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A Collection of Manuscripts Written in Honour of Alexander S. Merkurjev on the Occasion of His Sixtieth Birthday

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Documenta Mathematica
Extra Volume: Alexander S. Merkurjev's Sixtieth Birthday, 2015

Preface

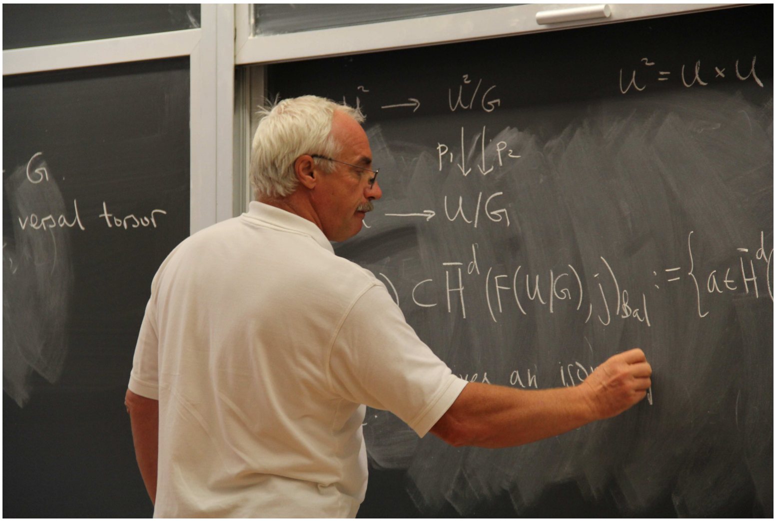
Alexander Sergeevich Merkurjev – or just Sasha to his friends – was born in 1955 in Leningrad (now St. Petersburg) Russia. His mathematical talents manifested themselves at an early age. In 1972 he was a part of the eight member Soviet team that won the first prize at the International Mathematics Olympiad for high school students. (Sasha also won a silver medal for his individual performance.)
In the early 1980s Sasha burst on the research scene, first with a proof of a conjecture of John Tate about the K-theory of local fields, then with a proof of a long-standing conjecture relating K_2 of a field to the 2-torsion in its Brauer group. Then, still in his 20s, Sasha (jointly with Andrei Suslin) strengthended the latter result to settle a key conjecture in the theory of central simple algebras. The theorem they proved, now known as the Merkurjev-Suslin theorem, is generally recognized as a high point of 20th century algebra. It can be found in many textbooks and has opened the door to many subsequent developments, including Vladimir Voevodsky's Fields medal winning proof of the Milnor Conjecture in the 1990s.
In the subsequent three decades Sasha has firmly established himself as one of the world's leading algebraists. He has made fundamental contributions in a number of areas, including algebraic K-theory, quadratic forms, Galois cohomology, algebraic groups, arithmetic and algebraic geometry (including higher class field theory and intersection theory), and essential dimension. His research accomplishments, too numerous to detail here, have been recognized with a prize of the St. Petersburg Mathematical Society (1982), a sectional lecture at the International Congress of Mathematicians (1986), the Humboldt Prize (1995), a plenary lecture at the European Congress of Mathematics (1996), the AMS Cole Prize in algebra (2012) and a Guggenheim Fellowship (2013-14).
At 60, Sasha is full of creative energy. His lectures are crystal clear and effortlessly delivered, his papers are efficiently written and uniformly of the highest quality. The three research monographs he has coauthored are standard references in the subject. Sasha has been an inspiring thesis advisor to many graduate students, both at St. Petersburg University and at UCLA, where he has been on the faculty since 1997. According to the Mathematics Genealogy Project, eight students have written their Ph.D. dissertations under his supervision at St. Petersburg University and fourteen at UCLA. Throughout his career Sasha devoted a great deal of his time to organizing and running high school mathematical competitions. He served as a member of the organizing committee for the St. Petersburg mathematical olympiad (in 1980-1999) as well as of the national Soviet – and then Russian – olympiad (8 times).

We are happy to dedicate this volume to Sasha on the occasion of his 60th birthday. DOCUMENTA MATHEMATICA is a particularly appropriate forum for this volume in view of Sasha's nearly 20 years of service as an editor, since the first issue of DOCUMENTA in 1996. In addition to peer-reviewed papers submitted by his friends and colleagues, this issue includes a new crossword by one of Sasha's PhD students who has published puzzles in venues such as the *New York Times*, and also the first English translation of a brief note by Merkurjev that has previously appeared only in Russian.

Happy birthday, Sasha!

P. Balmer, V. Chernousov, I. Fesenko, E. Friedlander,
S. Garibaldi, U. Rehmann, Z. Reichstein

At Mathematisches Forschungsinstitut Oberwolfach[1] in 1982

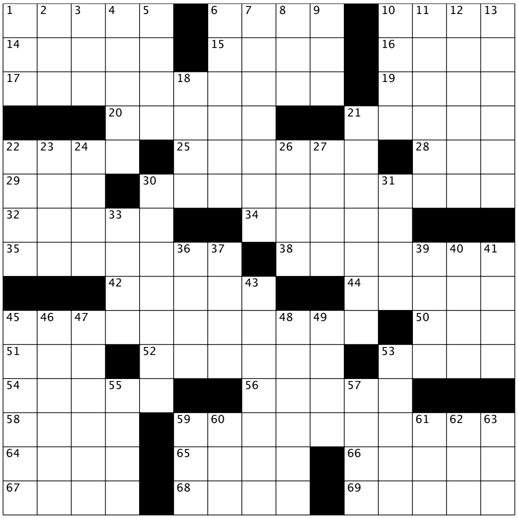


Lecturing at the Fields Institute thematic program *Torsors, Nonassociative Algebras and Cohomological Invariants* in 2013.[2]

[1]Author: George M. Bergman; Source: Archives of the Mathematisches Forschungsinstitut Oberwolfach
[2]Author: Nikolai Vavilov

Merkurjev's Faves

Alex Boisvert



(Published via Across Lite)

ACROSS

1. "___ mia!"
6. Burden
10. Singer Stefani
14. Sparkle, as in an eye
15. Prefix meaning "all"
16. "This is terrible!"
17. Merkurjev's favorite beer?
19. Encourage
20. Pelvic region
21. Mock-innocent question
22. Elman, Karpenko, and Merkurjev, e.g.
25. Immediately
28. Princess's bane, in a fairy tale
29. "The Annotated Flatland" author Stewart
30. Merkurjev's favorite toolbox item?
32. Ringmaster, for example
34. Lennart Carleson, for one
35. Brought into being
38. One doing the jitterbug, maybe
42. Goods for sale
44. Chutzpah
45. Merkurjev's favorite rural pastime?
50. Slippery swimmer
51. "___ to it!"
52. Where we meet the characters in a play
53. Berserk
54. Wedding locale, at times
56. "Man, it's sweltering today!"
58. Svelte
59. Merkurjev's favorite formal event?
64. Sao ___ and Principe
65. Iron, Bronze, and Space, notably
66. Root systems may be simply, doubly, or triply
67. Actor McGregor
68. Left, on a ship
69. Marine mammal that floats on its back

DOWN

1. ___ Grand (Las Vegas casino)
2. "Float like a butterfly, sting like a bee" speaker
3. Voice actor Blanc of "Looney Tunes"
4. Tropical smoothie staple
5. Love, in Latin
6. Nabokov novel
7. Foreboding
8. Little worker
9. Cube referenced in probability classes
10. Vincent van ___
11. Actress Goldberg of "Ghost"
12. Tooth covering
13. "I reject your offer!"
18. Lion's yell
21. Glass-stomping occasion
22. Knots, as shoes
23. Wheelchair-friendly feature
24. Ancient Peruvian
26. Goings-on
27. Workers on a ship
30. "Hit Me With Your Best Shot" singer Pat
31. Descartes's first name
33. "Groooooooss!"
36. "Etale Homotopy of Simplicial Schemes" author Friedlander
37. Section: Abbr.
39. Increased in size
40. At any time
41. Depend (on)
43. "Snape kills Dumbledore", e.g.
45. Pure
46. Empty on the inside
47. They may be global or local
48. Deepest
49. Soft drink brand with a "Blue Ice Cream" flavor
53. Plate appearance
55. Prayer ender
57. Site of an annual prize announcement
59. Something a proof should not have
60. In the past
61. Make a move
62. Jeans brand
63. One in charge, for short

Secondary Characteristic Classes and the Euler Class

Aravind Asok and Jean Fasel[1]

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Abstract. We discuss secondary (and higher) characteristic classes for algebraic vector bundles with trivial top Chern class. We then show that if X is a smooth affine scheme of dimension d over a field k of finite 2-cohomological dimension (with $\mathrm{char}(k) \neq 2$) and E is a rank d vector bundle over X, vanishing of the Chow-Witt theoretic Euler class of E is equivalent to vanishing of its top Chern class and these higher classes. We then derive some consequences of our main theorem when k is of small 2-cohomological dimension.

2010 Mathematics Subject Classification: 14F42, 14C15, 13C10, 55S20

Contents

1 Introduction

Suppose k is a field having characteristic unequal to 2, $X = \mathrm{Spec}(A)$ is a d-dimensional smooth affine k-scheme and $\mathcal{E}$ is a vector bundle of rank r over X. There is a well-defined primary obstruction to $\mathcal{E}$ splitting off a free rank 1 summand given by "the" Euler class $e(\mathcal{E})$ of $\mathcal{E}$ (see [Mor12, Theorem 8.2], [Fas08, Chapitre 13] and

[1]Aravind Asok was partially supported by National Science Foundation Awards DMS-0966589 and DMS-1254892. Jean Fasel was partially supported by the DFG Grant SFB Transregio 45.

[AF13], which shows two possible definitions coincide for oriented vector bundles). When $r = d$, Morel shows that this primary obstruction is the only obstruction to splitting off a trivial rank 1 summand, and we will focus on this case in this article.
Because the Euler class is defined using Chow-Witt theory, which is not part of an oriented cohomology theory (say in the sense of [LM07]), it is difficult to compute in general. The vanishing of the Euler class implies the vanishing of the top Chern class $c_d(\mathcal{E})$ in $CH^d(X)$ [AF14c, Proposition 6.3.1], though the converse is not true in general. It is therefore natural to try to approximate $e(\mathcal{E})$ using structures defined only in terms of oriented cohomology theories. More precisely, we now explain the strategy involved in studying such "approximations" as developed in Section 2.2.
If X is as above, let us fix a line bundle $\mathcal{L}$ on X. One can define the $\mathcal{L}$-twisted unramified Milnor-Witt K-theory sheaf $\mathbf{K}_d^{\mathrm{MW}}(\mathcal{L})$, which is a sheaf on the small Nisnevich site of X. The $\mathcal{L}$-twisted Chow-Witt group $\widetilde{CH}^d(X, \mathcal{L})$ can be defined as the Nisnevich cohomology group $H^d(X, \mathbf{K}_d^{\mathrm{MW}}(\mathcal{L}))$. With $\mathcal{E}$ as above, the Euler class $e(\mathcal{E})$ lives in this group with $\mathcal{L} = \det \mathcal{E}^\vee$.
If $\mathbf{K}_d^{\mathrm{M}}$ is the d-th unramified Milnor K-theory sheaf, then by Rost's formula $H^d(X, \mathbf{K}_d^{\mathrm{M}}) \cong CH^d(X)$. There is a natural morphism of sheaves on X of the form $\mathbf{K}_d^{\mathrm{MW}}(\mathcal{L}) \to \mathbf{K}_d^{\mathrm{M}}$, which furnishes a comparison morphism $\widetilde{CH}^d(X, \mathcal{L}) \to CH^d(X)$ whose study is the main goal of this paper.
By a result of F. Morel, the kernel of the morphism of sheaves $\mathbf{K}_d^{\mathrm{MW}}(\mathcal{L}) \to \mathbf{K}_d^{\mathrm{M}}$ is the $(d+1)$st power of the fundamental ideal in the Witt sheaf (twisted by $\mathcal{L}$), denoted $\mathbf{I}^{d+1}(\mathcal{L})$. The sheaf $\mathbf{I}^{d+1}(\mathcal{L})$ is filtered by subsheaves of the form $\mathbf{I}^r(\mathcal{L})$ for $r \geq d+1$:

$$\ldots \subset \mathbf{I}^{n+d}(\mathcal{L}) \subset \mathbf{I}^{n+d-1}(\mathcal{L}) \subset \ldots \subset \mathbf{I}^{d+1}(\mathcal{L}) \subset \mathbf{K}_d^{\mathrm{MW}}(\mathcal{L}).$$

This filtration induces associated long exact sequences in cohomology and gives rise to a spectral sequence $E(\mathcal{L}, \mathrm{MW})^{p,q}$ computing the cohomology groups with coefficients in $\mathbf{K}_d^{\mathrm{MW}}(\mathcal{L})$.
When $p = d = \dim(X)$, we obtain a filtration of the group $H^d(X, \mathbf{K}_d^{\mathrm{MW}}(\mathcal{L}))$ by subgroups $F^n H^d(X, \mathbf{K}_d^{\mathrm{MW}}(\mathcal{L}))$ for $n \in \mathbb{N}$ such that $F^0 H^d(X, \mathbf{K}_d^{\mathrm{MW}}(\mathcal{L})) = H^d(X, \mathbf{K}_d^{\mathrm{MW}}(\mathcal{L}))$ and where the successive subquotients $F^n H^d(X, \mathbf{K}_d^{\mathrm{MW}}(\mathcal{L}))/F^{n+1} H^d(X, \mathbf{K}_d^{\mathrm{MW}}(\mathcal{L}))$ are computed by the groups $E(\mathcal{L}, \mathrm{MW})_\infty^{d,d+n}$ arising in the spectral sequence. If furthermore k has finite 2-cohomological dimension, then only finitely many of the groups $E(\mathcal{L}, \mathrm{MW})_\infty^{d,d+n}$ are nontrivial and we obtain the following theorem.

THEOREM 1 (See Theorem 2.2.6). *Suppose k is a field having finite 2-cohomological dimension (and having characteristic unequal to 2). Suppose X is a smooth k-scheme of dimension d and suppose $\mathcal{L}$ is line bundle on X. For any $\alpha \in H^d(X, \mathbf{K}_d^{\mathrm{MW}}(\mathcal{L}))$, there are inductively defined obstructions $\Psi^n(\alpha) \in E(\mathcal{L}, \mathrm{MW})_\infty^{d,d+n}$ for $n \geq 0$ such that $\alpha = 0$ if and only if $\Psi^n(\alpha) = 0$ for any $n \geq 0$.*

The groups $E(\mathcal{L}, \mathrm{MW})_2^{p,q}$ are cohomology groups with coefficients either in $\mathbf{K}_d^{\mathrm{M}}$ or in $\mathbf{K}_j^{\mathrm{M}}/2$ for $j \geq d+1$, and thus they are theoretically easier to compute than the cohomology groups with coefficients in $\mathbf{K}_d^{\mathrm{MW}}$; this is the sense in which we have "approximated" our original non-oriented computation by "oriented" computations. The upshot is that if k has finite 2-cohomological dimension, we can use a vanishing result from [AF14b] (which appeals to Voevodsky's resolution of the Milnor conjecture on the mod 2 norm-residue homomorphism) to establish the following result.

Corollary 2. *Let k be a field having 2-cohomological dimension s (and having characteristic unequal to 2). If X is a smooth affine k-scheme of dimension d and $\xi : \mathcal{E} \to X$ is a rank d-vector bundle on X with $c_d(\mathcal{E}) = 0$, then $\mathcal{E}$ splits off a trivial rank 1 summand if and only if $\Psi^n(\mathcal{E}) = 0$ for $n \leq s - 1$.*

The problem that arises then is to identify the differentials in the spectral sequence, which provide the requisite "higher obstructions", in concrete terms. To this end, we first observe that there is a commutative diagram of filtrations by subsheaves

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & \mathbf{I}^{d+n}(\mathcal{L}) & \longrightarrow & \mathbf{I}^{d+n-1}(\mathcal{L}) & \longrightarrow & \cdots & \longrightarrow & \mathbf{I}^{d+1}(\mathcal{L}) & \longrightarrow & \mathbf{K}^{\mathrm{MW}}_d(\mathcal{L}) \\ & & \| & & \| & & & & \| & & \downarrow \\ \cdots & \longrightarrow & \mathbf{I}^{d+n}(\mathcal{L}) & \longrightarrow & \mathbf{I}^{d+n-1}(\mathcal{L}) & \longrightarrow & \cdots & \longrightarrow & \mathbf{I}^{d+1}(\mathcal{L}) & \longrightarrow & \mathbf{I}^{d}(\mathcal{L}). \end{array}$$

The filtration on the bottom gives rise to (a truncated version of) the spectral sequence Pardon studied [Par, 0.13]; this spectral sequence was further analyzed in [Tot03]. Totaro showed that the differentials on the main diagonal in the E_2-page of the Pardon spectral sequence are given by Voevodsky's Steenrod squaring operation Sq^2. Using the diagram above, we see that the differentials in the spectral sequence we define are essentially determined by the differentials in the Pardon spectral sequence, and we focus on the latter. We extend Totaro's results and obtain a description of the differentials just above the main diagonal as well and, more generally, the differentials in our $\mathcal{L}$-twisted spectral sequence (see Theorem 4.1.4).

We identify, using the Milnor conjecture on the mod 2 norm-residue homomorphism, the (mod 2) Milnor K-cohomology groups appearing in the pages of the spectral sequence above in terms of motivic cohomology groups. Via this identification, the differentials appearing just above the main diagonal in our spectral sequence can be viewed as operations on motivic cohomology groups. Bi-stable operations of mod 2 motivic cohomology groups have been identified by Voevodsky [Voe10] (if k has characteristic 0) or Hoyois-Kelly-Østvaer [HKØ13] (if k has characteristic unequal to 2). It follows from these identifications that the differentials in question are either the trivial operation or the (twisted) Steenrod square. In Section 3.3, we compute an explicit example to rule out the case that the operation is trivial. Finally, we put everything together in the last section to obtain, in particular, the following result.

Theorem 3. *Let k be a field having 2-cohomological dimension s (and having characteristic unequal to 2). Suppose X is a smooth affine k-scheme of dimension d and $\xi : \mathcal{E} \to X$ is a rank d-vector bundle on X with $c_d(\mathcal{E}) = 0$. The secondary obstruction $\Psi^1(\alpha)$ to $\mathcal{E}$ splitting off a trivial rank 1 summand is the class in the cokernel of the composite map*

$$H^{d-1}(X, \mathbf{K}^{\mathrm{M}}_d) \longrightarrow H^{d-1}(X, \mathbf{K}^{\mathrm{M}}_d/2) \xrightarrow{Sq^2 + c_1(\mathcal{L})\cup} H^d(X, \mathbf{K}^{\mathrm{M}}_{d+1}/2),$$

(the first map is induced by reduction mod 2*) defined as follows: choose a lift of the class* $e(\xi) \in H^d(X, \mathbf{I}^{d+1}(\det \mathcal{E}))$ *and look at its image in* $H^d(X, \mathbf{K}^{\mathrm{M}}_{d+1}/2)$ *under the map* $\mathbf{I}^{d+1}(\det \mathcal{E})) \to \mathbf{K}^{\mathrm{M}}_{d+1}/2$*. Furthermore: (i) if* k *has cohomological dimension* 1*, then the secondary (and all higher) obstructions are automatically trivial and (ii) if* k *has cohomological dimension* 2*, then the triviality of the secondary obstruction is the only obstruction to* $\mathcal{E}$ *splitting off a trivial rank* 1 *summand.*

For the sake of perspective, recall that Bhatwadekar and Sridharan asked whether the only obstruction to splitting a trivial rank 1 summand off a rank $(2n+1)$ vector bundle $\mathcal{E}$ on a smooth affine $(2n+1)$-fold $X = \operatorname{Spec} A$ is vanishing of a variant of the top Chern class living in a group $E_0(A)$ [BS00, Question 7.12]. The group $E_0(A)$ housing their obstruction class is isomorphic to the Chow group of 0-cycles on $\operatorname{Spec} A$ in some cases; see, e.g., [BS99, Remark 3.13 and Theorem 5.5]. It is an open problem whether the group $E_0(A)$ is isomorphic to the Chow group of zero cycles in general. A natural byproduct of their question is whether (or, perhaps, when) vanishing of the top Chern class is sufficient to guarantee that $\mathcal{E}$ splits off a free rank 1 summand. In view of Theorem 4.2.1, the sufficiency of the vanishing of the top Chern class is equivalent to all the higher obstructions vanishing, which from our point of view seems rather unlikely. Nevertheless, Bhatwadekar, Das and Mandal have shown that when $k = \mathbb{R}$, there are situations when vanishing of the top Chern class is sufficient to guarantee splitting [BDM06, Theorem 4.30].

Remark 4. Throughout this paper, we will assume that k has characteristic unequal to 2, but a result can be established if k has characteristic 2 as well. Indeed, one can first establish a much stronger version of Corollary 2. More precisely, suppose k is a perfect field having characteristic 2. If X is a smooth k-scheme of dimension d, and $\xi : \mathcal{E} \to X$ is a rank d vector bundle on X, then $e(\xi) = 0$ if and only if $c_d(\xi) = 0$. Establishing this result requires somewhat different arguments, and we will write a complete proof elsewhere.

Preliminaries

When mentioning motivic cohomology, we will assume k is perfect. Thus, for simplicity, the reader can assume that k is perfect and has characteristic unequal to 2 throughout the paper. The proof of Theorem 4.1.4 in positive characteristic depends on the main result of the preprint [HKØ13], which, at the time of writing, depends on several other pieces of work that are still only available in preprint form. We refer the reader to [Fas08] for results regarding Chow-Witt theory, [MVW06] for general properties of motivic cohomology, and [MV99] for results about $\mathbb{A}^1$-homotopy theory. We will consider cohomology of strictly $\mathbb{A}^1$-invariant sheaves on a smooth scheme X (see Section 2.1 for some recollections about the sheaves considered in this paper). In the introduction, we considered these sheaves on the small Nisnevich site of X, but below we will consider only sheaves in the Zariski topology. By, e.g., [Mor12, Corollary 5.43] the cohomology of a strictly $\mathbb{A}^1$-invariant sheaf computed in the Zariski topology coincides with cohomology computed in the Nisnevich topology.

ACKNOWLEDGEMENTS

We thank Burt Totaro for a discussion related to the proof of Theorem 4.1.4. We would also like to thank the referees for their thorough reading of the first version of this paper and a number of useful remarks.

2 A MODIFICATION OF THE PARDON SPECTRAL SEQUENCE

In this section, we recall the definition of twisted Milnor-Witt K-theory sheaves and various relatives. We then describe a standard filtration on twisted Milnor-Witt K-theory sheaves and analyze the associated spectral sequence.

2.1 UNRAMIFIED POWERS OF THE FUNDAMENTAL IDEAL AND RELATED SHEAVES

Let k be a field of characteristic different from 2 and let Sm_k be the category of schemes that are separated, smooth and have finite type over $\mathrm{Spec}(k)$. Let $\mathbf{W}$ be the (Zariski) sheaf on Sm_k associated with the presheaf $X \mapsto W(X)$, where $W(X)$ is the Witt group of X ([Kne77], [Knu91]). If X is a smooth connected k-scheme, then the restriction of $\mathbf{W}$ to the small Zariski site of X admits an explicit flasque resolution, the so called Gersten-Witt complex $C(X, \mathbf{W})$ ([BW02], [BGPW02]):

$$W(k(X)) \to \bigoplus_{x\in X^{(1)}} W_{fl}(k(x)) \xrightarrow{d_1} \bigoplus_{x\in X^{(2)}} W_{fl}(k(x)) \xrightarrow{d_2} \bigoplus_{x\in X^{(3)}} W_{fl}(k(x)) \to \dots .$$

Here, $W_{fl}(k(x))$ denotes the Witt group of finite length $\mathcal{O}_{X,x}$-modules ([Par82],[BO87]), which is a free $W(k(x))$-module of rank one.
For any $n \in \mathbb{Z}$, let $I^n(k(x)) \subset W(k(x))$ be the n-th power of the fundamental ideal (with the convention that $I^n(k(x)) = W(k(x))$ if $n \leq 0$) and let $I^n_{fl}(k(x)) := I^n(k(x)) \cdot W_{fl}(k(x))$. The differentials d_i of the Gersten-Witt complex respect the subgroups $I^n_{fl}(k(x))$ in the sense that $d_i(I^n_{fl}(k(x))) \subset I^{n-1}_{fl}(k(y))$ for any $i \in \mathbb{N}$, $x \in X^{(i)}$, $y \in X^{(i+1)}$ and $n \in \mathbb{Z}$ ([Gil07],[Fas08, Lemme 9.2.3]). This yields a Gersten-Witt complex $C(X, \mathbf{I}^j)$:

$$I^j(k(X)) \to \bigoplus_{x\in X^{(1)}} I^{j-1}_{fl}(k(x)) \xrightarrow{d_1} \bigoplus_{x\in X^{(2)}} I^{j-2}_{fl}(k(x)) \to \bigoplus_{x\in X^{(3)}} I^{j-2}_{fl}(k(x)) \to \dots$$

for any $j \in \mathbb{Z}$ which provides a flasque resolution of the sheaf $\mathbf{I}^j$, i.e., the sheaf associated with the presheaf $X \mapsto H^0(C(X, \mathbf{I}^j))$. There is an induced filtration of the sheaf $\mathbf{W}$ by subsheaves of the form:

$$\ldots \subset \mathbf{I}^j \subset \mathbf{I}^{j-1} \subset \ldots \subset \mathbf{I} \subset \mathbf{W};$$

the successive quotients are usually given special notation: $\bar{\mathbf{I}}^j := \mathbf{I}^j/\mathbf{I}^{j+1}$ for any $j \in \mathbb{N}$.

The exact sequence of sheaves

$$0 \longrightarrow \mathbf{I}^{j+1} \longrightarrow \mathbf{I}^j \longrightarrow \overline{\mathbf{I}}^j \longrightarrow 0$$

yields an associated flasque resolution of $\overline{\mathbf{I}}^j$ by complexes $C(X, \overline{\mathbf{I}}^j)$ [Fas07, proof of Theorem 3.24] of the form:

$$\overline{I}^j(k(X)) \to \bigoplus_{x \in X^{(1)}} \overline{I}^{j-1}(k(x)) \xrightarrow{d_1} \bigoplus_{x \in X^{(2)}} \overline{I}^{j-2}(k(x)) \to \bigoplus_{x \in X^{(3)}} \overline{I}^{j-2}(k(x)) \to \dots .$$

The subscript fl appearing in the notation above has been dropped in view of the canonical isomorphism

$$\overline{I}^j(k(x)) := I^j(k(x))/I^{j+1}(k(x)) \longrightarrow I^j_{fl}(k(x))/I^{j+1}_{fl}(k(x)) =: \overline{I}^j_{fl}(k(x))$$

induced by any choice of a generator of $W_{fl}(k(x))$ as $W(k(x))$-module ([Fas08, Lemme E.1.3, Proposition E.2.1]).

Suppose now that X is a smooth k-scheme and $\mathcal{L}$ is a line bundle on X. One may define the sheaf $\mathbf{W}(\mathcal{L})$ on the category of smooth schemes over X as the sheaf associated with the presheaf $\{f : Y \to X\} \to W(Y, f^*\mathcal{L})$, where the latter is the Witt group of the exact category of coherent locally free $\mathcal{O}_X$-modules equipped with the duality $\mathrm{Hom}_{\mathcal{O}_X}(\_, \mathcal{L})$. The constructions above extend to this "twisted" context and we obtain sheaves $\mathbf{I}^j(\mathcal{L})$ for any $j \in \mathbb{Z}$ and flasque resolutions of these sheaves by complexes that will be denoted $C(X, \mathbf{I}^j(\mathcal{L}))$.

There are canonical isomorphisms $\overline{\mathbf{I}}^j = \mathbf{I}^j(\mathcal{L})/\mathbf{I}^{j+1}(\mathcal{L})$ and we thus obtain a filtration $\ldots \subset \mathbf{I}^j(\mathcal{L}) \subset \mathbf{I}^{j-1}(\mathcal{L}) \subset \ldots \subset \mathbf{I}(\mathcal{L}) \subset \mathbf{W}(\mathcal{L})$ and long exact sequences

$$0 \longrightarrow \mathbf{I}^{j+1}(\mathcal{L}) \longrightarrow \mathbf{I}^j(\mathcal{L}) \longrightarrow \overline{\mathbf{I}}^j \longrightarrow 0. \qquad (2.1.1)$$

Let $\mathcal{F}_k$ be the class of finitely generated field extensions of k. As usual, write $K^{\mathrm{M}}_n(F)$ for the n-th Milnor K-theory group as defined in [Mil70] (with the convention that $K^{\mathrm{M}}_n(F) = 0$ if $n < 0$). The assignment $F \mapsto K^{\mathrm{M}}_n(F)$ defines a cycle module in the sense of [Ros96, Definition 2.1]. We denote by $\mathbf{K}^{\mathrm{M}}_n$ the associated Zariski sheaf ([Ros96, Corollary 6.5]), which has an explicit Gersten resolution by flasque sheaves ([Ros96, Theorem 6.1]). The same ideas apply for Milnor K-theory modulo some integer and, in particular, we obtain a sheaf $\mathbf{K}^{\mathrm{M}}_n/2$.

For any $F \in \mathcal{F}_k$ and any $n \in \mathbb{N}$, there is a surjective homomorphism $s_n : K^{\mathrm{M}}_n(F)/2 \to \overline{I}^n(F)$ which, by the affirmation of the Milnor conjecture on quadratic forms [OVV07], is an isomorphism. The homomorphisms s_n respect residue homomorphisms with respect to discrete valuations (e.g. [Fas08, Proposition 10.2.5]) and thus induce isomorphisms of sheaves $\mathbf{K}^{\mathrm{M}}_n/2 \to \overline{\mathbf{I}}^n$ for any $n \in \mathbb{N}$.

For any $n \in \mathbb{Z}$, the n-th Milnor-Witt K-theory sheaf $\mathbf{K}^{\mathrm{MW}}_n$ can (and will) be defined as the fiber product

$$\begin{array}{ccc} \mathbf{K}^{\mathrm{MW}}_n & \longrightarrow & \mathbf{I}^n \\ \downarrow & & \downarrow \\ \mathbf{K}^{\mathrm{M}}_n & \longrightarrow & \overline{\mathbf{I}}^n \end{array}$$

where the bottom horizontal morphism is the composite $\mathbf{K}^{\mathrm{M}}_n \to \mathbf{K}^{\mathrm{M}}_n/2 \xrightarrow{s_n} \overline{\mathbf{I}}^n$ and the right-hand vertical morphism is the quotient morphism. It follows from [Mor04, Théorème 5.3] that this definition coincides with the one given in [Mor12, §3.2].

If $\mathcal{L}$ is a line bundle on some smooth scheme X, then we define the $\mathcal{L}$-twisted sheaf $\mathbf{K}^{\mathrm{MW}}_n(\mathcal{L})$ on the small Zariski site of X analogously using $\mathcal{L}$-twisted powers of the fundamental ideal. Again, the resulting sheaf has an explicit flasque resolution obtained by taking the fiber products of the flasque resolutions mentioned above ([Fas07, Theorem 3.26]), or by using the Rost-Schmid complex of [Mor12, §5]. The above fiber product square yields a commutative diagram of short exact sequences of the following form:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{I}^{n+1}(\mathcal{L}) & \longrightarrow & \mathbf{K}^{\mathrm{MW}}_n(\mathcal{L}) & \longrightarrow & \mathbf{K}^{\mathrm{M}}_n & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbf{I}^{n+1}(\mathcal{L}) & \longrightarrow & \mathbf{I}^n(\mathcal{L}) & \longrightarrow & \overline{\mathbf{I}}^n & \longrightarrow & 0. \end{array} \qquad (2.1.2)$$

2.2 The Pardon spectral sequence

Continuing to assume k is a field having characteristic unequal to 2, let X be a smooth k-scheme and suppose $\mathcal{L}$ is a line bundle over X. The filtration

$$\ldots \subset \mathbf{I}^j(\mathcal{L}) \subset \mathbf{I}^{j-1}(\mathcal{L}) \subset \ldots \subset \mathbf{I}(\mathcal{L}) \subset \mathbf{W}(\mathcal{L})$$

yields a spectral sequence that we will refer to as the *Pardon spectral sequence*. We record the main properties of this spectral sequence here, following the formulation of [Tot03, Theorem 1.1].

Theorem 2.2.1. *Assume k is a field having characteristic unequal to 2, X is a smooth k-scheme, and $\mathcal{L}$ is a line bundle on X. There exists a spectral sequence $E(\mathcal{L})_2^{p,q} = H^p(X, \overline{\mathbf{I}}^q) \Rightarrow H^p(X, \mathbf{W}(\mathcal{L}))$. The differentials $d(\mathcal{L})_r$ are of bidegree $(1, r-1)$ for $r \geq 2$, and the groups $H^p(X, \overline{\mathbf{I}}^q)$ are trivial unless $0 \leq p \leq q$. There are identifications $H^p(X, \overline{\mathbf{I}}^p) = CH^p(X)/2$ and the differential $d_2^{pp} : H^p(X, \overline{\mathbf{I}}^p) \to H^{p+1}(X, \overline{\mathbf{I}}^{p+1})$ coincides with the Steenrod square operation Sq^2 as defined by Voevodsky ([Voe03b]) and Brosnan ([Bro03]) when $\mathcal{L}$ is trivial. Finally, if k has finite 2-cohomological dimension, the spectral sequence is bounded.*

Proof. All the statements are proved in [Tot03, proof of Theorem 1.1] except the last one, which follows from the cohomology vanishing statement contained in [AF14b, Proposition 5.1]. ☐

Remark 2.2.2. We will describe the differential $d(\mathcal{L})_2^{pp} : H^p(X, \overline{\mathbf{I}}^p) \to H^{p+1}(X, \overline{\mathbf{I}}^{p+1})$ for $\mathcal{L}$ nontrivial in Theorem 3.4.1.

Since $\mathbf{W}(\mathcal{L}) = \mathbf{I}^0(\mathcal{L})$ by convention, truncating the above filtration allows us to construct a spectral sequence abutting to the cohomology of $\mathbf{I}^j(\mathcal{L})$ for arbitrary $j \geq 0$:

$$\ldots \subset \mathbf{I}^{n+j}(\mathcal{L}) \subset \mathbf{I}^{n+j-1}(\mathcal{L}) \subset \ldots \subset \mathbf{I}^{j+1}(\mathcal{L}) \subset \mathbf{I}^j(\mathcal{L}).$$

The resulting spectral sequence $E(\mathcal{L},j)^{p,q}$ is very similar to the Pardon spectral sequence. Indeed, $E(\mathcal{L},j)_2^{p,q} = 0$ if $q < j$ and $E(\mathcal{L},j)_2^{p,q} = E(\mathcal{L})_2^{p,q}$ otherwise. Similarly $d(\mathcal{L},j)_2^{p,q} = 0$ if $q < j$ and $d(\mathcal{L},j)_2^{p,q} = d(\mathcal{L})_2^{p,q}$ otherwise. We call this spectral sequence the *j-truncated Pardon spectral sequence* and it will be one of the main objects of study in this paper. Using the description of the E_2-page of this spectral sequence and the associated differentials, the proof of the following lemma is straightforward (and left to the reader).

LEMMA 2.2.3. *Assume k is a field having characteristic unequal to 2 and suppose X is a smooth k-scheme of dimension d. There are identifications $E(\mathcal{L},d)_\infty^{d,d} = CH^d(X)/2$ and, for any $n \geq 1$, $E(\mathcal{L},d)_m^{d,d+n} = E(\mathcal{L})_m^{d,d+n}$ if $m \leq n+1$ and exact sequences*

$$E(\mathcal{L})_{n+1}^{d-1,d} \xrightarrow{d(\mathcal{L})_{n+1}^{d-1,d}} E(\mathcal{L})_{n+1}^{d,d+n} \longrightarrow E(\mathcal{L},d)_\infty^{d,d+n} \longrightarrow 0.$$

Using the monomorphism $\mathbf{I}^{j+1}(\mathcal{L}) \subset \mathbf{K}_j^{\mathrm{MW}}(\mathcal{L})$ described in the previous section, we can consider the filtration of $\mathbf{I}^{j+1}(\mathcal{L})$ as a filtration of $\mathbf{K}_j^{\mathrm{MW}}(\mathcal{L})$ of the form:

$$\ldots \subset \mathbf{I}^{n+j}(\mathcal{L}) \subset \mathbf{I}^{n+j-1}(\mathcal{L}) \subset \ldots \subset \mathbf{I}^{j+1}(\mathcal{L}) \subset \mathbf{K}_j^{\mathrm{MW}}(\mathcal{L}).$$

Once again, the spectral sequence $E(\mathcal{L},\mathrm{MW})^{p,q}$ associated with this filtration is very similar to the j-truncated Pardon spectral sequence. Indeed, there are identifications $E(\mathcal{L},\mathrm{MW})_2^{p,q} = E(\mathcal{L},j)_2^{p,q}$ if $q \neq j$ and $E(\mathcal{L},\mathrm{MW})_2^{p,j} = H^p(X,\mathbf{K}_j^{\mathrm{M}})$. In order to describe the terms $E(\mathcal{L},\mathrm{MW})_\infty^{j,q}$ in the situation of interest, we first need a few definitions.

Consider the commutative diagram of sheaves with exact rows from Diagram 2.1.2

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{I}^{j+1}(\mathcal{L}) & \longrightarrow & \mathbf{K}_j^{\mathrm{MW}}(\mathcal{L}) & \longrightarrow & \mathbf{K}_j^{\mathrm{M}} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbf{I}^{j+1}(\mathcal{L}) & \longrightarrow & \mathbf{I}^j(\mathcal{L}) & \longrightarrow & \overline{\mathbf{I}}^j & \longrightarrow & 0. \end{array}$$

The right vertical homomorphism $\mathbf{K}_j^{\mathrm{M}} \to \overline{\mathbf{I}}^j$ is described in the previous subsection and yields, in particular, a homomorphism $H^{j-1}(X,\mathbf{K}_j^{\mathrm{M}}) \to H^{j-1}(X,\overline{\mathbf{I}}^j)$ whose image we denote by $G_2(j)$. Now, $H^{j-1}(X,\overline{\mathbf{I}}^j) = E(\mathcal{L},j)_2^{j-1,j} = E(\mathcal{L})_2^{j-1,j}$ and there is a differential

$$d(\mathcal{L})_2^{j-1,j} : E(\mathcal{L})_2^{j-1,j} \longrightarrow E(\mathcal{L})_2^{j,j+1}.$$

We set $G_3(j) := G_2(j) \cap \ker(d(\mathcal{L})_2^{j-1,j})$ and write $\overline{G}_3(j)$ for its image in $E(\mathcal{L})_3^{j-1,j}$. There is also a differential

$$d(\mathcal{L})_3^{j-1,j} : E(\mathcal{L})_3^{j-1,j} \longrightarrow E(\mathcal{L})_3^{j,j+2}$$

and we set $G_4(j) := \overline{G}_3(j) \cap \ker(d(\mathcal{L})_3^{j-1,j})$ and define $\overline{G}_4(j)$ to be its image in $E(\mathcal{L})_4^{j-1,j}$. Continuing inductively, we can define a sequence of subgroups $\overline{G}_n(j) \subset E(\mathcal{L})_n^{j-1,j}$ for any $n \geq 2$.

LEMMA 2.2.4. *If k is a field having characteristic unequal to 2, and X is a smooth k-scheme of dimension d, then there are isomorphisms $E(\mathcal{L},\mathrm{MW})^{d,d}_{\infty} = CH^d(X)$, and $E(\mathcal{L},\mathrm{MW})^{d-1,d}_{2} = H^{d-1}(X,\mathbf{K}^M_d)$. Furthermore, for any integer $n \geq 1$, there are identifications $E(\mathcal{L},\mathrm{MW})^{d,d+n}_{m} = E(\mathcal{L})^{d,d+n}_{m}$ if $m \leq n+1$ and exact sequences of the form*

$$\overline{G}_{n+1}(d) \xrightarrow{d(\mathcal{L})^{d-1,d}_{n+1}} E(\mathcal{L})^{d,d+n}_{n+1} \longrightarrow E(\mathcal{L},\mathrm{MW})^{d,d+n}_{\infty} \longrightarrow 0.$$

Proof. The morphism of sheaves $\mathbf{K}^{\mathrm{MW}}_d(\mathcal{L}) \to \mathbf{I}^d(\mathcal{L})$ is compatible with the filtrations:

$$\begin{array}{ccccccccc} \ldots \longrightarrow & \mathbf{I}^{d+n}(\mathcal{L}) & \longrightarrow & \mathbf{I}^{d+n-1}(\mathcal{L}) & \longrightarrow \ldots \longrightarrow & \mathbf{I}^{d+1}(\mathcal{L}) & \longrightarrow & \mathbf{K}^{\mathrm{MW}}_d(\mathcal{L}) \\ & \| & & \| & & \| & & \downarrow \\ \ldots \longrightarrow & \mathbf{I}^{d+n}(\mathcal{L}) & \longrightarrow & \mathbf{I}^{d+n-1}(\mathcal{L}) & \longrightarrow \ldots \longrightarrow & \mathbf{I}^{d+1}(\mathcal{L}) & \longrightarrow & \mathbf{I}^d(\mathcal{L}) \end{array}$$

In particular, the induced maps of quotient sheaves are simply the identity map, except at the last spot where they fit into the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{I}^{d+1}(\mathcal{L}) & \longrightarrow & \mathbf{K}^{\mathrm{MW}}_d(\mathcal{L}) & \longrightarrow & \mathbf{K}^{\mathrm{M}}_d & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbf{I}^{d+1}(\mathcal{L}) & \longrightarrow & \mathbf{I}^d(\mathcal{L}) & \longrightarrow & \overline{\mathbf{I}}^d & \longrightarrow & 0 \end{array}$$

The result now follows from the definition of the groups $\overline{G}_i(d)$ and Lemma 2.2.3. $\square$

Remark 2.2.5. By construction, there are epimorphisms $E(\mathcal{L},\mathrm{MW})^{d,d+n}_{\infty} \to E(\mathcal{L},d)^{d,d+n}_{\infty}$ for any $n \geq 0$. Indeed, $\overline{G}_{n+1}(d)$ is, by definition, a subgroup of $E(\mathcal{L})^{d-1,d}_{n+1}$ and the diagram

$$\begin{array}{ccccccccc} \overline{G}_{n+1}(d) & \longrightarrow & E(\mathcal{L})^{d,d+n}_{n+1} & \longrightarrow & E(\mathcal{L},\mathrm{MW})^{d,d+n}_{\infty} & \longrightarrow & 0 \\ \downarrow & & \| & & \downarrow & & \\ E(\mathcal{L})^{d-1,d}_{n+1} & \longrightarrow & E(\mathcal{L})^{d,d+n}_{n+1} & \longrightarrow & E(\mathcal{L},d)^{d,d+n}_{\infty} & \longrightarrow & 0 \end{array}$$

commutes.

Suppose that X is a smooth k-scheme of dimension d such that the Chow group of 0-cycles $CH^d(X)$ is 2-torsion free. In that case, we claim that the dotted arrow in the above diagram is an isomorphism. To see this, observe that the exact sequence of sheaves

$$0 \longrightarrow 2\mathbf{K}^{\mathrm{M}}_d \longrightarrow \mathbf{K}^{\mathrm{M}}_d \longrightarrow \mathbf{K}^{\mathrm{M}}_d/2 \longrightarrow 0$$

yields an exact sequence

$$H^{d-1}(X,\mathbf{K}^{\mathrm{M}}_d) \longrightarrow H^{d-1}(X,\mathbf{K}^{\mathrm{M}}_d/2) \longrightarrow H^d(X,2\mathbf{K}^{\mathrm{M}}_d) \longrightarrow$$

$$\to H^d(X,\mathbf{K}^{\mathrm{M}}_d) \to H^d(X,\mathbf{K}^{\mathrm{M}}_d/2) \to 0.$$

The epimorphism $\mathbf{K}^{\mathrm{M}}_d \xrightarrow{2} 2\mathbf{K}^{\mathrm{M}}_d$ yields an isomorphism $H^d(X,\mathbf{K}^{\mathrm{M}}_d) \to H^d(X,2\mathbf{K}^{\mathrm{M}}_d)$ and we deduce the following exact sequence from Rost's formula and the definition of $G_2(d)$:

$$0 \to G_2(d) \to H^{d-1}(X,\mathbf{K}^{\mathrm{M}}_d/2) \to CH^d(X) \xrightarrow{2} CH^d(X) \longrightarrow CH^d(X)/2 \to 0.$$

Since $CH^d(X)$ is 2-torsion free, it follows that $G_2(d) = H^{d-1}(X,\mathbf{K}^{\mathrm{M}}_d/2)$ and by inspection we obtain an identification $\overline{G}_{n+1}(d) = E(\mathcal{L})^{d-1,d}_{n+1}$. We therefore conclude that the dotted arrow in the above diagram is an isomorphism.

Theorem 2.2.6. *Suppose k is a field having characteristic unequal to 2 and finite 2-cohomological dimension, X is a smooth k-scheme of dimension d and $\mathcal{L}$ is a line bundle over X. For any $\alpha \in H^d(X,\mathbf{K}^{\mathrm{MW}}_d(\mathcal{L}))$ there are inductively defined obstructions $\Psi^n(\alpha) \in E(\mathcal{L},\mathrm{MW})^{d,d+n}_{\infty}$ for $n \geq 0$ such that $\alpha = 0$ if and only if $\Psi^n(\alpha) = 0$ for any $n \geq 0$.*

Proof. The filtration

$$\ldots \subset \mathbf{I}^{n+d}(\mathcal{L}) \subset \mathbf{I}^{n+d-1}(\mathcal{L}) \subset \ldots \subset \mathbf{I}^{d+1}(\mathcal{L}) \subset \mathbf{K}^{\mathrm{MW}}_d(\mathcal{L})$$

to which the spectral sequence $E(\mathcal{L},\mathrm{MW})^{p,q}$ is associated yields a filtration $F^nH^d(X,\mathbf{K}^{\mathrm{MW}}_d(\mathcal{L}))$ for $n \geq 0$ of the cohomology group $H^d(X,\mathbf{K}^{\mathrm{MW}}_d(\mathcal{L}))$ with $F^0H^d(X,\mathbf{K}^{\mathrm{MW}}_d(\mathcal{L})) = H^d(X,\mathbf{K}^{\mathrm{MW}}_d(\mathcal{L}))$ and

$$F^nH^d(X,\mathbf{K}^{\mathrm{MW}}_d(\mathcal{L})) = \mathrm{Im}(H^d(X,\mathbf{I}^{d+n}(\mathcal{L})) \longrightarrow H^d(X,\mathbf{K}^{\mathrm{MW}}_d(\mathcal{L})))$$

for $n \geq 1$. Further, $F^nH^d(X,\mathbf{K}^{\mathrm{MW}}_d(\mathcal{L}))/F^{n+1}H^d(X,\mathbf{K}^{\mathrm{MW}}_d(\mathcal{L})) := E(\mathcal{L},\mathrm{MW})^{d,d+n}_{\infty}$ and the cohomological vanishing statement of [AF14b, Proposition 5.1] implies that only finitely many of the groups appearing above can be non-trivial. If we define the obstructions $\Psi^n(\alpha)$ to be the image of α in the successive quotients, the result is clear. □

The above result gives an inductively defined sequence of obstructions to decide whether an element of $H^d(X,\mathbf{K}^{\mathrm{MW}}_d(\mathcal{L}))$ is trivial. Our next goal is to provide a "concrete" description of the differentials appearing in the spectral sequence. Lemmas 2.2.3 and 2.2.4 imply that these differentials are essentially the differentials in the Pardon spectral sequence, and it is for that reason that we focus on the latter in the remaining sections.

3 Some properties of the differentials

In this section, we establish some properties of the differentials in the Pardon spectral sequence and thus the spectral sequence constructed in the previous section abutting to cohomology of twisted Milnor-Witt K-theory sheaves. We first recall how these differentials are defined and then show that, essentially, they can be viewed as bi-stable operations in motivic cohomology.

3.1 The operation $\Phi_{i,j}$

Suppose X is a smooth k-scheme and $\mathcal{L}$ is a line bundle on X. Recall that for any $j \in \mathbb{N}$, the sheaf $\mathbf{I}^j(\mathcal{L})$ comes equipped with a reduction map $\mathbf{I}^j(\mathcal{L}) \to \bar{\mathbf{I}}^j$ and that there is a canonical isomorphism $\mathbf{K}^{\mathrm{M}}_j/2 \to \bar{\mathbf{I}}^j$; we use this identification without mention in the sequel. The exact sequence

$$0 \longrightarrow \mathbf{I}^{j+1}(\mathcal{L}) \longrightarrow \mathbf{I}^j(\mathcal{L}) \longrightarrow \bar{\mathbf{I}}^j \longrightarrow 0$$

yields a connecting homomorphism

$$H^i(X, \bar{\mathbf{I}}^j) \xrightarrow{\partial_{\mathcal{L}}} \mathbf{H}^{i+1}(X, \mathbf{I}^{j+1}(\mathcal{L})).$$

The reduction map gives a homomorphism

$$H^{i+1}(X, \mathbf{I}^{j+1}(\mathcal{L})) \longrightarrow H^{i+1}(X, \bar{\mathbf{I}}^{j+1}).$$

Taking the composite of these two maps yields a homomorphism that is precisely the differential $d(\mathcal{L})^{i,j}_2$. We state the following definition in order to avoid heavy notation.

Definition 3.1.1. If X is a smooth scheme, and $\mathcal{L}$ is a line bundle on X, write

$$\Phi_{i,j,\mathcal{L}} : H^i(X, \bar{\mathbf{I}}^j) \longrightarrow H^{i+1}(X, \bar{\mathbf{I}}^{j+1}).$$

for the composite of the connecting homomorphism ∂_L and the reduction map just described. If $\mathcal{L}$ is trivial, suppress it from the notation and write $\Phi_{i,j}$ for the resulting homomorphism. Anticipating Theorem 4.1.4, we sometimes refer to $\Phi_{i,j,\mathcal{L}}$ as an operation.

When $i = j$, via the identification $\bar{\mathbf{I}}^j \cong \mathbf{K}^{\mathrm{M}}_j/2$, the map $\Phi_{i,i}$ can be viewed as a morphism $Ch^i(X) \to Ch^{i+1}(X)$, where $Ch^i(X) = CH^i(X)/2$. As stated in Theorem 2.2.1, Totaro identified this homomorphism as Sq^2. More generally, we observe that the homomorphisms $\Phi_{i,j,\mathcal{L}}$ are functorial with respect to pull-backs by definition.

3.2 Bi-stability of the operations $\Phi_{i,j}$

We now study bi-stability, i.e., stability with respect to $\mathbb{P}^1$-suspension, of the operations $\Phi_{i,j}$. If X is a smooth scheme, we then need to compare an operation on X and a corresponding operation on the space $X_+ {\wedge} \mathbb{P}^1$. The reader unfamiliar to this notation can take the following ad hoc definition. If $\mathbf{F}$ is a sheaf, then $H^i(X_+ {\wedge} \mathbb{P}^1, \mathbf{F})$ is defined to be the cokernel of the pull-back homomorphism

$$H^i(X, \mathbf{F}) \longrightarrow H^i(X \times \mathbb{P}^1, \mathbf{F}).$$

In case $\mathbf{F} = \bar{\mathbf{I}}^j$, we use the projective bundle formula in $\bar{\mathbf{I}}^j$-cohomology (see, e.g., [Fas13, §4]) to identify this group in terms of cohomology on X. Indeed, we have an identification

$$H^i(X \times \mathbb{P}^1, \bar{\mathbf{I}}^j) \cong H^i(X, \bar{\mathbf{I}}^j) \oplus H^{i-1}(X, \bar{\mathbf{I}}^{j-1}) \cdot \bar{c}_1(\mathcal{O}(-1)),$$

where $\bar{c}_1(\mathcal{O}(-1))$ is the first Chern class of $\mathcal{O}(-1)$ in $H^1(X, \mathbf{K}_1^{\mathrm{M}}/2) = CH^1(X)/2$. Unwinding the definitions, this corresponds to an isomorphism of the form

$$H^i(X_+ \wedge \mathbb{P}^1, \bar{\mathbf{I}}^j) \cong H^{i-1}(X, \bar{\mathbf{I}}^{j-1})$$

that is functorial in X. Using this isomorphism, we can compare the operation $\Phi_{i,j}$ on $H^i(X_+ \wedge \mathbb{P}^1, \bar{\mathbf{I}}^j)$ with the operation $\Phi_{i-1,j-1}$ on $H^{i-1}(X, \bar{\mathbf{I}}^{j-1})$.

Proposition 3.2.1. *There is a commutative diagram of the form*

$$\begin{array}{ccc} H^i(X_+ \wedge \mathbb{P}^1, \bar{\mathbf{I}}^j) & \xrightarrow{\Phi_{i,j}} & H^{i+1}(X_+ \wedge \mathbb{P}^1, \bar{\mathbf{I}}^{j+1}) \\ \downarrow & & \downarrow \\ H^{i-1}(X, \bar{\mathbf{I}}^{j-1}) & \xrightarrow[\Phi_{i-1,j-1}]{} & H^i(X, \bar{\mathbf{I}}^j), \end{array}$$

where the vertical maps are the isomorphisms described before the statement.

Proof. The operation $\Phi_{i,j}$ is induced by the composite morphism of the connecting homomorphism associated with the short exact sequence

$$0 \longrightarrow \mathbf{I}^{j+1} \longrightarrow \mathbf{I}^j \longrightarrow \bar{\mathbf{I}}^j \longrightarrow 0$$

and the reduction map $\mathbf{I}^{j+1} \to \bar{\mathbf{I}}^{j+1}$. The contractions of $\mathbf{I}^j$ and $\bar{\mathbf{I}}^j$ are computed in [AF14a, Lemma 2.7 and Proposition 2.8] and our result follows immediately from the proofs of those statements. ☐

Remark 3.2.2. Because of the above result, we will abuse terminology and refer to $\Phi_{i,j}$ as a bi-stable operation.

3.3 Non-triviality of the operation $\Phi_{i-1,i,\mathcal{L}}$

Our goal in this section is to prove that the operation $\Phi_{i-1,i}$ is nontrivial. By definition, the operation $\Phi_{i-1,i}$ can be computed as follows: given an element $\alpha \in H^{i-1}(X, \bar{\mathbf{I}}^i)$, we choose a lift to $C^{i-1}(X, \mathbf{I}^i)$, apply the boundary homomorphism to obtain an element $d_{i-1}(\alpha) \in C^i(X, \mathbf{I}^i)$ which becomes trivial under the homomorphism $C^i(X, \mathbf{I}^i) \to C^i(X, \bar{\mathbf{I}}^i)$ (since α is a cycle). There exists thus a unique lift of $d_{i-1}(\alpha) \in C^i(X, \mathbf{I}^{i+1})$, which is a cycle since $d_i d_{i-1} = 0$. Its reduction in $H^i(X, \bar{\mathbf{I}}^i)$ is $\Phi_{i-1,i}(\alpha)$ by definition. We use the identification $H^{i-1}(X, \bar{\mathbf{I}}^i) \cong H^{i-1}(X, \mathbf{K}_i^{\mathrm{M}}/2)$ and the computations of Suslin in the case where $X = SL_3$ to provide explicit generators. More precisely, [Sus91, Theorem 2.7] shows that $H^1(SL_3, \mathbf{K}_2^{\mathrm{M}}/2) = \mathbb{Z}/2$, $H^2(SL_3, \mathbf{K}_3^{\mathrm{M}}/2) = \mathbb{Z}/2$. We begin by finding explicit generators of the groups considered by Suslin and transfer those generators under the isomorphisms just described to obtain explicit representatives of classes in $H^1(SL_3, \bar{\mathbf{I}}^2)$ and $H^2(SL_3, \bar{\mathbf{I}}^3)$. Then, we explicitly compute the connecting homomorphism and the reduction. Our method and notation will follow closely [Sus91, §2].

For any $n \in \mathbb{N}$, let $Q_{2n-1} \subset \mathbb{A}^{2n}$ be the hypersurface given by the equation $\sum_{i=1}^n x_i y_i = 1$. Let $SL_n = \operatorname{Spec}(k[(t_{ij})_{1 \leq i,j \leq n}]/\langle \det(t_{ij}) - 1\rangle)$ and write

$\alpha_n = (t_{ij})_{1\leq i,j\leq n}$ for the universal matrix on SL_n, and $(t^{ij})_{1\leq i,j\leq n}$ for its inverse α_n^{-1}. For $n \geq 2$, we embed SL_{n-1} into SL_n as usual by mapping a matrix M to $\mathrm{diag}(1, M)$, and we observe that the quotient is precisely Q_{2n-1} by means of the homomorphism $f : SL_n \to Q_{2n-1}$ given by $f^*(x_i) = t_{1i}$ and $f^*(y_i) = t^{i1}$. Now Q_{2n-1} is covered by the affine open subschemes $U_i := D(x_i)$ and the projection $f : SL_n \to Q_{2n-1}$ splits over each U_i by means of a matrix $\gamma_i \in E_n(U_i)$ given for instance in [Sus91, §2]. The only properties that we will use here are that these sections induce isomorphisms $f^{-1}(U_i) \simeq U_i \times SL_{n-1}$ mapping $(\alpha_n)_{|f^{-1}(U_i)}\gamma_i^{-1}$ to $\mathrm{diag}(1, \alpha_{n-1})$. Recall next from [Gil81, §2], that one can define Chern classes

$$c_i : K_1(X) \longrightarrow H^i(X, \mathbf{K}^{\mathrm{M}}_{i+1}/2)$$

functorially in X. In particular, we have Chern classes $c_i : K_1(SL_n) \to H^i(SL_n, \mathbf{K}^{\mathrm{M}}_{i+1}/2)$ and we set $d_{i,n} := c_i(\alpha_n)$.

The stage being set, we now proceed to our computations. We will implicitly use the Gersten resolution of the sheaves $\mathbf{K}^{\mathrm{M}}_i/2$ in our computations below. Observe first that the equations $x_2 = \ldots = x_n = 0$ define an integral subscheme $Z_n \subset Q_{2n-1}$, and that the global section x_1 is invertible on Z_n. It follows that it defines an element in $(\mathbf{K}^{\mathrm{M}}_1/2)(k(Z_n))$ and a cycle $\theta_n \in H^{n-1}(Q_{2n-1}, \mathbf{K}^{\mathrm{M}}_n)$.

LEMMA 3.3.1. *For any smooth scheme X, the $H^*(X, \mathbf{K}^{\mathrm{M}}_*/2)$-module $H^*(Q_{2n-1} \times X, \mathbf{K}^{\mathrm{M}}_*/2)$ is free with basis $1, \theta_n$.*

Proof. Apply the proof of [Sus91, Theorem 1.5] mutatis mutandis. □

Since $Q_3 = SL_2$, we can immediately deduce a basis for the cohomology of SL_2. However, we can reinterpret θ_2 as follows.

LEMMA 3.3.2. *If X is a smooth scheme, then $H^*(SL_2 \times X, \mathbf{K}^{\mathrm{M}}_*/2)$ is a free $H^*(X, \mathbf{K}^{\mathrm{M}}_*/2)$-module generated by $1 \in H^0(X, \mathbf{K}^{\mathrm{M}}_0/2)$ and $d_{1,2} \in H^1(SL_2, \mathbf{K}^{\mathrm{M}}_2/2)$.*

Proof. Again, this is essentially [Sus91, proof of Proposition 1.6]. □

Before stating the next lemma, recall that we have a projection morphism $f : SL_3 \to Q_5$, yielding a structure of $H^*(Q_5, \mathbf{K}^{\mathrm{M}}_*/2)$-module on the cohomology of SL_3.

LEMMA 3.3.3. *The $H^*(Q_5, \mathbf{K}^{\mathrm{M}}_*/2)$-module $H^*(SL_3, \mathbf{K}^{\mathrm{M}}_*/2)$ is free with basis 1 and $d_{1,3}$.*

Proof. Using Mayer-Vietoris sequences in the spirit of [Sus91, Lemma 2.2], we see that it suffices to check locally that 1 and $d_{1,3}$ is a basis. Let $U_i \subset Q_{2n-1}$ be the open subschemes defined above. We know that we have an isomorphism $f^{-1}(U_i) \simeq U_i \times SL_2$ mapping $(\alpha_3)_{|f^{-1}(U_i)}\gamma_i^{-1}$ to $\mathrm{diag}(1, \alpha_2)$. The Chern class c_1 being functorial, we have a commutative diagram

$$\begin{array}{ccc} K_1(SL_3) & \xrightarrow{c_1} & H^1(SL_3, \mathbf{K}^{\mathrm{M}}_2/2) \\ \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} \\ K_1(f^{-1}(U_i)) & \xrightarrow[c_1]{} & H^1(f^{-1}(U_i), \mathbf{K}^{\mathrm{M}}_2/2) \end{array}$$

where the vertical homomorphisms are restrictions. We thus see that $i^*(d_{1,3}) = i^*(c_1(\alpha_3)) = c_1(i^*(\alpha_3))$. Since $\gamma_i \in E_3(U_i)$, we see that $c_1(i^*(\alpha_3)) = c_1(p^*\alpha_2) = p^*d_{1,2}$ where $p : f^{-1}(U_i) \to SL_2$ is the projection. The result now follows from Lemma 3.3.2. $\square$

Combining Lemmas 3.3.2 and 3.3.3, we immediately obtain the following result.

COROLLARY 3.3.4. *We have* $H^1(SL_3, \mathbf{K}_2^{\mathrm{M}}/2) = \mathbb{Z}/2 \cdot d_{1,3}$ *and* $H^2(SL_3, \mathbf{K}_3^{\mathrm{M}}/2) = \mathbb{Z}/2 \cdot f^*(\theta_3)$.

The cycle $f^*(\theta_3)$ is very explicit. Indeed, it can be represented by the class of the global section t_{11} in $(\mathbf{K}_1^{\mathrm{M}}/2)(k(z_1))$ where z_1 is given by the equations $t_{12} = t_{13} = 0$. We now make $d_{1,3}$ more explicit. Recall that $\alpha_3 = (t_{ij})$ is the universal matrix on SL_3 and $\alpha_3^{-1} = (t^{ij})$ is its inverse. In particular, we have $\sum_{j=1}^3 t_{ij}t^{jk} = \delta_{jk} = \sum_{j=1}^3 t^{ij}t_{jk}$.

LEMMA 3.3.5. *If* $y_1 \in SL_3^{(1)}$ *is defined by the ideal* $\langle t^{13}\rangle$ *and* $y_2 \in SL_3^{(1)}$ *is defined by the ideal* $\langle t_{12}\rangle$, *then a generator for the group* $H^1(SL_3, \mathbf{K}_2^{\mathrm{M}}/2) \cong \mathbb{Z}/2$, *is given by the class of the symbol*

$$\xi := \{t^{12}\} + \{t_{13}\}$$

in $\mathbf{K}_1^{\mathrm{M}}(k(y_1))/2 \oplus \mathbf{K}_1^{\mathrm{M}}(k(y_2))/2$.

Proof. The image of $\{t_{13}\}$ under the boundary map in the Gersten complex is the generator of $\mathbf{K}_0^M(k(z_1))/2$ where z_1 is the point defined by the ideal $I_1 := \langle t_{12}, t_{13}\rangle$, while the image of $\{t^{12}\}$ is the generator of $\mathbf{K}_0^M(k(z_2))/2$ where z_2 is the point defined by the ideal $I_2 := \langle t^{12}, t^{13}\rangle$. It suffices then to check that $z_1 = z_2$ to conclude that ξ is a cycle.
The equality $\sum_{j=1}^3 t^{1j}t_{j1} = 1$ shows that t^{11} is invertible modulo I_2 and we deduce from $\sum_{j=1}^3 t^{1j}t_{j2} = 0$ that $t_{12} \in I_2$. Similarly, we deduce from $\sum_{j=1}^3 t^{1j}t_{j3} = 0$ that $t_{13} \in I_2$ and therefore $I_1 \subset I_2$. Reasoning symmetrically we obtain that $I_2 \subset I_1$, proving the claim.
Since ξ is a cycle, it defines a class in $H^1(SL_3, \mathbf{K}_2^{\mathrm{M}}/2) \cong \mathbb{Z}/2$ and it suffices thus to show that the class of ξ is non trivial to conclude. Consider the embedding (of schemes, but not of group schemes) $g : SL_2 \to SL_3$ given by

$$\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \mapsto \begin{pmatrix} 0 & -1 & 0 \\ u_{11} & 0 & u_{12} \\ u_{21} & 0 & u_{22} \end{pmatrix}.$$

Since this morphism factors through the open subscheme $SL_3[t_{12}^{-1}] = f^{-1}(U_2)$ and the inverse of the above matrix is given by the matrix

$$\begin{pmatrix} 0 & u_{22} & -u_{12} \\ -1 & 0 & 0 \\ 0 & -u_{21} & u_{11} \end{pmatrix},$$

it follows that $g^*(\xi)$ is represented by the class of $\{u_{22}\}$ in $\mathbf{K}_1^{\mathrm{M}}(k(s))/2$, where s is given by $u_{12} = 0$. One can then verify directly that this cycle equals the generator $d_{1,2}$ given in Lemma 3.3.2, and it follows that $\xi \neq 0$. □

PROPOSITION 3.3.6. *The operation $\Phi_{i-1,i}$ is non-trivial.*

Proof. We compute the effect of the operation $\Phi_{1,2}$ on elements of $H^1(SL_3, \mathbf{K}_2^{\mathrm{M}}/2)$. By definition, $\Phi_{1,2}$ is the composite

$$H^1(SL_3, \mathbf{K}_2^{\mathrm{M}}/2) = H^1(SL_3, \bar{\mathbf{I}}^2) \longrightarrow H^2(SL_3, \mathbf{I}^3) \longrightarrow \\ \longrightarrow H^2(SL_3, \bar{\mathbf{I}}^3) = H^2(SL_3, \mathbf{K}_3^{\mathrm{M}}/2)$$

where the left-hand map is the boundary homomorphism associated with the exact sequence of sheaves

$$0 \longrightarrow \mathbf{I}^3 \longrightarrow \mathbf{I}^2 \longrightarrow \bar{\mathbf{I}}^2 \longrightarrow 0$$

and the right-hand map is the projection associated with the morphism of sheaves $\mathbf{I}^3 \to \bar{\mathbf{I}}^3$. We will show that $\Phi_{1,2}$ is an isomorphism by showing that the explicit generator of $H^1(SL_3, \mathbf{K}_2^{\mathrm{M}}/2)$ constructed in Lemma 3.3.5 is mapped to the explicit generator of $H^2(SL_3, \mathbf{K}_3^{\mathrm{M}}/2)$ constructed in Corollary 3.3.4.
Recall from Section 2.1 the Gersten resolution $C(X, \mathbf{I}^j)$ of the sheaf $\mathbf{I}^j$, which takes the form

$$I^j(k(X)) \to \bigoplus_{x \in X^{(1)}} I_{fl}^{j-1}(k(x)) \xrightarrow{d_1} \bigoplus_{x \in X^{(2)}} I_{fl}^{j-2}(k(x)) \to \bigoplus_{x \in X^{(3)}} I_{fl}^{j-3}(k(x)) \to \dots$$

where X is a smooth scheme, and $I_{fl}^{j-1}(k(x)) = I^{j-1}(k(x)) \cdot W_{fl}(\mathcal{O}_{X,x})$. Take $X = SL_3$.
An explicit lift of the generator of $H^1(SL_3, \mathbf{K}_2^{\mathrm{M}}/2)$ given in Lemma 3.3.5 is of the form

$$\langle -1, t^{12} \rangle \cdot \rho_1 + \langle -1, t_{13} \rangle \cdot \rho_2$$

where $\rho_1 : k(y_1) \to \mathrm{Ext}^1_{\mathcal{O}_{X,y_1}}(k(y_1), \mathcal{O}_{X,y_1})$ is defined by mapping 1 to the Koszul complex $Kos(t^{13})$ associated with the regular sequence t^{13}, and similarly $\rho_2 : k(y_2) \to \mathrm{Ext}^1_{\mathcal{O}_{X,y_2}}(k(y_2), \mathcal{O}_{X,y_2})$ is defined by $1 \mapsto Kos(t_{12})$. Using [Fas08, Section 3.5], the boundary d_1 of the above generator is of the form $\nu_1 + \nu_2$, where

$$\nu_1 : k(z) \longrightarrow \mathrm{Ext}^2_{\mathcal{O}_{X,z}}(k(z), \mathcal{O}_{X,z})$$

is defined by $1 \mapsto Kos(t^{13}, t^{12})$ and

$$\nu_2 : k(z) \longrightarrow \mathrm{Ext}^2_{\mathcal{O}_{X,z}}(k(z), \mathcal{O}_{X,z})$$

is defined by $1 \mapsto Kos(t_{12}, t_{13})$. Recall from the proof of Lemma 3.3.5 that $t^{11} \in \mathcal{O}_{X,z}^{\times}$ and it follows thus from the identities $\sum_{j=1}^{3} t^{1j} t_{jk} = 0$ for $k = 1, 2$ that we have

$$\begin{pmatrix} t_{12} \\ t_{13} \end{pmatrix} = \begin{pmatrix} -t_{32}/t^{11} & -t_{22}/t^{11} \\ -t_{33}/t^{11} & -t_{23}/t^{11} \end{pmatrix} \begin{pmatrix} t^{13} \\ t^{12} \end{pmatrix}.$$

Now $t_{32}t_{23} - t_{22}t_{33} = -t^{11}$ and $t^{11}t_{11} = 1$ modulo $\langle t^{12}, t^{13}\rangle$ and we therefore get

$$\nu_1 + \nu_2 = \langle 1, t_{11}\rangle \cdot \nu_1 = (\langle 1, 1\rangle + \langle -1, t_{11}\rangle) \cdot \nu_1$$

A simple computation shows that $\langle 1, 1\rangle \cdot \nu_1$ is the boundary of $(\langle 1, t_{13}\rangle \otimes \langle 1, t_{13}\rangle) \cdot \rho_2$ and therefore vanishes in $H^2(SL_3, \mathbf{I}^3)$. Now the class of $\langle -1, t_{11}\rangle \cdot \nu_1$ in $H^2(SL_3, \bar{\mathbf{I}}^3) = H^2(SL_3, \mathbf{K}_3^{\mathrm{M}}/2)$ is precisely a generator as shown by Corollary 3.3.4. Thus, $\Phi_{1,2} : H^1(SL_3, \mathbf{K}_2^{\mathrm{M}}/2) \to H^2(SL_3, \mathbf{K}_3^{\mathrm{M}}/2)$ is an isomorphism. □

3.4 Identification of $\Phi_{i-1,i,\mathcal{L}}$

If $\mathcal{L}$ is a line bundle over our smooth k-scheme X, we write $\bar{c}_1(\mathcal{L})$ for its first Chern class in $H^1(X, \mathbf{K}_1^{\mathrm{M}}/2) = CH^1(X)/2$.

Theorem 3.4.1. *For any smooth scheme X, any $i, j \in \mathbb{N}$ and any line bundle $\mathcal{L}$ over X, we have*

$$\Phi_{i,j,\mathcal{L}} = (\Phi_{i,j} + \bar{c}_1(\mathcal{L})\cup).$$

Proof. In outline, the proof will proceed as follows. We consider the total space of the line bundle $\mathcal{L}$ over X. By pull-back stability of the operation and homotopy invariance, we can relate the operation $\Phi_{i,j,\mathcal{L}}$ with the operation $\Phi_{i,j}$ on the total space of the line bundle $\mathcal{L}$, with a twist coming from the first Chern class of the line bundle via the various identifications. To establish the result, we track the action of $\Phi_{i,j,\mathcal{L}}$ on suitable explicit representatives of cohomology classes through the identifications just mentioned; for this, we use symmetric complexes and some ideas of Balmer.
As in the proof of Proposition 3.3.6, we consider the Gersten-Witt complex of X (filtered by powers of the fundamental ideal) $C(X, \mathbf{I}^j(\mathcal{L}))$:

$$I^j(\mathcal{L})(k(X)) \xrightarrow{d_0^{\mathcal{L}}} \bigoplus_{x\in X^{(1)}} I^{j-1}(\mathcal{L})_{fl}(k(x)) \xrightarrow{d_1^{\mathcal{L}}} \bigoplus_{x\in X^{(2)}} I^{j-2}(\mathcal{L})_{fl}(k(x)) \xrightarrow{d_2^{\mathcal{L}}} \ldots.$$

In the case where $\mathcal{L} = \mathcal{O}_X$, we will drop $\mathcal{L}$ from the notation. Recall that there is an exact sequence of complexes

$$0 \longrightarrow C(X, \mathbf{I}^{j+1}(\mathcal{L})) \longrightarrow C(X, \mathbf{I}^j(\mathcal{L})) \longrightarrow C(X, \bar{\mathbf{I}}^j) \longrightarrow 0.$$

If $\alpha \in H^i(X, \bar{\mathbf{I}}^j)$, then $\Phi_{i,j,\mathcal{L}}(\alpha)$ is defined as follows. If $\alpha' \in C^i(X, \mathbf{I}^j(\mathcal{L}))$ is any lift of α, then its boundary $d_i^{\mathcal{L}}(\alpha') \in C^{i+1}(X, \mathbf{I}^j(\mathcal{L}))$ is the image of a unique cycle $\beta \in C^{i+1}(X, \mathbf{I}^{j+1}(\mathcal{L}))$. The reduction of β in $C^{i+1}(X, \bar{\mathbf{I}}^{j+1})$ is precisely $\Phi_{i,j,\mathcal{L}}(\alpha)$.
Let us observe next that if $p : L \to X$ is the total space of $\mathcal{L}$, then p induces morphisms of complexes p^* ([Fas08, Corollaire 9.3.2]) fitting into the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C(X, \mathbf{I}^{j+1}(\mathcal{L})) & \longrightarrow & C(X, \mathbf{I}^j(\mathcal{L})) & \longrightarrow & C(X, \bar{\mathbf{I}}^j) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle p^*} & & \\
0 & \longrightarrow & C(L, \mathbf{I}^{j+1}(p^*\mathcal{L})) & \longrightarrow & C(L, \mathbf{I}^j(p^*\mathcal{L})) & \longrightarrow & C(L, \bar{\mathbf{I}}^j) & \longrightarrow & 0.
\end{array}$$

By homotopy invariance, the vertical morphisms induce isomorphisms on cohomology groups by [Fas08, Théorème 11.2.9]. We use these identifications to replace X by L in what follows.
Now, let us recall how to obtain explicit representatives for elements of $H^i(X, \bar{\mathbf{I}}^j)$; this involves the formalism of [BW02]. Let $\alpha \in H^i(X, \bar{\mathbf{I}}^j)$ and let $\alpha' \in C^i(X, \mathbf{I}^j)$ be a lift of α. Under the equivalences of [BW02, Theorem 6.1, Proposition 7.1], α' can be seen as a complex $P_\bullet$ of finitely generated $\mathcal{O}_X$-locally free modules, together with a symmetric morphism (for the i-th shifted duality)

$$\psi : P_\bullet \longrightarrow T^i \mathrm{Hom}(P_\bullet, \mathcal{O}_X)$$

whose cone is supported in codimension $\geq i+1$. By definition, $d_i(\alpha')$ is the localization at the points of codimension $i+1$ of the symmetric quasi-isomorphism on the cone of ψ (constructed for instance in [BW02, Proposition 1.2]), after dévissage ([BW02, Theorem 6.1, Proposition 7.1]).
The first Chern class of $p^*\mathcal{L}$ appears in a natural way using this language. We want to choose a representative of $\bar{c}_1(p^*\mathcal{L})$ in $H^1(L, \mathbf{K}_1^{\mathrm{M}}/2) \cong H^1(L, \bar{\mathbf{I}})$. A lift of this element to $H^1(L, \mathbf{I}(p^*\mathcal{L}))$ can be described as follows. The zero section

$$s : \mathcal{O}_L \longrightarrow p^*\mathcal{L}$$

can be seen as a symmetric morphism $\mathcal{O}_L \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_L, p^*\mathcal{L})$, which is an isomorphism after localization at the generic point of L, and whose cone is supported in codimension 1. It follows that s can be thought of as an element of $C^0(L, \mathbf{W}(p^*\mathcal{L}))$. The class of $d_0^{p^*\mathcal{L}}(s)$ can be viewed as an element of $H^1(L, \mathbf{I}(p^*\mathcal{L}))$, and its projection in $H^1(L, \bar{\mathbf{I}}) = Pic(L)/2$ is precisely the first Chern class of $p^*\mathcal{L}$ ([Fas13, proof of Lemma 3.1]).
To lift an element $\alpha \in H^i(X, \bar{\mathbf{I}}^j)$ to an element α' in $C^i(X, \mathbf{I}^j(\mathcal{L}))$, we will first find a lift $\alpha'' \in C^i(X, \mathbf{I}^j)$ and then multiply by s in a sense to be explained more carefully below to obtain our lift α'. A Leibniz-type formula can then be used to compute the boundary of this product and derive the formula in the statement of our theorem.
Using the product structure (say the left one) on derived categories with duality of [GN03], we can obtain an element of $C^i(L, \mathbf{I}^j(p^*\mathcal{L}))$ lifting $p^*\alpha \in H^i(L, \bar{\mathbf{I}}^j)$ using the symmetric morphism

$$p^*\psi \otimes s : p^*P_\bullet \longrightarrow T^i \mathrm{Hom}(p^*P_\bullet, p^*\mathcal{L}).$$

The degeneracy locus of $p^*\psi$ in the sense of [Bal05, Definition 3.2] is, by definition, the support of its cone, which has codimension $\geq i+1$ in L. The degeneracy locus of s has codimension 1 in L and intersects the degeneracy locus of $p^*\psi$ transversally.
Now, we are in a position to apply the Leibniz formula of [Bal05, Theorem 5.2] (while the hypotheses of the quoted result are not satisfied in our situation, the proof of [Fas07, Propostion 4.7] explains why the formula continues to hold in the case where the intersection of degeneracy loci is transversal). Since we will momentarily consider the sheaf $\bar{\mathbf{I}}^j$ whose cohomology groups are 2-torsion, we can ignore signs, in which case the Leibniz formula gives the equality:

$$d_i^{p^*\mathcal{L}}(p^*\psi \otimes s) = d_i(p^*\psi) \otimes s + p^*\psi \otimes d_0^{p^*\mathcal{L}}(s) \tag{3.4.1}$$

in $C^{i+1}(L, \mathbf{I}^j(p^*\mathcal{L}))$.
Since $p^*\alpha \in H^i(L, \bar{\mathbf{I}}^j)$ and $p^*\psi \otimes s$ lifts $p^*\alpha$ in $C^i(L, \mathbf{I}^j(p^*\mathcal{L}))$ it follows that $d_i^{p^*\mathcal{L}}(p^*\psi \otimes s)$ actually belongs to $C^{i+1}(L, \mathbf{I}^{j+1}(p^*\mathcal{L}))$. For the same reason, we have $d_i(p^*\psi) \in C^{i+1}(L, \mathbf{I}^{j+1})$ and then $d_i(p^*\psi) \otimes s \in C^{i+1}(L, \mathbf{I}^{j+1}(p^*\mathcal{L}))$. Thus $p^*\psi \otimes d_0^{p^*\mathcal{L}}(s)$ is in $C^{i+1}(L, \mathbf{I}^{j+1}(p^*\mathcal{L}))$ as well. It follows that all three terms in (3.4.1) define classes in $C^{i+1}(L, \bar{\mathbf{I}}^{j+1})$. The left term yields a class in $H^{i+1}(L, \bar{\mathbf{I}}^{j+1})$ which is $\Phi_{i,j,p^*\mathcal{L}}(p^*\alpha)$ by definition. The middle term projects to $\Phi_{i,j}(p^*\alpha)$ and the right-hand term to the class $p^*\alpha \cdot c_1(p^*\mathcal{L})$ in $H^{i+1}(L, \bar{\mathbf{I}}^{j+1})$. □

4 Differentials, cohomology operations and the Euler class

Having established the basic properties of the differentials in the Pardon spectral sequence, we now pass to their identification with known operations on motivic cohomology.

4.1 Differentials in terms of motivic cohomology

Let us first recall some notation. Write $\mathcal{H}^j$ for the Zariski sheaf associated with the presheaf $U \mapsto H^j_{\text{ét}}(U, \mathbb{Z}/2)$. For integers p, q, write $H^{p,q}(X, \mathbb{Z}/2)$ for the motivic cohomology groups with $\mathbb{Z}/2$ coefficients as defined by Voevodsky (see, e.g., [MVW06, Lecture 3]); these groups are by construction hypercohomology of certain complexes of Zariski sheaves. We begin by recalling a result of Totaro [Tot03, Theorem 1.3].

Theorem 4.1.1. *Suppose k is a field having characteristic unequal to 2, and X is a smooth k-scheme. For any integer $j \geq 0$, there is a long exact sequence of the form:*

$$\cdots \to H^{i+j,j-1}(X, \mathbb{Z}/2) \to H^{i+j,j}(X, \mathbb{Z}/2) \to$$
$$\to H^i(X, \mathcal{H}^j) \to H^{i+j+1,j-1}(X, \mathbb{Z}/2) \to \cdots;$$

this exact sequence is functorial in X.

Comments on the proof. This result requires Voevodsky's affirmation of Milnor's conjecture on the mod 2 norm residue homomorphism [Voe03a] as well as the Beilinson-Lichtenbaum conjecture, which is equivalent to the Milnor conjecture by results of Suslin-Voevodsky and Geisser-Levine. The functoriality assertion of the statement is evident from inspection of the proof (it appears by taking hypercohomology of a distinguished triangle). □

We will use the above exact sequence in the guise established in the following result.

Corollary 4.1.2. *For any $i \in \mathbb{N}$ and any smooth scheme X over a perfect field k with $\mathrm{char}(k) \neq 2$, the above sequence induces an isomorphism*

$$H^{2i+1,i+1}(X, \mathbb{Z}/2) \simeq H^i(X, \bar{\mathbf{I}}^{i+1})$$

that is functorial in X.

Proof. The exact sequence of Theorem 4.1.1 reads as follows for $j = i+1$

$$\ldots \to H^{2i+1,i}(X,\mathbb{Z}/2) \to H^{2i+1,i+1}(X,\mathbb{Z}/2) \to \\ \to \mathbf{H}^i(X,\mathcal{H}^{i+1}) \to H^{2i+2,i}(X,\mathbb{Z}/2) \to \ldots$$

Since k is perfect, we have $H^{p,i}(X,\mathbb{Z}/2) = 0$ for any $p \geq 2i+1$ by [MVW06, Theorem 19.3] and from this we can conclude that the middle arrow is an isomorphism. Since the exact sequence is functorial in X, it follows immediately that the isomorphism just mentioned has the same property. Now, the affirmation of the Milnor conjecture on the mod 2 norm residue homomorphism also implies that $\mathbf{K}^{\mathrm{M}}_{i+1}/2$ can be identified as a sheaf with $\mathcal{H}^{i+1}$, while the affirmation of the Milnor conjecture on quadratic forms yields an identification of sheaves $\mathbf{K}^{\mathrm{M}}_{i+1}/2 \cong \bar{\mathbf{I}}^{i+1}$. Combining these isomorphisms yields an isomorphism $\mathcal{H}^{i+1} \cong \bar{\mathbf{I}}^{i+1}$ and therefore an identification of cohomology with coefficients in these sheaves functorial in the input scheme. □

Voevodsky defined in [Voe03b, p. 33] motivic Steenrod operations Sq^{2i} : $H^{p-2i,q-i}(X,\mathbb{Z}/2) \to H^{p,q}(X,\mathbb{Z}/2)$. The resulting operations are bi-stable in the sense that they are compatible with $\mathbb{P}^1$-suspension in the same sense as described in the previous section. Via the isomorphism of Corollary 4.1.2, we can view Sq^2 as an operation

$$Sq^2 : H^{i-1}(X,\bar{\mathbf{I}}^i) \longrightarrow H^i(X,\bar{\mathbf{I}}^{i+1}),$$

which is again bi-stable in the sense that it is compatible with $\mathbb{P}^1$-suspension. The algebra of bistable cohomology operations in motivic cohomology with $\mathbb{Z}/2$-coefficients was determined by Voevodsky in characteristic 0, [Voe10] and extended to fields having characteristic unequal to 2 in [HKØ13]. Using these results, we may now identify the operation $\Phi_{i-1,i}$ described in Definition 3.1.1 in more explicit terms.

COROLLARY 4.1.3. *We have an identification* $\Phi_{i-1,i} = Sq^2$.

Proof. The operation $\Phi_{i-1,i}$ is bistable by Proposition 3.2.1, commutes with pullbacks by construction, and changes bidegree by $(2,1)$ so it is pulled back from a universal class on a motivic Eilenberg-Mac Lane space. On the other hand, the group of bi-stable operations of bidegree $(2,1)$ is isomorphic to $\mathbb{Z}/2$ generated by Sq^2: if k has characteristic zero, this follows from [Voe10, Theorem 3.49], while if k has characteristic unequal to 2, this follows from [HKØ13, Theorem 1.1]. Since the operation $\Phi_{i-1,i}$ is non-trivial by Proposition 3.3.6, it follows that it must be equal to Sq^2. □

The next result is an immediate consequence of Corollary 4.1.3 and Theorem 3.4.1.

THEOREM 4.1.4. *Suppose k is a field having characteristic unequal to 2, and X is a smooth k-scheme. For any integer $i > 0$, and any rank r vector bundle $\xi : \mathcal{E} \to X$, the operation $(Sq^2 + \bar{c}_1(\xi)\cup)$ coincides with $\Phi_{i-1,i,\det\xi}$.*

4.2 THE EULER CLASS AND SECONDARY CLASSES

The Euler class $e(\mathcal{E})$ of a rank d vector bundle $\xi : \mathcal{E} \to X$ is the only obstruction to splitting off a free rank 1 summand, and it lives in $H^d(X, \mathbf{K}_d^{\mathrm{MW}}(\mathcal{L}))$ where $\mathcal{L} = \det \mathcal{E}$. Now, the Euler class is mapped to the top Chern class $c_d(\mathcal{E})$ in $CH^d(X)$ under the homomorphism $H^d(X, \mathbf{K}_d^{\mathrm{MW}}(\mathcal{L})) \to H^d(X, \mathbf{K}_d^{\mathrm{M}}) = CH^d(X)$ induced by the morphism of sheaves $\mathbf{K}_d^{\mathrm{MW}}(\mathcal{L}) \to \mathbf{K}_d^{\mathrm{M}}$ and it follows that the vanishing of $e(\mathcal{E})$ guarantees vanishing of $c_d(\mathcal{E})$ in $CH^d(X)$ (see [AF14c, Proposition 6.3.1] for this statement).
The vanishing of the top Chern class does not, in general, imply vanishing of the Euler class, as shown by the example of the tangent bundle to the real algebraic sphere of dimension 2. For vector bundles with vanishing top Chern class, we can use Theorem 2.2.6 to decide whether its Euler class vanishes, provided we work over a field k of finite 2-cohomological dimension. In the next theorem, we denote by $\Psi^n(\mathcal{E})$ the obstruction classes $\Psi^n(e(\mathcal{E}))$ of Theorem 2.2.6 associated to the Euler class $e(\mathcal{E})$.

THEOREM 4.2.1. *Suppose k is a field having finite 2-cohomological dimension, X is a smooth k-scheme of dimension d and $\xi : \mathcal{E} \to X$ is a rank d vector bundle on X with $c_d(\mathcal{E}) = 0$. The vector bundle $\mathcal{E}$ splits off a trivial rank 1 summand if only if, in addition, $\Psi^n(\mathcal{E}) = 0$ for $n \geq 1$.*

As mentioned in the introduction, the advantage of the computation of these higher obstruction classes over the computation of the Euler class is that the cohomology groups involved are with coefficients in cycle modules in the sense of Rost, which are a priori more manageable than cohomology with coefficients in more exotic sheaves such as Milnor-Witt K-theory. Moreover, Corollary 4.1.2 shows that the differentials, at least in some range, can be identified with Steenrod operations, which are arguably more calculable. The obvious weakness of this approach is the appearance of the groups $\overline{G}_i$ defined in Section 2.2, though see Remark 2.2.5 for a counterpoint. Continuing with the assumption that our base field k has finite cohomological dimension one can show that establishing the vanishing of finitely many obstructions (depending on the cohomological dimension) are sufficient to guarantee vanishing of all obstructions. The next result completes the verification of Corollary 2 from the introduction.

COROLLARY 4.2.2. *Assume k is a field of 2-cohomological dimension s, X is a smooth k-scheme of dimension d and $\xi : \mathcal{E} \to X$ is a rank d-vector bundle over X with $c_d(\mathcal{E}) = 0$. The vector bundle $\mathcal{E}$ splits off a trivial rank 1 summand if and only if $\Psi^n(\mathcal{E}) = 0$ for $n \leq s - 1$.*

Proof. In view of the definition of the higher obstructions $\Psi^n(\mathcal{E})$, it suffices to show that $H^d(X, \bar{\mathbf{I}}^j)$ vanishes for $j \geq d + r$. This is [AF14b, Proposition 5.2], together with the identification of Nisnevich and Zariski cohomology with coefficients in $\bar{\mathbf{I}}^j$ explained in [AF14b, §2]. ☐

Finally, combining all of the results established so far, we can complete the verification of Theorem 3.

Completion of proof of Theorem 3. To identify the secondary obstruction Ψ^1 as the composition in the statement, we begin by observing that the group $E(\mathcal{L}, \mathrm{MW})^{d,d+1}_{\infty}$ is the cokernel of the composite map

$$H^{d-1}(X, \mathbf{K}^{\mathrm{M}}_d) \longrightarrow H^{d-1}(X, \mathbf{K}^{\mathrm{M}}_d/2) \xrightarrow{Sq^2+\bar{c}_1(\mathcal{L})} H^d(X, \mathbf{K}^M_{d+1}/2)$$

in view of Lemma 2.2.4 and Theorem 4.1.4.

If k has cohomological dimension ≤ 1, it follows immediately from Corollary 4.2.2 that the top Chern class is the only obstruction to splitting a free rank 1 summand.

If k has cohomological dimension ≤ 2, Theorem 4.2.1 says in this context that the Euler class of E (take $\mathcal{L} = \det(E)$) is trivial if and only if the top Chern class and the first obstruction class in $E(\mathcal{L}, \mathrm{MW})^{d,d+1}_{\infty}$ vanish. □

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Quadric Surface Bundles over Surfaces

Dedicated to Sasha Merkurjev on his 60th birthday

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Abstract. Let $f : T \to S$ be a finite flat morphism of degree 2 between regular integral schemes of dimension ≤ 2 with 2 invertible, having regular branch divisor $D \subset S$. We establish a bijection between Azumaya quaternion algebras on T and quadric surface bundles with simple degeneration along D. This is a manifestation of the exceptional isomorphism ${}^2\mathsf{A}_1 = \mathsf{D}_2$ degenerating to the exceptional isomorphism $\mathsf{A}_1 = \mathsf{B}_1$. In one direction, the even Clifford algebra yields the map. In the other direction, we show that the classical algebra norm functor can be uniquely extended over the discriminant divisor. Along the way, we study the orthogonal group schemes, which are smooth yet nonreductive, of quadratic forms with simple degeneration. Finally, we provide two applications: constructing counter-examples to the local-global principle for isotropy, with respect to discrete valuations, of quadratic forms over surfaces; and a new proof of the global Torelli theorem for very general cubic fourfolds containing a plane.

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Introduction

A quadric surface bundle $\pi : Q \to S$ over a scheme S is the flat fibration in quadrics associated to a line bundle-valued quadratic form $q : \mathscr{E} \to \mathscr{L}$ of rank 4 over S. A natural class of quadric surface bundles over $\mathbb{P}^2$ appearing in algebraic geometry arise from cubic fourfolds $Y \subset \mathbb{P}^5$ containing a plane. Projection from the plane $\pi : \widetilde{Y} \to \mathbb{P}^2$, where $\widetilde{Y}$ is the blow-up of Y along the plane, yields a quadric surface bundle with degeneration along a sextic curve $D \subset \mathbb{P}^2$. If Y is sufficiently general then D is smooth and the double cover $T \to \mathbb{P}^2$ branched along D is a smooth K3 surface of degree 2. Over the surface T, the even Clifford algebra $\mathscr{C}_0$ associated to π becomes an Azumaya quaternion algebra representing a Brauer class $\beta \in {}_2\mathrm{Br}(T)$. For Y even more sufficiently general, the association $Y \mapsto (T, \beta)$ is injective: smooth cubic fourfolds Y and Y' giving rise to isomorphic data $(T, \beta) \cong (T', \beta')$ are linearly isomorphic. This result was originally obtained via Hodge theory by Voisin [60] in the course of her proof of the global Torelli theorem for cubic fourfolds.
In this work, we provide a vast algebraic generalization of this result to any regular integral scheme T of dimension ≤ 2, which is a finite flat double cover $T \to S$ of a regular scheme S, such that the branch divisor $D \subset S$ is regular. We establish a bijection between the isomorphism classes of quadric surface bundles on S having simple degeneration (see §1) with discriminant $T \to S$ and the isomorphism classes of Azumaya quaternion algebras on T whose norm to S is split (see §5). In one direction, the even Clifford algebra $\mathscr{C}_0$, associated to a quadric surface bundle on S with simple degeneration along D, gives rise to an Azumaya quaternion algebra on T. In the other, a generalization of the classical algebra norm functor $N_{T/S}$, applied to an Azumaya quaternion algebra on T with split norm to S, gives rise to a quadric surface bundle on S. Our main result is the following.

Theorem 1. *Let S be a regular integral scheme of dimension ≤ 2 with 2 invertible and $T \to S$ a finite flat morphism of degree 2 with regular branch divisor $D \subset S$. Then the even Clifford algebra and norm functors*

$$\left\{ \begin{array}{c} \textit{quadric surface bundles with} \\ \textit{simple degeneration along } D \\ \textit{and discriminant } T \to S \end{array} \right\} \underset{N_{T/S}}{\overset{\mathscr{C}_0}{\rightleftarrows}} \left\{ \begin{array}{c} \textit{Azumaya quaternion} \\ \textit{algebras over } T \textit{ with} \\ \textit{split norm to } S \end{array} \right\}$$

give rise to mutually inverse bijections.

This result can be viewed as a significant generalization of the exceptional isomorphism ${}^2\mathsf{A}_1 = \mathsf{D}_2$ correspondence over fields and rings (cf. [41, IV.15.B] and [43, §10]) to the setting of line bundle-valued quadratic forms with simple degeneration over schemes. Most of our work goes toward establishing fundamental local results concerning quadratic forms with simple degeneration (see §3) and the structure of their orthogonal group schemes, which are nonreductive (see §2). In particular, we prove that these group schemes are smooth (see Proposition 2.3) and realize a degeneration of exceptional isomorphisms $^2\mathsf{A}_1 = \mathsf{D}_2$ to $\mathsf{A}_1 = \mathsf{B}_1$. We also establish fundamental structural results concerning quadric surface bundles over schemes (see §1) and the formalism of gluing tensors over surfaces (see §4).

We also present two surprisingly different applications of our results. First, in §6, we provide a class of geometrically interesting quadratic forms that are counter-examples to the local-global principle to isotropy, with respect to discrete valuations, over the function field of any surface over an algebraically closed field of characteristic zero. This is made possible by the tight control we have over the degeneration divisors of norm forms of unramified quaternion algebras over function fields of ramified double covers of surfaces. Moreover, our class of counter-examples exists even over rational function fields, where the existence of such counterexamples was an open question.

Second, in §7, combining our main result with tools from the theory of moduli of twisted sheaves, we are able to provide a new proof of the result of Voisin mentioned above, concerning general complex cubic fourfolds containing a plane. Our method is algebraic in nature and could lead to similar results for other classes of complex fourfolds birational to quadric surface bundles over surfaces.

Our perspective comes from the algebraic theory of quadratic forms. We employ the even Clifford algebra of a line bundle-valued quadratic form constructed by Bichsel [14]. Bichsel–Knus [15], Caenepeel–van Oystaeyen [16] and Parimala–Sridharan [51, §4] give alternate constructions, which are all detailed in [3, §1.8]. In a similar vein, Kapranov [39, §4.1] (with further developments by Kuznetsov [44, §3]) considered the *homogeneous* Clifford algebra of a quadratic form—this is related to the *generalized* Clifford algebra of [15] and the *graded* Clifford algebra of [16]—to study the derived category of projective quadrics and quadric bundles. We focus on the even Clifford algebra as a sheaf of algebras, rather than its geometric manifestation as a relative Hilbert scheme of lines in the quadric bundle, as in [60, §1] and [37, §5]. In this context, we refer to Hassett–Tschinkel [36, §3] for a version of our result over smooth projective curves over an algebraically closed field.

Finally, our work on degenerate quadratic forms may also be of independent interest. There has been a recent focus on classification of degenerate (quadratic) forms from various number theoretic directions. An approach to Bhargava's [13] seminal construction of moduli spaces of "rings of low rank" over arbitrary base schemes is developed by Wood [62] where line bundle-valued degenerate forms (of higher degree) are crucial ingredients. In this context, a correspondence such as ours, established over $\mathbb{Z}$, could facilitate density results for discriminants of quaternion orders over quadratic extensions of number fields. In related developments, building on the work of Delone–Faddeev [25] over $\mathbb{Z}$ and Gross–Lucianovic [31] over local rings, Venkata Balaji [9], and independently Voight [59], used Clifford algebras of degenerate ternary quadratic forms to classify degenerations of quaternion algebras over arbitrary bases. In this context, our main result can be viewed as a classification of quaternary quadratic forms with squarefree discriminant in terms of their even Clifford algebras.

1 Reflections on simple degeneration

Let S be a noetherian separated integral scheme. A *(line bundle-valued) quadratic form* on S is a triple $(\mathscr{E}, q, \mathscr{L})$, where $\mathscr{E}$ is a locally free $\mathscr{O}_S$-module of finite rank and $q : \mathscr{E} \to \mathscr{L}$ is a morphism of sheaves, homogeneous of degree 2 for the action of $\mathscr{O}_S$, and such that the associated morphism of sheaves $b_q : \mathscr{E} \times \mathscr{E} \to \mathscr{L}$, defined on sections by $b_q(v, w) = q(v+w) - q(v) - q(w)$, is $\mathscr{O}_S$-bilinear. Equivalently, a quadratic form is an $\mathscr{O}_S$-module morphism $q : S_2\mathscr{E} \to \mathscr{L}$, see [57, Lemma 2.1] or [3, Lemma 1.1]. Here, $S^2\mathscr{E}$ and $S_2\mathscr{E}$ denote the second symmetric power and the submodule of symmetric second tensors of $\mathscr{E}$, respectively. There is a canonical isomorphism $S^2(\mathscr{E}^\vee) \otimes \mathscr{L} \cong \mathscr{H}om(S_2\mathscr{E}, \mathscr{L})$. A line bundle-valued quadratic form then corresponds to a global section

$$q \in \Gamma(S, \mathscr{H}om(S_2\mathscr{E}, \mathscr{L})) \cong \Gamma(S, S^2(\mathscr{E}^\vee) \otimes \mathscr{L}) \cong \Gamma(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})/S}(2) \otimes p^*\mathscr{L}),$$

where $p : \mathbb{P}(\mathscr{E}) = \mathbf{Proj}\, S^\bullet(\mathscr{E}^\vee) \to S$. There is a canonical $\mathscr{O}_S$-module *polar* morphism $\psi_q : \mathscr{E} \to \mathscr{H}om(\mathscr{E}, \mathscr{L})$ associated to b_q. A line bundle-valued quadratic form $(\mathscr{E}, q, \mathscr{L})$ is *regular* if ψ_q is an $\mathscr{O}_S$-module isomorphism. Otherwise, the *radical* $\mathrm{rad}(\mathscr{E}, q, \mathscr{L})$ is the sheaf kernel of ψ_q, which is a torsion-free subsheaf of $\mathscr{E}$. We will mostly dispense with the adjective "line bundle-valued." We define the *rank* of a quadratic form to be the rank of the underlying module. A *similarity* $(\varphi, \lambda_\varphi) : (\mathscr{E}, q, \mathscr{L}) \to (\mathscr{E}', q', \mathscr{L}')$ consists of $\mathscr{O}_S$-module isomorphisms $\varphi : \mathscr{E} \to \mathscr{E}'$ and $\lambda_\varphi : \mathscr{L} \to \mathscr{L}'$ such that $q'(\varphi(v)) = \lambda_\varphi \circ q(v)$ on sections. A similarity $(\varphi, \lambda_\varphi)$ is an *isometry* if $\mathscr{L} = \mathscr{L}'$ and λ_φ is the identity map. We write $\simeq$ for similarities and $\cong$ for isometries. Denote by $\mathbf{GO}(\mathscr{E}, q, \mathscr{L})$ and $\mathbf{O}(\mathscr{E}, q, \mathscr{L})$ the presheaves, on the flat (fppf) site on S, of similitudes and isometries of a quadratic form $(\mathscr{E}, q, \mathscr{L})$, respectively. These are sheaves and are representable by affine group schemes of finite presentation over S, indeed closed subgroupschemes of $\mathbf{GL}(\mathscr{E})$. The similarity factor defines a homomorphism $\lambda : \mathbf{GO}(\mathscr{E}, q, \mathscr{L}) \to \mathbf{G}_\mathrm{m}$ with kernel $\mathbf{O}(\mathscr{E}, q, \mathscr{L})$. If $(\mathscr{E}, q, \mathscr{L})$ has even rank $n = 2m$, then there is a homomorphism $\det/\lambda^m : \mathbf{GO}(\mathscr{E}, b, \mathscr{L}) \to \boldsymbol{\mu}_2$, whose kernel is denoted by $\mathbf{GO}^+(\mathscr{E}, q, \mathscr{L})$ (this definition of $\mathbf{GO}^+$ assumes 2 is invertible on S; in general it is defined as the kernel of the Dickson invariant). The similarity factor $\lambda : \mathbf{GO}^+(\mathscr{E}, q, \mathscr{L}) \to \mathbf{G}_\mathrm{m}$ has kernel denoted by $\mathbf{O}^+(\mathscr{E}, q, \mathscr{L})$. Denote by $\mathbf{PGO}(\mathscr{E}, q, \mathscr{L})$ the sheaf cokernel of the central subgroup scheme $\mathbf{G}_\mathrm{m} \to \mathbf{GO}(\mathscr{E}, q, \mathscr{L})$ of homotheties; similarly define $\mathbf{PGO}^+(\mathscr{E}, q, \mathscr{L})$. At every point where $(\mathscr{E}, q, \mathscr{L})$ is regular, these group schemes are smooth and reductive (see [26, II.1.2.6, III.5.2.3]) though not necessarily connected. In §2, we will study their structure over points where the form is not regular.

The *quadric bundle* $\pi : Q \to S$ associated to a nonzero quadratic form $(\mathscr{E}, q, \mathscr{L})$ of rank $n \geq 2$ is the restriction of $p : \mathbb{P}(\mathscr{E}) \to S$ via the closed embedding $j : Q \to \mathbb{P}(\mathscr{E})$ defined by the vanishing of the global section $q \in \Gamma_S(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})/S}(2) \otimes p^*\mathscr{L})$. Write $\mathscr{O}_{Q/S}(1) = j^*\mathscr{O}_{\mathbb{P}(\mathscr{E})/S}(1)$. We say that $(\mathscr{E}, q, \mathscr{L})$ is *primitive* if $q_x \neq 0$ at every point x of S, i.e., if $q : \mathscr{E} \to \mathscr{L}$ is an epimorphism. If q is primitive then $Q \to \mathbb{P}(\mathscr{E})$ has relative codimension 1 over S and $\pi : Q \to S$ is flat of relative dimension $n - 2$, cf. [46, 8 Thm. 22.6].

We say that $(\mathscr{E}, q, \mathscr{L})$ is *generically regular* if q is regular over the generic point of S.
Define the *projective similarity* class of a quadratic form $(\mathscr{E}, q, \mathscr{L})$ to be the set of similarity classes of quadratic forms $(\mathscr{N} \otimes \mathscr{E}, \mathrm{id}_{\mathscr{N}^{\otimes 2}} \otimes q, \mathscr{N}^{\otimes 2} \otimes \mathscr{L})$ ranging over all line bundles $\mathscr{N}$ on S. Equivalently, this is the set of isometry classes $(\mathscr{N} \otimes \mathscr{E}, \phi \circ (\mathrm{id}_{\mathscr{N}^{\otimes 2}} \otimes q), \mathscr{L}')$ ranging over all isomorphisms $\phi : \mathscr{N}^{\otimes 2} \otimes \mathscr{L} \to \mathscr{L}'$ of line bundles on S. This is referred to as a *lax-similarity* class in [10]. The main result of this section shows that projectively similar quadratic forms yield isomorphic quadric bundles, while the converse holds under further hypotheses.
Let η be the generic point of S and $\pi : Q \to S$ a quadric bundle. Restriction to the generic fiber of π gives rise to a complex

$$0 \to \mathrm{Pic}(S) \xrightarrow{\pi^*} \mathrm{Pic}(Q) \to \mathrm{Pic}(Q_\eta) \to 0 \tag{1}$$

whose exactness we will study in Proposition 1.6 below.

Proposition 1.1. *Let $\pi : Q \to S$ and $\pi' : Q' \to S$ be quadric bundles associated to quadratic forms $(\mathscr{E}, q, \mathscr{L})$ and $(\mathscr{E}', q', \mathscr{L}')$. If $(\mathscr{E}, q, \mathscr{L})$ and $(\mathscr{E}', q', \mathscr{L}')$ are in the same projective similarity class then Q and Q' are S-isomorphic. The converse holds if q is assumed to be generically regular and (1) is assumed to be exact in the middle.*

Proof. Assume that $(\mathscr{E}, q, \mathscr{L})$ and $(\mathscr{E}', q', \mathscr{L}')$ are projectively similar with respect to an invertible $\mathscr{O}_S$-module $\mathscr{N}$ and $\mathscr{O}_S$-module isomorphisms $\varphi : \mathscr{E}' \to \mathscr{N} \otimes \mathscr{E}$ and $\lambda : \mathscr{L}' \to \mathscr{N}^{\otimes 2} \otimes \mathscr{L}$ preserving the quadratic forms. Let $p : \mathbb{P}(\mathscr{E}) \to S$ and $p' : \mathbb{P}(\mathscr{E}') \to S$ be the associated projective bundles and $h : \mathbb{P}(\mathscr{E}') \to \mathbb{P}(\mathscr{N} \otimes \mathscr{E})$ the S-isomorphism associated to $\varphi^\vee$. There is a natural S-isomorphism $g : \mathbb{P}(\mathscr{N} \otimes \mathscr{E}) \to \mathbb{P}(\mathscr{E})$ satisfying $g^*\mathscr{O}_{\mathbb{P}(\mathscr{E})/S}(1) \cong \mathscr{O}_{\mathbb{P}(\mathscr{E} \otimes \mathscr{N})/S}(1) \otimes p'^*\mathscr{N}$, see [34, II Lemma 7.9]. Denote by $f = g \circ h : \mathbb{P}(\mathscr{E}') \to \mathbb{P}(\mathscr{E})$ the composition. Then via the isomorphism

$$\Gamma\big(\mathbb{P}(\mathscr{E}'), f^*(\mathscr{O}_{\mathbb{P}(\mathscr{E})/S}(2) \otimes p^*\mathscr{L})\big) \to \Gamma\big(\mathbb{P}(\mathscr{E}'), \mathscr{O}_{\mathbb{P}(\mathscr{E}')/S}(2) \otimes p'^*\mathscr{L}'\big)$$

induced by $f^*\mathscr{O}_{\mathbb{P}(\mathscr{E})/S}(2) \cong \mathscr{O}_{\mathbb{P}(\mathscr{E}')}(2) \otimes (p'^*\mathscr{N})^{\otimes 2}$ and $p'^*\lambda^{-1} : (p'^*\mathscr{N})^{\otimes 2} \otimes p'^*\mathscr{L} \to p'^*\mathscr{L}'$, the global section f^*s_q is taken to the global section $s_{q'}$, hence f restricts to a S-isomorphism $Q' \to Q$. The proof of the first claim is complete.
Now assume that $(\mathscr{E}, q, \mathscr{L})$ is generically regular and that $f : Q' \to Q$ is an S-isomorphism. First, we will prove that f can be extended to a S-isomorphism $\tilde{f} : \mathbb{P}(\mathscr{E}') \to \mathbb{P}(\mathscr{E})$ satisfying $\tilde{f} \circ j' = j \circ f$. To this end, considering the long exact sequence associated to applying p_* to the short exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}(\mathscr{E})/S}(-1) \otimes p^*\mathscr{L}^\vee \xrightarrow{s_q} \mathscr{O}_{\mathbb{P}(\mathscr{E})/S}(1) \to j_*\mathscr{O}_{Q/S}(1) \to 0. \tag{2}$$

and keeping in mind that $R^i p_* \mathscr{O}_{\mathbb{P}(\mathscr{E})/S}(-1) = 0$ for $i = 0, 1$, we arrive at an isomorphism $p_*\mathscr{O}_{\mathbb{P}(\mathscr{E})/S}(1) \cong \pi_*\mathscr{O}_{Q/S}(1)$. In particular, we have a canonical identification $\mathscr{E}^\vee = \pi_*\mathscr{O}_{Q/S}(1)$. We have a similar identification $\mathscr{E}'^\vee = \pi'_*\mathscr{O}_{Q'/S}(1)$.

We claim that $f^*\mathscr{O}_{Q/S}(1) \cong \mathscr{O}_{Q'/S}(1) \otimes \pi'^*\mathscr{N}$ for some line bundle $\mathscr{N}$ on S. Indeed, over the generic fiber, we have $f^*\mathscr{O}_{Q/S}(1)_\eta = f_\eta^*\mathscr{O}_{Q_\eta}(1) \cong \mathscr{O}_{Q'_\eta}(1)$ by the case of smooth quadrics (as q is generically regular) over a field, cf. [28, Lemma 69.2]. Then the exactness of (1) in the middle finishes the proof of the present claim.

Finally, by the projection formula and our assumption that $\pi' : Q' \to S$ is of positive relative dimension, we have that f induces an $\mathscr{O}_S$-module isomorphism

$$\begin{aligned}\mathscr{E}^\vee \otimes \mathscr{N}^\vee &\cong \pi_*\mathscr{O}_{Q/S}(1) \otimes \pi'_*\pi'^*\mathscr{N}^\vee \\ &\cong \pi_* f_*(f^*\mathscr{O}_{Q/S}(1) \otimes \pi'^*\mathscr{N}^\vee) \cong \pi'_*\mathscr{O}_{Q'/S}(1) = \mathscr{E}'^\vee\end{aligned}$$

with induced dual isomorphism $\varphi : \mathscr{E}' \to \mathscr{N} \otimes \mathscr{E}$. Now define $\tilde{f} : \mathbb{P}(\mathscr{E}') \to \mathbb{P}(\mathscr{E})$ to be the composition of the morphism $\mathbb{P}(\mathscr{E}') \to \mathbb{P}(\mathscr{N} \otimes \mathscr{E})$ defined by $\varphi^\vee$ with the natural S-isomorphism $\mathbb{P}(\mathscr{N} \otimes \mathscr{E}) \to \mathbb{P}(\mathscr{E})$, as earlier in this proof. Then by the construction of $\tilde{f}$, we have that $\tilde{f}^*\mathscr{O}_{\mathbb{P}(\mathscr{E})/S}(1) \cong \mathscr{O}_{\mathbb{P}(\mathscr{E}')/S}(1) \otimes p'^*\mathscr{N}$ and that $j \circ f = \tilde{f} \circ j'$ (an equality that is checked on fibers using [28, Thm. 69.3]). Equivalently, there exists an isomorphism $\tilde{f}^*(\mathscr{O}_{\mathbb{P}(\mathscr{E})(2)/S} \otimes p^*\mathscr{L}) \cong \mathscr{O}_{\mathbb{P}(\mathscr{E}')/S}(2) \otimes p'^*\mathscr{L}'$ taking f^*s_q to $s_{q'}$. However, as $\tilde{f}^*(\mathscr{O}_{\mathbb{P}(\mathscr{E})(2)/S} \otimes p^*\mathscr{L}) \cong \mathscr{O}_{\mathbb{P}(\mathscr{E}')/S}(2) \otimes p'^*(\mathscr{N}^{\otimes 2} \otimes \mathscr{L}')$, we have an isomorphism $p'^*\mathscr{L}' \cong p'^*(\mathscr{N}^{\otimes 2} \otimes \mathscr{L})$. Upon taking pushforward, we arrive at an isomorphism $\lambda : \mathscr{L}' \to \mathscr{N}^{\otimes 2} \otimes \mathscr{L}$. By the construction of φ and λ, it follows that (φ, λ) is a similarity $(\mathscr{E}, q, \mathscr{L}) \to (\mathscr{E}', q', \mathscr{L}')$, proving the converse. □

DEFINITION 1.2. The determinant $\det \psi_q : \det \mathscr{E} \to \det \mathscr{E}^\vee \otimes \mathscr{L}^{\otimes n}$ gives rise to a global section of $(\det \mathscr{E}^\vee)^{\otimes 2} \otimes \mathscr{L}^{\otimes n}$, whose divisor of zeros is called the *discriminant divisor* D. The reduced subscheme associated to D is precisely the locus of points where the radical of q is nontrivial. If q is generically regular, then $D \subset S$ is closed of codimension one.

DEFINITION 1.3. We say that a quadratic form $(\mathscr{E}, q, \mathscr{L})$ has *simple degeneration* if

$$\mathrm{rk}_{\kappa(x)} \operatorname{rad}(\mathscr{E}_x, q_x, \mathscr{L}_x) \leq 1$$

for every point x of S, where $\kappa(x)$ is the residue field of $\mathscr{O}_{S,x}$.

Our first lemma concerns the local structure of simple degeneration.

