathcal{V}(E_{an})_y$$

is an isomorphism. As we observed above, if this is true for $\tilde{y}$, it is also true for every other $\tilde{y}'$ in the same log stratum. Thus we may assume that $\tilde{y}$ is the vertex of $\tilde{Y}^{log}$, where $Y := \mathsf{A}_{\mathsf{P/F},\mathsf{K/F}}$ for some face F of P. The map $P \to P/F$ induces a map $i \colon Y \to X$; note that i does not map y to x. The equivariant Riemann-Hilbert transform W of $E \otimes \mathbf{C}[P/F]$ can be identified with $V \otimes \mathbf{C}[-P/F]$, and i^*E_{an} is the analytic sheaf with connection corresponding to $E \otimes \mathbf{C}[P/F]$. Thus it follows from (2) of theorem (3.4.2) and the functoriality of the constructions of (3.3.5) that there is a commutative diagram

$$\begin{array}{ccccc}
V & \longrightarrow & \Gamma(\tilde{X}^{log}, \zeta^{-1}\mathcal{V}(E_{an})) & \longrightarrow & V_x \\
\downarrow & & \downarrow & & \\
W & \longrightarrow & \Gamma(\tilde{Y}^{log}, \zeta^{-1} i^* \mathcal{V}(E_{an})) & \longrightarrow & W_y
\end{array}$$

Lemma (3.4.3) tells us that the composed horizontal maps are isomorphisms, and it follows from the definitions that the resulting map $V_x \to V_y$ is the cospecialization map $cosp^*_{\tilde{x},\tilde{y}}$. In particular, condition (3) of the definition (3.2.4) is satisfied. Since these maps are automatically compatible with the operations of $\pi_1(P)$, the lemma is proved. □

Lemma 3.4.6 *Let E be a coherent sheaf on X_{an} and let*

$$\tilde{E} := \tau^{-1}E \otimes_{\tau^{-1}\mathcal{O}_X} \tilde{\mathcal{O}}^{log}_X.$$

Then the natural map $E \to R\tau^{\Lambda}_* \tilde{E}$ is a quasi-isomorphism.

Proof: It suffices to prove that the stalk of this natural map at every point x of X is an isomorphism. Since τ is a proper morphism of paracompact Hausdorff spaces, the natural map

$$(R\tau^{\Lambda}_* \tilde{E})_x \to R\Gamma^{\Lambda}(\tau^{-1}(x), i^{-1}\tilde{E})$$

is an isomorphism, where $i \colon \tau^{-1}(x) \to X_{log}$ is the inclusion. Recall that the superscript Λ means taking the degree zero part in the grading, which commutes with cohomology. The degree zero part of $\tilde{E}$ is just $\tau^{-1}E \otimes \mathcal{O}^{log}_X$. Furthermore, the fiber $\tau^{-1}(x)$ is a torus and $\tilde{E}$ is locally constant on the fiber, so the sheaf cohomology is the same as group cohomology, computed with respect to the action of the fundamental group I_x on any stalk. Thus it suffices to show that $H^q(I_x, E_x \otimes \mathcal{O}^{log}_{X,z}) = 0$ if $q > 0$ and is E_x if $q = 0$. We may assume that

$X = \mathsf{A_P}$, with P a sharp toric monoid, so that $\mathcal{O}^{log}_{X,z} \cong \mathcal{O}_{X,z} \otimes_{\mathbf{C}} \Gamma^{\cdot}(\Omega)$. The action of I_x is unipotent, and its logarithm is a nilpotent T-Higgs field. Thus by (1.4.4) the group cohomology can be identified with the Higgs cohomology. But this Higgs complex is just E_x tensored with the Higgs complex of $\Gamma^{\cdot}(\Omega)$, which is a resolution of $\mathbf{C}$. □

To prove that $\mathcal{V}$ is fully faithful, let E_i be objects of $MIC^{\Lambda}_{coh}(X_{an})$ and let $V_i := \mathcal{V}(E_i)$, for $i = 1, 2$. Since $\mathcal{V}$ is functorial, it induces a map of sheaves of $\mathbf{C}$-vector spaces $\mathcal{H}om(E_1, E_2) \to \tau_*\mathcal{H}om(V_1, V_2)$. It suffices to prove that this map of sheaves is an isomorphism, and to do this it suffices to check that its stalk at each point x is so. Then we may assume that $X = \mathsf{A_{P,K}}$, where P is a sharp toric monoid and that x is the vertex, and that each E_i comes from an object of $MIC^{\Lambda_x}_{coh}(P, K)$. Then the V_i can be identified with the corresponding equivariant Riemann-Hilbert transforms, and τ_* with the invariants under the log inertia group $\pi_1(P)$. Then the result follows from the full faithfulness of the equivariant Riemann-Hilbert transform.

To prove that $\mathcal{V}$ is essentially surjective, let V be an object of $L^{\Lambda}_{coh}(\mathbf{C}^{log}_X)$ and let x be a point of X. Then by (2.1.2) and (1.4.8), there exists an analytic neighborhood U of x and an object (E, ∇) of $MIC^{\Lambda}_{coh}(U)$ such that $\mathcal{V}(E, \nabla)_x \cong V_x$. Then in fact $\mathcal{V}(E, \nabla) \cong V$ on some possibly smaller neighborhood of x. We can glue these objects of MIC^{Λ}_{coh} using the gluing data coming from V and the full faithfulness of $\mathcal{V}$.

This completes the proof of (1) and (2) of the theorem. Parts (a) and (b) of (3) follow immediately from (3.4.3). For part (c), note that for each q, the natural map

$$E \otimes \Omega^q_{X/\mathbf{C}} \to R\tau^{\Lambda}_* \tilde{E} \otimes \tilde{\Omega}^{q,log}_{X/\mathbf{C}}$$

is a quasi-isomorphism, by (3.4.6), and hence the map in (c) is also a quasi-isomorphism. □

Associated to the Λ-grading of the category $L^{\Lambda}_{coh}(X)$ is a Λ-filtration, where Λ is regarded as a sheaf of partially ordered sets, with the partial ordering induced by the action of $-\overline{M}_X$. This filtration carries over to the equivalent category $MIC^{\Lambda}_{coh}(X)$. Matthew Emerton has pointed out that this gives a log construction of the "Kashiwara-Malgrange V-filtration."

Corollary 3.4.7 *Any object (E, ∇) of $MIC^{\Lambda}_{coh}(X_{an})$ admits a unique and functorial decreasing filtration indexed by the sheaf of partially ordered set Λ, such that $\mathcal{V}(F^{\lambda}E) = \oplus_{\lambda' \geq \lambda} \mathcal{V}_{\lambda'}(E, \nabla)$. If (E', ∇) is a subobject of (E, ∇), then $E'_x \subseteq F^{\lambda}E_x$ if and only if all the exponents of E' at x are greater than or equal to λ in the partial ordering on Λ induced by the action of $-\overline{M}_{X,x}$.*

Remark 3.4.8 It is easy to see, for example from the compatibility of the local and global Riemann-Hilbert correspondence, that if (E, ∇) is an object of $MIC_{coh}(X_{an})$, then E is locally free (resp. torsion free, resp. reflexive) over $\mathcal{O}_X$ if and only if $\mathcal{V}(E, \nabla)$ is locally free (resp. ...) over $\mathbf{C}^{log}_X$.

As an illustration of the content of the main theorem (3.4.2), let us show how it easily implies a logarithmic version of Deligne's comparison theorem [3, II, 3.13]

Theorem 3.4.9 *Let $X/\mathbf{C}$ be an fs smooth log scheme (with no idealized structure) and let (E,∇) be a torsion-free object of $MIC^{\Lambda}_{coh}(X_{an})$. At each point of x, let $S_x \subseteq \Lambda_x \subseteq \mathbf{C} \otimes \overline{M}^{gp}_{X,x}$ be the set of exponents of (E,∇) at x. Suppose that for each such x, $S_x \cap \overline{M}^{gp}_{X,x} \subseteq \overline{M}_{X,x}$. Then the natural map*

$$H^*_{DR}(X_{an},E) \to H^*_{DR}(X^*_{an},E)$$

is an isomorphism.

Proof: Let $E^* := j^*E$, let $V := \mathcal{V}(E,\nabla)$, and let $V^* := \mathcal{V}(E^*,\nabla)$, which we can regard as a locally constant sheaf of $\mathbf{C}$-vector spaces on X^*. Theorem (3.4.2) provides a commutative diagram in the derived category:

$$\begin{array}{ccc} E \otimes \Omega^{\cdot}_{X/\mathbf{C}} & \longrightarrow & Rj_*(E^* \otimes \Omega^{\cdot}_{X^*/\mathbf{C}}) \\ \downarrow & & \downarrow \\ R\tau^{\Lambda}_* V & \longrightarrow & Rj_*V^*, \end{array}$$

in which the vertical arrows are quasi-isomorphisms. Thus it suffices to show that the bottom horizontal arrow is a quasi-isomorphism.

By (3.1.2), $V' := j_{log*}V^*$ is a local system of $\mathbf{C}$-vector spaces on X_{log}, and $j_{log*}V^* \cong Rj_{log*}V^*$. Thus it suffices to show that the natural map

$$R\tau^{\Lambda}_* V \to R\tau_* V'$$

is a quasi-isomorphism. This is a local question, and so we can restrict our attention to a neighborhood of a point x of X. If $\tilde{x} \in \tau^{-1}(x)$, then $V_{\tilde{x}}$ and $V'_{\tilde{x}}$ are equipped with actions of I_x, and we have to prove that the maps

$$H^i(I_x, V_{0,\tilde{x}}) \to H^i(I_x, V'_{\tilde{x}}) \tag{3.4.3}$$

are isomorphisms. Here V is a Λ_x-graded $\mathbf{C}[-\overline{M}_{X,x}]$-module, and the 0 means the degree zero part.

It follows from the coherence of V that $V'_{\tilde{x}}$ can be identified with the tensor product of $V_{\tilde{x}}$ over the map $\mathbf{C}[-\overline{M}_{X,x}] \to \mathbf{C}$ sending $\overline{M}_{X,x}$ to 1. It follows from (2.1.3) and the hypothesis on the exponents that the set of degrees of a set of generators for V intersected with $\overline{M}^{gp}_{X,x}$ is contained in $\overline{M}_{X,x}$. Hence Corollary (1.4.6) implies that (3.4.3) is an isomorphism.

□

THEOREM 3.4.10 *Let $X/\mathbf{C}$ be a smooth fs log scheme (with no idealized structure) and let (E,∇) be an object of $MIC^{\Lambda}_{coh}(X_{an}/\mathbf{C})$. Let $j\colon X^* \to X$ be the inclusion of the maximal open set where the log structure is trivial. Then the natural map*

$$j_*j^*_mE\otimes\Omega^{\cdot}_{X/\mathbf{C}} \to j_*j^*E\otimes\Omega^{\cdot}_{X/\mathbf{C}}$$

is a quasi-isomorphism, where $j_*j^*_m$ means the sheaf of sections of E with meromorphic poles along $X \setminus X^*$.

Proof: Fix a point x in X. It suffice to prove the theorem in a neighborhood of x. Thus we may assume that $X = \mathsf{A}_\mathsf{P}$ for some toric monoid P. Let m be the sum of a minimal set of generators for P and let J be the ideal of P generated by m. The support of the corresponding closed subscheme of X is exactly the set where the log structure is nontrivial. The ideal I of $\mathcal{O}_X$ generated by $\beta(m)$ is an invertible sheaf of ideals, and its inverse defines an effective divisor D whose support is $X \setminus X^*$. Thus for any E, $j_*j^*_mE = \varinjlim E(nD)$. Since I comes from a sheaf of ideals in the monoid, it is stable under the connection d on $\mathcal{O}_X$. In particular, $\alpha(m)$ generates I and $d\alpha(m) = \alpha(m)dlogm \in I\otimes\Omega^1_{X/\mathbf{C}}$. By definition (2.1.1), $-\overline{m}_x$ is the unique exponent of this connection at x. Then the dual $\mathcal{O}_X(D)$ has a connection also, and its unique exponent is $\overline{m}_x$. If s is any element of $\overline{M}^{gp}_X$, $s + n\overline{m}_x \in \overline{M}_X$ for n sufficiently large. It follows that, locally on X, there exists an n such that $E(nD)$ satisfies the hypothesis of (3.4.9) for n sufficiently large. By the previous result, the map

$$E(nD)\otimes\Omega^{\cdot}_{X/\mathbf{C}} \to j_*j^*E\otimes\Omega^{\cdot}_{X/\mathbf{C}}$$

is a quasi-isomorphism for all n sufficiently large. Hence the same is true for the map from the direct limit. □

REFERENCES

[1] Pierre Berthelot. Immersions regulières et calcul du $K^{\cdot}$ d'un schéma éclaté. In *SGA 6, Exp. VII*, volume 225 of *Lecture Notes in Mathematics*, pages 416–465. Springer, 1971.

[2] Pierre Berthelot and Arthur Ogus. *Notes on Crystalline Cohomology.* Annals of Mathematics Studies. Princeton University Press, 1978.

[3] Pierre Deligne. *Equations Différentielles à Points Singuliers Réguliers*, volume 163 of *Lecture Notes in Mathematics*. Springer Verlag, 1970.

[4] Hélène Esnault and Eckart Viehweg. Logarithmic de Rham complexes and vanishing theorems. *Inv. Math.*, 86(1):161–194, 1986.

[5] Jean-Marc Fontaine. Arithmétique des représentations galoisiennes p-adiques. preprint.

[6] Kazuya Kato. Logarithmic structures of Fontaine-Illusie. In Jun-Ichi Igusa, editor, *Algebraic Analysis, Geometry, and Number Theory.* Johns Hopkins University Press, 1989.

[7] Kazuya Kato and Chikara Nakayama. Log Betti cohomology, log étale cohomology, and log De Rham cohomology of log schemes over C. *Kodai Math. J.*, 22(2):161–186, 1999.

[8] Pierre Lorenzon. Indexed algebras associated to a log structure and a theorem of p-descent on log schemes. *Manuscripta Mathematica*, 101:271–299, 2000.

[9] Nitin Nitsure. Moduli of semistable logarithmic connections. *Journal of the American Mathematical Society*, 6(3):597–609, 1993.

[10] Nitin Nitsure. Moduli of regular holonomic $\mathcal{D}$-modules with normal crossing singularities. *Duke Mathematical Journal*, 99(1):1–39, 1999.

[11] Arthur Ogus. Logarithmic de Rham cohomology. electronically published.

[12] Arthur Ogus. *F-Crystals, Griffiths transversality, and the Hodge decomposition*, volume 221 of *Astérisque*. Société Mathématique de France, 1994.

[13] Carlos Simpson. Higgs bundles and local systems. *Inst. Hautes Études Sci. Pub. Math.*, 75:5–95, 1992.

Arthur Ogus
Department of Mathematics
University of California
Berkeley
USA
ogus@math.berkeley.edu

Groupes de Selmer et Accouplements; Cas Particulier des Courbes Elliptiques

Bernadette Perrin-Riou

Received: October 25, 2002
Revised: September 27, 2003

Abstract. We give proofs of existence of alternating pairings on Selmer groups of p-ordinary elliptic curves on a $\mathbb{Z}_p^2$-extension by using the Cassels-Tate-Flach pairings for twists of the p-adic representation.

Soit E une courbe elliptique définie sur le corps des nombres rationnels $\mathbb{Q}$. D'après le théorème de Mordell, le groupe $E(\mathbb{Q})$ des points de E rationnels sur $\mathbb{Q}$ est un $\mathbb{Z}$-module de type fini. Nekovář a démontré que son rang est de même parité que la multiplicité du zéro en $s = 1$ de la fonction L complexe associée à $E/\mathbb{Q}$ lorsque la p-composante du groupe de Tate-Shafaravich est finie. La conjecture de Birch et Swinnerton-Dyer prédit qu'il y a égalité entre ces deux entiers attachés à E.
La démonstration de ce résultat par Nekovář utilise essentiellement trois arguments et se fait en introduisant un corps quadratique imaginaire K et un nombre premier p auxiliaires vérifiant certaines conditions. Le premier argument utilise le théorème de Cornut et Vatsal concernant les points de Heegner ([9], [10], [34], conjecture de Mazur [25]) : on en déduit que le rang de $E(K_n)$ tend vers l'infini avec n pour K_n l'extension de $\mathbb{Q}$ de degré $2p^n$ qui est diédrale sur $\mathbb{Q}$. Le deuxième argument est à base de théorie d'Iwasawa. Il s'agit de généraliser le théorème de Cassels qui affirme que le groupe de Tate-Shafaravich d'une courbe elliptique modulo sa partie divisible (noté div) est un carré ou, dans la version p-primaire, que le quotient du p-groupe de Selmer $S_p(E/K)$ de E/K modulo sa partie divisible est un carré : on peut construire une forme alternée et non dégénérée sur $S_p(E/K)/\operatorname{div}$. Dans cette généralisation, le groupe de Tate-Shafarevich devient le quotient de $S_p(E/K_\infty)$ par sa partie Λ-divisible $\operatorname{div}_\Lambda$ où $K_\infty = \cup K_n$, $\Gamma = \operatorname{Gal}(K_\infty/K)$ et Λ est l'algèbre de groupe continue $\mathbb{Z}_p[[\Gamma]]$. On peut construire sur $S_p(E/K_\infty)/\operatorname{div}_\Lambda$ une forme Λ-linéaire et alternée. Le troisième argument utilise les résultats de Kolyvagin généralisés par Bertolini et Darmon ([22], [5]) et des arguments de descente pour conclure.
Pour construire la forme alternée, Nekovář reprend complètement la théorie des groupes de Selmer en utilisant le formalisme des complexes. Il obtient ainsi d'autres applications en théorie de Hida et autres. On se contente ici de faire cette construction en allant au plus court.

Le principe de la démonstration est de faire grand usage du twist d'un Λ-module : l'adjoint d'un Λ-module de torsion se calcule en effet facilement lorsque ses coinvariants sont de torsion pour l'anneau quotient, ce qui est réalisable en faisant un twist convenable. Cette astuce permet d'éviter les difficultés dues au fait que le groupe de Mordell-Weil n'est pas fini. Ainsi, par exemple, l'accouplement de Cassels-Tate peut se calculer comme une "limite convenable" des accouplements relatifs aux représentations twistées par k.
Donnons une idée du contenu de l'article.
Dans le premier paragraphe, on fait quelques rappels d'algèbre commutative qu'on appliquera ensuite à la descente de modules d'Iwasawa relatifs à une $\mathbb{Z}_p^2$-extension à une sous-$\mathbb{Z}_p$-extension (passage à certains coinvariants). Dans ce genre de situation, on a l'habitude de négliger les modules pseudo-nuls. Mais la descente de tels modules peut donner des modules non pseudo-nuls sur la $\mathbb{Z}_p$-extension. Aussi, on introduit une notion de modules négligeables qui sont en gros les modules qui resteront négligeables par descente.
Le but est alors le §1.2 : on y montre comment à partir d'un isomorphisme entre un module M et son adjoint M' construire d'une part une forme bilinéaire sur la partie libre des coinvariants de M et M' et d'autre part une application bilinéaire sur leurs sous-modules de torsion.
Dans le paragraphe 2, on introduit les groupes de Selmer associés à une représentation p-adique ordinaire et on démontre les théorèmes tout à fait classiques de "contrôle" par descente (par exemple d'une $\mathbb{Z}_p^2$-extension à une $\mathbb{Z}_p$-extension). Remarquons qu'il n'y a pas d'hypothèses sur la finitude des places au dessus d'une place première à p dans les $\mathbb{Z}_p$-extensions et qu'on ne néglige pas les modules dont la série caractéristique est une puissance de p, ce qui permet ensuite de pouvoir traiter la μ-partie des modules de Selmer. On donne ensuite un moyen de calcul de l'adjoint à partir des modules de Selmer à un niveau fini relatif à un twist convenable de la représentation p-adique. On doit pour cela utiliser une condition technique (propriété (A) de §1) et la démontrer pour les modules d'Iwasawa utilisés (elle permet d'éviter le problème que les coinvariants d'un module pseudo-nul dans le cas de deux variables n'est pas toujours pseudo-nul.)
Dans le paragraphe 3, on rappelle les théorèmes de Cassels-Flach à un niveau fini. En twistant éventuellement la représentation V, on peut alors utiliser les résultats du paragraphe 2 pour construire un (presque)-isomorphisme entre le module de Selmer sur une $\mathbb{Z}_p^2$-extension et son adjoint. On démontre alors une propriété de "symétrie", c'est-à-dire un lien naturel entre ce pseudo-isomorphisme et celui construit pour le dual de Tate $V^*(1)$ (à ce stade, l'existence d'un isomorphisme alterné sur V n'a pas été supposé).
Il est alors possible d'appliquer les résultats d'algèbre commutative du premier paragraphe : par exemple reconstruire l'accouplement de Cassels et la hauteur p-adique pour les twists de la représentation p-adique, construire des formes bilinéaires sur la partie libre ou de torsion du module de Selmer relatif à une sous-$\mathbb{Z}_p$-extension...
Le dernier paragraphe est consacré aux applications à l'extension anti-

cyclotomique d'un corps quadratique imaginaire.

Table des matières

1 Préliminaires d'algèbre commutative

1.1 Adjoint et dualité

Soit Λ un anneau local noetherien complet de dimension r ; plus précisément Λ sera l'algèbre de groupes continue d'un groupe Γ topologiquement isomorphe à $\mathbb{Z}_p^{r-1}$: $\Lambda = \mathbb{Z}_p[[\Gamma]]$. Les Λ-modules considérés seront toujours de type fini. Dans ce texte, un homomorphisme entre deux tels modules est dit un quasi-isomorphisme si son noyau et conoyau sont finis. Un complexe (de longueur finie) de Λ-modules de type fini est dit suite quasi-exacte s'il est exact à des modules finis près.

Soit M un Λ-module de type fini. On pose pour tout entier $i \geq 0$

$$a^i_\Lambda(M) = \mathrm{Ext}^i_\Lambda(M, \Lambda) \ .$$

En particulier le Λ-module $a^1_\Lambda(M) = \mathrm{Ext}^1_\Lambda(M, \Lambda)$ est l'adjoint (d'Iwasawa) de M. Rappelons quelques faits d'algèbre commutative.

- Un module est dit pseudo-nul si la hauteur des idéaux associés est supérieure ou égale à 2 ;
- La hauteur des idéaux associés à $a^i_\Lambda(M)$ est supérieure ou égale à i ; aussi, $a^i_\Lambda(M)$ est pseudo-nul pour $i \geq 2$, $a^1_\Lambda(M)$ est un Λ-module de torsion et pour $\dim \Lambda = 3$, $a^3_\Lambda(M)$ est fini ;
- Lorsque M est de Λ-torsion, on peut interpréter $a^1_\Lambda(M)$ de la manière suivante. Soit $\mathrm{Frac}\,\Lambda$ le corps des fractions de Λ. Alors,

$$a^1_\Lambda(M) = \mathrm{Hom}_\Lambda(M, \mathrm{Frac}\,\Lambda/\Lambda) \ .$$

- Si M est un Λ-module de torsion, $a^1_\Lambda(M)$ n'a pas de sous-modules pseudo-nuls non nuls. En particulier, si M est un module pseudo-nul, $a^1_\Lambda(M) = 0$.
- Supposons Λ de dimension 3. Si M n'a pas de sous-modules pseudo-nuls non finis, alors $a^2_\Lambda(M)$ est fini.

Démontrons le dernier point. Il existe un homomorphisme $M \to E$ à conoyau pseudo-nul F et de noyau pseudo-nul avec E module élémentaire $E = \oplus \Lambda/(f_i)$. Un tel module E est de dimension projective 1. D'autre part, par hypothèse sur M, le noyau de $M \to E$ est fini. Ainsi, on a un quasi-isomorphisme

$$a^2_\Lambda(M) \xrightarrow{\sim} a^3_\Lambda(F) \ .$$

Comme la hauteur des idéaux associés à $a^3_\Lambda(F)$ est supérieure à 3, $a^3_\Lambda(F)$ est fini. On en déduit que $a^2_\Lambda(M)$ est fini.

DÉFINITION. Un Λ-module de type fini M est dit négligeable si la hauteur des idéaux associés à M est supérieure ou égale à 3. On dit que M vérifie la propriété (A) si $a^i_\Lambda(M)$ est négligeable pour $i \geq 2$.
Un complexe (de longueur finie) de Λ-modules de type fini est dit suite presque-exacte s'il est exact à des modules négligeables près.

Lorsque $\dim \Lambda = 2$, les modules négligeables sont les modules nuls. Lorsque $\dim \Lambda = 3$, les modules négligeables sont les modules finis. Une suite presque exacte est donc une suite quasi-exacte. Un Λ-module de type fini M vérifie la propriété (A) si $a^2_\Lambda(M)$ est fini (cela est automatique pour $a^3_\Lambda(M)$).
Ainsi, si M n'a pas de sous-modules pseudo-nuls non finis, M vérifie la propriété (A).
Si $\mathfrak{p}$ est un idéal de Λ, on note $M^{\mathfrak{p}}$ le sous-module des éléments de M annulés par $\mathfrak{p}$.

1.1.1 Proposition. *Soit M un Λ-module de type fini, de torsion et $\mathfrak{p}$ un idéal de Λ de hauteur* 1.
1) *Si $M/\mathfrak{p}$ est de $\Lambda/\mathfrak{p}$-torsion, on a la suite exacte naturelle*

$$0 \to a^1_\Lambda(M)/\mathfrak{p} \to a^1_{\Lambda/\mathfrak{p}}(M/\mathfrak{p}) \to a^2_\Lambda(M)^{\mathfrak{p}} \to 0\ .$$

2) *Si M vérifie la condition* (A), *on a une suite presque exacte*

$$0 \to a^1_{\Lambda/\mathfrak{p}}(M/\mathfrak{p}) \to a^1_\Lambda(M)/\mathfrak{p} \to a^0_{\Lambda/\mathfrak{p}}(M^{\mathfrak{p}}) \to 0\ .$$

Si M est de dimension projective inférieure ou égale à 1, *la suite*

$$0 \to a^1_{\Lambda/\mathfrak{p}}(M/\mathfrak{p}) \to a^1_\Lambda(M)/\mathfrak{p} \to a^0_{\Lambda/\mathfrak{p}}(M^{\mathfrak{p}}) \to a^2_{\Lambda/\mathfrak{p}}(M/\mathfrak{p}) \to 0$$

est exacte.

Démonstration. On déduit de la suite exacte $0 \to \Lambda \xrightarrow{f} \Lambda \to \Lambda/\mathfrak{p} \to 0$ avec $\mathfrak{p} = (f)$ la suite exacte

$$0 \to \mathrm{Ext}^1_\Lambda(M,\Lambda)/\mathfrak{p} \to \mathrm{Ext}^1_\Lambda(M,\Lambda/\mathfrak{p}) \to \mathrm{Ext}^2_\Lambda(M,\Lambda)^{\mathfrak{p}} \to 0\ .$$

D'autre part, on utilise une résolution de M par des modules de type fini

$$0 \to L' \to L \to M \to 0$$

avec L libre. Si L'' est le noyau de $L/\mathfrak{p} \to M/\mathfrak{p}$, on a la suite exacte

$$0 \to M^{\mathfrak{p}} \to L'/\mathfrak{p} \to L'' \to 0\ .$$

Le diagramme suivant est commutatif et ses lignes et colonnes sont exactes :

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
0 & \to & a^0_{\Lambda/\mathfrak{p}}(M/\mathfrak{p}) & \to & a^0_{\Lambda/\mathfrak{p}}(L/\mathfrak{p}) & \to & a^0_{\Lambda/\mathfrak{p}}(L'') & \to & a^1_{\Lambda/\mathfrak{p}}(M/\mathfrak{p}) \to 0 \\
 & & \| & & \| & & \downarrow & & \downarrow \\
0 & \to & \mathrm{Hom}_\Lambda(M,\Lambda/\mathfrak{p}) & \to & \mathrm{Hom}_\Lambda(L,\Lambda/\mathfrak{p}) & \to & \mathrm{Hom}_\Lambda(L',\Lambda/\mathfrak{p}) & \to & \mathrm{Ext}^1_\Lambda(M,\Lambda/\mathfrak{p}) \to 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & a^0_{\Lambda/\mathfrak{p}}(M^{\mathfrak{p}}) & & \\
 & & & & & & \downarrow & & \\
 & & & & & & a^1_{\Lambda/\mathfrak{p}}(L'') & & \\
 & & & & & & \downarrow & & \\
 & & & & & & a^1_{\Lambda/\mathfrak{p}}(L'/\mathfrak{p}) & &
\end{array}$$

Comme $a^i_{\Lambda/\mathfrak{p}}(L'') \cong a^{i+1}_{\Lambda/\mathfrak{p}}(M/\mathfrak{p})$ pour $i \geq 1$, on trouve la suite exacte

$$0 \to a^1_{\Lambda/\mathfrak{p}}(M/\mathfrak{p}) \to \mathrm{Ext}^1_\Lambda(M,\Lambda/\mathfrak{p}) \to a^0_{\Lambda/\mathfrak{p}}(M^{\mathfrak{p}}) \to a^2_{\Lambda/\mathfrak{p}}(M/\mathfrak{p})\ .$$

Lorsque $M/\mathfrak{p}$ est de $\Lambda/\mathfrak{p}$-torsion, il en est de même de $M^{\mathfrak{p}}$, $a^0_{\Lambda/\mathfrak{p}}(M^{\mathfrak{p}})$ est nul et on obtient l'assertion (1). En général, on peut résumer en les deux suites exactes

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ & & & & a^1_\Lambda(M)/\mathfrak{p} & & & & \\ & & & & \downarrow & & & & \\ 0\to & a^1_{\Lambda/\mathfrak{p}}(M/\mathfrak{p}) & \to & & \mathrm{Ext}^1_\Lambda(M,\Lambda/\mathfrak{p}) & \to & a^0_{\Lambda/\mathfrak{p}}(M^{\mathfrak{p}}) & \to & a^2_{\Lambda/\mathfrak{p}}(M/\mathfrak{p})\ . \\ & & & & \downarrow & & & & \\ & & & & a^2_\Lambda(M)^{\mathfrak{p}} & & & & \\ & & & & \downarrow & & & & \\ & & & & 0 & & & & \end{array}$$

Lorsque M vérifie la propriété (A), $a^2_\Lambda(M)$ est négligeable. Lorsque M est de dimension projective inférieure ou égale à 1, $a^2_\Lambda(M)=0$, le Λ-module L' est libre, donc le $\Lambda/\mathfrak{p}$-module $L'/\mathfrak{p}$ est libre et $a^1_{\Lambda/\mathfrak{p}}(L'/\mathfrak{p})=0$. D'où la suite exacte de la proposition. □

1.1.2 Remarques. (1) L'application $\mathrm{Ext}^1_\Lambda(M,\Lambda/\mathfrak{p})\to a^0_{\Lambda/\mathfrak{p}}(M^{\mathfrak{p}})$ dépend du choix d'un générateur f de $\mathfrak{p}$.
(2) Le $\Lambda/\mathfrak{p}$-module $a^1_\Lambda(M)/\mathfrak{p}$ contrôle à la fois la partie libre de $M^{\mathfrak{p}}$ et la partie de torsion de $M/\mathfrak{p}$. En particulier, si $M/\mathfrak{p}$ est de $\Lambda/\mathfrak{p}$-torsion, $a^1_\Lambda(M)/\mathfrak{p}$ est un module de torsion et on a la suite exacte

$$0\to a^1_\Lambda(M)/\mathfrak{p}\to a^1_{\Lambda/\mathfrak{p}}(M/\mathfrak{p})\to a^2_\Lambda(M)^{\mathfrak{p}}\to 0\ .$$

Si M est de dimension projective inférieure ou égale à 1, $a^1_\Lambda(M)/\mathfrak{p}$ est égal à $a^1_{\Lambda/\mathfrak{p}}(M/\mathfrak{p})$. Si M vérifie la propriété (A), ($a^2_\Lambda(M)$ est donc fini), le noyau de l'application $a^1_\Lambda(M)/\mathfrak{p}\to a^0_{\Lambda/\mathfrak{p}}(M^{\mathfrak{p}})$ est le sous-module de torsion de $a^1_\Lambda(M)/\mathfrak{p}$ et est quasi-isomorphe à $a^1_{\Lambda/\mathfrak{p}}(M/\mathfrak{p})$.

1.2 Construction d'accouplement

Soient deux Λ-modules M et M' de Λ-torsion et un Λ-homomorphisme

$$\Theta : M\to a^1_\Lambda(M')$$

autrement dit une application bilinéaire $M\times M'\to \mathrm{Frac}\,\Lambda/\Lambda$. Si $\mathfrak{p}$ est un idéal de Λ de hauteur 1, on en déduit un homomorphisme de $\Lambda/\mathfrak{p}$-modules

$$M/\mathfrak{p}\to a^1_\Lambda(M')/\mathfrak{p}\to \mathrm{Ext}^1_\Lambda(M',\Lambda/\mathfrak{p})\ .$$

L'image du $\Lambda/\mathfrak{p}$-module de torsion $t_{\Lambda/\mathfrak{p}}(M/\mathfrak{p})$ de $M/\mathfrak{p}$ est contenue dans le module de $\Lambda/\mathfrak{p}$-torsion de $\mathrm{Ext}^1_\Lambda(M',\Lambda/\mathfrak{p})$. On a donc le diagramme commutatif dont les lignes sont exactes :

$$\begin{array}{ccccccccc} 0 & \to & a^1_{\Lambda/\mathfrak{p}}(M'/\mathfrak{p}) & \to & \mathrm{Ext}^1_\Lambda(M',\Lambda/\mathfrak{p}) & \to & a^0_{\Lambda/\mathfrak{p}}(M'^{\mathfrak{p}}) & \to & a^2_{\Lambda/\mathfrak{p}}(M'/\mathfrak{p}) \\ & & & & \uparrow & & & & \\ & & \uparrow & & a^1_\Lambda(M')/\mathfrak{p} & & & & \\ & & & & \uparrow & & & & \\ 0 & \to & t_{\Lambda/\mathfrak{p}}(M/\mathfrak{p}) & \to & M/\mathfrak{p} & \to & (M/\mathfrak{p})^{**} & \to & \Lambda/\mathfrak{p}-\text{pseudo}-\text{nul} \end{array} \qquad (1)$$

avec $(M/\mathfrak{p})^{**} = \mathrm{Hom}_{\Lambda/\mathfrak{p}}(\mathrm{Hom}_{\Lambda/\mathfrak{p}}(M/\mathfrak{p},\Lambda/\mathfrak{p}),\Lambda/\mathfrak{p}) = a^0_{\Lambda/\mathfrak{p}}(a^0_{\Lambda/\mathfrak{p}}(M/\mathfrak{p}))$. On obtient ainsi des homomorphismes de $\Lambda/\mathfrak{p}$-modules

$$\ell_\mathfrak{p}(\Theta) : M/\mathfrak{p}/t_{\Lambda/\mathfrak{p}}(M/\mathfrak{p}) \to a^0_{\Lambda/\mathfrak{p}}(M'^{\mathfrak{p}})$$

et

$$t_\mathfrak{p}(\Theta) : t_{\Lambda/\mathfrak{p}}(M/\mathfrak{p}) \to a^1_{\Lambda/\mathfrak{p}}(M'/\mathfrak{p}) \to a^1_{\Lambda/\mathfrak{p}}(t_{\Lambda/\mathfrak{p}}(M'/\mathfrak{p}))\ .$$

Contrairement à $t_\mathfrak{p}(\Theta)$, $\ell_\mathfrak{p}(\Theta)$ dépend du choix d'un générateur de $\mathfrak{p}$.

1.2.1 Lemme. *Supposons Λ de dimension inférieure ou égale à 3. Si M et M' sont des Λ-modules de torsion vérifiant la propriété (A) et si Θ est un quasi-isomorphisme de Λ-modules, $\ell_\mathfrak{p}(\Theta)$ et $t_\mathfrak{p}(\Theta)$ sont des quasi-$\Lambda/\mathfrak{p}$-isomorphismes.*

Démonstration. L'hypothèse implique que $\Theta_\mathfrak{p} : M/\mathfrak{p} \to a^1_\Lambda(M')/\mathfrak{p}$ est un quasi-isomorphisme. D'autre part, $a^2_{\Lambda/\mathfrak{p}}(M'/\mathfrak{p})$ est fini. □

Gardons les hypothèses du lemme. On déduit de $t_\mathfrak{p}(\Theta)$ une forme bilinéaire quasi-non dégénérée :

$$t_{\Lambda/\mathfrak{p}}(M/\mathfrak{p}) \times t_{\Lambda/\mathfrak{p}}(M'/\mathfrak{p}) \to \mathrm{Frac}\,\Lambda/\mathfrak{p}/(\Lambda/\mathfrak{p})$$

ou

$$a^1_{\Lambda/\mathfrak{p}}(M/\mathfrak{p}) \times a^1_{\Lambda/\mathfrak{p}}(M'/\mathfrak{p}) \to \mathrm{Frac}\,\Lambda/\mathfrak{p}/(\Lambda/\mathfrak{p})\ .$$

Lorsqu'on tensorise $\ell_\mathfrak{p}(\Theta)$ par $\mathrm{Frac}\,\Lambda/\mathfrak{p}$, comme le conoyau de $M/\mathfrak{p} \to (M/\mathfrak{p})^{**}$ est fini, on en déduit un isomorphisme

$$\mathrm{Frac}\,\Lambda/\mathfrak{p} \otimes a^0_{\Lambda/\mathfrak{p}}(a^0_{\Lambda/\mathfrak{p}}(M/\mathfrak{p})) \to \mathrm{Frac}\,\Lambda/\mathfrak{p} \otimes a^0_{\Lambda/\mathfrak{p}}(M'^{\mathfrak{p}})\ .$$

En prenant l'inverse et en composant avec l'application induite par $M'^{\mathfrak{p}} \to M'/\mathfrak{p}$, on en déduit une forme bilinéaire

$$a^0_{\Lambda/\mathfrak{p}}(M/\mathfrak{p}) \times a^0_{\Lambda/\mathfrak{p}}(M'/\mathfrak{p}) \to \mathrm{Frac}\,\Lambda/\mathfrak{p}$$

qui est non dégénérée si et seulement si $\mathrm{Frac}\,\Lambda/\mathfrak{p} \otimes M'^{\mathfrak{p}} \to \mathrm{Frac}\,\Lambda/\mathfrak{p} \otimes M'/\mathfrak{p}$ est un isomorphisme, c'est-à-dire si et seulement si le noyau de $M^\mathfrak{p} \to M/\mathfrak{p}$ est de $\Lambda/\mathfrak{p}$-torsion.

Lorsque $\Lambda = \mathbb{Z}_p[[\Gamma]]$, Λ est muni d'une involution induite par $\tau \to \tau^{-1}$ et que l'on note avec un point. Si N est un Λ-module, $\dot{N}$ est le module N muni d'une nouvelle action de Γ : $\tau \cdot n = \tau^{-1} n$.
Supposons $\Gamma = \mathbb{Z}_p$. Reprenons les suites exactes et la construction : si $\Gamma_n = \Gamma^{p^n}$, les modules $\mathbb{Z}_p[\Gamma/\Gamma_n]$-pseudo-nuls sont nuls et $a^1_{\mathbb{Z}_p[\Gamma/\Gamma_n]}(\dot{N})$ est égal à $\widehat{t_{\mathbb{Z}_p}(N)}$ muni de l'action usuelle de Γ/Γ_n sur le dual de Pontryagin : $\tau(f)(n) = f(\tau^{-1}n)$.
Pour $\mathfrak{p}$ l'idéal engendré par $\gamma - 1$, $M^{\mathfrak{p}}$ est le module des invariants M^Γ et $M/\mathfrak{p}$ le module des coinvariants M_Γ. On obtient un diagramme commutatif dont les lignes sont exactes

$$\begin{array}{ccccccccc} & & & & & & \mathrm{Hom}_{\mathbb{Z}_p}(M'_\Gamma, \mathbb{Z}_p) & & \\ & & & & & & \downarrow & & \\ 0 & \to & \widehat{t_{\mathbb{Z}_p}(M'_\Gamma)} & \to & \mathrm{Ext}^1_\Lambda(M', \Lambda)_\Gamma & \to & \mathrm{Hom}_{\mathbb{Z}_p}(M'^\Gamma, \mathbb{Z}_p) & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & t_{\mathbb{Z}_p}(M_\Gamma) & \to & M_\Gamma & \to & L_{\mathbb{Z}_p}(M_\Gamma) & \to & 0 . \end{array}$$

Autrement dit, on obtient une forme bilinéaire à valeurs dans $\mathbb{Q}_p/\mathbb{Z}_p$

$$\widehat{t_{\mathbb{Z}_p}(M_\Gamma)} \times \widehat{t_{\mathbb{Z}_p}(M'_\Gamma)} \to \mathbb{Q}_p/\mathbb{Z}_p$$

et une forme bilinéaire à valeurs dans $\mathbb{Q}_p$

$${M'_\Gamma}^* \times M^*_\Gamma \to \mathbb{Q}_p$$

qui est non dégénérée si $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} M^\Gamma \to \mathbb{Q}_p \otimes_{\mathbb{Z}_p} M_\Gamma$ est un isomorphisme. En remplaçant γ par γ^{p^n}, on obtient de même une forme sesqui-linéaire

$$\dot{M'}^*_{\Gamma_n} \times M^*_{\Gamma_n} \to \mathbb{Q}_p[\Gamma/\Gamma_n] \ .$$

1.3 Calcul de l'adjoint

[24], [20], [2] Prenons $\Lambda = \mathbb{Z}_p[[\Gamma]]$ avec $\Gamma \cong \mathbb{Z}_p^r$. Iwasawa a donné un moyen explicite de calculer $a^1_\Lambda(M)$. Soit Γ' un sous-groupe isomorphe à $\mathbb{Z}_p$ de Γ. Soit γ un générateur topologique de Γ'. Posons $\Lambda_n = \mathbb{Z}_p[[\Gamma/\Gamma'_n]]$

1.3.1 Proposition. *Soit M un Λ-module de type fini de torsion tel que $M/(\gamma^{p^n} - 1)$ soit un Λ_n-module de torsion pour tout entier n. Alors*

$$a^1_\Lambda(M) = \varprojlim_n a^1_{\Lambda_n}(M/(\gamma^{p^n} - 1)) = \varprojlim_n a^1_{\Lambda_n}(M_{\Gamma'_n})$$

l'application de transition étant induite par la trace c'est-à-dire par la multiplication par $\sum_{i=0}^{p-1} \gamma^{ip^n}$.

Cela se déduit des suites exactes

$$0 \to a^1_\Lambda(M)_{\Gamma'_n} \to a^1_{\Lambda_n}(M_{\Gamma'_n}) \to a^2_\Lambda(M)^{\Gamma'_n}$$

et du fait que la limite projective des $a^2_\Lambda(M)^{\Gamma'_n}$ est nulle.
Dans le cas où $\Gamma = \Gamma' = \mathbb{Z}_p$, on obtient le résultat bien connu suivant : si M_{Γ_n} est fini pour tout entier n,

$$a^1_\Lambda(M) \cong \varprojlim_n \widehat{\dot{M}}^{\Gamma_n} = \varprojlim_n \widehat{\dot{M}}^{\Gamma_n} / p^n \widehat{\dot{M}}^{\Gamma_n} .$$

Nous renvoyons à [24] ou à [20] pour des précisions et une interprétation en termes de cohomologie locale.

2 Modules de Selmer et théorèmes de contrôle

2.1 Notations

Soit p un nombre premier impair, F un corps de nombres, S un nombre fini de places de F contenant les places à l'infini et les places au dessus de p. Si v est une place de F, on note G_v un groupe de décomposition de F en v. Si L est une extension de F, on note $S(L)$ l'ensemble des places de L au dessus de S. Si v est une place de F, la notation L_v/F_v signifie par abus de notation L_w/F_v où w est une place choisie de L au dessus de v (le contexte indiquant que le choix n'a alors pas d'importance).
Soit V une représentation p-adique du groupe de Galois absolu G_F de F, non ramifiée en dehors de S. Ainsi, si $G_{S,F}$ est le groupe de Galois de la plus grande extension de F non ramifiée en dehors de S, V est une représentation p-adique de $G_{S,F}$. Soit T un réseau de V stable par G_F. On note $V^* = \mathrm{Hom}(V, \mathbb{Q}_p)$, $T^* = \mathrm{Hom}(T, \mathbb{Z}_p)$, $\check{V} = V^*(1)$ le dual de Tate de V, $\check{T} = T^*(1)$. Si W est un $\mathbb{Z}_p$-module libre de type fini, on pose $\mathcal{U}(W) = (\mathbb{Q}_p \otimes W)/W$ et $\check{\mathcal{U}}(W) = (\mathbb{Q}_p \otimes \check{W})/\check{W}$.
Nous ferons désormais l'hypothèse suivante :

$$V \text{ est ordinaire aux places divisant } p. \qquad \text{(Ord)}$$

Rappelons que V est ordinaire aux places divisant p si pour tout $v \mid p$, il existe une filtration de G_v-modules $\mathrm{Fil}^j_v V$ associée à la représentation p-adique V telle que le groupe d'inertie en v agit sur le quotient $\mathrm{Fil}^j_v V/\mathrm{Fil}^{j+1}_v V$ par le caractère χ^j où χ est le caractère cyclotomique. On pose $\mathrm{Fil}^j_v T = \mathrm{Fil}^j_v V \cap T$.
Soit ρ un caractère continu de G_F à valeurs dans $\mathbb{Z}_p^*$. On note $V \otimes \rho$ la représentation V twistée par le caractère ρ. Lorsque ρ est la puissance k-ième du caractère cyclotomique, on trouve le twist à la Tate usuel noté $V(k)$. On pose pour simplifier $\mathcal{U}_\rho = \mathcal{U}(T \otimes \rho)$ et $\check{\mathcal{U}}_\rho = \check{\mathcal{U}}(T \otimes \rho) = \check{\mathcal{U}}(T) \otimes \rho^{-1}$.
Nous faisons plus loin l'hypothèse suivante pour certaines extensions L de F :

$$(V \otimes \rho)^{G_L} = 0 \text{ et pour } v \mid p,\ ((V/\mathrm{Fil}^1_v V) \otimes \rho)^{G_{L_v}} = 0. \qquad (\mathrm{Hyp}(L, V, \rho))$$

Nous faisons désormais les hypothèses

$$\mathrm{Hyp}(L, V) \text{ et } \mathrm{Hyp}(L, \check{V})$$

correspondant au caractère identité pour les extensions finies L utilisées dans la suite. Enfin, si F_∞ est une $\mathbb{Z}_p$ ou $\mathbb{Z}_p^2$-extension, on suppose qu'il n'y a qu'un nombre fini de places au dessus de p dans F_∞.

2.2 Groupes de Selmer

Sous les hypothèses faites sur V, (ordinarité, $\mathrm{Hyp}(F,V)$ et $\mathrm{Hyp}(F,\check{V})$), les modules de Selmer peuvent être définis en prenant la définition de Bloch-Kato ou en prenant celle de Greenberg. Soit

$$H^1_f(F_v,V)=\begin{cases} H^1(G_v/I_v,V^{I_v}) & \text{pour } v\nmid p\\ \operatorname{Im} H^1(F_v,\operatorname{Fil}^1_v V)\to H^1(F_v,V) & \\ =\ker H^1(F_v,V)\to H^1(F_v,V/\operatorname{Fil}^1_v V) & \text{pour } v\mid p\end{cases}$$

et

$$H^1_{/f}(F_v,V)=H^1(F_v,V)/H^1_f(F_v,V)\ .$$

On a alors

$$H^1_{/f}(F_v,V)=\begin{cases} H^1(I_v,V)^{G_v/I_v} & \text{pour } v\nmid p\\ H^1(F_v,V/\operatorname{Fil}^1_v V) & \text{pour } v\mid p\end{cases}$$

Soit $H^1_f(F_v,T)$ l'image réciproque de $H^1_f(F_v,V)$ dans $H^1(F_v,T)$ et

$$H^1_{/f}(F_v,\mathcal{U})=H^1(F_v,\mathcal{U})/\mathbb{Q}_p/\mathbb{Z}_p\otimes H^1_f(F_v,T)=H^1(F_v,\mathcal{U})/\operatorname{Im} H^1_f(F_v,V)\ .$$

On définit $H^1_f(F,T)$ comme le noyau de

$$H^1(G_{S,F},T)\to\prod_{v\in S}H^1_{/f}(F_v,V)\ .$$

Ensuite, $H^1_f(F,\mathcal{U}) = H^1_f(F,V/T)$ peut être défini comme le noyau de l'application

$$H^1(G_{S,F},\mathcal{U})\to\prod_{v\in S}H^1_{/f}(F_v,\mathcal{U})\ .$$

2.3 Modules d'Iwasawa

Soit F_∞/F une $\mathbb{Z}_p$-extension ou une $\mathbb{Z}_p^2$-extension. On pose $\Gamma=\mathrm{Gal}(F_\infty/F)$, $\Lambda=\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F)]]$ et on note $F_n=F_\infty^{\Gamma^{p^n}}$ le sous-corps de F_∞ fixe par Γ^{p^n}. Soit alors

$$X_{\infty,f}(F_\infty,\check{T})=\mathrm{Hom}_{\mathbb{Z}_p}(H^1_f(F_\infty,\mathcal{U}),\mathbb{Q}_p/\mathbb{Z}_p)$$

où $H^1_f(F_\infty,\mathcal{U})$ est la limite inductive des $H^1_f(L,\mathcal{U})$ pour L sous-extension de F_∞. C'est un Λ-module de type fini. Soit ρ un caractère continu de G_F à valeurs dans $\mathbb{Z}_p^*$ se factorisant par Γ.
Notons

$$H^1_*(F_v,V\otimes\rho)=\begin{cases}H^1(G_v/I_v,(V\otimes\rho)^{I_v}) & (v\nmid p)\\ \operatorname{Im}H^1(F_v,\operatorname{Fil}^1_v V\otimes\rho)\to H^1(F_v,V\otimes\rho) & \\ =\ker H^1(F_v,V\otimes\rho)\to H^1(F_v,V\otimes\rho/\operatorname{Fil}^1_v V\otimes\rho) & (v\mid p)\end{cases}$$

et

$$H^1_{/*}(F_v,\mathcal{U}_\rho)=H^1(F_v,\mathcal{U}_\rho)/\operatorname{Im}H^1_*(F_v,V\otimes\rho)$$

Soit $H^1_*(F,\mathcal{U}_\rho)$ le noyau de

$$H^1(G_{S,F},\mathcal{U}_\rho)\to\prod_{v\in S}H^1_{/*}(F_v,\mathcal{U}_\rho)=\prod_{v|p}H^1(F_v,\mathcal{U}_\rho)/\operatorname{Im}H^1(F_v,\operatorname{Fil}^1_v V\otimes\rho)$$
$$\prod_{v\in S,v\nmid p}H^1(F_v,\mathcal{U}_\rho)/\operatorname{Im}H^1(G_v/I_v,(V\otimes\rho)^{I_v}).$$

On définit H^1_*, $H^1_{/*}$ comme pour H^1_f, $H^1_{/f}$. Lorsque F_∞ contient le corps L_ρ fixé par le noyau de ρ, le module

$$X_{\infty,*}(F_\infty,\check{T}\otimes\rho^{-1})\overset{\text{déf}}{=}\operatorname{Hom}_{\mathbb{Z}_p}(H^1_*(F_\infty,\mathcal{U}_\rho),\mathbb{Q}_p/\mathbb{Z}_p)=X_{\infty,*}(F_\infty,\check{T})\otimes\rho^{-1}$$

est un twist de $X_{\infty,*}(F_\infty,\check{T})=X_{\infty,f}(F_\infty,\check{T})$ (sous les hypothèses $\operatorname{Hyp}(F_\infty,V)$ et $\operatorname{Hyp}(F_\infty,\check{V})$). [1]

2.4 Théorèmes de contrôle

On note

$$\mathfrak{Z}(L_v,T\otimes\rho)=\mathfrak{Z}_\rho(L_v)=\begin{cases}\mathcal{U}_\rho(T)^{G_{L_v}}/\operatorname{Im}(V\otimes\rho)^{G_{L_v}} & \text{si } v\nmid p\\ \mathcal{U}_\rho(T/\operatorname{Fil}^1_v T)^{G_{L_v}} & \text{si } v\mid p\end{cases}.$$

Considérons pour L contenu dans F_∞ les applications

$$\Xi_{F_\infty/L}(T\otimes\rho):H^1(F_\infty/L,\mathcal{U}_\rho^{G_{F_\infty}})\to\prod_{v\in S(L)}H^1(F_{\infty,v}/L_v,\mathfrak{Z}_\rho(F_{\infty,v}))\ .$$

Remarquons que seules les places de F qui ne sont pas totalement décomposées dans F_∞ interviennent réellement.

[1]remarquons que lorsque ρ est le caractère trivial, les notations * et f coïncident.

2.4.1 Proposition (théorème de contrôle, cas d'une $\mathbb{Z}_p$-extension). *Les applications induites par restriction*

$$H^1_*(F,\mathcal{U}_\rho) \to H^1_*(F_\infty,\mathcal{U}_\rho)^\Gamma = X_*(\widehat{F_\infty,\check{T}\otimes\rho^{-1})}^\Gamma$$

entrent dans une suite exacte naturelle

$$0 \to \ker \Xi_{F_\infty/F}(T\otimes\rho) \to H^1_*(F,\mathcal{U}_\rho) \to H^1_*(F_\infty,\mathcal{U}_\rho)^\Gamma$$
$$\to \operatorname{coker} \Xi_{F_\infty/F}(T\otimes\rho) \to H^1_*(\widehat{F,\check{T}\otimes}\rho^{-1}).$$

2.4.2 Corollaire. *Sous l'hypothèse* (Hyp(F,V,ρ))*, l'homomorphisme*

$$X_*(F_\infty,\check{T}\otimes\rho^{-1})_\Gamma \to H^1_*\widehat{(F,\mathcal{U}_\rho)}$$

a un noyau et conoyau finis. Sous l'hypothèse (Hyp(F_∞,V,ρ))*, les noyaux et les conoyaux des*

$$X_*(F_\infty,\check{T}\otimes\rho^{-1})_{\Gamma_n} \to H^1_*\widehat{(F_n,\mathcal{U}_\rho)}$$

sont d'ordre borné par rapport à n.

Il est commode d'introduire un sous-groupe de $H^1(G_{S,F},\mathcal{U}_\rho)$ un peu plus grand que $H^1_*(F,\mathcal{U}_\rho)$. Il s'agit du noyau de

$$H^1(G_{S,F},\mathcal{U}_\rho) \to \prod_{v\in S} \tilde{H}^1_{/*}(F_v,\mathcal{U}_\rho)$$

avec

$$\tilde{H}^1_{/*}(F_v,\mathcal{U}_\rho) = \begin{cases} (\prod_{w|v} H^1(F_{\infty,w},\mathcal{U}_\rho))^\Gamma & \text{si } v\nmid p \\ (\prod_{w|v} H^1(F_{\infty,v},\mathcal{U}_\rho/Im(\mathrm{Fil}^1_v V\otimes\rho)))^\Gamma & \text{si } v\mid p \\ = (\prod_{w|v} H^1(F_{\infty,v},\mathcal{U}_\rho(T/\mathrm{Fil}^1_v T)))^\Gamma & \end{cases}$$

On note aussi $\tilde{H}^1_*(F_v,\mathcal{U}_\rho)$ le noyau de $H^1(F_v,\mathcal{U}_\rho) \to \tilde{H}^1_{/*}(F_v,\mathcal{U}_\rho)$.
L'intérêt d'introduire ce module est le lemme suivant

2.4.3 Lemme. *Les suites suivantes sont exactes*

$$0 \to H^1(\Gamma,\mathcal{U}_\rho^{G_{F_\infty}}) \to \tilde{H}^1_*(F,\mathcal{U}_\rho) \to H^1_*(F_\infty,\mathcal{U}_\rho)^\Gamma \to 0$$

$$0 \to X_*(F_\infty,\check{T}\otimes\rho^{-1})_\Gamma \to \tilde{H}^1_*\widehat{(F,\mathcal{U}_\rho)} \to H^1\widehat{(\Gamma,\mathcal{U}_\rho^{G_{F_\infty}}})\to 0$$

Démonstration. On a en effet le diagramme commutatif dont les lignes et les colonnes sont exactes

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \uparrow & & \\ 0 & \to & H^1_*(F_\infty,\mathcal{U}_\rho)^\Gamma & \to & H^1(G_{S,F_\infty},\mathcal{U}_\rho)^\Gamma & \to & \left(\prod_{v\in S(F_\infty)} H^1_{/*}(F_{\infty,v},\mathcal{U}_\rho)\right)^\Gamma \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \to & \tilde{H}^1_*(F,\mathcal{U}_\rho) & \to & H^1(G_{S,F},\mathcal{U}_\rho) & \to & \prod_{v\in S} \tilde{H}^1_{/*}(F_v,\mathcal{U}_\rho) \\ & & & & \uparrow & & \\ & & & & H^1(\Gamma,\mathcal{U}_\rho^{G_{F_\infty}}) & & \\ & & & & \uparrow & & \\ & & & & 0 & & \end{array}$$

Il s'agit de voir que la flèche verticale de droite est injective. Lorsque $v \nmid p$ est totalement décomposée dans F_∞, $H^1_{/*}(F_v,\mathcal{U}_\rho) = (\prod_{w|v} H^1_{/*}(F_{\infty,v},\mathcal{U}_\rho))^\Gamma$ car Γ agit simplement par permutation des facteurs. Soit v ne divisant pas p et non totalement décomposée dans F_∞. Si w est une place de F_∞ au dessus de v, l'extension $F_{\infty,w}$ est l'unique extension de F_v non ramifiée et de groupe de Galois un pro-p-groupe. Donc, $H^1_{/*}(F_{\infty,w},\mathcal{U}_\rho)$ est égal à $H^1(F_{\infty,w},\mathcal{U}_\rho)^\Gamma$. On en déduit que $\tilde{H}^1_{/*}(F_v,\mathcal{U}_\rho) \to \left(\prod_{w|v} H^1_{/*}(F_{\infty,v},\mathcal{U}_\rho)\right)^\Gamma$ est un isomorphisme. Lorsque $v \mid p$, l'assertion est claire. □

Démonstration de la proposition. La différence entre $\tilde{H}^1_*(F,\mathcal{U}_\rho)$ et $H^1_*(F,\mathcal{U}_\rho)$ est calculée par la suite exacte suivante, conséquence de la suite exacte de Poitou-Tate :

$$0 \to H^1_*(F,\mathcal{U}_\rho) \to \tilde{H}^1_*(F,\mathcal{U}_\rho) \to \prod_{v\in S} \tilde{H}^1_*(F_v,\mathcal{U}_\rho)/H^1_*(F,\mathcal{U}_\rho) \to H^1_*(\widehat{F,\check{T}\otimes}\rho^{-1})$$

et il n'est pas difficile de voir que $\prod_{v\in S} \tilde{H}^1_*(F_v,\mathcal{U}_\rho)/H^1_*(F_v,\mathcal{U}_\rho)$ est exactement l'ensemble d'arrivée de $\Xi_{F_\infty/F}(T\otimes\rho)$. En effet, cela est clair pour la contribution des places totalement décomposées dans F_∞. Pour une place v non totalement décomposée dans F_∞ et ne divisant pas p, on a d'après le calcul précédent

$$\begin{aligned} \tilde{H}^1_*(F_v,\mathcal{U}_\rho) &= H^1(F_{\infty,v}/F_v,\mathcal{U}_\rho^{G_{F_{\infty,v}}}) \\ H^1_*(F_v,\mathcal{U}_\rho) &= \mathbb{Q}_p/\mathbb{Z}_p \otimes H^1(F_{\infty,v}/F_v,(T\otimes\rho)^{G_{F_{\infty,v}}}) \\ &= H^1(F_{\infty,v}/F_v,\mathcal{U}_\rho(T^{G_{F_{\infty,v}}})) \end{aligned}$$

car $\mathrm{Gal}(F_{\infty,v}/F_v)$ est de dimension cohomologique 1. Donc,

$$\tilde{H}^1_*(F_v,\mathcal{U}_\rho)/H^1_*(F_v,\mathcal{U}_\rho) = H^1(F_{\infty,v}/F_v,\mathfrak{Z}_\rho(F_{\infty,v}))\ .$$

Remarquons que par la dualité de Tate, c'est aussi le dual de Pontryagin du sous-$\mathbb{Z}_p$-module de torsion de $H^1(F_{\infty,v},\check{T}\otimes\rho^{-1})^{G_{F_v}}$ dont le cardinal est le nombre de Tamagawa local en v de $\check{T}\otimes\rho^{-1}$.

Soit maintenant une place v divisant p. On a par définition le diagramme commutatif et exact suivant

$$\begin{array}{ccccccc} & & & & 0 & & 0 & & \\ & & & & \uparrow & & \uparrow & & \\ 0 & \to & \tilde{H}^1_*(F_v,\mathcal{U}_\rho) & \to & H^1(F_v,\mathcal{U}_\rho) & \to & H^1(F_{\infty,v},\mathcal{U}_\rho(T/\operatorname{Fil}^1_v T))^{\Gamma_v} & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & H^1_*(F_v,\mathcal{U}_\rho) & \to & H^1(F_v,\mathcal{U}_\rho) & \to & H^1(F_v,\mathcal{U}_\rho(T/\operatorname{Fil}^1_v T)) & \to & H^2 . \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & 0 & & 0 & & H^1(F_{\infty,v}/F_v,(\mathcal{U}_\rho(T/\operatorname{Fil}^1_v T))^{G_{F_{\infty,v}}}) & & \\ & & & & & & \uparrow & & \\ & & & & & & 0 & & \end{array}$$

L'image de $H^1(F_{\infty,v}/F_v,\mathcal{U}_\rho(T/\operatorname{Fil}^1_v T)^{G_{F_{\infty,v}}})$ dans $H^2 = H^2(F_v,\mathcal{U}_\rho(\operatorname{Fil}^1_v T))$ est nulle. D'où l'assertion sur la contribution en p. $\square$

Pour démontrer le corollaire, on remarque que si $v \nmid p$ et que v n'est pas totalement décomposée dans F_∞, $H^1(F_{\infty,v}/F_v,\mathcal{U}_\rho^{G_{F_{\infty,v}}}/(V\otimes\rho)^{G_{F_{\infty,v}}})$ est dual du sous-groupe de torsion de $H^1(I_v,\check{T}\otimes\rho^{-1})^{G_v/I_v}$ et a comme cardinal le nombre de Tamagawa local $\operatorname{Tam}_v(\check{T}\otimes\rho^{-1})$. Ainsi, il vaut 0 si V a bonne réduction en v. Lorsqu'on remplace F par F_n, $\operatorname{Tam}_{F_n,v}(\check{T}\otimes\rho^{-1})$ est borné par rapport à n ([31, 2.2.6]). Pour $v \mid p$, l'hypothèse ($\operatorname{Hyp}(F_\infty,V,\rho)$) implique que $H^1(F_{\infty,v}/F_{n,v},\mathcal{U}_\rho(T/\operatorname{Fil}^1_v T)^{G_{F_{\infty,v}}})$ est fini et d'ordre borné par rapport à n.

2.5 Théorème de contrôle : cas d'une $\mathbb{Z}_p^2$-extension

On suppose maintenant que F_∞ est une $\mathbb{Z}_p^2$-extension de F. On a les théorèmes de contrôle suivants relatifs à la descente de F_∞ à une sous-$\mathbb{Z}_p$-extension L_∞ (on note alors $\Lambda_{L_\infty} = \mathbb{Z}_p[[\operatorname{Gal}(L_\infty/F)]]$). Notons $\Xi_{F_\infty/L_\infty}(T\otimes\rho)$ l'application

$$H^1(F_\infty/L_\infty,\mathcal{U}_\rho^{G_{F_\infty}}) \to \prod_{v\in S(L_\infty)} H^1(F_{\infty,v}/L_{\infty,v},\mathfrak{Z}_\rho(F_{\infty,v})) .$$

Seules les places totalement décomposées dans L_∞ et les places divisant p interviennent en fait. En effet, dans le cas contraire, elles sont nécessairement totalement décomposées dans F_∞/L_∞. Notons enfin $\check{H}^1_*(L_\infty,\check{T}\otimes\rho^{-1})$ la limite projective des $H^1_*(L_n,\check{T}\otimes\rho^{-1})$ pour les applications de corestriction.

2.5.1 Proposition (théorème de contrôle, cas d'une $\mathbb{Z}_p^2$-extension). *Soit L_∞ une sous-$\mathbb{Z}_p$-extension de F_∞. L'application de restriction induit par dualité un homomorphisme*

$$r_{F_\infty/L_\infty} : X_{\infty,*}(F_\infty,\check{T}\otimes\rho^{-1})_{\operatorname{Gal}(F_\infty/L_\infty)} \to X_{\infty,*}(L_\infty,\check{T}\otimes\rho^{-1})$$

qui se trouve dans une suite exacte naturelle de Λ_{L_∞}-modules

$$\begin{aligned}\check{H}^1_*(L_\infty,\check{T}\otimes\rho^{-1}) &\to \operatorname{coker}\widehat{\Xi_{F_\infty/L_\infty}}(T\otimes\rho) \to X_{\infty,*}(F_\infty,\check{T}\otimes\rho^{-1})_{\operatorname{Gal}(F_\infty/L_\infty)}\\ &\to X_{\infty,*}(L_\infty,\check{T}\otimes\rho^{-1}) \to \ker\widehat{\Xi_{F_\infty/L_\infty}}(T\otimes\rho) \to 0 .\end{aligned}$$

Sous l'hypothèse $(\mathrm{Hyp}(L_\infty, V, \rho))$*, le noyau de* r_{F_∞/L_∞} *est fini et son conoyau est annulé par une puissance de* p*. Lorsqu'il y a un nombre fini de places au dessus de* S *dans* L_∞ *et sous* $(\mathrm{Hyp}(F_\infty, V, \rho))$*, les noyaux et conoyaux des* r_{F_∞/F_nL_∞} *sont finis et d'ordre borné par rapport à* n*.*

Démonstration. La suite exacte se démontre en utilisant le diagramme exact et commutatif suivant avec $\Gamma' = \mathrm{Gal}(F_\infty/L_\infty)$,

$$\begin{array}{ccccccc}
 & & & & 0 & & 0\\
 & & & & \uparrow & & \uparrow\\
0 & \to & H^1_*(F_\infty,\mathcal{U}_\rho)^{\Gamma'} & \to & H^1(G_{S,F_\infty},\mathcal{U}_\rho)^{\Gamma'} & \to & \left(\prod_{w\in S(F_\infty)} H^1_{/*}(F_{\infty,w},\mathcal{U}_\rho)\right)^{\Gamma'}\\
 & & \uparrow & & & & \uparrow\\
0 & \to & H^1_*(L_\infty,\mathcal{U}_\rho) & \to & H^1(G_{S,L_\infty},\mathcal{U}_\rho) & \to & \prod_{v\in S(L_\infty)} H^1_{/*}(L_{\infty,v},\mathcal{U}_\rho)\\
 & & & & \uparrow & & \uparrow\\
 & & & & H^1(\Gamma',\mathcal{U}_\rho^{G_{F_\infty}}) & \to & \prod_{v\in S(L_\infty)} H^1(F_{\infty,v}/L_{\infty,v},\mathfrak{Z}_\rho(F_{\infty,v}))\\
 & & & & \uparrow & & \uparrow\\
 & & & & 0 & & 0
\end{array}$$

En effet, soit w une place de F_∞ ne divisant pas p et v sa restriction à L_∞. Si w n'est pas totalement décomposée dans F_∞, on a $H^1_{/*}(F_{\infty,w},\mathcal{U}_\rho) = H^1(F_{\infty,w},\mathcal{U}_\rho)$ car $H^1_*(F_{\infty,w},\mathcal{U}_\rho) = 0$. Si v n'est pas totalement décomposée non plus dans L_∞, on a alors aussi $H^1_{/*}(L_{\infty,w},\mathcal{U}_\rho) = H^1(L_{\infty,w},\mathcal{U}_\rho)$ et l'application d'inflation est un isomorphisme. Si v est totalement décomposée dans L_∞, le noyau de $H^1_{/*}(L_{\infty,w},\mathcal{U}_\rho) \to \prod_{w|v} H^1_{/*}(F_{\infty,w},\mathcal{U}_\rho)$ est égal au quotient

$$H^1(F_{\infty,v}/L_{\infty,v},\mathcal{U}_\rho)/\mathbb{Q}_p/\mathbb{Z}_p \otimes H^1(F_{\infty,v}/L_{\infty,v},(T\otimes\rho)^{G_{F_{\infty,v}}})$$

qui est isomorphe à $H^1(F_{\infty,v}/L_{\infty,v},\mathfrak{Z}_\rho(F_{\infty,v}))$. Si w est totalement décomposée dans F_∞, l'assertion est triviale. Si w est une place divisant p, le noyau de l'application inflation est isomorphe à $H^1(F_{\infty,v}/L_{\infty,v},\mathfrak{Z}_\rho(F_{\infty,v}))$. Cela démontre les assertions sur le diagramme précédent. Par une des variantes de la suite exacte de Poitou-Tate, le conoyau de $H^1(G_{S,L_\infty},\mathcal{U}_\rho) \to \prod_{v\in S(L_\infty)} H^1_{/*}(L_{\infty,v},\mathcal{U}_\rho)$ est contenu dans le dual de Pontryagin de $\check{H}^1_*(L_\infty,\check{T}\otimes\rho^{-1})$. On en déduit la proposition. □

Un cas particulier est le cas où ρ est le caractère trivial.

2.5.2 Corollaire. *Soit* L_∞ *une sous-*$\mathbb{Z}_p$*-extension de* F_∞*. Il existe une suite exacte naturelle de* Λ_{L_∞}*-modules*

$$\begin{aligned}\check{H}^1_f(L_\infty,\check{T}) &\to \operatorname{coker}\widehat{\Xi_{F_\infty/L_\infty}}(T)\\ &\to X_{\infty,f}(F_\infty,\check{T})_{\mathrm{Gal}(F_\infty/L_\infty)} \to X_{\infty,f}(L_\infty,\check{T}) \to \ker\widehat{\Xi_{F_\infty/L_\infty}}(T) \to 0.\end{aligned}$$

2.5.3 Corollaire. *Soit* L_∞ *une sous-*$\mathbb{Z}_p$*-extension de* F_∞*. Si*

$$X_{\infty,*}(L_\infty,\check{T}\otimes\rho^{-1})$$

est un Λ_{L_∞}*-module de torsion, alors* $X_{\infty,*}(F_\infty,\check{T}\otimes\rho^{-1})$ *est un* Λ*-module de torsion.*

2.6 Construction de l'adjoint

Supposons que F_∞ est une $\mathbb{Z}_p$-extension de F.

Définition. Nous dirons que ρ est admissible (pour V et F_∞) si pour tout entier n, les $X_{\infty,*}(F_\infty,\check{T}\otimes\rho^{-1})_{\Gamma_n}$ sont finis.

Si ρ est admissible, nécessairement $X_{\infty,*}(F_\infty,\check{T}\otimes\rho^{-1})$ est de Λ-torsion. Il existe un caractère de $\mathrm{Gal}(F_\infty/F)$ dans $\mathbb{Z}_p^*$ admissible pour V si et seulement si

$$X_f(F_\infty,\check{T}) \text{ est un } \Lambda\text{-module de torsion} \qquad (\mathrm{Tors}(F_\infty,V))$$

et il en existe alors une infinité. En effet, fixons un caractère non trivial de $\mathrm{Gal}(F_\infty/F)$ dans $\mathbb{Z}_p^*$; si M est un Λ-module de type fini et de torsion, pour tout entier k sauf un nombre fini, $(M\otimes\rho^k)_{\Gamma_n}$ est de torsion pour tout entier n. En effet, si H est une série caractéristique de M (en particulier H annule M), $(M\otimes\rho^k)_{\Gamma_n}$ est fini si et seulement si $H(u^k\zeta_n-1)$ est non nul pour ζ_n une racine de l'unité d'ordre p^n et $u=\rho(\gamma)$. Comme H n'a qu'un nombre fini de zéros par le théorème de préparation de Weierstrass, le fait précédent s'en déduit. Les applications

$$H^1_*(F_n,\mathcal{U}_\rho)\to X_*(\widehat{F_\infty,\check{T}}\otimes\rho^{-1})^{\Gamma_n}$$

induisent par passage à la limite projective pour les applications de corestriction un Λ-homomorphisme

$$\mathcal{A}^{(\rho)}_{F_\infty}:\varprojlim_n H^1_*(F_n,\mathcal{U}_\rho)\to a^1_\Lambda(\dot{X}_{\infty,*}(F_\infty,\check{T}\otimes\rho^{-1}))$$

(proposition 1.3.1). Soit

$$\xi_{F_\infty}(T\otimes\rho):\mathcal{U}_\rho^{G_{F_\infty}}\to\prod_{v\in S^{nd}(F_\infty)}\mathfrak{Z}_\rho(F_{\infty,v})$$

où $S^{nd}(F_\infty)$ désigne l'ensemble des places de $S(F_\infty)$ non totalement décomposées sur F.

2.6.1 Proposition. *Soit F_∞/F une $\mathbb{Z}_p$-extension telle que* $(\mathrm{Tors}(F_\infty,V))$ *soit vérifiée et soit ρ admissible pour F_∞. On a la suite exacte naturelle*

$$0\to\ker\xi_{F_\infty}(T\otimes\rho)\to\varprojlim_n H^1_*(F_n,\mathcal{U}_\rho)\xrightarrow{\mathcal{A}^{(\rho)}_{F_\infty}} a^1_\Lambda(\dot{X}_{\infty,*}(F_\infty,\check{T}\otimes\rho^{-1}))$$
$$\to\operatorname{coker}\xi_{F_\infty}(T\otimes\rho)\to H^1_*(\widehat{F_\infty,\check{T}}\otimes\rho^{-1}).$$

2.6.2 Corollaire. *Sous les hypothèses de la proposition, si de plus* $(\mathrm{Hyp}(F_\infty,V,\rho))$ *est vérifié, $\mathcal{A}^{(\rho)}_{F_\infty}$ est un quasi-isomorphisme.*

Le corollaire se déduit de la finitude du noyau et du conoyau de $\xi_{F_\infty}(T\otimes\rho)$ (il n'y a qu'un nombre fini de places dans $S^{nd}(F_\infty)$).

Démonstration de la proposition. La proposition se déduit de la proposition 1.3.1 et de la proposition 2.4.1 (remarquons que l'application de corestriction devient dans l'isomorphisme $H^1(\Gamma,S)\cong S_\Gamma$ l'application induite par l'identité sur S). □

2.6.3 Remarques. Supposons de plus que ρ^{-1} est admissible pour $\check{T}$ et pour F_∞. Alors, $H^1_*(F_n,\check{T}\otimes\rho^{-1})$ est fini pour tout entier n et est égal au sous-groupe de $\mathbb{Z}_p$-torsion de $H^1_*(F_n,\check{T}\otimes\rho^{-1})$, c'est-à-dire à $\check{\mathcal{U}}_\rho^{G_{F_n}}$. Donc, sous cette hypothèse,

$$H^1_*(F_\infty,\check{T}\otimes\rho^{-1})=\check{\mathcal{U}}_\rho^{G_{F_\infty}}=(\check{V}\otimes\rho^{-1}/\check{T}\otimes\rho^{-1})^{G_{F_\infty}}\ .$$

Prenons maintenant pour F_∞/F une $\mathbb{Z}_p^2$-extension vérifiant $(\mathrm{Hyp}(F_\infty,V,\rho))$. Soit F_n le corps fixe par $\Gamma_n=\Gamma^{p^n}$.
Si L_∞ est une sous-$\mathbb{Z}_p$-extension de F_∞/F, on note $L_{\infty,n}=L_\infty F_n$. Ainsi, $L_{\infty,n+1}/L_{\infty,n}$ est une extension d'ordre p (pour n assez grand). D'autre part, fixons une $\mathbb{Z}_p$-extension L'_∞ de F telle que $F_\infty=L_\infty L'_\infty$. On pose $\Lambda_{L_{\infty,n}}=\mathbb{Z}_p[[\mathrm{Gal}(L_{\infty,n}/L'_n)]]$. Si M est un $\mathbb{Z}_p[[\mathrm{Gal}(L_{\infty,n}/F)]]$-module, $a^1_{\mathbb{Z}_p[[\mathrm{Gal}(L_{\infty,n}/F)]]}(M)$ et $a^1_{\Lambda_{L_{\infty,n}}}(M)$ muni de sa structure naturelle de $\mathbb{Z}_p[[\mathrm{Gal}(L_{\infty,n}/F)]]$-modules s'identifient canoniquement ([20, Lemme 2.3]). La norme de $L_{\infty,n+1}$ à $L_{\infty,n}$ induit alors des homomorphismes naturels :

$$a^1_{\mathbb{Z}_p[[\mathrm{Gal}(L_{\infty,n+1}/F)]]}(M_{\Gamma_{n+1}})\to a^1_{\mathbb{Z}_p[[\mathrm{Gal}(L_{\infty,n}/F)]]}(M_{\Gamma_n})\ .$$

Choisissons une $\mathbb{Z}_p$-extension L_∞ telle que $X_{\infty,*}(F_\infty,\check{T}\otimes\rho^{-1})_{\mathrm{Gal}(F_\infty/L_{\infty,n})}$ soit de $\Lambda_{L_{\infty,n}}$-torsion pour tout entier n. Par la proposition 2.5.1, cela est équivalent à ce que $X_{\infty,*}(L_{\infty,n},\check{T}\otimes\rho^{-1})$ soit de $\Lambda_{L_{\infty,n}}$-torsion pour tout entier n. On dit alors que ρ est admissible pour F_∞/L_∞.

2.6.4 Proposition. *On suppose vérifiés* $(\mathrm{Hyp}(F_\infty,V,\rho))$*, que*

$$X_{\infty,*}(F_\infty,\check{T}\otimes\rho^{-1})\ \textit{est un}\ \Lambda\textit{-module de torsion}\qquad(\mathrm{Tors}(F_\infty,V,\rho))$$

et que ρ est admissible pour F_∞/L_∞. Les applications naturelles

$$r_n:a^1_{\Lambda_{L_{\infty,n}}}(X_{\infty,*}(L_{\infty,n},\check{T}\otimes\rho^{-1}))\to a^1_{\Lambda_{L_{\infty,n}}}(X_{\infty,*}(F_\infty,\check{T}\otimes\rho^{-1})_{\mathrm{Gal}(F_\infty/L_{\infty,n})})$$

induisent un Λ-homomorphisme r_∞ injectif

$$\varprojlim_n a^1_{\Lambda_{L_{\infty,n}}}(X_{\infty,*}(L_{\infty,n},\check{T}\otimes\rho^{-1}))\to a^1_\Lambda(X_{\infty,*}(F_\infty,\check{T}\otimes\rho^{-1}))$$

et on a la suite quasi-exacte

$$0\to\varprojlim_n a^1_{\Lambda_{L_{\infty,n}}}(X_{\infty,*}(L_{\infty,n},\check{T}\otimes\rho^{-1}))\to a^1_\Lambda(X_{\infty,*}(F_\infty,\check{T}\otimes\rho^{-1}))$$
$$\to\prod_{v\in S^{nd}(F_\infty/L_\infty)}\mathfrak{z}_\rho(F_{\infty,v})$$

Démonstration. Posons $\Lambda_n = \Lambda_{L_{\infty,n}}$, $\Gamma'_n = \mathrm{Gal}(F_\infty/L_{\infty,n})$ et $a^1_n = a^1_{\Lambda_n}$, $M_n = X_{\infty,*}(L_{\infty,n}, \check{T}\otimes\rho^{-1})$, $M = X_{\infty,*}(F_\infty, \check{T}\otimes\rho^{-1})$, $\mathfrak{Z} = \prod_{v\in S(F_\infty)} \mathfrak{Z}_\rho(F_{\infty,v})$, $\Xi_n = \Xi_{F_\infty/L_{\infty,n}}(T\otimes\rho)$. La suite exacte de la proposition 2.5.1 appliquée à $F_\infty/L_{\infty,n}$ devient

$$\widehat{\operatorname{coker}\Xi_n} \to M_{\Gamma'_n} \to M_n \to \widehat{\ker\Xi_n} \to 0$$

et on a la suite exacte tautologique

$$0 \to \widehat{\operatorname{coker}\Xi_n} \to \widehat{H^1(\Gamma'_n, \mathfrak{Z})} \to \widehat{H^1(\Gamma'_n, \mathcal{U}_\rho^{G_{F_\infty}})} \to \widehat{\ker\Xi_n} \to 0\ .$$

Soit M'_n l'image de $M_{\Gamma'_n}$ dans M_n. On a alors les suites exactes

$$0 \to a^1_n(M'_n) \to a^1_n(M_{\Gamma'_n}) \to a^1_n(\widehat{\operatorname{coker}\Xi_n})$$
$$0 \to a^1_n(M_n) \to a^1_n(M'_n) \to a^2_n(\widehat{\ker\Xi_n})\ .$$

On en déduit l'injectivité de $a^1_n(M_n) \to a^1_n(M_{\Gamma'_n})$ et par passage à la limite celle de $\varprojlim_n a^1_n(M_n) \to a^1_\Lambda(M_{\Gamma'_n})$.

D'autre part, on déduit de la suite exacte tautologique la suite exacte

$$0 \to a^1_n(\widehat{H^1(\Gamma'_n, \mathfrak{Z})}) \to a^1_n(\widehat{\operatorname{coker}\Xi_n}) \to R_n \to 0$$

avec R_n d'ordre borné par rapport à n (on utilise le fait que $a^1_n(R) = 0$ si R est un module fini).

Nous allons maintenant raisonner à des modules finis près d'ordre borné par rapport à n (on parle alors de suites quasi-exactes et de quasi-isomorphismes contrôlés): on a la suite quasi-exacte contrôlée:

$$0 \to a^1_n(M_n) \to a^1_n(M_{\Gamma'_n}) \to a^1_n(\widehat{\operatorname{coker}\Xi_n})$$

et le quasi-isomorphisme contrôlé

$$a^1_n(\widehat{H^1(\Gamma'_n, Z)}) \cong a^1_n(\widehat{\operatorname{coker}\Xi_n})$$

Comme $\mathfrak{Z}$ est annulé par une puissance de p, $a^1_n(\widehat{H^1(\Gamma'_n, \mathfrak{Z})}) \cong H^1(\Gamma'_n, \mathfrak{Z})$ et la limite projective des $a^1_n(\widehat{\operatorname{coker}\Xi_n})$ est quasi-isomorphe à $\mathfrak{Z}$. La proposition se déduit alors de la proposition 1.3.1 □

2.6.5 REMARQUES. On peut être plus précis sous une hypothèse dont on montrera plus tard qu'elle est vraie. Supposons que le plus grand sous-Λ_n-module fini de $M_n = X_{\infty,*}(L_{\infty,n}, \check{T}\otimes\rho^{-1})$ est d'ordre borné par rapport à n. Alors la dernière flèche est quasi-surjective. En effet, comme M'_n est contenu dans M_n, $a^2_n(M'_n)$ est fini d'ordre borné par rapport à n. On a donc la suite quasi-exacte

$$\begin{aligned} 0 \to \varprojlim_n a^1_{\Lambda_{L_{\infty,n}}}(X_{\infty,*}(L_{\infty,n}, \check{T}\otimes\rho^{-1})) \to & a^1_\Lambda(X_{\infty,*}(F_\infty, \check{T}\otimes\rho^{-1})) \\ & \to \prod_{v\in S^{nd}(F_\infty/L_\infty)} \mathfrak{Z}_\rho(F_{\infty,v}) \to 0 \end{aligned} \tag{2}$$

3 Construction d'accouplements entre modules de Selmer

3.1 Accouplements de Cassels-Tate

3.1.1 Théorème (Flach). *Il existe un homomorphisme naturel*

$$\mathrm{Cassels}_F(T \otimes \rho) : H^1_*(F,\check{\mathcal{U}}_\rho) \times H^1_*(F,\mathcal{U}_\rho) \to \mathbb{Q}_p/\mathbb{Z}_p$$

qui induit un isomorphisme

$$C_F(T \otimes \rho) : H^1_*(\widehat{F,\check{\mathcal{U}}_\rho)}/\,\mathrm{div} \to H^1_*(F,\mathcal{U}_\rho)/\,\mathrm{div}$$

où $M/\,\mathrm{div}$ *désigne le quotient d'un* $\mathbb{Z}_p$*-module* M *par sa partie divisible. En particulier, si* $H^1_*(F,\check{\mathcal{U}}_\rho)$ *et* $H^1_*(F,\mathcal{U}_\rho)$ *sont finis, on en déduit un isomorphisme*

$$C_F(T \otimes \rho) : \widehat{H^1_*(F,\check{\mathcal{U}}_\rho)} \to H^1_*(F,\mathcal{U}_\rho)\ .$$

On a les propriétés suivantes :

1. *Si L/F est une extension finie,*

$$\mathrm{Cassels}_F(T \otimes \rho)(x, \mathrm{cores}_{L/F}\, y) = \mathrm{Cassels}_L(\mathrm{res}_{L/F}\, x, y)$$

2. *Si F/F_1 est une extension galoisienne et $\sigma \in \mathrm{Gal}(F/F_1)$,*

$$\mathrm{Cassels}_F(T \otimes \rho)(\sigma x, \sigma y) = \mathrm{Cassels}_F(T \otimes \rho)(x,y)\ .$$

3. *Soit L un corps contenant le corps fixe par le noyau de ρ^{p^n}. Soit $x \in H^1_*(L,\check{\mathcal{U}}_{p^n})$ et $y \in H^1_*(L,\mathcal{U}_{p^n})$. Alors, si $Tw_\rho(x)$ (resp. $Tw_{\rho^{-1}}(x)$ désigne le ρ-ième twist de x (resp. le ρ^{-1}-ième twist de y), on a*

$$\mathrm{Cassels}_L(T \otimes \rho)(Tw_\rho(x), Tw_{\rho^{-1}}(x)) = \mathrm{Cassels}_L(T)(x,y)$$

4. *Le dual de $C_F(T \otimes \rho)$ par la dualité de Pontryagin est $C_F(\check{T}\otimes\rho^{-1})$.*

3.1.2 Remarques. 1) La partie divisible de $H^1_*(F,\mathcal{U}_\rho)$ est $\mathbb{Q}_p/\mathbb{Z}_p \otimes H^1_*(F, T \otimes \rho)$.
2) Pour ρ le caractère trivial, $V = V_p(E)$ et en utilisant l'accouplement de Weil pour identifier $V \to \check{V} = V^*(1)$, l'accouplement obtenu est l'accouplement de Cassels. L'accouplement de Weil étant alterné, l'accouplement de Cassels est une forme bilinéaire alternée (on utilise pour cela la propriété 4). C'est ce qui permet de montrer que l'ordre du quotient du groupe de Tate-Shafarevich par sa partie divisible est un carré.

La démonstration du théorème 3.1.1 est faite dans [11], les deux sous-espaces de $V \otimes \rho$ et $\check{V}\otimes\rho^{-1}$ que sont $\mathrm{Fil}^1_v V\otimes\rho$ et $\mathrm{Fil}^1_v \check{V}\otimes\rho^{-1}$ sont orthogonaux dans la dualité naturelle $V \otimes \rho \times \check{V}\otimes\rho^{-1} \to \mathbb{Q}_p(1)$ (voir aussi [13, §5.4]) Pour le comportement par twist, il suffit de reprendre la définition en remarquant que pour $\tau \in G_L$, $\rho(\tau) \equiv 1 \bmod p^n$. Les différentes cochaines construites diffèrent alors d'éléments de T et finalement l'image est la même dans $\frac{1}{p^n}\mathbb{Z}/\mathbb{Z}$.

3.2 Dualité : cas d'une $\mathbb{Z}_p$-extension

Soit F_∞/F une $\mathbb{Z}_p$-extension. On suppose toujours vérifiées les hypothèses $(\mathrm{Hyp}(F_\infty, V))$ et $(\mathrm{Hyp}(F_\infty, \check{V}))$.

3.2.1 Théorème. *Soit ρ un caractère continu de G_F à valeurs dans $\mathbb{Z}_p^*$ tel que $(\mathrm{Tors}(F_\infty, V, \rho))$ soit vérifiée et tel que ρ soit admissible pour F_∞ et V. Les applications $C_{F_n}(T\otimes\rho)$ induisent un Λ-homomorphisme quasi-injectif*

$$C_{F_\infty}(T\otimes\rho) : X_{\infty,*}(F_\infty, T\otimes\rho) \to a^1_\Lambda(\dot{X}_{\infty,*}(F_\infty, \check{T}\otimes\rho^{-1}))$$

et on a plus précisément la suite exacte

$$\begin{aligned}0 \to \ker \xi_{F_\infty}(T\otimes\rho) \to X_{\infty,*}(F_\infty, T\otimes\rho) &\to a^1_\Lambda(\dot{X}_{\infty,*}(F_\infty, \check{T}\otimes\rho^{-1}))\\ &\to \operatorname{coker} \xi_{F_\infty}(T\otimes\rho)\end{aligned}$$

où

$$\xi_{F_\infty}(T\otimes\rho) : \mathcal{U}_\rho^{G_{F_\infty}} \to \prod_{v\in S^{nd}(F_\infty)} \mathfrak{Z}_\rho(F_{\infty,v}) \ .$$

Si de plus ρ^{-1} est admissible pour V et F_∞, on a la suite exacte

$$\begin{aligned}0 \to \ker \xi_{F_\infty}(T\otimes\rho) \to X_{\infty,*}(F_\infty, T\otimes\rho) \to a^1_\Lambda(\dot{X}_{\infty,*}(F_\infty, \check{T}\otimes\rho^{-1}))&\\ \to \operatorname{coker} \xi_{F_\infty}(T\otimes\rho) \to \widehat{\check{\mathcal{U}}_\rho^{G_{F_\infty}}}&\ .\end{aligned}$$

En particulier, $X_{\infty,*}(F_\infty, T\otimes\rho)$ est lui aussi de Λ-torsion.

Démonstration. Par passage à la limite projective des isomorphismes

$$C_{F_n}(T\otimes\rho) : \widehat{H^1_*(F_n, \check{\mathcal{U}}_\rho)} \to H^1_*(F_n, \mathcal{U}_\rho) \ ,$$

on obtient un homomorphisme de Λ-modules

$$X_{\infty,*}(F_\infty, T\otimes\rho) \to \varprojlim_n H^1_*(F_n, \mathcal{U}_\rho) \to a^1_\Lambda(\dot{X}_{\infty,*}(F_\infty, \check{T}\otimes\rho^{-1})) \ .$$

La première flèche est bijective. Les noyau et conoyau de la seconde sont décrits en 2.6.1 ainsi que dans la remarque qui le suit. □

3.2.2 Corollaire. *On suppose vérifiée* $(\mathrm{Tors}(F_\infty, V))$. *Les applications $C_{F_\infty}(T\otimes\rho)$, pour un caractère continu admissible ρ de* $\mathrm{Gal}(F_\infty/F)$ *à valeurs dans $\mathbb{Z}_p^*$, induisent par twist par ρ^{-1} un quasi-isomorphisme indépendant de ρ*

$$X_{\infty,f}(F_\infty, T) \xrightarrow{\sim} a^1_\Lambda(\dot{X}_{\infty,f}(F_\infty, \check{T}))$$

et on a plus précisément la suite exacte

$$0\to\ker \xi_{F_\infty}(T)\to X_{\infty,f}(F_\infty, T)\to a^1_\Lambda(\dot{X}_{\infty,f}(F_\infty, \check{T}))\to\operatorname{coker} \xi_{F_\infty}(T)\to\widehat{\check{\mathcal{U}}^{G_{F_\infty}}} \quad (3)$$

où

$$\xi_{F_\infty}(T) : \mathcal{U}^{G_{F_\infty}} \to \prod_{v \in S^{nd}(F_\infty)} \mathfrak{Z}_\rho(F_{\infty,v})$$

En particulier, $X_{\infty,f}(F_\infty, T)$ est lui aussi de Λ-torsion.

Démonstration. Il existe sous ces hypothèses un caractère ρ de $\mathrm{Gal}(F_\infty/F)$ admissible pour T tel que ρ^{-1} soit admissible pour $\check{T}$. L'indépendance par rapport à ρ se déduit de 3.1.1 et du fait que le calcul de l'adjoint d'un module M peut se faire en utilisant uniquement les quotient $\widehat{M^{\Gamma_n}}/p^n\widehat{M^{\Gamma_n}}$. □

3.2.3 Corollaire (Greenberg). *Supposons vérifiée* (Tors(F_∞, V)). *Le plus grand sous-Λ-module fini de $X_{\infty,f}(F_\infty, T)$ est égal au noyau de $\xi_{F_\infty}(T)$. Si* $\ker \xi_{F_\infty}(T)$ *est nul, $X_{\infty,f}(F_\infty, T)$ n'a pas de sous-modules finis non nuls et est de dimension projective inférieure ou égale à* 1.

Ainsi, si $\mathcal{U}^{G_F}$ est nul, il en est de même de $\mathcal{U}^{G_{F_\infty}}$ (son dual de Pontryagin est alors un Λ-module de type fini de coinvariant nul, il est donc nul) et $X_{\infty,f}(F_\infty, T)$ n'a pas de sous-modules finis non nuls et est de dimension projective inférieure ou égale à 1. S'il existe une place $v \nmid p$ de S telle que $V^{G_{F_{\infty,v}}} = 0$, $\xi_{F_\infty}(T)$ est injective et $X_{\infty,f}(F_\infty, T)$ est de dimension projective inférieure ou égale à 1. Si V est la représentation p-adique associée à une courbe elliptique, cela est le cas s'il existe une place $v \nmid p$ où E a mauvaise réduction additive. On retrouve le résultat démontré par Greenberg ([14, Proposition 4.15]).
Il est commode de travailler avec l'application Λ-sesquilinéaire qui se déduit de $\mathcal{C}_{F_\infty}(T \otimes \rho)$:

$$\mathrm{Cassels}_{F_\infty}(T \otimes \rho) : X_{\infty,*}(F_\infty, T \otimes \rho) \times X_{\infty,*}(F_\infty, \check{T} \otimes \rho^{-1})) \to \mathrm{Frac}\,\Lambda/\Lambda$$

ou

$$\mathrm{Cassels}_{F_\infty}(T) : X_{\infty,f}(F_\infty, T) \times X_{\infty,f}(F_\infty, \check{T})) \to \mathrm{Frac}\,\Lambda/\Lambda\ .$$

On a donc pour l'une ou l'autre

$$\mathrm{Cassels}(\lambda x, y) = \mathrm{Cassels}(x, \dot{\lambda} y) = \lambda\,\mathrm{Cassels}(x, y)\ .$$

La proposition suivante est fondamentale :

3.2.4 Proposition. *On a*

$$\mathrm{Cassels}_{F_\infty}(T)(x, y) = \mathrm{Cassels}_{F_\infty}(\check{T})(y, x)^{\cdot}\ .$$

Démonstration. Il suffit de démontrer l'égalité

$$\mathrm{Cassels}_{F_\infty}(T \otimes \rho)(x, y) = \mathrm{Cassels}_{F_\infty}(\check{T} \otimes \rho^{-1})(y, x)^{\cdot}$$

pour ρ un caractère de $\mathrm{Gal}(F_\infty/F)$ admissible. Posons $M = X_*(F_\infty, T\otimes\rho)$ et $M' = \dot{X}_*(F_\infty, \check{T}\otimes\rho^{-1})$. L'application $C_{F_\infty}(T\otimes\rho)$ est définie par passage à la limite des Λ-homomorphismes

$$\widehat{H^1_*(F_n, \check{\mathcal{U}}_\rho)} \to H^1_*(F_n, \mathcal{U}_\rho)$$

et on a le diagramme commutatif

$$\begin{array}{ccc} \widehat{H^1_*(F_n,\check{\mathcal{U}}_\rho)} & \overset{C_{F_n}(T\otimes\rho)}{\to} & H^1_*(F_n,\mathcal{U}_\rho) \\ \uparrow & & \downarrow \\ M_{\Gamma_n} & \to & \widehat{\dot{M}'}^{\Gamma_n} \\ || & & \uparrow \\ M_{\Gamma_n} & \to & a^1_\Lambda(M')_{\Gamma_n} \end{array}$$

En prenant le dual de Pontryagin de ce diagramme, on obtient le diagramme commutatif

$$\begin{array}{ccc} \widehat{H^1_*(F_n,\check{\mathcal{U}}_\rho)} & \overset{\dot{C}_{F_n}(\check{T}\otimes\rho^{-1})}{\to} & H^1_*(F_n,\mathcal{U}_\rho) \\ \uparrow & & \downarrow \\ M'_{\Gamma_n} & \to & \widehat{\dot{M}}^{\Gamma_n} \\ \downarrow & & || \\ \widehat{a^1_\Lambda(\dot{M}')_{\Gamma_n}} & \to & \widehat{M_{\Gamma_n}} \end{array}$$

On passe ensuite à la limite projective. Les applications $M'_{\Gamma_n} \to \widehat{a_\Lambda(\dot{M}')_{\Gamma_n}}$ induisent alors l'homomorphisme naturel $M' \to a_\Lambda(a_\Lambda(M'))$ et les $C_{F_n}(\check{T}\otimes\rho^{-1})$ induisent l'application $C_{F_\infty}(\check{T}\otimes\rho^{-1})$. □

3.3 Dualité : cas d'une $\mathbb{Z}_p^2$-extension

Soit F_∞/F une $\mathbb{Z}_p^2$-extension. On suppose $(\mathrm{Hyp}(F_\infty, V))$ et $(\mathrm{Hyp}(F_\infty, \check{V}))$. Soit L_∞ une sous-$\mathbb{Z}_p$-extension de F_∞/F.

Définition. Disons que L_∞ est admissible si $X_{\infty,f}(L_{\infty,n}, \check{T})$ est un $\Lambda_{L_\infty,n}$-module de torsion pour tout entier n.

Une telle $\mathbb{Z}_p$-extension existe lorsque $(\mathrm{Tors}(F_\infty, V))$ est vérifiée. En effet, cela revient à montrer que si F est un élément de $\mathbb{Z}_p[[T_1, T_2]]$ (en l'occurrence la série caractéristique de $X_{\infty,f}(F_\infty\check{T})$), il existe un entier b tel que $F(\zeta_n(1+T_2)^b - 1, T_2) \neq 0$ pour tout entier n. Dans le cas contraire, $F(T_1, T_2)$ serait divisible par $1+T_1-\zeta_{n(b)}(1+T_2)^b$ pour tout entier b avec $n(b)$ entier dépendant de b. Ces éléments étant premiers entre eux, cela impliquerait que $F(T_1, T_2)$ est identiquement nul.
On peut alors appliquer le corollaire 3.2.2 à $L_{\infty,n}/L'_n$: il existe une famille de $\Lambda_{L_\infty,n}$-homomorphismes

$$\mathcal{C}_{L_{\infty,n}}(T) : X_{\infty,f}(L_{\infty,n}, T) \to a^1_{\Lambda_{L_\infty,n}}(\dot{X}_{\infty,f}(L_{\infty,n}, \check{T}))$$

puis définir par passage à la limite projective et en composant avec r_∞ (2.6.4) un homomorphisme de Λ-modules

$$X_{\infty,f}(F_\infty,T)\to a^1_\Lambda(\dot X_{\infty,f}(F_\infty,\check T))$$

3.3.1 Proposition. *On suppose vérifiés* $(\mathrm{Hyp}(F_\infty,T))$, $(\mathrm{Hyp}(F_\infty,\check V))$ *et* $(\mathrm{Tors}(F_\infty,V))$. *Le Λ-homomorphisme $\mathcal{C}_{F_\infty}(T)$ se trouve dans une suite quasi-exacte*

$$0\to X_{\infty,f}(F_\infty,T)\to a^1_\Lambda(\dot X_{\infty,f}(F_\infty,\check T))\to \prod_{w\in S^{nd}(F_\infty)}\mathfrak{Z}(F_{\infty,w})\to 0$$

Le noyau de $X_{\infty,f}(F_\infty,T)\to a^1_\Lambda(\dot X_{\infty,f}(F_\infty,\check T))$ est contenu dans $\widehat{\check{\mathcal{U}}^{G_{F_\infty}}}$.