LEMMA 1.4. *Let $(\mathscr{E}, q)$ be a quadratic form with simple degeneration over the spectrum of a local ring R with 2 invertible. Then $(\mathscr{E}, q) \cong (\mathscr{E}_1, q_1) \perp (R, <\pi>)$ where $(\mathscr{E}_1, q_1)$ is regular and $\pi \in R$.*

Proof. Over the residue field k, the form $(\mathscr{E}, q)$ has a regular subform $(\overline{\mathscr{E}}_1, \overline{q}_1)$ of corank one, which can be lifted to a regular orthogonal direct summand $(\mathscr{E}_1, q_1)$ of corank 1 of $(\mathscr{E}, q)$, cf. [8, Cor. 3.4]. This gives the required decomposition. Moreover, we can lift a diagonalization $\overline{q}_1 \cong <\overline{u}_1, \ldots, \overline{u}_{n-1}>$ with $\overline{u}_i \in k^\times$, to a diagonalization

$$q \cong <u_1, \ldots, u_{n-1}, \pi>,$$

with $u_i \in R^\times$ and $\pi \in R$. □

Let $D \subset S$ be a regular divisor. Assuming that S is normal, the local ring $\mathscr{O}_{S,D'}$ at the generic point of a component D' of D is a discrete valuation ring. When 2 is invertible on S, Lemma 1.4 shows that a quadratic form $(\mathscr{E}, q, \mathscr{L})$ with simple degeneration along D can be diagonalized over $\mathscr{O}_{S,D'}$ as

$$q \cong < u_1, \dots, u_{r-1}, u_r \pi^e >$$

where u_i are units and π is a parameter of $\mathscr{O}_{S,D'}$. We call $e \geq 1$ the *multiplicity* of the simple degeneration along D'. If e is even for every component of D, then there is a birational morphism $g : S' \to S$ such that the pullback of $(\mathscr{E}, q, \mathscr{L})$ to S' is regular. We will focus on quadratic forms with simple degeneration of multiplicity one along (all components of) D.

We can give a geometric interpretation of simple degeneration.

Proposition 1.5. *Let $\pi : Q \to S$ be the quadric bundle associated to a generically regular quadratic form $(\mathscr{E}, q, \mathscr{L})$ over S and $D \subset S$ its discriminant divisor. Then:*

a) q has simple degeneration if and only if the fiber Q_x of its associated quadric bundle has at worst isolated singularities for each closed point x of S;

b) if 2 is invertible on S and D is reduced, then any simple degeneration along D has multiplicity one;

c) if 2 is invertible on S and D is regular, then any degeneration along D is simple of multiplicity one;

d) if S is regular and q has simple degeneration, then D is regular if and only if Q is regular.

Proof. The first claim follows from the classical geometry of quadrics over a field: the quadric of a nondegenerate form is smooth while the quadric of a form with nontrivial radical has isolated singularity if and only if the radical has rank one. As for the second claim, the multiplicity of the simple degeneration is exactly the scheme-theoretic multiplicity of the divisor D. For the third claim, see [20, §3], [37, Rem. 7.1], or [4, Rem. 2.6]. The final claim is standard, cf. [11, I Prop. 1.2(iii)], [37, Lemma 5.2], or [4, Prop. 1.2.5]. □

We do not need the full flexibility of the following general result, but we include it for completeness.

Proposition 1.6. *Let $\pi : Q \to S$ be a flat proper separated morphism with geometrically integral fibers between noetherian integral separated locally factorial schemes and let η be the generic point of S. Then the complex of Picard groups (1) is exact.*

Proof. First, we argue that flat pullback and restriction to the generic fiber give rise to an exact sequence of Weil divisor groups

$$0 \to \mathrm{Div}(S) \xrightarrow{\pi^*} \mathrm{Div}(Q) \to \mathrm{Div}(Q_\eta) \to 0. \tag{3}$$

Indeed, as $\mathrm{Div}(Q_\eta) = \varinjlim \mathrm{Div}(Q_U)$, where the limit is taken over all dense open sets $U \subset S$ and we write $Q_U = Q \times_S U$, the exactness at right of sequence (3) then follows from the exactness of the excision sequence

$$Z^0(\pi^{-1}(S \smallsetminus U)) \to \mathrm{Div}(Q) \to \mathrm{Div}(Q_U) \to 0$$

cf. [30, 1 Prop. 1.8]. The sequence (3) is exact at left since π is surjective on codimension 1 points, providing a retraction of π^*. As for exactness in the middle, if a prime Weil divisor T on Q has trivial generic fiber then it is supported on the fibers over a closed subscheme of S not containing η. Since the fibers of π are irreducible, T must coincide with $\pi^{-1}(Z)$ for some prime Weil divisor Z of S. Thus T is in the image of π^*.
Second, we argue that there is an analogous exact sequence of principal Weil divisor groups

$$0 \to \mathrm{PDiv}(S) \xrightarrow{\pi^*} \mathrm{PDiv}(Q) \to \mathrm{PDiv}(Q_\eta) \to 0. \tag{4}$$

Indeed, since π is dominant, it induces an extension of function fields K_Q over K_S, and hence a well defined π^* on principal divisors, which is injective. Since $K_Q = K_{Q_\eta}$, restriction to the generic point is surjective on principal divisors. For the exactness in the middle, if $\mathrm{div}_Q(f)_\eta = 0$ then $f \in \Gamma(Q_\eta, \mathscr{O}_{Q_\eta}^\times)$, i.e., f has neither zeros nor poles on Q_η. Since Q_η is a proper geometrically integral K_S-scheme, $\Gamma(Q_\eta, \mathscr{O}_{Q_\eta}^\times) = K_S^\times$, and hence $f \in K_S^\times$. Thus $\mathrm{div}_Q(f)$ is in the image of π^*.
The snake lemma then induces an exact sequence of Weil divisor class groups

$$0 \to \mathrm{Cl}(S) \xrightarrow{\pi^*} \mathrm{Cl}(Q) \to \mathrm{Cl}(Q_\eta) \to 0.$$

As π is separated with geometrically integral fibers, Q_η is separated and integral. As Q is a noetherian locally factorial scheme, Q_η is as well. Hence all Weil divisor class groups coincide with Picard groups by [32, Cor. 21.6.10], immediately implying that the complex (1) is exact. □

COROLLARY 1.7. *Let S be a regular integral scheme with 2 invertible and $(\mathscr{E}, q, \mathscr{L})$ a quadratic form on S of rank ≥ 4 having at most simple degeneration along a regular divisor $D \subset S$. Let $\pi : Q \to S$ be the associated quadric bundle. Then the complex* (1) *is exact.*

Proof. First, recall that a quadratic form over a field contains a nondegenerate subform of rank ≥ 3 if and only if its associated quadric is irreducible, cf. [34, I Ex. 5.12]. Hence the fibers of π are geometrically irreducible. By Proposition 1.5, Q is regular. Quadratic forms with simple degeneration are primitive, hence π is flat. Thus we can apply all the parts of Proposition 1.6. □

We will define $\mathrm{Quad}_n^D(S)$ to be the set of projective similarity classes of line bundle-valued quadratic forms of rank n on S with simple degeneration of multiplicity one along an effective Cartier divisor D. An immediate consequence of Propositions 1.1 and 1.5 and Corollary 1.7 is the following.

Corollary 1.8. *For $n \geq 4$ and D reduced, the set $\mathrm{Quad}_n^D(S)$ is in bijection with the set of S-isomorphism classes of quadric bundles of relative dimension $n-2$ with isolated singularities in the fibers above D.*

Definition 1.9. Let $(\mathscr{E}, q, \mathscr{L})$ be a quadratic form of rank n on a scheme S, and $\mathscr{C}_0 = \mathscr{C}_0(\mathscr{E}, q, \mathscr{L})$ be its even Clifford algebra (see [15] or [3, §1.8]), and $\mathscr{Z} = \mathscr{Z}(\mathscr{E}, q, \mathscr{L})$ be its center. Then $\mathscr{C}_0$ is a locally free $\mathscr{O}_S$-algebra of rank 2^{n-1}, cf. [40, IV.1.6]. The associated finite morphism $f : T \to S$ is called the *discriminant cover*. We remark that if S is locally factorial and q is generically regular of even rank then $\mathscr{Z}$ is a locally free $\mathscr{O}_S$-algebra of rank two, by (the remarks preceding) [40, IV Prop. 4.8.3], hence the discriminant cover $f : T \to S$ is finite flat of degree two. Below, we will arrive at the same conclusion under weaker hypotheses on S but assuming that q has simple degeneration.

Lemma 1.10 ([4, App. B]). *Let $(\mathscr{E}, q, \mathscr{L})$ be a quadratic form of even rank with simple degeneration of multiplicity one along $D \subset S$ and $f : T \to S$ its discriminant cover. Then $f^*\mathscr{O}(D)$ is a square in $\mathrm{Pic}(T)$ and the branch divisor of f is precisely D.*

By abuse of notation, we also denote by $\mathscr{C}_0 = \mathscr{C}_0(\mathscr{E}, q, \mathscr{L})$ the $\mathscr{O}_T$-algebra associated to the $\mathscr{Z}$-algebra $\mathscr{C}_0 = \mathscr{C}(\mathscr{E}, q, \mathscr{L})$. The center $\mathscr{Z}$ is an étale algebra over every point of S where $(\mathscr{E}, q, \mathscr{L})$ is regular and $\mathscr{C}_0$ is an Azumaya algebra over every point of T lying over a point of S where $(\mathscr{E}, q, \mathscr{L})$ is regular. Now we prove [44, Prop. 3.13] over any integral scheme.

Proposition 1.11. *Let $(\mathscr{E}, q, \mathscr{L})$ be a quadratic form of even rank with simple degeneration over a scheme S with 2 invertible. Then the discriminant cover $T \to S$ is finite flat of degree two and $\mathscr{C}_0$ is an Azumaya $\mathscr{O}_T$-algebra.*

Proof. The desired properties are local for the étale topology, so we can assume that $S = \mathrm{Spec}\, R$ for a local ring R with 2 invertible, we can fix a trivialization of $\mathscr{L}$, and by Lemma 1.4 we can write $(\mathscr{E}, q) \cong (\mathscr{E}_1, q_1) \perp <\pi>$ with $\pi \in R$ (not necessarily nonzero) and $(\mathscr{E}_1, q_1) \cong <1, -1, \ldots, 1, -1, 1>$ a standard split quadratic form of odd rank. We have that $\mathscr{C}_0(\mathscr{E}_1, q_1)$ is (split) Azumaya over $\mathscr{O}_S$ and that $\mathscr{C}(<-\pi>)$ is $\mathscr{O}_S$-isomorphic to $\mathscr{Z}(\mathscr{E}, q)$.
Since $\mathscr{C}(<-\pi>) \cong R[\sqrt{-\pi}]$ is finite flat of degree two over S, the first claim is verified. For the second claim, by [40, IV Prop. 7.3.1], there are then $\mathscr{O}_S$-algebra isomorphisms

$$\mathscr{C}_0(\mathscr{E}, q) \cong \mathscr{C}_0(\mathscr{E}_1, q_1) \otimes_{\mathscr{O}_S} \mathscr{C}(<-\pi>) \cong \mathscr{C}_0(\mathscr{E}_1, q_1) \otimes_{\mathscr{O}_S} \mathscr{Z}(\mathscr{E}, q). \tag{5}$$

Thus étale locally, $\mathscr{C}_0(\mathscr{E}, q)$ is the base extension to $\mathscr{Z}(\mathscr{E}, q)$ of an Azumaya algebra over $\mathscr{O}_S$, hence can be regarded as an Azumaya $\mathscr{O}_T$-algebra. ☐

Over a field, we can now provide a strengthened version of [28, Prop. 11.6].

PROPOSITION 1.12. *Let (V, q) be a quadratic form of even rank $n = 2m$ over a field k of characteristic $\neq 2$. If $n \geq 4$ then the following are equivalent:*

a) The radical of q has rank at most 1.

b) The center $Z(q) \subset C_0(q)$ is a k-algebra of rank 2.

c) The algebra $C_0(q)$ is $Z(q)$-Azumaya of degree 2^{m-1}.

If $n = 2$, then $C_0(q)$ is always commutative.

Proof. If q is nondegenerate (i.e., has trivial radical), then it is classical that $Z(q)$ is an étale quadratic algebra and $C_0(q)$ is an Azumaya $Z(q)$-algebra. If $\mathrm{rad}(q)$ has rank 1, generated by $v \in V$, then a straightforward computation shows that $Z(q) \cong k[\varepsilon]/(\epsilon^2)$, where $\epsilon \in vC_1(q) \cap Z(q) \smallsetminus k$. Furthermore, we have that $C_0(q) \otimes_{k[\varepsilon]/(\varepsilon^2)} k \cong C_0(q)/vC_1(q) \cong C_0(q/\mathrm{rad}(q))$ where $q/\mathrm{rad}(q)$ is nondegenerate of rank $n-1$, cf. [28, II §11, p. 58]. Proposition 1.11 implies that $C_0(q)$ is $Z(q)$-Azumaya of degree 2^{m-1}, proving *a)* $\Rightarrow$ *c)*
The fact that *c)* $\Rightarrow$ *b)* is clear from a dimension count. To prove *b)* $\Rightarrow$ *a)*, suppose that $\mathrm{rk}_k \,\mathrm{rad}(q) \geq 2$. In this case, the embedding $\bigwedge^2 \mathrm{rad}(q) \subset C_0(q)$ is central (and does not contain the central subalgebra generated by $V^{\otimes n}$, as q has rank > 2). More explicitly, if $e_1, e_2, \ldots, e_n$ is an orthogonal basis of (V, q), then $k \oplus ke_1 \cdots e_n \oplus \bigwedge^2 \mathrm{rad}(q) \subset Z(q)$. Thus $Z(q)$ has k-rank at least $2 + \mathrm{rk}_k \bigwedge^2 \mathrm{rad}(q) \geq 3$. □

Finally, as a corollary of Proposition 1.12, we can deduce a converse to Proposition 1.11.

PROPOSITION 1.13. *Let $(\mathscr{E}, q, \mathscr{L})$ be a quadratic form of even rank on an integral scheme S with discriminant cover $f : T \to S$. Then $\mathscr{C}_0(\mathscr{E}, q, \mathscr{L})$ is an Azumaya $\mathscr{O}_T$-algebra if and only if $(\mathscr{E}, q, \mathscr{L})$ has simple degeneration or has rank 2 (and any degeneration).*

2 ORTHOGONAL GROUPS WITH SIMPLE DEGENERATION

The main results of this section concern the special (projective) orthogonal group schemes of quadratic forms with simple degeneration over semilocal principal ideal domains.
Let S be a regular integral scheme. Recall, from Proposition 1.11, that if $(\mathscr{E}, q, \mathscr{L})$ is a line bundle-valued quadratic form on S with simple degeneration along a closed subscheme D of codimension 1, then the even Clifford algebra $\mathscr{C}_0(q)$ is an Azumaya algebra over the discriminant cover $T \to S$. The main result of this section is the following.

Theorem 2.1. *Let S be a regular scheme with 2 invertible, D a regular divisor, $(\mathscr{E}, q, \mathscr{L})$ a quadratic form of rank 4 on S with simple degeneration along D, $T \to S$ its discriminant cover, and $\mathscr{C}_0(q)$ its even Clifford algebra over T. The canonical homomorphism*

$$c : \mathbf{PGO}^+(q) \to R_{T/S}\mathbf{PGL}(\mathscr{C}_0(q)),$$

induced from the functor $\mathscr{C}_0$, is an isomorphism of S-group schemes.

The proof involves several preliminary general results concerning orthogonal groups of quadratic forms with simple degeneration and will occupy the remainder of this section.

Let $S = \operatorname{Spec} R$ be an affine scheme with 2 invertible, $D \subset S$ be the closed scheme defined by an element π in the Jacobson radical of R, and let $(V, q) = (V_1, q_1) \perp (R, <\pi>)$ be a quadratic form of rank n over S with q_1 regular and V_1 free. Let Q_1 be a Gram matrix of q_1. Then as an S-group scheme, $\mathbf{O}(q)$ is the subvariety of the affine space of block matrices

$$\begin{pmatrix} A & v \\ w & u \end{pmatrix} \quad \text{satisfying} \quad \begin{array}{l} A^t Q_1 A + \pi\, w^t w = Q_1 \\ A^t Q_1 v + u\pi\, w^t = 0 \\ v^t Q_1 v = (1-u^2)\pi \end{array} \tag{6}$$

where A is an invertible $(n-1) \times (n-1)$ matrix, v is an $n \times 1$ column vector, w is a $1 \times n$ row vector, and u a unit. Note that since A and Q_1 are invertible, the second relation in (6) implies that v is determined by w and u and that $\bar{v} = 0$ over R/π. In particular, if $\pi \neq 0$ and R is a domain then the third relation implies that $\bar{u}^2 = 1$ in R/π. Define $\mathbf{O}^+(q) = \ker(\det : \mathbf{O}(q) \to \mathbf{G}_{\mathrm{m}})$. If R is an integral domain then det factors through $\boldsymbol{\mu}_2$ and $\mathbf{O}^+(q)$ is the irreducible component of the identity.

Proposition 2.2. *Let R be a regular local ring with 2 invertible, $\pi \in \mathfrak{m}$ a nonzero element in the maximal ideal, and $(V, q) = (V_1, q_1) \perp (R, <\pi>)$ a quadratic form with q_1 regular of rank $n-1$ of R. Then $\mathbf{O}(q)$ and $\mathbf{O}^+(q)$ are smooth R-group schemes.*

Proof. Let K be the fraction field of R and k its residue field. First, we'll show that the equations in (6) define a local complete intersection morphism in the affine space $\mathbb{A}_R^{n^2}$ of $n \times n$ matrices over R. Indeed, the condition that the generic $n \times n$ matrix M over $R[x_1, \ldots, x_{n^2}]$ is orthogonal with respect to a given symmetric $n \times n$ matrix Q over R can be written as the equality of symmetric matrices $M^t Q M = Q$ over $R[x_1, \ldots, x_{n^2}][(\det M)^{-1}]$, hence giving $n(n+1)/2$ equations. Hence, the orthogonal group is the scheme defined by these $n(n+1)/2$ equations in the Zariski open of $\mathbb{A}_R^{n^2}$ defined by $\det M$.
Since q is generically regular of rank n, the generic fiber of $\mathbf{O}(q)$ has dimension $n(n-1)/2$. By (6), the special fiber of $\mathbf{O}^+(q)$ is isomorphic to the group scheme of rigid motions of the regular quadratic space (V_1, q_1), which is the semidirect product

$$\mathbf{O}^+(q) \times_R k \cong \mathbf{G}_{\mathrm{a}}^{n-1} \rtimes \mathbf{O}(q_{1,k}) \tag{7}$$

where $\mathbf{G}_{\mathrm{a}}^{n-1}$ acts in V_1 by translation and $\mathbf{O}(q_{1,k})$ acts on $\mathbf{G}_{\mathrm{a}}^{n-1}$ by conjugation. In particular, the special fiber of $\mathbf{O}^+(q)$ has dimension $(n-1)(n-2)/2+(n-1) = n(n-1)/2$, and similarly with $\mathbf{O}(q)$.

In particular, $\mathbf{O}(q)$ is a local complete intersection morphism. Since R is Cohen–Macaulay (being regular local) then $R[x_1,\dots,x_{n^2}][(\det M)^{-1}]$ is Cohen–Macaulay, and thus $\mathbf{O}(q)$ is Cohen–Macaulay. By the "miracle flatness" theorem, equidimensional and Cohen–Macaulay over a regular base implies that $\mathbf{O}(q) \to \operatorname{Spec} R$ is flat, cf. [32, Prop. 15.4.2] or [46, 8 Thm. 23.1]. Also $\mathbf{O}^+(q) \to \operatorname{Spec} R$ is flat. The generic fiber of $\mathbf{O}^+(q)$ is smooth since q is generically regular while the special fiber is smooth since it is a (semi)direct product of smooth schemes (recall that $\mathbf{O}(q_1)$ is smooth since 2 is invertible). Hence $\mathbf{O}^+(q) \to \operatorname{Spec} R$ is flat and has geometrically smooth fibers, hence is smooth. □

Proposition 2.3. *Let S be a regular scheme with 2 invertible and $(\mathscr{E}, q, \mathscr{L})$ a quadratic form of even rank on S with simple degeneration. Then the group schemes $\mathbf{O}(q)$, $\mathbf{O}^+(q)$, $\mathbf{GO}(q)$, $\mathbf{GO}^+(q)$, $\mathbf{PGO}(q)$, and $\mathbf{PGO}^+(q)$ are S-smooth. If $T \to S$ is the discriminant cover and $\mathscr{C}_0(q)$ is the even Clifford algebra of $(\mathscr{E}, q, \mathscr{L})$ over T, then $R_{T/S}\mathbf{GL}_1(\mathscr{C}_0(q))$, $R_{T/S}\mathbf{SL}_1(\mathscr{C}_0(q))$, and $R_{T/S}\mathbf{PGL}_1(\mathscr{C}_0(q))$ are smooth S-schemes.*

Proof. The S-smoothness of $\mathbf{O}(q)$ and $\mathbf{O}^+(q)$ follows from the fibral criterion for smoothness, with Proposition 2.2 handling points of S contained in the discriminant divisor. As $\mathbf{GO} \cong (\mathbf{O}(q) \times \mathbf{G}_{\mathrm{m}})/\boldsymbol{\mu}_2$, $\mathbf{GO}^+(q) \cong (\mathbf{O}^+(q) \times \mathbf{G}_{\mathrm{m}})/\boldsymbol{\mu}_2$, $\mathbf{PGO}(q) \cong \mathbf{GO}(q)/\mathbf{G}_{\mathrm{m}}$, $\mathbf{PGO}^+(q) \cong \mathbf{GO}^+(q)/\mathbf{G}_{\mathrm{m}}$ are quotients of S-smooth group schemes by flat closed subgroups, they are S-smooth. Finally, $\mathscr{C}_0(q)$ is an Azumaya $\mathscr{O}_T$-algebra by Proposition 1.11, hence $\mathbf{GL}_1(\mathscr{C}_0(q))$, $\mathbf{SL}_1(\mathscr{C}_0(q))$, and $\mathbf{PGL}_1(\mathscr{C}_0(q))$ are smooth T-schemes, hence their Weil restrictions via the finite flat map $T \to S$ are S-smooth by [21, App. A.5, Prop. A.5.2]. □

Remark 2.4. If the radical of q_s has rank ≥ 2 at a point s of S, a calculation shows that the fiber of $\mathbf{O}(q) \to S$ over s has dimension $> n(n-1)/2$. In particular, if q is generically regular over S then $\mathbf{O}(q) \to S$ is not flat. The smoothness of $\mathbf{O}(q)$ is a special feature of quadratic forms q with simple degeneration. Over a complete discretely valued ring, such $\mathbf{O}^+(q)$ can be viewed as an explicit model for one of the quasisplit Bruhat–Tits groups of type ${}^2\mathsf{D}_m$.

We will also make frequent reference to the classical version of Theorem 2.1 in the regular case, when the discriminant cover is étale.

Theorem 2.5. *Let S be a scheme and $(\mathscr{E}, q, \mathscr{L})$ a regular quadratic form of rank 4 with discriminant cover $T \to S$ and even Clifford algebra $\mathscr{C}_0(q)$ over T. The canonical homomorphism*

$$c : \mathbf{PGO}^+(q) \to R_{T/S}\mathbf{PGL}_1(\mathscr{C}_0(q)),$$

induced from the functor $\mathscr{C}_0$, is an isomorphism of S-group schemes.

Proof. The proof over affine schemes S in [43, §10] carries over immediately. See [41, IV.15.B] for the particular case of S the spectrum of a field. Also see [3, §5.3]. □

Finally, we come to the proof of the main result of this section.

Proof of Theorem 2.1. We will use the following fibral criteria for relative isomorphisms (cf. [32, IV.4 Cor. 17.9.5]): let $g : X \to Y$ be a morphism of S-schemes locally of finite presentation over a scheme S and assume X is S-flat, then g is an S-isomorphism if and only if its fiber $g_s : X_s \to Y_s$ is an isomorphism over each geometric point s of S.
For each s in $S \smallsetminus D$, the fiber q_s is a regular quadratic form over $\kappa(s)$, hence the fiber $c_s : \mathbf{PGO}^+(q_s) \to R_{T/S}\mathbf{PGL}(\mathscr{C}_0(q_s))$ is an isomorphism by Theorem 2.5. We are thus reduced to considering the geometric fibers over points in D. Let $s = \operatorname{Spec} k$ be a geometric point of D. By Proposition 1.12, there is a natural identification of the fiber $T_s = \operatorname{Spec} k_\epsilon$, where $k_\epsilon = k[\epsilon]/(\epsilon^2)$.
We use the following criteria for isomorphisms of group schemes (cf. [41, VI Prop. 22.5]): let $g : X \to Y$ be a homomorphism of affine k-group schemes of finite type over an algebraically closed field k and assume that Y is smooth, then g is a k-isomorphism if and only if $g : X(k) \to Y(k)$ is an isomorphism on k-points and the Lie algebra map $\mathrm{d}g : \mathrm{Lie}(X) \to \mathrm{Lie}(Y)$ is an injective map of k-vector spaces.
First, we shall prove that c is an isomorphism on k points. Applying cohomology to the exact sequence

$$1 \to \boldsymbol{\mu}_2 \to \mathbf{O}^+(q) \to \mathbf{PGO}^+(q) \to 1,$$

we see that the corresponding sequence of k-points is exact since k is algebraically closed. Hence it suffices to show that $\mathbf{O}^+(q)(k) \to \mathbf{PGL}_1(\mathscr{C}_0(q))(k)$ is surjective with kernel $\boldsymbol{\mu}_2(k)$.
Write $q = q_1 \perp\!<0>$, where q_1 is regular over k. Denote by $\mathbf{E}$ the unipotent radical of $\mathbf{O}^+(q)$. We will now proceed to define the following diagram

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \boldsymbol{\mu}_2 & = & \boldsymbol{\mu}_2 & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbf{E} & \longrightarrow & \mathbf{O}^+(q) & \longrightarrow & \mathbf{O}(q_1) & \longrightarrow & 1 \\
 & & \vdots & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & I + \epsilon\,\mathfrak{c}_0(q) & \longrightarrow & \mathbf{PGL}_1(\mathscr{C}_0(q)) & \longrightarrow & \mathbf{PGL}_1(\mathscr{C}_0(q_1)) & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

of groups schemes over k, and verify that it is commutative with exact rows and columns. This will finish the proof of the statement concerning c being an isomorphism on k-points. We have $H^1_{\acute{e}t}(k, \mathbf{E}) = 0$ and also $H^1_{\acute{e}t}(k, \boldsymbol{\mu}_2) = 0$, as k is algebraically closed. Hence it suffices to argue after taking k-points in the diagram.

The central and right most vertical columns are induced by the standard action of the (special) orthogonal group on the even Clifford algebra. The right most column is an exact sequence

$$1 \to \boldsymbol{\mu}_2 \to \mathbf{O}(q_1) \cong \boldsymbol{\mu}_2 \times \mathbf{O}^+(q_1) \to \mathbf{PGL}_1(\mathscr{C}_0(q_1)) \to 1$$

arising from the split isogeny of type $\mathsf{A}_1 = \mathsf{B}_1$, cf. [41, IV.15.A]. The central row is defined by the map $\mathbf{O}^+(q)(k) \to \mathbf{O}(q_1)(k)$ defined by

$$\begin{pmatrix} A & v \\ w & u \end{pmatrix} \mapsto A$$

in the notation of (6). In particular, the group $\mathbf{E}(k)$ consists of block matrices of the form

$$\begin{pmatrix} I & 0 \\ w & 1 \end{pmatrix}$$

for $w \in \mathbb{A}^3(k)$. Since $\mathbf{O}(q_1)$ is semisimple, the kernel contains the unipotent radical $\mathbf{E}$, so coincides with it by a dimension count. The bottom row is defined as follows. By (5), we have $\mathscr{C}_0(q) \cong \mathscr{C}_0(q_1) \otimes_k \mathscr{Z}(q) \cong \mathscr{C}_0(q_1) \otimes_k k_\epsilon$. The map $\mathbf{PGL}_1(\mathscr{C}_0(q)) \to \mathbf{PGL}_1(\mathscr{C}_0(q_1)$ is thus defined by the reduction $k_\epsilon \to k$. This also identifies the kernel as $I + \epsilon\, \mathfrak{c}_0(q)$, where $\mathfrak{c}_0(q)$ is the affine scheme of reduced trace zero elements of $\mathscr{C}_0(q)$, which is identified with the Lie algebra of $\mathbf{PGL}_1(\mathscr{C}_0(q))$ in the usual way. The only thing to check is that the bottom left square commutes (since by (7), the central row is split). By the five lemma, it will then suffice to show that $\mathbf{E}(k) \to 1 + \epsilon\, \mathfrak{c}_0(q)(k)$ is an isomorphism.

To this end, we can diagonalize $q =< 1, -1, 1, 0 >$, since k is algebraically closed of characteristic $\neq 2$. Let $e_1, \dots, e_4$ be the corresponding orthogonal basis. Then $\mathscr{C}_0(q_1)(k)$ is generated over k by 1, e_1e_2, e_2e_3, and e_1e_3 and we have an identification $\varphi : \mathscr{C}_0(q_1)(k) \to M_2(k)$ given by

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1e_2 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2e_3 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_1e_3 \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Similarly, $\mathscr{C}_0(q)$ is generated over $\mathscr{Z}(q) = k_\epsilon$ by 1, e_1e_2, e_2e_3, and e_1e_3, since we have

$$e_1e_4 = \epsilon\, e_2e_3, \quad e_2e_4 = \epsilon\, e_1e_3, \quad e_3e_4 = \epsilon\, e_1e_2, \quad e_1e_2e_3e_4 = \epsilon.$$

and we have an identification $\psi : \mathscr{C}_0(q) \to M_2(k_\epsilon)$ extending φ. With respect to this k_ϵ-algebra isomorphism, we have a group isomorphism $\mathbf{PGL}_1(\mathscr{C}_0(q))(k) = \mathbf{PGL}_2(k_\epsilon)$ and a Lie algebra isomorphism $\mathfrak{c}_0(q)(k) \cong \mathfrak{sl}_2(k)$, where $\mathfrak{sl}_2$ is the scheme of traceless 2×2 matrices. We claim that the map $\mathbf{E}(k) \to I + \epsilon\, \mathfrak{sl}_2(k)$

is explicitly given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a & b & c & 1 \end{pmatrix} \mapsto I - \frac{1}{2}\epsilon \begin{pmatrix} a & -b+c \\ b+c & -a \end{pmatrix}. \tag{8}$$

Indeed, let $\phi_{a,b,c} \in \mathbf{E}(S_0)$ be the orthogonal transformation whose matrix is displayed in (8), and $\sigma_{a,b,c}$ its image in $I + \epsilon\,\mathfrak{sl}_2(k)$, thought of as an automorphism of $\mathscr{C}_0(q)(k_\epsilon)$. Then we have

$$\begin{aligned} \sigma_{a,b,c}(e_1e_2) &= e_1e_2 + b\epsilon\, e_2e_3 - a\epsilon\, e_1e_3 \\ \sigma_{a,b,c}(e_2e_3) &= e_2e_3 + c\epsilon\, e_1e_3 - b\epsilon\, e_1e_2 \\ \sigma_{a,b,c}(e_1e_3) &= e_1e_3 + c\epsilon\, e_2e_3 - a\epsilon\, e_1e_2 \end{aligned}$$

and $\sigma_{a,b,c}(\epsilon) = \epsilon$. It is then a straightforward calculation to see that

$$\sigma_{a,b,c} = \mathrm{ad}\big(1 - \frac{1}{2}\epsilon(c\, e_1e_2 + a\, e_2e_3 - b\, e_1e_3)\big),$$

where ad is conjugation in the Clifford algebra, and furthermore, that ψ takes $c\, e_1e_2 + a\, e_2e_3 - b\, e_1e_3$ to the 2×2 matrix displayed in (8). Thus the map $\mathbf{E}(k) \to I + \epsilon\,\mathfrak{sl}_2(k)$ is as stated, and in particular, is an isomorphism. Thus the diagram is commutative with exact rows and columns, and in particular, $c : \mathbf{PGO}^+(q) \to \mathbf{PGL}_1(\mathscr{C}_0(q))$ is an isomorphism on k-points.

Now we prove that the Lie algebra map dc is injective. Consider the commutative diagram

$$\begin{array}{ccccccccc} 1 & \to & I + x\,\mathfrak{so}(q)(k) & \longrightarrow & \mathbf{O}^+(q)(k[x]/(x^2)) & \longrightarrow & \mathbf{O}^+(q)(k) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle 1+x\,\mathrm{d}c} & & \downarrow{\scriptstyle c(k[x]/(x^2))} & & \downarrow & & \\ 1 & \longrightarrow & I + x\,\mathfrak{g}(k) & \longrightarrow & \mathbf{PGL}_1(\mathscr{C}_0(q))(k[\epsilon, x]/(\epsilon^2, x^2)) & \to & \mathbf{PGL}_1(\mathscr{C}_0(q))(k_\epsilon) & \to & 1 \end{array}$$

where $\mathfrak{so}(q)$ and $\mathfrak{g}$ are the Lie algebras of $\mathbf{O}^+(q)$ and $R_{k_\epsilon/k}\mathbf{PGL}_1(\mathscr{C}_0(q))$, respectively.

The Lie algebra $\mathfrak{so}(q_1)$ of $\mathbf{O}(q_1)$ is identified with the scheme of 3×3 matrices A such that AQ_1 is skew-symmetric, where $Q_1 = \mathrm{diag}(1, -1, 1)$. It is then a consequence of (6) that $I + x\,\mathfrak{so}(q)(k)$ consists of block matrices of the form

$$\begin{pmatrix} I + xA & 0 \\ xw & 1 \end{pmatrix}$$

for $w \in \mathbb{A}^3(k)$ and $A \in \mathfrak{so}(q_1)(k)$. Since

$$\begin{pmatrix} I + xA & 0 \\ xw & 1 \end{pmatrix} = \begin{pmatrix} I + xA & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I & 0 \\ xw & 1 \end{pmatrix} = \begin{pmatrix} I & 0 \\ xw & 1 \end{pmatrix} \begin{pmatrix} I + xA & 0 \\ 0 & 1 \end{pmatrix},$$

we see that $I + x\,\mathfrak{so}(q)$ has a direct product decomposition $\mathbf{E} \times \big(I + x\,\mathfrak{so}(q_1)\big)$. We claim that the map $\mathfrak{h} \to \mathfrak{g}$ is explicitly given by the product map

$$\begin{pmatrix} I + xA & 0 \\ xw & 1 \end{pmatrix} \mapsto \big(I - \epsilon\beta(xw)\big)\big(I - \alpha(xA)\big) = I - x(\alpha(A) + \epsilon\beta(w))$$

where $\alpha : \mathfrak{so}(q_1) \to \mathfrak{sl}_2$ is the Lie algebra isomorphism

$$\begin{pmatrix} 0 & a & -b \\ a & 0 & c \\ b & c & 0 \end{pmatrix} \mapsto \frac{1}{2}\begin{pmatrix} a & -b+c \\ b+c & -a \end{pmatrix}$$

induced from the isomorphism $\mathbf{PSO}(q_1) \cong \mathbf{PGL}_2$ and $\beta : \mathbb{A}^3 \to \mathfrak{sl}_2$ is the Lie algebra isomorphism

$$(a \;\; b \;\; c) \mapsto \frac{1}{2}\begin{pmatrix} a & -b+c \\ b+c & -a \end{pmatrix}$$

as above. Thus $\mathrm{d}c : \mathfrak{so}(q) \to \mathfrak{g}$ is an isomorphism. ☐

Remark 2.6. The isomorphism of algebraic groups in the proof of Theorem 2.1 can be viewed as a degeneration of an isomorphism of semisimple groups of type ${}^2\mathsf{A}_1 = \mathsf{D}_2$ (on the generic fiber) to an isomorphism of nonreductive groups whose semisimplification has type $\mathsf{A}_1 = \mathsf{B}_1 = \mathsf{C}_1$ (on the special fiber).

3 Simple degeneration over semi-local rings

The semilocal ring R of a normal scheme at a finite set of points of codimension 1 is a semilocal Dedekind domain, hence a principal ideal domain. Let R_i denote the (finitely many) discrete valuation overrings of R contained in the fraction field K (the localizations at the height one prime ideals), $\widehat{R}_i$ their completions, and $\widehat{K}_i$ their fraction fields. If $\widehat{R}$ is the completion of R at its Jacobson radical $\mathrm{rad}(R)$ and $\widehat{K}$ the total ring of fractions, then $\widehat{R} \cong \prod_i \widehat{R}_i$ and $\widehat{K} \cong \prod_i \widehat{K}_i$. We call an element $\pi \in R$ a *parameter* if $\pi = \prod_i \pi_i$ is a product of parameters π_i of R_i.
We first recall a well-known result, cf. [17, §2.3.1].

Lemma 3.1. *Let R be a semilocal principal ideal domain and K its field of fractions. Let q be a regular quadratic form over R and $u \in R^\times$ a unit. If u is represented by q over K then it is represented by q over R.*

We now provide a generalization of Lemma 3.1 to the case of simple degeneration.

Proposition 3.2. *Let R be a semilocal principal ideal domain with 2 invertible and K its field of fractions. Let q be a quadratic form over R with simple degeneration of multiplicity one and let $u \in R^\times$ be a unit. If u is represented by q over K then it is represented by q over R.*

For the proof, we'll first need to generalize, to the degenerate case, some standard results concerning regular forms. If (V, q) is a quadratic form over a ring R and $v \in V$ is such that $q(v) = u \in R^\times$, then the *reflection* $r_v : V \to V$ through v given by

$$r_v(w) = w - u^{-1} b_q(v, w)\, v$$

is an isometry over R satisfying $r_v(v) = -v$ and $r_v(w) = w$ if $w \in v^\perp$.

Lemma 3.3. *Let R be a semilocal ring with 2 invertible. Let (V, q) be a quadratic form over R and $u \in R^\times$. Then $\mathbf{O}(V, q)(R)$ acts transitively on the set of vectors $v \in V$ such that $q(v) = u$.*

Proof. Let $v, w \in V$ be such that $q(v) = q(w) = u$. We first prove the lemma over any local ring with 2 invertible. Without loss of generality, we can assume that $q(v-w) \in R^\times$. Indeed, $q(v+w)+q(v-w) = 4u \in R^\times$ so that, since R is local, either $q(v+w)$ or $q(v-w)$ is a unit. If $q(v-w)$ is not a unit, then $q(v+w)$ is and we can replace w by $-w$ using the reflection r_w. Finally, by a standard computation, we have $r_{v-w}(v) = w$. Thus any two vectors representing u are related by a product of at most two reflection.
For a general semilocal ring, the quotient $R/\mathrm{rad}(R)$ is a product of fields. By the above argument, $\overline{v}$ can be transported to $-\overline{w}$ in each component by a product $\overline{\tau}$ of at most two reflections. By the Chinese remainder theorem, we can lift $\overline{\tau}$ to a product of at most two reflections τ of (V, q) transporting v to $-w + z$ for some $z \in \mathrm{rad}(R) \otimes_R V$. Replacing v by $-w + z$, we can assume that $v + w = z \in \mathrm{rad}(R) \otimes_R V$. Finally, $q(v + w) + q(v - w) = 4u$ and $q(v + w) \in \mathrm{rad}(R)$, thus $q(v - w)$ is a unit. As before, $r_{v-w}(v) = w$. ☐

Corollary 3.4. *Let R be a semilocal ring with 2 invertible. Then regular forms can be canceled, i.e., if q_1 and q_2 are quadratic forms and q a regular quadratic form over R with $q_1 \perp q \cong q_2 \perp q$, then $q_1 \cong q_2$.*

Proof. Regular quadratic forms over a semilocal ring with 2 invertible are diagonalizable. Hence we can reduce to the case of rank one form $q = (R, <u>)$ for $u \in R^\times$. Let $\varphi : q_1 \perp (R\,w_1, <u>) \cong q_2 \perp (R\,w_2, <u>)$ be an isometry. By Lemma 3.3, there is an isometry ψ of $q_2 \perp (R\,w_2, <u>)$ taking $\varphi(w_1)$ to w_2, so that $\psi \circ \varphi$ takes w_1 to w_2. By taking orthogonal complements, φ thus induces an isometry $q_1 \cong q_2$. ☐

Lemma 3.5. *Let R be a complete discrete valuation ring with 2 invertible and K its fraction field. Let q be a quadratic form with simple degeneration of multiplicity one and let $u \in R^\times$. If u is represented by q over K then it is represented by q over R.*

Proof. For a choice of parameter π of R, write $(V, q) = (V_1, q_1) \perp (R\,e, <\pi>)$ with q_1 regular. There are two cases, depending on whether $\overline{q}_1$ is isotropic over the residue field. First, if $\overline{q}_1$ is anisotropic, then q_1 only takes values with even valuation. Let $v \in V_K$ satisfy $q|_K(v) = u$ and write $v = \pi^n v_1 + a\pi^m e$ with $v_1 \in V_1$ such that $\overline{v}_1 \neq 0$ and $a \in R^\times$. Then we have $\pi^{2n} q_1(v_1) + a\pi^{2m+1} = u$.

By parity considerations, we see that $n = 0$ and $m \geq 0$ are forced, thus $v \in V$ and u is represented by q. Second, R being complete, if $\overline{q}_1$ is isotropic, then it splits off a hyperbolic plane, so represents u. □

We now recall the theory of elementary hyperbolic isometries initiated by Eichler [27, Ch. 1] and developed in the setting of regular quadratic forms over rings by Wall [61, §5] and Roy [54, §5]. See also [48], [50], [56], and [8, III §2]. We will need to develop the theory for quadratic forms that are not necessarily regular.

Let R be a ring with 2 invertible, (V, q) a quadratic form over R, and (R^2, h) the hyperbolic plane with basis e, f. For $v \in V$, define E_v and E_v^* in $\mathbf{O}(q \perp h)(R)$ by

$$\begin{aligned} E_v(w) &= w + b(v,w)e & E_v^*(w) &= w + b(v,w)f, \quad \text{for } w \in V \\ E_v(e) &= e & E_v^*(e) &= -v - 2^{-1}q(v)f + e \\ E_v(f) &= -v - 2^{-1}q(v)e + f & E_v^*(f) &= f. \end{aligned}$$

Define the group of *elementary hyperbolic isometries* $\mathbf{EO}(q,h)(R)$ to be the subgroup of $\mathbf{O}(q \perp h)(R)$ generated by E_v and E_v^* for $v \in V$.

For $u \in R^\times$, define $\alpha_u \in \mathbf{O}(h)(R)$ by

$$\alpha_u(e) = ue, \qquad \alpha_u(f) = u^{-1}f$$

and $\beta_u \in \mathbf{O}(h)(R)$ by

$$\beta_u(e) = u^{-1}f, \qquad \beta_u(f) = ue.$$

Then $\mathbf{O}(h)(R) = \{\alpha_u : u \in R^\times\} \cup \{\beta_u : u \in R^\times\}$. One can verify the following identities:

$$\begin{aligned} \alpha_u^{-1}E_v\alpha_u &= E_{u^{-1}v}, & \beta_u^{-1}E_v\beta_u &= E_v^*, \\ \alpha_u^{-1}E_v^*\alpha_u &= E_{u^{-1}v}^*, & \alpha_u^{-1}E_v^*\alpha_u &= E_v. \end{aligned}$$

Thus $\mathbf{O}(h)(R)$ normalizes $\mathbf{EO}(q,h)(R)$.

If $R = K$ is a field and q is nondegenerate, then $\mathbf{EO}(q,h)(K)$ and $\mathbf{O}(h)(K)$ generate $\mathbf{O}(q \perp h)(K)$ (see [27, ch. 1]) so that

$$\mathbf{O}(q \perp h)(K) = \mathbf{EO}(q,h)(K) \rtimes \mathbf{O}(h)(K). \tag{9}$$

Proposition 3.6. *Let R be a semilocal principal ideal domain with 2 invertible and K its fraction field. Let $\widehat{R}$ be the completion of R at the radical and $\widehat{K}$ its fraction field. Let (V,q) be a quadratic form over R that is nondegenerate over K. Then every element $\varphi \in \mathbf{O}(q \perp h)(\widehat{K})$ is a product $\varphi_1\varphi_2$, where $\varphi_1 \in \mathbf{O}(q \perp h)(K)$ and $\mathbf{O}(q \perp h)(\widehat{R})$.*

Proof. We follow portions of the proof in [50, Prop. 3.1]. As topological rings, $\widehat{R}$ is open in $\widehat{K}$, and hence as topological groups, $\mathbf{O}(q \perp h)(\widehat{R})$ is open inside $\mathbf{O}(q \perp h)(\widehat{K})$. In particular, $\mathbf{O}(q \perp h)(\widehat{R}) \cap \mathbf{EO}(q,h)(\widehat{K})$ is open in $\mathbf{EO}(q,h)(\widehat{K})$. Since R is dense in $\widehat{R}$, K is dense in $\widehat{K}$, $V \otimes_R K$ is dense in $V \otimes_R \widehat{K}$, and hence $\mathbf{EO}(q,h)(K)$ is dense in $\mathbf{EO}(q,h)(\widehat{K})$.
Thus, by topological considerations, every element φ' of $\mathbf{EO}(q \perp h)(\widehat{K})$ is a product $\varphi'_1\varphi'_2$, where $\varphi'_1 \in \mathbf{EO}(q,h)(K)$ and $\varphi'_2 \in \mathbf{EO}(q,h)(\widehat{K}) \cap \mathbf{O}(q \perp h)(\widehat{R})$. Clearly, every element γ of $\mathbf{O}(h)(\widehat{K})$ is a product $\gamma_1\gamma_2$, where $\gamma_1 \in \mathbf{O}(h)(K)$ and $\gamma_2 \in \mathbf{O}(h)(\widehat{R})$.
The form $q \perp h$ is nondegenerate over $\widehat{K}$, so by (9), every $\varphi \in \mathbf{O}(q \perp h)(\widehat{K})$ is a product $\varphi'\gamma$, where $\varphi' \in \mathbf{EO}(q,h)(\widehat{K})$ and $\gamma \in \mathbf{O}(h)(\widehat{K})$. As above, we can write

$$\varphi = \varphi'\gamma = \varphi'_1\varphi'_2\gamma_1\gamma_2 = \varphi'_1\gamma_1(\gamma_1^{-1}\varphi'_2\gamma_1)\gamma_2.$$

Since $\mathbf{EO}(q,h)(\widehat{K})$ is a normal subgroup, $\gamma_1^{-1}\varphi'_2\gamma_1 \in \mathbf{EO}(q,h)(\widehat{K})$ and is thus a product $\psi_1\psi_2$, where $\psi_1 \in \mathbf{EO}(q,h)(K)$ and $\psi_2 \in \mathbf{EO}(q,h)(\widehat{K}) \cap \mathbf{O}(q \perp h)(\widehat{R})$. Finally, φ is a product $(\varphi'_1\gamma_1\psi_1)(\psi_2\gamma_2)$ of the desired form. □

Proof of Proposition 3.2. Let $\widehat{R}$ be the completion of R at the radical and $\widehat{K}$ the total ring of fractions. As $q|_K$ represents u, we have a splitting $q|_K \cong q_1 \perp <u>$. We have that $q|_{\widehat{R}} = \prod_i q|_{\widehat{R}_i}$ represents u over $\widehat{R} = \prod_i \widehat{R}_i$, by Lemma 3.5, since u is represented over $\widehat{K} = \prod_i \widehat{K}_i$. We thus have a splitting $q|_{\widehat{R}} \cong q_2 \perp <u>$. By Witt cancellation over $\widehat{K}$, we have an isometry $\varphi : q_1|_{\widehat{K}} \cong q_2|_{\widehat{K}}$, which by patching defines a quadratic form $\tilde{q}$ over R such that $\tilde{q}_K \cong q_1$ and $\tilde{q}|_{\widehat{R}} \cong q_2$. We claim that $q \perp <-u> \cong \tilde{q} \perp h$. Indeed, as $h \cong <u, -u>$, we have isometries

$$\psi^K : (q \perp <-u>)_K \cong (\tilde{q} \perp h)_K, \qquad \psi^{\widehat{R}} : (q \perp <-u>)_{\widehat{R}} \cong (\tilde{q} \perp h)_{\widehat{R}}.$$

By Proposition 3.6, there exists $\theta_1 \in \mathbf{O}(\tilde{q} \perp h)(\widehat{R})$ and $\theta_2 \in \mathbf{O}(\tilde{q} \perp h)(K)$ such that $\psi^{\widehat{R}}(\psi^K)^{-1} = \theta_1^{-1}\theta_2$. The isometries $\theta_1\psi^{\widehat{R}}$ and $\theta_2\psi^K$ then agree over $\widehat{K}$ and so patch to yield an isometry $\psi : q \perp <-u> \cong \tilde{q} \perp h$.
As $h \cong <u, -u>$, we have $q \perp <-u> \cong \tilde{q} \perp <u, -u>$. By Corollary 3.4, we can cancel the regular form $<-u>$, so that $q \cong \tilde{q} \perp <u>$. Thus q represents u over R. □

Lemma 3.7. *Let R be a discrete valuation ring and (E, q) a quadratic form of rank n over R with simple degeneration. If q represents $u \in R^\times$ then it can be diagonalized as $q \cong <u, u_2, \ldots, u_{n-1}, \pi>$ for $u_i \in R^\times$ and some parameter π.*

Proof. If $q(v) = u$ for some $v \in E$, then q restricted to the submodule $Rv \subset E$ is regular, hence (E, q) splits as $(R, <u>) \perp (Rv^\perp, q|_{Rv^\perp})$. Since $(Rv^\perp, q|_{Rv^\perp})$ has simple degeneration, we are done by induction. □

Corollary 3.8. *Let R be a semilocal principal ideal domain with 2 invertible and fraction field K. If quadratic forms q and q' with simple degeneration and multiplicity one over R are isometric over K, then they are isometric over R.*

Proof. Any quadratic form q with simple degeneration and multiplicity one has discriminant $\pi \in R/R^{\times 2}$ given by a parameter. Since $R^\times/R^{\times 2} \to K^\times/K^{\times 2}$ is injective, if q' is another quadratic form with simple degeneration and multiplicity one, such that $q|_K$ is isomorphic to $q|'_K$, then q and q' have the same discriminant.

Over each discrete valuation overring R_i of R, we thus have diagonalizations,

$$q|_{R_i} \cong < u_1, \ldots, u_{r-1}, u_1 \cdots u_{r-1}\pi_i >, \quad q'|_{R'_i} \cong < u'_1, \ldots, u'_{r-1}, u'_1 \cdots u'_{r-1}\pi_i >,$$

for a suitable parameter π_i of R_i, where $u_j, u'_j \in R_i^\times$. Now, since $q|_K$ and $q'|_K$ are isometric, $q'|_K$ represents u_1 over K, hence by Proposition 3.2, $q'|_{R_i}$ represents u_1 over R_i. Hence by Lemma 3.7, we have a further diagonalization

$$q'|_{R_i} \cong < u_1, u'_2, \ldots, u'_{r-1}, u_1 u'_2 \cdots u'_{r-1}\pi_i >$$

with possibly different units u'_i. By cancellation over K, we have

$$< u_2, \ldots, u_{r-1}, u_1 \cdots u_{r-1}\pi_i > |_K \cong < u'_2, \ldots, u'_{r-1}, u'_1 \cdots u'_{r-1}\pi_i > |_K.$$

By an induction hypothesis over the rank of q, we have that

$$< u_2, \ldots, u_{r-1}, u_1 \cdots u_{r-1}\pi_i > \cong < u'_2, \ldots, u'_{r-1}, u'_1 \cdots u'_{r-1}\pi_i >$$

over R. By induction, we have the result over each R_i.

Thus $q|_{\widehat{R}} \cong q'|_{\widehat{R}}$ over $\widehat{R} = \prod_i \widehat{R}_i$. Consider the induced isometry $\psi^{\widehat{R}} : (q \perp h)|_{\widehat{R}} \cong (q' \perp h)|_{\widehat{R}}$ as well as the isometry $\psi^K : (q \perp h)|_K \cong (q' \perp h)|_K$ induced from the given one. By Proposition 3.6, there exists $\theta^{\widehat{R}} \in \mathbf{O}(q \perp h)(\widehat{R})$ and $\theta^K \in \mathbf{O}(q \perp h)(K)$ such that $\psi^{\widehat{R}-1}\psi^K = \theta^{\widehat{R}}\theta^{K-1}$. The isometries $\psi^{\widehat{R}}\theta^{\widehat{R}}$ and $\psi^K\theta^K$ then agree over $\widehat{K}$ and so patch to yield an isometry $\psi : q \perp h \cong q' \perp h$ over R. By Corollary 3.4, we then have an isometry $q \cong q'$. □

Remark 3.9. Let R be a semilocal principal ideal domain with 2 invertible, closed fiber D, and fraction field K. Let $\mathrm{QF}^D(R)$ be the set of isometry classes of quadratic forms on R with simple degeneration of multiplicity one along D. Corollary 3.8 says that $\mathrm{QF}^D(R) \to \mathrm{QF}(K)$ is injective, which can be viewed as an analogue of the Grothendieck–Serre conjecture for the (nonreductive) orthogonal group of a quadratic form with simple degeneration of multiplicity one over a discrete valuation ring. One might wonder if such a statement is true for more general nonreductive smooth R-group schemes.

Corollary 3.10. *Let R be a complete discrete valuation ring with 2 invertible and K its fraction field. If quadratic forms q and q' of even rank $n = 2m \geq 4$ with simple degeneration and multiplicity one over R are similar over K, then they are similar.*

Proof. Let $\psi : q|_K \simeq q'|_K$ be a similarity with factor $\lambda = u\pi^e$ where $u \in R^\times$ and π is a parameter whose square class we can assume is the discriminant of q and q'. If e is even, then $\pi^{e/2}\psi : q|_K \simeq q'|_K$ has factor u, so defines an isometry $q|_K \cong uq'|_K$. Hence by Corollary 3.8, there is an isometry $q \cong uq'$, hence a similarity $q \simeq q'$. If e is odd, then $\pi^{(e-1)/2}\psi^K$ defines an isometry $q|_K \cong u\pi q'|_K$. Writing $q \cong q_1 \perp < a\pi >$ and $q' = q_1' \perp < b\pi >$ for regular quadratic forms q_1 and q_1' over R and $a, b \in R^\times$, then $\pi q|_K \cong u\pi q'|_K \cong u\pi q_1' \perp <bu>$. Comparing first residues, we have that $\overline{q}_1$ and $<\overline{bu}>$ are equal in $W(k)$, where k is the residue field of R. Since R is complete, q_1 splits off the requisite number of hyperbolic planes, and so $q_1 \cong h^{m-1} \perp <(-1)^{m-1}a>$. Now note that $(-1)^{m-1}\pi$ is a similarity factor of the form $q|_K$. Finally, we have $(-1)^{m-1}\pi q|_K \cong q|_K \cong u\pi q'|_K$, so that $q|_K \cong (-1)^{m-1}uq'|_K$. Thus by Corollary 3.8, $q \cong (-1)^m uq'$ over R, so there is a similarity $q \simeq q'$ over R. $\square$

We need the following relative version of Theorem 2.1.

Proposition 3.11. *Let R be a semilocal principal ideal domain with 2 invertible and K its fraction field. Let q and q' be quadratic forms of rank 4 over R with simple degeneration and multiplicity one. Given any R-algebra isomorphism $\varphi : \mathscr{C}_0(q) \cong \mathscr{C}_0(q')$ there exists a similarity $\psi : q \simeq q'$ such that $\mathscr{C}_0(\psi) = \varphi$.*

Proof. By Theorem 2.5, there exists a similarity $\psi^K : q \simeq q'$ such that $\mathscr{C}_0(\psi^K) = \varphi|_K$. Thus over $\widehat{R} = \prod_i \widehat{R}_i$, Corollary 3.10 applied to each component provides a similarity $\rho : q|_{\widehat{R}} \simeq q'|_{\widehat{R}}$. Now $\mathscr{C}_0(\rho)^{-1} \circ \varphi : \mathscr{C}_0(q)|_{\widehat{R}} \cong \mathscr{C}_0(q)|_{\widehat{R}}$ is a $\widehat{R}$-algebra isomorphism, hence by Theorem 2.1, is equal to $\mathscr{C}_0(\sigma)$ for some similarity $\sigma : q|_{\widehat{R}} \simeq q|_{\widehat{R}}$. Then $\psi^{\widehat{R}} = \rho \circ \sigma : q|_{\widehat{R}} \simeq q'|_{\widehat{R}}$ satisfies $\mathscr{C}_0(\psi^{\widehat{R}}) = \varphi|_{\widehat{R}}$. Let $\lambda \in K^\times$ and $u \in \widehat{R}^\times$ be the factor of ψ^K and $\psi^{\widehat{R}}$, respectively. Then $\psi^K|_{\widehat{K}}^{-1} \circ \psi^{\widehat{R}}|_{\widehat{K}} : q|_{\widehat{K}} \simeq q|_{\widehat{K}}$ has factor $u^{-1}\lambda \in \widehat{K}^\times$. But since $\mathscr{C}_0(\psi^K|_{\widehat{K}}^{-1} \circ \psi^{\widehat{R}}|_{\widehat{K}}) = \mathrm{id}$, we have that $\psi^K|_{\widehat{K}}^{-1} \circ \psi^{\widehat{R}}|_{\widehat{K}}$ is given by multiplication by $\mu \in \widehat{K}^\times$. In particular, $u^{-1}\lambda = \mu^2$ and thus the valuation of $\lambda \in K^\times$ is even in every R_i. Thus $\lambda = v\varpi^2$ with $v \in R^\times$ and so $\varpi\psi$ defines an isometry $q|_K \cong vq'|_K$. By Corollary 3.8, there's an isometry $\alpha : q \cong vq'$, i.e., a similarity $\alpha : q \simeq q'$. As before, $\mathscr{C}_0(\alpha)^{-1} \circ \varphi : \mathscr{C}_0(q) \cong \mathscr{C}_0(q)$ is a R-algebra isomorphism, hence by Theorem 2.1, is equal to $\mathscr{C}_0(\beta)$ for some similarity $\beta : q \simeq q$. Then we can define a similarity $\psi = \alpha \circ \beta : q \simeq q'$ over R, which satisfies $\mathscr{C}_0(\psi) = \varphi$. $\square$

Finally, we need the following generalization of [19, Prop. 2.3] to the setting of quadratic forms with simple degeneration.

Proposition 3.12. *Let S be the spectrum of a regular local ring $(R, \mathfrak{m})$ of dimension ≥ 2 with 2 invertible and $D \subset S$ a regular divisor. Let (V, q) be a quadratic form over S such that $(V, q)|_{S \smallsetminus \{\mathfrak{m}\}}$ has simple degeneration of multiplicity one along $D \smallsetminus \{\mathfrak{m}\}$. Then (V, q) has simple degeneration along D of multiplicity one.*

Proof. First note that the discriminant of (V, q) (hence the subscheme D) is represented by a regular element $\pi \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$. Now assume, to get a contradiction, that the radical of $(V, q)_{\kappa(\mathfrak{m})}$, where $\kappa(\mathfrak{m})$ is the residue field at $\mathfrak{m}$, has dimension $r > 1$ and let $\overline{e}_1, \dots, \overline{e}_r$ be a $\kappa(\mathfrak{m})$-basis of the radical. Lifting to unimodular elements $e_1, \dots, e_r$ of V, we can complete to a basis $e_1, \dots, e_n$. Since $b_q(e_i, e_j) \in \mathfrak{m}$ for all $1 \le i \le r$ and $1 \le j \le n$, inspecting the Gram matrix M_q of b_q with respect to this basis, we find that $\det M_q \in \mathfrak{m}^r$, contradicting the description of the discriminant above. Thus the radical of (V, q) has rank 1 at $\mathfrak{m}$ and (V, q) has simple degeneration along D. Similarly, (V, q) also has multiplicity one at $\mathfrak{m}$, hence on S by hypothesis. □

COROLLARY 3.13. *Let S be a regular integral scheme of dimension ≤ 2 with 2 invertible and D a regular divisor. Let $(\mathscr{E}, q, \mathscr{L})$ be a quadratic form over S that is regular over every codimension 1 point of $S \smallsetminus D$ and has simple degeneration of multiplicity one over every codimension one point of D. Then over S, the quadratic form q has simple degeneration along D of multiplicity one.*

Proof. Let $U = S \smallsetminus D$. The quadratic form $q|_U$ is regular except possibly at finitely many closed points. But regular quadratic forms over the complement of finitely many closed points of a regular surface extend uniquely by [19, Prop. 2.3]. Hence $q|_U$ is regular. The restriction $q|_D$ has simple degeneration at the generic point of D, hence along the complement of finitely many closed points of D. At each of these closed points, q has simple degeneration by Proposition 3.12. Thus q has simple degeneration along D. □

4 GLUING TENSORS

In this section, we reproduce some results on gluing (or patching) tensor structures on vector bundles communicated to us by M. Ojanguren and inspired by Colliot-Thélène–Sansuc [19, §2, §6]. As usual, any scheme S is assumed to be noetherian.

LEMMA 4.1. *Let S be a scheme of dimension n, $U \subset S$ a dense open subset, $x \in S \smallsetminus U$ a point of codimension 1 of S, $V \subset S$ a dense open neighborhood of x, and $W \subset U \cap V$ a dense open subset of S. Then there exists a dense open neighborhood V' of x such that $V' \cap U \subset W$.*

Proof. The closed set $Z = S \smallsetminus W$ is of dimension $n-1$, contains x, and has a decomposition into closed sets $Z = Z_1 \cup Z_2$, where $Z_1 = Z \cap (S \smallsetminus U)$ contains x and $Z_2 = \overline{Z \cap U}$. No irreducible component of Z_2 can contain x, otherwise it would contain (hence coincide with) the dimension $n-1$ set $\overline{\{x\}}$. Setting $V' = S \smallsetminus Z_2$, then $V' \subset S$ is a dense open neighborhood of x and satisfies $V' \cap U \subset W$. □

Let $\mathscr{V}$ be a locally free $\mathscr{O}_S$-module (of finite rank). A *tensorial construction* $t(\mathscr{V})$ in $\mathscr{V}$ is any locally free $\mathscr{O}_S$-module that is a tensor product of modules $\bigwedge^j(\mathscr{V})$, $\bigwedge^j(\mathscr{V}^\vee)$, $S^j(\mathscr{V})$, or $S^j(\mathscr{V}^\vee)$. Let $\mathscr{L}$ be a line bundle on S. An $\mathscr{L}$-*valued tensor* $(\mathscr{V}, q, \mathscr{L})$ of *type* $t(\mathscr{V})$ on S is a global section $q \in \Gamma(S, t(\mathscr{V}) \otimes \mathscr{L})$ for some tensorial construction $t(\mathscr{V})$ in $\mathscr{V}$. For example, an $\mathscr{L}$-valued quadratic form is an $\mathscr{L}$-valued tensor of type $t(\mathscr{V}) = S^2(\mathscr{V}^\vee)$; an $\mathscr{O}_S$-algebra structure on $\mathscr{V}$ is an $\mathscr{O}_S$-valued tensor of type $t(\mathscr{V}) = \mathscr{V}^\vee \otimes \mathscr{V}^\vee \otimes \mathscr{V}$. If $U \subset S$ is an open set, denote by $(\mathscr{V}, q, \mathscr{L})|_U = (\mathscr{V}|_U, q|_U, \mathscr{L}|_U)$ the restricted tensor over U. If $D \subset S$ is a closed subscheme, let $\mathscr{O}_{S,D}$ denote the semilocal ring at the generic points of D and $(\mathscr{V}, q, \mathscr{L})|_D = (\mathscr{V}, q, \mathscr{L}) \otimes_{\mathscr{O}_S} \mathscr{O}_{S,D}$ the associated tensor over $\mathscr{O}_{S,D}$. If S is integral and K its function field, we write $(\mathscr{V}, q, \mathscr{L})|_K$ for the stalk at the generic point.

A *similarity* between line bundle-valued tensors $(\mathscr{V}, q, \mathscr{L})$ and $(\mathscr{V}', q', \mathscr{L}')$ consists of a pair (φ, λ) where $\varphi : \mathscr{V} \cong \mathscr{V}'$ and $\lambda : \mathscr{L} \cong \mathscr{L}'$ are $\mathscr{O}_S$-module isomorphisms such that $t(\varphi) \otimes \lambda : t(\mathscr{V}) \otimes \mathscr{L} \cong t(\mathscr{V}') \otimes \mathscr{L}'$ takes q to q'. A similarity is an *isomorphism* if $\mathscr{L} = \mathscr{L}'$ and $\lambda = \mathrm{id}$.

Proposition 4.2. *Let S be an integral scheme, K its function field, $U \subset S$ a dense open subscheme, and $D \subset S \smallsetminus U$ a closed subscheme of codimension 1. Let $(\mathscr{V}^U, q^U, \mathscr{L}^U)$ be a tensor over U, $(\mathscr{V}^D, q^D, \mathscr{L}^D)$ a tensor over $\mathscr{O}_{S,D}$, and $(\varphi, \lambda) : (\mathscr{V}^U, q^U, \mathscr{L}^U)|_K \simeq (\mathscr{V}^D, q^D, \mathscr{L}^D)|_K$ a similarity of tensors over K. Then there exists a dense open set $U' \subset S$ containing U and the generic points of D and a tensor $(\mathscr{V}^{U'}, q^{U'}, \mathscr{L}^{U'})$ over U' together with similarities $(\mathscr{V}^U, q^U, \mathscr{L}^U) \cong (\mathscr{V}^{U'}, q^{U'}, \mathscr{L}^{U'})|_U$ and $(\mathscr{V}^D, q^D, \mathscr{L}^D) \cong (\mathscr{V}^{U'}, q^{U'}, \mathscr{L}^{U'})|_D$. A corresponding statement holds for isomorphisms of tensors.*

Proof. By induction on the number of irreducible components of D, gluing over one at a time, we can assume that D is irreducible. Choose an extension $(\mathscr{V}^V, q^V, \mathscr{L}^V)$ of $(\mathscr{V}^D, q^D, \mathscr{L}^D)$ to some open neighborhood V of D in S. Since $(\mathscr{V}^V, q^V, \mathscr{L}^V)|_K \simeq (\mathscr{V}^U, q^U, \mathscr{L}^U)|_K$, there exists an open subscheme $W \subset U \cap V$ over which $(\mathscr{V}^V, q^V, \mathscr{L}^V)|_W \simeq (\mathscr{V}^U, q^U, \mathscr{L}^U)|_W$. By Lemma 4.1, there exists an open neighborhood $V' \subset S$ of D such that $V' \cap U \subset W$. We can glue $(\mathscr{V}^U, q^U, \mathscr{L}^U)$ and $(\mathscr{V}^{V'}, q^{V'}, \mathscr{L}^{V'})$ over $U \cap V'$ to get a tensor $(\mathscr{V}^{U'}, q^{U'}, \mathscr{L}^{U'})$ over U' extending $(\mathscr{V}^U, q^U, \mathscr{L}^U)$, where $U' = U \cup V'$. But U' contains the generic points of D and we are done. ∎

For an open subscheme $U \subset S$, a closed subscheme $D \subset S \smallsetminus U$ of codimension 1, a *similarity gluing datum* (resp. *gluing datum*) is a triple $((\mathscr{V}^U, q^U, \mathscr{L}^U), (\mathscr{V}^D, q^D, \mathscr{L}^D), \varphi)$ consisting of a tensor over U, a tensor over $\mathscr{O}_{S,D}$, and a similarity (resp. an isomorphism) of tensors (φ, λ) : $(\mathscr{V}^U, q^U, \mathscr{L}^U)|_K \simeq (\mathscr{V}^D, q^D, \mathscr{L}^D)|_K$ over K. There is an evident notion of isomorphism between two (similarity) gluing data. Two isomorphic gluing data yield, by Proposition 4.2, tensors $(\mathscr{V}^{U'}, q^{U'}, \mathscr{L}^{U'})$ and $(\mathscr{V}^{U''}, q^{U''}, \mathscr{L}^{U''})$ over open dense subsets $U', U'' \subset S$ containing U and the generic points of D such that there is an open dense refinement $U''' \subset U' \cap U''$ over which we have $(\mathscr{V}^{U'}, q^{U'}, \mathscr{L}^{U'})|_{U'''} \simeq (\mathscr{V}^{U''}, q^{U''}, \mathscr{L}^{U''})|_{U'''}$.

Together with results of [19], we get a well-known result—*purity for division algebras* over surfaces—which we state in a precise way, due to Ojanguren, that is conducive to our usage. If K is the function field of a regular scheme S, we say that $\beta \in \mathrm{Br}(K)$ is *unramified* (along S) if it is contained in the image of the injection $\mathrm{Br}(\mathscr{O}_{S,x}) \to \mathrm{Br}(K)$ for all codimension 1 points x of S.

THEOREM 4.3. *Let S be a regular integral scheme of dimension ≤ 2, K its function field, $D \subset S$ a closed subscheme of codimension 1, and $U = S \smallsetminus D$.*

a) If $\mathscr{A}^U$ is an Azumaya $\mathscr{O}_U$-algebra such that $\mathscr{A}^U|_K$ is unramified along D then there exists an Azumaya $\mathscr{O}_S$-algebra $\mathscr{A}$ such that $\mathscr{A}|_U \cong \mathscr{A}^U$.

b) If a central simple K-algebra A has Brauer class unramified over S, then there exists an Azumaya $\mathscr{O}_S$-algebra $\mathscr{A}$ such that $\mathscr{A}|_K \cong A$.

Proof. For *a)*, since $\mathscr{A}^U|_K$ is unramified along D, there exists an Azumaya $\mathscr{O}_{S,D}$-algebra $\mathscr{B}^D$ with $\mathscr{B}^D|_K$ Brauer equivalent to A.
We argue that we can choose $\mathscr{B}^D$ such that $\mathscr{B}^D|_K = A$. Indeed, writing $\mathscr{B}^D|_K = M_m(\Delta)$ for a division K-algebra Δ and choosing a maximal $\mathscr{O}_{S,D}$-order $\mathscr{D}^D$ of Δ, then $M_m(\mathscr{D}^D)$ is a maximal order of $\mathscr{B}^D|_K$. Any two maximal orders are isomorphic by [7, Prop. 3.5], hence $M_m(\mathscr{D}^D) \cong \mathscr{B}^D$. In particular, $\mathscr{D}^D$ is an Azumaya $\mathscr{O}_{S,D}$-algebra. Finally writing $A = M_n(\Delta)$, then $M_n(\mathscr{D}^D)$ is an Azumaya $\mathscr{O}_{S,D}$-algebra and is our new choice for $\mathscr{B}^D$.
Applying Proposition 4.2 to $\mathscr{A}^U$ and $\mathscr{B}^D$, we get an Azumaya $\mathscr{O}_{U'}$-algebra $\mathscr{A}^{U'}$ extending $\mathscr{A}^U$, where U' contains all points of S of codimension 1. Finally, by [19, Thm. 6.13] applied to the group $\mathbf{PGL}_n$ (where n is the degree of A), $\mathscr{A}^{U'}$ extends to an Azumaya $\mathscr{O}_S$-algebra $\mathscr{A}$ such that $\mathscr{A}|_U = \mathscr{A}^U$.
For *b)*, the K-algebra A extends, over some open subscheme $U \subset S$, to an Azumaya $\mathscr{O}_U$-algebra $\mathscr{A}^U$. If U contains all codimension 1 points, then we apply [19, Thm. 6.13] as above. Otherwise, $D = S \smallsetminus U$ has codimension 1 and we apply part (1). ☐

Finally, we note that isomorphic Azumaya algebra gluing data on a regular integral scheme S of dimension ≤ 2 yield, by [19, Thm. 6.13], isomorphic Azumaya algebras on S.

5 THE NORM FORM $N_{T/S}$ FOR RAMIFIED COVERS

Let S be a regular integral scheme, $D \subset S$ a regular divisor, and $f : T \to S$ a ramified cover of degree 2 branched along D. Then T is a regular integral scheme. Let L/K be the corresponding quadratic extension of function fields. Let $U = S \smallsetminus D$, and for $E = f^{-1}(D)$, let $V = T \smallsetminus E$. Then $f|_V : V \to U$ is étale of degree 2. Let ι be the nontrivial Galois automorphism of T/S.
The following lemma is not strictly used in our construction but we need it for the applications in §6.

LEMMA 5.1. *Let S be a regular integral scheme and $f : T \to S$ a finite flat cover of prime degree ℓ with regular branch divisor $D \subset S$ on which ℓ is invertible. Let L/K be the corresponding extension of function fields. Let d be a positive integer invertible on D.*

a) The corestriction map $N_{L/K} : \mathrm{Br}(L) \to \mathrm{Br}(K)$ restricts to a well-defined map $N_{T/S} : {}_d\mathrm{Br}(T) \to {}_d\mathrm{Br}(S)$.

b) *If S has dimension ≤ 2 and $\mathscr{B}$ is an Azumaya $\mathscr{O}_T$-algebra of degree d representing $\beta \in \mathrm{Br}(T)$ then there exists an Azumaya $\mathscr{O}_S$-algebra of degree d^ℓ representing $N_{T/S}(\beta)$ whose restriction to U coincides with the classical étale norm algebra $N_{V/U}\mathscr{B}|_V$.*

Proof. The hypotheses imply that T is regular integral and so by [6], we can consider $\mathrm{Br}(S) \subset \mathrm{Br}(K)$ and $\mathrm{Br}(T) \subset \mathrm{Br}(L)$. Let $\mathscr{B}$ be an Azumaya $\mathscr{O}_T$-algebra of degree d representing $\beta \in \mathrm{Br}(T)$. As V/U is étale of degree ℓ, the classical norm algebra $N_{V/U}(\mathscr{B}|_V)$ is an Azumaya $\mathscr{O}_U$-algebra of degree d^ℓ representing the class of $N_{L/K}(\beta) \in \mathrm{Br}(K)$. In particular, $N_{L/K}(\beta)$ is unramified at every point (of codimension 1) in U. As D is regular, it is a disjoint union of irreducible divisors and let D' be one such irreducible component. If $E' = f^*D'$, then $\mathscr{O}_{T,E'}$ is totally ramified over $\mathscr{O}_{S,D'}$ (since it is ramified of prime degree). In particular, $E' \subset T$ is an irreducible component of $E = f^*D$. The commutative diagram

$$\begin{array}{ccccc} {}_d\mathrm{Br}(\mathscr{O}_{T,E'}) & \longrightarrow & {}_d\mathrm{Br}(L) & \xrightarrow{\partial} & H^1(\kappa(E'), \mathbb{Z}/d\mathbb{Z}) \\ \vdots & & \downarrow{\scriptstyle N_{L/K}} & & \| \\ {}_d\mathrm{Br}(\mathscr{O}_{S,D'}) & \longrightarrow & {}_d\mathrm{Br}(K) & \xrightarrow{\partial} & H^1(\kappa(D'), \mathbb{Z}/d\mathbb{Z}) \end{array}$$

of residue homomorphisms implies, since β is unramified along E', that $N_{L/K}(\beta)$ is unramified along D'. Thus $N_{L/K}(\beta)$ is an unramified class in $\mathrm{Br}(K)$, hence is contained in $\mathrm{Br}(S)$ by purity for the Brauer group (cf. [33, Cor. 1.10]). This proves part *a*.
By Theorem 4.3, $N_{V/U}(\mathscr{B}|_V)$ extends (since by part *a*, it is unramified along D) to an Azumaya $\mathscr{O}_S$-algebra of degree d^ℓ, whose generic fiber is $N_{L/K}(\beta)$. This proves part *b*. ☐

Remark 5.2. Following Deligne [2, Exp. XVII 6.3.13], for any finite flat morphism $f : T \to S$, there exists a natural trace morphism $\mathrm{Tr}_f : f_*\mathbf{G}_\mathrm{m} \to \mathbf{G}_\mathrm{m}$ of sheaves of abelian groups on X. Taking flat (fppf) cohomology, we arrive at a homomorphism $H^2(\mathrm{Tr}_f) : H^2(T, \mathbf{G}_\mathrm{m}) \to H^2(S, \mathbf{G}_\mathrm{m})$. If we assume that the Brauer group and cohomological Brauer group of S coincide (e.g., S has an ample invertible sheaf [22] or is regular of dimension ≤ 2 [33, Cor. 2.2]), then we can refine this to a map $\mathrm{Br}(T) \to \mathrm{Br}(S)$. This map can be seen to coincide with the one constructed in Lemma 5.1, under the different set of hypotheses imposed there. We do not know how these norm constructions coincide with that defined by Ferrand [29].

Suppose that S has dimension ≤ 2. We are interested in finding a good extension of $N_{V/U}(\mathscr{B}|_V)$ to S. We note that if $\mathscr{B}$ has an involution of the first kind τ, then the corestriction involution $N_{V/U}(\tau|_V)$, given by the restriction of $\iota_*\tau|_V \otimes \tau|_V$ to $N_{V/U}(\mathscr{A}|_V)$, is of orthogonal type. If $N_{V/U}(\mathscr{B}|_V) \cong \mathscr{E}nd(\mathscr{E}^U)$ is split, then $N_{V/U}(\tau|_V)$ is adjoint to a regular line bundle-valued quadratic form $(\mathscr{E}^U, q^U, \mathscr{L}^U)$ on U unique up to projective similarity. The main result of this section is that this extends to a line bundle-valued quadratic form $(\mathscr{E}, q, \mathscr{L})$ on S with simple degeneration along a regular divisor D satisfying $\mathscr{C}_0(\mathscr{E}, q, \mathscr{L}) \cong \mathscr{B}$.

THEOREM 5.3. *Let S be a regular integral scheme of dimension ≤ 2 with 2 invertible and $f : T \to S$ a finite flat cover of degree 2 with regular branch divisor D. Let $\mathscr{B}$ be an Azumaya quaternion $\mathscr{O}_T$-algebra with standard involution τ. Suppose that $N_{V/U}(\mathscr{B}|_V)$ is split and $N_{V/U}(\tau|_V)$ is adjoint to a regular line bundle-valued quadratic form $(\mathscr{E}^U, q^U, \mathscr{L}^U)$ on U. There exists a line bundle-valued quadratic form $(\mathscr{E}, q, \mathscr{L})$ on S with simple degeneration along D with multiplicity one, which restricts to $(\mathscr{E}^U, q^U, \mathscr{L}^U)$ on U and such that $\mathscr{C}_0(\mathscr{E}, q, \mathscr{L}) \cong \mathscr{B}$.*

First we need the following lemma. Let S be a normal integral scheme, K its function field, $D \subset S$ a regular divisor, and $\mathscr{O}_{S,D}$ the the semilocal ring at the generic points of D.

LEMMA 5.4. *Let S be a normal integral scheme with 2 invertible, $T \to S$ a finite flat cover of degree 2 with regular branch divisor $D \subset S$, and L/K the corresponding extension of function fields. Under the restriction map $H^1_{\text{ét}}(U, \mathbb{Z}/2\mathbb{Z}) \to H^1_{\text{ét}}(K, \mathbb{Z}/2\mathbb{Z}) = K^\times/K^{\times 2}$, the class of the étale quadratic extension $[V/U]$ maps to a square class represented by a parameter $\pi \in K^\times$ of the semilocal ring $\mathscr{O}_{S,D}$.*

Proof. Consider any $\pi \in K^\times$ with $L = K(\sqrt{\pi})$. For any irreducible component D' of D, if $v_{D'}(\pi)$ is even, then we can modify π up to squares in K so that $v_{D'}(\pi) = 0$. But then T/S would be étale at the generic point of D', which is impossible. Hence, $v_{D'}(\pi)$ is odd for every irreducible component D' of D. Since $\mathscr{O}_{S,D}$ is a principal ideal domain, we can modify π up to squares in K so that $v_{D'}(\pi) = 1$ for every component D' of D. Under the restriction map $H^1_{\text{ét}}(U, \mathbb{Z}/2\mathbb{Z}) \to H^1_{\text{ét}}(K, \mathbb{Z}/2\mathbb{Z}) = K^\times/K^{\times 2}$, the class $[V/U]$ is mapped to the class $[L/K]$, which corresponds via Kummer theory to the square class (π). ☐

Proof of Theorem 5.3. If $D = \cup D_i$ is the irreducible decomposition of D and π_i is a parameter of $\mathscr{O}_{S,D_i}$, then $\pi = \prod_i \pi_i$ is a parameter of $\mathscr{O}_{S,D}$. Choose a regular quadratic form $(\mathscr{E}^U, q^U, \mathscr{L}^U)$ on U adjoint to $N_{V/S}(\sigma|_V)$. Since $\mathscr{O}_{S,D}$ is a principal ideal domain, modifying by squares over K, the form $q^U|_K$ has a diagonalization $< a_1, a_2, a_3, a_4 >$, where $a_i \in \mathscr{O}_{S,D}$ are squarefree. By Lemma 5.4, we can choose $\pi \in K^\times$ so that $[V/U] \in H^1_{\text{ét}}(U, \mathbb{Z}/2\mathbb{Z})$ maps to the square class (π). By Theorem 2.5, the class $[V/U]$ maps to the discriminant of $q^U|_K$. Since $\mathscr{O}_{S,D}$ is a principal ideal domain, $a_1 \cdots a_4 = \mu^2\pi$, for some $\mu \in \mathscr{O}_{S,D}$. If π_i divides μ, then π_i divides exactly 3 of a_1, a_2, a_3, a_4, so that clearing squares from the entries of $\mu < a_1, a_2, a_3, a_4 >$ yields a form $< a'_1, a'_2, a'_3, a'_4 >$ over $\mathscr{O}_{S,D}$ with simple degeneration along D, which over K, is isometric to $\mu q^U|_K$. Define

$$(\mathscr{E}^D, q^D, \mathscr{L}^D) = (\mathscr{O}^4_{S,D}, < a'_1, a'_2, a'_3, a'_4 >, \mathscr{O}_{S,D}).$$

By definition, the identity map is a similarity $q^U|_K \simeq q^D|_K$ with similarity factor μ (up to $K^{\times 2}$). Our aim is to find a good similarity enabling a gluing to a quadratic form over S with simple degeneration along D and the correct even Clifford algebra.

First note that by the classical theory of ${}^2\mathsf{A}_1 = \mathsf{D}_2$ over V/U (cf. Theorem 2.5), we can choose an $\mathscr{O}_V$-algebra isomorphism $\varphi^U : \mathscr{C}_0(\mathscr{E}^U, q^U, \mathscr{L}^U) \to \mathscr{B}|_V$. Second, we can pick an $\mathscr{O}_{T,E}$-algebra isomorphism $\varphi^D : \mathscr{C}_0(q^D) \to \mathscr{B}|_E$, where $E = f^{-1}D$. Indeed, by the classical theory of ${}^2\mathsf{A}_1 = \mathsf{D}_2$ over L/K (cf. Theorem 2.5), the central simple algebras $\mathscr{C}_0(q^D)|_L$ and $\mathscr{B}|_L$ are isomorphic over L, hence they are isomorphic over the semilocal principal ideal domain $\mathscr{O}_{T,E}$. Now consider the L-isomorphism $\varphi^L = (\varphi^U|_L)^{-1} \circ \varphi^D|_L : \mathscr{C}_0(q^D)|_L \to \mathscr{C}_0(q^U)|_L$. Again by the classical theory of ${}^2\mathsf{A}_1 = \mathsf{D}_2$ over L/K (cf. Theorem 2.5), this is induced by a similarity $\psi^K : q^D|_K \to q^U|_K$, unique up to multiplication by scalars. By Proposition 4.2, the quadratic forms $(\mathscr{E}^U, q^U, \mathscr{L}^U)$ and $(\mathscr{E}^D, q^D, \mathscr{L}^D)$ glue, via the similarity ψ^K, to a quadratic form $(\mathscr{E}^{U'}, q^{U'}, \mathscr{L}^{U'})$ on a dense open subscheme $U' \subset S$ containing U and the generic points of D, hence all points of codimension 1. By [19, Prop. 2.3], the quadratic form $(\mathscr{E}^{U'}, q^{U'}, \mathscr{L}^{U'})$ extends uniquely to a quadratic form $(\mathscr{E}, q, \mathscr{L})$ on S since the underlying vector bundle $\mathscr{E}^{U'}$ extends to a vector bundle $\mathscr{E}$ on S (because S is a regular integral schemes of dimension ≤ 2). By Corollary 3.13, this extension has simple degeneration along D.

Finally, we argue that $\mathscr{C}_0(q) \cong \mathscr{B}$. We know that $q|_U = q^U$ and $q|_D = q^D$ and we have algebra isomorphisms $\varphi^U : \mathscr{C}_0(q)|_U \cong \mathscr{B}|_U$ and $\varphi^D : \mathscr{C}_0(q)|_D \cong \mathscr{B}|_D$ such that $\varphi^L = (\varphi^U|_L)^{-1} \circ \varphi^D|_L$. Hence the gluing data $(\mathscr{C}_0(q)|_U, \mathscr{C}_0(q)|_D, \varphi^L)$ is isomorphic to the gluing data $(\mathscr{B}|_U, \mathscr{B}|_D, \mathrm{id})$. Thus $\mathscr{C}_0(q)$ and $\mathscr{B}$ are isomorphic over an open subset $U' \subset S$ containing all codimension 1 points of S. Hence by [19, Thm. 6.13], these Azumaya algebras are isomorphic over S. □

Finally, we can prove our main result.

Proof of Theorem 1. Theorem 5.3 implies that $\mathscr{C}_0 : \mathrm{Quad}_2(T/S) \to \mathrm{Az}_2(T/S)$ is surjective. To prove the injectivity, let $(\mathscr{E}_1, q_1, \mathscr{L}_1)$ and $(\mathscr{E}_2, q_2, \mathscr{L}_2)$ be line bundle-valued quadratic forms of rank 4 on S with simple degeneration along D of multiplicity one such that there is an $\mathscr{O}_T$-algebra isomorphism $\varphi : \mathscr{C}_0(q_1) \cong \mathscr{C}_0(q_2)$. By the classical theory of ${}^2\mathsf{A}_1 = \mathsf{D}_2$ over V/U (cf. Theorem 2.5), we know that $\varphi|_U : \mathscr{C}_0(q_1)|_U \cong \mathscr{C}_0(q_2)|_U$ is induced by a similarity transformation $\psi^U : q_1|_U \simeq q_2|_U \otimes \mathscr{N}^U$, for some line bundle $\mathscr{N}^U$ on U, which we can assume is the restriction of a line bundle $\mathscr{N}$ on S. Replacing $(\mathscr{E}_2, q_2, \mathscr{L}_2)$ by $(\mathscr{E}_2 \otimes \mathscr{N}, q_2 \otimes \,{<}1{>}, \mathscr{L} \otimes \mathscr{N}^{\otimes 2})$, which is in the same projective similarity class, we can assume that $\psi^U : q_1|_U \simeq q_2|_U$. In particular, $\mathscr{L}_1|_U \cong \mathscr{L}_2|_U$ so that we have $\mathscr{L}_1 \cong \mathscr{L}_2 \otimes \mathscr{M}$ for some $\mathscr{M}|_U \cong \mathscr{O}_U$ by the exact excision sequence

$$A^0(D) \to \mathrm{Pic}(S) \to \mathrm{Pic}(U) \to 0$$

of Picard groups (really Weil divisor class groups), cf. [32, Cor. 21.6.10], [30, 1 Prop. 1.8].

By Theorem 3.11, we know that $\varphi|_D : \mathscr{C}_0(q_1)|_E \cong \mathscr{C}_0(q_2)|_E$ is induced by some similarity transformation $\psi^D : q_1|_D \simeq q_2|_D$. Thus $\psi^K = (\psi^D|_K)^{-1} \circ \psi^U|_K \in \mathbf{GO}(q_1|_K)$. Since $\mathscr{C}_0(\psi^U|_K) = \mathscr{C}_0(\psi^D|_K) = \varphi|_K$, we have that $\psi^K \in \mathbf{GO}(q_1|_K)$ is a homothety, multiplication by $\lambda \in K^\times$. As in §4, define a line bundle $\mathscr{P}$ on S by the gluing datum $(\mathscr{O}_U, \mathscr{O}_D, \lambda^{-1} : \mathscr{O}_U|_K \cong \mathscr{O}_D|_K)$. Then $\mathscr{P}$ comes equipped with isomorphisms $\rho^U : \mathscr{O}_U \cong \mathscr{P}|_U$ and $\rho^D : \mathscr{O}_D \cong \mathscr{P}|_D$ with $(\rho^D|_K)^{-1} \circ \rho^U|_K = \lambda^{-1}$. Then we have similarities $\psi^U \otimes \rho^U : q_1|_U \simeq q_2|_D \otimes \mathscr{P}|_U$ and $\psi^D \otimes \rho^D : q_1|_D \simeq q_2|_D \otimes \mathscr{P}|_D$ such that

$$(\psi^D \otimes \rho^D)|_K^{-1} \circ (\psi^U \otimes \rho^U)|_K = (\psi^D|_K^{-1}\psi^U|_K)(\rho^D|_K^{-1}\rho^U|_K) = \psi^K\lambda^{-1} = \mathrm{id}$$

in $\mathbf{GO}(q_1|_K)$. Hence, as in §4, $\psi^U \otimes \rho^U$ and $\psi^D \otimes \rho^D$ glue to a similarity $(\mathscr{E}_1, q_1, \mathscr{L}_1) \simeq (\mathscr{E}_2 \otimes \mathscr{P}, q_2 \otimes < 1 >, \mathscr{L}_2 \otimes \mathscr{P}^{\otimes 2})$. Thus $(\mathscr{E}_1, q_1, \mathscr{L}_1)$ and $(\mathscr{E}_2, q_2, \mathscr{L}_2)$ define the same element of $\mathrm{Quad}_2(T/S)$. ☐

6 Failure of the local-global principle for isotropy of quadratic forms over surfaces

In this section, we mention one application of the theory of quadratic forms with simple degeneration over surfaces. Let S be a regular proper integral scheme of dimension d over an algebraically closed field k of characteristic $\neq 2$. For a point x of X, denote by K_x the fraction field of the completion $\widehat{\mathscr{O}}_{S,x}$ of $\mathscr{O}_{S,x}$ at its maximal ideal.

Lemma 6.1. *Let S be a regular integral scheme of dimension d over an algebraically closed field k of characteristic $\neq 2$ and let $D \subset S$ be a divisor. Fix $i > 0$. If $(\mathscr{E}, q, \mathscr{L})$ is a quadratic form of rank $> 2^{d-i} + 1$ over S with simple degeneration along D then q is isotropic over K_x for all points x of S of codimension $\geq i$.*

Proof. The residue field $\kappa(x)$ of K_x has transcendence degree $\leq d - i$ over k and is hence a C_{d-i}-field. By hypothesis, q has, over K_x, a subform q_1 of rank $> 2^{d-i}$ that is regular over $\widehat{\mathscr{O}}_{S,x}$. Hence q_1 is isotropic over $\kappa(x)$, thus q is isotropic over the complete field K_x. ☐

As usual, denote by $K = k(S)$ the function field. We say that a quadratic form q over K is *locally isotropic* if q is isotropic over K_x for all points x of codimension one.

Corollary 6.2. *Let S be a proper regular integral surface over an algebraically closed field k of characteristic $\neq 2$ and let $D \subset S$ be a regular divisor. If $(\mathscr{E}, q, \mathscr{L})$ is a quadratic form of rank ≥ 4 over S with simple degeneration along D then q over K is locally isotropic.*

For a different proof of this corollary, see [49, §3]. However, quadratic forms with simple degeneration are mostly anisotropic.

Theorem 6.3. *Let S be a proper regular integral surface over an algebraically closed field k of characteristic $\neq 2$. Assume that ${}_2\mathrm{Br}(S)$ trivial. Let $T \to S$ be a finite flat morphism of degree 2 with regular branch divisor $D \subset S$. Then each nontrivial class in ${}_2\mathrm{Br}(T)$ gives rise to a locally isotropic, yet anisotropic, quadratic form over $k(S)$, unique up to similarity.*

Proof. Let $L = k(T)$ and $K = k(S)$. Let $\beta \in {}_2\mathrm{Br}(T)$ be nontrivial. By a result of Artin [1], the class $\beta|_L \in {}_2\mathrm{Br}(L)$ has index 2. Thus by purity for division algebras over regular surfaces (Theorem 4.3), there exists an Azumaya quaternion algebra $\mathscr{B}$ over T whose Brauer class is β. Since $N_{L/K}(\beta|_L)$ is unramified on S, by Lemma 5.1, it extends to an element of ${}_2\mathrm{Br}(S)$, which is assumed to be trivial. Hence $\mathscr{B} \in \mathrm{Az}_2(T/S)$.
By the classical theory of ${}^2\mathsf{A}_1 = \mathsf{D}_2$ over L/K (cf. Theorem 2.5), the quaternion algebra $\mathscr{B}|_L$ corresponds to a unique similarity class of quadratic form q^K of rank 4 on K. The crucial contribution of our work is that we can control the degeneration divisor of an extension of q^K to a quadratic form on S. Indeed, by Theorem 1, $\mathscr{B}$ corresponds to a unique projective similarity class of quadratic form $(\mathscr{E}, q, \mathscr{L})$ of rank 4 with simple degeneration along D that is generically similar, by the compatibility of the norm constructions in Theorems 2.5 and 5.3, to q^K. Thus by Corollary 6.2, q^K is locally isotropic.
A classical result in the theory of quadratic forms of rank 4 is that q^K is isotropic over K if and only if $\mathscr{C}_0(q^K)$ splits over L (since L/K is the discriminant extension of q^K), see [42, Thm. 6.3], [55, 2 Thm. 14.1, Lemma 14.2], or [8, II Prop. 5.3]. Hence q^K is anisotropic since $\mathscr{C}_0(q^K) = \mathscr{B}_L$ has nontrivial Brauer class β by construction. □

We can make Theorem 6.3 explicit as follows. Write $L = K(\sqrt{d})$. Let $\mathscr{B}$ be an Azumaya quaternion algebra over T, with $\mathscr{B}_L$ given by the quaternion symbol (a, b) over L. Since $N_{L/K}(\mathscr{B}_L)$ is trivial, the restriction-corestriction sequence shows that $\mathscr{B}|_L$ is the restriction of a class from ${}_2\mathrm{Br}(K)$, so we can choose $a, b \in K^\times$. The corresponding quadratic form over K (from Theorem 1) is then given, up to similarity, by $< 1, a, b, abd >$. Indeed, its similarity class is uniquely characterized by having discriminant d and even Clifford invariant (a, b) over L, see [42].
In order to produce counterexamples to the local-global principle for isotropy of quadratic forms over a given surface, we need branched double covers with nontrivial 2-torsion in their Brauer group. This always exists, at least assuming characteristic zero.

Proposition 6.4. *Let S be a smooth projective surface over an algebraically closed field k of characteristic zero. Then there exists a finite flat double cover $T \to S$ with smooth branch divisor $D \subset S$ such that ${}_2\mathrm{Br}(T) \neq 0$.*

Proof. Choose a very ample line bundle $\mathscr{N}$ on S. By Serre's theorem [34, II Thm. 5.17], there exists n_0 such that $\omega_S \otimes \mathscr{N}^{\otimes n}$ is generated by global sections for all $n \geq n_0$. We are free to enlarge n_0 as we wish. Write $\mathscr{M} = \mathscr{N}^{\otimes n_0}$.

Let $\varphi : S \to \mathbb{P}^N$ be the projective embedding associated to the very ample line bundle $\mathscr{M}^{\otimes 2}$. Then by Bertini's theorem, there exists a hyperplane $H \subset \mathbb{P}^N$ such that $D = H \cap S$ is a smooth divisor of S. As $\mathscr{M}^{\otimes 2} \cong \mathscr{O}_S(D)$, there exists a nonzero section $s \in \Gamma(S, \mathscr{M}^{\otimes 2})$ with D as divisor of zeros. Then s defines an $\mathscr{O}_S$-algebra structure on $\mathscr{O}_S \oplus \mathscr{M}^\vee$ and let $f : T \to S$ be the finite flat double cover associated to its relative spectrum, i.e., the cyclic double cover taking a square root of D. As D is smooth, T is a smooth projective surface. We will argue that taking the degree of the embedding φ large enough (i.e., taking n_0 large enough) will suffice.
The double cover is tame, so we have $\omega_T \cong f^*(\omega_S \otimes \mathscr{M})$. Then

$$H^0(T, \omega_T) \cong H^0(S, f_*\omega_T) \cong H^0(S, \omega_S) \oplus H^0(S, \omega_S \otimes \mathscr{M})$$

is a k-vector space of positive dimension, since $\omega_S \otimes \mathscr{M}$ is generated by global sections. Hence $h^{2,0}(T) = \dim_k H^0(T, \omega_{T/k}) > 0$. In general, the Hodge numbers are defined as $h^{p,q}(T) = \dim_k H^q(T, \Omega^p_{T/k})$. From the Kummer exact sequence, we derive a short exact sequence

$$0 \to \mathrm{Pic}(T) \otimes_{\mathbb{Z}} \mathbb{Z}_2 \to H^2_{\text{ét}}(T, \mathbb{Z}_2(1)) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}_2/\mathbb{Z}_2, H^2_{\text{ét}}(T, \mathbf{G}_{\mathrm{m}})) \to 0.$$

As T is smooth and proper over a field, the 2-adic cohomology groups are of cofinite type, thus we get isomorphisms of 2-primary torsion subgroups

$$\mathrm{Br}(T)[2^\infty] \cong H^2(T, \mathbf{G}_{\mathrm{m}})[2^\infty] \cong (\mathbb{Q}_2/\mathbb{Z}_2)^{b_2(T)-\rho(T)} \times G,$$

for some finite group G, where $b_2(T) = \dim_{\mathbb{Z}_2} H^2_{\text{ét}}(T, \mathbb{Z}_2)$ is the 2nd 2-adic Betti number, and $\rho(T)$ is the rank of the Néron–Severi group of T. By the degeneration of the Hodge–de Rham spectral sequence for smooth projective varieties in characteristic zero, the Hodge decomposition yields $b_2(T) = h^{2,0}(T) + h^{1,1}(T) + h^{0,2}(T)$ and we note that $\rho(T) \le h^{1,1}(T)$. Hence by construction, $b_2(T) - \rho(T) \ge 2h^{2,0}(T) > 0$. In particular, we have ${}_2\mathrm{Br}(T) \neq 0$. ☐

We remark that the characteristic zero hypothesis can be relaxed to the condition that the Hodge–de Rham spectral sequence degenerates at the first page, since all we used in the proof of Lemma 6.4 was the Hodge decomposition. By [24], for a smooth surface S over a perfect field of characteristic $\neq 2$, it is sufficient to assume that S admits a smooth lift to the Witt vectors $W_2(k)$ of length 2. In any case, we wonder whether it is possible to remove the characteristic zero hypothesis in general.

COROLLARY 6.5. *Let K be a field finitely generated of transcendence degree 2 over an algebraically closed field k of characteristic zero. Then there exist anisotropic quadratic forms q of rank 4 over K such that q_v is isotropic for every rank 1 discrete valuation v on K.*

Proof. By resolution of singularities and Chow's lemma, we can find a smooth projective connected surface S over k with function field K. If ${}_2\mathrm{Br}(S) \neq 0$, then as before, by purity for division algebras (Theorem 4.3) and Artin's result [1], any nontrivial β in ${}_2\mathrm{Br}(S)$ is represented by an Azumaya quaternion algebra $\mathscr{B}$ over S. Then the norm form $\mathrm{Nrd} : \mathscr{B} \to \mathscr{O}_S$ is locally isotropic by Tsen's theorem (cf. Lemma 6.1) yet is globally anisotropic. Hence we can assume that ${}_2\mathrm{Br}(S) = 0$. Appealing to Proposition 6.4, we have a finite flat morphism $T \to S$ of degree 2 with regular branch divisor such that ${}_2\mathrm{Br}(T) \neq 0$. We then apply Theorem 6.3 to provide the counterexamples. ☐

Example 6.6. Let $T \to \mathbb{P}^2$ be a double cover branched over a smooth sextic curve over an algebraically closed field of characteristic $\neq 2$. Then T is a smooth projective K3 surface of degree 2. We remark that $b_2(T) = 22$ and that $\rho(T) \leq 20$. In fact, S admits a smooth lift to the Witt vectors by [23]. In particular, ${}_2\mathrm{Br}(T) \cong (\mathbb{Z}/2\mathbb{Z})^{22-\rho} \neq 0$, so that T gives rise to $2^{22-\rho} - 1$ similarity classes of locally isotropic yet anisotropic quadratic forms of rank 4 over $K = k(\mathbb{P}^2)$. This proves that the explicit Brauer classes constructed in [37] and [5] give rise to explicit quadratic forms that are counterexamples to the local-global principle.

We remark that while counter-examples to the local-global principle for isotropy of quadratic forms over the function field of a surfaces S could have previously been constructed from unramified quaternion algebras on S (cf. [18, Prop. 11]), such an approach cannot be used, for example, over rational surfaces.

7 A Torelli theorem for general cubic fourfolds containing a plane

Let Y be a *cubic fourfold*, i.e., a smooth cubic hypersurface of $\mathbb{P}^5 = \mathbb{P}(V)$ over $\mathbb{C}$. Let $W \subset V$ be a vector subspace of dimension three, $P = \mathbb{P}(W) \subset \mathbb{P}(V)$ the associated plane, and $P' = \mathbb{P}(V/W)$. If Y contains P, let $\widetilde{Y}$ be the blow-up of Y along P and $\pi : \widetilde{Y} \to P'$ the projection from P. The blow-up of $\mathbb{P}^5$ along P is isomorphic to the total space of the projective bundle $p : \mathbb{P}(\mathscr{E}) \to P'$, where $\mathscr{E} = W \otimes \mathscr{O}_{P'} \oplus \mathscr{O}_{P'}(-1)$, and in which $\pi : \widetilde{Y} \to P'$ embeds as a quadric surface bundle. The degeneration divisor of π is a sextic curve $D \subset P'$. It is known that D is smooth and π has simple degeneration along D if and only if Y does not contain any other plane meeting P, cf. [60, §1, Lemme 2]. In this case, the discriminant cover $T \to P'$ is a K3 surface of degree 2. All K3 surfaces considered will be smooth and projective.

We choose an identification $P' = \mathbb{P}^2$ and suppose, for the rest of this section, that $\pi : \widetilde{Y} \to P' = \mathbb{P}^2$ has simple degeneration. If Y contains another plane R disjoint from P, then $R \subset \widetilde{Y}$ is the image of a section of π, hence $\mathscr{C}_0(\pi)$ has trivial Brauer class over T by a classical result concerning quadratic forms of rank 4, cf. proof of Theorem 6.3. Thus if $\mathscr{C}_0(\pi)$ has nontrivial Brauer class $\beta \in {}_2\mathrm{Br}(T)$, then P is the unique plane contained in Y.

Given a scheme T with 2 invertible and an Azumaya quaternion algebra $\mathscr{B}$ on T, there is a standard choice of lift $[\mathscr{B}] \in H^2_{\text{ét}}(T, \boldsymbol{\mu}_2)$ of the Brauer class of $\mathscr{B}$, defined in [52] by taking into account the standard symplectic involution on $\mathscr{B}$. Denote by $c_1 : \mathrm{Pic}(T) \to H^2_{\text{ét}}(T, \boldsymbol{\mu}_2)$ the mod 2 cycle class map arising from the Kummer sequence.

Definition 7.1. Let T be a K3 surface of degree 2 over k together with a *polarization* $\mathscr{F}$, i.e., an ample line bundle of self-intersection 2. For $\beta \in H^2_{\text{ét}}(T, \boldsymbol{\mu}_2)/\langle c_1(\mathscr{F})\rangle$, we say that a cubic fourfold Y *represents* β if Y contains a plane whose associated quadric bundle $\pi : \widetilde{Y} \to \mathbb{P}^2$ has simple degeneration and discriminant cover $f : T \to \mathbb{P}^2$ satisfying $f^*\mathscr{O}_{\mathbb{P}^2}(1) \cong \mathscr{F}$ and $[\mathscr{C}_0(\pi)] = \beta$.

Remark 7.2. For a K3 surface T of degree 2 with a polarization $\mathscr{F}$, not every class in $H^2_{\text{ét}}(T, \boldsymbol{\mu}_2)/\langle c_1(\mathscr{F})\rangle$ is represented by a cubic fourfold, though one can characterize such classes. Consider the cup product pairing in étale cohomology $H^2_{\text{ét}}(T, \boldsymbol{\mu}_2) \times H^2_{\text{ét}}(T, \boldsymbol{\mu}_2) \to H^4_{\text{ét}}(T, \boldsymbol{\mu}_2^{\otimes 2}) \cong \mathbb{Z}/2\mathbb{Z}$. Define

$$B(T, \mathscr{F}) = \{\overline{x} \in H^2_{\text{ét}}(T, \boldsymbol{\mu}_2)/\langle c_1(\mathscr{F})\rangle \mid x \cup c_1(\mathscr{F}) \neq 0\}.$$

Note that the natural map $B(T, \mathscr{F}) \to {}_2\mathrm{Br}(T)$ is injective if and only if $\mathrm{Pic}(T)$ is generated by $\mathscr{F}$. A consequence of the global description of the period domain for cubic fourfolds containing a plane is that for a K3 surface T of degree 2 with polarization $\mathscr{F}$, the subset of $H^2_{\text{ét}}(T, \boldsymbol{\mu}_2)/\langle c_1(\mathscr{F})\rangle$ represented by a cubic fourfolds containing a plane coincides with $B(T, f) \cup \{0\}$, cf. [58, §9.7] and [37, Prop. 2.1].

We can now state the main result of this section. Using Theorem 1 and results on twisted sheaves described below, we provide an algebraic proof of the following result, which is due to Voisin [60] (cf. [58, §9.7] and [37, Prop. 2.1]). See [12, Prop. 6.3] for a related result.

Theorem 7.3. *Let T be a general K3 surface of degree 2 with a polarization $\mathscr{F}$. Then each element of $B(T, \mathscr{F})$ is represented by a single cubic fourfold containing a plane up to isomorphism.*

We now explain the interest in this statement. The global Torelli theorem for cubic fourfolds states that a cubic fourfold Y is determined up to isomorphic by the polarized Hodge structure on $H^4(Y, \mathbb{Z})$. Here *polarization* means a class $h^2 \in H^4(Y, \mathbb{Z})$ of self-intersection 3. Voisin's approach [60] is to deal first with cubic fourfolds containing a plane, then apply a deformation argument to handle the general case. For cubic fourfolds containing a plane, we can give an alternate argument in the general case, assuming the global Torelli theorem for K3 surfaces of degree 2, which is a celebrated result of Piatetski-Shapiro and Shafarevich [53].

Proposition 7.4. *Assuming the global Torelli theorem holds for K3 surfaces of degree 2, the global Torelli theorem holds for general cubic fourfolds.*

Proof. Let Y be a cubic fourfold containing a plane P with discriminant cover $f : T \to \mathbb{P}^2$ and even Clifford algebra $\mathscr{C}_0$. Consider the cycle class of P

in $H^4(Y,\mathbb{Z})$. Then $\mathscr{F} = f^*\mathscr{O}_{\mathbb{P}^2}(1)$ is a polarization on T, which together with $[\mathscr{C}_0] \in H^2_{\text{ét}}(T,\boldsymbol{\mu}_2)$, determines the sublattice $\langle h^2, P\rangle^\perp \subset H^4(Y,\mathbb{Z})$. The key lattice-theoretic result we use is [60, §1, Prop. 3], which can be stated as follows: the polarized Hodge structure $H^2(T,\mathbb{Z})$ and the class $[\mathscr{C}_0] \in H^2_{\text{ét}}(T,\boldsymbol{\mu}_2)$ determines the Hodge structure of Y; conversely, the polarized Hodge structure $H^4(Y,\mathbb{Z})$ and the sublattice $\langle h^2, P\rangle$ determines the primitive Hodge structure of T, hence T itself by the global Torelli theorem for K3 surfaces of degree 2. Furthermore, if Y (and hence T) is general, then $H^4(Y,\mathbb{Z})$ and $\langle h^2, P\rangle$ determines the Brauer class $[\mathscr{C}_0]$ of the even Clifford algebra.
Now let Y and Y' be cubic fourfolds containing a plane P with associated discriminant covers T and T' and even Clifford algebras $\mathscr{C}_0$ and $\mathscr{C}_0'$. Assume that $\Psi : H^4(Y,\mathbb{Z}) \cong H^4(Y,\mathbb{Z})$ is an isomorphism of Hodge structures preserving the polarization h^2. By [35, Prop. 3.2.4], we can assume (by composing Ψ with a Hodge automorphism fixing h^2) that Ψ preserves the sublattice $\langle h^2, P\rangle$. By [60, §1, Prop. 3], Ψ induces an isomorphism $T \cong T'$, with respect to which $[\mathscr{C}_0] = [\mathscr{C}_0'] = \beta \in H^2_{\text{ét}}(T,\boldsymbol{\mu}_2) \cong H^2_{\text{ét}}(T',\boldsymbol{\mu}_2)$, for T general. Hence if there is at most a single cubic fourfold representing β up to isomorphism then $Y \cong Y'$. □

The following lemma, whose proof we could not find in the literature, holds for smooth cubic hypersurfaces $Y \subset \mathbb{P}^{2r+1}_k$ containing a linear subspace of dimension r over any field k. Since $\mathrm{Aut}(\mathbb{P}^{2r+1}_k) \cong \mathbf{PGL}_{2r+2}(k)$ acts transitively on the set of linear subspaces in $\mathbb{P}^{2r+1}_k$ of dimension r, any two cubic hypersurfaces containing linear subspaces of dimension r have isomorphic representatives containing a common such linear subspace.

Lemma 7.5. *Let Y_1 and Y_2 be smooth cubic hypersurfaces in $\mathbb{P}^{2r+1}_k$ containing a linear space P of dimension r. The associated quadric bundles $\pi_1 : \widetilde{Y}_1 \to \mathbb{P}^r_k$ and $\pi_2 : \widetilde{Y}_2 \to \mathbb{P}^r_k$ are $\mathbb{P}^r_k$-isomorphic if and only if the there is a linear isomorphism $Y_1 \cong Y_2$ fixing P.*

Proof. Any linear isomorphism $Y_1 \cong Y_2$ fixing P will induce an isomorphism of blow-ups $\widetilde{Y}_1 \cong \widetilde{Y}_2$ commuting with the projections from P. Conversely, assume that $\widetilde{Y}_1$ and $\widetilde{Y}_2$ are $\mathbb{P}^r_k$-isomorphic. Since $\mathbf{PGL}_{2r+2}(k)$ acts transitively on the set of linear subspaces of dimension r, without loss of generality, we can assume that $P = \{x_0 = \cdots = x_r = 0\}$ where $(x_0 : \cdots : x_r : y_0 : \cdots : y_r)$ are homogeneous coordinates on $\mathbb{P}^{2r+1}_k$. For $l = 1, 2$, write Y_l as

$$\sum_{0\le m\le n\le r} a^l_{mn} y_m y_n + \sum_{0\le p\le r} b^l_p y_p + c^l = 0$$

for homogeneous linear forms a^l_{mn}, quadratic forms b^l_p, and cubic forms c^l in $k[x_0,\dots,x_r]$. The blow-up of $\mathbb{P}^{2r+1}_k$ along P is identified with the total space of the projective bundle $\pi : \mathbb{P}(\mathscr{E}) \to \mathbb{P}^r_k$, where $\mathscr{E} = \mathscr{O}^{r+1}_{\mathbb{P}^r_k} \oplus \mathscr{O}_{\mathbb{P}^r_k}(-1)$. The homogeneous coordinates $y_0,\dots,y_r$ correspond, in the blow-up, to a basis of global sections of $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$. Let z be a nonzero global section of of line bundle

$\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1) \otimes \pi^* \mathscr{O}_{\mathbb{P}^r_k}(-1)$. Then z is unique up to scaling, as we have

$$\begin{aligned}\Gamma(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1) \otimes \pi^* \mathscr{O}_{\mathbb{P}^r_k}(-1)) &\cong \Gamma(\mathbb{P}^r_k, \pi_* \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1) \otimes \mathscr{O}_{\mathbb{P}^r_k}(-1)) \\ &= \Gamma(\mathbb{P}^r_k, \mathscr{E}^\vee \otimes \mathscr{O}_{\mathbb{P}^r_k}(-1)) = k\end{aligned}$$

by the projection formula. Thus $(y_0 : \cdots : y_r : z)$ forms a relative system of homogeneous coordinates on $\mathbb{P}(\mathscr{E})$ over $\mathbb{P}^r_k$. Then $\widetilde{Y}_l$ can be identified with the subscheme of $\mathbb{P}(\mathscr{E})$ defined by the global section

$$q_l(y_0, \ldots, y_r, z) = \sum_{0 \le m \le n \le r} a^l_{mn} y_m y_n + \sum_{0 \le p \le r} b^l_p y_p z + c^l z^2 = 0$$

of $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(2) \otimes \pi^* \mathscr{O}_{\mathbb{P}^r_k}(1)$. Under these identifications, $\pi_l : \widetilde{Y}_l \to \mathbb{P}^r_k$ can be identified with the restriction of π to $\widetilde{Y}_l$, hence with the quadric bundle associated to the line bundle-valued quadratic form $(\mathscr{E}, q_l, \mathscr{O}_{\mathbb{P}^r_k}(1))$. Since Y_l and P are smooth, so is $\widetilde{Y}_l$. Thus $\pi_l : \widetilde{Y}_l \to \mathbb{P}^r_k$ is flat, being a morphism from a Cohen–Macaulay scheme to a regular scheme. Thus by Propositions 1.1 and 1.6, the $\mathbb{P}^r_k$-isomorphism $\widetilde{Y}_1 \cong \widetilde{Y}_2$ induces a projective similarity ψ between q_1 and q_2. But as $\mathscr{E} \otimes \mathscr{N} \cong \mathscr{E}$ implies $\mathscr{N}$ is trivial in $\mathrm{Pic}(\mathbb{P}^r_k)$, we have that $\psi : q_1 \simeq q_2$ is, in fact, a similarity. In particular, $\psi \in \mathbf{GL}(\mathscr{E})(\mathbb{P}^r_k)$, hence consists of a block matrix of the form

$$\begin{pmatrix} H & v \\ 0 & u \end{pmatrix}$$

where $H \in \mathbf{GL}(\mathscr{O}^{r+1}_{\mathbb{P}^r_k})(\mathbb{P}^r_k) = \mathbf{GL}_{r+1}(k)$ is a constant invertible matrix and $u \in \mathbf{GL}(\mathscr{O}_{\mathbb{P}^r_k}(-1))(\mathbb{P}^r_k) = \mathbf{G}_{\mathrm{m}}(\mathbb{P}^r_k) = k^\times$ is an invertible constant element, while $v \in \mathrm{Hom}_{\mathscr{O}_{\mathbb{P}^r_k}}(\mathscr{O}_{\mathbb{P}^r_k}(-1), \mathscr{O}^{r+1}_{\mathbb{P}^r_k}) = \Gamma(\mathbb{P}^r_k, \mathscr{O}_{\mathbb{P}^r_k}(1))^{\oplus(r+1)}$ consists of a vector of linear forms in $k[x_0, \ldots, x_r]$. Let $v = G \cdot (x_0, \ldots, x_r)^t$ for a matrix $G \in M_{r+1}(k)$. Then writing $H = (h_{ij})$ and $G = (g_{ij})$, we have that ψ acts as

$$x_i \mapsto x_i, \qquad y_i \mapsto \sum_{0 \le j \le r} (h_{ij} y_j + g_{ij} x_j z), \qquad z \mapsto uz$$

and satisfies $q_2(\psi(y_0), \ldots, \psi(y_r), \psi(z)) = \lambda q_1(y_0, \ldots, y_r, z)$ for some $\lambda \in k^\times$. Considering the matrix $J \in M_{2r+2}(k)$ with $(r+1) \times (r+1)$ blocks

$$J = \begin{pmatrix} uI & 0 \\ G & H \end{pmatrix}$$

as a linear automorphism of $\mathbb{P}^{2r+1}_k$, then J acts on $(x_0 : \cdots : x_r : y_0 : \cdots : y_r)$ as

$$x_i \mapsto u x_i, \qquad y_i \mapsto \sum_{0 \le j \le r} (h_{ij} y_j + g_{ij} x_j),$$

and hence satisfies $q_2(J(y_0), \ldots, J(y_r), 1) = u\lambda q_1(y_0, \ldots, y_r, 1)$ due to the homogeneity properties of x_i and z. Thus J is a linear automorphism taking Y_1 to Y_2 and fixes P. □

Let T be a K3 surface. We shall freely use the notions of β-twisted sheaves, B-fields associated to β, the β-twisted Chern character, and β-twisted Mukai vectors from [38]. For a Brauer class $\beta \in {}_2\mathrm{Br}(T)$ we choose the rational B-field $\beta/2 \in H^2(T, \mathbb{Q})$. The β-twisted Mukai vector of a β-twisted sheaf $\mathscr{V}$ is

$$v^B(\mathscr{V}) = \mathrm{ch}^B(\mathscr{V})\sqrt{\mathrm{Td}_T} = \big(\mathrm{rk}\,\mathscr{V}, c_1^B(\mathscr{V}), \mathrm{rk}\,\mathscr{V} + \frac{1}{2}c_1^B(\mathscr{V}) - c_2^B(\mathscr{V})\big) \in H^*(T, \mathbb{Q})$$

where $H^*(T,\mathbb{Q}) = \bigoplus_{i=0}^{2} H^{2i}(T,\mathbb{Q})$. As in [47], one introduces the Mukai pairing

$$(v, w) = v_2 \cup w_2 - v_0 \cup w_4 - v_4 \cup w_0 \in H^4(T, \mathbb{Q}) \cong \mathbb{Q}$$

for Mukai vectors $v = (v_0, v_2, v_4)$ and $w = (w_0, w_2, w_4)$.
By [63, Thm. 3.16], the moduli space of stable β-twisted sheaves $\mathscr{V}$ with Mukai vector $v = v^B(\mathscr{V})$ satisfying $(v, v) = 2n$ is isomorphic to the Hilbert scheme Hilb_T^{n+1}. In particular, when $(v, v) = -2$, this moduli space consists of one point; we give a direct proof of this fact inspired by [47, Cor. 3.6].