Démonstration. Considérons les suites exactes (3) relatives à la $\mathbb{Z}_p$-extension $L_{\infty,n}/L'_n$ et passons à la limite projective. Les $\ker\xi_{L_{\infty,n}}(T)$ sont finis et d'ordre borné et leur limite projective est finie. La limite projective des $X_{\infty,f}(L_{\infty,n},\check T)$ est $X_{\infty,f}(F_\infty,\check T)$. La limite projective des $a^1_{\Lambda_{L_{\infty,n}}}(\dot X_{\infty,f}(L_{\infty,n},T))$ est étudiée dans la proposition 2.6.4 (on peut utiliser la remarque qui suit car le plus grand module fini de $X_{\infty,f}(L_{\infty,n},\check T)$ est $\ker\xi_{L_{\infty,n}}$ qui est d'ordre borné par rapport à n) : on a la suite quasi-exacte

$$\begin{aligned}0\to \varprojlim_n a^1_{\Lambda_{L_{\infty,n}}}(X_{\infty,*}(L_{\infty,n},\check T)) \to & a^1_\Lambda(X_{\infty,f}(F_\infty,\check T))\\ &\to \prod_{v\in S^{nd}(F_\infty/L_\infty)}\mathfrak{Z}(F_{\infty,v})\to 0\end{aligned}$$

où $S^{nd}(F_\infty/L_\infty)$ désigne les places de F_∞ non totalement décomposées dans F_∞/L_∞. Comme $\mathcal{U}^{G_{F_\infty}}$ est supposé fini, la limite projective W du quatrième terme de la suite exacte est quasi-isomorphe à la limite projective des $\prod_{v\in S^{nd}(L_{\infty,n})}\mathfrak{Z}(L_{\infty,n,v})$. Soit $v\in S^{nd}(L_{\infty,n})$ ne divisant pas p. Elle n'est pas totalement décomposée dans $L_{\infty,n}$, elle est donc totalement décomposée dans F_∞/L_∞. Comme d'autre part $\mathfrak{Z}(F_{\infty,w})$ est fini, les groupes $\mathfrak{Z}(L_{\infty,n,w})$ sont stationnaires pour $n\gg 0$ et l'application de corestriction de $\prod_{w\in S(L_{\infty,n}),w|v}\mathfrak{Z}(L_{\infty,n,w})$ est surjective. La limite projective est $\prod_{w\in S(F_\infty),w|v}\mathfrak{Z}(F_{\infty,w})$. On en déduit que W est quasi-isomorphe à $\prod_{w\in S(F_\infty),w|v\in S^{nd}(L_\infty)}\mathfrak{Z}(L_{\infty,n,w})$. Enfin, la limite projective des $\widehat{\check{\mathcal{U}}^{G_{L_{\infty,n}}}}$ est finie. □

3.3.2 Corollaire. *On suppose* $(\mathrm{Hyp}(F_\infty,V))$, $(\mathrm{Hyp}(F_\infty,\check V))$ *et* $(\mathrm{Tors}(F_\infty,V))$. *Le Λ-module $X_{\infty,f}(F_\infty,T)$ vérifie la propriété (A) : il n'a pas de sous-modules pseudo-nuls non finis et les $a^i_\Lambda(X_{\infty,f}(F_\infty,T))$ sont finis pour $i\geq 2$.*

4 Conséquences

Nous pouvons maintenant appliquer les résultats de §1.

4.1 DESCENTE : $\mathbb{Z}_p$-EXTENSION

Soit F_∞/F une $\mathbb{Z}_p$-extension. On suppose $(\mathrm{Hyp}(F_\infty,V))$, $(\mathrm{Hyp}(F_\infty,\check{V}))$ et $(\mathrm{Tors}(F_\infty,V))$. Réécrivons le diagramme (1) pour $M=X_{\infty,f}(F_\infty,T)$, $M'=\dot{X}_{\infty,f}(F_\infty,\check{T})$ et pour le Λ-homomorphisme

$$X_{\infty,f}(F_\infty,T)\to a^1_\Lambda(\dot{X}_{\infty,f}(F_\infty,\check{T}))$$

qu'on a construit dans les §3.2 et 3.3. Posons $S_p(T)=H^1_f(F,\mathcal{U})$, $S_p(\check{T})=H^1_f(F,\check{\mathcal{U}})$, $\check{S}_p(T)=H^1_f(F,T)$ et $\check{S}_p(\check{T})=H^1_f(F,\check{T})$.

$$\begin{array}{ccccccccc}
 & & & & & & \check{S}_p(\check{T}) & & \\
 & & & & & & \downarrow & & \\
0 & \to & S_p(\check{T})/\operatorname{div} & & & & \mathrm{Hom}_{\mathbb{Z}_p}(M'_\Gamma,\mathbb{Z}_p) & \to & 0\\
 & & \downarrow & & & & \downarrow & & \\
0 & \to & \widehat{t_{\mathbb{Z}_p}(M'_\Gamma)} & \to & a^1_\Lambda(M')_\Gamma & \to & \mathrm{Hom}_{\mathbb{Z}_p}({M'}^\Gamma,\mathbb{Z}_p) & \to & 0\\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & t_{\mathbb{Z}_p}(M_\Gamma) & \to & M_\Gamma & \to & L_{\mathbb{Z}_p}(M_\Gamma) & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & S_p\widehat{(T)/\operatorname{div}} & \to & \widehat{S_p(T)} & \to & \mathrm{Hom}_{\mathbb{Z}_p}(\check{S}(T),\mathbb{Z}_p) & \to & 0
\end{array}$$

On peut démontrer en suivant les flèches qu'on retrouve l'application de Cassels, autrement dit que l'on a le diagramme commutatif

$$C_F(T)\qquad \begin{array}{ccc}
S_p(\check{T})/\operatorname{div} & \to & \widehat{t_{\mathbb{Z}_p}(M'_\Gamma)}\\
\uparrow & & \uparrow\\
S_p\widehat{(T)/\operatorname{div}} & \leftarrow & t_{\mathbb{Z}_p}(M_\Gamma)
\end{array}$$

D'autre part, lorsque $\check{S}(T)$ n'est pas fini, on obtient une forme bilinéaire $\langle\cdot,\cdot\rangle_\gamma$

$$\check{S}(T)\times\check{S}(\check{T})\to\mathbb{Q}_p\ .$$

Elle dépend de γ et bien sûr de la $\mathbb{Z}_p$-extension F_∞. Pour ρ caractère non trivial de Γ à valeurs dans $\mathbb{Z}_p^*$, on note $\langle\cdot,\cdot\rangle_\rho=(\log_p\rho(\gamma))^{-1}\langle\cdot,\cdot\rangle_\gamma$. On peut démontrer que l'on retrouve la hauteur p-adique ordinaire associée à ρ (cf. [28] dans un cadre un peu différent). Nous n'en aurons pas besoin.

4.2 DESCENTE : $\mathbb{Z}_p^2$-EXTENSION

Maintenant qu'a été construit $\mathcal{C}_{F_\infty}(T)$ avec l'aide de coinvariants convenables (c'est-à-dire de torsion pour la $\mathbb{Z}_p$-extension corespondante), il est possible de redescendre en utilisant les homomorphismes fonctoriels construits dans le §1 et plus particulièrement §1.2. Ce qui permet d'obtenir des informations pour les $\mathbb{Z}_p$-extensions telles que $X_{\infty,f}(L_\infty,T)$ n'est pas de torsion.
On fait les hypothèses $(\mathrm{Hyp}(F_\infty,V))$, $(\mathrm{Hyp}(F_\infty,\check{V}))$ et $(\mathrm{Tors}(F_\infty,V))$. Soit $\mathfrak{Z}(T)=\mathfrak{Z}(F_\infty,T)=\prod_{w\in S^{nd}(F_\infty)}\mathfrak{Z}(F_{\infty,w},T)$ (pour le caractère ρ trivial). On

a contruit dans le §3.3 la suite quasi-exacte de Λ-modules suivante

$$0 \to X_{\infty,f}(F_\infty, T) \to a^1_\Lambda(\dot{X}_{\infty,f}(F_\infty, \check{T})) \to \mathfrak{Z}(F_\infty, T) \to 0$$

Soit L_∞ une sous-$\mathbb{Z}_p$-extension de F_∞/F. Posons $\Gamma' = \mathrm{Gal}(F_\infty/L_\infty)$, $\Lambda_{L_\infty} = \mathbb{Z}_p[[\mathrm{Gal}(L_\infty/F)]]$. Dans le cas où $X_{\infty,f}(L_\infty, T)$ est de torsion, on a alors le diagramme commutatif suivant dont les lignes et les colonnes sont quasi-exactes

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \uparrow & & \\
 & & & & \widehat{\mathfrak{Z}(\check{T})_{\Gamma'}} & & a^1_{\Lambda_{L_\infty}}(\widehat{\mathfrak{Z}(T)_{\Gamma'}}) & & \\
 & & & & \downarrow & & \uparrow & & \\
0 & \to & \mathfrak{Z}(T)^{\Gamma'} & \to & X_{\infty,f}(F_\infty, T)_{\Gamma'} & \to & a^1_\Lambda(\dot{X}_{\infty,f}(F_\infty, \check{T})_{\Gamma'}) & \to & \mathfrak{Z}(T)_{\Gamma'} \to 0 \cdot \\
 & & & & \downarrow & & \uparrow & & \\
 & & & & X_{\infty,f}(L_\infty, T) & \to & a^1_\Lambda(\dot{X}_{\infty,f}(L_\infty, \check{T})) & & \\
 & & & & \downarrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

et le quasi-isomorphisme du §3.2

$$X_{\infty,f}(L_\infty, T) \to a^1_{\Lambda_{L_\infty}}(\dot{X}_{\infty,f}(L_\infty, T))$$

Ne supposons plus $X_{\infty,f}(L_\infty, \check{T})$ de Λ_{L_∞}-torsion. On a alors le diagramme commutatif suivant dont les lignes sont quasi-exactes :

$$\begin{array}{ccccc}
a^1_{L_\infty}(t_{\Lambda_{L_\infty}}(\dot{X}_{\infty,f}(F_\infty, \check{T})_{\Gamma'})) & & & & \mathrm{Hom}_{\Lambda_{L_\infty}}(\dot{X}_{\infty,f}(F_\infty, \check{T})_{\Gamma'}, \Lambda_{L_\infty}) \\
\uparrow\sim & & & & \downarrow \\
0 \to a^1_{L_\infty}(\dot{X}_{\infty,f}(F_\infty, \check{T})) & \to & A & \to & \mathrm{Hom}_{\Lambda_{L_\infty}}(\dot{X}_{\infty,f}(F_\infty, \check{T})^{\Gamma'}, \Lambda_{L_\infty}) \to 0 \\
\uparrow & & \uparrow & & \uparrow \\
0 \to t_{\Lambda_{L_\infty}}(X_{\infty,f}(F_\infty, T)_{\Gamma'}) & \to & B & \to & L_{\Lambda_{L_\infty}}(X_{\infty,f}(F_\infty, T)_{\Gamma'}) \to 0
\end{array}$$

$$\text{avec } A := a^1_{L_\infty}(\dot{X}_{\infty,f}(F_\infty, \check{T})_{\Gamma'}), \quad B := X_{\infty,f}(F_\infty, T)_{\Gamma'}.$$

On en déduit comme en §1.2 des homomorphismes

$$t_{\Lambda_{L_\infty}}(X_{\infty,f}(F_\infty, T)_{\Gamma'}) \to \dot{a}_{L_\infty}(X_{\infty,f}(F_\infty, \check{T})_{\Gamma'})$$

et une forme sesqui-linéaire

$$\mathrm{Cassels}_{L_\infty}(T) : t_{\Lambda_{L_\infty}}(X_{\infty,f}(F_\infty, T)_{\Gamma'}) \times t_{\Lambda_{L_\infty}}(X_{\infty,f}(F_\infty, \check{T})_{\Gamma'}) \to \mathrm{Frac}(\Lambda_{L_\infty})/\Lambda_{L_\infty}$$

quasi non dégénérée vérifiant

$$\mathrm{Cassels}_{L_\infty}(T)(\gamma x, y) = \gamma\, \mathrm{Cassels}_{L_\infty}(T)(x, y) = \mathrm{Cassels}_{L_\infty}(T)(x, \gamma^{-1} y)$$
$$\mathrm{Cassels}_{L_\infty}(T)(x, y) = \mathrm{Cassels}_{L_\infty}(\check{T})(y, x) \ .$$

On obtient aussi une hauteur p-adique qui est un accouplement sur les quotients sans Λ_{L_∞}-torsion de $X_{\infty,f}(F_\infty, T)_{\Gamma'}$ et de $X_{\infty,f}(F_\infty, \check{T})_{\Gamma'}$ à valeurs dans Λ_{L_∞}.

Le noyau (resp. conoyau) de $X_{\infty,f}(F_\infty,T)_{\Gamma'} \to a^1_{L_\infty}(\dot{X}_{\infty,f}(F_\infty,\check{T})_{\Gamma'})$ est de torsion et quasi-isomorphe à $\mathfrak{Z}(T)^{\Gamma'}$(resp. $\mathfrak{Z}(T)_{\Gamma'}$) qui est d'ailleurs annulé par une puissance de p. On en déduit une suite exacte

$$0 \to \mathfrak{Z}(T)^{\Gamma'} \to t_{\Lambda_{L_\infty}}(X_{\infty,f}(F_\infty,T)_{\Gamma'}) \to a^1_{L_\infty}(\dot{X}_{\infty,f}(F_\infty,\check{T})_{\Gamma'}) \to Z \to 0$$

avec Z annulé par une puissance de p et de μ-invariant inférieur à celui de $\mathfrak{Z}(T)^{\Gamma'}$. La série caractéristique de $t_{\Lambda_{L_\infty}}(X_{\infty,f}(F_\infty,\check{T})_{\Gamma'})$ divise donc celle de $t_{\Lambda_{L_\infty}}(X_{\infty,f}(F_\infty,T)_{\Gamma'})$. Par symétrie, on en déduit qu'elles sont égales et que l'on a la suite quasi-exacte

$$0 \to \mathfrak{Z}(T)^{\Gamma'} \to t_{\Lambda_{L_\infty}}(X_{\infty,f}(F_\infty,T)_{\Gamma'}) \to a^1_{L_\infty}(\dot{X}_{\infty,f}(F_\infty,\check{T})_{\Gamma'}) \to \mathfrak{Z}(T)_{\Gamma'} \to 0$$

Comme $\widehat{\mathfrak{Z}(\check{T})_{\Gamma'}}$ et $\widehat{\mathfrak{Z}(T)_{\Gamma'}}$ sont de torsion, les suites suivantes sont quasi-exactes

$$0 \to \widehat{\mathfrak{Z}(\check{T})_{\Gamma'}} \to t_{\Lambda_{L_\infty}}(X_{\infty,f}(F_\infty,T)_{\Gamma'}) \to t_{\Lambda_{L_\infty}}(X_{\infty,f}(L_\infty,T)) \to 0$$

$$0 \to a^1_{\Lambda_{L_\infty}}(t_{\Lambda_{L_\infty}}(\dot{X}_{\infty,f}(L_\infty,\check{T}))) \\ \to a^1_{\Lambda_{L_\infty}}(t_{\Lambda_{L_\infty}}(\dot{X}_{\infty,f}(F_\infty,\check{T})_{\Gamma'})) \to a^1_{\Lambda_{L_\infty}}(\dot{\widehat{\mathfrak{Z}(T)_{\Gamma'}}}) \to 0\ .$$

En remarquant que $\mathfrak{Z}(T)^{\Gamma'}$ et $\widehat{\mathfrak{Z}(\check{T})_{\Gamma'}}$ ont même série caractéristique (cela peut se voir soit sur le diagramme, soit directement : seuls les nombres de Tamagawa aux places de S totalement décomposées dans L_∞ interviennent et on a alors $\mathrm{Tam}_v(T) = \mathrm{Tam}_v(\check{T})$ pour une place ne divisant pas p, en fait $\mathfrak{Z}(T)$ et $\mathfrak{Z}(\check{T})$ sont quasi-isomorphes), on en déduit le théorème :

4.2.1 Théorème. *Supposons* $(\mathrm{Hyp}(F_\infty,V))$, $(\mathrm{Hyp}(F_\infty,\check{V}))$ *et* $(\mathrm{Tors}(F_\infty,V))$. *Soit $f(F_\infty,T)$ la série caractéristique de $X_{\infty,f}(F_\infty,T)$ et $f(F_\infty,\check{T})$ la série caractéristique de $X_{\infty,f}(F_\infty,\check{T})$. Alors*

$$\dot{f}(F_\infty,\check{T})\Lambda = f(F_\infty,T)\Lambda$$

Pour toute sous-$\mathbb{Z}_p$-extension L_∞ de F_∞/F, soit $f^*(L_\infty,\check{T})$ (resp. $f^*(L_\infty,T)$) la série caractéristique du sous-module de torsion de $X_{\infty,f}(L_\infty,\check{T})$ (resp. $X_{\infty,f}(L_\infty,T)$). Alors

$$\dot{f}^*(L_\infty,\check{T})\Lambda_{L_\infty} = f^*(L_\infty,T)\Lambda_{L_\infty}$$

Autrement dit, pour tout caractère ρ de $\mathrm{Gal}(L_\infty/F)$ *à valeurs dans $\mathbb{C}_p^*$, on a*

$$\rho(f^*(L_\infty,T)) = \rho^{-1}(f^*(L_\infty,\check{T}))$$

5 La situation diédrale

5.1 Préliminaires

Soit K un corps quadratique imaginaire de discriminant d_K et p un nombre premier impair et premier à d_K. Il existe une unique extension K_∞ de K dont le groupe de Galois est topologiquement isomorphe à $\mathbb{Z}_p^2$. Elle contient deux $\mathbb{Z}_p$-extensions qui sont la sous-$\mathbb{Z}_p$-extension cyclotomique $K\mathbb{Q}_\infty$ de K et la sous-$\mathbb{Z}_p$-extension anti-cyclotomique (diédrale sur $\mathbb{Q}$) $H_\infty = D_\infty$ de $K[p^\infty] = \cup K[p^n]$, le Ringklasskörper de K de rayon une puissance de p. Soit $G_\infty = \mathrm{Gal}(K_\infty/K)$. L'algèbre d'Iwasawa associée est $\Lambda = \Lambda_{K_\infty} = \mathbb{Z}_p[[G_\infty]]$. On a une dualité

$$\Lambda_{K_\infty} \times \mathrm{Hom}(G_\infty, \mathbb{C}_p^\times) \to \mathbb{C}_p \ .$$

Ce qui permet de voir les éléments de Λ_{K_∞} comme des fonctions sur $\mathrm{Hom}(G_\infty, \mathbb{C}_p^\times)$ à valeurs dans $\mathbb{C}_p$. Tout élément de $\mathrm{Hom}(G_\infty, \mathbb{Z}_p^\times)$ est de la forme $\nu^a\chi^b$ avec χ la p-partie du caractère cyclotomique et ν un caractère diédral. On note χ_{cycl} le caractère cyclotomique et ν_{died} un caractère diédral. La théorie d'Iwasawa d'une courbe elliptique sur un corps quadratique imaginaire et des famille des points de Heegner tire ses origines de l'article de Mazur ([25], voir aussi Kurčanov, [23]). Grâce aux résultats récents de Cornut et Vatsal, un regain d'intérêt s'est manifesté. Mais il y a bien d'autres résultats montrés ou en voie de l'être et je voudrais les placer ici un peu plus dans le contexte de la théorie d'Iwasawa.

5.2 Théorie arithmétique

Soit E une courbe elliptique définie sur $\mathbb{Q}$ ou K de conducteur N_E premier à d_K et ayant bonne réduction ordinaire en p. Soit $T = T_p(E)$ son module de Tate, c'est-à-dire la limite projective des points de p^n-torsion pour n entier et $V_p(E) = \mathbb{Q}_p \otimes T_p(E)$. Soit L une extension finie de K. Le groupe de Selmer $S_p(E/L)$ de E/L vérifie la suite exacte

$$0 \to \mathbb{Q}_p/\mathbb{Z}_p \otimes E(L) \to S_p(E/L) \to \text{Ш}(E/L)(p) \to 0$$

Son dual de Pontryagin $\hat{S}_p(E/L)$ est $\mathrm{Hom}_{\mathbb{Z}_p}(S_p(E/L), \mathbb{Q}_p/\mathbb{Z}_p)$. Sa variante compacte (voir [3], [12]) $\check{S}_p(E/L)$ est la limite projective des groupes de Selmer relatif à la multiplication par p^n :

$$0 \to \mathbb{Z}_p \otimes_{\mathbb{Z}} E(L) \to \check{S}_p(E/L) \to T_p(\text{Ш}(E/L)) \to 0$$

Lorsque $\text{Ш}(E/L)(p)$ est fini, le dernier terme est nul. Avec les notations du §2, on a $\check{S}_p(E/L) = H^1_f(L, T_p(E))$ et $S_p(E/L) = H^1_f(L, V_p(E)/T_p(E))$. Le quotient de $\text{Ш}(E/L)(p)$ par sa partie divisible est le groupe de Tate-Shafarevich $\text{Ш}(T_p(E)/L)$ associé à la représentation p-adique $T_p(E)$ et vaut

$S_p(E/L)/\mathbb{Q}_p/\mathbb{Z}_p \otimes \check{S}_p(E/L)$. Ainsi,

$$\begin{aligned} S_p(E/K_\infty) &= \varinjlim_L S_p(E/L) = H^1_f(K_\infty, \mathcal{U}) \\ S_p(\widehat{E/K}_\infty) &= X_{\infty,f}(K_\infty, T_p(E)) \\ \check{S}_p(E/K_\infty) &= \varprojlim_L \check{S}_p(E/L) \end{aligned}$$

où la limite projective est prise relativement aux applications de norme (corestriction). Enfin, il n'est pas difficile de montrer [30] que pour une $\mathbb{Z}_p$-extension, par exemple D_∞, on a

$$\begin{aligned} \check{S}_p(E/D_\infty) &= \mathrm{Hom}_{\Lambda_{D_\infty}}(S_p(\widehat{E/D}_\infty), \Lambda_{D_\infty}) \\ &\cong \mathrm{Hom}_{\Lambda_{D_\infty}}(S_p(\widehat{E/D}_\infty)/t_{\Lambda_{D_\infty}}(S_p(\widehat{E/D}_\infty)), \Lambda_{D_\infty}) \end{aligned}$$

et que

$$S_p(\widehat{E/D}_\infty)/t_{\Lambda_{D_\infty}}(S_p(\widehat{E/D}_\infty)) \to \mathrm{Hom}(\check{S}_p(E/D_\infty), \Lambda_{D_\infty})$$

est injectif avec conoyau fini. En particulier, $\check{S}_p(E/D_\infty)$ est le Λ_{D_∞}-dual de $X_{\infty,f}(D_\infty, T_p(E)) = S_p(\widehat{E/D}_\infty)$ et est libre.

5.2.1 Théorème (Kato). *Si E est définie sur $\mathbb{Q}$, $\check{S}_p(E/K\mathbb{Q}_\infty)$ est de torsion sur $\Lambda_{\mathbb{Q}_\infty K}$ et $\check{S}_p(E/K_\infty)$ est de torsion sur Λ_{K_∞} (donc nuls). Il en est de même de $S_p(\widehat{E/K}_\infty)$.*

Il suffit d'appliquer le théorème démontré par Kato dans [21] à E et à sa tordue par le caractère quadratique de $K/\mathbb{Q}$.
Soit $\mathcal{L}_p(E/K_\infty)$ une série caractéristique du module de torsion $S_p(\widehat{E/K}_\infty)$.

5.3 Théorie analytique

Soit $L_p(E/K_\infty)$ la fonction L p-adique interpolant les valeurs $L(E,\rho,1)$ pour ρ caractère d'ordre fini de $\mathrm{Gal}(K_\infty/K)$. On peut trouver sa définition dans [29] qui suit de très près une construction antérieure de Hida. D'autres constructions ont été faites par Bertolini et Darmon [6]. Par un théorème de Rohrlich [32], $L_p(E/K_\infty)$ est non nulle.

Conjecture (Conjecture principale, [30]). *Les idéaux de Λ_{K_∞} engendrés par $L_p(E/K_\infty)$ et par $\mathcal{L}_p(E/K_\infty)$) sont égaux.*

Remarque. On peut utiliser le théorème de Kato pour obtenir une divisibilité lorsqu'on se restreint à $\mathrm{Gal}(K\mathbb{Q}_\infty/K)$.

5.4 Equation fonctionnelle

Soit c une conjugaison complexe induisant l'automorphisme non trivial de $\mathrm{Gal}(K/\mathbb{Q})$. Elle agit sur le groupe des caractères $\hat{G}_\infty$ de G_∞ : $\rho^c(\tau) = \rho(c\tau c^{-1})$. Ainsi, si χ se factorise par $\mathrm{Gal}(K\mathbb{Q}_\infty/K)$, on a $\chi^c = \chi$. Si ν est diédral, $\nu^c = \nu^{-1}$. On considère l'involution suivante sur $\hat{G}_\infty$: $\rho^\iota = \rho^{-c}$. Ainsi, $\rho^\iota(\tau) = \rho(c^{-1}\tau c)^{-1}$, $\chi_{cycl} = \chi_{cycl}^{-1}$, $\nu_{died}^\iota = \nu_{died}$.
Les deux fonctions vérifient une équation fonctionnelle

$$(L(\rho)) = (L(\rho^\iota))$$

Pour la première, cela se déduit de l'équation fonctionnelle complexe et on a en fait

$$L_p(E/K_\infty)(\rho^\iota) = \epsilon_D(-N_E)L_p(E/K_\infty)(\rho)\ .$$

En appliquant l'automorphisme non trivial c de $K/\mathbb{Q}$ qui laisse stable E ainsi que tous les modules définis et la proposition 4.2.1 on obtient la seconde équation fonctionnelle. On en déduit que l'on peut définir le signe de l'équation fonctionnelle de $\mathcal{L}_p(E/K_\infty)$: le groupe de cohomologie $H^1(\{1,\iota\}, \Lambda_{K_\infty}^\times)$ est d'ordre 2 et admet -1 comme élément non trivial. Ainsi, on peut choisir $\mathcal{L}_p(E/K_\infty)$ (qui est alors défini à une unité près de $\mathbb{Z}_p^*$) de manière à ce que

$$\mathcal{L}_p(E/K_\infty)(\rho^\iota) = \epsilon_p \mathcal{L}_p(E/K_\infty)(\rho)$$

avec $\epsilon_p = \pm 1$.

5.4.1 Proposition. *Soit $\lambda_0(D_\infty)$ le rang du Λ_{D_∞}-module $S(\widehat{E/D_\infty})$. Alors,*

$$\epsilon_p = (-1)^{\mathrm{rg}_{\mathbb{Z}_p} \check{S}_p(E/K)} = (-1)^{\lambda_0(D_\infty)}$$

Démonstration. On utilise le théorème de contrôle 2.5.1, l'existence de formes bilinéaires alternées montrées dans le paragraphe 3.3 et l'argument suivant de Guo [16] tel qu'il a été repris par Greenberg dans [14].
Soit $\Lambda = \mathbb{Z}_p[[\Gamma]]$, Γ' un sous-$\mathbb{Z}_p$-module de Γ isomorphe à $\mathbb{Z}_p$. On note $\Xi_n = \Gamma/\Gamma'_n$, $\Lambda_n = \mathbb{Z}_p[[\Xi_n]]$, Ξ un sous-$\mathbb{Z}_p$-module de Γ tel que $\Xi \cap \Gamma' = \{0\}$, $\Lambda_\Xi = \mathbb{Z}_p[[\Xi]]$. Soit M un Λ-module de torsion et de type fini ; $M_{\Gamma'_n}$ est un Λ_Ξ-module de type fini. Soit λ_n le Λ_Ξ-rang de $M_{\Gamma'_n}$. Alors λ_n est stationnaire et on a $\lambda_n \equiv \lambda_0 \bmod p-1$. En effet, les $\mathbb{Q}_p$-représentations irréductibles de Γ'/Γ'_n sont de degré divisible par $(p-1)p^{n-1}$.
Soit $\mathfrak{L}_n$ le quotient de M_n par son Λ_Ξ-module de torsion $\mathfrak{T}_n$ et $\mathfrak{T}_\infty$ le noyau de $M \to \mathfrak{L}_\infty$ où $\mathfrak{L}_\infty$ est la limite projective des $\mathfrak{L}_n$. Alors, l'application naturelle $\mathfrak{L}_\infty \to \mathfrak{L}_n$ est injective pour n assez grand, les rangs de $\mathfrak{L}_\infty$ et de $\mathfrak{L}_n$ sur Λ_Ξ sont égaux pour n assez grand et on a donc

$$\mathrm{rg}_{\Lambda_\Xi} \mathfrak{L}_\infty \equiv \lambda_0 \bmod p-1\ .$$

Supposons qu'il existe une forme bilinéaire alternée sur $\mathfrak{T}_n$ quasi non dégénérée (ou telle que les noyaux soient annulés par une puissance de p). Alors le rang de $\mathfrak{T}_\infty$ en tant que Λ_Ξ-module est pair et le rang de M en tant que Λ_Ξ-module est de même parité que λ_0. Pour le démontrer, on remarque, en utilisant le lemme du serpent et le fait que $\mathfrak{L}_\infty \to \mathfrak{L}_n$ est injective, que l'application $\mathfrak{T}_\infty \to \mathfrak{T}_n$ est surjective. Si $\mathfrak{p}$ est un idéal de Λ de hauteur 1 premier à p, la forme bilinéaire alternée sur $\mathfrak{T}_n$ induit par localisation une forme alternée non dégénérée sur $\Lambda_\mathfrak{p} \otimes \mathfrak{T}_n$. Le $\Lambda_\mathfrak{p}$-rang de $\Lambda_\mathfrak{p} \otimes \mathfrak{T}_\infty$ est alors pair. En effet, il suffit d'appliquer le lemme suivant :

Lemme (Guo). *Soit A un anneau de valuation discrète d'uniformisante π et soit M_n un système projectif de A-modules de longueur finie tel que $M_\infty = \varprojlim_n M_n$ soit de type fini et que l'application naturelle $M_\infty \to M_n$ soit surjective. Alors, il existe un entier d et des entiers $r_1(n), \cdots, r_d(n)$ tels que $M_n \cong A/\pi^{r_1(n)} \times \cdots \times A/\pi^{r_d(n)}$ avec $r_1(n) \geq \cdots \geq r_d(n)$ et le rang sur $\mathbb{Z}_p$ de M_∞ est égal au nombre d'entiers j tels que la suite $r_j(n)$ soit non bornée.*

Si maintenant M_n est muni d'une forme bilinéaire alternée pour tout entier n, on a $r_{2j-1}(n) = r_{2j}(n)$ pour $j \geq 1$ et le rang sur $\mathbb{Z}_p$ de M_∞ est donc pair.
Revenons à la démonstration de la proposition 5.4.1. Prenons $\Gamma = \mathrm{Gal}(K_\infty/K)$, $\Gamma' = \mathrm{Gal}(K_\infty/D_\infty)$ et Ξ le sous-groupe de Γ laissant invariant la sous-$\mathbb{Z}_p$-extension cyclotomique. Alors, l'équation fonctionnelle de $\mathcal{L}_p(E/K_\infty)$ implique que

$$\epsilon_p = (-1)^\lambda$$

avec λ le Λ_Ξ-invariant de la série caractéristique, autrement dit le Λ_Ξ-rang du module $S_p\widehat{(E/K_\infty)}$. En utilisant les théorèmes de contrôle et les formes bilinéaires alternées du paragraphe 3.2, on obtient l'existence des formes bilinéaires quasi-non dégénérées alternées nécessaires et on a donc :

$$\lambda \equiv \lambda_0(D_\infty) \mod 2 .$$

Ce qui démontre une des égalités de la proposition 5.4.1. L'autre égalité se démontre de la même manière (et était déjà montrée par Greenberg) en travaillant sur la $\mathbb{Z}_p$-extension cyclotomique (ici Ξ est nul) :

$$\epsilon_p = (-1)^{\mathrm{rg}_{\mathbb{Z}_p} \check{S}_p(E/K)}$$

□

5.5 La conjecture de Mazur

On suppose toujours que le discriminant de K est premier au conducteur N_E de E. Dans [25], Mazur conjecture que $S_p\widehat{(E/D_\infty)}$ est un Λ_{D_∞}-module de rang 1 si $\epsilon(-N_E) = -1$ et de rang 0 si $\epsilon(-N_E) = 1$.
Cette conjecture et la proposition 5.4.1 impliquent que les signes des équations fonctionnelles de $\mathcal{L}_p(E/K_\infty)$ et de $L_p(E/K_\infty)$ sont égaux.

Récemment, la situation de cette conjecture a énormément évolué dans le cas de l'hypothèse de Heegner mais aussi dans les autres cas. Nous appellerons hypothèses techniques des hypothèses qui devraient pouvoir être affaiblies ou évoluer rapidement jusqu'à disparaître et que l'on trouvera dans les articles originaux :

5.5.1 Théorème (Bertolini-Darmon+Vatsal, [7], [33]). *Supposons que* $\epsilon(-N_E) = 1$. *Supposons de plus que* $\ell^2 \nmid N_E$ *si* ℓ *est inerte dans* K + *des hypothèses techniques, alors* $S_p(\widehat{E/D}_\infty)$ *est de torsion.*

La démonstration est en deux parties :

- démontrer la non nullité de la fonction L p-adique $L_p(E/D_\infty)$ (théorème sur les familles de $L(E/K, \eta)$ pour η un caractère diédral d'ordre fini et de conducteur une puissance de p)
- démontrer que si $L_p(E/D_\infty)$ est non nul, $S_p(\widehat{E/D}_\infty)$ est de torsion

5.5.2 Théorème (Cornut-Vatsal + Bertolini-Darmon-Nekovář [5]). *Lorsque* $\epsilon(-N_E) = -1$ *et que tous les nombres premiers divisant* N_E *sont décomposés dans* K, $S_p(\widehat{E/D}_\infty)$ *est de rang* 1.

L'énoncé complet que l'on attend est montré ou sur le point de l'être :

5.5.3 Théorème. *Supposons que* $\epsilon(-N_E) = -1$ *et que* p^2 *ne divise pas* N_E. *Alors,* $S_p(\widehat{E/D}_\infty)$ *est de rang* 1.

La démonstration comporte plusieurs étapes :

- Construire des points x_n de $E(D_n)$ en utilisant une paramétrisation de E par une courbe modulaire ou une courbe de Shimura et les points de Heegner ou les points spéciaux provenant de la théorie de la multiplication complexe $x(p^n)$ (idée de Gross exploitée par Bertolini et Darmon, [6]). En modifiant légèrement ces points, on obtient des points compatibles pour les applications de trace et donc un élément z_∞^{spec} de $\check{S}_p(D_\infty)$ et un sous-module $\mathcal{H}_\infty$ de $\check{S}_p(E/D_\infty)$.
- Montrer que z_∞^{spec} est non nul (conjecture de Mazur), ce qui se ramène facilement à montrer qu'il existe un entier n tel que x_n est non nul. C'est le rôle des théorèmes de Cornut et Vatsal. On pourrait aussi peut-être utiliser les formules démontrées par Zhang ([35]) généralisant les formules de Gross-Zagier et qui sont du type :

  $$L'(E/K, \nu, 1) = C < x_n^{(\nu)}, x_n^{(\nu)} >$$

  avec C non nul et utiliser un théorème de non-annulation de la famille des $L'(E/K, \nu, 1)$ pour un caractère ν diédral d'ordre une puissance de p. On en déduit alors facilement que $\mathcal{H}_\infty$ est un module libre de rang 1.

- Utiliser les techniques de Kolyvagin [22] pour démontrer que le quotient $\check{S}_p(E/D_\infty)/\mathcal{H}_\infty$ est de torsion et donc que $\check{S}_p(E/D_\infty)$ et $S_p(\widehat{E/D_\infty})$ sont de Λ-rang 1. Ici, c'est la notion de système d'Euler qui est fondamentale. La définition précise des points x_n ne sert pas mais le fait qu'il existe des points $x(np^m)$ pour n sans facteurs carrés définis sur le Ringklasskörper de rayon np^m vérifiant des relations convenables.

5.6 Quelques remarques supplémentaires

Soit $u = (\sharp\mathcal{O}_K^*)/2$ et c_E la constante de Manin correspondant à la paramétrisation de E par $X_0(N_E)$. Soit $I(\mathcal{H}_\infty)$ une série caractéristique du Λ_{K_∞}-module de torsion $\check{S}_p(E/D_\infty)/\mathcal{H}_\infty$. Soit $\mathcal{T}_p(E/D_\infty)$ une série caractéristique du Λ_{D_∞}-module de torsion de $S_p(\widehat{E/D_\infty})$.

5.6.1 Conjecture ([30]). *Sous l'hypothèse de Heegner, les deux éléments $c_E^2u^2\mathcal{T}_p(E/D_\infty)$ et $I(\mathcal{H}_\infty)^2$ engendrent le même idéal de Λ_{D_∞}.*

Une version faible de cette conjecture avait été montrée par Bertolini [4] en utilisant les techniques de Kolyvagin. Récemment, Howard [18] a démontré la divisibilité de $\mathcal{T}_p(E/D_\infty)$ par $I(\mathcal{H}_\infty)^2$ lorsque l'application $\mathrm{Gal}(\overline{K}/K) \to Aut(T_p)$ est surjective ([1]).
On devrait d'autre part pouvoir remplacer l'hypothèse de Heegner par l'hypothèse que $\epsilon(-N_E) = -1$ en utilisant les points de Heegner-Shimura. [2]

Remarquons qu'on déduit des résultats du §4.2 que $\mathcal{T}_p(E/D_\infty)$ est un carré, ce qui est compatible avec la conjecture précédente. En effet, en composant avec l'involution c et en identifiant $T_p(E)$ avec $T_p(E)^*(1)$ par l'accouplement alterné de Weil, on obtient une forme bilinéaire alternée

$$t_{\Lambda_{D_\infty}}(X_{\infty,f}(K_\infty,T_p(E))_{\mathrm{Gal}(K_\infty/D_\infty)})\times t_{\Lambda_{D_\infty}}(X_{\infty,f}(K_\infty,T_p(E))_{\mathrm{Gal}(K_\infty/D_\infty)}) \to \mathrm{Frac}(\Lambda_{D_\infty})/\Lambda_{D_\infty}$$

En tenant compte des noyaux et de la différence entre les modules $X_{\infty,f}(K_\infty,T_p(E))_{\mathrm{Gal}(K_\infty/D_\infty)}$ et $X_{\infty,f}(D_\infty,T_p(E))$ dont la série caractéristique est une puissance de p, on en déduit que $\mathcal{T}_p(E/D_\infty)$ est un carré.
Une conséquence de la démonstration de Cornut [9, théorème B appliqué à $q=p$] est la suivante :

5.6.2 Proposition. *On suppose que p ne divise pas $N_E\varphi(N_Ed_K)$, ainsi que le nombre de composantes connexes du noyau de la paramétrisation modulaire choisie de E. Alors, $I(\mathcal{H}_\infty)$ n'est pas divisible par p.*

Supposons que $\epsilon(-N_E) = -1$. Nous avons défini précédemment une forme bilinéaire

$$\langle\langle\cdot,\cdot\rangle\rangle_{\chi_{cycl}} : \check{S}_p(E/D_\infty)\times\check{S}_p(E/D_\infty)\to\Lambda_{D_\infty}\ .$$

[2]Cela semble être fait maintenant, voir [19].

Elle peut s'écrire en termes des hauteurs p-adiques classiques de la manière suivante

$$\langle\langle x,y\rangle\rangle_{\chi_{cycl}} = \left(\frac{1}{[D_n:K]}\sum_{\sigma,\tau} < \sigma x_n, \tau y_n >_{\chi_n} \sigma\tau^{-1}\right)_n .$$

Ici, χ_n est un caractère de $\mathrm{Gal}(K_\infty/D_n)$ dont la restriction à $\mathrm{Gal}(K_\infty/D_\infty)$ est χ_{cycl}. On peut reprendre la démonstration de [30] pour démontrer :

5.6.3 Théorème. *Soit ρ un caractère diédral de* $\mathrm{Gal}(K_\infty/K)$ *à valeurs dans* $\mathbb{Z}_p^*$.
1) *Soit r_{D_∞} le rang de $\check{S}_p(E/D_\infty)$ en tant que Λ_{D_∞}-modules. Alors, $\mathcal{L}(E/K_\infty)(\rho\chi_{cycl}^s)$ a un zéro en χ_{cycl} de multiplicité supérieure ou égale à r_{D_∞}.*
2) *Ce zéro est d'ordre exactement r_{D_∞} si et seulement si $\langle\langle\cdot,\cdot\rangle\rangle_{\chi_{cycl}}$ est non dégénérée.*
3) *On a dans ce cas*

$$\lim_{s\to 0}\frac{\mathcal{L}(K_\infty/K)(\rho\chi_{cycl}^s)}{s^{r_{D_\infty}}} \sim disc_{\check{S}_p(E/D_\infty)}\langle\langle\cdot,\cdot\rangle\rangle_{\chi_{cycl}}(\rho)\mathcal{T}_p(E/D_\infty)(\rho).$$

Les théorèmes ou conjectures précédentes impliquent que r_{D_∞} est en fait égal à 1 lorsque $\epsilon(-N_E) = -1$. D'autre part, l'ordre du zéro de $\mathcal{L}(E/K_\infty)(\rho\chi_{cycl}^s)$ en $s = 0$ est impair. Il serait intéressant de montrer qu'il existe un point de Heegner z_n dont la hauteur p-adique $< z_n, z_n >_{\chi_n}$ est non nulle. Cela n'est connu que si E est à multiplication complexe ([8]).
Revenons sur le module des points de Heegner. Soit $\mathcal{H}_n$ le sous-module de $\check{S}_p(E/D_n)$ engendré par les traces de $K[p^n]$ à D_n des points de Heegner de niveau divisant p^{n+1}. On a alors la proposition :

5.6.4 Proposition. *La norme de D_{n+1} à D_n induit une application de $\mathcal{H}_{n+1}$ à $\mathcal{H}_n$. Elle est surjective pour $n \geq 1$. L'indice de $Tr_{n,0}(\mathcal{H}_n)$ dans $\mathcal{H}_0$ est égal à $L(E/K_p,1)^{-1}$ (facteur d'Euler local en p).*

5.6.5 Remarques. La définition couramment admise est de prendre le sous-module de $\check{S}_p(E/D_n)$ engendré par les traces de $K[p^n]$ à D_n des points de Heegner de niveau p^{n+1}. Malheureusement, ce n'est pas toujours gros ! L'énoncé de Mazur dans [25] est incorrect : la condition $a_p \equiv 2 \bmod p$ est inutile avec cette définition et la surjectivité affirmée est fausse pour $a_p \equiv 1 \bmod p$. Essentiellement, on a besoin des points de niveau p et de niveau 1 à la fois car la trace de $H[p]$ à K de y_p est un "multiple rationnel" (et non entier) de la trace de $H[1]$ à K.

5.6.6 Remarques. Plaçons-nous dans le cas où l'hypothèse de Heegner est vérifiée et où le rang de $E(K)$ est strictement supérieur à 1. L'image de $\mathcal{H}_\infty$ dans $E(K)$ est alors nulle. On peut construire un élément de $\mathbb{Z}_p \otimes E(K)$ de la manière suivante (à condition que $\text{Ш}(E/K)(p)$ soit fini, construction de Kolyvagin-Solomon) : on choisit un générateur γ de $\mathrm{Gal}(D_\infty/K)$ et γ_n sa restriction à D_n. Soit un élément $z_\infty = (z_n)$ de $\mathcal{H}_\infty$ dont la projection est nulle (les z_n se calculent en fonction des points de Heegner de niveau une puissance de p). Alors

$$\sum_{i=0}^{p^n-1} i\gamma_n^i z_n$$

converge dans $\varinjlim_n \check{S}(D_n)$ vers un élément de $\check{S}(K)$. Moins explicitement, cela revient à résoudre l'équation $z_\infty = (\gamma - 1)z'_\infty$ dans $\check{S}(D_\infty)$ et à regarder la projection de z'_∞ dans $\check{S}(K)$. Le fait que cette équation admet une solution vient de ce que l'application trace $\mathbb{Q}_p \otimes \check{S}(D_\infty)_{\mathrm{Gal}(D_\infty/K)} \to \mathbb{Q}_p \otimes \check{S}_p(K)$ est un isomorphisme. Il est possible que l'image de z'_∞ dans $E(K)$ soit encore nulle. Il existe alors un entier $r = r_K$ tel que $z_\infty = (\gamma-1)^r z_\infty^{(r)}$ et tel que la projection de $z_\infty^{(r)}$ dans $\check{S}_p(K)$ soit non nulle. Cette projection donne un point non trivial z_K de $\check{S}_p(K)$.
Soit ϵ le signe de l'équation fonctionnelle de $E/\mathbb{Q}$.

5.6.7 LEMME. *Avec les notations précédentes, on a* $c(z_K) = -\epsilon(-1)^{r_K} z_K$ mod *torsion.*

Cela se déduit de la relation $x^c = -\epsilon c$ modulo torsion pour un point de Heegner et du fait que $c\gamma c^{-1} = \gamma^{-1}$.
Ainsi, z_K appartient à $\mathbb{Z}_p \otimes E(K)^{-(-1)^{r_K}\epsilon}$. On peut alors se poser un certain nombre de questions. Entre autres :

1. Pour K fixé, peut-on relier les parités de $r_K = r$ et de $\mathrm{rg}\, E(\mathbb{Q})$?
2. Comment varient les r_K avec K ?
3. Est-il possible de trouver une base du $\mathbb{Z}_p$-module $\mathbb{Z}_p \otimes E(\mathbb{Q})$ formée des points z_K pour certains corps quadratiques imaginaires Kg Cela "signifierait" que "le groupe de Mordell-Weil est engendré par des limites p-adiques de points de Heegner même en rang supérieur à 1".
4. Si la réponse à la question précédente est vraie, peut-on espérer borner la taille des corps quadratiques ?
5. Y a-t-il des relations "intéressantes" entre les différents $z_{K_i} \in \mathbb{Z}_p \otimes E(\mathbb{Q})$ pour différents corps K_i ?

RÉFÉRENCES

[1] A. Agboola et B. Howard. Anticyclotomic Iwasawa theory of CM elliptic curves (2003).

[2] P. Billot. Quelques aspects de la descente sur une courbe elliptique dans le cas de réduction supersingulière. *Compositio Math.* 58 (1986), 341-369.

[3] M. I. Bashmakov. The cohomology of abelian varieties over a number field, *Russian Math. Survey* 27 (1972), 25-70.

[4] M. Bertolini. Selmer groups and Heegner points in anticyclotomic $\mathbb{Z}_p$-extensions. *Compositio Math.* 99 (1995), 153-182.

[5] M. Bertolini et H. Darmon. Kolyvagin's descent and Mordell-Weil groups over ring class fields. *J. reine angew. Math.* 412 (1990), 63-74.

[6] M. Bertolini et H. Darmon. Heegner points on Mumford-Tate curves. *Invent. Math.* 126 (1996), 413-456.

[7] M. Bertolini et H. Darmon. Iwasawa's main conjecture for elliptic curves over anticyclotomic $\mathbb{Z}_p$-extensions. Prépublication 2001

[8] Daniel Bertrand. Sous-groupes à un paramètre p-adique de variétés de groupe. *Invent. Math.* 40, 1977, 171-193.

[9] C. Cornut. Mazur's conjecture on higher Heegner points. *Invent. Math.* 148, 2002, 495-523.

[10] C. Cornut. Non trivialité des points de Heegner. *C.R. Acad. Sci. Paris* 334 (2002), no 12, 1039-1042

[11] M. Flach. A generalisation of the Cassels-Tate pairing *J. Reine Angew. Math.*, 412 (1990), 113-127

[12] J. W. S. Cassels. Arithmetic on curves of genus 1. IV. Proof of the Hauptvermutung. *J. Reine Angew. Math.* 211 (1962), 95-112

[13] J.-M. Fontaine et B. Perrin-Riou. Autour des conjectures de Bloch et Kato: cohomologie galoisienne et valeurs de fonctions L In *Motives* (Seattle, WA, 1991), Amer. Math. Soc., Providence, RI (1994), 599–706.

[14] R. Greenberg. Iwasawa theory for elliptic curves. In *Arithmetic theory of elliptic curves* (Cetraro, 1997), Lecture Notes in math. 1716, p. 51-144. Springer, Berlin, 1999.

[15] R. Greenberg. Introduction to Iwasawa theory for elliptic curves. In *Arithmetic algebraic geometry* (Park City, UT, 1999) Amer. Math. Soc.Providence s(2001), 407-464.

[16] L. Guo. On a generalisation of Tate dualities with applications to Iwasawa theory. *Compos. Math.* 85 (1993), 125-161.

[17] H. Hida. A p-adic measure attached to the zeta function associated with two elliptic modular forms I. *Invent. Math.* 79 (1985), 159-195.

[18] B. Howard. The Heegner point Kolyvagin system (2002).

[19] B. Howard. Iwasawa theory of Heegner points on abelian varieties of GL_2-type (2003).

[20] U. Jannsen. Iwasawa modules up to isomorphism. In *Algebraic number theory*, Adv. Stud. Pure Math., 17, Academic Press, Boston, pp. 171–207,

[21] K. Kato p-adic Hodge theory and values of zeta functions of modular forms prépublication (2001).

[22] V. A. Kolyvagin. Euler systems. In *The Grothendieck Festschrift*, Vol. II, Progress in Math. 87 (1990), Birkhäuser Boston, 435-483.

[23] P. F. Kurčanov. Elliptic curves of finite rank over Γ-extensions. *Mat. Sb. (N.S.)* 90 (132), (1973), 320-324.

[24] W. McCallum. A duality theorem in the multivariable Iwasawa theory of local fields. *J. Reine Angew. Math.* 464 (1995), 143-172.

[25] B. Mazur. Modular curves and arithmetic. in *Proceedings of the ICM* 1983 (Varsaw), Vol. 1, 185-211, PWN, Warsaw, 1984.

[26] J. Nekovář. On the parity of ranks of Selmer groups. II. *C. R. Acad. Sci. Paris Sér. I Math.* 332 (2001), no. 2, 99–104.

[27] J. Nekovář. Selmer complexes. preprint (2001)

[28] B. Perrin-Riou. Arithmétique des courbes elliptiques et théorie d'Iwasawa. Mém. Soc. Math. France (N.S.) No. 17 (1984), 130 pp

[29] B. Perrin-Riou. Fonctions L p-adiques associées à une forme modulaire et à un corps quadratique imaginaire, *J. London Math. Soc. (2)* 38 (1988), 1-32.

[30] B. Perrin-Riou. Fonctions L p-adiques, théorie d'Iwasawa et points de Heegner. *Bull. Soc Math. France* 115 (1987), 399-456.

[31] B. Perrin-Riou. Théorie d'Iwasawa et hauteurs p-adiques. *Invent. Math.* 109 (1992), 137-185.

[32] D. E. Rohrlich. On L-functions of elliptic curves and anticyclotomic towers. *Invent. Math.* 75 (1984), 383-408.

[33] V. Vatsal. Uniform distribution of Heegner points. *Invent. Math.* 148 (2002), 1-46.

[34] V. Vatsal. Special values of anticyclotomic L functions *Duke Math. J.* 116 (2003), 219-261.

[35] S-W Zhang. Gross-Zagier Formula for GL_2 *The Asian Journal of Mathematics* 5 (2001), 183-290.

Bernadette Perrin-Riou
Mathématiques, Bât. 425
Université Paris-Sud F-91405 Orsay
France
Bernadette.Perrin-Riou@math.u-psud.fr

A Bound for the Torsion in the K-Theory of Algebraic Integers

Christophe Soulé

Received: November 6, 2002
Revised: May 30, 2003

A Kazuya Kato, qui
marche sous la lune.

Abstract. Let m be an integer bigger than one, A a ring of algebraic integers, F its fraction field, and $K_m(A)$ the m-th Quillen K-group of A. We give a (huge) explicit bound for the order of the torsion subgroup of $K_m(A)$ (up to small primes), in terms of m, the degree of F over $\mathbf{Q}$, and its absolute discriminant.

Let F be a number field, A its ring of integers and $K_m(A)$ the m-th Quillen K-group of A. It was shown by Quillen that $K_m(A)$ is finitely generated. In this paper we shall give a (huge) explicit bound for the order of the torsion subgroup of $K_m(A)$ (up to small primes), in terms of m, the degree of F over $\mathbf{Q}$, and its absolute discriminant.

Our method is similar to the one developed in [13] for $F = \mathbf{Q}$. Namely, we reduce the problem to a bound on the torsion in the homology of the general linear group $\mathrm{GL}_N(A)$. Thanks to a result of Gabber, such a bound can be obtained by estimating the number of cells of given dimension in any complex of free abelian groups computing the homology of $\mathrm{GL}_N(A)$. Such a complex is derived from a contractible CW-complex $\widetilde{W}$ on which $\mathrm{GL}_N(A)$ with compact quotient. We shall use the construction of $\widetilde{W}$ given by Ash in [1] . It consists of those hermitian metrics h on A^N which have minimum equal to one and are such that their set $M(h)$ of minimal vectors has rank equal to N in F^N. To count cells in $\widetilde{W}/\mathrm{GL}_N(A)$, one will exhibit an explicit compact subset of $A^N \otimes_{\mathbf{Z}} \mathbf{R}$ which, for every $h \in \widetilde{W}$, contains a translate of $M(h)$ by some matrix of $\mathrm{GL}_N(A)$ (Proposition 2). The proof of this result relies on several arguments from the geometry of numbers using, among other things, the number field analog of Hermite's constant [4].

The bound on the K-theory of A implies a similar upper bound for the étale cohomology of Spec $(A[1/p])$ with coefficients in the positive Tate twists of $\mathbf{Z}_p$, for any (big enough) prime number p.
However, this bound is quite large since it is doubly exponential both in m and, in general, the discriminant of F. We expect the correct answer to be polynomial in the discriminant and exponential in m (see 5.1).
The paper is organized as follows. In Section 1 we prove a few facts on the geometry of numbers for A, including a result about the image of A^* by the regulator map (Lemma 3), which was shown to us by H. Lenstra. Using these, we study in Section 2 hermitian lattices over A, and we get a bound on $M(h)$ when h lies in $\widetilde{W}$. The cell structure of $\widetilde{W}$ is described in Section 3. The main Theorems are proved in Section 4. Finally, we discuss these results in Section 5, where we notice that, because of the Lichtenbaum conjectures, a lower bound for higher regulators of number fields would probably provide much better upper bounds for the étale cohomology of Spec $(A[1/p])$. We conclude with the example of $K_8(\mathbf{Z})$ and its relation to the Vandiver conjecture.

1 Geometry of algebraic numbers

1.1

Let F be a number field, and A its ring of integers. We denote by $r = [F : \mathbf{Q}]$ the degree of F over $\mathbf{Q}$ and by $D = |\mathrm{disc}\,(K/\mathbf{Q})|$ the absolute value of the discriminant of F over $\mathbf{Q}$. Let r_1 (resp. r_2) be the number of real (resp. complex) places of F. We have $r = r_1 + 2\,r_2$. We let $\Sigma = \mathrm{Hom}\,(F, \mathbf{C})$ be the set of complex embeddings of F. These notations will be used throughout.
Given a finite set X we let $\#\,(X)$ denote its cardinal.

1.2

We first need a few facts from the geometry of numbers applied to A and A^*. The first one is the following classical result of Minkowski:

Lemma 1. *Let L be a rank one torsion-free A-module. There exists a non zero element $x \in L$ such that the submodule spanned by x in L has index*

$$\#\,(L/Ax) \leq C_1\,,$$

where

$$C_1 = \frac{r!}{r^r} \cdot 4^{r_2}\,\pi^{-r_2}\,\sqrt{D}$$

in general, and $C_1 = 1$ when A is principal.

Proof. The A-module L is isomorphic to an ideal I in A. According to [7], V §4, p. 119, Minkowski's first theorem implies that there exists $x \in I$ the norm of which satisfies

$$|N(x)| \leq C_1\,N(I)\,.$$

Here $|N(x)| = \#(A/Ax)$ and $N(I) = \#(A/I)$, therefore $\#(I/Ax) \leq C_1$. The case where A is principal is clear. q.e.d.

1.3

The family of complex embeddings $\sigma : F \to \mathbf{C}$, $\sigma \in \Sigma$, gives rise to a canonical isomorphism of real vector spaces of dimension r

$$F \otimes_{\mathbf{Q}} \mathbf{R} = (\mathbf{C}^{\Sigma})^{+},$$

where $(\cdot)^{+}$ denotes the subspace invariant under complex conjugation. Given $\alpha \in F$ we shall write sometimes $|\alpha|_{\sigma}$ instead of $|\sigma(\alpha)|$.

Lemma 2. *Given any element $x = (x_{\sigma}) \in F \otimes_{\mathbf{Q}} \mathbf{R}$, there exists $a \in A$ such that*

$$\sum_{\sigma \in \Sigma} |x_{\sigma} - \sigma(a)| \leq C_2,$$

with

$$C_2 = \frac{4^{r_1}\, \pi^{r_2}}{r^{r-2}\, r!} \sqrt{D}$$

in general, and

$$C_2 = 1/2 \qquad \text{if} \qquad F = \mathbf{Q}.$$

Proof. Define a norm on $F \otimes_{\mathbf{Q}} \mathbf{R}$ by the formula

$$\|x\| = \sum_{\sigma \in \Sigma} |x_{\sigma}|.$$

The additive group A is a lattice in $F \otimes_{\mathbf{Q}} \mathbf{R}$, and we let $\mu_1, \ldots, \mu_r$ be its successive minima. In particular, there exist $a_1, \ldots, a_r \in A$ such that $\|a_i\| = \mu_i$, $1 \leq i \leq r$, and $\{a_1, \ldots, a_r\}$ are linearly independent over $\mathbf{Z}$. Any $x \in F \otimes_{\mathbf{Q}} \mathbf{R}$ can be written

$$x = \sum_{i=1}^{r} \lambda_i\, a_i\,, \quad \lambda_i \in \mathbf{R}.$$

Let $n_i \in \mathbf{Z}$ be such that $|n_i - \lambda_i| \leq 1/2$, for all $i = 1, \ldots, r$, and

$$a = \sum_{i=1}^{r} n_i\, a_i\,.$$

Clearly

$$\|x - a\| \leq \sum_{i=1}^{r} |\lambda_i - n_i|\, \|a_i\| \leq \frac{1}{2}\,(\mu_1 + \cdots + \mu_r) \leq \frac{r}{2}\,\mu_r\,. \tag{1}$$

On the other hand, we know from the product formula that, given any $a \in A - \{0\}$,

$$\prod_{\sigma \in \Sigma} |\sigma(a)| \geq 1 \, . \tag{2}$$

By the inequality between arithmetic and geometric means this implies

$$\|a\| = \sum_{\sigma \in \Sigma} |\sigma(a)| \geq r \, ,$$

hence

$$\mu_i \geq r \qquad \text{for all} \quad i = 1, \ldots, r \, . \tag{3}$$

Minkowski's second theorem tells us that

$$\mu_1 \ldots \mu_r \leq 2^r \, W \, 2^{-r_2} \sqrt{D} \tag{4}$$

([7], Lemma 2, p. 115), where W is the euclidean volume of the unit ball for $\| \cdot \|$ in $F \otimes_{\mathbf{Q}} \mathbf{R}$. (Note that the covolume of A is $\sqrt{D}$.) The volume W is the euclidean volume of those elements $(x_i, z_j) \in \mathbf{R}^{r_1} \times \mathbf{C}^{r_2}$ such that

$$\sum_{i=1}^{r_1} |x_i| + 2 \sum_{j=1}^{r_2} |z_j| \leq 1 \, .$$

One finds ([7], Lemma 3, p. 117)

$$W = 2^{r_1} \, 4^{-r_2} (2\pi)^{r_2} / r! \, . \tag{5}$$

From (3) and (4) we get

$$\mu_r \leq 2^r \, W \sqrt{D} \, 2^{-r_2} \, r^{-(r-1)} \, . \tag{6}$$

The lemma follows from (1), (5) and (6). q.e.d.

1.4

We also need a multiplicative analog of Lemma 2. Let $R(F)$ be the regulator of F, as defined in [7] V, § 1, p. 109. Let $s = r_1 + r_2 - 1$.

Lemma 3. *Let (λ_σ), $\sigma \in \Sigma$ be a family of positive real numbers such that $\lambda_{\overline{\sigma}} = \lambda_\sigma$ when $\overline{\sigma}$ is the complex conjugate of σ. There exists a unit $u \in A^*$ such that*

$$\operatorname*{Sup}_{\sigma \in \Sigma} (\lambda_\sigma \, |u|_\sigma) \leq C_3 \left(\prod_{\sigma \in \Sigma} \lambda_\sigma \right)^{1/r} ,$$

with

$$C_3 = \exp\left(s(4\,r\,(\log 3\,r)^3)^{s-1}\,2^{r_2-1}\,R(F)\right).$$

Proof. We follow an argument of H. Lenstra. Let $H \subset \mathbf{R}^{r_1+r_2}$ be the s-dimensional hyperplane consisting of vectors $(x_1,\ldots,x_{r_1+r_2})$ such that $x_1+x_2+\cdots+x_{r_1+r_2}=0$. Choose a subset $\{\sigma_1,\ldots,\sigma_{r_1+r_2}\} \subset \Sigma$ such that $\sigma_1,\ldots,\sigma_{r_1}$ are the real embeddings of Σ and $\sigma_i \neq \overline{\sigma}_j$ if $i \neq j$. Given $\lambda = (\lambda_\sigma)_{\sigma\in\Sigma}$ as in the lemma, we let

$$\rho(\lambda) = (\log(\lambda_{\sigma_1}),\ldots,\log(\lambda_{\sigma_{r_1}}), 2\log(\lambda_{\sigma_{r_1+1}}),\ldots,2\log(\lambda_{\sigma_{r_1+r_2}}))\,.$$

If $u \in A^*$ is a unit, and $\lambda = (|u|_\sigma)$, we have $\rho(\lambda) \in H$. We get this way a lattice

$$L = \{\rho(|u|_\sigma)\,,\ u \in A^*\}$$

in H.
Define a norm $\|\cdot\|$ on H by the formula

$$\|(x_i)\| = \operatorname{Sup}\left(\operatorname*{Sup}_{1\le i\le r_1}|x_i|, \operatorname*{Sup}_{r_1+1\le i\le r_1+r_2}|x_i|/2\right).$$

It is enough to show that, for any vector $x \in H$ there exists $a \in L$ such that

$$\|x-a\| \le \log(C_3)\,.$$

According to [14], Cor. 2, p. 84, we have (when $r \ge 2$)

$$\|a\| \ge \varepsilon$$

where

$$\varepsilon = r^{-1}(\log(3\,r))^{-3}\,,$$

for any $a \in L - \{0\}$. Therefore, using Minkowski's second theorem as in the proof of Lemma 2 we get that, for any $x \in H$ there exists $a \in L$ with

$$\|x-a\| \le s\,2^{s-1}\,\varepsilon^{1-s}\,W\operatorname{vol}(H/L)\,,$$

where W is the euclidean volume of the unit ball for $\|\cdot\|$, where we identify H with $\mathbf{R}^s$ by projecting on the first s coordinates. Clearly $W \le 2^{s+r_2-1}$ when, by definition (loc.cit.), $\operatorname{vol}(H/L)$ is equal to $R(F)$. The lemma follows.

1.5

We now give an upper bound for the constant C_3 of Lemma 3.

Lemma 4. *The following inequality holds*

$$R \le 11\,r^2\sqrt{D}\log(D)^{r-1}\,.$$

Proof. Let κ be the residue at $s=1$ of the zeta function of F. According to [11], Cor. 3, p. 333, we have

$$\kappa \leq 2^{r+1}\, D^a\, a^{1-r}$$

whenever $0 < a \leq 1$. Taking $a = \log(D)^{-1}$ we get

$$\kappa \leq e\, 2^{r+1} \log(D)^{r-1} \, . \tag{7}$$

On the other hand

$$\kappa = 2^{r_1}\, (2\pi)^{r_2}\, \frac{h(F)\, R(F)}{w(F)\, \sqrt{D}} \, , \tag{8}$$

where $h(F)$ is the class number of F and $w(F)$ the number of roots of unity in F ([7], Prop. 13, p. 300). Since $h(F) \geq 1$ we get

$$R(F) \leq w(F)\, 2^{-(r_1+r_2)}\, \pi^{-r_2}\, e\, 2^{r+1} \sqrt{D} \log(D)^{r-1} \, .$$

Since the degree over $\mathbf{Q}$ of $\mathbf{Q}(\sqrt[n]{1})$ is $\varphi(n)$, where φ is the Euler function, we must have

$$\varphi(w(F)) \leq r \, .$$

When $n = p^t$ is an odd prime power we have

$$\varphi(p^t) = (p-1)\, p^{t-1} \geq p^{t/2} \, .$$

Therefore

$$w(F) \leq 2\, r^2 \, .$$

Since

$$2^{-(r_1+r_2)}\, \pi^{-r_2}\, 2^r = \left(\frac{2}{\pi}\right)^{r_2} \leq 1$$

and $4\, e \leq 11$, the lemma follows.

2 Hermitian lattices

2.1

An *hermitian lattice* $\overline{M} = (M, h)$ is a torsion free A-module M of finite rank, equipped with an hermitian scalar product h on $M \otimes_{\mathbf{Z}} \mathbf{C}$ which is invariant under complex conjugation. In other words, if we let $M_\sigma = M \otimes_A \mathbf{C}$ be the complex vector space obtained from M by extension of scalars via $\sigma \in \Sigma$, h is given by a collection of hermitian scalar products h_σ on M_σ, $\sigma \in \Sigma$, such that $h_{\overline{\sigma}}(x, y) = h_\sigma(x, y)$ whenever x and y are in M.
We shall also write

$$h_\sigma(x) = h_\sigma(x, x)$$

and

$$\|x\|_\sigma = \sqrt{h_\sigma(x)}\,.$$

Lemma 5. *Let $\overline{M}$ be an hermitian lattice of rank N. Assume that M contains N vectors $e_1, \ldots, e_N$ which are F-linearly independent in $M \otimes_A F$ and such that*

$$\|e_i\| \leq 1$$

for all $i = 1, \ldots, N$. Then there exist a direct sum decomposition

$$M = L_1 \oplus \cdots \oplus L_N$$

where each L_i has rank one and contains a vector f_i such that

$$\#\,(L_i / A\, f_i) \leq C_1$$

and

$$\|f_i\| \leq (i-1)\, C_1\, C_2 + C_1^{1/r}\, C_3\,.$$

Here C_1, C_2, C_3 are the constants defined in Lemmas 1, 2, 3 respectively.

Proof. We proceed by induction on N. When $N = 1$, Lemma 1 tells us that $L_1 = M$ contains x_1 such that

$$\#\,(L_1 / A\, x_1) \leq C_1\,.$$

Let us write

$$x_1 = \alpha\, e_1$$

with $\alpha \in F^*$. Using Lemma 3, we can choose $u \in A^*$ such that

$$\operatorname*{Sup}_{\sigma \in \Sigma} |u\,\alpha|_\sigma \leq C_3 \left(\prod_\sigma |\alpha|_\sigma \right)^{1/r} = C_3\, N(\alpha)^{1/r} \leq C_3\, C_1^{1/r}\,.$$

The lemma follows with $f_1 = u\, x_1$.

Assume now that $N \geq 2$, and let $L_1 = M \cap F\, e_1$ in $M \otimes_A F$. As above, we choose $f_1 = a_{11}\, e_1$ in L_1 with $[L_1 : A\, f_1] \leq C_1$ and

$$\operatorname*{Sup}_{\sigma \in \Sigma} |a_{11}|_\sigma \leq C_3\, C_1^{1/r}\,.$$

The quotient $M' = L / L_1$ is torsion free of rank $N - 1$. We equip M' with the quotient metric induced by h, we let $p : M \to M'$ be the projection, and $e'_i = p(e_i)$, $i = 2, \ldots, N$. Clearly

$$\|e'_i\| \leq 1$$

for all $i = 2, \ldots, N$.

We assume by induction that M' can be written

$$M' = L_2' \oplus \cdots \oplus L_N'$$

and that L_i' contains a vector f_i' such that

$$n_i = \# \, (L_i' / A \, f_i') \leq C_1$$

with

$$f_i' = \sum_{2 \leq j \leq i} a_{ij} \, e_j' \, , \tag{9}$$

$a_{ij} \in F$, and, for all $\sigma \in \Sigma$,

$$|a_{ij}|_\sigma \leq C_1 \, C_2 \qquad \text{if } 2 \leq j < i \leq N \, ,$$

and

$$|a_{ii}|_\sigma \leq C_1^{1/r} \, C_3 \, , \ 2 \leq i \leq N \, .$$

Let $s : M' \to M$ be any section of the projection p. From (9) it follows that there exists $\mu_i \in F$ such that

$$s(f_i') - \sum_{2 \leq j < i} a_{ij} \, e_j = \mu_i \, e_1 \, .$$

Applying Lemma 2, we can choose $b_i \in A$ such that

$$\sum_{\sigma \in \Sigma} \left| \frac{\mu_i}{n_i} - b_i \right|_\sigma \leq C_2 \, .$$

Define $t : M' \to M$ by the formulae

$$t(x) = s(x) - a(x) \, b_i \, e_1$$

whenever $x \in L_i'$, hence

$$n_i \, x = a(x) \, f_i' \, ,$$

for some $a(x) \in A$, $2 \leq i \leq N$. If we take $f_i = t(f_i')$, we get

$$f_i = s(f_i') - n_i \, b_i \, e_1 = \sum_{2 \leq j < i} a_{ij} \, e_j + (\mu_i - n_i \, b_i) \, e_1$$

and, for all $\sigma \in \Sigma$,

$$|\mu_i - n_i \, b_i|_\sigma \leq n_i \, C_2 \leq C_1 \, C_2 \, .$$

We define $a_{i1} = \mu_i - n_i \, b_i$ and $L_i = t(L_i')$. Then

$$M = L_1 \oplus \cdots \oplus L_N$$

satisfies our induction hypothesis:

$$\# (L_i / A\, f_i) \leq C_1$$

$$f_i = \sum_{1 \leq j \leq i} a_{ij}\, e_j \, ,$$

$$|a_{ij}|_\sigma \leq C_1\, C_2 \qquad \text{when } j < i \, ,$$

and

$$|a_{ii}|_\sigma \leq C_1^{1/r}\, C_3 \, , \qquad \text{for all } \sigma \in \Sigma \, , \; i = 1, \ldots, N \, .$$

Since

$$\|e_i\| \leq 1 \qquad \text{for all } i = 1, \ldots, N \, ,$$

this implies

$$\|f_i\| \leq (i-1)\, C_1\, C_2 + C_1^{1/r}\, C_3 \, .$$

q.e.d.

2.2

Lemma 6. *Let $I \subset A$ be a nontrivial ideal. There exists a set of representatives $\mathcal{R} \subset A$ of A modulo I such that, for any x in $\mathcal{R}$,*

$$\sum_{\sigma \in \Sigma} |x|_\sigma \leq C_2 \left(\frac{r+3}{4} \right) N(I) \, ,$$

where $N(I) = \# (A/I)$ and C_2 is the constant in Lemma 2.

Proof. According to the proof of Lemma 2, the $\mathbf{Z}$-module A contains a basis of r elements $e_1, \ldots, e_r$ such that

$$\sum_\sigma |e_i|_\sigma \leq \mu_r \leq \frac{2}{r}\, C_2 \, .$$

Therefore, by Lemma 5 applied to the field $\mathbf{Q}$, in which case $C_1 = C_3 = 1$ and $C_2 = 1/2$, there exists a basis (f_i) of A over $\mathbf{Z}$ such that,

$$\sum_{\sigma \in \Sigma} |f_i|_\sigma \leq \frac{2}{r}\, C_2 \left(\frac{i-1}{2} + 1 \right) .$$

Since the integer $n = N(I)$ belongs to I, the map $A/I \to A/n\,A$ is injective and we can choose $\mathcal{R}$ among those

$$x = \sum_{i=1}^r x_i\, f_i$$

such that $x_i \in \mathbf{Z}$ and $|x_i| \leq n/2$. In that case, if $x \in \mathcal{R}$, we have

$$\sum_{\sigma \in \Sigma} |x|_\sigma \leq \frac{n}{2} \sum_{\sigma, i} |f_i|_\sigma \leq n\, C_2\, \frac{r+3}{4} \, .$$

q.e.d.

2.3

LEMMA 7. *Let $\overline{M}$ be an hermitian lattice and assume that $M = L_1 \oplus L_2$ is the direct sum of two lattices of rank one. Let $f_i \in L_i$ be a non zero vector, and $n_i = \#\,(L_i/A\,f_i)$, $i = 1, 2$. Then there exists a vector $e_1 \in M$, and an isomorphism*

$$\psi : A\,e_1 \oplus L \to M$$

such that L contains a vector e_2 with

$$\#\,(L/A\,e_2) \leq n_1\,n_2\,,$$

$$\|e_1\| \leq n_2\,C_2\,\|f_1\| + \left(1 + C_2^2\,\frac{r+3}{4}\,n_1^r\right)\|f_2\|\,,$$

and

$$\|\psi(e_2)\| \leq n_2\,\|f_1\| + C_2\,\frac{r+3}{4}\,n_1^r\,\|f_2\|\,.$$

PROOF. The algebraic content of this lemma is [9], Lemma 1.7, p. 12. To control the norms in this proof we first define an isomorphism

$$u_i : L_i \to I_i$$

where I_i is an ideal of A. If $x \in L_i$, $u_i(x) \in A$ is the unique element such that

$$n_i\,x = u_i(x)\,f_i\,, \qquad i = 1, 2\,.$$

In particular $n_i = u_i(f_i)$.
Next, we choose an ideal J_1 in the class of I_1 which is prime to I_2. According to [9], proof of Lemma 1.8, we can choose

$$J_1 = \frac{x_0}{a_0}\,I_1\,,$$

where a_0 is any element of $I_1 - \{0\}$ and x_0 belongs to a set of representatives of A modulo $I_1\,J$, where $I_1\,J = a_0\,A$.
According to Lemma 6 we can assume that

$$\sum_{\sigma\in\Sigma} |x_0|_\sigma \leq C_2\left(\frac{r+3}{4}\right)N(I_1\,J) = C_2\left(\frac{r+3}{4}\right)N(a_0)\,.$$

The composite isomorphism

$$v_1 : L_1 \to J_1 \to I_1$$

maps f_1 to $n_1\,x_0/a_0$. We choose $a_0 = n_1$, hence $v_1(f_1) = x_0$ and

$$\sum_{\sigma\in\Sigma} |x_0|_\sigma \leq C_2\left(\frac{r+3}{4}\right)n_1^r\,.$$

The direct sum of the inverses of v_1 and u_2 is an isomorphism

$$\varphi : J_1 \oplus I_2 \xrightarrow{\sim} L_1 \oplus L_2 = M\,.$$

Since J_1 and I_2 are prime to each other we have an exact sequence (as in [9] loc.cit.)

$$0 \longrightarrow J_1\, I_2 \longrightarrow J_1 \oplus I_2 \xrightarrow{p} A \longrightarrow 0$$

where p is the sum in A. Let

$$s : A \longrightarrow J_1 \oplus I_2$$

be any section of p and let $\alpha \in J_1$ be such that

$$s(1) = (\alpha, 1-\alpha)\,.$$

Let $\alpha = \lambda\, n_2\, x_0$ with $\lambda \in F$. Applying Lemma 2, we choose $a \in A$ such that

$$\sum_{\sigma\in\Sigma} |\lambda - a|_\sigma \leq C_2\,.$$

Since $n_2\, x_0$ lies in $J_1\, I_2$ the element

$$\beta = \alpha - a\, n_2\, x_0 = (\lambda - a)\, n_2\, x_0$$

lies in J_1, and $1-\beta$ lies in I_2. Since

$$\beta = v_1((\lambda - a)\, n_2\, f_1)$$

and

$$1 - \beta = u_2\left(\frac{1}{n_2} - (\lambda - a)\, x_0\, f_2\right)$$

we get

$$\begin{aligned} \|\varphi(\beta, 1-\beta)\| &\leq \|(\lambda - a)\, n_2\, f_1\| + \left\|\left(\frac{1}{n_2} - (\lambda - a)\, x_0\right) f_2\right\| \\ &\leq n_2\, C_2\, \|f_1\| + \left(1 + C_2^2\left(\frac{r+3}{4}\right) n_1^r\right) \|f_2\|\,. \end{aligned}$$

We let $e_1 = \varphi(\beta, 1-\beta)$. On the other hand we define

$$L = J_1\, I_2 (\simeq L_1 \otimes L_2)$$

and map L to M by the composite map

$$L \xrightarrow{i} J_1 \oplus I_2 \xrightarrow{\varphi} M$$

where $i(x) = (x, -x)$. We choose

$$e_2 = n_2\, x_0 \in L$$

so that $\varphi \circ i\,(e_2) = (n_2\, v_1(f_1), x_0\, u_2(f_2))$ has norm

$$\|\varphi \circ i\,(e_2)\| \leq n_2\, \|f_1\| + C_2 \left(\frac{r+3}{4} \right) n_1^r\, \|f_2\|\,.$$

Furthermore we have isomorphisms

$$L \oplus A \xrightarrow{(i,s)} J_1 \oplus I_2 \xrightarrow{\varphi} M$$

and

$$\#\,(L/A\,e_2) = \#\,(J_1\, I_2/A\,e_2) \leq \#\,(J_1/A\,x_0) \times \#\,(I_2/A\,n_2) = n_1\, n_2\,.$$

q.e.d.

2.4

Proposition 1. *Let $\overline{M}$ be a rank N hermitian free A-module such that its unit ball contains a basis of $M \otimes_A F$. Then M has a basis $(e_1, \dots, e_N)$ such that*

$$\|e_i\| \leq B_i$$

with $B_i = (i-1)\, C_2 + C_3$, $i = 1, \dots, N$, when A is principal and

$$B_i = (1 + C_1\, C_2)(N\, C_2 + C_3) \left(1 + C_2\, \frac{r+3}{4} \right)^{\log_2(N)+2} C_1^{2(r+1)N/i}$$

in general. Here $\log_2(N)$ is the logarithm of N in base 2.

Proof. When A is principal, $C_1 = 1$ and Proposition 1 follows from Lemma 5.
In general Lemma 5 tells us that

$$M = L_1 \oplus \cdots \oplus L_N$$

and L_i contains a vector f_i with $\#\,(L_i/A\, f_i) \leq C_1$ and

$$\|f_i\| \leq C_1((i-1)\, C_2 + C_3) \leq C_1\, (N\, C_2 + C_3)\,.$$

Let $k > 1$ be an integer and $\lambda > 0$ be a real number. We shall prove by induction N that, if M has a decomposition as above with

$$\#\,(L_i/A\, f_i) \leq k$$

and

$$\|f_i\| \leq k\, \lambda\,,$$

then M has a basis $(e_1, \dots, e_N)$ such that

$$\|e_i\| \leq \lambda \left(\frac{1}{k} + C_2 \right) \left(1 + C_2\, \frac{r+3}{4} \right)^t k^{(r+1)(1+2+\cdots+2^t)}\,, \tag{10}$$

for all $i = 1, \dots, N$, where $t \geq 1$ is such that

$$\frac{N}{2^t} < i \leq \frac{N}{2^{t-1}}\,.$$

The case $N \leq 2$ follows from Lemma 7. If $N > 2$, let N' be the integral part of $N/2$. Applying Lemma 7 to every direct sum $L_i \oplus L_{N-i}$, $N/2 < i \leq N$, we get

$$M = M' \oplus \left(\bigoplus_{i=N'+1}^{N} A\, e_i \right)$$

with

$$\begin{aligned} \|e_i\| &\leq k\lambda \left(1 + C_2\, k + C_2^2\, \frac{r+3}{4}\, k^r \right) \\ &\leq \lambda \left(\frac{1}{k} + C_2 \right) \left(1 + C_2\, \frac{r+3}{4} \right) k^{r+1} \end{aligned}$$

and M' is free, $M' = \overset{N'}{\underset{i=0}{\oplus}} L'_i$, and each L'_i contains a vector f'_i such that

$$[L'_i : A\, f'_i] \leq k^2$$

and

$$\|f'_i\| \leq \lambda \left(1 + C_2\, \frac{r+3}{4} \right) k^{r+1}\,.$$

By the induction hypothesis, M' has a basis (e_i), $1 \leq i \leq N'$, such that

$$\|e_i\| \leq \left(1 + C_2\, \frac{r+3}{4} \right) k^(r+1) \left(\frac{1}{k^2} + C_2 \right) \left(1 + C_2\, \frac{r+3}{4} \right)^t (k^2)^{(r+1)(1+\cdots+2^t)}$$

whenever

$$\frac{N'}{2^t} < i \leq \frac{N'}{2^{t-1}}\,.$$

If

$$\frac{N}{2^{t+1}} < i \leq \frac{N}{2^t}\,,$$

this inequality implies

$$\|e_i\| \leq \lambda \left(\frac{1}{k} + C_2 \right) \left(1 + C_2\, \frac{r+3}{4} \right)^{t+1} k^{(r+1)(1+\cdots+2^{t+1})}\,.$$

Therefore M satisfies the induction hypothesis (10).
Since

$$1 + 2 + \cdots + 2^t = 2^{t+1} - 1 \leq \frac{2N}{i}$$

and $t \leq \log_2(N) + 1$, Proposition 1 follows by taking $k = C_1$ and

$$\lambda = C_1(N\, C_2 + C_3)$$

in (10). q.e.d.

2.5

Let $\overline{M}$ be a rank N hermitian free A-module. We let

$$m(h) = \mathrm{Inf}\,\{h(x)\,,\ x \in M - \{0\}\}$$

be the minimum value of h on $M - \{0\}$ and

$$M(h) = \{x \in M / h(x) = m(h)\}$$

be the (finite) set of minimal vectors of M. Let ω_N be the standard volume of the unit ball in $\mathbf{R}^N$.

Proposition 2. *Let $\overline{M} = (M,h)$ be as above. Assume that $m(h) = 1$ and that $M(h)$ spans the F-vector space $M \otimes_A F$. Then M has a basis $f_1, \ldots, f_N$ such that any $x \in M(h)$ is of the form*

$$x = \sum_{i=1}^{N} y_i\, f_i$$

with

$$\sum_{\sigma \in \Sigma} |y_i|_\sigma^2 \leq T_i\,,$$

$$T_i = r^{rN}\, C_3^{2rN+2}\, \gamma^N \prod_{j \neq i} B_j^2\,,$$

and

$$\gamma = 4^{r_1+r_2}\, \omega_N^{-2r_1/N}\, \omega_{2N}^{-2r_2/N}\, D\,.$$

Proof. From Proposition 1 we know that M has a basis $(e_1, \ldots, e_N)$ with $\|e_i\| \leq B_i$. Let $x \in M(h)$ be a minimal vector and (x_i) its coordinates in the basis (e_i).
Fix $i \in \{1, \ldots, N\}$ and $\sigma \in \Sigma$. Consider the square matrix

$$H_i = (h_\sigma(v_k, v_\ell))\,,$$

where $v_k = e_k$ if $k \neq i$ and $v_i = x$. Furthermore, let

$$H_\sigma = (h_\sigma(e_k, e_\ell))\,.$$

Since

$$|x_i|_\sigma^2 = \det(H_i)\, \det(H_\sigma)^{-1}$$

the Hadamard inequality implies

$$|x_i|_\sigma^2 \leq h_\sigma(x) \prod_{j \neq i} h_\sigma(e_j)\, \det(H_\sigma)^{-1}\,.$$

For any unit $u \in A^*$ we can replace e_i by $u^{-1}\, e_i$, and x_i by $y_i = u\, x_i$. We then have

$$\sum_{\sigma \in \Sigma} |y_i|_\sigma^2 \leq \sum_{\sigma \in \Sigma} h_\sigma(x) \prod_{j \neq i} h_\sigma(e_j)\, |u|_\sigma^2 \, \det(H_\sigma)^{-1} . \tag{11}$$

Applying Lemma 3 to $\lambda_\sigma = \det(H_\sigma)^{-1/2}$ we find u such that, for all $\sigma \in \Sigma$,

$$|u|_\sigma^2 \det(H_\sigma)^{-1} \leq C_3^2 \prod_{\sigma \in \Sigma} \det(H_\sigma)^{-1} . \tag{12}$$

Since $\sum\limits_{\sigma} h_\sigma(x) = 1$ and $h_\sigma(e_j) \leq \|e_j\|^2 \leq B_j^2$, we deduce from (11) and (12) that

$$\sum_{\sigma \in \Sigma} |y_i|_\sigma^2 \leq C_3^2 \cdot \prod_{j \neq i} B_j^2 \cdot \prod_{\sigma \in \Sigma} \det(H_\sigma)^{-1} . \tag{13}$$

According to Icaza [4], Theorem 1, there exists $z \in L$ such that

$$\prod_{\sigma \in \Sigma} h_\sigma(z) \leq \gamma \prod_{\sigma \in \Sigma} \det(H_\sigma)^{1/N}$$

with

$$\gamma = 4^{r_1 + r_2}\, \omega_N^{-2r_1/N}\, \omega_{2N}^{-2r_2/N}\, D .$$

Using Lemma 3 again and the fact that $m(h) = 1$, we find $v \in A^*$ such that

$$\begin{aligned} 1 \ &\leq \ h(vz) \leq r\, C_3^2 \prod_{\sigma \in \Sigma} h_\sigma(z)^{1/r} \\ &\leq \ r\, C_3^2\, \gamma^{1/r} \prod_{\sigma \in \Sigma} \det(H_\sigma)^{1/rN} . \end{aligned}$$

From this it follows that

$$\prod_{\sigma \in \Sigma} \det(H_\sigma)^{-1} \leq (r\, C_3^2)^{rN}\, \gamma^N \tag{14}$$

and Proposition 2 follows from (13) and (14).

2.6

To count the number of vectors in $M(h)$ using Proposition 2 we shall apply the following lemma :

Lemma 8. *The number of elements a in A such that*

$$\sum_{\sigma \in \Sigma} |a|_\sigma^2 \leq T$$

is at most

$$B(T) = \mathrm{Sup}\,(T^{r/2}\,2^{r+3}, 1)\,.$$

Proof. When $r_2 > 0$, this follows from [7], V § 1, Theorem 0, p. 102, by noticing that one can take $C_3 = 2^{r+3}$ in loc.cit. When $r_2 = 0$, the argument is similar.

3 Reduction theory

3.1

Fix an integer $N \geq 2$. Let

$$\Gamma = \mathrm{GL}_N(A)$$

and

$$G = \mathrm{GL}_N(F \otimes_{\mathbf{Q}} \mathbf{R})\,.$$

On the standard lattice $L_0 = A^N$ consider the hermitian metric h_0 defined by

$$h_0(x,y) = \sum_{\sigma \in \Sigma} \sum_{i=1}^{N} x_{i\sigma}\,\overline{y_{i\sigma}}$$

for all vectors $x = (x_{i\sigma})$ and $y = (y_{i\sigma})$ in $L_0 \otimes_{\mathbf{Z}} \mathbf{C} = (\mathbf{C}^N)^\Sigma$. Any $g \in G$ defines an hermitian metric $h = g(h_0)$ on L_0 by the formula

$$g(h_0)\,(x,y) = h_0(g(x), g(y))\,.$$

Let K be the stabilizer of h_0 and G and $X = K\backslash G$. We can view each $h \in X$ as a metric on L_0.
Following Ash [1], we say that a finite subset $M \subset L_0$ is *well-rounded* when it spans the F-vector space $L_0 \otimes_A F$. We let $\widetilde{W} \subset X$ be the space of metrics h such that $m(h) = 1$ and $M(h)$ is well-rounded. Given a well-rounded set $M \subset L_0$ we let $C(M) \subset \widetilde{W}$ be the set of metrics h such that

- $h(x) = 1$ for all $x \in M$
- $h(x) > 1$ for all $x \in L_0 - (M \cup \{0\})$.

As explained in [1], proof of (iv), pp. 466-467, $C(M)$ is either empty or topologically a cell, and the family of closed cells $\overline{C(M)}$ gives a Γ-invariant cellular decomposition of $\widetilde{W}$, such that

$$\overline{C(M)} = \coprod_{M' \supset M} C(M')\,.$$

Furthermore $\widetilde{W}/\Gamma$ is compact, of dimension $\dim(X) - N$.

3.2

Proposition 3. i) *For any integer $k \geq 0$, the number of cells of codimension k in $\widetilde{W}$ is at most*

$$c(k,N) = \binom{a(N)}{N+k}$$

where

$$a(N) = 2^{N(r+3)} \left(\prod_{i=1}^{N} T_i \right)^{r/2} ,$$

and T_i is as in Proposition 2.
ii) *Given a cell in $\widetilde{W}$, its number of codimension one faces is at most $a(N)^{N+1}$.*

Proof. Let Φ be the set of vectors $x = (x_i)$ in A^N such that, for all $i = 1, \ldots, N$,

$$\sum_{\sigma \in \Sigma} |x_i|_\sigma^2 \leq T_i \, .$$

Given $h \in \widetilde{W}$, Proposition 2 says that we can find a basis (f_i) of L_0 such that any x in $M(h)$ has its coordinates (x_i) bounded as above. If $\gamma \in \Gamma$ is the matrix mapping the standard basis of A^N to (f_i), this means that $M(\gamma(h)) = \gamma^{-1}(M(h))$ is contained in Φ.
Let $\overline{C(M)}$ be a nonempty closed cell of codimension k in $\widetilde{W}$. For any $x \in L_0$, the equation $h(x) = 1$ defines a real affine hyperplane in the set of $N \times N$ hermitian matrices with coefficients in $(F \otimes_{\mathbf{Q}} \mathbf{C})^+$. The equations $h(x) = 1$, $x \in M$, may not be linearly independent, but, since $\overline{C(M)}$ has codimension k, M has at least $N + k$ elements. And since $M \subset M(h)$ for some $h \in \widetilde{W}$, there exists $\gamma \in \Gamma$ such that $\gamma^{-1}(M)$ is contained in Φ. Therefore, modulo the action of Γ, there are at most $\binom{\mathrm{card}(\Phi)}{N+k}$ cells $\overline{C(M)}$ of codimension k. From Lemma 7 we know that

$$\mathrm{card}(\Phi) \leq a(N) \, ,$$

therefore i) follows.
To prove ii), consider a cell $\overline{C(M)}$ and a codimension one face $\overline{C(M')}$ of $\overline{C(M)}$. We can write $M' = M \cup \{x\}$ for some vector x and there exists $\gamma \in \Gamma$ such that $\gamma(M') \subset \Phi$. Since M is well-rounded, the matrix γ is entirely determined by the set of vectors $\gamma(M)$, i.e. there are at most $\mathrm{card}(\Phi)^N$ matrices γ such that $\gamma(M) \subset \Phi$. Since $\gamma(x) \in \Phi$, there are at most $\mathrm{card}(\Phi)^{N+1}$ vectors x as above.
q.e.d.

3.3

Lemma 9. *Let $\gamma \in \Gamma - \{1\}$ and p be a prime number such that $\gamma^p = 1$. Then*

$$p \leq 1 + \mathrm{Sup}(r, N) \, .$$

Proof. Since γ is non trivial we have $P(\gamma) = 0$ where P is the cyclotomic polynomial

$$P(x) = X^{p-1} + X^{p-2} + \cdots + 1\,.$$

If F does not contain the p-th roots of one, P is irreducible, and therefore it divides the characteristic polynomial of the matrix γ over F, hence $p-1 \leq N$. Otherwise, F contains $Q(\mu_p)$, which is of degree $p-1$, therefore $p-1 \leq r$.

4 The main results

4.1

For any integer $n > 0$ and any finite abelian group A we let $\mathrm{card}_n(A)$ be the largest divisor of the integer $\#(A)$ such that no prime $p \leq n$ divides $\mathrm{card}_n(A)$. Let $N \geq 2$ be an integer. We keep the notation of § 3 and we let

$$\widetilde{w} = \dim(X) - N = r_1\,\frac{N(N+1)}{2} + r_2\,N^2 - N$$

be the dimension of $\widetilde{W}$. For any $k \leq \widetilde{w}$ we define

$$h(k,N) = a(N)^{(N+1)c(\widetilde{w}-k-1,N)}\,,$$

where $c(\cdot, N)$ and $a(N)$ are defined in Proposition 3.

Theorem 1. *The torsion subgroup of the homology of* $\mathrm{GL}_N(A)$ *is bounded as follows*

$$\mathrm{card}_{1+\sup(r,N)} H_k(\mathrm{GL}_N(A), \mathbf{Z})_{\mathrm{tors}} \leq h(k,N)\,.$$

Proof. We know from [1] that $\widetilde{W}$ is contractible and the stabilizer of any $h \in \widetilde{W}$ is finite. From Lemma 9 it follows that, modulo $\mathcal{S}_{1+\sup(r,N)}$, the homology of $\Gamma = \mathrm{GL}_N(A)$ is the homology of a complex $(C_., \partial)$, where C_k is the free abelian group generated by a set of Γ-representatives of those k-dimensional cells c in $\widetilde{W}$ such that the stabilizer of c does not change its orientation ([2], VII). According to Proposition 3, the rank of C_k is at most $c\,(\widetilde{w}-k, N)$ and any cell of $\widetilde{W}$ has at most $a(N)^{N+1}$ faces. Theorem 1 then follows from a general result of Gabber ([13], Proposition 3 and equation (18)).

4.2

For any integer $m \geq 1$ let

$$k(m) = h(m, 2\,m+1)\,.$$

Denote by $K_m(A)$ the m-th algebraic K-group of A.

Theorem 2. *The following inequality holds*

$$\mathrm{card}_{\sup(r+1,2m+2)} K_m(A)_{\mathrm{tors}} \leq k(m)\,.$$

Proof. As in [13], Theorem 2, we consider the Hurewicz map

$$H : K_m(A) \to H_m(\mathrm{GL}(A), \mathbf{Z})\,,$$

the kernel of which lies in $\mathcal{S}_n$, $n \leq (m+1)/2$. Since, according to Maazen and Van der Kallen,

$$H_m(\mathrm{GL}(A), \mathbf{Z}) = H_m(\mathrm{GL}_N(A), \mathbf{Z})$$

when $N \geq 2\,m+1$, Theorem 2 is a consequence of Theorem 1.

4.3

Let p be an odd prime and $n \geq 2$ an integer. For any $\nu \geq 1$ denote by $\mathbf{Z}/p^\nu(n)$ the étale sheaf $\mu_{p^\nu}^{\otimes n}$ on $\mathrm{Spec}(A[1/p])$, and let

$$H^2(\mathrm{Spec}(A[1/p]), \mathbf{Z}_p(n)) = \varprojlim_{\nu} H^2(\mathrm{Spec}(A[1/p]), \mathbf{Z}_{p^\nu}(n))\,.$$

From [12], we know that this group is finite and zero for almost all p.

Theorem 3. *The following inequality holds*

$$\prod_{\substack{p\geq 4n-1\\ p\geq r+2}} \mathrm{card}\, H^2(\mathrm{Spec}(A[1/p]), \mathbf{Z}_p(n)) \leq k(2n-2)\,.$$

Proof. According to [12], the cokernel of the Chern class

$$c_{n,2} : K_{2n-2}(A) \to H^2(\mathrm{Spec}(A[1/p]), \mathbf{Z}_p(n))$$

lies in $\mathcal{S}_{n+1}$ for all p. Furthermore, Borel proved that $K_{2m-2}(A)$ is finite. Therefore Theorem 3 follows from Theorem 2.

4.4

By Lemmas 1 to 7 and Propositions 1 to 3, the constant $k(m)$ is explicitly bounded in terms of m, r and D. We shall now simplify this upper bound.

Proposition 4. i) $\log\log k(m) \leq 220\, m^4 \log(m)\, r^{4r} \sqrt{D}\, \log(D)^{r-1}$

ii) *If F has class number one,*

$$\log\log k(m) \leq 210\, m^4 \log(m)\, r^{4r} \sqrt{D}\, \log(D)^{r-1}$$

iii) *If* $F = \mathbf{Q}(\sqrt{-D})$ *is imaginary quadratic*

$$\log\log k(m) \leq 1120\, m^4 \log(m) \log(D)\,;$$

if furthermore F *has class number one*

$$\log\log k(m) \leq 510\, m^4 \log(m) \log(D)\,.$$

iv) *When* $F = \mathbf{Q}$ *and* $m \geq 9$

$$\log\log k(m) \leq 8\, m^4 \log(m)\,;$$

furthermore

$$\log\log k(7) \leq 40\,545$$

and

$$\log\log k(8) \leq 70\,130\,.$$

PROOF. By definition

$$k(m) = h(m, 2\,m+1) = a(N)^{(N+1)c(\widetilde{w}-m-1,N)}$$

with $N = 2\,m+1$ and

$$c(\widetilde{w}-m-1, N) = \binom{a(N)}{N+\widetilde{w}-m-1}\,.$$

Since

$$\begin{aligned} N+\widetilde{w}-m-1 &= r_1\,\frac{N(N+1)}{2} + r_2\,N^2 - m - 1 \\ &\leq 2\,r\,m^2 + 3\,r\,m + r - 2\,m - 1 \end{aligned}$$

and since $a(2\,m+1)$ is very big, we get

$$\begin{aligned} \log\log k(m) &\leq (2\,r\,m^2 + 3\,r\,m + r - 2\,m - 1)\log a(2\,m+1) \\ &+ \log(2\,m+2) + \log\log a(2\,m+1) \\ &\leq r(2\,m^2 + 3\,m + 1)\log a(2\,m+1)\,. \end{aligned} \tag{15}$$

From Proposition 3 and Proposition 2 we get

$$a(N) = 2^{N(r+3)}\left(\prod_{i=1}^{N} T_i\right)^{r/2}, \tag{16}$$

and

$$\prod_{i=1}^{N} T_i = (r^{rN}\,\gamma^N\,C_3^{2rN+2})^N \prod_{i=1}^{N} B_i^{N-1}\,. \tag{17}$$

According to Proposition 1

$$\prod_{i=1}^{N} B_i = \left[(1 + C_1\, C_2)(N\, C_2 + C_3)\left(1 + C_2\, \frac{r+3}{4}\right)^{\log_2(N)+2}\right]^N \cdot C_1^{2(r+1)NH_N}\,, \quad (18)$$

where

$$H_N = \sum_{i=1}^{N} \frac{1}{i} \le 1 + \log(N)\,.$$

Assume $s \neq 0$. Then the upper bound C_3^* we get from Lemmas 3 and 4 for C_3 is much bigger than C_2. Therefore

$$\log(N\, C_2 + C_3) \le \log(N) + \log(C_3^*)\,. \quad (19)$$

We deduce from (15), (16), (17), (18), (19) that

$$\log\log k(m) \le X_1 + X_2$$

with

$$X_1 = r(2\, m^2 + 3\, m + 1)\, \frac{r}{2}\, (N(2\, r\, N + 2) + N(N-1)) \log(C_3^*)$$

and

$$\begin{aligned} X_2 &= r(2\, m^2 + 3\, m + 1)\Bigg(N(r+3)\log(2) \\ &+ \frac{r}{2}\, N\Bigg(N \log(\gamma) + (N-1)\Bigg[\log(1 + C_1\, C_2) + \log(N) \\ &+ (\log_2(N) + 2)\log\left(1 + C_2\, \frac{r+3}{4}\right) \\ &+ 2(r+1)(1 + \log(N))\log(C_1)\Bigg]\Bigg)\Bigg)\,. \end{aligned} \quad (20)$$

Since $s \le r - 1$, Lemma 3 and Lemma 4 imply

$$\log(C_3^*) \le 11 r^2 (r-1)(4\, r(\log 3\, r)^3)^{r-2}\, 2^{r-1} \sqrt{D}\, \log(D)^{r-1}\,,$$

from which it follows that

$$X_1 \quad \le \quad 208\, \log(m)\, m^4\, r^{4r} \sqrt{D}\, \log(D)^{r-1}$$

when $m \ge 2$ and $r \ge 2$.
To evaluate X_2 first notice that

$$4\, \omega_N^{-2/N} \le 1 + N/4$$

by [10], II, (1.5), Remark, hence

$$\begin{aligned}\log(\gamma) &\leq r_1 \log\left(1+\frac{N}{4}\right) + 2\, r_2 \log\left(1+\frac{N}{2}\right) + \log(D) \\ &\leq r\log(N) + \log(D) \qquad (21)\end{aligned}$$

since $N \geq 5$.
By the Stirling formula and Lemma 1, if $r \geq 2$,

$$\begin{aligned}\log(C_1) &= \log(r!) - r\log(r) + r_2 \log\left(\frac{4}{\pi}\right) + \frac{1}{2}\log(D) \\ &\leq 1 + \frac{1}{2}\log(r) + \frac{1}{2}\log(D)\,, \qquad (22)\end{aligned}$$

$$\log\left(1 + C_2\,\frac{r+3}{4}\right) \leq \mathrm{Sup}\left(\log(C_2) + \log\left(\frac{r+3}{4}\right) + 1, \log(2)\right)\,,$$

where

$$\begin{aligned}\log(C_2) + \log(\frac{r+3}{4}) &\leq r\log(4) - (r-2)\log(r) - \log(r!) \\ &+ \log(\frac{r+3}{4}) + \frac{1}{2}\log(D) \\ &\leq 2.4 + \frac{1}{2}\log(D)\,,\end{aligned}$$

so that

$$\log\left(1 + C_2\,\frac{r+3}{4}\right) \leq 3.4 + \frac{1}{2}\log(D)\,. \qquad (23)$$

We also have

$$\log(1 + C_1\,C_2) \leq \mathrm{Sup}(1 + \log(C_1) + \log(C_2), \log(2))$$

and

$$\begin{aligned}\log(C_1) + \log(C_2) &\leq -r\log(r) + r - (r-2)\log(r) + r\log(4) + \log(D) \\ &\leq 3.4 + \log(D)\,,\end{aligned}$$

so that

$$\log(1 + C_1\,C_2) \leq 4.4 + \log(D)\,. \qquad (24)$$

From (20), (21), (22), (23), (24) we get

$$X_2 \leq a\log(D) + b$$

with

$$\begin{aligned} a &= r(2\,m^2+3\,m+1)(2\,m+1)\Big(\frac{r}{2}\,((2\,m+1)+2\,m+m\log_2(2\,m+1)+m \\ &+ 2\,m\,(r+1)(1+\log(2\,m+1)))\Big) \le 75\,r^3\,m^4\log(m) \end{aligned}$$

if $r \geq 2$ and $m \geq 2$.
Finally

$$\begin{aligned} b &= r(2\,m^2+3\,m+1)(2\,m+1)\Big((r+3)\log(2)+\frac{r}{2}\,(2\,m+1)\,r(\log(r)+\log(2\,m+1)) \\ &+ \frac{r}{2}\,(2\,m)\Big(4.4+\log(2\,m+1)+3.4(\log_2(2\,m+1)+2) \\ &+ 2\,(r+1)\,(1+\log(2\,m+1))\left(1+\frac{1}{2}\log(r)\right)\Big)\Big) \le 148\,r^4\,m^4\log(m) \end{aligned}$$

when $r \geq 2$ and $m \geq 2$.
Therefore

$$\begin{aligned} \log\log k(m) &\le 208\log(m)\,m^4\,r^{4r}\sqrt{D}\log(D)^{r-1}+75\,r^3\,m^4\log(m)\log(D) \\ &+ 148\,r^4\,m^4\log(m) \le 220\,m^4\log(m)\,r^{4r}\sqrt{D}\log(D)^{r-1} \end{aligned}$$

when m, r and D are at least 2. This proves i).
If we assume that A is principal, we can take $C_1 = 1$ in Lemma 1 and $B_i = (i-1)\,C_2 + C_3$ in Proposition 1. Since $C_2 < C_3$ we get

$$\log\left(\prod_{i=1}^{N} B_i\right) \le \log(N!) + N\log(C_3)$$

and

$$\log\log k(m) \le X_1 + X_3$$

where

$$\begin{aligned} X_3 &= r(2\,m^2+3\,m+1)\Big[(r+3)(2\,m+1)\log(2)+\frac{r^2}{2}(2\,m+1)^2\log(r) \\ &+ \frac{r}{2}\,(2\,m+1)^2\log(\gamma)+\frac{r}{2}\,(2\,m)\log((2\,m+1)!)\Big] \\ &\le 6\,m^4\,r^2\log(D)+2\,r^{4r}\,m^4\log(m)\,. \end{aligned}$$

Therefore

$$X_1 + X_3 \le 210\,m^4\log(m)\,r^{4r}\sqrt{D}\log(D)^{r-1}\,.$$

Assume now that $r_1 + r_2 = 1$. Then $C_3 = 1$ and the term X_1 disappears from the above computation. Assume first that $F = \mathbf{Q}(\sqrt{-D})$. Since $r_2 = 1$ and $r_1 = 0$ we get

$$\log\log k(m) \le (4\,m^2+3\,m+1)\log a(2\,m+1)\,.$$

Furthermore (18) becomes

$$\prod_{i=1}^{N} B_i \leq \left[(1+C_1\,C_2)(1+N\,C_2)\left(1+\frac{5}{4}\,C_2\right)^{\log_2(N)+2}\right]^N \cdot C_1^{6N(1+\log(N))}\,.$$

Therefore

$$\begin{aligned}
\log\log k(m) &\leq (4\,m^2+3\,m+1)\Big[5\,N\log(2)+2N^2\log(2)\\
&+ N^2\log(\gamma)+N(N-1)\Big[\log(1+C_1\,C_2)\\
&+ \log(1+N\,C_2)+(\log_2(N)+2)\log\left(1+\frac{5}{4}\,C_2\right)\Big]\\
&+ 6\,N(1+\log N)\log(C_1)\Big]\,,
\end{aligned}$$

with $N=2\,m+1$. We have now

$$\gamma \leq \left(1+\frac{N}{2}\right)^2 D\,,$$

$$C_1=\frac{2}{\pi}\sqrt{D} \quad \text{and} \quad C_2=\frac{\pi}{2}\sqrt{D}\,.$$

This implies

$$\log\log k(m) \leq 597\,m^4\log(m)+256\,m^4\log(m)\log(D) \leq 1120\,m^4\log(m)\log(D)\,.$$

If $F=\mathbf{Q}(\sqrt{-D})$ is principal we can take $C_1=1$ and $B_i=(i-1)\,C_2+1$. We get

$$\log\log k(m) \leq 510\,m^4\log(m)\log(D)\,.$$

Finally, assume that $F=\mathbf{Q}$. Then

$$B_i=\frac{i+1}{2} \quad \text{since} \quad C_2=\frac{1}{2}\,, \quad \text{and} \quad \gamma \leq 1+\frac{N}{4}\,.$$

Therefore

$$\begin{aligned}
\log\log k(m) &\leq (2\,m^2+2\,m+1)\log a(2\,m+1)\\
&\leq (2\,m^2+2\,m+1)\Big[4\,N\log(2)+\frac{N^2}{2}\log\left(1+\frac{N}{4}\right)\\
&+ \frac{N-1}{2}\log\left(\prod_{i=1}^{N}\frac{i+1}{2}\right)\Big]\\
&\leq 8\,m^4\log(m)
\end{aligned}$$

if $m \geq 9$. We can also estimate $k(7)$ and $k(8)$ from this inequality above. This proves iv).

5 Discussion

5.1

The upper bound in Theorem 2 and Propostition 4 seems much too large. When $m = 0$, $\mathrm{card}\, K_0(A)_{\mathrm{tors}}$ is the class number $h(F)$, which is bounded as follows:

$$h(F) \leq \alpha \sqrt{D} \log(D)^{r-1}, \tag{25}$$

for some constant $\alpha(r)$ [11], Theorem 4.4, p. 153. Furthermore, when $F = \mathbf{Q}$, $m = 2n - 2$ and n is even, the Lichtenbaum conjecture predicts that $\mathrm{card}\, K_{2n-2}(\mathbf{Z})$ is the order of the numerator of B_n/n, where B_n is the n-th Bernoulli number. The upper bound

$$B_n \leq n^! \approx n^n$$

suggests, since the denominator of B_n/n is not very big, that $\mathrm{card}\, K_m(\mathbf{Z})_{\mathrm{tors}}$ should be exponential in m. We are thus led to the following:

Conjecture. *Fix $r \geq 1$. There exists positive constants α, β, γ such that, for any number field F of degree r on $\mathbf{Q}$,*

$$\mathrm{card}\, K_m(A)_{\mathrm{tors}} \leq \alpha \exp(\beta m^\gamma \log D).$$

Furthermore, we expect that γ does not depend on r.

5.2

As suggested by A. Chambert-Loir, it is interesting to consider the analog in positive characteristic of the conjecture above. Let X be a smooth connected projective curve of genus g over the finite field with q elements, $\zeta_X(s)$ its zeta function and

$$P(t) = \prod_{i=1}^{2g} (1 - \alpha_i\, t),$$

where α_i are the roots of Frobenius acting on the fist ℓ-adic cohomology group of X. When $n > 1$, it is expected that the finite group $K_{2n-2}(X)$ has order the numerator of $\zeta_X(1-n)$, i.e. $P(q^{n-1})$. Since $|\alpha_i| = q^{1/2}$ for all $i = 1 \cdots 2g$, we get

$$P(q^{n-1}) \leq (1 + q^{n-1/2})^{2g} \leq q^{2ng}.$$

In the analogy between number fields and function fields, the genus g is known to be an analog of $\log(D)$. Therefore the bound above is indeed analogous to the conjecture in §5.1.

5.3

The upper bound for $k(m)$ in Proposition 4 i) is twice exponential in D. One exponential is due to our use of Lemma 3, where C_3 is exponential in D. Maybe this can be improved in general, and not only when $s = 0$.
The exponential in D occuring in Proposition 4 ii) might be due to our use of the geometry of numbers. Indeed, if one evaluates the class number $h(F)$ by applying naively Minkowski's theorem (Lemma 1), the bound one gets is exponential in D; see however [8], Theorem 6.5., for a better proof.

5.4

One method to prove (25) consists in combining the class number formula (see (7) and (8)) with a lower bound for the regulator $R(F)$. This suggests replacing the arguments of this paper by analytic number theory, to get good upper bounds for étale cohomology.
More precisely, let $n \geq 2$ be an integer, and let $\zeta_F(1-n)^*$ be the leading coefficient of the Taylor series of $\zeta_F(s)$ at $s = 1-n$. Lichtenbaum conjectured that

$$\zeta_F(1-n)^* = \pm\, 2^{r_1}\, R_{2n-1}(F)\, \frac{\prod\limits_p \operatorname{card} H^2(\operatorname{Spec}(A[1/p]), \mathbf{Z}_p(n))}{\prod\limits_p \operatorname{card} H^1(\operatorname{Spec}(A[1/p]), \mathbf{Z}_p(n))_{\rm tors}}\,, \qquad (26)$$

where $R_{2n-1}(F)$ is the higher regulator for the group $K_{2n-1}(F)$. The equality (26) is known up a power of 2 when F is abelian over $\mathbf{Q}$ [5], [6], [3].
The order of the denominator on the right-hand side of (26) is easy to evaluate, as well as $\zeta_F(1-n)^*$ (since it is related by the functional equation to $\zeta_F(n)$).

PROBLEM. *Can one find a lower bound for $R_{2n-1}(F)$?*

If such a problem could be solved, the equality (26) is likely to produce a much better upper bound for étale cohomology than Theorem 3. Zagier's conjecture suggests that this problem could be solved if one knew that the values of the n-logarithm on F are $\mathbf{Q}$-linearly independent.

5.5

To illustrate our discussion, let $F = \mathbf{Q}$ and $n = 5$. Then we have

$$H^2(\operatorname{Spec}(\mathbf{Z}[1/p]), \mathbf{Z}_p(5))/p = C^{(p-5)}\,,$$

where C is the class group of $\mathbf{Q}(\sqrt[p]{1})$ modulo p, and $C^{(i)}$ is the eigenspace of C of the i-th power of the Teichmüller character. Vandiver's conjecture predicts that $C^{(p-5)} = 0$ when p is odd. It is true when $p \leq 4.10^6$. Theorem 3 and Proposition 4 tell us that

$$\prod_p H^2(\mathrm{Spec}(\mathbf{Z}[1/p], \mathbf{Z}_p(5)) \leq k(8) \leq \exp\exp(70130).$$

If one could find either a better upper bound for the order of $K_8(\mathbf{Z})$ or a good lower bound for $R_9(\mathbf{Q})$, this would get us closer to the expected vanishing of $C^{(p-5)}$.

Notice that, using knowledge on $K_4(\mathbf{Z})$, Kurihara has proved that $C^{(p-3)} = 0$.

References

[1] A. Ash, Deformation retracts with lowest possible dimension of arithmetic quotients of self-adjoint homogeneous cones, Math. Ann. 225, no. 1, 69–76 (1977).

[2] K. S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, 87. Springer-Verlag, New York-Berlin, 1982. x+306 pp.

[3] A.Huber, G.Kings, Bloch-Kato conjecture and main conjecture of Iwasawa theory for Dirichlet characters, preprint (2001).

[4] M.I. Icaza, Hermite constant and extreme forms for algebraic number fields, J. London Math. Soc. (2) 55, no. 1, 11–22 (1997).

[5] M. Kolster, T. Nguyen Quang Do, V. Fleckinger, Twisted S-units, p-adic class number formulas, and the Lichtenbaum conjectures, Duke Math. J. (3) 84, 679–717 (1996).

[6] M. Kolster, T. Nguyen Quang Do, V. Fleckinger, Correction to "Twisted S-units, p-adic class number formulas, and the Lichtenbaum conjectures", Duke Math. J. (3) 90, 641–643 (1997)

[7] S. Lang, *Algebraic number theory*, Addison-Wesley Publishing Co., Inc., Reading, Mass.-London-Don Mills, Ont. 1970 xi+354 pp.

[8] H.W. Lenstra, Jr., Algorithms in algebraic number theory, Bull. AMS 26, no. 2, 211–244 (1992).

[9] J. Milnor, *Introduction to algebraic K-theory*, Annals of Mathematics Studies, No. 72. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1971. xiii+184 pp.

[10] J. Milnor and D. Husemoller: *Symmetric bilinear forms*, Ergebnisse 73, 1973, Springer-Verlag.

[11] W. Narkiewicz, *Number theory*, Translated from the Polish by S. Kanemitsu. World Scientific Publishing Co., Singapore; distributed by Heyden & Son, Inc., Philadelphia, PA, 1983. xii+371 pp.

[12] C. Soulé, K-théorie des anneaux d'entiers de corps de nombres et cohomologie étale, Invent. Math. 55, no. 3, 251–295 (1979).

[13] C. Soulé, Perfect forms and the Vandiver conjecture, J. Reine Angew. Math. 517, 209–221 (1999).

[14] P.Voutier, An effective bound for the height of algebraic numbers, Acta Arith. (1) 74, 81–95 (1996).

Christophe Soulé
CNRS and IHES
35 route de Chartres
91440 Bures-sur-Yvette
France
soule@ihes.fr

Unramified Skolem Problems and Unramified Arithmetic Bertini Theorems in Positive Characteristic

Dedicated to Professor Kazuya Kato

on the occasion of his fiftieth birthday

Akio Tamagawa

Received: November 9, 2002
Revised: June 18, 2003

Abstract. In this paper, we prove unramified, positive-characteristic versions of theorems of Rumely and Moret-Bailly that generalized Skolem's classical problems, and unramified, positive-characteristic versions of arithmetic Bertini theorems. We also give several applications of these results.

2000 Mathematics Subject Classification: Primary 11G35; Secondary 11G25, 14G15, 14G25, 14H25
Keywords and Phrases: Skolem's problem, arithmetic Bertini theorem, positive characteristic.

§0. Introduction.

Let $f : X \to S$ be a morphism between schemes X and S. We refer to an S-morphism $\sigma : S' \to X$ from an S-scheme S' to X as a quasi-section of f, if the structure morphism $\pi : S' \to S$ is surjective. Moreover, for each property $\mathcal{P}$ of morphisms of schemes, we say that σ is a $\mathcal{P}$ quasi-section, if π is $\mathcal{P}$. In this terminology, Rumely's theorem, which generalized Skolem's classical problems and was augmented by the work of Moret-Bailly, can be stated as follows:

Theorem ([Ru1], [Mo2]). *Let S be a non-empty, affine, open subscheme of either the spectrum of the integer ring of an algebraic number field K or a proper, smooth, geometrically connected curve over a finite field with function field K. Let X be a scheme and $f : X \to S$ a morphism of schemes, such that X is irreducible, that $X_K \stackrel{\text{def}}{=} X \times_S \operatorname{Spec}(K)$ is geometrically irreducible over K, and that f is of finite type and surjective. Then, f admits a finite quasi-section.*

On the other hand, the following is a well-known fact in algebraic geometry:

Theorem ([EGA4], Corollaire (17.16.3)(ii)). *Let $f : X \to S$ be a morphism of schemes (with S arbitrary) which is smooth and surjective. Then, f admits an étale quasi-section.*

In the present paper, we prove, among other things, the following theorem in positive characteristic, which is a sort of mixture of the above two theorems:

Theorem (0.1). (See (3.1).) *Let S be a non-empty, affine, open subscheme of a proper, smooth, geometrically connected curve over a finite field with function field K. Let X be a scheme and $f : X \to S$ a morphism of schemes, such that X_K is geometrically irreducible over K, and that f is smooth and surjective. Then, f admits a finite étale quasi-section.*

Here, we would like to note that the validity of this theorem is typical of positive characteristic. For example, it is easy to observe that $\mathbf{P}^1_{\mathbb{Z}} - \{0, 1, \infty\} \to \mathrm{Spec}(\mathbb{Z})$ does not admit a finite étale quasi-section.

In the work of Rumely and Moret-Bailly, they also proved certain refined versions of the above theorem, which involve local conditions at a finite number of primes. To state these refined versions, let S and K be as in the theorem of Rumely and Moret-Bailly. Thus, K is either an algebraic number field or an algebraic function field of one variable over a finite field. We denote by Σ_K the set of primes of K, and we denote by Σ_S the set of closed points of S, which may be regarded as a subset of Σ_K. Moreover, let Σ be a (an automatically finite) subset of $\Sigma_K - \Sigma_S$, which is not the whole of $\Sigma_K - \Sigma_S$. (This last assumption is referred to as incompleteness hypothesis.) For each $v \in \Sigma$, let K_v denote the v-adic completion of K, and assume that a normal algebraic extension L_v/K_v (possibly of infinite degree) is given. Let X and $f : X \to S$ be as in the theorem of Rumely and Moret-Bailly, and assume that, for each $v \in \Sigma$, a non-empty, v-adically open, $\mathrm{Gal}(K_v^{\mathrm{sep}}/K_v)$-stable subset Ω_v of $X(L_v)$ is given.

Theorem ([Ru1], [Mo3]). *Notations and assumptions being as above, assume, moreover, either $L_v = \overline{K}_v$ ([Ru1]) or L_v is Galois over K_v ([Mo3]). Then, there exists a finite quasi-section $S' \to X$ of $f : X \to S$, such that, for each $v \in \Sigma$, $S'_{L_v} \stackrel{\mathrm{def}}{=} S' \times_S \mathrm{Spec}(L_v)$ is a direct sum of (a finite number of) copies of $\mathrm{Spec}(L_v)$, and the image of S'_{L_v} in $X(L_v) = X_{L_v}(L_v)$ is contained in Ω_v.*

Remark (0.2). In fact, Moret-Bailly's version implies Rumely's version. See [Mo3], Remarque 1.6 for this. Indeed, Moret-Bailly's version implies more, namely, that it suffices to assume that L_v is a normal algebraic extension of K_v such that $L_v \cap K_v^{\mathrm{sep}}$ is (v-adically) dense in L_v. (See the proof of [Mo3], Lemme 1.6.1, case (b).)

Remark (0.3). Here is a brief summary of the history (in the modern terminology) of Skolem's problems and its generalizations. Skolem [S] proved the existence of finite quasi-sections for rational varieties. Cantor and Roquette

[CR] proved it for unirational varieties. (A similar result was slightly later obtained in [EG].) Then, Rumely [Ru1] gave the first proof for arbitrary varieties (in the case of rings of algebraic integers). (See also [Ro].) Moret-Bailly (and Szpiro) [Mo2,3] gave an alternative proof of Rumely's result in stronger forms. (Another alternative proof was later given in [GPR].) Moret-Bailly also proved the existence of finite quasi-sections for algebraic stacks ([Mo5]).

We also prove the following refined version in the unramified setting. Unfortunately, in the unramified setting, our version for the present is weaker than Moret-Bailly's version (though it is stronger than Rumely's version). To state this, let S and K be as in (0.1). Thus, K is an algebraic function field of one variable over a finite field. We denote by Σ_K the set of primes of K, and we denote by Σ_S the set of closed points of S. Moreover, let Σ be a subset of $\Sigma_K - \Sigma_S$, which is not the whole of $\Sigma_K - \Sigma_S$. For each $v \in \Sigma$, let K_v denote the v-adic completion of K, and assume that a normal algebraic extension L_v/K_v is given. Let $f : X \to S$ be as in (0.1), and assume that, for each $v \in \Sigma$, a non-empty, v-adically open, $\mathrm{Gal}(K_v^{\mathrm{sep}}/K_v)$-stable subset Ω_v of $X(L_v)$ is given.

Theorem (0.4). (See (3.1).) *Notations and assumptions being as above, assume, moreover, that, for each $v \in \Sigma$, $L_v \cap K_v^{\mathrm{sep}}$ is dense in L_v, and that the residue field of L_v is infinite. Then, there exists a finite étale quasi-section $S' \to X$ of $f : X \to S$, such that, for each $v \in \Sigma$, S'_{L_v} is a direct sum of copies of* $\mathrm{Spec}(L_v)$*, and the image of S'_{L_v} in $X(L_v) = X_{L_v}(L_v)$ is contained in Ω_v.*

Roughly speaking, the proof of (0.4) goes as follows. Via some reduction steps, we may assume that X is quasi-projective over S. Then, by means of a version of arithmetic Bertini theorem, we take hyperplane sections successively to obtain a suitable quasi-section finally. More precisely, we use the following unramified version of arithmetic Bertini theorem, which is another main result of the present paper:

Theorem (0.5). (See (3.2).) *Let S, Σ, L_v be as in* (0.4). *Moreover, let $Y_1, \dots, Y_r$ be irreducible, reduced, closed subschemes of $\mathbf{P}^n_S$. For each $v \in \Sigma$, let $\check{\Omega}_v$ be a non-empty, v-adically open,* $\mathrm{Gal}(K_v^{\mathrm{sep}}/K_v)$*-stable subset of $\check{\mathbf{P}}^n_S(L_v)$. Then, there exist a connected, finite, étale covering $S' \to S$ such that, for each $v \in \Sigma$, S'_{L_v} is a direct sum of copies of* $\mathrm{Spec}(L_v)$*, and a hyperplane $H \subset \mathbf{P}^n_{S'}$, such that the following hold:* (a) *for each $i = 1, \dots, r$, each geometric point $\overline{s}$ of S' and each irreducible component P of $Y_{i,\overline{s}}$, we have $P \cap H_{\overline{s}} \subsetneq P$;* (b) *for each $i = 1, \dots, r$, the scheme-theoretic intersection $(Y_i^{\mathrm{sm}})_{S'} \cap H$ (in $\mathbf{P}^n_{S'}$) is smooth over S' (Here, Y_i^{sm} denotes the set of points of Y_i at which $Y_i \to S$ is smooth. This is an open subset of Y_i, and we regard it as an open subscheme of Y_i.);* (c) *for each $i = 1, \dots, r$ and each irreducible component P of $Y_{i,\overline{K}}$ (where we identify the algebraic closure of the function field of S' with that of S) with* $\dim(P) \geq 2$, *$P \cap H_{\overline{K}}$ is irreducible; and* (d) *for each $v \in \Sigma$, the image of S'_{L_v} in $\check{\mathbf{P}}^n(L_v)$ by the base change to L_v of the classifying morphism $[H] : S' \to \check{\mathbf{P}}^n_S$ over S is contained in $\check{\Omega}_v$.*

There remain, however, the following non-trivial problems. Firstly, a Bertini-type theorem is, after all, to find a (quasi-)section in an open subset of the (dual) projective space, which requires a Rumely-type theorem. Secondly, in the (most essential) case where X is of relative dimension 1 over S, the boundary of X (i.e., the closure of X minus X in the projective space) may admit vertical irreducible components (of dimension 1). Since a hyperplane intersects non-trivially with every positive-dimensional irreducible component, the hyperplane section does not yield a finite quasi-section of X (but merely of the closure of X).

To overcome the first problem, we have to prove a Rumely-type theorem for projective n-spaces directly. It is not difficult to reduce this problem to the case $n = 1$. First, we shall explain the proof of this last case assuming $\Sigma = \emptyset$. So, we have to construct a finite, étale quasi-section in an open subscheme X of $\mathbf{P}^1_S$. Moreover, for simplicity, we assume that X is a complement of the zero locus W of $w(T) \in R[T]$ in $\mathbf{A}^1_S = \mathrm{Spec}(R[T])$, where $R \stackrel{\mathrm{def}}{=} \Gamma(S, \mathcal{O}_S)$. (Since X is assumed to be surjectively mapped onto S, $w(T)$ is primitive.) The original theorem of Rumely and Moret-Bailly, together with some arguments from Moret-Bailly's proof, implies that there exists a monic polynomial $g(T) \in R[T]$ of positive degree, such that the zero locus of g in $\mathbf{A}^1_S$ is contained in X. Now, if the zero locus of g is étale over S, we are done. In general, we shall consider the following polynomial: $F(T) = g(T)^{pm} + w(T)^p T$ for sufficiently large $m > 0$. Then, $F(T)$ is a monic polynomial in $R[T]$, and its zero locus S' gives a closed subscheme of $\mathbf{A}^1_S$ which is finite, flat over S. Since $g(T)$ (resp. $w(T)$) is a unit (resp. zero) on W, $F(T)$ is a unit on W, or, equivalently, S' is contained in X. Moreover, since $F'(T) = w(T)^p$, the zero locus of F' coincides with W, hence is disjoint from the zero locus S' of F. This means that S' is étale over S, as desired. (This argument is inspired by an argument of Gabber in [G].) Next, assume $\Sigma \neq \emptyset$. Then, to find a finite, étale quasi-section with prescribed local conditions at Σ, we need to investigate local behaviors of roots of polynomials like the above F. Since it is easy to reduce the problem to the case where the above w is v-adically close to 1 (by means of a coordinate change), we see that it is essential to consider local behaviors of roots of polynomials in the form of

$$a_1 T + \sum_{i=0}^{m} a_{ip} T^{ip}$$

with $a_1 \neq 0$. (In the present paper, we refer to a polynomial in this form as a superseparable polynomial.) As a result of this investigation, we see that we can take the above F so that, for each $v \in \Sigma$, every root of F is contained in the given Ω_v. Also, in this investigation, the (hopefully temporary) condition that the residue field of L_v is infinite for each $v \in \Sigma$ arises.

To overcome the second problem, we take a finite, flat quasi-section with local conditions by means of Moret-Bailly's version of Rumely's theorem. Then, by using this (horizontal) divisor, we construct a (new) quasi-projective embedding of X. Now, in this projective space, we can construct a finite, étale quasi-section of X as a hyperplane section.

Here is one more ingredient of our proof that we have not yet mentioned:

Theorem (0.6). (See (2.1) and (2.2).) *Let S and Σ be as in* (0.4). *Assume, moreover, that, for each $v \in \Sigma$, a finite Galois extension L_v/K_v is given. Then, there exists a connected, finite, étale, Galois covering $S' \to S$, such that, for each $v \in \Sigma$, $S' \times_S \mathrm{Spec}(K_v)$ is isomorphic to a disjoint sum of copies of $\mathrm{Spec}(L_v)$ over K_v.*

We use this result in some reduction steps. See §3 for more details.

The author's original motivation to prove results like (0.1) arises from the study of coverings of curves in positive characteristic. For example, in the forthcoming paper, we shall prove the following result as an application of (0.1):

Theorem (0.7). *For each pair of affine, smooth, connected curves X, Y over $\overline{\mathbb{F}}_p$, there exists an affine, smooth, connected curve Z over $\overline{\mathbb{F}}_p$ that admits finite, étale morphisms $Z \to X$ and $Z \to Y$ over $\overline{\mathbb{F}}_p$.*

In other words, there exists an $\overline{\mathbb{F}}_p$-scheme H, such that, for every affine, smooth, connected curve X over $\overline{\mathbb{F}}_p$, the 'pro-finite-étale universal covering' $\tilde{X}$ of X is isomorphic to H over $\overline{\mathbb{F}}_p$.

For other applications of the above main results, see §4.

Finally, we shall explain the content of each § briefly. In §1, we investigate the above-mentioned class of polynomials in positive characteristic, namely, superseparable polynomials. The aim here is to control how a superseparable polynomial over a complete discrete valuation field in positive characteristic decomposes. Here, (1.18) is a final result, on which the arguments in §2 and §3 are based. In §2, we prove the existence of unramified extensions with prescribed local extensions, such as (0.6) above. The main results are (2.1) and (2.2). In the former, we treat an arbitrary Dedekind domain in positive characteristic, while, in the latter, we only treat a curve over a field of positive characteristic but we can impose (weaker) conditions on all the primes of the function field. The proofs of both results rely on the results of §1. In §3, we prove the main results of the present paper, namely, an unramified version of the theorem of Rumely and Moret-Bailly in positive characteristic (3.1), and an unramified version of the arithmetic Bertini theorem in positive characteristic (3.2). In §4, we give several remarks and applications of the main results. Some of these applications are essentially new features that only arise after our unramified versions.

Acknowledgment. Professor Kazuya Kato was the advisor of the author's master's thesis [T] at the University of Tokyo (May 1990 – March 1992). The author would like to express his sincerest gratitude to Professor Kato for having encouraged and stimulated the author constantly since then. Also, the author would like to thank Bjorn Poonen and Pierre Dèbes (resp. Osamu Fujino and Hiromichi Takagi) very much for various helpful information concerning the work of Rumely and Moret-Bailly (resp. [CFHR]). Finally, the author would like to thank one of the referees for a simplification of Step 2 of §3. (See (3.8).)

§1. Superseparable polynomials.

Throughout this §, we let K denote a field.

Definition. Let $f(T)$ be a polynomial in $K[T]$. We say that f is superseparable, if the derivative $f'(T)$ of $f(T)$ falls in $K[T]^\times = K^\times$.

Lemma (1.1). *For each $f(T) \in K[T]$, the following* (a)–(c) *are equivalent.*
(a) *f is superseparable.*
(b) *The K-morphism $\mathbf{A}^1_K \to \mathbf{A}^1_K$ associated to f is étale everywhere.*
(c) *f is in the form of*

$$f(T) = \begin{cases} a_1 T + a_0, & \textit{if } \mathrm{char}(K) = 0, \\ a_1 T + \sum_{i=0}^{m} a_{ip} T^{ip}, & \textit{if } \mathrm{char}(K) = p > 0, \end{cases}$$

where $a_j \in K$ and $a_1 \neq 0$.

Proof. Immediate. □

Remark (1.2). f is separable (i.e., $(f, f') = 1$) if and only if the associated K-morphism from $\mathbf{A}^1_{\mathrm{upper},K} \overset{\mathrm{def}}{=} \mathbf{A}^1_K$ to $\mathbf{A}^1_{\mathrm{lower},K} \overset{\mathrm{def}}{=} \mathbf{A}^1_K$ is étale at $0 \in \mathbf{A}^1_{\mathrm{lower},K}$.

From now on, let p denote a prime number, and we assume that K is of characteristic p and is equipped with a complete discrete valuation v, normalized as $v(K^\times) = \mathbb{Z}$. We denote by R, $\mathfrak{m}$, k, and t the valuation ring of v, the maximal ideal of R, the residue field $R/\mathfrak{m}$, and a prime element of R, respectively. We fix an algebraic closure $\overline{K}$ of K, and we denote again by v the unique valuation $\overline{K} \to \mathbb{Q} \cup \{\infty\}$ that extends v. Moreover, for each subfield L of $\overline{K}$ containing K, we denote by R_L, $\mathfrak{m}_L$, and k_L the integral closure of R in L, the maximal ideal of R_L, and the residue field $R_L/\mathfrak{m}_L$, respectively.

Now, consider a superseparable polynomial

$$f(T) = aT + h(T^p), \tag{1.3}$$

where $a \in K^\times$, $h \in K[T]$, and we put $m \overset{\mathrm{def}}{=} \deg(h)$. (We put $m = 0$ if $h = 0$.) The aim of this § is to describe how f decomposes and what is the Galois group associated with f.

Definition. (i) We say that a polynomial g in $\overline{K}[T]$ is integral, if all the coefficients of g belong to $R_{\overline{K}}$.

(ii) Let g be a non-zero polynomial in $\overline{K}[T]$. We denote by $\mathrm{roots}(g)$ the set of roots of g in $\overline{K}$. This is a finite subset of $\overline{K}$.
(iii) Let g be a separable polynomial in $K[T]$. Then, we denote by K_g the minimal splitting field of g in $\overline{K}$, i.e., the subfield of $\overline{K}$ generated by $\mathrm{roots}(g)$ over K. This is a Galois extension of K, and we put $G_g \stackrel{\mathrm{def}}{=} \mathrm{Gal}(K_g/K)$.
(iii) Let g be a polynomial in $\overline{K}[T]$, and α an element of $\mathrm{roots}(g)$. Then, we put

$$\mu(g,\alpha) \stackrel{\mathrm{def}}{=} \max\{v(\alpha'-\alpha) \mid \alpha' \in \mathrm{roots}(g) - \{\alpha\}\}.$$

Here, we put $\max\emptyset \stackrel{\mathrm{def}}{=} -\infty$.

The following is a version of Krasner's lemma.

Lemma (1.4). *Let f be a monic, integral, superseparable polynomial in $K[T]$ as in* (1.3).
(i) *For each $\alpha \in \mathrm{roots}(f)$, we have $\mu(f,\alpha) \leq \frac{1}{p-1}v(a)$.*
(ii) *Let $\epsilon(T) = \sum_{j=0}^{mp} \epsilon_j T^j$ be a polynomial in $K[T]$ (with degree $\leq mp$), such that $v(\epsilon_j) > \frac{p}{p-1}v(a)$ holds for all $j = 0, \dots, mp$. We put $f_1 \stackrel{\mathrm{def}}{=} f + \epsilon$. Then, f_1 is separable and we have $K_{f_1} = K_f$.*

Proof. (i) Observe the Newton polygon of $f(T+\alpha)$ (which is also a monic, integral, superseparable polynomial).
(ii) For each $\alpha \in \mathrm{roots}(f)$, put $g_\alpha(T) = f_1(T+\alpha)$, which is an integral polynomial in $\overline{K}[T]$. Then, we have $\mathrm{roots}(g_\alpha) = \{\beta - \alpha \mid \beta \in \mathrm{roots}(f_1)\}$. We have $g_\alpha(0) = f_1(\alpha) = \epsilon(\alpha)$ and $g'_\alpha(0) = f'_1(\alpha) = a + \epsilon'(\alpha)$, hence $v(g_\alpha(0)) > \frac{p}{p-1}v(a)$ and $v(g'_\alpha(0)) = v(a)$. Thus, by observing the Newton polygon of g_α, we see that there exists a unique $\beta = \beta_\alpha \in \mathrm{roots}(f_1)$, such that $v(\beta - \alpha) > \frac{1}{p-1}v(a)$. The map $\mathrm{roots}(f) \to \mathrm{roots}(f_1)$, $\alpha \mapsto \beta_\alpha$ is clearly $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-equivariant. Moreover, this map is injective, since, for each pair $\alpha, \alpha' \in \mathrm{roots}(f)$ with $\alpha \neq \alpha'$, we have $v(\alpha - \alpha') \leq \frac{1}{p-1}v(a)$ by (i). As $\sharp(\mathrm{roots}(f)) = mp \geq \sharp(\mathrm{roots}(f_1))$, this map must be a bijection. Thus, we obtain a $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-equivariant bijection $\mathrm{roots}(f) \xrightarrow{\sim} \mathrm{roots}(f_1)$, so that f_1 is separable and $K_f = K_{f_1}$, as desired. □

Definition. Let m and n be natural numbers.
(i) We put $I_n \stackrel{\mathrm{def}}{=} \{1, \dots, n\}$.
(ii) We denote by S_n the symmetric group on the finite set I_n. Moreover, identifying I_n with $\mathbb{Z}/n\mathbb{Z}$ naturally, we define

$$B_n \stackrel{\mathrm{def}}{=} \{\sigma \in S_n \mid \exists a \in (\mathbb{Z}/n\mathbb{Z})^\times,\ \exists b \in \mathbb{Z}/n\mathbb{Z},\ \forall i \in \mathbb{Z}/n\mathbb{Z},\ \sigma(i) = ai + b\}$$

and

$$C_n \stackrel{\mathrm{def}}{=} \{\sigma \in S_n \mid \exists b \in \mathbb{Z}/n\mathbb{Z},\ \forall i \in \mathbb{Z}/n\mathbb{Z},\ \sigma(i) = i + b\}.$$

Thus, $S_n \supset B_n \rhd C_n$, and B_n (resp. C_n) can be naturally identified with the semi-direct product $(\mathbb{Z}/n\mathbb{Z})^\times \ltimes (\mathbb{Z}/n\mathbb{Z})$ (resp. the cyclic group $\mathbb{Z}/n\mathbb{Z}$).

(ii) We denote by $S_{m\times n}$ the symmetric group on the finite set $I_m \times I_n$. (Thus, $S_{m\times n} \simeq S_{mn}$.) Let pr_1 denote the first projection $I_m \times I_n \to I_m$. We define

$$S_{m\ltimes n} \stackrel{\mathrm{def}}{=} \{\sigma \in S_{m\times n} \mid \exists \overline{\sigma} \in S_m,\ \forall (i,j) \in I_m \times I_n,\ \mathrm{pr}_1(\sigma((i,j))) = \overline{\sigma}(i)\}.$$

Thus, $S_{m\ltimes n}$ can be naturally identified with the semi-direct product $S_m \ltimes (S_n)^{I_m}$. Here, for a group G and a positive integer r, G^{I_r} denotes the direct product $\underbrace{G \times \cdots \times G}_{r \text{ times}}$. We adopt this slightly unusual notation to save the notation G^r for $\{g^r \mid g \in G\}$ (for a commutative group G).

The following proposition is a mere exercise in Galois theory over local fields in positive characteristic, but it is the starting point of our proofs of main results in later §§.

Proposition (1.5). *Let f be a superseparable polynomial as in* (1.3) *with $m \geq 1$. Moreover, we assume that* (a) *h is separable, and* (b) *we have $\delta(a,h,\alpha) > \mu(h,\alpha)$ for all $\alpha \in \mathrm{roots}(h)$, where*

$$\delta(a,h,\alpha) \stackrel{\mathrm{def}}{=} \min\left(v(a) - v(h'(\alpha)) + \frac{1}{p}v(\alpha), \frac{p}{p-1}(v(a) - v(h'(\alpha)))\right).$$

Then, by choosing a suitable bijection between $\mathrm{roots}(f)$ *and* $I_m \times I_p$*:*
(i) *The Galois group G_f can be identified with a subgroup of $S_{m\ltimes p}$ ($\subset S_{m\times p}$).*
(ii) $G_f \cap (S_p)^{I_m} \subset (B_p)^{I_m}$.
(iii) *The group filtration*

$$\{1\} \subset G_f \cap (C_p)^{I_m} \subset G_f \cap (S_p)^{I_m} \subset G_f$$

corresponds via Galois theory to the field filtration

$$K_f \supset M_f \supset K_h \supset K,$$

where M_f is the subfield of $\overline{K}$ generated by $\{(-a/h'(\alpha))^{\frac{1}{p-1}} \mid \alpha \in \mathrm{roots}(h)\}$ over K_h.

Proof. (i) First, we shall prove the following:

Claim (1.6). (i) For each $\alpha \in \mathrm{roots}(h)$, put

$$F_\alpha \stackrel{\mathrm{def}}{=} \{\beta \in \mathrm{roots}(f) \mid v(\beta^p - \alpha) \geq \delta(a,h,\alpha)\}.$$

Then, F_α has cardinality p for each $\alpha \in \mathrm{roots}(h)$.
(ii) For each $\beta \in \mathrm{roots}(f)$, there exists a unique $\alpha = \alpha_\beta \in \mathrm{roots}(h)$, such that $F_\alpha \ni \beta$.

Proof. (i) Observe the Newton polygon of $f(T+\alpha^{1/p}) = h(T^p+\alpha)+aT+a\alpha^{1/p}$ by using $\delta(a,h,\alpha) > \mu(h,\alpha)$. Then, we see that F_α has cardinality p, as

desired (and that the subset $\{\beta \in \mathrm{roots}(f) \mid v(\beta^p - \alpha) = \delta(a,h,\alpha)\}$ of F_α has cardinality $\geq p-1$).
(ii) First, we shall prove the uniqueness. Suppose that there exist $\alpha_1, \alpha_2 \in \mathrm{roots}(h)$, $\alpha_1 \neq \alpha_2$, such that $v(\beta^p - \alpha_i) \geq \delta(a,h,\alpha_i)$ holds for $i = 1,2$. Then, we have

$$v(\alpha_1 - \alpha_2) = v((\beta^p - \alpha_2) - (\beta^p - \alpha_1)) \geq \min(\delta(a,h,\alpha_1), \delta(a,h,\alpha_2)),$$

while, by assumption, we have

$$\min(\delta(a,h,\alpha_1), \delta(a,h,\alpha_2)) > \min(\mu(h,\alpha_1), \mu(h,\alpha_2)) \geq v(\alpha_1 - \alpha_2).$$

This is absurd.

By this uniqueness and (i), we have

$$\sharp(\bigcup_{\alpha \in \mathrm{roots}(h)} F_\alpha) = \sum_{\alpha \in \mathrm{roots}(h)} \sharp(F_\alpha) = mp = \sharp(\mathrm{roots}(f)),$$

hence $\bigcup_{\alpha \in \mathrm{roots}(h)} F_\alpha = \mathrm{roots}(f)$. This implies the existence of $\alpha = \alpha_\beta$ for each $\beta \in \mathrm{roots}(f)$. $\square$

By (1.6)(ii), we obtain a well-defined map $\pi : \mathrm{roots}(f) \to \mathrm{roots}(h)$, $\beta \mapsto \alpha_\beta$. By (1.6)(i), π is surjective and each fiber of π has cardinality p. Since π is $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-equivariant by definition, this implies (1.5)(i). (We may choose any bijections $\mathrm{roots}(h) \simeq I_m$ and $F_\alpha \simeq I_p$ ($\alpha \in \mathrm{roots}(h)$).)

Note that this construction already shows that the field extension of K corresponding to the subgroup $G_f \cap (S_p)^{I_m} = \mathrm{Ker}(G_f \to S_m)$ coincides with K_h.
(ii) We shall start with the following. From now on, for each $x, x' \in \overline{K}^\times$, we write $x \sim x'$ if $x'/x \in 1 + \mathfrak{m}_{\overline{K}}$, or, equivalently, $v(x' - x) > v(x)$.

Claim (1.7). (i) Let $\alpha, \alpha' \in \mathrm{roots}(h)$ and $\beta \in F_\alpha$. Assume $\alpha \neq \alpha'$. Then, we have $\beta^p - \alpha' \sim \alpha - \alpha'$. In particular, we have $v(\beta^p - \alpha') = v(\alpha - \alpha')$.
(ii) Let $\alpha, \alpha' \in \mathrm{roots}(h)$, $\beta \in F_\alpha$, and $\beta' \in F_{\alpha'}$. Assume $\beta \neq \beta'$. Then, we have

$$v((\beta')^p - \beta^p) = \begin{cases} v(\alpha' - \alpha) \ (\leq \mu(h,\alpha)), & \text{if } \alpha \neq \alpha', \\ \frac{p}{p-1}(v(a) - v(h'(\alpha))) \ (\geq \delta(a,h,\alpha)), & \text{if } \alpha = \alpha'. \end{cases}$$

Proof. (i) $v((\beta^p - \alpha') - (\alpha - \alpha')) = v(\beta^p - \alpha) \geq \delta(a,h,\alpha) > \mu(h,\alpha) \geq v(\alpha - \alpha')$.
(ii) If $\alpha \neq \alpha'$, we have $v((\beta')^p - \beta^p) = v(((\beta')^p - \alpha') - (\beta^p - \alpha')) = v(\beta^p - \alpha')$, since $v((\beta')^p - \alpha') > v(\beta^p - \alpha')$ by the definition of $F_{\alpha'}$. (Recall that $F_\alpha \cap F_{\alpha'} = \emptyset$ holds by (1.6)(ii).) Thus, in this case, $v((\beta')^p - \beta^p) = v(\alpha' - \alpha)$ holds by (i).

By using this and (i), observe the Newton polygon of $f(T + \beta) = h(T^p + \beta^p) + aT + a\beta$ and compare it with the Newton polygon of $f(T + \alpha^{1/p})$. Then, we can read off the value $v(\beta' - \beta)$ for $\alpha = \alpha'$. $\square$

For each $\alpha \in \mathrm{roots}(h)$, the subgroup $\mathrm{Gal}(K_f/K(\alpha))$ of G_f acts on F_α. In order to prove (1.5)(ii), it suffices to prove that the image $\mathrm{Gal}(K(\alpha)(F_\alpha)/K(\alpha))$ of this action is contained in B_p ($\subset S_p$), after choosing a suitable bijection $F_\alpha \simeq I_p$.

Claim (1.8). Let $\alpha \in \mathrm{roots}(h)$ and $\beta \in F_\alpha$.
(i) We have $K(\alpha)(F_\alpha) = K(\alpha)(\beta)((-a/h'(\alpha))^{1/(p-1)})$.
(ii) Let $\beta' \in F_\alpha$, $\beta' \neq \beta$. Then, we have $(\beta' - \beta)^{p-1} \sim -a/h'(\alpha)$. More precisely, we have

$$\{(\beta' - \beta) \bmod \sim \mid \beta' \in F_\alpha,\ \beta' \neq \beta\} = \{\zeta(-a/h'(\alpha))^{\frac{1}{p-1}} \bmod \sim \mid \zeta \in \mathbb{F}_p^\times\}.$$

Proof. As in the proof of (1.7)(ii), observe the Newton polygon of $f(T+\beta) = h(T^p + \beta^p) + aT + a\beta$. Then, observing the coefficients of $T^0, T^1, \ldots, T^p$, we see that $K(\alpha)(F_\alpha) = K(\alpha)(\beta)((-a/h'(\beta^p))^{1/(p-1)})$. Now, by (1.7)(i), we obtain $h'(\beta^p) \sim h'(\alpha)$, which implies $K(\alpha)(\beta)((-a/h'(\beta^p))^{1/(p-1)}) = K(\alpha)(\beta)((-a/h'(\alpha))^{1/(p-1)})$. These complete the proof of (i), and also show (ii). $\square$

Lemma (1.9). *Let G be a subgroup of S_p, and, for each $i = 1, \ldots, p$, we denote by G_i the stabilizer of i in G. Moreover, let $\phi : G \to \mathbb{F}_p^\times$ be a homomorphism, such that, for each $i = 1, \ldots, p$, there exists an identification $\sigma_i : I_p - \{i\} \xrightarrow{\sim} \mathbb{F}_p^\times$, such that $\sigma_i g_i \sigma_i^{-1}$ coincides with the $\phi(g_i)$-multiplication map on $\mathbb{F}_p^\times$ for each $g_i \in G_i$. Then, we have $G \subset B_p$ via a suitable identification $I_p \simeq \mathbb{F}_p$.*

Proof. Put $N \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\phi)$. Then, N is a normal subgroup of G. By using the identity $\phi(g_i)\cdot = \sigma_i g_i \sigma_i^{-1}$, we see that $N \cap G_i = \{1\}$ for all $i = 1, \ldots, p$. Namely, the action of N on I_p is free. Since $\sharp(I_p) = p$ is a prime number, this implies that either $N = \{1\}$ or $N = C_p$ (via some identification $I_p \simeq \mathbb{F}_p$). In the latter case, we obtain $G \subset B_p$, since the normalizer of C_p in S_p coincides with B_p. So, assume $N = \{1\}$. Then, $G = G/N$ is abelian with $\sharp(G) \mid p-1$.

Let X be any G-orbit of I_p. Suppose that X is not a one-point set. Then, there exist $i, j \in X$, $i \neq j$. Since G is abelian, this implies $G_i = G_j$. On the other hand, by the identity $\phi(g_i)\cdot = \sigma_i g_i \sigma_i^{-1}$, we see that $G_i \cap G_j = \{1\}$. Thus, we must have $G_i (= G_j) = \{1\}$.

By this consideration, we conclude that I_p is isomorphic as a G-set to a disjoint union of copies of G and copies of G/G. If a copy of G/G appears, then this means $G = G_i$ for some $i = 1, \ldots, p$, and, by using the unique extension of σ_i to $I_p \xrightarrow{\sim} \mathbb{F}_p$, we obtain $G \subset \mathbb{F}_p^\times \subset B_p$.

On the other hand, if no copy of G/G appears, we must have $\sharp(G) \mid p$. As $\sharp(G) \mid p-1$ also holds, we conclude $G = \{1\} \subset B_p$. This completes the proof. $\square$

By (1.8), we may apply (1.9) to $G = \mathrm{Gal}(K(\alpha)(F_\alpha)/K(\alpha))$ and the Kummer character $\phi : G \to \mathbb{F}_p^\times$ defined by $(-a/h'(\alpha))^{1/(p-1)}$, and conclude $G \subset B_p$, as desired.
(iii) This has been already done in the proofs of (i) and (ii). $\square$

Corollary (1.10). *Let m be an integer ≥ 1. Let $R = R^{\mathrm{univ}}$ be the completion of the discrete valuation ring $\mathbb{F}_p[s_0, s_p, s_{2p}, \ldots, s_{(m-1)p}, s_1]_{(s_1)}$ (where s_i's are algebraically independent indeterminates), i.e., $R = \mathbb{F}_p(s_0, s_p, \ldots, s_{(m-1)p})[[s_1]]$, and $K = K^{\mathrm{univ}}$ the field of fractions of R. Consider a superseparable polynomial $f(T)$ as in (1.3), where $a = s_1$ and $h(T) = \sum_{i=0}^{m} s_{ip}T^i$ ($s_{mp} \stackrel{\mathrm{def}}{=} 1$). Then:*

(i) *By choosing a suitable bijection between* $\mathrm{roots}(f)$ *and $I_m \times I_p$, the Galois group G_f can be identified with an extension group of S_m by a subgroup B of $(B_p)^{I_m}$. Here, B is an extension of a subgroup E of $(B_p)^{I_m}/(C_p)^{I_m} = (\mathbb{F}_p^\times)^{I_m}$ by $(C_p)^{I_m}$, where $E = (\mathbb{F}_2^\times)^{I_m} = \{1\}$ if $p = 2$,*

$$E = \begin{cases} \mathrm{Ker}((\mathbb{F}_p^\times)^{I_2} \twoheadrightarrow (\mathbb{F}_p^\times/\{\pm 1\})^{I_2}/\Delta(\mathbb{F}_p^\times/\{\pm 1\})), & m = 2, \\ \mathrm{Ker}((\mathbb{F}_p^\times)^{I_m} \twoheadrightarrow \mathbb{F}_p^\times/(\mathbb{F}_p^\times)^2), & m \equiv 0 \pmod 2,\ m \neq 2, \\ (\mathbb{F}_p^\times)^{I_m}, & m \not\equiv 0 \pmod 2, \end{cases}$$

if $p \equiv 1 \pmod 4$, and

$$E = \begin{cases} \mathrm{Ker}((\mathbb{F}_p^\times)^{I_2} \twoheadrightarrow (\mathbb{F}_p^\times/\{\pm 1\})^{I_2}/\Delta(\mathbb{F}_p^\times/\{\pm 1\})), & m = 2, \\ \mathrm{Ker}((\mathbb{F}_p^\times)^{I_m} \twoheadrightarrow \mathbb{F}_p^\times/(\mathbb{F}_p^\times)^2), & m \equiv 0 \pmod 4, \\ (\mathbb{F}_p^\times)^{I_m}, & m \not\equiv 0 \pmod 4,\ m \neq 2, \end{cases}$$

if $p \equiv 3 \pmod 4$. Here, for a commutative group G and a positive integer r, we define subgroups $\Delta(G)$ and $(G^{I_r})^0$ of G^{I_r} by $\Delta(G) = \{(g, \ldots, g) \in G^{I_r} \mid g \in G\}$ and $(G^{I_r})^0 = \mathrm{Ker}(G^{I_r} \to G,\ (g_1, \ldots, g_r) \mapsto g_1 \cdots g_r)$, respectively, and, in the case where either $p \equiv 1 \pmod 4$, $m \equiv 0 \pmod 2$, $m \neq 2$ or $p \equiv 3 \pmod 4$, $m \equiv 0 \pmod 4$ holds, the surjective homomorphism $(\mathbb{F}_p^\times)^{I_m} \twoheadrightarrow \mathbb{F}_p^\times/(\mathbb{F}_p^\times)^2$ is the composite of $(\mathbb{F}_p^\times)^{I_m} \twoheadrightarrow (\mathbb{F}_p^\times/(\mathbb{F}_p^\times)^2)^{I_m}$ and $(\mathbb{F}_p^\times/(\mathbb{F}_p^\times)^2)^{I_m} \twoheadrightarrow (\mathbb{F}_p^\times/(\mathbb{F}_p^\times)^2)^{I_m}/((\mathbb{F}_p^\times/(\mathbb{F}_p^\times)^2)^{I_m})^0 = \mathbb{F}_p^\times/(\mathbb{F}_p^\times)^2$. Moreover, the inertia subgroup of G_f corresponds to $\Delta(\mathbb{F}_p^\times) \ltimes (\mathbb{F}_p)^{I_m}$.

(ii) *k_{K_f} is generated by $\{(\alpha \bmod \mathfrak{m}_{K_f})^{1/p} \mid \alpha \in \mathrm{roots}(h)\} \cup \{(h'(\alpha)/h'(\alpha'))^{\frac{1}{p-1}} \bmod \mathfrak{m}_{K_f} \mid \alpha, \alpha' \in \mathrm{roots}(h)\}$ over k. Moreover, the algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$ in k_{K_f} coincides with $\mathbb{F}_2$ if $p = 2$,*

$$\mathbb{F} = \begin{cases} \mathbb{F}_{p^2}, & m = 2, \\ \mathbb{F}_p, & m \neq 2, \end{cases}$$

if $p \equiv 1 \pmod 4$, and

$$\mathbb{F} = \begin{cases} \mathbb{F}_{p^2}, & m \equiv 2 \pmod 4, \\ \mathbb{F}_p, & m \not\equiv 2 \pmod 4, \end{cases}$$

if $p \equiv 3 \pmod 4$.

Proof. (i) In order to apply (1.5), we have to check that conditions (a) and (b) of (1.5) hold. It is easy to see that (a) holds. Next, since $K_h = \mathbb{F}_p(\alpha_1, \ldots, \alpha_m)((s_1))$, where $\mathrm{roots}(h) = \{\alpha_1, \ldots, \alpha_m\}$, we have $\mu(h, \alpha) = 0$ for each $\alpha \in \mathrm{roots}(h)$, while $\delta(a, h, \alpha) = v(s_1) = 1$. Thus (b) holds, and we may apply (1.5).

It is easy to see that K_h/K is an unramified S_m-extension. Next we have $M_f = K_h((-s_1/h'(\alpha_1))^{1/(p-1)}, \ldots, (-s_1/h'(\alpha_m))^{1/(p-1)})$. Since $-1/h'(\alpha_i)$ is a unit of R_{K_h} and s_1 is a prime element of R_{K_h}, the inertia subgroup of $\mathrm{Gal}(M_f/K_h)$ corresponds to $\Delta(\mathbb{F}_p^\times)$, and the maximal unramified subextension $M_{0,f}/K_h$ in M_f/K_h is $K_h((h'(\alpha_i)/h'(\alpha_j))^{1/(p-1)} \mid i, j = 1, \ldots, m) = K_h((h'(\alpha_i)/h'(\alpha_1))^{1/(p-1)} \mid i = 2, \ldots, m)$.

Now, observing the subgroup of $K_h^\times/(K_h^\times)^{p-1}$ generated by the classes of $-a/h'(\alpha)$ ($\alpha \in$ roots) by using the divisor group of (the spectrum of) the polynomial ring $\mathbb{F}_p[\alpha_1, \ldots, \alpha_m]$ over $\mathbb{F}_p$, we obtain the desired description of E. (We leave the details to the readers.)

Finally, by (1.6)(i), k_{K_f} contains $\{(\alpha \bmod \mathfrak{m}_{K_f})^{1/p} \mid \alpha \in \mathrm{roots}(h)\}$. Since k_{K_h} is a purely transcendental extension of $\mathbb{F}_p$ generated by $\alpha \bmod \mathfrak{m}_{K_f}$ and k_{M_f} is separable over k_{K_h}, the inseparable degree of the extension k_{K_f}/k_{M_f} is at least p^m. Thus, the ramification index of the extension K_f/M_f is at least p^m. Therefore, K_f/M_f must be totally ramified with degree p^m and the Galois group $\mathrm{Gal}(K_f/M_f)$ must coincide with the whole of $(C_p)^{I_m}$.

These complete the proof of (i).

(ii) The above proof shows that $k_{K_h} = k(\alpha \bmod \mathfrak{m}_{K_h} \mid \alpha \in \mathrm{roots}(h))$, and that k_{K_f} contains the field k'_{K_f} generated by $\{(\alpha \bmod \mathfrak{m}_{K_f})^{1/p} \mid \alpha \in \mathrm{roots}(h)\} \cup \{(h'(\alpha)/h'(\alpha'))^{\frac{1}{p-1}} \bmod \mathfrak{m}_{K_f} \mid \alpha, \alpha' \in \mathrm{roots}(h)\}$ over k_{K_h} (or, equivalently, over k, as $\alpha = (\alpha^{1/p})^p$). Moreover, we can check $[k_{K_f} : k_{K_h}] = [k'_{K_f} : k_{K_h}]$, which implies $k_{K_f} = k'_{K_f}$, as desired.

Finally, k_{K_h} is a purely transcendental extension of $\mathbb{F}_p$ generated by $\{\alpha \bmod \mathfrak{m}_{K_h} \mid \alpha \in \mathrm{roots}(h)\}$. Moreover, observing the subgroup of $(K_h\overline{\mathbb{F}}_p)^\times/((K_h\overline{\mathbb{F}}_p)^\times)^{p-1}$ generated by the classes of $-a/h'(\alpha)$ ($\alpha \in \mathrm{roots}(h)$) by using the divisor group of (the spectrum of) the polynomial ring $\overline{\mathbb{F}}_p[\alpha_1, \ldots, \alpha_m]$ over $\overline{\mathbb{F}}_p$, and comparing the result with the above description of the subgroup of $K_h^\times/(K_h^\times)^{p-1}$ generated by the classes of $-a/h'(\alpha)$ ($\alpha \in \mathrm{roots}(h)$), we see that the algebraic closure of $\mathbb{F}_p$ in k_{K_h} is as described in the assertion. Since k_{K_f}/k_{M_f} is purely inseparable, this completes the proof. □

So far, we have only investigated superseparable polynomials over complete discrete valuation fields. Here, we shall introduce the following global situation and study superseparable polynomials in a moduli-theoretic fashion. We put

$$A^{mp}_{\mathrm{upper}} \stackrel{\mathrm{def}}{=} \mathrm{Spec}(\mathbb{F}_p[t_1, \ldots, t_{mp}]) \simeq \mathbf{A}^{mp}_{\mathbb{F}_p}$$

and

$$A^{mp}_{\mathrm{lower}} \stackrel{\mathrm{def}}{=} \mathrm{Spec}(\mathbb{F}_p[s_0, \ldots, s_{mp-1}]) \simeq \mathbf{A}^{mp}_{\mathbb{F}_p}.$$

Moreover, consider the morphism $E : A^{mp}_{\mathrm{upper}} \to A^{mp}_{\mathrm{lower}}$, defined by

$$\prod_{i=1}^{mp}(T - t_i) = \sum_{i=0}^{mp} s_i T^i,$$

where $s_{mp} \stackrel{\mathrm{def}}{=} 1$. Namely, for $i = 0, \dots, mp-1$, s_i is $(-1)^{mp-i}$ times the $(mp-i)$-th elementary symmetric polynomial in $t_1, \dots, t_{mp}$. It is well-known that E is finite flat of degree $(mp)!$, and that, if we delete the discriminant locus D_{lower} from A^{mp}_{lower} and the union D_{upper} of weak diagonals from A^{mp}_{upper}, E gives a finite, étale, Galois covering with Galois group S_{mp}.

Let A^{m+1}_{lower} be the closed subscheme of A^{mp}_{lower} defined by $s_i = 0$ for all i with $p \nmid i$ and $i \neq 1$. We define a divisor A^{m}_{lower} of A^{m+1}_{lower} by $s_1 = 0$. Observe that A^{m}_{lower} coincides with the non-étale locus of $E|_{A^{m+1}_{\mathrm{lower}}}$, and that we have $A^{m}_{\mathrm{lower}} = A^{m+1}_{\mathrm{lower}} \cap D_{\mathrm{lower}}$ set-theoretically. We also have $A^{m+1}_{\mathrm{lower}} \simeq \mathbf{A}^{m+1}_{\mathbb{F}_p}$ and $A^{m}_{\mathrm{lower}} \simeq \mathbf{A}^{m}_{\mathbb{F}_p}$ naturally.

Now, we have the following diagram:

$$\begin{array}{ccc} A^{mp}_{\mathrm{upper}} - D_{\mathrm{upper}} & \stackrel{E}{\to} & A^{mp}_{\mathrm{lower}} - D_{\mathrm{lower}} \\ \uparrow \text{c.i.} & \square & \uparrow \text{c.i.} \\ U_m & \to & A^{m+1}_{\mathrm{lower}} - A^{m}_{\mathrm{lower}} \end{array}$$

where $\square$ means a fiber product diagram, $\stackrel{\text{c.i.}}{\to}$ means a closed immersion, and $U_m \stackrel{\mathrm{def}}{=} A^{mp}_{\mathrm{upper}} \times_{A^{mp}_{\mathrm{lower}}} (A^{m+1}_{\mathrm{lower}} - A^{m}_{\mathrm{lower}})$.

We shall apply this moduli-theoretic situation to the study of superseparable polynomials over a (an arbitrary) complete discrete valuation field K of characteristic $p > 0$. From now, for each finite subset S of $\overline{K}$, we put $\phi_S(T) \stackrel{\mathrm{def}}{=} \prod_{\alpha \in S}(T - \alpha)$.

Proposition (1.11). *Let m be an integer ≥ 1. Assume that there exists a finite subset S of K with cardinality m, such that ϕ_S satisfies*

(1.12) $$\phi'_S(\gamma)/\phi'_S(\gamma') \in (K^\times)^{p-1} \textit{ for all } \gamma, \gamma' \in S.$$

Then, there exists a monic, superseparable polynomial $f(T) \in K[T]$ with $\deg(f) = mp$, such that f is completely splittable in K.

Proof. We consider the above moduli-theoretic situation $E : A^{mp}_{\mathrm{upper}} \to A^{mp}_{\mathrm{lower}}$ and $A^{m}_{\mathrm{lower}} \stackrel{\text{c.i.}}{\to} A^{m+1}_{\mathrm{lower}} \stackrel{\text{c.i.}}{\to} A^{mp}_{\mathrm{lower}}$. We have to show that $U \stackrel{\mathrm{def}}{=} U_m$ admits a K-rational point. Recall that E induces a finite, étale (not necessarily connected) S_{mp}-Galois covering $U \to A^{m+1}_{\mathrm{lower}} - A^{m}_{\mathrm{lower}}$. However, first we need to investigate the non-étale loci of E.

We put $A^m_{\text{upper}} \overset{\text{def}}{=} \operatorname{Spec}(\mathbb{F}_p[u_1, \dots, u_m]) \simeq \mathbf{A}^m_{\mathbb{F}_p}$, and define a morphism $D : A^m_{\text{upper}} \to A^{mp}_{\text{upper}}$ by

$$(u_1, \dots, u_m) \mapsto (\underbrace{u_1, \dots, u_1}_{p \text{ times}}, \dots, \underbrace{u_m, \dots, u_m}_{p \text{ times}}),$$

which is clearly a closed immersion. It is easy to see that $E \circ D : A^m_{\text{upper}} \to A^{mp}_{\text{lower}}$ factors through $A^m_{\text{lower}} \overset{\text{c.i.}}{\to} A^{mp}_{\text{lower}}$. More explicitly, $E \circ D$ induces a morphism $A^m_{\text{upper}} \to A^m_{\text{lower}}$, $(u_1, \dots, u_m) \mapsto ((v_1)^p, \dots, (v_m)^p)$, where v_i is $(-1)^{m-i}$ times the $(m-i)$-th elementary symmetric polynomial in $u_1, \dots, u_m$.

Now, we obtain the following diagram:

$$\begin{array}{ccccccc}
 & & & & A^{mp}_{\text{upper}} & \overset{E}{\to} & A^{mp}_{\text{lower}} \\
 & & & & \uparrow \text{c.i.} & \Box & \uparrow \text{c.i.} \\
 & & \tilde{X} & \to & X & \to & A^{m+1}_{\text{lower}} \\
 & & \uparrow \text{c.i.} & \Box & \uparrow \text{c.i.} & \Box & \uparrow \text{c.i.} \\
 & & Z' & \to & Z & \to & A^m_{\text{lower}} \\
 & & \uparrow \text{c.i.} & & \uparrow \text{c.i.} & & \\
\tilde{W} & \to & W & \to & A^m_{\text{upper}}. & &
\end{array}$$

Here, $X \overset{\text{def}}{=} A^{mp}_{\text{upper}} \times_{A^{mp}_{\text{lower}}} A^{m+1}_{\text{lower}}$, $\tilde{X}$ denotes the normalization of X in U, $Z \overset{\text{def}}{=} X \times_{A^{m+1}_{\text{lower}}} A^m_{\text{lower}}$, $Z' \overset{\text{def}}{=} \tilde{X} \times_X Z$, W denotes an irreducible component of $\tilde{X} \times_X A^m_{\text{upper}}$ (regarded as a reduced closed subscheme of $\tilde{X}$) that is surjectively mapped onto A^m_{upper}, and $\tilde{W}$ is the normalization of the integral scheme W.

Now, we are in the situation of (1.10). More explicitly, in the notation of (1.10), K^{univ} is just the field of fractions of the completed local ring of A^{m+1}_{lower} at the generic point of A^m_{lower}, $k_{K^{\text{univ}}} = \mathbb{F}_p(A^m_{\text{lower}})$, and $k_{K^{\text{univ}}_f} = \mathbb{F}_p(W)$. Moreover, we see that $\mathbb{F}_p(A^m_{\text{upper}}) = K^{\text{univ}}_h((\alpha_1)^{1/p}, \dots, (\alpha_m)^{1/p})$. Thus, (1.10)(ii) implies that $\mathbb{F}_p(W)$ is generated by $\{(h'(\alpha)/h'(\alpha'))^{1/(p-1)} \mid \alpha, \alpha' \in \operatorname{roots}(h)\}$ over $\mathbb{F}_p(A^m_{\text{upper}})$. Note that $u_i = \alpha_i^{1/p}$ holds for each $i = 1, \dots, m$. So, if we put $\mathbb{S} \overset{\text{def}}{=} \{u_1, \dots, u_m\}$, we have $\phi'_{\mathbb{S}}(u_i)^p = h'(\alpha_i)$, hence

$$\left(\frac{(h'(\alpha_i)/h'(\alpha_j))^{1/(p-1)}}{(\phi'_{\mathbb{S}}(u_i)/\phi'_{\mathbb{S}}(u_j))} \right)^{p-1} = \phi'_{\mathbb{S}}(u_i)/\phi'_{\mathbb{S}}(u_j).$$

Thus, we see that $\mathbb{F}_p(W)$ is generated by $\{(\phi'_{\mathbb{S}}(u_i)/\phi'_{\mathbb{S}}(u_j))^{1/(p-1)} \mid i, j = 1, \dots, m\}$ over $\mathbb{F}_p(A^m_{\text{upper}})$.

Let V be the complement of the union of weak diagonals defined by $u_i - u_j = 0$ for $i, j = 1, \dots, m$, $i \neq j$ in A^m_{upper}. Since $\tilde{W}$ coincides with the integral closure of A^m_{upper} in $\mathbb{F}_p(W)$, we now see that $\tilde{W}_V \stackrel{\mathrm{def}}{=} \tilde{W} \times_{A^m_{\mathrm{upper}}} V$ is finite étale covering generated by $\{(\phi'_{\mathbb{S}}(u_i)/\phi'_{\mathbb{S}}(u_j))^{1/(p-1)} \mid i, j = 1, \dots, m\}$ over V.

Now, take a finite set S as in our assumption, and put $S = \mathrm{roots}(\phi_S) = \{\gamma_1, \dots, \gamma_m\}$. Then, $x = (\gamma_1, \dots, \gamma_m)$ gives an element of $V(K) \subset A^m_{\mathrm{upper}}(K)$. Moreover, condition (1.12), together with the above description of $\tilde{W}_V$, implies that the fiber of $\tilde{W}_V \to V$ at x consists of K-rational points. In particular, we have $\tilde{W}_V(K) \neq \emptyset$. Note that $\tilde{W}_V$ is smooth over $\mathbb{F}_p$, as being étale over A^m_{upper}. Or, equivalently, $\tilde{W}_V$ is contained in the smooth locus $\tilde{W}^{\mathrm{sm}}$ of $\tilde{W}$. Now, as K is large, we conclude that $\tilde{W}(K)$ is Zariski dense in $\tilde{W}$. (See [Pop] for the definition and properties of large fields.) Accordingly, $W(K)$ is dense in W, a fortiori.

On the other hand, since $\tilde{X}$ is normal (and $\mathbb{F}_p$ is perfect), the complement of $\tilde{X}^{\mathrm{sm}}$ is of codimension ≥ 2 in $\tilde{X}$. It follows from this that $W \cap \tilde{X}^{\mathrm{sm}}$ is non-empty (and open in W). Moreover, since W is integral (and $\mathbb{F}_p$ is perfect), we have W^{sm} is also non-empty and open, hence so is $W' \stackrel{\mathrm{def}}{=} W^{\mathrm{sm}} \cap \tilde{X}^{\mathrm{sm}}$. As we have already seen, $W(K)$ is dense in W. Accordingly, we have $W'(K) \neq \emptyset$, hence, a fortiori, $\tilde{X}^{\mathrm{sm}}(K) \neq \emptyset$. As K is large, this implies that there exists a connected (or, equivalently, irreducible) component Y of $\tilde{X}$, such that $Y(K)$ is dense in Y. Now, observe that $Y \to A^{m+1}_{\mathrm{lower}}$ is (finite and) surjective. From this, $Y_U \stackrel{\mathrm{def}}{=} Y \times_X U = Y \cap U$ (where the last intersection is taken in $\tilde{X}$) is non-empty (and open in Y). Thus, $Y_U(K)$ is non-empty, hence, a fortiori, $U(K)$ is non-empty. This completes the proof. □

Lemma (1.13). *Let m be an integer ≥ 1. Assume that (K, m) satisfies:*

(1.14) *At least one of the following holds:*
$K \supset \mathbb{F}_{p^2}$*;* $p = 2$*;* $m \equiv \epsilon \pmod{p+1}$ *for some* $\epsilon \in \{0, \pm 1\}$.

Then, there exists a finite subset S of K with cardinality m, such that ϕ_S satisfies (1.12).

Proof. Let s denote any element of $\mathfrak{m} \cap (K^\times)^{p-1}$ (e.g., $s = t^{p-1}$).

Firstly, assume that either $K \supset \mathbb{F}_{p^2}$ or $p = 2$ holds. In this case, we take any integers $i_1, \dots, i_m$ with $i_1 < \cdots < i_m$ and put $S \stackrel{\mathrm{def}}{=} \{s^{i_k} \mid k = 1, \dots, m\}$. Then, $\sharp(S) = m$ clearly holds. Now, for $\gamma = s^{i_k} \in S$, we have

$$\begin{aligned}\phi'_S(\gamma) &= \prod_{j=1}^{k-1}(s^{i_k} - s^{i_j}) \prod_{j=k+1}^{m} (s^{i_k} - s^{i_j}) \\ &= (-1)^{k-1} s^{i_1 + \cdots + i_{k-1} + (m-k) i_k} \prod_{j \neq k} (1 - s^{|i_k - i_j|}).\end{aligned}$$

Note that we have $-1, s \in (K^\times)^{p-1}$ and $1 + \mathfrak{m} \subset (K^\times)^{p-1}$. (For -1, use the assumption that either $K \supset \mathbb{F}_{p^2}$ or $p = 2$ holds.) Thus, (1.12) holds.

Secondly, assume $m \equiv \epsilon \pmod{p+1}$ with $\epsilon \in \{0, \pm 1\}$. We may put $m = (p+1)n + \epsilon$. Moreover, we take any integers $i_1, j_1, i_2, j_2, \dots, i_n, j_n, i_{n+1}$ with $i_1 < j_1 < i_2 < j_2 < \cdots < i_n < j_n < i_{n+1}$. Now, we put $S_\epsilon = \{s^{i_k} \mid k \in I_\epsilon\} \cup \{s^{j_k} + cs^{j_k+1} \mid k = 1, \dots, n,\ c \in \mathbb{F}_p\}$, where

$$I_\epsilon = \begin{cases} \{2, \dots, n\}, & \epsilon = -1, \\ \{1, \dots, n\}, & \epsilon = 0, \\ \{1, \dots, n+1\}, & \epsilon = 1. \end{cases}$$

Then, by using $s \in (K^\times)^{p-1}$, $1 + \mathfrak{m} \subset (K^\times)^{p-1}$, and the fact $\prod_{j \in \mathbb{F}_p^\times} j = -1$, we can elementarily check that $\phi'_{S_\epsilon}(\gamma) \in (K^\times)^{p-1}$ (resp. $\phi'_{S_\epsilon}(\gamma) \in -(K^\times)^{p-1}$) holds for each $\gamma \in S_\epsilon$, if $\epsilon = 0, 1$ (resp. $\epsilon = -1$). Thus, (1.12) holds. □

DEFINITION. Let $f(T) = a_1 T + \sum_{i=0}^{m} a_{ip} T^{ip}$ be a superseparable polynomial (over some field of characteristic p).
(i) We say that f is of special type, if $a_{mp} = a_1 = 1,\ a_0 = 0$ holds.
(ii) We put $\operatorname{def}(f) \stackrel{\text{def}}{=} \sup\{r > 0 \mid a_j = 0 \text{ for all } j \text{ with } mp > j > mp - r\}$ and call it the defect of f. (Thus, we have $0 < \operatorname{def}(f) \leq mp - 1$, unless $m = 0$.)

COROLLARY (1.15). *Let m be a positive integer.*
(i) *Assume that (K, m) satisfies (1.14). Then, there exists a monic, integral, superseparable polynomial $f(T) \in K[T]$ with $f(0) = 0$ and $\deg(f) = mp$, such that f is completely splittable in K.*
(ii) *Assume that (K, m) satisfies one of the following: $K \supset \mathbb{F}_{p^2}$ and $(p-1, m-1) = (p+1, m+1) = 1$; $p = 2$; $m \equiv \epsilon \pmod{p+1}$ for some $\epsilon \in \{0, \pm 1\}$ and $(p-1, m-1) = 1$. Then, there exists a superseparable polynomial $f(T) \in K[T]$ of special type with $\deg(f) = mp$, such that f is completely splittable in K.*

Proof. (i) By (1.11) and (1.13), there exists a monic superseparable polynomial $f(T) \in K[T]$ with $\deg(f) = mp$, such that f is completely splittable in K. Replacing $f(T)$ by $f_c(T) \stackrel{\text{def}}{=} c^{mp} f(c^{-1}T)$ with $c \in K^\times$, $v(c) \gg 0$, we may assume that f is integral. (Observe that $\operatorname{roots}(f_c) = c \operatorname{roots}(f)$.) Finally, replacing $f(T)$ by $f(T + \alpha)$, where $\alpha \in \operatorname{roots}(f)$, we may assume $f(0) = 0$.
(ii) We have $K \supset \mathbb{F}_q((t))$ with $q = p^2$ (resp. $q = p$), if $K \supset \mathbb{F}_{p^2}$ (resp. either $p = 2$ or $m \equiv \epsilon \pmod{p+1}$ with $\epsilon \in \{0, \pm 1\}$). Thus, it suffices to prove the assertion in the case $K = \mathbb{F}_q((t))$. So, from now on, we assume that $K = \mathbb{F}_q((t))$.

By (1.11) and (1.13), there exists a monic, superseparable polynomial $f_1(T) \in K[T]$ with $\deg(f_1) = mp$, such that f_1 is completely splittable in K. Replacing $f_1(T)$ by $f_1(T + \alpha)$, where $\alpha \in \operatorname{roots}(f_1)$, we may assume that $f_1(0) = 0$. Moreover, we may put $f_1(T) = a_1(t)T + \sum_{i=0}^{m} a_{ip}(t) T^{ip}$, where

$a_{ip}(t) \in K = \mathbb{F}_q((t))$, $a_1(t) \in K^\times = \mathbb{F}_q((t))^\times$, and $a_{mp}(t) = 1$, $a_0(t) = 0$. Next, we put $f_2(T) \stackrel{\text{def}}{=} a_1(t^{mp-1})T + \sum_{i=1}^{m} a_{ip}(t^{mp-1})T^{ip}$. Then, $f_2(T)$ is completely splittable in $\mathbb{F}_q((t)) \supset \mathbb{F}_q((t^{mp-1}))$.

Put $a_1(t) = ct^r + \cdots$, where $c \in \mathbb{F}_q^\times$, $r \in \mathbb{Z}$, and '$\cdots$' means the higher order terms. Then, we have $a_1(t^{mp-1}) = ct^{r(mp-1)} + \cdots$. Here, observe that $(p-1, m-1) = (p+1, m+1) = 1$ (resp. $(p-1, m-1) = 1$) is equivalent to saying $(q-1, mp-1) = 1$, for $q = p^2$ (resp. $q = p$). So, we have $\mathbb{F}_q^\times = (\mathbb{F}_q^\times)^{mp-1}$. By using this fact (and the fact that $1 + \mathfrak{m} \subset (K^\times)^{mp-1}$ as $p \nmid mp-1$), we see that $a_1(t^{mp-1}) \in (K^\times)^{mp-1}$. So, write $a_1(t^{mp-1}) = b(t)^{mp-1}$. Now, it is easy to check that $f(T) \stackrel{\text{def}}{=} b(t)^{-mp} f_2(b(t)T)$ satisfies the desired conditions. This completes the proof. $\square$

Corollary (1.16). (i) *There exists a positive integer m_1 (which depends only on p), such that, for each positive integer m with $m_1 \mid m$, there exists a monic, integral, superseparable polynomial f in $K[T]$ with $f(0) = 0$ and $\deg(f) = mp$, such that f is completely splittable in K.*
(ii) *There exists a positive integer m_2 (which depends only on p), such that, for each positive integer m with $m_2 \mid m$, there exists a superseparable polynomial f in $K[T]$ of special type and with $\deg(f) = mp$, such that f is completely splittable in K.*

Proof. (i) (resp. (ii)) is a direct corollary of (1.15)(i) (resp. (ii)). We can take, for example, $m_1 = p + 1$ (resp. $m_2 = (p+1)(p-1)$). $\square$

Lemma (1.17). *Let F be a field of characteristic p, and L a Galois extension of F. Let A be an $\mathbb{F}_p[\mathrm{Gal}(L/F)]$-submodule of L with $\dim_{\mathbb{F}_p}(A) = r < \infty$, and put $\phi_A(T) \stackrel{\text{def}}{=} \prod_{\alpha \in A}(T - \alpha)$. Then:*
(i) *$\phi_A(T)$ is a monic superseparable polynomial in $F[T]$.*
(ii) *$F_{\phi_A} = F(A)$.*
(iii) *$\deg(\phi_A) = p^r$ and $\mathrm{def}(\phi_A) \geq p^r - p^{r-1}$.*
(iv) *$\phi'_A(T) = \prod_{\alpha \in A - \{0\}} \alpha$.*
(v) *If, moreover, $F = K$ and $A \subset R_L$, then ϕ_A is integral.*

Proof. (i) By definition, ϕ_A is monic and separable. It is well-known that ϕ_A is an additive polynomial, hence a superseparable polynomial. Since A is $\mathrm{Gal}(L/F)$-stable, $\phi_A(T) \in F[T]$.
(ii) Clear.
(iii) The first assertion is clear. The second assertion follows from the fact that ϕ_A is an additive polynomial.
(iv) Since ϕ_A is monic and superseparable, we obtain

$$\phi'_A(T) = \phi'_A(0) = \prod_{\alpha \in A - \{0\}} (-\alpha) = \prod_{\alpha \in A - \{0\}} \alpha,$$

as desired.
(v) Clear. □

The following corollary is a final result of this §, of which (i) (resp. (ii)) will play a key role in §2 (resp. §3). Note that one of the main differences between (i) and (ii) consists in the fact that, in (ii), the defect of the superseparable polynomial is estimated from below.

Corollary (1.18). (i) *Let L be a finite Galois extension of K. Then, there exists a positive integer $m_{L/K}$, such that, for each positive integer m with $m_{L/K} \mid m$, there exists a superseparable polynomial $f(T) \in K[T]$ of special type with $\deg(f) = mp$ and $K_f = L$.*
(ii) *We have:*

$\forall n$*: positive integer,*

$\exists m_n$*: positive integer (depending only on p and n),*

$\forall m$*: positive integer with $m_{K,n} \mid m$,*

$\exists c = c_{K,n,m}$*: positive real number,*

$\forall L$*: (possibly infinite) Galois extension of K,*

$\forall A$*: finite $\mathbb{F}_p[\mathrm{Gal}(L/K)]$-submodule of R_L with $A \cap \mathfrak{m}_L = \{0\}$,*

$\forall r$*: integer $> \dim_{\mathbb{F}_p}(A)$,*

$\forall \nu$*: integer,*

$\exists \delta$*: positive integer with $\delta \leq cmp^{r+1}/\sharp(A)$ and $\delta \equiv \nu \pmod{n}$,*

$\forall a \in K^\times$ *with $v(a) = \delta$,*

$\exists f(T)$*: monic, integral, superseparable polynomial in $K[T]$,*

s.t. $\deg(f) = mp^{r+1}$, $\mathrm{def}(f) \geq (p-1)p^r$, $f'(T) = a$ *and* $K_f = K(A) \subset L$.

Proof. (i) Since L/K is finite, we see that there exists a finite $\mathbb{F}_p[\mathrm{Gal}(L/K)]$-submodule $A_0 \neq \{0\}$ of L, such that $L = K(A_0)$. We put $q \stackrel{\mathrm{def}}{=} \sharp(A_0)$, which is a power of p. Then, by (1.17), $\phi_1 \stackrel{\mathrm{def}}{=} \phi_{A_0}$ is a monic, superseparable polynomial in $K[T]$ with $\deg(\phi_1) = q$, $\phi_1(0) = 0$, and $\phi_1'(T) = a_0 \stackrel{\mathrm{def}}{=} \prod_{\alpha \in A_0 - \{0\}} \alpha$, and $L = K_{\phi_1}$. On the other hand, take m_2 as in (1.16)(ii). Now, we put $m_{L/K} \stackrel{\mathrm{def}}{=} q(q-1)m_2$.

Let m be any positive integer divisible by $m_{L/K}$, and put $n \stackrel{\mathrm{def}}{=} m/m_{L/K}$. Then, by (1.16)(ii), there exists a superseparable polynomial $f_1(T) \in K[T]$ of special type and with degree $n(q-1)m_2p$, such that f_1 is completely splittable in K.

For each $b \in K^\times$, we put $f_b(T) \stackrel{\mathrm{def}}{=} b^{n(q-1)m_2p} f_1(b^{-1}T)$ (resp. $\phi_b(T) \stackrel{\mathrm{def}}{=} b^q \phi_1(b^{-1}T)$), so that f_b (resp. ϕ_b) is a monic, superseparable polynomial with $\deg(f_b) = n(q-1)m_2p$ (resp. $\deg(\phi_b) = q$), $f_b(0) = 0$ (resp. $\phi_b(0) = 0$), and $f_b'(T) = b^{n(q-1)m_2p-1}$ (resp. $\phi_b'(T) = b^{q-1}a_0$).

Now, by (1.4)(ii), every polynomial in $L[T]$ with degree q which is sufficiently close to $\phi_b(T)$ ($b \in K^\times$) is completely splittable in L. By using this, we see that $F_{b,b'} \stackrel{\text{def}}{=} f_{b'} \circ \phi_b \in K[T]$ satisfies $K_{F_{b,b'}} = L$ for all $b' \in K^\times$ with $v(b') \geq C(b)$, where $C(b)$ denotes a constant depending on b. Observe that $F_{b,b'}$ is a monic, superseparable polynomial with $\deg(F_{b,b'}) = n(q-1)m_2p \times q = mp$, $F_{b,b'}(0) = 0$, and $F'_{b,b'}(T) = (b')^{n(q-1)m_2p-1}b^{q-1}a_0$.

Now, take $b = a_0^{-nm_2p}$ and $b' = a_0 d^{mp-1}$ for any $d \in K^\times$ with $v(d)$ sufficiently large, then $f(T) \stackrel{\text{def}}{=} D^{-mp}F_{b,b'}(DT)$ with $D \stackrel{\text{def}}{=} d^{n(q-1)m_2p-1}$ satisfies all the desired properties.

(ii) Let m_1 be as in (1.16)(i), and choose any common multiple $m_n > 1$ of m_1, $p-1$, and n.

Let m be any positive integer with $m_n \mid m$. Then, by (1.16)(i), there exists a monic, integral, superseparable polynomial $f_1(T) \in K[T]$ with $\deg(f_1) = mp$, $f_1(0) = 0$, and $K_{f_1} = K$. Now, put $f_1'(T) = a_1 \in R$ and $c \stackrel{\text{def}}{=} \max(\frac{v(a_1)}{m}, \frac{np}{p-1})$.

Let L be any Galois extension of K, A any finite $\mathbb{F}_p[\mathrm{Gal}(L/K)]$-submodule of R_L with $A \cap \mathfrak{m}_L = \{0\}$, and r any integer $> r_0 \stackrel{\text{def}}{=} \dim_{\mathbb{F}_p}(A)$. Let ν be any integer. We define μ to be the unique integer with $0 < \mu \leq n$, such that $\mu \equiv v(a_1) - (r - r_0) - \nu \pmod n$. We put $\delta \stackrel{\text{def}}{=} v(a_1) + \mu(mp-1) + \sum_{j=1}^{r-r_0}(mp^{j+1} - 1)$. Then, we have

$$\begin{aligned}
\delta &\leq v(a_1) + \sum_{j=0}^{r-r_0} nmp^{j+1} \\
&= m\left(\frac{v(a_1)}{m} + \frac{np}{p-1}(p^{r-r_0+1} - 1)\right) \\
&\leq m(c + c(p^{r-r_0+1} - 1)) \\
&= cmp^{r+1}/\sharp(A)
\end{aligned}$$

and

$$\delta \equiv v(a_1) - \mu - (r - r_0) \equiv \nu \pmod n,$$

as desired.

Let a be any element of $K^\times$ with $v(a) = \delta$. For $j = 0, \ldots, r - r_0$, we shall inductively define a monic, integral, superseparable polynomial $f_{2,j}(T)$ with $\deg(f_{2,j}) = mp^{j+1}$, $f_{2,j}(0) = 0$, $f_{2,j}(T) \equiv T^{mp^{j+1}} \pmod{\mathfrak{m}}$, and $K_{f_{2,j}} = K$, as follows. First, for $j = 0$, we put $g_0(T) \stackrel{\text{def}}{=} f_1(T)$ and $f_{2,0}(T) \stackrel{\text{def}}{=} t^{\mu mp} g_0(t^{-\mu}T)$. Next, for j with $0 < j < r - r_0$, we put $g_j \stackrel{\text{def}}{=} f_{2,j-1} \circ \phi_{\mathbb{F}_p}$, where $\phi_{\mathbb{F}_p}(T) = T^p - T$, and $f_{2,j}(T) \stackrel{\text{def}}{=} t^{mp^{j+1}} g_j(t^{-1}T)$. Finally, for $j = r - r_0$, let u and u' be elements of $R^\times$, which we shall fix later, and we put $g_{r-r_0} \stackrel{\text{def}}{=} f_{2,r-r_0-1} \circ \phi_{u\mathbb{F}_p}$, where $\phi_{u\mathbb{F}_p}(T) = T^p - u^{p-1}T$, and $f_{2,r-r_0}(T) \stackrel{\text{def}}{=} (u't)^{mp^{r-r_0+1}} g_{r-r_0}((u't)^{-1}T)$.

We can check inductively that $f_{2,j}(T)$ is a monic, integral, superseparable polynomial with $\deg(f_{2,j}) = mp^{j+1}$, $f_{2,j}(0) = 0$, $f_{2,j}(T) \equiv T^{mp^{j+1}} \pmod{\mathfrak{m}}$, and $K_{f_{2,j}} = K$. Moreover, since $f'_{2,0} = t^{\mu(mp-1)}a_1$, $f'_{2,j} = (-t^{mp^{j+1}-1})f'_{2,j-1}$ $(0 < j < r - r_0)$, and $f'_{2,r-r_0} = (-u^{p-1})(u't)^{mp^{r-r_0+1}-1}$, we obtain

$$\begin{aligned} f'_{2,r-r_0} &= u^{p-1}(u')^{mp^{r-r_0+1}-1}(-1)^{r-r_0}a_1 t^{\mu(mp-1)+\sum_{j=1}^{r-r_0}(mp^{j+1}-1)} \\ &= u^{p-1}(u')^{mp^{r-r_0+1}-1}(-1)^{r-r_0}(a_1/t^{v(a_1)})t^{\delta}. \end{aligned}$$

So, put $u' = (-1)^{r-r_0}(a_1/t^{v(a_1)})(t^{\delta}/a)w$, where $w \overset{\text{def}}{=} \prod_{\alpha \in A-\{0\}} \alpha \in R^{\times}$, and $u = (u')^{-\frac{m}{p-1}p^{r-r_0+1}}$, then we have $f'_{2,r-r_0}(T) = aw^{-1}$. Now, we put $f_2 \overset{\text{def}}{=} f_{2,r-r_0}$.

Finally, put $f \overset{\text{def}}{=} f_2 \circ \phi_A$. Then, f is a monic, integral, superseparable polynomial in $K[T]$ with $\deg(f) = \deg(f_2)\deg(\phi_A) = mp^{r-r_0+1}\sharp(A) = mp^{r+1}$, $f' = f'_2\phi'_A = (aw^{-1})w = a$, and $K_f = K(A) \subset L$. Finally, by the above construction, we see that f is in the form of (a superseparable polynomial with degree mp) $\circ$ (an additive polynomial with degree p^r). As $m \geq m_{K,n} > 1$ and $r > r_0 \geq 0$, this implies $\operatorname{def}(f) \geq p^{r+1} - p^r$. This completes the proof. $\square$

Remark (1.19). So far, we have assumed that K is a complete discrete valuation field (of characteristic p). However, this assumption is superfluous. More specifically, (1.4), (1.5), (1.11), (1.13), (1.15), (1.16), and (1.18) remain valid if we replace this assumption by the weaker assumption that K is henselian (of characteristic p), and (1.10) remains valid if we replace the phrase 'completion' by 'henselization'. Indeed, the proof of the henselian case is just similar to the complete case.

Moreover, among these, (1.11), (1.13), (1.15) (except that we need to delete the phrase 'integral' in (i)), (1.16) (except that we need to delete the phrase 'integral' in (i)), and (1.18)(i) remain valid, if we only assume that K is a large field (of characteristic p) in the sense of [Pop]. (In particular, we do not have to assume that K is equipped with a discrete valuation.) Indeed, we see that these statements can be formulated in terms of the existence of K-rational points of K-varieties. The validity of the complete case implies that these varieties admit $K((t))$-rational points. Now, the large case follows directly from one of the equivalent definitions of large fields (see [Pop], Proposition 1.1, (5)).

§2. Unramified extensions with prescribed local extensions.

In this §, we use the following new notation. Let C be a noetherian, normal, integral, separated $\mathbb{F}_p$-scheme of dimension 1. We denote by K the rational function field of C, and fix an algebraic closure $\overline{K}$ of K. We denote by K^{sep} and $G = G_K$ the separable closure of K in $\overline{K}$ and the absolute Galois group $\operatorname{Gal}(K^{\text{sep}}/K)$ of K, respectively. Let Σ_C be the set of closed points of C. For each $v \in \Sigma_C$, we denote by R_v the completion of the local ring $\mathcal{O}_{C,v}$. This is a complete discrete valuation ring. We denote by K_v, $\mathfrak{m}_v$ and k_v the field of fractions of R_v, the maximal ideal of R_v and the residue field $R_v/\mathfrak{m}_v$ of R_v, respectively. We fix an algebraic closure $\overline{K}_v$ of K_v, and denote by K_v^{sep} and $G_v = G_{K_v}$ the separable closure of of K_v in $\overline{K}_v$ and the absolute Galois group $\mathrm{Gal}(K_v^{\mathrm{sep}}/K_v)$, respectively.

Definition. We refer to a tuple $\mathcal{C} = (C, \Sigma, \{L_v\}_{v\in\Sigma})$ as a base scheme data, if C is as above, Σ is a (possibly empty) finite subset of Σ_C; and, for each $v \in \Sigma$, L_v is a (possibly infinite) normal subextension of $\overline{K}_v$ over K_v, such that $L_v \cap K_v^{\mathrm{sep}}$ is v-adically dense in L_v. (For example, this last condition is satisfied if either L_v/K_v is Galois or $L_v = \overline{K}_v$.)

If, moreover, C is a normal, geometrically integral curve over a field k of characteristic p, we refer to $\mathcal{C}$ as a base curve data over k.

For a base scheme data $\mathcal{C} = (C, \Sigma, \{L_v\}_{v\in\Sigma})$, we put $B = B_{\mathcal{C}} \stackrel{\mathrm{def}}{=} C - \Sigma$. If, moreover, B is affine, then we put $R = R_{\mathcal{C}} \stackrel{\mathrm{def}}{=} \Gamma(B, \mathcal{O}_B)$, so that R is a Dedekind domain and that $B = \mathrm{Spec}(R)$.

We say that a base scheme data $\mathcal{C} = (C, \Sigma, \{L_v\}_{v\in\Sigma})$ is finite, if L_v is a finite extension of K_v for each $v \in \Sigma$. (In this case, L_v is automatically Galois over K_v.)

Definition. Let $\mathcal{C} = (C, \Sigma, \{L_v\}_{v\in\Sigma})$ be a base scheme data. Let K' be an extension of K contained in $\overline{K}$. Then, we say that K' is $\mathcal{C}$-distinguished (resp. $\mathcal{C}$-admissible), if the integral closure C' of C in K' is étale over B; and, for each $v \in \Sigma$ and each embedding $\iota : \overline{K} \hookrightarrow \overline{K}_v$ over K, we have $\iota(K')K_v = L_v$ (resp. $\iota(K')K_v \subset L_v$).

Theorem (2.1). *Let $\mathcal{C} = (C, \Sigma, \{L_v\}_{v\in\Sigma})$ be a finite base scheme data, and assume that C is affine. Then, there exists a $\mathcal{C}$-distinguished finite Galois extension K'/K.*

Proof. For each $v \in \Sigma$, take a positive integer m_{L_v/K_v} as in (1.18)(i), and let m be any common multiple of m_{L_v/K_v} $(v \in \Sigma)$. Then, for each $v \in \Sigma$, there exists a superseparable polynomial $f_v(T) \in K_v[T]$ of special type and with degree mp, such that $L_v = (K_v)_{f_v}$.

Now, observe that R is dense in $\prod_{v\in\Sigma} K_v$. (This follows essentially from the Chinese Remainder Theorem for the Dedekind domain $\Gamma(C, \mathcal{O}_C)$.) From this, we can take a superseparable polynomial $f(T) \in R[T]$ of special type and with degree mp, which is arbitrarily close to f_v for each $v \in \Sigma$. Then, we have $(K_v)_f = (K_v)_{f_v} = L_v$. Or, equivalently, $K_f \otimes_K K_v$ is isomorphic to a direct product of copies of L_v over K_v. On the other hand, for each $v \in \Sigma_C - \Sigma$, $f \bmod \mathfrak{m}_v$ is a separable polynomial over k_v, since f is of special type. From this, we see that K_f is unramified at v. Thus, $K' \stackrel{\mathrm{def}}{=} K_f$ satisfies all the desired properties. $\square$

Definition. Let F be a field. We denote by F^{sep} and G_F a separable closure of F and the absolute Galois group $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ of F, respectively. For each prime number l, we define $F(l)$ to be the union of finite Galois extensions F' of

F in F^{sep} with $\mathrm{Gal}(F'/F) \simeq (\mathbb{Z}/l\mathbb{Z})^n$ for some n. Thus, $F(l)$ corresponds via Galois theory to the closed subgroup $G_F(l) \stackrel{\mathrm{def}}{=} \overline{[G_F, G_F](G_F)^l}$ of G_F (which coincides with the kernel of $G_F \twoheadrightarrow G_F^{\mathrm{ab}}/(G_F^{\mathrm{ab}})^l$).

Definition. Let $\mathcal{C} = (C, \Sigma, \{L_v\}_{v\in\Sigma})$ be a base scheme data and Σ_∞ a subset of Σ. Let K' be an extension of K contained in $\overline{K}$. Then, we say that K' is nearly $\mathcal{C}$-distinguished (resp. nearly $\mathcal{C}$-admissible) with respect to Σ_∞, if the integral closure C' of C in K is étale over B; for each $v \in \Sigma - \Sigma_\infty$ and each embedding $\iota : \overline{K} \hookrightarrow \overline{K}_v$ over K, we have $\iota(K')K_v = L_v$ (resp. $\iota(K')K_v \subset L_v$); and, for each $v \in \Sigma_\infty$ and each embedding $\iota : \overline{K} \hookrightarrow \overline{K}_v$ over K, we have $L_v \subset \iota(K')K_v \subset L_v(p)$ (resp. $\iota(K')K_v \subset L_v(p)$).

Theorem (2.2). *Let k be a field of characteristic p, and $\mathcal{C} = (C, \Sigma, \{L_v\}_{v\in\Sigma})$ a finite base curve data over k. Let Σ_∞ be a subset of Σ, and assume that $C - \Sigma_\infty$ is affine. Then, there exists a finite Galois extension K'/K that is nearly $\mathcal{C}$-distinguished with respect to Σ_∞.*

Proof. Let C^* be the normal, geometrically integral compactification of C, and put $\Sigma^* \stackrel{\mathrm{def}}{=} \Sigma \cup (\Sigma_{C^*} - \Sigma_C)$ and $\Sigma^*_\infty \stackrel{\mathrm{def}}{=} \Sigma_\infty \cup (\Sigma_{C^*} - \Sigma_C)$. Moreover, for each $v \in \Sigma_{C^*} - \Sigma_C$, we choose any finite Galois extension L_v of K_v (say, $L_v = K_v$). Then, replacing $\mathcal{C} = (C, \Sigma, \{L_v\}_{v\in\Sigma})$ by $(C^*, \Sigma^*, \{L_v\}_{v\in\Sigma^*})$ and Σ_∞ by Σ^*_∞, we may assume that C is proper over k. In this case, we have $\Sigma_\infty \neq \emptyset$, since $C - \Sigma_\infty$ is affine.

For each $v \in \Sigma$, take a positive integer m_{L_v/K_v} as in (1.18)(i), and let m be any common multiple of m_{L_v/K_v} ($v \in \Sigma - \Sigma_\infty$), pm_{L_v/K_v} ($v \in \Sigma_\infty$) and 2. Then, for each $v \in \Sigma - \Sigma_\infty$ (resp. $v \in \Sigma_\infty$), there exists a superseparable polynomial $f_v(T) \in K_v[T]$ of special type and with degree mp (resp. m), such that $L_v = (K_v)_{f_v}$.

Now, let $v \in \Sigma - \Sigma_\infty$. Then, for each polynomial $f_{1,v}(T) \in K_v[T]$ with degree mp which is sufficiently close to $f_v(T)$, we have $(K_v)_{f_{1,v}} = (K_v)_{f_v} = L_v$. More precisely, we can take (sufficiently small) $n_v \in \mathbb{Z}$ and (sufficiently large) $m_v \in \mathbb{Z}$ with $n_v < m_v$, so that every coefficient of f_v belongs to $\mathfrak{m}_v^{n_v}$, and that, if every coefficient of $f_{1,v} - f_v$ belongs to $\mathfrak{m}_v^{m_v}$, then we have $(K_v)_{f_{1,v}} = (K_v)_{f_v}$.