Lemma 7.6. *Let T be a K3 surface and $\beta \in {}_2\mathrm{Br}(T)$ with chosen B-field. Let $v \in H^*(T,\mathbb{Q})$ with $(v, v) = -2$. If $\mathscr{V}$ and $\mathscr{V}'$ are stable β-twisted sheaves with $v^B(\mathscr{V}) = v^B(\mathscr{V}') = v$ then $\mathscr{V} \cong \mathscr{V}'$.*

Proof. Assume that β-twisted sheaves $\mathscr{V}$ and $\mathscr{V}'$ have the same Mukai vector $v \in H^2(T, \mathbb{Q})$. Since $-2 = (v, v) = \chi(\mathscr{V}, \mathscr{V}) = \chi(\mathscr{V}, \mathscr{V}')$, a Riemann–Roch calculation shows that either $\mathrm{Hom}(\mathscr{V}, \mathscr{V}') \neq 0$ or $\mathrm{Hom}(\mathscr{V}, \mathscr{V}') \neq 0$. Without loss of generality, assume $\mathrm{Hom}(\mathscr{V}, \mathscr{V}') \neq 0$. Since $\mathscr{V}$ is stable, a nonzero map $\mathscr{V} \to \mathscr{V}'$ must be injective. Since $\mathscr{V}'$ is stable, the map is an isomorphism. □

Lemma 7.7. *Let T be a K3 surface of degree 2 and $\beta \in {}_2\mathrm{Br}(T)$ with chosen B-field. Let Y be a smooth cubic fourfold containing a plane whose even Clifford algebra $\mathscr{C}_0$ represents $\beta \in {}_2\mathrm{Br}(T)$. If $\mathscr{V}_0$ is a β-twisted sheaf associated to $\mathscr{C}_0$ then $(v^B(\mathscr{V}_0), v^B(\mathscr{V}_0)) = -2$. Furthermore, if T is general then $\mathscr{V}_0$ is stable.*

Proof. By the β-twisted Riemann–Roch theorem, we have

$$-(v^B(\mathscr{V}_0), v^B(\mathscr{V}_0)) = \chi(\mathscr{V}_0, \mathscr{V}_0) = \sum_{i=0}^{2} \mathrm{Ext}_T^i(\mathscr{V}_0, \mathscr{V}_0).$$

Then $v^B(\mathscr{V}_0) = 2$ results from the fact that $\mathscr{V}_0$ is a *spherical* object, i.e., $\mathrm{Ext}_T^i(\mathscr{V}_0, \mathscr{V}_0) = \mathbb{C}$ for $i = 0, 2$ and $\mathrm{Ext}^1(\mathscr{V}_0, \mathscr{V}_0) = 0$. Indeed, as in [45, Rem. 2.1], we have $\mathrm{Ext}_T^i(\mathscr{V}_0, \mathscr{V}_0) = H^i(\mathbb{P}^2, \mathscr{C}_0)$, which can be calculated directly using the fact that, as $\mathscr{O}_{\mathbb{P}^2}$-algebras,

$$\mathscr{C}_0 \cong \mathscr{O}_{\mathbb{P}^2} \oplus \mathscr{O}_{\mathbb{P}^2}(-3) \oplus \mathscr{O}_{\mathbb{P}^2}(-1)^3 \oplus \mathscr{O}_{\mathbb{P}^2}(-2)^3 \tag{10}$$

If T is general, stability follows from [47, Prop. 3.14], cf. [63, Prop. 3.12]. □

Lemma 7.8. *Let T be a K3 surface of degree 2. Let Y and Y' be smooth cubic fourfolds containing a plane whose respective even Clifford algebras $\mathscr{C}_0$ and $\mathscr{C}_0'$ represent the same $\beta \in {}_2\mathrm{Br}(T)$. If T is general then $\mathscr{C}_0 \cong \mathscr{C}_0'$.*

Proof. Let $\mathscr{V}_0$ and $\mathscr{V}_0'$ be β-twisted sheaves associated to $\mathscr{C}_0$ and $\mathscr{C}_0$, respectively. A consequence of [45, Lemma 3.1] and (10) is that $v = v^B(\mathscr{V}_0) = v^B(\mathscr{V}_0' \otimes \mathscr{N})$ for some line bundle $\mathscr{N}$ on T. Replacing $\mathscr{V}_0'$ by $\mathscr{V}_0' \otimes \mathscr{N}^\vee$, we can assume that $v^B(\mathscr{V}_0) = v^B(\mathscr{V}_0')$. By Lemma 7.7, we have $(v, v) = -2$ and that $\mathscr{V}_0$ and $\mathscr{V}_0'$ are stable. Hence by Lemma 7.6, we have $\mathscr{V}_0 \cong \mathscr{V}_0'$ as β-twisted sheaves, hence $\mathscr{C}_0 \cong \mathscr{E}nd(\mathscr{V}_0) \cong \mathscr{E}nd(\mathscr{V}_0') \cong \mathscr{C}_0'$. □

Proof of Theorem 7.3. Suppose that Y and Y' are smooth cubic fourfolds containing a plane whose associated even Clifford algebras $\mathscr{C}_0$ and $\mathscr{C}_0'$ represent the same class $\beta \in B(T, \mathscr{F}) \subset H^2_{\text{ét}}(T, \boldsymbol{\mu}_2)/\langle c_1(\mathscr{F})\rangle \cong {}_2\mathrm{Br}(T)$. By Lemma 7.8, we have $\mathscr{C}_0 \cong \mathscr{C}_0'$. By Theorem 1, the quadric surface bundles $\pi : \widetilde{Y} \to \mathbb{P}^2$ and $\pi' : \widetilde{Y}' \to \mathbb{P}^2$ are $\mathbb{P}^2$-isomorphic. Finally, by Lemma 7.5, we have $Y \cong Y'$. □

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Hermitian Lattices and Bounds in K-Theory of Algebraic Integers

To Sasha Merkurjev, for his sixtieth birthday

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Abstract. Elaborating on a method of Soulé, and using better estimates for the geometry of hermitian lattices, we improve the upper bounds for the torsion part of the K-theory of the rings of integers of number fields.

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1 Introduction

Let F be a number field of degree d, with ring of integers $\mathcal{O}_F$ and discriminant D_F. We denote by $K_n(\mathcal{O}_F)$ the n-th K-theory group of $\mathcal{O}_F$, which was defined by Quillen and showed by him to be finitely generated. The rank of $K_n(\mathcal{O}_F)$ has been computed by Borel in [4]. In this article we consider the problem of finding an upper bound – in terms of n, d and D_F – for the order of the torsion part $K_n(\mathcal{O}_F)_{\text{tors}}$. Such general bounds have been obtained by Soulé in [11]. Our Theorem 1.1 below sharpens Soulé's results.

As in Soulé's paper, our inequalities hold "up to small torsion". To state this precisely, for a finite abelian group A let us write $\text{card}_\ell(A)$ for the order of A/B, where $B \subset A$ is the subgroup generated by elements of order $\leq \ell$.

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Theorem 1.1. *Let $F \neq \mathbb{Q}$ be a number field of degree d and with discriminant D_F. Then for any $n \geq 2$ we have*

$$\log \operatorname{card}_\ell K_n(\mathcal{O}_F)_{\text{tors}} \leq (2n+1)^{71n^4d^3} \cdot d^{293n^5d^5} \cdot |D_F|^{528n^5d^4},$$

where $\ell = \max(d+1, 2n+2)$.

To improve the readability, we have not tried to state here the best possible bounds that one could get with the method we use. We refer to the PhD thesis of the third author [7, Theorem 4.3] – in which the result was originally obtained – for slightly better estimates. However, it does not change the fact that the upper bounds are huge, and – although explicit – certainly unusable for practical computation. We shall insist here on the qualitative aspect of our result, which could be stated as follows.

Corollary 1.2. *There exist α and β, both polynomials in n and d, such that for any number field F of degree $d \geq 2$ we have*

$$\log \operatorname{card}_\ell K_n(\mathcal{O}_F)_{\text{tors}} \leq (nd)^\alpha |D_F|^\beta,$$

where $\ell = \max(d+1, 2n+2)$ and $n \geq 2$.

Compared to [11], our result improves the bound by an exponential factor: the previous bound for $\log \operatorname{card}_\ell K_n(\mathcal{O}_F)$ was at least $\exp(\alpha|D_F|^{1/2})$, for some polynomial $\alpha = \alpha(n,d)$ (see Proposition 4 in loc. cit. for the precise statement). The strategy is the following. The group $K_n(\mathcal{O}_F)$ can be related – via the Hurewicz map – to the integral homology $H_n(\mathrm{GL}(\mathcal{O}_F))$, and an upper bound (up to small torsion) for the order of $K_n(\mathcal{O}_F)$ can then be obtained through the study of the integral homology of $\mathrm{GL}_N(\mathcal{O}_F)$, with $N = 2n+1$ (cf. Section 3). The proof of Theorem 1.1 follows the method of Soulé, which uses Ash's well-rounded retract (cf. Section 2) to study these homology groups. This reduces the problem to finding good estimates concerning the geometry of hermitian lattices. Our approach to these estimates differs from that of Soulé (cf. Section 4), leading to the improved bounds in Theorem 1.1.
For $F = \mathbb{Q}$ our method does not bring any improvement, so that [11, Prop. 4 iv)] is still the best available general bound for $K_n(\mathbb{Z})$. We refer to [6, Theorem 1.3] for a different approach to the same problem for K_2, which gives better result than Corollary 1.2 in the case of totally imaginary fields. Note that all these results remain very far from the general bound conjectured by Soulé in [11, Sect. 5.1], which should take the following form for some constant $C(n,d)$:

$$\log \operatorname{card} K_n(\mathcal{O}_F)_{\text{tors}} \leq C(n,d) \log |D_F|. \tag{1.1}$$

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2 Hermitian metrics and the well-rounded retract

2.1 Notation

We keep the notation of the introduction. Let us denote by (r_1, r_2) the signature of the number field F. Let us write $F_\mathbb{R} = \mathbb{R} \otimes_\mathbb{Q} F$. If Σ denotes the set of field embeddings $\sigma : F \to \mathbb{C}$, then $F_\mathbb{R}$ can be identified with the subspace $(\mathbb{C}^\Sigma)^+ \subset \mathbb{C}^\Sigma$ invariant under the involution $(x_\sigma) \mapsto (\overline{x_{\overline{\sigma}}})$, where $\overline{a}$ denotes the complex conjugation. This also provides an isomorphism $F_\mathbb{R} \cong \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$.

For $x \in F_\mathbb{R}$, written as $x = (x_\sigma)_{\sigma \in \Sigma}$, we denote by $\overline{x} = (\overline{x_\sigma})$ the result of the complex conjugation applied component-wise. We denote by Tr the trace map from $F_\mathbb{R}$ to $\mathbb{R}$, defined by $\mathrm{Tr}(x) = \sum_{\sigma \in \Sigma} x_\sigma$. We will also use the absolute value of the norm map: $\mathcal{N}(x) = \prod_{\sigma \in \Sigma} |x_\sigma|$.

We fix a free $\mathcal{O}_F$-lattice L of finite rank $N \geq 1$. Let $V = F \otimes_{\mathcal{O}_F} L$ and $V_\mathbb{R} = \mathbb{R} \otimes_\mathbb{Q} V$, so that $V_\mathbb{R}$ can be seen as a (left) $F_\mathbb{R}$-module. Let Γ be the group $\mathrm{GL}(L)$ of automorphism of L. By fixing a basis of L, we have the identification $\Gamma = \mathrm{GL}_N(\mathcal{O}_F)$. Then Γ is a discrete subgroup of the reductive Lie group $\mathrm{GL}(V_\mathbb{R}) = \mathrm{GL}_N(F_\mathbb{R})$. We shall denote the latter by G, and we will let it act on $V_\mathbb{R}$ on the left (and similarly for Γ on L).

2.2 Hermitian metrics

Let $h : V_\mathbb{R} \times V_\mathbb{R} \to F_\mathbb{R}$ be a hermitian form on $V_\mathbb{R}$, that is, $h = (h_\sigma)_{\sigma \in \Sigma}$ is $F_\mathbb{R}$-linear in the first variable and $h(y,x) = \overline{h(x,y)}$. The pair (L,h) is called a *hermitian lattice*. When $x = y$, we also write $h(x) = h(x,x)$. Note that $h(x)$ has only real components, and we say that h is positive definite if $h_\sigma(x,x) > 0$ for any nonzero $x \in V_\mathbb{R}$ and all $\sigma \in \Sigma$.

Let X be the (topological) space of positive definite hermitian forms on $V_\mathbb{R}$. The group $G = \mathrm{GL}(V_\mathbb{R})$ acts transitively on X in the following way: the element $\gamma \in G$ maps the form $h \in X$ to

$$(\gamma \cdot h)(x,y) = h(\gamma^{-1}x, \gamma^{-1}y). \tag{2.1}$$

The space X can be identified with the set of positive definite symmetric $N \times N$ matrices with coefficients in $F_\mathbb{R}$. Using this identification, it is not difficult to see that X is contractible.

To each $h \in X$ we associate the real quadratic form q_h on $V_\mathbb{R}$ (seen as a real vector space) defined for $x \in V_\mathbb{R}$ by:

$$q_h(x) = \mathrm{Tr}(h(x)). \tag{2.2}$$

Such a quadratic form $q_h : V_\mathbb{R} \to \mathbb{R}$ for $h \in X$ will be called a *hermitian metric*. Given h, we will denote by $\|\cdot\|_h$ the norm on $V_\mathbb{R}$ induced by q_h.

2.3 Ash's well-rounded retract

For $h \in X$ we set $m(L,h) \in \mathbb{R}_{>0}$ to be the minimum of $q_h(x)$ over the nonzero $x \in L \subset V$, and define

$$M(L,h) = \{x \in L \mid q_h(x) = m(L,h)\}. \tag{2.3}$$

Definition 2.1. We say that $h \in X$ (or (L,h)) is *well rounded* if $m(L,h) = 1$ and $M(L,h)$ generates V (as a vector space over F).

Let $\widetilde{W} \subset X$ be the subspace of well-rounded hermitian forms. Note that the action defined by (2.1) restricts to an action of $\Gamma = \mathrm{GL}_N(\mathcal{O}_F)$ on $\widetilde{W}$. In [2, p. 466–467], Ash defined a CW-complex structure on $\widetilde{W}$ that has the following properties: two points h and h' of $\widetilde{W}$ belong to the interior of the same cell $C(h) = C(h')$ if and only if $M(L,h) = M(L,h')$, and $C(h') \subset C(h)$ if and only if $M(L,h') \supset M(L,h)$. Moreover, we have that $M(L,\gamma \cdot h) = \gamma M(L,h)$. Then the action of Γ on $\widetilde{W}$ is compatible with the cell structure: an element $\gamma \in \Gamma$ maps the cell $C(h)$ to $C(\gamma \cdot h)$.

Theorem 2.2 (Ash). *$\widetilde{W}$ is a deformation retract of X on which Γ acts with finite stabilizer Γ_σ for each cell σ of $\widetilde{W}$. The quotient $\Gamma\backslash\widetilde{W}$ is compact, of dimension $\dim(X) - N$.*

Proof. The proof of this statement follows from the argument of Ash given in the proof of the main theorem of [2], page 462. More precisely, the argument to prove that $\widetilde{W}$ is a deformation retract of X is the same as Ash uses for $W = \widetilde{W}/\Gamma$, in [2, §3 (i)]. The compactness of $\Gamma\backslash\widetilde{W}$ is proved in §3 (ii) of loc. cit., and the dimension is computed on page 466. □

Let $\mathcal{C}_\bullet$ be the complex of cellular chains on $\Gamma\backslash\widetilde{W}$. We can decompose it as $\mathcal{C}_\bullet = \mathcal{C}_\bullet^+ \cup \mathcal{C}_\bullet^-$, where Γ preserves (resp. does not preserve) the orientation of each $\sigma \in \mathcal{C}_\bullet^+$ (resp. $\sigma \in \mathcal{C}_\bullet^-$). It then follows from the spectral sequence described in [5, VII (7.10)] that up to prime divisors of the finite stabilizers Γ_σ, the homology of $\mathcal{C}_\bullet^+$ computes $H_\bullet(\Gamma)$. In particular, one has the following (cf. [11, Lemma 9]). See Section 1 for the definition of card_ℓ.

Corollary 2.3. *Let $\ell = 1 + \max(d,N)$. Then for any n we have:*

$$\mathrm{card}_\ell\, H_n(\Gamma)_{\mathrm{tors}} = \mathrm{card}_\ell\, H_n(\mathcal{C}_\bullet^+)_{\mathrm{tors}},$$

where $H_n(\cdot)_{\mathrm{tors}}$ denotes the torsion part of the integral homology.

3 Bounding torsion homology and K-theory

3.1 The Hurewicz map

For any $n > 1$ we consider the n-th Quillen K-group $K_n(\mathcal{O}_F) = \pi_n(B\,\mathrm{GL}(\mathcal{O}_F)^+)$ ("plus construction"). The Hurewicz map relating homotopy groups to homology provides a map $K_n(\mathcal{O}_F) \to H_n(\mathrm{GL}(\mathcal{O}_F)^+) =$

$H_n(\mathrm{GL}(\mathcal{O}_F))$. We know (see for instance [1, Theorem 1.5]) that its kernel does not contain elements of order p for $p > \frac{n+1}{2}$. Moreover, by a stability result of van der Kallen and Maazen (cf. [12, Theorem 4.11]) the homology of $\mathrm{GL}(\mathcal{O}_F) = \varinjlim \mathrm{GL}_N(\mathcal{O}_F)$ is equal to the homology of $\mathrm{GL}_N(\mathcal{O}_F)$ for any $N \geq 2n+1$. Let then $N = 2n+1$, and consider $\Gamma = \mathrm{GL}_N(\mathcal{O}_F)$. We deduce from Corollary 2.3 that for $\ell = \max(d+1, 2n+2)$ we have:

$$\mathrm{card}_\ell\, K_n(\mathcal{O}_F)_{\mathrm{tors}} \leq \mathrm{card}_\ell\, H_n(\mathcal{C}^+_\bullet)_{\mathrm{tors}}. \tag{3.1}$$

3.2 Gabber's lemma

The abstract result that allows to obtain a bound for the right hand side of (3.1) is the following lemma. It was discovered by Gabber, and first appeared in Soulé [10, Lemma 1]. See Sauer [9, Lemma 3.2] for a more elementary proof.

Lemma 3.1 (Gabber). *Let $A = \mathbb{Z}^a$ with the standard basis $(e_i)_{i=1,\ldots,a}$ and $B = \mathbb{Z}^b$, so that $B \otimes \mathbb{R}$ is equipped with the standard Euclidean norm $\|\cdot\|$. Let $\phi : A \to B$ be a $\mathbb{Z}$-linear map and let $\alpha \in \mathbb{R}$ such that $\|\phi(e_i)\| \leq \alpha$ for each $i = 1, \ldots, a$. If we denote by Q the cokernel of ϕ, then*

$$|Q_{\mathrm{tors}}| \leq \alpha^{\min(a,b)}\,.$$

Corollary 3.2. *Suppose that the cellular complex $\Gamma\backslash\widetilde{W}$ has at most α_k faces for any $k \geq 0$, and that any k-cell has at most β codimension 1 faces. Then*

$$H_k(\mathcal{C}^+_\bullet)_{\mathrm{tors}} \leq \beta^{\frac{1}{2}\min(\alpha_{k+1}, \alpha_k)}.$$

Proof. For a cell $c \in \mathcal{C}^+_{k+1}$, its image by the boundary map ∂ is a sum of at most β k-cells, so that $\|\partial c\| \leq \sqrt{\beta}$. Thus, by Lemma 3.1 $\mathrm{coker}(\partial)_{\mathrm{tors}}$ is bounded by $\beta^{\frac{1}{2}\min(\alpha_{k+1}, \alpha_k)}$ and a fortiori so is $H_k(\mathcal{C}^+_\bullet)_{\mathrm{tors}}$. □

3.3 Counting the cells

Suppose that the finite subset $\Phi \subset L$ has the following property:

> for any well-rounded pair (L, h), there exists $\gamma \in \Gamma = \mathrm{GL}_N(\mathcal{O}_F)$ such that $\gamma M(L, h) \subset \Phi$.

In other words, Φ contains a representative of every element of $\Gamma\backslash\widetilde{W}$. Since $C(h)$ has codimension j, where $N+j$ is the cardinality of $M(L,h)$, it follows immediately that the number of cells of codimension j in $\Gamma\backslash\widetilde{W}$ is bounded by the binomial coefficient $\binom{\mathrm{card}(\Phi)}{N+j}$. For large $\mathrm{card}(\Phi)$ we lose little by bounding this binomial coefficient by $\mathrm{card}(\Phi)^{N+j}$. Recall that $\Gamma\backslash\widetilde{W}$ has dimension

$\dim(X)-N$, so that for a k-cell of codimension j we have $N+j=\dim(X)-k$. For the dimension of X we have (where (r_1,r_2) is the signature of F):

$$\dim(X) = r_1\frac{N(N+1)}{2}+r_2N^2 \tag{3.2}$$

$$\leq d\,\frac{N(N+1)}{2}. \tag{3.3}$$

Thus, for the number of k-cells in $\Gamma\backslash\widetilde{W}$ we can use the following upper bound:

$$\alpha_k = \mathrm{card}(\Phi)^{d\cdot\frac{N(N+1)}{2}-k} \tag{3.4}$$

By a similar counting argument, Soulé shows in [11, proof of Prop. 3] that there are at most $\beta=\mathrm{card}(\Phi)^{N+1}$ faces of codimension 1 in any given cell (not necessarily top dimensional) on $\Gamma\backslash\widetilde{W}$.

3.4 Bounds for K-theory in terms of Φ

Let $\ell = \max(d+1, 2n+2)$. By Corollary 3.2 and (3.1) we have that $\mathrm{card}_\ell K_n(\mathcal{O}_F)_{\mathrm{tors}}$ is bounded by $\beta^{\frac{1}{2}\alpha_{n+1}}$, where α_{n+1} and β can be chosen as in Section 3.3 (with $N=2n+1$). This gives (using now logarithmic notation):

$$\log\mathrm{card}_\ell K_n(\mathcal{O}_F)_{\mathrm{tors}} \leq (n+1)\log(\mathrm{card}(\Phi))\,\mathrm{card}(\Phi)^{e(d,n)}, \tag{3.5}$$

where $\Phi\subset L$ has the property defined in Section 3.3, and

$$e(d,n) = d(2n^2+3n+1)-n-1. \tag{3.6}$$

This reduces the problem to finding such a set $\Phi\subset L$ of size as small as possible. In [11] Soulé constructed a suitable set Φ using the geometry of numbers. In what follows, we will exhibit a smaller Φ by using better estimates on hermitian lattices.

4 Hermitian lattices and bounded bases

The goal of this section is to construct in any well-rounded lattice (L,h) a basis whose vectors have bounded length, with respect to the norm induced by h. The method in an adaptation of the idea used by Soulé in [11] (see Section 4.3 below), in which we incorporate the results from [3], corresponding to the rank one case.

4.1 Geometry of ideal lattices

Let $I\subset F_{\mathbb{R}}$ be a nonzero $\mathcal{O}_F$-submodule of the form $I=x\mathfrak{a}$, where $x\in F_{\mathbb{R}}$ and $\mathfrak{a}$ is a fractional ideal of F. We define the *norm* of I by the rule $\mathcal{N}(I)=\mathcal{N}(x)\mathcal{N}(\mathfrak{a})$. Let q_0 be the standard (positive definite) hermitian metric on $F_{\mathbb{R}}$, i.e., for

$x \in F_{\mathbb{R}}$: $q(x) = \mathrm{Tr}(x\overline{x})$. The pair (I, q_0) is an *ideal lattice* (over F) in the sense of [3, Def. 2.2]. Its determinant is given by (see [3, Cor. 2.4]):

$$\det(I, q_0) = \mathcal{N}(I)^2\,|D_F|. \tag{4.1}$$

Let us denote by $\|\cdot\|$ the norm on $F_{\mathbb{R}}$ induced by the hermitian metric q_0. Estimates for the geometry of ideal lattices have been studied in [3]. For our particular case (I, q_0), Proposition 4.2 in loc. cit. takes the following form.

PROPOSITION 4.1. *Let F of degree d, with discriminant D_F, and consider the ideal lattice (I, q_0). Then for any $x \in F_{\mathbb{R}}$ there exists $y \in I$ such that $\|x - y\| \leq R$, where*

$$R = \frac{\sqrt{d}}{2}|D_F|^{1/d}\mathcal{N}(I)^{1/d}.$$

4.2 THREE CONSEQUENCES

From Proposition 4.1 we deduce the three following lemmas.

LEMMA 4.2. *Given $x = (x_\sigma) \in F_{\mathbb{R}}$, there exists $a \in \mathcal{O}_F$ such that*

$$\sum_{\sigma\in\Sigma} |x_\sigma - \sigma(a)| \leq C_2,$$

where

$$C_2 = \frac{d}{2}\,|D_F|^{1/d}. \tag{4.2}$$

Proof. First note the general inequality $(\sum_{i=1}^d b_i)^2 \leq d \cdot \sum_{i=1}^d b_i^2$, which follows from applying the summation $\sum_{i,j}$ on both sides of

$$2b_ib_j \leq b_i^2 + b_j^2.$$

This implies that for $a = y \in I$ as in Proposition 4.1 with $I = \mathcal{O}_F$, we have

$$\begin{aligned}\sum_{\sigma\in\Sigma} |x_\sigma - \sigma(a)| &\leq \sqrt{d \cdot \sum_{\sigma\in\Sigma} |x_\sigma - \sigma(a)|^2} \\ &= \sqrt{d}\,\|x - a\| \\ &\leq \frac{d}{2}|D_F|^{1/d}.\end{aligned}$$

□

LEMMA 4.3. *Given $x = (x_\sigma) \in F_{\mathbb{R}}$, there exists $a \in \mathcal{O}_F$ such that*

$$\sup_{\sigma\in\Sigma} |\sigma(a)\,x_\sigma| \leq C_3\,\mathcal{N}(x)^{1/d},$$

where

$$C_3 = \sqrt{d}\cdot|D_F|^{1/d}. \tag{4.3}$$

Proof. We can suppose that $x \neq 0$. We consider the ideal lattice (I, q_0) with $I = x\mathcal{O}_F$. For R as in Proposition 4.1, we have that $F_{\mathbb{R}} = I + B_R(0)$, where $B_R(0)$ is the closed ball of radius R with respect to $\|\cdot\|$. In particular, the smallest (nonzero) vector $xa \in I = x\mathcal{O}_F$ has length $\leq 2R$. That is, there exists $a \in \mathcal{O}_F$ such that

$$\begin{aligned}\sup_{\sigma\in\Sigma} |\sigma(a)x_\sigma| &\leq \|xa\| \\ &\leq 2R;\end{aligned}$$

and the result follows. ☐

Lemma 4.4. *Let $\mathfrak{a}$ be an ideal of $\mathcal{O}_F$. Then there exists a set $\mathcal{R} \subset \mathcal{O}_F$ of representatives of $\mathcal{O}_F/\mathfrak{a}$ such that for any $x \in \mathcal{R}$ we have*

$$\sum_{\sigma\in\Sigma} |\sigma(x)| \leq C_2 \mathcal{N}(\mathfrak{a})^{1/d}.$$

Proof. Let us consider the ideal lattice $(I, q) = (\mathfrak{a}, q_0)$, and let R be as in Proposition 4.1. Then for any $x \in \mathcal{O}_F \subset F_{\mathbb{R}}$, there exists $y \in I$ such that $\|x - y\| \leq R$. But $x - y \equiv x\ (I)$, so that the closed ball $B_R(0)$ contains a representative of each class of $\mathcal{O}_F/\mathfrak{a}$. The inequality is then obtained as in the proof of Lemma 4.2. ☐

4.3 Existence of bounded bases

Lemma 4.5 (Soulé). *Let $L = L_1 \oplus \cdots \oplus L_N$ be a decomposition of the hermitian lattice (L, h) into rank one lattices, and suppose that each L_i contains a vector f_i with $|L_i/\mathcal{O}_F f_i| \leq k$ and $\|f_i\|_h \leq k\lambda$, for some $k, \lambda > 1$. Then L has a basis $e_1, \ldots, e_N$ such that*

$$\|e_i\|_h \leq \lambda(1 + C_2)^{t+1} k^{(d+1)(4N-1)},$$

where $t = \lfloor \log_2(N) \rfloor + 1$.

Proof. The statement and its proof is essentially contained in the proof of [11, Prop. 1]. The main difference is that our constant C_2 is now smaller than C_2 in loc. cit. We can follow verbatim the same proof with the new C_2 except for the use of Lemma 6 (needed in Lemma 7) of loc. cit., which must be replaced by Lemma 4.4. Accordingly, the factor $(1 + C_2 \frac{r+3}{4})$ (where $r = d$) is replaced by $1 + C_2$. ☐

To obtain a bounded basis for (L, h) we need to find elements f_i that satisfy the condition of Lemma 4.5. This is done in the following proposition.

Proposition 4.6. *Let (L, h) be a free hermitian lattice over $\mathcal{O}_F$ of rank N, with $F \neq \mathbb{Q}$. We suppose that there exist $e_1, \ldots, e_N \in L$ that span $V = F \otimes_{\mathcal{O}_F} L$*

and such that $\|e_i\|_h \leq 1$ *for* $i = 1, \ldots, N$. *Then there exists a decomposition* $L = L_1 \oplus \cdots \oplus L_N$ *and elements* $f_i \in L_i$ *such that:*

$$|L_i / f_i \mathcal{O}_F| \leq C_1 C_3^d \,;$$
$$\|f_i\|_h \leq i C_1 C_2 C_3^d,$$

where $C_1 = |D_F|^{1/2}$, *and* C_2 *(resp.* C_3*) is defined in* (4.2) *(resp.* (4.3)*).*

Proof. The proof proceeds by induction, and follows the line of argument of [11, proof of Lemma 5]. Let $N = 1$. By Lemma 1 in loc. cit., there exists $x \in L$ such that $|L/\mathcal{O}_F x| \leq C_1 = |D_F|^{1/2}$. Let us write $x = \alpha \cdot e_1$ for $\alpha \in F^\times$, where by assumption $\|e_1\|_h \leq 1$. By Lemma 4.3 applied to $\alpha \in F_\mathbb{R}$, there exists $a \in \mathcal{O}_F$ such $\sup_\sigma |\sigma(a\alpha)| \leq C_3 |N(\alpha)|^{1/d}$. In particular,

$$\begin{aligned} |N(a\alpha)| &\leq \left(\sup_{\sigma \in \Sigma} |\sigma(a\alpha)| \right)^d \\ &\leq C_3^d \, |N(\alpha)|, \end{aligned}$$

so that $|N(a)| \leq C_3^d$. We set $f_1 = a \cdot x$. Then

$$\begin{aligned} |L/\mathcal{O}_F f_1| &= |N(a)| \cdot |L/\mathcal{O}_F x| \\ &\leq C_3^d C_1. \end{aligned}$$

For the norm we have:

$$\begin{aligned} \|f_1\|_h^2 &= \mathrm{Tr}(h(f_1, f_1)) \\ &= \sum_{\sigma \in \Sigma} |\sigma(\alpha)|^2 \, h_\sigma(e_1, e_1) \\ &\leq \left(\sup_{\sigma \in \Sigma} |\sigma(\alpha)| \right)^2 \, \|e_1\|_h^2 \\ &\leq C_3^2 \cdot |N(\alpha)|^{2/d}. \end{aligned}$$

Moreover, $|L/\mathcal{O}_F x| = |N(\alpha)| \cdot |L/\mathcal{O}_F e_1|$, so that $|N(\alpha)| \leq C_1$. This shows that $\|f_1\|_h \leq C_3 C_1^{1/d}$ and thus concludes the proof for $N = 1$.
The induction step is done exactly as in loc. cit., adapting the constants when necessary (C_1 to be replaced by $C_1 C_3^d$), to obtain the desired $f_i \in L_i$, i.e., with (using $F \neq \mathbb{Q}$ in the last inequality, so that $C_2 \geq 1$):

$$\begin{aligned} \|f_i\|_h &\leq (i-1) C_1 C_3^d C_2 + C_3 C_1^{1/d} \\ &\leq i C_1 C_2 C_3^d. \end{aligned}$$

□

We finally obtain the result about the existence of bounded bases. The assumption $N \geq 5$ is only here in order to simplify the statement.

Proposition 4.7. *Let (L,h) be a free hermitian lattice over $\mathcal{O}_F$ of rank $N \geq 5$, with $F \neq \mathbb{Q}$, and such that the subset $\{x \in L \mid \|x\|_h \leq 1\}$ spans $V = F \otimes_{\mathcal{O}_F} L$. Then there exists a basis $e_1, \ldots, e_N$ of L such that $\|e_i\|_h \leq B$ for every $i = 1, \ldots, N$, where*

$$B = \frac{4N^2}{2^N} d^{5Nd^2} |D_F|^{6N(d+1)}.$$

Proof. By Proposition 4.6 we can apply Lemma 4.5 with

$$k = C_1 C_3^d\,;$$
$$\lambda = N C_2.$$

This shows the existence of a basis $e_1, \ldots, e_N$ with

$$\|e_i\|_h \leq N C_2 (1 + C_2)^{\lfloor \log_2(N) \rfloor + 2} (C_1 C_3^d)^{(d+1)(4N-1)}.$$

Since $C_2 \geq 1$, we have $(1 + C_2)^n \leq 2^n C_2^n$. Moreover, for $N \geq 5$ we have $\lfloor \log_2(N) \rfloor + 3 \leq N$. We deduce:

$$\begin{aligned}\|e_i\|_h &\leq 4N^2 C_2^N (C_1 C_3^d)^{(d+1)(4N-1)}\\ &= \alpha d^\beta |D_F|^\gamma,\end{aligned}$$

with (using $N \geq 5$ and $d \geq 2$):

$$\begin{aligned}\alpha &= 4\,\frac{N^2}{2^N}\,;\\ \beta &= N + \frac{d}{2}(d+1)(4N-1)\\ &\leq 5Nd^2\,;\\ \gamma &= \frac{N}{d} + \frac{3}{2}(d+1)(4N-1)\\ &\leq 6(d+1)N.\end{aligned}$$

This finishes the proof. □

5 Improved estimates for K-groups

5.1 A Bounded set Φ

The construction of a bounded set $\Phi \subset L$ will follow from this proposition.

Proposition 5.1. *Let (L,h) be a free well-rounded $\mathcal{O}_F$-lattice of rank $N \geq 5$, with $F \neq \mathbb{Q}$. Let $e_1, \ldots, e_N$ and B be defined as in Proposition 4.7, and for $x \in M(L,h)$ write $x = \sum_i x_i e_i$, with $x_i \in \mathcal{O}_F$. Then for every $i = 1, \ldots, N$ we have:*

$$\sum_{\sigma \in \Sigma} |\sigma(x_i)|^2 \leq T,$$

where

$$T = N^{Nd} d^{\frac{3}{2}Nd+1} B^{2(Nd-1)} |D_F|^{2N}.$$

Proof. Let $x \in M(L,h)$, i.e., $h(x) = 1$. For each $\sigma \in \Sigma$ let us consider the matrix $H_\sigma = (h_\sigma(e_i, e_j))$. Then the first argument in [11, proof of Prop. 2], based on the Hadamard inequality for positive definite matrix, shows that

$$|\sigma(x_i)|^2 \leq \det(H_\sigma)^{-1} h_\sigma(x) \prod_{j \neq i} h_\sigma(e_j).$$

Since $h_\sigma(e_j) \leq \|e_j\|_h^2 \leq B^2$, and similarly $h_\sigma(x) \leq 1$, we obtain:

$$\sum_{\sigma \in \Sigma} |\sigma(x_i)|^2 \leq B^{2(N-1)} \sum_{\sigma \in \Sigma} \det(H_\sigma)^{-1}. \tag{5.1}$$

For $\sum_\sigma \det(H_\sigma)^{-1}$ we can write, using the Hadamard inequality:

$$\begin{aligned}
\sum_{\sigma \in \Sigma} \det(H_\sigma)^{-1} &= \sum_{\sigma \in \Sigma} \left(\prod_{\sigma' \neq \sigma} \det(H_{\sigma'}) \prod_{\sigma' \in \Sigma} \det(H_{\sigma'})^{-1} \right) \\
&\leq \left(\sum_{\sigma \in \Sigma} \prod_{\sigma' \neq \sigma} \prod_{j=1}^{N} h_{\sigma'}(e_j) \right) \cdot \left(\prod_{\sigma \in \Sigma} \det(H_\sigma)^{-1} \right) \\
&\leq d \cdot B^{2N(d-1)} \prod_{\sigma \in \Sigma} \det(H_\sigma)^{-1}.
\end{aligned} \tag{5.2}$$

According to Icaza [8, Theorem 1], there exists $z \in L$ such that

$$\prod_{\sigma \in \Sigma} \det(H_\sigma)^{-1} \leq \gamma^N \mathcal{N}(h(z))^{-N}, \tag{5.3}$$

where (cf. [11, Equ. (21)]):

$$\gamma \leq N^d |D_F|.$$

By applying Lemma 4.3 to $h(z) \in F_{\mathbb{R}}$, we find $a \in \mathcal{O}_F$ such that $h_\sigma(az) = \sigma(a) h_\sigma(z) \leq C_3 \mathcal{N}(h(z))^{1/d}$ for every $\sigma \in \Sigma$. Since (L,h) is well rounded, this implies:

$$d C_3 \mathcal{N}(h(z))^{1/d} \geq h(az) \geq 1, \tag{5.4}$$

so that $\mathcal{N}(h(z))^{-1} \leq d^d C_3^d = d^{\frac{3}{2}d} |D_F|$. Using this with (5.1), (5.2) and (5.3), this concludes the proof. ∎

Corollary 5.2. *Let L be a free $\mathcal{O}_F$-lattice of rank $N \geq 5$, with $F \neq \mathbb{Q}$. Then there exists a subset $\Phi \subset L$ with the property given in Section 3.3 and such that*

$$\operatorname{card}(\Phi) \leq N^{3N^2 d^2} \cdot d^{5N^3 d^4} \cdot |D_F|^{9N^3 d^3}.$$

Proof. Let $f_1,\dots,f_N$ be any basis of L, and set, for T as in Proposition 5.1:

$$\Phi = \left\{ \sum_{i=1}^{N} x_i f_i \;\middle|\; x_i \in \mathcal{O}_F \text{ with } \sum_{\sigma\in\Sigma} |\sigma(x_i)|^2 \leq T \right\}.$$

According to [11, Lemma 8], the number of elements $x_i \in \mathcal{O}_F$ with $\sum_{\sigma\in\Sigma}|\sigma(x_i)|^2 \leq T$ is at most $T^{d/2}2^{d+3}$, so that $\mathrm{card}(\Phi)$ is bounded above by $T^{Nd/2}2^{N(d+3)}$. Expanding the constants T and B as in the statements of Propositions 5.1 and 4.7, we obtain the stated upper bound for $\mathrm{card}(\Phi)$.
Let h be a well-rounded hermitian metric on L. We can apply Proposition 5.1 to write every $x \in M(L,h)$ as $x = \sum x_i e_i$ for a bounded basis $e_1,\dots,e_N$ of L. The proposition implies that the transformation $\gamma \in \Gamma = \mathrm{GL}_N(\mathcal{O}_F)$ that sends the basis (e_i) to (f_i) is such that $\gamma \cdot x \in \Phi$. This means that Φ has the property defined in Section 3.3. ☐

5.2 Upper bounds for $K_n(\mathcal{O}_F)$

We finally come to the bounds for the K-groups of $\mathcal{O}_F$, as stated in Theorem 1.1. Let $\ell = \max(d+1, 2n+2)$. From Equation (3.5) we obtain

$$\log \mathrm{card}_\ell K_n(\mathcal{O}_F)_{\mathrm{tors}} \leq \mathrm{card}(\Phi)^{e(d,n)+n+1},$$

and note that for $n \geq 2$ we have $e(d,n)+n+1 \leq \frac{15}{4}n^2 d$. Theorem 1.1 now follows directly from Corollary 5.2, applied with $N = 2n+1 \leq \frac{5}{2}n$.

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Stably Cayley Semisimple Groups

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Abstract. A linear algebraic group G over a field k is called a Cayley group if it admits a Cayley map, i.e., a G-equivariant birational isomorphism over k between the group variety G and its Lie algebra $\mathrm{Lie}(G)$. A prototypical example is the classical "Cayley transform" for the special orthogonal group $\mathbf{SO}_n$ defined by Arthur Cayley in 1846. A linear algebraic group G is called stably Cayley if $G \times S$ is Cayley for some split k-torus S. We classify stably Cayley semisimple groups over an arbitrary field k of characteristic 0.

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To Alexander Merkurjev on the occasion of his 60th birthday

0 Introduction

Let k be a field of characteristic 0 and $\bar{k}$ a fixed algebraic closure of k. Let G be a connected linear algebraic k-group. A birational isomorphism $\phi\colon G \overset{\simeq}{\dashrightarrow} \mathrm{Lie}(G)$ is called a *Cayley map* if it is equivariant with respect to the conjugation action of G on itself and the adjoint action of G on its Lie algebra $\mathrm{Lie}(G)$, respectively. A linear algebraic k-group G is called *Cayley* if it admits a Cayley map, and *stably Cayley* if $G \times_k (\mathbb{G}_{\mathrm{m},k})^r$ is Cayley for some $r \geq 0$. Here $\mathbb{G}_{\mathrm{m},k}$ denotes the multiplicative group over k. These notions were introduced by Lemire, Popov and Reichstein [LPR]; for a more detailed discussion and numerous classical examples we refer the reader to [LPR, Introduction]. The main results of [LPR] are the classifications of Cayley and stably Cayley *simple* groups over an algebraically closed field k of characteristic 0. Over an arbitrary field k of characteristic 0 stably Cayley *simple* k-groups, stably Cayley *simply connected* semisimple k-groups and stably Cayley *adjoint* semisimple k-groups were classified in the paper [BKLR] of Borovoi, Kunyavskiĭ, Lemire and Reichstein. In the present paper, building on results of [LPR] and [BKLR], we classify all stably Cayley *semisimple* k-groups (not necessarily simple, or simply connected, or adjoint) over an arbitrary field k of characteristic 0.

By a semisimple (or reductive) k-group we always mean a *connected* semisimple (or reductive) k-group. We shall need the following result of [BKLR] extending [LPR, Theorem 1.28].

Theorem 0.1 ([BKLR, Theorem 1.4]). *Let k be a field of characteristic* 0 *and G an absolutely simple k-group. Then the following conditions are equivalent:*

(a) *G is stably Cayley over k;*

(b) *G is an arbitrary k-form of one of the following groups:*

$$\mathbf{SL}_3,\ \mathbf{PGL}_2,\ \mathbf{PGL}_{2n+1}\ (n \geq 1),\ \mathbf{SO}_n\ (n \geq 5),\ \mathbf{Sp}_{2n}\ (n \geq 1),\ \mathbf{G}_2,$$

or an inner *k-form of* $\mathbf{PGL}_{2n}$ $(n \geq 2)$.

In this paper we classify stably Cayley semisimple groups over an *algebraically closed* field k of characteristic 0 (Theorem 0.2) and, more generally, over an *arbitrary* field k of characteristic 0 (Theorem 0.3). Note that Theorem 0.2 was conjectured in [BKLR, Remark 9.3].

Theorem 0.2. *Let k be an algebraically closed field of characteristic* 0 *and G a semisimple k-group. Then G is stably Cayley if and only if G decomposes into a direct product $G_1 \times_k \cdots \times_k G_s$ of its normal subgroups, where each G_i $(i = 1, \dots, s)$ either is a stably Cayley simple k-group (i.e., isomorphic to one of the groups listed in Theorem 0.1) or is isomorphic to the stably Cayley semisimple k-group* $\mathbf{SO}_4$.

Theorem 0.3. *Let G be a semisimple k-group over a field k of characteristic 0 (not necessarily algebraically closed). Then G is stably Cayley over k if and only if G decomposes into a direct product $G_1 \times_k \cdots \times_k G_s$ of its normal k-subgroups, where each G_i $(i = 1, \dots, s)$ is isomorphic to the Weil restriction $R_{l_i/k} G_{i,l_i}$ for some finite field extension l_i/k, and each G_{i,l_i} is either a stably Cayley absolutely simple group over l_i (i.e., one of the groups listed in Theorem 0.1) or an l_i-form of the semisimple group* $\mathbf{SO}_4$ *(which is always stably Cayley, but is not absolutely simple and can be not l_i-simple).*

Note that the "if" assertions in Theorems 0.2 and 0.3 follow immediately from the definitions.

The rest of the paper is structured as follows. In Section 1 we recall the definition of a quasi-permutation lattice and state some known results, in particular, an assertion from [LPR, Theorem 1.27] that reduces Theorem 0.2 to an assertion on lattices. In Sections 2 and 3 we construct certain families of non-quasi-permutation lattices. In particular, we correct an inaccuracy in [BKLR]; see Remark 2.5. In Section 4 we prove (in the language of lattices) Theorem

0.2 in the special case when G is isogenous to a direct product of simple groups of type $\mathbf{A}_{n-1}$ with $n \geq 3$. In Section 5 we prove (again in the language of lattices) Theorem 0.2 in the general case. In Section 6 we deduce Theorem 0.3 from Theorem 0.2. In Appendix A we prove in terms of lattices only, that certain quasi-permutation lattices are indeed quasi-permutation.

1 Preliminaries on quasi-permutation groups and on character lattices

In this section we gather definitions and known results concerning quasi-permutation lattices, quasi-invertible lattices and character lattices that we need for the proofs of Theorems 0.2 and 0.3. For details see [BKLR, Sections 2 and 10] and [LPR, Introduction].

1.1. By a *lattice* we mean a pair (Γ, L) where Γ is a finite group acting on a finitely generated free abelian group L. We say also that L is a Γ-lattice. A Γ-lattice L is called a *permutation* lattice if it has a $\mathbb{Z}$-basis permuted by Γ. Following Colliot-Thélène and Sansuc [CTS], we say that two Γ-lattices L and L' are *equivalent*, and write $L \sim L'$, if there exist short exact sequences

$$0 \to L \to E \to P \to 0 \qquad \text{and} \qquad 0 \to L' \to E \to P' \to 0$$

with the same Γ-lattice E, where P and P' are permutation Γ-lattices. For a proof that this is indeed an equivalence relation see [CTS, Lemma 8, p. 182] or [Sw, Section 8]. Note that if there exists a short exact sequence of Γ-lattices

$$0 \to L \to L' \to Q \to 0$$

where Q is a permutation Γ-lattice, then, taking in account the trivial short exact sequence

$$0 \to L' \to L' \to 0 \to 0,$$

we obtain that $L \sim L'$. If Γ-lattices L, L', M, M' satisfy $L \sim L'$ and $M \sim M'$, then clearly $L \oplus M \sim L' \oplus M'$.

Definition 1.2. A Γ-lattice L is called a *quasi-permutation* lattice if there exists a short exact sequence

$$0 \to L \to P \to P' \to 0, \tag{1.1}$$

where both P and P' are permutation Γ-lattices.

Lemma 1.3 (well-known). *A Γ-lattice L is quasi-permutation if and only if $L \sim 0$.*

Proof. If L is quasi-permutation, then sequence (1.1) together with the trivial short exact sequence

$$0 \to 0 \to P \to P \to 0$$

shows that $L \sim 0$. Conversely, if $L \sim 0$, then there are short exact sequences

$$0 \to L \to E \to P \to 0 \qquad \text{and} \qquad 0 \to 0 \to E \to P' \to 0,$$

where P and P' are permutation lattices. From the second exact sequence we have $E \cong P'$, hence E is a permutation lattice, and then the first exact sequence shows that L is a quasi-permutation lattice. □

Definition 1.4. A Γ-lattice L is called *quasi-invertible* if it is a direct summand of a quasi-permutation Γ-lattice.

Note that if a Γ-lattice L is not quasi-invertible, then it is not quasi-permutation.

Lemma 1.5 (well-known). *If a Γ-lattice L is quasi-permutation (resp., quasi-invertible) and $L' \sim L$, then L' is quasi-permutation (resp., quasi-invertible) as well.*

Proof. If L is quasi-permutation, then using Lemma 1.3 we see that $L' \sim L \sim 0$, hence L' is quasi-permutation. If L is quasi-invertible, then $L \oplus M$ is quasi-permutation for some Γ-lattice M, and by Lemma 1.3 we have $L \oplus M \sim 0$. We see that $L' \oplus M \sim L \oplus M \sim 0$, and by Lemma 1.3 we obtain that $L' \oplus M$ is quasi-permutation, hence L' is quasi-invertible. □

Let $\mathbb{Z}[\Gamma]$ denote the group ring of a finite group Γ. We define the Γ-lattice J_Γ by the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{N} \mathbb{Z}[\Gamma] \to J_\Gamma \to 0,$$

where N is the norm map, see [BKLR, before Lemma 10.4]. We refer to [BKLR, Proposition 10.6] for a proof of the following result, due to Voskresenskiĭ [Vo1, Corollary of Theorem 7]:

Proposition 1.6. *Let $\Gamma = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, where p is a prime. Then the Γ-lattice J_Γ is not quasi-invertible.*

Note that if $\Gamma = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, then $\operatorname{rank} J_\Gamma = 3$.

We shall use the following lemma from [BKLR]:

Lemma 1.7 ([BKLR, Lemma 2.8]). *Let $W_1, \dots, W_m$ be finite groups. For each $i = 1, \dots, m$, let V_i be a finite-dimensional $\mathbb{Q}$-representation of W_i. Set $V := V_1 \oplus \cdots \oplus V_m$. Suppose $L \subset V$ is a free abelian subgroup, invariant under $W := W_1 \times \cdots \times W_m$. If L is a quasi-permutation W-lattice, then for each $i = 1, \dots, m$ the intersection $L_i := L \cap V_i$ is a quasi-permutation W_i-lattice.*

We shall need the notion, due to [LPR] and [BKLR], of the character lattice of a reductive k-group G over a field k. Let $\bar{k}$ be a separable closure of k. Let $T \subset G$ be a maximal torus (defined over k). Set $\overline{T} = T \times_k \bar{k}$, $\overline{G} = G \times_k \bar{k}$. Let $\mathsf{X}(\overline{T})$

denote the character group of $\overline{T} := T \times_k \bar{k}$. Let $W = W(\overline{G}, \overline{T}) := \mathcal{N}_G(\overline{T})/\overline{T}$ denote the Weyl group, it acts on $\mathsf{X}(\overline{T})$. Consider the canonical Galois action on $\mathsf{X}(\overline{T})$, it defines a homomorphism $\mathrm{Gal}(\bar{k}/k) \to \mathrm{Aut}\,\mathsf{X}(\overline{T})$. The image $\mathrm{im}\,\rho \subset \mathrm{Aut}\,\mathsf{X}(\overline{T})$ normalizes W, hence $\mathrm{im}\,\rho \cdot W$ is a subgroup of $\mathrm{Aut}\,\mathsf{X}(\overline{T})$. By the character lattice of G we mean the pair $\mathcal{X}(G) := (\mathrm{im}\,\rho \cdot W,\, \mathsf{X}(\overline{T}))$ (up to an isomorphism it does not depend on the choice of T). In particular, if k is algebraically closed, then $\mathcal{X}(G) = (W, \mathsf{X}(T))$.

We shall reduce Theorem 0.2 to an assertion about quasi-permutation lattices using the following result due to [LPR]:

Proposition 1.8 ([LPR, Theorem 1.27], see also [BKLR, Theorem 1.3]). *A reductive group G over an algebraically closed field k of characteristic 0 is stably Cayley if and only if its character lattice $\mathcal{X}(G)$ is quasi-permutation, i.e., $\mathsf{X}(T)$ is a quasi-permutation $W(G,T)$-lattice.*

We shall use the following result due to Cortella and Kunyavskiĭ [CK] and to Lemire, Popov and Reichstein [LPR].

Proposition 1.9 ([CK], [LPR]). *Let D be a connected Dynkin diagram. Let $R = R(D)$ denote the corresponding root system, $W = W(D)$ denote the Weyl group, $Q = Q(D)$ denote the root lattice, and $P = P(D)$ denote the weight lattice. We say that L is an* intermediate lattice between Q and P *if $Q \subset L \subset P$ (note that $L = Q$ and $L = P$ are possible). Then the following list gives (up to an isomorphism) all the pairs (D, L) such that L is a quasi-permutation intermediate $W(D)$-lattice between $Q(D)$ and $P(D)$:*

$$Q(\mathbf{A}_n),\ Q(\mathbf{B}_n),\ P(\mathbf{C}_n),\ \mathcal{X}(\mathbf{SO}_{2n})\ \textit{(then } D = \mathbf{D}_n\textit{)},$$

or D is any connected Dynkin diagram of rank 1 or 2 (i.e. $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{B}_2$, or $\mathbf{G}_2$) and L is any lattice between $Q(D)$ and $P(D)$, (i.e., either $L = P(D)$ or $L = Q(D)$).

Proof. The positive result (the assertion that the lattices in the list are indeed quasi-permutation) follows from the assertion that the corresponding groups are stably Cayley (or that their generic tori are stably rational), see the references in [CK], Section 3. See Appendix A below for a proof of this positive result in terms of lattices only. The difficult part of Proposition 1.9 is the negative result (the assertion that all the other lattices are not quasi-permutation). This was proved in [CK, Theorem 0.1] in the cases when $L = Q$ or $L = P$, and in [LPR, Propositions 5.1 and 5.2] in the cases when $Q \subsetneq L \subsetneq P$ (this can happen only when $D = \mathbf{A}_n$ or $D = \mathbf{D}_n$). □

Remark 1.10. It follows from Proposition 1.9 that, in particular, the following lattices are quasi-permutation: $Q(\mathbf{A}_1)$, $P(\mathbf{A}_1)$, $P(\mathbf{A}_2)$, $P(\mathbf{B}_2)$, $Q(\mathbf{C}_2)$, $Q(\mathbf{G}_2) = P(\mathbf{G}_2)$, $Q(\mathbf{D}_3) = Q(\mathbf{A}_3)$, $\mathcal{X}(\mathbf{SL}_4/\mu_2) = \mathcal{X}(\mathbf{SO}_6)$.

2 A family of non-quasi-permutation lattices

In this section we construct a family of non-quasi-permutation (even non-quasi-invertible) lattices.

2.1. We consider a Dynkin diagram $D \sqcup \Delta$ (disjoint union). We assume that $D = \bigsqcup_{i\in I} D_i$ (a finite disjoint union), where each D_i is of type $\mathbf{B}_{l_i}$ ($l_i \geq 1$) or $\mathbf{D}_{l_i}$ ($l_i \geq 2$) (and where $\mathbf{B}_1 = \mathbf{A}_1$, $\mathbf{B}_2 = \mathbf{C}_2$, $\mathbf{D}_2 = \mathbf{A}_1 \sqcup \mathbf{A}_1$, and $\mathbf{D}_3 = \mathbf{A}_3$ are permitted). We denote by m the cardinality of the finite index set I. We assume that $\Delta = \bigsqcup_{\iota=1}^{\mu} \Delta_\iota$ (disjoint union), where Δ_ι is of type $\mathbf{A}_{2n_\iota - 1}$, $n_\iota \geq 2$ ($\mathbf{A}_3 = \mathbf{D}_3$ is permitted). We assume that $m \geq 1$ and $\mu \geq 0$ (in the case $\mu = 0$ the diagram Δ is empty).

For each $i \in I$ we realize the root system $R(D_i)$ of type $\mathbf{B}_{l_i}$ or $\mathbf{D}_{l_i}$ in the standard way in the space $V_i := \mathbb{R}^{l_i}$ with basis $(e_s)_{s\in S_i}$ where S_i is an index set consisting of l_i elements; cf. [Bou, Planche II] for $\mathbf{B}_l$ ($l \geq 2$) (the relevant formulas for $\mathbf{B}_1$ are similar) and [Bou, Planche IV] for $\mathbf{D}_l$ ($l \geq 3$) (again, the relevant formulas for $\mathbf{D}_2$ are similar). Let $M_i \subset V_i$ denote the lattice generated by the basis vectors $(e_s)_{s\in S_i}$. Let $P_i \supset M_i$ denote the weight lattice of the root system D_i. Set $S = \bigsqcup_i S_i$ (disjoint union). Consider the vector space $V = \bigoplus_i V_i$ with basis $(e_s)_{s\in S}$. Let $M_D \subset V$ denote the lattice generated by the basis vectors $(e_s)_{s\in S}$, then $M_D = \bigoplus_i M_i$. Set $P_D = \bigoplus_i P_i$.

For each $\iota = 1, \dots, \mu$ we realize the root system $R(\Delta_\iota)$ of type $\mathbf{A}_{2n_\iota - 1}$ in the standard way in the subspace V_ι of vectors with zero sum of the coordinates in the space $\mathbb{R}^{2n_\iota}$ with basis $\varepsilon_{\iota,1}, \dots, \varepsilon_{\iota,2n_\iota}$; cf. [Bou, Planche I]. Let Q_ι be the root lattice of $R(\Delta_\iota)$ with basis $\varepsilon_{\iota,1} - \varepsilon_{\iota,2},\ \varepsilon_{\iota,2} - \varepsilon_{\iota,3},\ \dots,\ \varepsilon_{\iota,2n_\iota-1} - \varepsilon_{\iota,2n_\iota}$, and let $P_\iota \supset Q_\iota$ be the weight lattice of $R(\Delta_\iota)$. Set $Q_\Delta = \bigoplus_\iota Q_\iota$, $P_\Delta = \bigoplus_\iota P_\iota$.

Set

$$W := \prod_{i\in I} W(D_i) \times \prod_{\iota=1}^{\mu} W(\Delta_\iota), \quad L' = M_D \oplus Q_\Delta = \bigoplus_{i\in I} M_i \oplus \bigoplus_{\iota=1}^{\mu} Q_\iota,$$

then W acts on L' and on $L' \otimes_{\mathbb{Z}} \mathbb{R}$. For each i consider the vector

$$x_i = \sum_{s\in S_i} e_s \in M_i,$$

then $\frac{1}{2}x_i \in P_i$. For each ι consider the vector

$$\xi_\iota = \varepsilon_{\iota,1} - \varepsilon_{\iota,2} + \varepsilon_{\iota,3} - \varepsilon_{\iota,4} + \dots + \varepsilon_{\iota,2n_\iota-1} - \varepsilon_{\iota,2n_\iota} \in Q_\iota,$$

then $\frac{1}{2}\xi_\iota \in P_\iota$; see [Bou, Planche I]. Write

$$\xi'_\iota = \varepsilon_{\iota,1} - \varepsilon_{\iota,2}, \quad \xi''_\iota = \varepsilon_{\iota,3} - \varepsilon_{\iota,4} + \dots + \varepsilon_{\iota,2n_\iota-1} - \varepsilon_{\iota,2n_\iota},$$

then $\xi_\iota = \xi'_\iota + \xi''_\iota$. Consider the vector

$$v = \frac{1}{2}\sum_{i\in I} x_i + \frac{1}{2}\sum_{\iota=1}^{\mu} \xi_\iota = \frac{1}{2}\sum_{s\in S} e_s + \frac{1}{2}\sum_{\iota=1}^{\mu} \xi_\iota \in P_D \oplus P_\Delta.$$

Set

$$L = \langle L', v \rangle, \tag{2.1}$$

then $[L : L'] = 2$ because $v \in \frac{1}{2}L' \smallsetminus L'$. Note that the sublattice $L \subset P_D \oplus P_\Delta$ is W-invariant. Indeed, the group W acts on $(P_D \oplus P_\Delta)/(M_D \oplus Q_\Delta)$ trivially.

Proposition 2.2. *We assume that $m \geq 1$, $m + \mu \geq 2$. If $\mu = 0$, we assume that not all of D_i are of types $\mathbf{B}_1$ or $\mathbf{D}_2$. Then the W-lattice L as in* (2.1) *is not quasi-invertible, hence not quasi-permutation.*

Proof. We consider a group $\Gamma = \{e, \gamma_1, \gamma_2, \gamma_3\}$ of order 4, where $\gamma_1, \gamma_2, \gamma_3$ are of order 2. The idea of our proof is to construct an embedding

$$j \colon \Gamma \to W$$

in such a way that L, viewed as a Γ-lattice, is equivalent to its Γ-sublattice L_1, and L_1 is isomorphic to a direct sum of a Γ-sublattice $L_0 \simeq J_\Gamma$ of rank 3 and a number of Γ-lattices of rank 1. Since by Proposition 1.6 J_Γ is not quasi-invertible, this will imply that L_1 and L are not quasi-invertible Γ-lattices, and hence L is not quasi-invertible as a W-lattice. We shall now fill in the details of this argument in four steps.

Step 1. We begin by partitioning each S_i for $i \in I$ into three (non-overlapping) subsets $S_{i,1}$, $S_{i,2}$ and $S_{i,3}$, subject to the requirement that

$$|S_{i,1}| \equiv |S_{i,2}| \equiv |S_{i,3}| \equiv l_i \pmod 2 \text{ if } D_i \text{ is of type } \mathbf{D}_{l_i}. \tag{2.2}$$

We then set U_1 to be the union of the $S_{i,1}$, U_2 to be the union of the $S_{i,2}$, and U_3 to be the union of the $S_{i,3}$, as i runs over I.

Lemma 2.3. (i) *If $\mu \geq 1$, the subsets $S_{i,1}$, $S_{i,2}$ and $S_{i,3}$ of S_i can be chosen, subject to* (2.2)*, so that $U_1 \neq \emptyset$.*

(ii) *If $\mu = 0$ (and $m \geq 2$), the subsets $S_{i,1}$, $S_{i,2}$ and $S_{i,3}$ of S_i can be chosen, subject to* (2.2)*, so that $U_1, U_2, U_3 \neq \emptyset$.*

To prove the lemma, first consider case (i). For all i such that D_i is of type $\mathbf{D}_{l_i}$ with *odd* l_i, we partition S_i into three non-empty subsets of odd cardinalities. For all the other i we take $S_{i,1} = S_i$, $S_{i,2} = S_{i,3} = \emptyset$. Then $U_1 \neq \emptyset$ (note that $m \geq 1$) and (2.2) is satisfied.

In case (ii), if one of the D_i is of type $\mathbf{D}_{l_i}$ where $l_i \geq 3$ is *odd,* then we partition S_i for each such D_i into three non-empty subsets of odd cardinalities. We partition all the other S_i as follows:

$$S_{i,1} = S_{i,2} = \emptyset \text{ and } S_{i,3} = S_i. \tag{2.3}$$

Clearly $U_1, U_2, U_3 \neq \emptyset$ and (2.2) is satisfied.

If there is no D_i of type $\mathbf{D}_{l_i}$ with odd $l_i \geq 3$, but one of the D_i, say for $i = i_0$, is $\mathbf{D}_l$ with *even* $l \geq 4$, then we partition S_{i_0} into two non-empty subsets $S_{i_0,1}$ and

$S_{i_0,2}$ of even cardinalities, and set $S_{i_0,3} = \emptyset$. We partition the sets S_i for $i \neq i_0$ as in (2.3) (note that by our assumption $m \geq 2$). Once again, $U_1, U_2, U_3 \neq \emptyset$ and (2.2) is satisfied.

If there is no D_i of type $\mathbf{D}_{l_i}$ with $l_i \geq 3$ (odd or even), but one of the D_i, say for $i = i_0$, is of type $\mathbf{B}_l$ with $l \geq 2$, we partition S_{i_0} into two non-empty subsets $S_{i_0,1}$ and $S_{i_0,2}$, and set $S_{i_0,3} = \emptyset$. We partition the sets S_i for $i \neq i_0$ as in (2.3) (again, note that $m \geq 2$). Once again, $U_1, U_2, U_3 \neq \emptyset$ and (2.2) is satisfied.

Since by our assumption not all of D_i are of type $\mathbf{B}_1$ or $\mathbf{D}_2$, we have exhausted all the cases. This completes the proof of Lemma 2.3. $\square$

Step 2. We continue proving Proposition 2.2. We construct an embedding $\Gamma \hookrightarrow W$.

For $s \in S$ we denote by c_s the automorphism of L taking the basis vector e_s to $-e_s$ and fixing all the other basis vectors. For $\iota = 1, \dots, \mu$ we define $\tau_\iota^{(12)} = \operatorname{Transp}((\iota,1),(\iota,2)) \in W_\iota$ (the transposition of the basis vectors $\varepsilon_{\iota,1}$ and $\varepsilon_{\iota,2}$). Set

$$\tau_\iota^{>2} = \operatorname{Transp}((\iota,3),(\iota,4)) \cdot \ \cdots \ \cdot \operatorname{Transp}((\iota,2n_\iota - 1),(\iota,2n_\iota)) \in W_\iota.$$

Write $\Gamma = \{e, \gamma_1, \gamma_2, \gamma_3\}$ and define an embedding $j\colon \Gamma \hookrightarrow W$ as follows:

$$j(\gamma_1) = \prod_{s \in S \smallsetminus U_1} c_s \ \cdot \ \prod_{\iota=1}^{\mu} \tau_\iota^{(12)} \tau_\iota^{>2}\,;$$

$$j(\gamma_2) = \prod_{s \in S \smallsetminus U_2} c_s \ \cdot \ \prod_{\iota=1}^{\mu} \tau_\iota^{(12)}\,;$$

$$j(\gamma_3) = \prod_{s \in S \smallsetminus U_3} c_s \ \cdot \ \prod_{\iota=1}^{\mu} \tau_\iota^{>2}\,.$$

Note that if D_i is of type $\mathbf{D}_{l_i}$, then by (2.2) for $\varkappa = 1,2,3$ the cardinality $\#(S_i \smallsetminus S_{i,\varkappa})$ is even, hence the product of c_s over $s \in S_i \smallsetminus S_{i,\varkappa}$ is contained in $W(D_i)$ for all such i, and therefore, $j(\gamma_\varkappa) \in W$. Since $j(\gamma_1)$, $j(\gamma_2)$ and $j(\gamma_3)$ commute, are of order 2, and $j(\gamma_1)j(\gamma_2) = j(\gamma_3)$, we see that j is a homomorphism. If $\mu \geq 1$, then, since $2n_1 \geq 4$, clearly $j(\gamma_\varkappa) \neq 1$ for $\varkappa = 1,2,3$, hence j is an embedding. If $\mu = 0$, then the sets $S \smallsetminus U_1$, $S \smallsetminus U_2$ and $S \smallsetminus U_3$ are nonempty, and again $j(\gamma_\varkappa) \neq 1$ for $\varkappa = 1,2,3$, hence j is an embedding.

Step 3. We construct a Γ-sublattice L_0 of rank 3. Write a vector $\mathbf{x} \in L$ as

$$\mathbf{x} = \sum_{s \in S} b_s e_s + \sum_{\iota=1}^{\mu} \sum_{\nu=1}^{2n_\iota} \beta_{\iota,\nu} \varepsilon_{\iota,\nu},$$

where $b_s\,, \beta_{\iota,\nu} \in \frac{1}{2}\mathbb{Z}$. Set $n' = \sum_{\iota=1}^{\mu}(n_\iota - 1)$. Define a Γ-equivariant homomorphism

$$\phi\colon L \to \mathbb{Z}^{n'}, \quad \mathbf{x} \mapsto (\beta_{\iota,2\lambda-1} + \beta_{\iota,2\lambda})_{\iota=1,\dots,\mu,\ \lambda=2,\dots,n_\iota}$$

(we skip $\lambda = 1$). We obtain a short exact sequence of Γ-lattices

$$0 \to L_1 \to L \xrightarrow{\phi} \mathbb{Z}^{n'} \to 0,$$

where $L_1 := \ker \phi$. Since Γ acts trivially on $\mathbb{Z}^{n'}$, we have $L_1 \sim L$. Therefore, it suffices to show that L_1 is not quasi-invertible.

Recall that

$$v = \frac{1}{2}\sum_{s\in S} e_s + \frac{1}{2}\sum_{\iota=1}^{\mu} \xi_\iota.$$

Set $v_1 = \gamma_1 \cdot v,\ v_2 = \gamma_2 \cdot v,\ v_3 = \gamma_3 \cdot v$. Set

$$L_0 = \langle v, v_1, v_2, v_3\rangle.$$

We have

$$v_1 = \frac{1}{2}\sum_{s\in U_1} e_s - \frac{1}{2}\sum_{s\in U_2\cup U_3} e_s - \frac{1}{2}\sum_{\iota=1}^{\mu} \xi_\iota,$$

whence

$$v + v_1 = \sum_{s\in U_1} e_s. \tag{2.4}$$

We have

$$v_2 = \frac{1}{2}\sum_{s\in U_2} e_s - \frac{1}{2}\sum_{s\in U_1\cup U_3} e_s + \frac{1}{2}\sum_{\iota=1}^{\mu}(-\xi'_\iota + \xi''_\iota),$$

whence

$$v + v_2 = \sum_{s\in U_2} e_s + \sum_{\iota=1}^{\mu} \xi''_\iota. \tag{2.5}$$

We have

$$v_3 = \frac{1}{2}\sum_{s\in U_3} e_s - \frac{1}{2}\sum_{s\in U_1\cup U_2} e_s + \frac{1}{2}\sum_{\iota=1}^{\mu}(\xi'_\iota - \xi''_\iota),$$

whence

$$v + v_3 = \sum_{s\in U_3} e_s + \sum_{\iota=1}^{\mu} \xi'_\iota. \tag{2.6}$$

Clearly, we have

$$v + v_1 + v_2 + v_3 = 0.$$

Since the set $\{v, v_1, v_2, v_3\}$ is the orbit of v under Γ, the sublattice $L_0 = \langle v, v_1, v_2, v_3\rangle \subset L$ is Γ-invariant. If $\mu \geq 1$, then $U_1 \neq \emptyset$, and we see from (2.4), (2.5) and (2.6) that $\operatorname{rank} L_0 \geq 3$. If $\mu = 0$, then $U_1, U_2, U_3 \neq \emptyset$, and again we see from (2.4), (2.5) and (2.6) that $\operatorname{rank} L_0 \geq 3$. Thus $\operatorname{rank} L_0 = 3$ and $L_0 \simeq J_\Gamma$, whence by Proposition 1.6 L_0 is not quasi-invertible.

Step 4. We show that L_0 is a direct summand of L_1. Set $m' = |S|$.

First assume that $\mu \geq 1$. Choose $u_1 \in U_1 \subset S$. Set $S' = S \smallsetminus \{u_1\}$. For each $s \in S'$ (i.e., $s \neq u_1$) consider the one-dimensional (i.e., of rank 1) lattice $X_s = \langle e_s \rangle$. We obtain $m' - 1$ Γ-invariant one-dimensional sublattices of L_1.

Denote by Υ the set of pairs (ι, λ) such that $1 \leq \iota \leq \mu$, $1 \leq \lambda \leq n_\iota$, and if $\iota = 1$, then $\lambda \neq 1, 2$. For each $(\iota, \lambda) \in \Upsilon$ consider the one-dimensional lattice

$$\Xi_{\iota,\lambda} = \langle \varepsilon_{\iota,2\lambda-1} - \varepsilon_{\iota,2\lambda} \rangle.$$

We obtain $-2 + \sum_{\iota=1}^{\mu} n_\iota$ one-dimensional Γ-invariant sublattices of L_1.

We show that

$$L_1 = L_0 \oplus \bigoplus_{s \in S'} X_s \oplus \bigoplus_{(\iota,\lambda)\in\Upsilon} \Xi_{\iota,\lambda}. \tag{2.7}$$

Set $L_1' = \left\langle\, L_0,\, (X_s)_{s\neq u_1},\, (\Xi_{\iota,\lambda})_{(\iota,\lambda)\in\Upsilon} \,\right\rangle$, then

$$\operatorname{rank} L_1' \leq 3 + (m'-1) - 2 + \sum_{\iota=1}^{\mu} n_\iota = m' + \sum_{\iota=1}^{\mu}(2n_\iota - 1) - \sum_{\iota=1}^{\mu}(n_\iota - 1) = \operatorname{rank} L_1. \tag{2.8}$$

Therefore, it suffices to check that $L_1' \supset L_1$. The set

$$\{v\} \cup \{e_s \mid s \in S\} \cup \{\varepsilon_{\iota,2\lambda-1} - \varepsilon_{\iota,2\lambda} \mid 1 \leq \iota \leq \mu, 1 \leq \lambda \leq n_\iota\}$$

is a set of generators of L_1. By construction $v, v_1, v_2, v_3 \in L_0 \subset L_1'$. We have $e_s \in X_s \subset L_1'$ for $s \neq u_1$. By (2.4) $\sum_{s\in U_1} e_s \in L_1'$, hence $e_{u_1} \in L_1'$. By construction

$$\varepsilon_{\iota,2\lambda-1} - \varepsilon_{\iota,2\lambda} \in L_1', \quad \text{for all } (\iota,\lambda) \neq (1,1),(1,2)\,.$$

From (2.6) and (2.5) we see that

$$\sum_{\iota=1}^{\mu}(\varepsilon_{\iota,1} - \varepsilon_{\iota,2}) \in L_1', \quad \sum_{\iota=1}^{\mu} \xi_\iota'' \in L_1'\,.$$

Thus

$$\varepsilon_{1,1} - \varepsilon_{1,2} \in L_1', \quad \varepsilon_{1,3} - \varepsilon_{1,4} \in L_1'\,.$$

We conclude that $L_1' \supset L_1$, hence $L_1 = L_1'$. From dimension count (2.8) we see that (2.7) holds.

Now assume that $\mu = 0$. Then for each $\varkappa = 1, 2, 3$ we choose an element $u_\varkappa \in U_\varkappa$ and set $U_\varkappa' = U_\varkappa \smallsetminus \{u_\varkappa\}$. We set $S' = U_1' \cup U_2' \cup U_3' = S \smallsetminus \{u_1, u_2, u_3\}$. Again for $s \in S'$ (i.e., $s \neq u_1, u_2, u_3$) consider the one-dimensional lattice $X_s = \langle e_s \rangle$. We obtain $m' - 3$ one-dimensional Γ-invariant sublattices of $L_1 = L$. We show that

$$L_1 = L_0 \oplus \bigoplus_{s\in S'} X_s\,. \tag{2.9}$$

Set $L_1' = \langle L_0, (X_s)_{s\in S'}\rangle$, then

$$\operatorname{rank} L_1' \leq 3 + m' - 3 = m' = \operatorname{rank} L_1. \tag{2.10}$$

Therefore, it suffices to check that $L_1' \supset L_1$. The set $\{v\} \cup \{e_s \mid s \in S\}$ is a set of generators of $L_1 = L$. By construction $v, v_1, v_2, v_3 \in L_1'$ and $e_s \in L_1'$ for $s \neq u_1, u_2, u_3$. We see from (2.4), (2.5), (2.6) that $e_s \in L_1'$ also for $s = u_1, u_2, u_3$. Thus $L_1' \supset L_1$, hence $L_1' = L_1$. From dimension count (2.10) we see that (2.9) holds.

We see that in both cases $\mu \geq 1$ and $\mu = 0$, the sublattice L_0 is a direct summand of L_1. Since by Proposition 1.6 L_0 is not quasi-invertible as a Γ-lattice, it follows that L_1 and L are not quasi-invertible as Γ-lattices. Thus L is not quasi-invertible as a W-lattice. This completes the proof of Proposition 2.2. ☐

Remark 2.4. Since $Ш^2(\Gamma, J_\Gamma) \cong \mathbb{Z}/2\mathbb{Z}$ (Voskresenskiĭ, see [BKLR, Section 10] for the notation and the result), our argument shows that $Ш^2(\Gamma, L) \cong \mathbb{Z}/2\mathbb{Z}$.

Remark 2.5. The proof of [BKLR, Lemma 12.3] (which is a version with $\mu = 0$ of Lemma 2.3 above) contains an inaccuracy, though the lemma as stated is correct. Namely, in [BKLR] we write that if there exists i such that Δ_i is of type $\mathbf{D}_{l_i}$ where $l_i \geq 3$ is odd, then we partition S_i for *one* such i into three non-empty subsets $S_{i,1}$, $S_{i,2}$ and $S_{i,3}$ of odd cardinalities, and we partition all the other S_i as in [BKLR, (12.4)]. However, this partitioning of the sets S_i into three subsets does not satisfy [BKLR, (12.3)] for *other* i such that Δ_i is of type $\mathbf{D}_{l_i}$ with odd l_i. This inaccuracy can be easily corrected: we should partition S_i for *each* i such that Δ_i is of type $\mathbf{D}_{l_i}$ with odd l_i into three non-empty subsets of odd cardinalities.

3 More non-quasi-permutation lattices

In this section we construct another family of non-quasi-permutation lattices.

3.1. For $i = 1, \dots, r$ let $Q_i = \mathbb{Z}\mathbf{A}_{n_i-1}$ and $P_i = \Lambda_{n_i}$ denote the root lattice and the weight lattice of $\mathbf{SL}_{n_i}$, respectively, and let $W_i = \mathfrak{S}_{n_i}$ denote the corresponding Weyl group (the symmetric group on n_i letters) acting on P_i and Q_i. Set $F_i = P_i/Q_i$, then W_i acts trivially on F_i. Set

$$Q = \bigoplus_{i=1}^{r} Q_i, \quad P = \bigoplus_{i=1}^{r} P_i, \quad W = \prod_{i=1}^{r} W_i,$$

then $Q \subset P$ and the Weyl group W acts on Q and P. Set

$$F = P/Q = \bigoplus_{i=1}^{r} F_i,$$

then W acts trivially on F.

We regard $Q_i = \mathbb{Z}\mathbf{A}_{n_i-1}$ and $P_i = \Lambda_{n_i}$ as the lattices described in Bourbaki [Bou, Planche I]. Then we have an isomorphism $F_i \cong \mathbb{Z}/n_i\mathbb{Z}$. Note that for each $1 \le i \le r$, the set $\{\alpha_{\varkappa,i} \mid 1 \le \varkappa \le n_i - 1\}$ is a $\mathbb{Z}$-basis of Q_i.

Set $c = \gcd(n_1, \dots, n_r)$; we assume that $c > 1$. Let $d > 1$ be a divisor of c. For each $i = 1, \dots, r$, let $\nu_i \in \mathbb{Z}$ be such that $1 \le \nu_i < d$, $\gcd(\nu_i, d) = 1$, and assume that $\nu_1 = 1$. We write $\boldsymbol{\nu} = (\nu_i)_{i=1}^r \in \mathbb{Z}^r$. Let $\overline{\boldsymbol{\nu}}$ denote the image of $\boldsymbol{\nu}$ in $(\mathbb{Z}/d\mathbb{Z})^r$. Let $S_{\boldsymbol{\nu}} \subset (\mathbb{Z}/d\mathbb{Z})^r \subset \bigoplus_{i=1}^r \mathbb{Z}/n_i\mathbb{Z} = F$ denote the cyclic subgroup of order d generated by $\overline{\boldsymbol{\nu}}$. Let $L_{\boldsymbol{\nu}}$ denote the preimage of $S_{\boldsymbol{\nu}} \subset F$ in P under the canonical epimorphism $P \twoheadrightarrow F$, then $Q \subset L_{\boldsymbol{\nu}} \subset P$.

Proposition 3.2. *Let W and the W-lattice $L_{\boldsymbol{\nu}}$ be as in Subsection 3.1. In the case $d = 2^s$ we assume that $\sum n_i > 4$. Then $L_{\boldsymbol{\nu}}$ is not quasi-permutation.*

This proposition follows from Lemmas 3.3 and 3.8 below.

Lemma 3.3. *Let $p|d$ be a prime. Then for any subgroup $\Gamma \subset W$ isomorphic to $(\mathbb{Z}/p\mathbb{Z})^m$ for some natural m, the Γ-lattices $L_{\boldsymbol{\nu}}$ and $L_{\mathbf{1}} := L_{(1,\dots,1)}$ are equivalent for any $\boldsymbol{\nu} = (\nu_1, \dots, \nu_r)$ as above (in particular, we assume that $\nu_1 = 1$).*

Note that this lemma is trivial when $d = 2$.

3.4. We compute the lattice $L_{\boldsymbol{\nu}}$ explicitly. First let $r = 1$. We have $Q = Q_1$, $P = P_1$. Then P_1 is generated by Q_1 and an element $\omega \in P_1$ whose image in P_1/Q_1 is of order n_1. We may take

$$\omega = \frac{1}{n_1}[(n_1 - 1)\alpha_1 + (n_1 - 2)\alpha_2 + \cdots + 2\alpha_{n_1-2} + \alpha_{n_1-1}],$$

where $\alpha_1, \dots, \alpha_{n_1-1}$ are the simple roots, see [Bou, Planche I]. There exists exactly one intermediate lattice L between Q_1 and P_1 such that $[L : Q_1] = d$, and it is generated by Q_1 and the element

$$w = \frac{n_1}{d}\omega = \frac{1}{d}[(n_1 - 1)\alpha_1 + (n_1 - 2)\alpha_2 + \cdots + 2\alpha_{n_1-2} + \alpha_{n_1-1}].$$

Now for any natural r, the lattice $L_{\boldsymbol{\nu}}$ is generated by Q and the element

$$w_{\boldsymbol{\nu}} = \frac{1}{d}\sum_{i=1}^{r} \nu_i[(n_i - 1)\alpha_{1,i} + (n_i - 2)\alpha_{2,i} + \cdots + 2\alpha_{n_i-2,i} + \alpha_{n_i-1,i}].$$

In particular, $L_{\mathbf{1}}$ is generated by Q and

$$w_{\mathbf{1}} = \frac{1}{d}\sum_{i=1}^{r}[(n_i - 1)\alpha_{1,i} + (n_i - 2)\alpha_{2,i} + \cdots + 2\alpha_{n_i-2,i} + \alpha_{n_i-1,i}].$$

3.5. *Proof of Lemma 3.3.* Recall that $L_{\boldsymbol{\nu}} = \langle Q, w_{\boldsymbol{\nu}} \rangle$ with

$$Q = \langle \alpha_{\varkappa,i} \rangle\,, \quad \text{where} \quad i = 1, \dots, r,\ \varkappa = 1, \dots, n_i - 1.$$

Set $Q_{\boldsymbol{\nu}} = \langle \nu_i \alpha_{\varkappa,i} \rangle$. Denote by $\mathfrak{T}_{\boldsymbol{\nu}}$ the endomorphism of Q that acts on Q_i by multiplication by ν_i. We have $Q_{\mathbf{1}} = Q,\ Q_{\boldsymbol{\nu}} = \mathfrak{T}_{\boldsymbol{\nu}} Q_{\mathbf{1}},\ w_{\boldsymbol{\nu}} = \mathfrak{T}_{\boldsymbol{\nu}} w_{\mathbf{1}}$. Consider

$$\mathfrak{T}_{\boldsymbol{\nu}} L_{\mathbf{1}} = \langle Q_{\boldsymbol{\nu}}, w_{\boldsymbol{\nu}} \rangle.$$

Clearly the W-lattices $L_{\mathbf{1}}$ and $\mathfrak{T}_{\boldsymbol{\nu}} L_{\mathbf{1}}$ are isomorphic. We have an embedding of W-lattices $Q \hookrightarrow L_{\boldsymbol{\nu}}$, in particular, an embedding $Q \hookrightarrow L_{\mathbf{1}}$, which induces an embedding $\mathfrak{T}_{\boldsymbol{\nu}} Q \hookrightarrow \mathfrak{T}_{\boldsymbol{\nu}} L_{\mathbf{1}}$. Set $M_{\boldsymbol{\nu}} = L_{\boldsymbol{\nu}} / \mathfrak{T}_{\boldsymbol{\nu}} L_{\mathbf{1}}$, then we obtain a homomorphism of W-modules $Q/\mathfrak{T}_{\boldsymbol{\nu}} Q \to M_{\boldsymbol{\nu}}$, which is an isomorphism by Lemma 3.6 below.

Now let $p|d$ be a prime. Let $\Gamma \subset W$ be a subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^m$ for some natural m. As in [LPR, Proof of Proposition 2.10], we use Roiter's version [Ro, Proposition 2] of Schanuel's lemma. We have exact sequences of Γ-modules

$$0 \to \mathfrak{T}_{\boldsymbol{\nu}} L_{\mathbf{1}} \to L_{\boldsymbol{\nu}} \to M_{\boldsymbol{\nu}} \to 0,$$
$$0 \to Q \xrightarrow{\mathfrak{T}_{\boldsymbol{\nu}}} Q \to M_{\boldsymbol{\nu}} \to 0.$$

Since all ν_i are prime to p, we have $|\Gamma| \cdot M_{\boldsymbol{\nu}} = p^m M_{\boldsymbol{\nu}} = M_{\boldsymbol{\nu}}$, and by [Ro, Corollary of Proposition 3] the morphisms of $\mathbb{Z}[\Gamma]$-modules $L_{\boldsymbol{\nu}} \to M_{\boldsymbol{\nu}}$ and $Q \to M_{\boldsymbol{\nu}}$ are projective in the sense of [Ro, § 1]. Now by [Ro, Proposition 2] there exists an isomorphism of Γ-lattices $L_{\boldsymbol{\nu}} \oplus Q \simeq \mathfrak{T}_{\boldsymbol{\nu}} L_{\mathbf{1}} \oplus Q$. Since Q is a quasi-permutation W-lattice, it is a quasi-permutation Γ-lattice, and by Lemma 3.7 below, $L_{\boldsymbol{\nu}} \sim \mathfrak{T}_{\boldsymbol{\nu}} L_{\mathbf{1}}$ as Γ-lattices. Since $\mathfrak{T}_{\boldsymbol{\nu}} L_{\mathbf{1}} \simeq L_{\mathbf{1}}$, we conclude that $L_{\boldsymbol{\nu}} \sim L_{\mathbf{1}}$. ☐

Lemma 3.6. *With the above notation* $L_{\boldsymbol{\nu}} / \mathfrak{T}_{\boldsymbol{\nu}} L_{\mathbf{1}} \simeq Q/\mathfrak{T}_{\boldsymbol{\nu}} Q = \bigoplus_{i=2}^r Q_i / \nu_i Q_i$.

Proof. We have $\mathfrak{T}_{\boldsymbol{\nu}} L_{\mathbf{1}} = \langle S_{\boldsymbol{\nu}} \rangle$, where $S_{\boldsymbol{\nu}} = \{\nu_i \alpha_{\varkappa,i}\}_{i,\varkappa} \cup \{w_{\boldsymbol{\nu}}\}$. Note that

$$d w_{\boldsymbol{\nu}} = \sum_{i=1}^r \nu_i [(n_i - 1)\alpha_{1,i} + (n_i - 2)\alpha_{2,i} + \dots + 2\alpha_{n_i-2,i} + \alpha_{n_i-1,i}].$$

We see that $d w_{\boldsymbol{\nu}}$ is a linear combination with integer coefficients of $\nu_i \alpha_{\varkappa,i}$ and that $\alpha_{n_1-1,1}$ appears in this linear combination with coefficient 1 (because $\nu_1 = 1$). Set $B'_{\boldsymbol{\nu}} = S_{\boldsymbol{\nu}} \smallsetminus \{\alpha_{n_1-1,1}\}$, then $\langle B'_{\boldsymbol{\nu}} \rangle \ni \alpha_{n_1-1,1}$, hence $\langle B'_{\boldsymbol{\nu}} \rangle = \langle S_{\boldsymbol{\nu}} \rangle = \mathfrak{T}_{\boldsymbol{\nu}} L_{\mathbf{1}}$, thus $B'_{\boldsymbol{\nu}}$ is a basis of $\mathfrak{T}_{\boldsymbol{\nu}} L_{\mathbf{1}}$. Similarly, the set $B_{\boldsymbol{\nu}} := \{\alpha_{\varkappa,i}\}_{i,\varkappa} \cup \{w_{\boldsymbol{\nu}}\} \smallsetminus \{\alpha_{n_1-1,1}\}$ is a basis of $L_{\boldsymbol{\nu}}$. Both bases $B_{\boldsymbol{\nu}}$ and $B'_{\boldsymbol{\nu}}$ contain $\alpha_{1,1}, \dots, \alpha_{n_1-2,1}$ and $w_{\boldsymbol{\nu}}$. For all $i = 2, \dots, r$ and all $\varkappa = 1, \dots, n_i - 1$, the basis $B_{\boldsymbol{\nu}}$ contains $\alpha_{\varkappa,i}$, while $B'_{\boldsymbol{\nu}}$ contains $\nu_i \alpha_{\varkappa,i}$. We see that the homomorphism of W-modules $Q/\mathfrak{T}_{\boldsymbol{\nu}} Q = \bigoplus_{i=2}^r Q_i / \nu_i Q_i \to L_{\boldsymbol{\nu}} / \mathfrak{T}_{\boldsymbol{\nu}} L_{\mathbf{1}}$ is an isomorphism. ☐

Lemma 3.7. *Let Γ be a finite group, A and A' be Γ-lattices. If $A\oplus B \sim A'\oplus B'$, where B and B' are quasi-permutation Γ-lattices, then $A \sim A'$.*

Proof. Since B and B' are quasi-permutation, by Lemma 1.3 they are equivalent to 0, and we have

$$A = A \oplus 0 \sim A \oplus B \sim A' \oplus B' \sim A' \oplus 0 = A'.$$

This completes the proof of Lemma 3.7 and hence of Lemma 3.3. □

To complete the proof of Proposition 3.2 it suffices to prove the next lemma.

Lemma 3.8. *Let $p|d$ be a prime. Then there exists a subgroup $\Gamma \subset W$ isomorphic to $(\mathbb{Z}/p\mathbb{Z})^m$ for some natural m such that the Γ-lattice $L_{\mathbf{1}} := L_{(1,\dots,1)}$ is not quasi-permutation.*

3.9. Denote by U_i the space $\mathbb{R}^{n_i}$ with canonical basis $\varepsilon_{1,i}\,,\ \varepsilon_{2,i}\,,\ \dots,\ \varepsilon_{n_i,i}\,$. Denote by V_i the subspace of codimension 1 in U_i consisting of vectors with zero sum of the coordinates. The group $W_i = \mathfrak{S}_{n_i}$ (the symmetric group) permutes the basis vectors $\varepsilon_{1,i}\,,\ \varepsilon_{2,i}\,,\ \dots,\ \varepsilon_{n_i,i}$ and thus acts on U_i and V_i. Consider the homomorphism of vector spaces

$$\chi_i\colon U_i \to \mathbb{R}, \quad \sum_{\lambda=1}^{n_i} \beta_{\lambda,i}\,\varepsilon_{\lambda,i} \ \mapsto \sum_{\lambda=1}^{n_i} \beta_{\lambda,i}$$

taking a vector to the sum of its coordinates. Clearly this homomorphism is W_i-equivariant, where W_i acts trivially on $\mathbb{R}$. We have short exact sequences

$$0 \to V_i \to U_i \xrightarrow{\chi_i} \mathbb{R} \to 0.$$

Set $U = \bigoplus_{i=1}^r U_i,\ V = \bigoplus_{i=1}^r V_i$. The group $W = \prod_{i=1}^r W_i$ naturally acts on U and V, and we have an exact sequence of W-spaces

$$0 \to V \to U \xrightarrow{\chi} \mathbb{R}^r \to 0, \tag{3.1}$$

where $\chi = (\chi_i)_{i=1,\dots,r}$ and W acts trivially on $\mathbb{R}^r$.

Set $n = \sum_{i=1}^r n_i$. Consider the vector space $\overline{U} := \mathbb{R}^n$ with canonical basis $\overline{\varepsilon}_1, \overline{\varepsilon}_2, \dots, \overline{\varepsilon}_n$. Consider the natural isomorphism

$$\varphi\colon U = \bigoplus_i U_i \xrightarrow{\sim} \overline{U}$$

that takes $\varepsilon_{1,1}, \varepsilon_{2,1}, \dots, \varepsilon_{n_1,1}$ to $\overline{\varepsilon}_1, \overline{\varepsilon}_2, \dots, \overline{\varepsilon}_{n_1}$, takes $\varepsilon_{1,2}, \varepsilon_{2,2}, \dots, \varepsilon_{n_2,2}$ to $\overline{\varepsilon}_{n_1+1}, \overline{\varepsilon}_{n_1+2}, \dots, \overline{\varepsilon}_{n_1+n_2}$, and so on. Let $\overline{V}$ denote the subspace of codimension 1 in $\overline{U}$ consisting of vectors with zero sum of the coordinates. Sequence (3.1) induces an exact sequence of W-spaces

$$0 \to \varphi(V) \to \overline{V} \xrightarrow{\psi} \mathbb{R}^r \xrightarrow{\Sigma} \mathbb{R} \to 0. \tag{3.2}$$

Here $\psi = (\psi_i)_{i=1,\dots,r}$, where ψ_i takes a vector $\sum_{j=1}^{n} \beta_j \overline{\varepsilon}_j \in \overline{V}$ to $\sum_{\lambda=1}^{n_i} \beta_{n_1+\dots+n_{i-1}+\lambda}$, and the map Σ takes a vector in $\mathbb{R}^r$ to the sum of its coordinates. Note that W acts trivially on $\mathbb{R}^r$ and $\mathbb{R}$.

We have a lattice $Q_i \subset V_i$ for each $i = 1, \dots, r$, a lattice $Q = \bigoplus_i Q_i \subset \bigoplus_i V_i$, and a lattice $\overline{Q} := \mathbb{Z}\mathbf{A}_{n-1}$ in $\overline{V}$ with basis $\overline{\varepsilon}_1 - \overline{\varepsilon}_2, \dots, \overline{\varepsilon}_{n-1} - \overline{\varepsilon}_n$. The isomorphism φ induces an embedding of $Q = \bigoplus_i Q_i$ into $\overline{Q}$. Under this embedding

$$\begin{aligned}
&\alpha_{1,1} \mapsto \overline{\alpha}_1,\ \alpha_{2,1} \mapsto \overline{\alpha}_2,\ \dots,\ \alpha_{n_1-1,1} \mapsto \overline{\alpha}_{n_1-1},\\
&\alpha_{1,2} \mapsto \overline{\alpha}_{n_1+1},\ \alpha_{2,2} \mapsto \overline{\alpha}_{n_1+2},\ \dots,\ \alpha_{n_2-1,2} \mapsto \overline{\alpha}_{n_1+n_2-1},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\alpha_{1,r} \mapsto \overline{\alpha}_{n_1+n_2+\dots+n_{r-1}+1},\ \dots,\ \alpha_{n_r-1,r} \mapsto \overline{\alpha}_{n-1},
\end{aligned}$$

while $\overline{\alpha}_{n_1}, \overline{\alpha}_{n_1+n_2},\ \dots,\ \overline{\alpha}_{n_1+n_2+\dots+n_{r-1}}$ are skipped.

3.10. We write L for $L_{\mathbf{1}}$ and w for $w_{\mathbf{1}} \in \frac{1}{d}Q$, where $Q = \bigoplus_i Q_i$. Then

$$w = \sum_{i=1}^{r} w_i, \quad w_i = \frac{1}{d}[(n_i - 1)\alpha_{1,i} + \dots + \alpha_{n_i-1,i}].$$

Recall that

$$Q_i = \mathbb{Z}\mathbf{A}_{n_i-1} = \{(a_j) \in \mathbb{Z}^{n_i} \mid \sum_{j=1}^{n_i} a_j = 0\}.$$

Set

$$\overline{w} = \frac{1}{d}\sum_{j=1}^{n-1}(n-j)\overline{\alpha}_j.$$

Set $\Lambda_n(d) = \langle \overline{Q}, \overline{w} \rangle$. Note that $\Lambda_n(d) = Q_n(n/d)$ with the notation of [LPR, Subsection 6.1]. Set

$$N = \varphi(Q \otimes_{\mathbb{Z}} \mathbb{R}) \cap \Lambda_n(d) = \varphi(V) \cap \Lambda_n(d).$$

Lemma 3.11. *$\varphi(L) = N$.*

Proof. Write $j_1 = n_1,\ j_2 = n_1 + n_2,\ \dots,\ j_{r-1} = n_1 + \dots + n_{r-1}$. Set $J = \{1, 2, \dots, n-1\} \smallsetminus \{j_1, j_2, \dots, j_{r-1}\}$. Set

$$\mu = \frac{1}{d}\sum_{j\in J}(n-j)\overline{\alpha}_j = \overline{w} - \sum_{i=1}^{r-1}\frac{n-j_i}{d}\overline{\alpha}_{j_i}.$$

Note that $d|n$ and $d|j_i$ for all i, hence the coefficients $(n - j_i)/d$ are integral, and therefore $\mu \in \Lambda_n(d)$. Since also $\mu \in \varphi(Q \otimes_{\mathbb{Z}} \mathbb{R})$, we see that $\mu \in N$.

Let $y \in N$. Then

$$y = b\overline{w} + \sum_{j=1}^{n-1} a_j\overline{\alpha}_j$$

where $b, a_j \in \mathbb{Z}$, because $y \in \Lambda_n(d)$. We see that in the basis $\overline{\alpha}_1, \dots, \overline{\alpha}_{n-1}$ of $\Lambda_n(d) \otimes_{\mathbb{Z}} \mathbb{R}$, the element y contains $\overline{\alpha}_{j_i}$ with coefficient

$$b\frac{n-j_i}{d} + a_{j_i}.$$

Since $y \in \varphi(Q \otimes_{\mathbb{Z}} \mathbb{R})$, this coefficient must be 0:

$$b\frac{n-j_i}{d} + a_{j_i} = 0.$$

Consider

$$\begin{aligned} y - b\mu = y - b\left(\overline{w} - \sum_{i=1}^{r-1} \frac{n-j_i}{d}\overline{\alpha}_{j_i}\right) &= y - b\overline{w} + \sum_{i=1}^{r-1} \frac{b(n-j_i)}{d}\overline{\alpha}_{j_i} \\ &= \sum_{j=1}^{n-1} a_j\overline{\alpha}_j + \sum_{i=1}^{r-1} \frac{b(n-j_i)}{d}\overline{\alpha}_{j_i} = \sum_{j\in J} a_j\overline{\alpha}_j\,, \end{aligned}$$

where $a_j \in \mathbb{Z}$. We see that $y \in \langle \overline{\alpha}_j\ (j \in J), \mu\rangle$ for any $y \in N$, hence $N \subset \langle \overline{\alpha}_j\ (j \in J), \mu\rangle$. Conversely, $\mu \in N$ and $\overline{\alpha}_j \in N$ for $j \in J$, hence $\langle \overline{\alpha}_j\ (j \in J), \mu\rangle \subset N$, thus

$$N = \langle \overline{\alpha}_j\ (j \in J), \mu\rangle. \tag{3.3}$$

Now

$$\varphi(w) = \frac{1}{d}\left[\sum_{j=1}^{n_1-1}(n_1-j)\overline{\alpha}_j + \sum_{j=1}^{n_2-1}(n_2-j)\overline{\alpha}_{n_1+j} + \cdots + \sum_{j=1}^{n_r-1}(n_r-j)\overline{\alpha}_{j_{r-1}+j}\right]$$

while

$$\mu = \frac{1}{d}\left[\sum_{j=1}^{n_1-1}(n-j)\overline{\alpha}_j + \sum_{j=1}^{n_2-1}(n-n_1-j)\overline{\alpha}_{n_1+j} + \cdots + \sum_{j=1}^{n_r-1}(n_r-j)\overline{\alpha}_{j_{r-1}+j}\right].$$

Thus

$$\mu = \varphi(w) + \frac{n-n_1}{d}\sum_{j=1}^{n_1-1}\overline{\alpha}_j + \frac{n-n_1-n_2}{d}\sum_{j=1}^{n_2-1}\overline{\alpha}_{n_1+j} + \cdots + \frac{n_r}{d}\sum_{j=1}^{n_r-1}\overline{\alpha}_{j_{r-1}+j}\,,$$

where the coefficients

$$\frac{n-n_1}{d}, \quad \frac{n-n_1-n_2}{d}, \quad \dots, \quad \frac{n_r}{d}$$

are integral. We see that

$$\langle \overline{\alpha}_j\ (j \in J),\ \mu\rangle = \langle \overline{\alpha}_j\ (j \in J),\ \varphi(w)\rangle. \tag{3.4}$$

From (3.3) and (3.4) we obtain that

$$N = \langle \overline{\alpha}_j\,(j \in J),\,\mu\rangle = \langle \overline{\alpha}_j\,(j \in J),\,\varphi(w)\rangle = \varphi(L). \qquad \square$$

3.12. Now let $p|\gcd(n_1,\dots,n_r)$. Recall that $W=\prod_{i=1}^r \mathfrak{S}_{n_i}$. Since $p|n_i$ for all i, we can naturally embed $(\mathfrak{S}_p)^{n_i/p}$ into $\mathfrak{S}_{n_i}$. We obtain a natural embedding

$$\Gamma := (\mathbb{Z}/p\mathbb{Z})^{n/p} \hookrightarrow (\mathfrak{S}_p)^{n/p} \hookrightarrow W.$$

In order to prove Lemma 3.8, it suffices to prove the next Lemma 3.13. Indeed, if n has an odd prime factor p, then by Lemma 3.13 L is not quasi-permutation. If $n=2^s$, then we take $p=2$. By the assumptions of Proposition 3.2, $n>4=2^2$, and again by Lemma 3.13 L is not quasi-permutation. This proves Lemma 3.8.

Lemma 3.13. *If either p odd or $n>p^2$, then L is not quasi-permutation as a Γ-lattice.*

Proof. By Lemma 3.11 it suffices to show that N is not quasi-permutation. Since $N=\Lambda_n(d)\cap\varphi(V)$, we have an embedding

$$\Lambda_n(d)/N \hookrightarrow \overline{V}/\varphi(V).$$

By (3.2) $\overline{V}/\varphi(V)\simeq\mathbb{R}^{r-1}$ and W acts on $\overline{V}/\varphi(V)$ trivially. Thus $\Lambda_n(d)/N\simeq\mathbb{Z}^{r-1}$ and W acts on $\mathbb{Z}^{r-1}$ trivially. We have an exact sequence of W-lattices

$$0\to N\to\Lambda_n(d)\to\mathbb{Z}^{r-1}\to 0,$$

with trivial action of W on $\mathbb{Z}^{r-1}$. We obtain that $N\sim\Lambda_n(d)$ as a W-lattice, and hence, as a Γ-lattice. Therefore, it suffices to show that $\Lambda_n(d)=Q_n(n/d)$ is not quasi-permutation as a Γ-lattice if either p is odd or $n>p^2$. This is done in [LPR] in the proofs of Propositions 7.4 and 7.8. This completes the proof of Lemma 3.13 and hence those of Lemma 3.8 and Proposition 3.2. ☐

4 Quasi-permutation lattices – case $\mathbf{A}_{n-1}$

In this section we prove Theorem 0.2 in the special case when G is isogenous to a direct product of groups of type $\mathbf{A}_{n-1}$ for $n\geq 3$.

We maintain the notation of Subsection 3.1. Let L be an intermediate lattice between Q and P, i.e., $Q\subset L\subset P$ ($L=Q$ are $L=P$ are possible). Let S denote the image of L in F, then L is the preimage of $S\subset F$ in P. Since W acts trivially on F, the subgroup $S\subset F$ is W-invariant, and therefore, the sublattice $L\subset P$ is W-invariant.

Theorem 4.1. *With the notation of Subsection 3.1 assume that $n_i\geq 3$ for all $i=1,2,\dots,r$. Let L between Q and P be an intermediate lattice, and assume that $L\cap P_i=Q_i$ for all i such that $n_i=3$ or $n_i=4$. If L is a quasi-permutation W-lattice, then $L=Q$.*

Proof. We prove the theorem by induction on r. The case $r=1$ follows from our assumptions if $n_1=3$ or $n_1=4$, and from Proposition 1.9 if $n_1>4$.

We assume that $r > 1$ and that the assertion is true for $r-1$. We prove it for r.

For i between 1 and r we set

$$Q'_i = \bigoplus_{j\neq i} Q_j, \quad P'_i = \bigoplus_{j\neq i} P_j, \quad F'_i = \bigoplus_{j\neq i} F_j, \quad W'_i = \prod_{j\neq i} W_j,$$

then $Q'_i \subset Q$, $P'_i \subset P$, $F'_i \subset F$ and $W'_i \subset W$. If L is a quasi-permutation W-lattice, then by Lemma 1.7 $L \cap P'_i$ is a quasi-permutation W'_i-lattice, and by the induction hypothesis $L \cap P'_i = Q'_i$.

Now let $Q \subset L \subset P$, and assume that $L \cap P'_i = Q'_i$ for all $i = 1, \dots, r$. We shall show that if $L \neq Q$ then L is not a quasi-permutation W-lattice. This will prove Theorem 4.1.

Assume that $L \neq Q$. Set $S = L/Q \subset F$, then $S \neq 0$. We first show that $(L \cap P'_i)/Q'_i = S \cap F'_i$. Indeed, clearly $(L \cap P'_i)/Q'_i \subset L/Q \cap P'_i/Q'_i = S \cap F'_i$. Conversely, let $f \in S \cap F'_i$, then f can be represented by some $l \in L$ and by some $p \in P'_i$, and $q := l - p \in Q$. Since $L \supset Q$, we see that $p = l - q \in L \cap P'_i$, hence $f \in (L \cap P'_i)/Q'_i$, and therefore $S \cap F'_i \subset (L \cap P'_i)/Q'_i$. Thus $(L \cap P'_i)/Q'_i = S \cap F'_i$.

By assumption we have $L \cap P'_i = Q'_i$, and we obtain that $S \cap F'_i = 0$ for all $i = 1, \dots, r$. Let $S_{(i)}$ denote the image of S under the projection $F \to F_i$. We have a canonical epimorphism $p_i \colon S \to S_{(i)}$ with kernel $S \cap F'_i$. Since $S \cap F'_i = 0$, we see that $p_i \colon S \to S_{(i)}$ is an isomorphism. Set $q_i = p_i \circ p_1^{-1} \colon S_{(1)} \to S_{(i)}$, it is an isomorphism.

We regard $Q_i = \mathbb{Z}\mathbf{A}_{n_i-1}$ and $P_i = \Lambda_{n_i}$ as the lattices described in [Bou, Planche I]. Then we have an isomorphism $F_i \cong \mathbb{Z}/n_i\mathbb{Z}$. Since $S_{(i)}$ is a subgroup of the cyclic group $F_i \cong \mathbb{Z}/n_i\mathbb{Z}$ and $S \cong S_{(i)}$, we see that S is a cyclic group, and we see also that $|S|$ divides n_i for all i, hence $d := |S|$ divides $c := \gcd(n_1, \dots, n_r)$.

We describe all subgroups S of order d in $\bigoplus_{i=1}^r \mathbb{Z}/n_i\mathbb{Z}$ such that $S \cap (\bigoplus_{j\neq i} \mathbb{Z}/n_j\mathbb{Z}) = 0$ for all i. The element $a_i := n_i/d + n_i\mathbb{Z}$ is a generator of $S_{(i)} \subset F_i = \mathbb{Z}/n_i\mathbb{Z}$. Set $b_i = q_i(a_1)$. Since b_i is a generator of $S_{(i)}$, we have $b_i = \overline{\nu}_i a_i$ for some $\overline{\nu}_i \in (\mathbb{Z}/d\mathbb{Z})^\times$. Let $\nu_i \in \mathbb{Z}$ be a representative of $\overline{\nu}_i$ such that $1 \leq \nu_i < d$, then $\gcd(\nu_i, d) = 1$. Moreover, since $q_1 = \mathrm{id}$, we have $b_1 = a_1$, hence $\overline{\nu}_1 = 1$ and $\nu_1 = 1$. We obtain an element $\boldsymbol{\nu} = (\nu_1, \dots, \nu_r)$. With the notation of Subsection 3.1 we have $S = S_{\boldsymbol{\nu}}$ and $L = L_{\boldsymbol{\nu}}$.

By Proposition 3.2 $L_{\boldsymbol{\nu}}$ is not a quasi-permutation W-lattice. Thus L is not quasi-permutation, which completes the proof of Theorem 4.1. □

5 Proof of Theorem 0.2

Lemma 5.1 (well-known). *Let P_1 and P_2 be abelian groups. Set $P = P_1 \oplus P_2 = P_1 \times P_2$, and let $\pi_1 \colon P \to P_1$ denote the canonical projection. Let $L \subset P$ be a* *subgroup. If* $\pi_1(L) = L \cap P_1$*, then*

$$L = (L \cap P_1) \oplus (L \cap P_2).$$

Proof. Let $x \in L$. Set $x_1 = \pi_1(x) \in \pi_1(L)$. Since $\pi_1(L) = L \cap P_1$, we have $x_1 \in L \cap P_1$. Set $x_2 = x - x_1$, then $x_2 \in L \cap P_2$. We have $x = x_1 + x_2$. This completes the proof of Lemma 5.1. ☐

5.2. Let I be a finite set. For any $i \in I$ let D_i be a connected Dynkin diagram. Let $D = \bigsqcup_i D_i$ (disjoint union). Let Q_i and P_i be the root and weight lattices of D_i, respectively, and W_i be the Weyl group of D_i. Set

$$Q = \bigoplus_{i \in I} Q_i, \quad P = \bigoplus_{i \in I} P_i, \quad W = \prod_{i \in I} W_i.$$

5.3. We construct certain quasi-permutation lattices L such that $Q \subset L \subset P$.

Let $\{\, \{i_1, j_1\}, \dots, \{i_s, j_s\} \,\}$ be a set of non-ordered pairs in I such that D_{i_l} and D_{j_l} for all $l = 1, \dots, s$ are of type $\mathbf{B}_1 = \mathbf{A}_1$ and all the indices $i_1, j_1, \dots, i_s, j_s$ are distinct. Fix such an l. We write $\{i, j\}$ for $\{i_l, j_l\}$ and we set $D_{i,j} := D_i \sqcup D_j$, $Q_{i,j} := Q_i \oplus Q_j$, $P_{i,j} := P_i \oplus P_j$. We regard $D_{i,j}$ as a Dynkin diagram of type $\mathbf{D}_2$, and we denote by $M_{i,j}$ the intermediate lattice between $Q_{i,j}$ and $P_{i,j}$ isomorphic to $\mathcal{X}(\mathbf{SO}_4)$, the character lattice of the group $\mathbf{SO}_4$; see Section 1, after Lemma 1.7. Let f_i be a generator of the lattice Q_i of rank 1, and let f_j be a generator of Q_j, then $P_i = \langle \frac{1}{2} f_i \rangle$ and $P_j = \langle \frac{1}{2} f_j \rangle$. Set $e_1^{(l)} = \frac{1}{2}(f_i + f_j)$, $e_2^{(l)} = \frac{1}{2}(f_i - f_j)$, then $\{e_1^{(l)}, e_2^{(l)}\}$ is a basis of $M_{i,j}$, and

$$M_{i,j} = \left\langle Q_{i,j}, e_1^{(l)} \right\rangle, \qquad P_{i,j} = \left\langle M_{i,j}, \frac{1}{2}(e_1^{(l)} + e_2^{(l)}) \right\rangle. \tag{5.1}$$

We have $M_{i,j} \cap P_i = Q_i$, $M_{i,j} \cap P_j = Q_j$, and $[M_{i,j} : Q_{i,j}] = 2$. Concerning the Weyl group, we have

$$W(D_{i,j}) = W(D_i) \times W(D_j) = W(\mathbf{D}_2) = \mathfrak{S}_2 \times \{\pm 1\},$$

where the symmetric group $\mathfrak{S}_2$ permutes the basis vectors $e_1^{(l)}$ and $e_2^{(l)}$ of $M_{i,j}$, while the group $\{\pm 1\}$ acts on $M_{i,j}$ by multiplication by scalars. We say that $M_{i,j}$ is an *indecomposable quasi-permutation lattice* (it corresponds to the semisimple Cayley group $\mathbf{SO}_4$ which does not decompose into a direct product of its normal subgroups).