On the other hand, let $v \in \Sigma_\infty$. Then, similarly as above, we can take (sufficiently small) $n_v \in \mathbb{Z}$ and (sufficiently large) $m_v \in \mathbb{Z}$ with $n_v < m_v$, so that every coefficient of f_v belongs to $\mathfrak{m}_v^{n_v}$, and that, for each polynomial $f_{1,v}(T) \in K_v[T]$ with degree m, if every coefficient of $f_{1,v} - f_v$ belongs to $\mathfrak{m}_v^{m_v}$, then we have $(K_v)_{f_{1,v}} = (K_v)_{f_v}$. Moreover, replacing n_v and m_v if necessary, we may assume that, for each monic polynomial $f_{1,v}(T) \in K_v[T]$ with degree m whose constant term is 0, if every coefficient of $f_{1,v} - f_v$ belongs to $\mathfrak{m}_v^{m_v}$, then we have $(K_v)_{f_{1,v}} = (K_v)_{f_v}$ and there exists a bijection $\iota : \mathrm{roots}(f_v) \stackrel{\sim}{\rightarrow} \mathrm{roots}(f_{1,v})$, such that, for each $\alpha \in \mathrm{roots}(f_v)$, $v(\iota\alpha) = v(\alpha)$, $\mu(f_{1,v}, \iota(\alpha)) = \mu(f_v, \alpha)$, and $f'_{1,v}(\iota(\alpha)) \sim f'_v(\alpha) = 1$. (For the notations $\mu(-,-)$ and $\sim$, see §1.) Now, we let d_v denote the minimal non-negative integer satisfying

$$\mu(f_v,\alpha) < \min\left(d_v - v(f_v'(\alpha)) + \frac{1}{p}v(\alpha), \frac{p}{p-1}(d_v - v(f_v'(\alpha)))\right)$$
$$= \min\left(d_v + \frac{1}{p}v(\alpha), \frac{p}{p-1}d_v\right)$$

for all $\alpha \in \mathrm{roots}(f_v)$.

Lemma (2.3). *Let $P_1,\dots,P_r$ be distinct closed points of C, and, for each $i = 1,\dots,r$, let a_i and b_i be integers with $a_i \geq b_i$. If $b_1[k_{P_1} : k] + \cdots + b_r[k_{P_r} : k] > 2p_a(C) - 2$, then the natural map $\Gamma(C, \mathcal{O}_C(a_1P_1 + \cdots + a_rP_r)) \to \mathfrak{m}_{P_1}^{-a_1}/\mathfrak{m}_{P_1}^{-b_1} \oplus \cdots \oplus \mathfrak{m}_{P_r}^{-a_r}/\mathfrak{m}_{P_r}^{-b_r}$ is surjective. Here, $p_a(C)$ denotes the arithmetic genus of C.*

Proof. This follows from [CFHR], Theorem 1.1. □

We fix a sufficiently large integer N satisfying

$$(2.4) \qquad \left(p(p-1)\sum_{v\in\Sigma_\infty}[k_v : k]\right)N - \sum_{v\in\Sigma}[k_v : k]m_v > 2p_a(C) - 2$$

and

$$(2.5) \qquad (mp-1)(p-1)N \geq \max\{d_v \mid v \in \Sigma_\infty\}(\geq 0),$$

and, for each $v \in \Sigma_\infty$, choose any $e_v \in K_v$ with $v(e_v) = N$.

Now, put

$$g_v(T) \stackrel{\mathrm{def}}{=} \begin{cases} f_v(T), & v \in \Sigma - \Sigma_\infty, \\ e_v^{-mp(p-1)}f_v(-e_v^{p(p-1)}T^p) + T, & v \in \Sigma_\infty. \end{cases}$$

Then, $g_v(T)$ is a superseparable polynomial in $K_v[T]$ of special type and with degree mp. (For $v \in \Sigma_\infty$, use the assumption that $2 \mid m$.) So, by (2.3) and (2.4), we see that there exists a superseparable polynomial $g(T) \in R[T]$ of special type and with degree mp, such that every coefficient of $g(T) - g_v(T)$ belongs to $\mathfrak{m}_v^{m_v}$ (resp. $\mathfrak{m}_v^{-p(p-1)N+m_v}$) for $v \in \Sigma - \Sigma_\infty$ (resp. $v \in \Sigma_\infty$).

We put $K' \stackrel{\mathrm{def}}{=} K_g$. Just as in the proof of (2.1), K' satisfies the desired property for $v \in \Sigma - \Sigma_\infty$ and $v \in \Sigma_C - \Sigma$. So, we shall observe what happens at $v \in \Sigma_\infty$. We put $g_{e_v}(T) \stackrel{\mathrm{def}}{=} e_v^{mp(p-1)}g(-e_v^{-(p-1)}T)$. Or, writing $g(T) = T + k(T^p)$, we have $g_{e_v}(T) \stackrel{\mathrm{def}}{=} -e_v^{(mp-1)(p-1)}T + k_v(T^p)$, where $k_v(T) \stackrel{\mathrm{def}}{=} e_v^{mp(p-1)}k(-e_v^{-(p-1)p}T^p)$. By the choice of g, every coefficient of $g(T) - g_v(T)$ belongs to $\mathfrak{m}_v^{-p(p-1)N+m_v}$, hence every coefficient of $g_{e_v}(T) - (f_v(T^p) - e_v^{(mp-1)(p-1)}T)$ belongs to $e_v^{p(p-1)}\mathfrak{m}_v^{-p(p-1)N+m_v} = \mathfrak{m}_v^{m_v}$. Or, equivalently, every coefficient of $k_v - f_v$ belongs to $\mathfrak{m}_v^{m_v}$. Thus, we may apply the preceding argument to $f_{1,v} = k_v$. Since, moreover, $v(-e_v^{(mp-1)(p-1)}) = (mp-1)(p-1)N \geq d_v$ by (2.5), we may apply (1.5) to $g_{e_v}(T) = -e_v^{(mp-1)(p-1)}T + k_v(T^p)$. Then, firstly, we have $(K_v)_{g_{e_v}} \supset (K_v)_{k_v} = (K_v)_{f_v} = L_v$. Secondly, for each $\alpha_1 \in \mathrm{roots}(k_v)$, $(-(-e_v^{(mp-1)(p-1)})/k_v'(\alpha_1)) \sim (e_v^{mp-1})^{p-1}$ belongs to $((K_v)_{k_v}^\times)^{p-1}$. Thus, we have $M_{g_{e_v}} = (K_v)_{k_v}$. Thirdly, since $\mathrm{Gal}((K_v)_{g_{e_v}}/M_{g_{e_v}})$ is a subgroup of $(C_p)^{I_m}$, we have $(K_v)_{g_{e_v}} \subset M_{g_{e_v}}(p)$. Combining these, we obtain $L_v \subset (K_v)_{g_{e_v}} \subset L_v(p)$. Finally, since $\mathrm{roots}(g_{e_v}) = -e_v^{p-1}\,\mathrm{roots}(g)$, we have $(K_v)_{g_{e_v}} = (K_v)_g$. Thus, $K' = K_g$ satisfies the desired property at $v \in \Sigma_\infty$. This completes the proof. $\square$

§3. Main results.

In this §, we use the following notation. Let k be an (a possibly infinite) algebraic extension of $\mathbb{F}_p$ and C a smooth, geometrically connected (or, equivalently, normal, geometrically integral) curve over k. In particular, C is a noetherian, normal, integral, separated $\mathbb{F}_p$-scheme of dimension 1, and we use the notations introduced at the beginning of §2 for this C. Among other things, see §2 for the definition of base curve data.

Definition. (i) We refer to a tuple $\mathcal{S} = (\mathcal{C}, f : X \to B, \{\Omega_v\}_{v\in\Sigma})$ as a (smooth) Skolem data, if $\mathcal{C} = (C, \Sigma, \{L_v\}_{v\in\Sigma})$ is a base curve data; $B = B_{\mathcal{C}}$; $f : X \to B$ is a smooth, surjective morphism whose generic fiber X_K is geometrically irreducible; and, for each $v \in \Sigma$, Ω_v is a non-empty, v-adically open, G_v-stable subset of $X(L_v)$. (Observe that X is automatically irreducible.)
(ii) We refer to a tuple $\mathcal{B} = (\mathcal{C}, Y_1, \dots, Y_r \subset \mathbf{P}(\mathcal{E}), \{\check{\Omega}_v\}_{v\in\Sigma})$ as a Bertini data, if $\mathcal{C} = (C, \Sigma, \{L_v\}_{v\in\Sigma})$ is a base curve data; $\mathcal{E}$ is a locally free sheaf of finite rank $\neq 0$ on B; $r \geq 0$; Y_i is an irreducible, reduced, closed subscheme of $\mathbf{P}(\mathcal{E})$; and, for each $v \in \Sigma$, $\check{\Omega}_v$ is a non-empty, v-adically open, G_v-stable subset of $\mathbf{P}(\check{\mathcal{E}})(L_v)$, where $\check{\mathcal{E}} \stackrel{\mathrm{def}}{=} \mathcal{H}om_{\mathcal{O}_B}(\mathcal{E}, \mathcal{O}_B)$.

For a Bertini data $\mathcal{B} = (\mathcal{C}, Y_1, \dots, Y_r \subset \mathbf{P}(\mathcal{E}), \{\check{\Omega}_v\}_{v\in\Sigma})$, we define Y_i^{sm} $(i = 1, \dots, r)$ to be the set of points of Y_i at which $Y_i \to B$ is smooth. This is an (a possibly empty) open subset of Y_i, and we regard it as an open subscheme of Y_i.

Definition. (i) Let $\mathcal{S} = (\mathcal{C}, f : X \to B, \{\Omega_v\}_{v\in\Sigma})$ be a Skolem data with $\mathcal{C} = (C, \Sigma, \{L_v\}_{v\in\Sigma})$. Then, an $\mathcal{S}$-admissible quasi-section is a B-morphism $s : B' \to X$, where B' is the integral closure of B in a finite, $\mathcal{C}$-admissible extension K' of K, such that, for each $v \in \Sigma$, the image of $B'_{L_v} \stackrel{\mathrm{def}}{=} B' \times_B L_v$ in $X_{L_v} \stackrel{\mathrm{def}}{=} X \times_B L_v$ is contained in Ω_v $(\subset X(L_v) = X_{L_v}(L_v))$.
(ii) Let $\mathcal{B} = (\mathcal{C}, Y_1, \dots, Y_r \subset \mathbf{P}(\mathcal{E}), \{\check{\Omega}_v\}_{v\in\Sigma})$ be a Bertini data with $\mathcal{C} = (C, \Sigma, \{L_v\}_{v\in\Sigma})$. Then, a $\mathcal{B}$-admissible quasi-hyperplane is a hyperplane H in $\mathbf{P}(\mathcal{E})_{B'}$, where B' is the integral closure of B in a finite, $\mathcal{C}$-admissible extension K' of K, such that (a) for each $i = 1, \dots, r$, each geometric point $\overline{b}$ of B' and each irreducible component P of $Y_{i,\overline{b}}$, we have $P \cap H_{\overline{b}} \subsetneq P$; (b) for each $i = 1, \dots, r$, the scheme-theoretic intersection $(Y_i^{\mathrm{sm}})_{B'} \cap H$ (in $\mathbf{P}(\mathcal{E})_{B'}$) is smooth over B'; (c) for each $i = 1, \dots, r$ and each irreducible component P

of $Y_{i,\overline{K}}$ with $\dim(P) \geq 2$, $P \cap H_{\overline{K}}$ is irreducible; and (d) for each $v \in \Sigma$, the image of B'_{L_v} in $\mathbf{P}(\check{\mathcal{E}})_{L_v}$ by the base change to L_v of the classifying morphism $[H] : B' \to \mathbf{P}(\check{\mathcal{E}})$ over B is contained in $\check{\Omega}_v$ ($\subset \mathbf{P}(\check{\mathcal{E}})(L_v) = \mathbf{P}(\check{\mathcal{E}})_{L_v}(L_v)$).

Definition. Let $\mathcal{C} = (C, \Sigma, \{L_v\}_{v\in\Sigma})$ be a base curve data over k.
(i) We denote by R_{L_v}, $\mathfrak{m}_{L_v}$, and k_{L_v} the integral closure of R_v in L_v, the maximal ideal of R_{L_v}, and the residue field $R_{L_v}/\mathfrak{m}_{L_v}$, respectively.
(ii) We say that $\mathcal{C}$ satisfies condition (RI), if $[k_{L_v} : \mathbb{F}_p] = \infty$ for all $v \in \Sigma$. (Here, 'RI' means 'residually infinite'.)

Now, the following are the main results of the present paper.

Theorem (3.1). *Let $\mathcal{S} = (\mathcal{C}, f : X \to B, \{\Omega_v\}_{v\in\Sigma})$ be a Skolem data with $\mathcal{C} = (C, \Sigma, \{L_v\}_{v\in\Sigma})$, and assume that C is affine and that* (RI) *holds. Then, there exists an $\mathcal{S}$-admissible quasi-section.*

Theorem (3.2). *Let $\mathcal{B} = (\mathcal{C}, Y_1, \dots, Y_r \subset \mathbf{P}(\mathcal{E}), \{\check{\Omega}_v\}_{v\in\Sigma})$ be a Bertini data with $\mathcal{C} = (C, \Sigma, \{L_v\}_{v\in\Sigma})$, and assume that C is affine and that* (RI) *holds. Then, there exists a $\mathcal{B}$-admissible quasi-hyperplane.*

The aim of the rest of this § is to prove these theorems, together and step by step. From now on, we put $\mathcal{C} = (C, \Sigma, \{L_v\}_{v\in\Sigma})$, $\mathcal{S} = (\mathcal{C}, f : X \to B, \{\Omega_v\}_{v\in\Sigma})$, and $\mathcal{B} = (\mathcal{C}, Y_1, \dots, Y_r \subset \mathbf{P}(\mathcal{E}), \{\check{\Omega}_v\}_{v\in\Sigma})$, and assume always that C is affine and that (RI) holds.

Definition. We say that a Skolem data $\mathcal{S} = (\mathcal{C}, f : X \to B, \{\Omega_v\}_{v\in\Sigma})$ is essentially rational, if $\Omega_v \cap X(K_v) \neq \emptyset$ for each $v \in \Sigma$.

Step 1. Assume that $\mathcal{S}$ is essentially rational, and that X is an open subscheme of $\mathbf{P}^1_B$. Then, there exists an $\mathcal{S}$-admissible quasi-section.

Proof. We put $W \stackrel{\text{def}}{=} \mathbf{P}^1_B - X$. By shrinking X if necessary, we may assume that W is purely of codimension 1 in $\mathbf{P}^1_B$ and that W contains the infinity section ∞_B of $\mathbf{P}^1_B$. Next, we put $\tilde{R} \stackrel{\text{def}}{=} \Gamma(C, \mathcal{O}_C)$, which is a Dedekind domain contained in $R = \Gamma(B, \mathcal{O}_B)$, such that $C = \mathrm{Spec}(\tilde{R})$.

Since $\mathrm{Pic}(C)$ is a torsion group (cf. [Mo2], 1.9), there exists $n > 0$, such that $(\mathfrak{m}_v \cap \tilde{R})^n$ is a principal ideal of $\tilde{R}$ for each $v \in \Sigma$. In particular, there exists $\varpi \in \tilde{R}$, such that $(\prod_{v\in\Sigma}(\mathfrak{m}_v \cap \tilde{R}))^n = \tilde{R}\varpi$. On the other hand, since $\mathbf{A}^1(R)$ is dense in $\prod_{v\in\Sigma} \mathbf{P}^1(K_v)$, there exists $x \in \mathbf{A}^1(R)(= R)$, such that $x \in \Omega_v \cap X(K_v)$ for each $v \in \Sigma$. (Here, we have used the assumption that $\mathcal{S}$ is essentially rational.) Since Ω_v is v-adically open in $X(L_v)$, there exists $l_v \geq 0$, such that $x + (\mathfrak{m}_v R_{L_v})^{l_v} \subset \Omega_v$. Finally, take a sufficiently large integer M, such that $nM \geq l_v$ for each $v \in \Sigma$ and that $nM > v(\omega - x)$ for each $v \in \Sigma$ and each $\omega \in W(\overline{K}_v) - \{\infty\}$.

Now, let S denote the coordinate of $\mathbf{A}^1_B$ that we are using. Since $\varpi \in R^\times$ and $x \in R$, the coordinate change $S \to T \stackrel{\text{def}}{=} (S - x)/\varpi^M$ gives an automorphism of $\mathbf{P}^1_B$ that fixes the infinity section ∞_B. (More sophisticatedly, this corresponds to a certain blowing-up(-and-down) process in the fibers of $\mathbf{P}^1_C \to C$ at Σ.)

From now, we shall use this new coordinate T. Then, by the choice of (ϖ, x, M), we have $R_{L_v} = \mathbf{A}^1(R_{L_v}) \subset \Omega_v$ for each $v \in \Sigma$ and $v(\omega) < 0$ for each $v \in \Sigma$ and $\omega \in W(\overline{K}_v) - \{\infty\}$.

We define $\tilde{W}$ to be the closure of W in $\mathbf{P}^1_C$, which contains the infinity section ∞_C of $\mathbf{P}^1_C$. By the above choice of coordinate, we have $\tilde{W} \cap \mathbf{P}^1_{k_v} \subset \infty_{k_v}$ for each $v \in \Sigma$. From now, we regard $\tilde{W}$ as a reduced closed subscheme (or, as a divisor) of $\mathbf{P}^1_C$. By [Mo2], Théorème 1.3, $\mathrm{Pic}(\tilde{W})$ is a torsion group. So, let s_0 be the order of the class of the line bundle $\mathcal{O}_{\mathbf{P}^1_C}(1)|_{\tilde{W}}$ on $\tilde{W}$. On the other hand, let e be the degree of $\tilde{W}$ over C. Now, choose a positive integer s which is divisible by s_0 and greater than $e-2$. As in [Mo2], proof of Théorème 1.7, Étape VIII, consider the exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^1_C}(s)(-\tilde{W}) \to \mathcal{O}_{\mathbf{P}^1_C}(s) \to \mathcal{O}_{\mathbf{P}^1_C}(s)|_{\tilde{W}} \to 0,$$

which induces the following long exact sequence:

$$\cdots \to H^0(\mathbf{P}^1_C, \mathcal{O}_{\mathbf{P}^1_C}(s)) \to H^0(\tilde{W}, \mathcal{O}_{\mathbf{P}^1_C}(s)|_{\tilde{W}}) \to H^1(\mathbf{P}^1_C, \mathcal{O}_{\mathbf{P}^1_C}(s)(-\tilde{W})) \to \cdots.$$

Since $s_0 \mid s$, we have $\mathcal{O}_{\mathbf{P}^1_C}(s)|_{\tilde{W}} \simeq \mathcal{O}_{\tilde{W}}$, so that there exists an element $g_0 \in H^0(\tilde{W}, \mathcal{O}_{\mathbf{P}^1_C}(s)|_{\tilde{W}})$ which generates $\mathcal{O}_{\mathbf{P}^1_C}(s)|_{\tilde{W}}$. On the other hand, since $s > e-2$, we see that $H^1(\mathbf{P}^1_C, \mathcal{O}_{\mathbf{P}^1_C}(s)(-\tilde{W}))$ (which is the dual of $H^0(\mathbf{P}^1_C, \mathcal{O}_{\mathbf{P}^1_C}(-2-s)(\tilde{W})))$ vanishes. Thus, there exists an element $g \in H^0(\mathbf{P}^1_C, \mathcal{O}_{\mathbf{P}^1_C}(s))$ that maps to g_0. Then, we have $\mathrm{Supp}(g) \cap \tilde{W} = \emptyset$. In particular, we have $\mathrm{Supp}(g) \cap \infty_C = \emptyset$.

We may identify $H^0(\mathbf{P}^1_C, \mathcal{O}_{\mathbf{P}^1_C}(s))$ with the set of polynomials in $\tilde{R}[T]$ with degree $\leq s$. Then, since $\mathrm{Supp}(g) \cap \infty_C = \emptyset$, we see that g is strictly of degree s and that the coefficient u of T^s in $g = g(T)$ is an element of $\tilde{R}^\times$. So, replacing g by $u^{-1}g$ (and g_0 by $u^{-1}g_0$), we may assume that g is monic.

Next, since $\mathrm{Pic}(\mathbf{A}^1_C) = \mathrm{Pic}(C)$ is a torsion group, there exists an element $w(T) \in \tilde{R}[T]$, such that the zero locus of $w(T)$ in $\mathbf{A}^1_C$ coincides (set-theoretically) with $\tilde{W} \cap \mathbf{A}^1_C$. Recall that, for each $v \in \Sigma$ and each root ω of w in $\overline{K}_v$, we have $v(\omega) < 0$. From this fact (and the fact that $\tilde{W} \cap \mathbf{A}^1_{k_v} \subsetneq \mathbf{A}^1_{k_v}$), we see that $w(0)$ is a unit in R_v and that $w(T) \equiv w(0) \pmod{\mathfrak{m}_v}$. Moreover, since $k_v^\times$ is a torsion group, we may assume that $w(0) \equiv 1 \pmod{\mathfrak{m}_v}$ for each $v \in \Sigma$, replacing w by a suitable power. Now, we define d to be the degree of w.

First, assume $\Sigma \neq \emptyset$, and we shall apply (1.18)(ii) carefully. Let n be as in the beginning of the proof. Then, there exists a positive integer m_n. We choose a positive integer m to be a common multiple of m_n and s. For this m, we obtain a positive real number $c_{K_v,n,m}$. We put $c_m \stackrel{\mathrm{def}}{=} \max\{c_{K_v,n,m} \mid v \in \Sigma\}$.

We put $D \stackrel{\mathrm{def}}{=} \frac{d}{p-1}$ and $E \stackrel{\mathrm{def}}{=} \frac{p}{p-1} c_m mp$. We take a positive integer t, such that $p^t > D$. Next, since we are assuming the condition (RI) that k_{L_v} is an infinite algebraic extension of $\mathbb{F}_p$, there exists a finite subfield of k_{L_v} with arbitrarily large cardinality. So, we may take a finite subfield $\mathbb{F}_v$ of k_{L_v}, such that $p^{r_v} \stackrel{\text{def}}{=} \sharp(\mathbb{F}_v) > Ep^t$. Since R_v is complete, $\mathbb{F}_v$ admits a canonical lifting in R_{L_v}, to which we refer again as $\mathbb{F}_v$. Now, take a positive integer $r > \max\{r_v(v \in \Sigma), t\}$.

Applying (1.18)(ii) to $L = L_v$, $A = \mathbb{F}_v$, r as above, and $\nu = 0$, we see that there exists a positive integer δ_v, such that $\delta_v \leq c_m mp^{r+1}/\sharp(\mathbb{F}_v) = c_m mp^{r-r_v+1}$ and that $\delta_v \equiv 0 \pmod{n}$. Since δ_v is divisible by n, $\mathfrak{a} \stackrel{\text{def}}{=} \prod_{v\in\Sigma}(\mathfrak{m}_v \cap \tilde{R})^{\delta_v}$ is a principal ideal of $\tilde{R}$. So, let $a \in \tilde{R}$ be a generator of $\mathfrak{a}$. Then, $v(a) = \delta_v$ for each $v \in \Sigma$. Now, the conclusion of (1.18)(ii) is that there exists a monic, integral, superseparable polynomial $f_v(T) \in K_v[T]$, such that $\deg(f_v) = mp^{r+1}$, $\operatorname{def}(f_v) \geq (p-1)p^r$, $f_v'(T) = a$ and $(K_v)_{f_v} = K_v\mathbb{F}_v \subset L_v$. Moreover, by using (1.4)(ii) and the Chinese Remainder Theorem (for the Dedekind domain $\tilde{R}$), we may assume that $f_v(T) \in \tilde{R}[T]$ and $f_v(T)$ does not depend on v. So, put $f(T) \stackrel{\text{def}}{=} f_v(T)$ for some (or, equivalently, all) $v \in \Sigma$, then, f is monic, superseparable polynomial in $\tilde{R}[T]$, such that $\deg(f) = mp^{r+1}$, $\operatorname{def}(f) \geq (p-1)p^r$, $f'(T) = a$ and $(K_v)_f = K_v\mathbb{F}_v \subset L_v$ for each $v \in \Sigma$.

Next, assume $\Sigma = \emptyset$. In this case, we define m to be any multiple of s, put $D \stackrel{\text{def}}{=} \frac{d}{p-1}$, take a positive integer t with $p^t > D$ and a positive integer r with $r > t$, and let a be any element of $\tilde{R}^\times$. Now, we choose a monic superseparable polynomial $f(T) \in \tilde{R}[T] = R[T]$, such that $\deg(f) = mp^{r+1}$, $\operatorname{def}(f) \geq (p-1)p^r$, and $f'(T) = a$. (For example, put $f(T) = T^{mp^{r+1}} + aT$.)

Finally, we put

$$F(T) \stackrel{\text{def}}{=} g(T)^{\frac{m}{s}p^{r+1}} + w(T)^{p^{r-t}}(f(T) - g(T)^{\frac{m}{s}p^{r+1}}) \in \tilde{R}[T].$$

Claim (3.3). F is monic of degree mp^{r+1}.

Proof. f is monic with $\deg(f) = mp^{r+1}$ and $\operatorname{def}(f) \geq (p-1)p^r$. On the other hand, since g is monic of degree s, $g^{\frac{m}{s}}$ is monic of degree m, hence $g^{\frac{m}{s}p^{r+1}}$ is monic of degree mp^{r+1} and with 'defect' $\geq p^{r+1} \geq (p-1)p^r$. From these, we see that $f - g^{\frac{m}{s}p^{r+1}}$ has degree $\leq mp^{r+1} - (p-1)p^r$. Thus,

$$\begin{aligned}\deg(w^{p^{r-t}}(f - g^{\frac{m}{s}p^{r+1}})) &\leq p^{r-t}d + mp^{r+1} - (p-1)p^r \\ &< p^r dD^{-1} + mp^{r+1} - (p-1)p^r = mp^{r+1}.\end{aligned}$$

Since $g^{\frac{m}{s}p^{r+1}}$ is monic of degree mp^{r+1} as we have already seen, we conclude that F is monic of degree mp^{r+1}. $\square$

Claim (3.4). For each $v \in \Sigma$, any root α of F in $\overline{K}_v$ is contained in R_{L_v}.

Proof. Since $w(T) \equiv 1 \pmod{\mathfrak{m}_v}$, we have $w(T)^{p^{r-t}} \equiv 1 \pmod{\mathfrak{m}_v^{p^{r-t}}}$. Thus,

$$F(T) \equiv g(T)^{\frac{m}{s}p^{r+1}} + (f(T) - g(T)^{\frac{m}{s}p^{r+1}}) = f(T) \pmod{\mathfrak{m}_v^{p^{r-t}}}.$$

Now, since $p^{r-t} > Ep^{r-r_v} = \frac{p}{p-1}c_m m p^{r-r_v+1} \geq \frac{p}{p-1}\delta_v$, we have $(K_v)_F = (K_v)_f \subset L_v$ by (1.4)(ii). This implies $\alpha \in L_v$. Since $F(T)$ is a monic polynomial in $\tilde{R}[T] \subset R_v[T]$, we have $\alpha \in R_{L_v}$, as desired. □

Let $\tilde{Z}$ be the zero locus of $F(T)$ in $\mathbf{A}^1_C$. By (3.3), $\tilde{Z}$ is closed in $\mathbf{P}^1_C$. We put $Z \stackrel{\text{def}}{=} \tilde{Z} \cap \mathbf{P}^1_B$.

Claim (3.5). (i) $Z \subset X$.
(ii) Z is finite étale over B.

Proof. (i) On $(W \cap \mathbf{A}^1_B)^{\text{red}}$, we have $F(T) \equiv g(T)^{\frac{m}{s}p^{r+1}}$. Now, since the zero locus of g in $\mathbf{A}^1_B$ is disjoint from $W \cap \mathbf{A}^1_B$, so is that of F, as desired.
(ii) By (3.3), $Z = \text{Spec}(R[T]/(F(T)))$ is finite (and flat) over R. Since $F' = w^{p^{r-t}}f' = w^{p^{r-t}}a$ and $a \in R^\times$, the zero locus of F' in $\mathbf{A}^1_B$ is (set-theoretically) contained in W. This, together with (i), implies that the zero loci of F and F' are disjoint from each other, as desired. □

Take an irreducible (or, equivalently, connected) component B' of Z. By (3.5), we have a natural immersion $B' \hookrightarrow X$ over B, which we regard as a finite étale quasi-section of $X \to B$. Since $R_{L_v} \subset \Omega_v$, (3.4) implies that this quasi-section is $\mathcal{S}$-admissible. This completes the proof. □

Step 1 is the main step, and, roughly speaking, the rest of proof is only concerning how to reduce general cases to Step 1.

Step 2. Assume that $\mathcal{S}$ is essentially rational, and that X is an open subscheme of $\mathbf{P}^n_B$ for some $n \geq 0$. Then, there exists an $\mathcal{S}$-admissible quasi-section.

Proof. If $n = 0$, we must have $X = B$, and the assertion clearly holds. So, assume $n \geq 1$.

Let A be a commutative ring. We define $\mathbf{P}^n(A)^0$ to be $(\mathbf{A}^{n+1} - 0)(A)/A^\times$, where 0 denotes the section $(0, \dots, 0)$, regarded as a closed subscheme of $\mathbf{A}^{n+1}$. We define $\mathbf{P}^n(A)^{00}$ to be $\cup_{i=0}^n U_i(A)$, where $U_i(\simeq \mathbf{A}^n)$ is the standard open subset of $\mathbf{P}^n$. Then, we have $\mathbf{P}^n(A)^{00} \subset \mathbf{P}^n(A)^0 \subset \mathbf{P}^n(A)$. If $\text{Pic}(A) = \{0\}$ (resp. A is a local ring), then we have $\mathbf{P}^n(A)^0 = \mathbf{P}^n(A)$ (resp. $\mathbf{P}^n(A)^{00} = \mathbf{P}^n(A)^0 = \mathbf{P}^n(A)$). If A is a Dedekind domain, we see that $\mathbf{P}^n(A)^0$ forms a $GL_{n+1}(A)$-orbit.

Now, observe that $\mathbf{P}^n(R)^{00} \cap X(K)$ is dense in $\prod_{v \in \Sigma} \mathbf{P}^n(K_v)$. ($X(R)$ may be empty, though.) So, there exists $x \in \mathbf{P}^n(R)^0 \cap X(K)$, such that $x \in \Omega_v \cap X(K_v)$ for each $v \in \Sigma$. (Note that $\mathcal{S}$ is essentially rational.) By changing the coordinates via the $GL_{n+1}(R)$-action, we may assume $x = [1 : 0 : \cdots : 0]$ $(\in U_0)$.

Let $e_1, \dots, e_{n-1}$ be positive integers, and consider the B-morphism $i_{e_1,\dots,e_{n-1}} : \mathbf{A}^1_B \to U_0 = \mathbf{A}^n_B$, $t \mapsto (t^{e_1}, \dots, t^{e_{n-1}}, t)$. It is easy to see that $i_{e_1,\dots,e_{n-1}}$ is a closed immersion.

Claim (3.6). For some choice of $e_1, \dots, e_{n-1}$, $(i_{e_1,\dots,e_{n-1}})^{-1}(X)$ surjects onto B.

Proof. Denote by $T_1, \dots, T_n$ the coordinates of $U_0 = \mathbf{A}^n_B$. Then, there exist a finite number of polynomials $f_1, \dots, f_r \in R[T_1, \dots, T_n]$, such that the closed subset $\mathbf{A}^n_B - X$ of $\mathbf{A}^n_B$ coincides with the common zero locus of $f_1, \dots, f_r$. Since $\mathbf{A}^n_B \cap X$ surjects onto B (as X surjects onto B), we see that, for each $b \in B$, there exists $i = i_b \in \{1, \dots, r\}$, such that the image of f_i in $k_b[T_1, \dots, T_r]$ is non-zero.

Lemma (3.7). *Let S be a finite subset of $\mathbb{Z}^n$. Then, there exist positive integers $e_1, \dots, e_{n-1}$, such that the map $S \to \mathbb{Z}$, $(k_1, \dots, k_n) \mapsto e_1k_1 + \dots + e_{n-1}k_{n-1} + k_n$ is injective.*

Proof. Put $T \stackrel{\text{def}}{=} \{s - s' \mid s, s' \in S, s \neq s'\}$. This is a finite subset of $\mathbb{Z}^n$ that does not contain $0 = (0, \dots, 0)$. For each $t = (l_1, \dots, l_n) \in T$, consider the linear subspace W_t of $\mathbf{A}^n_{\mathbb{Q}}$ defined by $l_1x_1 + \dots l_nx_n = 0$. On the other hand, consider the hyperplane $H = \{(x_1, \dots, x_n) \mid x_n = 1\}$ of $\mathbf{A}^n_{\mathbb{Q}}$. As $H \not\subset W_t$, $H' \stackrel{\text{def}}{=} H - \cup_{t \in T} W_t$ is a non-empty open subset of H. Since the set $\{(e_1, \dots, e_{n-1}, 1) \mid e_1, \dots, e_{n-1} \in \mathbb{Z}_{>0}\}$ is Zariski dense in H (as $\mathbb{Z}_{>0}$ is an infinite set), it must intersect non-trivially with H'. Take $(e_1, \dots, e_{n-1}, 1)$ in this intersection, then $e_1, \dots, e_{n-1}$ satisfies the desired property. □

We define S to be the set of elements $(k_1, \dots, k_n) \in (\mathbb{Z}_{\geq 0})^n$ such that the coefficient of $T_1^{k_1} \cdots T_n^{k_n}$ in f_i is non-zero for some $i = 1, \dots, r$. Applying (3.7) to this S, we obtain $e_1, \dots, e_{n-1} \in \mathbb{Z}_{>0}$. Then, we see that, for each $b \in B$, there exists $i = i_b \in \{1, \dots, r\}$, such that the image of $f_i(T^{e_1}, \dots, T^{e_{n-1}}, T)$ in $k_b[T]$ is non-zero. This means that $(i_{e_1,\dots,e_{n-1}})^{-1}(X)$ surjects onto B, as desired. □

Take $e_1, \dots, e_{n-1}$ as in (3.6), and put $\mathcal{S}' \stackrel{\text{def}}{=} (\mathcal{C}, (i_{e_1,\dots,e_{n-1}})^{-1}(X) \to B, \{(i_{e_1,\dots,e_{n-1}}(L_v))^{-1}(\Omega_v)\})$, where $i_{e_1,\dots,e_{n-1}}(L_v)$ denotes the map $\mathbf{A}^1(L_v) \to U_0(L_v) = \mathbf{A}^n(L_v)$ induced by $i_{e_1,\dots,e_{n-1}}$. Then, $\mathcal{S}'$ is an essentially rational Skolem data. (Observe that $0 \in \mathbf{A}^1(K)$ lies in $(i_{e_1,\dots,e_{n-1}}(L_v))^{-1}(\Omega_v)$.)

Now, by Step 1, there exists an $\mathcal{S}'$-admissible quasi-section. By composing this quasi-section with $i_{e_1,\dots,e_{n-1}}$, we obtain an $\mathcal{S}$-admissible quasi-section. This completes the proof. □

Remark (3.8). The above argument that involves rational curves with higher degree was communicated to the author by a referee. The author's original argument, which is slightly more complicated, uses lines over finite extensions.

Step 3. Assume that X is an open subscheme of $\mathbf{P}^n_B$ for some $n \geq 0$. Then, there exists an $\mathcal{S}$-admissible quasi-section.

Proof. For each $v \in \Sigma$, $\mathbf{P}^n(L_v \cap K_v^{\text{sep}})$ is v-adically dense in $\mathbf{P}^n(L_v)$. Accordingly, we have $\Omega_v \cap \mathbf{P}^n(L_v \cap K_v^{\text{sep}}) \neq \emptyset$. So, there exists a finite Galois subextension M_v/K_v of L_v/K_v, such that $\Omega_v \cap X(M_v)$ is non-empty. Now, put $\mathcal{C}_1 \stackrel{\text{def}}{=} (C, \Sigma, \{M_v\}_{v \in \Sigma})$, which is a finite base curve data. By (2.1), there exists a $\mathcal{C}_1$-distinguished finite Galois extension K' of K. We define C' (resp. B') to be the integral closure of C (resp. B) in K', and put $\Sigma' \stackrel{\text{def}}{=} C' - B'$, which is the inverse image of Σ in C'. Let v' be an element of Σ' and v the image of v' in Σ. Then, we have $(K')_{v'} = M_v \subset L_v$. So, put $\mathcal{C}' \stackrel{\text{def}}{=} (C', \Sigma', \{L_v\}_{v' \in \Sigma'})$ and $\mathcal{S}' \stackrel{\text{def}}{=} (\mathcal{C}', X_{B'} \to B', \{\Omega_v\}_{v' \in \Sigma'})$. Then, $\mathcal{C}'$ is a base curve data (over the algebraic closure of k in K') such that C' is affine and that (RI) holds, and $\mathcal{S}'$ is an essentially rational Skolem data such that $X_{B'}$ is an open subscheme of $\mathbf{P}^n_{B'}$. So, by Step 2, there exists an $\mathcal{S}'$-admissible quasi-section $B'' \to X_{B'}$. Now, the composite of this morphism and the natural projection $X_{B'} \to X$ gives an $\mathcal{S}$-admissible quasi-section. This completes the proof. □

Step 4. Assume that $\mathcal{E} \simeq \mathcal{O}_B^{n+1}$, where $n+1$ is the rank of $\mathcal{E}$. Then, there exists a $\mathcal{B}$-admissible quasi-hyperplane.

Proof. For simplicity, we put $\mathbf{P} = \mathbf{P}(\mathcal{E})$ and $\check{\mathbf{P}} = \mathbf{P}(\check{\mathcal{E}})$. Let I denote the incidence subscheme of $\mathbf{P} \times_B \check{\mathbf{P}}$, and let p and $\check{p}$ be the natural projections $\mathbf{P} \times_B \check{\mathbf{P}} \to \mathbf{P}$ and $\mathbf{P} \times \check{\mathbf{P}} \to \check{\mathbf{P}}$, respectively. Both $p|_I$ and $\check{p}|_I$ are $\mathbf{P}^{N-1}$-bundles, hence, a fortiori, smooth.

Let $i = 1, \dots, r$. Since Y_i is an integral scheme and B is a smooth curve, the morphism $Y_i \to B$ is either flat over B or flat over b_i for some closed point $b_i \in B$. We shall refer to the former (resp. latter) case as case 1 (resp. 2).

In case 1, let $\tilde{Y}_i$ be the normalization of Y_i and B_i the integral closure of B in $\tilde{Y}_i$. Then, since the generic fiber of $\tilde{Y}_i \to B_i$ is geometrically irreducible, there exists a non-empty open subset B_i' of B_i, such that each fiber of $\tilde{Y}_i \times_{B_i} B_i' \to B_i'$ is geometrically irreducible ([EGA4], Théorème (9.7.7)). We denote by Σ_i the image of $B_i - B_i'$ in B, which is a finite set. We define $U_{i,1}$ to be the image of $(\tilde{Y}_i \times_{B_i} \check{\mathbf{P}}_{B_i}$ minus the inverse image of $I_{B_i})$ in $\check{\mathbf{P}}_{B_i}$, which is an open subset of $\check{\mathbf{P}}_{B_i}$, and define $U_{i,2}$ to be the complement in $\check{\mathbf{P}}$ of the image of $\check{\mathbf{P}}_{B_i} - U_{i,1}$, which is an open subset of $\check{\mathbf{P}}$. Moreover, for each $b \in \Sigma_i$, fix a geometric point $\overline{b}$ on b. Then, for each irreducible component P of $Y_{i,\overline{b}}$, we put $U_{P,1}$ the image of $(P \times_{\overline{b}} \check{\mathbf{P}}_{\overline{b}}$ minus the inverse image of $I_{\overline{b}})$ in $\check{\mathbf{P}}_{\overline{b}}$, and define $T_{P,2}$ to be the image of $\check{\mathbf{P}}_{\overline{b}} - U_{P,1}$ in $\check{\mathbf{P}}_b$, which is a closed subset of $\check{\mathbf{P}}_b$. Now, put $U_i \stackrel{\text{def}}{=} U_{i,2} - \cup_{b \in \Sigma_i} \cup_P T_{P,2}$, which is an open subset of $\check{\mathbf{P}}$. In case 2, for each irreducible component P of $Y_{i,\overline{b}_i}$, we define a closed subset $T_{P,2}$ of $\check{\mathbf{P}}_{b_i}$ just as above, and put $U_i \stackrel{\text{def}}{=} \check{\mathbf{P}} - \cup_P T_{P,2}$.

Now, we put $U \stackrel{\text{def}}{=} \cap_{i=1}^r U_i$. Let $\overline{b}$ be a geometric point on B, then, we see that a point of $U_{\overline{b}}$ corresponds to a hyperplane $H_{\overline{b}}$ of $\mathbf{P}_{\overline{b}}$ that satisfies condition (a) in the definition of $\mathcal{B}$-admissible quasi-hyperplane. In particular, U_b is a non-empty open subset of $\check{\mathbf{P}}_b$ for each $b \in B$.

Next, for each $i = 1, \dots, r$, let $((p|_I)^{-1}(Y_i^{\text{sm}}))^{\text{non-sm}}$ be the set of points of $(p|_I)^{-1}(Y_i^{\text{sm}})$ at which $(p|_I)^{-1}(Y_i^{\text{sm}}) \to \check{\mathbf{P}}$ is not smooth. This is a closed subset of $(p|_I)^{-1}(Y_i^{\text{sm}})$. Let Z_i be the image of $((p|_I)^{-1}(Y_i^{\text{sm}}))^{\text{non-sm}}$ in $\check{\mathbf{P}}$. Chevalley's theorem implies that Z_i is a constructible subset of $\check{\mathbf{P}}$, and the usual Bertini theorem implies that, for each $b \in B$, $\check{\mathbf{P}}_b - Z_i$ contains a non-empty open subset of $\check{\mathbf{P}}_b$. From these, we observe that $V_i \stackrel{\text{def}}{=} \check{\mathbf{P}} - \overline{Z_i}$ satisfies that, for each $b \in B$, $(V_i)_b$ is a non-empty open subset of $\check{\mathbf{P}}_b$. We put $V \stackrel{\text{def}}{=} \cap_{i=1}^r V_i$. Then, for each $b \in B$, V_b is a non-empty open subset of $\check{\mathbf{P}}_b$.

Next, let P be an irreducible component of $Y_{i,\overline{K}}$ with $\dim(P) \geq 2$. Then, by a version of Bertini theorem ([J], Théorème 6.11, 3), there exists a non-empty open subset $W_{P,1}$ of $\check{\mathbf{P}}_{\overline{K}}$, such that, for each hyperplane $H_{\overline{K}}$ corresponding to a point of $W_{P,1}$, $P \cap H_{\overline{K}}$ is irreducible. We define W_1 to be the intersection of $W_{P,1}$ for irreducible components P of $Y_{i,\overline{K}}$ with $\dim(P) \geq 2$, which is a non-empty open subset of $\check{\mathbf{P}}_{\overline{K}}$, and T_2 the image of $\check{\mathbf{P}}_{\overline{K}} - W_1$ in $\check{\mathbf{P}}_K$, which is a proper closed subset of $\check{\mathbf{P}}_K$. Moreover, we denote by $\overline{T}_2$ the closure of T_2 in $\check{\mathbf{P}}$. We see that $(\overline{T}_2)_b$ is a proper closed subset of $\check{\mathbf{P}}_b$ for each $b \in B$. Now, we put $W = \check{\mathbf{P}} - \overline{T}_2$. Then, for each $b \in B$, W_b is a non-empty open subset of $\mathbf{P}_b$.

Now, we put $\check{X} \stackrel{\text{def}}{=} U \cap V \cap W$. This is an open subset of $\check{\mathbf{P}}$ that is surjectively mapped onto B. Put $\mathcal{S}' \stackrel{\text{def}}{=} (\mathcal{C}, \check{X} \to B, \{\check{\Omega}_v \cap \check{X}(L_v)\}_{v\in\Sigma})$. Then, $\mathcal{S}'$ is a Skolem data.

So, by Step 3 and the assumption that $\mathcal{E} \simeq \mathcal{O}_B^{n+1}$, there exists an $\mathcal{S}'$-admissible quasi-section $B' \to \check{X}$. By the choice of $\mathcal{S}'$, this section corresponds to a hyperplane H of $\mathbf{P}_{B'}$, which satisfies all the conditions (a)–(d) in the definition of $\mathcal{B}$-admissible quasi-hyperplane. This completes the proof. □

Step 5. Assume that X is quasi-projective of relative dimension 1 over B. Then, there exists an $\mathcal{S}$-admissible quasi-section.

Proof. By assumption, we may assume that X is a subscheme of $\mathbf{P}_B^n$ for some $n \geq 1$. We define $\overline{X}_1$ to be the closure of X in $\mathbf{P}_B^n$, regarded as a reduced scheme. $\overline{X}_1$ is a projective flat integral curve over B, and X is an open subscheme of $\overline{X}_1$. It is well-known that, after normalizations and blowing-ups outside X, $\overline{X}_1$ can be desingularized. Namely, there exists a birational projective morphism $\pi : \overline{X}_2 \to \overline{X}_1$, where $\overline{X}_2$ is a regular, integral scheme, such that $\pi^{-1}(X) \xrightarrow{\sim} X$. Since $X_{\overline{K}}$ is irreducible, so is $(\overline{X}_2)_{\overline{K}}$. Hence, by [EGA4], Théorème (9.7.7), there exists a non-empty open subset B_1 of B, such that each fiber of $X_{B_1} \to B_1$ is geometrically irreducible, and, in particular, irreducible. We put $\Sigma_1 \stackrel{\text{def}}{=} B - B_1$, which is a finite set.

Now, we introduce a new base curve data $\mathcal{C}_1 \stackrel{\text{def}}{=} (C, \Sigma \cup \Sigma_1, \{L_v\}_{v\in\Sigma} \cup \{K_v^{\text{ur}}\}_{v\in\Sigma_1})$, where K_v^{ur} denotes the maximal unramified extension of the complete discrete valuation field K_v. Note that $\mathcal{C}_1$ satisfies (RI) and that C is affine. Moreover, we put $\mathcal{S}_1 = \{\mathcal{C}_1, X_{B_1} \to B_1, \{\Omega_v\}_{v\in\Sigma} \cup \{X(R_{K_v^{\text{ur}}})\}_{v\in\Sigma_1}\}$. Since $X \to B$ is smooth surjective, we see that $X(R_{K_v^{\text{ur}}})$ is non-empty. Thus, $\mathcal{S}_1$ becomes a Skolem data. Now, suppose that there exists an $\mathcal{S}_1$-admissible quasi-section $B_1' \to X_{B_1}$. Then, firstly, the integral closure B' of B in B_1' is finite étale over B. Secondly, as $\overline{X}_2 \to B$ is proper, $B_1' \to X_{B_1}$ extends to $B' \to \overline{X}_2$. Now, since $B_1' \to X_{B_1}$ is $\mathcal{S}_1$-admissible, we see that the image of $B' \to \overline{X}_2$ must be contained in X. Thus, we obtain an $\mathcal{S}$-admissible quasi-section $B' \to X$.

So, replacing $\mathcal{S}$ by $\mathcal{S}_1$, we may assume that each fiber of $\overline{X}_2 \to B$ is geometrically irreducible. Now, we put $\overline{X} \stackrel{\text{def}}{=} \overline{X}_2$.

Lemma (3.9). *Let F be a field and X a projective, geometrically integral F-scheme of dimension 1. We denote by X' the normalization of $X_{\overline{F}}$, so that we have a natural morphism $\pi : X' \to X$. Then, there exists a natural number N, such that each invertible sheaf L on X with* $\deg(L) \geq N$ *is very ample, where* $\deg(L) \stackrel{\text{def}}{=} \deg(\pi^*(L))$.

Proof. This follows from [CFHR], Theorem 1.1. (We may take $N = 2p_a(X) + 1$.) $\square$

Now, take a natural number N for $\overline{X}_K$ as in (3.9). We shall choose horizontal divisors $Y_1, Y_2, \ldots$ of $\overline{X}$ inductively, as follows. Firstly, by [Mo3], Théorème 1.3, there exists a horizontal divisor Y_1 of $\overline{X}$, such that Y_1 is contained in X and that, for each $v \in \Sigma$, $Y_1 \times_B \mathrm{Spec}(L_v)$ is a disjoint union of copies of $\mathrm{Spec}(L_v)$ and is contained in Ω_v. Next, assume that we have defined $Y_1, \ldots, Y_r$. Then, again by [Mo3], Théorème 1.3, there exists a horizontal divisor Y_{r+1} of $\overline{X}$, such that Y_{r+1} is contained in $X - \cup_{i=1}^r Y_i$ and that, for each $v \in \Sigma$, $Y \times_B \mathrm{Spec}(L_v)$ is a disjoint union of copies of $\mathrm{Spec}(L_v)$ and is contained in $\Omega_v - \cup_{i=1}^r Y_i(L_v)$. By construction, $Y_1, Y_2, \ldots$ are disjoint from one another. Now, take n so large that $\deg(Y_{1,K} + \cdots + Y_{n,K}) \geq N$, and we put $Y \stackrel{\text{def}}{=} Y_1 + \cdots + Y_n$. (Note that each Y_i defines an invertible sheaf on $\overline{X}$, since it lies in the smooth locus.) Then, by (3.9), Y_K is very ample. On the other hand, since each fiber of $\overline{X} \to B$ is geometrically irreducible, Y itself is ample (cf. [Mo2], Proposition 4.3), hence there exists a natural number m such that mY is very ample. So, consider an embedding $\overline{X} \hookrightarrow \mathbf{P}^n_B$ with respect to the very ample divisor mY.

We put $D \stackrel{\text{def}}{=} \overline{X} - X$. Let $E_1, \ldots, E_h$ be the irreducible components of D, which must be either an isolated point or a horizontal divisor, as $X \to B$ is surjective and each fiber of $\overline{X} \to B$ is irreducible. Next, for each $v \in \Sigma$, we define $\check{\Omega}'_v$ to be the subset of $\check{\mathbf{P}}^n(L_v)$ consisting of points corresponding to L_v-rational hyperplanes H such that $\overline{X}_{L_v} \cap H$ is a disjoint union of L_v-rational points in Ω_v (whose cardinality must coincide with $\deg(mY_K)$). It is easy to show that $\check{\Omega}'_v$ is a v-adically open subset of $\check{\mathbf{P}}(L_v)$. Moreover, by using the fact that (not only mY_K but also) Y_K is very ample and that Y_{L_v} is a disjoint union of L_v-rational points in Ω_v, we see that $\check{\Omega}'_v$ is non-empty.

Now, we put $\mathcal{B}' \stackrel{\text{def}}{=} (\mathcal{C}, \overline{X}, E_1, \ldots, E_h \subset \mathbf{P}^n_B, \{\check{\Omega}'_v\}_{v \in \Sigma})$, which becomes a Bertini data. As $\mathbf{P}^n_B = \mathbf{P}(\mathcal{O}_B^{n+1})$, we may apply Step 4 to this Bertini data $\mathcal{B}'$, to conclude that there exists a $\mathcal{B}'$-admissible quasi-hyperplane $H \subset \mathbf{P}^n_{B'}$. By condition (a), we see that $\overline{X}_{B'} \cap H$ is finite (as proper and quasi-finite) over B, and that $E_{i,B'} \cap H = \emptyset$ for each $i = 1, \ldots, h$, hence $D_{B'} \cap H = \emptyset$, or, equivalently, $\overline{X}_{B'} \cap H = X_{B'} \cap H$. By condition (b), we see that $X_{B'} \cap H$ is smooth over B'. From these, $X_{B'} \cap H$ is finite étale over B', hence over B. Moreover, by condition (d), each component of $(\overline{X}_{B'} \cap H)_{L_v}$ is a disjoint union of L_v-rational point in Ω_v. Thus, any connected component of $X_{B'} \cap H$ gives an $\mathcal{S}$-admissible quasi-section. This completes the proof. $\square$

Step 6. Assume that X is quasi-projective over B. Then, there exists an $\mathcal{S}$-admissible quasi-section.

Proof. We shall prove this by using induction on the relative dimension d of X over B. If $d = 0$, this is clear. If $d = 1$, this is just the content of Step 5. So, assume $d > 1$. Since X is quasi-projective, we may choose an embedding $X \hookrightarrow \mathbf{P}^n_B$. We denote by $\overline{X}$ the closure of X in $\mathbf{P}^n_B$, and put $W \stackrel{\text{def}}{=} \overline{X} - X$. Next, for each $v \in \Sigma$, we define $\check{\Omega}'_v$ to be the subset of $\check{\mathbf{P}}^n(L_v)$ consisting of points corresponding to L_v-rational hyperplanes that meet transversally with a point of Ω_v. Then, it is easy to see that $\check{\Omega}'_v$ is a non-empty, v-adically open, G_v-stable subset of $\check{\mathbf{P}}^n(L_v)$. Thus, $\mathcal{B}' \stackrel{\text{def}}{=} (\mathcal{C}, \overline{X}, W \subset \mathbf{P}^n_B, \{\check{\Omega}'_v\}_{v \in \Sigma})$ becomes a Bertini data, and, by Step 4, there exists a $\mathcal{B}'$-admissible quasi-hyperplane $H \subset \mathbf{P}^n_{B'}$, where B' is the integral closure of B in some finite $\mathcal{C}$-admissible extension K' of K. By conditions (a) and (b) in the definition of $\mathcal{B}'$-admissibility, we see that $X'_{B'} \stackrel{\text{def}}{=} X_{B'} \cap H$ is smooth, surjective over B'. By condition (c), $X'_{B'} \times_{B'} \overline{K'}$ is irreducible. Moreover, by condition (d), $\Omega'_v \stackrel{\text{def}}{=} \Omega_v \cap H(L_v)$ is non-empty.

Now, we define C' (resp. B') to be the integral closure of C (resp. B) in K', and put $\Sigma' \stackrel{\text{def}}{=} C' - B'$, which is the inverse image of Σ in C'. Let v' be an element of Σ' and v the image of v' in Σ. Then, we have $(K')_{v'} \subset L_v$. So, put $\mathcal{C}' \stackrel{\text{def}}{=} (C', \Sigma', \{L_v\}_{v' \in \Sigma'})$ and $\mathcal{S}' \stackrel{\text{def}}{=} (\mathcal{C}', X'_{B'} \to B', \{\Omega'_v\}_{v' \in \Sigma'})$. Then, $\mathcal{C}'$ is a base curve data (over the algebraic closure of k in K') such that C' is affine and that (RI) holds, and $\mathcal{S}'$ is a Skolem data such that the relative dimension of $X_{B'}$ over B' is $d - 1$. Thus, by the assumption of induction, there exists an $\mathcal{S}'$-admissible quasi-section $B'' \to X'_{B'}$. Composing this quasi-section with the natural map $X'_{B'} \to X$, we obtain an $\mathcal{S}$-admissible quasi-section, as desired. This completes the proof. $\square$

Step 7. There exists an $\mathcal{S}$-admissible quasi-section. Namely, (3.1) holds.

Proof. Let X' be a non-empty affine open subset of X, and let B' denote the image of X' in B, which is a non-empty open subset of B. Put $\Sigma' \stackrel{\text{def}}{=} B - B'$. Then, $\mathcal{S}' \stackrel{\text{def}}{=} (\mathcal{C}', X' \to B', \{\Omega_v \cap X'(L_v)\}_{v \in \Sigma} \cup \{X'(K_v^{\text{ur}}) \cap X(R_{K_v^{\text{ur}}})\}_{v \in \Sigma'})$, where $\mathcal{C}' \stackrel{\text{def}}{=} (C, \Sigma \cup \Sigma', \{L_v\}_{v \in \Sigma} \cup \{K_v^{\text{ur}}\}_{v \in \Sigma'})$, becomes a Skolem data. Now, just as in the proof of Step 5, an $\mathcal{S}'$-admissible quasi-section (whose existence is assured by Step 6) induces an $\mathcal{S}$-admissible quasi-section. This completes the proof. $\square$

Step 8. There exists a $\mathcal{B}$-admissible quasi-hyperplane. Namely, (3.2) holds.

Proof. This is just similar to the proof of Step 4, except that we use Step 7 instead of Step 3. $\square$

§4. Some remarks and applications.

4.1. On condition (RI).
It is desirable to remove the disgusting condition (RI) in the main results (3.1) and (3.2). The main (and the only) technical difficulty in doing so appears in Step 1 of §3. More specifically, recall that we have applied (1.18)(ii) in Step 1. However, to apply (1.18)(ii), we need a finite submodule A of R_{L_v} with $A \cap \mathfrak{m}_{L_v} = \{0\}$ and with $\sharp(A)$ sufficiently large, which requires the infiniteness of the residue field of L_v. In fact, it is possible to modify (1.18)(ii) to include the case where $A \cap \mathfrak{m}_{L_v} = \{0\}$ does not hold, but then we cannot expect the valuation δ of a is sufficiently small compared to $\deg(f)$, and the proof of (3.4) fails when we try to apply (1.4)(ii).

4.2. On the incompleteness hypothesis.
In the main results (3.1) and (3.2), we have assumed the incompleteness hypothesis that the base curve C is affine. It is impossible to remove this condition entirely, but it is desirable to be able to control the objects at the points at infinity, even in some weaker sense. In this direction, we have a capacity-theoretic approach due to Rumely ([Ru1], [Ru2]) and another approach via small codimension arguments due to Moret-Bailly ([Mo5]). The author hopes for the following third approach (though it is only applicable to positive characteristic). More precisely, let $\mathcal{C} = (C, \Sigma, \{L_v\}_{v\in\Sigma})$ be a base curve data over an algebraic extension k of $\mathbb{F}_p$ (with C not necessarily affine), and Σ_∞ a non-empty subset of Σ. Then, even if C is proper over k, we might expect that the following version of (3.1) and (3.2) hold.

For (3.1), let $\mathcal{S} = (\mathcal{C}, f : X \to B, \{\Omega_v\}_{v\in\Sigma-\Sigma_\infty} \cup \{X(L_v)\}_{v\in\Sigma_\infty})$ be a Skolem data. (Thus, for $v \in \Sigma_\infty$, we just assume $X(L_v) \neq \emptyset$.) Then, we might expect that there exists a quasi-section $s : B' \to X$ of $f : X \to B$ which is nearly $\mathcal{S}$-admissible with respect to Σ_∞ in the following sense: K' is a finite extension of K which is nearly $\mathcal{C}$-admissible with respect to Σ_∞; B' is the integral closure of B in K'; and for each $v \in \Sigma - \Sigma_\infty$, the image of $B'_{L_v} \stackrel{\text{def}}{=} B' \times_B L_v$ in $X_{L_v} \stackrel{\text{def}}{=} X \times_B L_v$ is contained in Ω_v ($\subset X(L_v) = X_{L_v}(L_v)$). (For $v \in \Sigma_\infty$, the image of $B'_{L_v(p)}$ in $X_{L_v(p)}$ is automatically contained in $X(L_v(p)) = X_{L_v(p)}(L_v(p))$, and we do not impose any more condition.)

For (3.2), let $\mathcal{B} = (\mathcal{C}, Y_1, \ldots, Y_r \subset \mathbf{P}(\mathcal{E}), \{\check{\Omega}_v\}_{v\in\Sigma-\Sigma_\infty} \cup \{\mathbf{P}(\check{\mathcal{E}})(L_v)\}_{v\in\Sigma_\infty})$ be a Bertini data. Then, we might expect that there exists a quasi-hyperplane $H \subset \mathbf{P}(\mathcal{E})_{B'}$ which is nearly $\mathcal{B}$-admissible with respect to Σ_∞ in the following sense: K' is a finite extension of K which is nearly $\mathcal{C}$-admissible with respect to Σ_∞; B' is the integral closure of B in K'; (a), (b), (c) as in the definition of $\mathcal{B}$-admissibility; and (d) as in the definition of $\mathcal{B}$-admissibility only for $v \in \Sigma - \Sigma_\infty$. (For $v \in \Sigma_\infty$, the image of $B'_{L_v(p)}$ in $\mathbf{P}(\check{\mathcal{E}})_{L_v(p)}$ is automatically contained in $\mathbf{P}(\check{\mathcal{E}})(L_v(p)) = \mathbf{P}(\check{\mathcal{E}})_{L_v(p)}(L_v(p))$, and we do not impose any more condition.)

We might consider (2.2) as a weak evidence for this expectation.

4.3. Hopeful generalizations (mild).
Firstly, it should be possible to generalize the main results (3.1) and (3.2) to the case of algebraic spaces or even algebraic stacks, along the lines of [Mo5].

Secondly, it should be possible to prove qualitative versions of (3.1) and (3.2), in terms of heights (cf. [U], and [A1,2]) and/or degrees (cf. [Mi], [E1,2]). (See also [Poo].)

Thirdly, it is desirable to be able to prove that, in (3.1), we can choose an $\mathcal{S}$-admissible quasi-section $B' \to X$ which is a closed immersion (cf., e.g., [Mo2], Définition 1.4.), and, similarly, that, in (3.2), we can choose a $\mathcal{B}$-admissible quasi-hyperplane $H \subset \mathbf{P}(\mathcal{E})_{B'}$ such that the classifying morphism $[H] : B' \to \mathbf{P}(\check{\mathcal{E}})$ is a closed immersion. This third possible generalization was suggested to the author by the referee. Indeed, this generalization is possible in Steps 1 and 2 of §3. (For Step 2, this is possible by means of the simplification of the proof due to him or her. See (3.8).)

4.4. Hopeful generalizations (ambitious).
As we have mentioned in the Introduction, word-for-word translations of the main results (3.1) and (3.2) to the number field case, namely, to the case where C in the base scheme data is (an open subscheme of) the spectrum of the integer ring of an algebraic number field are false. However, it is very interesting (at least to the author) to ask if we might hope for any (modified) unramified versions of (3.1) and (3.2) also in the number field case.

Also, it might be interesting to investigate what happens in the case where C is a higher-dimensional (affine) scheme, even in positive characteristic. One of the main obstacles of this direction consists in the fact that the Picard group of C is no longer a torsion group, and word-for-word translations of (3.1) and (3.2) to the higher-dimensional case are false. However, there might exist some reasonable restrictions on the (Skolem or Bertini) data, with which (3.1) and (3.2) are valid.

4.5. An application to local-global principle and largeness in field theory.

Definition. Let $\mathcal{C} = (C, \Sigma, \{L_v\}_{v\in\Sigma})$ be a base scheme data. Then, we define $K^{\mathcal{C}}$ to be the maximal $\mathcal{C}$-admissible extension of K contained in the algebraic closure $\overline{K}$ of K, $C^{\mathcal{C}}$ (resp. $B^{\mathcal{C}}$) the integral closure of C (resp. B) in $K^{\mathcal{C}}$, and $\Sigma^{\mathcal{C}}$ to be the inverse image of Σ in $C^{\mathcal{C}}$. (Thus, $\Sigma^{\mathcal{C}} = C^{\mathcal{C}} - B^{\mathcal{C}}$.)

As an application of (3.1), we obtain the following local-global principle in field theory (cf. [Mo4]).

Theorem (4.1). *Let $\mathcal{C} = (C, \Sigma, \{L_v\}_{v\in\Sigma})$ be a base curve data over an algebraic extension of $\mathbb{F}_p$, and assume that C is affine and that* (RI) *holds. Then, $K^{\mathcal{C}}$ satisfies the local-global principle in the sense that, for each smooth, geometrically connected scheme X over $K^{\mathcal{C}}$, $X(K^{\mathcal{C}}) \neq \emptyset$ holds if and only if $X((K^{\mathcal{C}})_w) \neq \emptyset$ holds for every prime w of $K^{\mathcal{C}}$. Here, $(K^{\mathcal{C}})_w$ denotes the algebraic closure of K_v in the completion of $K^{\mathcal{C}}$ at w, where v is the prime of K that is below w.*

Proof. The 'only if' part is trivial. To prove the 'if' part, assume that $X((K^{\mathcal{C}})_w) \neq \emptyset$ holds for every prime w of $K^{\mathcal{C}}$. First, replacing X by any non-empty quasi-compact open subset, we may assume that X is of finite type over $K^{\mathcal{C}}$. (Observe that the image of $X((K^{\mathcal{C}})_w)$ in X is Zariski dense.) Then, replacing K and $\mathcal{C}$ by a suitable finite $\mathcal{C}$-admissible extension and a suitable base curve data, respectively, we may assume that X comes from a (smooth, geometrically connected) K-scheme X_K. Now, the following (4.2) implies $X(K^{\mathcal{C}}) = X_K(K^{\mathcal{C}}) \neq \emptyset$, as desired. (Put $\Omega_v = X_K(L_v)$.) □

Theorem (4.2). *Notations and assumptions being as in* (4.1)*, let X_K be a smooth, geometrically connected K-scheme. Assume that $X_K(K_b^{\mathrm{ur}}) \neq \emptyset$ holds for each closed point $b \in B$, and that a non-empty, v-adically open, G_v-stable subset Ω_v of $X_K(L_v)$ is given for each $v \in \Sigma$. Then, there exists a finite $\mathcal{C}$-admissible extension K' of K and $s_K \in X_K(K')$, such that, for each $v \in \Sigma$, the image by $s_K \times_K L_v$ of* $\mathrm{Spec}(K') \times_K L_v$ *in $X_{L_v} \stackrel{\mathrm{def}}{=} X_K \times_K L_v$ is contained in Ω_v ($\subset X_K(L_v) = X_{L_v}(L_v)$).*

Proof. First, assume that there exist a regular, integral scheme $\overline{X}$ proper, flat over B and an open immersion $X_K \hookrightarrow \overline{X}_K$ over K. (We shall refer to such an $\overline{X}$ as a regular, relative compactification over B.) Put $W_K \stackrel{\mathrm{def}}{=} \overline{X}_K - X_K$ and denote by W the closure of W_K in $\overline{X}$. Then, we see easily that $W \cap \overline{X}_K = W_K$ and that the fiber W_b of W at each closed point b of B has dimension strictly smaller than the dimension of the whole fiber $\overline{X}_b$ (which is automatically equidimensional). On the other hand, let $\overline{X}^{\mathrm{sm}}$ denote the set of points of $\overline{X}$ at which $\overline{X} \to B$ is smooth. This is an open subset of $\overline{X}$. Since $\overline{X}$ is regular and $\overline{X}(K_b^{\mathrm{ur}})(\supset X_K(K_b^{\mathrm{ur}})) \neq \emptyset$ for each closed point $b \in B$, we see that $\overline{X}^{\mathrm{sm}} \to B$ is surjective. From these, we conclude that $X \stackrel{\mathrm{def}}{=} \overline{X}^{\mathrm{sm}} - W$ surjects onto B and that $X \times_B K = X_K$ holds. Now, applying (3.1) to the Skolem data $\mathcal{S} \stackrel{\mathrm{def}}{=} (\mathcal{C}, X \to B, \{\Omega_v\}_{v \in \Sigma})$, we obtain an $\mathcal{S}$-admissible quasi-section $s : B' \to X$. Then, $s_K \stackrel{\mathrm{def}}{=} s \times_B K$ satisfies the desired properties.

In general, the above desingularization result may not be available, but we can proceed by using induction on $d \stackrel{\mathrm{def}}{=} \dim(X_K)$, as follows. The case $d = 0$ is trivial. In the case $d = 1$, the existence of a regular, relative compactification as above is well-known. So, we may assume $d \geq 2$. Replacing X_K by a suitable (say, non-empty and affine) open subset, we may also assume that X_K is quasi-projective over K. (Observe that the image of Ω_v in X_K is Zariski dense.) We choose an embedding $X_K \hookrightarrow \mathbf{P}^n_K$. Then, as in the proof of Step 4 of §3, there exists a non-empty open subset $\check{U}_K$ of the dual projective space $\check{\mathbf{P}}^n_K$, such that, for each hyperplane $H_{\overline{K}}$ that corresponds to a point of $\check{U}_K(\overline{K})$, $X_{\overline{K}} \cap H_{\overline{K}}$ is smooth, (geometrically) connected of dimension $d-1$. Moreover, as in the proof of Step 6 of §3, for each $v \in \Sigma$, we define $\check{\Omega}'_v$ to be the subset of $\check{\mathbf{P}}^n(L_v)$ consisting of points corresponding to L_v-rational hyperplanes that meet transversally with a point of Ω_v. Then, it is easy to see that $\check{\Omega}'_v$ is a non-empty, v-adically open, G_v-stable subset of $\check{\mathbf{P}}^n(L_v)$. Now, since $\check{U}_K$ admits a regular, relative compactification $\check{\mathbf{P}}^n_B$, we may apply the above argument to $(\check{U}_K, \{\Omega'_v \cap \check{U}_K(L_v)\}_{v\in\Sigma})$ to obtain a suitable quasi-section s'_K of $\check{U}_K$. (Note also that $\check{U}_K(K_b^{\mathrm{ur}}) \neq \emptyset$ holds for each closed point b of B.) Now, as in the proof of Step 6 of §3, we may reduce the problem to the case $d-1$ by cutting (the base change of) X_K with the quasi-hyperplane corresponding to s'_K. Thus, the proof by induction is completed. □

Corollary (4.3). *$K^{\mathcal{C}}$ is large (in the sense of* [Pop]*).* □

Proof. Immediate from (4.1). (See [Pop], Proposition 3.1.) □

This corollary gives an interesting new example of large fields. Indeed, as far as the author knows, in all the known examples of large fields which are algebraic extensions of either number fields K or function fields K over finite fields, we can control only finitely many primes of K. On the other hand, our $K^{\mathcal{C}}$ is defined by imposing restrictions at almost all primes of K.

In this sense, this corollary may be regarded as the first example of large fields which are not so large! (See also (4.11) below.)

4.6. An application to principal ideal theorem.
As an application of (3.1), we obtain the following (cf. [Mo1], 3.1):

Theorem (4.4). *Let $\mathcal{C} = (C, \Sigma, \{L_v\}_{v\in\Sigma})$ be a base curve data over an algebraic extension k of $\mathbb{F}_p$, and assume that C is affine and that* (RI) *holds. Then, we have* $\mathrm{Pic}(C^{\mathcal{C}}) = \{0\}$. *In particular, we have* $\mathrm{Pic}(B^{\mathcal{C}}) = \{0\}$.

Proof. Let $\mathcal{L}^{\mathcal{C}}$ be any invertible sheaf on $C^{\mathcal{C}}$. Then, there exists a finite subextension K_1 of $K^{\mathcal{C}}$ over K, such that $\mathcal{L}^{\mathcal{C}} = \mathcal{L}_1 \otimes_{\mathcal{O}_{C_1}} \mathcal{O}_{C^{\mathcal{C}}}$ holds for some invertible sheaf $\mathcal{L}_1$ on C_1, where C_1 is the integral closure of C in K_1. We define B_1 and Σ_1 to be the integral closure of B in K_1 and the inverse image of Σ in C_1, respectively, and, for each $v_1 \in \Sigma_1$, we put $L_{v_1} \stackrel{\mathrm{def}}{=} L_v$, where v is the image of v_1 in Σ. Then, observe that $\mathcal{C}_1 \stackrel{\mathrm{def}}{=} (C_1, \Sigma_1, \{L_{v_1}\}_{v_1\in\Sigma_1})$ becomes a base curve data (over the algebraic closure of k in K_1), such that $(C_1)^{\mathcal{C}_1} = C^{\mathcal{C}}$. Now, consider the Skolem data $\mathcal{S}_1 = (\mathcal{C}_1, (\mathbf{V}(\check{\mathcal{L}}_1) - 0_{C_1}) \times_{C_1} B_1 \to B_1, \{(\mathbf{V}(\check{\mathcal{L}}_1) - 0_{C_1})(R_{L_v})\}_{v_1\in\Sigma_1})$ (such that C_1 is affine and that (RI) holds), where $\mathbf{V}(\check{\mathcal{L}}_1)$ denotes the (geometric) line bundle on C_1 defined by the dual $\check{\mathcal{L}}_1$ of $\mathcal{L}_1$, and 0_{C_1} denotes the zero section of $\mathbf{V}(\check{\mathcal{L}}_1)$. Now, by (3.1), there exists an $\mathcal{S}_1$-admissible quasi-section $B' \to (\mathbf{V}(\check{\mathcal{L}}_1) - 0_{C_1}) \times_{C_1} B_1$. By the choice of $\mathcal{S}_1$, this quasi-section extends to a (unique) quasi-section $C' \to \mathbf{V}(\check{\mathcal{L}}_1) - 0_{C_1}$ of $\mathbf{V}(\check{\mathcal{L}}_1) - 0_{C_1} \to C_1$, where C' is the integral closure of C_1 in B'. This implies that $\mathcal{L}_1$ admits an everywhere non-vanishing section over C', or, equivalently, $\mathcal{L}_1$ becomes trivial after the base change to C'. Therefore, $\mathcal{L}^{\mathcal{C}}$ is trivial, a fortiori. This completes the proof of the first assertion.

The second assertion follows from the first, as the natural map $\mathrm{Pic}(C^{\mathcal{C}}) \to \mathrm{Pic}(B^{\mathcal{C}})$ is surjective. □

In the case where $\Sigma = \emptyset$, (4.4) directly follows from the principal ideal theorem in class field theory. In general, however, there exists an invertible sheaf that cannot be trivialized if we only consider abelian $\mathcal{C}$-admissible extensions. (See (4.11) below.) In this sense, we may regard (4.4) as a new (non-abelian) type of principal ideal theorem which cannot be covered by class field theory.

4.7. An application to torsors.
More generally than the second assertion of (4.4), we obtain the following:

Theorem (4.5). *Let $\mathcal{C} = (C, \Sigma, \{L_v\}_{v\in\Sigma})$ be a base curve data over an algebraic extension of $\mathbb{F}_p$, and assume that C is affine and that* (RI) *holds. Let $G^{\mathcal{C}}$ be a smooth, separated group scheme of finite type over $B^{\mathcal{C}}$, such that the generic fiber $G^{\mathcal{C}}_{K^{\mathcal{C}}}$ is connected. Then, we have* $\mathrm{Ker}(\check{H}^1_{fpqc}(B^{\mathcal{C}}, G^{\mathcal{C}}) \to \prod_{w\in\Sigma^{\mathcal{C}}} \check{H}^1_{fpqc}(\mathrm{Spec}((K^{\mathcal{C}})_w), G^{\mathcal{C}}))) = \{1\}$.

Proof. By [Ra], Théorème XI, 3.1, each class x of $\check{H}^1_{fpqc}(B^{\mathcal{C}}, G^{\mathcal{C}})$ corresponds to a representable $B^{\mathcal{C}}$-torsor $X^{\mathcal{C}}$. If, moreover, x belongs to the kernel in question, $X^{\mathcal{C}}$ admits a $(K^{\mathcal{C}})_w$-rational point for each $w \in \Sigma^{\mathcal{C}}$. Since $X^{\mathcal{C}}$ is of finite presentation over $B^{\mathcal{C}}$, it comes from a scheme over a finite ($\mathcal{C}$-admissible) extension of B. Now, as in the proof of (4.4), we can prove that $X^{\mathcal{C}}$ admits a $B^{\mathcal{C}}$-section, by using (3.1). This completes the proof. □

Note that the second assertion of (4.4) is a special case of (4.5), where $G^{\mathcal{C}} = \mathbb{G}_{m,B^{\mathcal{C}}}$. (The first assertion of (4.4) can also be generalized in a suitable sense. We leave it to the readers.)

As an interesting corollary of (4.5), we obtain:

Corollary (4.6). *Let C be an affine, smooth curve over an algebraic extension of $\mathbb{F}_p$, and K the function field of C. Let G be a smooth, separated, commutative group scheme of finite type over C, such that the generic fiber G_K is connected.*

Then, we have $H^1_{\text{ét}}(C, G) = H^1(\pi_1(C), G(\tilde{C}))$, where $\tilde{C} \stackrel{\text{def}}{=} B^{\mathcal{C}} (= C^{\mathcal{C}})$ for the base curve data $\mathcal{C} \stackrel{\text{def}}{=} (C, \emptyset, \emptyset)$.

Proof. This follows from (4.5), together with the Hochschild-Serre spectral sequence. □

4.8. A group-theoretical remark.
Recall that a quasi-p group (for a prime number p) is a finite group that does not admit a non-trivial quotient group of order prime to p.

Proposition (4.7). *Let B be a smooth, geometrically connected curve over a finite field k of characteristic p. We denote by B^* the smooth compactification of B and put $\Sigma \stackrel{\text{def}}{=} B^* - B$ (which we regard as a reduced closed subscheme of B^*). Then, there exists a natural number N (depending only on the genus g of B^* and the cardinality n of $\Sigma(\overline{k})$), such that, for each finite extension l of k with $\sharp(l) \geq N$, there is no non-trivial finite étale covering of $B_l \stackrel{\text{def}}{=} B \times_k l$ at most tamely ramified over $\Sigma_l \stackrel{\text{def}}{=} \Sigma \times_k l$ in which every point of $B(l) = B_l(l)$ splits completely.*

Proof. This is a rather well-known application of the Weil bound on the cardinality of rational points over finite fields. More specifically, suppose that l is a finite extension of k with cardinality q and that B' is a connected finite étale covering with degree d of B_l at most tamely ramified over Σ_l in which every point of $B(l)$ splits completely. We denote by l' the integral closure of l in B', and define $(B')^*$, g' and n' for B' just similarly to B^*, g and n, respectively, for B. Now, firstly, the tamely ramified condition, together with the Hurwitz' formula, implies

$$2g' - 2 \leq d_{\text{geom}}(2g-2) + (d_{\text{geom}} - 1)n \leq d(2g-2) + (d-1)n,$$

where $d_{\text{geom}} \stackrel{\text{def}}{=} d/[l' : l]$. Secondly, the complete splitting condition, together with the Weil (lower) bound for B_l, implies that

$$\sharp(B'(l)) = d\sharp(B_l(l)) \geq d(\sharp((B')^*(l)) - n) \geq d(1 + q - 2g\sqrt{q} - n).$$

Thirdly, the Weil (upper) bound for B' implies

$$\sharp(B'(l)) \leq \sharp((B')^*(l)) \leq 1 + q + 2g'\sqrt{q}.$$

(Note that this holds (trivially) even if $[l' : l] > 1$.) Combining these three inequalities together, we obtain

$$d(q - (4g + n - 2)\sqrt{q} - (n-1)) \leq q - (n-2)\sqrt{q} + 1.$$

From this, we see that $d < 2$ (or, equivalently, $d = 1$) must hold for sufficiently large q, as desired. □

PROPOSITION (4.8). *Let k and B be as in* (4.7). *Then, there exist finite sets Σ_1 and Σ_2 of closed points of B, disjoint from each other, such that the following holds: For each $v \in \Sigma_1$ (resp. $v \in \Sigma_2$), let L_v be any (possibly infinite) pro-p Galois extension of K_v (resp. Galois extension such that* $\mathrm{Gal}(L_v \cap K_v^{\mathrm{ur}}/K_v)$ *is a pro-prime-to-p group) and put $\mathcal{C} = (B, \Sigma_1 \cup \Sigma_2, \{L_v\}_{v \in \Sigma_1 \cup \Sigma_2})$. Then, for every $\mathcal{C}$-admissible, finite, Galois extension K' of K,* $\mathrm{Gal}(K'/K)$ *is a quasi-p group and the constant field of K' (i.e., the algebraic closure of k in K') coincides with k.*

Proof. Take a natural number N as in (4.7), and choose two finite extensions l_1 and l_1' of k with $\sharp(l_1) \geq N$ and $\sharp(l_1') \geq N$, such that $l_1 \cap l_1' = k$. We define Σ_1 to be the union of the images of $B(l_1)$ and $B(l_1')$ in B. Next, the Weil bound implies that there exists a natural number N', such that, for each finite extension l of k with $\sharp(l) \geq N'$, $(B - \Sigma_1)(l) \neq \emptyset$ holds. (Note that $B - \Sigma_1$ is geometrically connected over k.) So, we can choose a finite extension l_2 of k, such that $(B - \Sigma_1)(l_2) \neq \emptyset$ and that $[l_2 : k]$ is not divisible by p. We define Σ_2 to be any non-empty subset of the image of $(B - \Sigma_1)(l_2)$ in $B - \Sigma_1$. Now, for each $v \in \Sigma_1$ (resp. $v \in \Sigma_2$), let L_v be any pro-p Galois extension of K_v (resp.

Galois extension such that $\mathrm{Gal}(L_v \cap K_v^{\mathrm{ur}}/K_v)$ is a pro-prime-to-p group), and put $\mathcal{C} = (B, \Sigma_1 \cup \Sigma_2, \{L_v\}_{v \in \Sigma_1 \cup \Sigma_2})$.

Let K' be any $\mathcal{C}$-admissible finite Galois extension of K. We define $\mathrm{Gal}(K'/K)^{p'}$ (resp. $\mathrm{Gal}(K'/K)^{p}$) to be the maximal quotient group of $\mathrm{Gal}(K'/K)$ with order prime to p (resp. with order a power of p), and denote by M_1 (resp. M_2) the subextension of K' over K that corresponds via Galois theory the kernel of the natural surjective map $\mathrm{Gal}(K'/K) \twoheadrightarrow \mathrm{Gal}(K'/K)^{p'}$ (resp. $\mathrm{Gal}(K'/K) \twoheadrightarrow \mathrm{Gal}(K'/K)^{p}$). Thus we have $\mathrm{Gal}(M_1/K) = \mathrm{Gal}(K'/K)^{p'}$ (resp. $\mathrm{Gal}(M_2/K) = \mathrm{Gal}(K'/K)^{p}$) .

We shall first prove that $\mathrm{Gal}(K'/K)$ is a quasi-p group, or, equivalently, that $M_1 = K$ holds. As K' is $\mathcal{C}$-admissible, so is M_1. Since, moreover, $\mathrm{Gal}(M_1/K)$ has order prime to p and $\mathrm{Gal}(L_v/K_v)$ is pro-p for each $v \in \Sigma_1$, M_1/K must split completely at each $v \in \Sigma_1$. Now, by (4.7), we obtain $M_1 \subset Kl_1 \cap Kl'_1 = K$, as desired.

Next, we shall prove that the constant field of K' is k. Since the Galois group over k of a finite extension of k is cyclic (hence nilpotent, a fortiori), we see that the constant field of K' is the compositum of those of M_1 and M_2. Since we have already proved $M_1 = K$, it suffices to prove that the constant field of M_2 is k. As K' is $\mathcal{C}$-admissible, so is M_2. Since, moreover, $\mathrm{Gal}(M_2/K)$ has p-power order and $\mathrm{Gal}(L_v \cap K_v^{\mathrm{ur}}/K_v)$ is pro-prime-to-p for each $v \in \Sigma_2$, M_2/K does not admit a non-trivial residue field extension over Σ_2. In particular, the constant field of M_2 is contained in the residue field of each $v \in \Sigma_2$, hence in l_2 by the choice of Σ_2. Now, since $\mathrm{Gal}(M_2/K)$ has p-power order and $[l_2 : k]$ is prime to p, the constant field of M_2 must coincide with k, as desired.

This completes the proof. □

Proposition (4.9). *Let the notations and the assumptions be as in* (2.2)*, and assume, moreover, that k is an algebraic extension of $\mathbb{F}_p$. Then, in the conclusion of* (2.2)*, we may assume that* $\mathrm{Gal}(K'/K)$ *is a quasi-p group and that the constant field of K' coincides with k.*

Proof. We can choose a finite subextension k_0 of $\mathbb{F}_p$ in k, such that the curve C and the (reduced) closed subscheme Σ of C descend to C_0 and Σ_0, respectively. Replacing k_0 by a suitable finite extension (in k), we may assume $\sharp(\Sigma) = \sharp(\Sigma_0)$. Moreover, again replacing k_0 by a suitable finite extension, we may assume that, for each $v \in \Sigma$, the finite Galois extension L_v/K_v descends to a finite Galois extension $L_{0,v_0}/K_{0,v_0}$, where K_0 is the function field of C_0 and v_0 is the image of v in Σ_0. We define $\Sigma_{0,\infty}$ to be the image of Σ_∞ in Σ_0. We also put $B_0 \stackrel{\mathrm{def}}{=} C_0 - \Sigma_0$.

Now, take finite sets Σ_1 and Σ_2 of closed points of B_0 as in (4.8), and put $\mathcal{C}_0 \stackrel{\mathrm{def}}{=} (C_0, \Sigma_0 \cup \Sigma_1 \cup \Sigma_2, \{L_{0,v_0}\}_{v_0 \in \Sigma_0} \cup \{K_{0,v_0}\}_{v_0 \in \Sigma_1 \cup \Sigma_2})$. Applying (2.2) to $\mathcal{C}_0$ and $\Sigma_{\infty,0}$, we obtain a finite Galois extension K'_0 of K_0 that is nearly $\mathcal{C}_0$-distinguished with respect to $\Sigma_{\infty,0}$. By (4.8), $\mathrm{Gal}(K'_0/K_0)$ is a quasi-p group and the constant field of K'_0 coincides with k_0.

Finally, put $K' \stackrel{\mathrm{def}}{=} K'_0 k$. Then, we easily see that K' is a finite Galois extension of $K = K_0k$ that is nearly $\mathcal{C}$-distinguished with respect to Σ_∞, that $\mathrm{Gal}(K'/K)$ is a quasi-p group (as $\mathrm{Gal}(K'/K) \stackrel{\sim}{\to} \mathrm{Gal}(K'_0/K_0)$) and that the constant field of K' coincides with k. This completes the proof. □

Proposition (4.10). *In* (3.1) *and* (3.2), *we may choose* B' *such that* K' *is a finite Galois extension of* K *with quasi-*p *Galois group and that the constant field of* K' *coincides with* k.

Proof. The proof of this fact goes rather similarly as that of (4.9). The main difference between them consists in the fact that, in the proof of (4.9), we can use the trivial extension $K_{0,v_0}/K_{0,v_0}$ for each $v \in \Sigma_1 \cup \Sigma_2$, while, in the proof of (4.10), this is impossible, since we have to require that condition (RI) also holds for the (enlarged) base curve data. Here, however, we may take $K_{0,v_0}\mathbb{F}_p(p^\infty)/K_{0,v_0}$ (resp. $K_{0,v_0}\mathbb{F}_p(p')/K_{0,v_0}$) for $v_0 \in \Sigma_1$ (resp. $v_0 \in \Sigma_2$) instead of $K_{0,v_0}/K_{0,v_0}$, where $\mathbb{F}_p(p^\infty)$ (resp. $\mathbb{F}_p(p')$) denotes the maximal pro-p (resp. pro-prime-to-p) extension of $\mathbb{F}_p$. Details are left to the readers. □

Remark (4.11). The group-theoretical results of this subsection are also applicable to other results in this section.

For example, (4.8) implies that, for some base curve data $\mathcal{C}$, the field $K^{\mathcal{C}}$ that appears in (4.1) and (4.3) satisfies the following property: $\mathrm{Gal}(K^{\mathcal{C}}/K)$ is pro-quasi-p in the sense that its maximal pro-prime-to-p quotient is trivial.

A similar remark is also applicable to (4.4). Namely, for some base curve data $\mathcal{C}$, $\mathrm{Gal}(K^{\mathcal{C}}/K)(= \mathrm{Aut}(B^{\mathcal{C}}/B))$ is pro-quasi-p. In particular, the abelianization $\mathrm{Gal}(K^{\mathcal{C}}/K)^{\mathrm{ab}}$ of $\mathrm{Gal}(K^{\mathcal{C}}/K)$ is a pro-p group, or, equivalently, every finite abelian $\mathcal{C}$-admissible extension of K has p-power degree. Accordingly, if, moreover, we start with $\mathcal{C}$ such that $\mathrm{Pic}(C)$ admits a non-trivial torsion element $[L]$ whose order is prime to p, then L cannot be trivialized over any finite abelian $\mathcal{C}$-admissible extension, while it can be trivialized over some finite (necessarily non-abelian) $\mathcal{C}$-admissible extension by (4.4).

References

[A1] P. Autissier, *Points entiers sur les surfaces arithmétiques*, J. Reine Angew. Math. **531** (2001), 201–235.

[A2] ———, *Points entiers et théorèmes de Bertini arithmétiques*, Ann. Inst. Fourier (Grenoble) **51** (2001), 1507–1523; Corrigendum, ibid. **52** (2002), 303–304.

[CR] D. C. Cantor and P. Roquette, *On Diophantine equations over the ring of all algebraic integers*, J. Number Theory **18** (1984), 1–26.

[CFHR] F. Catanese, M. Franciosi, K. Hulek and M. Reid, *Embeddings of curves and surfaces*, Nagoya Math. J. **154** (1999), 185–220.

[EGA4] A. Grothendieck et al, *Éléments de géométrie algébrique IV, Étude locale des schémas et des morphismes de schémas,* Publications Mathématiques, 20, 24, 28, 32, IHES, 1964–1967.

[E1] R. Erné, *The degree of an integral point in* $\mathbf{P}^s$ *minus a hypersurface*, C. R. Acad. Sci. Paris, Sér. I, Math. **324** (1997), 1121–1126.

[E2] ———, *On the degree of integral points of a projective space minus a horizontal hypersurface*, J. Reine Angew. Math. **532** (2001), 151–177.

[EG] D. R. Estes and R. M. Guralnick, *Module equivalences: local to global when primitive polynomials represent units*, J. Algebra **77** (1982), 138–157.

[G] O. Gabber, *On space filling curves and Albanese varieties*, Geom. Funct. Anal. **11** (2001), 1192–1200.

[GPR] B. Green, F. Pop and P. Roquette, *On Rumely's local-global principle*, Jahresber. Deutsch. Math.-Verein. **97** (1995), 43–74.

[J] J.-P. Jouanolou, *Théorèmes de Bertini et applications,* Progress in Mathematics, 42, Birkhäuser Boston, Inc., Boston, 1983.

[Mi] P. Mikkelsen, *Effective bounds for integral points on arithmetic surfaces*, J. Reine Angew. Math. **496** (1998), 55–72.

[Mo1] L. Moret-Bailly, *Points entiers des variétés arithmétiques*, in Séminaire de Théorie des Nombres, Paris 1985–86, Progr. Math., 71, Birkhäuser Boston, Boston, 1987, pp. 147–154.

[Mo2] ———, *Groupes de Picard et problèmes de Skolem, I*, Ann. Sci. École Norm. Sup. (4) **22** (1989), 161–179.

[Mo3] ———, *Groupes de Picard et problèmes de Skolem, II*, Ann. Sci. École Norm. Sup. (4) **22** (1989), 181–194.

[Mo4] ———, *Applications of local-global principles to arithmetic and geometry*, in Hilbert's tenth problem: relations with arithmetic and algebraic geometry (Ghent, 1999), Contemp. Math., 270, Amer. Math. Soc., Providence, 2000, pp. 169–186.

[Mo5] ———, *Problèmes de Skolem sur les champs algébriques*, Compositio Math. **125** (2001), 1–30.

[Poo] B. Poonen, *Bertini theorems over finite fields*, Ann. of Math. (to appear).

[Pop] F. Pop, *Embedding problems over large fields*, Ann. of Math. (2) **144** (1996), 1–34.

[Ra] M. Raynaud, *Faisceaux amples sur les schémas en groupes et les espaces homogènes,* Lecture Notes in Mathematics, 119, Springer-Verlag, Berlin-New York, 1970.

[Ro] P. Roquette, *Solving diophantine equations over the ring of all algebraic integers*, in Atas da 8. Escola de Álgebra (Rio de Janeiro, 1984), 2, Coleção Atas, no. 16, Instituto de Mathemática Pura e Aplicada, 1985, pp. 1–26.

[Ru1] R. S. Rumely, *Arithmetic over the ring of all algebraic integers*, J. Reine Angew. Math. **368** (1986), 127–133.

[Ru2] ———, *Capacity theory on algebraic curves,* Lecture Notes in Mathematics, 1378, Springer-Verlag, Berlin, 1989.

[S] T. Skolem, *Lösung gewisser Gleichungen in ganzen algebraischen Zahlen, insbesondere in Einheiten*, Skr. Norske Vid.-Akad. Oslo I, Mat. Naturv. Kl. (1934), no. 10.

[T] A. Tamagawa, *The Eisenstein quotient of the Jacobian variety of a Drinfeld modular curve*, Publ. Res. Inst. Math. Sci. **31** (1995), 203–246.

[U] E. Ullmo, *Points entiers, points de torsion et amplitude arithmétique*, Amer. J. Math. **117** (1995), 1039-1055.

Akio Tamagawa
Research Institute for
Mathematical Sciences
Kyoto University
Kyoto 606-8502, Japan

On the Maximal Unramified Quotients of p-Adic Étale Cohomology Groups and Logarithmic Hodge–Witt Sheaves

Dedicated to Professor K. Kato on his 50th birthday

Takeshi Tsuji

Received: November 11, 2002
Revised: November 9, 2003

Abstract. Let O_K be a complete discrete valuation ring of mixed characteristic $(0,p)$ with perfect residue field. From the semi-stable conjecture (C_{st}) and the theory of slopes, we obtain isomorphisms between the maximal unramified quotients of certain Tate twists of p-adic étale cohomology groups and the cohomology groups of logarithmic Hodge-Witt sheaves for a proper semi-stable scheme over O_K. The object of this paper is to show that these isomorphisms are compatible with the symbol maps to the p-adic vanishing cycles and the logarithmic Hodge-Witt sheaves, and that they are compatible with the integral structures under certain restrictions. We also treats an open case and a proof of C_{st} in such a case is given for that purpose. The results are used in the work of U. Jannsen and S. Saito in this volume.

2000 Mathematics Subject Classification: 14F30, 14F20
Keywords and Phrases: p-adic étale cohomology, p-adic vanishing cycles, logarithmic Hodge-Witt sheaf, p-adic Hodge theory

We will study a description of the maximal unramified quotients of certain p-adic étale cohomology groups in terms of logarithmic Hodge-Witt sheaves. Let K be a complete discrete valuation field of mixed characteristic $(0,p)$ whose residue field is perfect, let O_K be the ring of integers of K, and let $\overline{K}$ be an algebraic closure of K. We consider a proper semi-stable scheme X over O_K, i.e. a regular scheme X proper and flat over O_K such that the special fiber Y of X is reduced and is a divisor with normal crossings on X. For such an X, we have a comparison theorem (the semi-stable conjecture by Fontaine-Jannsen) between the p-adic étale cohomology $H^r_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ and the logarithmic crystalline cohomology of X with some additional structures (Theorem 3.2.2). Combining this with the description of the maximal slope part of the logarithmic crystalline cohomology in terms of logarithmic Hodge-Witt sheaves (see §2.3 for details), we obtain canonical isomorphisms (=(3.2.6), (3.2.7)):

$$(0.1)\qquad \begin{aligned} &H^r_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(r))_I \cong \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^0_{\text{ét}}(\overline{Y}, W\omega^r_{\overline{Y}/\overline{s},\log}) \\ &H^r_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(d))_I \cong \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^{r-d}_{\text{ét}}(\overline{Y}, W\omega^d_{\overline{Y}/\overline{s},\log}) \qquad \text{if } r \geq d \end{aligned}$$

by a simple argument. Here $d = \dim X_K$, I denotes the inertia subgroup of $\mathrm{Gal}(\overline{K}/K)$ and the subscript I denotes the cofixed part by I, i.e. the maximal unramified quotient.

The purpose of this paper is to answer the following two questions (partially for the second one) on these isomorphisms.

First, if we denote by i and j the closed immersion and the open immersion $Y \to X$ and $X_K := X \times_{\mathrm{Spec}(O_K)} \mathrm{Spec}(K) \to X$ respectively, then we have a unique surjective homomorphism of sheaves on the étale site $Y_{\text{ét}}$ of Y:

$$(0.2)\qquad i^*R^rj_*\mathbb{Z}/p^n\mathbb{Z}(r) \longrightarrow W_n\omega^r_{Y/s,\log}$$

compatible with the symbol maps (see §3.1 for details), from which we obtain homomorphisms from the LHS's of (0.1) to the RHS's of (0.1). *Do these homomorphisms coincide with* (0.1) *constructed from the semi-stable conjecture?* In [Sat], it is stated without proof that they coincide. We will give its precise proof. This second construction via (0.2) is necessary in the applications [Sat] and [J-Sai].

Secondly, the both sides of the isomorphisms (0.1) have natural integral structures coming from $H^r_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p(s))$ and $H^{r-s}_{\text{ét}}(\overline{Y}, W\omega^s_{\overline{Y}/\overline{s},\log})$ for $s = r, d$. *Do the two integral structures coincide?* We will prove that it is true in the case $r \leq p-2$ and the base field K is absolutely unramified by using the comparison theorem of C. Breuil (Theorem 3.2.4) between the p-torsion étale cohomology and certain log crystalline cohomology.

By [Ts4], one can easily extend the theorem of C. Breuil to any fine and saturated smooth log scheme whose special fiber is reduced (including the case that the log structure on the generic fiber is non-trivial), and we will discuss on the second question under this more general setting. For the $\mathbb{Q}_p$ case, G. Yamashita [Y] recently proved the semi-stable conjecture in the open case (more precisely under the condition (3.1.2)) by the syntomic method. If we use his result, we can prove our result also in the open case. Considering the necessity in [J-Sai] of our result in the open case, we will give an alternative proof in §4 when the horizontal divisors at infinity do not have self-intersections by proving the compatibility of the comparison maps with the Gysin sequences.

This paper is organized as follows. In §1, we will give a description of the maximal unramified quotients of semi-stable $\mathbb{Q}_p$-representations and semi-stable

$\mathbb{Z}_p$-representations (in the sense of C. Breuil) in terms of the corresponding objects in $\underline{MF}_K(\varphi, N)$ and in $\underline{\mathcal{MF}}_{W,\mathrm{tor}}(\varphi, N)$ respectively. In §2, we study the relation between the maximal slope part of the log crystalline cohomology and the logarithmic Hodge-Witt sheaves taking care of their integral structures. In §3, we will state our main theorem, review the construction of the comparison map in the semi-stable conjecture and then prove the main theorem.

I dedicate this paper to Professor K. Kato, who guided me to the p-adic world, especially to the p-adic Hodge theory. This paper is based on his work at many points. I would like to thank Professors U. Jannsen and S. Saito for fruitful discussions on the subject of this paper.

Notation. Let K be a complete discrete valuation field of mixed characteristic $(0, p)$ whose residue field k is perfect, and let O_K denote the ring of integers of K. Let W be the ring of Witt vectors with coefficients in k, and let K_0 denote the field of fractions of W. We denote by σ the Frobenius endomorphisms of k, W and K_0. Let $\overline{K}$ be an algebraic closure of K, and let $\overline{k}$ be the residue field of $\overline{K}$, which is an algebraic closure of k. Let G_K (resp. G_k) be the Galois group $\mathrm{Gal}(\overline{K}/K)$ (resp. $\mathrm{Gal}(\overline{k}/k)$), and let I_K be the inertia group of G_K. We have $G_K/I_K \cong G_k$. We denote by P_0 the field of fractions of the ring of Witt vectors $W(\overline{k})$ with coefficients in $\overline{k}$.

§1. The maximal unramified quotients of semi-stable representations.

In this section, we study the maximal unramified quotients (i.e. the coinvariant by the inertia group I_K) of semi-stable p-adic or $\mathbb{Z}_p$-representations of G_K.

§1.1. Review on slopes.

Let D be a finite dimensional P_0-vector space with a semi-linear automorphism φ. For a rational number $\alpha = sr^{-1}$ $(s, r \in \mathbb{Z}, r > 0, (s, r) = 1)$, we denote by D_α the P_0-subspace of D generated by the $\mathrm{Frac}(W(\mathbb{F}_{p^r}))$-vector space $D^{\varphi^r = p^s}$. Then, the natural homomorphisms $D^{\varphi^r=p^s} \otimes_{\mathrm{Frac}(W(\mathbb{F}_{p^r}))} P_0 \to D_\alpha$ and $\oplus_{\alpha\in\mathbb{Q}} D_\alpha \to D$ are isomorphisms. We call D_α the subspace of D whose slope is α, and we say that α is a slope of D if $D_\alpha \neq 0$.

Let D be a finite dimensional K_0-vector space with a σ-semi-linear automorphism φ. Then the above slope decomposition of $(P_0 \otimes_{K_0} D, \varphi \otimes \varphi)$ descends to the decomposition $D = \oplus_{\alpha\in\mathbb{Q}} D_\alpha$ of D. We call D_α the subspace of D whose slope is α and we say that α is a slope of D if $D_\alpha \neq 0$. For a subset $I \subset \mathbb{Q}$, we denote the sum $\oplus_{\alpha\in I} D_\alpha$ by D_I.

§1.2. Review on semi-stable, crystalline and unramified p-adic representations ([Fo1], [Fo2]A1, [Fo3], [Fo4]).

Let $\underline{MF}_K(\varphi, N)$ denote the category of finite dimensional K_0-vector spaces endowed with σ-semilinear automorphisms φ, K_0-linear endomorphisms N satisfying $N\varphi = p\varphi N$, and exhaustive and separated descending filtrations $Fil^{\cdot}$ on $D_K := K \otimes_{K_0} D$ by K-subspaces. Let $\underline{MF}_K(\varphi)$ denote the full subcategory of $\underline{MF}_K(\varphi, N)$ consisting of the objects such that $N = 0$. We denote by

$\underline{\mathrm{M}}_{K_0}(\varphi)$ the category of a finite dimensional K_0-vector space endowed with a σ-semilinear automorphism φ whose slope is 0. We regard an object D of $\underline{\mathrm{M}}_{K_0}(\varphi)$ as an object of $\underline{MF}_K(\varphi)$ by giving the filtration $Fil^0 D_K = D_K$, $Fil^1 D_K = 0$. By a p-adic representation of G_K, we mean a finite dimensional $\mathbb{Q}_p$-vector space endowed with a continuous and linear action of G_K, and we denote by $\underline{\mathrm{Rep}}(G_K)$ the category of p-adic representations. We denote by $\underline{\mathrm{Rep}}_{\mathrm{st}}(G_K)$ (resp. $\underline{\mathrm{Rep}}_{\mathrm{crys}}(G_K)$, resp. $\underline{\mathrm{Rep}}_{\mathrm{ur}}(G_K)$) be the full subcategory of $\underline{\mathrm{Rep}}(G_K)$ consisting of semi-stable (resp. crystalline, resp. unramified) p-adic representations.

Choose and fix a uniformizer π of K. Then, by the theory of Fontaine, we have the following commutative diagram of categories and functors:

$$\begin{array}{ccc} \underline{\mathrm{Rep}}_{\mathrm{ur}}(G_K) & \xrightarrow{D_{\mathrm{ur}}} & \underline{\mathrm{M}}_{K_0}(\varphi) \\ \cap & & \cap \\ \underline{\mathrm{Rep}}_{\mathrm{crys}}(G_K) & \xrightarrow{D_{\mathrm{crys}}} & \underline{MF}_K(\varphi) \\ \cap & & \cap \\ \underline{\mathrm{Rep}}_{\mathrm{st}}(G_K) & \xrightarrow{D_{\mathrm{st}}} & \underline{MF}_K(\varphi, N) \end{array}$$

The functors D_{crys} and D_{st} are fully faithful and the functor D_{ur} is an equivalence of categories; they are defined by

$$D_\bullet(V) = (B_\bullet \otimes_{\mathbb{Q}_p} V)^{G_K} \quad (\bullet = \mathrm{st}, \mathrm{crys}, \mathrm{ur}),$$

where B_{st} and B_{crys} are the rings of Fontaine and $B_{\mathrm{ur}} = P_0$. A semi-stable representation V is crystalline if and only if $N = 0$ on $D_{\mathrm{st}}(V)$. Recall that the embedding $B_{\mathrm{st}} \hookrightarrow B_{\mathrm{dR}}$ and hence the functor D_{st} depends on the choice of π. The quasi-inverse of D_{ur} is given by

$$V_{\mathrm{ur}}(D) = (P_0 \otimes_{K_0} D)^{\varphi \otimes \varphi = 1}$$

We say that an object D of $\underline{MF}_K(\varphi, N)$ (resp. $\underline{MF}_K(\varphi)$) is *admissible* if there exists a semi-stable (resp. crystalline) representation V such that $D_{\mathrm{st}}(V) \cong D$ (resp. $D_{\mathrm{crys}}(V) \cong D$). We denote by $\underline{MF}_K^{\mathrm{ad}}(\varphi, N)$ (resp. $\underline{MF}_K^{\mathrm{ad}}(\varphi)$) the full subcategory of $\underline{MF}_K(\varphi, N)$ (resp. $\underline{MF}_K(\varphi)$) consisting of admissible objects. Then the quasi-inverse $V_{\mathrm{st}} \colon \underline{MF}_K^{\mathrm{ad}}(\varphi, N) \to \underline{\mathrm{Rep}}_{\mathrm{st}}(G_K)$ (resp. $V_{\mathrm{crys}} \colon \underline{MF}_K^{\mathrm{ad}}(\varphi) \to \underline{\mathrm{Rep}}_{\mathrm{crys}}(G_K)$) of the functor D_{st} (resp. D_{crys}) is given by

$$V_{\mathrm{st}}(D) = Fil^0(B_{\mathrm{dR}} \otimes_K D_K) \cap (B_{\mathrm{st}} \otimes_{K_0} D)^{\varphi \otimes \varphi = 1, N \otimes 1 + 1 \otimes N = 0}$$
$$(\text{resp. } V_{\mathrm{crys}}(D) = Fil^0(B_{\mathrm{dR}} \otimes_K D_K) \cap (B_{\mathrm{crys}} \otimes_{K_0} D)^{\varphi \otimes \varphi = 1})$$

For an object D of $\underline{MF}_K(\varphi, N)$ and an integer r, we denote by $D(r)$ the object of $\underline{MF}_K(\varphi, N)$ whose underlying K_0-vector space and monodromy operator N are the same as D and whose Frobenius endomorphism (resp. filtration) is

defined by $\varphi_{D(r)} = p^{-r}\varphi_D$ (resp. $Fil^i D(r)_K = Fil^{i+r} D_K$). If D is admissible, then $D(r)$ is also admissible and there is a canonical isomorphism $V_{\mathrm{st}}(D)(r) \cong V_{\mathrm{st}}(D(r))$ induced by $(B_{\mathrm{st}} \otimes_{K_0} D) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(r) \cong B_{\mathrm{st}} \otimes_{K_0} D(r); (a \otimes d) \otimes t^r \mapsto at^r \otimes d$. Here (r) in the left-hand sides means the usual Tate twist
For an object D of $\underline{MF}_K(\varphi, N)$, we define the integers $t_H(D)$ and $t_N(D)$ by

$$t_H(D) = \sum_{i \in \mathbb{Z}} (\dim_K \mathrm{gr}^i_{Fil} D_K) \cdot i$$

$$t_N(D) = \sum_{\alpha \in \mathbb{Q}} (\dim_{K_0} D_\alpha) \cdot \alpha,$$

where D_α denotes the subspace of D whose slope is α.
We say that an object D of $\underline{MF}_K(\varphi, N)$ is *weakly admissible* if $t_H(D) = t_N(D)$ and if, for any K_0-subspace D' of D stable under φ and N, $t_H(D') \leq t_N(D')$. Here we endow D'_K with the filtration $Fil^i D'_K = Fil^i D_K \cap D'_K$ ($i \in \mathbb{Z}$). Note that $t_H(D') \leq t_N(D')$ is equivalent to $t_H(D/D') \geq t_N(D/D')$ under the assumption $t_H(D) = t_N(D)$. The admissibility implies the weak admissibility. P. Colmez and J.-M. Fontaine proved that the converse is also true ([C-Fo]).

§1.3. THE MAXIMAL UNRAMIFIED QUOTIENTS OF SEMI-STABLE p-ADIC REPRESENTATIONS.

LEMMA 1.3.1. *Let D be a weakly admissible object of $\underline{MF}_K(\varphi, N)$. If $Fil^r D_K = D_K$ and $Fil^{s+1} D_K = 0$ for integers $r \leq s$, then the slopes of φ on D are contained in $[r, s]$.*

Proof. We prove that the slopes of φ on D are not less than r. The proof of slopes$\leq s$ is similar and is left to the reader. (Consider the projection $D \to D_\alpha$ for the largest slope α.) Let α be the smallest slope of D, and let D_α be the subspace of D whose slope is α. Then, D_α is stable under φ. By the formula $N\varphi = p\varphi N$ and the choice of α, we see $N = 0$ on D_α, especially D_α is stable under N. Hence $t_H(D_\alpha) \leq t_N(D_\alpha) = \alpha \cdot \dim_{K_0} D_\alpha$. Since $t_H(D_\alpha) \geq r \cdot \dim_{K_0} D_\alpha$ by the assumption on D, we obtain $\alpha \geq r$. $\square$

Let V be a semi-stable p-adic representation of G_K, and set $D = D_{\mathrm{st}}(V)$. Let s be an integer such that $Fil^{s+1} D_K = 0$, and let V_s be the quotient $V(s)_{I_K}(-s)$ of V. We will construct explicitly the corresponding admissible quotient of D. For $\alpha \in \mathbb{Q}$, let D_α denote the subspace of D whose slope is α. By Lemma 1.3.1, $D_\alpha = 0$ if $\alpha > s$ and hence $D = \oplus_{\alpha \in \mathbb{Q}, \alpha \leq s} D_\alpha$. We define the monodromy operator on D_s by $N = 0$ and the filtration on $(D_s)_K$ by $Fil^s(D_s)_K = (D_s)_K$ and $Fil^{s+1}(D_s)_K = 0$. Then, we see that $D_s(s)$ is an object of $\underline{\mathrm{M}}_{K_0}(\varphi)$. Especially D_s is admissible. Using the relation $N\varphi = p\varphi N$ on D and $Fil^{s+1} D_K = 0$, we see that the projection $D \to D_s$ is a morphism in the category $\underline{MF}^{\mathrm{ad}}_{K_0}(\varphi, N)$. Especially it is strictly compatible with the filtrations ([Fo4]4.4.4. Proposition i)), that is, the image of $Fil^i D_K$ is $Fil^i(D_s)_K$.

PROPOSITION 1.3.2. *Under the notation and the assumption as above, the quotient V_s of V corresponds to the admissible quotient D_s of D.*

Proof. Since $D_s(s)$ is contained in $\underline{\mathrm{M}}_{K_0}(\varphi)$, $V_{\mathrm{st}}(D_s)(s) \cong V_{\mathrm{st}}(D_s(s))$ is unramified. Hence the natural surjection $V \to V_{\mathrm{st}}(D_s)$ factors through V_s. On the other hand, since $V_s(s)$ is unramified, $D_{\mathrm{st}}(V_s)(s) \cong D_{\mathrm{st}}(V_s(s))$ is contained in $\underline{\mathrm{M}}_{K_0}(\varphi)$. Hence the unique slope of $D_{\mathrm{st}}(V_s)$ is s and the natural projection $D \to D_{\mathrm{st}}(V_s)$ factors as $D \to D_s \to D_{\mathrm{st}}(V_s)$. $D_s \to D_{\mathrm{st}}(V_s)$ is strictly compatible with the filtrations because $D \to D_s$ and $D \to D_{\mathrm{st}}(V_s)$ are strictly compatible with the filtrations. (Compatibility with φ and N is trivial). □

COROLLARY 1.3.3. *Under the above notations and assumptions, there exists a canonical G_k-equivariant isomorphism:*

$$V(s)_{I_K} \cong (P_0 \otimes_{K_0} D)^{\varphi = p^s}.$$

Proof. By Proposition 1.3.2, we have canonical G_k-equivariant isomorphisms:

$$\begin{aligned} V(s)_{I_K} &= V_s(s) \cong V_{\mathrm{st}}(D_s)(s) \cong V_{\mathrm{st}}(D_s(s)) = V_{\mathrm{ur}}(D_s(s)) \\ &= (P_0 \otimes_{K_0} D_s)^{\varphi \otimes p^{-s}\varphi = 1} = (P_0 \otimes_{K_0} D)^{\varphi \otimes \varphi = p^s}. \end{aligned}$$

□

§1.4. REVIEW ON SEMI-STABLE, CRYSTALLINE AND UNRAMIFIED p-TORSION REPRESENTATIONS ([Fo-L], [Fo2]A1, [Br2], [Br3]§3.2.1).

In this section, we assume $K = K_0$. Following [Br2], we denote by S the p-adic completion of the PD-polynomial ring in one variable $W\langle u\rangle = W\langle u-p\rangle$, and by f_0 (resp. f_p) the W-algebra homomorphism $S \to W$ defined by $u^{[n]} \mapsto 0$ (resp. $p^{[n]} = p^n/n!$) for $n \geq 1$. We define the filtration Fil^iS ($i \in \mathbb{Z}, i > 0$) to be the p-adic completion of the i-th divided power of the PD-ideal of $W\langle u-p\rangle$ generated by $u-p$. We set $Fil^iS = S$ for $i \in \mathbb{Z}$, $i \leq 0$. Let $\varphi_S \colon S \to S$ denote the lifting of Frobenius defined by σ on W and $u^{[n]} \mapsto (u^p)^{[n]}$. For an integer i such that $0 \leq i \leq p-2$, we have $\varphi(Fil^iS) \subset p^iS$ and we denote by $\varphi_i \colon Fil^iS \to S$ the homomorphism $p^{-i}\cdot\varphi|_{Fil^iS}$. Finally let N denote the W-linear derivation $N \colon S \to S$ defined by $N(u^{[n]}) = nu^{[n]}$ ($n \in \mathbb{N}$).

Let $\underline{MF}_{W,[0,p-2],\mathrm{tor}}(\varphi)$ be the category of Fontaine-Laffaille of level within $[0,p-2]$, let $\underline{\mathcal{MF}}_{W,[0,p-2],\mathrm{tor}}(\varphi,N)$ be the category $\underline{\mathcal{M}}^{p-2}$ of Breuil, and let $\underline{\mathrm{M}}_{W,\mathrm{tor}}(\varphi)$ be the category of W-modules of finite length endowed with σ-semilinear automorphisms. These categories are abelian and artinian.

An object of $\underline{MF}_{W,[0,p-2],\mathrm{tor}}(\varphi)$ is a W-module M of finite length endowed with a descending filtration Fil^iM ($i \in \mathbb{Z}$) by W-submodules such that $Fil^0M = M$, $Fil^{p-1}M = 0$ and σ-semilinear homomorphisms $\varphi_i \colon Fil^iM \to M$ ($0 \leq i \leq p-2$) such that $\varphi_i|_{Fil^{i+1}M} = p\varphi_{i+1}$ ($0 \leq i \leq p-3$) and $M = \sum_{0\leq i\leq p-2}\varphi_i(Fil^iM)$. We can prove that Fil^iM ($i \in \mathbb{Z}$) are direct summands of M. For an integer $0 \leq r \leq p-2$, we say that M is of level within $[0,r]$ if $Fil^{r+1}M = 0$. The sequence $M_1 \to M_2 \to M_3$ in $\underline{MF}_{W,[0,p-2],\mathrm{tor}}(\varphi)$ is exact if and only if it is exact as a sequence of W-modules. Furthermore, for an exact sequence $M_1 \to M_2 \to M_3$ in $\underline{MF}_{W,[0,p-2],\mathrm{tor}}(\varphi)$, the sequence $Fil^i M_1 \to Fil^i M_2 \to Fil^i M_3$ is exact for any $i \in \mathbb{Z}$.

An object of $\underline{\mathcal{MF}}_{W,[0,p-2],\mathrm{tor}}(\varphi, N)$ is an S-module $\mathcal{M}$ isomorphic to a finite sum of S-modules of the form $S/p^n S$ $(n \geq 1)$ endowed with the following three structures: A submodule $Fil^{p-2}\mathcal{M}$ such that $Fil^{p-2}S \cdot \mathcal{M} \subset Fil^{p-2}\mathcal{M}$. A φ_S-semi-linear homomorphism $\varphi_{p-2}\colon Fil^{p-2}\mathcal{M} \to \mathcal{M}$ such that $(\varphi_1(u-p))^{p-2}\varphi_{p-2}(ax) = \varphi_{p-2}(a)\varphi_{p-2}((u-p)^{p-2}x)$ $(a \in Fil^{p-2}S,\ x \in \mathcal{M})$ and that $\mathcal{M}$ is generated by $\varphi_{p-2}(Fil^{p-2}\mathcal{M})$ as an S-module. A W-linear map $N\colon \mathcal{M} \to \mathcal{M}$ such that $N(ax) = N(a)x + aN(x)$ $(a \in S, x \in \mathcal{M})$, $(u-p)N(Fil^{p-2}\mathcal{M}) \subset Fil^{p-2}\mathcal{M}$ and $\varphi_1(u-p)N\varphi_{p-2}(x) = \varphi_{p-2}((u-p)N(x))$ $(x \in Fil^{p-2}\mathcal{M})$. Note that $\varphi_1(u-p) = (p-1)!u^{[p]} - 1$ is invertible in S. For an object $\mathcal{M}$ of $\underline{\mathcal{MF}}_{W,[0,p-2],\mathrm{tor}}(\varphi, N)$, we define the filtration $Fil^i\mathcal{M}$ $(0 \leq i \leq p-2)$ by $Fil^i\mathcal{M} := \{x \in \mathcal{M} | (u-p)^{p-2-i}x \in Fil^{p-2}\mathcal{M}\}$ and the Frobenius $\varphi_i : Fil^i\mathcal{M} \to \mathcal{M}$ $(0 \leq i \leq p-2)$ by the formula $\varphi_i(x) = \varphi_1(u-p)^{-(p-2-i)}\varphi_{p-2}((u-p)^{p-2-i}x)$. We have $Fil^0\mathcal{M} = \mathcal{M}$ and $\varphi_i|Fil^{i+1}\mathcal{M} = p\varphi_{i+1}$ for $0 \leq i \leq p-3$. For an object $\mathcal{M}$ of $\underline{\mathcal{MF}}_{W,[0,p-2],\mathrm{tor}}(\varphi, N)$ and an integer $0 \leq r \leq p-2$, we say that $\mathcal{M}$ is of level within $[0,r]$ if $Fil^{p-2-r}S \cdot \mathcal{M} \supset Fil^{p-2}\mathcal{M}$. The sequence $\mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3$ in $\underline{\mathcal{MF}}_{W,[0,p-2],\mathrm{tor}}(\varphi, N)$ is exact if and only if it is exact as S-modules. Furthermore, for an exact sequence $\mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3$ in $\underline{\mathcal{MF}}_{W,[0,p-2],\mathrm{tor}}(\varphi, N)$ and an integer $0 \leq r \leq p-2$, if $\mathcal{M}_s$ $(s = 1, 2, 3)$ are of level within $[0, r]$, then the sequence $Fil^r\mathcal{M}_1 \to Fil^r\mathcal{M}_2 \to Fil^r\mathcal{M}_3$ is exact.

We regard an object M of $\underline{\mathrm{M}}_{W,\mathrm{tor}}(\varphi)$ as an object of $\underline{MF}_{W,[0,p-2],\mathrm{tor}}(\varphi)$ by setting $Fil^0 M = M$, $Fil^1 M = 0$ and $\varphi_0 = \varphi$. We have a canonical fully faithful exact functor $\underline{MF}_{W,[0,p-2],\mathrm{tor}}(\varphi) \to \underline{\mathcal{MF}}_{W,[0,p-2],\mathrm{tor}}(\varphi, N)$ defined as follows ([Br2]2.4.1): To an object M of $\underline{MF}_{W,[0,p-2],\mathrm{tor}}(\varphi)$, we associate the following object $\mathcal{M}$. The underlying S-module is $S \otimes_W M$ and $Fil^{p-2}\mathcal{M} = \sum_{0 \leq i \leq p-2} Fil^{p-2-i}S \otimes_W Fil^i M$. The Frobenius $\varphi_{p-2} : Fil^{p-2}\mathcal{M} \to \mathcal{M}$ is defined by the formula: $\varphi_{p-2}(a \otimes x) = \varphi_{p-2-i}(a) \otimes \varphi_i(x)$ $(0 \leq i \leq p-2, a \in Fil^{p-2-i}S, x \in Fil^i M)$. The monodromy operator is defined by $N(a \otimes x) = N(a) \otimes x$ $(a \in S, x \in M)$. To prove that φ_{p-2} is well-defined, we use the fact that $Fil^i M$ $(i \in \mathbb{Z})$ are direct summands of M. Note that, for an integer $0 \leq r \leq p-2$, $\mathcal{M}$ is of level within $[0, r]$ if and only if M is of level within $[0, r]$.

Let $\underline{\mathrm{Rep}}_{\mathrm{tor}}(G_K)$ be the category of $\mathbb{Z}_p$-modules of finite length endowed with continuous actions of G_K, and let $\underline{\mathrm{Rep}}_{\mathrm{tor,ur}}(G_K)$ be the full subcategory of $\underline{\mathrm{Rep}}_{\mathrm{tor}}(G_K)$ consisting of the objects such that the actions of G_K are trivial on I_K.

First we have an equivalence of categories:

$$T_{\mathrm{ur}}\colon \underline{\mathrm{M}}_{W,\mathrm{tor}}(\varphi) \to \underline{\mathrm{Rep}}_{\mathrm{tor,ur}}(G_K)$$

defined by $T_{\mathrm{ur}}(M) = (W(\overline{k}) \otimes_W M)^{\varphi \otimes \varphi = 1}$. Its quasi-inverse is given by $T \mapsto (W(\overline{k}) \otimes_{\mathbb{Z}_p} T)^{G_K}$ (cf. for example, [Fo2]A1).

J.-M. Fontaine and G. Laffaille constructed a covariant fully faithful exact functor ([Fo-L]):

$$T_{\mathrm{crys}}\colon \underline{MF}_{W,[0,p-2],\mathrm{tor}}(\varphi) \to \underline{\mathrm{Rep}}_{\mathrm{tor}}(G_K).$$

(Strictly speaking, they constructed a contravariant functor $\underline{U}_S$. We define T_{crys} to be its dual: $T_{\mathrm{crys}}(M) := \mathrm{Hom}(\underline{U}_S(M), \mathbb{Q}_p/\mathbb{Z}_p)$.) For an object M of $\underline{MF}_{W,[0,p-2],\mathrm{tor}}(\varphi)$ and an integer $0 \le r \le p-2$ such that M is of level within $[0,r]$, we have the following exact sequence functorial on M (cf. [Ka1]II §3):

$$(1.4.1) \quad 0 \longrightarrow T_{\mathrm{crys}}(M)(r) \longrightarrow Fil^r(A_{\mathrm{crys}} \otimes_W M) \overset{1-\varphi_r}{\longrightarrow} A_{\mathrm{crys}} \otimes_W M \longrightarrow 0.$$

Here $Fil^i(A_{\mathrm{crys}} \otimes_W M) = \sum_{0 \le j \le i} Fil^{i-j} A_{\mathrm{crys}} \otimes_W Fil^j M$ $(0 \le i \le p-2)$ and φ_i is defined by $\varphi_i(a \otimes x) = \varphi_{i-j}(a) \otimes \varphi_j(x)$ $(0 \le j \le i$, $a \in Fil^{i-j}A_{\mathrm{crys}}$, $x \in Fil^j M)$.
C. Breuil constructed a covariant fully faithful exact functor ([Br2]):

$$T_{\mathrm{st}}\colon \underline{\mathcal{MF}}_{W,[0,p-2],\mathrm{tor}}(\varphi, N) \to \underline{\mathrm{Rep}}_{\mathrm{tor}}(G_K).$$

(Strictly speaking, he constructed a contravariant functor V_{st}. We define T_{st} to be its dual: $T_{\mathrm{st}}(\mathcal{M}) := \mathrm{Hom}(V_{\mathrm{st}}(\mathcal{M}), \mathbb{Q}_p/\mathbb{Z}_p)$.) For an object $\mathcal{M}$ of $\underline{\mathcal{MF}}_{W,[0,p-2],\mathrm{tor}}(\varphi, N)$ and an integer $0 \le r \le p-2$ such that $\mathcal{M}$ is of level within $[0,r]$, we have the following exact sequence functorial on $\mathcal{M}$ ([Br3]§3.2.1):

$$(1.4.2) \quad 0 \longrightarrow T_{\mathrm{st}}(\mathcal{M})(r) \longrightarrow (Fil^r(\widehat{A_{\mathrm{st}}} \otimes_S \mathcal{M}))^{N=0} \overset{1-\varphi_r}{\longrightarrow} (\widehat{A_{\mathrm{st}}} \otimes_S \mathcal{M})^{N=0} \longrightarrow 0$$

Here we define $Fil^{\cdot}$, φ_r and N on $\widehat{A_{\mathrm{st}}} \otimes_S \mathcal{M}$ as follows (see [Br1]§2 for the definition of $\widehat{A_{\mathrm{st}}}$): We define $Fil^i(\widehat{A_{\mathrm{st}}} \otimes_S \mathcal{M})$ $(0 \le i \le p-2)$ to be the sum of the images of $Fil^{i-j}\widehat{A_{\mathrm{st}}} \otimes_S Fil^j\mathcal{M}$ $(0 \le j \le i)$. The homomorphism $\varphi_r\colon Fil^r(\widehat{A_{\mathrm{st}}} \otimes_S \mathcal{M}) \to \widehat{A_{\mathrm{st}}} \otimes_S \mathcal{M}$ is defined by $\varphi_r(a \otimes x) = \varphi_{r-i}(a) \otimes \varphi_i(x)$ $(0 \le i \le r, a \in Fil^{r-i}\widehat{A_{\mathrm{st}}}, x \in Fil^i\mathcal{M})$. (The well-definedness is non-trivial). The monodromy operator N is defined by $N(a \otimes x) = N(a) \otimes x + a \otimes N(x)$ $(a \in \widehat{A_{\mathrm{st}}}, x \in \mathcal{M})$.
Now we have the following diagram of categories and functors commutative up to canonical isomorphisms:

$$(1.4.3) \qquad \begin{array}{ccc} \underline{\mathrm{M}}_{W,\mathrm{tor}}(\varphi) & \overset{T_{\mathrm{ur}}}{\longrightarrow} & \underline{\mathrm{Rep}}_{\mathrm{tor,ur}}(G_K) \\ \cap & & \cap \\ \underline{MF}_{W,[0,p-2],\mathrm{tor}}(\varphi) & \overset{T_{\mathrm{crys}}}{\longrightarrow} & \underline{\mathrm{Rep}}_{\mathrm{tor}}(G_K) \\ \downarrow & & \| \\ \underline{\mathcal{MF}}_{W,[0,p-2],\mathrm{tor}}(\varphi, N) & \overset{T_{\mathrm{st}}}{\longrightarrow} & \underline{\mathrm{Rep}}_{\mathrm{tor}}(G_K). \end{array}$$

For an object M of $\underline{\mathrm{M}}_{W,\mathrm{tor}}(\varphi)$, the isomorphism $T_{\mathrm{ur}}(M) \cong T_{\mathrm{crys}}(M)$ is induced by the natural homomorphism $(W(\overline{k}) \otimes_W M)(r) \to Fil^r(A_{\mathrm{crys}} \otimes_W$

$M) = (Fil^r A_{\mathrm{crys}}) \otimes_W M$ $(0 \leq r \leq p-2)$ defined by the natural inclusion $W(\overline{k})(r) \subset Fil^r A_{\mathrm{crys}}$. (The isomorphism is independent of the choice of r.) For an object M of $\underline{MF}_{W,[0,p-2],\mathrm{tor}}(\varphi)$, if we denote by $\mathcal{M}$ the corresponding object of $\underline{\mathcal{MF}}_{W,[0,p-2],\mathrm{tor}}(\varphi, N)$, then the natural homomorphisms $Fil^i(A_{\mathrm{crys}} \otimes_W M) \to Fil^i(\widehat{A_{\mathrm{st}}} \otimes_S \mathcal{M})^{N=0}$ $(0 \leq i \leq p-2)$ are isomorphisms and we obtain the natural isomorphism $T_{\mathrm{crys}}(M) \cong T_{\mathrm{st}}(\mathcal{M})$ from the two exact sequences (1.4.1) and (1.4.2).

§1.5. Review of the comparison between $\mathbb{Q}_p$ and p-torsion theories ([Fo-L], [Br1], [Br2]).

We assume $K = K_0$ and keep the notation of §1.4. We review the relation between the functor $T_\bullet$ (§1.4) and the functor $V_\bullet$ (§1.2) ($\bullet \in \{\mathrm{ur}, \mathrm{crys}, \mathrm{st}\}$).

Let us begin with $\bullet = \mathrm{ur}$. Let $(M_n)_{n\geq 1}$ be a projective system of objects of $\underline{\mathrm{M}}_{W,\mathrm{tor}}(\varphi)$ such that the underlying W-module of M_n is a free W/p^nW-module of finite rank and the morphism $M_{n+1}/p^n M_{n+1} \to M_n$ is an isomorphism. Associated to such a system, we define the object D of $\underline{\mathrm{M}}_{K_0}(\varphi)$ to be $K_0 \otimes_W \varprojlim_n M_n$. Then, we have a canonical isomorphism defined in an obvious way:

$$V_{\mathrm{ur}}(D) \cong \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_n T_{\mathrm{ur}}(M_n). \tag{1.5.1}$$

Next consider the case $\bullet = \mathrm{crys}$. Let $(M_n)_{n\geq 1}$ be a projective system of objects of $\underline{MF}_{W,[0,p-2],\mathrm{tor}}(\varphi)$ such that the underlying W-module of M_n is a free W/p^nW-module of finite rank and the morphism $M_{n+1}/p^n M_{n+1} \to M_n$ is an isomorphism. We define the object D of $\underline{MF}_K(\varphi)$ associated to this system as follows: The underling vector space is $K_0 \otimes_W (\varprojlim_n M_n)$, the filtration is defined by $K_0 \otimes_W (\varprojlim_n Fil^i M_n)$ and the Frobenius endomorphism is defined to be the projective limit of φ_0 of M_n. Then D is admissible, and we have a canonical isomorphism ([Fo-L]§7, §8):

$$V_{\mathrm{crys}}(D) \cong \mathbb{Q}_p \otimes_{\mathbb{Z}_p} (\varprojlim_n T_{\mathrm{crys}}(M_n)). \tag{1.5.2}$$

The homomorphism from the RHS to the LHS is constructed as follows: By taking the projective limit of the exact sequence (1.4.1) for $M = M_n$ and $0 \leq r \leq p-2$ such that M_n are of level within $[0, r]$ for all $n \geq 1$ and tensoring with $\mathbb{Q}_p$, we obtain an isomorphism

$$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} (\varprojlim_n T_{\mathrm{crys}}(M_n))(r) \cong Fil^r(B^+_{\mathrm{crys}} \otimes_{K_0} D)^{\varphi = p^r},$$

whose RHS is contained in

$$\begin{aligned} V_{\mathrm{crys}}(D)(r) &\cong Fil^r(B_{\mathrm{crys}} \otimes_{K_0} D)^{\varphi=p^r} \\ v \otimes t^{\otimes r} &\mapsto t^r v \quad (v \in V_{\mathrm{crys}}(D), t \in \mathbb{Q}_p(1)). \end{aligned}$$

Finally let us explain the case $\bullet = \mathrm{st}$. In this case, we need to introduce another category $\underline{\mathcal{MF}}_K(\varphi, N)$ defined by C. Breuil [Br1]§6 ($\underline{\mathcal{MF}}_{S\otimes_W K_0}(\Phi, \mathcal{N})$

in his notation), which is categorically equivalent to $\underline{MF}_K(\varphi, N)$. Set $S_{K_0} := K_0 \otimes_W S$, and let f_0 (resp. f_p) also denote the surjective homomorphism $S_{K_0} \to K_0$ induced by f_0 (resp. f_p) $: S \to W$. The filtration $Fil^{\cdot}$, the Frobenius endomorphism φ and the monodromy operator N on S naturally induce those on S_{K_0}, which we also denote by $Fil^{\cdot}$, φ and N. An object of $\underline{\mathcal{MF}}_K(\varphi, N)$ is a free S_{K_0}-module $\mathcal{D}$ of finite rank endowed with the following three structures: A descending filtration $Fil^i\mathcal{D}$ ($i \in \mathbb{Z}$) by S_{K_0}-submodules such that $Fil^i S_{K_0} \cdot Fil^j \mathcal{D} \subset Fil^{i+j}\mathcal{D}$ ($i, j \in \mathbb{Z}$), $Fil^i\mathcal{D} = \mathcal{D}$ ($i << 0$) and $Fil^i\mathcal{D} \subset Fil^1 S_{K_0} \cdot \mathcal{D}$ ($i >> 0$). A $\varphi_{S_{K_0}}$-semilinear endomorphism $\mathcal{D} \to \mathcal{D}$ whose linearization $\mathcal{D} \otimes_{S_{K_0},\varphi} S_{K_0} \to \mathcal{D}$ is an isomorphism. A homomorphism $N \colon \mathcal{D} \to \mathcal{D}$ such that $N(ax) = N(a)x + aN(x)$ ($a \in S_{K_0}, x \in \mathcal{D}$), $N(Fil^i\mathcal{D}) \subset Fil^{i-1}\mathcal{D}$ ($i \in \mathbb{Z}$) and $N\varphi = p\varphi N$. We can construct a functor $\underline{MF}_K(\varphi, N) \to \underline{\mathcal{MF}}_K(\varphi, N)$ easily as follows: Let D be an object of $\underline{MF}_K(\varphi, N)$. The corresponding object $\mathcal{D}$ is the S_{K_0}-module $S_{K_0} \otimes_{K_0} D$ with the Frobenius $\varphi \otimes \varphi$ and the monodromy operator defined by $N(a \otimes x) = N(a) \otimes x + a \otimes N(x)$. The filtration is defined inductively by the following formula, where i_0 is an integer such that $Fil^{i_0} D = D$.

$$Fil^i\mathcal{D} = \mathcal{D} \quad (i \leq i_0)$$
$$Fil^i\mathcal{D} = \{x \in Fil^{i-1}\mathcal{D} | f_p(x) \in Fil^i D, N(x) \in Fil^{i-1}\mathcal{D}\} \quad (i > i_0)$$

Here f_p denotes the natural projection $\mathcal{D} \to \mathcal{D} \otimes_{S_{K_0},f_p} K_0$ and we identify $\mathcal{D} \otimes_{S_{K_0},f_p} K_0$ with D by the natural isomorphism $(D \otimes_{S_{K_0}} S_{K_0}) \otimes_{S_{K_0},f_p} K_0 \cong D$. To construct the quasi-inverse of the above functor, we need the following proposition. (Compare with $B^+_{\mathrm{st}} = \widehat{B^+_{\mathrm{st}}}^{N\text{-nilp}}$.)

PROPOSITION 1.5.3. ([Br1]§6.2.1). *Let $\mathcal{D}$ be an object of $\underline{\mathcal{MF}}_K(\varphi, N)$. Then $D := \mathcal{D}^{N\text{-nilp}}$ is a finite dimensional vector space over K_0 and the natural homomorphism $D \otimes_{K_0} S_{K_0} \to \mathcal{D}$ is an isomorphism. Here N-nilp denotes the part where N is nilpotent.*

With the notation of Proposition 1.5.3, we can verify easily that D is stable under N and φ and we can define an exhaustive and separated filtration on D by the image of $Fil^i\mathcal{D}$ under the homomorphism $\mathcal{D} \to \mathcal{D} \otimes_{S_{K_0},f_p} K_0 \cong D$. Thus we obtain an object D of $\underline{MF}_K(\varphi, N)$. The functor associating D to $\mathcal{D}$ is the quasi-inverse of the above functor.

Let $(\mathcal{M}_n)_{n \geq 1}$ be a projective system of objects of $\underline{\mathcal{MF}}_{W,[0,p-2],\mathrm{tor}}(\varphi, N)$ such that the underlying S-module of $\mathcal{M}_n$ is a free S/p^nS-module and the morphism $\mathcal{M}_{n+1}/p^n\mathcal{M}_{n+1} \to \mathcal{M}_n$ is an isomorphism for $n \geq 1$. Set $\mathcal{M} := \varprojlim_n \mathcal{M}_n$, $Fil^{p-2}\mathcal{M} := \varprojlim_n Fil^{p-2}\mathcal{M}_n$, and define the Frobenius $\varphi_{p-2} \colon Fil^{p-2}\mathcal{M} \to \mathcal{M}$ and the monodromy operator $N \colon \mathcal{M} \to \mathcal{M}$ by taking the projective limit of those on $\mathcal{M}_n$ ($n \in \mathbb{N}$). Then $\mathcal{M}$ is a free S-module of finite rank and the additional three structures satisfy the same conditions required in the definition of the category $\underline{\mathcal{MF}}_{W,[0,p-2],\mathrm{tor}}(\varphi, N)$. Furthermore $\mathcal{M}/Fil^{p-2}\mathcal{M}$ is p-torsion free. Indeed, we have the following injective morphism between two short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{M} & \xrightarrow{p} & \mathcal{M} & \longrightarrow & \mathcal{M}_1 & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & Fil^{p-2}\mathcal{M} & \xrightarrow{p} & Fil^{p-2}\mathcal{M} & \longrightarrow & Fil^{p-2}\mathcal{M}_1 & \longrightarrow & 0. \end{array}$$

(With the terminology of [Br2]Définition 4.1.1.1, $\mathcal{M}$ with the three additional structures is a strongly divisible S-module.) For each integer i such that $0 \leq i \leq p-2$, we define the S-submodule $Fil^i\mathcal{M}$ of $\mathcal{M}$ by $\{x \in \mathcal{M} | (u-p)^{p-2-i}x \in Fil^{p-2}\mathcal{M}\}$ and the Frobenius $\varphi_i \colon Fil^i\mathcal{M} \to \mathcal{M}$ by $\varphi_i(x) = \varphi_1(u-p)^{-(p-2-i)}\varphi_{p-2}((u-p)^{p-2-i}x)$. We have $Fil^0\mathcal{M} = \mathcal{M}$, $\varphi_i | Fil^{i+1}\mathcal{M} = p\varphi_{i+1}$ for $0 \leq i \leq p-3$ and $Fil^i\mathcal{M} = \varprojlim_n Fil^i\mathcal{M}_n$ $(0 \leq i \leq p-2)$. If $\mathcal{M}_n$ are of level within $[0,r]$ for an integer $0 \leq r \leq p-2$, then we see that $\mathcal{M}/Fil^r\mathcal{M}$ is p-torsion free. Now we define the object $\mathcal{D}$ of $\underline{\mathcal{MF}}_K(\varphi, N)$ associated to the projective system $(\mathcal{M}_n)_n$ as follows ([Br2]§4.1.1): The underlying S_{K_0}-module is $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathcal{M}$. The filtration is defined by $Fil^i\mathcal{D} = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} Fil^i\mathcal{M}$ $(0 \leq i \leq p-2)$ and $Fil^i\mathcal{D} = \sum_{0 \leq j \leq p-2} Fil^{i-j}S_{K_0} \cdot Fil^j\mathcal{D}$ $(i \geq p-1)$. The Frobenius and the monodromy operator on $\mathcal{D}$ are defined to be the endomorphisms induced by φ_0 and N on $\mathcal{M}$. Finally let D be the object of $\underline{MF}_K(\varphi, N)$ corresponding to $\mathcal{D}$. (If $\mathcal{M}_n$ are of level within $[0,r]$ for an integer $0 \leq r \leq p-2$, then so is D, that is, $Fil^0 D = D$ and $Fil^{r+1}D = 0$.) Then D is admissible and there is a canonical isomorphism

$$V_{\mathrm{st}}(D) \cong \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_n T_{\mathrm{st}}(\mathcal{M}_n) \tag{1.5.4}$$

functorial on $(\mathcal{M}_n)_n$ ([Br2]4.2).

This isomorphism is constructed as follows (cf. [Br3]4.3.2). First, since $\mathcal{M}$ is a free S-module of finite rank and $\mathcal{M}_n \cong \mathcal{M}/p^n\mathcal{M}$, we have

$$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} (\varprojlim_n (\widehat{A_{\mathrm{st}}} \otimes_S \mathcal{M}_n)) \cong \widehat{B^+_{\mathrm{st}}} \otimes_{S_{K_0}} \mathcal{D}. \tag{1.5.5}$$

See [Br1]§2 for the definition of $\widehat{B^+_{\mathrm{st}}}$.

Lemma 1.5.6. *Let r be an integer such that $0 \leq r \leq p-2$ and $\mathcal{M}_n$ $(n \geq 1)$ are of level within $[0,r]$. Then the above isomorphism induces an isomorphism:*

$$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} (\varprojlim_n (Fil^r(\widehat{A_{\mathrm{st}}} \otimes_S \mathcal{M}_n)^{N=0})) \cong Fil^r(\widehat{B^+_{\mathrm{st}}} \otimes_{S_{K_0}} \mathcal{D})^{N=0}.$$

Here $Fil^r(\widehat{A_{\mathrm{st}}} \otimes_S \mathcal{M}_n)$ is the $\widehat{A_{\mathrm{st}}}$-submodule of $\widehat{A_{\mathrm{st}}} \otimes_S \mathcal{M}_n$ defined after (1.4.2), *and $Fil^r(\widehat{B^+_{\mathrm{st}}} \otimes_{S_{K_0}} \mathcal{D})$ is the sum of the images of $Fil^{r-j}\widehat{B^+_{\mathrm{st}}} \otimes_{S_{K_0}} Fil^j\mathcal{D}$ $(0 \leq j \leq r)$ in $\widehat{B^+_{\mathrm{st}}} \otimes_{S_{K_0}} \mathcal{D}$.*

This is stated in [Br3] Lemme 4.3.2.2 in the special case that $(\mathcal{M}_n)$ comes from crystalline cohomology with only an outline of a proof, and his proof seems to work for a general $(\mathcal{M}_n)$. To make our argument certain, I will give a proof, which seems a bit different from his.

Proof. We use the terminology and the notation in [Br3]3.2.1 freely. We define another filtration $\overline{Fil}^i\mathcal{M}$ $(0 \leq i \leq r)$ of $\mathcal{M}$ by $\overline{Fil}^i\mathcal{M} := Fil^i\mathcal{D} \cap \mathcal{M}$. We see easily that this filtration satisfies the three conditions of [Br3] Définition 3.2.1.1, and $\overline{Fil}^i\mathcal{M} \supset Fil^i\mathcal{M}$. Note that $\mathcal{M}/Fil^r\mathcal{M}$ is p-torsion free. Define another filtration $\overline{Fil}^i\mathcal{M}_n$ $(0 \leq i \leq r)$ of $\mathcal{M}_n$ to be the images of $\overline{Fil}^i\mathcal{M}$. Then this filtration is admissible in the sense of [Br3] Définition 3.2.1.1. Define $\overline{Fil}^r(\widehat{A_{\rm st}} \otimes_S \mathcal{M}_n)$ using $\overline{Fil}^{\cdot}\mathcal{M}_n$ instead of $Fil^{\cdot}\mathcal{M}_n$, and $\overline{Fil}^r(\widehat{A_{\rm st}} \otimes_S \mathcal{M})$ similarly using $\overline{Fil}^{\cdot}\mathcal{M}$. Then, by [Br3] Proposition 3.2.1.4, we have

$$(\overline{Fil}^r(\widehat{A_{\rm st}} \otimes_S \mathcal{M}_n))^{N=0} = (Fil^r(\widehat{A_{\rm st}} \otimes_S \mathcal{M}_n))^{N=0}.$$

Hence it suffices to prove that (1.5.5) induces an isomorphism

$$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} (\varprojlim_n \overline{Fil}^r(\widehat{A_{\rm st}} \otimes_S \mathcal{M}_n)) \cong Fil^r(\widehat{B^+_{\rm st}} \otimes_{S_{K_0}} \mathcal{D}).$$

To prove this isomorphism, we choose a basis e_λ $(\lambda \in \Lambda)$ of $\mathcal{D}$ over S_{K_0} and integers $0 \leq r_\lambda \leq r$ such that

$$Fil^i\mathcal{D} = \oplus_\lambda Fil^{i-r_\lambda}S_{K_0}e_\lambda \quad (0 \leq i \leq r)$$

([Br1] A). Let $\mathcal{M}'$ be the free S-module generated by e_λ. By multiplying p^{-m} for some $m > 0$ if necessary, we may assume that there exists an integer $\nu \geq 0$ such that $p^\nu\mathcal{M}' \subset \mathcal{M} \subset \mathcal{M}'$. We define the filtration $\overline{Fil}^i\mathcal{M}'$ $(0 \leq i \leq r)$ and $\overline{Fil}^r(\widehat{A_{\rm st}} \otimes_S \mathcal{M}')$ in the same way as $\mathcal{M}$. Then, for $0 \leq i \leq r$, we have

$$\overline{Fil}^i\mathcal{M}' = \oplus_\lambda Fil^{i-r_\lambda}S \cdot e_\lambda$$

and hence

$$\overline{Fil}^r(\widehat{A_{\rm st}} \otimes_S \mathcal{M}') = \oplus_\lambda Fil^{r-r_\lambda}\widehat{A_{\rm st}} \cdot e_\lambda.$$

Especially $\overline{Fil}^r(\widehat{A_{\rm st}} \otimes_S \mathcal{M}')$ and $(\widehat{A_{\rm st}} \otimes_S \mathcal{M}')/(\overline{Fil}^r(\widehat{A_{\rm st}} \otimes_S \mathcal{M}'))$ are p-adically complete and separated, and p-torsion free. On the other hand, we have $p^\nu\overline{Fil}^i\mathcal{M}' \subset \overline{Fil}^i\mathcal{M} \subset \overline{Fil}^i\mathcal{M}'$ and hence

$$p^\nu\overline{Fil}^r(\widehat{A_{\rm st}} \otimes_S \mathcal{M}') \subset \overline{Fil}^r(\widehat{A_{\rm st}} \otimes_S \mathcal{M}) \subset \overline{Fil}^r(\widehat{A_{\rm st}} \otimes_S \mathcal{M}').$$

Since $\overline{Fil}^r(\widehat{A_{\rm st}} \otimes_S \mathcal{M}_n)$ is the image of $\overline{Fil}^r(\widehat{A_{\rm st}} \otimes_S \mathcal{M})$ and $(\widehat{A_{\rm st}} \otimes_S \mathcal{M})/p^n = \widehat{A_{\rm st}} \otimes_S \mathcal{M}_n$, we have the following commutative diagram:

$$\begin{array}{ccccc}
\overline{Fil}^r(\widehat{A_{\rm st}} \otimes_S \mathcal{M}')/p^n & \xrightarrow{\text{“}p^\nu\text{”}} & \overline{Fil}^r(\widehat{A_{\rm st}} \otimes_S \mathcal{M}_n) & \longrightarrow & \overline{Fil}^r(\widehat{A_{\rm st}} \otimes_S \mathcal{M}')/p^n \\
\cap & & \cap & & \cap \\
(\widehat{A_{\rm st}} \otimes_S \mathcal{M}')/p^n & \xrightarrow{\text{“}p^\nu\text{”}} & \widehat{A_{\rm st}} \otimes_S \mathcal{M}_n & \longrightarrow & (\widehat{A_{\rm st}} \otimes_S \mathcal{M}')/p^n.
\end{array}$$

By taking the projective limit with respect to n, we obtain the following commutative diagram:

$$\begin{array}{ccccc}
\overline{Fil}^r(\widehat{A_{\rm st}}\otimes_S\mathcal{M}') & \xrightarrow{p^\nu} & \varprojlim_n(\overline{Fil}^r(\widehat{A_{\rm st}}\otimes_S\mathcal{M}_n)) & \longrightarrow & \overline{Fil}^r(\widehat{A_{\rm st}}\otimes_S\mathcal{M}') \\
\cap & & \cap & & \cap \\
\widehat{A_{\rm st}}\otimes_S\mathcal{M}' & \xrightarrow{p^\nu} & \widehat{A_{\rm st}}\otimes_S\mathcal{M} & \longrightarrow & \widehat{A_{\rm st}}\otimes_S\mathcal{M}'.
\end{array}$$

By tensoring with $\mathbb{Q}_p$, we obtain the claim. $\square$

Let r be an integer as in Lemma 1.5.6. Then by taking the projective limit of the exact sequence (1.4.2) for $\mathcal{M}_n$ and r and using the isomorphism (1.5.5) and Lemma 1.5.6, we obtain an isomorphism:

$$\mathbb{Q}_p\otimes_{\mathbb{Z}_p}\varprojlim_n(T_{\rm st}(\mathcal{M}_n)(r))\cong Fil^r(\widehat{B_{\rm st}^+}\otimes_{S_{K_0}}\mathcal{D})^{N=0,\varphi=p^r},$$

where $Fil^r(\widehat{B_{\rm st}^+}\otimes_{S_{K_0}}\mathcal{D})$ is as in Lemma 1.5.6. Recall that D denotes the object of $\underline{MF}_K(\varphi,N)$ corresponding to $\mathcal{D}$. By definition, $D=\mathcal{D}^{N\text{-Nilp}}$, the canonical homomorphism $D\otimes_{K_0}S_{K_0}\to\mathcal{D}$ is an isomorphism, φ and N on D are induced from those on $\mathcal{D}$, and the filtration Fil^iD is the image of $Fil^i\mathcal{D}$ by $f_p\colon\mathcal{D}\to\mathcal{D}\otimes_{S_{K_0},f_p}K_0\cong D$. (Recall that we assume $K=K_0$.) On the other hand, $B_{\rm st}^+=\widehat{B_{\rm st}^+}^{N\text{-Nilp}}$ ([Ka3]Theorem (3.7)) and we have a canonical homomorphism $\widehat{B_{\rm st}^+}\to B_{\rm dR}^+$ ([Br1]§7, see also [Ts2]§4.6) compatible with the filtrations and $f_p\colon S_{K_0}\to K_0$ such that the composite with $B_{\rm st}\subset\widehat{B_{\rm st}^+}$ is the inclusion $B_{\rm st}^+\subset B_{\rm dR}^+$ (associated to p). Hence we have $(\widehat{B_{\rm st}^+}\otimes_{S_{K_0}}\mathcal{D})^{N\text{-Nilp}}=(\widehat{B_{\rm st}^+}\otimes_{K_0}D)^{N\text{-Nilp}}=B_{\rm st}^+\otimes_{K_0}D$ and the image of $Fil^i(\widehat{B_{\rm st}^+}\otimes_{S_{K_0}}\mathcal{D})\cap(B_{\rm st}^+\otimes_{K_0}D)$ by the homomorphism $B_{\rm st}^+\otimes_{K_0}D\to B_{\rm dR}^+\otimes_{K_0}D$ is contained in $Fil^i(B_{\rm dR}^+\otimes_{K_0}D)$. Thus we obtain an injective homomorphism

$$\mathbb{Q}_p\otimes_{\mathbb{Z}_p}(\varprojlim_n T_{\rm st}(\mathcal{M}_n)(r))\hookrightarrow(B_{\rm st}^+\otimes_{K_0}D)^{N=0,\varphi=p^r}\cap Fil^r(B_{\rm dR}^+\otimes_{K_0}D)$$

and the RHS is contained in

$$V_{\rm st}(D)(r)\cong(B_{\rm st}\otimes_{K_0}D)^{N=0,\varphi=p^r}\cap Fil^r(B_{\rm dR}\otimes_{K_0}D).$$

§1.6. Unramified quotients of semi-stable $\mathbb{Z}_p$-representations.

We assume $K=K_0$ and keep the notation of §1.4 and §1.5.

Let $(\mathcal{M}_n)_{n\geq1}$ be a projective system of objects of $\underline{\mathcal{MF}}_{W,[0,p-2],\rm tor}(\varphi,N)$ such that the underlying S-module $\mathcal{M}_n$ is a free S/p^nS-module of finite rank and the morphism $\mathcal{M}_{n+1}/p^n\mathcal{M}_{n+1}\to\mathcal{M}_n$ in $\underline{\mathcal{MF}}_{W,[0,p-2],\rm tor}(\varphi,N)$ is an isomorphism for every integer $n\geq1$. Let $\mathcal{D}$ and D be the objects of $\underline{\mathcal{MF}}_K(\varphi,N)$ and $\underline{MF}_K(\varphi,N)$ associated to $(\mathcal{M}_n)_{n\geq1}$. If we denote by $\mathcal{M}$ the projective limit of $\mathcal{M}_n$ with respect to n as S-modules, then $\mathcal{D}=\mathbb{Q}_p\otimes_{\mathbb{Z}_p}\mathcal{M}$, $D=\mathcal{D}^{N\text{-Nilp}}$ and

$D \otimes_{K_0} S_{K_0} \cong \mathcal{D}$. Set $T_n := T_{\mathrm{st}}(\mathcal{M}_n)$, $T := \varprojlim_n T_n$, and $V := V_{\mathrm{st}}(D)$. Recall that D is admissible. We have a canonical isomorphism $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} T \cong V$ (1.5.4) and we will regard T as a lattice of V by this isomorphism in the following. Let r be an integer such that $0 \leq r \leq p-2$ and let V' be a quotient of the representation V such that $V'(r)$ is unramified. Since $V'(r)$ is unramified and hence crystalline, V' is also crystalline, and if we denote by D' the corresponding quotient of D in the category $\underline{MF}_K(\varphi, N)$, then $Fil^r D' = D'$, $Fil^{r+1} D' = 0$, and the slope of the Frobenius φ is r. Furthermore, we have G_K-equivariant isomorphisms

(1.6.1)

$$V' \cong V_{\mathrm{crys}}(D') \cong V_{\mathrm{crys}}(D'(r))(-r) \cong V_{\mathrm{ur}}(D'(r))(-r) = ((P_0 \otimes_{K_0} D')^{\varphi=p^r})(-r).$$

Let T' be the image of T in V' and let M' be the image of $\mathcal{M}$ under the composite

$$\mathcal{D} \to \mathcal{D} \otimes_{S_{K_0}, f_0} K_0 \cong D \to D'.$$

Theorem 1.6.2. *Let the notation and the assumption be as above. Then the composite of the isomorphisms* (1.6.1) *induces an isomorphism*

$$T' \cong ((W(\overline{k}) \otimes_W M')^{\varphi \otimes \varphi = p^r})(-r).$$

Proof. Set $T'_n := T'/p^n T'$. Since $T'_n(r)$ is unramified, T'_n is crystalline. If we denote by $\mathcal{M}'_n$ the corresponding quotient of $\mathcal{M}_n$, the projective system $(\mathcal{M}'_n)_{n \geq 0}$ comes from the projective system $(M'_n)_{n \geq 1}$ in $\underline{MF}_{W,[0,p-2],\mathrm{tor}}(\varphi)$ such that, for every integer $n \geq 1$, $Fil^r M'_n = M'_n$ and $Fil^{r+1} M'_n = 0$, the underlying W_n-module of M'_n is free of rank $\mathrm{rank}_{\mathbb{Z}/p^n\mathbb{Z}} T'_n = \dim_{\mathbb{Q}_p} V'$, and the morphism $M'_{n+1}/p^n M'_{n+1} \to M'_n$ in $\underline{MF}_{W,[0,p-2],\mathrm{tor}}(\varphi)$ is an isomorphism. We define $M'_n(r)$ to be the object of $\underline{\mathrm{M}}_{W,\mathrm{tor}}(\varphi) \subset \underline{MF}_{W,[0,p-2],\mathrm{tor}}(\varphi)$ whose underlying W_n-module is the same as M'_n and whose Frobenius φ is φ_r. Then, we have isomorphisms:

$$\begin{aligned} T'_n &\cong T_{\mathrm{st}}(\mathcal{M}'_n) \cong T_{\mathrm{crys}}(M'_n) \cong T_{\mathrm{crys}}(M'_n(r))(-r) \\ &\cong T_{\mathrm{ur}}(M'_n(r))(-r) = ((W(\overline{k}) \otimes_W M'_n)^{\varphi \otimes \varphi_r = 1})(-r). \end{aligned}$$

The third isomorphism follows from the exact sequence (1.4.1). By taking the projective limit, we obtain an isomorphism

$$T' \cong (W(\overline{k}) \otimes_W (\varprojlim_n M'_n))^{\varphi \otimes \varphi_r = 1})(-r).$$

Let $\mathcal{M}'$ be the projective limit of $\mathcal{M}'_n$ as S-modules. By taking the projective limit of the surjective homomorphisms $\mathcal{M}_n \to \mathcal{M}'_n$ of $S/p^n S$-modules, we obtain a surjective homomorphisms $\mathcal{M} \to \mathcal{M}'$ of S-modules such that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{M} & \longrightarrow & \mathcal{M}' \\ \cap\downarrow & & \cap\downarrow \\ \mathcal{D} & \longrightarrow & \mathcal{D}', \end{array}$$

where $\mathcal{D}'$ denotes the quotient of $\mathcal{D}$ corresponding to the quotient D' of D. Note that $\mathcal{D}'$ and D' are the objects of $\underline{\mathcal{MF}}_{K_0}(\varphi, N)$ and $\underline{MF}_{K_0}(\varphi, N)$ associated to $(\mathcal{M}'_n)_n$. On the other hand, since $\mathcal{M}'_n \otimes_{S,f_0} W \cong M'_n$, we have $\mathcal{M}' \otimes_{S,f_0} W \cong \varprojlim_n M'_n$. Hence there exists an injective homomorphism $\varprojlim_n M'_n \to D'$ which makes the following diagram commutative:

$$\begin{array}{ccccc} \mathcal{M}' & \longrightarrow & \mathcal{M}' \otimes_{S_{K_0}, f_0} W & \cong & \varprojlim_n M'_n \\ \downarrow & & & & \downarrow \\ \mathcal{D}' & \longrightarrow & \mathcal{D}' \otimes_{S_{K_0}, f_0} K_0 & \cong & D'. \end{array}$$

By the definition of M', the image of $\varprojlim_n M'_n$ in D' is M', and φ_r on $\varprojlim_n M'_n$ is induced by $p^{-r}\varphi$ on D'.

Thus we obtain an isomorphism:

$$T' \cong ((W(\overline{k}) \otimes_W M')^{\varphi \otimes \varphi = p^r})(-r).$$

Now it remains to prove that this is compatible with (1.6.1), which is straightforward. $\square$

§2. The maximal slope of log crystalline cohomology.

Set $s := \operatorname{Spec}(k)$ and $\overline{s} := \operatorname{Spec}(\overline{k})$. Let L be any fine log structure on s, and let $\overline{L}$ denote its inverse image on $\overline{s}$. We consider a fine log scheme (Y, M_Y) smooth and of Cartier type over (s, L) such that Y is proper over s. Let $H^q((Y, M_Y)/(W, W(L)))$ be the crystalline cohomology $\varprojlim_n H^q_{\mathrm{crys}}((Y, M_Y)/(W_n, W_n(L)))$ defined by Hyodo and Kato in [H-Ka] (3.2), which is a finitely generated W-module endowed with a σ-semi-linear endomorphism φ called the Frobenius. The Frobenius φ becomes bijective after $\otimes_W K_0$. In this section, we will construct a canonical decomposition $M_1 \oplus M_2$ of $H^q((Y, M_Y)/(W, W(L)))$ stable under φ such that $K_0 \otimes_W M_1$ (resp. $K_0 \otimes_W M_2$) is the direct factor of slope q (resp. slopes$< q$), and a canonical isomorphism

$$W(\overline{k}) \otimes_W M_1 \cong W(\overline{k}) \otimes_{\mathbb{Z}_p} H^0_{\text{ét}}(\overline{Y}, W\omega^q_{\overline{Y}/\overline{s},\log}).$$

We will also construct a canonical decomposition $M'_1 \oplus M'_2$ stable under φ such that $K_0 \otimes_W M'_1$ (resp. $K_0 \otimes_W M'_2$) is the direct factor of slope d (resp. slopes$< d$), and a canonical isomorphism

$$W(\overline{k}) \otimes_W M'_1 \cong W(\overline{k}) \otimes_{\mathbb{Z}_p} H^{q-d}_{\text{ét}}(\overline{Y}, W\omega^d_{\overline{Y}/\overline{s},\log}).$$

Here $(\overline{Y}, M_{\overline{Y}}) := (Y, M_Y) \times_{(s,L)} (\overline{s}, \overline{L})$. See §2.2 for the definition of the RHS's.

§2.1. De Rham-Witt complex.

First, we will review the de Rham-Witt complex (with log poles) $W_\bullet\omega^\bullet_{Y/s}$ associated to $(Y, M_Y)/(s, L)$. We don't assume that Y is proper over s in §2.1. Noting the crystalline description of the de Rham-Witt complex given in [I-R] III (1.5) for a usual smooth scheme over s, we define the sheaf of W_n-modules $W_n\omega^i_{Y/s}$ on $Y_{\text{ét}}$ to be

$$\sigma^n_* R^i u_{(Y,M_Y)/(W_n,W_n(L))*}\mathcal{O}_{(Y,M_Y)/(W_n,W_n(L))}$$

for integers $i \geq 0$ and $n > 0$ ([H-Ka] (4.1)). Here $u_{(Y,M_Y)/(W_n,W_n(L))}$ denotes the canonical morphism of topoi: $((Y,M_Y)/(W_n,W_n(L)))^{\sim}_{\text{crys}} \to Y^{\sim}_{\text{ét}}$. These sheaves are endowed with the canonical projections $\pi\colon W_{n+1}\omega^i_{Y/s} \to W_n\omega^i_{Y/s}$ ([H-Ka] (4.2)) and the differentials $d\colon W_n\omega^i_{Y/s} \to W_n\omega^{i+1}_{Y/s}$ ([H-Ka] (4.1)) such that $\pi d = d\pi$ and $((W_n\omega^{\bullet}_{Y/s}, d)_{n\geq 1}, \pi)_n$ becomes a projective system of graded differential algebras over W. The projections π are surjective ([H-Ka] Theorem (4.4)). Furthermore, we have the operators $F\colon W_{n+1}\omega^i_{Y/s} \to W_n\omega^i_{Y/s}$ and $V\colon W_n\omega^i_{Y/s} \to W_{n+1}\omega^i_{Y/s}$ ([H-Ka] (4.1)) compatible with the projections π. As in the classical case (cf. [I1] I Introduction), one checks easily that we have
(1) $FV = VF = p$, $FdV = d$.
(2) $Fx.Fy = F(xy)$ $(x \in W_n\omega^i_{Y/s}, y \in W_n\omega^j_{Y/s})$,
$xVy = V(Fx.y)$ $(x \in W_{n+1}\omega^i_{Y/s}, y \in W_n\omega^j_{Y/s})$.
(3) $V(xdy) = Vx.dVy$ $(x \in W_n\omega^i_{Y/s}, y \in W_n\omega^j_{Y/s})$.
(The property (3) follows from (1) and (2): $V(xdy) = V(x.FdVy) = Vx.dVy$.)
We have W-algebra isomorphisms $\tau\colon W_n(\mathcal{O}_Y) \xrightarrow{\sim} W_n\omega^0_{Y/s}$ (see [H-Ka] (4.9) for the construction of τ and the proof of the isomorphism) compatible with π, F and V, where the projections π and the operators F and V on $W_n(\mathcal{O}_Y)$ are defined in the usual manner. With the notation in [H-Ka] (4.9), $C^{\bullet}_{Y/W_n}$ is a complex of quasi-coherent $W_n(\mathcal{O}_Y)$-modules and hence $W_n\omega^i_{Y/s} = \mathcal{H}^i(C^{\bullet}_{Y/W_n})$ are quasi-coherent $W_n(\mathcal{O}_Y)$-modules.
In the special case $n = 1$, we have an isomorphism $W_1\omega^i_{Y/s} \cong \mathcal{H}^i(\omega^{\bullet}_{Y/s}) \underset{C^{-1}}{\xleftarrow{\sim}} \omega^i_{Y/s}$ ([Ka2] (6.4), (4.12)), which is $\mathcal{O}_Y$-linear and compatible with the differentials. Recall that we regard $\mathcal{H}^i(\omega^{\bullet}_{Y/s})$ as an $\mathcal{O}_Y$-module by the action via $\mathcal{O}_Y \to \mathcal{O}_Y; x \mapsto x^p$. We will identify $(W_1\omega^{\bullet}_{Y/s}, d)$ with $(\omega^{\bullet}_{Y/s}, d)$ in the following by this isomorphism.
With the notation of [H-Ka] Definition (4.3) and Theorem (4.4) $B_{n+1}\omega^i_{Y/s} \oplus Z_n\omega^{i-1}_{Y/s}$ (resp. $B_1\omega^i_{Y/s}$) are coherent subsheaves of the coherent $\mathcal{O}_Y$-module $F^{n+1}_*(\omega^i_{Y/s} \oplus \omega^{i-1}_{Y/s})$ (resp. $F_*\omega^i_{Y/s}$), the homomorphism $(C^n, dC^n)\colon B_{n+1}\omega^i_{Y/s} \oplus Z_n\omega^{i-1}_{Y/s} \to B_1\omega^i_{Y/s}$ is $\mathcal{O}_Y$-linear and the isomorphism ([H-Ka] Theorem (4.4)):
(2.1.1)
$$(V^n, dV^n)\colon F^{n+1}_*(\omega^i_{Y/s} \oplus \omega^{i-1}_{Y/s})/\mathrm{Ker}(C^n, dC^n) \xrightarrow{\sim} \mathrm{Ker}(\pi\colon W_{n+1}\omega^i_{Y/s} \to W_n\omega^i_{Y/s})$$

is $\mathcal{O}_Y$-linear, where we regard the right-hand side as an $\mathcal{O}_Y$-module via $F\colon \mathcal{O}_Y = W_{n+1}\mathcal{O}_Y/VW_{n+1}\mathcal{O}_Y \to W_{n+1}\mathcal{O}_Y/pW_{n+1}\mathcal{O}_Y$. Especially this implies, by induction on n, that $W_n\omega^i_{Y/s}$ is a coherent $W_n(\mathcal{O}_Y)$-module.

We set $W\omega^i_{Y/s} := \varprojlim_n W_n\omega^i_{Y/s}$ and denote also by d, F and V the projective limits of d, F and V for $W_{\bullet}\omega^{\bullet}_{Y/s}$. Then, by the same argument as in the proof of [I1] II Proposition 2.1 (a) using the above coherence, we see that, if Y is proper over k, then $H^j(Y, W_n\omega^i_{Y/s})$ and $H^j(Y, W_n\omega^{\bullet}_{Y/s})$ are finitely generated

W_n-modules and the canonical homomorphisms

$$H^j(Y, W\omega^i_{Y/s}) \to \varprojlim_n H^j(Y, W_n\omega^i_{Y/s}) \tag{2.1.2}$$

$$H^j(Y, W\omega^\bullet_{Y/s}) \to \varprojlim_n H^j(Y, W_n\omega^\bullet_{Y/s}) \tag{2.1.3}$$

are isomorphisms.

Theorem 2.1.4. ([H-Ka] Theorem (4.19), cf. [I1] II Théorème 1.4). *There exists a canonical isomorphism in $D^+(Y_{\text{ét}}, W_n)$:*

$$Ru_{(Y,M_Y)/(W_n,W_n(L))*}\mathcal{O}_{(Y,M_Y)/(W_n,W_n(L))} \cong W_n\omega^\bullet_{Y/s}$$

functorial on (Y, M_Y) and compatible with the products, the Frobenius and the transition maps. Here the Frobenius on the RHS is defined by p^iF in degree i.

Recall that $p\colon W_n\omega^i_{Y/s} \to W_n\omega^i_{Y/s}$ factors through the canonical projection $\pi\colon W_n\omega^i_{Y/s} \to W_{n-1}\omega^i_{Y/s}$ and that the induced homomorphism $\underline{p}\colon W_{n-1}\omega^i_{Y/s} \to W_n\omega^i_{Y/s}$ is injective ([H-Ka] Corollary (4.5) (1), cf. [I1] I Proposition 3.4). Hence $p^iF\colon W_n\omega^i_{Y/s} \to W_n\omega^i_{Y/s}$ is well-defined for $i \geq 1$. For $i = 0$, it is defined by the usual F on $W_n\mathcal{O}_Y$.

Remark. Strictly speaking, the compatibility with the products is not mentioned in [H-Ka] Theorem (4.19), but one can verify it simply by looking at the construction of the map carefully as follows: We use the notation of the proof in [H-Ka]. We have a PD-homomorphism $\mathcal{O}_{D^\cdot_n} \to W_n(\mathcal{O}_{Y^\cdot})$ and $\omega^1_{Z^\cdot_n/(W_n,W_n(L))}|_{Y^\cdot} \to \omega^1_{W_n(Y^\cdot)/(W_n,W_n(L))}$, which induce a morphism of graded algebras $C_n = \mathcal{O}_{D^\cdot_n} \otimes \omega^\bullet_{Z^\cdot_n/(W_n,W_n(L))} \to \omega^\bullet_{W_n(Y^\cdot)/(W_n,W_n(L))}$. By taking the quotient $\omega^\bullet_{W_n(Y^\cdot)/(W_n,W_n(L),[\,])}$ of the target, we obtain a morphism $C_n \to \omega^\bullet_{W_n(Y^\cdot)/(W_n,W_n(L),[\,])}$ of differential graded algebras. The homomorphism $W_n(\mathcal{O}_{Y^\cdot}) \to W_n\omega^0_{Y^\cdot/s}$ is extended uniquely to a $W_n(\mathcal{O}_{Y^\cdot})$-linear morphism of differential graded algebras: $\omega^\bullet_{W_n(Y^\cdot)/(W_n,W_n(L))} \to W_n\omega^\bullet_{Y^\cdot}$ compatible with $d\log$'s from $W_n(M^\cdot) = M \oplus \mathrm{Ker}(W_n(\mathcal{O}_{Y^\cdot})^* \to \mathcal{O}^*_{Y^\cdot})$. It factors through the quotient $\omega^\bullet_{W_n(Y^\cdot)/(W_n,W_n(L),[\,])}$. Thus we see that the morphism $C_n \to W_n\omega^\bullet_{Y^\cdot/s}$ constructed in the proof of [H-Ka] Theorem (4.19) is a morphism of differential graded algebras.

From the definition, we immediately obtain the following exact sequences (cf. [I1] Remarques 3.21.1):

$$\begin{aligned}
&W_{2n+k}\omega^i_{Y/s} \xrightarrow{F^{n+k}} W_n\omega^i_{Y/s} \xrightarrow{dV^k} W_{n+k}\omega^{i+1}_{Y/s},\\
&W_k\omega^{i-1}_{Y/s} \xrightarrow{F^k dV^n} W_n\omega^i_{Y/s} \xrightarrow{V^k} W_{n+k}\omega^i_{Y/s},\\
&W_{2n+k}\omega^i_{Y/s} \xrightarrow{F^n} W_{n+k}\omega^i_{Y/s} \xrightarrow{F^k d} W_n\omega^{i+1}_{Y/s},\\
&W_k\omega^i_{Y/s} \xrightarrow{V^n} W_{n+k}\omega^i_{Y/s} \xrightarrow{F^k} W_n\omega^i_{Y/s}.
\end{aligned} \tag{2.1.5}$$

For example, the first exact sequence is obtained by considering the following morphism of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^\bullet_{n+k} & \xrightarrow[\underline{p}^{n+k}]{} & C^\bullet_{2n+2k} & \longrightarrow & C^\bullet_{n+k} & \longrightarrow & 0 \\ & & \| & & \uparrow \underline{p}^k & & \uparrow \underline{p}^k & & \\ 0 & \longrightarrow & C^\bullet_{n+k} & \xrightarrow[\underline{p}^{n}]{} & C^\bullet_{2n+k} & \longrightarrow & C^\bullet_{n} & \longrightarrow & 0, \end{array}$$

where $C^\bullet_n$ is the same complex as in [H-Ka] (4.1).

Finally we review a generalization of the Cartier isomorphism to the de Rham-Witt complex (cf. [I-R] III Proposition (1.4)) and the operator V' (cf. [I-R] III (1.3.2)), which will be used in the proof of our main result in §2.

By the exact sequences (2.1.5), $F^n\colon W_{2n}\omega^i_{Y/s} \to W_n\omega^i_{Y/s}$ induces an isomorphism $F^n\colon W_{2n}\omega^i_{Y/s}/V^nW_n\omega^i_{Y/s} \xrightarrow{\sim} ZW_n\omega^i_{Y/s}$. We define the operator V' on $ZW_\bullet\omega^i_{Y/s}$ by the following commutative diagram:

$$\text{(2.1.6)} \qquad \begin{array}{ccc} W_{2n+2}\omega^i_{Y/s}/V^{n+1}W_{n+1}\omega^i_{Y/s} & \xrightarrow{\pi^2} & W_{2n}\omega^i_{Y/s}/V^nW_n\omega^i_{Y/s} \\ \wr\downarrow F^{n+1} & & \wr\downarrow F^n \\ ZW_{n+1}\omega^i_{Y/s} & \xrightarrow{V'} & ZW_n\omega^i_{Y/s}. \end{array}$$

We see easily that this operator satisfies the relations:

$$\text{(2.1.7)} \qquad V'\pi = \pi V',\ FV' = V'F = \pi^2,\ V'd = dV\pi^2.$$

The last relation implies $V'(BW_{n+1}\omega^i_{Y/s}) \subset BW_n\omega^i_{Y/s}$ and hence V' induces a morphism $\mathcal{H}^i(W_{n+1}\omega^\bullet_{Y/s}) \to \mathcal{H}^i(W_n\omega^\bullet_{Y/s})$, which we will also denote by V'.

Using the property $F^ndV^n = d$, we also see that F^n induces an isomorphism $W_{2n}\omega^i_{Y/s}/(V^nW_n\omega^i_{Y/s}+dV^nW_n\omega^{i-1}_{Y/s}) \xrightarrow{\sim} \mathcal{H}^i(W_n\omega^\bullet_{Y/s})$. On the other hand, from [H-Ka] Theorem (4.4), we can easily derive

$$\text{(2.1.8)} \qquad \mathrm{Ker}(\pi^m\colon W_{n+m}\omega^i_{Y/s} \to W_n\omega^i_{Y/s}) = V^nW_m\omega^i_{Y/s} + dV^nW_m\omega^{i-1}_{Y/s}$$

by induction on m (cf. [I1] I Proposition 3.2). Hence F^n induces an isomorphism:

$$\text{(2.1.9)} \qquad C^{-n}\colon W_n\omega^i_{Y/s} \xrightarrow{\sim} \mathcal{H}^i(W_n\omega^\bullet_{Y/s}),$$

which we can regard as a generalization of the Cartier isomorphism. Indeed, if $n = 1$, this coincides with the Cartier isomorphism by the identification $W_1\omega^i_{Y/s} = \omega^i_{Y/s}$. (To prove this coincidence, we need $d\log$ which will be explained in the next subsection.)

We need the following lemma in the proof of Lemma 3.4.4.

Lemma 2.1.10. *The composite of the following homomorphisms is the identity.*

$$W_n\omega^r_{Y/s} \xrightarrow[C^{-n}]{\sim} \mathcal{H}^r(W_n\omega^\bullet_{Y/s}) \overset{2.1.4}{\cong} R^r u_{(Y,M_Y)/(W_n,W_n(L))*}\mathcal{O} = W_n\omega^r_{Y/s}.$$

Proof. Denote by α the homomorphism in question. Then α is compatible with the products. Since $W_n\omega^r_{Y/s}$ $(r \geq 2)$ is generated by $x_1.x_2\dots x_r$ $(x_i \in W_n\omega^1_{Y/s})$ as a sheaf of modules ([H-Ka] Proposition (4.6)), it suffices to prove the lemma in the case $r = 0,\ 1$. We use the notation in the proof of [H-Ka] Theorem (4.19). The lemma for $r = 0$ follows from the fact that the composite of $W_n\mathcal{O}_{Y^\bullet} \xrightarrow{\tau} \mathcal{O}_{D^\bullet} \to W_n\mathcal{O}_{Y^\bullet}$ coincides with F^n, where $\tau(x_0,\dots,x_{n-1}) = \sum_{i=0}^{n-1} p^i\tilde{x}_i^{p^{n-i}}$. By [H-Ka] Proposition (4.6), $W_n\omega^1_{Y/s}$ is generated by dx $(x \in W_n\mathcal{O}_Y)$ and $d\log(a)$ $(a \in M_Y^{\mathrm{gp}})$ as a $W_n\mathcal{O}_Y$-module. Using the quasi-isomorphism $W_{2n}\omega^\bullet_{Y/s}/\underline{p}^n W_n\omega^\bullet_{Y/s} \to W_n\omega^\bullet_{Y/s}$ ([H-Ka] Corollary (4.5)), we see that α commutes with the differentials. Hence $\alpha(dx) = d(\alpha(x)) = dx$. For $a \in M_Y^{\mathrm{gp}}$, we have $C^{-n}(d\log(a)) =$ the class of $d\log(a)$. On the other hand, by the construction of the quasi-isomorphism of [H-Ka] Theorem (4.19), the following diagram is commutative:

$$\begin{array}{ccc} M^{\mathrm{gp}}_{D^\bullet} & \longrightarrow & W_n(M_{Y^\bullet})^{\mathrm{gp}} \\ {\scriptstyle d\log}\downarrow & & {\scriptstyle d\log}\downarrow \\ C^1_n & \longrightarrow & W_n\omega^1_{Y^\bullet/s}. \end{array}$$

Choose a lifting $\tilde{a} \in M^{\mathrm{gp}}_{D^\bullet}$ of a and let au, $u \in \mathrm{Ker}(W_n\mathcal{O}^*_{Y^\bullet} \to \mathcal{O}^*_{Y^\bullet})$ be its image in $W_n(M^\bullet)^{\mathrm{gp}}$. Then the image of $d\log(\tilde{a}) \in C^1_n$ in $W_n\omega^1_{Y^\bullet/s}$ is $d\log(a) + d(\log(u))$, which is congruent to $d\log(a)$ modulo $dW_n(\mathcal{O}_{Y^\bullet})$. □

§2.2. Logarithmic Hodge-Witt sheaves.

We will review the logarithmic Hodge-Witt sheaves $W_\bullet\omega^\bullet_{Y/s,\,\log}$ associated to $(Y, M_Y)/(s, L)$ (See [I1] for usual smooth schemes and [H], [L] for log smooth schemes). We still don't assume that Y is proper over s.

As in [H-Ka] (4.9), we have natural homomorphisms

$$d\log\colon M_Y^{\mathrm{gp}} \to W_n\omega^1_Y \quad (n \geq 1),$$

which satisfy $d\log(g^{-1}(L^{\mathrm{gp}})) = 0$, $[b]d\log(a) = d([b])$ for $a \in M_Y$, its image b in $\mathcal{O}_Y$ and $[b] = (b, 0, 0, \cdots) \in W_n(\mathcal{O}_Y)$, $\pi d\log = d\log$, $Fd\log = d\log$, and $d(d\log(M_Y^{\mathrm{gp}})) = 0$. For $n = 1$, $d\log$ coincides with the usual $d\log\colon M_Y^{\mathrm{gp}} \to \omega^1_{Y/s}$. We define the *logarithmic Hodge-Witt sheaves* $W_n\omega^i_{Y/s,\,\log}$ to be the subsheaves of abelian groups of $W_n\omega^i_{Y/s}$ generated by local sections of the form $d\log(a_1) \wedge \cdots \wedge d\log(a_i)$ $(a_1, \dots, a_i \in M_Y^{\mathrm{gp}})$.

THEOREM 2.2.1. ([I1] 0 Théorème 2.4.2, [Ts1] Theorem (6.1.1)). *The following sequence is exact for any integer $i \geq 0$:*

$$0 \longrightarrow \omega^i_{Y/s,\,\log} \longrightarrow Z\omega^i_{Y/s} \xrightarrow{1-C} \omega^i_{Y/s} \longrightarrow 0.$$

Now, by the same argument as the proof of [I1] I (3.26) and [I1] I (5.7.2) plus some additional calculation, we can derive the following theorem from Theorem 2.2.1, the exact sequences (2.1.5) and the isomorphism (2.1.1) (cf. [L] 1.5.2).

THEOREM 2.2.2. (cf. [I1] I (3.26), (5.7.2)). *For any integers $n \geq 1$ and $i \geq 0$, the following sequence is exact:*

$$0 \longrightarrow W_n\omega^i_{Y/s,\,\log} + V^{n-1}W_1\omega^i_{Y/s} \longrightarrow W_n\omega^i_{Y/s} \xrightarrow{\pi-F} W_{n-1}\omega^i_{Y/s} \longrightarrow 0.$$

Note that we easily obtain

$$\begin{aligned} &\mathrm{Ker}(V^n\colon \omega^i_{Y/s} \to W_{n+1}\omega^i_{Y/s}) = B_n\omega^i_{Y/s}, \\ &\mathrm{Ker}(V^n\colon \omega^i_{Y/s} \to W_{n+1}\omega^i_{Y/s}/dV^n\omega^{i-1}_{Y/s}) = B_{n+1}\omega^i_{Y/s} \end{aligned} \tag{2.2.3}$$

(cf. [I1] I (3.8)) from the isomorphism (2.1.1) ([L] Proposition 1.2.7).

COROLLARY 2.2.4. (cf. [H] (2.6), [L] (1.5.4)). *The following sequence is exact for any integers $i \geq 0$, $n, m \geq 1$:*

$$0 \longrightarrow W_n\omega^i_{Y/s,\,\log} \xrightarrow{p^m} W_{n+m}\omega^i_{Y/s,\,\log} \longrightarrow W_m\omega^i_{Y/s,\,\log} \longrightarrow 0.$$

COROLLARY 2.2.5. (cf. [I-R] IV §3). *The homomorphism $W_n\omega^i_{Y/s,\,\log} \to \mathcal{H}^i(W_n\omega^\bullet_{Y/s})$ is injective and the following sequence is exact:*

$$0 \longrightarrow K^i_n \longrightarrow \mathcal{H}^i(W_n\omega^\bullet_{Y/s}) \xrightarrow{V'-\pi} \mathcal{H}^i(W_{n-1}\omega^\bullet_{Y/s}) \longrightarrow 0,$$

where K^i_n denotes the image of $W_n\omega^i_{Y/s,\,\log} + p^{n-1}F(W_{n+1}\omega^i_{Y/s})$ in $\mathcal{H}^i(W_n\omega^\bullet_{Y/s})$.

Proof. This immediately follows from Theorem 2.2.2 using the following commutative diagram:

$$\begin{array}{ccc} W_n\omega^i_{Y/s} & \xrightarrow{\pi-F} & W_{n-1}\omega^i_{Y/s} \\ \wr\big\downarrow C^{-n} & & \wr\big\downarrow C^{-(n-1)} \\ \mathcal{H}^i(W_n\omega^\bullet_{Y/s}) & \xrightarrow{V'-\pi} & \mathcal{H}^i(W_{n-1}\omega^\bullet_{Y/s}). \end{array}$$

□

Corollary 2.2.6. (cf. [I-R] IV §3). *The homomorphism* $V'-\pi\colon ZW_n\omega^i_{Y/s}\to ZW_{n-1}\omega^i_{Y/s}$ *is surjective and, if we denote its kernel by* L^i_n, *then* $W_n\omega^i_{Y/s,\,\log}\subset L^i_n$ *and* $L^i_n/W_n\omega^i_{Y/s,\,\log}$ *is killed by* π^3.

Proof. The surjectivity follows from Theorem 2.2.2 using the commutative diagram:

$$\begin{array}{ccc} W_{2n}\omega^i_{Y/s} & \xrightarrow{\pi^2-\pi F} & W_{2n-2}\omega^i_{Y/s} \\ {\scriptstyle F^n}\big\downarrow & & \big\downarrow{\scriptstyle F^{n-1}} \\ ZW_n\omega^i_{Y/s} & \xrightarrow{V'-\pi} & ZW_{n-1}\omega^i_{Y/s}. \end{array}$$

The assertion on the kernel follows from Theorem 2.2.2 and Lemma 2.2.7 below by considering the commutative diagram:

$$\begin{array}{ccc} ZW_{n+1}\omega^i_{Y/s} & \xrightarrow{\pi-F} & ZW_n\omega^i_{Y/s} \\ \big\| & & \big\downarrow{\scriptstyle V'} \\ ZW_{n+1}\omega^i_{Y/s} & \xrightarrow{\pi V'-\pi^2} & ZW_{n-1}\omega^i_{Y/s}. \end{array}$$

□

Lemma 2.2.7. *The homomorphism* $V'\colon ZW_{n+1}\omega^i_{Y/s}\to ZW_n\omega^i_{Y/s}$ *is surjective and its kernel is killed by* π^2.

Proof. In the diagram (2.1.6), the upper horizontal map is a surjection with its kernel $(V^nW_{n+2}\omega^i_{Y/s}+dV^{2n}W_2\omega^{i-1}_{Y/s})/V^{n+1}W_{n+1}\omega^i_{Y/s}$ (2.1.8). Hence V' is surjective and its kernel is $p^nFW_{n+2}\omega^i_{Y/s}+dV^{n-1}W_2\omega^{i-1}_{Y/s}$. □

§2.3. The maximal slope.

In §2.3, we always assume that Y is proper over s. For $i\in\mathbb{N}$, we define the projective system of morphisms of complexes (cf. [I-R] III (1.7)):

$$V'_{\le i}\colon \{\tau_{\le i}W_n\omega^\bullet_{Y/s}\to\tau_{\le i}W_{n-1}\omega^\bullet_{Y/s}\}_{n\ge1}$$

by the morphism $p^{i-j-1}\pi^2V$ in degree $j\le i-1$ and $V'\colon ZW_n\omega^i_{Y/s}\to ZW_{n-1}\omega^i_{Y/s}$ in degree i. If $j\le i$ or $i=d:=\dim Y$, then the natural homomorphism $H^j(Y,\tau_{\le i}W_n\omega^\bullet_{Y/s})\to H^j(Y,W_n\omega^\bullet_{Y/s})$ is an isomorphism and hence $V'_{\le i}$ induces an endomorphism on $H^j(Y,W\omega^\bullet_{Y/s})$, which we will also denote by $V'_{\le i}$. We need the following lemma, which is well-known for a σ-semilinear endomorphism and is proven in the same way as in the σ-semilinear case.

Lemma 2.3.1. *Let M be a finitely generated W-module and let V be a σ^{-1}-semilinear endomorphism of M.*

(1) *There exists a unique decomposition $M_{\rm bij}\oplus M_{\rm nil}$ of M stable under V such that V is bijective on $M_{\rm bij}$ and p-adically nilpotent on $M_{\rm nil}$.*

(2) *If k is algebraically closed, $V-1$ is surjective on M and bijective on M_{nil}. Furthermore, the natural homomorphism:*

$$W\otimes_{\mathbb{Z}_p} M^{V=1} = W\otimes_{\mathbb{Z}_p} (M_{\mathrm{bij}})^{V=1} \to M_{\mathrm{bij}}$$

is bijective.

Recall that we have canonical isomorphisms:

$$H^j(Y, W_n\omega^{\bullet}_{Y/s}) \cong H^j_{\mathrm{crys}}((Y, M_Y)/(W_n, W_n(L)))$$
$$H^j(Y, W\omega^{\bullet}_{Y}) \cong H^j_{\mathrm{crys}}((Y, M_Y)/(W, W(L)))$$

by Theorem 2.1.4 and the right hand sides are finitely generated modules over W_n and over W respectively. By applying Lemma 2.3.1 to $H^j(Y, W\omega^{\bullet}_{Y/s})$ and $V'_{\leq i}$ for $i \geq j$ or $i = d$, we obtain a decomposition

$$H^j(Y, W\omega^{\bullet}_{Y/s}) = H^j(Y, W\omega^{\bullet}_{Y/s})_{V'_{\leq i}-\mathrm{bij}} \oplus H^j(Y, W\omega^{\bullet}_{Y/s})_{V'_{\leq i}-\mathrm{nil}} \tag{2.3.2}$$

and a natural isomorphism

$$W\otimes_{\mathbb{Z}_p} H^j(Y, W\omega^{\bullet}_{Y/s})^{V'_{\leq i}=1} \xrightarrow{\sim} H^j(Y, W\omega^{\bullet}_{Y/s})_{V'_{\leq i}-\mathrm{bij}} \tag{2.3.3}$$

if k is algebraically closed.

Lemma 2.3.4. *For any integers i and j such that $i \geq j$ or $i = d$, we have*

$$K_0\otimes_W H^j(Y, W\omega^{\bullet}_{Y/s})_{V'_{\leq i}-\mathrm{bij}} = (K_0\otimes_W H^j(Y, W\omega^{\bullet}_{Y/s}))_{[i]},$$
$$K_0\otimes_W H^j(Y, W\omega^{\bullet}_{Y/s})_{V'_{\leq i}-\mathrm{nil}} = (K_0\otimes_W H^j(Y, W\omega^{\bullet}_{Y/s}))_{[0,i[}.$$

(*See* §1.1 *for the definition of D_I ($I\subset\mathbb{Q}$) for an F-isocrystal D over k.*)

Proof. Let $\mathcal{F}$ denote the morphism $\tau_{\leq i}W_n\omega^{\bullet}_{Y/s} \to \tau_{\leq i}W_{n-1}\omega^{\bullet}_{Y/s}$ whose degree q-part is p^qF, which induces the Frobenius endomorphism φ on $H^j(Y, W\omega^{\bullet}_{Y/s})$. Then $\mathcal{F}V'_{\leq i} = V'_{\leq i}\mathcal{F} = p^i\pi^2\colon \tau_{\leq i}W_n\omega^{\bullet}_{Y/s} \to \tau_{\leq i}W_{n-2}\omega^{\bullet}_{Y/s}$. Hence, we have $\varphi V'_{\leq i} = V'_{\leq i}\varphi = p^i$ on $H^j(Y, W\omega^{\bullet}_{Y/s})$, which implies the lemma. □

We set $H^j(Y, W\omega^i_{Y/s,\log}) := \varprojlim_n H^j(Y, W_n\omega^i_{Y/s,\log})$.

Proposition 2.3.5. *Assume that k is algebraically closed. Then, for any integers i and j, we have*

$$H^0(Y, W\omega^i_{Y/s,\log}) \xrightarrow{\sim} H^i(Y, W\omega^{\bullet}_{Y/s})^{V'_{\leq i}=1},$$
$$H^j(Y, W\omega^d_{Y/s,\log}) \xrightarrow{\sim} H^{j+d}(Y, W\omega^{\bullet}_{Y/s})^{V'_{\leq d}=1}.$$

Proof. First note that $V\pi$ is a nilpotent endomorphism of $W_n\omega^i_{Y/s}$. By Corollary 2.2.6, the morphism of complexes

$$V'_{\le i} - \pi\colon \tau_{\le i}W_n\omega^\bullet_{Y/s} \longrightarrow \tau_{\le i}W_{n-1}\omega^\bullet_{Y/s}$$

is surjective, and if we denote its kernel by $K^\bullet_{i,n}$, $K^j_{i,n} = \mathrm{Ker}(\pi\colon W_n\omega^j_{Y/s} \to W_{n-1}\omega^j_{Y/s})$ if $j \le i-1$, $W_n\omega^i_{Y/s,\,\log} \subset K^i_{i,n}$ and $K^i_{i,n}/W_n\omega^i_{Y/s,\,\log}$ is annihilated by π^3. Hence we have a long exact sequence

$$\cdots \to H^j(Y, K^\bullet_{n,i}) \longrightarrow H^j(Y, \tau_{\le i}W_n\omega^\bullet_{Y/s}) \xrightarrow{V'_{\le i}-\pi} H^j(Y, \tau_{\le i}W_{n-1}\omega^\bullet_{Y/s}) \longrightarrow \cdots$$

and the natural homomorphism

$$H^{j-i}(Y, W\omega^i_{Y/s,\,\log}) \longrightarrow \varprojlim_n H^j(Y, K^\bullet_{n,i})$$

is bijective. By Lemma 2.3.1 (2), if $j \le i$ or $i = d$, the endomorphism $V'_{\le i} - 1$ on $H^j(Y, W\omega^\bullet_{Y/s})$ is surjective. On the other hand, we have the following morphism of short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^j(Y/W)/p^n & \longrightarrow & H^j(Y/W_n) & \longrightarrow & H^{j+1}(Y/W)_{p^n} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \mathrm{proj}} & & \uparrow{\scriptstyle \mathrm{proj}} & & \uparrow{\scriptstyle p} & & \\
0 & \longrightarrow & H^j(Y/W)/p^{n+1} & \longrightarrow & H^j(Y/W_{n+1}) & \longrightarrow & H^{j+1}(Y/W)_{p^{n+1}} & \longrightarrow & 0,
\end{array}$$

where we abbreviate $(Y, M_Y)/(W, W(L))$ or $(W_n, W_n(L))$ to Y/W or W_n. Hence, if the torsion part of $H^{j+1}((Y, M_Y)/(W, W(L)))$ is killed by $p^{\nu_{j+1}}$, then the cokernel of the homomorphism

$$V'_{\le i} - \pi\colon H^j(Y, W_n\omega^\bullet_{Y/s}) \to H^j(Y, W_{n-1}\omega^\bullet_{Y/s})$$

is killed by $\pi^{\nu_{j+1}}$ if $j \le i$ or $i = d$. By taking $\varprojlim_n$ of the above long exact sequences, we obtain

$$\varprojlim_n H^i(Y, K^\bullet_{n,i}) \xrightarrow{\sim} H^i(Y, W\omega^\bullet_{Y/s})^{V'_{\le i}=1}$$
$$\varprojlim_n H^j(Y, K^\bullet_{n,d}) \xrightarrow{\sim} H^j(Y, W\omega^\bullet_{Y/s})^{V'_{\le d}=1}.$$

□

We also need the following lemma (in (3.4.5)):

Lemma 2.3.6. *Assume that k is algebraically closed. Then, for any integers $i \geq 0$ and $j \geq 0$, we have an isomorphism*

$$H^j(Y, W\omega^i_{Y/s,\,\log}) \xrightarrow{\sim} \varprojlim_n H^j(Y, \mathcal{H}^i(W_n\omega^\bullet_{Y/s}))^{V'=1}.$$

Proof. Set $M^j_n := H^j(Y, \mathcal{H}^i(W_n\omega^\bullet_{Y/s}))$, $M^j := \varprojlim_n M^j_n$ and let $M^{j\prime}_n$ be the image of M^j in M^j_n. By (2.1.9), M^j_n are finitely generated W_n-modules and hence $\{(V'-\pi)(M^j_{n+1})\}_{n\geq 1}$ satisfies the Mittag-Leffler condition. On the other hand, by Lemma 2.3.7 below, $(V'-\pi)(M^{j\prime}_{n+1}) = M^{j\prime}_n$, which implies

$$\varprojlim_n((V'-\pi)(M^j_{n+1})) \supset \varprojlim_n((V'-\pi)(M^{j\prime}_{n+1})) = \varprojlim_n M^{j\prime}_n = \varprojlim_n M^j_n.$$

Hence $\varprojlim_n(M^j_n/(V'-\pi)(M^j_{n+1})) = 0$. Since $\{M^{j-1}_n/(V'-\pi)(M^{j-1}_{n+1})\}_{n\geq 1}$ satisfies the Mittag-Leffler condition, Corollary 2.2.5 implies $\varprojlim H^j(Y, K^i_n) \cong (M^j)^{V'=1}$. Since $\pi(p^{n-1}FW_{n+1}\omega^i_{Y/s}) = 0$, the LHS is isomorphic to $H^j(Y, W\omega^i_{Y/s,\,\log})$. □

Lemma 2.3.7. *Let M_1 and M_2 be W-modules of finite length, let $\pi\colon M_1 \to M_2$ be a surjective W-linear homomorphism and let $V'\colon M_1 \to M_2$ be a σ^{-1}-linear homomorphism. If k is algebraically closed, then $V'-\pi\colon M_1 \to M_2$ is surjective.*

Proof. Using the short exact sequences $0 \to pM_i \to M_i \to M_i/pM_i \to 0$ $(i=1,2)$, we are easily reduced to the case $pM_i = 0$ $(i=1,2)$. In this case, π has a W-linear section $s\colon M_2 \to M_1$ and $V'\circ s - \pi\circ s = V'\circ s - 1$ is surjective by Lemma 2.3.1 (2). □

§3. The maximal unramified quotient of p-adic étale cohomology.

§3.1. Statement of the main theorem.

Let (S,N) be the scheme $\mathrm{Spec}(O_K)$ endowed with the canonical log structure (i.e. the log structure defined by its closed point). Let $f\colon (X,M) \to (S,N)$ be a smooth fs(=fine and saturated) log scheme and let $g\colon (Y,M_Y) \to (s,L)$ be the reduction of f modulo the maximal ideal of O_K. We assume that X is proper over S and that f is universally saturated, which is equivalent to saying that g is of Cartier type, or also to saying that Y is reduced ([Ts3]). Let X_{triv} denote the locus where the log structure M is trivial, which is open and contained in the generic fiber of X. Let $(\overline{s}, \overline{L})$ be the scheme $\mathrm{Spec}(\overline{k})$ endowed with the inverse image of L, and set $(\overline{Y}, M_{\overline{Y}}) := (Y, M_Y) \times_{(s,L)} (\overline{s}, \overline{L})$. Set $(X_{\mathrm{triv}})_{\overline{K}} := X_{\mathrm{triv}} \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\overline{K})$. We will describe the maximal unramified quotients

$$H^r_{\text{ét}}((X_{\mathrm{triv}})_{\overline{K}}, \mathbb{Q}_p(r))_{I_K}, \quad H^r_{\text{ét}}((X_{\mathrm{triv}})_{\overline{K}}, \mathbb{Q}_p(d))_{I_K} \quad (r \geq d)$$

of p-adic étale cohomology groups and the images of $H^r_{\text{ét}}((X_{\text{triv}})_{\overline{K}}, \mathbb{Z}_p(r'))$ ($r' = r$ or d) in them in terms of the logarithmic Hodge-Witt sheaves of $(\overline{Y}, M_{\overline{Y}})/(\overline{s}, \overline{L})$ (Theorem 3.1.11).
In the rest of §3.1, we choose and fix an integer $r \geq 0$, and assume that $(X, M)/(S, N)$ and r satisfy one of the following conditions:
(3.1.1) $\quad r \leq p-2$.
(3.1.2) Étale locally on X, there exists an étale morphism over S:

$$X \to \operatorname{Spec}((O_K[T_1, \ldots, T_u]/(T_1 \cdots T_u - \pi^e))[U_1, \ldots, U_s, V_1, \ldots, V_t])$$

for some integers $u \geq 1$, $s, t \geq 0$ and $e \geq 1$ such that $e \mid [K : K_0]$ and $X_{\text{triv}} = X - (Y \cup D)$, where D is the inverse image of $\{U_1 \cdots U_s = 0\}$.
Let i and j denote the immersions $Y \to X$ and $X_{\text{triv}} \to X$ respectively. By [Ka4]Theorem (11.6), we have

$$M = \mathcal{O}_X \cap j_*\mathcal{O}^*_{X_{\text{triv}}} \quad \text{and} \quad M^{\text{gp}} = j_*\mathcal{O}^*_{X_{\text{triv}}} \tag{3.1.3}$$

and, hence, from the Kummer sequence on $(X_{\text{triv}})_{\text{ét}}$, we obtain a symbol map:

$$(i^*M^{\text{gp}})^{\otimes r} \longrightarrow i^*R^r j_*\mathbb{Z}/p^n\mathbb{Z}(r). \tag{3.1.4}$$

Theorem 3.1.5. ([Bl-Ka]Theorem (1.4), [H](1.6.1), [Ts2], [Ts4]). *The homomorphism* (3.1.4) *is surjective.*

Proof. In the case (3.1.2) with $s = 0$, this is [H](1.6.1) (=[Bl-Ka]Theorem (1.4) in the good reduction case). The case (3.1.2) for a general s is reduced to the case $s = 0$ as in the proof of [Ts2] Lemma 3.4.7. (The proof of Lemma 3.4.7 (1) works without the assumption $\mu_{p^n} \subset K$). In the case (3.1.1), we are easily reduced to the case $n = 1$, and the theorem follows from [Ts3]Theorem 5.1 and Proposition A15 with $r = q$. □

We have a surjective homomorphism (§2.2):

$$(M_Y^{\text{gp}})^{\otimes r} \longrightarrow W_n\omega^r_{Y/s,\,\log}; a_1 \otimes \cdots \otimes a_r \mapsto d\log(a_1) \wedge \cdots \wedge d\log(a_r). \tag{3.1.6}$$

Proposition 3.1.7. (cf. [Bl-Ka]Theorem (1.4) (i), [H](1.6.2)). *There exists a unique surjective homomorphism:*

$$i^*R^r j_*\mathbb{Z}/p^n\mathbb{Z}(r) \longrightarrow W_n\omega^r_{Y/s,\,\log} \tag{3.1.8}$$

such that the following diagram commutes:

$$\begin{array}{ccc} (i^*M^{\text{gp}})^{\otimes r} & \longrightarrow & i^*R^r j_*\mathbb{Z}/p^n\mathbb{Z}(r) \\ \downarrow & & \downarrow \\ (M_Y^{\text{gp}})^{\otimes r} & \longrightarrow & W_n\omega^r_{Y/s,\,\log}. \end{array}$$

Proof. We have the required map $\nu^*(i^*R^rj_*\mathbb{Z}/p^n\mathbb{Z}(r)) \to \nu^*(W_n\omega^r_{Y/s,\,\log})$ for each generic point $\nu\colon \eta \to Y$ of Y ([Bl-Ka] (6.6)). Note that Y is reduced. Since the homomorphism $\omega^r_{Y/s} \to \oplus_\nu \nu_*\nu^*\omega^r_{Y/s}$ is injective, the homomorphism $W_n\omega^r_{Y/s,\,\log} \to \oplus_\nu\nu_*\nu^*W_n\omega^r_{Y/s,\,\log}$ is also injective by Corollary 2.2.4. Now the proposition follows from Theorem 3.1.5 (cf. [Bl-Ka] (6.6)). □

The condition (3.1.1) or (3.1.2) still holds for the base change of $(X,M)/(S,N)$ by any finite extension of K contained in $\overline{K}$. Hence, by taking the inductive limit, we obtain:

$$\overline{i}^*R^r\overline{j}_*\mathbb{Z}/p^n\mathbb{Z}(r) \longrightarrow W_n\omega^r_{\overline{Y}/\overline{s},\log} \tag{3.1.9}$$

Here $\overline{X} = X \times_{\mathrm{Spec}(O_K)} \mathrm{Spec}(O_{\overline{K}})$, and $\overline{i}$ and $\overline{j}$ denote the morphisms $\overline{Y} \to \overline{X}$ and $(X_{\mathrm{triv}})_{\overline{K}} \to \overline{X}$ respectively. Note that, for any fs log structure L' on s and a morphism $(s,L') \to (s,L)$, $W_n\omega^\bullet$ and $W_n\omega^\bullet_{\log}$ associated to $(Y,M_Y)/(s,L)$ and $(Y,M_Y)\times_{(s,L)}(s,L')/(s,L')$ coincide by the base change theorem [H-Ka]Proposition (2.23) (or [Ts2] Proposition 4.3.1).
Let $d = \dim X_K$. Then, for any affine scheme U étale over $(X_{\mathrm{triv}})_{\overline{K}}$, we have $H^i(U,\mathbb{Z}/p^n\mathbb{Z}(d)) = 0$ $(i>d)$. Hence we have $\overline{i}^*R^i\overline{j}_*\mathbb{Z}/p^n\mathbb{Z}(d) = 0$ $(i>d)$. By the proper base change theorem, we obtain from (3.1.9) homomorphisms:

$$\begin{aligned} &H^r_{\text{ét}}((X_{\mathrm{triv}})_{\overline{K}},\mathbb{Z}/p^n\mathbb{Z}(r)) \longrightarrow H^0_{\text{ét}}(\overline{Y},W_n\omega^r_{\overline{Y}/\overline{s},\log}),\\ &H^r_{\text{ét}}((X_{\mathrm{triv}})_{\overline{K}},\mathbb{Z}/p^n\mathbb{Z}(d)) \longrightarrow H^{r-d}_{\text{ét}}(\overline{Y},W_n\omega^d_{\overline{Y}/\overline{s},\log}) \end{aligned} \tag{3.1.10}$$

Theorem 3.1.11. (1) (cf. [Sat] Lemma 3.3). *Assume $K=K_0$ in the case* (3.1.1) *and $s=0$ in the case* (3.1.2)*. Then the homomorphisms* (3.1.10) *induce isomorphisms:*
(3.1.12)

$$\begin{aligned} &H^r_{\text{ét}}((X_{\mathrm{triv}})_{\overline{K}},\mathbb{Q}_p(r))_{I_K} \xrightarrow{\sim} \mathbb{Q}_p\otimes_{\mathbb{Z}_p} H^0_{\text{ét}}(\overline{Y},W\omega^r_{\overline{Y}/\overline{s}\,,\log})\\ &H^r_{\text{ét}}((X_{\mathrm{triv}})_{\overline{K}},\mathbb{Q}_p(d))_{I_K} \xrightarrow{\sim} \mathbb{Q}_p\otimes_{\mathbb{Z}_p} H^{r-d}_{\text{ét}}(\overline{Y},W\omega^d_{\overline{Y}/\overline{s}\,,\log}) \qquad \text{if } r\ge d \end{aligned}$$

(2) *In the case* (3.1.1)*, if $K=K_0$,* (3.1.10) *induce isomorphisms:*
(3.1.13)

$$\begin{aligned} &H^r_{\text{ét}}((X_{\mathrm{triv}})_{\overline{K}},\mathbb{Z}_p(r))_{I_K}/tor \xrightarrow{\sim} H^0_{\text{ét}}(\overline{Y},W\omega^r_{\overline{Y}/\overline{s},\,\log})\\ &H^r_{\text{ét}}((X_{\mathrm{triv}})_{\overline{K}},\mathbb{Z}_p(d))_{I_K}/tor \xrightarrow{\sim} H^{r-d}_{\text{ét}}(\overline{Y},W\omega^d_{\overline{Y}/\overline{s},\,\log})/tor \qquad \text{if } r\ge d \end{aligned}$$

Remark 3.1.14. For (1) in the case (3.1.2), if we use [Y] (resp. Theorem 4.1.2), the proof of Theorem 3.1.11 in 3.2–3.4 works without the assumption $s=0$ (resp. under the assumption (4.1.1)). See Remark 3.2.3.