Set $I' = I \smallsetminus \bigcup_{l=1}^{s} \{i_l, j_l\}$. For $i \in I'$ let M_i be any quasi-permutation intermediate lattice between Q_i and P_i (such an intermediate lattice exists if and only if D_i is of one of the types $\mathbf{A}_n,\ \mathbf{B}_n,\ \mathbf{C}_n,\ \mathbf{D}_n,\ \mathbf{G}_2$, see Proposition 1.9). We say that M_i is a *simple quasi-permutation lattice* (it corresponds to a stably Cayley simple group). We set

$$L = \bigoplus_{l=1}^{s} M_{i_l, j_l} \,\oplus\, \bigoplus_{i \in I'} M_i. \tag{5.2}$$

We say that a lattice L as in (5.2) is a *direct sum of indecomposable quasi-permutation lattices and simple quasi-permutation lattices.* Clearly L is a quasi-permutation W-lattice.

Theorem 5.4. *Let D, Q, P, W be as in Subsection 5.2. Let L be an intermediate lattice between Q and P, i.e., $Q \subset L \subset P$ (where $L = Q$ and $L = P$ are possible). If L is a quasi-permutation W-lattice, then L is as in* (5.2). *Namely, then L is a direct sum of indecomposable quasi-permutation lattices $M_{i,j}$ for some set of pairs $\{\{i_1, j_1\}, \ldots, \{i_s, j_s\}\}$ and some family of simple quasi-permutation intermediate lattices M_i between Q_i and P_i for $i \in I'$.*

Remark 5.5. The set of pairs $\{\{i_1, j_1\}, \ldots, \{i_s, j_s\}\}$ in Theorem 5.4 is uniquely determined by L. Namely, a pair $\{i, j\}$ belongs to this set if and only if the Dynkin diagrams D_i and D_j are of type $\mathbf{B}_1 = \mathbf{A}_1$ and

$$L \cap P_i = Q_i\,, \quad L \cap P_j = Q_j\,, \quad \text{while } L \cap (P_i \oplus P_j) \neq Q_i \oplus Q_j\,.$$

Proof of Theorem 5.4. We prove the theorem by induction on $m = |I|$, where I is as in Subsection 5.2. The case $m = 1$ is trivial.

We assume that $m \geq 2$ and that the theorem is proved for all $m' < m$. We prove it for m. First we consider three special cases.

Split case. Assume that for some subset $A \subset I$, $A \neq I$, $A \neq \emptyset$, we have $\pi_A(L) = L \cap P_A$, where $P_A = \bigoplus_{i \in A} P_i$ and $\pi_A \colon P \to P_A$ is the canonical projection. Then by Lemma 5.1 we have $L = (L \cap P_A) \oplus (L \cap P_{A'})$, where $A' = I \smallsetminus A$. By Lemma 1.7 $L \cap P_A$ is a quasi-permutation W_A-lattice, where $W_A = \prod_{i \in A} W_i$. By the induction hypothesis the lattice $L \cap P_A$ is a direct sum of indecomposable quasi-permutation lattices and simple quasi-permutation lattices. Similarly, $L \cap P_{A'}$ is such a direct sum. We conclude that $L = (L \cap P_A) \oplus (L \cap P_{A'})$ is such a direct sum, and we are done.

$\mathbf{A}_{n-1}$-*case.* Assume that all D_i are of type $\mathbf{A}_{n_i-1}$, where $n_i \geq 3$ (so $\mathbf{A}_1$ is not permitted). We assume also that when $n_i = 3$ and when $n_i = 4$ (that is, for $\mathbf{A}_2$ and for $\mathbf{A}_3 = \mathbf{D}_3$) we have $L \cap P_i = Q_i$ (for $n_i > 4$ this is automatic because $L \cap P_i$ is a quasi-permutation W_i-lattice, see Proposition 1.9). In this case by Theorem 4.1 we have $L = Q = \bigoplus Q_i$, hence L is a direct sum of simple quasi-permutation lattices, and we are done.

$\mathbf{A}_1$-*case.* Assume that all D_i are of type $\mathbf{A}_1$. Then by [BKLR, Theorem 18.1] the lattice L is a direct sum of indecomposable quasi-permutation lattices and simple quasi-permutation lattices, and we are done.

Now we shall show that these three special cases exhaust all the quasi-permutation lattices. In other words, we shall show that if $Q \subset L \subset P$ and L is not as in one of these three cases, then L is not quasi-permutation. This will complete the proof of the theorem.

For the sake of contradiction, let us assume that $Q \subset L \subset P$, that L is not in one of the three special cases above, and that L is a quasi-permutation W-lattice.

We shall show in three steps that L is as in Proposition 2.2. By Proposition 2.2, L is not quasi–permutation, which contradicts our assumptions. This contradiction will prove the theorem.

Step 1. For $i \in I$ consider the intersection $L \cap P_i$, it is a quasi-permutation W_i-lattice (by Lemma 1.7), hence D_i is of one of the types $\mathbf{A}_{n-1}$, $\mathbf{B}_n$, $\mathbf{C}_n$, $\mathbf{D}_n$, $\mathbf{G}_2$ (by Proposition 1.9). Note that $\pi_i(L) \neq L \cap P_i$ (otherwise we are in the split case).

Now assume that for some $i \in I$, the Dynkin diagram D_i is of type $\mathbf{G}_2$ or $\mathbf{C}_n$ for some $n \geq 3$, or D_i is of type $\mathbf{A}_2$ and $L \cap P_i \neq Q_i$. Then $L \cap P_i$ is a quasi-permutation W_i-lattice (by Lemma 1.7), hence $L \cap P_i = P_i$ (by Proposition 1.9). Since $P_i \supset \pi_i(L) \supset L \cap P_i$, we obtain that $\pi_i(L) = L \cap P_i$, which is impossible. Thus no D_i can be of type $\mathbf{G}_2$ or $\mathbf{C}_n$, $n \geq 3$, and if D_i is of type $\mathbf{A}_2$ for some i, then $L \cap P_i = Q_i$.

Thus all D_i are of types $\mathbf{A}_{n-1}$, $\mathbf{B}_n$ or $\mathbf{D}_n$, and if D_i is of type $\mathbf{A}_2$ for some $i \in I$, then $L \cap P_i = Q_i$. Since L is not as in the $\mathbf{A}_{n-1}$-case, we may assume that one of the D_i, say D_1, is of type $\mathbf{B}_n$ for some $n \geq 1$ ($\mathbf{B}_1 = \mathbf{A}_1$ is permitted), or of type $\mathbf{D}_n$ for some $n \geq 4$, or of type $\mathbf{D}_3$ with $L \cap P_1 \neq Q_1$. Indeed, otherwise *all* D_i are of type $\mathbf{A}_{n_i-1}$ for $n_i \geq 3$, and in the cases $\mathbf{A}_2$ ($n_i = 3$) and $\mathbf{A}_3$ ($n_i = 4$) we have $L \cap P_i = Q_i$, i.e., we are in the $\mathbf{A}_{n-1}$-case, which contradicts our assumptions.

Step 2. In this step, using the Dynkin diagram D_1 of type $\mathbf{B}_n$ or $\mathbf{D}_n$ from the previous step, we construct a quasi-permutation sublattice $L' \subset L$ of index 2 such that L' is as in (5.2). First we consider the cases $\mathbf{B}_n$ and $\mathbf{D}_n$ separately.

Assume that D_1 is of type $\mathbf{B}_n$ for some $n \geq 1$ ($\mathbf{B}_1 = \mathbf{A}_1$ is permitted). We have $[P_1 : Q_1] = 2$. Since $P_1 \supset \pi_1(L) \supsetneq L \cap P_1 \supset Q_1$, we see that $\pi_1(L) = P_1$ and $L \cap P_1 = Q_1$. Set $M_1 = Q_1$. We have $\pi_1(L) = P_1$, $L \cap P_1 = M_1$, and $[P_1 : M_1] = 2$.

Now assume that D_1 is of type $\mathbf{D}_n$ for some $n \geq 4$, or of type $\mathbf{D}_3$ with $L \cap P_1 \neq Q_1$. Set $M_1 = L \cap P_1$, then M_1 is a quasi-permutation W_1-lattice by Lemma 1.7, and it follows from Proposition 1.9 that $(W_1, M_1) \simeq \mathcal{X}(\mathbf{SO}_{2n})$, where $\mathcal{X}(\mathbf{SO}_{2n})$ denotes the character lattice of $\mathbf{SO}_{2n}$; see Section 1, after Lemma 1.7. It follows that $[M_1 : Q_1] = 2$ and $[P_1 : M_1] = 2$. Since $P_1 \supset \pi_1(L) \supsetneq L \cap P_1 = M_1$, we see that $\pi_1(L) = P_1$. Again we have $\pi_1(L) = P_1$, $L \cap P_1 = M_1$, and $[P_1 : M_1] = 2$.

Now we consider together the cases when D_1 is of type $\mathbf{B}_n$ for some $n \geq 1$ and when D_1 is of type $\mathbf{D}_n$ for some $n \geq 3$, where in the case $\mathbf{D}_3$ we have $L \cap P_1 \neq Q_1$. Set

$$L' := \ker[L \xrightarrow{\pi_1} P_1 \to P_1/M_1].$$

Since $\pi_1(L) = P_1$, and $[P_1 : M_1] = 2$, we have $[L : L'] = 2$. Clearly we have $\pi_1(L') = M_1$. Set

$$L_1^\dagger := \ker[\pi_1 \colon L \to P_1] = L \cap P_1',$$

where $P'_1 = \bigoplus_{i\neq 1} P_i$. Since L is a quasi-permutation W-lattice, by Lemma 1.7 the lattice $L_1^\dagger$ is a quasi-permutation W'_1-lattice, where $W'_1 = \prod_{i\neq 1} W_i$. By the induction hypothesis, $L_1^\dagger$ is a direct sum of indecomposable quasi-permutation lattices and simple quasi-permutation lattices as in (5.2). Since $M_1 = L\cap P_1$, we have $M_1 \subset L'\cap P_1$, and $L'\cap P_1 \subset L\cap P_1 = M_1$, hence $L'\cap P_1 = M_1 = \pi_1(L')$, and by Lemma 5.1 we have $L' = M_1 \oplus L_1^\dagger$. Since M_1 is a simple quasi-permutation lattice, we conclude that L' is a direct sum of indecomposable quasi-permutation lattices and simple quasi-permutation lattices as in (5.2), and $[L:L'] = 2$.

Step 3. In this step we show that L is as in Proposition 2.2. We write

$$L' = \bigoplus_{l=1}^{s}(L'\cap P_{i_l,j_l}) \,\oplus\, \bigoplus_{i\in I'}(L'\cap P_i),$$

where $P_{i_l,j_l} = P_{i_l}\oplus P_{j_l}$, the Dynkin diagrams D_{i_l} and D_{j_l} are of type $\mathbf{A}_1 = \mathbf{B}_1$, and $L'\cap P_{i_l,j_l} = M_{i_l,j_l}$ as in (5.1). For any $i\in I'$, we have $[\pi_i(L):\pi_i(L')] \le 2$, because $[L:L'] = 2$. Furthermore, for $i\in I'$ we have

$$\pi_i(L') = L'\cap P_i \subset L\cap P_i \subsetneq \pi_i(L),$$

hence $[\pi_i(L):(L\cap P_i)] = 2$ and $L'\cap P_i = L\cap P_i$. Similarly, for any $l = 1,\dots,s$, if we write $i = i_l,\ j = j_l$, then we have

$$M_{i,j} = L'\cap P_{i,j} \subset L\cap P_{i,j} \subsetneq \pi_{i,j}(L) \subset P_{i,j}, \qquad [P_{i,j}:M_{i,j}] = 2,$$

whence $\pi_{i,j}(L) = P_{i,j}$, $L\cap P_{i,j} = M_{i,j}$, and therefore $[\pi_{i,j}(L):(L\cap P_{i,j})] = [P_{i,j}:M_{i,j}] = 2$ and $L'\cap P_{i,j} = M_{i,j} = L\cap P_{i,j}$.

We view the Dynkin diagram $D_{i_l}\sqcup D_{j_l}$ of type $\mathbf{A}_1\sqcup\mathbf{A}_1$ corresponding to the pair $\{i_l,j_l\}$ $(l = 1,\dots,s)$ as a Dynkin diagram of type $\mathbf{D}_2$. Thus we view L' as a direct sum of indecomposable quasi-permutation lattices and simple quasi-permutation lattices corresponding to Dynkin diagrams of type $\mathbf{B}_n$, $\mathbf{D}_n$ and $\mathbf{A}_n$.

We wish to show that L is as in Proposition 2.2. We change our notation in order to make it closer to that of Proposition 2.2.

As in Subsection 2.1, we now write D_i for Dynkin diagrams of types $\mathbf{B}_{l_i}$ and $\mathbf{D}_{l_i}$ only, appearing in L', where $\mathbf{B}_1 = \mathbf{A}_1$, $\mathbf{B}_2 = \mathbf{C}_2$, $\mathbf{D}_2 = \mathbf{A}_1\sqcup\mathbf{A}_1$ and $\mathbf{D}_3 = \mathbf{A}_3$ are permitted, but for $\mathbf{D}_{l_i}$ with $l_i = 2,3$ we require that

$$L\cap P_i = M_i := \mathcal{X}(\mathbf{SO}_{2l_i}).$$

We write $L'_i := L\cap P_i = L'\cap P_i$. We have $[\pi_i(L):L'_i] = 2$, hence $[P_i:L'_i]\ge 2$. If D_i is of type $\mathbf{B}_{l_i}$, then $[P_i:L'_i] = 2$. If D_i is of type $\mathbf{D}_{l_i}$, then $L'_i = L\cap P_i \ne Q_i$, for $\mathbf{D}_2$ and $\mathbf{D}_3$ by our assumption and for $\mathbf{D}_{l_i}$ with $l_i\ge 4$ because $L\cap P_i$ is a quasi-permutation W_i-lattice (see Proposition 1.9); again we have $[P_i:L'_i] = 2$.

We see that for all i we have $[P_i : L'_i] = 2$, $\pi_i(L) = P_i$, and the lattice $L'_i = M_i$ is as in Subsection 2.1. We realize the root system $R(D_i)$ of type $\mathbf{B}_{l_i}$ or $\mathbf{D}_{l_i}$ in the standard way (cf. [Bou, Planches II, IV]) in the space $V_i := \mathbb{R}^{l_i}$ with basis $(e_s)_{s\in S_i}$, then L'_i is the lattice generated by the basis vectors $(e_s)_{s\in S_i}$ of V_i, and we have $P_i = \langle L'_i, \frac{1}{2}x_i\rangle$, where

$$x_i = \sum_{s\in S_i} e_s \in L'_i\,.$$

In particular, when D_i is of type $\mathbf{D}_2$ we have $x_i = e_1^{(l)} + e_2^{(l)}$ with the notation of formula (5.1).

As in Subsection 2.1, we write Δ_ι for Dynkin diagrams of type $\mathbf{A}_{n'_\iota - 1}$ appearing in L', where $n'_\iota \geq 3$ and for $\mathbf{A}_3 = \mathbf{D}_3$ we require that $L \cap P_\iota = Q_\iota$. We write $L'_\iota := L \cap P_\iota = L' \cap P_\iota$. Then $L'_\iota = Q_\iota$ for all ι, for $\mathbf{A}_2$ by Step 1, for $\mathbf{A}_3$ by our assumption, and for other $\mathbf{A}_{n'_\iota - 1}$ because L'_ι is a quasi-permutation W_ι-lattice; see Proposition 1.9. We have $\pi_\iota(L) \supsetneq L \cap P_\iota = L'_\iota$ and $[\pi_\iota(L) : L'_\iota] = [\pi_\iota(L) : \pi_\iota(L')] \leq 2$ (because $[L : L'] = 2$). It follows that $[\pi_\iota(L) : L'_\iota] = 2$, i.e., $[\pi_\iota(L) : Q_\iota] = 2$. We know that P_ι/Q_ι is a cyclic group of order n'_ι. Since it has a subgroup $\pi_\iota(L)/Q_\iota$ of order 2, we conclude that n'_ι is even, $n'_\iota = 2n_\iota$ (where $2n_\iota \geq 4$), and $\pi_\iota(L)/Q_\iota$ is the unique subgroup of order 2 of the cyclic group P_ι/Q_ι of order $2n_\iota$. As in Subsection 2.1, we realize the root system Δ_ι of type $\mathbf{A}_{2n_\iota - 1}$ in the standard way (cf. [Bou, Planche I]) in the subspace V_ι of vectors with zero sum of the coordinates in the space $\mathbb{R}^{2n_\iota}$ with basis $\varepsilon_{\iota,1}, \dots, \varepsilon_{\iota,2n_\iota}$. We set

$$\xi_\iota = \varepsilon_{\iota,1} - \varepsilon_{\iota,2} + \varepsilon_{\iota,3} - \varepsilon_{\iota,4} + \dots + \varepsilon_{\iota,2n_\iota - 1} - \varepsilon_{\iota,2n_\iota},$$

then $\xi_\iota \in L'_\iota$ and $\frac{1}{2}\xi_\iota \in \pi_\iota(L) \smallsetminus L'_\iota$ (cf. [Bou, Planche I, formula (VI)]), hence $\pi_\iota(L) = \langle L'_\iota, \frac{1}{2}\xi_\iota\rangle$.

Now we set

$$v = \frac{1}{2}\sum_{i\in I} x_i + \frac{1}{2}\sum_{\iota=1}^{\mu} \xi_\iota\,.$$

We claim that

$$L = \langle L', v\rangle.$$

Proof of the claim. Let $w \in L \smallsetminus L'$, then $L = \langle L', w\rangle$, because $[L : L'] = 2$. Set $z_i = \frac{1}{2}x_i - \pi_i(w)$, then $z_i \in L'_i \subset L'$, because $\frac{1}{2}x_i, \pi_i(w) \in \pi_i(L) \smallsetminus L'_i$. Similarly, we set $\zeta_\iota = \frac{1}{2}\xi_\iota - \pi_\iota(w)$, then $\zeta_\iota \in L'_\iota \subset L'$. We see that

$$v = w + \sum_i z_i + \sum_\iota \zeta_\iota,$$

where $\sum_i z_i + \sum_\iota \zeta_\iota \in L'$, and the claim follows. □

It follows from the claim that L is as in Proposition 2.2 (we use the assumption that we are not in the $\mathbf{A}_1$-case). Now by Proposition 2.2 L is not quasi-invertible, hence not quasi-permutation, which contradicts our assumptions. This contradiction proves Theorem 5.4. □

Proof of Theorem 0.2. Theorem 0.2 follows immediately from Theorem 5.4 by virtue of Proposition 1.8. □

6 Proof of Theorem 0.3

In this section we deduce Theorem 0.3 from Theorem 0.2.

Let G be a stably Cayley semisimple k-group. Then $\overline{G} := G \times_k \bar{k}$ is stably Cayley over an algebraic closure $\bar{k}$ of k. By Theorem 0.2, $G_{\bar{k}} = \prod_{j\in J} G_{j,\bar{k}}$ for some finite index set J, where each $G_{j,\bar{k}}$ is either a stably Cayley simple group or is isomorphic to $\mathbf{SO}_{4,\bar{k}}$. (Recall that $\mathbf{SO}_{4,\bar{k}}$ is stably Cayley and semisimple, but is not simple.) Here we write $G_{j,\bar{k}}$ for the factors in order to emphasize that they are defined over $\bar{k}$. By Remark 5.5 the collection of direct factors $G_{j,\bar{k}}$ is determined uniquely by $\overline{G}$. The Galois group $\mathrm{Gal}(\bar{k}/k)$ acts on $G_{\bar{k}}$, hence on J. Let Ω denote the set of orbits of $\mathrm{Gal}(\bar{k}/k)$ in J. For $\omega \in \Omega$ set $G^{\omega}_{\bar{k}} = \prod_{j\in\omega} G_{j,\bar{k}}$, then $\overline{G} = \prod_{\omega\in\Omega} G^{\omega}_{\bar{k}}$. Each $G^{\omega}_{\bar{k}}$ is $\mathrm{Gal}(\bar{k}/k)$-invariant, hence it defines a k-form G^{ω}_{k} of $G^{\omega}_{\bar{k}}$. We have $G = \prod_{\omega\in\Omega} G^{\omega}_{k}$.

For each $\omega \in \Omega$ choose $j = j_\omega \in \omega$. Let l_j/k denote the Galois extension in $\bar{k}$ corresponding to the stabilizer of j in $\mathrm{Gal}(\bar{k}/k)$. The subgroup $G_{j,\bar{k}}$ is $\mathrm{Gal}(\bar{k}/l_j)$-invariant, hence it comes from an l_j-form G_{j,l_j}. By the definition of Weil's restriction of scalars (see e.g. [Vo2, Subsection 3.12]) $G^{\omega}_{k} \cong R_{l_j/k} G_{j,l_j}$, hence $G \cong \prod_{\omega\in\Omega} R_{l_j/k} G_{j,l_j}$. Each G_{j,l_j} is either absolutely simple or an l_j-form of $\mathbf{SO}_4$.

We complete the proof using an argument from [BKLR, Proof of Lemma 11.1]. We show that G_{j,l_j} is a direct factor of $G_{l_j} := G \times_k l_j$. It is clear from the definition that $G_{j,\bar{k}}$ is a direct factor of $G_{\bar{k}}$ with complement $G'_{\bar{k}} = \prod_{i\in J\smallsetminus\{j\}} G_{i,\bar{k}}$. Then $G'_{\bar{k}}$ is $\mathrm{Gal}(\bar{k}/l_j)$-invariant, hence it comes from some l_j-group G'_{l_j}. We have $G_{l_j} = G_{j,l_j} \times_{l_j} G'_{l_j}$, hence G_{j,l_j} is a direct factor of G_{l_j}.

Recall that G_{j,l_j} is either a form of $\mathbf{SO}_4$ or absolutely simple. If it is a form of $\mathbf{SO}_4$, then clearly it is stably Cayley over l_j. It remains to show that if G_{j,l_j} is absolutely simple, then G_{j,l_j} is stably Cayley over l_j. The group $G_{\bar{k}}$ is stably Cayley over $\bar{k}$. Since $G_{j,\bar{k}}$ is a direct factor of the stably Cayley $\bar{k}$-group $G_{\bar{k}}$ over the algebraically closed field $\bar{k}$, by [LPR, Lemma 4.7] $G_{j,\bar{k}}$ is stably Cayley over $\bar{k}$. Comparing [LPR, Theorem 1.28] and [BKLR, Theorem 1.4], we see that G_{j,l_j} is either stably Cayley over l_j (in which case we are done) or an outer form of $\mathbf{PGL}_{2n}$ for some $n \geq 2$. Thus assume by the way of contradiction that G_{j,l_j} is an outer form of $\mathbf{PGL}_{2n}$ for some $n \geq 2$. Then by [BKLR, Example 10.7] the character lattice of G_{j,l_j} is not quasi-invertible, and by [BKLR, Proposition 10.8] the group G_{j,l_j} cannot be a direct factor of a stably Cayley l_j-group. This contradicts the fact that G_{j,l_j} is a direct factor of the stably Cayley l_j-group G_{l_j}. We conclude that G_{j,l_j} cannot be an outer form of $\mathbf{PGL}_{2n}$ for any $n \geq 2$. Thus G_{j,l_j} is stably Cayley over l_j, as desired. □

A Appendix: Some quasi-permutation character lattices

The positive assertion of Proposition 1.9 above is well known. It is contained in [CK, Theorem 0.1] and in [BKLR, Theorem 1.4]. However, [BKLR] refers to [CK, Theorem 0.1], and [CK] refers to a series of results on rationality (rather than only stable rationality) of corresponding generic tori. In this appendix for the reader's convenience we provide a proof of the following positive result in terms of lattices only.

Proposition A.1. *Let G be any form of one of the following groups*

$$\mathbf{SL}_3,\ \mathbf{PGL}_n\ (n \text{ odd}),\ \mathbf{SO}_n\ (n \geq 3),\ \mathbf{Sp}_{2n},\ \mathbf{G}_2$$

or an inner *form of $\mathbf{PGL}_n$ (n even). Then the character lattice of G is quasi-permutation.*

Proof. $\mathbf{SO}_{2n+1}$. Let L be the character lattice of $\mathbf{SO}_{2n+1}$ (including $\mathbf{SO}_3$). Then the Dynkin diagram is $D = \mathbf{B}_n$. The Weyl group is $W = \mathfrak{S}_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$. Then $L = \mathbb{Z}^n$ with the standard basis $e_1, \dots, e_n$. The group $\mathfrak{S}_n$ naturally permutes $e_1, \dots, e_n$, while $(\mathbb{Z}/2\mathbb{Z})^n$ acts by sign changes. Since W permutes the basis up to $\pm$ sign, the W-lattice L is quasi-permutation, see [Lo, § 2.8].

$\mathbf{SO}_{2n}$*, any form, inner or outer.* Let L be the character lattice of $\mathbf{SO}_{2n}$ (including $\mathbf{SO}_4$). Then the Dynkin diagram is $D = \mathbf{D}_n$, with root system $R = R(D)$. We consider the pair (A, L) where $A = \mathrm{Aut}(R, L)$, then (A, L) is isomorphic to the character lattice of $\mathbf{SO}_{2n+1}$, hence is quasi-permutation.

$\mathbf{Sp}_{2n}$. The character lattice of $\mathbf{Sp}_{2n}$ is isomorphic to the character lattice of $\mathbf{SO}_{2n+1}$, hence is quasi-permutation.

$\mathbf{PGL}_n$*, inner form.* The character lattice of $\mathbf{PGL}_n$ is the root lattice $L = Q$ of $\mathbf{A}_{n-1}$. It is a quasi-permutation $\mathfrak{S}_n$-lattice, cf. [Lo, Example 2.8.1].

$\mathbf{PGL}_n$*, outer form, n odd.* Let P be the *weight* lattice of $\mathbf{A}_{n-1}$, where $n \geq 3$ is odd. Then P is generated by elements $e_1, \dots, e_n$ subject to the relation

$$e_1 + \cdots + e_n = 0.$$

The automorphism group $A = \mathrm{Aut}(\mathbf{A}_{n-1})$ is the product of $\mathfrak{S}_n$ and $\mathfrak{S}_2$. The group A acts on P as follows: $\mathfrak{S}_n$ permutes $e_1, \dots, e_n$, and the nontrivial element of $\mathfrak{S}_2$ takes each e_i to $-e_i$.

We denote by M the A-lattice of rank $2n+1$ with basis $s_1, \dots, s_n, t_1, \dots, t_n, u$. The group $\mathfrak{S}_n$ permutes s_i and permutes t_i ($i = 1, \dots, n$), and the nontrivial element of $\mathfrak{S}_2$ permutes s_i and t_i for each i. The group A acts trivially on u. Clearly M is a permutation lattice.

We define an A-epimorphism $\pi\colon M \to P$ as follows:

$$\pi\colon \quad s_i \mapsto e_i, \quad t_i \mapsto -e_i, \quad u \mapsto 0.$$

Set $M' = \ker \pi$, it is an A-lattice of rank $n+2$. We show that it is a permutation lattice. We write down a set of $n+3$ generators of M':

$$\rho_i = s_i + t_i, \quad \sigma = s_1 + \cdots + s_n, \quad \tau = t_1 + \cdots + t_n, \quad u.$$

There is a relation

$$\rho_1 + \cdots + \rho_n = \sigma + \tau.$$

We define a new set of $n+2$ generators:

$$\tilde{\rho}_i = \rho_i + u, \quad \tilde{\sigma} = \sigma + \tfrac{n-1}{2}u, \quad \tilde{\tau} = \tau + \tfrac{n-1}{2}u,$$

where $\frac{n-1}{2}$ is integral because n is odd. We have

$$\tilde{\rho}_1 + \cdots + \tilde{\rho}_n - \tilde{\sigma} - \tilde{\tau} = u,$$

hence this new set indeed generates M', hence it is a basis. The group $\mathfrak{S}_n$ permutes $\tilde{\rho}_1, \dots, \tilde{\rho}_n$, while $\mathfrak{S}_2$ permutes $\tilde{\sigma}$ and $\tilde{\tau}$. Thus A permutes our basis, and therefore M' is a permutation lattice. We have constructed a left resolution of P:

$$0 \to M' \to M \to P \to 0,$$

(with permutation lattices M and M'), which by duality gives a right resolution of the *root* lattice $Q \cong P^\vee$ of $\mathbf{A}_{n-1}$:

$$0 \to Q \to M^\vee \to (M')^\vee \to 0$$

with permutation lattices $M^\vee$ and $(M')^\vee$. Thus the character lattice Q of $\mathbf{PGL}_n$ is a quasi-permutation A-lattice for odd n.

The assertion that the character lattice of G is quasi-permutation in the remaining cases $\mathbf{SL}_3$ and $\mathbf{G}_2$ follows from the next Lemma A.2.

Lemma A.2 ([BKLR, Lemma 2.5]). *Let Γ be a finite group and L be any Γ-lattice of rank $r = 1$ or 2. Then L is quasi-permutation.*

This lemma, which is a version of [Vo2, § 4.9, Examples 6 and 7], was stated in [BKLR] without proof. For the sake of completeness we supply a short proof here.

We may assume that Γ is a maximal finite subgroup of $\mathbf{GL}_r(\mathbb{Z})$. If $r = 1$, then $\mathbf{GL}_1(\mathbb{Z}) = \{\pm 1\}$, and the lemma reduces to the case of the character lattice of $\mathbf{SO}_3$ treated above.

Now let $r = 2$. Up to conjugation there are two maximal finite subgroups of $\mathbf{GL}_2(\mathbb{Z})$, they are isomorphic to the dihedral groups D_8 (of order 8) and to D_{12} (of order 12), resp., see e.g. [Lo, § 1.10.1, Table 1.2]. The group D_8 is the group of symmetries of a square, and in this case it suffices to show that the character lattice of $\mathbf{SO}_5$ is quasi-permutation, which we have done above. The group D_{12} is the group of symmetries of a regular hexagon, and in this case it suffices to show that the character lattice of $\mathbf{PGL}_3$ (outer form) is quasi-permutation, which we have done above as well. This completes the proofs of Lemma A.2 and Proposition A.1. □

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Equivariant Oriented Cohomology of Flag Varieties

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Abstract. Given an equivariant oriented cohomology theory $\mathtt{h}$, a split reductive group G, a maximal torus T in G, and a parabolic subgroup P containing T, we explain how the T-equivariant oriented cohomology ring $\mathtt{h}_T(G/P)$ can be identified with the dual of a coalgebra defined using exclusively the root datum of (G,T), a set of simple roots defining P and the formal group law of $\mathtt{h}$. In two papers [CZZ,CZZ2] we studied the properties of this dual and of some related operators by algebraic and combinatorial methods, without any reference to geometry. The present paper can be viewed as a companion paper, that justifies all the definitions of the algebraic objects and operators by explaining how to match them to equivariant oriented cohomology rings endowed with operators constructed using push-forwards and pull-backs along geometric morphisms. Our main tool is the pull-back to the T-fixed points of G/P which embeds the cohomology ring in question into a direct product of a finite number of copies of the T-equivariant oriented cohomology of a point.

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Contents

1. Introduction

Given an equivariant algebraic oriented cohomology theory $\mathtt{h}$ over a base field k, a split reductive group G over k, a maximal torus T in G and a parabolic subgroup P containing T, we explain how, as a ring, $\mathtt{h}_T(G/P)$ can naturally be identified with an algebraic object $\mathbf{D}_\Xi^\star$ introduced in [CZZ2]. This $\mathbf{D}_\Xi^\star$ is the dual of a coalgebra defined using exclusively the root datum of (G, T), a set of simple roots Ξ defining P and the formal group law F of $\mathtt{h}$. In [CZZ2], we studied the properties of this object and of some related operators by algebraic and combinatorial methods, without any reference to geometry. The present paper is a companion paper to [HMSZ, CZZ, CZZ2] that justifies the definitions of $\mathbf{D}_\Xi^\star$ and of other related algebraic objects and operators by explaining how to match them to equivariant cohomology rings endowed with operators constructed using push-forwards and pull-backs along geometric morphisms.

The starting point of our approach are celebrated papers by Bernstein-Gelfand-Gelfand and Demazure [BGG, D74] dedicated to (non-equivariant) Chow groups and K-theory, which then were extended to the respective T-equivariant setting by Arabia [Ar86, Ar89], Brion [Br97], Kostant, Kumar [KK86, KK90] and others. While the equivariant case looks more difficult, its big advantage is that the T-fixed points embedding injects $\mathtt{h}_T(G/P)$ into a very simple ring: a direct product of a finite number of copies of $\mathtt{h}_T(\mathrm{pt})$, where pt is $\mathrm{Spec}(k)$. This important property was already apparent in [Q71, Thm. 4.4] in the topological context (see also [CS74, AB84]). With this observation in hands, the study of the multiplication of Schubert classes (one of the major goals of Schubert calculus) turns into the study of the image of this injection, and then finding a good description of classes of geometric interest in this image, i.e. classes of Schubert varieties, or rather their Bott-Samelson desingularisations.

We would like to point out several places where the case of an oriented cohomology theory with an arbitrary formal group law is significantly more complicated than the two classical cases of the additive law (Chow groups) and the multiplicative one (K-theory). First of all, in these two classical cases, the formal group law is given by very simple polynomials; it is easy to conceive that the computations increase in complexity with other formal group laws given by powers series with an infinite number of nonzero coefficients. Secondly, in both of these classical cases, the (non-equivariant) cohomology ring of a point is $\mathbb{Z}$, which is a regular ring, while in general, this base ring can be arbitrary. In the work of Kostant and Kumar, the fraction field of the T-equivariant cohomology ring of the point is used as a crucial tool, but we are forced to invert fewer elements and use a more subtle localization process, for fear of killing everything in some cases (see the definition of Q from S in section 5). Thirdly, an important result by Bressler and Evens [BE90] shows that the additive and the multiplicative formal group laws are the only formal group laws for which the elements X_{I_w} and Y_{I_w} (see after Def. 5.2) are independent of the choice of a reduced decomposition I_w of w. Geometrically, this translates as the fact that for Chow groups or K-theory, the class of a Bott-Samelson desingularization corresponding to the reduced decomposition I_w only depends on w, and actually is the class of the (possibly singular) Schubert variety corresponding to w in Chow groups and the class of its structural sheaf in K-theory. This combinatorial/geometric independence plays a crucial role in the arguments dealing with Chow groups or K-theory: see [D73, Thm. 1] and how it is used in [D74, §4]; see also [KK86, Prop. 4.2] and its corollary Prop. 4.3. For an arbitrary oriented cohomology theory, for example for algebraic cobordism, this is simply not true: different desingularizations of the same Schubert variety give different classes.

Let us mention some of the literature on cohomology theories that go beyond Chow groups or K-theory. Using the Bernstein-Gelfand-Gelfand approach, Bressler and Evens [BE90, BE92] described bases of the (non equivariant) topological complex cobordism ring using Bott-Samelson classes and depending on choosing a reduced decomposition for each Weyl group element. These results were extended later to the algebro-geometric setting independently in [HK] and [CPZ]; in the latter, the approach is algebraic as in [D73, D74] and an efficient algorithm for multiplying Bott-Samelson classes [CPZ, §15] is provided. In [HHH], Harada, Henriques and Holm prove the injectivity of the pull-back to fixed points map and the characterization of its image in the topological context of generalized cohomology theories, under an assumption that certain characteristic classes are prime to each other. Our Theorem 9.2 gives the precise cases when this happens; as all of our statements and proofs, it only relies on algebro-geometric methods, with no input from topology.

In [KiKr, Thm. 5.1], a Borel-style presentation of equivariant algebraic cobordism is obtained after inverting the torsion index. The improvement of our Theorem 10.2 is that it applies to any oriented cohomology theory, and that, even over a field of characteristic zero, over which algebraic cobordism is the universal oriented cohomology theory, it gives a finer result than what one would get by specializing from cobordism, as one can see in the case of K-theory for which the Borel-style presentation always holds in the simply connected case, without inverting the torsion index.

The techniques developed in the present paper (together with [HMSZ], [CZZ] and [CZZ2]) have been successfully applied to elliptic cohomology: see [LZ14], where the Billey-Graham-Willems formulas for the localization of Schubert classes at torus fixed points were extended to degenerate elliptic cohomology case. In [ZZ14], the authors establish a residue interpretation of the formal affine Hecke algebra $\mathbf{H}_F$ (a deformation of $\mathbf{D}_\Xi$), which coincides with the residue construction of elliptic affine Hecke algebra of Ginzburg, Kapranov, and Vasserot [GKV97] for an arbitrary elliptic formal group law. They also constructed an isomorphism between $\mathbf{H}_F$ and the equivariant oriented cohomology of the Steinberg variety.

Our main results (Theorems 8.11 and 9.1) identify the ring $\mathbf{D}_\Xi^\star$ with the equivariant cohomology $\mathtt{h}_T(G/P)$, within the fixed points ring $S^\star_{W/W_\Xi}$ that is a direct product of copies of $\mathtt{h}_T(\mathrm{pt})$ and the image of the injective pull-back map $\mathtt{h}_T(G/P) \to \mathtt{h}_T(G/B)$ (B is a Borel subgroup) as the subring $\mathtt{h}_T(G/B)^{W_\Xi}$ of fixed elements under the parabolic Weyl group W_Ξ corresponding to P. In Theorem 10.2 we provide a Borel-style presentation $\mathtt{h}_T(\mathrm{pt}) \otimes_{\mathtt{h}_T(\mathrm{pt})^W} \mathtt{h}_T(\mathrm{pt}) \simeq \mathtt{h}_T(G/B)$ under certain conditions.

Other results are proved along the way: Theorem 9.2 gives an intrinsic characterization of the above mentioned image in the Borel case. Diagram (8.3) describes the push-forward map $\mathtt{h}_T(G/P') \to \mathtt{h}_T(G/P)$, induced by the projections $G/P' \to G/P$ for parabolic subgroups $P' \subseteq P$ of G. Lemma 7.6 describes the algebraic elements corresponding to Bott-Samelson classes, i.e. fundamental classes of desingularized Schubert varieties. Theorem 9.3 proves that the pairing defined by product and push-forward to $\mathtt{h}_T(\mathrm{pt})$ is non-degenerate.

The paper is organized as follows. First, we state the properties that we use from equivariant oriented cohomology theories, in section 2. Then, in section 3, we describe $\mathtt{h}_T(\mathrm{pt})$ as the formal group ring of [CPZ, Def. 2.4]. In section 4, we compute the case of $\mathtt{h}_T(\mathbb{P}^1)$ when the action of T on the projective line $\mathbb{P}^1 = (\mathbb{A}^2 \setminus \{0\})/\mathbb{G}_m$ is induced by a linear action of T on $\mathbb{A}^2$. It enables us to compute the pull-back of Bott-Samelson classes ζ_I to $h_T((G/B)^T)$ in Lemma 7.6. By localization, some of these classes generate $\mathtt{h}_T(G/B)$ and this lets us prove the Borel case of Theorem 8.11. The parabolic cases are then obtained in the remaining sections, as well as the Borel-style presentation. In the last section, we explain how equivariant groups under subgroups of T (and in particular the trivial group which gives the non-equivariant case) can be recovered out of the equivariant one.

2. Equivariant oriented cohomology theory

In the present section we recall the notion of an equivariant algebraic oriented cohomology theory, essentially by compiling definitions and results of [Des09], [EG98], [HM13], [KiKr], [Kr12], [LM07], [Pa09] and [To99]. We present it here in a way convenient for future reference.

In this paper, k is always a fixed base field, and pt denotes Spec(k). By a variety we mean a reduced separated scheme of finite type over k. Let G be a smooth linear algebraic group over k, abbreviated as *algebraic group*. In this paper we are mostly interested in the case $G = T$. Let G-Var be the category of smooth quasi-projective varieties over k endowed with an action of G, and with morphisms respecting this action (i.e. G-equivariant morphisms). The tangent sheaf $\mathcal{T}_X$ of any $X \in G$-Var is locally free and has a natural G-equivariant structure. The same holds for the (co)normal sheaf of any equivariant regular embedding of a closed subscheme.

An *equivariant oriented cohomology theory* over k is an additive contravariant functor $\mathtt{h}_G$ from the category G-Var to the category of commutative rings with unit for any algebraic group G (for an equivariant morphism f, the map $\mathtt{h}_G(f)$ is denoted by f^* and is called *pull-back*) together with

- a morphism $f_*\colon \mathtt{h}_G(X) \to \mathtt{h}_G(Y)$ of $\mathtt{h}_G(Y)$-modules (called *push-forward*) for any projective morphism $f\colon X \to Y$ in G-Var (here $\mathtt{h}_G(X)$ is an $\mathtt{h}_G(Y)$-module through f^*). That is, we have the projection formula

$$f_*(f^*(y)x) = yf_*(x), \quad x \in \mathtt{h}_G(X), y \in \mathtt{h}_G(Y). \tag{2.1}$$

- a natural transformation of functors $\mathrm{res}_\phi\colon \mathtt{h}_H \to \mathtt{h}_G \circ \mathrm{Res}_\phi$ (called *restriction*) for any morphism of algebraic groups $\phi\colon G \to H$ (here $\mathrm{Res}_\phi\colon H\text{-Var} \to G\text{-Var}$ simply restricts the action of H to an action of G through ϕ)
- a natural transformation of functors $c^G\colon K_G \to \tilde{\mathtt{h}}_G$ (called the *total equivariant characteristic class*), where $K_G(X)$ is the K-group of G-equivariant locally free sheaves over X and $\tilde{\mathtt{h}}_G(X)$ is the multiplicative group of the polynomial ring $\mathtt{h}_G(X)[t]$ (the coefficient at t^i is called the i-th equivariant characteristic class in the theory $\mathtt{h}$ and is denoted by c_i^G)

that satisfy the following properties:

A 1 (Compatibility for push-forwards). *The push-forwards respect composition and commute with pull-backs for transversal squares (a transversal square is a fiber product diagram with a nullity condition on* Tor*-sheaves, stated in* [LM07, Def. 1.1.1]*; in particular, this condition holds for any fiber product with a flat map).*

A 2 (Compatibility for restriction). *The restriction respects composition of morphisms of groups and commutes with push-forwards.*

A 3 (Localization). *For any smooth closed subvariety $i\colon Z \to X$ in G-*Var *with open complement $u\colon U \hookrightarrow X$, the sequence*

$$\mathtt{h}_G(Z) \xrightarrow{i_*} \mathtt{h}_G(X) \xrightarrow{u^*} \mathtt{h}_G(U) \to 0$$

is exact.

A 4 (Homotopy Invariance). *Let $p\colon X \times \mathbb{A}^n \to X$ be a G-equivariant projection with G acting linearly on $\mathbb{A}^n$. Then the induced pull-back $\mathtt{h}_G(X) \to \mathtt{h}_G(X \times \mathbb{A}^n)$ is an isomorphism.*

A 5 (Normalization). *For any regular embedding $i\colon D\subset X$ of codimension 1 in G-Var we have $c_1^G(\mathcal{O}(D)) = i_*(1)$ in $\mathtt{h}_G(X)$, where $\mathcal{O}(D)$ is the line bundle dual to the kernel of the map of G-equivariant sheaves $\mathcal{O}\to\mathcal{O}_D$.*

A 6 (Torsors). *Let $p\colon X\to Y$ be in G-Var and let H be a closed normal subgroup of G acting trivially on Y such that $p\colon X\to Y$ is a H-torsor. Consider the quotient map $\imath\colon G\to G/H$. Then the composite $p^*\circ \mathrm{res}_\imath\colon \mathtt{h}_{G/H}(Y)\to \mathtt{h}_G(X)$ is an isomorphism.*
In particular, if $H=G$ we obtain an isomorphism $\mathtt{h}_{\{1\}}(Y)\simeq \mathtt{h}_G(X)$ for a G-torsor X over Y.

A 7. *If $G=\{1\}$ is trivial, then $\mathtt{h}_{\{1\}}=\mathtt{h}$ defines an algebraic oriented cohomology in the sense of* [LM07, Def. 1.1.2] *(except that $\mathtt{h}$ takes values in rings, not in graded rings) with push-forwards and characteristic classes being as in* [LM07].

A 8 (Self-intersection formula). *Let $i: Y\subset X$ be a regular embedding of codimension d in G-Var. Then the normal bundle to Y in X, denoted by $\mathcal{N}_{Y/X}$ is naturally G-equivariant and there is an equality $i^*i_*(1) = c_d^G(\mathcal{N}_{Y/X})$ in $\mathtt{h}_G(Y)$.*

A 9 (Quillen's formula). *If $\mathcal{L}_1$ and $\mathcal{L}_2$ are locally free sheaves of rank one, then*
$$c_1(\mathcal{L}_1\otimes\mathcal{L}_2) = c_1(\mathcal{L}_1) +_F c_1(\mathcal{L}_2),$$
where F is the formal group law of $\mathtt{h}$ (here $G=\{1\}$).

As consequences of the projection formula (2.1), we have:

Lemma 2.1. *Let $p: X\to Y$ be a morphism in G-Var, with a section $s: Y\to X$. Then for any $u\in \mathtt{h}_G(Y)$, one has*

(a) $s^*s_*(u\cdot v) = u\cdot s^*s_*(v)$ if s is projective.
(b) $p_*(s_*(u)^n) = u\cdot s^*s_*(u)^{n-1}$ for any $n\geq 1$ if furthermore p is projective.

Proof. Part (a) follows from
$$s^*s_*(u\cdot v) = s^*s_*\big(s^*p^*(u)\cdot v\big) = s^*\big(p^*(u)\cdot s_*(v)\big) = s^*p^*(u)\cdot s^*s_*(v) = u\cdot s^*s_*(v)$$
and part (b) from
$$p_*(s_*(u)^n) = p_*\Big(s_*(u)\cdot s_*(u)^{n-1}\Big) = p_*\Big(s_*\big(u\cdot s^*(s_*(u)^{n-1})\big)\Big) = u\cdot s^*s_*(u)^{n-1}.$$
∎

This lemma applies in particular when $p: X\to \mathrm{pt}$ is the structural morphism of X and s is therefore a G-fixed point of X.

For any $X\in G$-Var consider the γ-filtration on $\mathtt{h}_G(X)$, whose i-th term $\gamma^i\mathtt{h}_G(X)$ is the ideal of $\mathtt{h}_G(X)$ generated by products of equivariant characteristic classes of total degree at least i. In particular, a G-equivariant locally free sheaf of rank n over pt is the same thing as an n-dimensional k-linear representation of G, so $\gamma^i\mathtt{h}_G(\mathrm{pt})$ is generated by characteristic classes of such representations. This can lead to concrete computations when the representations of G are well described.

We introduce the following important notion

Definition 2.2. An equivariant oriented algebraic cohomology theory is called *Chern-complete over the point* for G, if the ring $\mathtt{h}_G(\mathrm{pt})$ is separated and complete with respect to the topology induced by the γ-filtration.

Remark 2.3. Assume that the ring $\mathtt{h}_G(\mathrm{pt})$ is separated for all G, and let $\mathtt{h}_G(\mathrm{pt})^\wedge$ be its completion with respect to the γ-filtration. We can Chern-complete the equivariant cohomology theory by tensoring with $-\otimes_{\mathtt{h}_G(\mathrm{pt})} \mathtt{h}_G(\mathrm{pt})^\wedge$. In this way, we obtain a completed version of the cohomology theory, still satisfying the axioms. Note that this completion has no effect on the non-equivariant groups, since in $\mathtt{h}(\mathrm{pt})$, the characteristic classes are automatically nilpotent by [LM07, Lemma 1.1.3].

Here are three well-known examples of equivariant oriented cohomology theories.

Example 2.4. The equivariant Chow ring functor $\mathtt{h}_G = \mathrm{CH}_G$ was constructed by Edidin and Graham in [EG98], using an inverse limit process of Totaro [To99]. In this case the formal group law is the additive one $F(x,y) = x+y$, the base ring $\mathrm{CH}(\mathrm{pt})$ is $\mathbb{Z}$, and the theory is Chern-complete over the point for any group G by construction.

Example 2.5. Equivariant algebraic K-theory and, in particular, K_0 was constructed by Thomason [Th87] (see also [Me05] for a good survey). The formal group law is multiplicative $F(x,y) = x+y-xy$, the base ring $K_0(\mathrm{pt})$ is $\mathbb{Z}$, and the theory is *not* Chern complete: for example, $(K_0)_{\mathbb{G}_m}(\mathrm{pt}) \simeq \mathbb{Z}[t,t^{-1}]$ with the γ^i generated by $(1-t)^i$. Observe that $(K_0)_G(\mathrm{pt})$ consists of classes of k-linear finite dimensional representations of G.

Example 2.6 (Algebraic cobordism). Equivariant algebraic cobordism was defined by Deshpande [Des09], Malgón-López and Heller [HM13] and Krishna [Kr12]. The formal group law is the universal one over $\Omega(\mathrm{pt}) = \mathbb{L}$ the Lazard ring. The equivariant theory is Chern complete over the point for any group G by construction.

By Totaro's process one can construct many examples of equivariant theories, such as equivariant connective K-theory, equivariant Morava K-theories, etc. Moreover, in this way one automatically obtains Chern-complete theories.

3. Torus-equivariant cohomology of a point

From now on, T is always a split torus. In the present section we show that the completed T-equivariant oriented cohomology ring of a point can be identified with the formal group algebra of the respective group of characters (see Theorem 3.3).

Let Λ be the group of characters of T, which is therefore the Cartier dual of Λ. Let X be a smooth variety over k endowed with a trivial T-action. Consider the pull-back $p^*\colon \mathtt{h}_T(\mathrm{pt}) \to \mathtt{h}_T(X)$ induced by the structure map. Let $\gamma^i_{\mathrm{pt}}\, \mathtt{h}_T(X)$

denote the ideal in $\mathtt{h}_T(X)$ generated by elements from the image of $\gamma^i\,\mathtt{h}_T(\mathrm{pt})$ under the pull-back. Since any representation of T decomposes as a direct sum of one dimensional representations, $\gamma^i\,\mathtt{h}_T(\mathrm{pt})$ is generated by products of first characteristic classes $c_1^T(L_\lambda)$, $\lambda \in \Lambda$. Since characteristic classes commute with pull-backs, $\gamma^i_{\mathrm{pt}}\,\mathtt{h}_T(X)$ is also generated by products of first characteristic classes (of pull-backs p^*L_λ).

Let F be a one-dimensional commutative formal group law over a ring R. We often write $x +_F y$ (formal addition) for the power series $F(x, y)$ defining F. Following [CPZ, §2] consider the formal group algebra $R[\![\Lambda]\!]_F$. It is an R-algebra together with an augmentation map $R[\![\Lambda]\!]_F \to R$ with kernel denoted by $\mathcal{I}_F$, and it is complete with respect to the $\mathcal{I}_F$-adic topology. Thus

$$R[\![\Lambda]\!]_F = \varprojlim_i R[\![\Lambda]\!]_F/\mathcal{I}_F^i,$$

and it is topologically generated by elements of the form x_λ, $\lambda \in \Lambda$, which satisfy $x_{\lambda+\mu} = x_\lambda +_F x_\mu$. By definition (see [CPZ, 2.8]) the algebra $R[\![\Lambda]\!]_F$ is universal among R-algebras with an augmentation ideal I and a morphism of groups $\Lambda \to (I, +_F)$ that are complete with respect to the I-adic topology. The choice of a basis of Λ defines an isomorphism

$$R[\![\Lambda]\!]_F \simeq R[\![x_1, \ldots, x_n]\!],$$

where n is the rank of Λ.

Set $R = \mathtt{h}(X)$. Then $\mathtt{h}_T(X)$ is an R-algebra together with an augmentation map $\mathtt{h}_T(X) \to R$ via the restrictions induced by $\{1\} \to T \to \{1\}$. The assignment $\lambda \in \Lambda \mapsto c_1^T(L_\lambda)$ induces a group homomorphism $\Lambda \to (I, +_F)$, where I is the augmentation ideal. Therefore, by the universal property of $R[\![\Lambda]\!]_F$, there is a morphism of R-algebras

$$\phi\colon R[\![\Lambda]\!]_F/\mathcal{I}_F^i \to \mathtt{h}_T(X)/\gamma^i_{\mathrm{pt}}\,\mathtt{h}_T(X).$$

We claim that

Lemma 3.1. *The morphism ϕ is an isomorphism.*

Proof. We proceed by induction on the rank n of Λ.
For $n = 0$, we have $T = \{1\}$, $R = \mathtt{h}_T(X)$, $\mathcal{I}_F^i = \gamma^i_{\mathrm{pt}}\,\mathtt{h}_T(X) = \{0\}$ and the map ϕ turns into an identity on R.
For rank $n > 0$ we choose a basis $\{\lambda_1, \ldots, \lambda_n\}$ of Λ. Let $\{L_1, \ldots, L_n\}$ be the respective one-dimensional representations of T. This gives isomorphisms $\Lambda \simeq \mathbb{Z}^n$ and $T \simeq \mathbb{G}_m^n$ and $\mathbb{G}_m^n$ acts on L_i by multiplication by the i-th coordinate. Let $\mathbb{G}_m^n$ act on $\mathbb{A}^i$ by multiplication by the last coordinate. Consider the localization sequence (A3)

$$\mathtt{h}_{\mathbb{G}_m^n}(X) \longrightarrow \mathtt{h}_{\mathbb{G}_m^n}(X \times \mathbb{A}^i) \longrightarrow \mathtt{h}_{\mathbb{G}_m^n}(X \times (\mathbb{A}^i \setminus \{0\})) \longrightarrow 0.$$

After identifying

$$\mathtt{h}_{\mathbb{G}_m^n}(X) \xrightarrow{\sim} \mathtt{h}_{\mathbb{G}_m^n}(X \times \mathbb{A}^i) \text{ and } \mathtt{h}_{\mathbb{G}_m^{n-1}}(X \times \mathbb{P}^{i-1}) \xrightarrow{\sim} \mathtt{h}_{\mathbb{G}_m^n}(X \times (\mathbb{A}^i \setminus \{0\}))$$

via (A8) and (A6), we obtain an exact sequence

$$\mathtt{h}_{\mathbb{G}_m^n}(X) \xrightarrow{c_1(L_n)^i} \mathtt{h}_{\mathbb{G}_m^n}(X) \longrightarrow \mathtt{h}_{\mathbb{G}_m^{n-1}}(X \times \mathbb{P}^{i-1}) \longrightarrow 0.$$

where the first map is obtained by applying self-intersection (A5) and homotopy invariance (A4) properties.
By definition, all these maps are R-linear, and the action of $\mathbb{G}_m^{n-1}$ on $X \times \mathbb{P}^{i-1}$ is the trivial one. Since the last map is given by pull-back maps and restrictions (although not all in the same direction), and since equivariant characteristic classes commute with these, one checks that it sends $c_1(L_i)$ to $c_1(L_i)$ for any $i \leq n-1$ and $c_1(L_n)$ to $c_1(\mathcal{O}(1))$; this last case holds because $\mathcal{O}(1)$ on $\mathbb{P}^{i-1}$ goes (by restriction and pull-back) to the equivariant line bundle on $\mathbb{A}^i \setminus \{0\}$ with trivial underlying line bundle, but where $\mathbb{G}_m^n$ acts by λ_n on fibers.
By the projective bundle theorem, we have $R' := \mathtt{h}(X \times \mathbb{P}^{i-1}) \simeq R[y]/y^i$ with $c_1(\mathcal{O}(1)) = y$. By induction, we obtain for any i an isomorphism

$$\mathtt{h}_{\mathbb{G}_m^{n-1}}(X \times \mathbb{P}^{i-1})/\gamma_{\mathrm{pt}}^i \simeq R'[\![\Lambda']\!]_F/(\mathcal{I}'_F)^i,$$

where $\Lambda' = \mathbb{Z}^{n-1}$ and $\mathcal{I}'_F$ is the augmentation ideal of $R'[\![\Lambda']\!]_F$. Using the isomorphisms $R[\![\Lambda]\!]_F \simeq R[\![x_1, \dots, x_n]\!]$ and $R'[\![\Lambda']\!]_F \simeq R'[\![x_1, \dots, x_{n-1}]\!]$ induced by the basis of Λ, we are reduced to checking that

$$\begin{array}{ccc} R[\![x_1, \dots, x_n]\!]/\mathcal{I}_F^i & \longrightarrow & (R[y]/y^i)[\![x_1, \dots, x_{n-1}]\!]/\mathcal{J} \\ x_i & \longmapsto & \begin{cases} x_i & \text{if } i \leq n-1 \\ y & \text{if } i = n. \end{cases} \end{array}$$

is an isomorphism, when $\mathcal{J} = (\mathcal{I}'_F)^i + y \cdot (\mathcal{I}'_F)^{i-1} + \cdots + y^i$. The latter then follows by definition. □

Remark 3.2. Similar statements can be found in [HM13, 3.2.1] or [Kr12, 6.7], but we gave a full proof for the sake of completeness.

We obtain a natural map of R-algebras

$$\mathtt{h}_T(\mathrm{pt}) \to \varprojlim_i \mathtt{h}_T(\mathrm{pt})/\gamma^i\, \mathtt{h}_T(\mathrm{pt}) \simeq \varprojlim_i R[\![\Lambda]\!]_F/\mathcal{I}_F^i = R[\![\Lambda]\!]_F$$

and, therefore, by Lemma 3.1, we have:

Theorem 3.3. *If $\mathtt{h}$ is (separated and) Chern-complete over the point for T, then the natural map $\mathtt{h}_T(\mathrm{pt}) \to R[\![\Lambda]\!]_F$ is an isomorphism. It sends the characteristic class $c_1^T(L_\lambda) \in \mathtt{h}_T(\mathrm{pt})$ to $x_\lambda \in R[\![\Lambda]\!]_F$.*

4. Equivariant cohomology of $\mathbb{P}^1$

In the present section we compute equivariant cohomology $\mathtt{h}_T(\mathbb{P}(V_1 \oplus V_2))$ of a projective line, where a split torus T acts on one-dimensional representations V_1 and V_2 by means of characters λ_1 and λ_2.

Assumption 4.1. For the rest of the paper we assume that the equivariant cohomology of the point $\mathtt{h}_T(\mathrm{pt})$ is (separated and) complete for the γ-filtration in the sense of Definition 2.2.

Let X be a smooth T-variety. By section 3, the ring $\mathtt{h}_T(X)$ can be considered as a ring over $S := R[\![\Lambda]\!]_F$ via the identification $S \simeq \mathtt{h}_T(\mathrm{pt})$ of Theorem 3.3 and the pull-back map $\mathtt{h}_T(\mathrm{pt}) \to \mathtt{h}_T(X)$. By convention, we'll use the same notation for an element u of S and the element $u \cdot 1 \in \mathtt{h}_T(X)$, where 1 is the unit of $\mathtt{h}_T(X)$. Thus, for example, $x_\lambda = c_1^T(L_\lambda)$ in $\mathtt{h}_T(X)$.

Given a morphism $f : X \to Y$ in T-Var, the pull-back map f^* is a morphism of rings over S and the push-forward map f_* (when it exists) is a morphism of S-modules by the projection formula.

REMARK 4.2. Note that we are not claiming that S injects in $\mathtt{h}_T(X)$ for all $X \in T$-Var; it will nevertheless hold when X has a k-point that is fixed by T, as most of the schemes considered in this paper have.

We now concentrate on the following setting. Let λ_1 and λ_2 be characters of T, and let V_1 and V_2 be the corresponding one dimensional representations of T, i.e. $t \in T$ acts on $v \in V_i$ by $t \cdot v = \lambda_i(t)v$. Thus, the projective space $\mathbb{P}(V_1 \oplus V_2)$ is endowed with a natural T-action, induced by the action of T on the direct sum of representations $V_1 \oplus V_2$. Furthermore, the line bundle $\mathcal{O}(-1)$ has a natural T-equivariant structure, that can be described in the following way: The geometric points of the total space of $\mathcal{O}(-1)$ are pairs (W, w) where W is a rank one sub-vector space of $V_1 \oplus V_2$ and $w \in W$. The torus T acts by $t \cdot (W, w) = (t(W), t(w))$.
Two obvious embeddings $V_i \subseteq V_1 \oplus V_2$ induce two T-fixed points closed embeddings $\sigma_1, \sigma_2 \colon \mathrm{pt} \hookrightarrow \mathbb{P}(V_1 \oplus V_2)$. The open complement to σ_1 is an affine space isomorphic to $V_1 \otimes V_2^\vee$, with T-action by the character $\lambda_1 - \lambda_2$. We set $\alpha := \lambda_2 - \lambda_1$. By homotopy invariance (A4) applied to the pull-back induced by the structural morphism of V_1, we have $\mathtt{h}_T(\mathrm{pt}) \xrightarrow{\sim} \mathtt{h}_T(V_1)$ with inverse given by the pull-back σ_2^* (which actually lands in V_1). The exact localization sequence (A3) can therefore be rewritten as

$$\mathtt{h}_T(\mathrm{pt}) \xrightarrow{(\sigma_1)_*} \mathtt{h}_T(\mathbb{P}(V_1 \oplus V_2)) \xrightarrow{\sigma_2^*} \mathtt{h}_T(\mathrm{pt}) \longrightarrow 0$$

Using the structural map $p : \mathbb{P}(V_1 \oplus V_2) \to \mathrm{pt}$, we get a splitting p^* of σ_2^* and a retract p_* of $(\sigma_1)_*$. Thus, the exact sequence is in fact injective on the left, and we can decompose $\mathtt{h}_T(\mathbb{P}(V_1 \oplus V_2))$ using mutually inverse isomorphisms

$$\mathtt{h}_T(\mathrm{pt}) \oplus \mathtt{h}_T(\mathrm{pt}) \underset{((\sigma_1)_*,\ p^* - (\sigma_1)_* p_* p^*)}{\overset{\binom{p_*}{\sigma_2^*}}{\rightleftarrows}} \mathtt{h}_T(\mathbb{P}(V_1 \oplus V_2)) \tag{4.1}$$

LEMMA 4.3. *(a) As T-equivariant bundles, we have $\sigma_i^*(\mathcal{O}(-1)) = V_i$.*
(b) We have $(\sigma_1)_*(1) = c_1\big(\mathcal{O}(1) \otimes p^*(V_2)\big)$ and $(\sigma_2)_*(1) = c_1(\mathcal{O}(1) \otimes p^*(V_1))$ in $\mathtt{h}_T\big(\mathbb{P}(V_1 \oplus V_2)\big)$.
(c) For any $u \in \mathtt{h}_T(\mathrm{pt})$, we have $\sigma_1^*(\sigma_1)_*(u) = x_\alpha u$, $\sigma_2^*(\sigma_2)_*(u) = x_{-\alpha} u$ and $\sigma_1^*(\sigma_2)_*(u) = \sigma_2^*(\sigma_1)_*(u) = 0$.

Proof. The first part is easily checked on the geometric points of total spaces and is left to the reader. The second part follows from (A5), given the exact sequence of T-equivariant sheaves

$$0 \to \mathcal{O}(-1) \otimes p^*(V_2)^\vee \to \mathcal{O} \to \mathcal{O}_{\sigma_1} \to 0,$$

where $\mathcal{O}_{\sigma_1}$ is the structural sheaf of the closed subscheme given by σ_1. Again this exact sequence is easy to check and we leave it to the reader. In the third part, the last equality holds by transverse base change through the empty scheme, while the first two follow from Lemma 2.1 and

$$\sigma_1^*(\sigma_1)_*(1) = \sigma_1^* c_1\big(\mathcal{O}(1) \otimes p^*(V_2)\big) = \\ = c_1\Big(\sigma_1^*\big(\mathcal{O}(1) \otimes p^*(V_2)\big)\Big) = c_1\big(V_1^\vee \otimes V_2\big) = x_{\lambda_2 - \lambda_1}.$$

or a symmetric computation for $\sigma_2^*(\sigma_2)_*(1)$. □

LEMMA 4.4. *If x_α is not a zero divisor in S, then the push-forward*

$$p_*\colon \mathtt{h}_T(\mathbb{P}(V_1 \oplus V_2)) \to \mathtt{h}_T(\mathrm{pt}) \text{ satisfies } p_*(1) = \tfrac{1}{x_\alpha} + \tfrac{1}{x_{-\alpha}}.$$

(Observe that $p_*(1) \in S$ by [CPZ, 3.12]*, where it is denoted by e_α.)*

Proof. By Lemma 4.3, we have

$$\begin{aligned} x_\alpha &= c_1(p^*(V_2 \otimes V_1^\vee)) = c_1(\mathcal{O}(1) \otimes p^*(V_2) \otimes (\mathcal{O}(1) \otimes p^*(V_1))^\vee) \\ &= c_1\big(\mathcal{O}(1) \otimes p^*(V_2)\big) -_F c_1\big(\mathcal{O}(1) \otimes p^*(V_1)\big) = (\sigma_1)_*(1) -_F (\sigma_2)_*(1). \end{aligned}$$

By transverse base change, we have $(\sigma_1)_*(1) \cdot (\sigma_2)_*(1) = 0$, and therefore

$$(\sigma_1)_*(1) -_F (\sigma_2)_*(1) = (\sigma_1)_*(1) + \big(-_F (\sigma_2)_*(1)\big).$$

Since x_α is not a zero divisor in S, it suffices to prove that

$$x_\alpha \cdot p_*(1) = 1 + \tfrac{x_\alpha}{x_{-\alpha}},$$

where $\frac{x_\alpha}{x_{-\alpha}} \in S^\times$ is the power series $\frac{-_F(x)}{x}$ applied to $x = x_{-\alpha}$. Now,

$$\begin{aligned} x_\alpha p_*(1) &= p_*(x_\alpha) = p_*\big((\sigma_1)_*(1) + (-_F(\sigma_2)_*(1))\big) \\ &= 1 + p_*(-_F(\sigma_2)_*(1)) = 1 + \tfrac{x_\alpha}{x_{-\alpha}}. \end{aligned}$$

where the last equality follows from Lemma 2.1, part (b). □

Let $\sigma = \sigma_1 \sqcup \sigma_2 \colon \mathrm{pt} \sqcup \mathrm{pt} \to \mathbb{P}(V_1 \oplus V_2)$ be the inclusion of both T-fixed points.

LEMMA 4.5. *If x_α is not a zero divisor in S, the pull-back σ^* is injective, and*

$$\operatorname{im} \sigma^* = \{(u, v) \in \mathtt{h}_T(\mathrm{pt}) \oplus \mathtt{h}_T(\mathrm{pt}) \mid x_{-\alpha} u + x_\alpha v \in x_\alpha x_{-\alpha} \cdot \mathtt{h}_T(\mathrm{pt})\}.$$

Proof. Since $\mathtt{h}_T(\mathrm{pt} \sqcup \mathrm{pt}) = \mathtt{h}_T(\mathrm{pt}) \oplus \mathtt{h}_T(\mathrm{pt})$ identifies σ^* with (σ_1^*, σ_2^*), it suffices to check that the composition

$$\mathtt{h}_T(\mathrm{pt}) \oplus \mathtt{h}_T(\mathrm{pt}) \xrightarrow[\simeq]{((\sigma_1)_*,\ p^* - (\sigma_1)_* p_* p^*)} \mathtt{h}_T(\mathbb{P}(V_1 \oplus V_2)) \xrightarrow{\binom{\sigma_1^*}{\sigma_2^*}} \mathtt{h}_T(\mathrm{pt}) \oplus \mathtt{h}_T(\mathrm{pt})$$

is injective. Indeed, it is given by the matrix

$$\begin{pmatrix} \sigma_1^*(\sigma_1)_* & \sigma_1^* p^* - \sigma_1^*(\sigma_1)_* p_* p^* \\ \sigma_2^*(\sigma_1)_* & \sigma_2^* p^* - \sigma_2^*(\sigma_1)_* p_* p^* \end{pmatrix} = \begin{pmatrix} x_\alpha & 1 - x_\alpha \cdot p_*(1) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x_\alpha & -\frac{x_\alpha}{x_{-\alpha}} \\ 0 & 1 \end{pmatrix}$$

where in the first equality, we have used $p \circ \sigma_i = \mathrm{id}$, Lemma 4.3 part (c), to get the 1's and the 0, and then the projection formula $p_* p^*(u) = u \cdot p_*(1)$ and Lemma 2.1 to get $\sigma_1^*(\sigma_1)_* p_* p^*(u) = x_\alpha p_*(1) \cdot u$. The last equality holds by Lemma 4.4.
Finally, the image of this matrix is of the expected form. □

Let $S[\frac{1}{x_\alpha}]$ be the localization of S at the multiplicative subset generated by x_α. Since $\frac{x_\alpha}{x_{-\alpha}}$ is invertible, there is a canonical isomorphism $S[\frac{1}{x_\alpha}] \simeq S[\frac{1}{x_{-\alpha}}]$. We consider the $S[\frac{1}{x_\alpha}]$-linear operator

$$A\colon S[\tfrac{1}{x_\alpha}] \oplus S[\tfrac{1}{x_\alpha}] \longrightarrow S[\tfrac{1}{x_\alpha}] \text{ given by } (u,v) \mapsto \tfrac{u}{x_\alpha} + \tfrac{v}{x_{-\alpha}}.$$

Note that by the previous lemma, it sends the image of σ^* to S inside $S[\frac{1}{x_\alpha}]$.

LEMMA 4.6. *If x_α is not a zero divisor in S, the following diagram commutes.*

$$\begin{array}{ccccccc} \mathtt{h}_T\big(\mathbb{P}(V_1 \oplus V_2)\big) & \xrightarrow{\sigma^*} & \mathtt{h}_T(\mathrm{pt}) \oplus \mathtt{h}_T(\mathrm{pt}) & \xleftarrow{\simeq} & S \oplus S & \subseteq & S[\frac{1}{x_\alpha}] \oplus S[\frac{1}{x_\alpha}] \\ \downarrow p_* & & & & & & \downarrow A \\ \mathtt{h}_T(\mathrm{pt}) & & \xleftarrow{\qquad\simeq\qquad} & & S & \subseteq & S[\frac{1}{x_\alpha}] \end{array}$$

Proof. It suffices to check the equality of the two maps after precomposition by the isomorphism $\mathtt{h}_T(\mathrm{pt}) \oplus \mathtt{h}_T(\mathrm{pt}) \to \mathtt{h}_T\big(\mathbb{P}(V_1 \oplus V_2)\big)$ given in (4.1). Using the matrix already computed in the proof of Lemma 4.5, one obtains that the upper right composition sends (u,v) to u. The lower left composition sends (u,v) to

$$p_*\big((\sigma_1)_*(u) + p^*(v) - (\sigma_1)_* p_* p^*(v)\big) = u + p_* p^*(v) - p_* p^*(v) = u. \qquad \square$$

5. ALGEBRAIC AND COMBINATORIAL OBJECTS

Let us now introduce the main algebraic objects $\mathbf{D}^\star$, $\mathbf{D}^\star_\Xi$, $S^\star_W$ and $S^\star_{W/W_\Xi}$ that play the role of algebraic replacements for some equivariant cohomology groups in the remaining of this paper. These objects were discussed in detail in [CZZ] and [CZZ2], and we only give a brief overview here. Their geometric interpretation will be explained in the next sections.

Let $\Sigma \hookrightarrow \Lambda^\vee$, $\alpha \mapsto \alpha^\vee$ be a root datum in the sense of [SGA3, Exp. XXI, §1.1]. Thus, Λ is a lattice and Σ is a non-empty finite subset of Λ, called the set of roots. The rank of the root datum is the dimension of $\mathbb{Q} \otimes_{\mathbb{Z}} \Lambda$, and elements in Σ are called roots. The root lattice Λ_r is the subgroup of Λ generated by elements in Σ, and the weight lattice is defined as

$$\Lambda_w = \{\omega \in \mathbb{Q} \otimes_{\mathbb{Z}} \Lambda \mid \alpha^\vee(\omega) \in \mathbb{Z} \text{ for all } \alpha \in \Sigma\}.$$

We have $\Lambda_r \subseteq \Lambda \subseteq \Lambda_w$. We always assume that the root datum is *semisimple* (the ranks of Λ, Λ_r, Λ_w are equal and no root is twice any other root). The root datum is called *simply connected* (resp. *adjoint*) if $\Lambda = \Lambda_w$ (resp. $\Lambda = \Lambda_r$) and if it is furthermore irreducible of rank n, we use the notation $\mathcal{D}_n^{sc}$ (resp. $\mathcal{D}_n^{ad}$) for its Dynkin type, with $\mathcal{D}$ among A, B, C, D, G, F, E.

The Weyl group W of the root datum is the subgroup of $\mathrm{Aut}_{\mathbb{Z}}(\Lambda)$ generated by simple reflections

$$s_\alpha(\lambda) = \lambda - \alpha^\vee(\lambda)\alpha, \ \lambda \in \Lambda.$$

Fixing a set of simple roots $\Pi = \{\alpha_1, ..., \alpha_n\}$ induces a partition $\Sigma = \Sigma^+ \cup \Sigma^-$, where Σ^+ is the set of positive roots and $\Sigma^- = -\Sigma^+$ is the set of negative roots. The Weyl group W is actually generated by $s_i := s_{\alpha_i}$, $i = 1, ..., n$.

Let F be a one-dimensional commutative formal group law over a commutative ring R. Let $S = R[\![\Lambda]\!]_F$. From now on we always assume that

Assumption 5.1. The algebra S is Σ-regular, that is, x_α is regular in S for all $\alpha \in \Sigma$ (see [CZZ, Def. 4.4]).

This holds if 2 is regular in R, or if the root datum does not contain an irreducible component of type C_k^{sc} [CZZ, Rem. 4.5].

The action of W on Λ induces an action of W on S, and let S_W be the R-algebra defined as $S \otimes_R R[W]$ as an R-module, and with product given by

$$q\delta_w q'\delta_{w'} = qw(q')\delta_{ww'}, \quad q, q' \in S, \ w, w' \in W.$$

Let $Q = S[\frac{1}{x_\alpha} | \alpha \in \Sigma]$ and $Q_W = Q \otimes_S S_W$, with ring structure given by the same formula with $q, q' \in Q$. Then $\{\delta_w\}_{w\in W}$ is an S-basis of S_W and a Q-basis of Q_W. There is an action of Q_W on Q, restricting to an action of S_W on S, and given by

$$q\delta_w \cdot q' = qw(q'), \quad q, q' \in Q, \ w \in W.$$

For each $\alpha \in \Sigma$, we define $\kappa_\alpha = \frac{1}{x_\alpha} + \frac{1}{x_{-\alpha}} \in S$.

Definition 5.2. For any $\alpha \in \Sigma$, let

$$X_\alpha = \tfrac{1}{x_\alpha} - \tfrac{1}{x_\alpha}\delta_{s_\alpha}, \quad Y_\alpha = \kappa_\alpha - X_\alpha = \tfrac{1}{x_{-\alpha}} + \tfrac{1}{x_\alpha}\delta_{s_\alpha},$$

in Q_W, respectively called a *formal Demazure element* and a *formal push-pull element*.

For each sequence $(i_1, ..., i_k)$ with $1 \leq i_j \leq n$, we define $X_I = X_{\alpha_{i_1}} \cdots X_{\alpha_{i_k}}$ and $Y_I = Y_{\alpha_{i_1}} \cdots Y_{\alpha_{i_k}}$.

Definition 5.3. Let $\mathbf{D}$ be the R-subalgebra of Q_W generated by elements from S and the elements X_α, $\alpha \in \Sigma$.

Since $\delta_{s_i} = 1 - x_{\alpha_i}X_{\alpha_i}$, we have $S_W \subseteq \mathbf{D}$. By [CZZ, Prop. 7.7], $\mathbf{D}$ is a free S-module and for any choice of reduced decompositions I_w for every element $w \in W$ the family $\{X_{I_w}\}_{w\in W}$ is an S-basis of $\mathbf{D}$.

There is a coproduct structure on the Q-module Q_W defined by

$$Q_W \to Q_W \otimes_Q Q_W,\ q\delta_w \mapsto q\delta_w \otimes \delta_w,$$

with counit $Q_W \to Q, q\delta_w \mapsto q$. Here $Q_W \otimes_Q Q_W$ is the tensor product of left Q-modules. By the same formula, one can define a coproduct structure on the S-module S_W. The coproduct on Q_W induces a coproduct structure on $\mathbf{D}$ as a left S-module.

On duals $S_W^\star = \mathrm{Hom}_S(S_W, S)$, $\mathbf{D}^\star = \mathrm{Hom}_S(\mathbf{D}, S)$ and $Q_W^* = \mathrm{Hom}_Q(Q_W, Q)$ (notice the different stars $\star$ for S-duality and $*$ for Q-duality), the respective coproducts induce products. In $S_W^\star$ or Q_W^*, this product is given by the simple formula

$$f_v f_w = \delta^{\mathrm{Kr}}_{v,w} f_v$$

on the dual basis $\{f_v\}_{w\in W}$ to $\{\delta_w\}_{w\in W}$, with $\delta^{\mathrm{Kr}}_{v,w}$ the Kronecker delta. The multiplicative identity is $\mathbf{1} = \sum_{v\in W} f_v$. Let η be the inclusion $S_W \subseteq \mathbf{D}$. It induces an S-algebra map $\eta^\star : \mathbf{D}^\star \to S_W^\star$, which happens to be injective [CZZ2, Lemma 10.2]. Furthermore, after localization, $\eta_Q : Q_W \to Q \otimes_S \mathbf{D}^\star$ is an isomorphism and by freeness, we have $Q \otimes_S \mathbf{D}^\star \simeq \mathrm{Hom}_Q(Q \otimes_S \mathbf{D}, Q)$ and thus $Q \otimes_S \mathbf{D}^\star \simeq Q_W^*$, as left Q-rings.

There is a Q-linear action of the R-algebra Q_W on Q_W^* given by

$$(z \bullet f)(z') = f(z'z), \quad z, z' \in Q_W, f \in Q_W^*.$$

as well as S-linear actions of S_W on $S_W^\star$ and of $\mathbf{D}$ on $\mathbf{D}^\star$, given by the same formula. With this action, it is proved in [CZZ2, Theorem 10.13] that $\mathbf{D}^\star$ is a free $\mathbf{D}$-module of rank 1 and any $w \in W$ gives a one-element basis $\{x_\Pi \bullet f_w\}$ of it, where $x_\Pi = \prod_{\alpha\in\Sigma^-} x_\alpha$.
The map $c_S : S \to \mathbf{D}^\star$ sending s to $s \bullet \mathbf{1}$ is called the *algebraic (equivariant) characteristic map*, and it is of special importance (see section 10).

We now turn to the setting related to parabolic subgroups. Let $\Xi \subseteq \Pi$ be a subset and let W_Ξ be the subgroup of W generated by the s_i with $\alpha_i \in \Xi$. Let $\Sigma_\Xi = \{\alpha \in \Sigma | s_\alpha \in W_\Xi\}$, and define $\Sigma_\Xi^+ = \Sigma^+ \cap \Sigma_\Xi$ and $\Sigma_\Xi^- = \Sigma^- \cap \Sigma_\Xi$. For $\Xi' \subseteq \Xi \subseteq \Pi$, let $\Sigma^+_{\Xi/\Xi'} = \Sigma^+_\Xi \backslash \Sigma^+_{\Xi'}$ and $\Sigma^-_{\Xi/\Xi'} = \Sigma^-_\Xi \backslash \Sigma^-_{\Xi'}$. In S, we set

$$x_{\Xi/\Xi'} = \prod_{\alpha \in \Sigma^-_{\Xi/\Xi'}} x_\alpha \quad \text{and} \quad x_\Xi = x_{\Xi/\emptyset}.$$

Let S_{W/W_Ξ} be the free S-module with basis $\{\delta_{\bar w}\}_{\bar w \in W/W_\Xi}$ and let $Q_{W/W_\Xi} = Q \otimes_S S_{W/W_\Xi}$ be its localization.

As on Q_W, one can define a coproduct structure on Q_{W/W_Ξ} and S_{W/W_Ξ}, by the same diagonal formula. Let

$$S^\star_{W/W_\Xi} = \mathrm{Hom}_S(S_{W/W_\Xi}, S) \quad \text{and} \quad Q^*_{W/W_\Xi} = \mathrm{Hom}_Q(Q_{W/W_\Xi}, Q)$$

be the respective dual rings of the corings S_{W/W_Ξ} and Q_{W/W_Ξ}. On the basis $\{f_{\bar v}\}_{\bar v \in W/W_\Xi}$ dual to the basis $\{\delta_{\bar w}\}_{\bar w \in W/W_\Xi}$, the unit element is $\mathbf{1}_\Xi = \sum_{\bar v \in W/W_\Xi} f_{\bar v}$, both in $S^\star_{W/W_\Xi}$ and in Q^*_{W/W_Ξ}.

Assume $\Xi' \subseteq \Xi$. Let $\bar{w} \in W/W_{\Xi'}$ and let $\hat{w}$ denote its class in W/W_Ξ. Consider the projection and the sum over orbits

$$p_{\Xi/\Xi'} : \begin{array}{ccc} S_{W/W_{\Xi'}} & \to & S_{W/W_\Xi} \\ \delta_{\bar{w}} & \mapsto & \delta_{\hat{w}} \end{array} \quad \text{and} \quad d_{\Xi/\Xi'} : \begin{array}{ccc} S_{W/W_\Xi} & \to & S_{W/W_{\Xi'}} \\ \delta_{\hat{w}} & \mapsto & \sum_{\substack{\bar{v} \in W/W_{\Xi'} \\ \hat{v}=\hat{w}}} \delta_{\bar{v}} \end{array}$$

with S-dual maps

$$p^\star_{\Xi/\Xi'} : \begin{array}{ccc} S^\star_{W/W_\Xi} & \to & S^\star_{W/W_{\Xi'}} \\ f_{\hat{w}} & \mapsto & \sum_{\substack{\bar{v} \in W/W_{\Xi'} \\ \hat{v}=\hat{w}}} f_{\bar{v}} \end{array} \quad \text{and} \quad d^\star_{\Xi/\Xi'} : \begin{array}{ccc} S^\star_{W/W_{\Xi'}} & \to & S^\star_{W/W_\Xi} \\ f_{\bar{w}} & \mapsto & f_{\hat{w}} \end{array}.$$

Note that $p_{\Xi/\Xi'}$ respects coproducts, so $p^\star_{\Xi/\Xi'}$ is a ring map while $d^\star_{\Xi/\Xi'}$ isn't.

We set $p_\Xi = p_{\Xi/\emptyset}$. Let $\mathbf{D}_\Xi$ denote the image of $\mathbf{D}$ via p_Ξ. The coproduct structure on Q_{W/W_Ξ} induces an S-linear coproduct structure on $\mathbf{D}_\Xi$, so its S-dual $\mathbf{D}^\star_\Xi$ has a ring structure.
In summary, we have the following diagram followed by its dualization

$$\begin{array}{ccccc} S_{W/W_{\Xi'}} & \xrightarrow{\eta_{\Xi'}} & \mathbf{D}_{\Xi'} & \hookrightarrow & Q_{W/W_{\Xi'}} \\ \downarrow{\scriptstyle p_{\Xi/\Xi'}} & & \downarrow{\scriptstyle p_{\Xi/\Xi'}} & & \downarrow{\scriptstyle p_{\Xi/\Xi'}} \\ S_{W/W_\Xi} & \xrightarrow{\eta_\Xi} & \mathbf{D}_\Xi & \hookrightarrow & Q_{W/W_\Xi} \end{array} \qquad \begin{array}{ccccc} \mathbf{D}^\star_{\Xi'} & \xrightarrow{\eta^\star_{\Xi'}} & S^\star_{W/W_{\Xi'}} & \hookrightarrow & Q^*_{W/W_{\Xi'}} \\ \uparrow{\scriptstyle p^\star_{\Xi/\Xi'}} & & \uparrow{\scriptstyle p^\star_{\Xi/\Xi'}} & & \uparrow{\scriptstyle p^\star_{\Xi/\Xi'}} \\ \mathbf{D}^\star_\Xi & \xrightarrow{\eta^\star_\Xi} & S^\star_{W/W_\Xi} & \hookrightarrow & Q^*_{W/W_\Xi} \end{array}$$

in which all horizontal maps become isomorphisms after tensoring by Q on the left. It will receive a geometric interpretation as Diagram (8.2). Moreover, by [CZZ2, Lemma 11.7], the image of $p^\star_\Xi$ in $\mathbf{D}^\star$ (or $S^\star_W$, Q^*_W) is the subset of W_Ξ-invariant elements.

There is no '$\bullet$'-action of S_{W/W_Ξ} on $S^\star_{W/W_\Xi}$ because S_{W/W_Ξ} is not a ring. But since $x_{\Pi/\Xi} \in S^{W_\Xi}$, the element $x_{\Pi/\Xi} \bullet f$ is well-defined for any $f \in S^\star_{W/W_\Xi}$ and actually belongs to $\mathbf{D}^\star_\Xi$ inside $S^\star_{W/W_\Xi}$, by [CZZ2, Lemma 15.3]. This defines a map $\mathbf{D}^\star_\Xi \to S_{W/W_\Xi}$, interpreted geometrically in Diagram (8.1).

For a given set of representatives of $W_\Xi/W_{\Xi'}$ we define the *push-pull element* by

$$Y_{\Xi/\Xi'} = \sum_{w \in W_{\Xi/\Xi'}} \delta_w \frac{1}{x_{\Xi/\Xi'}} \in Q_W.$$

We set $Y_\Xi = Y_{\Xi/\emptyset}$. If $\Xi = \{\alpha_i\}$, then $Y_\Xi = Y_{\alpha_i}$. By [CZZ2, Lemma 10.12], $Y_\Xi \in \mathbf{D}$.

Let

$$A_{\Xi/\Xi'} : \begin{array}{ccc} (Q^*_W)^{W_{\Xi'}} & \to & (Q^*_W)^{W_\Xi} \\ f & \mapsto & Y_{\Xi/\Xi'} \bullet f \end{array} \quad \text{and} \quad \mathcal{A}_{\Xi/\Xi'} : \begin{array}{ccc} Q^*_{W/W_{\Xi'}} & \to & Q^*_{W/W_\Xi} \\ f & \mapsto & d^\star_{\Xi/\Xi'}(\frac{1}{x_{\Xi/\Xi'}} \bullet f) \end{array}$$

and respectively call them *push-pull operator* and *push-forward operator*. The operator $\mathcal{A}_{\Xi/\Xi'}$ is actually independent of the choice of representatives [CZZ2, Lem. 6.5]. We have $A_{\Xi/\Xi'}((\mathbf{D}^\star)^{W_{\Xi'}}) = (\mathbf{D}^\star)^{W_\Xi}$ by [CZZ2, Cor. 14.6] and $\mathcal{A}_{\Xi/\Xi'}$ induces a map $\mathcal{A}_{\Xi/\Xi'} : \mathbf{D}^\star_{\Xi'} \to \mathbf{D}^\star_{\Xi}$ by [CZZ2, Lemma 15.1]. These two operators are related by the commutative diagram on the left below, becoming the one on the right after tensoring by Q.

$$\begin{array}{ccc} \mathbf{D}^\star_{\Xi'} & \xrightarrow[\simeq]{p^\star_{\Xi'}} & (\mathbf{D}^\star)^{W_{\Xi'}} \\ {\scriptstyle \mathcal{A}_{\Xi/\Xi'}}\big\downarrow & & \big\downarrow{\scriptstyle A_{\Xi/\Xi'}} \\ \mathbf{D}^\star_{\Xi} & \xrightarrow[\simeq]{p^\star_{\Xi}} & (\mathbf{D}^\star)^{W_{\Xi}} \end{array} \qquad\qquad \begin{array}{ccc} Q^*_{W/W_{\Xi'}} & \xrightarrow[\simeq]{p^\star_{\Xi'}} & (Q^*_W)^{W_{\Xi'}} \\ {\scriptstyle \mathcal{A}_{\Xi/\Xi'}}\big\downarrow & & \big\downarrow{\scriptstyle A_{\Xi/\Xi'}} \\ Q^*_{W/W_{\Xi}} & \xrightarrow[\simeq]{p^\star_{\Xi}} & (Q^*_W)^{W_{\Xi}} \end{array}$$

Again, when $\Xi' = \emptyset$, we set $A_\Xi = A_{\Xi/\emptyset}$ and $\mathcal{A}_\Xi = \mathcal{A}_{\Xi/\emptyset}$.

6. Fixed points of the torus action

We now consider a split semi-simple algebraic group G over k containing T as a maximal torus, with character group Λ. Let W be the Weyl group associated to (G,T), with roots $\Sigma \subseteq \Lambda$. We choose a Borel subgroup B of G containing T. It defines a set Π of simple roots in W. Given a subset $\Xi \subseteq \Pi$, the subgroup generated by B and representatives in $G(k)$ of reflections with respect to roots in Ξ is a parabolic subgroup, denoted by P_Ξ. The map sending Ξ to P_Ξ is a bijection between subsets of Π and parabolic subgroups of G containing B. Let W_Ξ be the subgroup of W generated by reflections with respect to roots in Ξ. We will abuse the notation by also writing W (or W_Ξ, etc.) when referring to the constant finite algebraic group over pt whose set of points over any field is W.

For any parabolic subgroup P, the quotient variety G/P is projective and we consider it in T-Var by letting T act on G by multiplication on the left. After identifying $W \simeq \mathrm{N}_G(T)/T$, the Bruhat decomposition says that $G/P = \bigsqcup_{w\in W^\Xi} BwP_\Xi/P_\Xi$, where the union is taken over the set W^Ξ of minimal left coset-representatives of W/W_Ξ. It induces a bijection between k-points of G/P_Ξ that are fixed by T and the set W^Ξ (or W/W_Ξ). In particular, fixed k-points of G/B are in bijection with elements of W.

Let $(G/P_\Xi)^T = \bigsqcup_{\bar{w}\in W/W_\Xi} \mathrm{pt}_{\bar{w}}$ denote the closed subvariety of T-fixed k-points, then by additivity there is an $S = \mathtt{h}_T(\mathrm{pt})$-algebra isomorphism

$$\Theta_\Xi \colon \mathtt{h}_T((G/P_\Xi)^T) \xrightarrow{\simeq} \prod_{\bar{w}\in W/W_\Xi} \mathtt{h}_T(\mathrm{pt}_{\bar{w}}) = \prod_{\bar{w}\in W/W_\Xi} S \cong S^\star_{W/W_\Xi}.$$

If $\Pi = \emptyset$, we denote $\Theta : \mathtt{h}_T((G/B)^T) = \mathtt{h}_T(W) \to S^\star_W$.

Let $\imath_\Xi \colon (G/P_\Xi)^T \hookrightarrow G/P_\Xi$ denote the (closed) embedding of the T-fixed locus, and let $\imath^{\bar{w}}_\Xi \colon \mathrm{pt}_{\bar{w}} \hookrightarrow G/P_\Xi$ denote the embedding corresponding to $\bar{w}$. Given $\Xi' \subseteq \Xi \subseteq \Pi$, we define projections

$$\pi_{\Xi/\Xi'} \colon G/P_{\Xi'} \to G/P_\Xi \qquad \text{and} \qquad \rho_{\Xi/\Xi'} \colon W/W_{\Xi'} \to W/W_\Xi$$

(here we view W/W_Ξ as a variety that is a disjoint union of copies of pt indexed by cosets). If $\Xi = \{\alpha\}$ consists of a single simple root α, we omit the brackets in the indices, i.e. we abbreviate $W_{\{\alpha\}}$ as W_α, $P_{\{\alpha\}}$ as P_α, etc. If $\Xi' = \emptyset$, we omit the $\emptyset$ in the notation, i.e. $\pi_{\Xi/\emptyset} = \pi_\Xi$, $\rho_{\Xi/\emptyset} = \rho_\Xi$, etc. By definition, we have

$$\Theta_\Xi \circ (\rho_{\Xi/\Xi'})_* = d^\star_{\Xi/\Xi'} \circ \Theta_{\Xi'} \quad \text{and} \quad \Theta_{\Xi'} \circ (\rho_{\Xi/\Xi'})^* = p^\star_{\Xi/\Xi'} \circ \Theta_\Xi. \tag{6.1}$$

The following lemma is easy and well-known. We include a proof for the sake of completeness.

Lemma 6.1. *Let $w \in W$ be a representative of $\bar{w} \in W/W_\Xi$. The pull-pack $(\imath_\Xi^{\bar{w}})^* \mathcal{T}_{G/P_\Xi}$ of the tangent bundle $\mathcal{T}_{G/P_\Xi}$ of G/P_Ξ is the representation of T (the T-equivariant bundle over a point) with weights $\{w(\alpha) \mid \alpha \in \Sigma^-_{\Pi/\Xi}\}$. (This set is indeed independent of the choice of a representative w, e.g. by* [CZZ2, Lemma 5.1]*.)*

Proof. Consider the exact sequence of T-representations at the neutral element $e \in G$

$$0 \to \mathcal{T}_{P_\Xi,e} \to \mathcal{T}_{G,e} \to \mathcal{T}_{G/P_\Xi,e} \to 0$$

(it is exact by local triviality of the right P_Ξ-torsor $G \to G/P_\Xi$). By definition of the root system associated to (G,T), the roots Σ are the characters of $\mathcal{T}_{G,e}$. By definition of the parabolic subgroup P_Ξ, the characters of $\mathcal{T}_{P_\Xi,e}$ are $\Sigma^+ \sqcup \Sigma^-_\Xi$. This proves the lemma when $w = e$. For an arbitrary w, we consider the diagram

$$\begin{array}{ccccc}
\mathrm{pt}_e & \xrightarrow{\imath_\Xi^e} & G & \xrightarrow{w\cdot} & G \\
 & \searrow^{\imath_\Xi^{\bar{e}}} & \downarrow & & \downarrow \\
 & & G/P_\Xi & \xrightarrow{w\cdot} & G/P_\Xi
\end{array}$$

which is T-equivariant if T acts by multiplication on the left on the right column and through conjugation by w^{-1} and then by multiplication on the left on the left column. Since $\imath_\Xi^{\bar{w}}$ is the bottom composite from pt_e to G/P_Ξ, the fiber of $\mathcal{T}_{G/P_\Xi}$ at $\bar{w}$ is isomorphic to its fiber at e, but for every character α, the action of T is now by $t(v) = \alpha(\bar{w}^{-1}t\bar{w})\cdot v = \alpha(w^{-1}(t))\cdot v = w(\alpha)(t)\cdot v$, in other words by the character $w(\alpha)$. □

Proposition 6.2. *We have $(\imath_\Xi^{\bar{w}})^*(\imath_\Xi^{\bar{w}'})_*(1) = 0$ if $\bar{w} \neq \bar{w}' \in W/W_\Xi$ and*

$$(\imath_\Xi^{\bar{w}})^*(\imath_\Xi^{\bar{w}})_*(1) = \prod_{\alpha \in \Sigma^-_{\Pi/\Xi}} x_{w(\alpha)} = w(x_{\Pi/\Xi}).$$

Proof. The case $\bar{w} \neq \bar{w}'$ holds by transverse base change through the empty scheme. Since the normal bundle to a point in G/P_Ξ is the tangent bundle of G/P_Ξ pulled back to that point, and since any T-representation splits into one-dimensional ones, the case $\bar{w} = \bar{w}'$ follows from (A8) using Lemma 6.1 to identify the characters. □

Remark 6.3. Note that in the Borel case, the inclusion of an individual fixed point is local complete intersection as any other morphism between smooth varieties, but not “global” complete intersection, in the sense that it is not the zero locus of transverse sections of a globally defined vector bundle. Otherwise, for Chow groups, such a point would be in the image of the characteristic map as a product of first characteristic classes, and it isn't for types for which the simply connected torsion index isn't 1. Locally, on an open excluding other fixed points, it becomes such a product, as the previous proposition shows.

Corollary 6.4. *We have* $\Theta_\Xi(\imath_\Xi)^*(\imath_\Xi)_*(1) = x_{\Pi/\Xi} \bullet \mathbf{1}_\Xi$.

Proof. Since $\imath_\Xi = \coprod_{\bar w \in W/W_\Xi} \imath_\Xi^{\bar w}$, we have

$$\Theta_\Xi(\imath_\Xi)^*(\imath_\Xi)_*(1) = \Theta_\Xi\Big(\sum_{\bar v,\, \bar w \in W/W_\Xi} (\imath_\Xi^{\bar v})^*(\imath_\Xi^{\bar w})_*(1)\Big) = \Theta_\Xi\Big(\sum_{\bar w \in W/W_\Xi} w\big(x_{\Pi/\Xi}\big)1_{\mathrm{pt}_{\bar w}}\Big)$$
$$= \sum_{\bar w \in W/W_\Xi} w\big(x_{\Pi/\Xi}\big) f_{\bar w} = x_{\Pi/\Xi} \bullet \mathbf{1}_\Xi. \qquad \square$$

7. Bott-Samelson classes

In the present section we describe the Bott-Samelson classes in the T-equivariant cohomology of G/P_Ξ.

Let $\Xi \subseteq \Pi$ as before. For each $\bar w \in W/W_\Xi$ consider the B-orbit BwP_Ξ/P_Ξ of the point in G/P_Ξ corresponding to $\bar w$. It is isomorphic to the affine space $\mathbb{A}^{l(v)}$ where $v \in W^\Xi$ is the representative of $\bar w$ of minimal length $l(v)$. Its closure $\overline{BwP_\Xi/P_\Xi}$ is called the Schubert variety at $\bar w$ with respect to Ξ and is denoted by $\mathcal{X}_{\bar w}^\Xi$. If $\Xi = \emptyset$, we write $\mathcal{X}_w$ for $\mathcal{X}_w^\emptyset$. Moreover, by Bruhat decomposition the closed complement of BwP_Ξ/P_Ξ is the union of Schubert varieties $\mathcal{X}_{\bar u}^\Xi$ with $\bar u < \bar w$ for the Bruhat order on W/W_Ξ. For any $w \in W$, the projection map $G/B \to G/P_\Xi$ induces a projective map $\mathcal{X}_w \to \mathcal{X}_{\bar w}^\Xi$. Moreover, if $w \in W^\Xi$, then the projection $\mathcal{X}_w \to \mathcal{X}_{\bar w}^\Xi$ is (projective and) birational.

The variety $\mathcal{X}_{\bar w}^\Xi$ is not smooth in general, but it admits nice desingularizations, that we now recall, following [D74]. Given a sequence of simple reflections $I = (s_1, \dots, s_l)$ corresponding to simple roots $(\alpha_1, \dots, \alpha_l)$, the Bott-Samelson desingularization of $\mathcal{X}_I$ is defined as

$$\hat{\mathcal{X}}_I = P_{\alpha_1} \times^B P_{\alpha_2} \times^B \cdots \times^B P_{\alpha_l}/B$$

where $\times^B$ means the quotient by the action of B given on points by $b \cdot (x, y) = (xb^{-1}, by)$. By definition, the multiplication of all factors induces a map $q_I\colon \hat{\mathcal{X}}_I \to G/B$ which factors through a map $\mu_I\colon \hat{\mathcal{X}}_I \to \mathcal{X}_{w(I)}$ where

$w(I) = s_1 \cdots s_l$. It is easy to see that if $I' = (s_1, \ldots, s_{l-1})$, the diagram

(7.1)
$$\begin{array}{ccc} \hat{\mathcal{X}}_I & \xrightarrow{q_I} & G/B \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle \pi_{\alpha_l}} \\ \hat{\mathcal{X}}_{I'} & \xrightarrow{\pi_{\alpha_l} \circ q_{I'}} & G/P_{\alpha_l} \end{array}$$

is cartesian, when p' is projection on the first $l-1$ factors. By induction on l, the variety $\hat{\mathcal{X}}_I$ is smooth projective and the morphism μ_I is projective. When furthermore I is a reduced decomposition of $w(I)$, meaning that it is of minimal length among the sequences J such that $w(J) = w(I)$, the map μ_I is birational (still by Bruhat decomposition). We can compose this map with the projection to get a map $\hat{\mathcal{X}}_w \to \mathcal{X}^{\Xi}_{\bar{w}}$ and thus when $w \in W^{\Xi}$, we obtain a (projective birational) desingularization $\hat{\mathcal{X}}_w \to \mathcal{X}^{\Xi}_{\bar{w}}$. It shows that, G/P_{Ξ} has a cellular decomposition with desingularizations, as considered just before [CPZ, Thm. 8.8], with cells indexed by elements of W/W_{Ξ}.

REMARK 7.1. The flag varieties, the Schubert varieties, their Bott-Samelson desingularizations and the various morphisms between them that we have just introduced are all B-equivariant when B acts on the left, and therefore are T-equivariant.

DEFINITION 7.2. Let $q^{\Xi}_I = \pi_{\Xi} \circ q_I$, let ζ^{Ξ}_I be the push-forward $(q^{\Xi}_I)_*(1)$ in $\mathtt{h}_T(G/P_{\Xi})$, and let $\zeta_I = \zeta^{\emptyset}_I$ in $\mathtt{h}_T(G/B)$.

Note that by definition, we have $(\pi_{\Xi})_*(\zeta_I) = \zeta^{\Xi}_I$.

LEMMA 7.3. *For any choice of reduced sequences* $\{I_w\}_{w \in W^{\Xi}}$*, the classes* $\zeta^{\Xi}_{I_w}$ *generate* $\mathtt{h}_T(G/P_{\Xi})$ *as an* S*-module.*

Proof. The proof of [CPZ, Theorem 8.8] goes through when $\mathtt{h}$ is replaced by $\mathtt{h}_T$, since all morphisms involved are T-equivariant; it only uses homotopy invariance and localization. $\square$

Let V_0 (resp. V_α) be the 1-dimensional representation of T corresponding to the 0 (resp. α) character. Let σ_0 and σ_α be the inclusions of T-fixed points corresponding to V_0 and V_α in $\mathbb{P}(V_0 \oplus V_\alpha)$ as in the setting of Section 4.
Consider the projection $\pi_\alpha \colon G/B \to G/P_\alpha$. Given an element $w \in W$, with image $\bar{w}$ in W/W_α and any lifting w' of w in G, the fiber over the fixed point $\imath^{\bar{w}}_\alpha \colon \mathrm{pt}_{\bar{w}} \to G/P_\alpha$ is $w'P_\alpha/B$.

LEMMA 7.4. *There is a T-equivariant isomorphism* $w'P_\alpha/B \simeq \mathbb{P}(V_0 \oplus V_{-w(\alpha)})$*, such that the closed fixed point* $\imath^w \colon \mathrm{pt}_w \to w'P_\alpha/B \hookrightarrow G/B$ *(resp.* $\imath^{ws_\alpha}$*) is sent to* $\sigma_0 \colon \mathrm{pt} \to \mathbb{P}(V_0 \oplus V_{-w(\alpha)})$ *(resp. to* $\sigma_{-w(\alpha)}$*).*

Proof. Multiplication on the left by w' defines an isomorphism $P_\alpha/B \to w'P_\alpha/B$ and it is T-equivariant if T acts by multiplication on the left on $w'P_\alpha/B$ and through conjugation by $(w')^{-1}$ and then by multiplication on

the left on P_α/B. Thus, we can reduce to the case where $w' = e$: the general case follows by replacing the character α by $w(\alpha)$.
First, let us observe that PGL_2 acts on the projective space $\mathbb{P}^1$ by projective transformations, i.e.

$$\overline{\begin{pmatrix} t & b \\ c & d \end{pmatrix}}[x : y] = [tx + by : cx + dy]$$

with its Borel subgroup B_{PGL_2} of upper triangular matrices fixing the point $[1 : 0]$, which therefore gives an identification $\mathrm{PGL}_2/B_{\mathrm{PGL}_2} \simeq \mathbb{P}^1$. So, its maximal torus $\mathbb{G}_m$ of matrices such that $b = c = 0$ and $d = 1$ acts by $t[x : y] = [tx : y] = [x : t^{-1}y]$. Thus, as a $\mathbb{G}_m$-variety, this $\mathbb{P}^1$ is actually $\mathbb{P}(V_1 \oplus V_0) \simeq \mathbb{P}(V_0 \oplus V_{-1})$. The adjoint semi-simple quotient of P_α is of rank one, so it is isomorphic to PGL_2. The maximal torus T maps to a maximal torus $\mathbb{G}_m$ and the Borel B to a Borel in this PGL_2. Up to modification of the isomorphism by a conjugation, we can assume that this Borel of PGL_2 is indeed B_{PGL_2} as above. The map $T \to \mathbb{G}_m$ is $\pm\alpha$ (the sign depends on how the maximal torus of PGL_2 is identified with $\mathbb{G}_m$). Since $P_\alpha/B \simeq \mathrm{PGL}_2/B_{\mathrm{PGL}_2}$, we are done by the PGL_2 case. □

Recall the notation from section 5.

Lemma 7.5. *The following diagram commutes.*

$$\begin{array}{ccccccc}
\mathtt{h}_T(G/B) & \xrightarrow{\imath^*} & \mathtt{h}_T(W) & \xrightarrow[\simeq]{\Theta} & S_W^\star & \subseteq & Q_W^* \\
\Big\downarrow{\scriptstyle \pi_\alpha^*(\pi_\alpha)_*} & & & & & & \Big\downarrow{\scriptstyle A_\alpha} \\
\mathtt{h}_T(G/B) & \xrightarrow{\imath^*} & \mathtt{h}_T(W) & \xrightarrow[\simeq]{\Theta} & S_W^\star & \subseteq & Q_W^*
\end{array}$$

Proof. In view of Lemma 7.4, the strategy is to reduce to the case of Lemma 4.6 by restricting to the fiber over one fixed point of G/P_α at a time.
We decompose $Q_W^* = \bigoplus_{w \in W^\alpha}(Q \cdot f_w \oplus Q \cdot f_{ws_\alpha})$ and note that A_α preserves this decomposition since

$$A_\alpha(f_w) = \frac{1}{x_{-w(\alpha)}}(f_w + f_{ws_\alpha}), \qquad A_\alpha(f_{ws_\alpha}) = \frac{1}{x_{w(\alpha)}}(f_w + f_{ws_\alpha})$$

and A_α is Q-linear. It therefore suffices to check the commutativity of the diagram after extending both rows on the right by a projection $Q_W^* \to Q \cdot f_w \oplus Q \cdot f_{ws_\alpha}$, for all $w \in W^\alpha$. But then, the composite horizontal maps $\mathtt{h}_T(G/B) \to Q \cdot f_w \oplus Q \cdot f_{ws_\alpha}$ factor as

$$\mathtt{h}_T(G/B) \to \mathtt{h}_T(P_\alpha wB/B) \to \mathtt{h}_T(\mathrm{pt}) \oplus \mathtt{h}_T(\mathrm{pt}) \simeq$$
$$\simeq S \oplus S \subseteq S[\tfrac{1}{x_{w(\alpha)}}] \oplus S[\tfrac{1}{x_{w(\alpha)}}] \subseteq Q \oplus Q.$$

Using proper base change on the diagram

$$\begin{array}{ccc} G/B & \hookleftarrow & w'P_\alpha/B \\ {\scriptstyle \pi_\alpha}\downarrow & & \downarrow \\ G/P_\alpha & \overset{\imath_\alpha^{\bar{w}}}{\hookleftarrow} & \mathrm{pt} \end{array}$$

and identifying $w'P_\alpha/B$ with $\mathbb{P}(V_0 \oplus V_{-w(\alpha)})$ by Lemma 7.4, we are reduced to proving the commutativity of

$$\begin{array}{ccccccc} \mathtt{h}_T\big(\mathbb{P}(V_0 \oplus V_{-w(\alpha)})\big) & \xrightarrow{\sigma^*} & \mathtt{h}_T(\mathrm{pt}) \oplus \mathtt{h}_T(\mathrm{pt}) & \xleftarrow{\simeq} & S \oplus S & \subseteq & S[\frac{1}{x_{w(\alpha)}}] \oplus S[\frac{1}{x_{w(\alpha)}}] \\ {\scriptstyle p^*p_*}\downarrow & & & & & & \downarrow{\scriptstyle A_\alpha} \\ \mathtt{h}_T\big(\mathbb{P}(V_0 \oplus V_{-w(\alpha)})\big) & \xrightarrow{\sigma^*} & \mathtt{h}_T(\mathrm{pt}) \oplus \mathtt{h}_T(\mathrm{pt}) & \xleftarrow{\simeq} & S \oplus S & \subseteq & S[\frac{1}{x_{w(\alpha)}}] \oplus S[\frac{1}{x_{w(\alpha)}}] \end{array}$$

which immediately reduces to the diagram of Lemma 7.5 followed by an obvious commutative diagram involving pull-backs

$$\begin{array}{ccccccc} \mathtt{h}_T\big(\mathbb{P}(V_0 \oplus V_{-w(\alpha)})\big) & \xrightarrow{\sigma^*} & \mathtt{h}_T(\mathrm{pt}) \oplus \mathtt{h}_T(\mathrm{pt}) & \xleftarrow{\simeq} & S \oplus S & \subseteq & S[\frac{1}{x_{w(\alpha)}}] \oplus S[\frac{1}{x_{w(\alpha)}}] \\ {\scriptstyle p^*}\uparrow & \nearrow{\scriptstyle \Delta} & & & \uparrow{\scriptstyle \Delta} & & \uparrow{\scriptstyle \Delta} \\ \mathtt{h}_T(\mathrm{pt}) & & \xleftarrow{\simeq} & & S & \subseteq & S[\frac{1}{x_{w(\alpha)}}] \end{array}$$

in which Δ is the diagonal morphism. □

Lemma 7.6. *For any sequence $I = (i_1, \dots, i_l)$, the Bott-Samelson class $\zeta_I \in \mathtt{h}_T(G/B)$ maps to*

$$\Theta \circ \imath^*(\zeta_I) = A_{I^{\mathrm{rev}}}\big(x_\Pi \cdot f_e\big)$$

in $S_W^\star$.

Proof. By induction using diagram (7.1), we have

$$\zeta_I = \pi^*_{\alpha_{i_l}}(\pi_{\alpha_{i_l}})_* \circ \cdots \circ \pi^*_{\alpha_{i_1}}(\pi_{\alpha_{i_1}})_* \circ (\imath^e)_*(1).$$

Since $\Theta\imath^*(\imath^e)_*(1) = x_\Pi \cdot f_e$ by Proposition 6.2, the conclusion follows from Lemma 7.5. □

8. Pull-back to T-fixed points

In the present section we describe the T-equivariant cohomology of an arbitrary split flag variety G/P_Ξ via the pull-back map to the cohomology of T-fixed points.

First, consider the complete flag variety G/B.

Proposition 8.1. *For any choice of reduced decompositions $(I_w)_{w\in W}$, the family $(\zeta_{I_w})_{w\in W}$ form a basis of $\mathtt{h}_T(G/B)$ over $S = \mathtt{h}_T(\mathrm{pt})$.*

Proof. By Lemma 7.6, the element ζ_{I_w} pulls-back to $A_{I^{\mathrm{rev}}}(x_\Pi \cdot f_e)$ in $S_W^\star$ and these are linearly independent over S by [CZZ2, Theorem 12.4]. They generate $\mathtt{h}_T(G/B)$ by Lemma 7.3. □

Theorem 8.2. *The pull-back map to fixed points $\imath^* : \mathtt{h}_T(G/B) \to \mathtt{h}_T(W)$ is injective, and the isomorphism $\Theta : \mathtt{h}_T(W) \simeq S_W^\star$, identifies its image to $\mathbf{D}^\star \subseteq S_W^\star$.*

Proof. This follows from Lemma 8.1 and the fact that the $A_{I^{\mathrm{rev}}}(x_\Pi \cdot f_e)$ form a basis of $\mathbf{D}^\star$ as a submodule of $S_W^\star$, still by [CZZ2, Theorem 12.4]. □

Remark 8.3. We do not know a direct geometric proof that $\mathtt{h}_T(G/B)$ injects into $\mathtt{h}_T((G/B)^T)$, which is of course well known for Chow groups or K-theory. To prove injectivity for Chow groups, one usually argues along the following lines:

(a) the composition $\imath^* \circ \imath_*$ becomes an isomorphism over Q (see Prop. 6.2);
(b) $\mathrm{CH}_T(G/B)$ is a free $\mathrm{CH}_T(\mathrm{pt})$-module of rank $|W|$ and so is $\mathrm{CH}_T((G/B)^T)$.
(c) the pull-back $Q \otimes_S \mathrm{CH}_T(G/B) \to Q \otimes_S \mathrm{CH}_T((G/B)^T)$ is an isomorphism as any surjection of free modules of the same rank over a noetherian ring (Q is a localization of $\mathrm{CH}_T(\mathrm{pt}) = \mathbb{Z}[\![x_1, \ldots, x_n]\!]$).

However, in the general case, localization arguments only give generating families so the freeness part of (b) does not follow, and in (c), Q is not noetherian (e.g. the Lazard ring is not noetherian), so we need to look more carefully into the structure of the image as a submodule of $\mathtt{h}_T((G/B)^T)$. This is done in the algebraic world: the Bott-Samelson classes considered are linearly independent when pulled to $S_W^\star$ and the map $\mathbf{D}^\star \to S_W^\star$ is indeed an injection by [CZZ2, Lemma 10.2].

Corollary 8.4. *The pull-back map $\imath^* : \mathtt{h}_T(G/B) \to \mathtt{h}_T(W)$ becomes an isomorphism after localization at the multiplicative subset generated by all x_α where α is a root.*

Proof. After localization at this subset, the inclusion $\mathbf{D}^\star \subseteq S_W^\star$ becomes an isomorphism (see [CZZ2, Lemma 10.2]). □

Lemma 8.5. *The following diagram commutes*

$$\begin{array}{ccccc}
\mathtt{h}_T(W) & \xrightarrow{\imath_*} & \mathtt{h}_T(G/B) & \xhookrightarrow{\imath^*} & \mathtt{h}_T(W) \\
\simeq \downarrow \Theta & & \simeq \downarrow \Theta & & \simeq \downarrow \Theta \\
S_W^\star & \xrightarrow{x_\Pi \bullet (-)} & \mathbf{D}^\star & \xhookrightarrow{\eta^*} & S_W^\star
\end{array}$$

Proof. This follows from Corollary 6.4 and Theorem 8.2. □

We now consider an arbitrary flag variety G/P_Ξ.