§3.2. Review of the semi-stable conjecture.

We will review comparison theorems between p-adic étale cohomology and crystalline cohomology.

Let $f\colon (X,M)\to(S,N)$ and $g\colon (Y,M_Y)\to(s,L)$ be the same as in §3.1. Then the crystalline cohomology $D^q := K_0\otimes_W \varprojlim_n H^q((Y,M_Y)/(W_n,W_n(L))$ ([H-Ka] (3.2)) is a finite dimensional K_0-vector space endowed with a σ-semi-linear automorphism φ and a linear endomorphism N ([H-Ka] (3.4), (3.5), (3.6)) satisfying the relation $N\varphi = p\varphi N$. We choose and fix a uniformizer π of K. Then there exists a canonical isomorphism ([H-Ka] Theorem (5.1)):

$$\rho_\pi\colon K\otimes_{K_0} D^q \cong H^q(X_K, \Omega^\bullet_{X_K}(\log M_K)). \tag{3.2.1}$$

Using the Hodge filtration on the RHS, the crystalline cohomology D^q becomes an object of $\underline{MF}_K(\varphi,N)$ (§1.2). Set $V^q := H^q_{\text{ét}}((X_{\text{triv}})_{\overline{K}},\mathbb{Q}_p)$, which is a finite dimensional $\mathbb{Q}_p$-vector space endowed with a continuous and linear action of G_K.

Theorem 3.2.2. (The semi-stable conjecture by Fontaine-Jannsen:[Ka3], [Ts2]). *Assume that $(X,M)/(S,N)$ satisfies the condition* (3.1.2) *with $s=0$. Then, for any integer $q\geq 0$, V^q is a semi-stable p-adic representation and there exists a canonical isomorphism $D_{\text{st}}(V^q)\cong D^q$ in $\underline{MF}_K(\varphi,N)$. Here we define D_{st} using the same uniformizer π as* (3.2.1).

Remark 3.2.3. G. Faltings ([Fa]) proved the theorem without the assumption on $(X,M)/(S,N)$. However his construction of the comparison map is different from that in [Ka3], [Ts2] via syntomic cohomology, and we will use the latter construction in the proof of Theorem 3.1.11. Recently, G. Yamashita [Y] proved that the comparison map via syntomic cohomology is an isomorphism for any (X,M) satisfying (3.1.2). We give an alternative proof in §4 when the horizontal divisors at infinity do not have self-intersections.

To prove Theorem 3.1.11 (2), we need the following refinement by C. Breuil for the integral p-adic étale cohomology $T^q := H^q_{\text{ét}}((X_{\text{triv}})_{\overline{K}},\mathbb{Z}_p)/\text{tor}$. We assume $K=K_0$ and $q\leq p-2$. Let (E_n,M_{E_n}) be the scheme $\text{Spec}(W_n\langle u\rangle)=\text{Spec}(W_n\langle u-p\rangle)$ endowed with the log structure associated to $\mathbb{N}\to W_n\langle u\rangle; 1\mapsto u$. We have a closed immersion $i_p\colon (S_n,N_n)\hookrightarrow (E_n,M_{E_n})$ defined by $u\mapsto p$. Then the crystalline cohomology $\mathcal{M}^q_n := H^q((X_n,M_n)/(E_n,M_{E_n}))\cong H^q((X_1,M_1)/(E_n,M_{E_n}))$ is naturally regarded as an object of $\underline{\mathcal{MF}}_{W,[0,q],\text{tor}}(\varphi,N)$ ([Br3] Théorème 2.3.2.1). Set $\mathcal{M}^q := (\varprojlim_n \mathcal{M}^q_n)/\text{tor}$ and $\mathcal{M}'^q_n := \mathcal{M}^q/p^n\mathcal{M}^q$. Then $\{\mathcal{M}'^q_n\}_n$ becomes a projective system of objects of $\underline{\mathcal{MF}}_{W,[0,p-2],\text{tor}}(\varphi,N)$ satisfying the condition in the beginning of §1.6 ([Br3] 4.1).

Theorem 3.2.4. ([Br3] Théorème 3.2.4.7, §4.2, [Ts4]). *Assume $K=K_0$ and let q be any integer such that $0\leq q\leq p-2$. Then there exist canonical G_K-equivariant isomorphisms:*

$$T_{\text{st}}(\mathcal{M}^q_n)\cong H^q_{\text{ét}}((X_{\text{triv}})_{\overline{K}},\mathbb{Z}/p^n\mathbb{Z}),\quad T_{\text{st}}(\mathcal{M}'^q_n)\cong T^q/p^nT^q$$

The object of $\underline{MF}_K(\varphi,N)$ associated to the projective system $\{\mathcal{M}'^q_n\}_{n\geq 0}$ (§1.6) is canonically isomorphic to D^q ([Br3] Proposition 4.3.2.3 and the remark after Corollaire 4.3.2.4). Hence, Theorem 3.2.4 implies (§1.5):

Theorem 3.2.5. ([Br3]Corollaire 4.3.2.4 and the following remark). *If $K = K_0$, for any integer $0 \leq q \leq p-2$, the p-adic étale cohomology V^q is a semi-stable p-adic representation and there exists a canonical isomorphism $D_{\mathrm{st}}(V^q) \cong D^q$ in $\underline{MF}_K(\varphi, N)$*

By Corollary 1.3.3, Lemma 2.3.4 and Proposition 2.3.5, we obtain the following isomorphisms form Theorem 3.2.2 (resp. 3.2.5) for an integer $r \geq 0$ such that the condition (3.1.2) with $s = 0$ is satisfied (resp. the condition (3.1.1) is satisfied and $K = K_0$):

(3.2.6) $$V^r(r)_{I_K} \cong \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^0_{\text{ét}}(\overline{Y}, W\omega^r_{\overline{Y}/\overline{s},\log})$$

(3.2.7) $$V^r(d)_{I_K} \cong \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^{r-d}_{\text{ét}}(\overline{Y}, W\omega^d_{\overline{Y}/\overline{s},\log}) \qquad \text{if } r \geq d.$$

Here $d = \dim(X_K)$. See the proof of Lemma 2.3.4 for the relation between φ and $V'_{\leq i}$. In the case (3.1.2), if we use the result in [Y] (resp. Theorem 4.1.2), we obtain the isomorphisms without the assumption $s = 0$ (resp. under the condition (4.1.1)).

In the case (3.1.1) and $K = K_0$, the image of $\mathcal{M}^r$ under the projection $\mathcal{M}^r \to D^r$ given by $u \mapsto 0$ coincides with the image of $H^r((Y,M)/(W,W(L)))(\cong H^r(Y, W\omega^\bullet_{Y/s}))$ ([Br3] Proposition 4.3.1.3). Hence, by Corollary 1.3.3, Theorem 1.6.2, (2.3.2), Lemma 2.3.4, Proposition 2.3.5 and Theorem 3.2.4, we see that the isomorphisms (3.2.6) and (3.2.7) induce isomorphisms:

(3.2.8) $$T^r(r)_{I_K}/\mathrm{tor} \cong H^0_{\text{ét}}(\overline{Y}, W\omega^r_{\overline{Y}/\overline{s},\log})$$

(3.2.9) $$T^r(d)_{I_K}/\mathrm{tor} \cong H^{r-d}_{\text{ét}}(\overline{Y}, W\omega^d_{\overline{Y}/\overline{s},\log})/\mathrm{tor} \quad \text{if } r \geq d.$$

To prove Theorem 3.1.11, it remains to prove that (3.2.6) and (3.2.7) are induced by (3.1.10).

§3.3. Review of the construction of the comparison map.

We will review the construction of the comparison map in Theorems 3.2.2 and 3.2.5. First recall that to give an isomorphism $D_{\mathrm{st}}(V^q) \cong D^q$ is equivalent to give a B_{st}-linear isomorphism:

(3.3.1) $$B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V^q \xrightarrow{\sim} B_{\mathrm{st}} \otimes_{K_0} D^q$$

compatible with φ, N, the actions of G_K and $Fil^\cdot$ after $B_{\mathrm{dR}} \otimes_{B_{\mathrm{st}}}$. Recall also that we fixed a uniformizer π of K (in the case $K = K_0$, we choose p as π) in order to define the functor D_{st} (or equivalently the embedding $B_{\mathrm{st}} \hookrightarrow B_{\mathrm{dR}}$) and to define the filtration on $K \otimes_{K_0} D^q$.

Let (E_n, M_{E_n}) be the PD-envelope of the exact closed immersion $(S_n, N_n) \hookrightarrow (\mathrm{Spec}(W_n[u]), \mathcal{L}(u))$ defined by $u \mapsto \pi$ compatible with the canonical PD-structure on pW_n. Here $\mathcal{L}(u)$ denotes the log structure defined by $\mathbb{N} \to W[u]; 1 \mapsto u$. Since (E_n, M_{E_n}) is isomorphic to the PD-envelope of $(S_1, N_1) \hookrightarrow (\mathrm{Spec}(W_n[u]), \mathcal{L}(u))$, the lifting of Frobenius of $(\mathrm{Spec}(W_n[u]), \mathcal{L}(u))$ defined by

$\sigma\colon W_n \to W_n$ and $u \mapsto u^p$ induces that of (E_n, M_{E_n}), which is compatible with the PD-structure $\overline{\delta}$ on the ideal $\overline{J}_{E_n}$ of $\mathcal{O}_{E_n}$ defining S_1. We denote by $i_{E_n,\pi}$ the canonical exact closed immersion $(S_n, N_n) \hookrightarrow (E_n, M_{E_n})$. We have $\Gamma(E_n, \mathcal{O}_{E_n}) = W[u, \frac{u^{en}}{n!}\ (n \in \mathbb{N})]/p^n$ $(e = [K : K_0])$ and (E_n, M_{E_n}) coincides with the log scheme appearing before Theorem 3.2.4 when $K = K_0$ and $\pi = p$. Let $W_n(L)$ be the "Teichmüller lifting" ([H-Ka] Definition (3.1)) of the log structure L on s to $\mathrm{Spec}(W_n)$, which already appeared in the definition of D^q. Then, we have a closed immersion $i_{E_n,0}\colon (\mathrm{Spec}(W_n), W_n(L)) \hookrightarrow (E_n, M_{E_n})$ defined by $u \mapsto 0$, which is compatible with the lifting of Frobenius.

First we review the crystalline interpretation of $(B^+_{\mathrm{st}} \otimes_{K_0} D)^{N=0}$. We set $R_{E,\mathbb{Q}_p} := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_n \Gamma(E_n, \mathcal{O}_{E_n})$. We define the crystalline cohomology $\mathcal{D}^q$ to be

$$\begin{aligned}&\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_n H^q_{\mathrm{crys}}((X_n, M_n)/(E_n, M_{E_n}, \overline{J}_{E_n}, \overline{\delta}))\\ \cong &\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_n H^q_{\mathrm{crys}}((X_1, M_1)/(E_n, M_{E_n}, \overline{J}_{E_n}, \overline{\delta})),\end{aligned}$$

which is an $R_{E,\mathbb{Q}_p}$-module endowed with φ and N satisfying $N\varphi = p\varphi N$ ([Ts2] 4.3). The projection $\mathrm{pr}_0\colon \mathcal{D}^q \to D^q$ induced by the exact closed immersions $\{i_{E_n,0}\}$, which is compatible with φ, N, has a unique K_0-linear section $s\colon D^q \to \mathcal{D}^q$ compatible with φ and N, and it induces an isomorphism $R_{E,\mathbb{Q}_p} \otimes_{K_0} D^q \xrightarrow{\sim} \mathcal{D}^q$ ([H-Ka] Lemma (5.2), [Ts2] Propositions 4.4.6, 4.4.9).

We define $H^q_{\mathrm{crys}}((\overline{X}_n, \overline{M}_n)/(E_n, M_{E_n}))$ to be the inductive limit of $H^q_{\mathrm{crys}}((X'_n, M'_n)/(E_n, M_{E_n}))$, where (X', M') ranges over the base changes of (X, M) by all finite extensions K' of K contained in $\overline{K}$. We define $\overline{\mathcal{D}}^q$ to be $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_n H^q_{\mathrm{crys}}((\overline{X}_n, \overline{M}_n)/(E_n, M_{E_n}))$, which is also endowed with φ and N ([Ts2] 4.3). In the special case $(X, M) = (S, N)$, we have $\mathcal{D}^q = 0$ $(q > 0)$ ([Ka3] Proposition (3.1), [Ts2] Lemma 1.6.7) and $\widehat{B^+_{\mathrm{st}}} = \mathcal{D}^0$ by definition ([Br1]§2). Furthermore, there exists a canonical isomorphism (the crystalline interpretation of B^+_{st}) $\iota\colon B^+_{\mathrm{st}} \cong \widehat{B^+_{\mathrm{st}}}^{N\text{-nil}}$ compatible with the actions of G_K, φ and N ([Ka3] Theorem (3.7)). Here N-nil denotes the part where N is nilpotent. Let us return to a general (X, M). Then we have a Künneth isomorphism $\widehat{B^+_{\mathrm{st}}} \otimes_{R_{E,\mathbb{Q}_p}} \mathcal{D}^q \xrightarrow{\sim} \overline{\mathcal{D}}^q$ ([Ka3]§4, [Ts2]Proposition 4.5.4). By taking $N = 0$, we obtain the following crystalline interpretation of $(B^+_{\mathrm{st}} \otimes_{K_0} D^q)^{N=0}$ ([Ka3]§4, [Ts2]§4.5).

$$(3.3.2) \qquad (B^+_{\mathrm{st}} \otimes_{K_0} D^q)^{N=0} \xrightarrow[\iota\otimes s]{\sim} (\widehat{B^+_{\mathrm{st}}} \otimes_{R_{E,\mathbb{Q}_p}} \mathcal{D}^q)^{N=0} \xrightarrow{\sim} (\overline{\mathcal{D}}^q)^{N=0}.$$

To compare $\overline{\mathcal{D}}^q$ with V^q, we use syntomic cohomology. The syntomic complex $\mathcal{S}^{\sim}_n(r)_{(X,M)}$ ([Ts2]§2.1) is an object of $D^+(Y_{\mathrm{\acute{e}t}}, \mathbb{Z}/p^n\mathbb{Z})$ such that there exists a canonical distinguished triangle:

$$(3.3.3) \qquad \to \mathcal{S}^{\sim}_n(r)_{(X,M)} \to Ru_{(X_n,M_n)/W_n*} J^{[r]}_{(X_n,M_n)/W_n} \xrightarrow{p^r-\varphi} Ru_{(X_n,M_n)/W_n*} \mathcal{O}_{(X_n,M_n)/W_n},$$

where $u_{(X_n,M_n)/W_n}$ denotes the canonical morphism of topoi $((X_n,M_n)/W_n)^{\sim}_{\rm crys} \to (X_n)^{\sim}_{\text{ét}}$. We define the syntomic cohomology $H^q(\overline{Y}, \mathcal{S}_{\mathbb{Q}_p}(r)_{(\overline{X},\overline{M})})$ to be $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_n (\varinjlim_{K'} H^q_{\text{ét}}(Y', \mathcal{S}^{\sim}_n(r)_{(X',M')}))$, where K' ranges over all finite extensions of K contained in $\overline{K}$ and (X',M') denotes the base change of (X,M) by $O_K \to O_{K'}$. From the above distinguished triangle (3.3.3), we obtain a natural homomorphism

$$(3.3.4) \quad H^q(\overline{Y}, \mathcal{S}_{\mathbb{Q}_p}(r)_{(\overline{X},\overline{M})}) \to (\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_n H^q_{\rm crys}((\overline{X}_n, \overline{M}_n)/W_n))^{\varphi=p^r} \to (\overline{\mathcal{D}}^q)^{N=0,\varphi=p^r}.$$

On the other hand, if we set $\mathbb{Z}/p^n\mathbb{Z}(r)' = (\frac{1}{p^a a!}\mathbb{Z}_p(r))/p^n$ $(r = a(p-1)+b, a \geq 0, 0 \leq b \leq p-2)$, we have a canonical map

$$(3.3.5) \qquad \mathcal{S}^{\sim}_n(r)_{(X,M)} \longrightarrow i^*Rj_*\mathbb{Z}/p^n\mathbb{Z}(r)'$$

in $D^+(Y_{\text{ét}}, \mathbb{Z}/p^n\mathbb{Z})$ ([Ts2] §3.1), which induces a homomorphism

$$(3.3.6) \qquad H^q(\overline{Y}, \mathcal{S}_{\mathbb{Q}_p}(r)_{(\overline{X},\overline{M})}) \longrightarrow V^q(r)$$

by taking $H^q_{\text{ét}}(Y,-)$ and the inductive limit with respect to all finite base changes of (X,M). We do not identify $\mathbb{Z}/p^n\mathbb{Z}(r)'$ with $\mathbb{Z}/p^n\mathbb{Z}(r)$ by the multiplication by $p^a a!$ because (3.3.5) is compatible with the products and the Tate twist if we use the natural maps $\mathbb{Z}/p^n\mathbb{Z}(r)' \otimes \mathbb{Z}/p^n\mathbb{Z}(s)' \to \mathbb{Z}/p^n\mathbb{Z}(r+s)'$ and $\mathbb{Z}/p^n\mathbb{Z}(r)'(1) \to \mathbb{Z}/p^n\mathbb{Z}(r+1)'$.

Theorem 3.3.7. ([Ku], [Ka3]Theorem (5.4), [Ts2]Theorem 3.4.4, [Ts4]Theorem 5.1). *Assume* $(X,M)/(S,N)$ *and* r *satisfy one of the conditions* (3.1.1) *and* (3.1.2). *Then the homomorphism* (3.3.6) *is an isomorphism if* $q \leq r$ *or* $r = d = \dim X_K$.

For the case $r = d$, note $\overline{i}^* R^i \overline{j}_* \mathbb{Z}/p^n\mathbb{Z}(d) = 0$ $(i > d)$, $\mathcal{H}^i(\mathcal{S}^{\sim}_n(d)_{(\overline{X},\overline{M})}) = 0$ $(i > d+1)$ ([Ts2] (2.3.3) and Lemma 2.3.4), and $p^N \mathcal{H}^{d+1}(\mathcal{S}^{\sim}_n(d)_{(\overline{X},\overline{M})}) = 0$ for some $N > 0$ independent of n ([Ts2] Lemma 2.3.19: $\mathcal{H}^{d+1}(C_n(d)) = 0$ and the proof of [Ts2] Theorem 2.3.2).
Thus, under the condition (3.1.1) or (3.1.2), we obtain a homomorphism

$$(3.3.8) \quad V^q(r) \underset{p^{-r}\cdot(3.3.6)}{\overset{\sim}{\longleftarrow}} H^q(\overline{Y}, \mathcal{S}_{\mathbb{Q}_p}(r)_{(\overline{X},\overline{M})}) \underset{(3.3.4)}{\longrightarrow} (\overline{\mathcal{D}}^q)^{N=0,\varphi=p^r} \underset{(3.3.2)}{\overset{\sim}{\longleftarrow}} (B^+_{\rm st} \otimes_{K_0} D^q)^{N=0,\varphi=p^r}.$$

if $q \leq r$ or $r = d$. By using $\mathbb{Q}_p(-r) \subset B_{\rm st}$, we obtain a homomorphism

$$(3.3.9) \qquad B_{\rm st} \otimes_{\mathbb{Q}_p} V^q \longrightarrow B_{\rm st} \otimes_{K_0} D^q,$$

which is independent of r ([Ts2] Corollary 4.8.8 and Lemma 4.9.1). If $K = K_0$ in the case (3.1.1) and $s = 0$ in the case (3.1.2), then (3.3.9) gives the required isomorphism $D_{\rm st}(V^q) \cong D^q$. In the case (3.1.2), the result of [Y] (resp. Theorem 4.1.2) says that (3.3.9) still gives the isomorphism without the condition $s = 0$ (resp. under (4.1.1)).

§3.4. Proof of the main theorem.

Let us prove the main theorem: Theorem 3.1.11. Let $(X,M)/(S,N)$ be as in the beginning of §3.1. We fix an integer $r \geq 0$ and assume that $(X,M)/(S,N)$ and r satisfy (3.1.1) and $K = K_0$, or (3.1.2) with $s = 0$. (In the case (3.1.2), we can remove (resp. replace) the condition $s = 0$ (resp. by (4.1.1)) if we use [Y] (resp. Theorem 4.1.2).) The remaining problem is only to prove that the isomorphisms (3.2.6) and (3.2.7) induced by the comparison isomorphism $D_{\mathrm{st}}(V^r) \cong D^r$ (Theorems 3.2.2 and 3.2.5) coincide with the homomorphisms induced by (3.1.10).

First we define canonical projections $\overline{f}_0 \colon \widehat{B^+_{\mathrm{st}}} \to P_0$ and $\overline{\mathrm{pr}}_0 \colon \overline{\mathcal{D}}^q \to P_0 \otimes_{K_0} D^q$ compatible with φ. Let K' be any finite extension of K contained in $\overline{K}$, let (S', N') be the scheme $\mathrm{Spec}(O_{K'})$ endowed with the canonical log structure, and set $(X', M') := (X, M) \times_{(S,N)} (S', N')$. Then we have a commutative diagram:

$$
(3.4.1) \qquad
\begin{array}{ccccc}
(X'_n, M'_n) & \longleftarrow & & & (Y', M_{Y'}) \\
\downarrow & & & & \downarrow \\
(S'_n, N'_n) & \longleftarrow & & & (s', L') \\
\downarrow & & & \swarrow & \downarrow \\
(S_n, N_n) & \longleftarrow & (s, L) & & \\
{\scriptstyle i_{E_n,\pi}}\downarrow & & \downarrow & & \downarrow \\
(E_n, M_{E_n}) & \xleftarrow{i_{E_n,0}} & (W_n(s), W_n(L)) & \longleftarrow & (W_n(s'), W_n(L')),
\end{array}
$$

which induces a homomorphism

$$
\begin{aligned}
H^q((X'_n, M'_n)/(E_n, M_{E_n})) &\longrightarrow H^q((Y', M_{Y'})/(W_n(s'), W_n(L'))) \\
&\xleftarrow{\sim} W_n(k') \otimes_{W_n} H^q((Y, M_Y)/(W_n, W_n(L))).
\end{aligned}
$$

By taking the inductive limit with respect to K' and $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_n$, we obtain

$$\overline{\mathrm{pr}}_0 \colon \overline{\mathcal{D}}^q \longrightarrow P_0 \otimes_W D^q.$$

This is compatible with φ and the action of G_K, but not with N. In the special case $(X,M) = (S,N)$ and $q = 0$, this becomes a ring homomorphism

$$\overline{f}_0 \colon \widehat{B^+_{\mathrm{st}}} \longrightarrow P_0$$

and the above homomorphism $\overline{\mathrm{pr}}_0$ is compatible with $\overline{f}_0$. By definition, $\overline{\mathrm{pr}}_0$ and $\overline{f}_0$ are compatible with $\mathrm{pr}_0 \colon \mathcal{D}^q \to D^q$ and $f_0 \colon R_{E,\mathbb{Q}_p} \to K_0$ induced by $\{i_{E_n,0}\}$.

LEMMA 3.4.2. *Let V be a semi-stable p-adic representation of G_K and set $D := D_{\mathrm{st}}(V)$. Let s be a positive integer such that $Fil^{s+1}D_K = 0$ and let D' be the quotient of D corresponding to the quotient $V(s)_{I_K}(-s)$ of V (§1.3). Then the following diagram is commutative:*

$$\begin{array}{ccc} Fil^s(B^+_{\mathrm{st}} \otimes_{K_0} D)^{N=0,\varphi=p^s} & \xrightarrow[\overline{f}_0\otimes 1]{} & (P_0 \otimes_{K_0} D)^{\varphi=p^s} \\ \downarrow & & \downarrow\wr \\ Fil^s(B^+_{\mathrm{st}} \otimes_{K_0} D')^{N=0,\varphi=p^s} & \xleftarrow{\sim} & (P_0 \otimes_{K_0} D')^{\varphi=p^s}. \end{array}$$

Recall $D' = D_{[s]}$, $Fil^s D'_K = D'_K$, $Fil^{s+1}D'_K = 0$ and $N = 0$ on D'.

Proof. We are reduced to the case $D = D'$, in which case the claim is trivial because $\overline{f}_0$ is a P_0-algebra homomorphism. □

Note that, with the notation of Lemma 3.4.2, the isomorphism in Corollary 1.3.3 is characterized by the following commutative diagram:
(3.4.3)

$$\begin{array}{ccccc} V(s) & ===== & & & Fil^s(B_{\mathrm{st}} \otimes_{K_0} D)^{N=0,\varphi=p^s} \\ \downarrow & & & & \downarrow \\ V(s)_{I_K} & \underset{\text{Cor. 1.3.3}}{\cong} & (P_0 \otimes_{K_0} D')^{\varphi=p^s} & \xrightarrow{\sim} & Fil^s(B_{\mathrm{st}} \otimes_{K_0} D')^{N=0,\varphi=p^s}. \end{array}$$

LEMMA 3.4.4. *Let K', (S', N') and (X', M') be as in the definition of $\overline{pr}_0$ above. Then the composites of the following two sequences of homomorphisms coincide:*

$$\mathcal{H}^r(\mathcal{S}^{\sim}_n(r)_{(X',M')}) \xrightarrow{p^r\cdot(\mathrm{A})} R^r u_{(X'_n,M'_n)/W_n*}\mathcal{O} \xrightarrow{(\mathrm{B})} R^r u_{(X'_n,M'_n)/(E_n,M_{E_n})*}\mathcal{O}$$
$$\xrightarrow{(\mathrm{C})} R^r u_{(Y',M_{Y'})/(W_n(s'),W_n(L'))*}\mathcal{O} \overset{(\mathrm{D})}{\cong} \mathcal{H}^r(W_n\omega^{\bullet}_{Y'/s'}) \xrightarrow{p^m} \mathcal{H}^r(W_n\omega^{\bullet}_{Y'/s'}),$$
$$\mathcal{H}^r(\mathcal{S}^{\sim}_n(r)_{(X',M')}) \xrightarrow{(\mathrm{E})} i'^* R^r j'_*\mathbb{Z}/p^n\mathbb{Z}(r)' \xrightarrow{\text{“}p^m\text{”}} i'^* R^r j'_*\mathbb{Z}/p^n\mathbb{Z}(r)$$
$$\xrightarrow{(\mathrm{F})} W_n\omega^r_{Y'/s',\log} \longrightarrow \mathcal{H}^r(W_n\omega^{\bullet}_{Y'/s'}).$$

Here the integer m is defined by $p^a a! = p^m\cdot$unit in $\mathbb{Z}_p$, $r = a(p-1)+b$, $a, b \in \mathbb{Z}$, $0 \le b \le p-2$. The homomorphisms (A), (B), (C), (D), (E) and (F) are induced by the distinguished triangle (3.3.3), *the morphism $(E_n, M_{E_n}) \to Spec(W_n)$, the diagram* (3.4.1), *Theorem* 2.1.4, (3.3.5) *and* (3.1.8) *respectively.*

Proof. The question is étale local on X'. Étale locally on X', by choosing a closed immersion of (X', M') into a smooth fine log scheme (Z, M_Z) over W with liftings of Frobenius $\{F_{Z_n}\}$ of $\{(Z_n, M_{Z_n})\}$ satisfying the condition [Ts2] (2.1.1), we can define a complex $\mathcal{S}'_n(r)$ ([Ts2]§2.1) and a natural homomorphism $\mathcal{S}^{\sim}_n(r) \to \mathcal{S}'_n(r)$ ([Ts2](2.1.2)). By the definition of the morphisms [Ts2](2.1.2) and [Ts2](3.1.1) (=(3.3.5)), the homomorphisms $p^r\cdot$ (A) and (E) canonically factor through $\mathcal{H}^r(\mathcal{S}'_n(r))$. Hence we may replace $\mathcal{H}^r(\mathcal{S}^{\sim}_n(r))$ with $\mathcal{H}^r(\mathcal{S}'_n(r))$.

We denote by $\{a_1, \dots, a_r\}_{\text{ét}}$ (resp. $\{a_1, \dots, a_r\}_{\text{syn}}$) the image of $a_1 \otimes \cdots \otimes a_r$ ($a_i \in M'^{\text{gp}}$) under the symbol map (3.1.4) (resp. $(i'^{*}M'^{\text{gp}})^{\otimes r} \to \mathcal{H}^r(\mathcal{S}'_n(r))$ ([Ts2](2.2.1)). Then the local section $\{a_1, \dots, a_r\}_{\text{syn}}$ is sent to $\{a_1, \dots, a_r\}_{\text{ét}}$ by (E) ([Ts2] Proposition 3.2.4 (2)) and hence to $p^m d\log(\overline{a_1}) \wedge \cdots \wedge d\log(\overline{a_r})$ by the composite of the second sequence. On the other hand, by the explicit description of the syntomic symbol map [Ts2] Lemma 2.4.6, which is still valid for $\mathcal{S}_n(r)'$, and Lemma 2.1.10, the image of $\{a_1, \dots, a_r\}_{\text{syn}}$ under the composite of the first sequence is also $p^m d\log(\overline{a_1}) \wedge \cdots \wedge d\log(\overline{a_r})$. By [Ts2] Proposition 2.4.1 (1), we see that the symbol map $(i'^{*}M'^{\text{gp}})^{\otimes r} \to \mathcal{H}^r(\mathcal{S}'_n(r))$ is surjective by induction on n. □

By taking the inductive limit with respect to K' and $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_n$, we obtain the following commutative diagram from Lemma 3.4.4:
(3.4.5)

$$\begin{array}{ccccc}
V^r(s) & \xleftarrow[p^{-s}\cdot(3.3.6)]{\sim} & H^r(\overline{Y}, \mathcal{S}_{\mathbb{Q}_p}(s)_{(\overline{X},\overline{M})}) & \longrightarrow & (\overline{\mathcal{D}}^r)^{N=0,\varphi=p^s} \\
\downarrow & & & & \downarrow{\scriptstyle \overline{\mathrm{pr}}_0} \\
H^{r-s}(\overline{Y}, W\omega^s_{\overline{Y}/\overline{s},\,\log})_{\mathbb{Q}_p} & \xrightarrow{\sim} & (\varprojlim_n H^{r-s}(\overline{Y}, \mathcal{H}^s(W_n\omega^{\bullet}_{\overline{Y}/\overline{s}})))^{\varphi=p^s}_{\mathbb{Q}_p} & \longleftarrow & (P_0 \otimes_{K_0} D)^{\varphi=p^s}.
\end{array}$$

where $s = r$ or $s = d = \dim X_K$ and we assume $r \geq d$ in the latter case. The left vertical (resp. lower right) homomorphism is induced by (3.1.10) (resp. $\tau_{\leq s} W_n\omega^{\bullet}_{Y/s} \to \mathcal{H}^s(W_n\omega^{\bullet}_{Y/s})$). Note $W_n\omega^{\bullet}_{Y/s} = \tau_{\leq d} W_n\omega^{\bullet}_{Y/s}$ in the case $s = d$. The lower left one is an isomorphism by Lemma 2.3.6. See (2.1.7) for the relation between φ and V'. To prove the commutativity in the case $s = d$, we need the remark after Theorem 3.3.7.
On the other hand, we have a commutative diagram:
(3.4.6)

$$\begin{array}{ccccc}
(\overline{\mathcal{D}}^r)^{N=0,\varphi=p^s} & \xleftarrow{\sim} & (\widehat{B^+_{\text{st}} \otimes_{R_{E,\mathbb{Q}_p}}} \mathcal{D}^r)^{N=0,\varphi=p^s} & \xleftarrow[\iota\otimes s]{\sim} & (B^+_{\text{st}} \otimes_{K_0} D^r)^{N=0,\varphi=p^s} \\
& {\scriptstyle \overline{\mathrm{pr}}_0}\searrow & \downarrow{\scriptstyle \overline{f}_0\otimes \mathrm{pr}_0} & \swarrow{\scriptstyle \overline{f}_0\otimes 1} & \\
& & (P_0 \otimes_{K_0} D^r)^{\varphi=p^s} & &
\end{array}$$

Note that the first line is (3.3.2). Combining the above two commutative diagrams, we obtain a commutative diagram:

$$\text{(3.4.7)}\qquad \begin{array}{ccc}
V^r(s) & \xrightarrow[(3.3.8)]{\sim} & Fil^r(B^+_{\text{st}} \otimes_{K_0} D^r)^{N=0,\varphi=p^s} \\
\downarrow & & \downarrow{\scriptstyle \overline{f}_0\otimes 1} \\
\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^{r-s}(\overline{Y}, W\omega^s_{\overline{Y}/\overline{s},\,\log}) & \xrightarrow[\text{Prop. 2.3.5}]{\sim} & (P_0 \otimes_{K_0} D^r)^{\varphi=p^s}.
\end{array}$$

Note that the composite of the lower horizontal map of (3.4.7) with the lower right one of (3.4.5) coincides with the lower left one of (3.4.5). By Lemma 3.4.2, (3.4.3) and (3.4.7), we see that the isomorphisms (3.2.6) and (3.2.7) induced by the comparison isomorphism $D_{\text{st}}(V^r) \cong D^r$ are induced by (3.1.10).

§4. The semi-stable conjecture in the open case.

§4.1. Statement of the theorem.

Recently, G. Yamashita [Y] gave a proof of the semi-stable conjecture via syntomic cohomology for $(X,M)/(S,N)$ satisfying (3.1.2) in the open case, i.e. without assuming $s=0$. (Moreover he proved it also for cohomologies with partial compact supports.) In this section, we will give an alternative proof in the special case that the "horizontal divisors at infinity do not have self-intersections" i.e. when $(X,M)/(S,N)$ satisfies the following condition:

(4.1.1) There exist a finite number of divisors D_i $(i \in I)$ on X such that $X_{\rm triv} = X-(Y\cup(\cup_{i\in I}D_i))$ and an étale covering $\{X_\lambda \to X\}_{\lambda\in\Lambda}$ satisfying the following condition. For each $\lambda\in\Lambda$, there exists an étale morphism over S:

$$X \to \operatorname{Spec}((O_K[T_1,\ldots,T_u]/(T_1\cdots T_u-\pi^e))[U_i\ (i\in I_\lambda),V_1,\ldots,V_t])$$

for some integers $u\geq 1$, $t\geq 0$ and $e\geq 1$ such that $e|[K:K_0]$ and $D_i\times_X X_\lambda$ is the inverse image of $\{U_i=0\}$ for each $i\in I_\lambda$. Here I_λ denotes the set of all $i\in I$ such that $D_i\times_X X_\lambda\neq\emptyset$.

Theorem 4.1.2. *Assume that* $(X,M)/(S,N)$ *satisfies the condition* (4.1.1). *Then, for any integer* $q\geq 0$*, the homomorphism* (3.3.9) *is an isomorphism preserving the filtrations after taking* $B_{\rm dR}\otimes_{B_{\rm st}}$. *Hence* V^q *is a semi-stable p-adic representation and* (3.3.9) *induces an isomorphism:* $D_{\rm st}(V^q)\cong D^q$ *in* $\underline{MF}_K(\varphi,N)$.

We will prove this theorem by removing the divisors D_i at infinity one by one and using the Gysin exact sequences.

§4.2 Gysin sequence for crystalline cohomology.

Associated to an effective Cartier divisor $\mathcal{X}'$ on a scheme $\mathcal{X}$, one can construct a log structure on $\mathcal{X}'$ as follows: If $\mathcal{X}'$ is defined by a global section $f\in\Gamma(\mathcal{X},\mathcal{O}_\mathcal{X})$, then f is a non-zero divisor and unique up to multiplication by units. Hence the fine log structure on $\mathcal{X}$ associated to the pre-log structure $\mathbb{N}_\mathcal{X}\to\mathcal{O}_\mathcal{X}; 1\mapsto f$ is independent on the choice of f up to canonical isomorphisms. In the general case, one obtains a fine log structure by gluing the above log structures étale locally. For a fine log scheme $(\mathcal{X},\mathcal{M})$, we define the fine log structure associated to a Cartier divisor $\mathcal{X}'\subset\mathcal{X}$ to be the co-product of $\mathcal{M}$ and the log structure constructed above.

We say that a morphism of fine log scheme $f\colon(\mathcal{X},\mathcal{M})\to(\mathcal{Y},\mathcal{N})$ is *syntomic* ([Ka3] (2.5)) if it is integral, the underlying morphism of schemes $\mathcal{X}\to\mathcal{Y}$ is flat and locally of finite presentation, and étale locally on $\mathcal{X}$, there exists a $(\mathcal{Y},\mathcal{N})$-exact closed immersion of $(\mathcal{X},\mathcal{M})$ into a smooth fine log scheme $(\mathcal{Z},\mathcal{L})$ over $(\mathcal{Y},\mathcal{N})$ such that the underlying closed immersion of schemes is transversally regular relative to $\mathcal{Y}$ ([EGA IV] Définition (19.2.2). See also Proposition (19.2.4)). If f is syntomic, for any $(\mathcal{Y},\mathcal{N})$-exact closed immersion of $(\mathcal{X},\mathcal{M})$ into a smooth fine log scheme $(\mathcal{Z},\mathcal{L})$ over $(\mathcal{Y},\mathcal{N})$, the underlying closed immersion is transversally regular relative to $\mathcal{S}$. Syntomic morphisms are stable under base changes and compositions.

Proposition 4.2.1. *Let $(\mathcal{S},\mathcal{N})$ be a fine log scheme endowed with a PD-ideal (I,γ) such that p is nilpotent on $\mathcal{O}_{\mathcal{S}}$, let $\mathcal{S}_0$ be the closed subscheme of $\mathcal{S}$ defined by a sub PD-ideal of I, and let $\mathcal{N}_0$ be the inverse image of $\mathcal{N}$. Let $(\mathcal{X}_0,\mathcal{M}_0)$ be a syntomic fine log scheme over $(\mathcal{S}_0,\mathcal{N}_0)$, let $\mathcal{X}_0'\subset\mathcal{X}_0$ be a Cartier divisor flat over $\mathcal{S}_0$, and let $\mathcal{M}_0'$ be the inverse image of $\mathcal{M}_0$ on $\mathcal{X}_0'$.*
(1) Étale locally on $\mathcal{X}_0$, there exist an $(\mathcal{S},\mathcal{N})$-closed immersion of $(\mathcal{X}_0,\mathcal{M}_0)$ into a smooth fine log scheme $(\mathcal{Y},\mathcal{L})$ over $(\mathcal{S},\mathcal{N})$ and a Cartier divisor $\mathcal{Y}'\subset\mathcal{Y}$ such that $\mathcal{X}_0'$ is the inverse image of $\mathcal{Y}'$ and $\mathcal{Y}'$ endowed with the inverse image $\mathcal{L}'$ of $\mathcal{L}$ is smooth over $(\mathcal{S},\mathcal{N})$.
(2) Suppose that we are given $i\colon(\mathcal{X}_0,\mathcal{M}_0)\hookrightarrow(\mathcal{Y},\mathcal{L})$ and $\mathcal{Y}'$ as in (1) globally. Let $(\mathcal{D},\mathcal{M}_{\mathcal{D}})$ (resp. $(\mathcal{D}',\mathcal{M}_{\mathcal{D}'})$) be the PD-envelope of $(\mathcal{X}_0,\mathcal{M}_0)$ in $(\mathcal{Y},\mathcal{L})$ (resp. $(\mathcal{X}_0',\mathcal{M}_0')$ in $(\mathcal{Y}',\mathcal{L}')$) compatible with the PD-structure (I,γ). Let $J_{\mathcal{D}}$ (resp. $J_{\mathcal{D}'}$) be the PD-ideal of $\mathcal{O}_{\mathcal{D}}$ (resp. $\mathcal{O}_{\mathcal{D}'}$). If $\mathcal{Y}'$ is defined by a global section $f\in\Gamma(\mathcal{Y},\mathcal{O}_{\mathcal{Y}})$, then f is a non-zero divisor on $\mathcal{O}_{\mathcal{D}}$ and we have isomorphisms $\mathcal{O}_{\mathcal{D}}/f\mathcal{O}_{\mathcal{D}}\cong\mathcal{O}_{\mathcal{D}'}$ and $J_{\mathcal{D}}^{[r]}/fJ_{\mathcal{D}}^{[r]}\cong J_{\mathcal{D}'}^{[r]}$ $(r\geq 1)$.
(3) Under the notation of (2), let $\mathcal{L}^\circ$ (resp. $\mathcal{M}_0^\circ$) be the log structure on $\mathcal{Y}$ (resp. $\mathcal{X}_0$) defined by the log structure $\mathcal{L}$ (resp. $\mathcal{M}_0$) and the Cartier divisor $\mathcal{Y}'$ (resp. $\mathcal{X}_0'$). Then the PD-envelope of $(\mathcal{X}_0,\mathcal{M}_0^\circ)$ in $(\mathcal{Y},\mathcal{L}^\circ)$ compatible with the PD-structure (I,γ) has the same underlying scheme as $(\mathcal{D},\mathcal{M}_{\mathcal{D}})$. Furthermore $(\mathcal{Y},\mathcal{L}^\circ)$ is smooth over $(\mathcal{S},\mathcal{N})$, and, for each integer $r\geq 0$, we have a canonical exact sequence:

$$0\to J_{\mathcal{D}}^{[r-\bullet]}\otimes_{\mathcal{O}_{\mathcal{Y}}}\Omega^\bullet_{\mathcal{Y}/\mathcal{S}}(\log(\mathcal{L}/\mathcal{N}))\to J_{\mathcal{D}}^{[r-\bullet]}\otimes_{\mathcal{O}_{\mathcal{Y}}}\Omega^\bullet_{\mathcal{Y}/\mathcal{S}}(\log(\mathcal{L}^\circ/\mathcal{N}))$$
$$\overset{(*)}{\to}(J_{\mathcal{D}'}^{[r-1-\bullet]}\otimes_{\mathcal{O}_{\mathcal{Y}'}}\Omega^\bullet_{\mathcal{Y}'/\mathcal{S}}(\log(\mathcal{L}'/\mathcal{N})))[-1]\to 0$$

such that $(*)$ sends $\omega_1+d\log(g)\wedge\omega_2$ to $\overline{\omega_2}$ for $\omega_1\in J_{\mathcal{D}}^{[r-q]}\otimes_{\mathcal{O}_{\mathcal{Y}}}\Omega^q_{\mathcal{Y}/\mathcal{S}}(\log(\mathcal{L}/\mathcal{N}))$, $\omega_2\in J_{\mathcal{D}}^{[r-q]}\otimes_{\mathcal{O}_{\mathcal{Y}}}\Omega^{q-1}_{\mathcal{Y}/\mathcal{S}}(\log(\mathcal{L}/\mathcal{N}))$ and a local equation $g=0$ defining $\mathcal{Y}'$ in $\mathcal{Y}$, where $\overline{\omega_2}$ denotes the image of ω_2 in $J_{\mathcal{D}'}^{[r-q]}\otimes_{\mathcal{O}_{\mathcal{Y}'}}\Omega^{q-1}_{\mathcal{Y}'/\mathcal{S}}(\log(\mathcal{L}'/\mathcal{N}))$.

Proof. (1) Étale locally on $\mathcal{X}_0$, there exists an $(\mathcal{S},\mathcal{N})$-closed immersion i of $(\mathcal{X}_0,\mathcal{M}_0)$ into a smooth fine log scheme $(\mathcal{Z},\mathcal{M}_{\mathcal{Z}})$ over $(\mathcal{S},\mathcal{N})$, and $\mathcal{X}_0'$ is defined by the global equation $f=0$ for some $f\in\Gamma(\mathcal{X}_0,\mathcal{O}_{\mathcal{X}_0})$. Set $(\mathcal{Y},\mathcal{L}):=(\mathcal{Z},\mathcal{M}_{\mathcal{Z}})\times_{\mathrm{Spec}(\mathbb{Z})}\mathrm{Spec}(\mathbb{Z}[T])$ an let $\mathcal{Y}'$ be the closed subscheme of $\mathcal{Y}$ defined by $T=0$. Then the closed immersion $(\mathcal{X}_0,\mathcal{M}_0)\hookrightarrow(\mathcal{Y},\mathcal{L})$ defined by i and $T\mapsto f$ satisfies the required condition.
(2) Since the question is étale local on $\mathcal{X}_0$, we may assume that we have a factorization $(\mathcal{X}_0,\mathcal{M}_0)\overset{j}{\hookrightarrow}(\mathcal{Z},\mathcal{M}_{\mathcal{Z}})\overset{\alpha}{\to}(\mathcal{Y},\mathcal{L})$ such that j is an exact closed immersion and α is étale, and that $\mathcal{Y}'$ is defined by the global equation $g=0$ for some $g\in\Gamma(\mathcal{Y},\mathcal{O}_{\mathcal{Y}})$. Since $(\mathcal{X}_0,\mathcal{M}_0)$ is integral over $(\mathcal{S}_0,\mathcal{N}_0)$, we may also assume that $(\mathcal{Z},\mathcal{M}_{\mathcal{Z}})$ is integral over $(\mathcal{S},\mathcal{N})$. Let $\mathcal{Z}'\subset\mathcal{Z}$ be the pull-back of $\mathcal{Y}'\subset\mathcal{Y}$ and let h be the inverse image of g in $\Gamma(\mathcal{Z},\mathcal{O}_{\mathcal{Z}})$. Since $(\mathcal{Z},\mathcal{M}_{\mathcal{Z}})$ and $\mathcal{Z}'$ endowed with the pull-back of $\mathcal{M}_{\mathcal{Z}}$ are smooth and integral over $(\mathcal{S},\mathcal{N})$, the morphism $(\mathcal{Z},\mathcal{M}_{\mathcal{Z}})\to(\mathcal{S},\mathcal{N})\times_{\mathrm{Spec}(\mathbb{Z})}\mathrm{Spec}(\mathbb{Z}[T])$ defined by $T\mapsto g$ is smooth

and integral on a neighbourhood of $\mathcal{Z}'$. Especially the underlying morphism of schemes is flat on the neighbourhood. Hence h is a non-zero divisor. Thus, by the construction of PD-envelopes ([Ka2] (5.6)), we may replace $(\mathcal{Y},\mathcal{L})$ with $(\mathcal{Z},\mathcal{M}_{\mathcal{Z}})$ and assume that i is an exact closed immersion and $(\mathcal{Y},\mathcal{L})$ is integral over $(\mathcal{S},\mathcal{N})$. Since $(\mathcal{X}_0,\mathcal{M}_0)$ is syntomic over $(\mathcal{S}_0,\mathcal{N}_0)$, the closed immersion $\mathcal{X}_0 \hookrightarrow \mathcal{Y}_0 := \mathcal{Y}\times_{\mathcal{S}}\mathcal{S}_0$ is transversally regular relative to $\mathcal{S}_0$ and hence we may assume that there exists a sequence $g_1,\dots,g_d\in\Gamma(\mathcal{Y},\mathcal{O}_{\mathcal{Y}})$ whose image in $\mathcal{O}_{\mathcal{Y}_0}$ is transversally $\mathcal{O}_{\mathcal{Y}_0}$-regular relative to $\mathcal{S}_0$, and $\mathcal{X}_0\subset\mathcal{Y}_0$ is defined by the ideal $\sum_{1\leq i\leq d} g_i\cdot\mathcal{O}_{\mathcal{Y}_0}$. Since $\mathcal{S}_0\hookrightarrow\mathcal{S}$ is a nilimmersion and $\mathcal{Y}$ is flat over $\mathcal{S}$, we see that the sequence $g_1,\dots,g_d$ is transversally $\mathcal{O}_{\mathcal{Y}}$-regular relative to $\mathcal{S}$. (Since $\mathcal{Y}$ is locally of finite presentation over $\mathcal{S}$, we are reduced to the case $\mathcal{S}$ is noetherian and then to the case $\mathcal{S}_0$ is defined by an ideal J of $\mathcal{O}_{\mathcal{S}}$ such that $J^2=0$). Let $\mathcal{X}$ be the closed subscheme of $\mathcal{Y}$ defined by the ideal $\sum_{1\leq i\leq d} g_i\cdot\mathcal{O}_{\mathcal{Y}}$ and set $\mathcal{X}':=\mathcal{X}\times_{\mathcal{Y}}\mathcal{Y}'(\subset\mathcal{Y}')$. Since the image of the sequence $g_1,\cdots,g_d,g$ in $\mathcal{O}_{\mathcal{Y}_0}$ is transversally $\mathcal{O}_{\mathcal{Y}_0}$-regular relative to $\mathcal{S}_0$ and $\mathcal{S}_0\hookrightarrow\mathcal{S}$ is a nilimmersion, the sequence $g_1,\dots,g_d,g$ is transversally $\mathcal{O}_{\mathcal{Y}}$-regular relative to $\mathcal{S}$ and hence the image of $g_1,\dots,g_d$ in $\mathcal{O}_{\mathcal{Y}'}$ is transversally $\mathcal{O}_{\mathcal{Y}'}$-regular relative to $\mathcal{S}$. Hence the morphism $\mathcal{Y}\to\mathcal{S}[T_1,\dots,T_d]$ (resp. $\mathcal{Y}'\to\mathcal{S}[T_1,\dots,T_d]$) defined by $T_i\mapsto g_i$ (resp. $T_i\mapsto$ the image of g_i in $\mathcal{O}_{\mathcal{Y}'}$) is flat on a neighbourhood of $\mathcal{X}$ (resp. $\mathcal{X}'$). (Since $\mathcal{Y}$ and $\mathcal{Y}'$ are locally of finite presentation over $\mathcal{S}$, we are easily reduced to the case $\mathcal{S}$ is noetherian, where we can use [EGA IV] Chap. 0 Proposition (15.1.21).) Furthermore, since $\mathcal{X}_0=\mathcal{X}\times_{\mathcal{S}}\mathcal{S}_0$ (resp. $\mathcal{X}'_0=\mathcal{X}'\times_{\mathcal{S}}\mathcal{S}_0$), $\mathcal{D}$ (resp. $\mathcal{D}'$) is isomorphic to the PD-envelope of $\mathcal{X}$ in $\mathcal{Y}$ (resp. $\mathcal{X}'$ in $\mathcal{Y}'$). Hence, by [Be-O] 3.2.1, we have $\mathcal{D}\cong\mathcal{Y}\times_{\mathcal{S}[T_1,\dots,T_d]}\mathcal{S}<T_1,\dots,T_d>$, $\mathcal{D}'\cong\mathcal{Y}'\times_{\mathcal{S}[T_1,\dots,T_d]}\mathcal{S}<T_1,\dots,T_d>$, which implies the claim.

(3) As in (2), we may assume that i is an exact closed immersion, and $(\mathcal{Y},\mathcal{L})$ is integral over $(\mathcal{S},\mathcal{N})$, and $\mathcal{Y}'\subset\mathcal{Y}$ is defined by the global equation $g=0$ for some $g\in\Gamma(\mathcal{Y},\mathcal{O}_{\mathcal{Y}})$. Then $(\mathcal{X},\mathcal{M}^{\circ})\hookrightarrow(\mathcal{Y},\mathcal{L}^{\circ})$ is an exact closed immersion and we obtain the first claim. For the second claim, we may replace $\mathcal{Y}$ with a neighbourhood of $\mathcal{Y}'$. Hence, we may assume $(\mathcal{Y},\mathcal{L})\to(\mathcal{S},\mathcal{N})\times_{\mathrm{Spec}(\mathbb{Z})}\mathrm{Spec}(\mathbb{Z}[T])$ defined by $T\mapsto g$ is smooth, and there exists a chart $P\to\Gamma(\mathcal{S},\mathcal{N})$, $Q\to\Gamma(\mathcal{Y},\mathcal{L})$, $P\to Q$ of $(\mathcal{Y},\mathcal{L})\to(\mathcal{S},\mathcal{N})$ such that $(\mathcal{Y},\mathcal{L})\to(\mathcal{S},\mathcal{N})\times_{\mathrm{Spec}(\mathbb{Z}[P])}\mathrm{Spec}(\mathbb{Z}[Q][T])$ is étale and the kernel and the torsion part of the cokernel of $P^{\mathrm{gp}}\to Q^{\mathrm{gp}}$ have orders invertible on $\mathcal{S}$. Then $Q\oplus\mathbb{N}\to\Gamma(\mathcal{Y},\mathcal{L}^{\circ});(0,1)\mapsto g$ becomes a chart of $\mathcal{L}^{\circ}$. Hence $(\mathcal{Y},\mathcal{L}^{\circ})$ is smooth over $(\mathcal{S},\mathcal{N})$, and we have

$$\begin{aligned}\Omega_{\mathcal{Y}/\mathcal{S}}(\log(\mathcal{L}^{\circ}))&\cong\mathcal{O}_{\mathcal{Y}}\otimes_{\mathbb{Z}}P^{\mathrm{gp}}\oplus\mathcal{O}_{\mathcal{Y}}\cdot d\log(g),\\ \Omega_{\mathcal{Y}/\mathcal{S}}(\log(\mathcal{L}))&\cong\mathcal{O}_{\mathcal{Y}}\otimes_{\mathbb{Z}}P^{\mathrm{gp}}\oplus\mathcal{O}_{\mathcal{Y}}\cdot dg,\\ \Omega_{\mathcal{Y}'/\mathcal{S}'}(\log(\mathcal{L}'))&\cong\mathcal{O}_{\mathcal{Y}'}\otimes_{\mathbb{Z}}P^{\mathrm{gp}}.\end{aligned}$$

Now the claim follows from (2). □

Proposition 4.2.2. *Let $(\mathcal{S},\mathcal{N},I,\gamma)$, $(\mathcal{S}_0,\mathcal{N}_0)$, $(\mathcal{X}_0,\mathcal{M}_0)$ and $\mathcal{X}'_0$ be the same as in Proposition* 4.2.1. *Assume that $\mathcal{X}_0$ is quasi-compact and separated. Then, there exist an étale hypercovering $\mathcal{X}_0^{\bullet}$ of $\mathcal{X}_0$, a simplicial smooth and integral* *fine log scheme* $(\mathcal{Y}^\bullet, \mathcal{L}^\bullet)$ *over* $(\mathcal{S}, \mathcal{N})$ *with a Cartier divisor* $\mathcal{Y}'^\nu \subset \mathcal{Y}^\nu$ *for each* $\nu \geq 0$ *and an* $(\mathcal{S}, \mathcal{N})$*-closed immersion of* $(\mathcal{X}_0^\bullet, \mathcal{M}|_{\mathcal{X}_0^\bullet})$ *into* $(\mathcal{Y}^\bullet, \mathcal{L}^\bullet)$ *such that* $\mathcal{X}_0^\nu$ $(\nu \geq 0)$ *is affine,* $\mathcal{Y}'^\nu$ $(\nu \geq 0)$ *endowed with the inverse image of* $\mathcal{L}^\nu$ *is smooth over* $(\mathcal{S}, \mathcal{N})$*,* $\mathcal{Y}'^0$ *is defined by the global equation* $g = 0$ *for some* $g \in \Gamma(\mathcal{Y}^0, \mathcal{O}_{\mathcal{Y}^0})$*, for any non-decreasing map* $s\colon \{0, 1, \dots, \nu\} \to \{0, 1, \dots, \mu\}$*,* $\mathcal{Y}'_\mu$ *is the pull-back of* $\mathcal{Y}'_\nu$ *by the morphism* $\mathcal{Y}_\mu \to \mathcal{Y}_\nu$ *corresponding to* s*, and* $\mathcal{X}_0'^\nu := \mathcal{X}_0' \times_{\mathcal{X}_0} \mathcal{X}_0^\nu$ $(\nu \geq 0)$ *is the pull-back of* $\mathcal{Y}'^\nu$ *by the closed immersion* $\mathcal{X}_0^\nu \hookrightarrow \mathcal{Y}^\nu$*.*

Proof. We will write $\mathcal{X}$ for $\mathcal{X}_0$ to simplify the notation. Since $\mathcal{X}$ is quasi-compact, by Proposition 4.2.1 (1), there exist an étale covering $\mathcal{X}^0 \to \mathcal{X}$ with $\mathcal{X}^0$ affine, an $(\mathcal{S}, \mathcal{N})$-closed immersion of $(\mathcal{X}^0, \mathcal{M}|_{\mathcal{X}^0})$ into a fine log scheme $(\mathcal{Y}^0, \mathcal{L}^0)$ smooth and integral over $(\mathcal{S}, \mathcal{N})$, and a Cartier divisor $\mathcal{Y}'^0 \subset \mathcal{Y}^0$ defined by the global equation $g = 0$ for some $g \in \Gamma(\mathcal{Y}^0, \mathcal{O}_{\mathcal{Y}^0})$ such that $\mathcal{X}'^0 \subset \mathcal{X}^0$ is the pull-back of $\mathcal{Y}'^0$ by the closed immersion $\mathcal{X}^0 \hookrightarrow \mathcal{Y}^0$ and $\mathcal{Y}'^0$ endowed with the inverse image of $\mathcal{L}^0$ is smooth over $(\mathcal{S}, \mathcal{N})$. For each $\nu \geq 0$, we define $\mathcal{X}^\nu$ to be the fiber product of $\nu+1$ copies of $\mathcal{X}^0$ over $\mathcal{X}$, which is affine, and $\mathcal{X}'^\nu$ to be $\mathcal{X}' \times_{\mathcal{X}} \mathcal{X}^\nu$. We define $(\mathcal{Y}(\nu), \mathcal{L}(\nu))$ (resp. $\mathcal{Y}'(\nu)$) to be the fiber product of $\nu+1$ copies of $(\mathcal{Y}^0, \mathcal{L}^0)$ (resp. $\mathcal{Y}'^0$) over $(\mathcal{S}, \mathcal{N})$ (resp. $\mathcal{S}$), and $\overline{\mathcal{Y}^\nu}$ to be the blowing-up of $\mathcal{Y}(\nu)$ along $\mathcal{Y}'(\nu)$. We define $\mathcal{Y}''(\nu)$ to be the sum of the pull-backs of $\mathcal{Y}'^0$ by the $\nu+1$ projections $\mathcal{Y}(\nu) \to \mathcal{Y}$ and $\mathcal{Y}^\nu$ to be the complement of the strict transform of $\mathcal{Y}''(\nu) \subset \mathcal{Y}(\nu)$ on $\overline{\mathcal{Y}^\nu}$. We denote by $\overline{\mathcal{L}^\nu}$ and $\mathcal{L}^\nu$ the inverse images of $\mathcal{L}(\nu)$ on $\overline{\mathcal{Y}^\nu}$ and $\mathcal{Y}^\nu$ respectively. We define $\mathcal{Y}'^\nu$ to be $\mathcal{Y}'(\nu) \times_{\mathcal{Y}(\nu)} \mathcal{Y}^\nu$, which is a Cartier divisor on $\mathcal{Y}^\nu$. Let $\mathrm{pr}_i^\nu\colon \mathcal{Y}(\nu) \to \mathcal{Y}^1$ $(0 \leq i \leq \nu)$ be the projection to the $(i+1)$-th component and set $g_i^\nu := (\mathrm{pr}_i^\nu)^*(g)$. Then the morphism $(\mathcal{Y}(\nu), \mathcal{L}(\nu)) \to (\mathcal{S}, \mathcal{N})[T_0, \dots, T_\nu]$ defined by $T_i \mapsto g_i^\nu$ is smooth and integral on a neighbourhood of $\mathcal{Y}'(\nu)$ in $\mathcal{Y}(\nu)$, especially, the underlying morphism of scheme is flat on the neighbourhood. Hence $\overline{\mathcal{Y}^\nu}$ is the pull-back of the blowing-up of $\mathcal{S}[T_0, \dots, T_\nu]$ along $T_0 = T_1 = \cdots = T_\nu = 0$. If we choose an integer i_0 such that $0 \leq i_0 \leq \nu$, $\mathcal{Y}^\nu$ is the pull-back of $\mathcal{S}[T_{i_0}, U_i, U_i^{-1} (0 \leq i \leq \nu, i \neq i_0)] \to \mathcal{S}[T_0, \dots, T_\nu]$ where $T_i = T_{i_0} U_i$ $(i \neq i_0)$.This implies that $(\overline{\mathcal{Y}^\nu}, \overline{\mathcal{L}^\nu})$, $(\mathcal{Y}^\nu, \mathcal{L}^\nu)$ and $\mathcal{Y}'^\nu$ endowed with the inverse images of $\mathcal{L}^\nu$ are smooth and integral over $(\mathcal{S}, \mathcal{N})$. We also see that $\mathcal{Y}'^\nu \subset \mathcal{Y}^\nu$ is defined by the equation $g_{i_0}^\nu = 0$ and $g_{i_0}^\nu$ is a non-zero divisor on $\mathcal{Y}^\nu$. By the universality of blowing-up, the closed immersion $i(\nu)\colon (\mathcal{X}^\nu, \mathcal{M}|_{\mathcal{X}^\nu}) \hookrightarrow (\mathcal{Y}(\nu), \mathcal{L}(\nu))$ canonically factors through a closed immersion $\overline{i^\nu}\colon (\mathcal{X}^\nu, \mathcal{M}|_{\mathcal{X}^\nu}) \hookrightarrow (\overline{\mathcal{Y}^\nu}, \overline{\mathcal{L}^\nu})$. If we denote by h_i^ν the inverse image of g_i^ν in $\mathcal{O}_{\mathcal{X}^\nu}$, then, for each i, the closed subscheme $\mathcal{X}'^\nu$ of $\mathcal{X}^\nu$ is defined by $h_i^\nu = 0$. Hence $h_i^\nu = h_{i_0}^\nu \cdot u_i$ for some $u_i \in \mathcal{O}_{\mathcal{X}^\nu}^*$. This implies that $\overline{i^\nu}$ factors through $(\mathcal{Y}^\nu, \mathcal{L}^\nu)$, which we denote by i^ν. Furthermore we see that $\mathcal{X}'^\nu$ is the pull-back of $\mathcal{Y}'^\nu$. Let $s\colon \{0, 1, \cdots, \nu\} \to \{0, 1, \dots, \mu\}$ be a non-decreasing map. By the universality of blowing-up, the composite of $(\mathcal{Y}^\mu, \mathcal{L}^\mu) \to (\mathcal{Y}(\mu), \mathcal{L}(\mu))$ with the morphism $(\mathcal{Y}(\mu), \mathcal{L}(\mu)) \to (\mathcal{Y}(\nu), \mathcal{L}(\nu))$ corresponding to s uniquely factors through $(\overline{\mathcal{Y}^\nu}, \overline{\mathcal{L}^\nu})$. The inverse images of g_i^ν $(0 \leq i \leq \nu)$ in $\mathcal{O}_{\mathcal{Y}^\mu}$ are $g_{s(i)}^\mu$ and coincide up to the multiplication by units. Hence it further factors through $(\mathcal{Y}^\nu, \mathcal{L}^\nu)$ and $\mathcal{Y}'^\mu$ is the pull-back of $\mathcal{Y}'^\nu$. Thus

$\{(\mathcal{Y}^\nu, \mathcal{L}^\nu)\}_{\nu\geq 0}$ become a simplicial fine log scheme and $\{i^\nu\}$ are compatible with the simplicial structures. □

Corollary 4.2.3. *Let $(\mathcal{S}, \mathcal{N}, I, \gamma)$, $(\mathcal{S}_0, \mathcal{N}_0)$, $(\mathcal{X}_0, \mathcal{M}_0)$ and $(\mathcal{X}'_0, \mathcal{M}'_0)$ be as in Proposition* 4.2.1 *and let $\mathcal{M}_0^\circ$ be the fine log structure on $\mathcal{X}_0$ defined by $\mathcal{M}_0$ and the Cartier divisor $\mathcal{X}'_0 \subset \mathcal{X}_0$. Assume that $\mathcal{X}_0$ is quasi-compact and separated. Let $u_{(\mathcal{X}_0,\mathcal{M}_0)/(\mathcal{S},\mathcal{N})}$ denote the morphism of topoi*

$$((\mathcal{X}_0, \mathcal{M}_0)/(\mathcal{S}, \mathcal{N}, I, \gamma))^\sim_{\text{crys}} \longrightarrow (\mathcal{X}_0)^\sim_{\text{ét}},$$

and define $u_{(\mathcal{X}_0,\mathcal{M}_0^\circ)/(\mathcal{S},\mathcal{N})}$ and $u_{(\mathcal{X}'_0,\mathcal{M}'_0)/(\mathcal{S},\mathcal{N})}$ similarly. Then we have a canonical distinguished triangle:

$$Ru_{(\mathcal{X}_0,\mathcal{M}_0)/(\mathcal{S},\mathcal{N}),*}\mathcal{J}^{[r]} \to Ru_{(\mathcal{X}_0,\mathcal{M}_0^\circ)/(\mathcal{S},\mathcal{N}),*}\mathcal{J}^{[r]} \to Ru_{(\mathcal{X}'_0,\mathcal{M}'_0)/(\mathcal{S},\mathcal{N}),*}\mathcal{J}^{[r-1]}[-1] \to$$

for each integer r. Here $\mathcal{O}$ denotes the structure sheaf on the relevant crystalline site, $\mathcal{J}$ denotes the PD-ideal of $\mathcal{O}$ and, for an integer r, $\mathcal{J}^{[r]}$ denotes the r-th divided power of $\mathcal{J}$ if $r \geq 1$ and $\mathcal{O}$ if $r \leq 0$.

Proof. Choose $(\mathcal{X}_0^\bullet, \mathcal{M}_0|_{\mathcal{X}_0^\bullet}) \hookrightarrow (\mathcal{Y}^\bullet, \mathcal{L}^\bullet)$, $\mathcal{Y}'^\bullet \subset \mathcal{Y}^\bullet$ and g as in Proposition 4.2.2 Then we can apply Proposition 4.2.1 (3) to $(\mathcal{X}_0^\nu, \mathcal{M}_0|_{\mathcal{X}_0^\nu}) \hookrightarrow (\mathcal{Y}^\nu, \mathcal{L}^\nu)$, $\mathcal{X}_0'^\nu$ and $\mathcal{Y}'^\nu$ for each $\nu \geq 0$. Furthermore, for each non-decreasing map $s\colon \{0, 1, \ldots, \nu\} \to \{0, 1, \ldots, \mu\}$, since $\mathcal{Y}'^\mu$ is the pull-back of $\mathcal{Y}'^\nu$ by the morphism $f_s\colon \mathcal{Y}^\mu \to \mathcal{Y}^\nu$ corresponding to s, the short exact sequences are functorial with respect to f_s. Hence, by the cohomological descent ([Ka3](2.18)–(2.21)), we obtain the required distinguished triangles. If we are given another $(\tilde{\mathcal{X}}_0^\bullet, \mathcal{M}_0|_{\tilde{\mathcal{X}}_0^\bullet}) \hookrightarrow (\tilde{\mathcal{Y}}^\bullet, \tilde{\mathcal{L}}^\bullet)$, $\tilde{\mathcal{Y}}'^\bullet \subset \tilde{\mathcal{Y}}^\bullet$ and $\tilde{g}$, we define $\overline{\mathcal{Z}^\nu}$ to be the blowing-up of $\mathcal{Y}^\nu \times_{\mathcal{S}} \tilde{\mathcal{Y}}^\nu$ along $\mathcal{Y}'^\nu \times_{\mathcal{S}} \tilde{\mathcal{Y}}'^\nu$ and let $\mathcal{Z}^\nu$ be the complement of the strict transform of $\mathcal{Y}'^\nu \times \tilde{\mathcal{Y}}^\nu \cup \mathcal{Y}^\nu \times \tilde{\mathcal{Y}}'^\nu$ on $\overline{\mathcal{Z}^\nu}$. Let $\mathcal{M}_{\mathcal{Z}^\nu}$ be the inverse image of the log structure of $(\mathcal{Y}^\nu, \mathcal{L}^\nu) \times_{(\mathcal{S},\mathcal{N})} (\tilde{\mathcal{Y}}^\nu, \tilde{\mathcal{L}}^\nu)$ to $\mathcal{Z}^\nu$, and let $\mathcal{Z}'^\nu \subset \mathcal{Z}^\nu$ be the pull-back of $\mathcal{Y}'^\nu \times \tilde{\mathcal{Y}}'^\nu$. Then, similarly as the proof of Proposition 4.2.2, using g and $\tilde{g}$, we see that $\{(\mathcal{Z}^\nu, \mathcal{M}_{\mathcal{Z}^\nu})\}_{\nu\geq 0}$ naturally become a simplicial fine log scheme, there exists a closed immersion $(\mathcal{X}_0^\nu \times_{\mathcal{X}_0} \tilde{\mathcal{X}}_0^\nu$, the inverse image of $\mathcal{M}_0) \hookrightarrow (\mathcal{Z}^\nu, \mathcal{M}_{\mathcal{Z}^\nu})$ inducing a morphism between simplicial fine log schemes, and this closed immersion with $\mathcal{Z}'^\nu \subset \mathcal{Z}^\nu$ satisfies the conditions in Proposition 4.2.2. Furthermore, we have natural morphisms $(\mathcal{Z}^\bullet, \mathcal{M}_{\mathcal{Z}^\bullet}) \to (\mathcal{Y}^\bullet, \mathcal{L}^\bullet)$ and $(\mathcal{Z}^\bullet, \mathcal{M}_{\mathcal{Z}^\bullet}) \to (\tilde{\mathcal{Y}}^\bullet, \tilde{\mathcal{L}}^\bullet)$ compatible with the closed immersions, and isomorphisms $\mathcal{Z}'^\nu \cong \mathcal{Y}'^\nu \times_{\mathcal{Y}^\nu} \mathcal{Z}^\nu \cong \tilde{\mathcal{Y}}'^\nu \times_{\tilde{\mathcal{Y}}^\nu} \mathcal{Z}^\nu$. Hence the distinguished triangle is independent of the choice of $\mathcal{X}_0^\bullet$ etc. □

The distinguished triangle in Corollary 4.2.3 is functorial with respect to $(\mathcal{X}_0, \mathcal{M}_0) \to (\mathcal{S}_0, \mathcal{N}_0) \hookrightarrow (\mathcal{S}, \mathcal{N}, I, \gamma)$ and $\mathcal{X}'_0$ as follows: Suppose that we are given another $(\tilde{\mathcal{X}}_0, \tilde{\mathcal{M}}_0) \to (\tilde{\mathcal{S}}_0, \tilde{\mathcal{N}}_0) \hookrightarrow (\tilde{\mathcal{S}}, \tilde{\mathcal{N}}, \tilde{I}, \tilde{\gamma})$ and $\tilde{\mathcal{X}}'_0$, a morphism $\alpha\colon (\tilde{\mathcal{X}}_0, \tilde{\mathcal{M}}_0) \to (\mathcal{X}_0, \mathcal{M}_0)$ and a PD-morphism $\beta\colon (\tilde{\mathcal{S}}, \tilde{\mathcal{N}}) \to (\mathcal{S}, \mathcal{N})$ inducing a morphism $\beta_0\colon (\tilde{\mathcal{S}}_0, \tilde{\mathcal{N}}_0) \to (\mathcal{S}_0, \mathcal{N}_0)$ in a compatible manner in the obvious sense.

We further assume that $\tilde{\mathcal{X}}_0' = \mathcal{X}_0' \times_{\mathcal{X}_0} \tilde{\mathcal{X}}_0$. Then the distinguished triangles in Corollary 4.2.3 for $(\mathcal{X}_0, \mathcal{M}_0)/(\mathcal{S}, \mathcal{N}, I, \gamma)$, $\mathcal{X}_0'$ and for $(\tilde{\mathcal{X}}_0, \tilde{\mathcal{M}}_0)/(\tilde{\mathcal{S}}, \tilde{\mathcal{N}}, \tilde{I}, \tilde{\gamma})$, $\tilde{\mathcal{X}}_0'$ are compatible with the morphisms between the each component induced by α and β: Choose $(\mathcal{X}_0^\bullet, \mathcal{M}_0^\bullet) \hookrightarrow (\mathcal{Y}^\bullet, \mathcal{L}^\bullet)$, $\mathcal{Y}'^\bullet \subset \mathcal{Y}^\bullet$ for $(\mathcal{X}_0, \mathcal{M}_0)/(\mathcal{S}, \mathcal{N})$, and $(\tilde{\mathcal{X}}_0^\bullet, \tilde{\mathcal{M}}_0^\bullet) \hookrightarrow (\tilde{\mathcal{Y}}^\bullet, \tilde{\mathcal{L}}^\bullet)$, $\tilde{\mathcal{Y}}'^\bullet \subset \tilde{\mathcal{Y}}^\bullet$ for $(\tilde{\mathcal{X}}_0, \tilde{\mathcal{M}}_0)/(\tilde{\mathcal{S}}, \tilde{\mathcal{N}})$ as in Proposition 4.2.2. Let $\overline{\mathcal{Z}^\nu}$ be the blowing-up of $\tilde{\mathcal{Y}}^\nu \times_{\mathcal{S}} \mathcal{Y}^\nu$ along $\tilde{\mathcal{Y}}'^\nu \times_{\mathcal{S}} \mathcal{Y}'^\nu$, let $\mathcal{Z}^\nu$ be the complement of the strict transform of $\tilde{\mathcal{Y}}'^\nu \times_{\mathcal{S}} \mathcal{Y}^\nu \cup \tilde{\mathcal{Y}}^\nu \times_{\mathcal{S}} \mathcal{Y}'^\nu$ on $\overline{\mathcal{Z}^\nu}$, and let $\mathcal{Z}'^\nu$ be the inverse image of $\tilde{\mathcal{Y}}'^\nu \times_{\mathcal{S}} \mathcal{Y}'^\nu$. Let $\mathcal{M}_{\mathcal{Z}^\nu}$ be the inverse image of the log structure on $(\tilde{\mathcal{Y}}^\nu, \tilde{\mathcal{L}}^\nu) \times_{(\mathcal{S}, \mathcal{N})} (\mathcal{Y}^\nu, \mathcal{L}^\nu)$ to $\mathcal{Z}^\nu$. Then similarly as in the proof of Corollary 4.2.3, we see that $\{(\mathcal{Z}^\nu, \mathcal{M}_{\mathcal{Z}^\nu})\}_{\nu \geq 0}$ naturally become a simplicial fine log scheme smooth and integral over $(\tilde{\mathcal{S}}, \tilde{\mathcal{N}})$, there exists a closed immersion of $(\tilde{\mathcal{X}}_0^\nu \times_{\mathcal{X}_0} \mathcal{X}_0^\nu,$ the inverse image of $\tilde{\mathcal{M}}_0)$ into $(\mathcal{Z}^\nu, \mathcal{M}_{\mathcal{Z}^\nu})$ over $(\tilde{\mathcal{S}}, \tilde{\mathcal{N}})$ compatible with the simplicial structures, and this closed immersion with $\mathcal{Z}'^\nu \subset \mathcal{Z}^\nu$ satisfies the conditions in Proposition 4.2.2. Furthermore we have a natural morphism to the closed immersion $(\mathcal{X}_0^\bullet, \mathcal{M}_0^\bullet) \hookrightarrow (\mathcal{Y}^\bullet, \mathcal{L}^\bullet)$ (resp. $(\tilde{\mathcal{X}}_0^\bullet, \tilde{\mathcal{M}}_0^\bullet) \hookrightarrow (\tilde{\mathcal{Y}}^\bullet, \tilde{\mathcal{L}}^\bullet)$) such that $\mathcal{Z}'^\nu$ is the pull-back of $\mathcal{Y}'^\nu$ (resp. $\tilde{\mathcal{Y}}'^\nu$). This implies the required functoriality.

§4.3. Gysin sequence for syntomic cohomology.

Let the notation and the assumption as in §4.1. Assume that I is non-empty and choose one $i_0 \in I$. We will change the notation as follows: We write M° for M, and M will denote the log structure defined by the union of the special fiber of X and the divisors D_i $(i \in I, i \neq i_0)$. We define (X', M') to be D_{i_0} endowed with the inverse image of M. Then (X, M), (X, M°) and (X', M') satisfies the condition (4.1.1). Note that X' is a Cartier divisor on X and M° is the co-product of M and the log structure on X defined by X' (cf. §4.2).

We can construct Gysin sequence for syntomic cohomology as follows. We choose an affine étale covering $X^0 \to X$, a closed immersion of $(X^0, M|_{X^0})$ into a fine log scheme (Z^0, M_{Z^0}) smooth over $\mathrm{Spec}(W)$ endowed with a Cartier divisor Z'^0 defined by a global equation $g = 0$ and with a compatible system of liftings of Frobenius $\{F_{Z_n^0} \colon (Z_n^0, M_{Z_n^0}) \to (Z_n^0, M_{Z_n^0})\}_{n \geq 1}$ such that $X' \times_X X^0$ is the pull-back of Z'^0 and $F_{Z_n^0}^*(g) = g^p \cdot (1 + py)$ for some $y \in \mathcal{O}_{Z_n^0}$. Here the subscript n denotes the reduction mod p^n. Such a covering and an embedding exist by a similar argument as the proof of Proposition 4.2.1 (1). Starting from this embedding, we can construct an étale hypercovering $X^\bullet \to X$, a closed immersion $(X^\bullet, M^\bullet) \hookrightarrow (Z^\bullet, M_{Z^\bullet})$ and a Cartier divisor $Z'^\bullet \subset Z^\bullet$ as in Proposition 4.2.2 endowed with a compatible system of liftings of Frobenius on $\{(Z_n^\bullet, M_{Z_n^\bullet})\}_{n \geq 1}$. By taking the PD-envelope of $(X_n^\bullet, M_n^\bullet)$ in $(Z_n^\bullet, M_{Z_n^\bullet})$ compatible with the canonical PD-structure on pW_n and applying Proposition 4.2.1 (3), we obtain a short exact sequence on the étale site of the simplicial scheme $X_1^\bullet$ for each $r \geq 0$. By using the property $F_{Z_n^0}^*(g) = g^p \cdot (1 + py)$, we see that the short exact sequences are compatible with the Frobenius induced by $F_{Z_n^\bullet}$ and obtain a short exact sequence:

$$0 \to \mathcal{S}_n^{\sim}(r)_{(X^\bullet, M^\bullet), (Z^\bullet, M_{Z^\bullet})} \to \mathcal{S}_n^{\sim}(r)_{(X^\bullet, M^{\circ\bullet}), (Z^\bullet, M^\circ_{Z^\bullet})} \to \mathcal{S}_n^{\approx}(r-1)_{(X'^\bullet, M'^\bullet), (Z'^\bullet, M_{Z'^\bullet})}[-1] \to 0.$$

Here $\mathcal{S}_n^{\sim}(s)$ denotes the syntomic complex defined in [Ts2] §2.1, and $\mathcal{S}_n^{\approx}(s)$ denotes the complex obtained by replacing $p^s - \varphi$ with $p^{s+1} - p\varphi$ in the definition of $\mathcal{S}_n^{\sim}(s)$. By taking the derived direct image by the morphism of topoi $(X_1^\bullet)_{\text{ét}}^{\sim} \to (X_1)_{\text{ét}}^{\sim}$. We obtain the required distinguished triangle:

$$\to \mathcal{S}_n^{\sim}(r)_{(X,M)} \to \mathcal{S}_n^{\sim}(r)_{(X,M^\circ)} \to \mathcal{S}_n^{\approx}(r-1)_{(X',M')}[-1] \tag{4.3.1}$$

on $(X_1)_{\text{ét}}$. We can verify the independence and the functoriality similarly as in the crystalline case by taking care of liftings of Frobenius.

We define $V^q_{\text{syn},1}(r)$ to be the syntomic cohomology $H^q(\overline{Y}, \mathcal{S}_{\mathbb{Q}_p}(r)_{(\overline{X},\overline{M})})$ (cf.§3.3) of $(\overline{X}, \overline{M})$. We define $V^q_{\text{syn},2}(r)$ and $V^q_{\text{syn},3}(r)$ to be the syntomic cohomology of $(\overline{X}, \overline{M^\circ})$ and $(\overline{X'}, \overline{M'})$ respectively. Then, by taking $\mathbb{Q} \otimes \varprojlim_n \varinjlim_{K'} H^*((X_1 \times_{\text{Spec}(O_K)} \text{Spec}(O_{K'}))_{\text{ét}}, -)$ of the triangle (4.3.1) for the base changes of (X_n, M_n), (X_n, M_n°) and (X'_n, M'_n) by $O_K \to O_{K'}$, we obtain a complex:

$$\cdots \to V^q_{\text{syn},1}(r) \to V^q_{\text{syn},2}(r) \to V^{q-1}_{\text{syn},3}(r-1) \to V^{q+1}_{\text{syn},1}(r) \to \cdots \tag{4.3.2}$$

§4.4 COMPATIBILITY OF GYSIN SEQUENCES 1.

We will prove the compatibility of Gysin sequences with the isomorphisms (3.2.1), (3.3.2) and the homomorphism (3.3.4). We follow the notation in §3.3. We denote by D_1^q (resp. $\mathcal{D}_1^q$, resp. $\overline{\mathcal{D}}_1^q$) for the crystalline cohomology D^q (resp. $\mathcal{D}^q$, resp. $\overline{\mathcal{D}}^q$) for $(X, M)/(S, N)$ defined in §3.2 (resp. §3.3, resp. §3.3). We denote by D_2^q, $\mathcal{D}_2^q$ and $\overline{\mathcal{D}}_2^q$ for the cohomologies of $(X, M^\circ)/(S, N)$, and D_3^q, $\mathcal{D}_3^q$ and $\overline{\mathcal{D}}_3^q$ for the cohomologies of $(X', M')/(S, N)$. We denote by $D_3^q(-1)$, $\mathcal{D}_3^q(-1)$ and $\overline{\mathcal{D}}_3^q(-1)$ the same modules as D_3^q, $\mathcal{D}_3^q$ and $\overline{\mathcal{D}}_3^q$ whose Frobenius endomorphisms φ are replaced with $p\varphi$. By taking $\mathbb{Q} \otimes \varprojlim_n$ of the Gysin sequences for the crystalline cohomologies over the bases $(W_n, W_n(L), pW_n, \gamma)$ and $(E_n, M_{E_n}, \overline{J_{E_n}}, \overline{\delta})$, we obtain an exact sequence:

$$\cdots \to D_1^q \to D_2^q \to D_3^{q-1}(-1) \to D_1^{q+1} \to \cdots \tag{4.4.1}$$

and a complex:

$$\cdots \to \mathcal{D}_1^q \to \mathcal{D}_2^q \to \mathcal{D}_3^{q-1}(-1) \to \mathcal{D}_1^{q+1} \to \cdots . \tag{4.4.2}$$

LEMMA 4.4.3. *Let $\mathcal{D}_i$ $(i = 1, 2)$ be finitely generated free $R_{E,\mathbb{Q}_p}$-modules endowed with φ_E-semilinear endomorphisms $\varphi_{\mathcal{D}_i}$ whose linearizations $R_{E,\varphi} \otimes_{R_E} \mathcal{D}_i \to \mathcal{D}_i$ are isomorphisms. Let D_i be the reduction of $\mathcal{D}_i$ with respect to $R_{E,\mathbb{Q}_p} \to K_0$ induced by $\{i_{E_n,0}\}$ (§3.3) and let φ_{D_i} be the σ-semilinear automorphism of D_i induced by $\varphi_{\mathcal{D}_i}$. Suppose that we are given an $R_{E,\mathbb{Q}_p}$-linear homomorphism $f\colon \mathcal{D}_1 \to \mathcal{D}_2$ compatible with $\varphi_{\mathcal{D}_i}$ and K_0-linear sections $s_i\colon D_i \to \mathcal{D}_i$ of the canonical surjections $p_i\colon \mathcal{D}_i \to D_i$ compatible with $\varphi_{\mathcal{D}_i}$ and φ_{D_i}. Let $g\colon D_1 \to D_2$ be the K_0-linear homomorphism induced by f. Then we have $f \circ s_1 = s_2 \circ g$.*

Proof. We apply [Ts2] Lemma 4.4.11. Let I_n be as in loc. cit. For any $a \in D_1$, $f(s_1(\varphi_{D_1}^{-n}(a)))$ is a lifting of $g(\varphi_{D_1}^{-n}(a)) = \varphi_{D_2}^{-n}(g(a)) \in D_2$ in $\mathcal{D}_2$. Hence, by loc. cit., $s_2(g(a)) \equiv \varphi_{\mathcal{D}_2}^{n}(f(s_1(\varphi_{D_1}^{-n}(a)))) = f(s_1(a)) \mod I_n\mathcal{D}_2$. Since $\cap_n(I_n \otimes \mathbb{Q}) = 0$ and $\mathcal{D}_2$ is a free $R_{E,\mathbb{Q}_p}$-module, this implies $s_2(g(a)) = f(s_1(a))$. □

By functoriality, the projections $\mathcal{D}_i^q \to D_i^q$ (cf. §3.3) are compatible with (4.4.1) and (4.4.2). On the other hand, by a similar argument as the proof of the functoriality of the Gysin sequences, we also see that the sequences (4.4.1) and (4.4.2) are compatible with the Frobenius endomorphisms. Hence, from Lemma 4.4.3, we obtain the following:

Lemma 4.4.4. *The isomorphisms* $R_{E,\mathbb{Q}_p} \otimes_{K_0} D_i^q \cong \mathcal{D}_i^q$ $(i = 1, 2, 3)$ (§3.3) *are compatible with the sequences* (4.4.1) *and* (4.4.2).

Note that this and the exactness of (4.4.1) implies that (4.4.2) is also exact. We denote by $D_{\mathrm{dR},1}^q$ the de Rham cohomology $H^q(X_K, \Omega_{X_K}^{\bullet}(\log(M_K)))$ endowed with the Hodge filtration (cf. (3.2.1)), which is canonically isomorphic to the projective limit of the crystalline cohomology of (X_n, M_n) over $(S_n, N_n, p\mathcal{O}_{S_n}, \gamma)$ with respect to n tensored with K over O_K. We denote by $D_{\mathrm{dR},2}^q$ and $D_{\mathrm{dR},3}^q$ the de Rham cohomology of (X_K, M_K°) and $(X'_K, M_{K'})$ respectively. Recall that the Hodge spectral sequences for $D_{\mathrm{dR},i}^q$ degenerate (cf. [Ts2] Proposition 4.7.9). We denote by $D_{\mathrm{dR},3}^q(-1)$ the same K-vector space as $D_{\mathrm{dR},3}^q$ whose filtration is defined by $Fil^r(D_{\mathrm{dR},3}^q(-1)) = Fil^{r-1}D_{\mathrm{dR},3}^q$. Then, by taking $\mathbb{Q} \otimes \varprojlim_n$ of the Gysin sequence for the crystalline cohomology over the base $(S_n, N_n, p\mathcal{O}_{S_n}, \gamma)$, we obtain an exact sequence of filtered K-vector spaces:

$$\cdots \to D_{\mathrm{dR},1}^q \to D_{\mathrm{dR},2}^q \to D_{\mathrm{dR},3}^{q-1}(-1) \to D_{\mathrm{dR},1}^{q+1} \to \cdots \tag{4.4.5}$$

By functoriality, the projections $\mathcal{D}_i^q \to D_{\mathrm{dR},i}^q$ induced by $\{i_{E_n,\pi}\}$ (§3.3) are compatible with the exact sequences (4.4.2) and (4.4.5). Hence by combining with Lemma 4.4.4, we obtain the following compatibility:

Lemma 4.4.6. *The isomorphisms* $\rho_\pi \colon K \otimes_{K_0} D_i^q \cong D_{\mathrm{dR},i}^q$ (3.2.1) *are compatible with the exact sequences* (4.4.1) *and* (4.4.5).

By taking $\mathbb{Q} \otimes \varprojlim_n \varinjlim_{K'}$ of the Gysin sequence for the base changes of (X_n, M_n), (X_n, M_n°) and (X'_n, M'_n) by $(S', N') \to (S, N)$ over (E_n, M_{E_n}), we obtain a complex:

$$\cdots \to \overline{\mathcal{D}}_1^q \to \overline{\mathcal{D}}_2^q \to \overline{\mathcal{D}}_3^{q-1}(-1) \to \overline{\mathcal{D}}_1^{q+1} \to \cdots . \tag{4.4.7}$$

Here $S' = \mathrm{Spec}(O_{K'})$ and N' is the log structure defined by the closed point. By functoriality, the natural homomorphisms $\mathcal{D}_i^q \to \overline{\mathcal{D}}_i^q$ are compatible with (4.4.2) and (4.4.7). Hence, by Lemma 4.4.4, we obtain the following compatibility:

Lemma 4.4.8. *The isomorphisms* $\widehat{B_{\mathrm{st}}^+} \otimes_{K_0} D_i^q \cong \overline{\mathcal{D}}_i^q$ ([Ts2] *Proposition* 4.4.6) *are compatible with the sequence* (4.4.1) *and* (4.4.7).

Note that this lemma and the exactness of (4.4.1) imply the exactness of (4.4.7). By construction, it is clear that the distinguished triangle (4.3.1) is compatible with the distinguished triangle of Corollary 4.2.3 for (X_n, M_n), (X'_n, M'_n), ... over (W_n, pW_n, γ). Hence, by the functoriality of the Gysin sequence for crystalline cohomology, we obtain:

Lemma 4.4.9. *The natural homomorphisms* $V^q_{\mathrm{syn},i} \to \overline{\mathcal{D}}^q_i$ (3.3.4) *are compatible with the sequences* (4.3.2) *and* (4.4.7).

§4.5. Compatibility of Gysin sequences 2.

To prove Theorem 4.1.2, we also need to verify the compatibility of the Gysin sequence (4.3.2) of the syntomic cohomology with that of the étale cohomology. For simplicity, we omit the log structures from the notation of log schemes; we simply write X, X° and X' for the log schemes (X, M), (X, M°) and (X', M') appearing in §4.4. As in [O], we denote by $\underline{X}$, $\underline{S}$,... the underlying schemes of log schemes X, S,... (We do not adopt the notation $\overset{\circ}{X}$, $\overset{\circ}{S}$,... in [Na] Notation (1.1.2) and [I2]1.2 to avoid the confusion with the notation X°.)

Let X_{triv} (resp. $(X^\circ)_{\mathrm{triv}}$, resp. X'_{triv}) be the maximal open subschemes of X (resp. X°, resp. X') on which the log structure is trivial. We have $(X^\circ)_{\mathrm{triv}} = X_{\mathrm{triv}} \backslash X'_{\mathrm{triv}}$. We denote by V^q_1, V^q_2 and V^q_3 the q-th étale cohomology of $(X_{\mathrm{triv}})_{\overline{K}}$, $((X^\circ)_{\mathrm{triv}})_{\overline{K}}$ and $(X'_{\mathrm{triv}})_{\overline{K}}$ with coefficients $\mathbb{Q}_p$ respectively. Then we have the Gysin exact sequence:

$$(4.5.1) \qquad \cdots \to V^q_1 \to V^q_2 \to V^{q-1}_3(-1) \to V^{q+1}_1 \to \cdots$$

Proposition 4.5.2. *For any integer* $r \geq 0$*, the homomorphisms* $V^q_{\mathrm{syn},i}(r) \to V^q_i(r)$ *defined by* $p^{-r} \cdot$ (3.3.6) *are compatible with the sequences* (4.5.1) *and* (4.3.2).

Let i and i' denote the closed immersions $\underline{Y} \to \underline{X}$ and $\underline{Y'} \to \underline{X'}$ and let j, j° and j' denote the open immersions $X_{\mathrm{triv}} \to X$, $(X^\circ)_{\mathrm{triv}} \to X$ and $X'_{\mathrm{triv}} \to X'$ respectively. Proposition 4.5.2 follows from the following local version:

Proposition 4.5.3. *For any integer* $r \geq 0$*, the following diagram is commutative:*

$$\begin{array}{ccccccc}
\longrightarrow & \mathcal{S}^{\sim}_n(r)_X & \longrightarrow & \mathcal{S}^{\sim}_n(r)_{X^\circ} & \longrightarrow & \mathcal{S}^{\approx}_n(r-1)_{X'}[-1] \\
& \downarrow{\scriptstyle (3.3.5)} & & \downarrow{\scriptstyle (3.3.5)} & & \downarrow \\
\longrightarrow & i^* Rj_* \mathbb{Z}/p^n\mathbb{Z}(r)' & \longrightarrow & i^* Rj^\circ_* \mathbb{Z}/p^n\mathbb{Z}(r)' & \longrightarrow & i'^* Rj'_* \mathbb{Z}/p\mathbb{Z}(r)'(-1)[-1].
\end{array}$$

Here the right vertical homomorphism is the composite of

(4.5.4)

$$\mathcal{S}^{\approx}_n(r-1)_{X'} \to \mathcal{S}^{\sim}_n(r-1)_{X'} \xrightarrow{(3.3.5)} i'^* Rj'_* \mathbb{Z}/p^n(r-1)' \to i'^* Rj'_* \mathbb{Z}/p^n(r)'(-1),$$

where the first map is defined by the multiplication by p *on* $J^{[r-1-\bullet]}_{D'} \otimes \Omega^\bullet$ *and the identity map on* $\mathcal{O}_{D'} \otimes \Omega^\bullet$.

We will prove Proposition 4.5.3 in §4.8 after some preliminaries in §4.6 and §4.7. We will prove it along the following lines. By using the Gysin sequence (4.3.1) and explicit descriptions of $i^*Rj_*\mathbb{Z}/p^n\mathbb{Z}(r)'$ and $i^*Rj^\circ_*\mathbb{Z}/p^n\mathbb{Z}(r)'$ as complexes in terms of Godment resolutions, we construct a map

$$\alpha\colon \mathcal{S}_n^{\approx}(r-1)_{X'} \to i'^*Rj'_*\mathbb{Z}/p^n\mathbb{Z}(r)'(-1) \tag{4.5.5}$$

such that the diagram in Proposition 4.5.3 with (4.5.4) replaced by α is commutative. The main difficulty to compare α with (4.5.4) comes from the fact that the resolution $\overline{\mathcal{S}}_n(r)$ of $\mathbb{Z}/p^n\mathbb{Z}(r)'$ relating $\mathbb{Z}/p^n\mathbb{Z}(r)'$ with $\mathcal{S}_n^{\sim}(r)$ (cf. [Ts2] §3.1) does not behave well with respect to the closed immersion $X' \hookrightarrow X$. We overcome this problem by replacing $X' \to X \leftarrow X^\circ$ with $X' \stackrel{\mathrm{id}}{\to} X' \leftarrow X'^\circ$, where X'° is the scheme $\underline{X'}$ endowed with the inverse image of M_{X°. Although X'° is not log smooth, we still have a Gysin sequence for $\mathcal{S}_n^{\sim}(r)$ (4.7.11) and we can construct a map

$$\beta\colon \mathcal{S}_n^{\approx}(r-1)_{X'} \to i'^*Rj'_*\mathbb{Z}/p^n\mathbb{Z}(r)'(-1) \tag{4.5.6}$$

in the same way as α, which is easily seen to be equal to α above. (For the étale side, we need to use the Kummer étale sites of fs log schemes ([Na]).) For $X' \stackrel{\mathrm{id}}{\to} X' \leftarrow X'^\circ$, we also have a Gysin sequence for $\overline{\mathcal{S}}_n(r)$ (4.7.10), which allows us to compare the morphism β with (4.5.4).

§4.6. Preliminaries on log fundamental groups.

We will summarize some basic facts on log fundamental groups ([I2]§4) which we use in the proof of the compatibility of Gysin sequences for syntomic and étale cohomologies. We leave the most of their proofs to the readers. We continue to omit the log structures in the notation of log schemes.

A *logarithmic point* s is $\mathrm{Spec}(k)$ for a separably closed field k with a saturated log structure such that the multiplication by n is bijective on M_s/k^* for any positive integer n prime to the characteristic of k. A *log geometric point* of an fs log scheme S is a morphism $s \to S$ for a log geometric point s ([Na] Definition (2.5), [I2] Definition 4.1).

Lemma 4.6.1. *Let $T \to S$ be a Kummer étale morphism* ([Na] *Definition* (2.1.2), [I2] 1.6) *and let $\tilde{s} \to S$ be a log geometric point. If the image of $\tilde{s}$ in S is contained in the image of T in S, then there exists a lifting $\tilde{s} \to T$.*

Using the fact that a Kummer étale closed immersion is an open immersion, we can prove the following lemma in the same way as [SGA1] 5.3, 5.4.

Lemma 4.6.2. *Let $T \to S$ be a Kummer étale separated morphism of fs log schemes, let U be an fs log scheme over S and let $\varphi\colon \tilde{u} \to U$ be a log geometric point of U. Then the map $Hom_S(U,T) \to Hom_S(\tilde{u},T); f \mapsto f\circ\varphi$ is injective.*

Let S be a locally noetherian fs log scheme and let $\tilde{s} \to S$ be a log geometric point. Then the category $\mathrm{Kcov}(S)$ of Kummer étale covers of S ([I2] 3.1) with the fiber functor $F_{\tilde{s}}\colon \mathrm{Kcov}(S) \to (\mathrm{Sets})$ satisfies the axioms (G1) to (G6) of [SGA1] V§4. We define the log fundamental group of S at $\tilde{s}$ to be $\mathrm{Aut}(F_{\tilde{s}})$ ([I2]4.6). By Lemma 4.6.2, we see that the fiber functor $F_{\tilde{s}}(-)$ is canonically identified with $\mathrm{Hom}_S(\tilde{s}, -)$.