Lemma 8.6. *The following diagram commutes.*

$$\begin{array}{ccccccc} \mathtt{h}_T(G/B) & \xrightarrow{\imath^*} & \mathtt{h}_T(W) & \underset{\simeq}{\xrightarrow{\Theta}} & S^\star_W & \subseteq & Q^*_W \\ \downarrow{\scriptstyle (\pi_\Xi)_*} & & & & & & \downarrow{\scriptstyle \mathcal{A}_\Xi} \\ \mathtt{h}_T(G/P_\Xi) & \xrightarrow{\imath^*_\Xi} & \mathtt{h}_T(W/W_\Xi) & \underset{\simeq}{\xrightarrow{\Theta_\Xi}} & S^\star_{W/W_\Xi} & \subseteq & Q^*_{W/W_\Xi} \end{array}$$

Proof. After tensoring the whole diagram with Q over S, the morphism $\imath^*$ becomes an isomorphism by Corollary 8.4. The family $\big((\imath^w)_*(1)\big)_{w\in W}$ is a Q-basis of $Q \otimes_S \mathtt{h}_T(G/B)$, since by Proposition 6.2, $\Theta \circ \imath^* \circ (\imath^w)_*(1)$ is f_w multiplied by an element that is invertible (in Q). It therefore suffices to check the equality of both compositions in the diagram when applied to all $(\imath^w)_*(1)$ with $w \in W$:

$$\mathcal{A}_\Xi \circ \Theta \circ \imath^* \circ (\imath^w)_*(1) = \mathcal{A}_\Xi(w(x_\Pi)f_w) = w(x_\Pi)\mathcal{A}_\Xi(f_w) \overset{(*)}{=}$$
$$\overset{(*)}{=} w(x_{\Pi/\Xi})f_{\bar{w}} = \Theta_\Xi(\imath_\Xi)^*(\imath_\Xi^{\bar{w}})_*(1) = \Theta_\Xi(\imath_\Xi)^*(\pi_\Xi)_*(\imath^w)_*(1)$$

where equality $(*)$ follows from the definition of $\mathcal{A}_\Xi$. □

Corollary 8.7. *The following diagram commutes.*

$$\begin{array}{ccccccc} \mathtt{h}_T(G/B) & \xrightarrow{\imath^*} & \mathtt{h}_T(W) & \underset{\simeq}{\xrightarrow{\Theta}} & S^\star_W & \subseteq & Q^*_W \\ \downarrow{\scriptstyle (\pi_\Xi)^*(\pi_\Xi)_*} & & & & & & \downarrow{\scriptstyle \mathcal{A}_\Xi} \\ \mathtt{h}_T(G/B) & \xrightarrow{\imath^*} & \mathtt{h}_T(W) & \underset{\simeq}{\xrightarrow{\Theta}} & S^\star_W & \subseteq & Q^*_W \end{array}$$

Proof. Using equation (6.1), one easily checks the commutativity of diagram involving pull-backs

$$\begin{array}{ccccccc} \mathtt{h}_T(G/B) & \xrightarrow{\imath^*} & \mathtt{h}_T(W) & \underset{\simeq}{\xrightarrow{\Theta}} & S^\star_W & \subseteq & Q^*_W \\ \uparrow{\scriptstyle \pi^\star_\Xi} & & & & & & \uparrow{\scriptstyle p^\star_\Xi} \\ \mathtt{h}_T(G/P_\Xi) & \xrightarrow{\imath^*} & \mathtt{h}_T(W/W_\Xi) & \underset{\simeq}{\xrightarrow{\Theta_\Xi}} & S^\star_{W/W_\Xi} & \subseteq & Q^*_{W/W_\Xi} \end{array}$$

where $p^\star_\Xi$ is the sum over orbits: $p^\star_\Xi(f_{\bar{w}}) = \sum_{\bar{v}=\bar{w}} f_v$. The result follows from the combination of this diagram and the one of Lemma 8.6. □

Lemma 8.8. *For any sequence $I = (i_1, \ldots, i_l)$, the Bott-Samelson class $\zeta^\Xi_I \in \mathtt{h}_T(G/P_\Xi)$ maps to*

$$\Theta \circ (\imath_\Xi)^*(\zeta^\Xi_I) = \mathcal{A}_\Xi \circ A_{I^{\mathrm{rev}}}\big(x_\Pi f_e\big)$$

in $S^\star_W$.

Proof. We have

$$\Theta(\imath_\Xi)^*(\zeta^\Xi_{I_w}) = \Theta(\imath_\Xi)^*(\pi_\Xi)_*(\zeta_{I_w}) = \mathcal{A}_\Xi \circ \Theta \circ \imath^*(\zeta_{I_w}) = \mathcal{A}_\Xi \circ A_{I_w^{\mathrm{rev}}}\big(x_\Pi f_e\big)$$

using Lemma 8.6 and Lemma 7.6 for the last two equalities. □

Proposition 8.9. *For any choice of reduced decompositions $(I_w)_{w\in W^\Xi}$ for elements minimal in their W_Ξ-cosets, the classes $\zeta^\Xi_{I_w}$ form an S-basis of $\mathtt{h}_T(G/P_\Xi)$.*

Proof. By Lemma 7.3, the classes $\zeta^\Xi_{I_w}$ generate $\mathtt{h}_T(G/P_\Xi)$ as an S-module. We have

$$\Theta\imath^*(\pi_\Xi)^*(\zeta^\Xi_{I_w}) = \Theta\imath^*(\pi_\Xi)^*(\pi_\Xi)_*(\zeta_{I_w}) = A_\Xi\Theta\imath^*(\zeta_{I_w}) = A_\Xi A_{I_w^{\mathrm{rev}}}(x_\Pi f_e)$$

and these elements are linearly independent by [CZZ2, Theorem 14.3]. □

Let $\Xi' \subseteq \Xi \subseteq \Pi$.

Corollary 8.10. *The push-forward map $(\pi_{\Xi/\Xi'})_* : \mathtt{h}_T(G/P_{\Xi'}) \to \mathtt{h}_T(G/P_\Xi)$ is surjective and the pull-back map $(\pi_{\Xi/\Xi'})^* : \mathtt{h}_T(G/P_\Xi) \to \mathtt{h}_T(G/P_{\Xi'})$ is injective.*

Proof. Surjectivity is obvious from the fact that ζ_{I_w} maps to the basis element $\zeta^\Xi_{I_{\bar w}}$ for any $w \in W^\Xi$ and injectivity can be seen in the proof of Proposition 8.9: the elements $\zeta^\Xi_{I_{\bar w}}$ stay independent when pulled back all the way to $\mathtt{h}_T(W)$ through $\mathtt{h}_T(G/B)$. □

Theorem 8.11. *The pull-back map $\imath^*_\Xi : \mathtt{h}_T(G/P_\Xi) \to \mathtt{h}_T(W/W_\Xi)$ is injective and the isomorphism $\Theta_\Xi : \mathtt{h}_T(W/W_\Xi) \xrightarrow{\sim} S^\star_{W/W_\Xi}$ identifies its image to $\mathbf{D}^\star_\Xi \subseteq S^\star_{W/W_\Xi}$.*

Proof. As seen in the proof of Corollary 8.10, pulling back further to $\mathtt{h}_T(W)$ is injective, so injectivity of $\imath^*_\Xi$ is clear. By Lemma 8.8, for any $w \in W^\Xi$, the Bott-Samelson class $\zeta^\Xi_{I_w}$ is sent to $\mathcal{A}_\Xi A_{I_w^{\mathrm{rev}}}(x_{\Pi/\Xi} f_e)$. These elements form a basis of $\mathbf{D}^\star_\Xi$ by [CZZ2, Theorem 14.3 and Lemma 15.1]. □

Corollary 8.12. *The pull-back map $\imath^*_\Xi : \mathtt{h}_T(G/P_\Xi) \to \mathtt{h}_T(W/W_\Xi)$ becomes an isomorphism after localization at the multiplicative subset generated by all x_α where α is a root.*

Proof. After localization at this subset, the inclusion $\mathbf{D}^\star_\Xi \subseteq S^\star_{W/W_\Xi}$ becomes an isomorphism (see [CZZ2, Lemma 11.5]). □

As for G/B, we have the following commutative diagram

$$\text{(8.1)}\qquad \begin{array}{ccccc}
\mathtt{h}_T(W/W_\Xi) & \xrightarrow{(\imath_\Xi)_*} & \mathtt{h}_T(G/P_\Xi) & \xhookrightarrow{(\imath_\Xi)^*} & \mathtt{h}_T(W/W_\Xi) \\
\simeq\big\downarrow \Theta_\Xi & & \simeq\big\downarrow \Theta_\Xi & & \simeq\big\downarrow \Theta_\Xi \\
S^\star_{W/W_\Xi} & \xrightarrow{x_{\Pi/\Xi}\bullet(-)} & \mathbf{D}^\star_\Xi & \xhookrightarrow{\eta^*_\Xi} & S^\star_{W/W_\Xi}
\end{array}$$

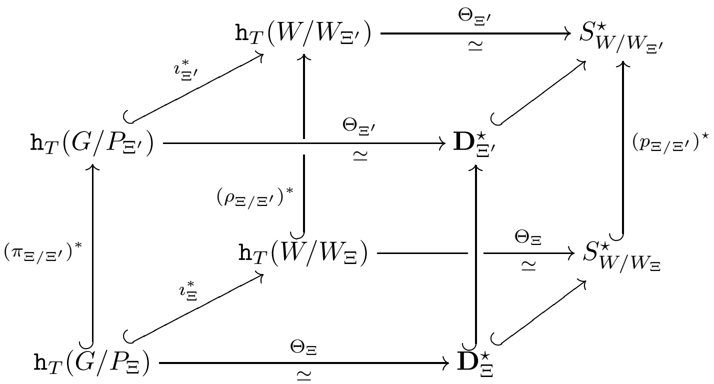
Lemma 8.13. *The following diagram commutes.*

$$\begin{array}{ccccccc}
\mathtt{h}_T(G/P_{\Xi'}) & \xrightarrow{(\imath_{\Xi'})^*} & \mathtt{h}_T(W/W_{\Xi'}) & \xrightarrow[\simeq]{\Theta_{\Xi'}} & S^\star_{W/W_{\Xi'}} & \subseteq & Q^*_{W/W_{\Xi'}} \\
\Big\downarrow{\scriptstyle (\pi_{\Xi/\Xi'})_*} & & & & & & \Big\downarrow{\scriptstyle \mathcal{A}_{\Xi/\Xi'}} \\
\mathtt{h}_T(G/P_{\Xi}) & \xrightarrow{(\imath_{\Xi})^*} & \mathtt{h}_T(W/W_{\Xi}) & \xrightarrow[\simeq]{\Theta_{\Xi}} & S^\star_{W/W_{\Xi}} & \subseteq & Q^*_{W/W_{\Xi}}
\end{array}$$

Proof. By the surjectivity claim in Corollary 8.10, we can precompose the diagram by $\pi_{\Xi'}$. Since $\mathcal{A}_\Xi = \mathcal{A}_{\Xi/\Xi'} \circ \mathcal{A}_{\Xi'}$, the result follows from Lemma 8.6 applied first to Ξ' and then to Ξ. □

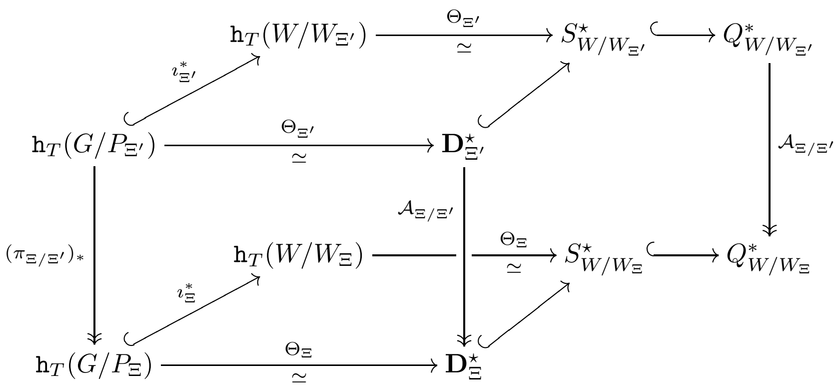
Summarizing, we have the following commutative diagrams describing the correspondence between the cohomology rings and their algebraic counterparts:

(8.2)



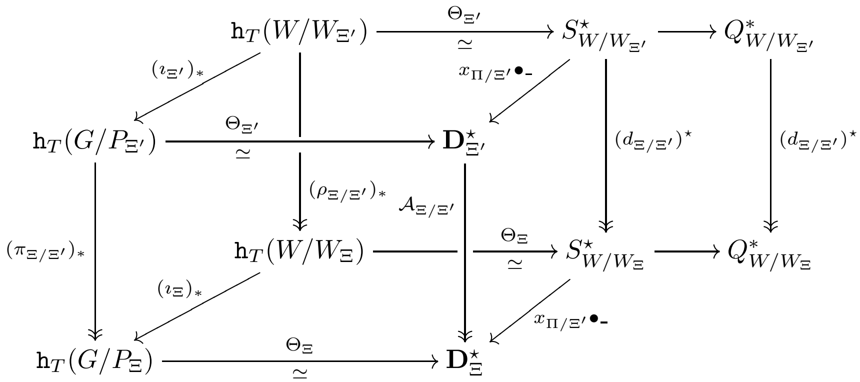
For push-forwards, instead, the morphism $\mathcal{A}_{\Xi/\Xi'} : Q^*_{W/W_{\Xi'}} \to Q^*_{W/W_{\Xi}}$ induces a map $\mathcal{A}_{\Xi/\Xi'} : \mathbf{D}^\star_{\Xi'} \to \mathbf{D}^\star_{\Xi}$ by [CZZ2, Lemma 15.1], and we have:

(8.3)



Notice that on this diagram, there is no map from $\mathtt{h}_T(W/W_{\Xi'})$ to $\mathtt{h}_T(W/W_{\Xi})$, nor from $S^\star_{W/W_{\Xi'}}$ to $S^\star_{W/W_{\Xi}}$ because the operator $\mathcal{A}_{\Xi/\Xi'}$ is not defined at that level.

By (8.1) and the identity $x_{\Pi/\Xi'} = x_{\Pi/\Xi} x_{\Xi/\Xi'}$, we finally have the following.
(8.4)



9. Invariant subrings and push-forward pairings

We now describe how the Weyl group W, as an abstract group, acts on $\mathtt{h}_T(G/B)$, and how W_Ξ-invariant elements of this action are related to $\mathtt{h}_T(G/P_\Xi)$.

Since the projection $G/T \to G/B$ is an affine bundle, by homotopy invariance the induced pull-back $\mathtt{h}_T(G/B) \xrightarrow{\sim} \mathtt{h}_T(G/T)$ is an isomorphism. The Weyl group action is easier to describe geometrically on $\mathtt{h}_T(G/T)$. Since $W \simeq N_G(T)/T$, multiplication on the right by $w \in W$ defines a right action of W on G/T, by T-equivariant morphisms. Action by induced pull-backs, therefore, defines a left action of W on $\mathtt{h}_T(G/T)$. Similarly, a right action of W on the T-fixed points $(G/T)^T = W$ induces a left action of W on $\mathtt{h}_T(W)$, and the pull-back $\mathtt{h}_T(G/T) \to \mathtt{h}_T(W)$ is W-equivariant. Identifying $\mathtt{h}_T(G/T) \simeq \mathtt{h}_T(G/B)$, we obtain the Weyl group action on $\mathtt{h}_T(G/B)$ with $\imath^* \colon \mathtt{h}_T(G/B) \to \mathtt{h}_T(W)$ being W-equivariant.

One easily checks on S-basis elements f_w that through Θ, this W-action on $\mathtt{h}_T(W)$ corresponds to the W-action on $S^\star_W$ by the Hecke action $w(z) = \delta_w \bullet z$, as described in [CZZ2, §4] (by definition, we have $\delta_w \bullet f_v = f_{vw^{-1}}$).

Theorem 9.1. *The image of the injective pull-back map* $\mathtt{h}_T(G/P_\Xi) \to \mathtt{h}_T(G/B)$ *is* $\mathtt{h}_T(G/B)^{W_\Xi}$.

Proof. In Diagram (8.2), the upper square is W-equviariant. Since $\imath^*$ is both W-equivariant and injective, we are reduced to showing that $p^\star_\Xi$ identifies $S^\star_{W/W_\Xi}$ to $(S^\star_W)^{W_\Xi}$, which follows from [CZZ2, Lemma 11.7]. □

The following theorem generalizes [Br97, Proposition 6.5.(i)]. According to the irreducible Dynkin types of the group, regularity assumptions on elements of the base ring R (or weaker assumptions on elements in $R[\![x]\!]$) are needed. They are carefully summarized in [CZZ2, Lemma 2.7], but as a first approximation, regularity in R of 2, 3 and divisors of $|\Lambda_w/\Lambda_r|$ cover all types, except the C_n^{sc} case, in which one needs 2 to be invertible.[4]

Theorem 9.2. *Under the regularity assumptions of* [CZZ2, Lemma 2.7], *the image of the injective pull-back* $\imath^* : \mathtt{h}_T(G/B) \to \mathtt{h}_T(W) \cong S_W^\star$ *is the set of element* $\sum_{w\in W} q_w f_w$ *such that* $x_\alpha | (q_w - q_{s_\alpha w})$ *for all roots* α.

Proof. If follows from [CZZ2, Theorem 10.7]. □

We now describe the pairing given by multiplication and then push-forward to the point, that we call the *push-forward pairing*. Let

$$\begin{array}{ccc} \mathtt{h}_T(G/P_\Xi) \otimes_S \mathtt{h}_T(G/P_\Xi) & \overset{\langle -,-\rangle_\Xi}{\longrightarrow} & S \\ \xi \otimes \xi' & \longmapsto & \langle \xi, \xi'\rangle_\Xi = (\pi_{\Pi/\Xi})_*(\xi \cdot \xi') \end{array}$$

It is clearly S-bilinear and symmetric. Through the isomorphism Θ, this pairing corresponds to

$$\langle \xi, \xi'\rangle_\Xi = \mathcal{A}_{\Pi/\Xi}(\Theta_\Xi(\xi) \cdot \Theta_\Xi(\xi'))$$

by Diagram (8.3).

Theorem 9.3. *The push-forward pairing* $\mathtt{h}_T(G/P_\Xi) \otimes_S \mathtt{h}_T(G/P_\Xi) \to \mathtt{h}_T(\mathrm{pt}) \simeq S$, *sending* (ξ, ξ') *to* $\langle \xi, \xi'\rangle_\Xi$ *is non-degenerate.*

Proof. This follows from [CZZ2, Theorem 15.6]. □

Remark 9.4. Note that in [CZZ2, Theorem 15.5], we describe a basis that is dual to the basis of Bott-Samelson classes for the push-forward pairing on G/B. That dual basis can be very useful for algorithmic computations. However, it is given in combinatorial terms, and we do not have a geometric interpretation of its elements. When the formal group law is additive, this problem disappears since the basis is auto-dual (up to a permutation), see [D74, Prop. 1, p. 69], but for general formal group laws, this is not the case.

10. Borel-style presentation

The *geometric (equivariant) characteristic map* $c_g : \mathtt{h}_T(\mathrm{pt}) \to \mathtt{h}_T(G/B)$ is defined as the composition

$$\mathtt{h}_T(\mathrm{pt}) \overset{\sim}{\to} \mathtt{h}_{T\times G}(G) \overset{\sim}{\leftarrow} \mathtt{h}_G(G/T) \to \mathtt{h}_T(G/T) \overset{\sim}{\leftarrow} \mathtt{h}_T(G/B)$$

where the first two maps are isomorphisms from Axiom (A6), the third is the restriction to the subgroup T of G and the fourth is the pull-back map, an isomorphism by Axiom (A4) of homotopy invariance. In $\mathtt{h}_{T\times G}(G)$, the action of $T \times G$ on G is by $(t, g) \cdot g' = gg't^{-1}$, and the other non-trivial actions are by multiplication on the left. Note that c_g is $R = \mathtt{h}(\mathrm{pt})$-linear, although

[4] Regarding these assumptions, there is a slight omission in the statement of [Br97, Proposition 6.5.(i)]. One needs to add that no root is divisible in the lattice for the statement to hold integrally. Otherwise, for example, the product of all roots divided by 2 gives a counter-example in the C_2^{sc} case.

not $\mathtt{h}_T(\mathrm{pt})$-linear. By restricting further to $\mathtt{h}(G/B)$, one obtains the non-equivariant characteristic map $c : \mathtt{h}_T(\mathrm{pt}) \to \mathtt{h}(G/B)$. Recall the algebraic characteristic map $c_S : S \to \mathbf{D}^\star$, sending $s \to s \bullet \mathbf{1}$, defined in section 5.

Lemma 10.1. *The algebraic and geometric characteristic maps coincide with each other, up to the identifications $S \simeq \mathtt{h}_T(\mathrm{pt})$ of Theorem 3.3 and $\Theta : \mathtt{h}_T(G/B) \simeq \mathbf{D}^\star$ of Theorem 8.2.*

Proof. It suffices to show the equality after embedding in $S_W^\star \simeq \mathtt{h}_T(W)$, which decomposes as copies of S. In other words, it suffices to compare, for every $w \in W$, a map ϕ_w from S to itself, and a map ψ_w from $\mathtt{h}_T(\mathrm{pt})$ to itself. Both are continuous R-algebra maps, ψ_w for the topology induced by the γ-filtration and ϕ_w for the $\mathcal{I}_F$-adic topology, which correspond to each other through $S \simeq \mathtt{h}_T(\mathrm{pt})$. Since S is (topologically) generated by elements x_λ, corresponding to first characteristic classes of line bundles $c_1^T(L_\lambda)$ in $\mathtt{h}_T(\mathrm{pt})$, it suffices to compare $\phi_w(x_\lambda)$ and $\psi_w(c_1^T(L_\Lambda))$. By definition of c_S, we have $\phi(x_\lambda) = x_{w(\lambda)}$. Since c_g is defined using only pull-back and restriction maps, both commuting with taking characteristic classes, it suffices to verify that when $\mathtt{h} = K$, the Grothendieck group, we have $\psi_w([L_\lambda]) = [L_{w(\lambda)}]$. This is easily checked by using total spaces of bundles, and the formalism of points. For this purpose, let us consider the following equivariant bundles:

- M_λ, the $T \times G$-equivariant line bundle over G, whose total space is $L_\lambda \times G$ mapping by the second projection to G, and with action given on points by $(t, g) \cdot (v, g') = (\lambda(t)v, gg't^{-1})$;
- N_λ, the G-equivariant line bundle over G/T, whose total space is $G \times^T L_\lambda$, the quotient of $G \times L_\lambda$ by the relation $(gt, v) = (g, \lambda(t)v)$, mapping to G/T by the first projection, and with G action by $g \cdot (g', v) = (gg', v)$;
- M'_λ, the $T \times G$-equivariant line bundle over G, whose total space is $G \times_{G/T} G \times^T L_\lambda$, mapping to G by the first projection, with action of $T \times G$ given by $(t, g) \cdot (g_1, g_2, v) = (gg_1t^{-1}, gg_2, v)$.

It is clear that L_λ restricts to $T \times G$ and pulls-back over G to M_λ. Similarly, N_λ restricts and pulls-back to M'_λ. But M_λ maps isomorphically to M'_λ by the map $(v, g) \mapsto (g, g, v)$. Therefore, $[L_\lambda]$ maps to $[N_\lambda]$ by the map $K_T(\mathrm{pt}) \xrightarrow{\sim} K_{T\times G}(G) \xleftarrow{\sim} K_G(G/T)$. Furthermore, N_λ restricts and pulls-back as a T-equivariant bundle to the fixed point w in G/T (or G/B) as $wT \times^T L_\lambda$ with T-action on the left, isomorphic to $L_{w(\lambda)}$. This completes the proof. □

Let $\mathfrak{t}$ be the torsion index of the root datum, as defined in [D73, §5]. See also [CPZ, 5.1] for a table giving the values of its prime divisors for each simply connected type. For other types, one just needs to add the prime divisors of $|\Lambda_w/\Lambda|$ by [D73, §5, Prop. 6]. Together with the previous lemma, [CZZ, Thm. 11.4] immediately implies a Borel-style presentation of $\mathtt{h}_T(G/B)$. Let $\pi : G/B \to \mathrm{pt}$ be the structural map.

Theorem 10.2. *If $2\mathfrak{t}$ is regular in R, then the map $\mathtt{h}_T(pt) \otimes_{\mathtt{h}_T(\mathrm{pt})^W} \mathtt{h}_T(\mathrm{pt}) \to \mathtt{h}_T(G/B)$ sending $a \otimes b$ to $\pi^*(a)c_g(b)$ is an $\mathtt{h}_T(\mathrm{pt})$-linear ring isomorphism if* *and only if the (non-equivariant) characteristic map $c : \mathtt{h}_T(\mathrm{pt}) \to \mathtt{h}(G/B)$ is surjective.*

In particular, it will hold for K-theory, since the characteristic map is always surjective for K-theory. It will also hold for any cohomology theory if $\mathfrak{t}$ is invertible in R, as [CPZ, Cor. 13.9] shows that the non-equivariant characteristic map is then surjective.
As mentioned in the introduction, this presentation was obtained in [KiKr] for algebraic cobordism, with the torsion index inverted, and by using comparisons with complex cobordism.

11. Subgroups of T

Let H be a subgroup of T given by the embedding $h : H \hookrightarrow T$. For example H could be the trivial group, a finite multiplicative group or a subtorus of T. For any $X \in T$-Var, and thus in H-Var by restriction, there is a restriction ring map $\mathrm{res}_h : \mathtt{h}_T(X) \to \mathtt{h}_H(X)$, in particular if $X = \mathrm{pt}$, which induces a canonical morphism $\mathtt{h}_H(\mathrm{pt}) \otimes_{\mathtt{h}_T(\mathrm{pt})} \mathtt{h}_T(X) \to \mathtt{h}_H(X)$ of rings over $\mathtt{h}_H(\mathrm{pt})$, sending $a \otimes b$ to $a \cdot \mathrm{res}_h(b)$. This "change of coefficients" morphism is compatible with pull-backs and push-forwards.

Lemma 11.1. *The morphism $\mathtt{h}_H(\mathrm{pt}) \otimes_{\mathtt{h}_T(\mathrm{pt})} \mathtt{h}_T(X) \to \mathtt{h}_H(X)$ is an isomorphism when $X = G/P_\Xi$ or $X = W/W_\Xi$.*

Proof. The case of $X = W/W_\Xi$ is obvious, since as as scheme, it is simply a disjoint union of copies of pt. If $X = G/P_\Xi$, the left-hand side is free, with a basis of Bott-Samelson classes. So is the right-hand side: it is still generated as an $\mathtt{h}_H(\mathrm{pt})$-module by the corresponding Bott-Samelson classes because the proof of Lemma 7.3 works for H as well as for T. Thus, the change of coefficients is surjective. The push-forward pairing is perfect and commutes to the restriction map from T to H, so these classes stay independent in $\mathtt{h}_H(G/P_\Xi)$ (they have a dual family). Thus, the change of coefficients is injective. □

This shows that Diagram (8.2) for H is obtained by change of coefficients, as well as Diagram (8.3) and Diagram (8.4) except their rightmost columns involving Q. Theorem 9.3 on the bilinear pairing stays valid for H instead of T.

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Integral Mixed Motives in Equal Characteristic

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Abstract. For noetherian schemes of finite dimension over a field of characteristic exponent p, we study the triangulated categories of $\mathbf{Z}[1/p]$-linear mixed motives obtained from cdh-sheaves with transfers. We prove that these have many of the expected properties. In particular, the formalism of the six operations holds in this context. When we restrict ourselves to regular schemes, we also prove that these categories of motives are equivalent to the more classical triangulated categories of mixed motives constructed in terms of Nisnevich sheaves with transfers. Such a program is achieved by comparing these various triangulated categories of motives with modules over motivic Eilenberg-MacLane spectra.

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Contents

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The main advances of the actual theory of mixed motivic complexes over a field come from the fact they are defined integrally. Indeed, this divides the theory in two variants, the Nisnevich one and the étale one. With rational coefficients, the two theories agree and share their good properties. But with integral coefficients, their main success comes from the comparison of these two variants, the so-called Beilinson-Lichtenbaum conjecture which was proved by Voevodsky and gave the solution of the Bloch-Kato conjecture.

One of the most recent works in the theory has been devoted to extend the definitions in order to get the 6 operations of Grothendieck and to check they satisfy the required formalism; in chronological order: an unpublished work of Voevodsky, [Ayo07a], [CDa]. While the project has been finally completely realized with rational coefficients in [CDa], the case of integral coefficients remains unsolved. In fact, this is half true: the étale variant is now completely settled: see [Ayo14], [CDb].

But the Nisnevich variant is less mature. Markus Spitzweck [Spi] has constructed a motivic ring spectrum over any Dedekind domain, which allows to define motivic cohomology of arbitrary schemes, and even triangulated categories of motives on a general base (under the form of modules over the pullbacks of the motivic ring spectrum over $\mathrm{Spec}(\mathbf{Z})$). However, at this moment, there is no proof that Spitzweck's motivic cohomology satisfies the absolute purity theorem, and we do not know how to compare Spitzweck's construction with triangulated categories of motives constructed in the language of algebraic correspondences (except for fields). What is concretely at stake is the theory of algebraic cycles: we expect that motivic cohomology of a regular scheme in degree $2n$ and twist n agrees with the Chow group of n-codimensional cycles of X. Let us recall for example that the localization long exact sequence for higher Chow groups and the existence of a product of Chow groups of regular schemes are still open questions in the arithmetic case (*i.e.* for schemes of unequal residual characteristics). For sake of completeness, let us recall that the localization long exact sequence in equal characteristic already is the fruit of non trivial contributions of Spencer Bloch [Blo86, Blo94] and Marc Levine [Lev01]. Their work involves moving lemmas which are generalizations of the classical moving lemma used to understand the intersection product of cycles [Ful98].

Actually, Suslin and Voevodsky have already provided an intersection theoretic basis for the integral definition of Nisnevich motivic complexes: the theory of relative cycles of [VSF00, chap. 2]. Then, along the lines drawn by Voevodsky, and especially the homotopy theoretic setting realized by Morel and Voevodsky, it was at least possible to give a reasonable definition of such a theory over an arbitrary base, using Nisnevich sheaves with transfers over this base, and the methods of $\mathbf{A}^1$-homotopy and $\mathbf{P}^1$-stabilization: this was done in [CDa, Sec. 7]. Interestingly enough, the main technical issue of this construction is to prove that these motivic complexes satisfy the existence of the localization triangle:

$$j_!\, j^*(M) \to M \to i_*\, i^*(M) \to j_!\, j^*(M)[1]$$

for any closed immersion i with open complement j. This echoes much with the question of localization sequence for higher Chow groups.
In our unsuccessful efforts to prove this property with integral coefficients, we noticed two things: the issue of dealing with singular schemes (the property is true for smooth schemes over any base, and, with rational coefficients, for any closed immersion between excellent geometrically unibranch scheme); the fact this property implies cdh-descent (i.e. Nisnevich descent together with descent by blow ups). Moreover, in [CDb], we show that, at least for torsion coefficients, the localization property for étale motivic complexes is true without any restriction, but this is due to rigidity properties (*à la* Suslin) which only hold étale locally, and for torsion coefficients.
Therefore, the idea of replacing Nisnevich topology by a finer one, which allows to deal with singularities, but remains compatible with algebraic cycles, becomes obvious. The natural choice goes to the cdh-topology: in Voevodsky's work [VSF00], motivic (co)homology of smooth schemes over a field is naturally extended to schemes of finite type by cdh-descent in characteristic zero (or, more generally, if we admit resolution of singularities), and S. Kelly's thesis [Kel12] generalizes this result to arbitrary perfect fields of characteristic $p > 0$, at least with $\mathbf{Z}[1/p]$-linear coefficients.

In this work, we prove that if one restricts to noetherian schemes of finite dimension over a prime field (in fact, an arbitrary perfect field) k, and if we invert solely the characteristic exponent of k, then mixed motives built out of cdh-sheaves with transfers (Definition 1.5) do satisfy the localization property: Theorem 5.11. Using the work of Ayoub, it is then possible to get the complete 6 functors formalism for these cdh-motives. Note that we also prove that these cdh-motives agree with the Nisnevich ones for regular k-schemes – hence proving that the original construction done in [CDa, Def. 11.1.1] is meaningful if one restricts to regular schemes of equal characteristic and invert the residue characteristic (see Corollary 3.2 for a precise account).
The idea is to extend a result of Röndigs and Østvær, which identifies motivic complexes with modules over the motivic Eilenberg-MacLane spectrum over a field of characteristic 0. This was recently generalized to perfect fields of characteristic $p > 0$, up to inverting p, by Hoyois, Kelly and Østvær [HKØ]. Our

main result, proved in Theorem 5.1, is that this property holds for arbitrary noetherian k-schemes of finite dimension provided we use cdh-motives and invert the exponent characteristic p of k in their coefficients. For any noetherian k-scheme of finite dimension X with structural map $f : X \to \mathrm{Spec}(k)$, let us put $H\mathbf{Z}_{X/k} = \mathbf{L}f^*(H\mathbf{Z}_k)$. Then there is a canonical equivalence of triangulated categories

$$H\mathbf{Z}_{X/k}[1/p]\text{-Mod} \simeq \mathrm{DM}_{\mathrm{cdh}}(X, \mathbf{Z}[1/p])\,.$$

One of the ingredients is to prove this result for Nisnevich motivic complexes with $\mathbf{Z}[1/p]$-coefficients if one restricts to noetherian regular k-schemes of finite dimension: see Theorem 3.1. The other ingredient is to use Gabber's refinement of de Jong resolution of singularities by alteration via results and methods from Kelly's thesis.

We finally prove the stability of the notion of constructibility for cdh-motives up to inverting the characteristic exponent in Theorem 6.4. While the characteristic 0 case can be obtained using results of [Ayo07a], the positive characteristic case follows from a geometrical argument of Gabber (used in his proof of the analogous fact for torsion étale sheaves). We also prove a duality theorem for schemes of finite type over a field (7.3), and describe cycle cohomology of Friedlander and Voevodsky using the language of the six functors (8.11). In particular, Bloch's higher Chow groups and usual Chow groups of schemes of finite type over a field are are obtained via the expected formulas (see 8.12 and 8.13).

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Conventions

We will fix a perfect base field k of characteristic exponent p – the case where k is a prime field is enough. All the schemes appearing in the paper are assumed to be noetherian of finite dimension.

We will fix a commutative ring R which will serve as our coefficient ring.

1. Motivic complexes and spectra

In [VSF00, chap. 5], Voevodsky introduced the category of motivic complexes $\mathrm{DM}^{\mathit{eff}}_{-}(S)$ over a perfect field with integral coefficients, a candidate for a conjectural theory described by Beilinson. Since then, several generalizations to more general bases have been proposed.

In [CDa], we have introduced the following ones over a general base noetherian scheme S:

1.1. *The Nisnevich variant.*– Let Λ be the localization of $\mathbf{Z}$ by the prime numbers which are invertible in R. The first step is to consider the category $Sm^{cor}_{\Lambda,S}$

whose ojects are smooth separated S-schemes of finite type and morphisms between X and Y are *finite S-correspondences from X to Y with coefficients in* Λ (see [CDa, Def. 9.1.8] with $\mathscr{P}$ the category of smooth separated morphisms of finite type).[2] Taking the graph of a morphism between smooth S-schemes, one gets a faithful functor γ from the usual category of smooth S-schemes to the category $Sm^{cor}_{\Lambda,S}$.
Then one defines the category $\mathrm{Sh}^{tr}_{\mathrm{Nis}}(S,R)$ of *sheaves with transfers over S* as the category of presheaves F of R-modules over $Sm^{cor}_{\Lambda,S}$ whose restriction to the category of smooth S-schemes $F \circ \gamma$ is a sheaf for the Nisnevich topology. Essentially according to the original proof of Voevodsky over a field (see [CDa, 10.3.3 and 10.3.17] for details), this is a symmetric monoidal Grothendieck abelian category.
The category $\mathrm{DM}(S,R)$ of Nisnevich motivic spectra over S is defined by applying the process of $\mathbf{A}^1$-localization, and then of $\mathbf{P}^1$-stabilization, to the (adequate model category structure corresponding to the) derived category of $\mathrm{Sh}^{tr}_{\mathrm{Nis}}(S,R)$; see [CDa, Def. 11.1.1]. By construction, any smooth S-scheme X defines a (homological) motive $M_S(X)$ in $\mathrm{DM}(S,R)$ which is a compact object. Moreover, the triangulated category $\mathrm{DM}(S,R)$ is generated by Tate twists of such homological motives, i.e. by objects of the form $M_S(X)(n)$ for a smooth S-scheme X, and an integer $n \in \mathbf{Z}$.

Remark 1.2. When $S = \mathrm{Spec}(K)$ is the spectrum of a perfect field, the triangulated category $\mathrm{DM}(S,\mathbf{Z})$ contains as a full and faithful subcategory the category $\mathrm{DM}^{eff}_{-}(K)$ defined in [VSF00, chap. 5]. This follows from the description of $\mathbf{A}^1$-local objects in this case and from the cancellation theorem of Voevodsky [Voe10] (see for example [Dég11, Sec. 4] for more details).

1.3. *The generalized variants.*– This variant is an *enlargement*[3] of the previous context. However, at the same time, one can consider several possible Grothendieck topologies t: the Nisnevich topology $t = \mathrm{Nis}$, the cdh-topology $t = \mathrm{cdh}$, the étale topology $t = \text{ét}$, or the h-topology $t = \mathrm{h}$.
Instead of using the category $Sm^{cor}_{\Lambda,S}$, we consider the larger category $\mathscr{S}^{ft,cor}_{\Lambda,S}$ made by all separated S-schemes of finite type whose morphisms are made by the finite S-correspondences with coefficients in Λ as in the previous paragraph

[2]Recall: a finite S-correspondence from X to Y with coefficients in Λ is an algebraic cycle in $X \times_S Y$ with Λ-coefficients such that:

(1) its support is finite equidimensional over X,
(2) it is a relative cycles over X in the sense of Suslin and Voevodsky (cf. [VSF00, chap. 2]) - equivalently it is a special cycle over X (cf. [CDa, def. 8.1.28]),
(3) it is Λ-universal (cf. [CDa, def. 8.1.48]).

When X is geometrically unibranch, condition (2) is always fulfilled (cf. [CDa, 8.3.26]). When X is regular of the characteristic exponent of any residue field of X is invertible in Λ, condition (3) is always fulfilled (cf. [CDa, 8.3.29] in the first case). Everything gets much simpler when we work locally for the cdh-topology; see [VSF00, Chap. 2, 4.2].

Recall also for future reference this definition makes sense even if X and Y are singular of finite type over S.

[3]See [CDa, 1.4.13] for a general definition of this term.

(see again [CDa, 9.1.8] with $\mathscr{P}$ the class of all separated morphisms of finite type).
Then we can still define the category $\underline{\mathrm{Sh}}_t^{tr}(S,R)$ of generalized t-sheaves with transfers over S as the category of additive presheaves of R-modules over $\mathscr{S}^{ft,cor}_{\Lambda,S}$ whose restriction to $\mathscr{S}^{ft}_S$ is a sheaf for the cdh topology. This is again a well suited Grothendieck abelian category (by which we mean that, using the terminology of [CDa], when we let S vary, we get an abelian premotivic category which is compatible with the topology t; see [CDa, Sec. 10.4]). Moreover we have natural adjunctions:

$$\mathrm{Sh}^{tr}_{\mathrm{Nis}}(S,R) \underset{\rho^*}{\overset{\rho_!}{\rightleftarrows}} \underline{\mathrm{Sh}}^{tr}_{\mathrm{Nis}}(S,R) \overset{a^*_{\mathrm{cdh}}}{\rightleftarrows} \underline{\mathrm{Sh}}^{tr}_{\mathrm{cdh}}(S,R) \tag{1.3.1}$$

where ρ^* is the natural restriction functor and a^*_{cdh} is the associated cdh-sheaf with transfers functor (see *loc. cit.*)
Finally, one defines the category $\underline{\mathrm{DM}}_t(S,R)$ of generalized motivic t-spectra over S and coefficients in R as the triangulated category obtained by $\mathbf{P}^1$-stabilization and $\mathbf{A}^1$-localization of the (adequate model category structure corresponding to the) derived category of $\underline{\mathrm{Sh}}_t^{tr}(S,R)$.
Note that in the generalized context, any S-scheme X defines a (homological) t-motive $M_S(X)$ in $\underline{\mathrm{DM}}_t(S,R)$ which is a compact object and depends covariantly on X. This can even be extended to simplicial S-schemes (although we might then obtain non compact objects). Again, the triangulated category $\underline{\mathrm{DM}}_t(S,R)$ is generated by objects of the form $M_S(X)(n)$ for a smooth S-scheme X and an integer $n \in \mathbf{Z}$.
Thus, we have three variants of motivic spectra. Using the adjunctions (1.3.1) (which are Quillen adjunctions for suitable underlying model categories), one deduces adjunctions made by exact functors as follows:

$$\mathrm{DM}(S,R) \underset{\mathbf{R}\rho^*}{\overset{\mathbf{L}\rho_!}{\rightleftarrows}} \underline{\mathrm{DM}}(S,R) \overset{\mathbf{L}a^*_{\mathrm{cdh}}}{\rightleftarrows} \underline{\mathrm{DM}}_{\mathrm{cdh}}(S,R) \tag{1.3.2}$$

The following assertions are consequences of the model category structures used to get these derived functors:

(1) for any smooth S-scheme X and any integer $n \in \mathbf{Z}$, $\mathbf{L}\rho_!\big(M_S(X)(n)\big) = \underline{M}_S(X)(n)$.
(2) for any S-scheme X and any integer $n \in \mathbf{Z}$, $\mathbf{L}a^*_{\mathrm{cdh}}\big(\underline{M}_S(X)(n)\big) = \underline{M}_S(X)(n)$.

The following proposition is a formal consequence of these definitions:

PROPOSITION 1.4. *The category $\underline{\mathrm{DM}}_{\mathrm{cdh}}(S,R)$ is the localization of $\underline{\mathrm{DM}}(S,R)$ obtained by inverting the class of morphisms of the form:*

$$\underline{M}_S(X_\bullet) \xrightarrow{p_*} \underline{M}_S(X)$$

for any cdh*-hypercover p of any S-scheme X. Moreover, the functor a_{cdh} is the canonical projection functor.*

The definition that will prove most useful is the following one.

Definition 1.5. Let S be any noetherian scheme.
One defines the triangulated category $\mathrm{DM}_{\mathrm{cdh}}(S,R)$ of cdh-motivic spectra, as the full localizing triangulated subcategory of $\underline{\mathrm{DM}}_{\mathrm{cdh}}(S,R)$ generated by motives of the form $\underline{M}_S(X)(n)$ for a smooth S-scheme X and an integer $n \in \mathbf{Z}$.

1.6. These categories for various base schemes S are equipped with a basic functoriality (f^*, f_*, $f_\sharp$ for f smooth, $\otimes$ and $\underline{\mathrm{Hom}}$) satisfying basic properties. In [CDa], we have summarized these properties saying that $\mathrm{DM}(-,R)$ is a *premotivic triangulated category* – see 1.4.2 for the definition and 11.1.1 for the construction.

2. Modules over motivic Eilenberg-MacLane spectra

2.a. Symmetric Tate spectra and continuity.

2.1. Given a scheme X we write Sp_X for the category of symmetric T-spectra, where T denotes a cofibrant resolution of the projective line $\mathbf{P}^1$ over X (with the point at infinity as a base point, say) in the projective model structure of pointed Nisnevich simplicial sheaves of sets. We will consider Sp_X as combinatorial stable symmetric monoidal model category, obtained as the T-stabilization of the $\mathbf{A}^1$-localization of the projective model category structure on the category of pointed Nisnevich simplicial sheaves of sets on the site Sm_X of smooth separated X-schemes of finite type. The corresponding homotopy category

$$\mathrm{Ho}(\mathrm{Sp}_X) = \mathrm{SH}(X)$$

is thus the stable homotopy category of schemes over X, as considered by Morel, Voevodsky and various other authors. This defines a motivic triangulated category in the sense of [CDa]: in other words, thanks to Ayoub's thesis [Ayo07a, Ayo07b], we have the whole formalism of the six operations in SH. We note that the categories $\mathrm{SH}(X)$ can be defined as the homotopy categories of their $(\infty,1)$-categorical counterparts; see [Rob15, 2.3] and [Hoy14, Appendix C].

2.2. In [CDa], we have introduced the notion of continuity for a premotivic category $\mathcal{T}$ which comes from the a premotivic model category. In the sequel, we will need to work in a more slightly general context, in which we do not consider a monoidal structure. Therefore, we will recast the definition of continuity for complete triangulated Sm-fibred categories over Sch (see [CDa, 1.1.12, 1.3.13] for the definitions; in particular, the adjective 'complete' refers to the existence of right adjoints for the pullback functors).
Here Sch will be a full subcategory of the category of schemes stable by smooth base change and $\mathcal{F}$ will be a class of affine morphisms in Sch.[4]

[4]The examples we will use here are: Sch is the category of regular (excellent) k-schemes or the category of all noetherian finite dimensional (excellent) k-schemes; $\mathcal{F}$ is the category of dominant affine morphisms or the category of all affine morphisms.

Definition 2.3. Let $\mathcal{T}$ be a complete triangulated *Sm*-fibred category over *Sch* and c be a small family of cartesian sections $(c_i)_{i\in I}$ of $\mathcal{T}$.
We will say that $\mathcal{T}$ is *c-generated* if, for any scheme X in *Sch*, the family of objects $c_{i,X}$, $i \in I$, form a generating family of the triangulated category. We will then define $\mathcal{T}_c(X)$ as the smallest thick subcategory of $\mathcal{T}(X)$ which contains the elements of of the form $f_\sharp f^*(c_{i,X}) = f_\sharp(c_{i,Y})$, for any separated smooth morphism $f : Y \to X$ and any $i \in I$. The objects of $\mathcal{T}_c(X)$ will be called *c-constructible* (or simply *constructible*, when c is clearly determined by the context).

Remark 2.4. If for any $i \in I$, the objects $c_{i,X}$ are compact, then $\mathcal{T}_c(X)$ is the category of compact objects of $\mathcal{T}(X)$ and so does not depend on c.
When $\mathcal{T}$ has a symmetric monoidal structure, or in other words, is a premotivic category, and if we ask that c is stable by tensor product, then c is what we call a *set of twists* in [CDa, 1.1.d]. This is what happens in practice (e.g. for $\mathcal{T} = \mathrm{SH}$, DM or $\mathrm{DM}_{\mathrm{cdh}}$), and the family c consists of the Tate twist $\mathbb{1}_X(n)$ of the unit object for $n \in \mathbf{Z}$. Moreover, constructible objects coincide with compact objects for SH, DM and $\mathrm{DM}_{\mathrm{cdh}}$.

For short, a $(Sch, \mathcal{F})$-pro-scheme will be a pro-scheme $(S_\alpha)_{\alpha\in A}$ with values in *Sch*, whose transition morphisms are in $\mathcal{F}$, which admits a projective limit S in the category of schemes such that S belongs to *Sch*. The following definition is a slightly more general version of [CDa, 4.3.2].

Definition 2.5. Let $\mathcal{T}$ be a c-generated complete triangulated *Sm*-fibred category over *Sch*.
We say that $\mathcal{T}$ is *continuous with respect to* $\mathcal{F}$, if given any $(Sch, \mathcal{F})$-pro-scheme (X_α) with limit S, for any index α_0, any object E_{α_0} in $\mathcal{T}(X_{\alpha_0})$, and any $i \in I$, the canonical map

$$\varinjlim_{\alpha\geq\alpha_0} \mathrm{Hom}_{\mathcal{T}(X_\alpha)}(c_{i,X_\alpha}, E_\alpha) \to \mathrm{Hom}_{\mathcal{T}(X)}(c_{i,S}, E),$$

is bijective, where E_α is the pullback of E_{α_0} along the transition morphism $X_\alpha \to X_{\alpha_0}$, while E is the pullback of E_{α_0} along the projection $X \to X_{\alpha_0}$

Example 2.6.
(1) The premotivic category SH on the category of noetherian finite dimensional schemes satisfies continuity without restriction (*i.e.* $\mathcal{F}$ is the category of all affine morphisms). This is a formal consequence of [Hoy14, Proposition C.12] and of [Lur09, Lemma 6.3.3.6], for instance.
(2) According to [CDa, 11.1.4], the premotivic triangulated categories DM and $\mathrm{DM}_{\mathrm{cdh}}$, defined over the category of all schemes, are continuous with respect to dominant affine morphisms. (Actually, this example is the only reason why we introduce a restriction on the transition morphisms in the previous continuity property.)

The following proposition is a little variation on [CDa, 4.3.4], in the present slightly generalized context:

Proposition 2.7. *Let $\mathcal{T}$ be a c-generated complete triangulated Sm-fibred category over Sch which is continuous with respect to $\mathcal{F}$. Let (X_α) be a $(Sch, \mathcal{F})$-pro-scheme with projective limit X and let $f_\alpha : X \to X_\alpha$ be the canonical projection.*
For any index α_0 and any objects M_{α_0} and E_{α_0} in $\mathcal{T}(S_{\alpha_0})$, if M_{α_0} is c-constructible, then the canonical map

$$\varinjlim_{\alpha \geq \alpha_0} \operatorname{Hom}_{\mathcal{T}(S_\alpha)}(M_\alpha, E_\alpha) \to \operatorname{Hom}_{\mathcal{T}(S)}(M, E),$$

is bijective, where M_α and E_α are the respective pullbacks of M_{α_0} and E_{α_0} along the transition morphisms $S_\alpha \to S_{\alpha_0}$, while $M = f_{\alpha_0}^*(M_{\alpha_0})$ and $E = f_{\alpha_0}^*(E_{\alpha_0})$. Moreover, the canonical functor:

$$2\text{-}\varinjlim_{\alpha} \mathcal{T}_c(X_\alpha) \xrightarrow{2\text{-}\varinjlim_\alpha (f_\alpha^*)} \mathcal{T}_c(X)$$

is an equivalence of triangulated categories.

The proof is identical to that of *loc. cit.*

Proposition 2.8. *Let $f : X \to Y$ be a regular morphism of schemes. Then the pullback functor*

$$f^* : \mathrm{Sp}_Y \to \mathrm{Sp}_X$$

of the premotivic model category of Tate spectra (relative to simplicial sheaves) preserves stable weak $\mathbf{A}^1$-equivalences as well as $\mathbf{A}^1$-local fibrant objects.

Proof. This property is local in X so that replacing X (resp. Y) by a suitable affine open neighbourhood of any point $x \in X$ (resp. $f(x)$), we can assume that X and Y are affine.
Then, according to Popescu's theorem (as stated in Spivakovsky's article [Spi99, Th. 1.1]), the morphism f can be written as a projective limit of smooth morphisms $f_\alpha : X_\alpha \to Y$. By a continuity argument (in the context of sheaves of sets!), as each functor f_α^* commutes with small limits and colimits, we see that the functor f^* commutes with small colimits as well as with finite limits. These exactness properties imply that the functor f^* preserves stalkwise simplicial weak equivalences. One can also check that, for any Nisnevich sheaves E and F on Sm_Y, the canonical map

$$f^* \underline{\operatorname{Hom}}(E, F) \to \underline{\operatorname{Hom}}(f^*(E), f^*(F)) \tag{2.8.1}$$

is an isomorphism (where $\underline{\operatorname{Hom}}$ denotes the internal Hom of the category of sheaves), at least when E is a finite colimit of representable sheaves. Since the functor f^* preserves projections of the form $\mathbf{A}^1 \times U \to U$, this readily implies that, if L denotes the explicit $\mathbf{A}^1$-local fibrant replacement functor defined in [MV99, Lemma 3.21, page 93], then, for any simplicial sheaf E on Sm_Y, the map $f^*(E) \to f^*(L(E))$ is an $\mathbf{A}^1$-equivalence with fibrant $\mathbf{A}^1$-local codomain. Therefore, the functor f^* preserves both $\mathbf{A}^1$-equivalences and $\mathbf{A}^1$-local fibrant objects at the level of simplicial sheaves. Using the isomorphism (2.8.1), it is easy to see that f^* preserves $\mathbf{A}^1$-local motivic Ω-spectra. Given that one can turn a levelwise $\mathbf{A}^1$-local fibrant Tate spectrum into a motivic Ω-spectrum by a suitable filtered colimit of iterated T-loop space functors, we see that there exists a fibrant replacement functor R in Sp_Y such that, for any Tate spectrum E over Y, the map $f^*(E) \to f^*(R(E))$ is a stable $\mathbf{A}^1$-equivalence with fibrant codomain. This implies that f^* preserves stable $\mathbf{A}^1$-equivalences. □

Corollary 2.9. *Let A be a commutative monoid in Sp_k. Given a regular k-scheme X with structural map $f : X \to \mathrm{Spec}(k)$, let us put $A_X = f^*(R)$. Then, for any k-morphism between regular k-schemes $\varphi : X \to Y$, the induced map $\mathbf{L}\varphi^*(A_Y) \to A_X$ is an isomorphism in $\mathrm{SH}(X)$.*

Proof. It is clearly sufficient to prove this property when $Y = \mathrm{Spec}(k)$, in which case this is a direct consequence of the preceding proposition. □

We will use repeatedly the following easy fact to get the continuity property.

Lemma 2.10. *Let*

$$\varphi^* : \mathcal{T} \rightleftarrows \mathcal{T}' : \varphi_*$$

be an adjunction of complete triangulated Sm-fibred categories. We make the following assumptions:

(i) *There is a small family c of cartesian sections of $\mathcal{T}$ such that $\mathcal{T}$ is c-generated.*
(ii) *The functor φ_* is conservative (or equivalently, $\mathcal{T}'$ is $\varphi^*(c)$-generated; by abuse, we will then write $\varphi^*(c) = c$ and will say that $\mathcal{T}'$ is c-generated).*
(iii) *The functor φ_* commutes with the operation f^* for any morphism $f \in \mathcal{F}$.*

Then, if $\mathcal{T}$ is continuous with respect to $\mathcal{F}$, the same is true for $\mathcal{T}'$.

Proof. Let $c = (c_{i,?})_{i \in I}$. For any morphism $f : Y \to X$ in $\mathcal{F}$, any object $E \in \mathcal{T}'(X)$ and any $i \in I$, one has a canonical isomorphism:

$$\begin{aligned} \mathrm{Hom}_{\mathcal{T}'(Y)}(c_{i,Y}, f^*(E)) &= \mathrm{Hom}_{\mathcal{T}'(Y)}(\varphi^*(c_{i,Y}), f^*(E)) \\ &\simeq \mathrm{Hom}_{\mathcal{T}(Y)}(c_{i,Y}, \varphi_* f^*(E)) \\ &\simeq \mathrm{Hom}_{\mathcal{T}(Y)}(c_{i,Y}, f^* \varphi_*(E)) \,. \end{aligned}$$

This readily implies the lemma. □

Example 2.11. Let Reg_k be the category of regular k-schemes with morphisms all morphisms of k-schemes.
Let $(A_X)_{X \in \mathit{Reg}_k}$ be a cartesian section of the category of commutative monoids in the category of Tate spectra (*i.e.* a strict commutative ring spectrum stable by pullbacks with respect to morphisms in Reg_k). In this case, we have defined in [CDa, 7.2.11] a premotivic model category over Reg_k whose fiber A_X-Mod over a scheme X in Reg_k is the homotopy category of the symmetric monoidal

stable model category of A_X-modules[5] (i.e. of Tate spectra over S, equiped with an action of the commutative monoid A_X). Since Corollary 2.9 ensures that $(A_X)_{X\in Reg_k}$ is a homotopy cartesian section in the sense of [CDa, 7.2.12], according to [CDa, 7.2.13], there exists a premotivic adjunction:

$$L_A : \mathrm{SH} \rightleftarrows A\text{-Mod} : \mathcal{O}_A$$

of triangulated premotivic categories over Reg_k, such that $L_A(E) = A_S \wedge E$ for any spectrum E over a scheme S in Reg_k. Lemma 2.10 ensures that A-Mod is continuous with respect to affine morphisms in Reg_k.

2.b. Motivic Eilenberg-MacLane spectra over regular k-schemes.

2.12. There is a canonical premotivic adjunction:

$$\varphi^* : \mathrm{SH} \rightleftarrows \mathrm{DM} : \varphi_* \tag{2.12.1}$$

(see [CDa, 11.2.16]). It comes from an adjunction of the premotivic model categories of Tate spectra built out of simplicial sheaves of sets and of complexes of sheaves with transfers respectively (see 1.1):

$$\tilde{\varphi}^* : \mathrm{Sp} \rightleftarrows \mathrm{Sp}^{tr} : \tilde{\varphi}_*. \tag{2.12.2}$$

In other words, we have $\varphi^* = \mathbf{L}\tilde{\varphi}^*$ and $\varphi_* = \mathbf{R}\tilde{\varphi}_*$ (strictly speaking, we can construct the functors $\mathbf{L}\tilde{\varphi}^*$ and $\mathbf{R}\tilde{\varphi}_*$ so that these equalities are true at the level of objects). Recall in particular from [CDa, 10.2.16] that the functor $\tilde{\varphi}_*$ is composed by the functor $\tilde{\gamma}_*$ with values in Tate spectra of Nisnevich sheaves of R-modules (without transfers), which *forgets transfers* and by the functor induced by the right adjoint of the Dold-Kan equivalence. We define, for any scheme X:

$$HR_X = \tilde{\varphi}_*(R_X)\,. \tag{2.12.3}$$

This is Voevodsky's motivic Eilenberg-MacLane spectrum over X, originally defined in [Voe98, 6.1]. In the case where $X = \mathrm{Spec}(K)$ for some commutative ring K, we sometimes write

$$HR_K = HR_{\mathrm{Spec}\,K}\,. \tag{2.12.4}$$

According to [CDa, 6.3.9], the functor $\tilde{\gamma}_*$ preserves (and detects) stable $\mathbf{A}^1$-equivalences. We deduce that the same fact is true for $\tilde{\varphi}_*$. Therefore, we have a canonical isomorphism

$$HR_X \simeq \varphi_*(R_X) \simeq \mathbf{R}\tilde{\varphi}_*(R_X)\,.$$

The Tate spectrum HR_X is a commutative motivic ring spectrum in the strict sense (*i.e.* a commutative monoid in the category Sp_X). We denote by HR_X-Mod the homotopy category of HR_X-modules. This defines a fibred triangulated category over the category of schemes; see [CDa, Prop. 7.2.11]. The functor $\tilde{\varphi}_*$ being weakly monoidal, we get a natural structure of a commutative monoid on $\tilde{\varphi}_*(M)$ for any symmetric Tate spectrum with transfers M.

[5] In order to apply this kind of construction, we need to know that the model category of symmetric Tate spectra in simplicial sheaves satisfies the monoid axiom of Schwede and Shipley [SS00]. This is proved explicitely in [Hoy15, Lemma 4.2], for instance.

This means that the Quillen adjunction (2.12.2) induces a Quillen adjunction from the fibred model category of HR-modules to the premotivic model category of symmetric Tate spectra with transfers[6], and thus defines an adjunction

$$t^* : HR_X\text{-Mod} \rightleftarrows \mathrm{DM}(X,R) : t_* \tag{2.12.5}$$

for any scheme X. For any object E of $\mathrm{SH}(X)$, there is a canonical isomorphism $t^*(HR_X \otimes^{\mathbf{L}} E) = \varphi^*(E)$. For any object M of $\mathrm{DM}(X,R)$, when we forget the HR_X-module structure on $t_*(M)$, we simply obtain $\varphi_*(M)$.
Let $f : X \to S$ be a regular morphism of schemes. Then according to Proposition 2.8, $f^* = \mathbf{L}f^*$. In particular, the isomorphism τ_f of $\mathrm{SH}(X)$ can be lifted as a morphism of strict ring spectra:

$$\tilde{\tau}_f : f^*(HR_S) \to HR_X \,. \tag{2.12.6}$$

Let Reg_k be the category of regular k-schemes as in Example 2.11.

Proposition 2.13. *The adjunctions* (2.12.5) *define a premotivic adjunction*

$$t^* : HR\text{-Mod} \rightleftarrows \mathrm{DM}(-,R) : t_*$$

over the category Reg_k of regular k-schemes.

Proof. We already know that this is a an adjunction of fibred categories over Reg_k and that t^* is (strongly) symmetric monoidal. Therefore, it is sufficient to check that t^* commutes with the operations $f_\sharp$ for any smooth morphism between regular k-scheme $f : X \to S$ (via the canonical exchange map). For this, it is sufficient to check what happens on free HR_X-modules (because we are comparing exact functors which preserve small sums, and because the smallest localizing subcategory of HR_X-Mod containing free HR_X-modules is HR_X-Mod). For any object E of $\mathrm{SH}(X)$, we have, by the projection formula in SH, a canonical isomorphism in HZ_S-Mod:

$$\mathbf{L}f_\sharp(HR_X \otimes^{\mathbf{L}} E) \simeq HR_S \otimes^{\mathbf{L}} \mathbf{L}f_\sharp(E)\,.$$

Therefore, formula $t^*(HR_X \otimes^{\mathbf{L}} E) = \varphi^*(E)$ tells us that t^* commutes with $f_\sharp$ when restricted to free HR_X-modules, as required. □

3. Comparison theorem: regular case

The aim of this section is to prove the following result:

Theorem 3.1. *Let R be a ring in which the characteristic exponent of k is invertible. Then the premotivic adjunction of Proposition 2.13 is an equivalence of premotivic categories over Reg_k. In particular, for any regular noetherian scheme of finite dimension X over k, we have a canonical equivalence of symmetric monoidal triangulated categories*

$$HR_X\text{-Mod} \simeq \mathrm{DM}(X,R)\,.$$

[6] The fact that the induced adjunction is a Quillen adjunction is obvious: this readily comes from the fact that the forgetful functor from HR-modules to symmetric Tate spectra preserves and detects weak equivalences as well as fibrations (by definition).

The preceding theorem tells us that the 6 operations constructed on $\mathrm{DM}(-,R)$ in [CDa, 11.4.5], behave appropriately if one restricts to regular noetherian schemes of finite dimension over k:

Corollary 3.2. *Consider the notations of paragraph 2.12.*

(1) *The functors φ^* and φ_* commute with the operations f^*, f_* (resp. $p_!$, $p^!$) for any morphism f (resp. separated morphism of finite type p) between regular k-schemes.*
(2) *The premotivic category* $\mathrm{DM}(-,R)$ *over Reg_k satisfies:*
   - *the localization property;*
   - *the base change formula ($g^*f_! \simeq f'_!g'^*$, with notations of* [CDa, 11.4.5, (4)]*);*
   - *the projection formula ($f_!(M \otimes f^*(N)) \simeq f_!(M) \otimes N$, with notations of* [CDa, 11.4.5, (5)]*).*

Proof. Point (1) follows from the fact the premotivic adjunction $(L_{HR}, \mathcal{O}_{HR})$ satisfies the properties stated for (φ^*, φ_*) and that they are true for (t^*, t_*) because it is an equivalence of premotivic categories, due to Theorem 3.1. The first statement of Point (2) follows from the fact that the localization property over Reg_k holds in HR-Mod, and from the equivalence $HR\text{-Mod} \simeq \mathrm{DM}(-,R)$ over Reg_k; the remaining two statements follow from Point (2) and the fact they are true for SH (see [Ayo07a] in the quasi-projective case and [CDa, 2.4.50] in the general case). ☐

The proof of Theorem 3.1 will be given in Section 3.c (page 165), after a few preparations. But before that, we will explain some of its consequences.

3.3. Let $f : X \to S$ be a morphism of schemes. Since (2.12.1) is an adjunction of fibred categories over the category of schemes, we have a canonical exchange transformation (see [CDa, 1.2.5]):

$$Ex(f^*, \varphi_*) : \mathbf{L}f^*\varphi_* \to \varphi_*\mathbf{L}f^*. \tag{3.3.1}$$

Evaluating this natural transformation on the object $\mathbb{1}_S$ gives us a map:

$$\tau_f : \mathbf{L}f^*(HR_S) \to HR_X.$$

Voevodsky conjectured in [Voe02] the following property:

Conjecture (Voevodsky). The map τ_f is an isomorphism.

When f is smooth, the conjecture is obviously true as $Ex(f^*, \varphi_*)$ is an isomorphism.

Remark 3.4. The preceding conjecture of Voevodsky is closely related to the localization property for DM. In fact, let us also mention the following result which was implicit in [CDa] – as it will not be used in the sequel we leave the proof as an exercise for the reader.[7]

[7]Hint: use the fact that φ_* commutes with $j_\sharp$ ([CDa, 6.3.11] and [CDa, 11.4.1]).

Proposition 3.5. *We use the notations of Par. 3.3. Let $i : Z \to S$ be a closed immersion. Then the following properties are equivalent:*

(i) *The premotivic triangulated category* DM *satisfies the localization property with respect to i (see* [CDa, 2.3.2]*).*
(ii) *The natural transformation $Ex(i^*, \varphi_*)$ is an isomorphism.*

From the case of smooth morphisms, we get the following particular case of the preceding conjecture.

Corollary 3.6. *The conjecture of Voevodsky holds for any morphism $f : X \to S$ of regular k-schemes.*

Proof. By transitivity of pullbacks, it is sufficient to consider the case where $f = p$ is the structural morphism of the k-scheme S, with k a prime field (in particular, with k perfect). Since DM is continuous with respect to projective systems of regular k-schemes with affine transition maps (because this is the case for HR-modules, using Theorem 3.1), we are reduced to the case where S is smooth over k, which is trivial. □

Remark 3.7. The previous result is known to have interesting consequences for the motivic Eilenberg-MacLane spectrum HR_X where X is an arbitrary noetherian regular k-scheme of finite dimension.
For example, we get the following extension of a result of Hoyois on a theorem first stated by Hopkins and Morel (for $p = 1$). Given a scheme X as above, the canonical map

$$MGL_X/(a_1, a_2, \ldots)[1/p] \to H\mathbf{Z}_X[1/p]$$

from the algebraic cobordism ring spectrum modulo generators of the Lazard ring is an isomorphism up to inverting the characteristic exponent of k. This was proved in [Hoy15], for the base field k, or, more generally, for any essentially smooth k-scheme X.
This shows in particular that $H\mathbf{Z}_X[1/p]$ is the universal oriented ring $\mathbf{Z}[1/p]$-linear spectrum over X with additive formal group law.
All this story remains true for arbitrary noetherian k-schemes of finite dimension if we are eager to replace $H\mathbf{Z}_X$ by its cdh-local version: this is one of the meanings of Theorem 5.1 below. Note that, since Spitweck's version of the motivic spectrum has the same relation with algebraic cobordism (see [Spi, Theorem 11.3]), it coincides with the cdh-local version of $H\mathbf{Z}_X$ as well, at least up to p-torsion.

Definition 3.8. Let X be a regular k-scheme with structural map $f : X \to \mathrm{Spec}(k)$. We define the *relative motivic Eilenberg-MacLane spectrum* of X/k by the formula

$$HR_{X/k} = f^*(HR_{\mathrm{Spec}(k)})$$

(where $f^* : \mathrm{Sp}_k \to \mathrm{Sp}_X$ is the pullback functor at the level of the model categories).

Remark 3.9. By virtue of Propositions 2.8 and Corollary 3.6, we have canonical isomorphisms

$$\mathbf{L}f^*(HR_{\mathrm{Spec}\,(k)}) \simeq HR_{X/k} \simeq HR_X\,.$$

Note that, the functor f^* being symmetric monoidal, each relative motivic Eilenberg-MacLane spectrum $HR_{X/k}$ is a commutative monoid in Sp_X. This has the following consequences.

Proposition 3.10. *For any regular k-scheme X, there is a canonical equivalence of symmetric monoidal triangulated categories*

$$HR_{X/k}\text{-Mod} \simeq HR_X\text{-Mod}\,.$$

In particular, the assignment $X \mapsto HR_X$-Mod defines a premotivic symmetric monoidal triangulated category HR-Mod over Reg_k, which is continuous with respect to any projective system of regular k-schemes with affine transition maps.
Moreover the forgetful functor

$$HR\text{-Mod} \to \mathrm{SH}$$

commutes with $\mathbf{L}f^*$ for any k-morphism $f : X \to Y$ between regular schemes, and with $\mathbf{L}f_\sharp$ for any smooth morphism of finite type between regular schemes.

Proof. Since the canonical morphism of commutative monoids $HR_{X/k} \to HR_X$ is a stable $\mathbf{A}^1$-equivalence the first assertion is a direct consequence of [CDa, Prop. 7.2.13]. The property of continuity is a particular case of Example 2.11, with $R_X = HR_{X/k}$. For the last part of the proposition, by virtue of the last assertion of [CDa, Prop. 7.1.11 and 7.2.12] we may replace (coherently) HR_X by a cofibrant monoid R_X (in the model category of monoids in Sp_X), in order to apply [CDa, Prop. 7.2.14]: The forgetful functor from R_X-modules to Sp_X is a left Quillen functor which preserves weak equivalences and commutes with f^* for any map f in Reg_k: therefore, this relation remains true after we pass to the total left derived functors. The case of $\mathbf{L}f_\sharp$ is similar. □

We now come back to the aim of proving Theorem 3.1.

3.a. Some consequences of continuity.

Lemma 3.11. *Consider the cartesian square of schemes below.*

$$\begin{array}{ccc} X' & \xrightarrow{q} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{p} & Y \end{array}$$

We assume that Y' is the projective limit of a projective system of Y-schemes (Y_α) with affine flat transition maps, and make the following assumption. For any index α, if $p_\alpha : Y_\alpha \to Y$ denotes the structural morphism, the base change *morphism associated to the pullback square*

$$\begin{array}{ccc} X_\alpha & \xrightarrow{q_\alpha} & X \\ {\scriptstyle g_\alpha}\downarrow & & \downarrow{\scriptstyle f} \\ Y_\alpha & \xrightarrow{p_\alpha} & Y \end{array}$$

in $\mathrm{DM}(Y_\alpha, R)$ *is an isomorphism:* $\mathbf{R}p_\alpha^* \, \mathbf{R}f_* \simeq \mathbf{R}g_{\alpha,*} \, \mathbf{L}q_\alpha^*$.
Then the base change morphism $\mathbf{L}p^* \, \mathbf{R}f_* \to \mathbf{R}g_* \, \mathbf{L}q^*$ *is invertible in* $\mathrm{DM}(Y', R)$.

Proof. We want to prove that, for any object E of $\mathrm{DM}(X, R)$, the map

$$\mathbf{L}p^* \, \mathbf{R}f_*(E) \to \mathbf{R}g_* \, \mathbf{L}q^*(E)$$

is invertible. For this, it is sufficient to prove that, for any constructible object M of $\mathrm{DM}(Y', R)$, the map

$$\mathrm{Hom}(M, \mathbf{L}p^* \, \mathbf{R}f_*(E)) \to \mathrm{Hom}(M, \mathbf{R}g_* \, \mathbf{L}q^*(E))$$

is bijective. Since $\mathrm{DM}(-, R)$ is continuous with respect to dominant affine morphisms, we may assume that there exists an index α_0 and a constructible object M_{α_0}, such that $M \simeq \mathbf{L}p_{\alpha_0}^*(M_{\alpha_0})$. For $\alpha > \alpha_0$, we will write M_α for the pullback of M_{α_0} along the transition map $Y_\alpha \to Y_{\alpha_0}$. By continuity, we have a canonical identification

$$\varinjlim_\alpha \mathrm{Hom}(M_\alpha, \mathbf{L}p_\alpha^* \, \mathbf{R}f_*(E)) \simeq \mathrm{Hom}(M, \mathbf{L}p^* \, \mathbf{R}f_*(E)) \,.$$

On the other hand, by assumption, we also have:

$$\begin{aligned} \varinjlim_\alpha \mathrm{Hom}(M_\alpha, \mathbf{L}p_\alpha^* \, \mathbf{R}f_*(E)) &\simeq \varinjlim_\alpha \mathrm{Hom}(M_\alpha, \mathbf{R}g_{\alpha,*} \, \mathbf{L}q_\alpha^*(E)) \\ &\simeq \varinjlim_\alpha \mathrm{Hom}(\mathbf{L}g_\alpha^*(M_\alpha), \mathbf{L}q_\alpha^*(E)) \,. \end{aligned}$$

The flatness of the maps $p_{\beta\alpha}$ ensures that the transition maps of the projective system (X_α) are also affine and dominant, so that, by continuity, we get the isomorphisms

$$\begin{aligned} \varinjlim_\alpha \mathrm{Hom}(\mathbf{L}g_\alpha^*(M_\alpha), \mathbf{L}q_\alpha^*(E)) &\simeq \mathrm{Hom}(\mathbf{L}g^*(M), \mathbf{L}q^*(E)) \\ &\simeq \mathrm{Hom}(M, \mathbf{R}g_* \, \mathbf{L}q^*(E)) \,, \end{aligned}$$

and this achieves the proof. □

PROPOSITION 3.12. *Let* $i : Z \to S$ *be a closed immersion between regular* k*-schemes. Assume that* S *is the limit of a projective system of smooth separated* k*-schemes of finite type, with affine flat transition maps. Then* $\mathrm{DM}(-, R)$ *satisfies the localization property with respect to* i *(cf.* [CDa, Def. 2.3.2]*).*

Proof. According to [CDa, 11.4.2], the proposition holds when S is smooth of finite type over k – the assumption then implies that Z is smooth of finite type over k.

According to [CDa, 2.3.18], we have only to prove that for any smooth S-scheme X, putting $X_Z \times_S Z$, the canonical map in $\mathrm{DM}(S, R)$

$$M_S(X/X - X_Z) \to i_*\big(M_Z(X_Z)\big) \tag{3.12.1}$$

is an isomorphism. This property is clearly local for the Zariski topology, so that we can even assume that both X and S are affine.

Lifting the ideal of definition of Z, one can assume that Z lifts to a closed subscheme $i_\alpha : Z_\alpha \hookrightarrow S_\alpha$. We can also assume that i_α is regular (apply [EGA4, 9.4.7] to the normal cone of the i_α). Thus Z_α is smooth over k. Because X/S is affine of finite presentation, it can be lifted to a smooth scheme X_α/S_α and because X/S is smooth we can assume X_α/S_α is smooth.
Put $X_{Z,\alpha} = X_\alpha \times_{S_\alpha} Z_\alpha$. Then, applying localization with respect to i_α, we obtain that the canonical map:

$$M_{S_\alpha}(X_\alpha/X_\alpha - X_{Z,\alpha}) \to i_{\alpha*}\big(M_{Z_\alpha}\big(X_{Z,\alpha}\big)\big) \tag{3.12.2}$$

is an isomorphism in $\mathrm{DM}(S_\alpha, R)$. Of course the analogue of (3.12.2) remains an isomorphism for any $\alpha' > \alpha$. Given $\alpha' > \alpha$, let us consider the cartesian square

$$\begin{array}{ccc} Z_{\alpha'} & \xrightarrow{i_{\alpha'}} & S_{\alpha'} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z_\alpha & \xrightarrow{i_\alpha} & S_\alpha \end{array}$$

in which $f : X_{\alpha'} \to X_\alpha$ denotes the transition map. Then according to [CDa, Prop. 2.3.11(1)], the localization property with respect to i_α and $i_{\alpha'}$ implies that the canonical base change map $f^* i_{\alpha,*} \to i_{\alpha',*} g^*$ is an isomorphism. By virtue of Lemma 3.11, if $\varphi : S \to S_\alpha$ denote the canonical projection, the pullback square

$$\begin{array}{ccc} Z & \xrightarrow{i} & S \\ {\scriptstyle \psi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ Z_\alpha & \xrightarrow{i_\alpha} & S_\alpha \end{array}$$

induces a base change isomorphism $\mathbf{L}\varphi^* i_{\alpha,*} \to i_* \mathbf{L}\psi^*$. Therefore, the image of the map (3.12.2) by $\mathbf{L}\varphi^*$ is isomorphic to the map (3.12.1), and this ends the proof. □

3.b. Motives over fields. This section is devoted to prove Theorem 3.1 when one restricts to field extensions of k:

Proposition 3.13. *Consider the assumptions of 3.1 and let K be an extension field of k. Then the functor*

$$t^* : HR_K\text{-Mod} \to \mathrm{DM}(K, R)$$

is an equivalence of symmetric monoidal triangulated categories.

In the case where K is a perfect field, this result is proved in [HKØ, 5.8] in a slightly different theoretical setting. The proof will be given below (page 164), after a few steps of preparation.

3.14. In the end, the main theorem will prove the existence of very general trace maps, but the proof of this intermediate result requires that we give a preliminary construction of traces in the following case.
Let K be an extension field of k, and $f : Y \to X$ be a flat finite surjective morphism of degree d between integral K-schemes. There is a natural morphism ${}^tf : R_X \to f_\sharp(R_Y)$ in $\underline{\mathrm{DM}}(X,R)$, defined by the transposition of the graph of f. The composition

$$f_\sharp(R_Y) \to R_X \xrightarrow{{}^tf} f_\sharp(R_Y)$$

is d times the identity of $f_\sharp(R_Y)$; see [CDa, Prop. 9.1.13]. Moreover, if f is radicial (i.e. if the field of functions on Y is a purely inseparable extension of the field of functions of X), then the composition

$$R_X \xrightarrow{{}^tf} f_\sharp(R_Y) \xrightarrow{f} R_X$$

is d times the identity of R_X; see [CDa, Prop. 9.1.14]. In other words, in the latter case, since p is invertible, the co-unit map $f_\sharp(R_Y) \to R_X$ is an isomorphism in $\underline{\mathrm{DM}}(X,R)$.

Lemma 3.15. *Under the assumptions of the previous paragraph, if f is radicial, then the pullback functor*

$$\mathbf{L}f^* : \mathrm{DM}(X,R) \to \mathrm{DM}(Y,R)$$

is fully faithful.

Proof. As the inclusion $\mathrm{DM}(-,R) \subset \underline{\mathrm{DM}}(-,R)$ is fully faithful and commutes with $\mathbf{L}f^*$, it is sufficient to prove that the functor

$$f^* : \underline{\mathrm{DM}}(X,R) \to \underline{\mathrm{DM}}(Y,R)$$

is fully faithful. In other words, we must see that the composition of f^* with its left adjoint $f_\sharp$ is isomorphic to the identity functor (through the co-unit of the adjunction). For any object M of $\underline{\mathrm{DM}}(X,R)$, we have a projection formula:

$$f_\sharp f^*(M) \simeq f_\sharp(R_Y) \otimes^{\mathbf{L}}_R M\,.$$

Therefore, it is sufficient to check that the co-unit

$$f_\sharp(R_Y) \simeq R_X$$

is an isomorphism. Since f is radicial, its degree must be a power of p, hence must be invertible in R. An inverse is provided by the map ${}^tf : R_X \to f_\sharp(R_Y)$. □

3.16. These computations can be interpreted in terms of HR-modules as follows (we keep the assumptions of 3.14).
Using the internal Hom of $\underline{\mathrm{DM}}(X,R)$, one gets a morphism

$$Tr_f : \mathbf{R}f_*(R_Y) \to R_X$$

Since the right adjoint of the inclusion $\mathrm{DM}(-,R) \subset \underline{\mathrm{DM}}(-,R)$ commutes with $\mathbf{R}f_*$, the map Tr_f above can be seen as a map in $\mathrm{DM}(X,R)$. Similarly, since the functor $t_* : \mathrm{DM}(-,R) \to HR\text{-Mod}$ commutes with $\mathbf{R}f_*$, we get a trace morphism

$$Tr_f : \mathbf{R}f_* HR_Y \to HR_X$$

in HR_X-Mod. For any HR_X-module E, we obtain a trace morphism

$$Tr_f : \mathbf{R}f_*\mathbf{L}f^*(E) \to E$$

as follows. Since we have the projection formula

$$\mathbf{R}f_*(HR_Y) = \mathbf{R}f_*\mathbf{L}f^*(HR_X) \simeq \mathbf{R}f_*(\mathbb{1}_Y) \otimes^{\mathbf{L}} HR_X\,,$$

the unit $\mathbb{1}_X \to HR_X$ induces a map

$$\widetilde{Tr}_f : \mathbf{R}f_*(\mathbb{1}_Y) \to \mathbf{R}f_*(\mathbb{1}_Y)\otimes^{\mathbf{L}} HR_X \simeq \mathbf{R}f_*\mathbf{L}f^*(HR_X) \simeq \mathbf{R}f_*(HR_Y) \overset{Tr_f}{\to} HR_X.$$

For any HR_X-module E, tensoring the map $\widetilde{Tr}_f$ with identity of E and composing with the action $HR_X \otimes^{\mathbf{L}} E \to E$ leads to a canonical morphism in HR_X-Mod:

$$Tr_f : \mathbf{R}f_*\mathbf{L}f^*(E) \simeq \mathbf{R}f_*(\mathbb{1}_Y) \otimes^{\mathbf{L}} E \to E\,.$$

By construction of these trace maps, we have the following lemma.

Lemma 3.17. *Under the assumptions of paragraph 3.14, for any HR_X-module E, the composition of Tr_f with the unit of the adjunction between $\mathbf{L}f^*$ and $\mathbf{R}f_*$*

$$E \to \mathbf{R}f_*\mathbf{L}f^*(E) \xrightarrow{Tr_f} E$$

is d times the identity of E. If, moreover, f is radicial, then the composition

$$\mathbf{R}f_*\mathbf{L}f^*(E) \xrightarrow{Tr_f} E \to \mathbf{R}f_*\mathbf{L}f^*(E)$$

is also d times the identity of $\mathbf{R}f_*\mathbf{L}f^*(E)$.

This also has consequences when looking at the HR_K-modules associated to X and Y. To simplify the notations, we will write

$$HR(U) = HR_K \otimes^{\mathbf{L}} \Sigma^\infty(U_+)$$

for any smooth K-scheme U.

Lemma 3.18. *Under the assumptions of paragraph 3.14, if d is invertible in R, and if both X and Y are smooth over K, then $HR(X)$ is a direct factor of $HR(Y)$ in HR_K-Mod.*

Proof. Let $p : X \to \mathrm{Spec}(K)$ and $q : Y \to \mathrm{Spec}(K)$ be the structural maps of X and Y, respectively. Since $pf = q$, for any HR_K-module E, we have:

$$\mathrm{Hom}(HR(X), E) = \mathrm{Hom}(HR_X, p^*(E))$$
$$\mathrm{Hom}(HR(Y), E) = \mathrm{Hom}(HR_X, \mathbf{R}f_*\mathbf{L}f^*p^*(E))\,.$$

Therefore, this lemma is a translation of the first assertion of Lemma 3.17 and of the Yoneda Lemma. □

Proof of Proposition 3.13. We first consider the case of a perfect field K. The reference is [HKØ, 5.8]. We use here a slightly different theoretical setting than these authors so we give a proof to convince the reader.
Because t^* preserves the canonical compact generators of both categories, we need only to prove it is fully faithful on a family of compact generators of HR_K-Mod (see [CDa, Corollary 1.3.21]). For any HR_K-modules E, F belonging to a suitable generating family of HR_K-Mod, and and any integer n, we want to prove that the map

$$\mathrm{Hom}_{HR_K\text{-Mod}}(E, F[n]) \xrightarrow{t^*} \mathrm{Hom}_{\mathrm{DM}(K,R)}(t^*(E), t^*(F)[n]) \tag{3.18.1}$$

For this purpose, using the method of [Rio05, Sec. 1], with a small change indicated below, we first prove that HR_K-Mod is generated by objects of the form $HR(X)(i)$ for a smooth projective K-scheme X and an integer i. Since these are compact, it is sufficient to prove the following property: for any HR_K-module M such that

$$\mathrm{Hom}_{HR_K\text{-Mod}}(HR(X)(p)[q], M) = 0$$

for any integers p and q, we must have $M \simeq 0$. To prove the vanishing of M, it is sufficient to prove the vanishing of $M \otimes \mathbf{Z}_{(\ell)}$ for any prime $\ell \neq p$. On the other hand, for any compact object C, the formation of $\mathrm{Hom}(C, -)$ commutes with tensoring by $\mathbf{Z}_{(\ell)}$; therefore, we may assume R to be a $\mathbf{Z}_{(\ell)}$-algebra for some prime number $\ell \neq p$. Under this additional assumption, we will prove that, for any smooth connected K-scheme X, the object $HR(X) = HR_k \otimes^{\mathbf{L}} \Sigma^\infty(X_+)$ is in the thick subcategory $\mathcal{P}$ generated by Tate twists of HR_K-modules of the form $HR(W)$ for W a smooth projective K-scheme. Using the induction principle explained by Riou in *loc. cit.*, on the dimension d of X, we see that, given any couple (Y, V), where Y is a smooth K-scheme of dimension d, and V is a dense open subscheme of Y, the property that $HR(Y)$ belongs to $\mathcal{P}$ is equivalent to the property $HR(V)$ belongs to $\mathcal{P}$. Therefore, it is enough to consider the case of a dense open subscheme of X which we can shrink at will. In particular, applying Gabber's theorem [ILO14, IX, 1.1], we can assume there exists a flat, finite, and surjective morphism, $f : Y \to X$ which is of degree prime to ℓ, and such that Y is a dense open subscheme of a smooth projective k-scheme. Since $HR(Y) \in \mathcal{P}$, Lemma 3.18 concludes.
We now are reduced to prove that the map (3.18.1) is an isomorphism when $E = HR(X)(i)$ and $F = HR(Y)(j)$ for X and Y smooth and projective over K. Say d is the dimension of Y. Then according to [Dég08a, Sec. 5.4], $HR_K(Y)$ is strongly dualizable with strong dual $HR_K(Y)(-d)[-2d]$. Then the result follows from the fact that the two members of (3.18.1) compute the motivic cohomology group of $X \times_K Y$ in degree $(n - 2d, j - i - d)$ (in a compatible way, because the functor t^* is symmetric monoidal). This achieves the proof of Proposition 3.13 in the case where the ground field K is perfect.

Let us now consider the general case. Again, we are reduced to prove the map (3.18.1) is an isomorphism whenever E and F are compact (hence constructible). Let K be a finite extension of k, and let L/K be a finite totally inseparable extension of fields, with corresponding morphism of schemes $f : \mathrm{Spec}(L) \to \mathrm{Spec}(K)$. According to Lemma 3.15, the functor $\mathbf{L}f^* : \mathrm{DM}(K,R) \to \mathrm{DM}(L,R)$ is fully faithful. Moreover, the pullback functor $\mathbf{L}f^* : HR_K\text{-Mod} \to HR_L\text{-Mod}$ is fully faithful as well; see the last assertion of Lemma 3.17 (and recall that the degree of the extension L/K must be a power of p, whence is invertible in R). Thus, by continuity of the premotivic categories $\mathrm{DM}(-,R)$ and HR-Mod (see Examples 2.6(2) and 2.11), Proposition 2.7 gives the following useful lemma:

Lemma 3.19. *Let K^s be the inseparable closure of K (i.e. the biggest purely inseparable extension of K in some algebraic closure of K). Then the following pullback functors:*

$$\mathrm{DM}_c(K,R) \to \mathrm{DM}_c(K^s,R) \quad \textit{and} \quad HR_K\text{-Mod}_c \to HR_{K^s}\text{-Mod}_c$$

are fully faithful.

With this lemma in hands, to prove that (3.18.1) is an isomorphism for constructible HR_K-modules E and F, we can replace the field K by the perfect field K^s, and this proves Proposition 3.13 in full generality. □

3.c. Proof in the regular case. In the course of the proof of Theorem 3.1, we wil use the following lemma:

Lemma 3.20. *Let T and S be regular k-schemes and $f : T \to S$ be a morphism of k-schemes.*

1. *If T is the limit of a projective system of S-schemes with dominant affine smooth transition morphisms, then t_* commutes with f^*.*
2. *If f is a closed immersion, and if S is the limit of a projective system of smooth separated k-schemes of finite type with flat affine transition morphims, then t_* commutes with f^*.*
3. *If f is an open immersion, then t_* commutes with $f_!$.*

Proof. The forgetful functor $\mathcal{O}_{HR} : HR\text{-Mod} \to \mathrm{SH}$ is conservative, and it commutes with f^* for any morphism f and with $j_!$ for any open immersion; see the last assertion of [CDa, Prop. 7.2.14]. Therefore, it is sufficient to check each case of this lemma after replacing t_* by φ_*.
Then, case (1) follows easily by continuity of DM and SH with respect to dominant maps, and from the case where f is a smooth morphism. Case (2) was proved in Proposition 3.12. (taking into account 3.5). Then case (3) finally follows from results of [CDa]: in fact φ_* is defined as the following composition:

$$\mathrm{DM}(S,R) \xrightarrow{\mathbf{L}\gamma_*} \mathrm{D}_{\mathbf{A}^1}(S,R) \xrightarrow{K} \mathrm{SH}(S)$$

with the notation of [CDa, 11.2.16] ($\Lambda = R$). The fact K commutes with $j_!$ is obvious and for $\mathbf{L}\gamma_*$, this is [CDa, 6.3.11]. □

To be able to use the refined version of Popescu's theorem proved by Spivakovsky (see [Spi99, Th. 10.1], "resolution by smooth sub-algebras"), we will need the following esoteric tool extracted from an appendix of Bourbaki (see [Bou93, IX, Appendice] and, in particular, Example 2).

Definition 3.21. Let A be a local ring with maximal ideal $\mathfrak{m}$.
We define the *∞-gonflement* (resp. *n-gonflement*) of A as the localization of the polynomial A-algebra $A[(x_i)_{i\in\mathbf{N}}]$ (resp. $A[(x_i)_{0\leq i\leq n}]$) with respect to the prime ideal $\mathfrak{m}.A[x_i, i\in\mathbf{N}]$ (resp. $\mathfrak{m}.A[x_i, 0\leq i\leq n]$).

3.22. Let B (resp. B_n) be the ∞-gonflement (resp. n-gonflement) of a local noetherian ring A. We will use the following facts about this construction, which are either obvious or follow from *loc. cit.*, Prop. 2:

(1) The rings B and B_n are noetherian.
(2) The A-algebra B_n is the localization of a smooth A-algebra.
(3) The canonical map $B_n \to B_{n+1}$ is injective.
(4) $B = \varinjlim_{n\in\mathbf{N}} B_n$, with the obvious transition maps.

We will need the following easy lemma:

Lemma 3.23. *Consider the notations above. Assume that A is a local henselian ring with infinite residue field. Then for any integer $n\geq 0$, the A-algebra B_n is a filtered inductive limit of its smooth and split sub-A-algebras.*

Proof. We know that B_n is the union of A-algebras of the form $A[x_1,\dots,x_n][1/f]$ for a polynomial $f\in A[x_1,...,x_n]$ whose reduction modulo $\mathfrak{m}$ is non zero. Let us consider the local scheme $X=\mathrm{Spec}(A)$, s be its closed point and put $U_n(f)=\mathrm{Spec}(A[x_1,\dots,x_n][1/f])$ for a polynomial f as above. To prove the lemma, it is sufficient to prove that $U_n(f)/X$ admits a section. By definition, the fiber $U_n(f)_s$ of $U_n(f)$ at the point s is a non empty open subscheme. As $\kappa(s)$ is infinite by assumption, $U_n(f)_s$ admits a $\kappa(s)$-rational point. Thus $U_n(f)$ admits an S-point because X is henselian and $U_n(f)/X$ is smooth (see [EGA4, 18.5.17]). □

Combining properties (1)-(4) above with the preceding lemma, we get the following property:

(G) Let A be a noetherian local henselian ring with infinite residue field, and B be its ∞-gonflement. Then B is a noetherian A-algebra which is the filtering union of a family $(B_\alpha)_{\alpha\in I}$ of smooth split sub-A-algebras of B.

Lemma 3.24. *Consider the notations of property (G). Then the pullback along the induced map $p: X'=\mathrm{Spec}(B)\to X=\mathrm{Spec}(A)$ defines a conservative functor $\mathbf{L}p^*:\mathrm{SH}(X)\to\mathrm{SH}(X')$.*

Proof. Let E be an object of $\mathrm{SH}(X)$ such that $\mathbf{L}p^*(E)=0$ in $\mathrm{SH}(X')$. We want to prove that $E=0$. For this, it is sufficient to prove that, for any constructible object C of $\mathrm{SH}(X)$, we have

$$\mathrm{Hom}(C,E)=0\,.$$

Given the notations of property (G), and any index $\alpha \in I$, let C_i and E_i be the respective pullbacks of C and E along the structural map $p_\alpha : \mathrm{Spec}(B_\alpha) \to \mathrm{Spec}(A)$. Then, by continuity, the map

$$\varinjlim_{\alpha} \mathrm{Hom}(C_\alpha, E_\alpha) \to \mathrm{Hom}(\mathbf{L}p^*(C), \mathbf{L}p^*(E))$$

is an isomorphism, and thus, according to property (G), the map

$$\mathrm{Hom}(C, E) \to \mathrm{Hom}(\mathbf{L}p^*(C), \mathbf{L}p^*(E))$$

is injective because each map p_α is a split epimorphism. □

In order to use ∞-gonflements in HR-modules without any restriction on the size of the ground field, we will need the the following trick, which makes use of transfers up to homotopy:

Lemma 3.25. *Let L/K be a purely transcendental extension of fields of transcendence degree 1, with K perfect, and let $p : \mathrm{Spec}(L) \to \mathrm{Spec}(K)$ be the induced morphism of schemes. Then, for any objects M and N of $\mathrm{DM}(K, R)$, if M is compact, then the natural map*

$$\mathrm{Hom}_{\mathrm{DM}(K,R)}(M, N) \to$$
$$\to \mathrm{Hom}_{\mathrm{DM}(K,R)}(M, \mathbf{R}p_*\, p^*(N)) = \mathrm{Hom}_{\mathrm{DM}(K,R)}(\mathbf{L}p^*(M), \mathbf{L}p^*(N))$$

is a split embedding. In particular, the pullback functor

$$\mathbf{L}p^* : \mathrm{DM}(K, R) \to \mathrm{DM}(L, R)$$

is conservative.

Proof. Let I be the cofiltering set of affine open neighbourhoods of the generic point of $\mathbf{A}^1_K$ ordered by inclusion. Obviously, $\mathrm{Spec}(L)$ is the projective limit of these open neighbourhoods. Thus, using continuity for DM with respect to dominant maps, we get that:

$$\mathrm{Hom}(M, \mathbf{R}p_*\, \mathbf{L}p^*(N)) = \varinjlim_{V \in I^{op}} \mathrm{Hom}(M(V), \underline{\mathrm{Hom}}(M, N))\,.$$

We will use the language of generic motives from [Dég08b]. Recall that $M(L) =$ "$\varprojlim M(V)$" is a pro-motive in $\mathrm{DM}(K)$, so that the preceding identification now takes the following form.

$$\mathrm{Hom}(M, \mathbf{R}p_*\, \mathbf{L}p^*(N)) = \mathrm{Hom}(M(L), \underline{\mathrm{Hom}}(M, N))\,.$$

Since, according to [Dég08b, Cor. 6.1.3], the canonical map $M(L) \to M(K)$ is a split epimorphism of pro-motives, this proves the first assertion of the lemma. The second assertion is a direct consequence of the first and of the fact that the triangulated category $\mathrm{DM}(K, R)$ is compactly generated. □

Proof of Theorem 3.1. We want to prove that for a regular noetherian k-scheme of finite dimension S, the adjunction:

$$t^* : HR_S\text{-Mod} \rightleftarrows \mathrm{DM}(S, R) : t_*$$

is an equivalence of triangulated categories. Since the functor t^* preserves compact objects, and since there is a generating family of compact objects of $\mathrm{DM}(S,R)$ in the essential image of the functor t^*, it is sufficient to prove that t^* is fully faithful on compact objects (see [CDa, Corollary 1.3.21]): we have to prove that, for any compact HR_S-module M, the adjunction map $\eta_M : M \to t_*t^*(M)$ is an isomorphism.

First case: We first assume that S is essentially smooth – *i.e.* the localization of a smooth k-scheme. We proceed by induction on the dimension of S. The case of dimension 0 follows from Proposition 3.13.
We recall that the category HR-Mod is continuous on Reg_k (3.10). Let x be a point of S and S_x be the localization of S at x, $p_x : S_x \to S$ the natural projection. Then it follows from [CDa, Prop. 4.3.9] that the family of functors:

$$p_x^* : HR_S\text{-Mod} \to HR_{S_x}\text{-Mod}, x \in S$$

is conservative.
Since p_x^* commutes with t^* (trivial) and with t_* (according to Lemma 3.20), we can assume that S is a local essentially smooth k-scheme.
To prove the induction case, let i (resp. j) be the immersion of the closed point x of S (resp. of the open complement U of the closed point of S). Since the localization property with respect to i is true in HR-Mod (because it is true in SH, using the last assertions of Proposition 3.10) and in DM (because of Proposition 3.12 that we can apply because we have assumed that S is essentially smooth), we get two morphisms of distinguished triangles:

$$\begin{array}{ccccccc}
j_!j^*(M) & \longrightarrow & M & \longrightarrow & i_*i^*(M) & \longrightarrow & j_!j^*(M)[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
j_!j^*(t_*t^*(M)) & \to & t_*t^*(M) & \to & i_*i^*(t_*t^*(M)) & \to & j_!j^*(t_*t^*(M))[1] \\
\downarrow\wr & & \| & & \downarrow\wr & & \downarrow\wr \\
j_!t_*t^*j^*(M) & \longrightarrow & t_*t^*(M) & \longrightarrow & i_*t_*t^*i^*(M) & \longrightarrow & j_!t_*t^*j^*(M)[1]
\end{array}$$

The vertical maps on the second floor are isomorphisms: both functors t^* and t_* commute with j^* (as t^* is the left adjoint in a premotivic adjunction, it commutes with $j_!$ and j^*, and this implies that t_* commutes with j^*, by transposition); the functor t^* commutes with i_* because it commutes with $j_!$, j^* and i^*, and because the localization property with respect to i is verified in HR-Mod as well as in DM); finally, applying the third assertion of Lemma 3.20 for $f = j$, this implies that the functor t_* commutes with i^*. To prove that the map η_M is an isomorphism, it is thus sufficient to treat the case of $j_!\eta_{j^*(M)}$ and of $i_*\eta_{i^*(M)}$. This means we are reduced to the cases of U and $\mathrm{Spec}(\kappa(x))$, which follow respectively from the inductive assumption and from the case of dimension zero.

General case: Note that the previous case shows in particular the theorem for any smooth k-scheme. Assume now that S is an arbitrary regular noetherian k-scheme. Using [CDa, Prop. 4.3.9] again, and proceeding as we already did above (but considering limits of Nisnevich neighbourhoods instead of Zariski ones), we may assume that S is henselian. Let $L = k(t)$ be the field of rational functions, and let us form the following pullback square.

$$\begin{array}{ccc} S' & \xrightarrow{q} & S \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(k(t)) & \xrightarrow{p} & \mathrm{Spec}(k) \end{array}$$

Then the functor

$$\mathbf{R}p_* \, \mathbf{L}p^* : HR_k\text{-Mod} \to HR_k\text{-Mod}$$

is conservative: this follows right away from Lemma 3.25 and Proposition 3.13. This implies that the functor

$$\mathbf{L}q^* : HR_S\text{-Mod} \to HR_{S'}\text{-Mod}$$

is conservative. To see this, let us consider an object E of HR_S-Mod such that $\mathbf{L}q^*(E) = 0$. To prove that $E = 0$, it is sufficient to prove that $\mathrm{Hom}(M, E) = 0$ for any compact object M of HR_S-Mod. Formula

$$\mathrm{Hom}(HR_k, \mathbf{R}f_* \, \underline{\mathrm{Hom}}(M, E)) \simeq \mathrm{Hom}(M, E)$$

implies that it is sufficient to check that $\mathbf{R}f_* \, \underline{\mathrm{Hom}}(M, E) = 0$ for any compact object M (where $\underline{\mathrm{Hom}}$ is the internal Hom of HR_S-Mod).
Since the functor $\mathbf{R}p_* \, \mathbf{L}p^*$ is conservative, it is thus sufficient to prove that

$$\mathbf{R}p_* \, \mathbf{L}p^* \mathbf{R}f_* \, \underline{\mathrm{Hom}}(M, E) = 0\,.$$

We thus conclude with the following computations (see [CDa, Propositions 4.3.11 and 4.3.14]).

$$\begin{aligned} \mathbf{R}p_* \, \mathbf{L}p^* \mathbf{R}f_* \, \underline{\mathrm{Hom}}(M, E) &\simeq \mathbf{R}p_* \, \mathbf{R}g_* \, \mathbf{L}q^* \, \underline{\mathrm{Hom}}(M, E) \\ &\simeq \mathbf{R}p_* \, \mathbf{R}g_* \, \underline{\mathrm{Hom}}(\mathbf{L}q^*(M), \mathbf{L}q^*(E)) = 0 \end{aligned}$$

In conclusion, since the functor $\mathbf{L}q^*$ commutes with t_* (see Lemma 3.20 (1)), we may replace S by S' and thus assume that the residue field of S is infinite. Let B be the ∞-gonflement of $A = \Gamma(S, \mathcal{O}_S)$ (Definition 3.21), and $f : T = \mathrm{Spec}(B) \to S$ be the map induced by the inclusion $A \subset B$. We know that the functor

$$\mathbf{L}f^* : HR_S\text{-Mod} \to HR_T\text{-Mod}$$

is conservative: as the forgetful functor HR-Mod $\to$ SH is conservative and commutes with $\mathbf{L}f^*$, this follows from Lemma 3.24 (or one can reproduce the proof of this lemma, which only used the continuity property of SH with respect to projective systems of schemes with dominant affine transition morphisms). Similarly, it follows again from Lemma 3.20 (1) that the functor t_* commutes with $\mathbf{L}f^*$. As the functor t^* commutes with $\mathbf{L}f^*$, it is sufficient to prove that the functor t^* is fully faithful over T, and it is still sufficient to check this property on compact objects. Since the ring B is noetherian and regular, and has a field of functions with infinite transcendance degree over the perfect field k (see 3.22), it follows from Spivakovsky's refinement of Popescu's Theorem [Spi99, 10.1] that B is the filtered union of its smooth subalgebras of finite type over k. In other terms, T is the projective limit of a projective system of smooth affine k-schemes of finite type (T_α) with dominant transition maps. Therefore, by continuity (see Examples 2.11 and 2.6(2)), we can apply Proposition 2.7 twice and see that the functor

$$2\text{-}\varinjlim_{\alpha} HR_{T_\alpha}\text{-Mod}_c \simeq HR_T\text{-Mod}_c \to 2\text{-}\varinjlim_{\alpha} \mathrm{DM}_c(T_\alpha, R) \simeq \mathrm{DM}_c(T, R)$$

is fully faithful, as a filtered 2-colimit of functors having this property. □

4. More modules over motivic Eilenberg-MacLane spectra

4.1. Given a scheme X, let $Mon(X)$ be the category of unital associative monoids in the category of symmetric Tate spectra Sp_X. The forgetful functor

$$U : Mon(X) \to \mathrm{Sp}_X$$

has a left adjoint, the free monoid functor:

$$F : \mathrm{Sp}_X \to Mon(X)\,.$$

Since the stable model category of symmetric Tate spectra satisfies the monoid axiom (see [Hoy15, Lemma 4.2]), by virtue of a well known theorem of Schwede and Shipley [SS00, Theorem 4.1(3)], the category $Mon(X)$ is endowed with a combinatorial model category structure whose weak equivalences (fibrations) are the maps whose image by U are weak equivalences (fibrations) in Sp_X; furthermore, any cofibrant monoid is also cofibant as an object of Sp_X.

4.2. We fix once and for all a cofibrant resolution

$$HR' \to HR_k$$

of the motivic Eilenberg-MacLane spectrum HR_k in the model category $Mon(k)$. Given a k-scheme X with structural map $f : X \to \mathrm{Spec}(k)$, we define

$$HR_{X/k} = f^*(HR')$$

(where f^* denotes the pullback functor in the premotivic model category Sp). The family $(HR_{X/k})_X$ is a cartesian section of the Sm-fibred category of monoids in Sp which is also homotopy cartesian (as we have an equality $\mathbf{L}f^*(HR_k) = HR_{X/k}$). We write $HR_{X/k}$-Mod for the homotopy category of (left) $HR_{X/k}$-modules.

This notation is in conflict with the one introduced in Definition 3.8. This conflict disappears up to weak equivalence[8]: when X is regular, the comparison

[8] In the proof of Theorem 3.1, we used the fact that the spectra $HR_{X/k}$, as defined in Definition 3.8, are *commutative* monoids of the model category of symmetric Tate spectra (because we used Poincaré duality in an essential way, in the case where X is the spectrum of a perfect field). This new version of motivic Eilenberg-MacLane spectra $HR_{X/k}$ is not required to be commutative anymore (one could force this property by working with fancier model categories of motivic spectra (some version of the 'positive model structure', as discussed in [Hor13] for instance), but these extra technicalities are not necessary for our purpose. We shall use Theorem 3.1 in a crucial way, though.

map

$$f^*(HR') \to f^*(HR_k)$$

is a weak equivalence (Proposition 2.8). For X regular, $HR_{X/k}$ is thus a cofibrant resolution of HR_X in the model category $Mon(X)$. In particular, in the case where X is regular, we have a canonical equivalence of triangulated categories:

$$HR_{X/k}\text{-Mod} \simeq HR_X\text{-Mod}\,.$$

Proposition 4.3. *The assignment $X \mapsto HR_{X/k}$-Mod defines a motivic category over the category of noetherian k-schemes of finite dimension which has the property of continuity with respect to arbitrary projective systems with affine transition maps. Moreover, when we let X vary, both the free $HR_{X/k}$-algebra (derived) functor*

$$L_{HR_{X/k}} : \mathrm{SH}(X) \to HR_{X/k}\text{-Mod}$$

and its right adjoint

$$\mathcal{O}_{HR_{X/k}} : HR_{X/k}\text{-Mod} \to \mathrm{SH}(X)$$

are morphisms of premotivic triangulated categories over the category of k-schemes. In other words both functors commute with $\mathbf{L}f^*$ for any morphism of k-schemes f, and with $\mathbf{L}g_\sharp$ for any separated smooth morphism of k-schemes g.

Proof. The first assertion comes from [CDa, 7.2.13 and 7.2.18], the one about continuity is a direct application of Lemma 2.10, and the last one comes from [CDa, 7.2.14]. □

Remark 4.4. Since the functor $\mathcal{O}_{HR_{X/k}} : HR_{X/k}\text{-Mod} \to \mathrm{SH}(X)$ is conservative and preserves small sums, the family of objects of the form $HR_{X/k} \otimes^{\mathbf{L}} \Sigma^\infty(Y_+)(n)$, for any separated smooth X-scheme Y and any integer n, do form a generating family of compact objects. In particular, the notions of constructible object and of compact object coincide in $HR_{X/k}$-Mod (see for instance [CDb, Remarks 5.4.10 and 5.5.11], for a context in which these two notions fail to coincide).

4.5. For any k-scheme X, we have canonical morphisms of monoids in Sp_X:

$$HR_{X/k} \to f^*(HR_k) \to HR_X\,.$$

In particular, we have a canonical functor

$$HR_{X/k}\text{-Mod} \to HR_X\text{-Mod}\,, \quad E \mapsto HR_X \otimes^{\mathbf{L}}_{HR_{X/k}} E\,.$$

If we compose the latter with the functor

$$HR_X\text{-Mod} \xrightarrow{t^*} \mathrm{DM}(X,R) \xrightarrow{\mathbf{L}\rho_!} \underline{\mathrm{DM}}(X,R) \xrightarrow{a^*_{\mathrm{cdh}}} \underline{\mathrm{DM}}_{\mathrm{cdh}}\,,$$

we get a functor

$$HR_{X/k}\text{-Mod} \to \underline{\mathrm{DM}}(X,R)$$

which defines a morphism a premotivic categories. In particular, this functor takes it values in $\mathrm{DM}_{\mathrm{cdh}}(X, R)$, and we obtain a functor

$$\tau^* : HR_{X/k}\text{-Mod} \to \mathrm{DM}_{\mathrm{cdh}}(X, R)\,.$$

As τ^* preserves small sums, it has a right adjoint τ_*, and we finally get a premotivc adjunction

$$\tau^* : HR_{(-)/k}\text{-Mod} \rightleftarrows \mathrm{DM}_{\mathrm{cdh}}(-, R) : \tau_*\,.$$

Moreover, the functor τ^* preserves the canonical generating families of compact objects. Therefore, the functor τ_* is conservative and commutes with small sums.

5. Comparison theorem: general case

The aim of this section is to prove:

THEOREM 5.1. *Let k be a perfect field of characteristic exponent p. Assume that p is invertible in the ring of coefficients R. For any noetherian k-scheme of finite dimension X, the canonical functor*

$$\tau^* : HR_{X/k}\text{-Mod} \to \mathrm{DM}_{\mathrm{cdh}}(X, R)$$

is an equivalence of categories.

The proof will take the following path: we will prove this statement in the case where X is separated and of finite type over k. For this, we will use Gabber's refinement of de Jong's resolution of singularities by alterations, as well as descent properties for HR_k-modules proved by Shane Kelly to see that it is sufficient to consider the case of a smooth k-scheme. In this situation, Theorem 5.1 will be a rather formal consequence of Theorem 3.1. The general case will be obtained by a continuity argument.

5.2. Let ℓ be a prime number. Following S. Kelly [Kel12], one defines the ℓdh-topology on the category of noetherian schemes as the coarsest Grothendieck topology such that any cdh-cover is an ℓdh-cover and any morphism of the form $f : X \to Y$, with f finite, surjective, flat, and of degree prime to ℓ is an ℓdh-cover. For instance, if $\{U_i \to X\}_{i\in I}$ is a cdh-cover, and if, for each i one has a finite surjective flat morphism $V_i \to U_i$ of degree prime to ℓ, we get an ℓdh-cover $\{V_i \to X\}_{i\in I}$. In the case where X is noetherian, one can show that, up to refinement, any ℓdh-cover is of this form; see [Kel12, Prop. 3.2.5]. We will use several times the following non-trivial fact, which is a direct consequence of Gabber's theorem of uniformization prime to ℓ [ILO14, Exp. IX, Th. 1.1]: locally for the ℓdh-topology, any quasi-excellent scheme is regular. In other words, for any noetherian quasi-excellent scheme X (e.g. any scheme of finite type over field), there exists a morphism of finite type $p : X' \to X$ which is a covering for the ℓdh-topology and has a regular domain.

PROPOSITION 5.3. *Let F be a* cdh*-sheaf with transfers over X which is $\mathbf{Z}_{(\ell)}$-linear. Then F is an* ℓdh*-sheaf and, for any integer n, the map*

$$H^n_{\mathrm{cdh}}(X, F) \to H^n_{\ell\mathrm{dh}}(X, F)$$

is an isomorphism.

Proof. See [Kel12, Theorem 3.4.17]. □

Corollary 5.4. *Assume that X is of finite dimension, and let C be a complex of $\mathbf{Z}_{(\ell)}$-linear* cdh-*sheaves with transfers over X. Then the comparison map of hypercohomologies*

$$H^n_{\mathrm{cdh}}(X, C) \to H^n_{\ell\mathrm{dh}}(X, C)$$

is an isomorphism for all n.

Proof. Note that, for $t = \mathrm{cdh}$ or $t = \ell\mathrm{dh}$, the forgetful functor from $\mathbf{Z}_{(\ell)}$-linear t-sheaves with tranfers to $\mathbf{Z}_{(\ell)}$-linear t-sheaves on the big site of X is exact (this follows from the stronger results given by [Kel12, Prop. 3.4.15 and 3.4.16] for instance). Therefore, we have a canonical spectral sequence of the form

$$E_2^{p,q} = H^p_t(X, H^q(C)_t) \Rightarrow H^{p+q}_t(X, C)\,.$$

As the cohomological dimension with respect to the cdh-topology is bounded by the dimension, this spectral sequence strongly converges for $t = \mathrm{cdh}$. Proposition 5.3 thus implies that, for $t = \ell\mathrm{dh}$, the groups $E_2^{p,q}$ vanish for $p < 0$ or $p > \dim X$, so that this spectral sequence also converges in this case. Therefore, as these two spectral sequences agree on the E_2 term, we conclude that they induce an isomorphism on E_∞. □

Corollary 5.5. *For X of finite dimension and R an $\mathbf{Z}_{(\ell)}$-algebra, any object of the triangulated category $\underline{\mathrm{DM}}_{\mathrm{cdh}}(X, R)$ satisfies ℓ*dh*-descent (see* [CDa, Definition 3.2.5]*).*

Lemma 5.6. *Assume that X is of finite type over the perfect field k. Consider a prime ℓ which is distinct from the characteristic exponent of k. If R is a $\mathbf{Z}_{(\ell)}$-algebra, then any compact object of $HR_{X/k}$*-Mod *satisfies ℓ*dh*-descent.*

Proof. As X is allowed to vary, it is sufficient to prove that, for any constructible $HR_{X/k}$-modules M and any ℓdh-hypercover $p_\bullet : U_\bullet \to X$, the map

(5.6.1) $$\mathbf{R}\Gamma(X, M) \to \mathbf{R}\varprojlim_{\Delta_n} \mathbf{R}\Gamma(U_n, p_n^* M)$$

is an isomorphism. The category of compact objects of HR_X-Mod is the thick subcategory generated by objects of the form $\mathbf{R}f_* HR_{Y/k}(p)$ for $f : Y \to X$ a projective map and p an integer (this follows right away from the fact that te analogous property is true in SH). We may thus assume that $M = \mathbf{R}f_* HR_{Y/k}(p)$. We can then form the following pullback in the category of simplicial schemes.

$$\begin{array}{ccc} V_\bullet & \xrightarrow{g} & U_\bullet \\ {\scriptstyle q_\bullet}\downarrow & & \downarrow{\scriptstyle p_\bullet} \\ Y & \xrightarrow{f} & X \end{array}$$

Using the proper base change formula for $HR_{(-)/k}$-modules, we see that the map (5.6.1) is isomorphic to the map

$$\mathbf{R}\Gamma(Y, HR_{Y/k}(p)) \to \mathbf{R}\varprojlim_{\Delta_n} \mathbf{R}\Gamma(V_n, HR_{V_n/k}(p))\,. \tag{5.6.2}$$

By virtue of Kelly's ℓdh-descent theorem [Kel12, Theorem 5.3.7], the map (5.6.2) is an isomorphism. □

Lemma 5.7. *Let X be a k-scheme of finite type. Assume that R is a $\mathbf{Z}_{(\ell)}$-algebra for ℓ a prime number distinct from the characteristic exponent of k. Let M be an object of $\underline{\mathrm{DM}}(X,R)$ satisfying ℓdh-descent on the site of smooth k-schemes over X: for any X-scheme of finite type Y which is smooth over k and any ℓdh-hypercover $p : U_\bullet \to Y$ such that U_n is smooth over k for any $n \geq 0$, the map*

$$\mathbf{R}\,\mathrm{Hom}_{\underline{\mathrm{DM}}(X,R)}(R(Y), M(p)) \to \mathbf{R}\varprojlim_{\Delta_n} \mathbf{R}\,\mathrm{Hom}_{\underline{\mathrm{DM}}(X,R)}(R(U_n), M(p))$$

is an isomorphism in the derived category of R-modules. Then, for any X-scheme Y which is smooth over k and any integer p, the canonical map

$$\mathbf{R}\,\mathrm{Hom}_{\underline{\mathrm{DM}}(X,R)}(R(Y), M(p)) \to \mathbf{R}\,\mathrm{Hom}_{\underline{\mathrm{DM}}_{\mathrm{cdh}}(X,R)}(R(Y), M_{\mathrm{cdh}}(p))$$

is an isomorphism.

Proof. Let us denote by $R\{1\}$ the complex

$$R\{1\} = R(1)[1] = \ker(R(\mathbf{A}^1_X - \{0\}) \to R)$$

induced by the structural map $\mathbf{A}^1 - \{0\} \times X \to X$. We may consider that the object M is a fibrant $R\{1\}$-spectrum in the category of complexes of R-linear sheaves with transfers on the category of X-schemes of finite type. In particular, M corresponds to a collection of complexes of R-linear sheaves with transfers $(M_n)_{\geq 0}$ together with maps $R\{1\} \otimes_R M_n \to M_{n+1}$ such that we have the following properties.

(i) For any integer $n \geq 0$ and any X-scheme of finite type Y, the map

$$\Gamma(Y, M_n) \to \mathbf{R}\Gamma(Y, M_n)$$

is an isomorphism in the derived category of R-modules (where $\mathbf{R}\Gamma$ stands for the derived global section with respect to the Nisnevich topology).

(ii) For any integer $n \geq 0$, the map

$$M_n \to \mathbf{R}\,\underline{\mathrm{Hom}}(R\{1\}, M_{n+1})$$

is an isomorphism in the derived category of Nisnevich sheaves with tranfers (where $\mathbf{R}\,\underline{\mathrm{Hom}}$ stands for the derived internal Hom).

We can choose another $R\{1\}$-spectum $N = (N_n)_{n \geq 0}$ of cdh-sheaves with transfers, together with a cofibration of spectra $M \to N$ such that $M_n \to N_n$ is a quasi-isomorphism locally for the cdh-topology, and such that each N_n satisfies cdh-descent: we do this by induction as follows. First, N_0 is any fibrant resolution of $(M_0)_{\mathrm{cdh}}$ for the cdh-local model structure on the category of complexes of cdh-sheaves with transfers. If N_n is already constructed, we denote by E the pushout of M_n along the map $R\{1\} \otimes_R M_n \to R\{1\} \otimes_R N_n$, and we factor the map $E_{\mathrm{cdh}} \to 0$ into a trivial cofibration followed by a fibration in the cdh-local model structure.

Note that, for any X-scheme Y which is smooth over k, the map

$$H^i(Y, M_n) \to H^i(Y, N_n)$$

is an isomorphism of R-modules for any integers $i \in \mathbf{Z}$ and $n \geq 0$. Indeed, as, by virtue of Gabber's theorem of resolution of singularities by ℓdh-alterations [ILO14, Exp. IX, Th. 1.1], one can write both sides with the Verdier formula in the following way (because of our hypothesis on M and by construction of N):

$$H^i(Y, E) \simeq \varinjlim_{U_\bullet \to Y} H^i(\mathbf{R} \varprojlim_{\Delta_j} \Gamma(U_j, M_n)) \text{ for } E = M_n \text{ or } E = N_n,$$

where $U_\bullet \to Y$ runs over the filtering category of ℓdh-hypercovers of Y such that each U_j is smooth over k. It is also easy to see from this formula that each N_n is $\mathbf{A}^1$-homotopy invariant and that the maps

$$N_n \to \underline{\mathrm{Hom}}(R\{1\}, N_{n+1})$$

are isomorphisms. In other words, N satisfies the analogs of properties (i) and (ii) above with respect to the cdh-topology. We thus get the following identifications for $p \geq 0$:

$$\begin{aligned}\Gamma(Y, M_p) &= \mathbf{R}\,\mathrm{Hom}_{\underline{\mathrm{DM}}(X,R)}(R(Y), M(p))\\ \Gamma(Y, N_p) &= \mathbf{R}\,\mathrm{Hom}_{\underline{\mathrm{DM}}_{\mathrm{cdh}}(X,R)}(R(Y), M_{\mathrm{cdh}}(p))\,.\end{aligned}$$

The case where $p < 0$ follows from the fact that, for $d = -p$, $R(Y)(d)[2d]$ is then a direct factor of $R(Y \times \mathbf{P}^d)$ (by the projective bundle formula in $\underline{\mathrm{DM}}_{\mathrm{cdh}}(X, R)$). □

Lemma 5.8. *Let X be a smooth separated k-scheme of finite type. Assume that R is a $\mathbf{Z}_{(\ell)}$-algebra for ℓ a prime number distinct from the characteristic exponent of k. If M and N are two constructible objects of* $\mathrm{DM}(X, R)$*, then the comparison map*

$$\mathbf{R}\,\mathrm{Hom}_{\mathrm{DM}(X,R)}(M, N) \to \mathbf{R}\,\mathrm{Hom}_{\mathrm{DM}_{\mathrm{cdh}}(X,R)}(M, N)$$

is an isomorphism in the derived category of R-modules.

Proof. It is sufficient to prove this in the case where $M = R(Y)(p)$ for Y a smooth X-scheme and p any integer. By virtue of Lemma 5.7, it is sufficient to prove that any constructible object of $\mathrm{DM}(X, R)$ satisfies ℓdh-descent on the site of X-schemes which are smooth over k. By virtue of Theorem 3.1, it is thus sufficient to prove the analogous property for constructible HR_X-modules, which follows from Lemma 5.6. □

Proof of Theorem 5.1. It is sufficient to prove that the restriction of the comparison functor

$$HR_{X/k}\text{-Mod} \to \mathrm{DM}_{\mathrm{cdh}}(X,R)\ ,\quad M \mapsto \tau^*(M) \tag{5.8.1}$$

to constructible $HR_{X/k}$-modules is fully faithful (by virtue of [CDa, Corollary 1.3.21], this is because both triangulated categories are compactly generated and because the functor (5.8.1) preserves the canonical compact generators). It is easy to see that this functor is fully faithful (on constructible objects) if and only if, for any prime $\ell \neq p$, its $R \otimes \mathbf{Z}_{(\ell)}$-linear version has this property (this is because the functor (5.8.1) preserves compact objects, which implies that its right adjoint commutes with small sums, hence both functors commute with the operation of tensoring by $\mathbf{Z}_{(\ell)}$). Therefore, we may assume that a prime number $\ell \neq p$ is given and that R is a $\mathbf{Z}_{(\ell)}$-algebra. We will then prove the property of being fully faithful first in the case where X is of finite type over k, and then, by a limit argument, in general.
Assume that X is of finite type over k, and consider constructible $HR_{X/k}$-modules M and N. We want to prove that, the map

$$\mathbf{R}\operatorname{Hom}_{HR_{X/k}\text{-Mod}}(M,N) \to \mathbf{R}\operatorname{Hom}_{\mathrm{DM}_{\mathrm{cdh}}(X,R)}(\tau^*(M),\tau^*(N)) \tag{5.8.2}$$

is an isomorphism (here all the $\mathbf{R}\operatorname{Hom}$'s take their values in the triangulated category of topological S^1-spectra; see [CDa, Theorem 3.2.15] for the existence (and uniqueness) of such an enrichment). By virtue of Gabber's theorem of resolution of singularities by ℓdh-alterations [ILO14, Exp. IX, Th. 1.1], we can choose an ℓdh-hypercover $p_\bullet : U_\bullet \to X$, with U_n smooth, separated, and of finite type over k for any non negative integer n. We then have the following chain of isomorphisms, justified respectively by ℓdh-descent for constructible $HR_{X/k}$-modules (Lemma 5.6), by the comparison theorem relating the category of HR-modules with DM over regular k-schemes (Theorem 3.1), by Lemma 5.8, and finally by the fact that any complex of R-modules with transfers on the category of separated X-schemes of finite type which satisfies cdh-descent must satisfy ℓdh-descent as well (Corollary 5.4):

$$\begin{aligned}
\mathbf{R}\operatorname{Hom}_{HR_{X/k}\text{-Mod}}(M,N) &\simeq \mathbf{R}\varprojlim_{\Delta_n}\mathbf{R}\operatorname{Hom}_{HR_{U_n}\text{-Mod}}(\mathbf{L}p_n^*M,\mathbf{L}p_n^*N)\\
&\simeq \mathbf{R}\varprojlim_{\Delta_n}\mathbf{R}\operatorname{Hom}_{\mathrm{DM}(U_n,R)}(\mathbf{L}p_n^*\,t^*(M),\mathbf{L}p_n^*\,t^*(N))\\
&\simeq \mathbf{R}\varprojlim_{\Delta_n}\mathbf{R}\operatorname{Hom}_{\mathrm{DM}_{\mathrm{cdh}}(U_n,R)}(\mathbf{L}p_n^*\,\tau^*(M),\mathbf{L}p_n^*\,\tau^*(N))\\
&\simeq \mathbf{R}\operatorname{Hom}_{\mathrm{DM}_{\mathrm{cdh}}(X,R)}(\tau^*(M),\tau^*(N))\,.
\end{aligned}$$

It remains to treat the case of an arbitrary noetherian k-scheme X. It is easy to see that the property that the functor (5.8.1) is fully faithful (on constructible objects) is local on X with respect to the Zariski topology. Therefore, we may assume that X is affine with structural ring A. We can then write A as a filtering colimit of k-algebras of finite type $A_i \subset A$, so that we obtain a projective system of k-schemes of finite type $\{X_i = \operatorname{Spec} A_i\}_i$ with affine and dominant transition maps, such that $X = \varprojlim_i X_i$. But then, by continuity (applying Proposition 2.7 twice, using Lemma 2.10 for $HR_{X/k}$-Mod, and Example 2.6(2) for $\mathrm{DM}_{\mathrm{cdh}}(X, R)$), we have canonical equivalences of categories at the level of constructible objects:

$$\begin{aligned} HR_{X/k}\text{-Mod}_c &\simeq 2\text{-}\varinjlim_i HR_{X_i/k}\text{-Mod}_c \\ &\simeq 2\text{-}\varinjlim_i \mathrm{DM}_{\mathrm{cdh}}(X_i, R)_c \\ &\simeq \mathrm{DM}_{\mathrm{cdh}}(X, R)_c\,. \end{aligned}$$

In particular, the functor (5.8.1) is fully faithful on constructible objects, and this ends the proof. □

COROLLARY 5.9. *Let X be a regular noetherian k-scheme of finite dimension. Then the canonical functor*

$$\mathrm{DM}(X, R) \to \mathrm{DM}_{\mathrm{cdh}}(X, R)$$

is an equivalence of symmetric monoidal triangulated categories.

Proof. This is a combination of Theorems 3.1 and 5.1, and of Proposition 3.10. □

Remark that we get for free the following result, which generalizes Kelly's ℓdh-descent theorem:

THEOREM 5.10. *Let k be a field of characteristic exponent p, ℓ a prime number distinct from p, and R a $\mathbf{Z}_{(\ell)}$-algebra. Then, for any noetherian k-scheme of finite dimension X, any object of $HR_{X/k}$-Mod satisfies ℓdh-descent.*

Proof. This follows immediately from Theorem 5.1 and from Corollary 5.5. □

Similarly, we see that $\mathrm{DM}_{\mathrm{cdh}}$ is continuous is a rather general sense.

THEOREM 5.11. *The motivic category $\mathrm{DM}_{\mathrm{cdh}}(-, R)$ has the properties of localization with respect to any closed immersion as well as the property of continuity with respect to arbitrary projective systems with affine transition maps over the category of noetherian k-schemes of finite dimension.*

Proof. Since $HR_{(-)/k}$-Mod has these properties, Theorem 5.1 allows to transfer it to $\mathrm{DM}_{\mathrm{cdh}}(-, R)$. □

6. FINITENESS

6.1. In this section, all the functors are derived functors, but we will drop $\mathbf{L}$ or $\mathbf{R}$ from the notations. The triangulated motivic category $\mathrm{DM}_{\mathrm{cdh}}(-, R)$ is endowed with the six operations $\otimes_R$, $\underline{\mathrm{Hom}}_R$, f^*, f_*, $f_!$ and $f^!$ which satisfy the usual properties; see [CDa, Theorem 2.4.50] for a summary.
Recall that an object of $\mathrm{DM}_{\mathrm{cdh}}(X, R)$ is constructible if and only if it is compact. Here is the behaviour of the six operations with respect to constructible objects in $\mathrm{DM}_{\mathrm{cdh}}(-, R)$, when we restrict ourselves to k-schemes (see [CDa, 4.2.5, 4.2.6, 4.2.10, 4.2.12]):

(i) constructible objects are stable by tensor products;
(ii) for any morphism $f : X \to Y$, the functor $f^* : \mathrm{DM}_{\mathrm{cdh}}(Y,R) \to \mathrm{DM}_{\mathrm{cdh}}(X,R)$ preserves constructible objects;
(iii) The property of being constructible is local for the Zariski topology;
(iii) given a closed immersion $i : Z \to X$ with open complement $j : U \to X$, an object M of $\mathrm{DM}_{\mathrm{cdh}}(X,R)$ is constructible if and only if $i^*(M)$ and $j^*(M)$ are constructible;
(iv) the functor $f_! : \mathrm{DM}_{\mathrm{cdh}}(X,R) \to \mathrm{DM}_{\mathrm{cdh}}(Y,R)$ preserves constructible objects for any separated morphism of finite type $f : X \to Y$.

Proposition 6.2. *Let $i : Z \to X$ be a closed immersion of codimension c between regular k-schemes. Then there is a canonical isomorphism $i^!(R_X) \simeq R_Z(-c)[-2c]$ in $\mathrm{DM}_{\mathrm{cdh}}(Z,R)$.*

Proof. In the case where X and Z are smooth over k, this is a direct consequence of the relative purity theorem. For the general case, using the reformulation of the absolute purity theorem of [CDb, Appendix, Theorem A.2.8(ii)], we see that it is sufficient to prove this proposition locally for the Zariski topology over X. Therefore we may assume that X is affine. Since $\mathrm{DM}_{\mathrm{cdh}}(-,R)$ is continuous (5.11), using Popescu's theorem and [CDa, 4.3.12], we see that it is sufficient to treat the case where X is smooth of finite type over k. But then, this is a direct consequence of the relative purity theorem. □

Proposition 6.3. *Let $f : X \to Y$ be a morphism of noetherian k-schemes. Assume that both X and Y are integral and that f is finite and flat of degree d. Then, there is a canonical natural transformation*

$$Tr_f : \mathbf{R}f_*\mathbf{L}f^*(M) \to M$$

for any object M of $\mathrm{DM}_{\mathrm{cdh}}(X,R)$ such that the composition with the unit of the adjunction $(\mathbf{L}f^*, \mathbf{R}f_*)$

$$M \to \mathbf{R}f_*\mathbf{L}f^*(M) \xrightarrow{Tr_f} M$$

is d times the identity of M.

Proof. As in paragraphs 3.14 and 3.16 (simply replacing $\underline{\mathrm{DM}}(X,R)$ and $\mathrm{DM}(X,R)$ by $\underline{\mathrm{DM}}_{\mathrm{cdh}}(X,R)$ and $\mathrm{DM}_{\mathrm{cdh}}(X,R)$, respectively), we construct

$$Tr_f : \mathbf{R}f_*(R_X) = \mathbf{R}f_*\mathbf{L}f^*(R_Y) \to R_Y$$

such that the composition with the unit

$$R \to \mathbf{R}f_*(R_X) \xrightarrow{Tr_f} R_Y$$

is d. Then, since f is proper, we have a projection formula

$$\mathbf{R}f_*(R_X) \otimes^{\mathbf{L}}_R M \simeq \mathbf{R}f_*\mathbf{L}f^*(M)$$

and we construct

$$Tr_f : \mathbf{R}f_*\mathbf{L}f^*(M) \to M$$

as

$$M \otimes^{\mathbf{L}}_R \left(\mathbf{R}f_*(R_X) \xrightarrow{Tr_f} R_Y\right).$$

This ends the construction of Tr_f and the proof of this proposition. □

Theorem 6.4. *The six operations preserve constructible objects in* $\mathrm{DM}_{\mathrm{cdh}}(-,R)$ *over quasi-excellent k-schemes. In particular, we have the following properties.*

(a) *For any morphism of finite type between quasi-excellent k-schemes, the functor* $f_* : \mathrm{DM}_{\mathrm{cdh}}(X,R) \to \mathrm{DM}_{\mathrm{cdh}}(Y,R)$ *preserves constructible objects.*

(b) *For any separated morphism of finite type between quasi-excellent k-schemes* $f : X \to Y$*, the functor* $f^! : \mathrm{DM}_{\mathrm{cdh}}(Y,R) \to \mathrm{DM}_{\mathrm{cdh}}(X,R)$ *preserves constructible objects.*

(c) *If X is a quasi-excellent k-scheme, for any constructible objects M and N of* $\mathrm{DM}_{\mathrm{cdh}}(M,N)$*, the object* $\underline{\mathrm{Hom}}_R(M,N)$ *is constructible.*

Sketch of proof. It is standard that properties (b) and (c) are corollaries of property (a); see the proof of [CDb, Cor. 6.2.14], for instance. Also, to prove (a), the usual argument (namely [Ayo07a, Lem. 2.2.23]) shows that it is sufficient to prove that, for any morphism of finite type $f : X \to Y$, the object $f_*(R_X)$ is constructible. As one can work locally for the Zariski topology on X and on Y, one may assume that f is separated (e.g. affine) and thus that $f = pj$ with j an open immersion and p a proper morphism. As $p_! = p_*$ is already known to preserve constructible objects, we are thus reduced to prove that, for any dense open immersion $j : U \to X$, the object $j_*(R_U)$ is constuctible. This is where the serious work begins. First, using the fact that constructible objects are compact, for any prime $\ell \neq p$, the triangulated category $\mathrm{DM}_{\mathrm{cdh}}(X, R\otimes\mathbf{Z}_{(\ell)})$ is the idempotent completion of the triangulated category $\mathrm{DM}_{\mathrm{cdh}}(X,R)\otimes\mathbf{Z}_{(\ell)}$. Therefore, using [CDb, Appendix, Prop. B.1.7], we easily see that it is sufficient to consider the case where R is a $\mathbf{Z}_{(\ell)}$-algebra for some prime $\ell \neq p$. The rest of the proof consists to follow word for word a beautiful argument of Gabber: the very proof of [CDb, Lem. 6.2.7]. Indeed, the only part of the proof of *loc. cit.* which is not meaningful in an abstract motivic triangulated category is the proof of the sublemma [CDb, 6.2.12], where we need the existence of trace maps for flat finite surjective morphisms satisfying the usual degree formula. In the case of $\mathrm{DM}_{\mathrm{cdh}}(X,R)$, we have such trace maps natively: see Proposition 6.3. □

7. Duality

In this section, we will consider a field K of exponential characteristic p, and will focus our attention on K-schemes of finite type. As anywhere else in this article, the ring of coefficients R is assumed to be a $\mathbf{Z}[1/p]$-algebra.

Proposition 7.1. *Let $f : X \to Y$ be a surjective finite radicial morphism of noetherian K-schemes of finite dimension. Then the functor*

$$\mathbf{L}f^* : \mathrm{DM}_{\mathrm{cdh}}(Y,R) \to \mathrm{DM}_{\mathrm{cdh}}(X,R)$$

is an equivalence of categories and is canonically isomorphic to the functor $f^!$.

Proof. By virtue of [CDa, Prop. 2.1.9], it is sufficient to prove that pulling back along such a morphism f induces a conservative functor $\mathbf{L}f^*$ (the fact that $\mathbf{L}f^* \simeq f^!$ come from the fact that if $\mathbf{L}f^*$ is an equivalence of categories, then so is its right adjoint $f_! \simeq \mathbf{R}f_*$, so that $\mathbf{L}f^*$ and $f^!$ must be quasi-inverses of the same equivalence of categories). Using the localization property as well as a suitable noetherian induction, it is sufficient to check this property generically on Y. In particular, we may assume that Y and X are integral and that f is moreover flat. Then the degree of f must be some power of p, and Proposition 6.3 then implies that the functor $\mathbf{L}f^*$ is faithful (and thus conservative). □

Proposition 7.2. *Let X be a scheme of finite type over K, and Z a fixed nowhere dense closed subscheme of X. Then the category of constructible motives $\mathrm{DM}_{\mathrm{cdh},c}(X,R)$ is the smallest thick subcategory containing objects of the form $f_!(R_Y)(n)$, where $f : Y \to X$ is a projective morphism with Y smooth over a finite purely inseparable extension of K and such that $f^{-1}(Z)$ is either empty, the whole scheme Y itself, or the support of a strict normal crossing divisor, while n is any integer.*

Proof. Let $\mathcal{G}$ be the family of objects of the form $f_!(R_Y)(n)$, with $f : Y \to X$ a projective morphism, Y smooth over a finite purely inseparable extension of K, $f^{-1}(Z)$ either empty or the support of a strict normal crossing divisor, and n any integer. We already know that any element of $\mathcal{G}$ is constructible. Since the constructible objects of $\mathrm{DM}_{\mathrm{cdh}}(X,R)$ precisely are the compact objects, which do form a generating family of the triangulated category $\mathrm{DM}_{\mathrm{cdh}}(X,R)$, it is sufficient to prove that the family $\mathcal{G}$ is generating. Let M be an object of $\mathrm{DM}_{\mathrm{cdh}}(X,R)$ such that $\mathrm{Hom}(C,M[i]) = 0$ for any element C of $\mathcal{G}$ and any integer i. We want to prove that $M = 0$. For this, it is sufficient to prove that $M \otimes \mathbf{Z}_{(\ell)} = 0$ for any prime ℓ which not invertible in R (hence, in particular, is prime to p). Since, for any compact object C of $\mathrm{DM}_{\mathrm{cdh}}(X,R)$, we have

$$\mathrm{Hom}(C,M[i]) \otimes \mathbf{Z}_{(\ell)} \simeq \mathrm{Hom}(C, M \otimes \mathbf{Z}_{(\ell)}[i])\,,$$

and since $f_!$ commutes with tensoring with $\mathbf{Z}_{(\ell)}$ (because it commutes with small sums), we may assume that R is a $\mathbf{Z}_{(\ell)}$-algebra for some prime number $\ell \neq p$. Under this extra hypothesis, we will prove directly that $\mathcal{G}$ generates the thick category of compact objects. Let T be the smallest thick subcategory of $\mathrm{DM}_{\mathrm{cdh}}(X,R)$ which contains the elements of $\mathcal{G}$.

For Y a separated X-scheme of finite type, we put

$$M^{BM}(Y/X) = f_!(R_Y)$$

with $f : Y \to X$ the structural morphism. If Z is a closed subscheme of Y with open complement U, we have a canonical distinguished triangle

$$M^{BM}(U/X) \to M^{BM}(Y/X) \to M^{BM}(Z/X) \to M^{BM}(Z/X)[1]\,.$$

We know that the subcategory of constructible objects of $\mathrm{DM}_{\mathrm{cdh}}(X,R)$ is the smallest thick subcategory which contains the objects of the form $M^{BM}(Y/X)(n)$ for $Y \to X$ projective, and $n \in \mathbf{Z}$; see [Ayo07a, Lem. 2.2.23].

By cdh-descent (as formulated in [CDa, Prop. 3.3.10 (i)]), we easily see that objects of the form $M^{BM}(Y/X)(n)$ for $Y \to X$ projective, Y integral, and $n \in \mathbf{Z}$, generate the thick subcategory of constructible objects of $\mathrm{DM}_{\mathrm{cdh}}(X,R)$. By noetherian induction on the dimension of such a Y, it is sufficient to prove that, for any projective X-scheme Y, there exists a dense open subscheme U in Y such that $M^{BM}(U/X)$ belongs to T. By virtue of Gabber's refinement of de Jong's theorem of resolution of singularities by alterations [ILO14, Exp. X, Theorem 2.1], there exists a projective morphism $Y' \to Y$ which is generically flat, finite surjective of degree prime to ℓ, such that Y' is smooth over a finite purely inseparable extension of K, and such that the inverse image of Z in Y' is either empty, the whole scheme Y', or the support of a strict normal crossing divisor. Thus, by induction, for any dense open subscheme $V \subset Y'$, the motive $M^{BM}(V/X)$ belongs to T. But, by assumption on $Y' \to Y$, there exists a dense open subscheme U of Y such that, if V denote the pullback of U in Y', the induced map $V \to U$ is a finite, flat and surjective morphism between integral K-schemes and is of degree prime to ℓ. By virtue of Proposition 6.3, the motive $M^{BM}(U/X)$ is thus a direct factor of $M^{BM}(V/X)$, and since the latter belongs to T, this shows that $M^{BM}(Y/X)$ belongs to T as well, and this achieves the proof. □

Theorem 7.3. *Let X be a separated K-scheme of finite type, with structural morphism $f : X \to \mathrm{Spec}(K)$. Then the object $f^!(R)$ is dualizing. In other words, for any constructible object M in $\mathrm{DM}_{\mathrm{cdh}}(X,R)$, the natural map*

$$M \to \mathbf{R}\,\underline{\mathrm{Hom}}_R(\mathbf{R}\,\underline{\mathrm{Hom}}_R(M, f^!(R), f^!(R))) \tag{7.3.1}$$

is an isomorphism. In particular, the natural map

$$R_X \to \mathbf{R}\,\underline{\mathrm{Hom}}_R(f^!(R), f^!(R)) \tag{7.3.2}$$

is an isomorphism in $\mathrm{DM}_{\mathrm{cdh}}(X,R)$.

Proof. By virtue of Proposition 7.2, it is sufficient to prove that the map (7.3.1) is an isomorphism for $M = p_!(R_Y)$ with $p : Y \to X$ projective and Y smooth over a finite purely inseparable extension of K. We then have

$$\mathbf{R}\,\underline{\mathrm{Hom}}_R(M, f^!(R)) \simeq p_!\mathbf{R}\,\underline{\mathrm{Hom}}_R(R_Y, p^!f^!(R)) \simeq p_!p^!(f^!(R))\,,$$

hence

$$\begin{aligned}\mathbf{R}\,\underline{\mathrm{Hom}}_R(\mathbf{R}\,\underline{\mathrm{Hom}}_R(M, f^!(R), f^!(R)) &\simeq \mathbf{R}\,\underline{\mathrm{Hom}}_R(p_!p^!(f^!(R)), f^!(R))\\ &\simeq p_!\mathbf{R}\,\underline{\mathrm{Hom}}_R(p^!f^!(R), p^!f^!(R))\,.\end{aligned}$$

The map (7.3.1) is thus, in this case, the image by the functor $p_!$ of the map $R_Y \to \mathbf{R}\,\underline{\mathrm{Hom}}_R(p^!f^!(R), p^!f^!(R))$. In other words, it is sufficient to prove that the map (7.3.2) is an isomorphism in the case where X is projective over K, and smooth over a finite purely inseparable field extension L/K. In particular we get the following factorization of f

$$X \xrightarrow{g} \mathrm{Spec}(L) \xrightarrow{h} \mathrm{Spec}(K)$$

such that g is smooth and h surjective finite radicial. By virtue of Proposition 7.1, $h^!(R) = R_L$. Moreover, if d is the dimension of X, since $\mathrm{DM}_{\mathrm{cdh}}$ is oriented, we have a purity isomorphism $g^!(R_L) \simeq R_X(d)(2d]$, Thus we get an isomorphism $f^!(R) \simeq R_X(d)[2d]$. Since we obviously have the identification, $R_X \simeq \mathbf{R}\,\underline{\mathrm{Hom}}_R(R_X(d), R_X(d))$, this achieves the proof. □

Remark 7.4. The preceding theorem means that, if we restrict to separated K-schemes of finite type, the whole formalism of Grothendieck-Verdier duality holds in the setting of R-linear cdh-motives. In other words, for a separated K-scheme of finite type X with structural map $f : X \to \mathrm{Spec}(K)$, we define the functor D_X by

$$D_X(M) = \mathbf{R}\,\underline{\mathrm{Hom}}_R(M, f^!(R))$$

for any object M of $\mathrm{DM}_{\mathrm{cdh}}(X, R)$. We already know that D_X preserves constructible objects and that the natural map $M \to D_X(D_X(M))$ is invertible for any constructible object M of $\mathrm{DM}_{\mathrm{cdh}}(X, R)$. For any objects M and N of $\mathrm{DM}_{\mathrm{cdh}}(X, R)$, if N is constructible, we have a natural isomorphism

$$\mathbf{R}\,\underline{\mathrm{Hom}}_R(M, N) \simeq D_X(M \otimes^{\mathbf{L}}_R D_X(N))\,. \tag{7.4.1}$$

For any K-morphism between separated K-schemes of finite type $f : Y \to X$, and for any constructible objects M and N in $\mathrm{DM}_{\mathrm{cdh}}(X, R)$ and $\mathrm{DM}_{\mathrm{cdh}}(Y, R)$, respectively, we have the following natural identifications.

$$D_Y(f^*(M)) \simeq f^!(D_X(M)) \tag{7.4.2}$$

$$f^*(D_X(M)) \simeq D_Y(f^!(M)) \tag{7.4.3}$$

$$D_X(f_!(N)) \simeq f_*(D_Y(N)) \tag{7.4.4}$$

$$f_!(D_Y(N)) \simeq D_X(f_*(N)) \tag{7.4.5}$$

8. Bivariant cycle cohomology

Proposition 8.1. *Let K be a field of characteristic exponent p, and K^s its inseparable closure.*

(a) *The map $u : \mathrm{Spec}(K^s) \to \mathrm{Spec}(K)$ induces fully faithful functors*

$$u^* : \underline{\mathrm{DM}}^{\mathit{eff}}(K, R) \to \underline{\mathrm{DM}}^{\mathit{eff}}(K^s, R) \text{ and } u^* : \underline{\mathrm{DM}}^{\mathit{eff}}_{\mathrm{cdh}}(K, R) \to \underline{\mathrm{DM}}^{\mathit{eff}}_{\mathrm{cdh}}(K^s, R)\,.$$

(b) *We have a canonical equivalence of categories*

$$\mathrm{DM}^{\mathit{eff}}(K^s, R) \simeq \underline{\mathrm{DM}}^{\mathit{eff}}_{\mathrm{cdh}}(K^s, R)\,.$$

(c) *At the level of non-effective motives, we have canonical equivalences of categories*

$$\mathrm{DM}(K, R) \simeq \mathrm{DM}_{\mathrm{cdh}}(K, R) \simeq \underline{\mathrm{DM}}_{\mathrm{cdh}}(K, R)\,.$$

(d) *The pullback functor*

$$u^* : \mathrm{DM}(K, R) \to \mathrm{DM}(K^s, R)$$

is an equivalence of categories.

Proof. In all cases, u^* has a right adjoint $\mathbf{R}u_*$ which preserves small sums (because u^* preserves compact objects, which are generators).
Let us prove that the functor

$$u^* : \underline{\mathrm{DM}}^{\mathit{eff}}(K) \to \underline{\mathrm{DM}}^{\mathit{eff}}(K^s)$$

is fully faithful. By continuity (see [CDa, Example 11.1.25]), it is sufficient to prove that, for any finite purely inseparable extension L/K, the pullback functor along the map $v : \mathrm{Spec}(L) \to \mathrm{Spec}(K)$,

$$v^* : \mathrm{DM}^{\mathit{eff}}(K, R) \to \mathrm{DM}^{\mathit{eff}}(L, R)\,,$$

is fully faithful. As, for any field E, we have a fully faithful embedding

$$\mathrm{DM}^{\mathit{eff}}(E, R) \to \underline{\mathrm{DM}}^{\mathit{eff}}(E, R)$$

which is compatible with pulbacks (see [CDa, Prop. 11.1.19]), it is sufficient to prove that the pullback functor

$$v^* : \underline{\mathrm{DM}}^{\mathit{eff}}(K, R) \to \underline{\mathrm{DM}}^{\mathit{eff}}(L, R)$$

is fully faithful. In this case, the functor v^* has a left adjoint $v_\sharp$, and we must prove that the co-unit

$$v_\sharp\, v^*(M) \to M$$

is fully faithful for any object M of $\underline{\mathrm{DM}}^{\mathit{eff}}(K)$. The projection formula $v_\sharp\, v^*(M) = v_\sharp(R) \otimes^{\mathbf{L}}_R M$ reduces to prove that the co-unit $v_\sharp\, v^*(R) \to R$ is an isomorphism, which follows right away from [CDa, Prop. 9.1.14]. The same arguments show that the functor

$$u^* : \underline{\mathrm{DM}}^{\mathit{eff}}_{\mathrm{cdh}}(K, R) \to \underline{\mathrm{DM}}^{\mathit{eff}}_{\mathrm{cdh}}(K^s, R)$$

is fully faithful.
The canonical functor

$$\mathrm{DM}^{\mathit{eff}}(L, R) \to \underline{\mathrm{DM}}^{\mathit{eff}}_{\mathrm{cdh}}(L, R)$$

is an equivalence of categories for any perfect field L of exponent characteristic p by a result in Kelly's thesis (more precisely the right adjoint of this functor is an equivalence of categories; see the last assertion of [Kel12, Cor. 5.3.9]).
The fact that the functor

$$u^* : \mathrm{DM}_c(K, R) \to \mathrm{DM}_c(K^s, R)$$

is an equivalence of categories follows by continuity from the fact that the pullback functor

$$\mathrm{DM}_c(K, R) \to \mathrm{DM}_c(L, R)$$

is an equivalence of categories for any finite purely inseparable extension L/K (see [CDa, Prop. 2.1.9 and 2.3.9]). As the right adjoint of u^* preserves small sums, this implies that $u^* : \mathrm{DM}(K, R) \to \mathrm{DM}(K^s, R)$ is fully faithful. Since any compact object of $\mathrm{DM}(K^s, R)$ is in the essential image and since $\mathrm{DM}(K^s, R)$ is compactly generated, this proves that $u^* : \mathrm{DM}(K, R) \to \mathrm{DM}(K^s, R)$ is an equivalence of categories; see [CDa, Corollary 1.3.21].

As we already know that the functor

$$\mathrm{DM}(K,R)\to \mathrm{DM}_{\mathrm{cdh}}(K,R)$$

is an equivalence of categories (Cor. 5.9), it remains to prove that the functor

$$\mathrm{DM}_{\mathrm{cdh}}(K,R)\to \underline{\mathrm{DM}}_{\mathrm{cdh}}(K,R)$$

is an equivalence of categories (or even an equality). Note that we have

$$\mathrm{DM}_{\mathrm{cdh}}(L,R)=\underline{\mathrm{DM}}_{\mathrm{cdh}}(L,R)$$

for any perfect field of exponent characteristic p. This simply means that motives of the form $M(X)(n)$, for X smooth over L and $n\in\mathbf{Z}$, do form a generating family of $\underline{\mathrm{DM}}(L,R)$. To prove this, let us consider an object C of $\mathrm{DM}_{\mathrm{cdh}}(L,R)$ such that

$$\mathrm{Hom}(M(X)(n),C[i])=0$$

for any smooth L-scheme X and any integers n and i. To prove that $C=0$, since, for any compact object E and any localization A of the ring $\mathbf{Z}$, the functor $\mathrm{Hom}(E,-)$ commutes with tensoring by A, we may assume that R is a $\mathbf{Z}_{(\ell)}$-algebra for some prime number $\ell\neq p$. Under this extra assumption, we know that the object C satisfies ℓdh-descent (see Corollary 5.5). Since, by Gabber's theorem, any scheme of finite type over L is smooth locally for the ℓdh-topology, this proves that $C=0$.
Finally, let us consider an object C of $\underline{\mathrm{DM}}_{\mathrm{cdh}}(K,R)$ such that $\mathrm{Hom}(M,C)=0$ for any object M of $\mathrm{DM}_{\mathrm{cdh}}(K,R)$. Then, for any object N of $\mathrm{DM}_{\mathrm{cdh}}(K^s,R)$, we have $\mathrm{Hom}(N,u^*(C))=0$: indeed, such an N must be of the form $u^*(M)$ for some M in $\mathrm{DM}_{\mathrm{cdh}}(K,R)$, and the functor u^* is fully faithful on $\underline{\mathrm{DM}}_{\mathrm{cdh}}(-,R)$. Since K^s is a perfect field, this proves that $u^*(C)=0$, and using the fully faithfulness of u^* one last time implies that $C=0$. This proves that $\mathrm{DM}_{\mathrm{cdh}}(K,R)=\underline{\mathrm{DM}}_{\mathrm{cdh}}(K,R)$ and achieves the proof of the proposition. □

Corollary 8.2. *Let K be a field of exponent characteristic p. Then the infinite suspension functor*

$$\Sigma^\infty:\underline{\mathrm{DM}}^{\mathit{eff}}_{\mathrm{cdh}}(K,R)\to \underline{\mathrm{DM}}_{\mathrm{cdh}}(K,R)=\mathrm{DM}_{\mathrm{cdh}}(K,R)$$

is fully faithful.

Proof. Let K^s be the inseparable closure of K. The functor

$$\Sigma^\infty:\underline{\mathrm{DM}}^{\mathit{eff}}_{\mathrm{cdh}}(K^s,R)\to \underline{\mathrm{DM}}_{\mathrm{cdh}}(K^s,R)=\mathrm{DM}_{\mathrm{cdh}}(K^s,R)$$

is fully faithful: this follows from the fact that the functor

$$\Sigma^\infty:\mathrm{DM}^{\mathit{eff}}(K^s,R)\to \mathrm{DM}(K^s,R)$$

is fully faithful (which is a reformulation of Voevodsky's cancellation theorem [Voe10]) and from assertions (b) and (c) in Proposition 8.1.

Pulling back along the map $u : \mathrm{Spec}(K^s) \to \mathrm{Spec}(K)$ induces an essentially commutative diagram of the form

$$\begin{array}{ccccc}
\underline{\mathrm{DM}}^{eff}_{\mathrm{cdh}}(K) & \xrightarrow{\Sigma^\infty} & \underline{\mathrm{DM}}_{\mathrm{cdh}}(K) & = & \mathrm{DM}_{\mathrm{cdh}}(K, R) \\
\downarrow{\scriptstyle u^*} & & \downarrow{\scriptstyle u^*} & & \downarrow{\scriptstyle u^*} \\
\underline{\mathrm{DM}}^{eff}_{\mathrm{cdh}}(K^s) & \xrightarrow{\Sigma^\infty} & \underline{\mathrm{DM}}_{\mathrm{cdh}}(K^s) & = & \mathrm{DM}_{\mathrm{cdh}}(K^s, R)
\end{array}$$

and thus, Proposition 8.1 allows to conclude. □

8.3. The preceding proposition and its corollary explain why it is essentially harmless to only work with perfect ground fields[9]. From now on, we will focus on our fixed perfect field k of characteristic exponent p, and will work with separated k-schemes of finite type.

Let X be a separated k-scheme of finite type and $r \geq 0$ an integer. Let $z_{equi}(X, r)$ be the presheaf with transfers of equidimensional relative cycles of dimension r over k (see [VSF00, Chap. 2, page 36]); its evaluation at a smooth k-scheme U is the free group of cycles in $U \times X$ which are equidimensionnal of relative dimension r over k; see [VSF00, Chap. 2, Prop. 3.3.15]. If $\Delta^\bullet$ denotes the usual cosimplicial k-scheme,

$$\Delta^n = \mathrm{Spec}\left(k[t_0, \dots, t_n]/(\sum_i t_i = 1)\right),$$

then, for any presheaf of ablian groups F, the Suslin complex $\underline{C}_*(F)$ is the complex associated to the simplicial presehaf of abelian groups $F((-) \times \Delta^\bullet)$. Let Y be another k-scheme of finite type. After Friedlander and Voevodsky, for $r \geq 0$, the (R-linear) *bivariant cycle cohomology of Y with coefficients in cycles on X* is defined as the following cdh-hypercohomology groups:

$$A_{r,i}(Y, X)_R = H^{-i}_{\mathrm{cdh}}(Y, \underline{C}_*(z_{equi}(X, r))_{\mathrm{cdh}} \otimes^{\mathbf{L}} R)\,. \tag{8.3.1}$$

Since $\mathbf{Z}(Y)$ is a compact object in the derived category of cdh-sheaves of abelian groups, we have a canonical isomorphism

$$\mathbf{R}\Gamma(Y, \underline{C}_*(z_{equi}(X, r))_{\mathrm{cdh}} \otimes^{\mathbf{L}} R) \simeq \mathbf{R}\Gamma(Y, \underline{C}_*(z_{equi}(X, r))_{\mathrm{cdh}}) \otimes^{\mathbf{L}} R \tag{8.3.2}$$

in the derived category of R-modules. We also put $A_{r,i}(Y, X)_R = 0$ for $r < 0$.

Recall that, for any separated k-scheme of finite type X, we have its motive $M(X)$ and its motive with compact support $M^c(X)$. Seen in $\mathrm{DM}(k, R)$, they are the objects associated to the presheaves with transfers $R(X)$ and $R^c(X)$ on smooth k-schemes: for a smooth k-scheme U, $R(X)(U)$ (resp. $R^c(X)(U)$) is the free R-module on the set of cycles in $U \times X$ which are finite (resp. quasi-finite) over U and dominant over an irreducible component of U. We will also denote by $M(X)$ and $M^c(X)$ the corresponding objects in $\mathrm{DM}_{\mathrm{cdh}}(k, R)$ through the equivalence $\mathrm{DM}(k, R) \simeq \mathrm{DM}_{\mathrm{cdh}}(k, R)$.

[9]Note however that the recent work of Suslin [Sus13] should provide explicit formulas such as the one of Theorem 8.11 for separated schemes of finite type over non-perfect infinite fields.

Theorem 8.4 (Voevodsky, Kelly). *For any integers $r, i \in \mathbf{Z}$, there is a canonical isomorphism of R-modules*

$$A_{r,i}(Y,X)_R \simeq \mathrm{Hom}_{\mathrm{DM}(k,R)}(M(Y)(r)[2r+i], M^c(X)) .$$

Proof. For $R = \mathbf{Z}$, in view of Voevodsky's cancellation theorem, this is a reformulation of [VSF00, Chap. 5, Prop.4.2.3] in characteristic zero; the case where the exponent characteristic is p, with $R = \mathbf{Z}[1/p]$, is proved by Kelly in [Kel12, Prop. 5.5.11]. This readily implies this formula for a general $\mathbf{Z}[1/p]$-algebra R as ring of coefficients, using (8.3.2). □

Remark 8.5. Let $g : Y \to \mathrm{Spec}(k)$ be a separated morphism of finite type. The pullback functor

$$\mathbf{L}g^* : \mathrm{DM}_{\mathrm{cdh}}(k,R) \to \mathrm{DM}_{\mathrm{cdh}}(Y,R) \tag{8.5.1}$$

has a left adjoint

$$\mathbf{L}g_\sharp : \mathrm{DM}_{\mathrm{cdh}}(Y,R) \to \mathrm{DM}_{\mathrm{cdh}}(k,R) . \tag{8.5.2}$$

Indeed, this is obviously true if we replace $\mathrm{DM}_{\mathrm{cdh}}(-,R)$ by $\underline{\mathrm{DM}}_{\mathrm{cdh}}(-,R)$. Since we have $\mathrm{DM}(k,R) \simeq \mathrm{DM}_{\mathrm{cdh}}(k,R) = \underline{\mathrm{DM}}_{\mathrm{cdh}}(k,R)$ (8.1 (c)), the restriction of the functor

$$\mathbf{L}g_\sharp : \underline{\mathrm{DM}}_{\mathrm{cdh}}(Y,R) \to \underline{\mathrm{DM}}_{\mathrm{cdh}}(k,R)$$

to $\mathrm{DM}_{\mathrm{cdh}}(Y,R) \subset \underline{\mathrm{DM}}_{\mathrm{cdh}}(Y,R)$ provides the left adjoint of the pullback functor $\mathbf{L}g^*$ in the fibred category $\mathrm{DM}_{\mathrm{cdh}}(-,R)$. This construction does not only provide a left adjoint, but also computes it: the motive of Y is the image by this left adjoint of the constant motive on Y:

$$M(Y) = \mathbf{L}g_\sharp(R_Y) . \tag{8.5.3}$$

We also deduce from this description of $\mathbf{L}g_\sharp$ that, for any object M of $\mathrm{DM}_{\mathrm{cdh}}(k,R)$, we have a canonical isomorphism

$$\mathbf{R}g_*\mathbf{L}g^*(M) \simeq \mathbf{R}\,\underline{\mathrm{Hom}}_R(M(Y),M) \tag{8.5.4}$$

(where $\underline{\mathrm{Hom}}_R$ is the internal Hom of $\mathrm{DM}_{\mathrm{cdh}}(k,R)$): again, this readily follows from the analogous formula in $\underline{\mathrm{DM}}_{\mathrm{cdh}}(-,R)$).
If we wite $z(X,r)$ for the cdh-sheaf asociated to $z_{equi}(X,r)$ (which is compatible with the notations of Suslin and Voevodsky, according to [VSF00, Chap. 2, Thm. 4.2.9]), we thus have another way of expressing the preceding theorem.

Corollary 8.6. *With the notations of Remark 8.5, we have a canonical isomorphism of R-modules:*

$$A_{r,i}(Y,X)_R \simeq \mathrm{Hom}_{\mathrm{DM}_{\mathrm{cdh}}(Y,R)}(R_Y(r)[2r+i], \mathbf{L}g^*(M^c(X))) .$$

8.7. The preceding corollary is not quite the most natural way to express bivariant cycle cohomology $A_{r,i}(Y,X)$. Keeping track of the notations of Remark 8.5, we can see that there is a canonical isomorphism

$$g_!\, g^!(R) \simeq M(Y) . \tag{8.7.1}$$

Indeed, we have:

$$\mathbf{R}\,\underline{\mathrm{Hom}}_R(g_!g^!(R),R) = \mathbf{R}g_*\mathbf{R}\,\underline{\mathrm{Hom}}_R(g^!(R),g^!(R))\,.$$

But Grothendieck-Verdier duality (7.3) implies that

$$R_Y = \mathbf{R}\,\underline{\mathrm{Hom}}_R(g^!(R),g^!(R))\,,$$

and thus (8.5.4) gives:

$$\mathbf{R}\,\underline{\mathrm{Hom}}_R(g_!g^!(R),R) \simeq \mathbf{R}g_*\mathbf{L}g^*(R) \simeq \mathbf{R}\,\underline{\mathrm{Hom}}_R(M(Y),R)\,.$$

Since the natural map

$$M \to \mathbf{R}\,\underline{\mathrm{Hom}}_R(\mathbf{R}\,\underline{\mathrm{Hom}}_R(M,R),R)$$

is invertible for any constructible motive M in $\mathrm{DM}_{\mathrm{cdh}}(k,R)$, we obtain the identification (8.7.1) (note that $M(Y)$ is constructible; see [Kel12, Lemma 5.5.2]).

Corollary 8.8. *With the notations of Remark 8.5, we have a canonical isomorphism of R-modules:*

$$A_{r,i}(Y,X)_R \simeq \mathrm{Hom}_{\mathrm{DM}_{\mathrm{cdh}}(Y,R)}(g^!(R)(r)[2r+i], g^!(M^c(X)))\,.$$

8.9. Let $f : X \to \mathrm{Spec}(k)$ be a separated morphism of finite type. We want to describe $M^c(X)$ in terms of the six operations in $\mathrm{DM}_{\mathrm{cdh}}(-,R)$.

Proposition 8.10. *With the notations of 8.9, there are canonical isomorphisms*

$$M^c(X) \simeq \mathbf{R}f_*\, f^!(R) \simeq \mathbf{R}\,\underline{\mathrm{Hom}}_R(f_!(R_X),R)$$

in the triangulated category $\mathrm{DM}_{\mathrm{cdh}}(k,R)$.

Proof. If f is proper, then $f_!(R_X) = \mathbf{R}f_*(R_X)$, while $M^c(X) = M(X)$ (we really mean equality here, in both cases). Therefore, we also have

$$\mathbf{R}\,\underline{\mathrm{Hom}}_R(M^c(X),R) = \mathbf{R}\,\underline{\mathrm{Hom}}_R(M(X),R) \simeq \mathbf{R}f_*(R_X) = f_!(R_X)$$

in a rather canonical way: the identification $\mathbf{R}\,\underline{\mathrm{Hom}}_R(M(X),R) \simeq \mathbf{R}f_*(R_X)$ can be constructed in $\underline{\mathrm{DM}}_{\mathrm{cdh}}(K,R)$, in which case it can be promoted to a canonical weak equivakence at the level of the model category of symmetric Tate spectra of complexes of (R-linear) cdh-sheaves with transfers over the category of separated K-schemes of finite type. In particular, for any morphism $i : Z \to X$ with $g = fi$ proper, we have a commutative diagram of the form

$$\begin{array}{ccc}
\mathbf{R}\,\underline{\mathrm{Hom}}_R(M(X),R) & \xrightarrow{\sim} & \mathbf{R}f_*(R_X) \\
{\scriptstyle i^*}\downarrow & & \downarrow{\scriptstyle i^*} \\
\mathbf{R}\,\underline{\mathrm{Hom}}_R(M(Z),R) & \xrightarrow{\sim} & \mathbf{R}g_*(R_Z)
\end{array}$$

in the (stable model category underlying the) triangulated category $\mathrm{DM}_{\mathrm{cdh}}(X,R)$.

In the general case, let us choose an open embedding $j : X \to \bar{X}$ with a proper k-scheme $q : \bar{X} \to \mathrm{Spec}(k)$, such that $f = qj$. Let $\partial\bar{X}$ be a closed subscheme of $\bar{X}$ such that $\bar{X} \setminus \partial\bar{X}$ is the image of j, and write $r : \partial\bar{X} \to \mathrm{Spec}(k)$ for the structural map. What precedes means that there is a canonical identification between the homotopy fiber of the restriction map

$$\mathbf{R}q_*(R_{\bar{X}}) \to \mathbf{R}r_*(R_{\partial\bar{X}})$$

and the homotopy fiber of the restriction map

$$\mathbf{R}\,\underline{\mathrm{Hom}}_R(M(\bar{X}), R) \to \mathbf{R}\,\underline{\mathrm{Hom}}_R(M(\partial\bar{X}), R)\,.$$

But, by definition of $f_!(R_X)$, and by virtue of [VSF00, Chap. 5, Prop. 4.1.5] in characteristic zero, and of [Kel12, Prop. 5.5.5] in general, this means that we have a canonical isomorphism

$$\mathbf{R}\,\underline{\mathrm{Hom}}_R(M^c(X), R) \simeq f_!(R_X)\,.$$

By duality (7.3), taking the dual of this identification leads to a canonical isomorphism $\mathbf{R}f_*\,f^!(R) \simeq M^c(X)$. □

Theorem 8.11. *Let Y and X be two separated k-schemes of finite type with structural maps $g : Y \to \mathrm{Spec}(k)$ and $f : X \to \mathrm{Spec}(k)$. Then, for any $r \geq 0$, there is a natural identification*

$$A_{r,i}(Y, X)_R \simeq \mathrm{Hom}_{\mathrm{DM}_{\mathrm{cdh}}(k,R)}(g_!g^!(R)(r)[2r+i], \mathbf{R}f_*f^!(R))\,.$$

Proof. We simply put Corollary 8.8 and Proposition 8.10 together. □

Corollary 8.12. *Let X be an equidimensional quasi-projective k-scheme of dimension n, with structural morphism $f : X \to \mathrm{Spec}(k)$, and consider any subring $\Lambda \subset \mathbf{Q}$ in which the characteristic exponent of k is invertible. Then, for any integers i and j, we have a natural isomorphism*

$$\mathrm{Hom}_{\mathrm{DM}_{\mathrm{cdh}}(X,\Lambda)}(\Lambda_X(i)[j], f^!\Lambda) \simeq \mathrm{CH}^{n-i}(X, j-2i) \otimes \Lambda$$

(where $\mathrm{CH}^{n-i}(X, j-2i)$ is Bloch's higher Chow group.

Proof. In the case where k is of characteristic zero, this is a reformulation of the preceding theorem and of [VSF00, Chap. 5, Prop. 4.2.9]. For the proof of *loc. cit.* to hold *mutatis mutandis* for any perfect field k of characteristic $p > 0$ (and with $\mathbf{Z}[1/p]$-linear coefficients), we see that apart from Proposition 8.1 and Theorem 8.4 above, the only ingredient that we need is the $\mathbf{Z}[1/p]$-linear version of [VSF00, Theorem 4.2.2], which is provided by results of Kelly [Kel12, Theorems 5.4.19 and 5.4.21]. □

Corollary 8.13. *Let X be a separated k-scheme of finite type, with structural morphism $f : X \to \mathrm{Spec}(k)$. For any subring $\Lambda \subset \mathbf{Q}$ in which p is invertible, there is a natural isomorphism*

$$\mathrm{CH}_n(X) \otimes \Lambda \simeq \mathrm{Hom}_{\mathrm{DM}_{\mathrm{cdh}}(X,\Lambda)}(\Lambda_X(n)[2n], f^!\Lambda)$$

for any integer n (where $\mathrm{CH}_n(X)$ is the usual Chow group of cycles of dimension n on X, modulo rational equivalence).

Proof. Thanks to [VSF00, Chap. 4, Theorem 4.2] and to [Kel12, Theorem 5.4.19], we know that

$$\mathrm{CH}_n(X) \otimes \Lambda \simeq A_{n,0}(\mathrm{Spec}(k), X)_\Lambda \,.$$

We thus conclude with Theorem 8.11 for $r = n$ and $i = 0$. □

9. Realizations

9.1. Recall from paragraph 1.3 that, for a noetherian scheme X, and a ring a coefficients Λ, one can define the Λ-linear triangulated category of mixed motives over X associated to the h-topology $\mathrm{DM}_{\mathrm{h}}(X, \Lambda)$. The latter construction is the subject of the article [CDb], in which we see that $\mathrm{DM}_{\mathrm{h}}(X, \Lambda)$ is a suitable version of the theory of étale mixed motives. In particular, we have a natural functor induced by the h-sheafification functor:

$$\mathrm{DM}_{\mathrm{cdh}}(X, \Lambda) \to \mathrm{DM}_{\mathrm{h}}(X, \Lambda) \ , \quad M \mapsto M_{\mathrm{h}} \,. \tag{9.1.1}$$

These functors are part of a premotivic adjunction in the sense of [CDa, Def. 1.4.6].

From now on, we assume that the schemes X are defined over a given field k and that the characteristic exponent of k is invertible in Λ. Since both $\mathrm{DM}_{\mathrm{cdh}}$ and DM_{h} are motivic categories over k-schemes in the sense of [CDa, Def. 2.4.45] (see Theorem 5.11 above and [CDb, Theorem 5.6.2], respectively), we have the following formulas (see [CDa, Prop. 2.4.53]):

(9.1.2)
$$(M \otimes^{\mathbf{L}}_\Lambda N)_{\mathrm{h}} \simeq M_{\mathrm{h}} \otimes^{\mathbf{L}}_\Lambda N_{\mathrm{h}}$$

(9.1.3) $(\mathbf{L}f^*(M))_{\mathrm{h}} \simeq \mathbf{L}f^*(M_{\mathrm{h}})$ (for any morphism f)

(9.1.4) $(\mathbf{L}f_\sharp(M))_{\mathrm{h}} \simeq \mathbf{L}f_\sharp(M_{\mathrm{h}})$ (for any smooth separated morphism f)

(9.1.5) $(f_!(M))_{\mathrm{h}} \simeq f_!(M_{\mathrm{h}})$ (for any separated morphism of finite type f)

Note finally that the functor (9.1.1) has fully faithful right adjoint; its essential image consists of objects of $\mathrm{DM}_{\mathrm{cdh}}$ which satisfy the property of cohomological h-descent (see [CDa, Def. 3.2.5]).

Lemma 9.2. *Let $f : X \to \mathrm{Spec}\, k$ be a separated morphism of finite type. Then the natural morphism*

$$(\mathbf{R}f_*(\Lambda_X))_{\mathrm{h}} \to \mathbf{R}f_*((\Lambda_X)_{\mathrm{h}})$$

is invertible in $\mathrm{DM}_{\mathrm{h}}(k, \Lambda)$.

Proof. We may assume that k is a perfect field (using Prop. 8.1 (d) as well as its analogue for the h-topology (which readily follows from [CDb, Prop. 6.3.16])). We know that $\mathrm{DM}_{\mathrm{cdh}}(k, \Lambda) = \underline{\mathrm{DM}}_{\mathrm{cdh}}(k, \Lambda)$ by Prop. 8.1 (c), and similarly that $\mathrm{DM}_{\mathrm{h}}(k, \Lambda) = \underline{\mathrm{DM}}_{\mathrm{h}}(k, \Lambda)$ (since, by virtue of de Jong's theorem of resolution of singularities by alterations, locally for the h-topology, any k-scheme of finite type is smooth). The functor

$$\underline{\mathrm{DM}}_{\mathrm{cdh}}(k, \Lambda) \to \underline{\mathrm{DM}}_{\mathrm{h}}(k, \Lambda) \ , \quad M \mapsto M_{\mathrm{h}}$$

is symmetric monoidal and sends $\mathbf{L}f_\sharp(\Lambda_X)$ to $\mathbf{L}f_\sharp((\Lambda_X)_\mathrm{h})$. On the other hand, the motive $\mathbf{L}f_\sharp(\Lambda_X) \simeq f_!(f^!(\Lambda))$ is constructible (see (8.7.1) for $g = f$ and Theorem 6.4), whence has a strong dual in $\mathrm{DM}_\mathrm{cdh}(k, \Lambda)$ (since objects with a strong dual form a thick subcategory, this follows from Proposition 7.2, by Poincaré duality; see [CDa, Theorems 2.4.42 and 2.4.50]). The functor $M \mapsto M_\mathrm{h}$ being symmetric monoidal, it preserves the property of having a strong dual and preserves strong duals. Since $\mathbf{R}f_*(\Lambda_X)$ is the (strong) dual of $\mathbf{L}f_\sharp(\Lambda_X)$ both in $\underline{\mathrm{DM}}_\mathrm{cdh}(k, \Lambda)$ and in $\underline{\mathrm{DM}}_\mathrm{h}(k, \Lambda)$, this proves this lemma. □

Lemma 9.3. *Let $f : X \to Y$ be a k-morphism between separated k-schemes of finite type. Then the functors*

$$\mathbf{R}f_* : \mathrm{DM}_\mathrm{cdh}(X, \Lambda) \to \mathrm{DM}_\mathrm{cdh}(Y, \Lambda) \quad \textit{and} \quad \mathbf{R}f_* : \mathrm{DM}_\mathrm{h}(X, \Lambda) \to \mathrm{DM}_\mathrm{h}(Y, \Lambda)$$

commute with small sums.

Proof. In the case of cdh-motives follows from the fact that the functor

$$\mathbf{L}f^* : \mathrm{DM}_\mathrm{cdh}(Y, \Lambda) \to \mathrm{DM}_\mathrm{cdh}(X, \Lambda)$$

sends a family of compact generators into a family of compact objects. The case of h-motives is proven in [CDb, Prop. 5.5.10]. □

Proposition 9.4. *Let $f : X \to Y$ be a k-morphism between separated k-schemes of finite type. Then, for any object M of $\mathrm{DM}_\mathrm{cdh}(X, \Lambda)$, the natural map*

$$\mathbf{R}f_*(M)_\mathrm{h} \to \mathbf{R}f_*(M_\mathrm{h})$$

is invertible in $\mathrm{DM}_\mathrm{h}(Y, \Lambda)$.

Proof. The triangulated category $\mathrm{DM}_\mathrm{cdh}(X, \Lambda)$ is compactly generated by objects of the form $\mathbf{R}g_*(\Lambda_{X'}(n)$ for $g : X' \to X$ a proper morphism and n any integer; see [CDa, Prop. 4.2.13], for instance. Since the lemma is already known in the case of proper maps (see equation (9.1.5)), we easily deduce from Lemma 9.3 that we may assume M to be isomorphic to the constant motive Λ_X. In this case, we conclude with Lemma 9.2. □

Corollary 9.5. *Under the assumptions of paragraph 9.1, the restriction of the motivic functor $M \mapsto M_\mathrm{h}$ (9.1.1) to constructible objects commutes with the six operations of Grothendieck over the category of separated k-schemes of finite type.*

Proof. After Proposition 9.4, we see that it is sufficient to prove the compatibility with internal Hom and with operations of the form $g^!$ for any morphism g between separated k-schemes of finite type.

Let us prove that, for any separated k-scheme of finite type Y and any constructible objects A and N of $\mathrm{DM}_\mathrm{cdh}(Y, \Lambda)$, the natural map

$$\mathbf{R}\,\underline{\mathrm{Hom}}(A, N)_\mathrm{h} \to \mathbf{R}\,\underline{\mathrm{Hom}}(A_\mathrm{h}, N_\mathrm{h})$$

is invertible in $\mathrm{DM}_{\mathrm{h}}(Y,\Lambda)$. We may assume that $A=f_\sharp(\Lambda_X)$ for some smooth morphism $f:X\to Y$. Since we have the canonical identification

$$\mathbf{R}\,\underline{\mathrm{Hom}}(\mathbf{L}f_\sharp(\Lambda_X),N)\simeq\mathbf{R}f_*\,f^*(N)\,,$$

we conclude by using the isomorphism provided by Proposition 9.4 in the case where $M=f^*(N)$.

Consider now a separated morphism of finite type $f:X\to\operatorname{Spec}k$. For any constructible objects M and N of $\mathrm{DM}_{\mathrm{cdh}}(X,\Lambda)$ and $\mathrm{DM}_{\mathrm{cdh}}(k,\Lambda)$, respectively, we have:

$$\begin{aligned}\mathbf{R}f_*(\mathbf{R}\,\underline{\mathrm{Hom}}(M_{\mathrm{h}},f^!(N)_{\mathrm{h}}))&\simeq\mathbf{R}f_*(\mathbf{R}\,\underline{\mathrm{Hom}}(M,f^!(N))_{\mathrm{h}})\\&\simeq(\mathbf{R}f_*\mathbf{R}\,\underline{\mathrm{Hom}}(M,f^!(N)))_{\mathrm{h}}\\&\simeq\mathbf{R}\,\underline{\mathrm{Hom}}(f_!(M),N)_{\mathrm{h}}\\&\simeq\mathbf{R}\,\underline{\mathrm{Hom}}(f_!(M_{\mathrm{h}}),N_{\mathrm{h}})\\&\simeq\mathbf{R}f_*(\mathbf{R}\,\underline{\mathrm{Hom}}(M_{\mathrm{h}},f^!(N_{\mathrm{h}})))\,.\end{aligned}$$

Therefore, for any object C of $\mathrm{DM}_{\mathrm{h}}(k,\Lambda)$, there is an isomorphism:

$$\mathbf{R}\,\mathrm{Hom}(\mathbf{L}f^*(C)\otimes^{\mathbf{L}}_{\Lambda}M_{\mathrm{h}},f^!(N)_{\mathrm{h}})\simeq\mathbf{R}\,\mathrm{Hom}(\mathbf{L}f^*(C)\otimes^{\mathbf{L}}_{\Lambda}M_{\mathrm{h}},f^!(N_{\mathrm{h}}))\,.$$

Since the constructible objects of the form M_{h} are a generating family of $\mathrm{DM}_{\mathrm{h}}(k,\Lambda)$, this proves that the natural map

$$f^!(N)_{\mathrm{h}}\to f^!(N_{\mathrm{h}})$$

is an isomorphism. The functor $M\mapsto M_{\mathrm{h}}$ preserves internal Hom's of constructible objects, whence it follows from Formula (7.4.1) that it preserves duality. Therefore, Formula (7.4.2) shows that it commutes with operations of the form $g^!$ for any morphism g between separated k-schemes of finite type. □

Remark 9.6. In the case where Λ is of positive characteristic, the trianguated category $\mathrm{DM}_{\mathrm{h}}(X,\Lambda)$ is canonically equivalent to the derived category $\mathrm{D}(X_{\text{ét}},\Lambda)$ of the abelian category of sheaves of Λ-modules on the small étale site of X; see [CDb, Cor. 5.4.4]. Therefore, Corollary 9.5 then provides a system of triangulated functors

$$\mathrm{DM}_{\mathrm{cdh}}(X,\Lambda)\to\mathrm{D}(X_{\text{ét}},\Lambda)$$

which preserve the six operations when restricted to constructible objects. Moreover, constructible objects of $\mathrm{DM}_{\mathrm{h}}(X,\Lambda)$ correspond to the full subcategory $\mathrm{D}^b_{ctf}(X_{\text{ét}},\Lambda)$ of the category $\mathrm{D}(X_{\text{ét}},\Lambda)$ which consists of bounded complexes of sheaves of Λ-modules over $X_{\text{ét}}$ with constructible cohomology, and which are of finite tor-dimension; see [CDb, Cor. 5.5.4 (and Th. 6.3.11)]. Therefore, for $\ell\neq p$, using [CDb, Prop. 7.2.21], we easily get ℓ-adic realizations which are compatible with the six operations (on constructible objects) over separated k-schemes of finite type:

$$\mathrm{DM}_{\mathrm{cdh},c}(X,\mathbf{Z}[1/p])\to\mathrm{D}^b_c(X_{\text{ét}},\mathbf{Z}_\ell)\to\mathrm{D}^b_c(X_{\text{ét}},\mathbf{Q}_\ell)\,.$$

For instance, this gives an alternative proof of some of the results of Olsson (such as [Ols15, Theorem 1.2]).

Together with Theorem 8.11, Corollary 9.5 is thus a rather functorial way to construct cycle class maps in étale cohomology (and in any mixed Weil cohomology, since they define realization functors of $\mathrm{DM}_{\mathrm{h}}(-, \mathbf{Q})$ which commute with the six operations on constructible objects; see [CDa, 17.2.5] and [CDb, Theorem 5.2.2]). This provides a method to prove independence of ℓ results as follows. Let X be a separated k-scheme of finite type, with structural map $a : X \to \operatorname{Spec} k$, and $f : X \to X$ any k-morphism. Then f induces an endomorphism of $\mathbf{R}a_*(\mathbf{Z}[1/p]_X)$ in $\mathrm{DM}_{\mathrm{cdh}}(k, \mathbf{Z}[1/p])$. Since the latter object is constructible (by Theorem 6.4 (a)), it has a strong dual (as explained in the proof of Lemma 9.2), and thus one can define the trace of the morphism induced by f, which is an element of $\mathbf{Z}[1/p]$ (since one can identify $\mathbf{Z}[1/p]$ with the ring of endomorphisms of the constant motive $\mathbf{Z}[1/p]$ in $\mathrm{DM}_{\mathrm{cdh}}(k, \mathbf{Z}[1/p])$ using Corollary 8.13). Let ℓ be a prime number distinct from the characteristic exponent of k. Since the ℓ-adic realization functor is symmetric monoidal, it preserves the property of having a strong dual and preserves traces of endomorphisms of objects with strong duals. Therefore, if $\bar{k}$ is any choice of an algebraic closure of k, and if $\bar{X} = \bar{k} \otimes_k X$, the number

$$\sum_i (-1)^i \mathrm{Tr}\big[f^* : H^i_{\text{ét}}(\bar{X}, \mathbf{Q}_\ell) \to H^i_{\text{ét}}(\bar{X}, \mathbf{Q}_\ell)\big]$$

is independent of ℓ and belongs to $\mathbf{Z}[1/p]$: Corollary 9.5 implies that it is the image through the unique morphism of rings $\mathbf{Z}[1/p] \to \mathbf{Q}_\ell$ of the trace of the endomorphism of the motive $\mathbf{R}a_*(\mathbf{Z}[1/p]_X)$ induced by f. This might be compared with Olsson's proof in the case where f is finite; see [Ols, Theorem 1.2]. One may also replace $H^i(\bar{X}, \mathbf{Q}_\ell)$ with the evaluation at X of any mixed Weil cohomology defined on smooth k-schemes, and still use the same argument.

Remark 9.7. If the ring Λ is a $\mathbf{Q}$-algebra, the functor $M \mapsto M_{\mathrm{h}}$ defines an equivalence of categories $\mathrm{DM}_{\mathrm{cdh}}(X, \Lambda) \simeq \mathrm{DM}_{\mathrm{h}}(X, \Lambda)$ (so that Corollary 9.5 becomes a triviality). This is because, under the extra hypothesis that $\mathbf{Q} \subset \Lambda$, the abelian categories of cdh-sheaves of Λ-modules with transfers and of h-sheaves of Λ-modules are equivalent: by a limit argument, it is sufficient to prove this when X is excellent, and then, this is an exercise which consists to put together [CDa, Prop. 10.4.8, Prop. 10.5.8, Prop. 10.5.11 and Th. 3.3.30].

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Descente galoisienne sur le second groupe de Chow : mise au point et applications

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Résumé. Le troisième groupe de cohomologie étale non ramifié d'une variété projective et lisse, à coefficients dans les racines de l'unité tordues deux fois, intervient dans plusieurs articles récents, en particulier en relation avec le groupe de Chow de codimension 2. Des résultats généraux ont été obtenus à ce sujet par B. Kahn en 1996. De récents travaux, du côté des groupes algébriques linéaires d'une part, du côté de la géométrie algébrique complexe d'autre part, m'incitent à les passer en revue, et à les spécialiser aux variétés proches d'être rationnelles.

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Dans tout cet article, on note F un corps de caractéristique zéro, $\overline{F}$ une clôture algébrique de F et $G = \mathrm{Gal}(\overline{F}/F)$. Soit X une F-variété lisse et géométriquement intègre. On note $\overline{X} = X \times_F \overline{F}$. On note $F(X)$ le corps des fonctions rationnelles de X et $\overline{F}(X)$ le corps des fonctions rationnelles de $\overline{X}$. L'application naturelle entre groupes de Chow de codimension 2

$$CH^2(X) \to CH^2(\overline{X})^G$$

n'est en général ni injective ni surjective, même si l'on suppose que X est projective et que l'ensemble $X(F)$ des points rationnels de X est non vide – à la différence du cas bien connu de $CH^1(X)$.

Plusieurs travaux ont été consacrés à l'étude des noyau et conoyau de cette application et aux liens entre le groupe de Chow de codimension deux et le troisième groupe de cohomologie non ramifiée de X à valeurs dans $\mathbb{Q}/\mathbb{Z}(2)$, groupe noté $H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2))$. Citons en particulier [6], Raskind et l'auteur [10], Lichtenbaum [23], Kahn [19, 20], C. Voisin et l'auteur [11], Pirutka [29], Kahn et l'auteur [8], Merkurjev [4, 24, 25, 26], Voisin [34].
Une des raisons de s'intéresser au groupe $H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2))$ est que c'est un invariant F-birationnel des F-variétés projectives et lisses, réduit à $H^3(F, \mathbb{Q}/\mathbb{Z}(2))$ si la F-variété X est F-birationnelle à un espace projectif.

Le résultat principal du présent article est le Théorème 4.1, qui s'applique à toute variété projective et lisse géométriquement rationnellement connexe, et qui dans le cas particulier des variétés géométriquement rationnelles établit (Corollaire 4.2) une suite exacte

$$\mathrm{Ker}[CH^2(X) \to CH^2(\overline{X})^G] \to H^1(G, \mathrm{Pic}(\overline{X}) \otimes \overline{F}^\times) \to$$

$$\to H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2))/H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \to \mathrm{Coker}[CH^2(X) \to CH^2(\overline{X})^G] \to$$

$$H^2(G, \mathrm{Pic}(\overline{X}) \otimes \overline{F}^\times)$$

sous l'une des deux hypothèses supplémentaires :
(i) La F-variété X possède un F-point.
(ii) La dimension cohomologique de F est au plus 3.

Décrivons la structure de l'article.

Le §1 est consacré à des rappels de résultats fondamentaux sur la $\mathcal{K}$-cohomologie, la cohomologie non ramifiée et la cohomologie motivique. On y rappelle aussi (Prop. 1.3) un résultat de [8] apportant une correction à [20].

Au §2, sous l'hypothèse que le groupe $H^0(\overline{X}, \mathcal{K}_2)$ est uniquement divisible, on établit par deux méthodes différentes (l'une K-théorique, l'autre motivique) une suite exacte générale (Propositions 2.4 et 2.6). On suppose ici la variété X lisse et géométriquement intègre, mais non nécessairement propre. Ceci s'applique en particulier aux espaces classifiants de groupes semisimples considérés par Merkurjev [24].
La première méthode, à l'ancienne, via la K-cohomologie, est celle des articles [10], [11]. La seconde méthode fait usage des groupes de cohomologie motivique à coefficients $\mathbb{Z}(2)$, comme dans l'article [20] de Bruno Kahn. De ce point de vue, on ne fait que généraliser [20, Thm. 1, Corollaire], avec la correction mentionnée ci-dessus. Lorsque le corps de base est de dimension cohomologique au plus 1, auquel cas la correction n'est pas utile, et lorsque de plus les variétés considérées sont projectives, ces suites exactes ont déjà été utilisées dans [11] et [8].

Au §3, pour X projective et lisse, on donne des conditions permettant de contrôler le groupe $H^1(\overline{X}, \mathcal{K}_2)$ apparaissant dans les suites exactes du §2. On donne une application aux surfaces $K3$ définies sur $\mathbb{C}((t))$.

Au §4, on combine les résultats des paragraphes précédents pour établir les résultats principaux de l'article, le théorème 4.1 et son corollaire 4.2 cité cidessus.

Au §5, on applique les résultats du §4 aux hypersurfaces de Fano complexes. Pour $X \subset \mathbf{P}^n_{\mathbb{C}}$ hypersurface lisse de degré $d \leq n$ et F corps quelconque contenant $\mathbb{C}$, on établit $H^3_{nr}(X_F, \mathbb{Q}/\mathbb{Z}(2)) = H^3(F, \mathbb{Q}/\mathbb{Z}(2))$ dans chacun des cas suivants : pour $n > 5$; pour $n = 5$ sous réserve que l'on ait $H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2)) = 0$; pour $n = 4$ lorsqu'il existe un cycle universel de codimension 2. On fait le lien avec les résultats de Auel, Parimala et l'auteur [2] et de C. Voisin [33, 34] sur les hypersurfaces cubiques et sur les cycles universels de codimension 2, résultats sur lesquels on donne un nouvel éclairage – la K-théorie algébrique remplaçant certains arguments de géométrie complexe (voir la démonstration du théorème 5.4).

Par rapport à la première version de cet article, mise sur arXiv en février 2013, cet article diffère essentiellement par le contenu du présent §5, motivé par le travail [2] et par les articles [33, 34] de C. Voisin.

Terminons cette introduction en indiquant ce qui n'est pas fait dans cet article.

(i) Je n'ai pas vérifié que les arguments dans la littérature utilisant les complexes $\mathbb{Z}(2)$ de Voevodsky sont compatibles avec ceux utilisant le complexe $\Gamma(2)$ de Lichtenbaum ou avec ceux utilisant les groupes de cycles supérieurs de Bloch, dont il est fait usage dans [8]. Et je n'ai pas vérifié que dans les suites exactes des Propositions 2.4 et 2.6, dont les termes sont identiques, les flèches aussi coïncident. Ceci n'affecte pas les principaux résultats de l'article. Le lecteur vérifiera en effet que la Proposition 2.4, établie par des méthodes à l'ancienne via la Proposition 1.3, suffit à établir tous les résultats des paragraphes 3, 4, 5, à l'exception du lemme 5.7 (ii), du théorème 5.6 (viii) et de l'assertion de surjectivité de l'application $CH^2(X_F) \to CH^2(X_{\overline{F}})^G$ dans le théorème 5.8 (iii).

(ii) Les longues suites exactes des Propositions 2.4 et 2.6, le théorème 4.1 et le corollaire 4.2 devraient se spécialiser en un certain nombre des longues suites exactes pour les variétés classifiantes de groupes algébriques linéaires connexes établies par Blinstein-Merkurjev [4] et par Merkurjev [24, 25]. Je me suis contenté d'allusions à ces articles en divers points du texte.

(iii) Sur un corps de base de caractéristique positive, l'utilisation de la cohomologie de Hodge-Witt logarithmique permet de donner des analogues de certains des résultats du présent travail. Nous renvoyons pour cela aux articles [20] et [8].

Remerciements. Cet article fait suite à des travaux et discussions avec Bruno Kahn, et à des travaux de A. Merkurjev et de C. Voisin. Je remercie le rapporteur pour sa lecture critique du tapuscrit.

1 Rappels, propriétés générales

On utilise dans cet article le complexe motivique $\mathbb{Z}(2)$ de faisceaux de cohomologie étale sur les variétés lisses sur un corps, tel qu'il a été défini par Lichtenbaum [22, 23].
Les groupes de cohomologie à valeurs dans le complexe $\mathbb{Z}(2)$ sont dans tout cet article les groupes d'hypercohomologie étale. Ils sont notés $\mathbb{H}^i(X, \mathbb{Z}(2))$.
Sur un schéma X, on note $H^i(X, \mathcal{K}_j)$ les groupes de cohomologie de Zariski à valeurs dans le faisceau $\mathcal{K}_j$ sur X associé au préfaisceau $U \mapsto K_i(H^0(U, \mathcal{O}_X))$, où la K-théorie des anneaux est la K-théorie de Quillen.
Étant donné un module galoisien M, c'est-à-dire un G-module continu discret, on note tantôt $H^i(G, M)$ tantôt $H^i(F, M)$ les groupes de cohomologie galoisienne à valeurs dans M.
On note $\mathbb{Q}/\mathbb{Z}(2)$ le module galoisien $\varinjlim_n \mu_n^{\otimes 2}$.
On note $K_3F_{indec} := \text{Coker}[K_3^{Milnor}F \to K_3^{Quillen}F]$.
On a les propriétés suivantes, conséquences de travaux de Merkurjev et Suslin [27], de A. Suslin [30], de M. Levine [21], de S. Lichtenbaum [23], de B. Kahn [19], [20, Thm. 1.1, Lemme 1.4].
$\mathbb{H}^0(F, \mathbb{Z}(2)) = 0$.
$\mathbb{H}^1(F, \mathbb{Z}(2)) = K_3F_{indec}$.
$\mathbb{H}^2(F, \mathbb{Z}(2)) = K_2F$.
$\mathbb{H}^3(F, \mathbb{Z}(2)) = 0$.
$\mathbb{H}^i(F, \mathbb{Z}(2)) = H^{i-1}(F, \mathbb{Q}/\mathbb{Z}(2))$ si $i \geq 4$.
$\mathbb{H}^i(\overline{F}, \mathbb{Z}(2)) = 0$ si $i \neq 1, 2$.
$\mathbb{H}^1(\overline{F}, \mathbb{Z}(2)) = K_3(\overline{F})_{indec}$ est divisible, et sa torsion est $\mathbb{Q}/\mathbb{Z}(2)$ (cf. [19, (1.2)]). Il est donc extension d'un groupe uniquement divisible par $\mathbb{Q}/\mathbb{Z}(2)$.
$\mathbb{H}^2(\overline{F}, \mathbb{Z}(2)) = K_2(\overline{F})$ est uniquement divisible.

Soit X une F-variété lisse géométriquement intègre, non nécessairement projective. On a :
$\mathbb{H}^0(X, \mathbb{Z}(2)) = 0$.
$\mathbb{H}^1(X, \mathbb{Z}(2)) = K_{3,indec}F(X)$.
$\mathbb{H}^1(\overline{X}, \mathbb{Z}(2)) = K_{3,indec}\overline{F}(X)$ est extension d'un groupe uniquement divisible par $\mathbb{Q}/\mathbb{Z}(2)$. Ceci résulte de la suite exacte [19, (1.2)] et de [30, Thm. 3.7]).
$\mathbb{H}^2(X, \mathbb{Z}(2)) = H^0(X, \mathcal{K}_2)$.
$\mathbb{H}^3(X, \mathbb{Z}(2)) = H^1(X, \mathcal{K}_2)$.
On a la suite exacte fondamentale (Lichtenbaum, Kahn [20, Thm. 1.1])

$$0 \to CH^2(X) \to \mathbb{H}^4(X, \mathbb{Z}(2)) \to H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2)) \to 0 \tag{1.1}$$

où

$$H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2)) = H^0(X, \mathcal{H}^3(X, \mathbb{Q}/\mathbb{Z}(2)))$$

est le sous-groupe de $H^3(F(X), \mathbb{Q}/\mathbb{Z}(2))$ formé des éléments non ramifiés en tout point de codimension 1 de X.

Pour toute F-variété projective, lisse et géométriquement intègre X, dans l'article [10] avec W. Raskind, on a établi que les groupes $H^0(\overline{X},\mathcal{K}_2)$ et $H^1(\overline{X},\mathcal{K}_2)$ sont chacun extension d'un groupe fini par un groupe divisible. Si la dimension cohomologique de F satisfait $\mathrm{cd}(F)\leq i$, ceci implique que les groupes de cohomologie galoisienne $H^r(G,H^0(\overline{X},\mathcal{K}_2))$ et $H^r(G,H^1(\overline{X},\mathcal{K}_2))$ sont nuls pour $r\geq i+1$.

On a une suite spectrale

$$E_2^{pq}=H^p(G,\mathbb{H}^q(\overline{X},\mathbb{Z}(2)))\Longrightarrow \mathbb{H}^n(X,\mathbb{Z}(2)).$$

Remarque 1.1. Pour $X=\mathrm{Spec}(F)$, compte tenu des identifications ci-dessus, cette suite spectrale donne une suite exacte

$$H^1(G,\mathbb{Q}/\mathbb{Z}(2))\to K_2F\to K_2\overline{F}^G\to H^2(G,\mathbb{Q}/\mathbb{Z}(2))\to 0.$$

Ceci est un cas particulier de [19, Thm. 2.1].

En comparant la suite exacte fondamentale (1.1) au niveau F et au niveau $\overline{F}$, en prenant les points fixes de G agissant sur la suite au niveau $\overline{F}$, et en utilisant le lemme du serpent, on obtient :

Proposition 1.2. *Soit X une F-variété lisse et géométriquement intègre. Soit $\varphi:\mathbb{H}^4(X,\mathbb{Z}(2))\to\mathbb{H}^4(\overline{X},\mathbb{Z}(2))^G$. On a alors une suite exacte*

$$\begin{aligned}0\to \mathrm{Ker}[CH^2(X)\to CH^2(\overline{X})^G]\to \mathrm{Ker}(\varphi)\to\\ \to \mathrm{Ker}[H^3_{nr}(X,\mathbb{Q}/\mathbb{Z}(2))\to H^3_{nr}(\overline{X},\mathbb{Q}/\mathbb{Z}(2))]\to\\ \to \mathrm{Coker}[CH^2(X)\to CH^2(\overline{X})^G]\to \mathrm{Coker}(\varphi).\end{aligned}$$

Notons

$$\mathcal{N}(X):=\mathrm{Ker}\Big[H^2(G,K_2(\overline{F}(X))\to H^2(G,\bigoplus_{x\in\overline{X}^{(1)}}\overline{F}(x)^\times)\Big] \qquad (1.2)$$

L'énoncé suivant est essentiellement établi dans [8].

Proposition 1.3. *Soit X une F-variété lisse et géométriquement intègre.*
(a) On a une suite exacte

$$\begin{aligned}H^3(F,\mathbb{Q}/\mathbb{Z}(2))\to \mathrm{Ker}[H^3_{nr}(X,\mathbb{Q}/\mathbb{Z}(2))\to H^3_{nr}(\overline{X},\mathbb{Q}/\mathbb{Z}(2))]\to\\ \to\mathcal{N}(X)\to \mathrm{Ker}[H^4(F,\mathbb{Q}/\mathbb{Z}(2))\to H^4(F(X),\mathbb{Q}/\mathbb{Z}(2))].\end{aligned}$$

(b) Si $X(F)\neq\emptyset$ ou si $\mathrm{cd}(F)\leq 3$, on a un isomorphisme

$$\mathrm{Ker}[H^3_{nr}(X,\mathbb{Q}/\mathbb{Z}(2))/H^3(F,\mathbb{Q}/\mathbb{Z}(2))\to H^3_{nr}(\overline{X},\mathbb{Q}/\mathbb{Z}(2))]\xrightarrow{\simeq}\mathcal{N}(X).$$

(c) Si X est de dimension au plus 2, on a une suite exacte

$$\begin{aligned}H^3(F,\mathbb{Q}/\mathbb{Z}(2))\to H^3_{nr}(X,\mathbb{Q}/\mathbb{Z}(2)\to\mathcal{N}(X)\to\\ \to H^4(F,\mathbb{Q}/\mathbb{Z}(2))\to H^4(F(X),\mathbb{Q}/\mathbb{Z}(2)).\end{aligned}$$

Démonstration. L'énoncé (a) est [8, Prop. 6.1, Prop. 6.2]. L'énoncé (b) est une conséquence facile de (a). La proposition 6.1 de [8] montre aussi que, si X est de dimension au plus 2, alors le complexe

$$H^3(F,\mathbb{Q}/\mathbb{Z}(2) \to \mathrm{Ker}[H^3_{nr}(X,\mathbb{Q}/\mathbb{Z}(2)) \to H^3_{nr}(\overline{X},\mathbb{Q}/\mathbb{Z}(2))] \to \\ \to \mathcal{N}(X) \to H^4(F,\mathbb{Q}/\mathbb{Z}(2)) \to H^4(F(X),\mathbb{Q}/\mathbb{Z}(2))$$

est une suite exacte

$$H^3(F,\mathbb{Q}/\mathbb{Z}(2) \to H^3_{nr}(X,\mathbb{Q}/\mathbb{Z}(2)) \to \\ \to \mathcal{N}(X) \to H^4(F,\mathbb{Q}/\mathbb{Z}(2)) \to H^4(F(X),\mathbb{Q}/\mathbb{Z}(2)).$$

En effet les groupes $H^3(A_s,\mathbb{Q}/\mathbb{Z}(2))$ intervenant dans la proposition 6.1 de [8] sont alors nuls : via la conjecture de Gersten, cela résulte du fait que le corps des fractions de A_s est de dimension cohomologique 2, si bien que le complexe de la proposition 6.1 de [8] est alors exact. □

2 Le cas où le groupe $H^0(\overline{X},\mathcal{K}_2)$ est uniquement divisible

Le but de ce pagragraphe est d'établir la proposition 2.4. On le fait d'abord par une méthode "K-théorique" (paragraphe 2.1) qui se prête plus aux calculs explicites des flèches intervenant dans les suites exactes. La version "motivique" (paragraphe 2.2) est plus souple quand il s'agit d'étudier la fonctorialité en la F-variété X des suites concernées.

Dans ce paragraphe, on considère une F-variété X lisse et géométriquement intègre, telle que le groupe $H^0(\overline{X},\mathcal{K}_2)$ est uniquement divisible, mais on ne suppose pas X projective.

2.1 Méthode K-théorique

Pour $i \geq 1$, les flèches naturelles

$$H^i(G,K_2\overline{F}(X)) \to H^i(G,K_2\overline{F}(X)/K_2\overline{F}) \to H^i(G,K_2\overline{F}(X)/H^0(\overline{X},\mathcal{K}_2))$$

sont alors des isomorphismes.

D'après un théorème de Quillen (conjecture de Gersten pour la K-théorie), le complexe

$$K_2\overline{F}(X) \to \bigoplus_{x\in\overline{X}^{(1)}} \overline{F}(x)^\times \to \bigoplus_{x\in\overline{X}^{(2)}} \mathbb{Z}$$

est le complexe des sections globales d'une résolution flasque du faisceau $\mathcal{K}_2$ sur la $\overline{F}$-variété lisse $\overline{X}$.

Ce complexe donne donc naissance à trois suites exactes courtes de modules galoisiens :

$$0 \to K_2\overline{F}(X)/H^0(\overline{X}, \mathcal{K}_2) \to Z \to H^1(\overline{X}, \mathcal{K}_2) \to 0$$
$$0 \to Z \to \oplus_{x \in \overline{X}^{(1)}} \overline{F}(x)^\times \to I \to 0$$
$$0 \to I \to \oplus_{x \in \overline{X}^{(2)}} \mathbb{Z} \to CH^2(\overline{X}) \to 0.$$

En utilisant le théorème 90 de Hilbert et le lemme de Shapiro, le théorème de Merkurjev–Suslin et en particulier sa conséquence [6, Thm. 1] [30, 1.8]

$$K_2F(X)/K_2F = (K_2\overline{F}(X)/K_2\overline{F})^G,$$

par des arguments classiques (cf. [10, 11]) de cohomologie galoisienne, on obtient :

Proposition 2.1. *Soit X une F-variété lisse et géométriquement intègre telle que le groupe $H^0(\overline{X}, \mathcal{K}_2)$ soit uniquement divisible. Soit $\mathcal{N}(X)$ comme en (1.2). On a alors une suite exacte*

$$\begin{aligned}0 \to H^1(X, \mathcal{K}_2) \to H^1(\overline{X}, \mathcal{K}_2)^G \to \\ \to H^1(G, K_2\overline{F}(X)) \to \mathrm{Ker}[CH^2(X) \to CH^2(\overline{X})] \to \\ \to H^1(G, H^1(\overline{X}, \mathcal{K}_2)) \to \mathcal{N}(X) \to \\ \to \mathrm{Coker}[CH^2(X) \to CH^2(\overline{X})^G] \to H^2(G, H^1(\overline{X}, \mathcal{K}_2)).\end{aligned}$$

Pour toute F-variété X géométriquement intègre, un théorème de B. Kahn [19, Cor. 2, p. 70] donne un isomorphisme

$$H^1(G, K_2\overline{F}(X)) \xrightarrow{\simeq} \mathrm{Ker}[H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \to H^3(F(X), \mathbb{Q}/\mathbb{Z}(2))].$$

On a donc établi :

Proposition 2.2. *Soit X une F-variété lisse et géométriquement intègre telle que le groupe $H^0(\overline{X}, \mathcal{K}_2)$ soit uniquement divisible. Soit $\mathcal{N}(X)$ comme en (1.2). On a alors une suite exacte*

$$\begin{aligned}0 \to H^1(X, \mathcal{K}_2) \to H^1(\overline{X}, \mathcal{K}_2)^G \to \\ \to \mathrm{Ker}[H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \to H^3(F(X), \mathbb{Q}/\mathbb{Z}(2))] \to \\ \to \mathrm{Ker}[CH^2(X) \to CH^2(\overline{X})] \to H^1(G, H^1(\overline{X}, \mathcal{K}_2)) \to \mathcal{N}(X) \to \\ \to \mathrm{Coker}[CH^2(X) \to CH^2(\overline{X})^G] \to H^2(G, H^1(\overline{X}, \mathcal{K}_2)).\end{aligned}$$

Remarque 2.3. Soit X un espace principal homogène d'un F-groupe semisimple simplement connexe absolument presque simple. On a $K_2(\overline{F}) = H^0(\overline{X}, \mathcal{K}_2)$, et ce groupe est donc uniquement divisible. On a par ailleurs $H^1(\overline{X}, \mathcal{K}_2) = \mathbb{Z}$ avec action triviale du groupe de Galois. L'image de 1 par l'application

$$H^1(\overline{X}, \mathcal{K}_2)^G \to H^3(F, \mathbb{Q}/\mathbb{Z}(2))$$

est (au signe près) l'invariant de Rost de X. Pour tout ceci, voir [16, Part II, §6].

En combinant les propositions 2.2 et 1.3 on trouve :

Proposition 2.4. *Soit X une F-variété lisse et géométriquement intègre. Supposons le groupe $H^0(\overline{X}, \mathcal{K}_2)$ uniquement divisible. Sous l'une des hypothèses $X(F) \neq \emptyset$ ou $\mathrm{cd}(F) \leq 3$, on a une suite exacte*

$$\begin{aligned} 0 &\to H^1(X, \mathcal{K}_2) \to H^1(\overline{X}, \mathcal{K}_2)^G \to \\ &\to \mathrm{Ker}[H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \to H^3(F(X), \mathbb{Q}/\mathbb{Z}(2))] \to \\ &\to \mathrm{Ker}[CH^2(X) \to CH^2(\overline{X})^G] \to H^1(G, H^1(\overline{X}, \mathcal{K}_2)) \to \\ &\to \mathrm{Ker}[H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2))/H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \to H^3_{nr}(\overline{X}, \mathbb{Q}/\mathbb{Z}(2))] \to \\ &\to \mathrm{Coker}[CH^2(X) \to CH^2(\overline{X})^G] \to H^2(G, H^1(\overline{X}, \mathcal{K}_2)). \end{aligned}$$

Sous l'hypothèse $X(F) \neq \emptyset$, le groupe

$$\mathrm{Ker}[H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \to H^3(F(X), \mathbb{Q}/\mathbb{Z}(2))]$$

est nul.

Remarque 2.5. Sous l'hypothèse $K_2(\overline{F}) = H^0(\overline{X}, \mathcal{K}_2)$ et $H^3_{nr}(\overline{X}, \mathbb{Q}/\mathbb{Z}(2)) = 0$, on retrouve l'énoncé de B. Kahn [20, Thm. 1, Corollaire].

2.2 Méthode motivique

Toujours *sous l'hypothèse que le groupe $H^0(\overline{X}, \mathcal{K}_2) \simeq \mathbb{H}^2(\overline{X}, \mathbb{Z}(2))$ est uniquement divisible*, étudions la suite spectrale

$$E_2^{pq} = H^p(G, \mathbb{H}^q(\overline{X}, \mathbb{Z}(2))) \Longrightarrow \mathbb{H}^n(X, \mathbb{Z}(2)).$$

La page E_2^{pq} contient un certain nombre de zéros. Tous les termes E_2^{p0} sont nuls. Comme $\mathbb{H}^2(\overline{X}, \mathbb{Z}(2))$ est supposé uniquement divisible, tous les termes $E_2^{p2} = H^p(G, \mathbb{H}^2(\overline{X}, \mathbb{Z}(2)))$ pour $p \geq 1$ sont nuls. Les termes E_2^{p1} sont égaux à $H^p(F, \mathbb{Q}/\mathbb{Z}(2))$ pour $p \geq 2$, groupe qui coïncide avec $H^{p+1}(F, \mathbb{Z}(2))$ pour $p \geq 3$. La flèche $E_2^{02} \to E_2^{21}$, soit $H^0(\overline{X}, \mathcal{K}_2)^G \to H^2(F, \mathbb{Q}/\mathbb{Z}(2))$, est surjective, car il en est déjà ainsi de $K_2\overline{F}^G \to H^2(F, \mathbb{Q}/\mathbb{Z}(2))$ (Remarque 1.1).
Notons comme ci-dessus $\varphi : \mathbb{H}^4(X, \mathbb{Z}(2)) \to \mathbb{H}^4(\overline{X}, \mathbb{Z}(2))^G$. L'analyse de la suite spectrale donne les énoncés suivants.

1) Il y a une suite exacte

$$\begin{aligned} 0 &\to \mathbb{H}^3(X, \mathbb{Z}(2)) \to (\mathbb{H}^3(\overline{X}, \mathbb{Z}(2))^G \to \mathbb{H}^4(F, \mathbb{Z}(2)) \to \mathrm{Ker}(\varphi) \to \\ &\to H^1(G, H^1(\overline{X}, \mathcal{K}_2)) \to \mathrm{Ker}[\mathbb{H}^5(F, \mathbb{Z}(2)) \to \mathbb{H}^5(X, \mathbb{Z}(2))]. \end{aligned}$$

Ainsi il y a une suite exacte

$$\begin{aligned} 0 &\to H^1(X, \mathcal{K}_2) \to (H^1(\overline{X}, \mathcal{K}_2))^G \to H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \to \mathrm{Ker}(\varphi) \to \\ &\to H^1(G, \mathbb{H}^3(\overline{X}, \mathbb{Z}(2))) \to \mathrm{Ker}[H^4(F, \mathbb{Q}/\mathbb{Z}(2)) \to H^4(F(X), \mathbb{Q}/\mathbb{Z}(2))]. \end{aligned}$$

En particulier, si $X(F) \neq \emptyset$ ou si l'on a $\mathrm{cd}(F) \leq 3$, alors on a une suite exacte

$$0 \to H^1(X, \mathcal{K}_2) \to (H^1(\overline{X}, \mathcal{K}_2))^G \to$$
$$\to H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \to \mathrm{Ker}(\varphi) \to H^1(G, H^1(\overline{X}, \mathcal{K}_2)) \to 0.$$

La flèche $H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \to \mathrm{Ker}(\varphi)$ est injective si $X(F) \neq \emptyset$, ou si $H^3(F, \mathbb{Q}/\mathbb{Z}(2))$ est nul, par exemple si $\mathrm{cd}(F) \leq 2$.

2) Pour le conoyau de φ, on trouve une suite exacte

$$0 \to D \to \mathrm{Coker}(\varphi) \to H^2(G, H^1(\overline{X}, \mathcal{K}_2))$$

où D est un sous-quotient de $\mathrm{Ker}[\mathbb{H}^5(F, \mathbb{Z}(2)) \to \mathbb{H}^5(X, \mathbb{Z}(2))]$. Ce dernier groupe est nul si le noyau de $H^4(F, \mathbb{Q}/\mathbb{Z}(2)) \to H^4(F(X), \mathbb{Q}/\mathbb{Z}(2))$ est nul. En particulier $D = 0$ si $X(F) \neq \emptyset$, ou si $\mathbb{H}^5(F, \mathbb{Z}(2)) = H^4(F, \mathbb{Q}/\mathbb{Z}(2))$ est nul, par exemple si $\mathrm{cd}(F) \leq 3$.

En utilisant la proposition 1.2, on voit que pour toute F-variété X lisse géométriquement intègre avec $H^0(\overline{X}, \mathcal{K}_2)$ uniquement divisible, sous l'hypothèse que soit $X(F) \neq \emptyset$ soit $\mathrm{cd}(F) \leq 3$, on a une suite exacte

$$0 \to \mathrm{Ker}[CH^2(X) \to CH^2(\overline{X})^G] \to \mathrm{Ker}(\varphi) \to$$
$$\to \mathrm{Ker}[H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2)) \to H^3_{nr}(\overline{X}, \mathbb{Q}/\mathbb{Z}(2))] \to$$
$$\to \mathrm{Coker}[CH^2(X) \to CH^2(\overline{X})^G] \to H^2(G, H^1(\overline{X}, \mathcal{K}_2)).$$

et une suite exacte

$$H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \to \mathrm{Ker}(\varphi) \to H^1(G, H^1(\overline{X}, \mathcal{K}_2)) \to 0.$$

Si l'on quotiente les deux termes $\mathrm{Ker}(\varphi) \subset \mathbb{H}^4(X, \mathbb{Z}(2))$ et $H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2))$ par l'image de $\mathbb{H}^4(F, \mathbb{Z}(2)) \simeq H^3(F, \mathbb{Q}/\mathbb{Z}(2))$, ce qui par fonctorialité de la suite spectrale appliquée au morphisme structural $X \to \mathrm{Spec}(F)$ induit une flèche $\mathrm{Ker}(\varphi)/\mathbb{H}^4(F, \mathbb{Z}(2)) \to H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2))/H^3(F, \mathbb{Q}/\mathbb{Z}(2))$, on trouve :

Proposition 2.6. *Soit X une F-variété lisse et géométriquement intègre. Supposons que $H^0(\overline{X}, K_2)$ est uniquement divisible. Supposons en outre que l'on a $X(F) \neq \emptyset$ ou $\mathrm{cd}(F) \leq 3$. On a alors une suite exacte*

$$0 \to H^1(X, \mathcal{K}_2) \to H^1(\overline{X}, \mathcal{K}_2)^G \to$$
$$\to \mathrm{Ker}[H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \to H^3(F(X), \mathbb{Q}/\mathbb{Z}(2))] \to$$
$$\to \mathrm{Ker}[CH^2(X) \to CH^2(\overline{X})^G] \to H^1(G, H^1(\overline{X}, \mathcal{K}_2)) \to$$
$$\to \mathrm{Ker}[H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2))/H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \to H^3_{nr}(\overline{X}, \mathbb{Q}/\mathbb{Z}(2))] \to$$
$$\to \mathrm{Coker}[CH^2(X) \to CH^2(\overline{X})^G] \to H^2(G, H^1(\overline{X}, \mathcal{K}_2)).$$

Sous l'hypothèse $X(F) \neq \emptyset$, le groupe

$$\mathrm{Ker}[H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \to H^3(F(X), \mathbb{Q}/\mathbb{Z}(2))]$$

est nul.

Remarques 2.7. (a) La démonstration n'utilise ni le groupe $\mathcal{N}(X)$ défini en (1.2) ni la proposition 1.3.
(b) L'énoncé de cette proposition est identique à celui de la proposition 2.4, mais il n'est pas clair a priori que les flèches intervenant dans ces deux suites exactes coïncident.

2.3 Comparaison entre les deux méthodes

Supposons $H^0(\overline{X},\mathcal{K}_2)$ uniquement divisible. On a une suite exacte

$$\mathrm{Ker}[CH^2(X)\to CH^2(\overline{X})]\to H^1(G,H^1(\overline{X},\mathcal{K}_2))\xrightarrow{\rho}$$
$$\xrightarrow{\rho}\mathcal{N}(X)\to \mathrm{Coker}[CH^2(X)\to CH^2(\overline{X})^G]$$

extraite de la proposition 2.2, et utilisée dans la démonstration de la proposition 2.4. On a une suite exacte

$$\mathrm{Ker}[CH^2(X)\to CH^2(\overline{X})]\to \mathrm{Ker}(\varphi)\xrightarrow{\sigma}$$
$$\xrightarrow{\sigma}\mathrm{Ker}[H^3_{nr}(X,\mathbb{Q}/\mathbb{Z}(2))\to H^3_{nr}(\overline{X},\mathbb{Q}/\mathbb{Z}(2))]\to \mathrm{Coker}[CH^2(X)\to CH^2(\overline{X})^G]$$

extraite de la proposition 1.2 et utilisée dans la démonstration de la proposition 2.6. Les termes de gauche et de droite dans ces deux suites exactes coïncident. Sous réserve de vérification des commutativités des diagrammes, sur tout corps F (sans condition de dimension cohomologique), le lien entre ces deux suites est fourni par le diagramme de suites exactes verticales

$$\begin{array}{ccc}
H^3(F,\mathbb{Q}/\mathbb{Z}(2)) & \xrightarrow{=} & H^3(F,\mathbb{Q}/\mathbb{Z}(2)) \\
\downarrow & & \downarrow \\
\mathrm{Ker}(\varphi) & \xrightarrow{\sigma} & \mathrm{Ker}\left[\begin{smallmatrix} H^3_{\mathrm{nr}}(X,\mathbb{Q}/\mathbb{Z}(2))\to \\ \to H^3_{\mathrm{nr}}(\overline{X},\mathbb{Q}/\mathbb{Z}(2))\end{smallmatrix}\right] \\
\downarrow & & \downarrow \\
H^1(G,H^1(\overline{X},\mathcal{K}_2)) & \xrightarrow{\rho} & \mathcal{N}(X) \\
\downarrow & & \downarrow \\
\mathrm{Ker}\left[\begin{smallmatrix} H^4(F,\mathbb{Q}/\mathbb{Z}(2)\to \\ \to H^4(F(X),\mathbb{Q}/\mathbb{Z}(2))\end{smallmatrix}\right] & \xrightarrow{=} & \mathrm{Ker}\left[\begin{smallmatrix} H^4(F,\mathbb{Q}/\mathbb{Z}(2)\to \\ \to H^4(F(X),\mathbb{Q}/\mathbb{Z}(2))\end{smallmatrix}\right]
\end{array}$$

où la suite verticale de droite vaut pour toute F-variété lisse et géométriquement intègre X ([8], voir la proposition 1.3 ci-dessus), et où celle de gauche est établie au début de la section 2.2 pour les F-variétés X telles que $H^0(\overline{X},\mathcal{K}_2)$ est uniquement divisible.

2.4 Variétés avec $H^0(\overline{X},\mathcal{K}_2)$ uniquement divisible

2.4.1 Les espaces classifiants de groupes semisimples

Soit H un F-groupe semisimple connexe, soit V une représentation linéaire génériquement libre de H possédant un ouvert H-stable $U \subset V$, de complémentaire un fermé de codimension au moins 3 dans V, et tel que que l'on dispose d'une application quotient $U \to U/H$ qui soit un H-torseur. Soit $X := U/H$. Soit H_{sc} le revêtement simplement connexe de H et soit C le noyau de l'isogénie $H_{sc} \to H$, puis $\hat{C}$ le module galoisien fini défini par son groupe des caractères. Comme le montre Merkurjev dans [24, Thm. 5.3], on a des identifications

$$K_2(\overline{F}) = H^0(\overline{X},\mathcal{K}_2)$$

et

$$\hat{C}(1) := \mathrm{Tor}_1^{\mathbb{Z}}(\hat{C},\mathbb{Q}/\mathbb{Z}(1)) \simeq H^1(\overline{X},\mathcal{K}_2).$$

Le groupe $K_2(\overline{F})$ est uniquement divisible. La F-variété X possède un point F-rationnel.
La proposition 2.4 et la proposition 2.6 donnent donc chacune une suite exacte longue

$$\begin{aligned}0 \to \mathrm{Ker}[CH^2(X) \to CH^2(\overline{X})^G] \to H^1(G,\hat{C}(1)) \to \\ \to \mathrm{Ker}[H^3_{nr}(X,\mathbb{Q}/\mathbb{Z}(2))/H^3(F,\mathbb{Q}/\mathbb{Z}(2)) \to H^3_{nr}(\overline{X},\mathbb{Q}/\mathbb{Z}(2))] \to \\ \to \mathrm{Coker}[CH^2(X) \to CH^2(\overline{X})^G] \to H^2(G,\hat{C}(1)).\end{aligned}$$

Il serait intéressant de déterminer le lien entre la suite exacte à 5 termes obtenue par Merkurjev [25, Thm. 3.9] et les suites exactes à 5 termes ci-dessus. Elles ont en commun leurs deux premiers termes, et leur dernier terme.

2.4.2 Variétés projectives

Pour $\overline{F}$ un corps algébriquement clos – toujours supposé de caractéristique nulle – et Y une $\overline{F}$-variété intègre, projective et lisse, les propriétés suivantes sont équivalentes :
(i) Le groupe de Picard $\mathrm{Pic}(Y)$ est sans torsion.
(ii) Pour tout entier $n > 0$, $H^1_{\text{ét}}(Y,\mu_n) = 0$.
(iii) $H^1(Y,\mathcal{O}_Y) = 0$ et le groupe de Néron–Severi $\mathrm{NS}(Y)$ est sans torsion.
(iv) Le groupe $H^0(Y,\mathcal{K}_2)$ est uniquement divisible.
L'équivalence des trois premières propriétés est classique. Pour l'équivalence avec la quatrième, voir [10, Prop. 1.13], qui s'appuie sur des résultats de Merkurjev et de Suslin.

Les propriétés ci-dessus sont satisfaites par toute $\overline{F}$-variété projective et lisse géométriquement unirationnelle, mais aussi par toute surface $K3$ et par toute surface projective et lisse dans l'espace projectif $\mathbf{P}^3$.
Pour une $\overline{F}$-surface Y projective et lisse satisfaisant ces propriétés, la dualité de Poincaré implique la nullité des groupes $H^3_{\text{ét}}(Y,\mu_n)$ pour tout $n > 0$. On sait

(Bloch, Merkurjev–Suslin, cf. [10, (2.1)]) que la nullité de ces groupes implique que le groupe de Chow $CH^2(Y)$ n'a pas de torsion.
Pour une F-surface X projective, lisse et géométriquement intègre telle que $\overline{X}$ satisfasse ces propriétés, le groupe $\mathrm{Ker}[CH^2(X) \to CH^2(\overline{X})]$ coïncide donc avec le sous-groupe de torsion $CH^2(X)_{tors}$ de $CH^2(X)$.

Sans hypothèse supplémentaire sur X, il est difficile de contrôler le module galoisien $H^1(\overline{X}, K_2)$ et l'application

$$CH^2(X)_{tors} = \mathrm{Ker}[CH^2(X) \to CH^2(\overline{X})] \to H^1(G, H^1(\overline{X}, \mathcal{K}_2)).$$

Renvoyons ici le lecteur au délicat travail d'Asakura et Saito [1] qui établit que pour un corps p-adique F et une surface lisse dans $\mathbf{P}^3_F$, de degré au moins 5 "très générale", le groupe

$$CH^2(X)_{tors} \subset H^1(G, H^1(\overline{X}, \mathcal{K}_2))$$

est infini.
Au paragraphe suivant, on donnera des hypothèses restrictives permettant de facilement contrôler le module $H^1(\overline{X}, \mathcal{K}_2)$ et sa cohomologie galoisienne.

3 Le module galoisien $H^1(\overline{X}, \mathcal{K}_2)$

On considère la flèche naturelle

$$\mathrm{Pic}(\overline{X}) \otimes \overline{F}^\times \to H^1(\overline{X}, \mathcal{K}_2).$$

Proposition 3.1. *Soit X une F-variété projective, lisse et géométriquement intègre. Supposons $H^2(X, \mathcal{O}_X) = 0$ et supposons que les groupes $H^3_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell)$ sont sans torsion. Alors pour tout $i \geq 2$, la flèche*

$$H^i(G, \mathrm{Pic}(\overline{X}) \otimes \overline{F}^\times) \to H^i(G, H^1(\overline{X}, \mathcal{K}_2))$$

est un isomorphisme.

Démonstration. D'après [10, Thm. 2.12], la flèche Galois équivariante

$$\mathrm{Pic}(\overline{X}) \otimes \overline{F}^\times \to H^1(\overline{X}, \mathcal{K}_2)$$

a alors noyau et conoyau uniquement divisibles. Elle induit donc un isomorphisme sur $H^i(G, \bullet)$ pour $i \geq 2$. □

Remarque 3.2. L'hypothèse que les groupes $H^3_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell)$ sont sans torsion est équivalente à l'hypothèse que le groupe de Brauer $\mathrm{Br}(\overline{X})$ est un groupe divisible.

Proposition 3.3. *Soit X une F-variété projective, lisse et géométriquement intègre. Supposons qu'il existe une courbe $V \subset X$ telle que sur un domaine* *universel Ω l'application $CH_0(V_\Omega) \to CH_0(X_\Omega)$ est surjective, et supposons que les groupes $H^3_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell)$ sont sans torsion. Alors pour tout $i \geq 1$, la flèche*

$$H^i(G, \operatorname{Pic}(\overline{X}) \otimes \overline{F}^\times) \to H^i(G, H^1(\overline{X}, \mathcal{K}_2))$$

est un isomorphisme.

Démonstration. D'après [10, Thm. 2.12 ; Prop. 2.15], sous les hypothèses de la proposition, la flèche Galois-équivariante

$$\operatorname{Pic}(\overline{X}) \otimes \overline{F}^\times \to H^1(\overline{X}, \mathcal{K}_2)$$

est surjective et a un noyau uniquement divisible. Elle induit donc un isomorphisme sur $H^i(G, \bullet)$ pour $i \geq 1$. □

Remarques 3.4. Rappelons que l'on suppose $\operatorname{car}(F) = 0$.
(a) L'hypothèse sur le groupe de Chow des zéro-cycles faite dans la proposition 3.3 implique $H^i(X, \mathcal{O}_X) = 0$ pour $i \geq 2$. Elle implique que le groupe de Brauer $\operatorname{Br}(\overline{X})$ est un groupe fini. Elle est satisfaite pour les variétés $\overline{X}$ dominées rationnellement par le produit d'une courbe et d'un espace projectif, en particulier elle est satisfaite pour les variétés géométriquement unirationnelles.
(b) Sous les hypothèses de la proposition 3.3, on a $\operatorname{Br}(\overline{X}) = 0$.
(c) Toutes les hypothèses de la proposition 3.3 sont satisfaites pour une variété $\overline{X}$ qui est facteur direct birationnel d'une variété rationnelle.

La proposition suivante (cf. [11, Prop. 8.10]) s'applique par exemple aux surfaces $K3$ sur F corps de fonctions d'une variable sur $\mathbb{C}$, ou sur $F = \mathbb{C}((t))$.

Proposition 3.5. *Supposons le corps F de dimension cohomologique au plus 1. Soit X une F-surface projective, lisse, géométriquement connexe, satisfaisant $H^1(X, O_X) = 0$. Supposons le groupe $\operatorname{Pic}(\overline{X}) = \operatorname{NS}(\overline{X})$ sans torsion. On a alors un homomorphisme surjectif*

$$H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2)) \to \operatorname{Coker}[CH^2(X) \to CH^2(\overline{X})^G].$$

Si l'indice $I(X)$ de X, qui est le pgcd des degrés sur F des points fermés de X, n'est pas égal à 1, alors $H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2)) \neq 0$.

Démonstration. Sous les hypothèses de la proposition, le groupe $H^0(\overline{X}, \mathcal{K}_2)$ est uniquement divisible [10, Cor. 1.12]. Le groupe $H^1(\overline{X}, \mathcal{K}_2)$ est extension d'un groupe fini par un groupe divisible [10, Thm. 2.2], donc $H^2(G, H^1(\overline{X}, \mathcal{K}_2)) = 0$. Comme X est une surface, on a $H^3_{nr}(\overline{X}, \mathbb{Q}/\mathbb{Z}(2)) = 0$. La surjection résulte alors de la proposition 2.4 (ou de la proposition 2.6). Pour la surface X, on a la suite exacte de modules galoisiens

$$0 \to A_0(\overline{X}) \to CH^2(\overline{X}) \to \mathbb{Z} \to 0,$$

où la flèche $CH^2(\overline{X}) \to \mathbb{Z}$ est donnée par le degré des zéro-cycles. L'hypothèse $H^1(X, O_X) = 0$ implique que le groupe $A_0(\overline{X})$ est uniquement divisible (théorème de Roitman). L'application induite $CH^2(\overline{X})^G \to \mathbb{Z}$ est donc surjective, et le groupe $\operatorname{Coker}[CH^2(X) \to CH^2(\overline{X})^G]$ a pour quotient le groupe $\mathbb{Z}/I(X)$. □

Exemple 3.6. Soit $F = \mathbb{C}((t))$. Soient $n > 0$ un entier et $X \subset \mathbf{P}^3_F$ la surface définie par l'équation homogène

$$x_0^n + tx_1^n + t^2x_2^n + t^3x_3^n = 0.$$

D'après [13, Prop. 4.4], pour $n = 4$ (surface $K3$) et pour n premier à 6, on a $I(X) \neq 1$. La proposition ci-dessus donne alors $H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2)) \neq 0$.

4 Variétés à petit motif sur un corps non algébriquement clos

Commençons par un énoncé général mais peut-être un peu lourd.