We consider an equi-characteristic connected normal scheme X with an fs log structure such that there exists a global chart $\alpha\colon \mathbb{N} \to \Gamma(X, M_X)$ whose composite with $\Gamma(X, M_X) \to \Gamma(X, \mathcal{O}_X)$ sends $\mathbb{N}\backslash\{0\}$ to 0.

Let $x = \mathrm{Spec}(k)$ be the generic point of X with the inverse image log structure. Let k^{sep} be a separable closure of k and let $\overline{x}$ be $\mathrm{Spec}(k^{\mathrm{sep}})$ endowed with the inverse image of M_x. Choose a chart α as above and define a log geometric point $\tilde{x}$ of X to be $\mathrm{Spec}(k^{\mathrm{sep}})$ with the log structure associated to $\widetilde{\mathbb{N}} \to k; a \neq 0 \mapsto 0$. Here $\widetilde{\mathbb{N}} = \cup_{n\in\mathbb{N}, n\in k^*} \frac{1}{n}\mathbb{N}$. We define the morphism $\tilde{x} \to \overline{x} \to x \to X$ by the natural inclusion $\mathbb{N} \to \widetilde{\mathbb{N}}$ and the chart α. $\tilde{x}$ is independent of the choice of the chart α up to non-canonical isomorphisms over $\overline{x}$.

For $n \in \mathbb{N}$ invertible in k, let t_n denote the image of $\frac{1}{n} \in \widetilde{\mathbb{N}}$ in $\Gamma(\tilde{x}, M_{\tilde{x}})$. Then the homomorphism $\mathrm{Aut}(\tilde{x}/\overline{x}) \to \hat{\mathbb{Z}}'(1) := \varprojlim_{n\in\mathbb{N}, n\in k^*} \mu_n(k^{\mathrm{sep}})$ defined by $\sigma \mapsto (\sigma(t_n)t_n^{-1})_{n\geq 1}$ is an isomorphism. Any automorphism of $\tilde{x}$ over x induces an automorphism of $\overline{x}$ over x and thus we obtain a surjective homomorphism $\mathrm{Aut}(\tilde{x}/x) \to \mathrm{Aut}(\overline{x}/x) = \mathrm{Gal}(k^{\mathrm{sep}}/k)^\circ$ with kernel $\mathrm{Aut}(\tilde{x}/\overline{x}) \cong \hat{\mathbb{Z}}'(1)$. Here $(-)^\circ$ denotes the the opposite group.

We define k^{ur} to be the union of all finite extensions k' of k contained in k^{sep} such that the normalizations of the underlying scheme of X in k' are unramified. We define x^{ur} and $\widetilde{x^{\mathrm{ur}}}$ similarly as $\overline{x}$ and $\tilde{x}$ using k^{ur} instead of k^{sep}. We have a canonical surjective homomorphism $\mathrm{Aut}(\tilde{x}/x) \to \mathrm{Aut}(\widetilde{x^{\mathrm{ur}}}/x)$ inducing an isomorphism $\hat{\mathbb{Z}}'(1) \cong \mathrm{Aut}(\tilde{x}/\overline{x}) \xrightarrow{\sim} \mathrm{Aut}(\widetilde{x^{\mathrm{ur}}}/x^{\mathrm{ur}})$. We also have a natural surjection $\mathrm{Aut}(\widetilde{x^{\mathrm{ur}}}/x) \to \mathrm{Aut}(x^{\mathrm{ur}}/x) \cong \mathrm{Gal}(k^{\mathrm{ur}}/k)^\circ$.

The fiber functor $F_{\tilde{x}}\colon \mathrm{Kcov}(X) \to (\mathrm{Sets})$ is explicitly pro-represented as follows. For each finite extension k' of k contained in k^{ur}, let $X_{k'}$ be a strict étale cover of X whose function field is k', and for a positive integer n invertible on X, we define $X_{k',n}$ to be the Kummer étale cover $X_{k'} \times_{\mathrm{Spec}(\mathbb{Z}[\mathbb{N}])} \mathrm{Spec}(\mathbb{Z}[\frac{1}{n}\mathbb{N}])$ of X. Then, the inclusions $k' \hookrightarrow k^{\mathrm{sep}}$ and $\frac{1}{n}\mathbb{N} \hookrightarrow \widetilde{\mathbb{N}}$ define a morphism $\tilde{x} \to X_{k',n}$. If k'/k is Galois and $\mu_n(k^{\mathrm{sep}}) \subset k'$, then $X_{k',n}/X$ is Galois i.e. $\mathrm{Aut}(X_{k',n}/X)$ acts transitively on $F_{\tilde{x}}(X_{k',n}) = \mathrm{Hom}_X(\tilde{x}, X_{k',n})$. We assert that $F_{\tilde{x}}$ is pro-represented by $\{X_{k',n}\}$ i.e. $\varinjlim_{k',n} \mathrm{Hom}_X(X_{k',n}, Y) \to \mathrm{Hom}_X(\tilde{x}, Y)$ is an isomorphism for any $Y \in \mathrm{Kcov}(X)$. The injectivity follows from Lemma 4.6.2. For the surjectivity, by Lemma 4.6.1 and Lemma 4.6.3 below, we may replace Y by $X_{k',n}$ with k'/k Galois and $\mu_n(k^{\mathrm{sep}}) \subset k'$. In this case $\mathrm{Hom}_X(X_{k',n}, X_{k',n}) \to \mathrm{Hom}_X(\tilde{x}, X_{k',n})$ is surjective.

Lemma 4.6.3. *For any Kummer étale cover $Y \to X$, there exists $n \in \mathbb{N}$ invertible on X and a strict étale cover $X' \to X$ such that $Y \times_X X'_n \to X'_n$ is trivial. Here $X'_n = X_n \times_{\mathrm{Spec}(\mathbb{Z}[\mathbb{N}])} Spec(\mathbb{Z}[\frac{1}{n}\mathbb{N}])$.*

The automorphism group $\mathrm{Aut}(\tilde{x}/x)$ naturally acts on the fiber functor $F_{\tilde{x}}\colon \mathrm{Kcov}(X) \to (\mathrm{Sets})$ and we obtain a homomorphism $\mathrm{Aut}(\tilde{x}/x)^\circ \to \pi_1(X, \tilde{x})$.

Proposition 4.6.4. *The above homomorphism factors through an isomorphism* $Aut(\widetilde{x^{\mathrm{ur}}}/x)^{\circ} \xrightarrow{\cong} \pi_1(X, \tilde{x})$.

Proof. Since $F_{\tilde{x}}$ is pro-represented by $\{X_{k',n}\}$ and $\tilde{x} \to X_{k',n}$ factors through $\widetilde{x^{\mathrm{ur}}}$, we see that the action of $\mathrm{Aut}(\tilde{x}/x)$ on $F_{\tilde{x}}$ factors through $\mathrm{Aut}(\widetilde{x^{\mathrm{ur}}}/x)$. For $m \in \mathbb{N}$ invertible on X, we denote by t_m the image of $(\frac{1}{m}, 1)$ by the (non-canonical) isomorphism $\widetilde{\mathbb{N}} \oplus (k^{\mathrm{ur}})^* \cong M_{\widetilde{x^{\mathrm{ur}}}}$. We have a bijection as sets $\mathrm{Aut}(\widetilde{x^{\mathrm{ur}}}/x) \to \mathrm{Gal}(k^{\mathrm{ur}}/k) \times \hat{\mathbb{Z}}'(1)(k^{\mathrm{ur}})$ sending σ to the pair of $\sigma^* \colon k^{\mathrm{ur}} \to k^{\mathrm{ur}}$ and $(\sigma^*(t_m)t_m^{-1})_m$. On the other hand, we have $\mathrm{Aut}(F_{\tilde{x}})^{\circ} \cong \varprojlim_{k',n} \mathrm{Aut}(X_{k',n})$, where (k', n) ranges over all finite Galois extensions k' of k contained in k^{ur} and $n \in \mathbb{N}$ invertible on X such that $\mu_n(k^{\mathrm{ur}}) \subset k'$. For such (k', n), we have a bijection $\mathrm{Aut}(X_{k',n}) \to \mathrm{Gal}(k'/k) \times \mu_n(k')$ sending τ to the pair of $\tau^* \colon k' \to k'$ and $\tau^*(t'_n)(t'_n)^{-1}$, where t'_n denotes the image of $\frac{1}{n}$ by the chart $\frac{1}{n}\mathbb{N} \to \Gamma(X_{k',n}, M_{X_{k',n}})$. Hence $\mathrm{Aut}(\widetilde{x^{\mathrm{ur}}}/x)^{\circ} \cong \pi_1(X, \tilde{x})$. $\square$

Next we consider an equi-characteristic connected regular scheme Z with the fs log structure associated to a regular divisor defined by the equation $t = 0$ for some $t \in \Gamma(Z, \mathcal{O}_Z)$, and assume that X is the divisor with the inverse image of M_Z. Set $Z_{\mathrm{triv}} = Z \backslash X$. Then the functor $\mathrm{Kcov}(Z) \to \mathrm{Etcov}(Z_{\mathrm{triv}}); W \mapsto W \times_Z Z_{\mathrm{triv}}$ induces an equivalence of categories from $\mathrm{Kcov}(Z)$ to the subcategory consisting of étale covers of Z_{triv} tamely ramified along X ([I2] Theorem 7.6). Let $z = \mathrm{Spec}(K)$ be the generic point of Z, choose a separable closure K^{sep} of K and set $\overline{z} := \mathrm{Spec}(K^{\mathrm{sep}})$. Let K^{ur} be the union of all finite extensions K' of K contained in K^{sep} such that the normalizations of Z_{triv} in K' are unramified and tamely ramified along X. Set $z^{\mathrm{ur}} := \mathrm{Spec}(K^{\mathrm{ur}})$.
We will give a way to construct a path from $\overline{z}$ to $\tilde{x}$. For a finite extension K' of K contained in K^{ur}, we denote by $Z_{K'}$ a Kummer étale cover of Z whose function field is K'. Then we have a natural morphism $\overline{z} \to Z_{K'}$ and the fiber functor $F_{\overline{z}} \colon \mathrm{Kcov}(Z) \to (\mathrm{Sets})$ is pro-represented by $\{Z_{K'}\}$. By Lemma 4.5.1, $\varprojlim_{K'} \mathrm{Hom}_Z(\tilde{x}, Z_{K'})$ is non-empty. An element $\varphi = \{\varphi_{K'}\}_{K'}$ of this set defines a path from $\overline{z}$ to $\tilde{x}$; it induces a map

$$\mathrm{Hom}_Z(\overline{z}, W) \xleftarrow{\sim} \varinjlim_{K' \subset K^{\mathrm{ur}}} \mathrm{Hom}_Z(Z_{K'}, W) \xrightarrow{\varphi \circ -} \mathrm{Hom}_Z(\tilde{x}, W)$$

for $W \in \mathrm{Kcov}(Z)$. For any $\sigma \in \mathrm{Aut}(\widetilde{x^{\mathrm{ur}}}/x)$, there exists a unique automorphism $\sigma_{K'}$ of $Z_{K'}$ such that $\sigma_{K'} \circ \varphi_{K'} = \varphi_{K'} \circ \sigma$ for each finite Galois extension K' of K contained in K^{ur}, and $\{\sigma_{K'}\}$ defines an automorphism $\tau \in \mathrm{Aut}(z^{\mathrm{ur}}/z)$. The homomorphism $\mathrm{Aut}(\widetilde{x^{\mathrm{ur}}}/x)^{\circ} \cong \pi_1(X, \tilde{x}) \to \pi_1(Z, \overline{z}) \cong \mathrm{Aut}(z^{\mathrm{ur}}/z)^{\circ}$ induced by the above path sends σ to τ. For another φ', there exists a unique $\tau \in \mathrm{Aut}(z^{\mathrm{ur}}/z)$ such that $\varphi' = \sigma \circ \varphi$.

§4.7. The complexes $\overline{\mathcal{S}}_n(r)$ and $\mathcal{S}_n^{\sim}(r)$.

We keep the notation of §4.5. Working étale locally on $\underline{X}$, we assume that $\underline{X}$ is affine and we are given a W-closed immersion of X into a fine log scheme Z smooth over W endowed with a Cartier divisor $\underline{Z'} \subset \underline{Z}$ defined by a global equation $g = 0$ ($g \in \Gamma(Z, \mathcal{O}_Z)$) and with a compatible system of liftings of Frobenius $\{F_{Z_n} \colon Z_n \to Z_n\}_{n\geq 1}$ such that $\underline{X'}$ is the pull-back of $\underline{Z'}$ and $\underline{Z'}$ endowed with the pull-back of M_Z is smooth over W. We denote by Z° the scheme $\underline{Z}$ endowed with the log structure defined by M_Z and the Cartier divisor $\underline{Z'}$ (cf. the beginning of §4.2), and by Z' (resp. Z'°) the scheme $\underline{Z'}$ with the the inverse image of M_Z (resp. M_{Z°). Note that Z'° is not smooth over W. We further assume that there exists $t_1, \dots, t_d \in \Gamma(Z, M_Z)$ and $t \in \Gamma(Z, \mathcal{O}_Z)$ such that $\underline{Z'}$ is defined by $t = 0$ in $\underline{Z}$, $\{d\log(t_i)(1 \leq i \leq d), dt\}$ (resp. $\{d\log(t_i)(1 \leq i \leq d), d\log(t)\}$, resp. $\{d\log(t_i)\}$) form a basis of $\Omega_{Z/W}$ (resp. $\Omega_{Z^\circ/W}$, resp. $\Omega_{Z'/W}$), and $F^*_{Z_n}(t_i) = t_i^p$, $F^*_{Z_n}(t) = t^p$ for each $n \geq 1$. Choose and fix such t_i and t. We have closed immersions $X^\circ \hookrightarrow Z^\circ$, $X' \hookrightarrow Z'$ and $X'^\circ \hookrightarrow Z'^\circ$. Recall that X'° is $\underline{X'}$ with the inverse image of M_{X°.

Let $U = \mathrm{Spec}(A) \to X$ be a strict étale morphism and set $U^\circ := X^\circ \times_X U$, $U' := X' \times_X U$ and $U'^\circ := X'^\circ \times_X U$. By replacing U with a suitable affine open covering, we assume that U, U° and U' satisfy the condition [Ts2] (1.5.2). (See [Ts2] Lemma 1.3.3). We may further assume that U'° also satisfies the equivalent conditions in [Ts2] Lemma 1.3.2 and $\Gamma(U', M_{U'})/\Gamma(U', \mathcal{O}^*_{U'}) \to \Gamma(U', M_{U'}/\mathcal{O}^*_{U'})$ is an isomorphism. (See the proof of [Ts2] Lemma 1.3.3 for the latter.) Let A' be the coordinate ring of U'. As in [Ts2] 1.4, let A^h be the henselization of A with respect to the ideal pA.

Let U^h be $\mathrm{Spec}(A^h)$ with the inverse image of M_U and set $U^{h\circ} := X^\circ \times_X U^h$, $U'^h := X' \times_X U^h$ and $U'^{h\circ} := X'^\circ \times_X U^h$. The coordinate ring of $\underline{U'^h} = \underline{U'^{h\circ}}$ is the henselization of A' with respect to pA', which we denote by A'^h. Let U^h_{triv}, $(U^{h\circ})_{\mathrm{triv}}$ and U'^h_{triv} denote the maximal open subschemes of U^h, $U^{h\circ}$ and U'^h respectively on which the log structures are trivial and let A^h_{triv}, $(A^{h\circ})_{\mathrm{triv}}$ and A'^h_{triv} denote their coordinate rings. Finally we define $(X_{\mathrm{triv}})^\circ$ (resp. $(X'_{\mathrm{triv}})^\circ$) to be X_{triv} (resp. X'_{triv}) endowed with the inverse image of M_{X°, and define $(U^h_{\mathrm{triv}})^\circ$ and $(U'^h_{\mathrm{triv}})^\circ$ similarly. Note that the log structure of $(X_{\mathrm{triv}})^\circ$ is the one defined by the divisor $X'_{\mathrm{triv}} \hookrightarrow X_{\mathrm{triv}}$.

Now we have the following commutative diagrams:

$$
\begin{array}{ccccc}
((X^\circ)_{\mathrm{triv}})_{\mathrm{\acute{e}t}} & \to & ((X_{\mathrm{triv}})^\circ)_{\mathrm{K\acute{e}t}} & \overset{\varepsilon}{\to} & (X_{\mathrm{triv}})_{\mathrm{\acute{e}t}} \\
 & & \uparrow & & \uparrow \\
 & & ((X'_{\mathrm{triv}})^\circ)_{\mathrm{K\acute{e}t}} & \overset{\varepsilon'}{\to} & (X'_{\mathrm{triv}})_{\mathrm{\acute{e}t}}
\end{array}
\tag{4.7.1}
$$

$$
\begin{array}{ccccc}
((U^{h\circ})_{\mathrm{triv}})_{\mathrm{\acute{e}t}} & \to & ((U^h_{\mathrm{triv}})^\circ)_{\mathrm{K\acute{e}t}} & \overset{\varepsilon_U}{\to} & (U^h_{\mathrm{triv}})_{\mathrm{\acute{e}t}} \\
 & & \uparrow & & \uparrow \\
 & & ((U'^h_{\mathrm{triv}})^\circ)_{\mathrm{K\acute{e}t}} & \overset{\varepsilon'_U}{\to} & (U'^h_{\mathrm{triv}})_{\mathrm{\acute{e}t}}
\end{array}
\tag{4.7.2}
$$

Here Két denotes the Kummer étale site ([Na]). Note ét = Két for schemes with trivial log structures. We have natural morphisms from (4.7.2) to (4.7.1). We will construct a resolution $\overline{\mathcal{S}}_n(r)$ of $\mathbb{Z}/p^n\mathbb{Z}(r)'$ on each site in the diagram (4.7.2) in a compatible manner. For $(U^{h\circ})_{\mathrm{triv}}$, $(U^h_{\mathrm{triv}})^\circ$ and U'^h_{triv}, we can directly apply [Ts2]§3.1, but for the other two, we need some modifications.

Let $\eta := \mathrm{Spec}(\mathcal{K})$ be the generic point of U^h_{triv}. Choose an algebraic closure $\overline{\mathcal{K}}$ of $\mathcal{K}$ and set $\overline{\eta} := \mathrm{Spec}(\overline{\mathcal{K}})$. We define $\mathcal{K}^{\mathrm{ur}}$ (resp. $\mathcal{K}^{\circ\mathrm{ur}}$) to be the union of all finite extensions $\mathcal{L}$ of $\mathcal{K}$ contained in $\overline{\mathcal{K}}$ such that the normalizations of U^h_{triv} (resp. $(U^{h\circ})_{\mathrm{triv}}$) in $\mathcal{L}$ are unramified. We define η', $\mathcal{K}'$, $\overline{\mathcal{K}'}$, $\overline{\eta'}$ and $\mathcal{K}'^{\mathrm{ur}}$ similarly using U'^h_{triv}. We set

$$\begin{aligned} G_U &:= \mathrm{Gal}(\mathcal{K}^{\mathrm{ur}}/\mathcal{K}) \cong \pi_1(U^h_{\mathrm{triv}}, \overline{\eta}), \\ G_U^\circ &:= \mathrm{Gal}(\mathcal{K}^{\circ\mathrm{ur}}/\mathcal{K}) \cong \pi_1((U^{h\circ})_{\mathrm{triv}}, \overline{\eta}) \cong \pi_1((U^h_{\mathrm{triv}})^\circ, \overline{\eta}), \\ G'_U &:= \mathrm{Gal}(\mathcal{K}'^{\mathrm{ur}}/\mathcal{K}') \cong \pi_1(U'^h_{\mathrm{triv}}, \overline{\eta'}). \end{aligned}$$

See [I2]Theorem 7.6 for the last isomorphism in the second line. We define η'° to be $\mathrm{Spec}(\mathcal{K}')$ with the inverse image of $M_{(U'^h_{\mathrm{triv}})^\circ}$ and define $\widetilde{\eta'^\circ}$ and $\widetilde{\eta'^{\circ\mathrm{ur}}}$ similarly as $\tilde{x}$ and $\widetilde{x^{\mathrm{ur}}}$ in §4.6 using $\overline{\mathcal{K}'}$ and $\mathcal{K}'^{\mathrm{ur}}$. We set

$$G'^\circ_U := \mathrm{Aut}(\widetilde{\eta'^{\circ\mathrm{ur}}}/\eta'^\circ)^\circ \cong \pi_1((U'^h_{\mathrm{triv}})^\circ, \widetilde{\eta'^\circ}).$$

See Proposition 4.6.4 for the second isomorphism.
For a finite extension $\mathcal{L}$ of $\mathcal{K}$ contained in $\mathcal{K}^{\circ\mathrm{ur}}$, denote by $V_{\mathcal{L}}$ a Kummer étale cover of $(U^h_{\mathrm{triv}})^\circ$ whose function field is $\mathcal{L}$. We choose and fix a compatible system $\{f_{\mathcal{L}} \colon \widetilde{\eta'^\circ} \to V_{\mathcal{L}}\}_{\mathcal{L}\subset\mathcal{K}^{\circ\mathrm{ur}}}$, which gives a path from $\widetilde{\eta'^\circ} \to (U'^h_{\mathrm{triv}})^\circ \to (U^h_{\mathrm{triv}})^\circ$ to $\overline{\eta} \to (U^h_{\mathrm{triv}})^\circ$ (§4.6). It also induces a compatible system $\{\underline{f_{\mathcal{L}}} \colon \overline{\eta'} \to \underline{V_{\mathcal{L}}}\}_{\mathcal{L}\subset\mathcal{K}^{\mathrm{ur}}}$, which gives a path from $\overline{\eta'}$ to $\overline{\eta}$. These paths induce homomorphisms $G'^\circ_U \to G^\circ_U$ and $G'_U \to G_U$ which are compatible with the natural homomorphisms $G^\circ_U \to G_U$ and $G'^\circ_U \to G'_U$.
We define $\overline{A^h}$ (resp. $\overline{A^h_{\mathrm{triv}}}$) to be the normalization of A^h (resp. A^h_{triv}) in $\mathcal{K}^{\mathrm{ur}}$. Similarly, we define $\overline{A^{h\circ}}$ and $\overline{(A^{h\circ})_{\mathrm{triv}}}$ (resp. $\overline{A'^h}$ and $\overline{A'^h_{\mathrm{triv}}}$) using A^h, $(A^{h\circ})_{\mathrm{triv}}$ and $\mathcal{K}^{\circ\mathrm{ur}}$ (resp. A'^h, A'^h_{triv} and $\mathcal{K}'^{\mathrm{ur}}$). By applying [Ts2]§1.4 and §1.5 to U, U°, U' and $\overline{A^h}$, $\overline{A^{h\circ}}$, $\overline{A'^h}$, we obtain a commutative diagram:

$$\begin{array}{ccccc} U'^h & \leftarrow & \overline{U'} & \hookrightarrow & \overline{D'} \\ \downarrow & & \downarrow & & \downarrow \\ U^h & \leftarrow & \overline{U} & \hookrightarrow & \overline{D} \\ \uparrow & & \uparrow & & \uparrow \\ U^{h\circ} & \leftarrow & \overline{U^\circ} & \hookrightarrow & \overline{D^\circ} \end{array}$$

compatible with the actions of G^*_U on $\overline{U^*}$ and $\overline{D^*}$ and with the liftings of Frobenius on $\overline{D^*}$ ($* = \emptyset, \prime, \circ$). The upper vertical maps are induced by the path from $\overline{\eta'}$ to $\overline{\eta}$ chosen above.
Since Z° and Z' satisfy the condition [Ts2] (2.1.1), we can construct resolutions $\overline{\mathcal{S}}_n(r)_{U^\circ,Z^\circ}$ and $\overline{\mathcal{S}}_n(r)_{U',Z'}$ of $\mathbb{Z}/p^n\mathbb{Z}(r)'$ on $((U^{h\circ})_{\mathrm{triv}})_{\mathrm{\acute{e}t}}$ (or $((U^h_{\mathrm{triv}})^\circ)_{\mathrm{K\acute{e}t}}$) and on $(U'^h_{\mathrm{triv}})_{\mathrm{\acute{e}t}}$ as in [Ts2] §3.1. Although Z does not satisfy [Ts2] (2.1.1), the construction in [Ts2] §3.1 still works as follows.

LEMMA 4.7.3. (1) *The absolute Frobenius of* $\overline{A^h}/p\overline{A^h}$ *is surjective.*

(2) *The homomorphism* $Fil^1 A_{\mathrm{crys}}(\overline{A^h}) \to \widehat{\overline{A^h}}$ *defined by* $x \mapsto p^{-1}\varphi(x) \mod Fil^1$ *is surjective.*

Proof. (1) For any $a \in \overline{A^h}$, there exists $u \in (\overline{A^h})^*$ such that $1 + p^{1/2}a = u^p$ and hence $p^{1/2}a \equiv (u-1)^p \mod p\overline{A^h}$. Set $v = (u-1)\cdot p^{-1/2p}$. Then $v^p \in \overline{A^h}$ and hence $v \in \overline{A^h}$. Thus we obtain $a \equiv v^p \mod p^{1/2}\overline{A^h}$. Set $b = (a - v^p)\cdot p^{-1/2}$. Then, by the same argument, there exists $w \in \overline{A^h}$ such that $b \equiv w^p \mod p^{1/2}\overline{A^h}$. Now we have $a \equiv (v + p^{1/2p}w)^p \mod p\overline{A^h}$. (2) By (1), the homomorphism $A_{\mathrm{crys}}(\overline{A^h}) \to \widehat{\overline{A^h}}$ is surjective (cf. [Ts2] Lemma A1.1). Hence the claim follows from the surjectivity of $1 - p^{-1}\varphi\colon Fil^1 A_{\mathrm{crys}}(\overline{A^h}) \to A_{\mathrm{crys}}(\overline{A^h})$ (cf. [Ts2] Theorem A3.26 and Proposition A3.33). □

As in [Ts2] §3.1, let $\overline{E}_n$ be the PD-envelope of $\overline{U}_n \hookrightarrow \overline{D}_n \times_{W_n} Z_n$ compatible with the PD-structure on $J_{\overline{D}_n} + p\mathcal{O}_{\overline{D}_n}$. For each $t_i \in \Gamma(Z, M_Z)$, choose and fix a lifting $a_i \in \Gamma(\overline{D}, M_{\overline{D}})$ of the image of t_i in $\Gamma(\overline{U}, M_{\overline{U}})$ such that $F^*_{\overline{D}}(a_i) = a_i^p$ ([Ts2] Lemma 3.1.5). Let $u_i \in \Gamma(\overline{E}_n, 1 + J_{\overline{E}_n})$ be the unique element such that $t_i = a_i \cdot u_i$ in $\Gamma(\overline{E}_n, M_{\overline{E}_n})$. For $t \in \Gamma(Z, \mathcal{O}_Z)$, we choose a lifting $a \in \Gamma(\overline{D}, \mathcal{O}_{\overline{D}}) = A_{\mathrm{crys}}(\overline{A^h})$ of the image of t in $\Gamma(\overline{U}, \mathcal{O}_{\overline{U}}) = \widehat{\overline{A^h}}$. By Lemma 4.7.3 (2), we may assume that $a^p - \varphi(p) \in pFil^1 A_{\mathrm{crys}}(\overline{A^h})$. Similarly as [Ts2] Lemma 3.1.4, we see that there exists a PD-isomorphism over $\mathcal{O}_{\overline{D}_n}$:

$$\mathcal{O}_{\overline{D}_n}\langle V_1, \dots, V_d, V\rangle \xrightarrow{\sim} \mathcal{O}_{\overline{E}_n}; V_i \mapsto u_i - 1, V \mapsto t - a$$

We define $\widetilde{J^{[r]\prime}_{\overline{D}_n}}$ and $\widetilde{J^{[r]\prime}_{\overline{E}_n}}$ as in [Ts2] §3.1.

LEMMA 4.7.4. (cf. [Ts2] Lemma 3.1.6). *For each r, we have*

$$\widetilde{J^{[r]\prime}_{\overline{E}_n}} = \bigoplus_{\underline{m}\in\mathbb{N}^{d+1}} \widetilde{J^{[r]\prime}_{\overline{D}_n}} \prod_{1\le i\le d} (u_i - 1)^{[m_i]} \cdot (t-a)^{[m]},$$

where $\underline{m} = (m_1, \dots, m_d, m)$.

Proof. Since we can apply the same argument as the proof of [Ts2] Lemma 3.1.6 to the ring $R_n := \mathcal{O}_{\overline{D}_n}\langle u_1 - 1, \dots, u_d - 1\rangle$, it suffices to show the following: For $x = \sum_{m\in\mathbb{N}} x_m (t-a)^{[m]} \in \mathcal{O}_{\overline{E}_n} = R_n\langle t - a\rangle$, we have $\varphi(x) \in p^r\mathcal{O}_{\overline{E}_n}$ if and only if $\varphi(x_m) \in p^{\max\{r-m,0\}} R_n$ for all $m \in \mathbb{N}$. The sufficiency follows from

$$\begin{aligned}\varphi(t-a) &= t^p - \varphi(a) = (t - a + a)^p - \varphi(a)\\ &= p\big\{(p-1)!(t-a)^{[p]} + \sum_{\nu=1}^{p-1} \frac{1}{p}\binom{p}{\nu}(t-a)^{p-\nu}a^\nu\big\} + a^p - \varphi(p) \in pJ_{\overline{E}_n}.\end{aligned}$$

Assume $x \neq 0$ and $\varphi(x) \in p^r R_n\langle t-a\rangle$, and let M be the largest integer such that $x_M \neq 0$. Then the coefficient of $(t-a)^{[Mp]}$ in $\varphi(x)$ is $\varphi(x_M)p^M c$ for some $c \in \mathbb{Z}_p^* \cap \mathbb{Q}$. Hence $p^M\varphi(x_M) \in p^r R_n$, which implies $\varphi(x_M) \in p^{\max\{r-M,0\}}R_n$. By the sufficiency, we can subtract $x_M(t-a)^{[M]}$, repeat the argument and show $\varphi(x_m) \in p^{\max\{r-m,0\}}R_n$ for all m. □

By Lemma 4.7.4, we can construct a complex $J^{[r]\prime}_{\overline{E}_n} \otimes_{\mathcal{O}_{Z_n}} \Omega^\bullet_{Z_n/W_n}$ and show that it gives a resolution of $J^{[r]\prime}_{\overline{D}_n}$ as in [Ts2] Lemma 3.1.7. (Since $d(t-a)^{[m]} = (t-a)^{[m-1]}dt$, it is enough to use dt instead of $u_i d\log(t_i)$ for the indeterminate $t-a$.) Thus we obtain a resolution $\overline{\mathcal{S}}_n(r)_{U,Z}$ of $\mathbb{Z}/p^n\mathbb{Z}(r)'$ on $(U^h_{\mathrm{triv}})_{\text{ét}}$ by the same method as [Ts2] §3.1. We have natural maps from the pull-backs of $\overline{\mathcal{S}}_n(r)_{U,Z}$ to $\overline{\mathcal{S}}_n(r)_{U',Z'}$ and $\overline{\mathcal{S}}_n(r)_{U^\circ,Z^\circ}$. We will need the following lemma to construct the map α (4.5.5).

Lemma 4.7.5. *The natural map $\overline{\mathcal{S}}_n(r)_{U,Z} \to \varepsilon_{U*}\overline{\mathcal{S}}_n(r)_{U^\circ,Z^\circ}$ is injective.* (*See* (4.7.2) *for* ε_U.)

Let $\overline{E^\circ}_n$ be the PD-envelope of $\overline{U^\circ}_n \hookrightarrow \overline{D^\circ}_n \times Z^\circ_n$. Choose a lifting $a' \in \Gamma(\overline{D^\circ}, M_{\overline{D^\circ}})$ of the image of t in $\Gamma(\overline{U^\circ}, M_{\overline{U^\circ}})$ such that $F^*_{\overline{D^\circ}}(a') = (a')^p$ ([Ts2] Lemma 3.1.5) and let $u \in \Gamma(\overline{E^\circ}_n, 1 + J_{\overline{E^\circ}_n})$ be the unique element such that $a' \cdot u = t$ in $\Gamma(\overline{E^\circ}_n, M_{\overline{E^\circ}_n})$. We denote the image of $u_i \in \Gamma(\overline{E}_n, 1 + J_{\overline{E}_n})$ (defined above) in $\Gamma(\overline{E^\circ}_n, 1 + J_{\overline{E^\circ}_n})$ by the same letter u_i. Then we have an isomorphism:

$$\mathcal{O}_{\overline{E^\circ}_n} \cong \mathcal{O}_{\overline{D^\circ}_n}\langle u_1 - 1, \ldots, u_d - 1, u - 1\rangle.$$

Proof Lemma 4.7.5. It suffices to prove that the natural map $J^{[r]\prime}_{\overline{E}_n} \to J^{[r]\prime}_{\overline{E^\circ}_n}$ and the multiplication by t on $J^{[r]\prime}_{\overline{E^\circ}_n}$ are injective. We first prove the claim for $r = 0$. Since the map $\mathcal{O}_{\overline{E}_n} \to \mathcal{O}_{\overline{E^\circ}_n}$ factors through $\mathcal{O}_{\overline{D^\circ}_n}\langle u_i - 1, t - a\rangle \cong \mathcal{O}_{\overline{D^\circ}_n}\langle u_i - 1, t - a'\rangle$ and $t = a'u$, it is enough to prove that $\mathcal{O}_{\overline{D}_n} \to \mathcal{O}_{\overline{D^\circ}_n}$ and the multiplication by a' on $\mathcal{O}_{\overline{D^\circ}_n}$ are injective. We are easily reduced to the case $n = 1$. Define $R_{\overline{A^h}}$ and $R_{\overline{A^{h\circ}}}$ as in [Ts2] §1.1 and let z be a generator of the kernel of $W(R_{\overline{A^h}}) \to \widehat{\overline{A^h}}$ ([Ts2] Corollary A2.2). Then, by [Ts2] Lemma A2.11, we see that $\Gamma(\overline{D}_1, \mathcal{O}_{\overline{D}_1}) = A_{\mathrm{crys}}(\overline{A^h})/p$ (resp. $\Gamma(\overline{D^\circ}_1, \mathcal{O}_{\overline{D^\circ}_1}) = A_{\mathrm{crys}}(\overline{A^{h\circ}})/p$) is a free $R_{\overline{A^h}}/z^p$ (resp. $R_{\overline{A^{h\circ}}}/z^p$)-module with a base $\{z^{[pn]}|n \geq 0\}$. Especially, the filtration is separated and each graded quotient is isomorphic to $R_{\overline{A^h}}/z \cong \overline{A^h}/p$ (resp. $R_{\overline{A^{h\circ}}}/z \cong \overline{A^{h\circ}}/p$) (Lemma 4.7.3 (1) and [Ts2] Lemma A2.1). Hence the claim follows from the injectivity of the natural map $\overline{A^h}/p \to \overline{A^{h\circ}}/p$ and the multiplication by t on $\overline{A^{h\circ}}/p$. Next we consider the case $r \geq 1$. Let $x \in \widetilde{J^{[r]\prime}_{\overline{E}_{n+r}}}$ and assume that its image in $\widetilde{J^{[r]\prime}_{\overline{E^\circ}_{n+r}}}$ is contained in $p^n\widetilde{J^{[r]\prime}_{\overline{E^\circ}_{n+r}}}$. Then it is contained in $p^n J^{[r]}_{\overline{E^\circ}_{n+r}}$ and hence $x \in p^n J^{[r]}_{\overline{E}_{n+r}}$ by the case $r = 0$. Choose

$y\in J^{[r]}_{\overline{E}_{n+r}}$ such that $x=p^ny$ and let y° be the image of y in $J^{[r]}_{\overline{E^\circ}_{n+r}}$. Then $p^ny^\circ\in p^n\widetilde{J^{[r]\prime}_{\overline{E^\circ}_{n+r}}}$, which implies $y^\circ\in \widetilde{J^{[r]\prime}_{\overline{E^\circ}_{n+r}}}$ i.e. $\varphi(y^\circ)\in p^r\mathcal{O}_{\overline{E^\circ}_{n+r}}$. By the case $r=0$, this implies $\varphi(y)\in p^r\mathcal{O}_{\overline{E}_{n+r}}$ and hence $y\in \widetilde{J^{[r]\prime}_{\overline{E}_{n+r}}}$. We can prove the second assertion similarly. Note $\varphi(a')=(a')^p$. $\square$

Now it remains to construct a resolution $\overline{\mathcal{S}}_n(r)_{U'^\circ,Z'^\circ}$ of $\mathbb{Z}/p^n\mathbb{Z}(r)'$ on $((U'_{\mathrm{triv}})^\circ)_{\mathrm{K\acute{e}t}}$. To do this we need to construct $\overline{U'}^\circ\hookrightarrow\overline{D'}^\circ$ for the non-smooth U'° modifying the construction in [Ts2] §1.4, §1.5.
The underlying scheme of $\overline{D'}^\circ$ is the same as $\overline{D'}$ i.e. $\mathrm{Spec}(A_{\mathrm{crys}}(\overline{A'^h}))$. For the log structure, we use the fiber product Q'° of the diagram of monoids:

$$\varprojlim(M_{\widetilde{\eta'^\circ\mathrm{ur}}}\overset{f}{\leftarrow}M_{\widetilde{\eta'^\circ\mathrm{ur}}}\overset{f}{\leftarrow}M_{\widetilde{\eta'^\circ\mathrm{ur}}}\overset{f}{\leftarrow}\cdots)\to M_{\widetilde{\eta'^\circ\mathrm{ur}}}\leftarrow\Gamma(U'^\circ,M_{U'^\circ}),$$

where $f(x)=x^p$ and the left map is the projection to the first component. Choose a chart $\widetilde{\mathbb{N}}\to M_{\widetilde{\eta'^\circ\mathrm{ur}}}$ compatible with the chart $\mathbb{N}\to\Gamma(U'^\circ,M_{U'^\circ})$ sending 1 to the image of t (cf. the definition of $\widetilde{x^{\mathrm{ur}}}$ in §4.6). The chart induces an isomorphism $\widetilde{\mathbb{N}}\oplus(\mathcal{K}'^{\mathrm{ur}})^*\cong M_{\widetilde{\eta'^\circ\mathrm{ur}}}$. By $\Gamma(U',M_{U'}/\mathcal{O}^*_{U'})\oplus\mathbb{N}\cong\Gamma(U'^\circ,M_{U'^\circ}/\mathcal{O}^*_{U'^\circ})$ and the assumption $\Gamma(U',M_{U'})/\Gamma(U',\mathcal{O}^*_{U'})\cong\Gamma(U',M_{U'}/\mathcal{O}^*_{U'})$, we see $\Gamma(U',M_{U'})/\Gamma(U',\mathcal{O}^*_{U'})\oplus\mathbb{N}\cong\Gamma(U'^\circ,M_{U'^\circ})/\Gamma(U'^\circ,\mathcal{O}^*_{U'^\circ})$ and hence $\Gamma(U'^\circ,M_{U'^\circ})$ is generated by the images of t and $\Gamma(U',M_{U'})$. This implies that the image of $\Gamma(U'^\circ,M_{U'^\circ})$ in $M_{\widetilde{\eta'^\circ\mathrm{ur}}}\cong\widetilde{\mathbb{N}}\oplus(\mathcal{K}'^{\mathrm{ur}})^*$ is contained in $\mathbb{N}\oplus((A'^h_{\mathrm{triv}})^*\cap A'^h)$. Hence Q'° coincides with the fiber product of

$$\varprojlim_f\big(\widetilde{\mathbb{N}}\oplus((\overline{A'^h_{\mathrm{triv}}})^*\cap\overline{A'^h})\big)\to\widetilde{\mathbb{N}}\oplus((\overline{A'^h_{\mathrm{triv}}})^*\cap\overline{A'^h})\leftarrow\Gamma(U'^\circ,M_{U'^\circ})$$

and the morphism $Q'^\circ\to\varprojlim_f M_{\widetilde{\eta'^\circ\mathrm{ur}}}\to\varprojlim_f\mathcal{K}'^{\mathrm{ur}}$ factors through $\varprojlim_f\overline{A'^h}$. We define the log structure of $\overline{D'}^\circ$ to be the one associated to

$$Q'^\circ\to\varprojlim\overline{A^h}\to R_{\overline{A^h}}\overset{[\,]}{\to}W(R_{\overline{A'^h}})\subset A_{\mathrm{crys}}(\overline{A'^h}).$$

Using the natural action of G'°_U on Q'° and the multiplication by p on Q'°, we can define the action of G'°_U and the lifting of Frobenius on $\overline{D'}^\circ$.
We define $\overline{U'}^\circ$ to be $\mathrm{Spec}(\widehat{\overline{A'^h}})$ with the log structure associated to $\Gamma(U'^\circ,M_{U'^\circ})\to\widehat{\overline{A'^h}}$. We have a natural action of G'°_U (through G'_U) on $\overline{U'}^\circ$ and G'°_U-equivariant morphism $\overline{U'}^\circ\to\overline{D'}^\circ$ induced by the surjection $A_{\mathrm{crys}}(\overline{A'^h})\to\widehat{\overline{A'^h}}$ (cf. Lemma 4.7.3) and $Q'^\circ\to\Gamma(U'^\circ,M_{U'^\circ})$.
We assert that $\overline{U'}^\circ$ and $\overline{D'}^\circ$ are fs log schemes and the morphism $\overline{U'}^\circ\to\overline{D'}^\circ$ is an exact closed immersion. To prove this, we choose a chart $P\to\Gamma(U'^\circ,M_{U'^\circ})$ such that $P\to\Gamma(U'^\circ,M_{U'^\circ})/\Gamma(U'^\circ,\mathcal{O}^*_{U'^\circ})$ is an isomorphism ([Ts2] Lemma

1.3.2). Then the composite $P \to \Gamma(U'^{\circ}, M_{U'^{\circ}}) \to \widetilde{\mathbb{N}} \oplus ((\overline{A'^{h}_{\mathrm{triv}}})^* \cap \overline{A'^{h}})$ can be lifted to the projective limit $\varprojlim_f (\widetilde{\mathbb{N}} \oplus ((\overline{A'^{h}_{\mathrm{triv}}})^* \cap \overline{A'^{h}}))$. Hence $P \to \Gamma(U'^{\circ}, M_{U'^{\circ}})$ can be lifted to $P \to Q'^{\circ}$. On the other hand, since the image of $\Gamma(U'^{\circ}, \mathcal{O}^*_{U'^{\circ}})$ in $\widetilde{\mathbb{N}} \oplus (\overline{A'^{h}_{\mathrm{triv}}}^* \cap \overline{A'^{h}})$ is contained in $\overline{A'^{h}}^*$, the inverse image G of $\{1\}$ under $Q'^{\circ} \to \Gamma(U'^{\circ}, M_{U'^{\circ}})/\Gamma(U'^{\circ}, \mathcal{O}^*_{U'^{\circ}})$ is a group and we have $Q'^{\circ}/G \cong \Gamma(U'^{\circ}, M_{U'^{\circ}})/\Gamma(U'^{\circ}, \mathcal{O}^*_{U'^{\circ}}) \cong P$. Hence, by [Ts2] Lemma 1.3.1, $P \to \Gamma(U'^{\circ}, M_{U'^{\circ}}) \to \Gamma(\overline{U'}^{\circ}, M_{\overline{U'}^{\circ}})$ and $P \to Q'^{\circ} \to \Gamma(\overline{D'}^{\circ}, M_{\overline{D'}^{\circ}})$ are charts, and $\overline{U'}^{\circ} \to \overline{D'}^{\circ}$ is an exact closed immersion.
Next we compare $\overline{U'}^{\circ} \to \overline{D'}^{\circ}$ with $\overline{U'} \to \overline{D'}$ and $\overline{U^{\circ}} \to \overline{D^{\circ}}$. The fiber product Q' of $\varprojlim_f \overline{A'^{h}} \to \overline{A'^{h}} \leftarrow \Gamma(U', M_{U'})$ used in the definition of the log structure of $\overline{D'}$ is the same as the fiber product of the diagram with $\overline{A'^{h}}$ replaced by $\overline{A'^{h}_{\mathrm{triv}}}^* \cap \overline{A'^{h}}$. Hence, there exists a natural map $Q' \to Q'^{\circ}$ compatible with the actions of G'_U and G'°_U. Using this and the natural map $\Gamma(U', M_{U'}) \to \Gamma(U'^{\circ}, M_{U'^{\circ}})$, we obtain a commutative diagram:

$$\begin{array}{ccccc} U'^{h\circ} & \leftarrow & \overline{U'}^{\circ} & \hookrightarrow & \overline{D'}^{\circ} \\ \downarrow & & \downarrow & & \downarrow \\ U'^{h} & \leftarrow & \overline{U'} & \hookrightarrow & \overline{D'} \end{array} \tag{4.7.6}$$

compatible with the actions of G'_U, G'°_U and the liftings of Frobenius.
Similarly, the fiber product Q° of $\varprojlim_f \overline{A^{h\circ}} \to \overline{A^{h\circ}} \leftarrow \Gamma(U^{\circ}, M_{U^{\circ}})$ used in the construction of the log structure of $\overline{D^{\circ}}$ is the same as the the fiber product of the diagram with $\overline{A^{h\circ}}$ replaced with $\overline{(A^{h\circ})_{\mathrm{triv}}}^* \cap \overline{A^{h\circ}}$. On the other hand, we have $\overline{(A^{h\circ})_{\mathrm{triv}}}^* \cap \overline{A^{h\circ}} \subset \varinjlim_{\mathcal{L} \subset \mathcal{K}^{\circ\mathrm{ur}}} \Gamma(V_{\mathcal{L}}, M_{\mathcal{L}})$, where $V_{\mathcal{L}}$ is as in the construction of a path from $\widetilde{\eta'^{\circ}}$ to $\overline{\eta}$. The fixed system of morphisms $\{f_{\mathcal{L}} : \widetilde{\eta'^{\circ}} \to V_{\mathcal{L}}\}$ induces a morphism $Q^{\circ} \to Q'^{\circ}$ compatible with the actions of G°_U and G'°_U. Using this and the natural map $\Gamma(U^{\circ}, M_{U^{\circ}}) \to \Gamma(U'^{\circ}, M_{U'^{\circ}})$, we obtain a commutative diagram:

$$\begin{array}{ccccc} U^{h\circ} & \leftarrow & \overline{U^{\circ}} & \hookrightarrow & \overline{D^{\circ}} \\ \uparrow & & \uparrow & & \uparrow \\ U'^{h\circ} & \leftarrow & \overline{U'}^{\circ} & \hookrightarrow & \overline{D'}^{\circ} \end{array} \tag{4.7.7}$$

compatible with the actions of G'°_U, G°_U and the liftings of Frobenius.
Now we are ready to construct $\overline{\mathcal{S}}_n(r)_{U'^{\circ}, Z'^{\circ}}$. Let $\overline{E'^{\circ}}_n$ be the PD-envelope of $\overline{U'}^{\circ}_n \hookrightarrow \overline{D'}^{\circ}_n \times Z^{\circ}_n$, which is endowed with an action of G'°_U and a lifting of Frobenius by a natural way. The diagram (4.7.7) and id: $Z^{\circ} \to Z^{\circ}$ induce a PD-morphism $\overline{E'^{\circ}}_n \to \overline{E^{\circ}}_n$ compatible with the actions of G°_U and G'°_U and with the liftings of Frobenius. If we denote by the images of $u_i, u \in \Gamma(\overline{E^{\circ}}_n, \mathcal{O}^*_{\overline{E^{\circ}}_n})$ in $\Gamma(\overline{E'^{\circ}}_n, \mathcal{O}^*_{\overline{E'^{\circ}}_n})$ by the same symbols, we have (cf. [Ts2] Lemma 3.1.4)

$$\mathcal{O}_{\overline{E'^{\circ}}_n} \cong \mathcal{O}_{\overline{D'}^{\circ}_n} \langle u_i - 1, u - 1 \rangle.$$

Hence, if we define $\widetilde{J^{[r]\prime}_{\overline{E'^{\circ}}_n}}$ similarly as in [Ts2] §3.1, [Ts2] Lemma 3.1.6 still holds. Similarly as in [Ts2] Lemma 3.1.7, we obtain a resolution $J^{[r]\prime}_{\overline{D'}^{\circ}_n} \to J^{[r-\bullet]\prime}_{\overline{E'^{\circ}}_n} \otimes \Omega^{\bullet}_{Z^{\circ}_n/W_n}$. Thus, as in [Ts2] §3.1, we obtain the required resolution $\overline{\mathcal{S}}_n(r)_{U'^{\circ},Z'^{\circ}}$ of $\mathbb{Z}/p^n\mathbb{Z}(r)'$ on $((U'^{h}_{\mathrm{triv}})^{\circ})_{\mathrm{K\acute{e}t}}$ by taking the global section of the mapping fiber of

$$1-\varphi_r\colon J^{[r-\bullet]\prime}_{\overline{E'^{\circ}}_n} \otimes \Omega^{\bullet}_{Z^{\circ}_n/W_n} \to \mathcal{O}_{\overline{E'^{\circ}}_n} \otimes \Omega^{\bullet}_{Z^{\circ}_n/W_n}.$$

The morphism $\overline{E'^{\circ}}_n \to \overline{E^{\circ}}_n$ induces a map from the pull-back of $\overline{\mathcal{S}}_n(r)_{U^{\circ},Z^{\circ}}$ to $\overline{\mathcal{S}}_n(r)_{U'^{\circ},Z'^{\circ}}$. The comparison with $\overline{\mathcal{S}}_n(r)_{U',Z'}$ is non-trivial. Consider the morphism $\overline{D'}^{\circ} \times Z^{\circ} \to \mathrm{Spec}(\mathbb{Z}[\mathbb{N}\oplus\mathbb{N}])$ defined by sending $(1,0),(0,1)\in\mathbb{N}\oplus\mathbb{N}$ to the images of $a' \in \Gamma(\overline{D^{\circ}}, M_{\overline{D^{\circ}}})$ and $t \in \Gamma(Z^{\circ}, M_{Z^{\circ}})$ in $\Gamma(\overline{D'}^{\circ} \times Z^{\circ}, M)$. Define the log étale morphism $\mathrm{Spec}(\mathbb{Z}[\mathbb{N}\oplus\mathbb{Z}]) \to \mathrm{Spec}(\mathbb{Z}[\mathbb{N}\oplus\mathbb{N}])$ by $\mathbb{N}\oplus\mathbb{N}\to\mathbb{N}\oplus\mathbb{Z}; (m,n)\mapsto(m,m-n)$, and consider the following cartesian diagrams:

$$\begin{array}{ccccc}
(\overline{D'}^{\circ}\times Z'^{\circ})^{\sim} & \to & (\overline{D'}^{\circ}\times Z^{\circ})^{\sim} & \to & \mathrm{Spec}(\mathbb{Z}[\mathbb{N}\oplus\mathbb{Z}]) \\
\downarrow & & \downarrow & & \downarrow \\
\overline{D'}^{\circ}\times Z'^{\circ} & \to & \overline{D'}^{\circ}\times Z^{\circ} & \to & \mathrm{Spec}(\mathbb{Z}[\mathbb{N}\oplus\mathbb{N}]).
\end{array}$$

Then, the closed immersion $\overline{U'}^{\circ} \hookrightarrow \overline{D'}^{\circ}\times Z'^{\circ}$ naturally factors through $(\overline{D'}^{\circ}\times Z'^{\circ})^{\sim}$ because the images of a' and t coincide in $\Gamma(\overline{U'}^{\circ}, M)$. On the other hand, the vanishing of a' in $\Gamma(\overline{D'}^{\circ}\times Z^{\circ},\mathcal{O})$ implies that $(\overline{D'}^{\circ}\times Z'^{\circ})^{\sim} \to (\overline{D'}^{\circ}\times Z^{\circ})^{\sim}$ is an isomorphism. Hence $\overline{E'^{\circ}}_n$ is isomorphic to the PD-envelope of $\overline{U'}^{\circ}_n \hookrightarrow \overline{D'}^{\circ}_n \times Z'^{\circ}_n$, and the diagram (4.7.6) and $Z'^{\circ}\to Z'$ induce a PD-morphism $\overline{E'^{\circ}}_n \to \overline{E'}_n$. Here $\overline{E'}_n$ denotes the PD-envelope of $\overline{U'}_n \hookrightarrow \overline{D'}_n \times Z'_n$ used in the construction of $\overline{\mathcal{S}}_n(r)_{U',Z'}$. One can verify the compatibility with the actions of G'_U, G'°_U, with the liftings of Frobenius and with the connections. Thus we obtain a canonical map $\overline{\mathcal{S}}_n(r)_{U',Z'} \to \varepsilon'_{U*}\overline{\mathcal{S}}_n(r)_{U'^{\circ},Z'^{\circ}}$.

Lemma 4.7.8. *The map $\overline{\mathcal{S}}_n(r)_{U',Z'} \to \varepsilon'_{U*}\overline{\mathcal{S}}_n(r)_{U'^{\circ},Z'^{\circ}}$ is injective.*

Proof. This follows from $\mathcal{O}_{\overline{E'}_n} \cong \mathcal{O}_{\overline{D'}_n}\langle u_i-1\rangle$ and $\mathcal{O}_{\overline{E'^{\circ}}_n} \cong \mathcal{O}_{\overline{D'}_n}\langle u_i-1, u-1\rangle$. □

Next we discuss on Gysin sequence for $\overline{\mathcal{S}}_n(r)$ on $(U'_{\mathrm{triv}})_{\mathrm{\acute{e}t}}$. Recall that Z'° is not smooth over W.

Lemma 4.7.9. (1) *The natural map $\Omega_{Z^{\circ}/W}\otimes_{\mathcal{O}_{Z^{\circ}}}\mathcal{O}_{Z'^{\circ}} \to \Omega_{Z'^{\circ}/W}$ is an isomorphism.*
(2) *Let $\mathcal{F}$ be an $\mathcal{O}_{Z'}(=\mathcal{O}_{Z'^{\circ}})$-module with an integrable connection $\nabla:\mathcal{F}\to\mathcal{F}\otimes\Omega_{Z'/W}$. Then the composite*

$$\mathcal{F}\to\mathcal{F}\otimes_{\mathcal{O}_{Z'}}\Omega_{Z'/W}\to\mathcal{F}\otimes_{\mathcal{O}_{Z'}}\Omega_{Z'^{\circ}/W}\cong\mathcal{F}\otimes_{\mathcal{O}_Z}\Omega_{Z^{\circ}/W}$$

is an integrable connection on $\mathcal{F}$ as an $\mathcal{O}_{Z^\circ}$-module. Furthermore, the natural maps $\mathcal{F}\otimes\Omega^q_{Z'} \to \mathcal{F}\otimes\Omega^q_{Z'^\circ} \cong \mathcal{F}\otimes\Omega^q_{Z^\circ}$ induce a morphism between the de Rham complexes: $\mathcal{F}\otimes\Omega^\bullet_{Z'} \to \mathcal{F}\otimes\Omega^\bullet_{Z^\circ}$.

Proof. Straightforward. □

By Lemma 4.7.9 above, we can replace $\Omega^q_{Z'}$ with $\Omega^q_{Z'^\circ}$ in $\overline{\mathcal{S}}_n(r)_{U',Z'}$, and obtain a complex on $(U'^h_{\mathrm{triv}})_{\text{ét}}$, which we denote by $\overline{\mathcal{S}}_n(r)^\circ_{U',Z'}$. We can construct a short exact sequence:

$$(4.7.10)\qquad 0 \longrightarrow \overline{\mathcal{S}}_n(r)_{U',Z'} \longrightarrow \overline{\mathcal{S}}_n(r)^\circ_{U',Z'} \longrightarrow \overline{\mathcal{S}}_n(r-1)_{U',Z'}[-1] \longrightarrow 0$$

in an obvious way. On the other hand, the PD-morphism $\overline{E'^\circ}_n \to \overline{E'}_n$ induces a map $\overline{\mathcal{S}}_n(r)^\circ_{U',Z'} \to \varepsilon'_{U*}\overline{\mathcal{S}}_n(r)_{U'^\circ,Z'^\circ}$ compatible with the map in Lemma 4.7.8. Finally, we discuss on the complex $\mathcal{S}^\sim_n(r)$. As in [Ts2] §2.1, using the PD-envelopes of $X \hookrightarrow Z$, $X^\circ \hookrightarrow Z^\circ$ and $X' \hookrightarrow Z'$, we can define complexes $\mathcal{S}^\sim_n(r)_{X,Z}$, $\mathcal{S}^\sim_n(r)_{X^\circ,Z^\circ}$ and $\mathcal{S}^\sim_n(r)_{X',Z'}$ on $(\underline{X_1})_{\text{ét}}$, $(\underline{X^\circ_1})_{\text{ét}} = (\underline{X_1})_{\text{ét}}$ and $(\underline{X'_1})_{\text{ét}}$. We have natural maps from the first complex to the latter two. We also have natural maps from the sections over $\underline{U_1}$, $\underline{U^\circ_1} = \underline{U_1}$ and $\underline{U'_1}$ to the global sections of $\overline{\mathcal{S}}_n(r)_{U,Z}$, $\overline{\mathcal{S}}_n(r)_{U^\circ,Z^\circ}$ and $\overline{\mathcal{S}}_n(r)_{U',Z'}$ (cf. [Ts2] (3.1.9) and (2.1.2)). For $X'^\circ \hookrightarrow Z'^\circ$, we define the complex $\mathcal{S}^\sim_n(r)_{X'^\circ,Z'^\circ}$ on $(\underline{X'^\circ_1})_{\text{ét}} = (\underline{X'_1})_{\text{ét}}$ to be the one obtained by replacing $\Omega^q_{Z'}$ with $\Omega^q_{Z'^\circ}$ in $\mathcal{S}^\sim_n(r)_{X',Z'}$, using Lemma 4.7.9. We can construct a short exact sequence:

$$(4.7.11)\qquad 0 \longrightarrow \mathcal{S}^\sim_n(r)_{X',Z'} \longrightarrow \mathcal{S}^\sim_n(r)_{X'^\circ,Z'^\circ} \longrightarrow \mathcal{S}^\approx_n(r-1)_{X',Z'}[-1] \longrightarrow 0.$$

Here $\mathcal{S}^\approx_n(r-1)_{X',Z'}$ is the complex obtained from $\mathcal{S}^\sim_n(r-1)_{X',Z'}$ by replacing $p^{r-1}-\varphi$ with $p^r - p\varphi$. We have a natural map from $\Gamma(\underline{U'_1}, \mathcal{S}^\sim_n(r)_{X'^\circ,Z'^\circ})$ to the global section of $\overline{\mathcal{S}}_n(r)^\circ_{U',Z'}$, and (4.7.11) is compatible with (4.7.10). Noting that the PD-envelope of $U'^\circ \hookrightarrow Z'^\circ$ is isomorphic to the PD-envelope of $U' \hookrightarrow Z'$, one can also construct a natural map from the pull-back of $\mathcal{S}^\sim_n(r)_{X^\circ,Z^\circ}$ to $\mathcal{S}^\sim_n(r)_{X'^\circ,Z'^\circ}$ and (4.7.11) is compatible with the short exact sequence (cf. (4.3.1)):

$$(4.7.12)\qquad 0 \longrightarrow \mathcal{S}^\sim_n(r)_{X,Z} \longrightarrow \mathcal{S}^\sim_n(r)_{X^\circ,Z^\circ} \longrightarrow \mathcal{S}^\approx_n(r-1)_{X',Z'}[-1] \longrightarrow 0.$$

§4.8. Proof of Proposition 4.5.3.

We keep the notation and the assumption in §4.7. We first construct the morphisms α (4.5.5) and β (4.5.6).

Choose sufficiently large algebraically closed fields Ω of characteristic 0 and Ω' of characteristic p. Let $\mathcal{S}$ be the set of all isomorphic classes of fs monoids P such that $P^* = \{1\}$. For each isomorphic class $c \in \mathcal{S}$, choose a representative P_c of c and define the log geometric point Ω_c to be $\mathrm{Spec}(\Omega)$ with $M_{\Omega_c} = \Omega \oplus \cup_{n\in\mathbb{N},n\neq 0}\frac{1}{n}P_c$ and Ω'_c to be $\mathrm{Spec}(\Omega')$ with $M_{\Omega'_c} = \Omega' \oplus \cup_{n\in\mathbb{N},p\nmid n}\frac{1}{n}P_c$. In the following, we denote by C^* the Godement resolution with respect to all log geometric points whose sources are Ω_c or Ω'_c for some $c \in \mathcal{S}$. Note that such log geometric points form a set.

To simplify the notation, we write Θ for the operation $C^*i_*i^*j_*C^*$ and Θ' for $C^*i'_*i'^*j'_*C^*$. Denote by i_U and i'_U the closed immersions $\underline{U}\otimes k = \underline{U^h}\otimes k \to \underline{U^h}$ and $\underline{U'}\otimes k = \underline{U'^h}\otimes k \to \underline{U'^h}$, and by j_U, j°_U, j'_U the open immersions $U^h_{\mathrm{triv}} \to \underline{U^h}$, $(U^{h\circ})_{\mathrm{triv}} \to \underline{U^h}$ and $U'^h_{\mathrm{triv}} \to \underline{U'^h}$. Similarly, as above, we denote by Θ_U and Θ'_U the operations $C^*i_{U*}i^*_U j_{U*}C^*$ and $C^*i'_{U*}i'^*_U j'_{U*}C^*$.

Since the derived direct images of $\mathbb{Z}/p^n\mathbb{Z}(r)'$ by the left morphisms in the first lines of the diagrams (4.7.1) and (4.7.2) are again $\mathbb{Z}/p^n\mathbb{Z}(r)'$ ([I2] Theorem 7.4), we see that the left and middle vertical morphisms in the diagram in Proposition 4.5.3 are induced by sheafifying the following morphisms of presheaves on $\underline{X}_{\text{ét}}$:

$$\Gamma(\underline{U}, \Theta(\Lambda)) \overset{\text{q.i.}}{\to} \Gamma(\underline{U^h}, \Theta_U(\Lambda)) \overset{\text{q.i.}}{\to} \Gamma(\underline{U^h}, \Theta_U(\overline{\mathcal{S}}_n(r)_{U,Z})) \leftarrow \Gamma(\underline{U}, i_*\mathcal{S}^{\sim}_n(r)_{X,Z})$$

$$\Gamma(\underline{U}, \Theta(\varepsilon_*C^*(\Lambda))) \overset{\text{q.i}}{\to} \Gamma(\underline{U^h}, \Theta_U(\varepsilon_{U*}C^*(\Lambda))) \overset{\text{q.i.}}{\to}$$
$$\Gamma(\underline{U^h}, \Theta_U(\varepsilon_{U*}C^*(\overline{\mathcal{S}}_n(r)_{U^\circ,Z^\circ}))) \leftarrow \Gamma(\underline{U}, i_*\mathcal{S}^{\sim}_n(r)_{X^\circ,Z^\circ})$$

Here $\Lambda = \mathbb{Z}/p^n\mathbb{Z}(r)'$ and q.i. means a quasi-isomorphism. See (4.7.1) and (4.7.2) for ε and ε_U. Let $K^\bullet$, $K^\bullet_U$ and $L^\bullet_U$ be the cokernels of the injective homomorphisms $\Lambda_{X_{\mathrm{triv}}} \to \varepsilon_*C^*(\Lambda_{(X_{\mathrm{triv}})^\circ})$, $\Lambda_{U^h_{\mathrm{triv}}} \to \varepsilon_{U*}C^*(\Lambda_{(U^h_{\mathrm{triv}})^\circ})$, and $\overline{\mathcal{S}}_n(r)_{U,Z} \to \varepsilon_{U*}C^*(\overline{\mathcal{S}}_n(r)_{U^\circ,Z^\circ})$ (Lemma 4.7.5). We have a natural injective homomorphism from the first line to the second one. Taking its quotient and using (4.7.12), we obtain

$$(4.8.1)\qquad \Gamma(\underline{U},\Theta(K^\bullet)) \overset{\text{q.i.}}{\to} \Gamma(\underline{U^h},\Theta_U(K^\bullet_U)) \overset{\text{q.i.}}{\to} \Gamma(\underline{U^h},\Theta_U(L^\bullet_U)) \leftarrow \Gamma(\underline{U}, i_*\mathcal{S}^{\approx}_n(r)_{X',Z'}[-1])$$

We have quasi-isomorphisms $K^\bullet \leftarrow \tau_{\leq 1}K^\bullet \to \mathcal{H}^1(K^\bullet)[-1] \cong \Lambda(-1)_{X'_{\mathrm{triv}}}[-1]$ ([I2]Theorem 7.4). Hence varying U and sheafifying, we obtain the required morphism α (4.5.5).

We can apply the same argument to $\varepsilon'\colon ((X'_{\mathrm{triv}})^\circ)_{\text{Két}} \to (X'_{\mathrm{triv}})_{\text{ét}}$, $\varepsilon'_U\colon ((U'^h_{\mathrm{triv}})^\circ)_{\text{Két}} \to (U'^h_{\mathrm{triv}})_{\text{ét}}$ and the resolutions $\Lambda_{U'^h_{\mathrm{triv}}} \to \overline{\mathcal{S}}_n(r)_{U',Z'}$ and $\Lambda_{(U'^h_{\mathrm{triv}})^\circ} \to \overline{\mathcal{S}}_n(r)_{U'^\circ,Z'^\circ}$. We define $K'^\bullet$, $K'^\bullet_U$ and $L'^\bullet_U$ to be the cokernels of the injective homomorphisms $\Lambda_{X'_{\mathrm{triv}}} \to \varepsilon'_*C^*(\Lambda_{(X'_{\mathrm{triv}})^\circ})$, $\Lambda_{U'^h_{\mathrm{triv}}} \to \varepsilon'_{U*}C^*(\Lambda_{(U'^h_{\mathrm{triv}})^\circ})$ and $\overline{\mathcal{S}}_n(r)_{U',Z'} \to \varepsilon'_{U*}C^*(\overline{\mathcal{S}}_n(r)_{U'^\circ,Z'^\circ})$ (Lemma 4.7.8). Then using (4.7.11), we obtain

$$(4.8.2)\qquad \Gamma(\underline{U'},\Theta'(K'^\bullet)) \overset{\text{q.i.}}{\to} \Gamma(\underline{U'^h},\Theta'_U(K'^\bullet_U)) \overset{\text{q.i.}}{\to} \Gamma(\underline{U'^h},\Theta'_U(L'^\bullet_U)) \leftarrow \Gamma(\underline{U'}, i'_*\mathcal{S}^{\approx}_n(r)_{X',Z'}[-1])$$

By [I2]Theorem 7.4 and [Na] Theorem (5.1), we have quasi-isomorphisms $K'^\bullet \leftarrow \tau_{\leq 1}K'^\bullet \to \mathcal{H}^1(K'^\bullet)[-1] \cong \Lambda(-1)_{X'_{\mathrm{triv}}}[-1]$. Hence, varying U and sheafifying, we obtain the required β. We have a natural map from (4.8.1) to (4.8.2) and hence the two maps α and β coincide.

Let us compare β with the map (4.5.4). By (4.7.10), we have a morphism $\overline{\mathcal{S}}_n(r-1)_{U',Z'}[-1] \to L'^\bullet_U$ and the last map of (4.8.2) factors through

$\Gamma(\underline{U'^h}, \Theta'_U(\overline{\mathcal{S}}_n(r)_{U',Z'}[-1]))$. We have the following commutative diagram of complexes on $(U'^h_{\mathrm{triv}})_{\text{ét}}$:

$$\begin{array}{ccccc} K'^{\bullet}_U & \overset{\text{q.i.}}{\leftarrow} & \tau_{\le 1} K'^{\bullet}_U & \overset{\text{q.i.}}{\to} & \mathbb{Z}/p^n(r)'(-1)[-1] \\ \text{q.i.} \downarrow & & \text{q.i.} \downarrow & & \| \\ L'^{\bullet}_U & \overset{\text{q.i.}}{\leftarrow} & \tau_{\le 1} L'^{\bullet}_U & \overset{\text{q.i.}}{\to} & \mathbb{Z}/p^n(r)'(-1)[-1] \\ \uparrow & & \uparrow & & (*) \uparrow \\ \overline{\mathcal{S}}_n(r-1)_{U',Z'}[-1] & \overset{\text{q.i.}}{\leftarrow} & \tau_{\le 1}(\overline{\mathcal{S}}_n(r-1)_{U',Z'}[-1]) & \cong & \mathbb{Z}/p^n(r-1)'[-1] \end{array}$$

Here the morphism $(*)$ is the composite

(4.8.3)

$$\mathbb{Z}/p^n(r-1)' \cong \mathcal{H}^0(\overline{\mathcal{S}}_n(r-1)_{U',Z'}) \to \mathcal{H}^1(L'^{\bullet}_U) \overset{\sim}{\leftarrow} \mathcal{H}^1(K'^{\bullet}_U) \cong \mathbb{Z}/p^n(r)'(-1)$$

Hence to prove the coincidence of β and (4.5.4), it suffices to prove the following:

Proposition 4.8.4. *The map* (4.8.3) *is the natural map.*

Proof. By the definition of $K'^{\bullet}_U$ and $L'^{\bullet}_U$, the map (4.8.3) coincides with the composite of

$$\mathbb{Z}/p^n(r-1)' \cong \mathcal{H}^0(\overline{\mathcal{S}}_n(r-1)_{U',Z'}) \overset{\cong}{\leftarrow} \mathcal{H}^1(\overline{\mathcal{S}}_n(r)^{\circ}_{U',Z'}) \to R^1\varepsilon'_{U*}\overline{\mathcal{S}}_n(r)_{U'^{\circ},Z'^{\circ}}$$
$$\overset{\cong}{\leftarrow} R^1\varepsilon'_{U*}\mathbb{Z}/p^n(r)' \overset{(**)}{\cong} \mathbb{Z}/p^n(r)'(-1)$$

where the second isomorphism is defined by (4.7.10). Note that all sheaves appearing above are ind locally constant. Let I be the kernel of the surjection $G'^{\circ}_U \to G'_U$, which is canonically isomorphic to $\hat{\mathbb{Z}}(1)$. Then we have a natural isomorphism $H^1(I, \mathbb{Z}/p^n(r)') \cong \mathrm{Hom}(\hat{\mathbb{Z}}(1), \mathbb{Z}/p^n(r)') \cong \mathbb{Z}/p^n(r)'(-1)$, and it is compatible with the isomorphism $(**)$ above. Hence we may replace $R^1\varepsilon'_{U*}(-)$ with $H^1(I, -)$ regarding locally constant sheaves on $((U'^h_{\mathrm{triv}})^{\circ})_{\text{Két}}$ and $(U'^h_{\mathrm{triv}})_{\text{ét}}$ as G'°_U and G'_U-modules. Let $\alpha \in \mathbb{Z}/p^n(r-1)' \cong \mathcal{H}^0(\overline{\mathcal{S}}_n(r-1)_{U',Z'})$. Since $F^*_{Z_n}(t) = t^p$, we see that the image of α in $\mathcal{H}^1(\overline{\mathcal{S}}_n(r)^{\circ}_{U',Z'})$ is the class of $(\alpha \cdot d\log(t), 0)$. Choose an isomorphism $M_{\widetilde{\eta'^{\circ\mathrm{ur}}}} \cong \widetilde{\mathbb{N}} \oplus (\mathcal{K}'^{\mathrm{ur}})^*$ as in the definition of $\overline{D'}^{\circ}$ in §4.7. Then the pair $(t, \{(1/p^n, 1)\}_{n\in\mathbb{N}})$ defines an element of Q'° and we denote its image under $Q'^{\circ} \to \Gamma(\overline{D'}^{\circ}_n, M) \to \Gamma(\overline{E'^{\circ}}_n, M)$ by $[\underline{t}]$. Since the images of t and $[\underline{t}]$ in $\Gamma(\overline{U'}^{\circ}_n, M)$ coincide, there exists a unique $u \in \Gamma(\overline{E'^{\circ}}_n, 1 + J_{\overline{E'^{\circ}}_n})$ such that $u \cdot t = [\underline{t}]$ in $\Gamma(\overline{E'^{\circ}}_n, M)$. We have $\varphi(u) = u^p$ and $d\log(u) = -d\log(t)$. Hence $(\alpha \cdot d\log(t), 0) \in (\overline{\mathcal{S}}_n(r)_{U'^{\circ},Z'^{\circ}})^1$ is the image of $-\alpha \cdot \log(u) \in (\overline{\mathcal{S}}_n(r)_{U'^{\circ},Z'^{\circ}})^0$ by the differential map. Hence the image of $(\alpha \cdot d\log(t), 0)$ in $H^1(I, \mathbb{Z}/p^n\mathbb{Z}(r)')$ is given by the cocycle $\sigma \mapsto -(\sigma(-\alpha\log(u)) - (-\alpha\log(u))) = \alpha \cdot \log(\sigma([\underline{t}])[\underline{t}]^{-1})$. This completes the proof because $I \cong \hat{\mathbb{Z}}(1) \to \mathbb{Z}/p^n(1)' \subset \Gamma(\overline{E'^{\circ}}_n, J_{\overline{E'^{\circ}}_n})$ is given by $\sigma \mapsto \log(\sigma([\underline{t}])[\underline{t}]^{-1})$. □

In the case that X does not have a global embedding into Z as in the beginning of §4.7, we choose a strict étale covering $X^0 \to X$, and $X^0 \hookrightarrow Z^0$, $\underline{Z'^0} \subset$

$\underline{Z^0}$, and $\{F_{Z_n^0}: Z_n^0 \to Z_n^0\}$ satisfying the conditions in the beginning of §4.7. Such a covering and an embedding exist by a similar argument as the proof of Proposition 4.2.1 (1). From this embedding, we can construct $X^\bullet \hookrightarrow Z^\bullet$ and $\underline{Z'^\bullet} \subset \underline{Z^\bullet}$ as in Proposition 4.2.2 endowed with $\{F_{Z_n^\bullet}\}$. We can verify that $X^\nu \hookrightarrow Z^\nu$, $\underline{Z'^\nu} \subset \underline{Z^\nu}$ and $\{F_{Z_n^\nu}\}$ satisfy the conditions in the beginning of §4.7 for each $\nu \in \mathbb{N}$. By applying the above argument to each level, we can construct α and β on $(\underline{X^\bullet})_{\text{ét}}$, which coincide with each other, and show that β coincides with (4.5.4) on $(\underline{X^\bullet})_{\text{ét}}$. Note that our construction does not depend on $\{t_1, \ldots, t_d, t\}$ chosen in the beginning of §4.7. By taking $R\theta_*$ for the morphism of topoi $\theta: (\underline{X^\bullet})^\sim_{\text{ét}} \to (\underline{X})^\sim_{\text{ét}}$, we obtain Proposition 4.5.3 for a general X.

§4.9. Proof of Theorem 4.1.2.

We will prove Theorem 4.1.2 by the induction on the number of elements of I. In the case that I is empty, the theorem is nothing but Theorem 3.2.2. Assume that I is non-empty, choose $i_0 \in I$ and we define (X, M), (X, M°) and (X', M') as in the beginning of §4.3. As the induction hypothesis, we assume that Theorem 4.1.2 is true for (X, M) and (X', M').

By Lemma 4.4.8, Lemma 4.4.9 and Proposition 4.5.2, for an integer $r \geq 2\dim(X_K)$, the comparison maps $B_{\text{st}} \otimes_{\mathbb{Q}_p} V_i^q(r) \to B_{\text{st}} \otimes_{K_0} D_i^q(r)$ $(i = 1, 2)$ and $B_{\text{st}} \otimes_{\mathbb{Q}_p} V_3^q(r-1) \to B_{\text{st}} \otimes_{K_0} D_3^q(r-1)$ are compatible with the Gysin exact sequences (4.5.1) and (4.4.1). Since the comparison maps are isomorphisms for (X, M) and (X', M') for every q by the induction hypothesis, we see that the comparison map for (X, M°) is also an isomorphism for every q. Furthermore, by Lemma 4.4.6, the comparison maps above tensored with B_{dR} over B_{st} send Fil^i to Fil^i and are compatible with the Gysin exact sequences (4.5.1) and (4.4.5). By the induction hypothesis, the comparison maps tensored with B_{dR} are filtered isomorphisms for (X, M) and (X', M'). Hence by five lemma, it also holds for (X, M°). Thus we see that Theorem 4.1.2 is true for (X°, M°).

References

[Be-O] Berthelot, P. and Ogus, A., *Notes on crystalline cohomology*, Princeton University Press, Princeton, 1978.

[Bl-Ka] Bloch, S. and Kato, K., *p-adic etale cohomology*, Publ. Math. IHES. **63** (1986), 107-152.

[Br1] Breuil, C., *Representations p-adiques semi-stables et transversalite de Griffiths*, Math. Ann. **307** (1997), 191–224.

[Br2] Breuil, C., *Construction de représentations p-adiques semi-stables*, Ann. scient. E. N. S. **31** (1998), 281–327.

[Br3] Breuil, C., *Cohomologie étale de p-torsion et cohomologie cristalline en réduction semi-stable*, Duke Math. J **95** (1998), 523–620.

[C-Fo] Colmez, P. and Fontaine, J.-M., *Construction des représentations p-adiques semi-stables*, Invent. math. **140** (2000), 1–43.

[Fa] Faltings, G., *Almost étale extensions,* Cohomologies p-adiques et applications arithmétiques (II), Astérisque **279** (2002), 185–270.

[Fo1] Fontaine, J.-M., *Sur certains types de représentations p-adiques du groupe de Galois d'un corps local; construction d'un anneau de Barsotti-Tate*, Ann. of Math. **115** (1982), 529–577.

[Fo2] Fontaine, J.-M., *Représentations p-adiques des corps locaux*, The Grothendieck Festschrift vol II, Birkhäuser, Boston, 1991, pp. 249–309.

[Fo3] Fontaine, J.-M., *Le corps des périodes p-adiques,* Périodes p-adiques, Séminaire de Bures, 1988, Astérisque **223** (1994), 59–111.

[Fo4] Fontaine, J.-M., *Représentations p-adiques semi-stables,* Périodes p-adiques, Séminaire de Bures, 1988, Astérisque **223** (1994), 113–183.

[Fo-L] Fontaine, J.-M. and Laffaille, G., *Constructions de représentations p-adiques*, Ann. scient. E. N. S. **15** (1982), 547–608.

[H] Hyodo, O., *A note on p-adic etale cohomology in the semi-stable reduction case*, Invent. math. **91** (1988), 543-557.

[H-Ka] Hyodo, O. and Kato, K., *Semi-stable reduction and crystalline cohomology with logarithmic poles,* Périodes p-adiques, Séminaire de Bures, 1988, Astérisque **223** (1994), 221–268.

[I1] Illusie, L., *Complexe de de Rham-Witt et cohomologie cristalline*, Ann. scient. E. N. S. **12** (1979), 501–661.

[I2] Illusie, L., *An overview of the work of K. Fujiwara, K. Kato, and C. Nakayama on logarithmic étale cohomology,* Cohomologies p-adiques et applications arithmétiques (II), Astérisque **279** (2002), 271–322.

[I-R] Illusie, L. and Raynaud, M., *Les suites spectrales associées au complexe de de Rham-Witt*, Publ. Math. IHES. **57** (1983), 73–212.

[J-Sai] Jannsen, U. and Saito, S., *Kato homology of arithmetic schemes and higher class field theory over local fields*, in this volume.

[Ka1] Kato, K., *On p-adic vanishing cycles (Application of ideas of Fontaine-Messing)*, Advanced studies in Pure Math. **10** (1987), 207-251.

[Ka2] Kato, K., *Logarithmic structures of Fontaine-Illusie*, Algebraic analysis, geometry, and number theory, Johns Hopkins University Press, Baltimore, 1989, pp. 191–224.

[Ka3] Kato, K., *Semi-stable reduction and p-adic etale cohomology,* Périodes p-adiques, Séminaire de Bures, 1988, Astérisque **223** (1994), 269–293.

[Ka4] Kato, K., *Toric singularities*, Amer. J. Math. **116** (1994), 1073-1099.

[Ku] Kurihara, M., *A note on p-adic etale cohomology*, Proc. Japan Academy **63** (1987), 275–278.

[L] Lorenzon, P., *Logarithmic Hodge-Witt forms and Hyodo-Kato cohomology*, J. Algebra **249** (2002), 247–265.

[O] Ogus, A., *F-crystals on schemes with constant log structure.* Special issue in honour of Frans Oort, Compositio Math. **97** (1995), 187–225.

[Na] Nakayama, C., *Logarithmic étale cohomology.*, Math. Ann. **308** (1997), 365–404.

[Sat] Sato, K., *Injectivity of the torsion cycle map of codimension two of varieties over p-adic fields with semi-stable reduction*, J. reine angew. Math. **501** (1998), 221–235.

[Ts1] Tsuji, T., *Syntomic complexes and p-adic vanishing cycles*, J. reine angew. Math. **472** (1996), 69–138.

[Ts2] Tsuji, T., *p-adic étale cohomology and crystalline cohomology in the semi-stable reduction case*, Invent. math. **137** (1999), 233-411.

[Ts3] Tsuji, T., *Saturated morphisms of logarithmic schemes*, preprint 1997.

[Ts4] Tsuji, T., *On p-adic nearby cycles of log smooth families*, Bull. Soc. Math. France **128** (2000), 529–575.

[Y] Yamashita, G., *p-adic étale cohomology and crystalline cohomology for open varieties with semi-stable reduction*, master's thesis, University of Tokyo 2003.

[EGA IV] Grothendieck, A. (avec la collaboration de J. Dieudonné), *Éléments de géométrie algébrique,* IV Étude locale des schémas et des morphismes de schémas, Publ. Math. IHES 20 (1964), 24(1965), 28(1966), 32(1967).

[SGA 1] *Revêtments étales et groupe fondamental,* Séminaire de géométrie algébrique du Bois-Marie 1960-61, dirigé par A. Grothendieck, Lecture Notes in Math. 224, Springer, 1971.

Takeshi Tsuji
Graduate School of Mathematical Sciences
University of Tokyo
3-8-1 Komaba, Meguro
Tokyo 153-8914
Japan
t-tsuji@ms.u-tokyo.ac.jp

On Base Change Theorem and Coherence in Rigid Cohomology

Nobuo Tsuzuki[1]

Received: October 10, 2002
Revised: February 5, 2003

Abstract. We prove that the base change theorem in rigid cohomology holds when the rigid cohomology sheaves both for the given morphism and for its base extension morphism are coherent. Applying this result, we give a condition under which the rigid cohomology of families becomes an overconvergent isocrystal. Finally, we establish generic coherence of rigid cohomology of proper smooth families under the assumption of existence of a smooth lift of the generic fiber. Then the rigid cohomology becomes an overconvergent isocrystal generically. The assumption is satisfied in the case of families of curves. This example relates to P. Berthelot's conjecture of the overconvergence of rigid cohomology for proper smooth families.

2000 Mathematics Subject Classification: 14F30, 14F20, 14D15
Keywords and Phrases: rigid cohomology, overconvergent isocrystal, base change theorem, Gauss-Manin connection, coherence

1 Introduction

Let p be a prime number and let $\mathcal{V}$ (resp. k, resp. K) be a complete discrete valuation ring (resp. the residue field of $\mathcal{V}$ with characteristic p, resp. the quotient field of $\mathcal{V}$ with characteristic 0). Let $f : X \to \operatorname{Spec} k$ be a separated morphism of schemes of finite type. The finiteness of rigid cohomology $H^*_{\mathrm{rig}}(X/K, E)$ for an overconvergent F-isocrystal E on X/K are proved by recent developments [2] [6] [8] [9] [11] [18] [19] [20] [21]. However, if one takes another embedding $\operatorname{Spec} k \to \mathcal{S}$ for a smooth $\mathcal{V}$-formal scheme $\mathcal{S}$, we do not know whether the "same" rigid cohomology, $\mathbb{R}^* f_{\mathrm{rig}\mathfrak{S}*}E$ in our notation, with respect to the base $\mathfrak{S} = (\operatorname{Spec} k, \operatorname{Spec} k, \mathcal{S})$ becomes a sheaf of coherent $\mathcal{O}_{]\operatorname{Spec} k[_{\mathcal{S}}}$-modules or not, and whether the base change homomorphism

$$H^*_{\mathrm{rig}}(X/K, E) \otimes_K \mathcal{O}_{]\operatorname{Spec} k[_{\mathcal{S}}} \to \mathbb{R}^* f_{\mathrm{rig}\mathfrak{S}*}E$$

[1] Supported by JSPS.

is an isomorphism or not. In this case, if one knows the coherence of $\mathbb{R}^* f_{\mathrm{rig}\mathfrak{S}*}E$, then the homomorphism above is an isomorphism. Moreover, if the coherence holds for any $\mathcal{S}$, then there exists a rigid cohomology isocrystal $\mathbb{R}^* f_{\mathrm{rig}*}E$ on $\mathrm{Spec}\, k/K$ and $\mathbb{R}^* f_{\mathrm{rig}\mathfrak{S}*}E$ is a realization with respect to the base $\mathfrak{S}$.
In this paper we discuss the coherence, base change theorems, and the overconvergence of the Gauss-Manin connections, for rigid cohomology of families. Up to now, only few results are known. One of the difficulities to see the coherence of rigid cohomology comes from the reason that there is no global lifting. If a proper smooth family over $\mathrm{Spec}\, k$ admits a proper smooth formal lift over $\mathrm{Spf}\, \mathcal{V}$, then the rigid cohomology of the family is coherent by R. Kiehl's finiteness theorem for proper morphisms in rigid analytic geometry. Hence it becomes an overconvergent isocrystal. This was proved by P. Berthelot [4, Théorème 5]. (See 4.1.)
In general it is too optimistic to believe the existence of a proper smooth lift for a proper smooth family. So we present a problem on the existence of a projective smooth lift of the generic fiber up to "alteration" (Problem 4.2.1). Assuming a positive solution of this problem, we have generic coherence of rigid cohomology. This means that the rigid cohomology becomes an overconvergent isocrystal on a dense open subscheme. In the case of families of curves this problem is solved [12, Exposé III, Corollaire 7.4], so the rigid cohomology sheaves become overconvergent isocrystals generically.
In [1] Y. André and F. Baldassarri had a result on the generic overconvergence of Gauss-Manin connections of de Rham cohomologies for overconvergent isocrystals on families of smooth varieties (not necessary proper) which come from algebraic connections of characteristic 0.

Now let us explain the contents. See the notation in the convention.
Let

$$\begin{array}{ccc} \mathfrak{X} & \xleftarrow{v} & \mathfrak{Y} \\ f\downarrow & & \downarrow g \\ \mathfrak{S} & \xleftarrow[u]{} & \mathfrak{T} \end{array}$$

be a cartesian square of $\mathcal{V}$-triples separated of finite type such that $\widehat{f} : \mathcal{X} \to \mathcal{S}$ is smooth around X. In section 2 we discuss base change homomorphisms

$$\widetilde{u}^* \mathbb{R}^q f_{\mathrm{rig}\mathfrak{S}*}E \to \mathbb{R}^q g_{\mathrm{rig}\mathfrak{T}*} v^* E$$

such that $\mathcal{T} \to \mathcal{S}$ is flat. In a rigid analytic space one can not compare sheaves by stalks because of G-topology. Only coherent sheaves can be compared by stalks. The base change homomorphism is an isomorphism if both the source and the target are coherent (Proposition 2.3.1). By the hypothesis we can use the stalk argument.
In section 3 we review the Gauss-Manin connection on the rigid cohomology sheaf and give a condition under which the Gauss-Manin connection becomes overconvergent. Let $f : \mathfrak{X} \to \mathfrak{T}$ and $u : \mathfrak{T} \to \mathfrak{S}$ be morphisms of $\mathcal{V}$-triples such that $\widehat{f} : \mathcal{X} \to \mathcal{T}$ and $\widehat{u} : \mathcal{T} \to \mathcal{S}$ are smooth around X and T, respectively. Then

the Gauss-Manin connection ∇^{GM} on the rigid cohomology sheaf $\mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}*}E$ for an overconvergent isocrystal E on $(X, \overline{X})/\mathcal{S}_K$ is overconvergent if $\mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}'*}E$ is coherent for any triple $\mathfrak{T}' = (T, \overline{T}, \mathcal{T}')$ over $\mathfrak{T}$ such that $\mathcal{T}' \to \mathcal{T}$ is smooth around T (Theorem 3.3.1). If the Gauss-Manin connection is overconvergent, then there exists an overconvergent isocrystal $\mathbb{R}^q f_{\mathrm{rig}*}E$ on $(T, \overline{T})/\mathcal{S}_K$ such that the rigid cohomology sheaf $\mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}'*}E$ is the realization of $\mathbb{R}^q f_{\mathrm{rig}*}E$ on $\mathfrak{T}'$ for any embedding $\overline{T} \to \mathcal{T}'$ such that $\mathcal{T}' \to \mathcal{S}$ is smooth around T. We also prove the existence of the Leray spectral sequence (Theorem 3.4.1).
In section 4 we discuss Berthelot's conjecture [4, Sect. 4.3]. Let $f : (X, \overline{X}) \to (T, \overline{T})$ be a proper smooth family of k-pairs of finite type over a triple $\mathfrak{S}$. We give a proof of Berthelot's theorem using the result in the previous sections (Theorems 4.1.1, 4.1.4). Finally, we discuss the generic coherence of rigid cohomology of proper smooth families as mentioned above.

Convention. The notation follows [5] and [9].
Throughout this paper, k is a field of characteristic $p > 0$, K is a complete discrete valuation field of characteristic 0 with residue field k and $\mathcal{V}$ is the ring of integers of K. $|\ |$ is denoted an p-adic absolute value on K.
A k-pair $(X, \overline{X})$ consists of an open immersion $X \to \overline{X}$ over $\mathrm{Spec}\, k$. A $\mathcal{V}$-triple $\mathfrak{X} = (X, \overline{X}, \mathcal{X})$ separated of finite type consists of a k-pair $(X, \overline{X})$ and a formal $\mathcal{V}$-scheme $\mathcal{X}$ separated of finite type with a closed immersion $\overline{X} \to \mathcal{X}$ over $\mathrm{Spf}\, \mathcal{V}$. Let $\mathfrak{X} = (X, \overline{X}, \mathcal{X})$ and $\mathfrak{Y} = (Y, \overline{Y}, \mathcal{Y})$ be $\mathcal{V}$-triples separated of finite type. A morphism $f : \mathfrak{Y} \to \mathfrak{X}$ of $\mathcal{V}$-triples is a commutative diagram

$$\begin{array}{ccccc} Y & \to & \overline{Y} & \to & \mathcal{Y} \\ \overset{\circ}{f} \downarrow & & \overline{f} \downarrow & & \downarrow \widehat{f} \\ X & \to & \overline{X} & \to & \mathcal{X}. \end{array}$$

over $\mathrm{Spf}\, \mathcal{V}$. The associated morphism between tubes denotes $\widetilde{f} :]\overline{Y}[_{\mathcal{Y}} \to]\overline{X}[_{\mathcal{X}}$.
A Frobenius endomorphism over a formal $\mathcal{V}$-scheme is a continuous lift of p-power endomorphisms.

Acknowledgements. The author thanks Professor K. Kato, who invited the author to the wonderful world of "Mathematics", for his constant advice. This article is dedicated to him.
The author also thanks Professor H. Sumihiro for useful discussions.

2 Base change theorems

2.1 Base change homomorphisms

We recall the definition of rigid cohomology in [9, Sect. 10] and introduce base change homomorphisms. Let $\mathcal{V} \to \mathcal{W}$ be a ring homomorphism of complete discrete valuation rings whose valuations are extensions of that of the ring $\mathbb{Z}_p$ of p-adic integers and let k and K (resp. l and L) be the residue field and the quotient field of $\mathcal{V}$ (resp. $\mathcal{W}$), respectively. Let $\mathfrak{S} = (S, \overline{S}, \mathcal{S})$ (resp. $\mathfrak{T} = (T, \overline{T}, \mathcal{T})$) be a $\mathcal{V}$-triple (resp. a $\mathcal{W}$-triple) separated of finite type and let $u : \mathfrak{T} \to \mathfrak{S}$ be a morphism of triples. Let

$$\begin{array}{ccc} (X, \overline{X}) & \stackrel{v}{\longleftarrow} & (Y, \overline{Y}) \\ f \downarrow & & \downarrow g \\ (S, \overline{S}) & \underset{u}{\longleftarrow} & (T, \overline{T}) \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \longleftarrow & \operatorname{Spec} l \end{array}$$

be a commutative diagram of pairs such that the vertical arrows are separated of finite type and the upper square is cartesian. Then there always exists a Zariski covering $\mathfrak{X}'$ of $(X, \overline{X})$ over $\mathfrak{S}$ (resp. $\mathfrak{Y}'$ of $(Y, \overline{Y})$ over $\mathfrak{T}$) with a commutative diagram

$$\begin{array}{ccc} \mathfrak{X}' & \stackrel{v'}{\longleftarrow} & \mathfrak{Y}' \\ f' \downarrow & & \downarrow g' \\ \mathfrak{S} & \underset{u}{\longleftarrow} & \mathfrak{T} \end{array}$$

as triples such that the induced morphism $\mathcal{Y}' \to \mathcal{X}' \times_{\mathcal{S}} \mathcal{T}'$ is smooth around Y. Let $\mathfrak{X}'_{\bullet}$ be the Čech diagram as $(X, \overline{X})$-triples over $\mathfrak{S}$ and let $\mathrm{DR}^{\dagger}(\mathfrak{X}'_{\bullet}/\mathfrak{S}, (E_{\mathfrak{X}'_{\bullet}}, \nabla_{\mathfrak{X}'_{\bullet}}))$ be the de Rham complex

$$E_{\mathfrak{X}'_{\bullet}} \xrightarrow{\nabla_{\mathfrak{X}'_{\bullet}}} E_{\mathfrak{X}'_{\bullet}} \otimes_{j^{\dagger}\mathcal{O}_{]\overline{X}'_{\bullet}[_{\mathcal{X}'_{\bullet}}}} j^{\dagger}\Omega^1_{]\overline{X}'_{\bullet}[_{\mathcal{X}'_{\bullet}}/]\overline{S}[_{\mathcal{S}}} \xrightarrow{\nabla_{\mathfrak{X}'_{\bullet}}} E_{\mathfrak{X}'_{\bullet}} \otimes_{j^{\dagger}\mathcal{O}_{]\overline{X}'_{\bullet}[_{\mathcal{X}'_{\bullet}}}} j^{\dagger}\Omega^2_{]\overline{X}'_{\bullet}[_{\mathcal{X}'_{\bullet}}/]\overline{S}[_{\mathcal{S}}} \to \cdots$$

on $]\overline{X}[_{\mathcal{X}'_{\bullet}}$ associated to the realization $(E_{\mathfrak{X}'_{\bullet}}, \nabla_{\mathfrak{X}'_{\bullet}})$ of E with respect to $\mathfrak{X}'_{\bullet}$. Since $\mathfrak{X}'_{\bullet}$ is a universally de Rham descendable hypercovering of $(X, \overline{X})$ over $\mathfrak{S}$, one can calculate the q-th rigid cohomology $\mathbb{R}^q f_{\mathrm{rig}\mathfrak{S}*}E$ with respect to $\mathfrak{S}$ as the q-th hypercohomology of the total complex of $\mathrm{DR}^{\dagger}(\mathfrak{X}'_{\bullet}/\mathfrak{S}, (E_{\mathfrak{X}'_{\bullet}}, \nabla_{\mathfrak{X}'_{\bullet}}))$. From our choice of $\mathfrak{X}'$ and $\mathfrak{Y}'$, there is a canonical homomorphism

$$\mathbb{L}\widetilde{u}^*\mathbb{R}f_{\mathrm{rig}\mathfrak{S}*}E \to \mathbb{R}g_{\mathrm{rig}\mathfrak{T}*}v^*E$$

in the derived category of complexes of sheaves of abelian groups on $]\overline{T}[_{\mathcal{T}}$. The canonical homomorphism does not depend on the choices of $\mathfrak{X}'$ and $\mathfrak{Y}'$. If $\widehat{u} : \mathcal{T} \to \mathcal{S}$ is flat around T, we have a base change homomorphism

$$\widetilde{u}^*\mathbb{R}^q f_{\mathrm{rig}\mathfrak{S}*}E \to \mathbb{R}^q g_{\mathrm{rig}\mathfrak{T}*}v^*E$$

of sheaves of $j^{\dagger}\mathcal{O}_{]\overline{T}[_{\mathcal{T}}}$-modules for any q.
The following is the finite flat base change theorem in rigid cohomology.

2.1.1 THEOREM [9, Theorem 11.8.1]

With notation as above, we assume furthermore that $\widehat{u} : \mathcal{T} \to S$ is finite flat, $\widehat{u}^{-1}(\overline{S}) = \overline{T}$ and $\overline{u}^{-1}(S) = T$. Then the base change homomorphism

$$\widetilde{u}^*\mathbb{R}^q f_{\mathrm{rig}\mathfrak{S}*}E \to \mathbb{R}^q g_{\mathrm{rig}\mathfrak{T}*}v^*E$$

is an isomorphism for any q.

2.2 The condition (F)

Let $\mathfrak{T} = \mathfrak{S} \times_{(\mathrm{Spec}\, k, \mathrm{Spec}\, k, \mathrm{Spf}\, \mathcal{V})} (\mathrm{Spec}\, l, \mathrm{Spec}\, l, \mathrm{Spf}\, \mathcal{W})$ and let q be an integer. For an overconvergent isocrystal E on $(X, \overline{X})/\mathcal{S}_K$, we say that the condition $(\mathrm{F})^q_{f,\mathcal{W}/\mathcal{V},E}$ holds if and only if the base change homomorphism

$$\widetilde{u}^* \mathbb{R}^q f_{\mathrm{rig}\mathfrak{S}*} E \to \mathbb{R}^q g_{\mathrm{rig}\mathfrak{T}*} v^* E$$

is an isomorphism. By Theorem 2.1.1 we have

2.2.1 Proposition

If l is finite over k, then the condition $(\mathrm{F})^q_{f,\mathcal{W}/\mathcal{V},E}$ holds for any q and any E.

2.2.2 Example

Let $\mathfrak{S} = (\mathrm{Spec}\, k, \mathrm{Spec}\, k, \mathrm{Spf}\, \mathcal{V})$ and let $j^\dagger \mathcal{O}_{]\overline{X}[}$ be the overconvergent isocrystal on $(X, \overline{X})/K$ associated to the structure sheaf with the natural connection. If $\overline{X}$ is proper over $\mathrm{Spec}\, k$, then the condition $(\mathrm{F})^q_{f,\mathcal{W}/\mathcal{V},j^\dagger \mathcal{O}_{]\overline{X}[}}$ holds for any q and any extension $\mathcal{W}/\mathcal{V}$ of complete discrete valuation rings.

Proof. Using an alteration [14, Theorem 4.1] and the spectral sequence for proper hypercoverings [21, Theorem 4.5.1], we may assume that $\overline{X}$ is smooth. Note that the Gysin isomorphism [20, Theorem 4.1.1] commutes with any base extension. The assertion follows from induction on the dimension of X by a similar method of Berthelot's proof of finiteness of the rigid cohomology [6, Théorème 3.1] since the crystalline cohomology satisfies the base change theorem [5, Chap. 5, Théorème 3.5.1]. □

2.3 A base change theorem

We give a sufficient condition for a base change homomorphism to be an isomorphism.

2.3.1 Proposition

With notation in 2.1, assume furthermore that $\mathcal{W} = \mathcal{V}$ and $\widehat{u} : \mathcal{T} \to \mathcal{S}$ is smooth around T. Let q be an integer and suppose that $\mathbb{R}^q f_{\mathrm{rig}\mathfrak{S}*} E$ (resp. $\mathbb{R}^q g_{\mathrm{rig}\mathfrak{T}*} v^* E$) is a sheaf of coherent $j^\dagger \mathcal{O}_{]\overline{S}[_\mathcal{S}}$-modules (resp. a sheaf of coherent $j^\dagger \mathcal{O}_{]\overline{T}[_\mathcal{T}}$-modules) for an overconvergent isocrystal E on $(X, \overline{X})/\mathcal{S}_K$. Then the base change homomorphism

$$\widetilde{u}^* \mathbb{R}^q f_{\mathrm{rig}\mathfrak{S}*} E \to \mathbb{R}^q g_{\mathrm{rig}\mathfrak{T}*} v^* E$$

is an isomorphism.

PROOF. Since both sheaves are coherent, we may assume $T = \overline{T}$ by the faithfulness of the forgetful functor from the category of sheaves of coherent $j^\dagger\mathcal{O}_{]\overline{T}[_\mathcal{T}}$-modules to the category of sheaves of coherent $\mathcal{O}_{]T[_\mathcal{T}}$-modules [5, Corollaire 2.1.11]. Then we have only to compare stalks of both sides at each closed point of $]T[_\mathcal{T}$ by [7, Corollary 9.4.7] since both sides are coherent. Hence we may assume that $T = \overline{T}$ consists of a k-rational point by Proposition 2.2.1. Then the assertion follows from the following lemma. □

2.3.2 LEMMA

Under the assumption of Proposition 2.3.1, assume furthermore that $T = \overline{T} = \operatorname{Spec} k$. *Then the base change homomorphism*

$$\widetilde{u}^*\mathbb{R}^q f_{\mathrm{rig}\mathfrak{S}*}E \to \mathbb{R}^q g_{\mathrm{rig}\mathfrak{T}*}v^*E$$

is an isomorphism.

PROOF. We may assume that $S = \overline{S} = \operatorname{Spec} k$. By the fibration theorem [5, Théorème 1.3.7] we may assume that $\mathcal{T} = \widehat{\mathbb{A}}^d_\mathcal{S}$ is a formal affine space over $\mathcal{S}$ with coordinates $x_1, \cdots, x_d$ such that $T = S$ is included in the zero section of $\mathcal{T}$ over $\mathcal{S}$. Applying Proposition 2.2.1, we have only to compare stalks of both sides at a K-rational point $t \in]T[_\mathcal{T}$ with $x_i(t) = 0$ for all i after a suitable change of coordinates.

Let $\mathcal{T}_n = \operatorname{Spf} \mathcal{V}[x_1, \cdots, x_d]/(x_1^{n_1}, \cdots, x_d^{n_d}) \times_{\operatorname{Spf}\mathcal{V}} \mathcal{S}$ for $n = (n_1, \cdots, n_d)$ with $n_i > 0$ for all i and denote by $u_n : \mathfrak{T}_n = (T, \overline{T}, \mathcal{T}_n) \to \mathfrak{S}$ (resp. $w_n : \mathfrak{T}_n \to \mathfrak{T}$) the natural structure morphism. Observe a sequence of base change homomorphisms:

$$\widetilde{u}^*\mathbb{R}^q f_{\mathrm{rig}\mathfrak{S}*}E \to \mathbb{R}^q g_{\mathrm{rig}\mathfrak{T}*}v^*E \to \widetilde{w}_{n*}\mathbb{R}^q g_{\mathrm{rig}\mathfrak{T}_n*}v_n^*E.$$

By the finite flat base change theorem (Theorem 2.1.1) the induced homomorphism

$$\widetilde{u}_n^*\mathbb{R}^q f_{\mathrm{rig}\mathfrak{S}*}E \to \mathbb{R}^q g_{\mathrm{rig}\mathfrak{T}_n*}v_n^*E$$

is an isomorphism since the rigid cohomology is determined by the reduced subscheme. Hence, the base change homomorphism $\widetilde{u}^*\mathbb{R}^q f_{\mathrm{rig}\mathfrak{S}*}E \to \mathbb{R}^q g_{\mathrm{rig}\mathfrak{T}*}v^*E$ is injective.

Let us define an overconvergent isocrystal $F = v^*E/(x_1, \cdots, x_d)v^*E$ on $(Y, \overline{Y})/\mathcal{T}_K$ and observe a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\oplus_i \widetilde{u}^*\mathbb{R}^q f_{\mathrm{rig}\mathfrak{S}*}E & \xrightarrow{\oplus_i x_i} & \widetilde{u}^*\mathbb{R}^q f_{\mathrm{rig}\mathfrak{S}*}E & \longrightarrow & \widetilde{w}_{1*}\widetilde{u}_1^*\mathbb{R}^q f_{\mathrm{rig}\mathfrak{S}*}E & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\oplus_i \mathbb{R}^q g_{\mathrm{rig}\mathfrak{T}*}v^*E & \xrightarrow{\oplus_i x_i} & \mathbb{R}^q g_{\mathrm{rig}\mathfrak{T}*}v^*E & \longrightarrow & \mathbb{R}^q g_{\mathrm{rig}\mathfrak{T}*}F, & &
\end{array}$$

where the subscript 1 of u_1 and w_1 means the multi-index $(1, \cdots, 1)$. Indeed, one can prove $\mathbb{R}^q g_{\mathrm{rig}\mathfrak{T}*}((x_1, \cdots, x_d)v^*E) \cong (x_1, \cdots, x_d)\mathbb{R}^q g_{\mathrm{rig}\mathfrak{T}*}v^*E$ inductively since $x_i v^*E \cong v^*E$ as overconvergent isocrystals. Hence the bottom row is exact. By the finite flat base change theorem we have $\widetilde{u}_1^*\mathbb{R}^q f_{\mathrm{rig}\mathfrak{S}*}E \cong$

$\mathbb{R}^q g_{\mathrm{rig}\mathfrak{T}_1 *} v^* E$. Since $\widetilde{w}_1 :]\overline{T}[_{\mathcal{T}_1} \to]\overline{T}[_{\mathcal{T}}$ is a closed immersion of rigid analytic spaces, $R^q \widetilde{w}_{1*} \mathcal{F} = 0\, (q > 0)$ for any sheaf $\mathcal{F}$ of coherent $\mathcal{O}_{]\overline{T}[_{\mathcal{T}_1}}$-modules. Hence the right vertical arrow is an isomorphism.

Let $\mathcal{G}$ be a cokernel of the base change homomorphism $\widetilde{u}^* \mathbb{R}^q f_{\mathrm{rig}\mathfrak{S}*} E \to \mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}*} v^* E$. By the snake lemme we have

$$\mathcal{G} = (x_1, \cdots, x_d)\mathcal{G}.$$

Since $\mathcal{G}$ is coherent and the ideal $(x_1, \cdots, x_d)\mathcal{O}_{]\overline{T}[_{\mathcal{T}},t}$ is included in the unique maximal ideal of the stalk $\mathcal{O}_{]\overline{T}[_{\mathcal{T}},t}$ of $\mathcal{O}_{]\overline{T}[_{\mathcal{T}}}$ at t, the stalk $\mathcal{G}_t$ vanishes by Nakayama's lemma. Hence the homomorphism between stalks at t which is induced by the base change homomorphism is an isomorphism. This completes the proof. □

2.3.3 Corollary

With notation in 2.1, assume furthermore that the induced morphism $\mathcal{T} \to \mathcal{S} \times_{\mathrm{Spf}\, \mathcal{V}} \mathrm{Spf}\, \mathcal{W}$ is smooth around T. Suppose that, for an integer q and an overconvergent isocrystal E on $(X, \overline{X})/\mathcal{S}_K$, the condition $(\mathrm{F})^q_{f,\mathcal{W}/\mathcal{V},E}$ holds and $\mathbb{R}^q f_{\mathrm{rig}\mathfrak{S}*} E$ (resp. $\mathbb{R}^q g_{\mathrm{rig}\mathfrak{T}*} v^* E$) is a sheaf of coherent $j^\dagger \mathcal{O}_{]\overline{S}[_{\mathcal{S}}}$-modules (resp. a sheaf of coherent $j^\dagger \mathcal{O}_{]\overline{T}[_{\mathcal{T}}}$-modules). Then the base change homomorphism

$$\widetilde{u}^* \mathbb{R}^q f_{\mathrm{rig}\mathfrak{S}*} E \to \mathbb{R}^q g_{\mathrm{rig}\mathfrak{T}*} v^* E$$

is an isomorphism.

3 A condition for the overconvergence of Gauss-Manin connections

3.1 The condition (C)

Let $\mathfrak{S} = (S, \overline{S}, \mathcal{S})$ be a $\mathcal{V}$-triple separated of finite type and let $(X, \overline{X})$ be a pair separated of finite type over $(S, \overline{S})$ with structure morphism $f : (X, \overline{X}) \to (S, \overline{S})$.

Let E be an overconvergent isocrystal on $(X, \overline{X})/\mathcal{S}_K$ and let q be an integer. We say that the condition $(\mathrm{C})^q_{f,\mathfrak{S},E}$ holds if and only if, for any $\mathcal{V}$-morphism $\widehat{u} : \mathcal{T} \to \mathcal{S}$ separated of finite type with a closed immersion $\overline{S} \to \mathcal{T}$ over $\mathcal{S}$ such that $\widehat{u}$ is smooth around S, the rigid cohomology $\mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}*} v^* E$ with respect to $\mathfrak{T} = (S, \overline{S}, \mathcal{T})$ is a sheaf of coherent $j^\dagger \mathcal{O}_{]\overline{S}[_{\mathcal{T}}}$-modules.

Since an open covering (resp. a finite closed covering) of $\overline{S}$ induces an admissible covering of $]\overline{S}[_{\mathcal{S}}$ [5, Proposition 1.1.14], we have the proposition below by the gluing lemma.

3.1.1 Proposition

Let $u : \mathfrak{S}' \to \mathfrak{S}$ be a separated morphism of $\mathcal{V}$-triples locally of finite type such that $S' = \overline{u}^{-1}(S)$, and let

$$\begin{array}{ccc} (X,\overline{X}) & \stackrel{v}{\longleftarrow} & (X',\overline{X}') \\ f\downarrow & & \downarrow f' \\ (S,\overline{S}) & \underset{u}{\longleftarrow} & (S',\overline{S}') \end{array}$$

be a cartesian diagram of pairs. Let E be an overconvergent isocrystal on $(X,\overline{X})/\mathcal{S}_K$ and let $E' = v^*E$ be the inverse image on $(X',\overline{X}')/\mathcal{S}'_K$.

(1) *Suppose one of the situations* (i) *and* (ii)*.*

(i) *$\widehat{u} : \mathcal{S}' \to \mathcal{S}$ is an open immersion and $\overline{S}' = \widehat{u}^{-1}(\overline{S})$.*

(ii) *$\overline{u} : \overline{S}' \to \overline{S}$ is a closed immersion and $\overline{S}' \to \mathcal{S}' = \mathcal{S}$ is the natural closed immersion.*

Then, the condition $(\mathrm{C})^q_{f,\mathfrak{S},E}$ implies the condition $(\mathrm{C})^q_{f',\mathfrak{S}',E'}$.

(2) *Suppose one of the situations* (iii) *and* (iv)*.*

(iii) *$\widehat{u} : \mathcal{S}' \to \mathcal{S}$ is an open covering and $\overline{S}' = \widehat{u}^{-1}(\overline{S})$.*

(iv) *$\overline{u} : \overline{S}' \to \overline{S}$ is a finite closed covering and the closed immersion $\overline{S}' \to \mathcal{S}'$ is a disjoint sum of the natural closed immersion into $\mathcal{S}$ for each component of $\overline{S}'$.*

Then, the condition $(\mathrm{C})^q_{f,\mathfrak{S},E}$ holds if and only if the condition $(\mathrm{C})^q_{f',\mathfrak{S}',E'}$ holds.

3.2 The overconvergence of Gauss-Manin connections

Let $\mathfrak{S} = (S,\overline{S},\mathcal{S})$ be a $\mathcal{V}$-triple separated of finite type and let $\mathfrak{T} = (T,\overline{T},\mathcal{T})$ be a $\mathfrak{S}$-triple separated of finite type such that $\mathcal{T} \to \mathcal{S}$ is smooth around T. Let $f : (X,\overline{X}) \to (T,\overline{T})$ be a morphism of pairs separated of finite type. Then, for an overconvergent isocrystal E on $(X,\overline{X})/\mathcal{S}_K$, we have an integrable connection

$$\nabla^{\mathrm{GM}} : \mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}*}E \to \mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}*}E \otimes_{j^\dagger\mathcal{O}_{]\overline{T}[_\mathcal{T}}} j^\dagger\Omega^1_{]\overline{T}[_\mathcal{T}/]\overline{S}[_\mathcal{S}}$$

of sheaves of $j^\dagger\mathcal{O}_{]\overline{T}[_\mathcal{T}}$-modules over $j^\dagger\mathcal{O}_{]\overline{S}[_\mathcal{S}}$, which is called the Gauss-Manin connection and constructed as follows (cf. [16]). Here $\mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}*}E$ needs not be coherent and the integrable connection means a $j^\dagger\mathcal{O}_{]\overline{S}[_\mathcal{S}}$-homomorphism ∇ such that $\nabla(ae) = a\nabla(e) + e\otimes da$ for $e \in E, a \in j^\dagger\mathcal{O}_{]\overline{T}[_\mathcal{T}}$ and such that $\nabla^2 = 0$. Let us take a formal $\mathcal{V}$-scheme $\mathcal{X}$ separated of finite type over $\mathcal{T}$ with a $\mathcal{T}$-closed immersion $\overline{X} \to \mathcal{X}$ such that the structure morphism $\widehat{f} : \mathcal{X} \to \mathcal{T}$ is smooth around X. In general, one can not take such a global $\mathcal{X}$ and one needs to take a Zariski covering $\mathfrak{U}$ of $(X, \overline{X})$ over $\mathfrak{S}$ in order to define the rigid cohomology. For simplicity, we assume here that there exists a global $\mathcal{X}$. The following construction also works if one replaces the triple $\mathfrak{X} = (X, \overline{X}, \mathcal{X})$ by the Čech diagram $\mathfrak{U}_{\bullet}$ of $\mathfrak{U}$ as $(X, \overline{X})$-triples over $\mathfrak{S}$. (See [9, Sect. 10].)

Let $\mathrm{DR}^{\dagger}(\mathfrak{X}/\mathfrak{S}, (E_{\mathfrak{X}}, \nabla_{\mathfrak{X}}))$ be the de Rham complex associated to the realization $(E_{\mathfrak{X}}, \nabla_{\mathfrak{X}})$ of E with respect to $\mathfrak{X}$ and let us define a decreasing filtration $\{\mathrm{Fil}^q\}_q$ of $\mathrm{DR}^{\dagger}(\mathfrak{X}/\mathfrak{S}, (E_{\mathfrak{X}}, \nabla_{\mathfrak{X}}))$ by

$$\mathrm{Fil}^q = \mathrm{Image}(\mathrm{DR}^{\dagger}(\mathfrak{X}/\mathfrak{S}, (E_{\mathfrak{X}}, \nabla_{\mathfrak{X}}))[-q] \otimes_{\widetilde{f}^{-1} j^{\dagger}\mathcal{O}_{]\overline{T}[_{\mathcal{T}}}} \widetilde{f}^{-1} j^{\dagger}\Omega^q_{]\overline{T}[_{\mathcal{T}}/]\overline{S}[_{\mathcal{S}}} \to \mathrm{DR}^{\dagger}(\mathfrak{X}/\mathfrak{S}, (E_{\mathfrak{X}}, \nabla_{\mathfrak{X}})))$$

for any q, where $[-q]$ means the $-q$-th shift of the complex. Since

$$0 \to \widetilde{f}^{*} j^{\dagger}\Omega^1_{]\overline{T}[_{\mathcal{T}}/]\overline{S}[_{\mathcal{S}}} \to j^{\dagger}\Omega^1_{]\overline{X}[_{\mathcal{X}}/]\overline{S}[_{\mathcal{S}}} \to j^{\dagger}\Omega^1_{]\overline{X}[_{\mathcal{X}}/]\overline{T}[_{\mathcal{T}}} \to 0$$

is an exact sequence of sheaves of locally free $j^{\dagger}\mathcal{O}_{]\overline{X}[_{\mathcal{X}}/]\overline{S}[_{\mathcal{S}}}$-modules of finite type, the filtration $\{\mathrm{Fil}^q\}_q$ is well-defined and we have

$$\mathrm{gr}^q_{\mathrm{Fil}} = \mathrm{Fil}^q/\mathrm{Fil}^{q+1} = \mathrm{DR}^{\dagger}(\mathfrak{X}/\mathfrak{T}, (E_{\mathfrak{X}}, \overline{\nabla}_{\mathfrak{X}}))[-q] \otimes_{\widetilde{f}^{-1} j^{\dagger}\mathcal{O}_{]\overline{T}[_{\mathcal{T}}}} \widetilde{f}^{-1} j^{\dagger}\Omega^q_{]\overline{T}[_{\mathcal{T}}/]\overline{S}[_{\mathcal{S}}},$$

where $\overline{\nabla}_{\mathfrak{X}}$ is the connection induced by the composition

$$E_{\mathfrak{X}} \xrightarrow{\nabla_{\mathfrak{X}}} E_{\mathfrak{X}} \otimes_{j^{\dagger}\mathcal{O}_{]\overline{X}[_{\mathcal{X}}}} j^{\dagger}\Omega^1_{]\overline{X}[_{\mathcal{X}}/]\overline{S}[_{\mathcal{S}}} \longrightarrow E_{\mathfrak{X}} \otimes_{j^{\dagger}\mathcal{O}_{]\overline{X}[_{\mathcal{X}}}} j^{\dagger}\Omega^1_{]\overline{X}[_{\mathcal{X}}/]\overline{T}[_{\mathcal{T}}}.$$

From this decreasing filtration we have a spectral sequence

$$\underline{E}_1^{qr} = \mathbb{R}^{q+r}\widetilde{f}_{*}\mathrm{gr}^q_{\mathrm{Fil}} \Rightarrow \mathbb{R}^{q+r}\widetilde{f}_{*}\mathrm{DR}^{\dagger}(\mathfrak{X}/\mathfrak{S}, (E_{\mathfrak{X}}, \nabla_{\mathfrak{X}})),$$

where

$$\begin{aligned} \underline{E}_1^{qr} &= \mathbb{R}^r\widetilde{f}_{*}(\mathrm{DR}^{\dagger}(\mathfrak{X}/\mathfrak{T}, (E_{\mathfrak{X}}, \overline{\nabla}_{\mathfrak{X}})) \otimes_{\widetilde{f}^{-1} j^{\dagger}\mathcal{O}_{]\overline{T}[_{\mathcal{T}}}} \widetilde{f}^{-1} j^{\dagger}\Omega^q_{]\overline{T}[_{\mathcal{T}}/]\overline{S}[_{\mathcal{S}}}) \\ &\cong \mathbb{R}^r f_{\mathrm{rig}\mathfrak{T}*}E \otimes_{j^{\dagger}\mathcal{O}_{]\overline{T}[_{\mathcal{T}}}} j^{\dagger}\Omega^q_{]\overline{T}[_{\mathcal{T}}/]\overline{S}[_{\mathcal{S}}}. \end{aligned}$$

Then the Gauss-Manin connection $\nabla^{\mathrm{GM}} : \mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}*}E \to \mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}*}E \otimes_{j^{\dagger}\mathcal{O}_{]\overline{T}[_{\mathcal{T}}}} j^{\dagger}\Omega^1_{]\overline{T}[_{\mathcal{T}}/]\overline{S}[_{\mathcal{S}}}$ is defined by the differential

$$d_1^{0r} : \underline{E}_1^{0r} \to \underline{E}_1^{1r}.$$

Indeed, one can check that d_1^{0r} is an integrable connection by an explicit calculation (see [16, Sect. 3]).

3.2.1 Theorem

Let E be an overconvergent isocrystal on $(X, \overline{X})/\mathcal{S}_K$ and let q be an integer. If the condition $(\mathrm{C})^q_{f,\mathfrak{T},E}$ holds, then the Gauss-Manin connection $\nabla^{\mathrm{GM}} : \mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}*}E \to \mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}*}E \otimes_{j^\dagger\mathcal{O}_{]\overline{T}[_\mathcal{T}}} j^\dagger\Omega^1_{]\overline{T}[_\mathcal{T}/]\overline{S}[_\mathcal{S}}$ is overconvergent along $\partial T = \overline{T} \setminus T$.

Proof. Let us put $\mathfrak{X}^2 = (X, \overline{X}, \mathcal{X} \times_\mathcal{S} \mathcal{X})$, denote by $p_{\mathfrak{X}i} : \mathfrak{X}^2 \to \mathfrak{X}$ the i-th projection for $i = 1, 2$, and the same for $\mathfrak{T}$. By definition, the overconvergent connection $\nabla_\mathfrak{X}$ is induced from an isomorphism

$$\epsilon_\mathfrak{X} : \widetilde{p}^*_{\mathfrak{X}1}E \to \widetilde{p}^*_{\mathfrak{X}2}E$$

of sheaves of $j^\dagger\mathcal{O}_{]\overline{X}[_{\mathcal{X}^2}}$-modules which satisfies the cocycle condition [5, Definition 2.2.5]. Consider the commutative diagram

$$\begin{array}{ccccccc} \mathfrak{X}_{\mathfrak{T}1} & = & (X, \overline{X}, \mathcal{X} \times_\mathcal{S} \mathcal{T}) & \to & \mathfrak{X}^2 & & \\ & & \downarrow & & \downarrow & & \\ & & \mathfrak{T}^2 & \to & (X, \overline{X}, \mathcal{T} \times_\mathcal{S} \mathcal{X}) & = & \mathfrak{X}_{\mathfrak{T}2} \end{array}$$

of triples. Then the rigid cohomology $\mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}^2*}E$ can be calculated as the hypercohomology of the de Rham complex by using any of $\mathfrak{X}^2$, $\mathfrak{X}_{\mathfrak{T}1}$ and $\mathfrak{X}_{\mathfrak{T}2}$. Hence, we have an isomorphism

$$\epsilon_\mathfrak{T} : \widetilde{p}^*_{\mathfrak{T}1}\mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}*}E \cong \mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}^2*}E \xrightarrow[\cong]{\mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}^2*}(\epsilon_\mathfrak{X})} \mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}^2*}E \cong \widetilde{p}^*_{\mathfrak{T}2}\mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}*}E$$

of sheaves of $j^\dagger\mathcal{O}_{]\overline{T}[_{\mathcal{T}^2}}$-modules which satisfies the cocycle condition by $(\mathrm{C})^q_{f,\mathfrak{T},E}$ (Proposition 2.3.1). By an explicit calculation, the Gauss-Manin connection ∇^{GM} is induced from the isomorphism $\epsilon_\mathfrak{T}$ (see [3, Capt.4, Proposition 3.6.4]). Therefore, ∇^{GM} is an overconvergent connection along $\partial T = \overline{T} \setminus T$. □

3.2.2 Proposition

Let $w : \mathfrak{T}' \to \mathfrak{T}$ be a morphism separated of finite type over $\mathfrak{S}$ which satisfies the conditions

(i) $\mathring{w} : T' \to T$ *is an isomorphism;*

(ii) $\overline{w} : \overline{T}' \to \overline{T}$ *is proper;*

(iii) $\widehat{w} : \mathcal{T}' \to \mathcal{T}$ *is smooth around T',*

and let $f' : (X', \overline{X}') \to (T', \overline{T}')$ be the base extension of $f : (X, \overline{X}) \to (T, \overline{T})$ by $w : (T, \overline{T}) \to (T', \overline{T}')$. Let q be an integer and let E (resp. E') be an overconvergent isocrystal on $(X, \overline{X})/\mathcal{S}_K$ (resp. the inverse image of E on $(X', \overline{X}')/\mathcal{S}_K$).

(1) *If the condition $(\mathrm{C})^q_{f,\mathfrak{T},E}$ holds, then the condition $(\mathrm{C})^q_{f',\mathfrak{T}',E'}$ holds.*

(2) *If the condition* $(\mathrm{C})^{q'}_{f',\mathfrak{T}',E'}$ *holds for all* $q' \leq q$*, then the condition* $(\mathrm{C})^{q}_{f,\mathfrak{T},E}$ *holds.*

In both cases, the base change homomorphism

$$\widetilde{w}^*\mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}*}E \to \mathbb{R}^q f'_{\mathrm{rig}\mathfrak{T}'*}E'$$

is an isomorphism with respect to connections.

Proof. If $\widehat{w} : \mathcal{T}' \to \mathcal{T}$ is etale around T', then there are strict neighborhoods of $]T[_{\mathcal{T}}$ and $]T'[_{\mathcal{T}'}$ (resp. $]X[_{\mathcal{X}}$ and $]X'[_{\mathcal{X}'}$, where $\mathcal{X}' = \mathcal{X} \times_{\mathcal{T}} \mathcal{T}'$) which are isomorphic [5, Théorème 1.3.5]. Using the argument of the proof of [5, Théorème 2.3.5], we may assume that $\mathcal{T}'$ is a formal affine space over $\mathcal{T}$ and $\overline{w} : \overline{T}' \to \overline{T}$ is an isomorphism by Proposition 3.1.1. Then the assertion (1) follows from Proposition 2.3.1.

Now we prove the assertion (2). We may assume that $\mathcal{T}'$ is a formal affine line over $\mathcal{T}$ by induction. Since the equivalence between categories of realizations of overconvegent isocrystals with respect to $\mathcal{T}$ and $\mathcal{T}'$ is given by the functors w^* and $\mathbb{R}^0 w_{\mathrm{rig}\mathfrak{T}*}$ [5, Théorème 2.3.5] [9, Proposition 8.3.5], $\mathbb{R}^0 w_{\mathrm{rig}\mathfrak{T}*}\mathbb{R}^{q'} f'_{\mathrm{rig}\mathfrak{T}'*}E'$ is a sheaf of coherent $j^\dagger\mathcal{O}_{]\overline{T}[_{\mathcal{T}}}$-modules with an overconvergent connection for $q' \leq q$ by Theorem 3.2.1. Moreover, the canonical homomorphism

$$\mathbb{R}^0 w_{\mathrm{rig}\mathfrak{T}*}\mathbb{R}^{q'} f'_{\mathrm{rig}\mathfrak{T}'*}E' \to \mathbb{R}\widetilde{w}_*\mathrm{DR}^\dagger(\mathfrak{T}'/\mathfrak{T}, (\mathbb{R}^{q'} f'_{\mathrm{rig}\mathfrak{T}'*}E', \nabla^{\mathrm{GM}}))$$

is an isomorphism for $q' \leq q$.

Let us put $C^\cdot = \mathbb{R}\widetilde{f'}_*\mathrm{DR}^\dagger(\mathfrak{X}'/\mathfrak{T}', (E'_{\mathfrak{X}'}, \nabla_{\mathfrak{X}'}))$ and $D^\cdot = C^\cdot \otimes_{j^\dagger\mathcal{O}_{]\overline{T}'[_{\mathcal{T}'}}} j^\dagger\Omega^1_{]\overline{T}'[_{\mathcal{T}'}/]\overline{T}[_{\mathcal{T}}}$. Observe the filtration of $\mathrm{DR}^\dagger(\mathfrak{X}'/\mathfrak{T}, (E'_{\mathfrak{X}'}, \nabla_{\mathfrak{X}'}))$ with respect to w and f' in 3.2. Then

$$\mathbb{R}\widetilde{f'}_*\mathrm{DR}^\dagger(\mathfrak{X}'/\mathfrak{T}, (E'_{\mathfrak{X}'}, \nabla_{\mathfrak{X}'})) \cong \mathrm{Cone}(C^\cdot \to D^\cdot)[-1].$$

since $\mathcal{T}'$ is an affine line over $\mathcal{T}$. Let us denote by $C^{\cdot > i}$ (resp. $C^{\cdot \geq i}$) a subcomplex of $(C^\cdot, d^\cdot)$ defined by $(C^{\cdot > i})^j = 0\,(j < i-1), (C^{\cdot > i})^i = C^i/\mathrm{Ker}\,d^i$ and $(C^{\cdot > i})^j = C^j\,(j > i)$ (resp. $(C^{\cdot \geq i})^j = 0\,(j < i-1), (C^{\cdot \geq i})^i = C^i/\mathrm{Im}\,d^{i-1}$ and $(C^{\cdot \geq i})^j = C^j\,(j > i)$) and the same for $D^\cdot$. Then

$$\begin{aligned}&\mathrm{DR}^\dagger(\mathfrak{T}'/\mathfrak{T}, (\mathbb{R}^t f'_{\mathrm{rig}\mathfrak{T}'*}E', \nabla^{\mathrm{GM}}))[-t] \\ &\quad\cong \mathrm{Cone}(\mathrm{Cone}(C^{\cdot \geq t} \to D^{\cdot \geq t})[-1] \to \mathrm{Cone}(C^{\cdot > t} \to D^{\cdot > t})[-1])[-1]\end{aligned}$$

for any t. Hence we have an isomorphism

$$\mathbb{R}^{q'}(wf')_{\mathrm{rig}\mathfrak{T}*}E' \cong \mathbb{R}^0 w_{\mathrm{rig}\mathfrak{T}*}\mathbb{R}^{q'} f'_{\mathrm{rig}\mathfrak{T}'*}E'$$

for any $q' \leq q$ inductively.

On the contrary, if we denote by $v : (X', \overline{X}') \to (X, \overline{X})$ the structure morphism, then the spectral sequence

$$\underline{E}_1^{st} = \mathbb{R}^t v_{\mathrm{rig}\mathfrak{X}*}E' \otimes_{j^\dagger\mathcal{O}_{]\overline{X}[_{\mathcal{X}}}} j^\dagger\Omega^s_{]\overline{X}[_{\mathcal{X}}/]\overline{T}[_{\mathcal{T}}} \Rightarrow \mathbb{R}^{s+t}\widetilde{v}_*\mathrm{DR}^\dagger(\mathfrak{X}'/\mathfrak{T}, (E'_{\mathfrak{X}'}, \nabla_{\mathfrak{X}'}))$$

with respect to f and v in 3.2 induces an isomorphism

$$\mathbb{R}^q(wf')_{\mathrm{rig}\mathfrak{T}*}E' \cong \mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}*}E$$

since $\mathbb{R}^0 v_{\mathrm{rig}\mathfrak{X}*}E' = E$ and $\mathbb{R}^t v_{\mathrm{rig}\mathfrak{X}*}E' = 0\,(t > 0)$. Hence,

$$\mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}*}E \cong \mathbb{R}^0 w_{\mathrm{rig}\mathfrak{T}*}\mathbb{R}^q f'_{\mathrm{rig}\mathfrak{T}'*}E'$$

and $\mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}*}E$ is a sheaf of coherent $j^\dagger\mathcal{O}_{]\overline{T}[_{\mathcal{T}}}$-modules. The same holds for any triple $\mathfrak{T}'' = (T, \overline{T}, \mathcal{T}'')$ separated of finite type over $\mathfrak{T}$ such that $\mathcal{T}'' \to \mathcal{T}$ is smooth around T. Therefore, the condition $(\mathrm{C})^q_{f,\mathfrak{T},E}$ holds. □

3.3 Rigid cohomology as overconvergent isocrystals

Let $\mathfrak{S} = (S, \overline{S}, \mathcal{S})$ be a $\mathcal{V}$-triple separated of finite type and let

$$(X, \overline{X}) \xrightarrow{f} (T, \overline{T}) \xrightarrow{u} (S, \overline{S})$$

be morphisms of pairs separated of finite type over $\operatorname{Spec} k$.

As a consequence of Theorem 3.2.1, we have a criterion of the overconvergence of Gauss-Manin connections by the gluing lemma and Proposition 3.2.2.

3.3.1 Theorem

Let E be an overconvergent isocrystal on $(X, \overline{X})/\mathcal{S}_K$ and let q be an integer. Suppose that, for each $q' \leq q$, there exists a triple $\mathfrak{T}' = \coprod (T_i, \overline{T}_i, \mathcal{T}_i)$ separated of finite type over $\mathfrak{S}$ which satisfies the conditions

(i) $\overline{T}' = \coprod \overline{T}_i \to \overline{T}$ *is an open covering;*

(ii) T' *is the pull back of* T *in* $\overline{T}'$*;*

(iii) $\mathcal{T}' = \coprod \mathcal{T}_i \to \mathcal{T}$ *is smooth around* T'*,*

(iv) *the condition* $(\mathrm{C})^{q'}_{f',\mathfrak{T}',E'}$ *holds, where* $f' : (X', \overline{X}') \to (T', \overline{T}')$ *denotes the extension of* f *and* E' *is the inverse image of* E *on* $(X', \overline{X}')/\mathcal{S}_K$*.*

Then the rigid cohomology $\mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}*}E$ with the Gauss-Manin connection ∇^{GM} is a realization of an overconvergent isocrystal on $(T, \overline{T})/\mathcal{S}_K$. Moreover, the overconvergent isocrystal on $(T, \overline{T})/\mathcal{S}_K$ does not depend on the choice of $\mathfrak{T}'$.

Under the assumption of Theorem 3.3.1, we define the q-th rigid cohomology overconvergent isocrystal $R^q f_{\mathrm{rig}*}E$ as the overconvergent isocrystal on $(T, \overline{T})/\mathcal{S}_K$ in the theorem above.

3.3.2 Proposition

With notation as before, we have the following results.

(1) *Let* $E \to F$ *be a homomorphism of overconvergent isocrystals on* $(X,\overline{X})/\mathcal{S}_K$ *and let* q *be an integer. Suppose that, for each* $q' \leq q$ *and each* E *and* F*, there exists a triple* $\mathfrak{T}'$ *such that the conditions* (i) - (iv) *in Theorem 3.3.1 holds. Then there is a homomorphism* $R^q f_{\mathrm{rig}*}E \to R^q f_{\mathrm{rig}*}F$ *of overconvergent isocrystals on* $(T,\overline{T})/\mathcal{S}_K$*. This homomorphism commutes with the composition.*

(2) *Let* $0 \to E \to F \to G \to 0$ *be an exact sequence of overconvergent isocrystals on* $(X,\overline{X})/\mathcal{S}_K$*. Suppose that, for each* q *and each* E, F *and* G*, there exists a triple* $\mathfrak{T}'$ *such that the conditions* (i) - (iv) *in Theorem 3.3.1 holds. Then there is a connecting homomorphism* $R^q f_{\mathrm{rig}*}G \to R^{q+1} f_{\mathrm{rig}*}E$ *of overconvergent isocrystals on* $(T,\overline{T})/\mathcal{S}_K$*. This connecting homomorphism is functorial. Moreover, there is a long exact sequence*

$$\begin{array}{ccccccc} 0 & \to & f_{\mathrm{rig}*}E & \to & f_{\mathrm{rig}*}F & \to & f_{\mathrm{rig}*}G \\ & \to & R^1 f_{\mathrm{rig}*}E & \to & R^1 f_{\mathrm{rig}*}F & \to & R^1 f_{\mathrm{rig}*}G \\ & \to & R^2 f_{\mathrm{rig}*}E & \to & \cdots & & \end{array}$$

of overconvergent isocrystals on $(T,\overline{T})/\mathcal{S}_K$*.*

Proof. Since the induced homomorphism (resp. the connecting homomorphism) commutes with the isomorphism ϵ in the proof of Theorem 3.2.1 by [9, Propositions 4.2.1, 4.2.3], the assertions hold by [5, Corollaire 2.1.11, Propositions 2.2.7]. □

3.3.3 Proposition

With the situation as in Theorem 3.3.1, assume furthermore that the residue field k *of* K *is perfect, there is a Frobenius endomorphism* σ *on* K*, and* $\mathfrak{S} = (\mathrm{Spec}\, k, \mathrm{Spec}\, k, \mathrm{Spf}\, \mathcal{V})$*. Let* E *be an overconvergent* F*-isocrystal on* $(X,\overline{X})/K$ *and let* q *be an integer. Suppose that, for each* $q' \leq q$*, there exists a triple* $\mathfrak{T}'$ *such that the conditions* (i) - (iv) *in Theorem 3.3.1 holds and suppose that, for each closed point* t*, the Frobenius endomorphism*

$$\sigma_t^* H^q_{\mathrm{rig}}((X_t,\overline{X}_t)/K_t, E_t) \to H^q_{\mathrm{rig}}((X_t,\overline{X}_t)/K_t, E_t)$$

is an isomorphism. Then the rigid cohomology sheaf $R^q f_{\mathrm{rig}*}E$ *is an overconvergent* F*-isocrystal on* $(T,\overline{T})/K$*. Here* $f_t : (X_t,\overline{X}_t) \to (t,t)$ *is the fiber of* $f : (X,\overline{X}) \to (T,\overline{T})$ *at* t*,* K_t *is a unramified extension of* K *corresponding to the residue extension* $k(t)/k$*,* $\sigma_t : K_t \to K_t$ *is the unique extension of the Frobenius endomorphism* σ *and* E_t *is the inverse image of* E *on* $(X_t,\overline{X}_t)/K_t$*.*

Proof. Let σ_X (resp. σ_T) be an absolute Frobenius on $(X, \overline{X})$ (resp. $(T, \overline{T})$). Then the Frobenius isomorphism $\sigma_X^* E \to E$ induces a Frobenius homomorphism

$$\sigma_T^* R^q f_{\mathrm{rig}*} E \to R^q f_{\mathrm{rig}*} E$$

of overconvergent isocrystals on $(T, \overline{T})/K$ by Theorem 3.3.1 and Proposition 3.3.2. We have only to prove that the Frobenius homomorphism is an isomorphism when $\overline{T} = T = t$ for a k-rational point t and $K_t = K$ by Proposition 3.2.2 and the same reason as in the proof of Proposition 2.3.1.
Let us put $\mathcal{T} = (t, t, \mathrm{Spf}\, \mathcal{V})$. The realization of the overconvergent isocrystal $R^q f_{\mathrm{rig}*} E$ on t/K with respect to $\mathfrak{T}$ is $H^q_{\mathrm{rig}}((X, \overline{X})/K, E)$. Hence, the assertion follows from the hypothesis. □

3.4 The Leray spectral sequence

We apply the construction of the Leray spectral sequence in [15, Remark 3.3] to our relative rigid cohomology cases.

3.4.1 Theorem

With notation as in 3.3, suppose that, for each integer q, there exists a triple $\mathfrak{T}'$ such that the conditions (i) - (iv) *in Theorem 3.3.1 hold. Then there exists a spectral sequence*

$$\underline{E}_2^{qr} = \mathbb{R}^q u_{\mathrm{rig}\mathfrak{S}*}(\mathbb{R}^r f_{\mathrm{rig}*} E) \Rightarrow \mathbb{R}^{q+r}(u \circ f)_{\mathrm{rig}\mathfrak{S}*} E$$

of sheaves of $j^\dagger \mathcal{O}_{]\overline{S}[_{\mathcal{S}}}$-modules.

Proof. Let $\mathfrak{Y} = (Y, \overline{Y}, \mathcal{Y})$ (resp. $\mathfrak{U} = (U, \overline{U}, \mathcal{U})$) be a Zariski covering of $(X, \overline{X})$ (resp. $(T, \overline{T})$) over $\mathfrak{S}$ with a morphism $\mathfrak{Y} \to \mathfrak{U}$ of triples over $\mathfrak{S}$ such that $\mathcal{Y} \to \mathcal{U}$ is smooth around Y. Let $\mathfrak{Y}_\bullet$ (resp. $\mathfrak{U}_\bullet$) be the Čech diagram as $(X, \overline{X})$-triples (resp. $(T, \overline{T})$-triples) over $\mathfrak{S}$ associated to the $(X, \overline{X})$-triple $\mathfrak{Y}$ (resp. the $(T, \overline{T})$-triple $\mathfrak{U}$) over $\mathfrak{S}$ and let us denote by

$$\mathfrak{Y}_\bullet \xrightarrow{g_\bullet} \mathfrak{U}_\bullet \xrightarrow{v_\bullet} \mathfrak{S}$$

the structure morphisms. The Čech diagram $\mathfrak{Y}_\bullet$ (resp. $\mathfrak{U}_\bullet$) is a universally de Rham descendable hypercovering of $(X, \overline{X})$ (resp. $(T, \overline{T})$) over $\mathfrak{S}$ [9, Sect. 10.1].
Let us consider the filtration $\{\mathrm{Fil}^q\}_q$ of $\mathrm{DR}^\dagger(\mathfrak{Y}_\bullet/\mathfrak{S}, (E_{\mathfrak{Y}_\bullet}, \nabla_{\mathfrak{Y}_\bullet}))$ which is defined in 3.2 and take a finitely filtered injective resolution

$$\mathrm{DR}^\dagger(\mathfrak{Y}_\bullet/\mathfrak{S}, (E_{\mathfrak{Y}_\bullet}, \nabla_{\mathfrak{Y}_\bullet})) \to I^\cdot_\bullet$$

as complexes of abelian sheaves on $]\overline{Y}_\bullet[_{\mathcal{Y}_\bullet}$, that is, $\mathrm{Fil}^q I^\cdot_\bullet$ (resp. $\mathrm{gr}^q_{\mathrm{Fil}} I^\cdot_\bullet$) is an injective resolution. Let

$$\widetilde{g}_{\bullet *} I^\cdot_\bullet \to M^{\cdot\cdot}_\bullet$$

be a finitely filtered resolution as complexes of abelian sheaves on $]\overline{U}_{\bullet}[_{\mathcal{U}_{\bullet}}$ such that

(i) $M_s^{qr} = 0$ if one of q, r and s is less than 0;

(ii) $\mathrm{Fil}^i \widetilde{g}_{\bullet *} I_{\bullet}^r \to \mathrm{Fil}^i M_{\bullet}^{\bullet r}$ (resp. $\mathrm{gr}^i_{\mathrm{Fil}} \widetilde{g}_{\bullet *} I_{\bullet}^r \to \mathrm{gr}^i_{\mathrm{Fil}} M_{\bullet}^{\bullet r}$) is a resolution by $\widetilde{v}_{\bullet *}$-acyclic sheaves for any r;

(iii) the complex

$$\underline{H}^r(\mathrm{Fil}^i M_{\bullet}^{0\bullet}) \to \underline{H}^r(\mathrm{Fil}^i M_{\bullet}^{1\bullet}) \to \underline{H}^r(\mathrm{Fil}^i M_{\bullet}^{2\bullet}) \to \cdots$$

is a resolution of $\underline{H}^r(\mathrm{Fil}^i I_{\bullet}^{\bullet})$ by $\widetilde{v}_{\bullet *}$-acyclic sheaves for any r, and the same for $\mathrm{gr}^i_{\mathrm{Fil}} I_{\bullet}^{\bullet} \to \mathrm{gr}^i_{\mathrm{Fil}} M_{\bullet}^{\bullet\bullet}$.

One can construct such a resolution $\widetilde{g}_{\bullet *} I_{\bullet}^{\bullet} \to M_{\bullet}^{\bullet\bullet}$ inductively on degrees and it is called a filtered C-E resolution in [15].
Now we define a filtration $\{F^i\}_i$ of $\widetilde{v}_{\bullet, *} M_{\bullet}^{\bullet\bullet}$ by

$$F^i M_{\bullet}^{q\bullet} = \mathrm{Fil}^{i-q} M_{\bullet}^{q\bullet}.$$

Let us consider a spectral sequence

$$(*) \qquad {}^F\underline{E}_1^{qr} = \underline{H}^{q+r}(\mathrm{gr}^q_F \mathrm{tot}(\widetilde{v}_{\bullet *} M_{\bullet}^{\bullet\bullet})) \Rightarrow \underline{H}^{q+r}(\mathrm{tot}(\widetilde{v}_{\bullet *} M_{\bullet}^{\bullet\bullet}))$$

for the total complex of $\widetilde{v}_{\bullet *} M_{\bullet}^{\bullet\bullet}$ with respect to the filtration $\{F^i\}_i$. Since $\mathfrak{Y}_{\bullet}$ is a universally de Rham descendable hypercovering of $(X, \overline{X})$ over $\mathfrak{S}$, we have

$$\mathbb{R}^r(u \circ f)_{\mathrm{rig}\mathfrak{S}*} E \cong \underline{H}^r(\mathrm{tot}(\widetilde{v}_{\bullet *} M_{\bullet}^{\bullet\bullet})).$$

by the definition of rigid cohomology in [9, Sect 10.4]. Let $({}^{\mathrm{Fil}}\underline{E}_1^{\bullet r}, d_1^{\bullet r})$ be the complex induced by the edge homomorphism of the spectral sequence

$${}^{\mathrm{Fil}}\underline{E}_1^{qr} = \underline{H}^{q+r}(\mathrm{gr}^q_{\mathrm{Fil}} \widetilde{g}_{\bullet *} I_{\bullet}^{\bullet}) \ \Rightarrow \ \underline{H}^{q+r}(\widetilde{g}_{\bullet *} I_{\bullet}^{\bullet}).$$

Then there is a resolution

$$({}^{\mathrm{Fil}}\underline{E}_1^{\alpha r}, d_1^{\alpha r})_{\alpha} \to \{\underline{H}^{\alpha + r}(\mathrm{gr}^{\alpha}_{\mathrm{Fil}} M_{\bullet}^{\beta\bullet})\}_{\alpha,\beta}$$

by the double complex on $]\overline{U}_{\bullet}[_{\mathcal{U}_{\bullet}}$ by the condition (iii). The complex induced by the edge homomorphisms in the ${}^F\underline{E}_1$-stage of the spectral sequence $(*)$ is isomorphic to the total complex of $\{\widetilde{v}_{\bullet *}\underline{H}^{\alpha+r}(\mathrm{gr}^{\alpha}_{\mathrm{Fil}} M_{\bullet}^{\beta\bullet})\}_{\alpha,\beta}$. Hence there is a spectral sequence

$${}^F\underline{E}_2^{qr} = \underline{H}^q(\mathrm{tot}(\mathbb{R}\widetilde{v}_{\bullet *} {}^{\mathrm{Fil}}\underline{E}_1^{\bullet r})) \ \Rightarrow \ \underline{H}^{q+r}(\mathrm{tot}(\widetilde{v}_{\bullet, *} M_{\bullet}^{\bullet\bullet})).$$

Since the direct image overconvergent isocrystal $\mathbb{R}^r f_{\mathrm{rig}*} E$ on $(T, \overline{T})/\mathcal{S}_K$ exists by Theorem 3.3.1, we have

$$({}^{\mathrm{Fil}}\underline{E}_1^{\bullet r}, d_1^{\bullet r}) \ \cong \ \mathrm{DR}^{\dagger}(\mathfrak{U}_{\bullet}/\mathfrak{S}, ((\mathbb{R}^r f_{\mathrm{rig}*} E)_{\mathfrak{U}_{\bullet}}, \nabla^{\mathrm{GM}}_{\mathfrak{U}_{\bullet}}))$$

by Theorem 3.2.1, Proposition 3.2.2 and the definition of rigid cohomology. Here we also use the fact that an injective sheaf on $]\overline{U}_{\boldsymbol{.}}[_{\mathcal{U}_{\boldsymbol{.}}}$ consists of injective sheaves at each stage [9, Corollary 3.8.7]. Therefore, we have the Leray spectral sequence

$$\underline{E}_2^{qr} = \mathbb{R}^q u_{\mathrm{rig}\mathfrak{S}*}(\mathbb{R}^r f_{\mathrm{rig}*}E) \Rightarrow \mathbb{R}^{q+r}(u \circ f)_{\mathrm{rig}\mathfrak{S}*}E$$

in rigid cohomology. □

4 Examples of coherence

Let $\mathfrak{S} = (S, \overline{S}, \mathcal{S})$ be a $\mathcal{V}$-triple separated of finite type and let $(X, \overline{X}) \xrightarrow{f} (T, \overline{T}) \longrightarrow (S, \overline{S})$ be a sequence of morphisms of pairs separated of finite type over $\operatorname{Spec} k$. In order to see the existence of the overconvergent isocrystal $R^q f_{\mathrm{rig}*}E$, one has to show the coherence of direct images. Berthelot's conjecture [4, Sect. 4.3] asserts that, if $\overline{f}$ is proper, $X = \overline{f}^{-1}(T)$ and $\overset{\circ}{f}$ is smooth, then the rigid cohomology overconvergent $(F\text{-})$isocrystal $R^q f_{\mathrm{rig}*}E$ exists for any q and any overconvergent $(F\text{-})$isocrystal E on $(X, \overline{X})/\mathcal{S}_K$. In this section we discuss a generic coherence which relates to Berthelot's conjecture.

4.1 Liftable cases

The following Theorems 4.1.1 and 4.1.4 are due to Berthelot [4, Théorème 5]. We give a proof of the theorems along our studies in the previous sections.

4.1.1 Theorem

Suppose that there exists a commutative diagram

$$\begin{array}{ccccc} X & \rightarrow & \overline{X} & \rightarrow & \mathcal{X} \\ \overset{\circ}{f}\downarrow & & \overline{f}\downarrow & & \downarrow\widehat{f} \\ T & \rightarrow & \overline{T} & \rightarrow & \mathcal{T} \end{array}$$

of $\mathfrak{S}$-triples such that both squares are cartesian, $\widehat{f} : \mathcal{X} \to \mathcal{T}$ is proper and smooth around X and $\widehat{g} : \mathcal{T} \to \mathcal{S}$ is smooth around T. Then the condition $(\mathrm{C})^q_{f,\mathfrak{T},E}$ holds for any q and any overconvergent isocrystal E on $(X, \overline{X})/\mathcal{S}_K$. In particular, the rigid cohomology overconvergent isocrystal $R^q f_{\mathrm{rig}*}E$ on $(T, \overline{T})/\mathcal{S}_K$ exists. If $\mathfrak{S} = (\operatorname{Spec} k, \operatorname{Spec} k, \operatorname{Spf} \mathcal{V})$ and the relative dimension of X over T is less than or equal to d, then $R^q f_{\mathrm{rig}*}E = 0$ for $q > 2d$.
Moreover, the base change homomorphism is an isomorphism of overconvergent isocrystals for any base extension $(T', \overline{T}') \to (T, \overline{T})$ separated of finite type over $(S, \overline{S})$.

Proof. We may assume that $\mathcal{T}$ is affine. First we prove the coherence of $\mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}*}E$. By the Hodge-de Rham spectral sequence

$$\underline{E}_1^{qr} = R^r \widetilde{f}_*(E \otimes_{j^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}}}} j^\dagger \Omega^q_{]\overline{X}[_{\mathcal{X}}/]\overline{T}[_{\mathcal{T}}}) \Rightarrow \mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}*}E$$

we have only to prove that $R^r\widetilde{f}_*(E\otimes_{j^\dagger\mathcal{O}_{]\overline{X}[_\mathcal{X}}} j^\dagger\Omega^q_{]\overline{X}[_\mathcal{X}/]\overline{T}[_\mathcal{T}})$ is a sheaf of coherent $j^\dagger\mathcal{O}_{]\overline{T}[_\mathcal{T}}$-modules for any q and r. Since the second square is cartesian, the associated analytic map $\widetilde{f}:]\overline{X}[_\mathcal{X}\to]\overline{T}[_\mathcal{T}$ is quasi-compact. If $\{V\}$ is a filter of a fundamental system of strict neighbourhoods of $]T[_\mathcal{T}$ in $]\overline{T}[_\mathcal{T}$, then $\{\widetilde{f}^{-1}(V)\}$ is a filter of a fundamental system of strict neighbourhoods of $]\overline{X}[_\mathcal{X}$ since both squares are cartesian. Let us take a sheaf $\mathcal{E}$ of coherent $j^\dagger\mathcal{O}_{]\overline{X}[_\mathcal{X}}$-modules with $E=j^\dagger\mathcal{E}$ ($\mathcal{E}$ is defined only on a strict neighbourhood in general). One can take a filter of a fundamental system $\{V\}$ of strict neighbourhoods of $]T[_\mathcal{T}$ in $]\overline{T}[_\mathcal{T}$ such that, if $j_V: V\to]\overline{T}[_\mathcal{T}$ (resp. $j_{\widetilde{f}^{-1}(V)}:\widetilde{f}^{-1}(V)\to]\overline{X}[_\mathcal{X}$) denotes the open immersion, then $R^qj_{V*}\mathcal{E}=0$ (resp. $R^qj_{\widetilde{f}^{-1}(V)*}\widetilde{f}^*\mathcal{E}=0$) by [9, Sect. 2.6, Proposition 5.1.1]. Since the direct limit commutes with cohomological functors by quasi-separatedness and quasi-compactness, we have

$$\begin{aligned}
&R^r\widetilde{f}_*(E\otimes_{j^\dagger\mathcal{O}_{]\overline{X}[_\mathcal{X}}} j^\dagger\Omega^q_{]\overline{X}[_\mathcal{X}/]\overline{T}[_\mathcal{T}})\\
&\quad\cong R^r\widetilde{f}_*j^\dagger(\mathcal{E}\otimes_{\mathcal{O}_{]\overline{X}[_\mathcal{X}}}\Omega^q_{]\overline{X}[_\mathcal{X}/]\overline{T}[_\mathcal{T}})\\
&\quad\cong R^r\widetilde{f}_*(\varinjlim_V j_{\widetilde{f}^{-1}(V)*}j^{-1}_{\widetilde{f}^{-1}(V)}(\mathcal{E}\otimes_{\mathcal{O}_{]\overline{X}[_\mathcal{X}}}\Omega^q_{]\overline{X}[_\mathcal{X}/]\overline{T}[_\mathcal{T}}))\\
&\quad\cong \varinjlim_V R^r\widetilde{f}_*(j_{\widetilde{f}^{-1}(V)*}j^{-1}_{\widetilde{f}^{-1}(V)}(\mathcal{E}\otimes_{\mathcal{O}_{]\overline{X}[_\mathcal{X}}}\Omega^q_{]\overline{X}[_\mathcal{X}/]\overline{T}[_\mathcal{T}}))\\
&\quad\cong \varinjlim_V R^r(\widetilde{f}j_{\widetilde{f}^{-1}(V)})_*j^{-1}_{\widetilde{f}^{-1}(V)}(\mathcal{E}\otimes_{\mathcal{O}_{]\overline{X}[_\mathcal{X}}}\Omega^q_{]\overline{X}[_\mathcal{X}/]\overline{T}[_\mathcal{T}})\\
&\quad\cong \varinjlim_V j_V^{-1}R^r(j_{V*}\widetilde{f})_*(\mathcal{E}\otimes_{\mathcal{O}_{]\overline{X}[_\mathcal{X}}}\Omega^q_{]\overline{X}[_\mathcal{X}/]\overline{T}[_\mathcal{T}})\\
&\quad\cong \varinjlim_V j_V^{-1}j_{V*}R^r\widetilde{f}_*(\mathcal{E}\otimes_{\mathcal{O}_{]\overline{X}[_\mathcal{X}}}\Omega^q_{]\overline{X}[_\mathcal{X}/]\overline{T}[_\mathcal{T}})\\
&\quad\cong j^\dagger R^r\widetilde{f}_*(\mathcal{E}\otimes_{\mathcal{O}_{]\overline{X}[_\mathcal{X}}}\Omega^q_{]\overline{X}[_\mathcal{X}/]\overline{T}[_\mathcal{T}}).
\end{aligned}$$

Here $R^r\widetilde{f}_*(\mathcal{E}\otimes_{\mathcal{O}_{]\overline{X}[_\mathcal{X}}}\Omega^q_{]\overline{X}[_\mathcal{X}/]\overline{T}[_\mathcal{T}})$ is coherent on a strict neighborhood of $]T[_\mathcal{T}$ in $]\overline{T}[_\mathcal{T}$ by Kiehl's finiteness theorem of cohomology of coherent sheaves [17, Theorem 3.3] and Chow's lemma. Hence, each E_1-term is coherent. The situation is unchanged after any extension $\mathcal{T}'\to\mathcal{T}$ smooth around T. Therefore, the condition $(\mathrm{C})^q_{f,\mathfrak{T},E}$ holds for any q and any E. Hence, the rigid cohomology overconvergent isocrystal $R^qf_{\mathrm{rig}*}E$ exists by Theorem 3.3.1.

Suppose that $\mathfrak{S}=(\operatorname{Spec}k,\operatorname{Spec}k,\operatorname{Spf}\mathcal{V})$ and the relative dimension of X over T is less than or equal to d. Then $R^qf_{\mathrm{rig}*}E$ is an overconvergent isocrystal on $(T,\overline{T})/K$. In order to prove the vanishing of $R^qf_{\mathrm{rig}*}E$ for $q>2d$, we have only to prove the assertion when $\overline{T}=T=t$ for a k-rational point t by Proposition 3.2.2 and the same reason as in the proof of Proposition 2.3.1. Let us put $\mathfrak{T}=(\operatorname{Spec}k,\operatorname{Spec}k,\operatorname{Spf}\mathcal{V})$. The realization of the overconvergent isocrystal $R^qf_{\mathrm{rig}*}E$ on t/K with respect to $\mathfrak{T}$ is $H^q_{\mathrm{rig}}((X,\overline{X})/K,E)$. Hence, the vanishing follows from Lemma 4.1.2 below.

Since the liftable situation is unchanged locally on base schemes, the base change homomorphism for $(T', \overline{T}') \to (T, \overline{T})$ is isomorphic as overconvergent isocrystals by Proposition 2.3.1. □

4.1.2 LEMMA

Let X be a smooth separated scheme of finite type over $\operatorname{Spec} k$, let Z be a closed subscheme of X, and let E be an overconvergent isocrystal on X/K. Suppose that X is of dimension d and Z is of codimension greater than or equal to e. Then the rigid cohomology $H^q_{Z,\mathrm{rig}}(X/K, E)$ with supports in Z vanishes for any $q > 2d$ and any $q < 2e$.

PROOF. We use double induction on d and e, similar to the proof of [6, Théorème 3.1]. If $d = 0$, then the assertion is trivial. Suppose that the assertion holds for X with dimension less than d and suppose that $e = d$. Then we may assume that Z is a finite set of k-rational points by Proposition 2.2.2. Hence the assertion follows from the Gysin isomorphism

$$H^q_{Z,\mathrm{rig}}(X/K, E) \cong H^{q-2e}_{\mathrm{rig}}(Z/K, E_Z)$$

[20, Theorem 4.1.1]. Suppose that the assertion holds for any closed subscheme with codimension greater than e. By using the excision sequence

$$\begin{array}{llll} \cdots \to H^q_{Z',\mathrm{rig}}(X/K, E) & \to & H^q_{Z,\mathrm{rig}}(X/K, E) & \to & H^q_{Z \setminus Z',\mathrm{rig}}(X \setminus Z'/K, E) \\ \quad \to H^{q+1}_{Z',\mathrm{rig}}(X/K, E) & \to & \cdots, & & \end{array}$$

for a closed subscheme Z' of Z (see [6, Proposition 2.5] for the constant coefficients; the general case is similar), we may assume Z is irreducible. We may also assume that Z is absolutely irreducible by Proposition 2.2.2. Then there is an affine open subscheme U of X such that the inverse image Z_U of Z in U is smooth over $\operatorname{Spec} k$ after a suitable extension of k. Since $Z \setminus Z_U$ is of codimension greater than e, we may assume that Z is smooth over $\operatorname{Spec} k$ by the excision sequence and Proposition 2.2.2. Applying the Gysin isomorphism to $Z \subset X$, we have the assertion by the induction hypothesis if $e > 0$. Now suppose that $e = 0$. We may assume $X = Z$ by induction on the number of generic points of X. We may also assume that $X = Z$ is an affine smooth scheme over $\operatorname{Spec} k$ and we can find an affine smooth lift $\widetilde{X}$ of X over $\operatorname{Spec} \mathcal{V}$ by [10, Théorème 6]. Let $\mathcal{X}$ be the p-adic completion of the Zariski closure of $\widetilde{X}$ in a projective space over $\operatorname{Spec} \mathcal{V}$ and put $\overline{X} = \mathcal{X} \times_{\mathrm{Spf}\,\mathcal{V}} \operatorname{Spec} k$. Then $H^q(]\overline{X}[_{\mathcal{X}}, \mathcal{E}) = 0\,(q > 0)$ for any sheaf $\mathcal{E}$ of coherent $j^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$-modules by [9, Corollary 5.1.2]. Hence one can calculate the rigid cohomology by the complex of global sections of the de Rham complex associated to a realization of E. Therefore, $H^q_{\mathrm{rig}}(X/K, E) = 0$ for any $q > 2d$. This completes the proof. □

Since the situation in Theorem 4.1.1 is unchanged by any extension $\mathcal{V} \to \mathcal{W}$ of complete discrete valuation rings, we have

4.1.3 Proposition

Under the assumption in Theorem 4.1.1, the condition $(\mathrm{F})^q_{f,\mathcal{W}/\mathcal{V},E}$ *holds for any* q*, any extension* $\mathcal{W}$ *of complete discrete valuation ring over* $\mathcal{V}$ *and any overconvergent isocrystal* E *on* $(X, \overline{X})/\mathcal{S}_K$*.*

We mention overconvergent F-isocrystals.

4.1.4 Theorem

With the situation as in Theorem 4.1.1, suppose $\mathfrak{S} = (\mathrm{Spec}\, k, \mathrm{Spec}\, k, \mathrm{Spf}\, \mathcal{V})$ *and let* σ *be a Frobenius endomorphism on* K*. Then, for an overconvergent* F*-isocrystal* E *on* $(T, \overline{T})/K$*, the rigid cohomology overconvergent isocrystal* $R^q f_{\mathrm{rig}*}E$ *is an overconvergent* F*-isocrystal on* $(T, \overline{T})/K$ *for any* q*.*
Moreover, the base change homomorphism is an isomorphism as overconvergent F*-isocrystals for any base extension* $(T', \overline{T}') \to (T, \overline{T})$ *separated of finite type over* $(S, \overline{S})$*.*

Proof. Let us put $\mathcal{V}' = \varinjlim (\mathcal{V} \xrightarrow{\sigma} \mathcal{V} \xrightarrow{\sigma} \cdots)$. Then $\mathcal{V}'$ is a complete discrete valuation ring over $\mathcal{V}$ whose residue field k' is the perfection of k. Applying Proposition 4.1.3 to the base extension by $\mathrm{Spec}\, k' \to \mathrm{Spec}\, k$, we may assume that k is algebraically closed. Let $\mathcal{W}$ be the ring of Witt vectors with coefficients in k and let L be the quotient field of $\mathcal{W}$. Since the residue field of L is algebraically closed, the restriction of σ on L is the canonical Frobenius endomorphism. Regarding an overconvergent F-isocrystal E on $(X, \overline{X})/K$ as an overconvergent F-isocrystal E on $(X, \overline{X})/L$, we may assume that $K = L$ by the argument of [20, Sect. 5.1].
Let t be a closed point of T. Since the associated convergent F-isocrystal E_t on X_t/K comes from an F-crystal on $X_t/\mathcal{W}$ [5, Théorème 2.4.2], the Frobenius homomorphism

$$\sigma_t^* H^q_{\mathrm{rig}}(X_t/K, E_t) \to H^q_{\mathrm{rig}}(X_t/K, E_t)$$

is an isomorphism by the Poincaré duality of the crystalline cohomology theory [3, Théorème 2.1.3] and the comparison theorem between the rigid cohomology and the crystalline cohomology [6, Proposition 1.9] [20, Theorem 5.2.1]. Therefore, the assertion follows from Proposition 3.3.3. □

4.2 Generic coherence

First we present a problem concerning an existence of smooth liftings.

4.2.1 Problem

Let $\mathcal{V}$ be a complete discrete valuation ring of mixed characteristics with residue field k and let X be a proper smooth connected scheme over $\mathrm{Spec}\, k$. Are there a finite extension $\mathcal{W}$ of complete discrete valuation ring over $\mathcal{V}$ and a projective smooth scheme Y over $\operatorname{Spec}\mathcal{W}$ with a proper surjective and generically finite morphism

$$Y \times_{\operatorname{Spec}\mathcal{W}} \operatorname{Spec} l \to X$$

over $\operatorname{Spec} k$? Here l is the residue field of $\mathcal{W}$.

4.2.2 Remark

We give two remarks on the problem above.

(1) When X is a proper smooth curve over $\operatorname{Spec} k$, one can take a projective smooth scheme Y over $\operatorname{Spec}\mathcal{V}$ such that $Y \times_{\operatorname{Spec}\mathcal{V}} \operatorname{Spec} k = X$ [12, Exposé III, Corollaire 7.4].

(2) In Proposition 4.2.3 and its application to Theorem 4.2.4 and Corollary 4.2.5 we use only the smoothness of the formal completion $\widehat{Y} \to \operatorname{Spf}\mathcal{W}$. Hence, it is sufficient to resolve a weaker version of the problem which asks for the existence of a formal $\mathcal{W}$-scheme $\mathcal{Y}$ which is projective smooth over $\operatorname{Spf}\mathcal{W}$ with a proper surjective and generically finite morphism

$$\mathcal{Y} \times_{\operatorname{Spec}\mathcal{W}} \operatorname{Spec} l \to X$$

over $\operatorname{Spec} k$. If such a $\mathcal{Y}$ exists, then there is a projective scheme over $\operatorname{Spec}\mathcal{W}$ whose formal completion over $\operatorname{Spf}\mathcal{W}$ is $\mathcal{Y}$ by [13, Chap. 3, Corollaire 5.1.8].

Let $\boldsymbol{m}$ be the maximal ideal of $\mathcal{V}$ and let

$$\begin{array}{ccc} X & \to & \overline{X} \\ \overset{\circ}{f}\downarrow & & \downarrow \overline{f} \\ T & \to & \overline{T} \end{array}$$

be a cartesian square of k-schemes such that $\overline{T}$ is an affine and integral scheme which is smooth over $\operatorname{Spec} k$, $\overline{f}$ is proper and $\overset{\circ}{f}$ is smooth with a connected generic fiber X_η, where η denotes the generic point of $\overline{T}$. Let us take a smooth lift $\mathcal{T} = \operatorname{Spec} R$ of $\overline{T}$ over $\operatorname{Spec}\mathcal{V}$ and put $\mathfrak{T} = (T, \overline{T}, \mathcal{T})$ with $\mathcal{T} = \operatorname{Spf}\widehat{R}$. Here $\widehat{Z}$ (resp. $\widehat{A}$) means the p-adic formal completion for a $\mathcal{V}$-scheme Z (resp. a $\mathcal{V}$-algebra A). Let us denote by $R_{\boldsymbol{m}}$ the localization of R at the prime ideal $\boldsymbol{m}R$. Note that $\widehat{R}$ is integral and $\widehat{R_{\boldsymbol{m}}}$ is a complete discrete valuation ring over $\mathcal{V}$ with special point η.

4.2.3 Proposition

With notation as above, suppose that there is a projective smooth scheme Y over $\operatorname{Spec}\widehat{R_{\boldsymbol{m}}}$ whose reduction is the generic fiber X_η of X. Then there exist

(i) *an open dense subscheme* U *of* T *(we define the associated* $\mathcal{V}$*-triple* $\mathfrak{U} = (U, \overline{T}, \mathcal{T})$*);*

(ii) *a formal* $\mathcal{V}$*-scheme* $\mathcal{T}'$ *which is finite flat over* $\mathcal{T}$ *(*$\mathfrak{U}' = (U', \overline{T}', \mathcal{T}')$*), where* U' *(resp.* $\overline{T}'$*) is the inverse image of* U *(resp.* $\overline{T}$*) in* $\mathcal{T}'$*);*

(iii) *a formal* $\mathcal{V}$*-scheme* $\mathcal{T}''$ *separated of finite type over* $\mathcal{T}'$ *with a closed immersion* $\overline{T}' \to \mathcal{T}''$ *over* $\mathcal{T}'$ *such that* $\mathcal{T}'' \to \mathcal{T}'$ *is etale around* U' *(*$\mathfrak{U}'' = (U', \overline{T}', \mathcal{T}'')$*);*

(iv) *a formal* $\mathcal{V}$*-scheme* $\mathcal{X}''$ *projective over* $\mathcal{T}''$ *with a natural isomorphism*

$$\mathcal{X}'' \times_{\mathcal{T}''} U' \cong X \times_T U'$$

such that $\mathcal{X}'' \to \mathcal{T}''$ *is smooth around* $\mathcal{X}'' \times_{\mathcal{T}''} U'$*.*

Proof. Since Y is projective over $\operatorname{Spec} \widehat{R_{\boldsymbol{m}}}$, one can take $a_1, \cdots, a_n \in \boldsymbol{m}\widehat{R_{\boldsymbol{m}}}$ for any i such that there exists a projective scheme Z over the n-dimensional $\widehat{R}$-affine space $\mathbb{A}^n_{\widehat{R}}$ with the natural cartesian square

$$\begin{array}{ccc} Y & \rightarrow & Z \\ \downarrow & & \downarrow \\ \operatorname{Spec} \widehat{R_{\boldsymbol{m}}} & \rightarrow & \mathbb{A}^n_{\widehat{R}} \end{array} \qquad a_i \leftarrow x_i.$$

Indeed, if one considers the sheaf $\mathcal{I}$ of ideals of definition of Y in a projective space over $\operatorname{Spec} \widehat{R_{\boldsymbol{m}}}$, then the Serre twist $\mathcal{I}(r)$ of $\mathcal{I}$ is generated by global sections for sufficiently large r. Then all coefficients which appear in the generators belong to $R + \boldsymbol{m}\widehat{R_{\boldsymbol{m}}}$.

Let $I \subset \widehat{R}[x]$ be an ideal of definition of the image of $\operatorname{Spec} \widehat{R_{\boldsymbol{m}}}$ in $\mathbb{A}^n_{\widehat{R}}$. Choose $b \in \widehat{R_{\boldsymbol{m}}}^n$ with $|b| < 1$ such that $g(b) = 0$ for all $g \in I$ and denote by Z_b the pull back of $Z \to \mathbb{A}^n_{\widehat{R}}$ by the natural morphism $\operatorname{Spec} \widehat{R}[b]^{\mathrm{nor}} \to \mathbb{A}^n_{\widehat{R}}$, where $\operatorname{Spec} \widehat{R}[b]^{\mathrm{nor}}$ is the normalization of $\operatorname{Spec} \widehat{R}[b]$. Note that $\widehat{R}[b]^{\mathrm{nor}}$ is finitely generated over $\widehat{R}$ since the characteristic of the field of fractions of $\widehat{R}$ is 0. The map defined by $b \mapsto 0$ determines a closed immersion $\overline{T} \to \operatorname{Spf} \widehat{\widehat{R}[b]^{\mathrm{nor}}}$ over $\operatorname{Spf} \widehat{R}$ since $\widehat{R}[b]^{\mathrm{nor}}$ is included in $\widehat{R_{\boldsymbol{m}}}$. $\overline{T}$ is a connected component of $\operatorname{Spf} \widehat{\widehat{R}[b]^{\mathrm{nor}}} \times_{\operatorname{Spf} \mathcal{V}} k$. The generic fiber of $\widehat{Z}_b \times_{\operatorname{Spf} \widehat{\widehat{R}[b]^{\mathrm{nor}}}} \overline{T} \to \overline{T}$ is X_η by our construction of Z. Hence, there are an open dense subset of X containing the generic fiber and an open dense subset of $\widehat{Z}_b \times_{\operatorname{Spf} \widehat{\widehat{R}[b]^{\mathrm{nor}}}} T$ containing the generic fiber which are isomorphic to each other.

Put $a = (a_1, \cdots, a_n)$. Since smoothness is an open condition, $\widehat{Z}_a \to \operatorname{Spf} \widehat{\widehat{R}[a]^{\mathrm{nor}}}$ is smooth around X_η by applying the Jacobian criterion to the cartesian squares

$$\begin{array}{ccccc} X_\eta & \rightarrow & \widehat{Y} & \rightarrow & \widehat{Z}_a \\ \downarrow & & \downarrow & & \downarrow \\ \eta & \rightarrow & \operatorname{Spf} \widehat{R_{\boldsymbol{m}}} & \rightarrow & \operatorname{Spf} \widehat{\widehat{R}[a]^{\mathrm{nor}}}. \end{array}$$

Now we consider the associated analytic morphism to $\widehat{Z} \to \widehat{\mathbb{A}}^n_R$. Since smoothness is an open condition for morphisms of rigid analytic spaces, there exists $0 < \lambda < 1$ such that $\widehat{Z}_b \to \operatorname{Spf} \widehat{\widehat{R}[b]^{\mathrm{nor}}}$ is smooth around X_η for any $b \in \widehat{R_{\boldsymbol{m}}}^n$ with $|b - a| = \max_i |b_i - a_i| \le \lambda$ such that $g(b) = 0$ for any $g \in I$ by the quasi-compactness of $\widehat{Z}^{\mathrm{an}}_K$.

Let us take an element $b \in (\operatorname{Frac}(\widehat{R})^{\mathrm{alg}} \cap \widehat{R_{\boldsymbol{m}}})^n$ with $|b - a| \le \lambda$ such that $g(b) = 0$ for any $g \in I$. Here $\operatorname{Frac}(\widehat{R})^{\mathrm{alg}}$ is the algebraic closure of $\operatorname{Frac}(\widehat{R})$ in an algebraic closure $\widehat{R_{\boldsymbol{m}}}[p^{-1}]^{\mathrm{alg}}$ of $\widehat{R_{\boldsymbol{m}}}[p^{-1}]$. It is possible to choose such a b using Noether's normalization theorem and the approximation by Newton's method. Consider an extension $\widehat{R}[b]^{\mathrm{nor}} \widehat{\otimes}_{\widehat{R}} \widehat{R_{\boldsymbol{m}}}$ over $\widehat{R_{\boldsymbol{m}}}$. Then $\widehat{R}[b]^{\mathrm{nor}} \widehat{\otimes}_{\widehat{R}} \widehat{R_{\boldsymbol{m}}}$ is finite over $\widehat{R_{\boldsymbol{m}}}$ for $|b| < 1$ and it has no p-torsion. We denote by $\widehat{R_{\boldsymbol{m}}}'$ a finite extension as a complete discrete valuation ring over $\widehat{R_{\boldsymbol{m}}}$ which contains $\widehat{R}[b]^{\mathrm{nor}}$. Then there are a sequence $\widehat{R} = \widehat{R}_0, \widehat{R}_1, \cdots, \widehat{R}_s$ of finite extensions of $\widehat{R}$ in $\operatorname{Frac}(\widehat{R})^{\mathrm{alg}}$ and a sequence $q_1(z), \cdots, q_s(z)$ of monic polynomials with $q_i(z) \in \widehat{R}_{i-1}[z]$ such that $\widehat{R}_i \cong \widehat{R}_{i-1}[z]/(q_i(z))$ and $\widehat{R_{\boldsymbol{m}}}'$ is generated by $\widehat{R}_s$ over $\widehat{R_{\boldsymbol{m}}}$ using the approximation by Newton's method.

Now we define $\mathcal{V}$-formal schemes separated of finite type

$$\begin{array}{rcl} \mathcal{T}' & = & \operatorname{Spf} \widehat{R}_s \\ \mathcal{T}'' & = & \text{the Zariski closure of the image of the diagonal morphism} \\ & & \operatorname{Spf} \widehat{R_{\boldsymbol{m}}}' \to \mathcal{T}' \times_{\mathcal{T}} \operatorname{Spf} \widehat{\widehat{R}[b]^{\mathrm{nor}}} \text{ as a } \mathcal{V}\text{-formal scheme} \\ \mathcal{X}'' & = & \widehat{Z}_b \times_{\operatorname{Spf} \widehat{\widehat{R}[b]^{\mathrm{nor}}}} \mathcal{T}'' \end{array}$$

and define the k-scheme $\overline{T}' = \overline{T} \times_{\mathcal{T}} \mathcal{T}'$. Then there is a natural closed immersion $\overline{T}' \to \mathcal{T}''$ over $\mathcal{T}'$. By our construction of $\mathcal{T}''$, $\mathcal{T}'' \to \mathcal{T}'$ is etale around all generic points of $\mathcal{T}'' \times_{\mathcal{V}} k$ above on $\eta \times_T T'$ since all the localizations of $\widehat{R}[b]^{\mathrm{nor}} \widehat{\otimes}_{\widehat{R}} \widehat{R_{\boldsymbol{m}}}$ at the prime ideal above $\boldsymbol{m}$ is contained in $\widehat{R_{\boldsymbol{m}}}'$. Hence, $\mathcal{T}'' \to \mathcal{T}'$ is etale around a dense open subscheme of $\overline{T}'$. By the property of $\widehat{Z}_b$, $\mathcal{X}'' \to \mathcal{T}''$ is smooth around $X_\eta \times_{\overline{T}} \overline{T}'$. Moreover, there are an open dense subset of $X \times_{\overline{T}} \overline{T}'$ containing all of the generic fibers over $\overline{T}'$ and an open dense subset of $\mathcal{X}'' \times_{\mathcal{T}''} \overline{T}'$ containing all of the generic fibers over $\overline{T}'$ which are isomorphic to each other. Therefore, there is an open dense subscheme U of T such that $\mathfrak{U} = (U, \overline{T}, \mathcal{T})$, $\mathfrak{U}' = (U', \overline{T}', \mathcal{T}')$ with $U' = U \times_{\overline{T}} \overline{T}'$, $\mathfrak{U}'' = (U', \overline{T}', \mathcal{T}'')$ and $\mathcal{X}''$ are the desired objects. □

Now we apply the study in 4.1. Let us keep the notation as in the beginning of this section. Suppose that the diagram

$$\begin{array}{ccc} X & \to & \overline{X} \\ \overset{\circ}{f} \downarrow & & \downarrow \overline{f} \\ T & \to & \overline{T} \end{array}$$

is a cartesian square of k-schemes such that $\overline{f}$ is proper and $\overset{\circ}{f}$ has smooth generic fibers.

4.2.4 Theorem

Under the assumption as above, assume furthermore that $\overline{T}$ is an affine integral scheme which is smooth over $\mathrm{Spec}\, k$ and the generic fiber of $\overset{\circ}{f} : X \to T$ is connected. Suppose that there exists a projective smooth lift of the generic fiber of $\overset{\circ}{f}$ over the spectrum of a complete discrete valuation ring which is induced by the localization of a smooth lift of $\overline{T}$ over $\mathrm{Spec}\, \mathcal{V}$. Then there exists an open dense subscheme U of T and a formal $\mathcal{V}$-scheme $\mathcal{T}$ over $\mathcal{S}$ which is smooth around T such that, if one denotes by $f_U : (\overset{\circ}{f}{}^{-1}(U), \overline{X}) \to (U, \overline{T})$ the restriction of f and puts $\mathfrak{U} = (U, \overline{T}, \mathcal{T})$, then the condition $(\mathrm{C})^q_{f_U, \mathfrak{U}, E}$ holds for any q and any overconvergent isocrystal E on $(\overset{\circ}{f}{}^{-1}(U), \overline{X})/\mathcal{S}_K$. In particular, the rigid cohomology overconvergent isocrystal $R^q f_{U\mathrm{rig}*}E$ on $(U, \overline{T})/\mathcal{S}_K$ exists for any q and any E.

Proof. Let us take U, $\mathfrak{U}$, $\mathfrak{U}'$, $\mathfrak{U}''$ and $\mathcal{X}''$ as in Proposition 4.2.3 except for the formal schemes $\mathcal{U}$, $\mathcal{U}'$, $\mathcal{U}''$ and $\mathcal{X}''$. We replace $\mathcal{U}$, $\mathcal{U}'$, $\mathcal{U}''$ and $\mathcal{X}''$ by $\mathcal{U} \times_{\mathrm{Spf}\, \mathcal{V}} \mathcal{S}$, $\mathcal{U}' \times_{\mathrm{Spf}\, \mathcal{V}} \mathcal{S}$, $\mathcal{U}'' \times_{\mathrm{Spf}\, \mathcal{V}} \mathcal{S}$ and $\mathcal{X}'' \times_{\mathrm{Spf}\, \mathcal{V}} \mathcal{S}$. Let us denote by $f'_U : (X \times_T U', \overline{X} \times_{\overline{T}} \overline{T}') \to (U', \overline{T}')$ (resp. $f''_U : (\mathcal{X}'' \times_{\mathcal{T}''} U', \mathcal{X}'' \times_{\mathcal{T}''} \overline{T}') \to (U', \overline{T}')$) the induced morphism from the conclusion of Proposition 4.2.3, and by E' (resp. E'') the inverse image of E on $(X \times_T U', \overline{X} \times_{\overline{T}} \overline{T}')/\mathcal{T}'_K$ (resp. $(\mathcal{X}'' \times_{\mathcal{T}''} U', \mathcal{X}'' \times_{\mathcal{T}''} \overline{T}')/\mathcal{T}'_K$). Then the condition $(\mathrm{C})^q_{f_U, \mathfrak{U}, E}$ is equivalent to the condition $(\mathrm{C})^q_{f'_U, \mathfrak{U}', E'}$ by the finite flat base change theorem (Theorem 2.1.1) and the faithfully flat descent theorem for finitely generated modules. Since the rigid cohomology is independent of the choices of compactification [4, Sect. 2, Théorème 2], the condition $(\mathrm{C})^q_{f'_U, \mathfrak{U}', E'}$ is equivalent to the condition $(\mathrm{C})^q_{f''_U, \mathfrak{U}', E''}$. Then the assertion follows from Proposition 3.3.1. □

With the situation of Theorem 4.2.4, a smooth lift of $\overline{T}$ over $\mathrm{Spec}\, \mathcal{V}$ always exists by [10, Theorem 6] since $\overline{T}$ is affine and smooth.

4.2.5 Corollary

With notation as above, suppose that Problem 4.2.1 admits an affirmative solution in general. Then there exists an open dense subscheme U of T such that, if $f_U : (\overset{\circ}{f}{}^{-1}(U), \overline{X}) \to (U, \overline{T})$ denotes the morphism of pairs induced by f, the rigid cohomology overconvergent isocrystal $R^q f_{U\mathrm{rig}*}E$ on $(U, \overline{T})/\mathcal{S}_K$ exists for any q and any overconvergent isocrystal E on $(\overset{\circ}{f}{}^{-1}(U), \overline{X})/\mathcal{S}_K$. If

$\mathfrak{S} = (\operatorname{Spec} k, \operatorname{Spec} k, \operatorname{Spf}\mathcal{V})$ and the relative dimension X over T is d, then $R^q f_{U\mathrm{rig}*}E = 0$ for any $q > 2d$.
Moreover, the base change homomorphism is an isomorphism of overconvergent isocrystals for any base extension $(T', \overline{T}') \to (T, \overline{T})$ separated of finite type over $(S, \overline{S})$ (resp. any extension $\mathcal{W}$ of complete discrete valuation ring over $\mathcal{V}$).
Suppose that $\mathfrak{S} = (\operatorname{Spec} k, \operatorname{Spec} k, \operatorname{Spf}\mathcal{V})$ and there exists a Frobenius endomorphism on K. Then $R^q f_{U\mathrm{rig}*}E$ is an overconvergent F-isocrystal on $(U, \overline{T})/K$.

Proof. Note that the rigid cohomology is determined by the reduced structures of $\overline{X}$ and $\overline{T}$. We may assume that $\overline{T}$ is affine. Since we can replace K by finite extensions of K by Proposition 2.2.1 and faithfully flat descent of finitely generated modules, we may assume that there exists a generically etale and proper surjective morphism $\overline{T}' \to \overline{T}$ such that $\overline{T}'$ is smooth over $\operatorname{Spec} k$ by applying an alteration [14, Theorem 4.1]. Let us denote by T' the inverse image of T in $\overline{T}'$. Shrinking T, we can find a finite flat extension $\overline{T}''$ of $\overline{T}$ such that T' is an open subscheme of $\overline{T}''$. Indeed, such $\overline{T}''$ exists if one considers an open dense subscheme of $\overline{T}'$ which is standard etale over $\overline{T}$. Applying the finite flat base change theorem (Theorem 2.1.1) to the extension $(T', \overline{T}'')/(T, \overline{T})$, we may assume that $T' = T$ by the faithfully flat descent of finitely generated modules. Since the category of overconvergent isocrystals is independent of the choices of compactification [5, Théorème 2.3.5], we may assume that $\overline{T}' = \overline{T}$. Hence, we may assume that $\overline{T}$ is an affine and integral scheme which is smooth over $\operatorname{Spec} k$.
Let $\mathcal{T}$ be a formal $\mathcal{V}$-scheme separated of finite type with closed immersion $\overline{T} \to \mathcal{T}$ over $\mathcal{S}$ such that $\mathcal{T} \to \mathcal{S}$ is smooth around T. Then one can take a decreasing sequence $\{T^{(r)}\}_{r \geq 0}$ of open dense subschemes of T and an r-truncated proper hypercovering

$$f_{\bullet}^{(r)} : (X_{\bullet}^{(r)}, \overline{X}_{\bullet}^{(r)}) \to (X^{(r)}, \overline{X})$$

of pairs over $(T^{(r)}, \overline{T})$ for each r such that

(i) $X^{(r)} = \overset{\circ}{f}{}^{-1}(T^{(r)})$ and $X_n^{(r)}$ is smooth over $T^{(r)}$;

(ii) if we put $\mathfrak{T}^{(r)} = (T^{(r)}, \overline{T}, \mathcal{T})$, the condition $(\mathrm{C})^q_{f_n^{(r)}, \mathfrak{T}^{(r)}, G}$ holds for any q and any overconvergent isocrystal G on $(X_n^{(r)}, \overline{X}_n^{(r)})/\mathcal{T}_K$;

for any $n \leq r$. Indeed, one can construct a proper hypercovering inductively on n by [21, Lemma 4.2.3] such that the generic fiber of $\overline{X}_n^{(r)}$ is projective smooth over the generic point of T after taking a finite extension $\overline{T}' \to \overline{T}$ such that $\Gamma(\overline{T}', \mathcal{O}_{\overline{T}'})$ is free over $\Gamma(\overline{T}, \mathcal{O}_{\overline{T}})$ by Problem 4.2.1. Take a lift $\mathcal{T}'$ of $\overline{T}'$ over $\mathcal{T}$ such that $\mathcal{T}' \to \mathcal{T}$ is finite flat and $\overline{T}' = \overline{T} \times_{\mathcal{T}} \mathcal{T}'$. Then there is an open dense subscheme $T_n^{(r)'}$ of $\overline{T}'$ such that, if we put $\overline{X}_n^{(r)'} = \overline{X}_n^{(r)} \times_{\overline{T}} \overline{T}'$ (resp. $X_n^{(r)'}$ to be the inverse image of $T_n^{(r)'}$ in $\overline{X}_n^{(r)'}$, resp. $f_n^{(r)'} : (X_n^{(r)'}, \overline{X}_n^{(r)'}) \to$

$(T_n^{(r)'}, \overline{T}')$ to be the induced morphism, resp. $\mathfrak{T}^{(r)'} = (T_n^{(r)'}, \overline{T}', \mathcal{T}_n^{(r)'}))$, then the condition $(\mathrm{C})^q_{f_n^{(r)'}, \mathfrak{T}^{(r)'}, G'}$ holds for any q and any overconvergent isocrystal G' on $(X_n^{(r)'}, \overline{X}_n^{(r)'})/\mathcal{T}'_K$ by Theorem 4.2.4. By the finite flat base change theorem and faithfully flat descent of finitely generated modules, there is an open dense subscheme $T_n^{(r)}$ of $\overline{T}'$ such that the condition $(\mathrm{C})^q_{f_n^{(r)}, \mathfrak{T}^{(r)}, G}$ holds for a suitable choice of $f_n^{(r)}$ and $\mathfrak{T}^{(r)}$. Hence, by shrinking T, such a proper hypercovering exists by Proposition 3.1.1.

Let E be an overconvergent isocrystal on $(X^{(r)}, \overline{X})/\mathcal{S}_K$ and let $f^{(q)} : (\overline{f}^{-1}(U^{(q)}), \overline{X}) \to (U^{(q)}, \overline{T})$ be the induced structure morphism. Completing the truncated proper hypercovering $(X_{\bullet}^{(r)}, \overline{X}_{\bullet}^{(r)}) \to (X^{(r)}, \overline{X})$ as a full simplicial proper hypercovering [21, Proposition 4.3.1] and using the spectral sequence for the proper hypercovering

$$\underline{E}_1^{qs} = \mathbb{R}^s f^{(r)}_{q\mathrm{rig}\mathfrak{T}^{(r)}*}(f_q^{(r)*}E) \Rightarrow \mathbb{R}^{q+s} f^{(r)}_{\mathrm{rig}\mathfrak{T}^{(r)}*}E$$

[21, Theorem 4.1.1], the sheaf $\mathbb{R}^q f^{(r)}_{\mathrm{rig}\mathfrak{T}^{(r)}*}E$ is a sheaf of coherent $j^\dagger\mathcal{O}_{]\overline{T}[_{\mathcal{T}^{(r)}}}$-modules if $q \leq (r-1)/2$ since the category of coherent sheaves is abelian and any extension of a coherent sheaf by a coherent sheaf in the category of sheaves of $j^\dagger\mathcal{O}_{]\overline{T}[_{\mathcal{T}^{(r)}}}$-modules is coherent.

Since the condition $(\mathrm{C})^q_{f_n^{(r)}, \mathfrak{T}^{(r)}, G}$ holds for $n \leq r$, it also holds after any base extension by a morphism $\mathcal{T}' \to \mathcal{T}$ separated of finite type over $\mathcal{T}$ with a closed immersion $\overline{T} \to \mathcal{T}'$ such that $\mathcal{T}' \to \mathcal{T}$ is smooth around T. Hence, for any q, there exists an open dense subscheme $U^{(q)}$ of T such that the rigid cohomology overconvergent isocrystal $R^q f^{(q)}_{\mathrm{rig}*}E$ on $(U^{(q)}, \overline{T})/\mathcal{S}_K$ exists for any overconvergent isocrystal E on $(\overline{f}^{-1}(U^{(q)}), \overline{X})/\mathcal{S}_K$ by Theorem 3.3.1.

For an open subscheme W of T, we define a morphism $f_W : (\overline{f}^{-1}(W), \overline{X}) \to (W, \overline{T})$ of pairs by the induced structure morphism. By the proposition below, there exists an integer q_0 such that $R^q f_{V\mathrm{rig}\mathfrak{V}*}E = 0$ for any open subscheme V of T, any $q > q_0$ and any overconvergent isocrystal E on $(\overline{f}^{-1}(V), \overline{X})/\mathcal{S}_K$, where $\mathfrak{V} = (V, \overline{T}, \mathcal{T})$. Hence, there exists an open dense subscheme U of T such that the rigid cohomology overconvergent isocrystal $R^q f_{U\mathrm{rig}*}E$ on $(U, \overline{T})/\mathcal{S}_K$ exists for any overconvergent isocrystal E on $(\overline{f}^{-1}(U), \overline{X})/\mathcal{S}_K$. Indeed, we can shrink T by the vanishing of rigid cohomology sheaves.

The rest is same as in Theorems 4.1.1 and 4.1.4. □

4.2.6 Proposition

Let $\mathfrak{S} = (S, \overline{S}, \mathcal{S})$ be a $\mathcal{V}$-triple separated of finite type and let $\overline{f} : \overline{X} \to \overline{T}$ be a morphism of k-schemes separated of finite type over $\overline{S}$. Then there exists an integer q_0 such that, for

(i) *any open subscheme* X *(resp.* T*) of* $\overline{X}$ *(resp.* $\overline{T}$*) with* $\overline{f}^{-1}(T) = X$ *(we denote by* $f : (X, \overline{X}) \to (T, \overline{T})$ *the structure morphism of* k*-pairs and put* $\mathfrak{T} = (T, \overline{T}, \mathcal{T})$*);*

(ii) *any* $\mathcal{V}$*-formal scheme* $\mathcal{T}$ *separated of finite type over* $\mathcal{S}$ *with an* $\mathcal{S}$*-closed immersion* $\overline{T} \to \mathcal{T}$ *which is smooth over* $\mathcal{S}$ *around* T*;*

(iii) *any overconvergent isocrystal* E *on* $(X, \overline{X})/\mathcal{T}_K$*,*

the rigid cohomology $\mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}*}E$ *vanishes for any* $q > q_0$*.*

Proof. We may assume that $\mathcal{T}$ is affine. Since $\overline{X}$ is of finite type over $\operatorname{Spec} k$, there is a finite open covering $\{\overline{X}_\alpha\}_\alpha$ of $\overline{X}$ such that there exists a smooth formal scheme separated $\mathcal{X}_\alpha$ of finite type over $\operatorname{Spf} \mathcal{V}$ with a $\mathcal{V}$-closed immersion $\overline{X}_\alpha \to \mathcal{X}_\alpha$ for any α. We use induction on the minimal cardinality n of such a finite open covering of $\overline{X}$.
Suppose that $n = 1$. Let us take a finite affine open covering $\{\mathcal{U}_\beta\}_\beta$ of $\mathcal{X}$, put $\overline{U}_\beta$ (resp. U_β) to be the inverse image of $\overline{X}$ (resp. X) in $\mathcal{U}_\beta$ for each β, and denote by $\mathfrak{U}_\beta = (U_\beta, \overline{U}_\beta, \mathcal{U}_\beta \times_{\operatorname{Spf}\mathcal{V}} \mathcal{T})$ (resp. $f_\beta : \mathfrak{U}_\beta \to \mathfrak{T}$) the associated triple (resp. the structure morphism of triples). For a multi-index $\underline{\beta} = (\beta_0, \beta_1, \cdots, \beta_r)$ ($\beta_0 < \beta_1 < \cdots < \beta_r$), let $\mathfrak{U}_{\underline{\beta}}$ be the intersection of $\mathfrak{U}_{\beta_0}, \mathfrak{U}_{\beta_1}, \cdots, \mathfrak{U}_{\beta_r}$ and let $f_{\underline{\beta}} : \mathfrak{U}_{\underline{\beta}} \to \mathfrak{T}$ be the structure morphism. Let $\mathcal{E}$ be a sheaf of coherent $j^\dagger\mathcal{O}_{]\overline{X}[_\mathcal{X}}$-modules. We denote by $\mathcal{C}^\bullet_{\mathrm{alt}}(\{\mathfrak{U}_\beta\}_\beta, \mathcal{E})$ the alternating sheaf-valued Čech complex of $\mathcal{E}$ with respect to the covering $\{\mathfrak{U}_\beta\}_\beta$ of $\mathcal{X}$ [9, 2.11]. Then we have a natural isomorphism

$$R^q\widetilde{f}_*\mathcal{E} \cong \mathbb{R}^q\widetilde{f}_*\mathcal{C}^\bullet_{\mathrm{alt}}(\{\mathfrak{U}_\beta\}_\beta, \mathcal{E})$$

for any q by [9, Lemma 2.11.1]. Let us fix a multi-index $\underline{\beta}$. Since $\overline{f}_{\underline{\beta}}^{-1}(T) = U_{\underline{\beta}}$, there exists an admissible affinoid covering $\{W_\gamma\}_\gamma$ of $]\overline{T}[_\mathcal{T}$ such that

$$H^q(\widetilde{f}_{\underline{\beta}}^{-1}(W_\gamma), \mathcal{E}) = 0 \quad (q > 1)$$

for any γ. Indeed, we take an admissible affinoid covering by [9, Sect. 2.6] and prove the vanishing by a method similar to [21, Proposition 3.2.3] using [9, Proposition 5.1.1]. Hence, the direct image sheaf $R^q\widetilde{f}_{\underline{\beta}*}\mathcal{E}|_{]\overline{U}_{\underline{\beta}}[_{\mathcal{U}_{\underline{\beta}}}}$ vanishes for $q > 1$. By the Čech spectral sequence $R^q\widetilde{f}_*\mathcal{E} = 0$ for $q > \operatorname{card}(\{\mathfrak{U}_\beta\}_\beta)$. By the Hodge-de Rham spectral sequence there exists an integer q_0 such that $\mathbb{R}^q f_{\mathrm{rig}\mathfrak{T}*}E$ vanishes for any $q > q_0$. This q_0 is independent of the choices of an open subscheme T of X, a smooth formal scheme $\mathcal{T}$ and an overconvergent isocrystal E on $(X, \overline{X})/\mathcal{T}_K$.
Suppose that n is general. Let us put $\overline{X}' = \cup_{\alpha=2}^n \overline{X}_\alpha$ and $\overline{X}'_1 = \overline{X} \cap \overline{X}'$, denote by X_1 (resp. X', resp X'_1) the inverse image of T in $\overline{X}_1$ (resp. $\overline{X}'$, resp. $\overline{X}'_1$) and define $f_1 : (X_1, \overline{X}_1) \to (T, \overline{T})$ (resp. $f' : (X', \overline{X}') \to (T, \overline{T})$, resp. $f'_1 : (X'_1, \overline{X}'_1) \to (T, \overline{T})$) as the structure morphism. Then, for any

overconvergent isocrystal on $(X, \overline{X})/\mathcal{T}_K$, there exists a natural commutative diagram

$$\begin{array}{ccccccc} \mathbb{R}^q f'_{\mathrm{rig}\mathfrak{T}*}E' & \to & C^{q+1} & \to & \mathbb{R}^{q+1}f_{\mathrm{rig}\mathfrak{T}*}E & \to & \mathbb{R}^{q+1}f'_{\mathrm{rig}\mathfrak{T}*}E' \\ \downarrow & & \downarrow\cong & & \downarrow & & \downarrow \\ \mathbb{R}^q f'_{1\mathrm{rig}\mathfrak{T}*}E'_1 & \to & C^{q+1} & \to & \mathbb{R}^{q+1}f_{1\mathrm{rig}\mathfrak{T}*}E_1 & \to & \mathbb{R}^{q+1}f'_{1\mathrm{rig}\mathfrak{T}*}E'_1 \end{array}$$

of exact rows with a vertical isomorphism in the second terms for any q since $\{\overline{X}_1, \overline{X}'\}$ is an open covering of $\overline{X}$ with $\overline{X}_1 \cap \overline{X}' = \overline{X}'_1$. Here E_1 (resp. E', resp. E'_1) is the inverse image of E on $(X_1, \overline{X}_1)/\mathcal{T}_K$ (resp. $(X', \overline{X}')/\mathcal{T}_K$, resp. $(X'_1, \overline{X}'_1)/\mathcal{T}_K$). Since $\overline{X}'_1$ is an open subscheme of $\overline{X}_1$, it is embedded into a smooth formal scheme separated of finite type over $\mathrm{Spf}\,\mathcal{V}$. Therefore, the assertion follows from the induction hypothesis. □

In the case of families of curves one can take a lift of the generic fiber without any extension (Remark 4.2.2 (1)). Hence, we do not need to take a proper hypercovering.

4.2.7 THEOREM

If X is a proper smooth family of curves over T, then the conclusions of Theorem 4.2.4 hold.

REFERENCES

[1] André, Y., Baldassarri, F., *Geometric theory of G-functions*, Arithmetic Geometry edited by Catanese, F., Cambridge Univ. Press 1997.

[2] Baldassarri, F., Chiarellotto, B., *Algebraic versus rigid cohomology with logarithmic coefficients*, Barsotti Symposium in Algebraic Geometry, Perspective in Math. 15, Academic press (1994), 11–50.

[3] Berthelot, P., *Cohomologie Cristalline des schémas de caractéristique $p >$* 0, Lecture Notes in Math. 407, Springer-Verlag 1974.

[4] Berthelot, P., *Géométrie rigide et cohomologie des variétés algébriques de caractéristique p*, Bull. Soc. Math. France, Mémoire 23 (1986), 7-32.

[5] Berthelot, P., *Cohomologie rigide et cohomologie rigide à supports propres. Première Partie*, preprint, Université de Rennes (1996).

[6] Berthelot, P., *Finitude et pureté cohomologique en cohomologie rigide (avec un appendice par Aise Johan de Jong)*, Invent. math. 128 (1997), 329-377.

[7] Bosch, S., Güntzer U., Remmert, R., *Non-Archimedean Analysis*, Grundlehren der math. Wissenschaften 261, Springer-Verlag 1984.

[8] Crew, R., *Finiteness theorem for the cohomology of an overconvergent isocrystal on a curve*, Ann. Scient. École Norm. Sup. 31 (1998), 717-763.

[9] Chiarellotto, B., Tsuzuki, N., *Cohomological descent of rigid cohomology for etale coverings*, preprint.

[10] Elkik, R., *Solutions d'équations à coefficients dans un anneau hensélian*, Ann. Scient. École Norm. Sup. 6 (1973), 553-604.

[11] Grosse-Klönne, E., *Finiteness of de Rham cohomology in rigid analysis*, Duke Math. J. 113 (2002), 57-91.

[12] Grothendieck, A., *Revêtements Etales et Groupe Fondamental*, SGA1, Lecture Notes in Math. 224, Springer-Verlag 1971.

[13] Grothendieck, A., Dieudonné, J., *Eléments de Géométrie Algébrique III*, Publ. Math. I.H.E.S. 11 (1961).

[14] de Jong, A.J., *Smoothness, semi-stability and alterations*, Publ. Math. I.H.E.S. 83 (1996), 51-93.

[15] Katz, N.M., *Nilpotent connections and the monodromy theorem : application of a result of Turrittin*, Publ. Math. I.H.E.S. 39 (1970), 175-232.

[16] Katz, N.M., Oda, T., *On the differential of de Rham cohomology classes with respect to parameters*, J. Math. Kyoto Univ. 8 (1968), 199-213.

[17] Kiehl, R., *Der Endlichkeitssatz für Abbildungen in der nichtarchimedischen Funktionentheorie*, Invent. math. 2 (1967), 191-214.

[18] Kedlaya, K.S., *Finiteness of rigid cohomology with coefficients*, preprint.

[19] Shiho, A., *Crystalline fundamental groups. II. Log convergent cohomology and rigid cohomology*, J. Math. Sci. Univ. Tokyo 9 (2002), 1-163.

[20] Tsuzuki, N., *On the Gysin isomorphism of rigid cohomology*, Hiroshima Math. J. 29 (1999), 479-527.

[21] Tsuzuki, N., *Cohomological descent of rigid cohomology for proper coverings*, Invent. math. 151 (2003), 101-133.

Tsuzuki Nobuo
Department of Mathematics
Faculty of Science
Hiroshima University
Higashi-Hiroshima 739-8526
Japan