Théorème 4.1. *Soit X une F-variété projective, lisse et géométriquement intègre.*
Supposons satisfaites les conditions :
(i) Sur un domaine universel Ω, le degré $CH_0(X_\Omega) \to \mathbb{Z}$ est un isomorphisme.
(ii) Le groupe $\mathrm{Pic}(\overline{X}) = \mathrm{NS}(\overline{X})$ est sans torsion.
(iii) Pour tout ℓ premier, le groupe $H^3_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell)$ est sans torsion.
(iv) On a au moins l'une des propriétés : $X(F) \neq \emptyset$ ou $\mathrm{cd}(F) \leq 3$.
Alors on a une suite exacte

$$\mathrm{Ker}[CH^2(X) \to CH^2(\overline{X})^G] \xrightarrow{\alpha} H^1(G, \mathrm{Pic}(\overline{X}) \otimes \overline{F}^\times) \to$$
$$\to \mathrm{Ker}[H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2))/H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \to H^3_{nr}(\overline{X}, \mathbb{Q}/\mathbb{Z}(2))] \to$$
$$\to \mathrm{Coker}[CH^2(X) \to CH^2(\overline{X})^G] \xrightarrow{\beta} H^2(G, \mathrm{Pic}(\overline{X}) \otimes \overline{F}^\times).$$

Sous l'hypothèse $X(F) \neq \emptyset$ ou $\mathrm{cd}(F) \leq 2$, la flèche α est injective.

Démonstration. Comme on a supposé $\mathrm{car}(F) = 0$, d'après [5], l'hypothèse (i) implique que tous les groupes $H^i(X, \mathcal{O}_X)$ pour $i \geq 1$ sont nuls, que l'on a $\mathrm{Pic}(\overline{X}) = \mathrm{NS}(\overline{X})$, et que le groupe de Brauer $\mathrm{Br}(\overline{X})$ s'identifie au groupe fini $\oplus_\ell H^3(\overline{X}, \mathbb{Z}_\ell)_{tors}$. Sous l'hypothèse (i), l'hypothèse (iii) est donc équivalente à $\mathrm{Br}(\overline{X}) = 0$.
Sous les hypothèses (i) et (iii), la proposition 3.3 donne

$$H^i(G, \mathrm{Pic}(\overline{X}) \otimes \overline{F}^\times) \stackrel{\simeq}{\to} H^i(G, H^1(\overline{X}, \mathcal{K}_2))$$

pour tout $i \geq 1$.
Sous les hypothèses (i) et (ii), d'après [10, Prop. 1.14], on a $K_2\overline{F} = H^0(\overline{X}, \mathcal{K}_2)$. Le groupe $K_2\overline{F}$ étant uniquement divisible, on peut appliquer la Proposition 2.4 (ou la proposition 2.6). ☐

Corollaire 4.2. *Soit X une F-variété projective, lisse et géométriquement intègre.*
Supposons $X(F) \neq \emptyset$ ou $\mathrm{cd}(F) \leq 3$.
Supposons satisfaite l'une des hypothèses suivantes :
(i) la variété $\overline{X}$ est rationnelle ;

(ii) la variété $\overline{X}$ *est rationnellement connexe, et l'on a* $\mathrm{Br}(\overline{X}) = 0$ *et* $H^3_{nr}(\overline{X}, \mathbb{Q}/\mathbb{Z}(2)) = 0$ *;*
(iii) la variété $\overline{X}$ *est de dimension 3, rationnellement connexe, et* $\mathrm{Br}(\overline{X}) = 0$ *;*
(iv) la variété $\overline{X}$ *est de dimension 3, unirationnelle, et* $\mathrm{Br}(\overline{X}) = 0$*.*
Alors on a une suite exacte

$$\mathrm{Ker}[CH^2(X) \to CH^2(\overline{X})^G] \xrightarrow{\alpha} H^1(G, \mathrm{Pic}(\overline{X}) \otimes \overline{F}^{\times}) \to$$

$$\to H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2))/H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \to$$

$$\to \mathrm{Coker}[CH^2(X) \to CH^2(\overline{X})^G] \xrightarrow{\beta} H^2(G, \mathrm{Pic}(\overline{X}) \otimes \overline{F}^{\times}).$$

Sous l'hypothèse $X(F) \neq \emptyset$ *ou* $\mathrm{cd}(F) \leq 2$*, la flèche* α *est injective.*

Démonstration. Le cas (iv) est un cas particulier du cas (iii). Sous l'hypothèse (i), tous les groupes $H^i_{nr}(\overline{X}, \mathbb{Q}/\mathbb{Z}(2))$ sont nuls pour $i \geq 1$. Pour $i = 1$, cela établit que $\mathrm{Pic}(\overline{X})$ est sans torsion et donc $\mathrm{Pic}(\overline{X}) = \mathrm{NS}(\overline{X})$. Pour $i = 2$, cela établit $\mathrm{Br}(\overline{X}) = 0$ et donc $H^3_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell)_{tors} = 0$ pour tout premier ℓ.
Sous l'hypothèse (iii), on a $H^3_{nr}(\overline{X}, \mathbb{Q}/\mathbb{Z}(2)) = 0$. Cette annulation vaut en effet pour tout solide uniréglé [11, Cor. 6.2], c'est un corollaire d'un théorème de C. Voisin.
L'énoncé est alors une conséquence immédiate du théorème 4.1. □

Remarques 4.3. (a) Dans le cas particulier où X est une F-compactification lisse équivariante d'un F-tore, le corollaire 4.2 est très proche d'un résultat de Blinstein et Merkurjev ([4, Prop. 5.9]). Dans ce cas, le groupe $CH^2(\overline{X})$ est sans torsion, le groupe

$$\mathrm{Ker}[CH^2(X) \to CH^2(\overline{X})^G]$$

coïncide donc avec $CH^2(X)_{tors}$. Par ailleurs, l'intersection des cycles

$$\mathrm{Pic}(\overline{X}) \times \mathrm{Pic}(\overline{X}) \to CH^2(\overline{X})$$

induit une application naturelle surjective ([15, §5.2, Proposition, p. 106])

$$\mathrm{Sym}^2(\mathrm{Pic}(\overline{X})) \to CH^2(\overline{X}).$$

(b) Soit X une F-compactification lisse d'un F-tore. La flèche

$$H^1(G, \mathrm{Pic}(\overline{X}) \otimes \overline{F}^{\times}) \to H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2))/H^3(F, \mathbb{Q}/\mathbb{Z}(2))$$

intervient dans l'étude de l'approximation faible pour X sur le corps F des fonctions d'une courbe sur un corps p-adique (Harari, Scheiderer, Szamuely [18, Thm. 4.2]). Pour X une F-compactification lisse d'un espace principal homogène d'un F-tore, il conviendrait de comparer la flèche

$$H^1(G, \mathrm{Pic}(\overline{X}) \otimes \overline{F}^{\times}) \to H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2))/H^3(F, \mathbb{Q}/\mathbb{Z}(2))$$

ici obtenue (le corps F satisfaisant $cd(F) \leq 3$) avec l'application (19) utilisée dans [17, Thm. 5.1].

(c) Soit X/F une surface projective, lisse, géométriquement rationnelle possédant un zéro-cycle de degré 1, et telle que le module galoisien $\mathrm{Pic}(\overline{X})$ soit un facteur direct d'un module de permutation. Le corollaire ci-dessus implique alors $H^3(F,\mathbb{Q}/\mathbb{Z}(2)) \xrightarrow{\simeq} H^3_{nr}(X,\mathbb{Q}/\mathbb{Z}(2))$. C'est un cas particulier d'une remarque générale pour toute telle surface. Si le module galoisien $\mathrm{Pic}(\overline{X})$ est un facteur direct d'un module de permutation, alors, d'après [6, Prop. 4, p. 12], sur tout corps L contenant F, l'application degré $CH_0(X_L) \to \mathbb{Z}$ est un isomorphisme. Ceci implique $H^i(F,\mathbb{Q}/\mathbb{Z}(2)) \xrightarrow{\simeq} H^i_{nr}(X,\mathbb{Q}/\mathbb{Z}(2))$ pour tout entier i (cas particulier d'un théorème de Merkurjev, cf. [2, Thm. 1.4]).

(d) Dans l'article [9] avec Madore, on a construit des exemples de corps F de dimension cohomologique 1 et de surfaces X/F projectives, lisses, géométriquement rationnelles sans zéro-cycle de degré 1. Pour de telles surfaces, le corollaire 4.2 ci-dessus donne

$$H^3_{nr}(X,\mathbb{Q}/\mathbb{Z}(2)) = H^3_{nr}(X,\mathbb{Q}/\mathbb{Z}(2))/H^3(F,\mathbb{Q}/\mathbb{Z}(2)) \neq 0.$$

(e) Pour X une F-variété projective, lisse, géométriquement connexe quelconque, chacun des trois groupes suivants est un invariant F-birationnel de X :
– le groupe $\mathrm{Ker}[CH^2(X) \to CH^2(\overline{X})^G]$
– le groupe $H^1(G,\mathrm{Pic}(\overline{X}) \otimes \overline{F}^{\times})$
– le groupe $H^3_{nr}(X,\mathbb{Q}/\mathbb{Z}(2))$.
Si la dimension cohomologique de F est au plus 1, le groupe

$$\mathrm{Coker}[CH^2(X) \to CH^2(\overline{X})^G]$$

est un invariant F-birationnel, comme on voit en considérant la situation de l'éclatement en une sous-variété lisse. En général, ce groupe n'est pas un invariant birationnel, comme on peut voir en éclatant $\mathbf{P}^3_F$ en une F-conique lisse sans F-point. Ceci montre aussi que l'application

$$\beta : \mathrm{Coker}[CH^2(X) \to CH^2(\overline{X})^G] \longrightarrow H^2(G,\mathrm{Pic}(\overline{X}) \otimes \overline{F}^{\times})$$

n'est pas toujours nulle.

5 Variétés à petit motif sur le corps des complexes

5.1 Rappels

Pour tout corps F contenant $\mathbb{C}$, on note $A^2(X_F)$ le sous-groupe de $CH^2(X_F)$ formé des classes de cycles qui sur une clôture algébrique $\overline{F}$ de F sont algébriquement équivalents à zéro.
La proposition suivante rassemble des résultats connus, utiles pour la suite de ce paragraphe.

Proposition 5.1. *Soit X une variété connexe, projective et lisse sur le corps des complexes. Supposons que l'application degré $CH_0(X) \to \mathbb{Z}$ est un isomorphisme.*
Alors
(i) On a $H^i(X, \mathcal{O}_X) = 0$ pour $i \geq 1$.
(ii) Pour tout corps F contenant $\mathbb{C}$, les applications de restriction

$$\mathrm{Pic}(X) \to \mathrm{Pic}(X_F) \to \mathrm{Pic}(X_{\overline{F}})$$

sont des isomorphismes, et $\mathrm{Pic}(X) = \mathrm{NS}^1(X) = H^2_{Betti}(X, \mathbb{Z})$.
(iii) Équivalence homologique et équivalence algébrique coïncident sur le groupe de Chow $CH^2(X)$.
(iv) Le quotient $\mathrm{NS}^2(X) := CH^2(X)/A^2(X) \subset H^4_{Betti}(X, \mathbb{Z})$ *est un groupe abélien de type fini. Pour tout corps algébriquement clos F contenant $\mathbb{C}$, on a* $\mathrm{NS}^2(X) \stackrel{\simeq}{\to} \mathrm{NS}^2(X_F)$.
(v) Il existe une variété abélienne B sur $\mathbb{C}$ qui est un représentant algébrique de $A^2(X)$, au sens de Murre ([28], cf. [3, Déf. 3.2]). Pour tout corps F contenant $\mathbb{C}$, on a un homomorphisme $A^2(X_F) \to B(F)$ fonctoriel en F, et cet homomorphisme est un isomorphisme si F est algébriquement clos.
(vi) S'il existe un premier l avec $H^3_{Betti}(X, \mathbb{Z}/l) = 0$, alors $A^2(X) = 0$, on a une inclusion $CH^2(X) \hookrightarrow H^4_{Betti}(X, \mathbb{Z})$, et ces groupes sont sans l-torsion.

Démonstration. Pour les énoncés (i), (iii), (iv), (v), dus essentiellement à Bloch et Srinivas, et reposant sur des théorèmes de Merkujev–Suslin et de Murre [28], voir [5, Thm. 1] et [31]. L'énoncé (ii) est une conséquence connue de $H^1(X, \mathcal{O}_X) = 0$. Le dernier énoncé de (iv) est une propriété générale des quotients des groupes de Chow modulo l'équivalence algébrique. Pour l'énoncé (vi), les travaux de Bloch et de Merkurjev–Suslin montrent que le sous-groupe de l-torsion $CH^2(X)[l]$ de $CH^2(X)$ est un sous-quotient de $H^3_{Betti}(X, \mathbb{Z}/l)$. On a donc $CH^2(X)[l] = 0$ et a fortiori $A^2(X)[l] = 0$, donc $B[l] = 0$, donc la variété abélienne B est triviale et $A^2(X) = 0$. □

Remarques 5.2. (a) Si X est une variété rationnellement connexe, alors l'application degré $CH_0(X) \to \mathbb{Z}$ est un isomorphisme, les propriétés (i) à (v) sont donc satisfaites.
(b) Les énoncés (iii) à (v) valent sous l'hypothèse plus faible qu'il existe une courbe projective et lisse C et un morphisme $C \to X$ qui induise une surjection $CH_0(C) \to CH_0(X)$.

5.2 Cycle de codimension 2 universel

Soient F un corps, X et Y deux F-variétés projectives, lisses, géométriquement connexes. Soit $z \in CH^2(X \times_F Y)$ une classe de cycle de codimension 2. La théorie des correspondances [14] donne une application bilinéaire

$$CH_0(Y) \times CH^2(Y \times_F X) \to CH^2(X).$$

Le sous-groupe $A_0(Y)$ des zéro-cycles de degré 0 est formé de classes géométriquement algébriquement équivalentes à zéro. Un élément $z \in CH^2(Y \times_F X)$ définit donc un homomorphisme

$$CH_0(Y) \to CH^2(X)$$

envoyant le groupe $A_0(Y)$ dans le sous-groupe $A^2(X) \subset CH^2(X)$ défini au début du §5. Cette application est fonctorielle en le corps de base F. Via la flèche évidente $Y(F) \to CH_0(Y)$ envoyant un point rationnel sur sa classe dans le groupe de Chow, elle induit une application $Y(F) \to CH^2(X)$ qui ne saurait être qu'ensembliste. Si Y est muni d'un point rationnel noté O, en envoyant P sur la classe de $P - O$, on définit une flèche ensembliste

$$\theta_z : Y(F) \to A^2(X)$$

envoyant O sur 0.

Soient X et B comme dans la proposition 5.1. On note O l'élément neutre de de $B(\mathbb{C})$. La définition suivante est une variante de celle donnée par Claire Voisin [34, Déf. 0.5].

Définition 5.3. Pour X et B comme ci-desssus, on dit qu'il existe un cycle de codimension 2 universel sur X s'il existe un cycle $z \in CH^2(B \times X)$ tel que, sur tout corps F contenant $\mathbb{C}$, l' application ensembliste

$$\theta_z : B(F) \to A^2(X_F)$$

définie ci-dessus satisfasse la propriété : L'application composée

$$B(F) \to A^2(X_F) \to B(F)$$

est l'identité sur $B(F)$.

Le théorème ci-dessous est une variante d'un résultat de C. Voisin [34, Thm. 2.1, Cor. 2.3]. La démonstration ici proposée diffère sensiblement de celle donnée dans [34].

Théorème 5.4. *Soit X une variété connexe, projective et lisse sur $\mathbb{C}$. Supposons les conditions suivantes satisfaites.*
(i) L'application degré $CH_0(X) \to \mathbb{Z}$ est un isomorphisme.
(ii) Les groupes $H^2_{Betti}(X, \mathbb{Z})$ et $H^3_{Betti}(X, \mathbb{Z})$ sont sans torsion.
(iii) On a $H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2)) = 0$.
Alors : (1) Pour tout corps F contenant $\mathbb{C}$, on a une suite exacte

$$0 \to H^3_{nr}(X_F, \mathbb{Q}/\mathbb{Z}(2))/H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \to \mathrm{Coker}[CH^2(X_F) \to CH^2(X_{\overline{F}})^G] \xrightarrow{\beta} H^2(G, \mathrm{Pic}(X) \otimes \overline{F}^{\times}). \tag{5.4}$$

(2) Soit B le représentant algébrique de $A^2(X)$ (Prop. 5.1 (v)). S'il existe un cycle de codimension 2 universel dans $CH^2(B \times X)$, alors pour tout corps F contenant $\mathbb{C}$, on a $H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \xrightarrow{\simeq} H^3_{nr}(X_F, \mathbb{Q}/\mathbb{Z}(2))$.

Note : Sous l'hypothèse $CH_0(X) = \mathbb{Z}$, la condition $H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2)) = 0$ est, d'après [11, Thm. 1.1], équivalente au fait que la conjecture de Hodge entière vaut en degré 4, i.e. pour les cycles de codimension 2.

Démonstration. Soit F un corps contenant $\mathbb{C}$. Soit $\overline{F}$ une clôture algébrique de F et $G = \mathrm{Gal}(\overline{F}/F)$. D'après le théorème 4.1 appliqué à la F-variété $X_F := X \times_{\mathbb{C}} F$, on a une suite exacte

$$H^1(G, \mathrm{Pic}(X_{\overline{F}}) \otimes \overline{F}^{\times}) \to$$

$$\to \mathrm{Ker}[H^3_{nr}(X_F, \mathbb{Q}/\mathbb{Z}(2))/H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \to H^3_{nr}(X_{\overline{F}}, \mathbb{Q}/\mathbb{Z}(2))] \to$$

$$\to \mathrm{Coker}[CH^2(X_F) \to CH^2(X_{\overline{F}})^G] \xrightarrow{\beta} H^2(G, \mathrm{Pic}(X_{\overline{F}}) \otimes \overline{F}^{\times})$$

On sait [7, Thm. 4.4.1] que la cohomologie non ramifiée est invariante par extension de corps de base algébriquement clos. Sous l'hypothèse (iii), on a donc $H^3_{nr}(X_{\overline{F}}, \mathbb{Q}/\mathbb{Z}(2)) = 0$. Sous les hypothèses (i) et (ii), les applications de restriction $\mathrm{Pic}(X) \to \mathrm{Pic}(X_F) \to \mathrm{Pic}(X_{\overline{F}})$ sont des isomorphismes de réseaux (Proposition 5.1 (ii)). L'action de $\mathrm{Gal}(\overline{F}/F)$ sur le réseau $\mathrm{Pic}(X_{\overline{F}})$ est donc triviale. Le théorème 90 de Hilbert donne alors

$$H^1(G, \mathrm{Pic}(X) \otimes \overline{F}^{\times}) = 0.$$

Ceci donne la suite exacte (5.4).
Supposons qu'il existe un cycle de codimension 2 universel. Alors, sur tout corps F contenant $\mathbb{C}$, on dispose de l'application ensembliste $B(F) \to A^2(X_F)$ qui composée avec l'application $A^2(X_F) \to B(F)$ est l'identité. Ceci implique que l'homomorphisme $A^2(X_F) \to A^2(X_{\overline{F}})^G$ est une surjection. L'application composée $\mathrm{NS}^2(X) \to \mathrm{NS}^2(X_{\overline{F}})^G$ est surjective, car $\mathrm{NS}^2(X) \to \mathrm{NS}^2(X_{\overline{F}})$ est un isomorphisme (Prop. 5.1 (iv)). Ainsi $CH^2(X) \to CH^2(X_{\overline{F}})^G$ est surjectif, et de la suite exacte (5.4) on déduit $H^3(F, \mathbb{Q}/\mathbb{Z}(2)) = H^3_{nr}(X_F, \mathbb{Q}/\mathbb{Z}(2))$. □

Remarque 5.5. Sous des hypothèses additionnelles, C. Voisin [34, Thm. 2.1, Cor. 2.3] établit une réciproque du théorème 5.4. Il serait souhaitable d'établir une telle réciproque par les méthodes plus K-théoriques du présent article, en utilisant la suite exacte (5.4) pour le corps des fonctions $F = \mathbb{C}(B)$ du représentant algébrique B de $A^2(X)$.

5.3 Troisième groupe de cohomologie non ramifiée des hypersurfaces de Fano

THÉORÈME 5.6. *Soit $n \geq 4$. Soit $X \subset \mathbf{P}^n_{\mathbb{C}}$ une hypersurface lisse de degré $d \leq n$.*
(i) La flèche degré $CH_0(X) \to Z$ est un isomorphisme.
(ii) On a $\mathrm{Pic}(X) = \mathrm{NS}(X) = H^2_{Betti}(X, \mathbb{Z}) = \mathbb{Z}$, et ce groupe est engendré par la classe d'une section hyperplane.
(iii) Le groupe $H^3_{Betti}(X, \mathbb{Z})$ est sans torsion, et nul pour $n \geq 5$.

(iv) Pour $n \geq 5$, équivalences rationnelle, algébrique et homologique coïncident sur les cycles de codimension 2 sur X, et on a une injection de réseaux $CH^2(X) \hookrightarrow H^4_{Betti}(X,\mathbb{Z})$.
(v) Pour $n \neq 5$, $H^4_{Betti}(X,\mathbb{Z}) = \mathbb{Z}$, et l'application

$$CH^2(X) \to H^4_{Betti}(X,\mathbb{Z}) = \mathbb{Z}$$

est surjective, et est un isomorphisme pour $n > 5$.
(vi) Pour $n = 4$ et $n > 5$, on a $H^3_{nr}(X,\mathbb{Q}/\mathbb{Z}(2)) = 0$.
(vii) Pour $n \geq 5$, pour tout corps F contenant $\mathbb{C}$, de clôture algébrique $\overline{F}$, avec $G := \mathrm{Gal}(\overline{F}/F)$, la flèche naturelle

$$CH^2(X_F) \to CH^2(X_{\overline{F}})^G$$

est surjective, et on a une suite exacte naturellement scindée

$$0 \to H^3(F,\mathbb{Q}/\mathbb{Z}(2)) \to H^3_{nr}(X_F,\mathbb{Q}/\mathbb{Z}(2)) \to H^3_{nr}(X_{\overline{F}},\mathbb{Q}/\mathbb{Z}(2)) \to 0.$$

Pour $n > 5$,on a

$$H^3(F,\mathbb{Q}/\mathbb{Z}(2)) \xrightarrow{\simeq} H^3_{nr}(X_F,\mathbb{Q}/\mathbb{Z}(2)).$$

(viii) Pour $n = 4$, soit B le représentant algébrique de $A^2(X)$. S'il existe un cycle universel de codimension 2 dans $CH^2(B \times X)$, alors pour tout corps F contenant $\mathbb{C}$, on a $H^3(F,\mathbb{Q}/\mathbb{Z}(2)) \xrightarrow{\simeq} H^3_{nr}(X_F,\mathbb{Q}/\mathbb{Z}(2))$, et l'application $CH^2(X_F) \to CH^2(X_{\overline{F}})^G$ est surjective.

Démonstration. Les énoncés (i) à (v) sont bien connus. Comme ils sont utilisés pour établir les points suivants, donnons quelques rappels à leur sujet.
L'hypothèse $d \leq n$ assure $CH_0(X) \xrightarrow{\simeq} \mathbb{Z}$, soit (i). C'est un théorème de Roitman, que l'on peut aussi voir comme un cas particulier du théorème de Campana et Kollár-Miyaoka-Mori assurant qu'une variété de Fano est rationnellement connexe. L'énoncé (ii) vaut pour toute hypersurface lisse dans $\mathbf{P}^n_{\mathbb{C}}$, $n \geq 4$.
Pour $n \geq 5$, les théorèmes de Lefschetz donnent $H^3_{Betti}(X,\mathbb{Z}) = 0$ et $H^3_{Betti}(X,\mathbb{Z}/l) = 0$ pour tout l premier. L'énoncé (i) et la proposition 5.1 donnent alors (iv).
Pour $n = 4$, $H^3_{Betti}(X,\mathbb{Z})$ est sans torsion. Par ailleurs $H^4_{Betti}(X,\mathbb{Z}) = \mathbb{Z}$ (par dualité de Poincaré), la restriction $\mathbb{Z} = H^4_{Betti}(\mathbf{P}^4,\mathbb{Z}) \to H^4_{Betti}(X,\mathbb{Z}) = \mathbb{Z}$ est l'identité sur $\mathbb{Z}$.
Pour $n \geq 3$, toute hypersurface $X \subset \mathbf{P}^n_{\mathbb{C}}$ de degré $d \leq n$ contient une droite de $\mathbf{P}^n_{\mathbb{C}}$. C'est un résultat classique mais délicat dans le cas $d = n$ (voir [12]). Pour $d < n$, cela résulte d'un calcul immédiat de dimension, qui montre que par tout point de X il passe une droite de $\mathbf{P}^n_{\mathbb{C}}$ contenue dans X.
Soit $n = 4$. L'hypersurface X contient une droite de $\mathbf{P}^4_{\mathbb{C}}$. La classe de cette droite dans $CH^2(X)$ engendre donc $H^4_{Betti}(X,\mathbb{Z}) = \mathbb{Z}$.
Pour $n \geq 6$, les théorèmes de Lefschetz donnent que la flèche de restriction

$$\mathbb{Z} = H^4_{Betti}(\mathbf{P}^n_{\mathbb{C}},\mathbb{Z}) \to H^4_{Betti}(X,\mathbb{Z})$$

est un isomorphisme. Le diagramme commutatif

$$\begin{array}{ccc} CH^2(X) & \hookrightarrow & H^4_{Betti}(X,\mathbb{Z}) \\ \uparrow & & \uparrow \\ CH^2(\mathbf{P}^n_{\mathbb{C}}) & \stackrel{\simeq}{\to} & H^4_{Betti}(\mathbf{P}^n_{\mathbb{C}},\mathbb{Z}) \end{array}$$

donne alors $CH^2(X) \stackrel{\simeq}{\to} H^4_{Betti}(X,\mathbb{Z}) = \mathbb{Z}$, la conjecture de Hodge entière en degré 4 vaut donc pour X, et la théorie de Bloch-Ogus ou [11, Thm. 1.1] donnent alors $H^3_{nr}(X,\mathbb{Q}/\mathbb{Z}(2)) = 0$ soit (vi) pour $n \geq 6$. Le même argument vaut pour $n = 4$ et $d \leq 4$, puisque l'application $CH^2(X) \to H^4_{Betti}(X,\mathbb{Z}) = \mathbb{Z}$ est surjective. Ceci établit (v) et (vi). Pour $n = 4$, (vi) est un cas particulier d'un résultat de C. Voisin [11, Cor. 6.2].
Établissons les points (vii) et (viii).
Pour tout $n \geq 4$, Pour tout corps F contenant $\mathbb{C}$, on a

$$\mathrm{Pic}(X) = \mathrm{Pic}(X_F) = \mathbb{Z},$$

le groupe étant engendré par la classe d'une section hyperplane (théorème de Max Noether). On a donc $H^1(G,\mathrm{Pic}(X_{\overline{F}}) \otimes \overline{F}^\times) = H^1(G,\overline{F}^\times) = 0$ (théorème 90 de Hilbert). Les énoncés déjà établis et le le théorème 4.1 donnent alors une suite exacte

$$0 \to H^3(F,\mathbb{Q}/\mathbb{Z}(2)) \to \mathrm{Ker}[H^3_{nr}(X_F,\mathbb{Q}/\mathbb{Z}(2)) \to H^3_{nr}(X_{\overline{F}},\mathbb{Q}/\mathbb{Z}(2))] \to$$

$$\to \mathrm{Coker}[CH^2(X_F) \to CH^2(X_{\overline{F}})^G] \stackrel{\beta}{\to} H^2(G,\mathrm{Pic}(X_{\overline{F}}) \otimes \overline{F}^\times) \qquad (5.6)$$

Pour $n \geq 5$, d'après (iv), équivalence rationnelle et équivalence algébrique sur les cycles de codimension 2 de X coïncident sur un corps algébriquement clos. Pour une variété projective, lisse, connexe, sur un corps algébriquement clos, les groupes d'équivalence de cycles modulo l'équivalence algébrique sont, comme c'est bien connu et facile à établir, invariants par extension du corps de base à un autre corps algébriquement clos. Ainsi la flèche composée

$$CH^2(X) \to CH^2(X_F) \to CH^2(X_{\overline{F}})$$

est l'identité, donc l'application $CH^2(X_F) \to CH^2(X_{\overline{F}})^G$ est surjective. On obtient donc dans ce cas une suite exacte

$$0 \to H^3(F,\mathbb{Q}/\mathbb{Z}(2)) \to H^3_{nr}(X_F,\mathbb{Q}/\mathbb{Z}(2)) \to H^3_{nr}(X_{\overline{F}},\mathbb{Q}/\mathbb{Z}(2)).$$

D'après [7, Thm. 4.4.1], on a $H^3_{nr}(X,\mathbb{Q}/\mathbb{Z}(2)) \stackrel{\simeq}{\to} H^3_{nr}(X_{\overline{F}},\mathbb{Q}/\mathbb{Z}(2))$, si bien que la suite ci-dessus se complète en une suite exacte naturellement scindée

$$0 \to H^3(F,\mathbb{Q}/\mathbb{Z}(2)) \to H^3_{nr}(X_F,\mathbb{Q}/\mathbb{Z}(2)) \to H^3_{nr}(X_{\overline{F}},\mathbb{Q}/\mathbb{Z}(2)) \to 0.$$

Pour $n > 5$, une application de (vi) achève alors d'établir l'énoncé (vii).

Pour $n = 4$, on a déjà établi $H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2)) = 0$ et

$$H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \xrightarrow{\simeq} H^3_{nr}(X_F, \mathbb{Q}/\mathbb{Z}(2))$$

pour tout F contenant $\mathbb{C}$. La deuxième partie de l'énoncé (viii) résulte alors de la suite exacte (5.6) et du lemme 5.7 (b) ci-après. □

Lemme 5.7. *Soient $n \geq 4$ et $X \subset \mathbf{P}^n_{\mathbb{C}}$ une hypersurface lisse de degré $d \leq n$.*
(a) Pour tout corps F contenant $\mathbb{C}$, la flèche naturelle

$$\mathrm{Pic}(X_{\overline{F}}) \otimes \overline{F}^{\times} \to H^1(X_{\overline{F}}, \mathcal{K}_2)$$

est un isomorphisme

$$\overline{F}^{\times} \xrightarrow{\simeq} H^1(X_{\overline{F}}, \mathcal{K}_2).$$

(b) On a un isomorphisme

$$\mathrm{Ker}[H^3_{nr}(X_F, \mathbb{Q}/\mathbb{Z}(2))/H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \to H^3_{nr}(X_{\overline{F}}, \mathbb{Q}/\mathbb{Z}(2))] \xrightarrow{\simeq}$$
$$\xrightarrow{\simeq} \mathrm{Coker}[CH^2(X_F) \to CH^2(X_{\overline{F}})^G].$$

Démonstration. Pour tout corps F contenant $\mathbb{C}$, on a

$$\mathrm{Pic}(X) = \mathrm{Pic}(X_F) = \mathbb{Z},$$

le groupe étant engendré par la classe d'une section hyperplane (théorème de Max Noether). Comme on a $CH_0(X) \xrightarrow{\simeq} \mathbb{Z}$ et que les groupes $H^3_{Betti}(X, \mathbb{Z})$ sont sans torsion, d'après [10, Thm. 2.12 ; Prop. 2.15], la flèche naturelle

$$\mathrm{Pic}(X_{\overline{F}}) \otimes \overline{F}^{\times} \to H^1(X_{\overline{F}}, \mathcal{K}_2)$$

est surjective.
On a vu ci-dessus que X contient une droite de $\mathbf{P}^n$, soit $Y \subset X \subset \mathbf{P}^n_{\mathbb{C}}$. La restriction

$$\mathbb{Z} = \mathrm{Pic}(X_{\overline{F}}) \to \mathrm{Pic}(Y_{\overline{F}}) = \mathbb{Z}$$

est l'identité sur $\mathbb{Z}$, car le groupe $\mathrm{Pic}(X_{\overline{F}})$ est engendré par la classe d'une section hyperplane. Donc la flèche

$$\mathrm{Pic}(X_{\overline{F}}) \otimes \overline{F}^{\times} \to \mathrm{Pic}(Y_{\overline{F}}) \otimes \overline{F}^{\times}$$

est un isomorphisme. Pour la droite Y, l'application

$$\overline{F}^{\times} = \mathrm{Pic}(Y_{\overline{F}}) \otimes \overline{F}^{\times} \to H^1(Y_{\overline{F}}, \mathcal{K}_2)$$

est un isomorphisme. L'inclusion $Y \subset X$ induit un diagramme commutatif

$$\begin{array}{ccc} \mathrm{Pic}(X_{\overline{F}}) \otimes \overline{F}^{\times} & \longrightarrow & \mathrm{Pic}(Y_{\overline{F}}) \otimes \overline{F}^{\times} \\ \downarrow & & \downarrow \\ H^1(X_{\overline{F}}, \mathcal{K}_2) & \longrightarrow & H^1(Y_{\overline{F}}, \mathcal{K}_2) \end{array}$$

dans lequel la flèche horizontale supérieure est un isomorphisme, la flèche verticale de droite aussi, et la flèche verticale de gauche est surjective. La flèche $\mathrm{Pic}(X_{\overline{F}}) \otimes \overline{F}^\times \to H^1(X_{\overline{F}}, \mathcal{K}_2)$ est donc un isomorphisme $\overline{F}^\times \xrightarrow{\simeq} H^1(X_{\overline{F}}, \mathcal{K}_2)$, ce qui établit (a) et montre que la flèche de restriction

$$H^1(X_{\overline{F}}, \mathcal{K}_2) \to H^1(Y_{\overline{F}}, \mathcal{K}_2)$$

est un isomorphisme.
Considérons la suite exacte (5.6). Pour $n \geq 5$, nous avons établi $\mathrm{Coker}[CH^2(X_F) \to CH^2(X_{\overline{F}})^G] = 0$, et donc la flèche

$$\beta : \mathrm{Coker}[CH^2(X_F) \to CH^2(X_{\overline{F}})^G] \xrightarrow{\beta} H^2(G, \mathrm{Pic}(X_{\overline{F}}) \otimes \overline{F}^\times)$$

dans cette suite est nulle.
Montrons que l'on a encore $\beta = 0$ dans le cas $n = 4$. Nous avons ici recours au point de vue motivique, i.e. à la proposition 2.6. L'application β est induite par l'application composée

$$CH^2(X_{\overline{F}})^G \to \mathbb{H}^4(X_{\overline{F}}, \mathbb{Z}(2))^G \to H^2(G, \mathbb{H}^3(X_{\overline{F}}, \mathbb{Z}(2))).$$

Chacune des deux applications intervenant ici est définie pour toute variété lisse X, et leur formation est fonctorielle en la variété lisse X : la seconde application vient de la suite spectrale considérée à la section 2.2.
Soit $Y \subset X \subset \mathbf{P}^n_{\mathbb{C}}$ une droite. Comme la restriction

$$H^1(X_{\overline{F}}, \mathcal{K}_2) \to H^1(Y_{\overline{F}}, \mathcal{K}_2)$$

est un isomorphisme, la flèche

$$\beta : \mathrm{Coker}[CH^2(X_F) \to CH^2(X_{\overline{F}})^G] \to H^2(G, H^1(X_{\overline{F}}, \mathcal{K}_2))$$

se factorise par $\mathrm{Coker}[CH^2(Y_F) \to CH^2(Y_{\overline{F}})^G] = 0$ et donc est nulle, ce qui via la suite exacte (5.6) établit l'énoncé (b). □

Pour les hypersurfaces cubiques, un résultat de Claire Voisin permet de compléter le théorème 5.6 dans le cas $n = 5$.

Théorème 5.8. *Soit $X \subset \mathbf{P}^n_{\mathbb{C}}$, $n \geq 4$ une hypersurface cubique lisse.*
(i) On a $H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2)) = 0$.
(ii) Pour tout entier $n \geq 5$, pour tout corps F contenant $\mathbb{C}$, la flèche

$$H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \to H^3_{nr}(X_F, \mathbb{Q}/\mathbb{Z}(2))$$

est un isomorphisme, et l'application

$$CH^2(X_F) \to CH^2(X_{\overline{F}})^G$$

est surjective.
(iii) Pour $n = 4$, soit B le représentant algébrique de $A^2(X)$ (Prop. 5.1 (v)). S'il existe un cycle universel de codimension 2 dans $CH^2(B \times X)$, alors pour tout corps F contenant $\mathbb{C}$ on a $H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \xrightarrow{\simeq} H^3_{nr}(X_F, \mathbb{Q}/\mathbb{Z}(2))$, et l'application $CH^2(X_F) \to CH^2(X_{\overline{F}})^G$ est surjective.

Démonstration. Pour $n \neq 5$, ceci est un cas particulier du théorème 5.6. Soit donc $n = 5$. C. Voisin a établi la conjecture de Hodge entière en degré 4 pour toute hypersurface cubique lisse $X \subset \mathbf{P}^5_{\mathbb{C}}$ [32], [33, Thm. 0.4, Thm. 2.11]. D'après le théorème [11, Thm. 1.1], ceci implique $H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2)) = 0$, et donc, d'après [7, Thm. 4.4.1], $H^3_{nr}(X_{\overline{F}}, \mathbb{Q}/\mathbb{Z}(2)) = 0$ pour tout corps algébriquement clos $\overline{F}$ contenant $\mathbb{C}$. L'énoncé (ii) est alors une conséquence du théorème 5.6 (vii). □

Remarque 5.9. Soit $n = 5$. Si l'hypersurface cubique $X \subset \mathbf{P}^5_{\mathbb{C}}$ contient un plan, on peut fibrer X en quadriques au-dessus du plan. L'énoncé (i) résulte alors de [11, Cor. 8.2], qui repose seulement sur le calcul de la cohomologie non ramifiée des quadriques de dimension 2 sur un corps quelconque (cas particulier des résultats de Kahn, Rost, Sujatha sur les quadriques de dimension quelconque). Pour les *hypersurfaces cubiques lisses* $X \subset \mathbf{P}^5_{\mathbb{C}}$ *très générales contenant un plan*, l'isomorphisme

$$H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \xrightarrow{\simeq} H^3_{nr}(X_F, \mathbb{Q}/\mathbb{Z}(2))$$

dans la proposition 5.8 (ii) fut d'abord établi par des méthodes de K-théorie et de formes quadratiques, en collaboration avec Auel et Parimala [2]. Pour *toute hypersurface cubique lisse* $X \subset \mathbf{P}^5_{\mathbb{C}}$, il fut ensuite établi par C. Voisin [34, Thm. 2.1, Example 2.2], par une méthode différente de celle proposée ici.

Remarque 5.10. Soit $n = 4$. Si pour une hypersurface cubique $X \subset \mathbf{P}^4_{\mathbb{C}}$ et un corps F on avait $H^3_{nr}(X_F, \mathbb{Q}/\mathbb{Z}(2)) \neq H^3(F, \mathbb{Q}/\mathbb{Z}(2))$, alors X ne serait pas stablement rationnelle. Un tel exemple n'est pas connu. Dans [35, Thm. 4.5], C. Voisin montre qu'il existe des hypersurfaces cubiques dans $\mathbf{P}^4_{\mathbb{C}}$ pour lesquelles le groupe de Chow des zéro-cycles est universellement trivial, résultat plus fort que $H^3_{nr}(X_F, \mathbb{Q}/\mathbb{Z}(2)) = H^3(F, \mathbb{Q}/\mathbb{Z}(2))$ pour tout F.

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Limit Mordell–Weil Groups and their p-Adic Closure

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Abstract. This is a twin article of [H14b], where we study the projective limit of the Mordell–Weil groups (called pro Λ-MW groups) of modular Jacobians of p-power level. We prove a control theorem of an ind-version of the K-rational Λ-MW group for a number field K. In addition, we study its p-adic closure in the group of $K_{\mathfrak{p}}$-valued points of the modular Jacobians for a $\mathfrak{p}$-adic completion $K_{\mathfrak{p}}$ for a prime $\mathfrak{p}|p$ of K. As a consequence, if $K_{\mathfrak{p}} = \mathbb{Q}_p$, we give an exact formula for the rank of the ordinary/co-ordinary part of the closure.

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1. Introduction

Consider a p-adic ordinary family of modular eigenforms of prime-to-p level N. This is an irreducible scheme $\mathrm{Spec}(\mathbb{I})$ which is finite torsion-free over the Iwasawa algebra $\mathbb{Z}_p[[T]]$, and whose points P of codimension one and not in the special fiber correspond to ordinary p-adic modular eigenforms f_P. Among those points, many corresponds to modular classical eigenforms of weight 2 and level Np^r (for variable r), and such points are Zariski dense in $\mathrm{Spec}(\mathbb{I})$. An old, well-known, and fundamental construction of Eichler–Shimura attaches to any modular cuspidal eigenform f of weight 2 an abelian variety A_f defined over $\mathbb{Q}$, of dimension the degree of the field generated by the coefficients of f over $\mathbb{Q}$. For these abelian varieties A_f, one can consider the Mordell–Weil group $A_f(\mathbb{Q})$ and more generally, $A_f(k)$ for k a fixed number field, which are finitely generated abelian groups. Let us set $\widehat{A}_f(k) = A_f(k) \otimes_{\mathbb{Z}} \mathbb{Z}_p$. We consider the following natural question: how does the Mordell–Weil group $\widehat{A}_f(k)$ varies as f varies among those cuspidal eigenforms of weight 2 in the family? We give a partial answer to this question in the form of control theorems (Theorems 1.1 and 6.6) for these Mordell–Weil groups. An analogous result is proved when the number field k is replaced by an l-adic field $k_{\mathfrak{l}}$, and also a consequence concerning the image of $A_f(k)$ in $A_f(k_{\mathfrak{l}})$.

Fix a prime p. This article concerns the p-slope 0 Hecke eigen cusp forms of level Np^r for $r > 0$ and $p \nmid N$, and for small primes $p = 2, 3$, they exists only when $N > 1$; thus, we may assume $Np^r \geq 4$. Then the open curve $Y_1(Np^r)$ (obtained from $X_1(Np^r)$ removing all cusps) gives the fine smooth moduli scheme classifying elliptic curves E with an embedding $\mu_{Np^r} \hookrightarrow E$. Anyway for simplicity, we assume that $p \geq 3$, although we indicate often any modification necessary for $p = 2$. A main difference in the case $p = 2$ is that we need to consider the level Np^r with $r \geq 2$, and whenever the principal ideal $(\gamma^{p^{r-1}} - 1)$ shows up in the statement for $p > 2$, we need to replace it by $(\gamma^{p^{r-2}} - 1)$ (assuming $r \geq 2$), as the maximal torsion-free subgroup of $\mathbb{Z}_2^\times$ is $1+2^2\mathbb{Z}_2$. We applied in [H86b] and [H14a] the techniques of $U(p)$-isomorphisms to p-divisible Barsotti–Tate groups of modular Jacobian varieties of all p-power level (with a fixed prime-to-p level N) in order to get coherent control under diamond operators. In this article, we apply the same techniques to Mordell–Weil groups of the Jacobians and see what we can say. We hope to study $U(p)$-isomorphisms of the Tate–Shafarevich groups of the Jacobians in a future article.

Let $X_r = X_1(Np^r)_{/\mathbb{Q}}$ be the compactified moduli of the classification problem of pairs (E, ϕ) of elliptic curves E and an embedding $\phi : \mu_{Np^r} \hookrightarrow E[Np^r]$ as finite flat group schemes. Since $\mathrm{Aut}(\mu_{p^r}) = (\mathbb{Z}/p^r\mathbb{Z})^\times$, $z \in \mathbb{Z}_p^\times$ acts on X_r via $\phi \mapsto \phi \circ \overline{z}$ for the image $\overline{z} \in (\mathbb{Z}/p^r\mathbb{Z})^\times$. We write X_s^r $(s > r)$ for the quotient curve $X_s/(1 + p^r\mathbb{Z}_p)$. The complex points $X_s^r(\mathbb{C})$ contains $\Gamma_s^r\backslash\mathfrak{H}$ as an open Riemann surface for $\Gamma_s^r = \Gamma_0(p^s) \cap \Gamma_1(Np^r)$. Write $J_{r/\mathbb{Q}}$ (resp. $J^r_{s/\mathbb{Q}}$) for the Jacobian of X_r (resp. X_s^r) whose origin is given by the infinity cusp ∞ of the modular curves. We regard J_r as the degree 0 component of the Picard scheme of X_r. For a number field k, we consider the group of k-rational points $J_r(k)$. The Hecke operator $U(p)$ and its dual $U^*(p)$ act on $J_r(k)$ and their p-adic limit $e = \lim_{n\to\infty} U(p)^{n!}$ and $e^* = \lim_{n\to\infty} U^*(p)^{n!}$ are well defined on the Barsotti–Tate group $J_r[p^\infty]$. For a general abelian variety over a number field k, we put $\widehat{X}(k) = X(k) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ (though we give the definition of the sheaf $\widehat{X}$ in the following section for global and local field k and if k is local, $\widehat{X}$ may not be the tensor product as above).

By Picard functoriality, we have injective limits $J_\infty(k) = \varinjlim_r \widehat{J}_r(k)$ and $J_\infty[p^\infty](k) = \varinjlim_r J_r[p^\infty](k)$, on which e acts. Here $J_r[p^\infty]$ is the p-divisible Barsotti–Tate group of J_r over $\mathbb{Q}$). Write $\mathcal{G} = e(J_\infty[p^\infty])$, which is called the Λ-adic Barsotti–Tate group in [H14a] and whose integral property was scrutinized there. We define the p-adic completion of $J_\infty(k)$:

$$\check{J}_\infty(k) = \varprojlim_n J_\infty(k)/p^n J_\infty(k).$$

These groups we call ind (limit) MW-groups. Since projective limit and injective limit are left-exact, the functor $R \mapsto J_\infty(R)$ is a sheaf with values in abelian groups on the fppf site over $\mathbb{Q}$ (we call such a sheaf an fppf abelian sheaf).

Adding superscript or subscript "ord" (resp. "co-ord"), we indicate the image of e (resp. e^*). The compact cyclic group $\Gamma = 1 + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times$ acts on these modules by the diamond operators. In other words, we identify canonically $\mathrm{Gal}(X_r/X_0(Np^r))$ for modular curves X_r and $X_0(Np^r)$ with $(\mathbb{Z}/Np^r\mathbb{Z})^\times$, and the group Γ acts on J_r through its image in $\mathrm{Gal}(X_r/X_0(Np^r))$. We study control of $\check{J}_\infty(k)^{\mathrm{ord}}$ under diamond operators.

A compact or discrete $\mathbb{Z}_p$-module M is called an Iwasawa module if it has a continuous action of the multiplicative group $\Gamma = 1 + p\mathbb{Z}_p$ with a topological generator $\gamma = 1 + p$. If M is given by a projective or an injective limit of naturally defined compact $\mathbb{Z}_p[\Gamma/\Gamma^{p^r}]$-modules M_r, we say M has exact control if $M_r = M/(\gamma^{p^r}-1)M$ in the case of a projective limit and $M_r = M[\gamma^{p^r}-1] = \{x \in M | (\gamma^{p^r}-1)x = 0\}$ in the case of an injective limit. If M is compact and $M/(\gamma-1)M$ is finite (resp. of finite type over $\mathbb{Z}_p$), M is Λ-torsion (resp. of finite type over Λ), where $\Lambda = \mathbb{Z}_p[[\Gamma]] = \varprojlim_r \mathbb{Z}_p[\Gamma/\Gamma^{p^r}]$ (the Iwasawa algebra). When $p = 2$, we need to take $\Gamma = 1 + p^2\mathbb{Z}_2$ and $\gamma = 1 + 4 = 5 \in \Gamma$. In addition, we need to assume often $s > r > 1$ in place of $s > r > 0$ for odd primes.

The big ordinary Hecke algebra $\mathbf{h}$ (whose properties we recall at the end of this section) acts on $\check{J}_\infty^{\mathrm{ord}}$ and J_∞^{ord} as endomorphisms of functors. Let k be a number field or a finite extension of $\mathbb{Q}_l$ for a prime l. Write B_P for Shimura's abelian variety quotient of J_r in [Sh73] and A_P for his abelian subvariety $A_P \subset J_r$ [IAT, Theorem 7.14] associated to a Hecke eigenform f_P in an analytic family of slope 0 Hecke eigenforms $\{f_P | P \in \mathrm{Spec}(\mathbb{I})\}$ (for an irreducible component $\mathrm{Spec}(\mathbb{I})$ of $\mathrm{Spec}(\mathbf{h})$ for the big ordinary Hecke algebra $\mathbf{h}$). Here we assume that f_P has weight 2 and is a p-stabilized new form of level Np^r with $r = r(P) > 0$. Let $\mathrm{Spec}(\mathbb{T}) \subset \mathrm{Spec}(\mathbf{h})$ be the connected component containing $\mathrm{Spec}(\mathbb{I})$. For any $\mathbf{h}$-module, we write $M_\mathbb{T}$ (or $M^\mathbb{T}$) for the $\mathbb{T}$-eigen component $1_\mathbb{T} \cdot M = M \otimes_\mathbf{h} \mathbb{T}$ for the idempotent $1_\mathbb{T}$ of T in $\mathbf{h}$. Suppose that P is a principal ideal generated by $\alpha \in \mathbb{T}$ (regarding as $P \in \mathrm{Spec}(\mathbb{T})$). This principality assumption holds most of the cases (see Proposition 5.1). Then we may assume that $\alpha = \varprojlim_s \alpha_s$ (as an endomorphism of the fppf abelian sheaf J_∞) for $\alpha_s \in \mathrm{End}(J_s)$, $B_P = J_r/\alpha_r(J_r)$, and the abelian variety A_P is the connected component of $J_r[\alpha_r] = \mathrm{Ker}(\alpha_r)$. For a finite extension k of $\mathbb{Q}$ or $\mathbb{Q}_l$ (for a prime l), we show in Section 4 that the Pontryagin dual $\mathcal{G}_\mathbb{T}(k)^\vee$ is often a finite module and at worst is a torsion Λ-module of finite type.

In this paper as Proposition 6.4, we prove the following exact sequence:

$$\widehat{A}_P(k)^{\mathrm{ord},\mathbb{T}} \xrightarrow{\iota_\infty} \check{J}_\infty(k)^{\mathrm{ord},\mathbb{T}} \xrightarrow{\alpha} \check{J}_\infty(k)^{\mathrm{ord},\mathbb{T}}, \tag{1.1}$$

where $\mathrm{Ker}(\iota_\infty)$ is finite and $\mathrm{Coker}(\alpha)$ is a $\mathbb{Z}_p$-module of finite type with free rank less than or equal to $\dim_{\mathbb{Q}_p} \widehat{B}_r(k) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. The main result Theorem 6.6 of this paper is basically the $\mathbb{Z}_p$-dual version of Proposition 6.4 for $\check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}} := \mathrm{Hom}_{\mathbb{Z}_p}(\check{J}_\infty(k)^{\mathrm{ord},\mathbb{T}}, \mathbb{Z}_p)$. Here is a shortened statement of our main theorem (Theorem 6.6 in the text):

Theorem 1.1. *The sequence $\mathbb{Z}_p$-dual to the one in* (1.1)*:*

$$0 \to \mathrm{Coker}(\alpha)^*_{\mathbb{T}} \to \check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}} \to \check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}} \to \widehat{A}_P(k)^*_{\mathrm{ord},\mathbb{T}} \to 0 \tag{1.2}$$

is exact up to finite error.

In Theorem 6.6, we give many control sequences similar to (1.2) for other incarnations of $\check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}}$.

These modules $\check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}}$ are modules over the big ordinary Hecke algebra $\mathbf{h}$. We cut down these modules to an irreducible component $\mathrm{Spec}(\mathbb{I})$ of $\mathrm{Spec}(\mathbf{h})$. In other words, we study the following $\mathbb{I}$-modules:

$$\check{J}_\infty(k)^{\mathrm{ord}}_{\mathbb{I}} := \check{J}_\infty(k)^{\mathrm{ord}} \otimes_{\mathbf{h}} \mathbb{I}.$$

We could ask diverse questions out of our control theorem. For example, when is $A_P(\kappa)$ dense in $A_P(\kappa_{\mathfrak{p}})$ for a prime $\mathfrak{p}|p$ of a number field κ? We can answer this question for almost all P if $\kappa_{\mathfrak{p}} = \mathbb{Q}_p$ and $\dim_{\mathbb{Q}} A_{P_0}(\kappa) \otimes_{\mathbb{Z}} \mathbb{Q} > 0$ for one sufficiently generic P_0 (see Corollary 7.2). In [H14b], we extend the control result to the projective limit $\varprojlim_r \widehat{J}_r(k)^{\mathrm{ord}}_{\mathbb{T}}$. In a forthcoming paper [H14c], we prove "almost" constancy of the Mordell–Weil rank of Shimura's abelian variety in a p-adic analytic family.

Our point is that we have a control theorem of the limit Mordell–Weil groups (under mild assumptions) which is possibly smaller than the Selmer groups studied more often. We hope to discuss the relation of our result to the limit Selmer group studied by Nekovár in [N06] in our future paper.

The control theorems for $\mathbf{h}$ proven for $p \geq 5$ in [H86a] and [H86b] and in [GME, Corollary 3.2.22] for general p assert that, for $p > 2$, the quotient $\mathbf{h}/(\gamma^{p^{r-1}} - 1)\mathbf{h}$ is canonically isomorphic to the Hecke algebra $\mathbf{h}_r$ $(r > 0)$ in $\mathrm{End}_{\mathbb{Z}_p}(J_r[p^\infty]^{\mathrm{ord}})$ generated over $\mathbb{Z}_p$ by Hecke operators $T(n)$ (while for $p = 2$, $\mathbf{h}/(\gamma^{p^{r-2}} - 1)\mathbf{h} \cong \mathbf{h}_r$ for $r \geq 2$). By this control result, we showed that $\mathbf{h}$ is a free of finite rank over Λ (see [GK13] for the treatment for $p = 2$).

We recall succinctly how these control theorems were proven in [H86b] (and in [H86a]) for $p \geq 5$, as it gives a good introduction to the methods used in the present paper. The arguments in these papers work well for $p = 2, 3$ assuming that $Np^r \geq 4$ (see [GK13] for details in the case of $p = 2$). We have a well known commutative diagram of $U(p^{s-r})$-operators:

$$\begin{array}{ccc} J_{r,R} & \xrightarrow{\pi^*} & J^r_{s,R} \\ \downarrow u & \swarrow u' & \downarrow u'' \\ J_{r,R} & \xrightarrow{\pi^*} & J^r_{s,R}, \end{array} \tag{1.3}$$

where the middle u' is given by $U^s_r(p^{s-r})$ and u and u'' are $U(p^{s-r})$. These operators comes from the double coset $\Gamma \begin{pmatrix} 1 & 0 \\ 0 & p^{s-r} \end{pmatrix} \Gamma'$ for $\Gamma = \Gamma^r_s = \Gamma_0(p^s) \cap \Gamma_1(Np^r)$ and $\Gamma' = \Gamma^{r'}_{s'}$ for suitable $s \geq r, s' \geq r'$. Note that $U(p^n) = U(p)^n$. Then the above diagram implies

$$J_{r/\mathbb{Q}}[p^\infty]^{\mathrm{ord}} \cong J^r_{s/\mathbb{Q}}[p^\infty]^{\mathrm{ord}} \ \text{ and } \ \widehat{J}_{r/\mathbb{Q}}(k)^{\mathrm{ord}} \cong \widehat{J}^r_{s/\mathbb{Q}}(k)^{\mathrm{ord}}. \tag{1.4}$$

The commutativity of the diagram (1.3) and the level lowering (1.4) are universally true even when we replace the fppf abelian sheaf J_r by any fppf sheaf with reasonable $U(p)$-action compatible with the modular tower $\cdots \to X_r \to \cdots \to X_1$.

For computational purpose, in [H86b], we identified $J(\mathbb{C})$ with a subgroup of $H^1(\Gamma, \mathbf{T})$ (for the Γ-module $\mathbf{T} := \mathbb{R}/\mathbb{Z}$ with trivial Γ-action). Since $\Gamma_s^r \triangleright \Gamma_1(Np^s)$, we may consider the finite cyclic quotient group $C := \frac{\Gamma_s^r}{\Gamma_1(Np^s)} = \Gamma^{p^{r-1}}/\Gamma^{p^{s-1}}$. By the inflation restriction sequence, we have the following commutative diagram with exact rows, writing $H^\bullet(?, \mathbf{T})$ as $H^\bullet(?)$:

$$\begin{array}{ccccccc} H^1(C) & \xrightarrow{\hookrightarrow} & H^1(\Gamma_s^r) & \longrightarrow & H^1(\Gamma_1(Np^s))^{\gamma^{p^{r-1}}=1} & \longrightarrow & H^2(C) \\ \uparrow & & \cup\uparrow & & \uparrow\cup & & \uparrow \\ ? & \longrightarrow & J_s^r(\mathbb{C}) & \longrightarrow & J_s(\mathbb{C})[\gamma^{p^{r-1}}-1] & \longrightarrow & ?. \end{array}$$

Since $H^2(C, \mathbf{T}) = 0$ and $U(p)^{s-r}(H^1(C, \mathbf{T})) = 0$, we have the control of Barsotti–Tate groups (see [H86b] and more recent [H14a, §4–5]):

$$J_s[p^\infty][\gamma^{p^{r-1}}-1]^{\mathrm{ord}}_{/\mathbb{C}} \cong J_r[p^\infty]^{\mathrm{ord}}_{/\mathbb{C}}.$$

Out of this control by the Γ-action of the ordinary Barsotti–Tate groups $J_r[p^\infty]^{\mathrm{ord}}$, we proved the control of $\mathbf{h}$ (cited above) by the diamond operators.

A suitable power of $U(p)$-operator killing the kernel and cokernel of the restriction maps in (1) should be also universally true not just over $\mathbb{C}$ but over smaller rings. We will study almost the same diagram obtained by replacing $H^1(?, \mathbf{T})$ for $? = \Gamma_1(Np^s)$ and Γ_s^r by $H^1_{\mathrm{fppf}}(X_{/\mathbb{Q}}, \mathcal{O}_X^\times) = \mathrm{Pic}_{X/\mathbb{Q}}$ for $X = X_s$ and X_s^r. In an algebro-geometric way, we verify that an appropriate power of the $U(p)$-operator kills the corresponding kernel and cokernel. Technical points aside, this is a key to the proof of Theorem 1.1. This principle should hold for more general sheaves (under a Grothendieck topology) with $U(p)$-action compatible with the modular tower, and the author plans to present many other examples of such in his forthcoming papers.

We call a point $P \in \mathrm{Spec}(\mathbf{h})(\overline{\mathbb{Q}}_p)$ an arithmetic point of weight 2 if $P(\gamma^{p^j}-1) = 0$ for some integer $j \geq 0$. Though the construction of the big Hecke algebra is intrinsic, to relate an algebra homomorphism $\lambda : \mathbf{h} \to \overline{\mathbb{Q}}_p$ killing $\gamma^{p^{r-1}}-1$ for sufficiently large $r > 0$ to a classical Hecke eigenform, we need to fix (once and for all) an embedding $\overline{\mathbb{Q}} \xrightarrow{i_p} \overline{\mathbb{Q}}_p$ of the algebraic closure $\overline{\mathbb{Q}}$ in $\mathbb{C}$ into a fixed algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$.

Contents

2. Sheaves associated to abelian varieties

Let k be a finite extension of $\mathbb{Q}$ or the l-adic field $\mathbb{Q}_l$. In this section, we set the notation used in the rest of the paper and present a general fact about an exact sequence of abelian varieties. Let $0 \to A \to B \to C \to 0$ be an exact sequence of algebraic groups proper over the field k. We assume that B and C are abelian varieties. However A can be an extension of an abelian variety by a finite (étale) group.

If k is a number field, let S be a finite set of places where all members of the above exact sequence have good reduction outside S; so, all archimedean places are included in S. Let $K = k^S$ (the maximal extension unramified outside S). If k is a finite extension of $\mathbb{Q}_l$, we put $K = \overline{k}$ (an algebraic closure of k). A general field extension of k is denoted by κ. We consider the étale topology, the smooth topology and the fppf topology on the small site over $\mathrm{Spec}(k)$. Here under the smooth topology, covering families are made of faithfully flat smooth morphisms.

We want to define p-adically completed sheaves $\widehat{X}$ for $X = A, B, C$ as above defined over these sites. The word "p-adically completed" does not always mean $\widehat{X}(R)$ is given by the projective limit $\varprojlim_n X(R)/p^n X(R)$, and the definition depends on the choice of k. For the moment, assume that k is a number field. In this case, for an extension X of abelian variety defined over k by a finite flat group scheme, we define $\widehat{X}(F) := X(F) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ for an fppf extension F over k. We may regard its p-adic "completion" $0 \to \widehat{A} \to \widehat{B} \to \widehat{C} \to 0$ as an exact sequence of fppf/smooth/étale abelian sheaves over k (or over any subring of k over which B and C extend to abelian schemes). Here the word "completion" means tensoring with $\mathbb{Z}_p$ over $\mathbb{Z}$. Indeed, for any ring R of finite type over k, $R \mapsto \widehat{C}(R) := C(R) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is an exact functor from the category of abelian fppf/smooth/étale sheaves into itself; therefore, the tensor construction gives $\widehat{C}(\kappa) = \varprojlim_n C(\kappa)/p^n C(\kappa)$ if κ is a field of finite type, since $C(\kappa)$ is an abelian group of finite type by a generalized Mordell-Weil theorem (e.g., [RTP, IV]). Let ϵ denote the dual number. Then we have a canonical identification $Lie(C)_{/\kappa} = \mathrm{Ker}(C(\kappa[\epsilon]) \to C(\kappa))$ (e.g. [EAI, §10.2.4]), and hence $Lie(C) \otimes_{\mathbb{Z}} \mathbb{Z}_p = \mathrm{Ker}(\widehat{C}(\kappa[\epsilon]) \to \widehat{C}(\kappa))$ is the p-adic completion of the κ-vector space $Lie(C)$ if κ is a finite extension of k. Since we find a complementary abelian subvariety C' of B such that C' is isogenous to C and $B = A + C'$ with finite $A \cap C'$, adding the primes dividing the order $|A \cap C'|$ to S, the intersection $A \cap C' \cong \mathrm{Ker}(C' \to C)$ extends to an étale finite group scheme outside S; so, $C'(K) \to C(K)$ is surjective. Thus we have an exact sequence of $\mathrm{Gal}(K/k)$-modules

$$0 \to A(K) \to B(K) \to C(K) \to 0.$$

Note that $\widehat{A}(K) = A(K) \otimes_{\mathbb{Z}} \mathbb{Z}_p := \bigcup_F \widehat{A}(F)$ for F running over all finite extensions of k inside K. Then we have an exact sequence

$$0 \to \widehat{A}(K) \to \widehat{B}(K) \to \widehat{C}(K) \to 0. \tag{2.1}$$

Now assume that k is a finite extension of $\mathbb{Q}_l$. We put $K = \overline{k}$ (an algebraic closure of k). Suppose that F is a finite extension of k. Then $A(F) = O_F^{\dim A} \oplus \Delta_F$ for a finite group Δ_F and the l-adic integer ring O_F of F (see [M55] ot [T66]). Now suppose $l \neq p$. For an fppf extension $R_{/k}$, we define again $\widehat{A}(R) := A[p^\infty](R) = \varinjlim_n A[p^n]$ for $A[p^n] := \mathrm{Ker}(A(R) \xrightarrow{p^n} A(R))$. Then we have $\widehat{A}(F) = \varprojlim_n A(F)/p^n A(F) = \Delta_{F,p} := \Delta_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$, and we have $\widehat{A}(K) = \varinjlim_F \widehat{A}(F) = A[p^\infty](K)$, and $\widehat{A}$, $\widehat{B}$ and $\widehat{C}$ are identical to the fppf/smooth/étale abelian sheaves $A[p^\infty]$, $B[p^\infty]$ and $C[p^\infty]$, where $X[p^\infty] := \varinjlim_n X[p^n]$ as an ind finite flat group scheme with $X[p^n] = \mathrm{Ker}(p^n : X \to X)$ for $X = A, B, C$. We again have the exact sequence (2.1) of $\mathrm{Gal}(\overline{k}/k)$-modules:

$$0 \to \widehat{A}(K) \to \widehat{B}(K) \to \widehat{C}(K) \to 0$$

and an exact sequence of fppf/smooth/étale abelian sheaves

$$0 \to \widehat{A} \to \widehat{B} \to \widehat{C} \to 0$$

whose value at finite extension $\kappa/\mathbb{Q}_l$ coincides with the projective limit $\widehat{X}(\kappa) = \varprojlim_n X(\kappa)/p^n X(\kappa)$ for $X = A, B, C$.

Suppose $l = p$. For any module M, we define $M^{(p)}$ by the maximal prime-to-p torsion submodule of M. For $X = A, B, C$ and an fppf extension $R_{/k}$, the sheaf $R \mapsto X^{(p)}(R) = \varinjlim_{p \nmid N} X[N](R)$ is an fppf/smooth/étale abelian sheaf. Then we define the fppf/smooth/étale abelian sheaf $\widehat{X}$ by the sheaf quotient $X/X^{(p)}$. Since $X(F) = O_F^{\dim X} \oplus X[p^\infty](F) \oplus X^{(p)}(F)$ for a finite extension $F_{/k}$, on the étale site over k, $\widehat{X}$ is the sheaf associated to a presheaf $R \mapsto X(R)/X^{(p)}(R) = O_F^{\dim X} \oplus X[p^\infty](R)$. If X has semi-stable reduction over O_F, we have $\widehat{X}(F) = X^\circ(O_F) + X[p^\infty](F) \subset X(F)$ for the formal group X° of the identity connected component of the Néron model of X over O_F. Since X becomes semi-stable over a finite Galois extension F_0/k, in general $\widehat{X}(F) = H^0(\mathrm{Gal}(F_0F/F), X(F_0F))$ for any finite extension $F_{/K}$ (or more generally for each finite 'etale extension $F_{/k}$); so, $F \mapsto \widehat{X}(F)$ is a sheaf over the étale site over k. Thus by [ECH, II.1.5], the sheafication coincides over the étale site with the presheaf $F \mapsto \varprojlim_n X(F)/p^n X(F)$. Thus we conclude $\widehat{X}(F) = \varprojlim_n X(F)/p^n X(F)$ for any étale finite extensions $F_{/k}$. Moreover $\widehat{X}(K) = \bigcup_F \widehat{X}(F)$. Applying the snake lemma to the commutative diagram with exact rows (in the category of fppf/smooth/étale abelian sheaves):

$$\begin{array}{ccccc} A^{(p)} & \xrightarrow{\hookrightarrow} & B^{(p)} & \xrightarrow{\twoheadrightarrow} & C^{(p)} \\ \cap\downarrow & & \cap\downarrow & & \cap\downarrow \\ A & \xrightarrow[\hookrightarrow]{} & B & \xrightarrow[\twoheadrightarrow]{} & C, \end{array}$$

the cokernel sequence gives rise to an exact sequence of fppf/smooth/étale abelian sheaves over k:

$$0 \to \widehat{A} \to \widehat{B} \to \widehat{C} \to 0$$

and an exact sequence of $\mathrm{Gal}(\overline{k}/k)$-modules

$$0 \to \widehat{A}(K) \to \widehat{B}(K) \to \widehat{C}(K) \to 0.$$

In this way, we extended the étale sheaves $\widehat{A}, \widehat{B}, \widehat{C}$ defined on the étale site over $\mathrm{Spec}(k)$ to an abelian sheaves on the smooth, fppf and étale sites keeping the exact sequence $\widehat{A} \hookrightarrow \widehat{B} \twoheadrightarrow \widehat{C}$ intact. However note that our way of defining $\widehat{X}$ for $X = A, B, C$ depends on the base field $k = \mathbb{Q}, \mathbb{Q}_p, \mathbb{Q}_l$. In summary, we have, for fppf algebras $R_{/k}$:

$$\text{(S)} \qquad \widehat{X}(R) = \begin{cases} X(R) \otimes_{\mathbb{Z}} \mathbb{Z}_p & \text{if } [k:\mathbb{Q}] < \infty, \\ X[p^\infty](R) & \text{if } [k:\mathbb{Q}_l] < \infty \ (l \neq p), \\ (X/X^{(p)})(R) \text{ as a sheaf quotient} & \text{if } [k:\mathbb{Q}_p] < \infty. \end{cases}$$

Lemma 2.1. *Let the notation be as above (in particular, X is an extension of an abelian variety over k by a finite étale group scheme). If κ is either an integral domain or a field of finite type over k and either k is a number field or a local field with residual characteristic $l \neq p$, we have $\widehat{X}(\kappa) = \varprojlim_n \widehat{X}(\kappa)/p^n\widehat{X}(\kappa)$. If κ is an étale extension of finite type over k and k is a p-adic field, we again have $\widehat{X}(\kappa) = \varprojlim_n \widehat{X}(\kappa)/p^n\widehat{X}(\kappa)$.*

Proof. First suppose that k is a number field. If κ is a field extension of finite type over k, by [RTP, IV], $X(\kappa)$ is a $\mathbb{Z}$-module of finite type; so, we have $\widehat{X}(\kappa) = X(\kappa) \otimes_{\mathbb{Z}} \mathbb{Z}_p = \varprojlim X(\kappa)/p^n X(\kappa)$. Here the first identity is just by the definition. More generally, if κ/k is a Krull domain of finite type over k, κ is a normal noetherian domain; and $\kappa = \bigcap_V V$ for discrete valuation ring V in $Q(\kappa)$ containing κ. By projectivity of the abelian variety, we have $X(V) = X(Q(\kappa)$ (by the valuative criterion of properness), which implies $X(\kappa) = \bigcap_V X(V) = X(Q(\kappa))$ (so, $\widehat{X}(\kappa) = \widehat{X}(Q(\kappa))$) for the quotient field $Q(\kappa)$ of κ. In particular, if κ is a smooth extension of finite type, an the result follows, Since the normalization $\widetilde{\kappa}$ of κ in $Q(\kappa)$ is a Krull domain, we find $\widehat{X}(\kappa) \subset \widehat{X}(\widetilde{\kappa}) = \widehat{X}(Q(\kappa))$; so, $\widehat{X}(\kappa)$ is an abelian group of finite type as long as κ is an integral domain of finite type over k (and hence is a reduced algbera of finite type over k).

If k is local of residual characteristic $l \neq p$, we have $\widehat{X} = X[p^\infty]$. If κ is an integral domain of finite type over k, then $\widehat{X}(\kappa)$ is a finite p-group, and the result is obvious.

The case where k is local of residual characteristic p is already dealt with before the lemma. $\square$

For a sheaf X under the topology ?, we write $H^\bullet_?(X)$ for the cohomology group $H^1_?(\mathrm{Spec}(\kappa), X)$ under the topology ?. If we have no subscript, $H^1(X)$ means the Galois cohomology $H^\bullet(\mathrm{Gal}(K/\kappa), X)$ for the $\mathrm{Gal}(K/\kappa)$-module X.

Lemma 2.2. *Let X be an extension of an abelian variety over k by a finite étale group scheme of order prime to p. For any intermediate extension $K/\kappa/k$, We have a canonical injection*

$$\varprojlim_n \widehat{X}(\kappa)/p^n\widehat{X}(\kappa) \hookrightarrow \varprojlim_n H^1(X[p^n]).$$

Similarly, for any fppf, smooth or étale extension κ/k of finite type which is an integral domain, we have an injection

$$\varprojlim_n \widehat{X}(\kappa)/p^n\widehat{X}(\kappa) \hookrightarrow \varprojlim_n H^1_?(X[p^n])$$

for ? = fppf, sm *or* ét *according as κ/k is an fppf extension or a smooth extension.*

By Lemma 2.1, we have $\widehat{X}(\kappa) = \varprojlim_n \widehat{X}(\kappa)/p^n\widehat{X}(\kappa)$ in the following cases:
(2.2)

$$\begin{cases} [k:\mathbb{Q}] < \infty \text{ and } \kappa \text{ is an integral domain of finite type over } k \\ [k:\mathbb{Q}_l] < \infty \text{ with } l \neq p \text{ and } \kappa \text{ is an integral domain of finite type over } k \\ [k:\mathbb{Q}_p] < \infty \text{ and } \kappa \text{ is a finite algebraic extension over } k. \end{cases}$$

Proof. We consider the sheaf exact sequence under the topology ? = fppf or sm or étale on $\mathrm{Spec}(\kappa)$

$$0 \to X[p^n] \to \widehat{X} \xrightarrow{p^n} \widehat{X}.$$

We want to show that the multiplication by p^n is surjective. If our cohomology theory is Galois cohomology (or equivalently ? = étale), we have an exact sequence

$$0 \to X[p^n](K) \to X(K) \xrightarrow{p^n} X(K) \to 0.$$

Since $\widehat{X}(K) = X(K) \otimes_{\mathbb{Z}} \mathbb{Z}_p$, the desired exactness follows.
Let κ be an fppf extension of k. Then for each $x \in X(\kappa)$, we consider the Cartesian diagram

$$\begin{array}{ccc} X_x & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle p^n} \\ \mathrm{Spec}(\kappa) & \xrightarrow[x]{} & X. \end{array}$$

Then $X_x \cong X[p^n]$ as schemes over κ; so, $X_x = \mathrm{Spec}(R)$ for an étale finite extension R of κ, which is obviously smooth and also fppf extension of κ. Thus over the covering R/κ, x is the image of the point given by $\mathrm{Spec}(R) \hookrightarrow X$. Then by [ECH, II.2.5 (c)], $X \xrightarrow{p^n} X$ is an epimorphism of sheaves under étale, smooth

and also fppf topology. If k is a number field, we have $\widehat{X}(\kappa) = X(\kappa) \otimes_{\mathbb{Z}} \mathbb{Z}_p$, we get the exactness of $X[p^n] \hookrightarrow \widehat{X} \twoheadrightarrow \widehat{X}$ from the exactness of $X[p^n] \hookrightarrow X \twoheadrightarrow X$. If k is a finite extension of $\mathbb{Q}_l$ for $l \neq p$, we can argue as above replacing X by $\widehat{X} = X[p^\infty]$ and get the exactness of $X[p^n] \hookrightarrow \widehat{X} \twoheadrightarrow \widehat{X}$. Suppose that k is a finite extension of $\mathbb{Q}_p$. Then $\widehat{X} = X/X^{(p)}$ as a ?-sheaf. Take $x \in \widehat{X}(\kappa)$. Then by definition, we have an ?-extension R of κ such that x is the image of $y \in X(R)$. Then as above we can find a ?-extension R'/R such that $y = p^n y'$ for $y' \in X(R')$. Then for the image x' of $y' \in X(R')$ in $\widehat{X}(R')$, we have $p^n x' = x$. Thus again $\widehat{X} \xrightarrow{p^n} \widehat{X}$ is an epimorphism of sheaves under the topology ?.
Thus we can apply Kummer theory to the sheaf exact sequence

$$0 \to X[p^n] \hookrightarrow \widehat{X} \xrightarrow{p^n} \widehat{X} \to 0$$

with respect to the topology given by ?, we have an inclusion $\widehat{X}(\kappa)/p^n\widehat{X}(\kappa) \hookrightarrow H^1_?(X[p^n])$. Passing to the limit with respect to n, we have $\delta : \varprojlim_n X(\kappa)/p^n X(\kappa) \to \varprojlim_n H^1_?(X[p^n])$. Since taking projective limit is a left exact functor, δ is injective as desired. □

Taking instead an injective limit, we get

Lemma 2.3. *Let A be an abelian variety over k. For any intermediate extension $K/\kappa/k$, we have an exact sequence*

$$0 \to \widehat{A}(\kappa) \otimes_{\mathbb{Z}_p} \mathbb{T}_p \to H^1_?(A[p^\infty]) \to H^1_?(\widehat{A}) \to 0$$

for ? = fppf, sm *or* ét *according as κ/k is an fppf extension, a smooth extension or an étale extension. In particular, the Pontryagin dual of $H^1_?(\widehat{A})$ is a $\mathbb{Z}_p$-module of finite type; so, $H^1_?(\widehat{A})$ has the form $(\mathbb{Q}_p/\mathbb{Z}_p)^j \oplus \Delta$ for some $0 \le j \in \mathbb{Z}$ and a finite p-group Δ.*

Proof. Since any smooth covering has finer étale covering, we have $H^\bullet_{\mathrm{sm}}(\widehat{A}) = H^\bullet_{\text{ét}}(\widehat{A})$ (cf. [ECH, III.3.4 (c)]). Since an étale covering is covered by a finer étale finite coverings, $H^q_{\text{ét}}(\widehat{A})$ and $H^q(A)$ for $q > 0$ is a torsion module. This torsionness is well known for Galois cohomology (as the Galois group is profinite; see [CNF, (1.6.1)]).
Pick any $x \in \widehat{A}(\kappa)$. We can find an étale finite extension κ'/κ such that $p^n y = x$ for some $y \in \widehat{A}(\kappa')$. Then y is unique modulo $\widehat{A}[p^n](\kappa')$. Therefore, the sheaf quotient $(\widehat{A}/A[p^\infty])(\kappa)$ is p-divisible and torsion-free; so, is a sheaf of $\mathbb{Q}_p$-vector spaces. In other words, $\widehat{A}/A[p^\infty]$ is isomorphic to the sheaf tensor product $\widehat{A} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Thus we have an exact sequence

$$0 \to A[p^\infty] \to \widehat{A} \to \widehat{A} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to 0.$$

Since $H^1_?(\widehat{A} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$ is a $\mathbb{Q}_p$-vector space, the image in $H^1_?(\widehat{A} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$ of the torsion module $H^1(\widehat{A})$ vanishes. Thus we have an exact sequence

$$0 \to \widehat{A}(\kappa) \otimes_{\mathbb{Z}_p} \mathbb{T}_p \to H^1_?(A[p^\infty]) \to H^1_?(\widehat{A}) \to 0.$$

Since $0 \to \widehat{A}(\kappa) \otimes_{\mathbb{Z}_p} \mathbb{Z}/p\mathbb{Z} \to H^1_?(\widehat{A}[p]) \to H^1_?(\widehat{A})[p] \to 0$ is exact, by the finiteness of $H^1(\widehat{A}[p]) = H^1(A[p])$ (see [ADT, I.5]), the last assertion for Galois cohomology follows. Then using the comparison theorem (cf. [ECH, III.3.4 (c) and III.3.9]), we conclude the same for other topologies. □

3. $U(p)$-isomorphisms

We recall the results in [H14b, §3] with detailed proofs for some results and a brief account for some others (as [H14b] is being written along with this paper). For $Z[U]$-modules X and Y, we call a $\mathbb{Z}[U]$-linear map $X \xrightarrow{f} Y$ a U-injection (resp. a U-surjection) if $\mathrm{Ker}(f)$ is killed by a power of U (resp. $\mathrm{Coker}(f)$ is killed by a power of U). If f is both U-injection and U-surjection, we call f is a U-isomorphism. Thus, f is a U-injection (resp. a U-surjection, a U-isomorphism) if after tensoring $\mathbb{Z}[U, U^{-1}]$, it becomes an injection (resp. a surjection, an isomorphism). In terms of U-isomorphisms, we describe briefly the facts we study in this article (and in later sections, we fill in more details in terms of the ordinary projector e).

Let N be a positive integer prime to p. We assume $Np^r \geq 4$ (without losing any generality as remarked in the introduction). We consider the (open) modular curve $Y_1(Np^r)_{/\mathbb{Q}}$ which classifies elliptic curves E with an embedding $\phi : \mu_{p^r} \hookrightarrow E[p^r] = \mathrm{Ker}(p^r : E \to E)$. Let $R_i = \mathbb{Z}_{(p)}[\mu_{p^i}]$, $K_i = \mathbb{Q}[\mu_{p^i}]$, $R_\infty = \bigcup_i R_i \subset \overline{\mathbb{Q}}$ and $K_\infty = \bigcup_i K_i \subset \overline{\mathbb{Q}}$. For a valuation subring or a subfield R of K_∞ over $\mathbb{Z}_{(p)}$ with quotient field K, we write $X_{r/R}$ for the normalization of the j-line $\mathbf{P}(j)_{/R}$ in the function field of $Y_1(Np^r)_{/K}$. The group $z \in (\mathbb{Z}/p^r\mathbb{Z})^\times$ acts on X_r by $\phi \mapsto \phi \circ z$, as $\mathrm{Aut}(\mu_{Np^r}) \cong (\mathbb{Z}/Np^r\mathbb{Z})^\times$. Thus $\Gamma = 1 + p\mathbb{Z}_p = \gamma^{\mathbb{Z}_p}$ acts on X_r (and its Jacobian) through its image in $(\mathbb{Z}/Np^r\mathbb{Z})^\times$. Only in the following section, we need the result over a discrete valuation ring R. Hereafter, in most cases, we take $U = U(p)$ for the Hecke-Atkin operator $U(p)$ (though we take $U = U^*(p)$ sometimes for the dual $U^*(p)$ of $U(p)$).

Let $J_{r/R} = \mathrm{Pic}^0_{X_r/R}$ be the connected component of the Picard scheme. We state a result comparing $J_{r/R}$ and the Néron model of $J_{r/K}$ over R. Thus we assume that R is a valuation ring. By [AME, 13.5.6, 13.11.4], $X_{r/R}$ is regular; the reduction $X_r \otimes_R \mathbb{F}_p$ is a union of irreducible components, and the component containing the ∞ cusp has geometric multiplicity 1. Then by [NMD, Theorem 9.5.4], $J_{r/R}$ gives the identity connected component of the Néron model of the Jacobian of $X_{r/R}$. We write $X^s_{r/R}$ for the normalization of the j-line in the function field of the canonical $\mathbb{Q}$-curve associated to the modular curve of the congruence subgroup $\Gamma^r_s = \Gamma_1(Np^r) \cap \Gamma_0(p^s)$ (for $0 < r \leq s$). The open curve $Y^r_{s/\mathbb{Q}} = X^r_{s/\mathbb{Q}} - \{\text{cusps}\}$ classifies triples $(E, C, \phi : \mu_{Np^r} \hookrightarrow E)$ with a cyclic subgroup C of order p^s containing the image $\phi(\mu_{p^r})$.

We denote $\mathrm{Pic}^0_{X^r_s/R}$ by $J^r_{s/R}$. Similarly, as above, $J^r_{s/R}$ is the connected component of the Néron model of $X^r_{s/K}$. Note that

$$\text{(3.1)}\quad \Gamma^r_s\backslash\Gamma^r_s\begin{pmatrix}1&0\\0&p^{s-r}\end{pmatrix}\Gamma_1(Np^r) \\ =\left\{\begin{pmatrix}1&a\\0&p^{s-r}\end{pmatrix}\Big|a \bmod p^{s-r}\right\}=\Gamma_1(Np^r)\backslash\Gamma_1(Np^r)\begin{pmatrix}1&0\\0&p^{s-r}\end{pmatrix}\Gamma_1(Np^r).$$

Write $U^s_r(p^{s-r}):J^s_{r/R}\to J_{r/R}$ for the Hecke operator of $\Gamma^s_r\alpha_{s-r}\Gamma_1(Np^r)$ for $\alpha_m=\left(\begin{smallmatrix}1&0\\0&p^m\end{smallmatrix}\right)$. Strictly speaking, the Hecke operator induces a morphism of the generic fiber of the Jacobians and then extends to their connected components of the Néron models by the functoriality of the model (or equivalently by Picard functoriality). Then we have the following commutative diagram from the above identity, first over $\mathbb{C}$, then over K and by Picard functoriality over R:

$$\text{(3.2)}\qquad \begin{array}{ccc} J_{r/R} & \xrightarrow{\pi^*} & J^r_{s/R}\\ \downarrow u & \swarrow u' & \downarrow u''\\ J_{r/R} & \xrightarrow{\pi^*} & J^r_{s/R},\end{array}$$

where the middle u' is given by $U^s_r(p^{s-r})$ and u and u'' are $U(p^{s-r})$. Thus

(u1) $\pi^*:J_{r/R}\to J^r_{s/R}$ is a $U(p)$-isomorphism (for the projection $\pi:X^r_s\to X_r$).

Taking the dual $U^*(p)$ of $U(p)$ with respect to the Rosati involution associated to the canonical polarization on the Jacobians. We have a dual version of the above diagram for $s>r>0$:

$$\text{(3.3)}\qquad \begin{array}{ccc} J_{r/R} & \xleftarrow{\pi_*} & J^r_{s/R}\\ \uparrow u^* & \nearrow u'^* & \uparrow u''^*\\ J_{r/R} & \xleftarrow{\pi_*} & J^r_{s/R}.\end{array}$$

Here the superscript "$*$" indicates the Rosati involution corresponding to the canonical divisor on the Jacobians, and $u^*=U^*(p)^{s-r}$ for the level $\Gamma_1(Np^r)$ and $u''^*=U^*(p)^{s-r}$ for Γ^r_s. Note that these morphisms come from the following double coset identity:

$$\text{(3.4)}\quad \Gamma^r_s\backslash\Gamma^r_s\begin{pmatrix}p^{s-r}&0\\0&1\end{pmatrix}\Gamma_1(Np^r) \\ =\left\{\begin{pmatrix}p^{s-r}&a\\0&1\end{pmatrix}\Big|a \bmod p^{s-r}\right\}=\Gamma_1(Np^r)\backslash\Gamma_1(Np^r)\begin{pmatrix}p^{s-r}&0\\0&1\end{pmatrix}\Gamma_1(Np^r).$$

From this, we get

(u*1) $\pi_*:J^r_{s/R}\to J_{r/R}$ is a $U^*(p)$-isomorphism, where π_* is the dual of π^*.

In particular, if we take the ordinary and the co-ordinary projector $e=\lim_{n\to\infty}U(p)^{n!}$ and $e^*=\lim_{n\to\infty}U^*(p)^{n!}$ on $J[p^\infty]$ for $J=J_{r/R},J_{s/R},J^r_{s/R}$, noting $U(p^m)=U(p)^m$, we have

$$\pi^*:J^{\mathrm{ord}}_{r/R}[p^\infty]\cong J^{r,\mathrm{ord}}_{s/R}[p^\infty]\text{ and }\pi_*:J^{r,\text{co-ord}}_{s/R}[p^\infty]\cong J^{\text{co-ord}}_{r/R}[p^\infty]$$

where "ord" (resp. "co-ord") indicates the image of the projector e (resp. e^*). For simplicity, we write $\mathcal{G}_{r/R} := J^{\mathrm{ord}}_{r/R}[p^\infty]_{/R}$.

Suppose that we have morphisms of three noetherian schemes $X \xrightarrow{\pi} Y \xrightarrow{g} S$ with $f = g \circ \pi$. We look into

$$H^0_{\mathrm{fppf}}(T, R^1 f_* \mathbb{G}_m) = R^1 f_* O_X^\times(T) = \mathrm{Pic}_{X/S}(T)$$

for S-scheme T and the structure morphism $f : X \to S$ (see [NMD, Chapter 8]). Suppose that f and g have compatible sections $S \xrightarrow{s_g} Y$ and $S \xrightarrow{s_f} X$ so that $\pi \circ s_f = s_g$. Then we get (e.g., [NMD, Section 8.1])

$$\mathrm{Pic}_{X/S}(T) = \mathrm{Ker}(s^1_f : H^1_{\mathrm{fppf}}(X_T, O_X^\times) \to H^1_{\mathrm{fppf}}(T, O_T^\times))$$
$$\mathrm{Pic}_{Y/S}(T) = \mathrm{Ker}(s^1_g : H^1_{\mathrm{fppf}}(Y_T, O_{Y_T}^\times) \to H^1_{\mathrm{fppf}}(T, O_T^\times))$$

for any S-scheme T, where $s^q_f : H^q_{\mathrm{fppf}}(X_T, O^\times_{X_T}) \to H^q_{\mathrm{fppf}}(T, O^\times_T)$ and $s^n_g : H^n_{\mathrm{fppf}}(Y_T, O^\times_{Y_T}) \to H^n_{\mathrm{fppf}}(T, O^\times_T)$ are morphisms induced by s_f and s_g, respectively. Here we wrote $X_T = X \times_S T$ and $Y_T = Y \times_S T$. We suppose that the functors $\mathrm{Pic}_{X/S}$ and $\mathrm{Pic}_{Y/S}$ are representable by smooth group schemes (for example, if X, Y are curves and $S = \mathrm{Spec}(k)$ for a field k; see [NMD, Theorem 8.2.3 and Proposition 8.4.2]). We then put $J_? = \mathrm{Pic}^0_{?/S}$ $(? = X, Y)$. Anyway we suppose hereafter also that X, Y, S are varieties (in the sense of [ALG, II.4]).

For an fppf covering $\mathcal{U} \to Y$ and a presheaf $P = P_Y$ on the fppf site over Y, we define via Čech cohomology theory an fppf presheaf $\mathcal{U} \mapsto \check{H}^q(\mathcal{U}, P)$ denoted by $\underline{\check{H}}^q(P_Y)$ (see [ECH, III.2.2 (b)]). The inclusion functor from the category of fppf sheaves over Y into the category of fppf presheaves over Y is left exact. The derived functor of this inclusion of an fppf sheaf $F = F_Y$ is denoted by $\underline{H}^\bullet(F_Y)$ (see [ECH, III.1.5 (c)]). Thus $\underline{H}^\bullet(\mathbb{G}_{m/Y})(\mathcal{U}) = H^\bullet_{\mathrm{fppf}}(\mathcal{U}, O^\times_{\mathcal{U}})$ for a Y-scheme $\mathcal{U}$ as a presheaf (here $\mathcal{U}$ varies in the small fppf site over Y).

Assuming that f, g and π are all faithfully flat of finite presentation, we use the spectral sequence of Čech cohomology of the flat covering $\pi : X \twoheadrightarrow Y$ in the fppf site over Y [ECH, III.2.7]:

$$\check{H}^p(X_T/Y_T, \underline{H}^q(\mathbb{G}_{m/Y})) \Rightarrow H^n_{\mathrm{fppf}}(Y_T, O^\times_{Y_T}) \xrightarrow[\iota]{\sim} H^n(Y_T, O^\times_{Y_T}) \tag{3.5}$$

for each S-scheme T. Here $F \mapsto H^n_{\mathrm{fppf}}(Y_T, F)$ (resp. $F \mapsto H^n(Y_T, F)$) is the right derived functor of the global section functor: $F \mapsto F(Y_T)$ from the category of fppf sheaves (resp. Zariski sheaves) over Y_T to the category of abelian groups. The canonical isomorphism ι is the one given in [ECH, III.4.9]. By the sections $s_?$, we have a splitting $H^q(X_T, O^\times_{X_T}) = \mathrm{Ker}(s^q_f) \oplus H^q(T, O^\times_T)$ and $H^n(Y_T, O^\times_{Y_T}) = \mathrm{Ker}(s^n_g) \oplus H^n(T, O^\times_T)$. Write $\underline{H}^\bullet_{Y_T}$ for $\underline{H}^\bullet(\mathbb{G}_{m/Y_T})$ and $\check{H}^\bullet(\underline{H}^0_{Y_T})$ for $\check{H}^\bullet(Y_T/X_T, \underline{H}^0_{Y_T})$. Since

$$\mathrm{Pic}_{X/S}(T) = \mathrm{Ker}(s^1_{f,T} : H^1(X_T, O^\times_{X_T}) \to H^1(T, O^\times_T))$$

for the morphism $f : X \to S$ with a section [NMD, Proposition 8.1.4], from this spectral sequence, we have the following commutative diagram with exact

rows, writing $\check H^0(\frac{X_T}{Y_T},?)$ as $\check H^0(?)$ and $H^1(T,O_T^\times)$ as $H^1(O_T^\times)$:
(3.6)

$$\begin{array}{ccccc}
?_1 & \longrightarrow & \check H^1(\underline{H}^0_{Y_T}) & = & \check H^1(\underline{H}^0_{Y_T}) \\
\downarrow & & \downarrow & & \downarrow\cap \\
\mathrm{Pic}_T \oplus J_Y(T) & \stackrel{\hookrightarrow}{\longrightarrow} & \mathrm{Pic}_T \oplus \mathrm{Pic}_{Y/S}(T) & \stackrel{\sim}{\longrightarrow} & H^1(O_T^\times) \oplus \mathrm{Ker}(s^1_{g,T}) \\
c\downarrow & & b\downarrow & & a\downarrow \\
\mathrm{Pic}_T \oplus \check H^0(J_X(T)) & \stackrel{\hookrightarrow}{\longrightarrow} & \check H^0(\mathrm{Pic}_Y(T)) & = & \check H^0(\underline{H}^1(\mathbb{G}_{m,Y})) \\
\downarrow & & \downarrow & & \downarrow \\
?_2 & \longrightarrow & \check H^2(\underline{H}^0_{Y_T}) & = & \check H^2(\underline{H}^0_{Y_T}),
\end{array}$$

where we have written $J_? = \mathrm{Pic}^0_{?/S}$ (the identity connected component of $\mathrm{Pic}_{?/S}$). Here the vertical exactness at the right two columns follows from the spectral sequence (3.5) (see [ECH, Appendix B]).

We now recall the definition of the Čech cohomology: for a general S-scheme T and Čech cochain $c_{i_0,\dots,i_q} \in H^0(X_T^{(q+1)}, O^\times_{X_T^{(q+1)}})$,

$$(3.7)\quad \check H^q(\frac{X_T}{Y_T}, \underline{H}^0(\mathbb{G}_{m/Y})) = \frac{\{(c_{i_0,\dots,i_q})|\prod_j (c_{i_0\dots\check{i}_j\dots i_{q+1}} \circ p_{i_0\dots\check{i}_j\dots i_{q+1}})^{(-1)^j} = 1\}}{\{db_{i_0\dots i_q} = \prod_j (b_{i_0\dots\check{i}_j\dots i_q} \circ p_{i_0\dots\check{i}_j\dots i_q})^{(-1)^j} | b_{i_0\dots\check{i}_j\dots i_q} \in H^0(X_T^{(q)}, O^\times_{X_T^{(q)}})\}}$$

where we agree to put $H^0(X_T^{(0)}, O^{(0)}_{X_T}) = 0$ as a convention,

$$X_T^{(q)} = \overbrace{X \times_Y X \times_Y \cdots \times_Y X}^{q} \times_S T,$$

$$O_{X_T^{(q)}} = \overbrace{O_X \times_{O_Y} O_X \times_{O_Y} \cdots \times_{O_Y} O_X}^{q} \times_{O_S} O_T,$$

the identity $\prod_j (c \circ p_{i_0\dots\check{i}_j\dots i_{q+1}})^{(-1)^j} = 1$ takes place in $O_{X_T^{(q+2)}}$ and $p_{i_0\dots\check{i}_j\dots i_{q+1}} : X_T^{(q+2)} \to X_T^{(q+1)}$ is the projection to the product of X the j-th factor removed.

Since $T \times_T T \cong T$ canonically, we have $X_T^{(q)} \cong \overbrace{X_T \times_T \cdots \times_T X_T}^{q}$ by transitivity of fiber product.

Take a correspondence $U \subset Y \times_S Y$ given by two finite flat projections $\pi_1, \pi_2 : U \to Y$ of constant degree (i.e., $\pi_{j,*}\mathcal{O}_U$ is locally free of finite rank $\deg(\pi_j)$ over

$\mathcal{O}_Y$). Consider the pullback $U_X \subset X \times_S X$ given by the Cartesian diagram:

$$\begin{array}{ccc} U_X = U \times_{Y \times_S Y} (X \times_S X) & \longrightarrow & X \times_S X \\ \downarrow & & \downarrow \\ U & \stackrel{\hookrightarrow}{\longrightarrow} & Y \times_S Y \end{array}$$

Let $\pi_{j,X} = \pi_j \times_S \pi : U_X \twoheadrightarrow X$ $(j = 1, 2)$ be the projections.
We describe the correspondence action of U on $H^0(X, \mathcal{O}_X^\times)$ in down-to-earth terms. Consider $\alpha \in H^0(X, \mathcal{O}_X)$. Then we lift $\pi_{1,X}^*\alpha = \alpha \circ \pi_{1,X} \in H^0(U_X, \mathcal{O}_{U_X})$. Put $\alpha_U := \pi_{1,X}^*\alpha$. Note that $\pi_{2,X,*}\mathcal{O}_{U_X}$ is locally free of rank $d = \deg(\pi_2)$ over $\mathcal{O}_X$, the multiplication by α_U has its characteristic polynomial $P(T)$ of degree d with coefficients in $\mathcal{O}_X$. We define the norm $N_U(\alpha_U)$ to be the constant term $P(0)$. Since α is a global section, $N_U(\alpha_U)$ is a global section, as it is defined everywhere locally. If $\alpha \in H^0(X, \mathcal{O}_X^\times)$, $N_U(\alpha_U) \in H^0(X, \mathcal{O}_X^\times)$. Then define $U(\alpha) = N_U(\alpha_U)$, and in this way, U acts on $H^0(X, \mathcal{O}_X^\times)$.
For a degree q Čech cohomology class $[c] \in \check{H}^q(X_{/Y}, \underline{H}^0(\mathbb{G}_{m/Y}))$ of a Čech q-cocycle $c = (c_{i_0,\dots,i_q})$, $U([c])$ is given by the cohomology class of the Čech cocycle $U(c) = (U(c_{i_0,\dots,i_q}))$, where $U(c_{i_0,\dots,i_q})$ is the image of the global section $c_{i_0,\dots,i_q}$ under U. Indeed, $(\pi_{1,X}^* c_{i_0,\dots,i_q})$ plainly satisfies the cocycle condition, and $(N_U(\pi_{1,X}^* c_{i_0,\dots,i_q}))$ is again a Čech cocycle as N_U is a multiplicative homomorphism. By the same token, we see that this operation sends coboundaries to coboundaries and obtain the action of U on the cohomology group.

Lemma 3.1. *Let the notation and the assumption be as above. In particular, $\pi : X \to Y$ is a finite flat morphism of geometrically reduced proper schemes over $S = \mathrm{Spec}(k)$ for a field k. Suppose that X and U_X are proper schemes over a field k satisfying one of the following conditions:*

1. *U_X is geometrically reduced, and for each geometrically connected component X° of X, its pull back to U_X by $\pi_{2,X}$ is also connected; i.e., $\pi^0(X) \xrightarrow[\sim]{\pi_{2,X}^*} \pi^0(U_X)$;*
2. *$(f \circ \pi_{2,X})_* \mathcal{O}_{U_X} = f_* \mathcal{O}_X$.*

If $\pi_2 : U \to Y$ has constant degree $\deg(\pi_2)$, the action of U on $H^0(X, \mathcal{O}_X^\times)$ factors through the multiplication by $\deg(\pi_2) = \deg(\pi_{2,X})$.

Proof. By properness, under (1) or (2), $H^0(U_X, \mathcal{O}_{U_X}) \stackrel{\pi_{2,X,*}}{=} H^0(X, \mathcal{O}_X) (\stackrel{(1)}{=} k^{\pi^0(X)})$ for the set of connected components $\pi^0(X)$ of X. In particular, we see $\alpha_U \in H^0(U_X, \mathcal{O}_{U_X}) = H^0(X, \mathcal{O}_X)$, which tells us that $N_U(\alpha_U) = \alpha_U^{\deg(\pi_2)}$, and the result follows. ☐

Consider the iterated product $\pi_{i,X^{(q)}} = \pi_{i,X} \times_Y \cdots \times_Y \pi_{i,X} : U_X^{(q)} \to X^{(q)}$ $(i = 1, 2)$. Here $U_X^{(q)} = \overbrace{U_X \times_Y U_X \times_Y \cdots \times_Y U_X}^{q}$. We plug in $U_X^{(j)}$ in the first

j slots of the fiber product (for $0 < j \leq q$) and consider

$$U_X^{(j-1)} \times_Y X^{(q-j+1)} \xleftarrow{\pi_{1,j}^{(q)}} U_j := U_X^{(j)} \times_Y X^{(q-j)} \xrightarrow{\pi_{2,j}^{(q)}} U_X^{(j-1)} \times_Y X^{(q-j+1)}$$

which induces a correspondence U_j in $(U_X^{(j-1)} \times_Y X^{(q-j+1)}) \times_Y (U_X^{(j-1)} \times_Y X^{(q-j+1)})$. Here $\pi_{i,j}$ restricted to first $j-1$-factors U_X is the identity id_{U_X}; the last $q-j$ factors is the identity id_X and at the j-th factor, it is the projection π_i $(i = 1, 2)$. For example, if $q = 3$ and $i = 2$, we have

$$U_X \times_Y U_X \times_Y U_X \xrightarrow[\mathrm{id}_U \times \mathrm{id}_U \times \pi_2]{\pi_{2,3}^{(q)}} U_X \times_Y U_X \times_Y X$$
$$\xrightarrow[\mathrm{id}_U \times \pi_2 \times \mathrm{id}_X]{\pi_{2,2}^{(q)}} U_X \times_Y X \times_Y X \xrightarrow[\pi_2 \times \mathrm{id}_X \times \mathrm{id}_X]{\pi_{2,1}^{(q)}} X \times_Y \times_Y X.$$

Naturally $\pi_{2,X^{(q)}}$ factors through the following q consecutive coverings $U_q \xrightarrow{\rho_q} U_{q-1} \xrightarrow{\rho_{q-1}} \cdots \xrightarrow{\rho_1} X^{(q)}$ for $\rho_j = \pi_{2,j}^{(q)}$. Note that the norm map $N_{U_q} = N_{\pi_{2,X^{(q)}}} : \pi_{2,X^{(q)},*}\mathcal{O}_{U_q}^\times \to \mathcal{O}_{X^{(q)}}^\times$ factors through the corresponding norm maps:

$$N_{U_q} = N_q \circ N_{q-1} \circ \cdots \circ N_1,$$

where N_j is the norm map with respect to $U_j \to U_{j-1}$. The last norm is essentially the product of N_U and the identity of $X^{(q-1)}$ corresponding to $U \times_Y X^{(q-1)} \twoheadrightarrow X^{(q)}$. In particular, $\rho_{1,*}(\mathcal{O}_{U_1}) = \pi_{2,X,*}(\mathcal{O}_{U_X}) \otimes_{\mathcal{O}_Y} \mathcal{O}_{X^{(q-1)}}$ and

$$(f \circ \rho_1)_*(\mathcal{O}_{U_1}) = (f \circ \pi_{2,X})_*(\mathcal{O}_{U_X}) \otimes_{\mathcal{O}_Y} \overbrace{f_*\mathcal{O}_X \otimes_{\mathcal{O}_Y} \cdots \otimes_{\mathcal{O}_Y} f_*\mathcal{O}_X}^{q-1}.$$

Thus if the assumption (2) in Lemma 3.1 is satisfied, the corresponding assumption for U_1 is satisfied. The assumption (1) implies (2) which is really necessary for the proof of Lemma 3.1. Applying the argument proving Lemma 3.1 to the correspondence U_1 and the last factor N_1 of the norm, we get

Corollary 3.2. *Let the notation and the assumption be as in* Lemma 3.1. *Then the action of* $U^{(q)}$ *on* $H^0(X, \mathcal{O}_{X^{(q)}}^\times)$ *factors through the multiplication by* $\deg(\pi_2) = \deg(\pi_{2,X})$.

Here is a main result of this section:

Proposition 3.3. *Suppose that* $S = \mathrm{Spec}(k)$ *for a field* k. *Let* $\pi : X \to Y$ *be a finite flat covering of (constant) degree* d *of geometrically reduced proper varieties over* k, *and let* $Y \xleftarrow{\pi_1} U \xrightarrow{\pi_2} Y$ *be two finite flat coverings (of constant degree) identifying the correspondence* U *with a closed subscheme* $U \overset{\pi_1 \times \pi_2}{\hookrightarrow} Y \times_S Y$. *Write* $\pi_{j,X} : U_X = U \times_Y X \to X$ *be the base-change to* X. *Suppose one of the conditions* (1) *and* (2) *of* Lemma 3.1 *for* (X, U). *Then*

(1) *The correspondence* $U \subset Y \times_S Y$ *sends* $\check{H}^q(\underline{H}_Y^0)$ *into* $\deg(\pi_2)(\check{H}^q(\underline{H}_Y^0))$ *for all* $q > 0$.

(2) *If* d *is a* p*-power and* $\deg(\pi_2)$ *is divisible by* p, $\check{H}^q(\underline{H}_Y^0)$ *for* $q > 0$ *is killed by* U^M *if* $p^M \geq d$.

(3) *The cohomology $\check{H}^q(\underline{H}^0_Y)$ with $q > 0$ is killed by d.*

Proof. The first assertion follows from Corollary 3.2. Indeed, by (3.7), $U^{(q)}$ acts on each Čech q-cocycle, through an action factoring through the multiplication by $\deg(\pi_{2,X}) = \deg(\pi_2)$ by Corollary 3.2.
Now we regard $X \xrightarrow{\pi} Y$ as a correspondence of Y (with multiplicity d) by the projection $\pi_1 = \pi_2 = \pi : X \to Y$. Then $[X](c) = dc$ for $c \in \check{H}^q(X/Y, \underline{H}^0(\mathbb{G}_{m/Y}))$. On the other hand, by the definition of the correspondence action, $[X]$ factors through $\check{H}^q(X/X, \underline{H}^0(\mathbb{G}_{m/Y})) = 0$, and hence $dx = 0$. This shows that if X/Y is a covering of degree d, $\check{H}^q(X/Y, \underline{H}^0(\mathbb{G}_{m/Y}))$ is killed by d proving (3), and the assertion (2) follows from the first (1). □

We apply the above proposition to $(U, X, Y) = (U(p), X_s, X_s^r)$ with U given by $U(p) \subset X_s^r \times X_s^r$ over $\mathbb{Q}$. Indeed, $U := U(p) \subset X_s^r \times X_s^r$ corresponds to $X(\Gamma)$ given by $\Gamma = \Gamma_1(Np^r) \cap \Gamma_0(p^{s+1})$ and U_X is given by $X(\Gamma')$ for $\Gamma' = \Gamma_1(Np^s) \cap \Gamma_0(p^{s+1})$ both geometrically irreducible curves. In this case π_1 is induced by $z \mapsto \frac{z}{p}$ on the upper complex plane and π_2 is the natural projection of degree p. In this case, $\deg(X_s/X_s^r) = p^{s-r}$ and $\deg(\pi_2) = p$.

Assume that a finite group G acts on $X_{/Y}$ faithfully. Then we have a natural morphism $\phi : X \times G \to X \times_Y X$ given by $\phi(x, \sigma) = (x, \sigma(x))$. In other words, we have a commutative diagram

$$\begin{array}{ccc} X \times G & \xrightarrow{(x,\sigma)\mapsto\sigma(x)} & X \\ {\scriptstyle (x,\sigma)\mapsto x}\downarrow & & \downarrow \\ X & \longrightarrow & Y, \end{array}$$

which induces $\phi : X \times G \to X \times_Y X$ by the universality of the fiber product. Suppose that ϕ is surjective; for example, if Y is a geometric quotient of X by G; see [GME, §1.8.3]). Under this map, for any fppf abelian sheaf F, we have a natural map $\check{H}^0(X/Y, F) \to H^0(G, F(X))$ sending a Čech 0-cocycle $c \in H^0(X, F) = F(X)$ (with $p_1^*c = p_2^*c$) to $c \in H^0(G, F(X))$. Obviously, by the surjectivity of ϕ, the map $\check{H}^0(X/Y, F) \to H^0(G, F(X))$ is an isomorphism (e.g., [ECH, Example III.2.6, page 100]). Thus we get

Lemma 3.4. *Let the notation be as above, and suppose that ϕ is surjective. For any scheme T fppf over S, we have a canonical isomorphism: $\check{H}^0(X_T/Y_T, F) \cong H^0(G, F(X_T))$.*

We now assume $S = \mathrm{Spec}(k)$ for a field k and that X and Y are proper reduced connected curves. Then we have from the diagram (3.6) with the exact middle two columns and exact horizontal rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathbb{Z} & = & \mathbb{Z} & \longrightarrow & 0 \\
\uparrow & & {\scriptstyle\deg}\uparrow{\scriptstyle\text{onto}} & & {\scriptstyle\deg}\uparrow{\scriptstyle\text{onto}} & & \uparrow \\
\check{H}^1(\underline{H}^0_Y) & \longrightarrow & \mathrm{Pic}_{Y/S}(T) & \xrightarrow{b} & \check{H}^0(\frac{X_T}{Y_T}, \mathrm{Pic}_{Y/S}(T)) & \longrightarrow & \check{H}^2(\underline{H}^0_Y) \\
\uparrow & & \cup\uparrow & & \uparrow\cup & & \uparrow \\
?_1 & \longrightarrow & J_Y(T) & \xrightarrow[c]{} & \check{H}^0(\frac{X_T}{Y_T}, J_X(T)) & \longrightarrow & ?_2,
\end{array}$$

Thus we have $?_j = \check{H}^j(\underline{H}^0_Y)$ $(j = 1, 2)$.
By Proposition 3.3, if $q > 0$ and X/Y is of degree p-power and $p|\deg(\pi_2)$, $\check{H}^q(\underline{H}^0_Y)$ is a p-group, killed by U^M for $M \gg 0$. Taking $(X, Y, U)_{/S}$ to be $(X_{s/\mathbb{Q}}, X^r_{s/\mathbb{Q}}, U(p))_{/\mathbb{Q}}$ for $s > r \geq 1$ for p odd and $s > r \geq 2$ for $p = 2$, we get for the projection $\pi : X_s \to X^r_s$

COROLLARY 3.5. *Let F be a number field or a finite extension of $\mathbb{Q}_l$ (for a prime l not necessarily equal to p). Then we have*

(u) $\pi^* : J^r_{s/\mathbb{Q}}(F) \to \check{H}^0(X_s/X^r_s, J_{s/\mathbb{Q}}(F)) \overset{(*)}{=} J_{s/\mathbb{Q}}(F)[\gamma^{p^{r-1}} - 1]$ *is a $U(p)$-isomorphism,*

where $J_{s/\mathbb{Q}}(F)[\gamma^{p^{r-1}} - 1] = \mathrm{Ker}(\gamma^{p^{r-1}} - 1 : J_s(F) \to J_s(F))$.

From these, we got the following facts as [H14b, Lemma 3.7]

LEMMA 3.6. *We have morphisms*

$$\iota^r_s : J_{s/\mathbb{Q}}[\gamma^{p^{r-1}} - 1] \to J^r_{s/\mathbb{Q}} \quad \textit{and} \quad \iota^{r,*}_s : J^r_{s/\mathbb{Q}} \to J_{s/\mathbb{Q}}/(\gamma^{p^{r-1}} - 1)(J_{s/\mathbb{Q}})$$

satisfying the following commutative diagrams:

$$\begin{array}{ccc}
J^r_{s/\mathbb{Q}} & \xrightarrow{\pi^*} & J_{s/\mathbb{Q}}[\gamma^{p^{r-1}} - 1] \\
\downarrow u & \swarrow \iota^r_s & \downarrow u'' \\
J^r_{s/\mathbb{Q}} & \xrightarrow{\pi^*} & J_{s/\mathbb{Q}}[\gamma^{p^{r-1}} - 1],
\end{array} \tag{3.8}$$

and

$$\begin{array}{ccc}
J^r_{s/\mathbb{Q}} & \xleftarrow{\pi_*} & J_{s/\mathbb{Q}}/(\gamma^{p^{r-1}} - 1)(J_{s/\mathbb{Q}}) \\
\uparrow u^* & \nearrow \iota^{r,*}_s & \uparrow u''^* \\
J^r_{s/\mathbb{Q}} & \xleftarrow{\pi^*} & J_{s/\mathbb{Q}}/(\gamma^{p^{r-1}} - 1)(J_{s/\mathbb{Q}}),
\end{array} \tag{3.9}$$

where u and u'' are $U(p^{s-r}) = U(p)^{s-r}$ and u^* and u''^* are $U^*(p^{s-r}) = U^*(p)^{s-r}$. In particular, for an fppf extension $T_{/\mathbb{Q}}$, the evaluated map at T: $(J_{s/\mathbb{Q}}/(\gamma^{p^{r-1}} - 1)(J_{s/\mathbb{Q}}))(T) \xrightarrow{\pi_*} J^r_s(T)$ (resp. $J^r_s(T) \xrightarrow{\pi^*} J_s[\gamma^{p^{r-1}} - 1](T)$) is a $U^*(p)$-isomorphism (resp. $U(p)$-isomorphism).

REMARK 3.7. Note here that the natural morphism:

$$\frac{J_s(T)}{(\gamma^{p^{r-1}} - 1)(J_s(T))} \to (J_s/(\gamma^{p^{r-1}} - 1)(J_s))(T)$$

may have non-trivial kernel and cokernel which may not be killed by a power of $U^*(p)$. In other words, the left-hand-side is an fppf presheaf (of T) and the right-hand-side is its sheafication. On the other hand, $T \mapsto J_s[\gamma^{p^{r-1}} - 1](T)$ is already an fppf abelian sheaf; so, $J_r(T) \xrightarrow{\pi^*} J_s[\gamma^{p^{r-1}} - 1](T)$ is a $U(p)$-isomorphism without ambiguity by the above Lemma 3.6 and Corollary 3.5 combined. Also, as remarked in the introduction, we need to replace $\gamma^{p^{r-1}} - 1$ in the above statement by $\gamma^{p^{r-2}} - 1$ if $p = 2$.

4. Structure of Λ-BT groups over number fields and local fields

Let $\mathcal{G}_{/R_\infty} := \varinjlim_r J_r[p^\infty]^{\mathrm{ord}}_{/R_\infty}$, which is a Λ-BT group in the sense of [H14a, Sections 3 and 5] with a canonical $\mathbf{h}$-action. Here for an abelian variety $A_{/R}$, $A[p^n] = \mathrm{Ker}(A \xrightarrow{p^n} A)$ and $A[p^\infty]_{/R} = \varinjlim_n A[p^n]$ (the p-divisible Barsotti–Tate group of A over R). For an $\mathbf{h}$-algebra A, we put $\mathcal{G}_A = \mathcal{G} \otimes_{\mathbf{h}} A$. Pick a reduced local ring $\mathbb{T}$ of $\mathbf{h}$ and write $a(l^m)$ for the image in $\mathbb{T}$ of $U(l^m)$ or $T(l^m)$ for a prime l according as $l|Np$ or $l \nmid Np$ and $\mathfrak{m}_{\mathbb{T}}$ for the maximal ideal of $\mathbb{T}$. Since $\mathcal{G}_{\mathbb{T}}$ is a Λ-BT group in the sense of [H14a, Theorem 5.4, Remark 5.5], we have the connected-étale exact sequence over $\mathbb{Z}_p[\mu_{p^\infty}]$:

$$0 \to \mathcal{G}^\circ_{\mathbb{T}} \to \mathcal{G}_{\mathbb{T}} \to \mathcal{G}^{\text{ét}}_{\mathbb{T}} \to 0,$$

where $\mathcal{G}^\circ_{\mathbb{T}}$ is the connected component of the flat group $\mathcal{G}_{\mathbb{T}}$ and $\mathcal{G}^{\text{ét}}_{\mathbb{T}}$ is the quotient of $\mathcal{G}_{\mathbb{T}}$ by $\mathcal{G}^\circ_{\mathbb{T}}$. The étale group $\mathcal{G}_{\mathbb{T}/\mathbb{Q}}$ over $\mathbb{Q}$ is a Λ-BT group over $\mathbb{Q}$ (in the sense of [H14a, §4]) on which $\mathbb{Z}_p^\times$ act by diamond operators. The entire group $\mathcal{G}_{\mathbb{T}}$ extends to a Λ-BT group over $\mathbb{Z}_p[\mu_{p^\infty}]$ (see [H14a, Remark 5.5]). The $\overline{\mathbb{Q}}_p$-points of this sequence descent to $\mathbb{Q}_p$ giving an exact sequence:

$$0 \to \mathcal{G}^\circ_{\mathbb{T}}(\overline{\mathbb{Q}}_p) \to \mathcal{G}_{\mathbb{T}}(\overline{\mathbb{Q}}_p) \to \mathcal{G}^{\text{ét}}_{\mathbb{T}}(\overline{\mathbb{Q}}_p) \to 0$$

with $\mathcal{G}^{\text{ét}}_{\mathbb{T}}(\overline{\mathbb{Q}}_p) = H_0(I_p, \mathcal{G}_{\mathbb{T}}(\overline{\mathbb{Q}}_p))$ for the inertia group $I_p \subset \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$.
We know that $\mathcal{G}^\circ_{\mathbb{T}}$ and $\mathcal{G}^{\text{ét}}_{\mathbb{T}}$ are well controlled, and the Pontryagin dual modules of $\mathcal{G}^\circ_{\mathbb{T}}(\overline{\mathbb{Q}})$ and $\mathcal{G}^{\text{ét}}_{\mathbb{T}}(\overline{\mathbb{Q}})$ are Λ-free modules of (equal) finite rank (see [H86b, §9] or [H14a, Sections 4–5]). Here we equip these Λ-divisible modules with the discrete topology. Take a field k as a base field. Pick a $\mathbb{T}$-ideal $\mathfrak{a}$. Write $\mathcal{G}_{\mathbb{T}}[\mathfrak{a}]$ for the kernel of $\mathfrak{a}$:

$$\mathcal{G}_{\mathbb{T}}[\mathfrak{a}](R) = \{x \in \mathcal{G}_{\mathbb{T}}(R) | ax = 0\ \forall a \in \mathfrak{a}\},$$

where R is an fppf extension of k. Write $a(p)$ for the image of $U(p)$ in $\mathbb{T}$.
For the moment, assume that k is a finite extension k of $\mathbb{Q}_p$ with p-adic integer ring W. If the residual degree of k is f and $a(p)^f \not\equiv 1 \mod \mathfrak{m}_{\mathbb{T}}$ for the maximal ideal $\mathfrak{m}_{\mathbb{T}}$ of $\mathbb{T}$, we have

$$\mathcal{G}_{\mathbb{T}}[\mathfrak{m}_{\mathbb{T}}]^{\text{ét}}(k) = 0,$$

since the action of $Frob_p$ on $\mathcal{G}_{\mathbb{T}}[\mathfrak{m}_{\mathbb{T}}]^{\text{ét}}(\overline{\mathbb{Q}}_p)$ is given by multiplication by $a(p)$. On the other hand, the action of $\mathrm{Gal}(\overline{k}/k)$ on $e \cdot J_\infty[p^\infty]^\circ(\overline{k}) \otimes_{\mathbf{h}} \mathbb{T}$ factors through $\mathrm{Gal}(k[\mu_{p^\infty}]/k) \hookrightarrow \mathbb{Z}_p^\times \to \Lambda^\times$, where the factor $\Gamma = 1 + p\mathbb{Z}_p$ of $\mathbb{Z}_p^\times = \Gamma \times \mu_{p-1}$ is

embedded into $\Lambda = \mathbb{Z}_p[[\Gamma]]$ by natural inclusion and $\zeta \in \mu_{p-1}$ is sent to ζ^a for some $0 \le a =: a(\mathbb{T}) = a_k(\mathbb{T}) < p-1$. Thus if $a(\mathbb{T}) \neq 0$, we have $\mathcal{G}^\circ_{\mathbb{T}}[\mathfrak{m}_{\mathbb{T}}](k) = 0$. We have a natural projection $\pi = \pi^r_s : \mathcal{G}_s := J_s[p^\infty]^{\mathrm{ord}}_{/\mathbb{Q}} \to \mathcal{G}_r$ for $s > r$ (see [H13a, Section 4] where π^r_s is written as N^s_r). This induces a projective system of Tate modules $\{T\mathcal{G}_{s,\mathbb{T}} := T\mathcal{G}_s \otimes_{\mathbf{h}} \mathbb{T}\}_s$ and $\{T\mathcal{G}^?_{s,\mathbb{T}}\}$ for $? = \circ, \text{ét}$. We put $T\mathcal{G}^?_{\mathbb{T}} = \varprojlim_s T\mathcal{G}^?_{s,\mathbb{T}}(\overline{\mathbb{Q}})$ for ? = nothing, $\circ$ or ét. They are Λ-free modules with a continuous action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Write $\rho_{\mathbb{T}}$ for the Galois representation realized on $T\mathcal{G}_{\mathbb{T}}$, and put $\rho_P = \rho_{\mathbb{T}} \mod P$ acting on $T\mathcal{G}_{\mathbb{T}}/PT\mathcal{G}_{\mathbb{T}}$ for $P \in \mathrm{Spec}(\mathbb{T})$. In particular, we simply write $\overline{\rho} = \overline{\rho}_{\mathbb{T}} = \rho_{\mathfrak{m}_{\mathbb{T}}}$ for the maximal ideal $\mathfrak{m}_{\mathbb{T}}$ of $\mathbb{T}$.
If $\mathbb{T}$ is a Gorenstein ring, then for the Tate modules $T\mathcal{G}_{\mathbb{T}}$, $T\mathcal{G}^\circ_{\mathbb{T}}$ and $T\mathcal{G}^{\text{ét}}_{\mathbb{T}}$ as above, we have

$$T\mathcal{G}_{\mathbb{T}} \cong \mathbb{T}^2 \text{ and } T\mathcal{G}^\circ_{\mathbb{T}} \cong \mathbb{T} \cong T\mathcal{G}^{\text{ét}}_{\mathbb{T}}$$

as $\mathbb{T}$-modules (e.g., [H13a, Section 4]), and if $\overline{\rho}_{\mathbb{T}}(I_p)$ contains a non-trivial unipotent element for the inertia group I_p in $\mathrm{Gal}(\overline{\mathbb{Q}}_p/k)$, again we have $\mathcal{G}^{\text{ét}}_{\mathbb{T}}[\mathfrak{m}_{\mathbb{T}}](k) = 0$. Thus we get

LEMMA 4.1. *Let $k_{/\mathbb{Q}_p}$ in $\overline{\mathbb{Q}}_p$ be a finite extension and $\mathbb{T}$ be a reduced local ring of $\mathbf{h}$. Assume that k has residual degree f and one of the following two conditions:*

(1) $a_k(\mathbb{T}) \neq 0$ *and* $a(p)^f \not\equiv 1 \mod \mathfrak{m}_{\mathbb{T}}$*,*
(2) $\mathbb{T}$ *is a Gorenstein ring, and* $\overline{\rho}_{\mathbb{T}}(I_p)$ *has non-trivial unipotent element for the inertia group* I_p *of* $\mathrm{Gal}(\overline{\mathbb{Q}}_p/k)$*.*

Then we have $\mathcal{G}_{\mathbb{T}}(k) = 0$*.*

Proof. Let V be the Λ-dual of $T\mathcal{G}_{\mathbb{T}}$, which is also the Pontryagin dual of $\mathcal{G}_{\mathbb{T}}$. Then we have $H_0(k, V/\mathfrak{m}_{\mathbb{T}}V) \cong \mathcal{G}_{\mathbb{T}}[\mathfrak{m}_{\mathbb{T}}](k) = H^0(k, \mathcal{G}_{\mathbb{T}}[\mathfrak{m}_{\mathbb{T}}])$. By the assumption (1) or (2), we have the vanishing $\mathcal{G}_{\mathbb{T}}[\mathfrak{m}_{\mathbb{T}}](k) = 0$. Look into the following exact sequence of sheaves

$$0 \to \mathcal{G}_{\mathbb{T}}[\mathfrak{m}_{\mathbb{T}}] \to \mathcal{G}_{\mathbb{T}} \xrightarrow{\varphi} \bigoplus_{\alpha \in I} \mathcal{G}_{\mathbb{T}}$$

with $\varphi(x) = (\alpha x)_\alpha$ for a finite set $I = \{\alpha\}_\alpha$ of generators of $\mathfrak{m}_{\mathbb{T}}$. Taking the $\mathrm{Gal}(\overline{k}/k)$-invariant, we get another exact sequence

$$0 \to \mathcal{G}_{\mathbb{T}}[\mathfrak{m}_{\mathbb{T}}](k) \to \mathcal{G}_{\mathbb{T}}(k) \xrightarrow{\varphi_k} \bigoplus_{\alpha \in I} \mathcal{G}_{\mathbb{T}}(k).$$

Since $\mathrm{Ker}(\varphi_k) = \mathcal{G}_{\mathbb{T}}(k)[\mathfrak{m}_{\mathbb{T}}]$, we conclude $(\mathcal{G}_{\mathbb{T}}(k))[\mathfrak{m}_{\mathbb{T}}] = \mathcal{G}_{\mathbb{T}}[\mathfrak{m}_{\mathbb{T}}](k) = 0$. Taking the Pontryagin dual module written as $M^\vee$ for a compact or discrete module M, we have, setting $V = \mathcal{G}_{\mathbb{T}}(\overline{\mathbb{Q}})^\vee$,

$$H_0(k, V)/\mathfrak{m}_{\mathbb{T}} H_0(k, V) \cong (\mathcal{G}_{\mathbb{T}}(k))^\vee / \mathfrak{m}_{\mathbb{T}}(\mathcal{G}_{\mathbb{T}}(k))^\vee = (\mathcal{G}_{\mathbb{T}}(k)[\mathfrak{m}_{\mathbb{T}}])^\vee = 0,$$

which implies $\mathcal{G}_{\mathbb{T}}(k)^\vee = H_0(k, V) = 0$ by Nakayama's lemma, and hence $\mathcal{G}_{\mathbb{T}}(k) = 0$. This proves the assertion under (1) or (2). □

In the $l \neq p$ case, we remark the following fact:

Lemma 4.2. *Let $k_{/\mathbb{Q}_l}$ in $\overline{\mathbb{Q}}_l$ be a finite extension for a prime $l \neq p$ and $\mathbb{T}$ be a reduced local ring of* $\mathbf{h}$. *If the semi-simplification of $\mathcal{G}_{\mathbb{T}}[\mathfrak{m}_{\mathbb{T}}]$ as a representation of* $\mathrm{Gal}(\overline{\mathbb{Q}}_l/k)$ *does not contain the identity representation, then $\mathcal{G}_{\mathbb{T}}(k) = 0$. In general, $\mathcal{G}_{\mathbb{T}}(k)^{\vee}$ is always a torsion Λ-module of finite type.*

Proof. If the semi-simplification of $\mathcal{G}_{\mathbb{T}}[\mathfrak{m}_{\mathbb{T}}]$ as a representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_l/k)$ does not contain the identity representation, we have $H^0(k, \mathcal{G}_{\mathbb{T}}[\mathfrak{m}_{\mathbb{T}}]) = 0$; so, $H_0(k, V/\mathfrak{m}_{\mathbb{T}}V) = 0$ for $V = \mathcal{G}_{\mathbb{T}}(\overline{\mathbb{Q}})^{\vee}$. Writing $\mathfrak{m}_{\mathbb{T}} = (\alpha_i)_{i\in I}$ for $\alpha_i \in \mathbb{T}$ with a finite index set I, we have an exact sequence:

$$0 \to \mathcal{G}_{\mathbb{T}}[\mathfrak{m}_{\mathbb{T}}](\overline{\mathbb{Q}}) \to \mathcal{G}_{\mathbb{T}}(\overline{\mathbb{Q}}) \xrightarrow{x\mapsto(\alpha_i x)_i} \bigoplus_{i\in I} \mathcal{G}_{\mathbb{T}}(\overline{\mathbb{Q}}).$$

Taking the Pontryagin dual we have another exact sequence of Galois modules:

$$0 \leftarrow V/\mathfrak{m}_{\mathbb{T}}V \leftarrow V \xleftarrow{x\mapsto(\alpha_i x)_i} \prod_{i\in I} V.$$

Since Galois homology functor is right exact, the above exact sequence implies

$$H_0(k, V) \otimes_{\mathbb{T}} \mathbb{T}/\mathfrak{m}_{\mathbb{T}} = H_0(k, V/\mathfrak{m}_{\mathbb{T}}V) = 0.$$

Then by Nakayama's lemma, we get $H_0(k, V) = 0$, which implies $\mathcal{G}_{\mathbb{T}}(k) = 0$.
Let f be the residual degree of k as before. Consider the Hecke polynomial $H_{f,l}(X) = X^2 - A(l^f)X + l^f\langle l\rangle^f$, where $A(l^f)$ is determined by the following recurrence relation: $A(l) = a(l)$ and $A(l^m) = a(l^m) - l\langle l\rangle a(l^{m-1})$ for $m \geq 2$. If $l \nmid Np$, $\mathcal{G}_{\mathbb{T}}$ is unramified over k. By the Eichler–Shimura congruence relation (e.g. [GME, Theorem 4.2.1]), if $l \nmid Np$, for the l-Frobenius element $\phi \in \mathrm{Gal}(\overline{k}/k)$, the linear operator $H_{f,l}(\phi)$ annihilates $\mathcal{G}_{\mathbb{T}}$. Thus if $H_{f,l}(X)$ mod $\mathfrak{m}_{\mathbb{T}}$ is not divisible by $X-1$, $\mathcal{G}_{\mathbb{T}}[\mathfrak{m}_{\mathbb{T}}]$ as a representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_l/k)$ does not contain the identity representation.
For an arithmetic prime P, $H_{f,l}(X) \mod P$ does not have a factor $X-1$. Thus after the localization at P of the Pontryagin dual $(\mathcal{G}_{\mathbb{T}}(k)^{\vee})_P$ is killed by $H_{f,l}(\phi)$ and $\phi - 1$, and hence $\mathcal{G}_{\mathbb{T}}(k)^{\vee}$ is a torsion Λ-module.
Now assume that $l|N$. By the solution of the local Langlands conjecture (see [C86] and [AAG]), after replacing k by its finite extension, the Galois module $\mathcal{G}_{\mathbb{T}}[P]$ for an arithmetic point P becomes unramified unless ρ_P is Steinberg at l (i.e., is multiplicative type at l). Suppose that we have a non-Steinberg P. Then characteristic polynomial $H(X)$ of ϕ modulo P is prime to $X-1$ (as $H(X) \mod P$ has Weil numbers of weight f as its roots). Then by the same argument, we conclude the torsion property.
Suppose that all arithmetic point of $\mathrm{Spec}(\mathbb{T})$ is Steinberg at l (this often happens; see a remark below Conjecture 3.4 of [H11, §3]). Write ρ_P for the 2-dimensional Galois representation realized on $(\mathcal{G}_{\mathbb{T}}(\overline{\mathbb{Q}}_l)^{\vee}) \otimes_{\mathbb{T}} \kappa(P)$. Again by Langlands-Carayol, $\rho_P(I_l)$ for the inertia group $I_l \subset \mathrm{Gal}(\overline{\mathbb{Q}}_l/k)$ contains a non-trivial unipotent element. Thus ρ_P does not have a quotient on which I_l acts trivially. This shows again the Λ-torsion property. □

Let $\mathrm{Spec}(\mathbb{I}) \subset \mathrm{Spec}(\mathbb{T})$ be an irreducible component. Without assuming the Gorenstein condition, we have $(T\mathcal{G}_{\mathbb{I}})_P \cong \mathbb{I}_P^2$ for almost all height one primes $P \in \mathrm{Spec}(\Lambda)$; so, we have $\rho_{\mathbb{I}}$ with values in $GL_2(\mathbb{I}_P)$ for most of P. We call $\mathbb{I}$ a CM component if $\rho_{\mathbb{I}} \cong \mathrm{Ind}_M^{\mathbb{Q}} \Psi$ for a Galois character $\Psi : \mathrm{Gal}(\overline{\mathbb{Q}}/M) \to \mathbb{I}_P^\times$ (for an imaginary quadratic field M). If $\mathbb{I}$ is not a CM component, again for almost all P, by [Z14], $\rho_{\mathbb{T}}(I_p)$ contains an unipotent element conjugate to $\left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)$ with non-zero-divisor $u \in \mathbb{T}_P^\times$. In this case, we have $H_0(\mathrm{Gal}(\overline{k}/k), T\mathcal{G}_{\mathbb{I}})_P = 0$; so, $\mathcal{G}_{\mathbb{I}}(k)$ is a co-torsion Λ-module.

Lemma 4.3. *Let $k_{/\mathbb{Q}_p}$ in $\overline{\mathbb{Q}}_p$ be a finite extension with residual degree f and $\mathbb{T}$ be a reduced local ring of $\mathbf{h}$. Then the Pontryagin dual $\mathcal{G}_{\mathbb{T}}(k)^\vee$ of $\mathcal{G}_{\mathbb{T}}(k)$ is a torsion Λ–module of finite type.*

Proof. We may suppose either $a(p)^f \equiv 1 \mod \mathfrak{m}_{\mathbb{T}}$ or $a_k(\mathbb{T}) = 0$, as otherwise $\mathcal{G}_{\mathbb{T}}(k) = 0$ by Lemma 4.1. Replacing $\mathbb{T}$ by its irreducible component $\mathbb{I}$, we only need to prove torsion-ness for $\mathcal{G}_{\mathbb{I}}(k)^\vee$. Write V for the Λ-torsion free quotient of $T\mathcal{G}_{\mathbb{I}}$. Then for any $P \in \mathrm{Spec}(\Lambda)(\overline{\mathbb{Q}}_p)$, we have $V_P = (T\mathcal{G}_{\mathbb{I}})_P$ (as the reflexive closure in [BCM, Chapter 7] of $\mathbb{I}$ is Λ-free).

If $\mathbb{I}$ is not a CM component (i.e., $\rho_{\mathbb{I}}$ is not an induced representation from the Galois group over an imaginary quadratic field), the assertion follows from the same argument proving Lemma 4.1 replacing $\mathfrak{m}_{\mathbb{T}}$ by $P\mathbb{T}_P$ and $\mathbb{T}$ by $\mathbb{T}_P$. Indeed, taking an arithmetic point P of weight 2. Then by [Z14], we have $u \in \mathbb{T}_P^\times$. Then $H^0(k, V_P/PV_P)$ is a submodule of $H^0(I, V_P/PV_P)$ (for the inertia group I at p) killed by $a(p)^f - 1$. Since P is an arithmetic point of weight 1, we may choose P so that $a(p) \mod P$ is a Weil number of weight 1 (indeed, we only need to assume that the Neben character of f_P is non-trivial at p; see [MFM, Theorem 4.6.17]), and hence $a(p)^f \not\equiv 1 \mod P$. Thus $H^0(k, V_P/PV_P) = 0$. This implies $\mathcal{G}_{\mathbb{I}}(k)[P]$ is a finite module; so, $\mathcal{G}_{\mathbb{I}}(k)^\vee$ is a torsion Λ-module.

Now assume that $\mathbb{I}$ is a CM component with $\rho_{\mathbb{I}} = \mathrm{Ind}_M^{\mathbb{Q}} \Psi$. Define $\Psi^c(\sigma) = \Psi(c\sigma c^{-1})$ for a complex conjugation c. In the imaginary quadratic field M, p splits into a product of two primes $\mathfrak{p}\overline{\mathfrak{p}}$ as $\rho_{\mathbb{I}}$ is ordinary. For any arithmetic point $\mathfrak{P} \in \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$ $\Psi_{\mathfrak{P}} := \Psi \mod \mathfrak{P}$ ramifies at $\mathfrak{p}$ and its restriction to the inertia group at $\mathfrak{p}$ has infinite order, and Ψ^c is unramified at $\mathfrak{p}$ with infinite order $\Psi^c(Frob_p)$ (from an explicit description of Ψ; cf, [H13a, §3]). Then we have $V_{\mathfrak{P}} = V \otimes_{\mathbb{I}} \mathbb{I}_{\mathfrak{P}} \cong \mathbb{I}_{\mathfrak{P}}^2$. Thus replacing k by the composite $kM_{\mathfrak{p}}$, we have $V_{\mathfrak{P}} \cong \Psi \oplus \Psi^c$ over $\mathrm{Gal}(\overline{\mathbb{Q}}_p/k)$. Since Ψ^c is unramified at $\mathfrak{p}$ and $\Psi^c_{\mathfrak{P}}(Frob_p)$ has infinite order. This shows that $H^0(k, V_{\mathfrak{P}}/\mathfrak{P}V_{\mathfrak{P}}) = 0$, and again we find that $\mathcal{G}_{\mathbb{I}}^\vee(k)$ is a torsion $\mathbb{I}$-module and hence a torsion Λ-module. □

Corollary 4.4. *If k is a number field or a finite extension of $\mathbb{Q}_l$, the localization of $\mathcal{G}_{\mathbb{T}}(k)^\vee$ at an arithmetic prime of weight 2 vanishes.*

Proof. We only need to prove this for a finite extension k of $\mathbb{Q}_l$. Write W for the integer ring of k. Replacing k by its finite extension, we may assume that A_P has semi-stable reduction over W for an arithmetic prime at P. If A_P has good reduction and $l \neq p$, the l-Frobenius acts on T_pA_P by a Weil number of weight ≥ 1, and then $A_P[p^\infty](k)$ is finite; so, $\mathcal{G}_{\mathbb{T}}[P](k)$ is finite. If $l = p$, by [Z14], the inertia image in $\mathrm{Aut}(T_p A_P)$ contains a non-trivial unipotent element, and hence again $A_P[p^\infty](k)$ is finite, and the result follows. If A_P has multiplicative reduction, $A_P[p^\infty](k)$ is finite by a theorem of Tate–Mumford as the Tate period of A_P is non-trivial. This shows that $\mathcal{G}_{\mathbb{T}}[P](k)$ is finite, and hence the result follows. □

5. Abelian factors of modular Jacobians

Let $h_r(\mathbb{Z})$ be the subalgebra generated by $T(n)$ (including $U(l)$ for $l|Np$) of $\mathrm{End}(J_{r/\mathbb{Q}})$. Then $h_r(\mathbb{Z}_p) = h_r(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is canonically isomorphic to the $\mathbb{Z}_p$-subalgebra of $\mathrm{End}(J_r[p^\infty])$ generated by $T(n)$ (including $U(l)$ for $l|Np$). Then $\mathbf{h}_r = h_r(\mathbb{Z}_p)^{\mathrm{ord}}$ by the control theorems in [H86a] and [H86b].
As before, let k be a finite extension of $\mathbb{Q}$ inside $\overline{\mathbb{Q}}$ or a finite extension of $\mathbb{Q}_l$ inside $\overline{\mathbb{Q}}_l$. Let A_r be a abelian subvariety of J_r defined over k. Write A_s $(s > r)$ for the image of A_r in J_s under the morphism $\pi^* : J_r \to J_s$ given by Picard functoriality from the projection $\pi : X_s \to X_r$. If A_r is Shimura's abelian subvariety attached to a Hecke eigenform f, we sometimes write $A_{f,s}$ for A_s to indicate this fact. Hereafter we assume

- (A) We have a coherent sequence $\alpha_s \in \mathrm{End}(J_{s/\mathbb{Q}})$ (for all $s \geq r$) having the limit $\alpha = \varprojlim_s \alpha_s \in \mathrm{End}(J_{\infty/\mathbb{Q}})$ such that
  - (a) A_s is the connected component of $J_s[\alpha_s]$ with $J_s = A_s + \alpha_s(J_s)$ so that the inclusion: $A_s[p^\infty] \cong J_s[\alpha_s][p^\infty]$ is a $U(p)$-isomorphism,
  - (b) the restriction $\alpha_s|_{\alpha_s(J_s)} \in \mathrm{End}(\alpha_s(J_s))$ is a self-isogeny.

Here for $s' > s$, coherency of α_s means the following commutative diagram:

$$\begin{array}{ccc} J_s & \xrightarrow{\pi^*} & J_{s'} \\ {\scriptstyle \alpha_s}\downarrow & & \downarrow{\scriptstyle \alpha_{s'}} \\ J_s & \xrightarrow[\pi^*]{} & J_{s'} \end{array}$$

The Rosati involution $h \mapsto h^*$ and $T(n) \mapsto T^*(n)$ (with respect to the canonical divisor on J_r) brings $h_r(\mathbb{Z})$ to $h_r^*(\mathbb{Z}) \subset \mathrm{End}(J_{r/\mathbb{Q}})$. Define A_s^* to be the identity connected component of $J_s[\alpha^*]$. The condition (A) is equivalent to

- (B) The abelian quotient map $J_s \twoheadrightarrow B_s = \mathrm{Coker}(\alpha_s)$ dual to $A_s^* \subset J_s$ induces an $U(p)$-isomorphism of Tate modules: $T_p(J_s/\alpha_s(J_s)) \to T_p B_s$ and α_s induces an automorphism of the $\mathbb{Q}_p$-vector space $T_p\alpha_s(J_s) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Again if A_r is Shimura's abelian subvariety of J_r associated to a Hecke eigenform f, we sometimes write $B_{f,s}$ for B_s as above. The condition (A) (and hence (B)) is a mild condition. Here are sufficient conditions for (α, A_s, B_s) to satisfy (A) (and (B)):

Proposition 5.1. *Let* $\mathrm{Spec}(\mathbb{T})$ *be a connected component of* $\mathrm{Spec}(\mathbf{h})$ *and* $\mathrm{Spec}(\mathbb{I})$ *be an irreducible component of* $\mathrm{Spec}(\mathbb{T})$. *Then the condition* (A) *holds for the following choices of* (α, A_s, B_s):

(P1) *Fix $r > 0$. Then $\alpha_s = \alpha$ for a factor $\alpha|\gamma^{p^{r-1}} - 1$ in Λ, $A_s = J_s[\alpha]^\circ$ (the identity connected component) and $B_s = \mathrm{Pic}^0_{A_s/\mathbb{Q}}$ for all $s \geq r$.*

(P2) *Suppose that an eigen cusp form $f = f_P$ new at each prime $l|N$ belongs to $\mathrm{Spec}(\mathbb{T})$ and that $\mathbb{T} = \mathbb{I}$ is regular (or more generally a unique factorization domain). Then writing the level of f_P as Np^r, the algebra homomorphism $\lambda : \mathbb{T} \to \overline{\mathbb{Q}}_p$ given by $f|T(l) = \lambda(T(l))f$ gives rise to the prime ideal $P = \mathrm{Ker}(\lambda)$. Since P is of height 1, it is principal generated by $\varpi \in \mathbb{T}$. This ϖ has its image $\varpi_s \in \mathbb{T}_s = \mathbb{T} \otimes_\Lambda \Lambda_s$ for $\Lambda_s = \Lambda/(\gamma^{p^{s-1}} - 1)$. Since $\mathbf{h}_s = \mathbf{h} \otimes_\Lambda \Lambda_s = \mathbb{T}_s \oplus X_s$ as an algebra direct sum, $\mathrm{End}(J_{s/\mathbb{Q}}) \otimes_\mathbb{Z} \mathbb{Z}_p \supset h_s(\mathbb{Z}_p) = \mathbb{T}_s \oplus Y_s$ with Y_s projecting down onto X_s. Then, we can approximate $a_s = \varpi_s \oplus 1_s \in h_s(\mathbb{Z}_p)$ for the identity 1_s of Y_s by $\alpha_s \in h_s(\mathbb{Z})$ so that $\alpha_s h_s(\mathbb{Z}_p) = a_s h_s(\mathbb{Z}_p)$ (hereafter we call α_s "sufficiently close" to a_s if $\alpha_s h_s(\mathbb{Z}_p) = a_s h_s(\mathbb{Z}_p)$). For this choice of α_s, $A_s := A_{f,s}$ and $B_s := B_{f,s}$.*

(P3) *More generally than (P2), we pick a general connected component $\mathrm{Spec}(\mathbb{T})$ of $\mathrm{Spec}(\mathbf{h})$. Pick a (classical) Hecke eigenform $f = f_P$ (of weight 2) for $P \in \mathrm{Spec}(\mathbb{T})$. Assume that $\mathbf{h}_s$ (for every $s \geq r$) is reduced and $P = (\varpi)$ for $\varpi \in \mathbb{T}$, and write ϖ_s for the image of ϖ in $h_s(\mathbb{Z}_p)$. Take the complementary direct summand Y_s of $\mathbb{T}_s$ in $h_s(\mathbb{Z}_p)$ and approximate $a_s := \varpi_s \oplus 1_s$ in $h_s(\mathbb{Z}_p)$ to get α_s sufficiently close to a_s. Then for this choice of α_s, $A_s := A_{f,s}$ and $B_s := B_{f,s}$.*

(P4) *Suppose that $\mathbb{T}/(\varpi)$ for a non-zero divisor $\varpi \in \mathbb{T}$ is a reduced algebra of characteristic 0 factoring through $\mathbf{h}_r := \mathbf{h}/(\gamma^{p^{r-1}} - 1)\mathbf{h}$ for some $r > 0$. Assume that $\mathbb{T}_s$ is reduced for every $s \geq r$, and write ϖ_s for the image of ϖ in $\mathbb{T}_s$. Then approximating $a_s = \varpi_s \oplus 1_s$ by $\alpha_s \in h_s(\mathbb{Z})$ sufficiently closely for each $s \geq r$, we define A_s to be the connected component of $J_s[\alpha_s]$ and B_s to be its dual quotient.*

Proof. We first prove (P4). Since α_s is sufficiently close to a_s, we have the identity $\alpha_s h_s(\mathbb{Z}_p) = a_s h_s(\mathbb{Z}_p)$ of ideals. By reducedness of $\mathbb{T}_s$, we have an algebra product decomposition: $h_s(\mathbb{Q}_p) := h_s(\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \alpha_s(\mathbb{T}_s) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \times Z_s$ for the complementary $\mathbb{Q}_p$-subalgebra Z_s, which is given by $(\mathbb{T}_s/(\varpi_s)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Write the idempotent of Z_s as $\epsilon_s \in Z_s$. Then $\epsilon_s + a_s$ is invertible in $h_s(\mathbb{Q}_p)$. For some positive integer M_s, $\beta_s := U(p)^{M_s} \epsilon_s \in h_s(\mathbb{Z}) \subset \mathrm{End}(J_s)$. Then by $\epsilon_s + a_s \in h_s(\mathbb{Q}_p)^\times$, the connected component A_s of $J_s[\alpha_s]$ is given by $\beta_s(J_s)$, $J_s = \beta_s(J_s) + \alpha_s(J_s) = A_s + \alpha(J_s)$, and the inclusion map $A_s \hookrightarrow J_s[\alpha_s]$ is an $U(p)$-isomorphism. Since α_s is invertible in $\alpha_s(h_s(\mathbb{Q}_p))$, α_s induces a self-isogeny of $\alpha(J_s)$. Thus the triple satisfies (A). Since $\varpi_{s'} h_{s'}(\mathbb{Z}_p)$ surjects down to $\varpi_s h_s(\mathbb{Z}_p)$ for all $s' \geq s$, we can adjust α_s inductively to have a projective system $\{\alpha_s \in \mathrm{End}(J_s)\}_{s \geq r}$. Thus $\alpha = \varprojlim_s \alpha_s \in \mathrm{End}(J_\infty)$ does the job. This proves (P4). The assertions (P2) and (P3) are direct consequences of (P4).
As for (P1), since $\alpha|(\gamma^{p^{r-1}} - 1)|(\gamma^{p^{s-1}} - 1)$ in the unique factorization domain Λ, factoring $\gamma^{p^s} - 1 = \alpha_s \beta_s$, the ideals (α_s) and (β_s) are co-prime in the unique-factorization domain Λ. From this, we have $J_s = \beta_s(J_s) + \alpha(J_s) =$

$A_s + \alpha(J_s)$, and $\alpha|_{\alpha(J_s)}$ is a self isogeny of $\alpha(J_s)$ as $\alpha|_{\alpha(J_s)}$ is a non-zero-divisor in $\mathrm{End}(\alpha(J_s))$. □

Remark 5.2. (i) Under (P2), all arithmetic points P of weight 2 in $\mathrm{Spec}(\mathbb{I})$ satisfies (A).
(ii) For a given weight 2 Hecke eigenform f, for density 1 primes $\mathfrak{p}$ of $\mathbb{Q}(f)$, f is ordinary at $\mathfrak{p}$ (i.e., $a(p,f) \not\equiv 0 \mod \mathfrak{p}$; see [H13b, §7]). Except for finitely many primes $\mathfrak{p}$ as above, f belongs to a connected component $\mathbb{T}$ which is regular (see [F02, §3.1]); so, (P2) is satisfied for such $\mathbb{T}$.
(iii) If N is square-free (as assumed for simplicity in the introduction), $\mathbf{h}_s$ is reduced [H13a, Corollary 1.3]; so, if an arithmetic prime $P \in \mathrm{Spec}(\mathbf{h}_r)$ is principal, α_s as in (P3) satisfies (A).

If $A_r = A_{f,r}$ is Shimura's abelian subvariety associated to a primitive form f as in [IAT, Theorem 7.14], its dual quotient $J_r \twoheadrightarrow B_r = B_{f,r}$ is also associated to f in the sense of [Sh73]. However, if A_r is not associated to a new form, the dual quotient may not be associated to the Hecke eigen form f. To clarify this point, we introduce an involution of J_s. We fix a generator ζ of the $\mathbb{Z}_p$-module $\mathbb{Z}_p(1) = \varprojlim_n \mu_{p^n}(\overline{\mathbb{Q}})$; so, ζ is a coherent sequence of generators ζ_{p^n} of $\mu_{p^n}(\overline{\mathbb{Q}})$ (i.e., $\zeta_{p^{n+1}}^p = \zeta_{p^n}$ for all $n > 0$). We also fix a generator ζ_N of $\mu_N(\overline{\mathbb{Q}})$, and put $\zeta_{Np^r} := \zeta_N\zeta_{p^r}$. Identify the étale group scheme $\mathbb{Z}/Np^n\mathbb{Z}_{/\mathbb{Q}[\zeta_N,\zeta_{p^n}]}$ with μ_{Np^n} by sending $m \in \mathbb{Z}$ to $\zeta_{Np^n}^m$. Then for a couple $(E, \phi_{Np^r} : \mu_{Np^r} \hookrightarrow E)_{/K}$ for a $\mathbb{Q}[\mu_{Np^r}]$-algebra K, let $\phi^* : E[Np^r] \twoheadrightarrow \mathbb{Z}/Np^r\mathbb{Z}$ be the Cartier dual of ϕ_{Np^r}. Then ϕ^* induces $E[Np^r]/\mathrm{Im}(\phi_{Np^r}) \cong \mathbb{Z}/Np^r\mathbb{Z}$. Define $i : \mathbb{Z}/p^r\mathbb{Z} \cong (E/\mathrm{Im}(\phi_{Np^r}))[Np^r]$ by the inverse of ϕ^*. Then we define $\varphi_{Np^r} : \mu_{Np^r} \hookrightarrow E/\mathrm{Im}(\phi_{Np^r})$ by $\varphi_{Np^r} : \mu_{Np^r} \cong \mathbb{Z}/Np^r\mathbb{Z} \xrightarrow{i} (E/\mathrm{Im}(\phi_{Np^r}])[p^r] \subset E/\mathrm{Im}(\phi_{Np^r})$. This induces an involution w_r of X_r defined over $\mathbb{Q}[\mu_{Np^r}]$, which in turn induces an automorphism w_r of $J_{r/\mathbb{Q}[\zeta_{Np^r}]}$.
Let $P \in \mathrm{Spec}(\mathbf{h})(\overline{\mathbb{Q}}_p)$ be an arithmetic point of weight 2. Then we have a p-stabilized Hecke eigenform form f_P associated to P; i.e., $f_P|T(n) = P(T(n))f_P$ for all n. Suppose $f = f_P$ and write $A_{f,r} = A_P$. Then $f_P^* = w_r(f_P)$ is the dual common eigenform of $T^*(n)$. If f_P is new at every prime $l|Np$, f_P^* is a constant multiple of the complex conjugate f_P^c of f_P (but otherwise, it could be different). Then the abelian quotient associated to f_P^* is the dual abelian variety of A_P. Thus if f_P^* is not constant multiple of f_P^c, $B_{f,r}$ is not assocaited to f_P^* (see a remark at the end of [H14b, §6] for more details of this fact).
Pick an automorphism $\sigma \in \mathrm{Gal}(\mathbb{Q}(\mu_{Np^r})/\mathbb{Q})$ with $\zeta_{Np^r}^\sigma = \zeta_{Np^r}^z$ for $z \in (\mathbb{Z}/Np^r\mathbb{Z})^\times$. Since w_r^σ is defined with respect to $\zeta_{Np^r}^\sigma = \zeta_{Np^r}^z$, we find $w_r^\sigma = \langle z\rangle \circ w_r$. By this formula, if $x \in A_P(\overline{\mathbb{Q}})$ and $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with $\zeta^\sigma = \zeta^z$ for $z \in \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times = \varprojlim_s (\mathbb{Z}/Np^s\mathbb{Z})^\times$, we have $w_r(x)^\sigma = \langle z\rangle(w_r(x^\sigma))$. Thus $w_r^\sigma = \langle z\rangle \circ w_r = w_r \circ \langle z^{-1}\rangle$ (see [MW86, page 237] and [MW84, 2.5.6]).
Let $\pi_{s,r,*} : J_s \to J_r$ for $s > r$ be the morphism induced by the covering map $X_s \twoheadrightarrow X_r$ through Albanese functoriality. Then we define $\pi_s^r = w_r \circ \pi_{s,r,*} \circ w_s$. Then $(\pi_s^r)^\sigma = w_r\langle z^{-1}\rangle\pi_{s,r,*}\langle z\rangle w_s = \pi_s^r$ for all $\sigma \in \mathrm{Gal}(\mathbb{Q}(\mu_{Np^s})/\mathbb{Q})$; thus, π_s^r is well defined over $\mathbb{Q}$, and satisfies $T(n) \circ \pi_s^r = \pi_s^r \circ T(n)$ for all n prime to

Np and $U(q) \circ \pi^r_s = \pi^r_s \circ U(q)$ for all $q|Np$ (as $w_? \circ h \circ w_? = h^*$ for $h \in h_?(\mathbb{Z})$ $(? = s, r)$ by [MFM, Section 4.6]. Since $w_r^2 = 1$, $\{J_s, \pi^r_s\}_{s>r}$ form a Hecke equivariant projective system of abelian varieties defined over $\mathbb{Q}$. We then define as described in (S) just above Lemma 2.1 an fppf abelian sheaf $\widehat{X}$ for any abelian variety quotient or subvariety X of $J_{s/k}$ over the fppf site over $k = \mathbb{Q}$ and $\mathbb{Q}_l$ (note here the definition of $\widehat{X}$ depends on k).
In general, for A_s in (A), we have $A^*_s = w_s(A_s) \subset J_s$ because $T(n) \circ w_s = w_s \circ T^*(n)$ for all n (see [MFM, Theorem 4.5.5]). Thus (B_s, π^r_s) in (B) gives rise to a natural projective system of abelian variety quotients of J_s.

6. Structure of ind-Λ-MW groups over number fields and local field

We return to the setting of Section 2; so, K/k is the infinite Galois extension defined there. In this section, unless otherwise mentioned, we often let κ denote an intermediate finite extension of k inside K (although the results in this section are valid for κ satisfying (2.2) unless otherwise mentioned).
We assume (A) in Section 5 for (α_s, A_s, B_s). By (A), the inclusion $A_s[p^\infty] \hookrightarrow J_s[\alpha_s][p^\infty]$ is a $U(p)$-isomorphism; so, we have the identity of the ordinary parts: $\widehat{A}^{\mathrm{ord}}_s = \widehat{J}^{\mathrm{ord}}_s[\alpha_s]$. From the exact sequence

$$0 \to J_s[\alpha_s] \to J_s \xrightarrow{\alpha_s} J_s \to B_s \to 0,$$

we get the following exact sequence of sheaves:

$$0 \to \widehat{A}^{\mathrm{ord}}_s \to \widehat{J}^{\mathrm{ord}}_s \xrightarrow{\alpha_s} \widehat{J}^{\mathrm{ord}}_s \to \widehat{B}^{\mathrm{ord}}_s \to 0. \tag{6.1}$$

This is because tensoring $\mathbb{Z}_p$ (or taking the p-primary part $X/X^{(p)}$ as in (S)) is an exact functor. Since taking injective limit is an exact functor, writing $X^{\mathrm{ord}}_\infty = \varinjlim_s \widehat{X}^{\mathrm{ord}}_s$, we get the following exact sequence of sheaves:

$$0 \to A^{\mathrm{ord}}_\infty \to J^{\mathrm{ord}}_\infty \xrightarrow{\alpha} J^{\mathrm{ord}}_\infty \to B^{\mathrm{ord}}_\infty \to 0. \tag{6.2}$$

First, we shall describe A^{ord}_∞ and B^{ord}_∞ in terms of $\widehat{A}_r$ and $\widehat{B}_r$. The Picard functoriality induces a morphism $\pi^*_{r,s} : J_r \to J_s$. This gives a Hecke equivariant inductive system $\{J_s, \pi^*_{r,s}\}_{s>r}$ of abelian varieties defined over $\mathbb{Q}$. Since the two morphisms $J_r \to J^r_s$ and $J^r_s \to J_s[\gamma^{p^{r-1}} - 1]$ (Picard functoriality) are $U(p)$-isomorphisms of fppf abelian sheaves by (u1) and Corollary 3.5 (see also Remark 3.7), we get the following two isomorphisms of fppf abelian sheaves:

$$A_r[p^\infty]^{\mathrm{ord}} \xrightarrow[\pi^*_{r,s}]{\sim} A_s[p^\infty]^{\mathrm{ord}} \text{ and } \widehat{A}^{\mathrm{ord}}_r \xrightarrow[\pi^*_{r,s}]{\sim} \widehat{A}^{\mathrm{ord}}_s, \tag{6.3}$$

since $\widehat{A}^{\mathrm{ord}}_s$ is the isomorphic image of $\widehat{A}^{\mathrm{ord}}_r \subset \widehat{J}_r$ in $\widehat{J}_s[\gamma^{p^{r-1}} - 1]$. Since $w_r \circ T(n) = T^*(n) \circ w_r$ (by [MFM, Theorem 4.5.5]), twisting Cartier duality pairing $[\cdot, \cdot] : J_r[p^r] \times J_r[p^r] \to \mu_{p^r}$ coming from the canonical polarization, we get a perfect pairing $(\cdot, \cdot) : J_r[p^r] \times J_r[p^r] \to \mu_{p^r}$ with $(x|T(n), y) = (x, y|T(n))$

(e.g., [H14a, Section 4]). By this w-twisted Cartier duality applied to the first identity of (6.3), we have

$$B_s[p^\infty]^{\mathrm{ord}} \xrightarrow[\pi_s^r]{\sim} B_r[p^\infty]^{\mathrm{ord}}. \tag{6.4}$$

Thus, by Kummer sequence, we have the following commutative diagram

$$\begin{array}{ccc} \widehat{B}_s^{\mathrm{ord}}(\kappa)\otimes\mathbb{Z}/p^m\mathbb{Z} = (B_s(\kappa)\otimes\mathbb{Z}/p^m\mathbb{Z})^{\mathrm{ord}} & \hookrightarrow & H^1(B_s[p^m]^{\mathrm{ord}}) \\ \pi_s^r\downarrow & & \wr\downarrow(6.4) \\ \widehat{B}_r^{\mathrm{ord}}(\kappa)\otimes\mathbb{Z}/p^m\mathbb{Z} = (B_r(\kappa)\otimes\mathbb{Z}/p^m\mathbb{Z})^{\mathrm{ord}} & \hookrightarrow & H^1(B_r[p^m]^{\mathrm{ord}}) \end{array}$$

This shows

$$\widehat{B}_s^{\mathrm{ord}}(\kappa)\otimes\mathbb{Z}/p^m\mathbb{Z} \cong \widehat{B}_r^{\mathrm{ord}}(\kappa)\otimes\mathbb{Z}/p^m\mathbb{Z}.$$

Passing to the limit, we get

$$\widehat{B}_s^{\mathrm{ord}} \xrightarrow[\pi_s^r]{\sim} \widehat{B}_r^{\mathrm{ord}} \text{ and } (B_s\otimes_{\mathbb{Z}}\mathbb{T}_p)^{\mathrm{ord}} \xrightarrow[\pi_s^r]{\sim} (B_r\otimes_{\mathbb{Z}}\mathbb{T}_p)^{\mathrm{ord}} \tag{6.5}$$

as fppf abelian sheaves. As long as κ is either a field extension of finite type of a number field or a finite extension of $\mathbb{Q}_l$ ($l \neq p$) or a finite algebraic extension of $\mathbb{Q}_p$, the projective limit of $B_?(\kappa)\otimes\mathbb{Z}/p^m\mathbb{Z}$ (with respect to m) is equal to $\widehat{B}_?$ (by Lemma 2.1). In short, we get

Lemma 6.1. *Assume κ to be given either by a field extension of finite type of k if k is a finite extension of $\mathbb{Q}$ or $\mathbb{Q}_l$ ($l \neq p$) or by a finite algebraic extension of k if $[k:\mathbb{Q}_p]<\infty$. Then we have the following isomorphism*

$$\widehat{A}_r(\kappa)^{\mathrm{ord}} \xrightarrow[\pi_{s,r}^*]{\sim} \widehat{A}_s(\kappa)^{\mathrm{ord}} \quad \text{and} \quad \widehat{B}_s(\kappa)^{\mathrm{ord}} \xrightarrow[\pi_s^r]{\sim} \widehat{B}_r(\kappa)^{\mathrm{ord}}$$

for all $s>r$ including $s=\infty$.

By computation, we get $\pi_s^r\circ\pi_{r,s}^* = p^{s-r}U(p^{s-r})$. To see this, as Hecke operators, $\pi_{r,s}^* = [\Gamma_s^r]$, $\pi_{r,s,*} = [\Gamma_r]$. Thus we have

$$\begin{aligned} \pi_s^r\circ\pi_{r,s}^* &= [\Gamma_s^r]\cdot w_s\cdot[\Gamma_r]\cdot w_r = [\Gamma_s]\cdot[w_sw_r]\cdot[\Gamma_r] \\ &= [\Gamma_s^r:\Gamma_s][\Gamma_s^r\left(\begin{smallmatrix}1 & 0\\ 0 & p^{s-r}\end{smallmatrix}\right)\Gamma_r] = p^{s-r}U(p^{s-r}). \end{aligned} \tag{6.6}$$

Then we have the commutative diagram of fppf abelian sheaves for $s'>s$

$$\begin{array}{ccc} \widehat{A}_{s'}^{\mathrm{ord}} & \xleftarrow[\pi_{r,s'}^*]{\sim} & \widehat{A}_r^{\mathrm{ord}} \\ \pi_{s'}^s\downarrow & & \downarrow p^{s'-s}U(p)^{s'-s} \\ \widehat{A}_s^{\mathrm{ord}} & \xleftarrow[\pi_{r,s}^*]{\sim} & \widehat{A}_r^{\mathrm{ord}}. \end{array} \tag{6.7}$$

Note that A_s and B_s are mutually (w-twisted) dual as abelian varieties (see Section 5), and the w-twisted duality is compatible with Hecke operators. Thus $B_s[p^n]$ is the w-twisted Cartier dual of $A_s[p^n]$. The w-twisted Cartier duality pairing in [H14a, Section 4] satisfies $(x|X,y) = (x,y|X)$ for $X=T(n),U(q)$, and π_r^s and $\pi_{r,s}^*$ are adjoint each other under this duality. Then we have the dual commutative diagram of fppf abelian sheaves:

$$\begin{array}{ccc} \widehat{B}_{s'}^{\mathrm{ord}} & \xrightarrow[\pi_{s'}^r]{\sim} & \widehat{B}_r^{\mathrm{ord}} \\ {\scriptstyle \pi_{s,s'}^*}\uparrow & & \uparrow{\scriptstyle p^{s'-s}U(p)^{s'-s}} \\ \widehat{B}_s^{\mathrm{ord}} & \xrightarrow[\pi_s^r]{\sim} & \widehat{B}_r^{\mathrm{ord}}. \end{array} \tag{6.8}$$

By (6.7) and (6.8), we have the following four exact sequences of fppf abelian sheaves:

$$\begin{aligned} 0 \to & A_s[p^{s-r}]^{\mathrm{ord}} \to A_s[p^\infty]^{\mathrm{ord}} \xrightarrow{\pi_s^r} A_r[p^\infty]^{\mathrm{ord}} \to 0, \\ 0 \to & B_r[p^{s-r}]^{\mathrm{ord}} \to B_r[p^\infty]^{\mathrm{ord}} \xrightarrow{\pi_{r,s}^*} B_s[p^\infty]^{\mathrm{ord}} \to 0 \end{aligned} \tag{6.9}$$

and

$$\begin{aligned} 0 \to & A_s[p^{s-r}]^{\mathrm{ord}} \to \widehat{A}_s^{\mathrm{ord}} \xrightarrow{\pi_s^r} \widehat{A}_r^{\mathrm{ord}} \to 0, \\ 0 \to & B_r[p^{s-r}]^{\mathrm{ord}} \to \widehat{B}_r^{\mathrm{ord}} \xrightarrow{\pi_{r,s}^*} \widehat{B}_s^{\mathrm{ord}} \to 0. \end{aligned} \tag{6.10}$$

LEMMA 6.2. *Let the notation and assumtions be as in* Lemma 6.1. *Then we have a canonical isomorphism*

$$\varinjlim_{s,\pi_{r,s}^*} \widehat{B}_s^{\mathrm{ord}}(\kappa) \cong \varinjlim_{s,p^{s-r}U(p)^{s-r}} \widehat{B}_r^{\mathrm{ord}}(\kappa) \cong \widehat{B}_r(\kappa)^{\mathrm{ord}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Proof. Identifying the left and the right column of (6.8), we have the cohomology exact sequence of the second exact sequence of (6.10):

$$0 \to B_r[p^{s-r}]^{\mathrm{ord}}(\kappa) \xrightarrow{\pi_r^s} \widehat{B}_r^{\mathrm{ord}}(\kappa) \xrightarrow{\pi_{r,s}^*} \widehat{B}_s^{\mathrm{ord}}(\kappa) \to H_?^1(B_r[p^{s-r}]^{\mathrm{ord}}). \tag{6.11}$$

Passing to the inductive limit of $\{B_r[p^{s-r}]^{\mathrm{ord}}, p^{s-r}U(p)^{s-r}\}_s$, $\{\widehat{B}_r(\kappa)^{\mathrm{ord}}, p^{s-r}U(p)^{s-r}\}_s$ and $\{\widehat{B}_s, \pi_{r,s}^*\}_s$, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccc} \varinjlim_s \widehat{B}_r^{\mathrm{ord}}(\kappa) & \to & \varinjlim_s \widehat{B}_s^{\mathrm{ord}}(\kappa) & \to & \varinjlim_s H_?^1(B_r[p^{s-r}]^{\mathrm{ord}}) \\ \| & & \| & & \wr\downarrow \\ \varinjlim_s \widehat{B}_r^{\mathrm{ord}}(\kappa) & \to & \varinjlim_s \widehat{B}_s^{\mathrm{ord}}(\kappa) & \to & H_?^1(\varinjlim_s B_r[p^{s-r}]^{\mathrm{ord}}). \end{array} \tag{6.12}$$

Here the last isomorphism comes from the commutativity of injective limit and cohomology.

For a free $\mathbb{Z}_p$-module F of finite rank, we suppose to have a commutative diagram:

$$\begin{array}{ccc} F & \xrightarrow{p^n} & F \\ \|\downarrow & & \downarrow{\scriptstyle p^{-n}} \\ F & \xrightarrow[\hookrightarrow]{} & p^{-n}F. \end{array}$$

Thus we have $\varinjlim_{n,x\mapsto p^n x} F = \varinjlim_{n,x\mapsto p^{-n}x} p^{-n}F \cong F\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$. If T is a torsion $\mathbb{Z}_p$-module with $p^B T = 0$ for $B \gg 0$, we have $\varinjlim_{n,x\mapsto p^n x} T = 0$. Thus for general $M = F\oplus T$, we have $\varinjlim_{n,x\mapsto p^n x} M \cong M\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$. Applying this consideration to $M = \widehat{B}_r(\kappa)$, we get

$$\varinjlim_{s,x\mapsto p^s U(p)^s x} \widehat{B}_r(\kappa) \cong \widehat{B}_r(\kappa)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p.$$

Similarly, $\varinjlim_{n,x\mapsto p^n U(p)^n x} B_r[p^n](\kappa) = \varinjlim_{n,x\mapsto p^n U(p)^n x} B_r[p^n](K) = 0$. Thus from the above diagram (6.12), we conclude the lemma. □

Consider the composite morphism $\varpi_s : A_s \hookrightarrow J_s \twoheadrightarrow B_s$ of fppf abelian sheaves. Since $B_s = J_s/\alpha_s(J_s)$ and $J_s = A_s + \alpha_s(J_s)$ with finite intersection $J_s = A_s \times_{J_s} \alpha_s(J_s)$, we have a commutative diagram with exact rows in the category of fppf abelian sheaves:

$$\begin{array}{ccccc} \alpha(J_s) & \xrightarrow{\hookrightarrow} & J_s & \underset{\rho_s}{\xrightarrow{\twoheadrightarrow}} & B_s \\ \cup\uparrow & & \cup\uparrow & & \uparrow\| \\ 0 \to \alpha(J_s)\times_{J_s} A_s & \underset{\hookrightarrow}{\longrightarrow} & A_s & \underset{\twoheadrightarrow}{\xrightarrow{\varpi_s}} & B_s. \end{array} \tag{6.13}$$

We have this diagram over $R_s := \mathbb{Z}_{(p)}[\mu_{p^s}]$ (not just over $\mathbb{Q}$) by taking the connected components of the Néron models of J_s, A_s and B_s. The intersection $\alpha(J_s)\times_{J_s} A_s = \mathrm{Ker}(\varpi_s)$ is an étale finite group scheme over $\mathbb{Q}$. These abelian varieties are known to have semi-stable reduction over R_s by the good reduction theorem of Carayol–Langlands. If the character $\mathbb{Z}_p^\times \ni z \mapsto \langle z\rangle \in \mathrm{End}(A_s)^\times$ is non-trivial, we may replace J_s by its complement $J_s^{(0)}$ of the image of J_s^0 in J_s. Under this circumstance, $\alpha(J_s)\times_{J_s} A_s = \mathrm{Ker}(\varpi_s)$ is a finite flat group scheme over R_s. Since A_s and B_s has good reduction over R_r, $\mathrm{Ker}(\varpi_s)$ is a finite flat group scheme defined over R_r. We consider the exact sequence

$$0 \to \mathrm{Ker}(\varpi_s) \to A_s \xrightarrow{\varpi_s} B_s \to 0.$$

which is an exact sequence of fppf abelian sheaves over R_r (and smooth abelian sheaves over $\mathbb{Q}$ or $\mathbb{Z}[\frac{1}{p}]$). From this, writing C_s for the p-primary part of $\mathrm{Ker}(\varpi_s)$, we have an exact sequence of fppf abelian sheaves over R_r (and smooth abelian sheaves over $\mathbb{Q}$ or $\mathbb{Z}[\frac{1}{p}]$):

$$0 \to C_s \to \widehat{A}_s \to \widehat{B}_s \to 0.$$

We have the following commutative diagram with exact rows:

$$\begin{array}{ccccc}
A_s[p^{s-r}]^{\mathrm{ord}} & \xrightarrow{\sim} & A_s[p^{s-r}]^{\mathrm{ord}} & \underset{\twoheadrightarrow}{\longrightarrow} & 0 \\
\downarrow & & \downarrow & & \downarrow \\
C_s^{\mathrm{ord}} & \underset{\hookrightarrow}{\longrightarrow} & \widehat{A}_s^{\mathrm{ord}} & \underset{\twoheadrightarrow}{\longrightarrow} & \widehat{B}_s^{\mathrm{ord}} \\
\downarrow & & \downarrow & & \wr\downarrow \\
C_r^{\mathrm{ord}} & \underset{\hookrightarrow}{\longrightarrow} & \widehat{A}_r^{\mathrm{ord}} & \underset{\twoheadrightarrow}{\longrightarrow} & \widehat{B}_r^{\mathrm{ord}}.
\end{array}$$

By the snake lemma applied to the right two exact columns of the above diagram, we get the following exact sequence:

$$0 \to A_r[p^{s-r}]^{\mathrm{ord}} \to C_s^{\mathrm{ord}} \to C_r^{\mathrm{ord}} \to 0 \tag{6.14}$$

with $C_s^{\mathrm{ord}} \hookrightarrow A_s[p^\infty]^{\mathrm{ord}} \underset{\sim}{\xleftarrow{\pi_{r,s}^*}} A_r[p^\infty]^{\mathrm{ord}}$.

Proposition 6.3. *We have the following exact sequence under the ?-topology over k, where ? = sm, étale, nothing and fppf:*

$$0 \to \widehat{A}_r^{\mathrm{ord}} \to J_\infty^{\mathrm{ord}} \xrightarrow{\alpha} J_\infty^{\mathrm{ord}} \xrightarrow{\rho_\infty} \widehat{B}_r^{\mathrm{ord}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to 0 \tag{6.15}$$

with $\widehat{A}_r^{\mathrm{ord}}/\widehat{A}_r^{\mathrm{ord}}[p^\infty] \cong \widehat{B}_r^{\mathrm{ord}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Proof. By (6.13), C_s^{ord} is equal to $\widehat{A}_s^{\mathrm{ord}} \cap \widehat{\alpha}(\widehat{J}_s^{\mathrm{ord}})$. Since A_s is the connected component of $J_s[\alpha]$ with $U(p)$-isomorphism $A_s \hookrightarrow J_s[\alpha_s]$, we have $C_s^{\mathrm{ord}} = \alpha(\widehat{J}_s^{\mathrm{ord}})[\alpha]$. Since α is an isogeny on $\alpha(J_s)$, we have an exact sequence of sheaves indexed by s under ?-topology

$$0 \to C_s^{\mathrm{ord}} \to \alpha(\widehat{J}_s^{\mathrm{ord}}) \xrightarrow{\alpha_s} \alpha(\widehat{J}_s^{\mathrm{ord}}) \to 0.$$

Passing to the inductive limit of these exact sequences (and noting $\varinjlim_s C_s^{\mathrm{ord}} = A_r[p^\infty]^{\mathrm{ord}}$ by (6.14)), we get another exact sequences:

$$0 \to \widehat{A}_r^{\mathrm{ord}}[p^\infty] \to \alpha(J_\infty^{\mathrm{ord}}) \xrightarrow{\alpha} \alpha(J_\infty^{\mathrm{ord}}) \to 0.$$

Therefore by (6.14), we get the following exact sequences (indexed by s) of sheaves under ?-topology:

$$0 \to C_s^{\mathrm{ord}} \to (\widehat{A}_s^{\mathrm{ord}} \times \alpha(\widehat{J}_s^{\mathrm{ord}})) \to \widehat{J}_s^{\mathrm{ord}} \to 0. \tag{6.16}$$

Passing again to the inductive limit of these exact sequences (and noting $\widehat{A}_r^{\mathrm{ord}} \cong \widehat{A}_s^{\mathrm{ord}}$ by $\pi_{r,s}^*$ and $\varinjlim_s C_s^{\mathrm{ord}} = A_r[p^\infty]^{\mathrm{ord}}$), we get the top and the bottom exact sequences of the following commutative diagram:

$$\begin{array}{ccccc}
\widehat{A}_r^{\mathrm{ord}}[p^\infty] & \xrightarrow{\hookrightarrow} & (\widehat{A}_r^{\mathrm{ord}} \times \alpha(J_\infty^{\mathrm{ord}})) & \xrightarrow{\twoheadrightarrow} & J_\infty^{\mathrm{ord}} \\
{\scriptstyle 0}\downarrow & & {\scriptstyle \alpha}\downarrow & & {\scriptstyle \alpha}\downarrow \\
\widehat{A}_r^{\mathrm{ord}}[p^\infty] & \xrightarrow{\hookrightarrow} & (\widehat{A}_r^{\mathrm{ord}} \times \alpha(J_\infty^{\mathrm{ord}})) & \xrightarrow{\twoheadrightarrow} & J_\infty^{\mathrm{ord}}.
\end{array}$$

Applying the snake lemma (noting that the connection map is the zero map), we get

$$\mathrm{Coker}(J_\infty^{\mathrm{ord}} \xrightarrow{\alpha} J_\infty^{\mathrm{ord}}) = \widehat{A}_r^{\mathrm{ord}}/\widehat{A}_r^{\mathrm{ord}}[p^\infty].$$

Thus we have the following exact sequence of sheaves:

$$0 \to \widehat{A}_r^{\mathrm{ord}} \to J_\infty^{\mathrm{ord}} \to J_\infty^{\mathrm{ord}} \to \widehat{A}_r^{\mathrm{ord}}/\widehat{A}_r^{\mathrm{ord}}[p^\infty] \to 0. \tag{6.17}$$

There is another way to see (6.17). Passing to the inductive limit of the exact sequences of sheaves

$$0 \to \widehat{A}_s^{\mathrm{ord}} \to \widehat{J}_s^{\mathrm{ord}} \xrightarrow{\alpha_s} \widehat{J}_s^{\mathrm{ord}} \xrightarrow{\rho_s} \widehat{B}_s^{\mathrm{ord}} \to 0,$$

we get the following exact sequence of sheaves:

$$0 \to \widehat{A}_r^{\mathrm{ord}} \to J_\infty^{\mathrm{ord}} \xrightarrow{\alpha} J_\infty^{\mathrm{ord}} \xrightarrow{\rho_\infty} \varinjlim_{s, x \mapsto p^{s-r}U(p)^{s-r}} \widehat{B}_r^{\mathrm{ord}} \to 0$$

as $\widehat{A}_r^{\mathrm{ord}} \cong \widehat{A}_s^{\mathrm{ord}}$ by $\pi_{r,s}^*$. This combined with (6.17) and Lemma 6.2 proves the exact sequence in (6.15). By (6.16), we have $\widehat{A}_s^{\mathrm{ord}} \cap \alpha(\widehat{J}_s^{\mathrm{ord}}) \cong C_s^{\mathrm{ord}}$; thus $\mathrm{Ker}(\widehat{A}_s^{\mathrm{ord}} \to \widehat{B}_s^{\mathrm{ord}}) \cong C_s^{\mathrm{ord}}$ with $\varinjlim_s C_s^{\mathrm{ord}} = A_r[p^\infty]^{\mathrm{ord}}$, passing to the inductive limit we again get the identity of sheaves:

$$\varinjlim_{s, x \mapsto p^{s-r}U(p)^{s-r}} \widehat{B}_r^{\mathrm{ord}} \cong \widehat{A}_r^{\mathrm{ord}}/\widehat{A}_r[p^\infty]^{\mathrm{ord}} \cong \widehat{B}_r^{\mathrm{ord}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

This finishes the proof. □

We have two exact sequences of sheaves:

$$\begin{aligned} &0 \to \widehat{A}_r^{\mathrm{ord}} \to J_\infty^{\mathrm{ord}} \xrightarrow{\alpha} \alpha(J_\infty^{\mathrm{ord}}) \to 0, \\ &0 \to \alpha(J_\infty^{\mathrm{ord}}) \to J_\infty^{\mathrm{ord}} \xrightarrow{\rho_\infty} \widehat{B}_r^{\mathrm{ord}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to 0. \end{aligned} \tag{6.18}$$

These leave us to study the two error terms

$$E_1(\kappa) := \alpha(J_\infty^{\mathrm{ord}})(\kappa)/\alpha(J_\infty^{\mathrm{ord}}(\kappa)) \quad \text{and} \quad E_2(\kappa) := \widehat{B}_r^{\mathrm{ord}}(\kappa) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\rho_\infty(J_\infty^{\mathrm{ord}}(\kappa)).$$

Let $E_1^s(\kappa) := \alpha(J_s^{\mathrm{ord}})(\kappa)/\alpha(J_s^{\mathrm{ord}}(\kappa))$ and $E_2^s(\kappa) := \widehat{B}_s^{\mathrm{ord}}(\kappa)/\rho_s(\widehat{J}_s^{\mathrm{ord}}(\kappa)) = \mathrm{Coker}(\rho_s)$ for $\rho_s : J_s^{\mathrm{ord}}(\kappa) \to \widehat{B}_s^{\mathrm{ord}}(\kappa)$. Note that

$$E_1^s(\kappa)(\hookrightarrow H^1_?(\widehat{A}_r^{\mathrm{ord}}) = H^1_?(A_r^{\mathrm{ord}}) \otimes_{\mathbb{Z}} \mathbb{Z}_p)$$
$$\text{and} \quad E_2^s(\kappa) = B_s^{\mathrm{ord}}(\kappa)/\rho_s(\widehat{J}_s^{\mathrm{ord}}(\kappa))(\hookrightarrow H^1_?(\alpha(\widehat{J}_s^{\mathrm{ord}}))[\alpha])$$

are p-torsion finite modules as long as s is finite. Note that $\alpha|_{\alpha(J_s)}$ is a self isogeny; so,

$$0 \to \alpha(J_s)[\alpha]^{\mathrm{ord}} \to \alpha(\widehat{J}_s^{\mathrm{ord}}) \xrightarrow{\alpha_s} \alpha(\widehat{J}_s^{\mathrm{ord}}) \to 0$$

is an exact sequence of sheaves. Since $\alpha(J_s)[\alpha]^{\mathrm{ord}} = C_s^{\mathrm{ord}}$, we have another exact sequence:

$$0 \to \alpha(\widehat{J}_s^{\mathrm{ord}})(\kappa)/\alpha(\alpha(\widehat{J}_s^{\mathrm{ord}})(\kappa)) \to H^1_?(\widehat{C}_s^{\mathrm{ord}}) \to H^1_?(\alpha(\widehat{J}_s^{\mathrm{ord}}))[\alpha] \to 0.$$

We have the following commutative diagram with exact rows and exact columns:

$$\begin{array}{ccccc}
E_1^s(\kappa) & \xrightarrow{\hookrightarrow} & H^1_?(\widehat{A}_r^{\mathrm{ord}}) & \longrightarrow & H^1_?(\widehat{J}_s^{\mathrm{ord}}) \\
{\scriptstyle \mathrm{onto}}\Big\uparrow & & \Big\uparrow & & \Big\uparrow \\
\dfrac{\alpha(\widehat{J}_s^{\mathrm{ord}})(\kappa)}{\alpha(\alpha(\widehat{J}_s^{\mathrm{ord}})(\kappa))} & \xrightarrow{\hookrightarrow} & H^1_?(C_s^{\mathrm{ord}}) & \xrightarrow{\twoheadrightarrow} & H^1_?(\alpha(\widehat{J}_s^{\mathrm{ord}}))[\alpha] \\
{\scriptstyle \overline{\alpha}_s}\Big\uparrow & & \Big\uparrow & & \Big\uparrow{\scriptstyle \cup} \\
\dfrac{\widehat{J}_s^{\mathrm{ord}}(\kappa)}{\alpha(\widehat{J}_s^{\mathrm{ord}})(\kappa)} & \xrightarrow[\hookrightarrow]{\rho_s} & \widehat{B}_s^{\mathrm{ord}}(\kappa) & \xrightarrow[\twoheadrightarrow]{} & E_2^s(\kappa).
\end{array}$$

The left column is exact by definition. The middle column is the part of the long exact sequence attached to the short one $C_s^{\mathrm{ord}} \hookrightarrow \widehat{A}_s^{\mathrm{ord}} \twoheadrightarrow \widehat{B}_s^{\mathrm{ord}}$, and the right column is the same for $\alpha(\widehat{J}_s^{\mathrm{ord}}) \hookrightarrow \widehat{J}_s^{\mathrm{ord}} \twoheadrightarrow \widehat{B}_s^{\mathrm{ord}}$. Note $\varinjlim_s C_s^{\mathrm{ord}} = A_r[p^\infty]^{\mathrm{ord}}$. Passing to the limit, we have the limit commutative diagram with exact rows and exact columns:

$$\begin{array}{ccccc}
E_1(\kappa) & \xrightarrow{\hookrightarrow} & H^1_?(\widehat{A}_r^{\mathrm{ord}}) & \longrightarrow & H^1_?(J_\infty^{\mathrm{ord}}) \\
{\scriptstyle \mathrm{onto}}\Big\uparrow & & {\scriptstyle \pi_B}\Big\uparrow & & \Big\uparrow \\
\dfrac{\alpha(J_\infty^{\mathrm{ord}})(\kappa)}{\alpha(\alpha(J_\infty^{\mathrm{ord}})(\kappa))} & \xrightarrow{\hookrightarrow} & H^1_?(A_r[p^\infty]^{\mathrm{ord}}) & \xrightarrow{\twoheadrightarrow} & H^1_?(\alpha(J_\infty^{\mathrm{ord}}))[\alpha] \\
{\scriptstyle \overline{\alpha}}\Big\uparrow & & {\scriptstyle \delta_B}\Big\uparrow & & \Big\uparrow{\scriptstyle \cup} \\
\dfrac{J_\infty^{\mathrm{ord}}(\kappa)}{\alpha(J_\infty^{\mathrm{ord}})(\kappa)} & \xrightarrow[\hookrightarrow]{\rho_\infty} & \widehat{B}_r^{\mathrm{ord}}(\kappa) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p & \xrightarrow[\twoheadrightarrow]{} & E_2(\kappa).
\end{array} \tag{6.19}$$

We have seen, $\widehat{A}_r^{\mathrm{ord}}/A_r[p^\infty]^{\mathrm{ord}} \cong \widehat{B}_r \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ as sheaves of $\mathbb{Q}_p$-vector space; so, $H^1_?(\widehat{A}_r^{\mathrm{ord}}/A_r[p^\infty]^{\mathrm{ord}})$ is a $\mathbb{Q}_p$-vector space. On the other hand, $H^1_?(\widehat{A}_r^{\mathrm{ord}})$ is a p-torsion module (e.g., Lemma 2.2). Therefore the natural map $H^1_?(\widehat{A}_r^{\mathrm{ord}}) \to H^1_?(\widehat{A}^{\mathrm{ord}}/A_r[p^\infty]^{\mathrm{ord}})$ is the zero map. Thus by long exact sequence attached to $0 \to A_r[p^\infty]^{\mathrm{ord}} \to \widehat{A}_r^{\mathrm{ord}} \to \widehat{A}_r^{\mathrm{ord}}/A[p^\infty] \to 0$, the morphism π_B is onto. Since $A_r(\kappa) \otimes_{\mathbb{Z}} \mathbb{T}_p = B_r(\kappa) \otimes_{\mathbb{Z}} \mathbb{T}_p$, the map δ_B factors through the Kummer map $A_r(\kappa) \otimes_{\mathbb{Z}} \mathbb{T}_p \hookrightarrow H^1(A_r[p^\infty]^{\mathrm{ord}})$. Thus

$$\mathrm{Ker}(\delta_B) = \mathrm{Im}(\widehat{A}_r(\kappa) \to \widehat{B}_r(\kappa) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \cong \mathrm{Ker}(\overline{\alpha}),$$

where the last identity follows from the snake lemma applied to the above diagram.

Consider the following exact sequence:

$$\begin{aligned}
E_1(\kappa)[p^n] = \mathrm{Tor}_1^{\mathbb{Z}}(E_1(\kappa), \mathbb{Z}/p^n\mathbb{Z}) \xrightarrow{i_n} \alpha(J_\infty^{\mathrm{ord}}(\kappa)) \otimes \mathbb{Z}/p^n\mathbb{Z} \\
\to \alpha(J_\infty^{\mathrm{ord}})(\kappa) \otimes \mathbb{Z}/p^n\mathbb{Z} \to E_1(\kappa) \otimes \mathbb{Z}/p^n\mathbb{Z} \to 0,
\end{aligned}$$

which produces the following commutative diagram with exact rows for $n > m$:

$$\begin{array}{ccccccc}
E_1(\kappa)[p^n] & \xrightarrow{i_n} & \frac{\alpha(J^{\mathrm{ord}}_\infty(\kappa))}{p^n\alpha(J^{\mathrm{ord}}_\infty(\kappa))} & \xrightarrow{j_n} & \frac{\alpha(J^{\mathrm{ord}}_\infty)(\kappa)}{p^n\alpha(J^{\mathrm{ord}}_\infty)(\kappa)} & \twoheadrightarrow & E_1(\kappa)\otimes\mathbb{Z}/p^n\mathbb{Z} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
E_1(\kappa)[p^m] & \xrightarrow{i_m} & \frac{\alpha(J^{\mathrm{ord}}_\infty(\kappa))}{p^m\alpha(J^{\mathrm{ord}}_\infty(\kappa))} & \xrightarrow{j_m} & \frac{\alpha(J^{\mathrm{ord}}_\infty)(\kappa)}{p^m\alpha(J^{\mathrm{ord}}_\infty)(\kappa)} & \twoheadrightarrow & E_1(\kappa)\otimes\mathbb{Z}/p^m\mathbb{Z}
\end{array}$$

This in turn produces two commutative diagrams with exact rows:

$$\begin{array}{ccccc}
E_1(\kappa)[p^n] & \xrightarrow{i_n} & \frac{\alpha(J^{\mathrm{ord}}_\infty(\kappa))}{p^n\alpha(J^{\mathrm{ord}}_\infty(\kappa))} & \longrightarrow & \mathrm{Coker}(i_n) = \mathrm{Im}(j_n) \to 0 \\
\downarrow & & \downarrow & & \downarrow \\
E_1(\kappa)[p^m] & \xrightarrow{i_m} & \frac{\alpha(J^{\mathrm{ord}}_\infty(\kappa))}{p^m\alpha(J^{\mathrm{ord}}_\infty(\kappa))} & \longrightarrow & \mathrm{Coker}(i_n) = \mathrm{Im}(j_m) \to 0
\end{array} \tag{6.20}$$

and

$$\begin{array}{ccccc}
0 \to \mathrm{Ker}(i_n) & \longrightarrow & E_1(\kappa)[p^n] & \xrightarrow{i_n} & \mathrm{Im}(i_n) \to 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 \to \mathrm{Ker}(i_m) & \longrightarrow & E_1(\kappa)[p^m] & \xrightarrow{i_m} & \mathrm{Im}(i_m) \to 0.
\end{array} \tag{6.21}$$

Since the diagram of (6.21) is made of finite modules (as $E_1(\kappa) \subset H^1(\widehat{A}^{\mathrm{ord}}_r)$; Lemma 2.3), projective limit is an exact functor (from the category of compact modules), and passing to the limit, we get

$$\varprojlim_n \mathrm{Im}(i_n) = \mathrm{Im}(i_\infty : \varprojlim_n E_1(\kappa)[p^n] \to \varprojlim_n \frac{\alpha(J^{\mathrm{ord}}_\infty(\kappa))}{p^n\alpha(J^{\mathrm{ord}}_\infty(\kappa))}).$$

By the snake lemma (cf. [BCM, I.1.4.2 (2)]) applied to (6.20), $\mathrm{Im}(j_n) \to \mathrm{Im}(j_m)$ is a surjection for all $n > m$. Thus the projective system of the following exact sequences:

$$\{0 \to \mathrm{Im}(j_n) \to \frac{\alpha(J^{\mathrm{ord}}_\infty)(\kappa)}{p^n\alpha(J^{\mathrm{ord}}_\infty)(\kappa)} \to E_1(\kappa)\otimes\mathbb{Z}/p^n\mathbb{Z} \to 0\}_n$$

satisfies the Mittag–Leffler condition. Passing to the projective limit, we get the exact sequence

$$0 \to \alpha(\check{J}^{\mathrm{ord}}_\infty(\kappa)) = \mathrm{Im}(j_\infty) \to \alpha(\check{J}^{\mathrm{ord}}_\infty)(\kappa) \to \varprojlim_n E_1(\kappa)\otimes\mathbb{Z}/p^n\mathbb{Z} \to 0.$$

Since $E_1(\kappa) = (\mathbb{Q}_p/\mathbb{Z}_p)^R \oplus \Delta \hookrightarrow H^1_?(\widehat{A}_r)^{\mathrm{ord}}$ for a finite group Δ and an integer $R \geq 0$ (by Lemma 2.3), $\varprojlim_n E_1(\kappa)\otimes\mathbb{Z}/p^n\mathbb{Z}$ is a finite group isomorphic to the torsion subgroup Δ of $E_1(\kappa)$. Thus

$$\check{J}^{\mathrm{ord}}_\infty(\kappa) \xrightarrow{\alpha} \alpha(\check{J}^{\mathrm{ord}}_\infty)(\kappa) \text{ has finite cokernel } \Delta, \tag{6.22}$$

and Δ is isomorphic to the maximal torsion submodule of $E_1(\kappa)^\vee$.

Consider the "big" ordinary Hecke algebra $\mathbf{h}$ given by $\varprojlim_s \mathbf{h}_s$ as in the introduction. For a Λ-algebra homomorphism $\mathbf{h} \to R$ and an $\mathbf{h}$-module M, we put $M_R = M \otimes_{\mathbf{h}} R$. Take a connected component $\mathrm{Spec}(\mathbb{T})$ of $\mathrm{Spec}(\mathbf{h})$

such that α in (A) restricted to Spec($\mathbb{T}$) is a non-unit; so, $\widehat{A}^{\mathrm{ord}}_s(K)_{\mathbb{T}} \neq 0$. Note that $M_{\mathbb{T}}$ is a direct summand of M; so, the above diagrams and exactness are valid after tensoring $\mathbb{T}$ over $\mathbf{h}$ (attaching subscript $\mathbb{T}$). Note that $\alpha(J_\infty)^{\mathrm{ord}}[p^\infty](\kappa) \subset J^{\mathrm{ord}}_\infty[p^\infty](\kappa) = \mathcal{G}(\kappa)$.
Since $\mathrm{Im}(\rho_\infty)_{\mathbb{T}}$ is a direct summand in $J_\infty(\kappa)_{\mathbb{T}}$ and $\alpha(J^{\mathrm{ord}}_\infty(\kappa)_{\mathbb{T}})[p^n] \cong \mathrm{Tor}_1^{\mathbb{Z}_p}(\alpha(J^{\mathrm{ord}}_\infty(\kappa)_{\mathbb{T}}), \mathbb{Z}/p^n\mathbb{Z})$, we have the following exact sequences:

(6.23)

$$\alpha(J^{\mathrm{ord}}_\infty(\kappa)_{\mathbb{T}})[p^n] \to \frac{\widehat{A}^{\mathrm{ord}}_r(\kappa)_{\mathbb{T}}}{p^n\widehat{A}^{\mathrm{ord}}_r(\kappa)_{\mathbb{T}}} \to \frac{J^{\mathrm{ord}}_\infty(\kappa)_{\mathbb{T}}}{p^nJ^{\mathrm{ord}}_\infty(\kappa)_{\mathbb{T}}} \xrightarrow{\alpha} \frac{\alpha(J^{\mathrm{ord}}_\infty(\kappa)_{\mathbb{T}})}{p^n\alpha(J^{\mathrm{ord}}_\infty(\kappa)_{\mathbb{T}})} \to 0$$
$$0 \to \alpha(J^{\mathrm{ord}}_\infty(\kappa)_{\mathbb{T}}) \otimes \mathbb{Z}/p^n\mathbb{Z} \to J^{\mathrm{ord}}_\infty(\kappa)_{\mathbb{T}} \otimes \mathbb{Z}/p^n\mathbb{Z} \to \mathrm{Im}(\rho_\infty)_{\mathbb{T}} \otimes \mathbb{Z}/p^n\mathbb{Z} \to 0.$$

The module $\alpha(J^{\mathrm{ord}}_\infty(\kappa)_{\mathbb{T}})[p^n]$ is killed by the annihilator $\mathfrak{a}$ of $\mathcal{G}_{\mathbb{T}}(\kappa)^\vee$ in Λ which is prime to $\gamma^{p^r} - 1$ (note that $\gamma^{p^r} - 1$ kills $\widehat{A}^{\mathrm{ord}}_r(\kappa)$). Thus the image of $\alpha(J^{\mathrm{ord}}_\infty(\kappa)_{\mathbb{T}})[p^n]$ in $\widehat{A}^{\mathrm{ord}}_r(\kappa)_{\mathbb{T}} \otimes \mathbb{Z}/p^n\mathbb{Z}$ is killed by $\mathfrak{A} = \mathfrak{a} + (\gamma^{p^{r-1}} - 1) \subset \Lambda$. Since $\Lambda/\mathfrak{A}$ is a finite ring and $\mathcal{G}^\vee_{\mathbb{T}}$ is a Λ-module of finite type, we get

$$|\,\mathrm{Ker}(\widehat{A}^{\mathrm{ord}}_r(\kappa)_{\mathbb{T}} \otimes \mathbb{Z}/p^n\mathbb{Z} \to J^{\mathrm{ord}}_\infty(\kappa)_{\mathbb{T}} \otimes \mathbb{Z}/p^n\mathbb{Z})| < B \tag{6.24}$$

for a constant $B > 0$ independent of n.
Applying the snake lemma to the following commutative diagram with exact rows:

$$\begin{array}{ccccc} p^n\widehat{A}^{\mathrm{ord}}_r(\kappa)_{\mathbb{T}} & \xrightarrow{\hookrightarrow} & \widehat{A}^{\mathrm{ord}}_r(\kappa)_{\mathbb{T}} & \xrightarrow{\twoheadrightarrow} & \widehat{A}^{\mathrm{ord}}_r(\kappa)_{\mathbb{T}} \otimes \mathbb{Z}/p^n\mathbb{Z} \\ \downarrow & & \downarrow & & \downarrow \\ p^nJ^{\mathrm{ord}}_\infty(\kappa)_{\mathbb{T}} & \xrightarrow[\hookrightarrow]{} & J^{\mathrm{ord}}_\infty(\kappa)_{\mathbb{T}} & \xrightarrow[\twoheadrightarrow]{} & J^{\mathrm{ord}}_\infty(\kappa)_{\mathbb{T}} \otimes \mathbb{Z}/p^n\mathbb{Z}, \end{array}$$

we have an isomorphism, for $\mathcal{F}_n := p^nJ^{\mathrm{ord}}_\infty(\kappa)_{\mathbb{T}} \cap \widehat{A}^{\mathrm{ord}}_r(\kappa)_{\mathbb{T}}$,

$$\mathcal{F}_n/p^n\widehat{A}^{\mathrm{ord}}_r(\kappa)_{\mathbb{T}} \cong \mathrm{Ker}(\widehat{A}^{\mathrm{ord}}_r(\kappa)_{\mathbb{T}} \otimes \mathbb{Z}/p^n\mathbb{Z} \to J^{\mathrm{ord}}_\infty(\kappa)_{\mathbb{T}} \otimes \mathbb{Z}/p^n\mathbb{Z})$$

whose right-hand-side is finite with bounded order independent of n by (6.24).
Consider the two filters on $\widehat{A}^{\mathrm{ord}}_r(\kappa)_{\mathbb{T}}$:

$$\mathcal{F} := \{\mathcal{F}_n = (p^nJ^{\mathrm{ord}}_\infty(\kappa)_{\mathbb{T}} \cap \widehat{A}^{\mathrm{ord}}_r(\kappa)_{\mathbb{T}})\}_n \text{ and } \{p^n\widehat{A}^{\mathrm{ord}}_r(\kappa)_{\mathbb{T}}\}_n$$

with $\mathcal{F}_n \supset p^n\widehat{A}^{\mathrm{ord}}_r(\kappa)_{\mathbb{T}}$. On the free quotient $\widehat{A}^{\mathrm{ord}}_r(\kappa)_{\mathbb{T}}/\widehat{A}^{\mathrm{ord}}_r[p^\infty](\kappa)_{\mathbb{T}}$, the two filters induce the same p-adic topology. Writing $\widetilde{A}^{\mathrm{ord}}_r(\kappa)_{\mathbb{T}}$ for the completion of $\widehat{A}^{\mathrm{ord}}_r(\kappa)_{\mathbb{T}}$ with respect to $\mathcal{F}$, therefore we find

(6.25)

the natural surjective morphism: $\widehat{A}^{\mathrm{ord}}_r(\kappa)_{\mathbb{T}} \twoheadrightarrow \widetilde{A}^{\mathrm{ord}}_r(\kappa)_{\mathbb{T}}$ has finite kernel.

This shows that the following sequence is exact by [CRT, Theorem 8.1 (ii)]:

$$0 \to \widetilde{A}^{\mathrm{ord}}_r(\kappa)_{\mathbb{T}} \to \check{J}^{\mathrm{ord}}_\infty(\kappa)_{\mathbb{T}} \xrightarrow{\alpha} \alpha(\check{J}^{\mathrm{ord}}_\infty(\kappa)_{\mathbb{T}}) \to 0. \tag{6.26}$$

By this sequence combined with finiteness of $\mathrm{Ker}(\widetilde{A}^{\mathrm{ord}}_r(\kappa)_{\mathbb{T}} \to \widehat{A}^{\mathrm{ord}}_r(\kappa)_{\mathbb{T}})$, we get

Proposition 6.4. *Take a connected component* $\mathrm{Spec}(\mathbb{T})$ *of* $\mathrm{Spec}(\mathbf{h})$ *with* $\widehat{A}^{\mathrm{ord}}_{s,\mathbb{T}} \neq 0$. *Then we have the following exact sequence:*

$$0 \to \widetilde{A}^{\mathrm{ord}}_r(\kappa)_{\mathbb{T}} \to \check{J}^{\mathrm{ord}}_\infty(\kappa)_{\mathbb{T}} \xrightarrow{\alpha} \check{J}^{\mathrm{ord}}_\infty(\kappa)_{\mathbb{T}},$$

where $\mathrm{Coker}(\alpha)$ *is a* $\mathbb{Z}_p$*-module of finite type with* $\dim_{\mathbb{Q}_p} \mathrm{Coker}(\alpha) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \leq \dim_{\mathbb{Q}_p} \widehat{B}_r(\kappa)_{\mathbb{T}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. *Moreover we have a natural surjection:* $\widehat{A}^{\mathrm{ord}}_r(\kappa)_{\mathbb{T}} \to \widetilde{A}^{\mathrm{ord}}_r(\kappa)_{\mathbb{T}}$ *with finite kernel. If* $\mathcal{G}_{\mathbb{T}}(\kappa) = 0$, *then* $\widehat{A}^{\mathrm{ord}}_r(\kappa)_{\mathbb{T}} \cong \widetilde{A}^{\mathrm{ord}}_r(\kappa)_{\mathbb{T}}$

We will see that the torsion submodule of $\mathrm{Coker}(\alpha)$ is isomorphic to the maximal p-torsion submodule of $E_1(\kappa)^\vee$.

Proof. The second sequence of (6.18) evaluated at κ produces the following exact sequence:

$$0 \to \alpha(J^{\mathrm{ord}}_\infty)(\kappa)_{\mathbb{T}} \to J^{\mathrm{ord}}_\infty(\kappa)_{\mathbb{T}} \xrightarrow{\rho_\infty} \widehat{B}^{\mathrm{ord}}_r(\kappa)_{\mathbb{T}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

By the exact sequence of the bottom row in the diagram (6.19), the image $\mathrm{Im}(\rho_\infty)$ is embedded into $(\widehat{B}_r \otimes \mathbb{Q}_p)(\kappa)_{\mathbb{T}}$, and thus $\mathrm{Im}(\rho_\infty) \cong \mathbb{Q}_p^i \oplus \mathbb{Z}_p^j$ with $i + j \leq \dim \widehat{B}_r(\kappa)_{\mathbb{T}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Thus we get the following exact sequences indexed by n:

$$\begin{aligned}0 = \mathrm{Im}(\rho_\infty)[p^n] &\cong \mathrm{Tor}_1^{\mathbb{Z}_p}(\mathrm{Im}(\rho_\infty), \mathbb{Z}/p^n\mathbb{Z}) \to \alpha(J^{\mathrm{ord}}_\infty)(\kappa)_{\mathbb{T}} \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^n\mathbb{Z} \\ &\to J^{\mathrm{ord}}_\infty(\kappa)_{\mathbb{T}} \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^n\mathbb{Z} \to \mathrm{Im}(\rho_\infty)_{\mathbb{T}} \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^n\mathbb{Z} \cong (\mathbb{Z}/p^n\mathbb{Z})^j \to 0.\end{aligned}$$

Since these sequences satisfy the Mittag–Leffler condition, passing to the limit, we get another exact sequence:

$$0 \to \alpha(\check{J}^{\mathrm{ord}}_\infty)(\kappa)_{\mathbb{T}} \to \check{J}^{\mathrm{ord}}_\infty(\kappa)_{\mathbb{T}} \xrightarrow{\rho_\infty} \mathbb{Z}_p^j \to 0.$$

Then the assertion follows from (6.22).

We can check the last assertion by scrutinizing our computation, but here is a short cut. Since $\mathrm{Ker}(\widehat{A}^{\mathrm{ord}}_r(\kappa)_{\mathbb{T}} \to \widetilde{A}^{\mathrm{ord}}_r(\kappa)_{\mathbb{T}})$ is a submodule of $A[p^\infty]^{\mathrm{ord}}(\kappa) \subset \mathcal{G}_{\mathbb{T}}(\kappa) = 0$. Thus the morphism has to be an isomorphism. □

Lemma 6.5. *Let* κ *be as in* Lemma 6.1. *Then the maximal torsion submodule of* $\check{J}_\infty(\kappa)^{\mathrm{ord}}_{\mathbb{T}}$ *is equal to* $\mathcal{G}_{\mathbb{T}}(\kappa)$ *if* $\mathcal{G}_{\mathbb{T}}(\kappa)$ *is finite. Otherwise, it is killed by* p^B *for some* $0 < B \in \mathbb{Z}$.

Proof. By definition, the maximal torsion submodule of $\widehat{J}_s(\kappa)^{\mathrm{ord}}_{\mathbb{T}} = (J_s(\kappa) \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\mathrm{ord}}_{\mathbb{T}}$ for finite s is given by $\mathcal{G}_s(\kappa)_{\mathbb{T}} := J_s[p^\infty](\kappa)^{\mathrm{ord}}_{\mathbb{T}}$. For $s = \infty$, the maximal torsion submodule of $J_\infty(\kappa)_{\mathbb{T}} = \varinjlim_s \widehat{J}_s(\kappa)^{\mathrm{ord}}_{\mathbb{T}}$ is given by $\mathcal{G}(\kappa)_{\mathbb{T}}$. Thus we have an exact sequence for finite s:

$$0 \to \mathcal{G}_s(\kappa)_{\mathbb{T}} \to \widehat{J}_s(\kappa)^{\mathrm{ord}}_{\mathbb{T}} \to F_s \to 0$$

for the maximal $\mathbb{Z}_p$-free quotient $F_s := \widehat{J}_s(\kappa)^{\mathrm{ord}}_{\mathbb{T}}/\mathcal{G}_s(\kappa)_{\mathbb{T}}$. This is a split exact sequence as the right term $\widehat{J}_s(\kappa)^{\mathrm{ord}}_{\mathbb{T}}/\mathcal{G}_s(\kappa)_{\mathbb{T}}$ is $\mathbb{Z}_p$-free. By taking p-adic completion: $M \mapsto \check{M} = \varprojlim_n M/p^nM$, we get a split exact sequence for finite s:

$$0 \to \check{\mathcal{G}}_s(\kappa)_{\mathbb{T}} \to \widehat{J}_s(\kappa)^{\mathrm{ord}}_{\mathbb{T}} \to F_s \to 0.$$

This shows $\check{\mathcal{G}}_s(\kappa)_{\mathbb{T}} = \mathcal{G}_s(\kappa)_{\mathbb{T}}$ for finite s, and $\check{\mathcal{G}}_s(\kappa)_{\mathbb{T}}$ is a finite module if κ is as in Lemma 6.1. Since F_s is $\mathbb{Z}_p$-flat for all $s \geq r$, $F = \varinjlim_s F_s$ is a $\mathbb{Z}_p$-flat module. For $s = \infty$, we have the limit exact sequence (noting $\mathcal{G}(\kappa)_{\mathbb{T}} = \mathcal{G}_\infty(\kappa)_{\mathbb{T}}$)

$$0 \to \mathcal{G}(\kappa)_{\mathbb{T}} \to J_\infty(\kappa)^{\mathrm{ord}}_{\mathbb{T}} \to F \to 0,$$

and $F = J_\infty(\kappa)^{\mathrm{ord}}_{\mathbb{T}}/\mathcal{G}(\kappa)_{\mathbb{T}}$. By $\mathbb{Z}_p$-flatness of F, after tensoring $\mathbb{Z}/p^n\mathbb{Z}$ over $\mathbb{Z}_p$, we still have an exact sequence (cf. [BCM, I.2.5]) indexed by $0 < n \in \mathbb{Z}$:

$$0 \to \mathcal{G}(\kappa)_{\mathbb{T}}/p^n\mathcal{G}(\kappa)_{\mathbb{T}} \to J_\infty(\kappa)^{\mathrm{ord}}_{\mathbb{T}} \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^n\mathbb{Z} \to F/p^nF \to 0,$$

which obviously satisfies the Mittag–Leffler condition (with respect to n). Passing to the projective limit with respect to n, we get the limit exact sequence:

$$0 \to \check{\mathcal{G}}(\kappa)_{\mathbb{T}} \to \check{J}_\infty(\kappa)^{\mathrm{ord}}_{\mathbb{T}} \to \check{F} \to 0,$$

Since F is $\mathbb{Z}_p$-flat, $\check{F}$ is torsion-free (and hence $\mathbb{Z}_p$-flat by [BCM, I.2.4]). Indeed, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\mathrm{Tor}_{\mathbb{Z}_p}(F/pF, \mathbb{Z}/p^n\mathbb{Z}) & \xrightarrow{\hookrightarrow} & F/p^nF & \xrightarrow{x\mapsto px} & F/p^nF & \xrightarrow{\twoheadrightarrow} & F/pF \\
\wr\downarrow & & \Vert\downarrow & & \Vert\downarrow & & \downarrow\Vert \\
F/pF & \xrightarrow[\hookrightarrow]{} & F/p^nF & \xrightarrow{x\mapsto px} & F/p^nF & \xrightarrow[\twoheadrightarrow]{} & F/pF.
\end{array}$$

Regard this as a projective system of exact sequences indexed by $0 < n \in \mathbb{Z}$. Then the transition maps of F/pF at the extreme right end is the identity and at the extreme left end is multiplication by p (i.e., the zero map). Passing to the limit, from left exactness of projective limit, we get an exact sequence

$$0 = \varprojlim_{x\mapsto px} F/pF \to \check{F} \xrightarrow{f\mapsto pf} \check{F},$$

and hence $\check{F}$ is p-torsion-free.

If $\mathcal{G}(\kappa)$ is killed by p^B for some $0 < B \in \mathbb{Z}$, we still have $\check{\mathcal{G}}(\kappa)_{\mathbb{T}} = \mathcal{G}(\kappa)_{\mathbb{T}}$. Otherwise, for some $0 < j \in \mathbb{Z}$, $\mathcal{G}(\kappa)_{\mathbb{T}}$ fits into the following split exact sequence by Lemmas in Section 4,

$$0 \to (\mathbb{Q}_p/\mathbb{Z}_p)^j \to \mathcal{G}(\kappa)_{\mathbb{T}} \to \mathcal{G}(\kappa)^{tor}_{\mathbb{T}} \to 0$$

for $\mathcal{G}(\kappa)^{tor}_{\mathbb{T}}$ killed by p^B for some $0 < B \in \mathbb{Z}$. Thus $\check{\mathcal{G}}(\kappa)_{\mathbb{T}} = \mathcal{G}(\kappa)^{tor}_{\mathbb{T}}$, which is the maximal torsion submodule of $\check{J}_\infty(\kappa)^{\mathrm{ord}}_{\mathbb{T}}$. □

We put $M^* = \mathrm{Hom}_{\mathbb{Z}_p}(M, \mathbb{Z}_p)$ for a $\mathbb{Z}_p$-module M and

$$\check{X}_{s,\mathbb{T}}(k)^*_{\mathrm{ord}} := \mathrm{Hom}_{\mathbb{Z}_p}(\check{X}_s(k)^{\mathrm{ord}}_{\mathbb{T}}, \mathbb{Z}_p) \quad \text{and} \quad \widehat{X}_{s,\mathbb{T}}(k)^*_{\mathrm{ord}} := \mathrm{Hom}_{\mathbb{Z}_p}(\widehat{X}_s(k)^{\mathrm{ord}}_{\mathbb{T}}, \mathbb{Z}_p)$$

with $s = r, r+1, \ldots, \infty$ for $X = J, A, B$. The algebra $\mathbf{h}$ acts on $\check{J}_\infty$ naturally. As before, we write for an $\mathbf{h}$-algebra R, $\check{J}_\infty(k)^{\mathrm{ord}}_R = \check{J}_\infty(k)^{\mathrm{ord}} \otimes_{\mathbf{h}} R$ and $\check{J}_\infty(k)^*_{\mathrm{ord},\, R} = \check{J}_\infty(k)^*_{\mathrm{ord}} \otimes_{\mathbf{h}} R$.

Assume the condition (A) in Section 5 for (α, A_s, B_s). Take a connected component $\mathrm{Spec}(\mathbb{T})$ of $\mathrm{Spec}(\mathbf{h})$ in which the image of α is non-unit. Replacing α by $1_{\mathbb{T}}\alpha$ for the idempotent $1_{\mathbb{T}}$ of $\mathbb{T}$, we may assume that $\alpha \in \mathbb{T}$ as in the setting of

(P4) in Proposition 5.1. Recall $\mathcal{G}_{\mathbb{T}}(k)^{\vee}_{tor}$ is the maximal $\mathbb{Z}_p$-torsion submodule of $\mathcal{G}_{\mathbb{T}}(k)^{\vee}$. We now state the principal result of this paper:

Theorem 6.6. *Let k be either a number field or a finite extension of $\mathbb{Q}_l$ for a prime l. Then we get*

(1) *Consider the following sequence $\mathbb{Z}_p$-dual to the one in* Proposition 6.4*:*

$$0 \to \mathrm{Coker}(\alpha)^*_{\mathbb{T}} \to \check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}} \xrightarrow{\alpha^*} \check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}} \xrightarrow{\iota^*_\infty} \widehat{A}_r(k)^*_{\mathrm{ord},\mathbb{T}} \to 0.$$

Then

- (a) *If $\mathcal{G}_{\mathbb{T}}(k) = 0$, the sequence is exact except that* $\mathrm{Ker}(\iota^*_\infty)/\mathrm{Im}(\alpha^*)$ *is finite;*
- (b) *If $\mathcal{G}_{\mathbb{T}}(k)^{\vee}_{tor} = 0$, the sequence is exact except that* $\mathrm{Ker}(\iota^*_\infty)/\mathrm{Im}(\alpha^*)$ *and* $\mathrm{Coker}(\iota^*_\infty)$ *are both finite;*
- (c) *If $\mathcal{G}_{\mathbb{T}}(k)^{\vee}_{tor} \neq 0$, the sequence is exact up to finite error.*
- (d) *The module $\mathcal{G}_{\mathbb{T}}(k)^{\vee}_{tor}$ is killed by p^B for some finite $0 \leq B \in \mathbb{Z}$, and the cokernel* $\mathrm{Coker}(\iota^*_\infty)$ *is finite and is killed by p^B. In particular, after localizing the sequence by any prime divisor $P \in \mathrm{Spec}(\Lambda)$, the sequence is exact.*

(2) *After tensoring $\mathbb{Q}_p$ with the sequence* (1)*, the following sequence*

$$0 \to \mathrm{Coker}(\alpha)^*_{\mathbb{T}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \widehat{A}_r(k)^*_{\mathrm{ord},\,\mathbb{T}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to 0$$

is an exact sequence of p-adic $\mathbb{Q}_p$-Banach spaces (with respect to the Banach norm having the image of $\check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}}$ in $\check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ as its closed unit ball).

(3) *The compact module $\check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}}$ is a Λ-module of finite type, and $\check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a $\Lambda[\frac{1}{p}]$-module of finite type.*

Proof. We prove the exactness of the sequence (1). Since $\widehat{A}^{\mathrm{ord}}_r(\kappa)_{\mathbb{T}} \to \widetilde{A}^{\mathrm{ord}}_r(\kappa)_{\mathbb{T}}$ has finite kernel and is an isomorphism if $\mathcal{G}_{\mathbb{T}}(k) = 0$ by Proposition 6.4, we only need to prove the various exactness of (1). By Proposition 6.4, the following sequence is exact:

$$0 \to \widetilde{A}^{\mathrm{ord}}_r(k)_{\mathbb{T}} \xrightarrow{\iota_\infty} \check{J}_\infty(k)^{\mathrm{ord}}_{\mathbb{T}} \xrightarrow{\alpha} \check{J}_\infty(k)^{\mathrm{ord}}_{\mathbb{T}} \xrightarrow{\pi^*_\infty} X \to 0$$

for $X = \mathrm{Coker}(\alpha)$. We consider the short exact sequence:

$$0 \to \widetilde{A}^{\mathrm{ord}}_r(k)_{\mathbb{T}} \xrightarrow{\iota_\infty} \check{J}_\infty(k)^{\mathrm{ord}}_{\mathbb{T}} \to \mathrm{Coker}(\iota_\infty) \to 0$$

and another exact sequence:

$$0 \to \mathrm{Coker}(\iota_\infty) \xrightarrow{\alpha} \check{J}_\infty(k)^{\mathrm{ord}}_{\mathbb{T}} \xrightarrow{\pi^*_\infty} X \to 0.$$

Applying the dualizing functor: $M \mapsto M^* := \mathrm{Hom}_{\mathbb{Z}_p}(M, \mathbb{Z}_p)$, we get the following exact sequences:

$$0 \to \mathrm{Coker}(\iota_\infty)^* \to \check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}} \xrightarrow{\iota^*_\infty} \widetilde{A}_r(k)^*_{\mathrm{ord},\mathbb{T}} \to \mathrm{Ext}^1_{\mathbb{Z}_p}(\mathrm{Coker}(\iota_\infty), \mathbb{Z}_p),$$

$$X^* \hookrightarrow \check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}} \xrightarrow{\alpha^*} \mathrm{Coker}(\iota_\infty)^* \to \mathrm{Ext}^1_{\mathbb{Z}_p}(X,\mathbb{Z}_p) \to \mathrm{Ext}^1_{\mathbb{Z}_p}(\check{J}_\infty(k)^{\mathrm{ord},\mathbb{T}},\mathbb{Z}_p).$$

Thus $\mathrm{Ext}^1_{\mathbb{Z}_p}(X,\mathbb{Z}_p)$ contains $\mathrm{Ker}(\iota^*_\infty)/\mathrm{Im}(\alpha^*)$. Computing $\mathrm{Ext}^1_{\mathbb{Z}_p}(M,\mathbb{Z}_p)$ by the injective resolution $0 \to \mathbb{Z}_p \to \mathbb{Q}_p \xrightarrow{\pi} \mathbb{Q}_p/\mathbb{Z}_p \to 0$ (see [MFG, (4.10)]), we find

$$\mathrm{Ext}^1_{\mathbb{Z}_p}(M,\mathbb{Z}_p) = \mathrm{Coker}(\mathrm{Hom}_{\mathbb{Z}_p}(X,\mathbb{Q}_p) \xrightarrow{\pi_*} \mathrm{Hom}_{\mathbb{Z}_p}(X,\mathbb{Q}_p/\mathbb{Z}_p)) = M[p^\infty]^\vee.$$

Since X is a $\mathbb{Z}_p$-module of finite type by Proposition 6.4, $\mathrm{Ext}^1_{\mathbb{Z}_p}(X,\mathbb{Z}_p) = X[p^\infty]^\vee$ is finite. Similarly $\mathrm{Ext}^1_{\mathbb{Z}_p}(\check{J}_\infty(k)^{\mathrm{ord},\mathbb{T}},\mathbb{Z}_p) = \check{J}_\infty(k)^{\mathrm{ord},\mathbb{T}}[p^\infty]^\vee = \mathcal{G}_\mathbb{T}(k)^\vee_{tor}$ and hence if $\mathcal{G}_\mathbb{T}(k)^\vee_{tor} = 0$, $\mathrm{Ext}^1_{\mathbb{Z}_p}(X,\mathbb{Z}_p) = \mathrm{Ker}(\iota^*_\infty)/\mathrm{Im}(\alpha^*)$. Anyway, $\mathrm{Ker}(\iota^*_\infty)/\mathrm{Im}(\alpha^*)$ is finite.

We have $\mathrm{Coker}(\iota_\infty) \hookrightarrow \check{J}_\infty(k)^{\mathrm{ord}}_\mathbb{T}$. Again, we get, as Λ-modules,

$$\mathrm{Ext}^1_{\mathbb{Z}_p}(\mathrm{Coker}(\iota_\infty),\mathbb{Z}_p) \cong \mathrm{Coker}(\iota_\infty)[p^\infty]^\vee$$

which is a quotient of $\mathcal{G}_\mathbb{T}(k)^\vee_{tor}$ (see Lemmas 4.2, 4.3 and 6.5). Indeed, assuming finiteness of $\mathcal{G}_\mathbb{T}(k)$, the torsion part of $\check{J}_\infty(k)^{\mathrm{ord}}_\mathbb{T}$ is isomorphic to a submodule of $\mathcal{G}_\mathbb{T}(k)$ by Lemma 6.5; in particular, it has finite torsion (this proves (1a)). Without assuming finiteness of $\mathcal{G}_\mathbb{T}(k)$, the p-torsion part of $\mathrm{Coker}(\iota_\infty)$ is a Λ-submodule of a bounded p-torsion Λ–module $\mathcal{G}_\mathbb{T}(k)_{tor}$ by Lemma 6.5. Thus $\mathrm{Ext}^1_{\mathbb{Z}_p}(\mathrm{Coker}(\iota_\infty),\mathbb{Z}_p)$ is is a quotient of $\mathcal{G}_\mathbb{T}(k)^\vee_{tor}$ and killed by p^B for some $0 \geq B \in \mathbb{Z}$ (and this proves (1b)). In addition, $\mathrm{Coker}(i^*_\infty) = \mathrm{Coker}(\check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}} \to \widetilde{A}_r(k)^*_{\mathrm{ord},\mathbb{T}})$ factors through the $\mathbb{Z}_p$-module $\widetilde{A}_r(k)^*_{\mathrm{ord},\mathbb{T}}$ of finite type, which lands in the bounded p-torsion module $\mathrm{Coker}(\iota_\infty)[p^\infty]^\vee$ (by Lemma 6.5); so, $\mathrm{Coker}(i^*_\infty)$ must have finite order (this shows (1c)). Therefore, the error term $\mathrm{Coker}(i^*_\infty)$ is a pseudo-null Λ-module, it is killed after localization at prime divisors of $\mathrm{Spec}(\Lambda)$. Thus we get all the assertions in (1).

The exact sequence in (1) tells us that $\check{J}_\infty(k)^*_{\mathrm{ord},\ \mathbb{T}}/\alpha(\check{J}_\infty(k)^*_{\mathrm{ord},\ \mathbb{T}})$ is isomorphic (up to finite modules) to the $\mathbb{Z}_p$-module $\widehat{A}_r(k)^*_{\mathrm{ord},\ \mathbb{T}}$ of finite type, which is a torsion Λ-module of finite type. Then by Nakayama's lemma, $\check{J}_\infty(k)^*_{\mathrm{ord},\ \mathbb{T}}$ is a Λ-module of finite type. This proves the assertion (3).

The extension modules appearing in the above proof of (1) is p-torsion Λ-module of finite type. Thus the sequence

$$0 \to X^* \to \check{J}_\infty(k)^*_{\mathrm{ord},\ \mathbb{T}} \to \check{J}_\infty(k)^*_{\mathrm{ord},\ \mathbb{T}} \to \widehat{A}_r(k)^*_{\mathrm{ord},\ \mathbb{T}} \to 0$$

is exact up to p-torsion error. By tensoring $\mathbb{Q}$ over $\mathbb{Z}$, we get the exact sequence (2):

$$X^* \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \hookrightarrow \check{J}_\infty(k)^*_{\mathrm{ord},\ \mathbb{T}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \check{J}_\infty(k)^*_{\mathrm{ord},\ \mathbb{T}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \twoheadrightarrow \widehat{A}_r(k)^*_{\mathrm{ord},\ \mathbb{T}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

The above sequence is the p-adic Banach dual sequence of the following exact sequence obtained from the sequence in (1) by tensoring $\mathbb{Q}$:

$$\widehat{A}_r(k)^{\mathrm{ord}}_\mathbb{T} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \underset{\hookrightarrow}{\xrightarrow{\iota_\infty}} \check{J}_\infty(k)^{\mathrm{ord}}_\mathbb{T} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\alpha} \check{J}_\infty(k)^{\mathrm{ord}}_\mathbb{T} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \underset{\twoheadrightarrow}{\xrightarrow{\pi^*_\infty}} X \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Indeed, equipping $\check{J}_\infty(k)^{\mathrm{ord}}_\mathbb{T} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ with the Banach p-adic norm so that the closed unit ball is given by the image of $\check{J}_\infty(k)^{\mathrm{ord}}_\mathbb{T}$ in $\check{J}_\infty(k)^{\mathrm{ord}}_\mathbb{T} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, the sequence is continuous (the first and the last term are finite dimensional $\mathbb{Q}_p$-vector spaces; so, there is a unique p-adic Banach space structure on them). The dual space of bounded functionals of each term is given by the $\mathbb{Q}_p$-dual of the corresponding space before tensoring $\mathbb{Q}$, which is given by $Y \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ for $Y = \widehat{A}_r(k)^*_{\mathrm{ord},\,\mathbb{T}}$, $\check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}}$ and X^*, respectively. This proves (2). □

Corollary 6.7. *Let the notation be as in* (1) *of* Theorem 6.6. *Consider the set* $\Omega \subset \mathbb{T}$ *of prime factors* (*in* Λ) *of* $\gamma^{p^n} - 1$ *for* $n = 0, 1, 2, \ldots, \infty$. *Except for finitely many* $\alpha \in \Omega$, *we have* $\mathrm{Coker}(\alpha)^*_{\mathbb{T}} \otimes_\Lambda \Lambda_P = 0$ *for* $P = (\alpha) \in \mathrm{Spec}(\Lambda)$, *where* Λ_P *is the localization of* Λ *at* P.

Proof. Note that $\alpha \in \Omega$ (regarded as $\alpha \in \mathbb{T}$) satisfies the assumption (A) by Proposition 5.1 (P1) and that $\Lambda[\frac{1}{p}]$ is a principal ideal domain (as Λ is a unique factorization domain of dimension 2; see [CRT, Chapter 7]). Pick an isomorphism $\check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \Lambda[\frac{1}{p}]^R \oplus X^*_k$ of $\Lambda[\frac{1}{p}]$-modules with torsion $\Lambda[\frac{1}{p}]$-module X^*_k. Then for P outside the support of the $\Lambda[\frac{1}{p}]$-module X^*_k, by Theorem 6.6 (2),

$$K := \mathrm{Ker}(\alpha : \check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}} \to \check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}})$$

is killed by some p-power. Then by the assertion (1) of the above theorem, K is a $\mathbb{Z}_p$-module of finite type; hence K is finite. This shows the result. □

7. Closure of the global Λ-MW group in the local one

Let κ be a number field and $k = \kappa_{\mathfrak{p}}$ be the $\mathfrak{p}$-adic completion of κ for a prime $\mathfrak{p}|p$ of κ. Write W for the p-adic integer ring of k, and let Q be the quotient field of Λ. By [M55] or [T66], for an abelian variety $A_{/k}$ of dimension g, $\widehat{A}(k) = A(k) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ has torsion free part $\widehat{A}(k)_f$ isomorphic to the additive group W^g, and the torsion part $\widehat{A}(k)_{tor}$ is a finite group.
Write $F = \kappa$ or k. Recall the $\mathbb{T}$-component

$$\check{J}_\infty(F)^*_{\mathrm{ord},\mathbb{T}} := \check{J}_\infty(F)^*_{\mathrm{ord}} \otimes_{\mathbf{h}} \mathbb{T}$$

for a connected component $\mathrm{Spec}(\mathbb{T})$ of $\mathrm{Spec}(\mathbf{h})$. By Theorem 6.6 (3), $\check{J}_\infty(F)^*_{\mathrm{ord},\mathbb{T}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a $\mathbb{T} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$-module of finite type. For simplicity, write $\mathbb{J}_{\mathbb{T}}(F) := \check{J}_\infty(F)^*_{\mathrm{ord},\mathbb{T}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.
Let the notation be as in Corollary 6.7; in particular, Ω is the set of prime factors (in Λ) of $\gamma^{p^n} - 1$ for $n = 1, 2, \ldots, \infty$. Note that $\alpha \in \Omega \subset \mathbb{T}$ satisfies the condition (A) by Proposition 5.1 (P1). Then by Theorem 6.6 (2), this implies

$$\mathbb{J}_{\mathbb{T}}(F)/\alpha(\mathbb{J}_{\mathbb{T}}(F)) \cong \widehat{A}^{\mathrm{ord}}_r(F)^*_{\mathbb{T}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Further localizing at each arithmetic point $P \in \mathrm{Spec}(\mathbf{h})(\overline{\mathbb{Q}}_p)$ with $P|(\gamma^{p^{r-1}} - 1)$, we get, for $\mathbb{J}_{\mathbb{T}_P}(F) = \mathbb{J}_{\mathbb{T}}(F) \otimes_{\mathbb{T}} \mathbb{T}_P$ for the localization $\mathbb{T}_P$ at P,

$$\mathbb{J}_{\mathbb{T}_P}(F)/\alpha(\mathbb{J}_{\mathbb{T}_P}(F)) \cong \widehat{A}^{\mathrm{ord}}_r(F)^*_{\mathbb{T}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Since $\Lambda[\frac{1}{p}]$ is a principal ideal domain, $\mathbb{J}_{\mathbb{T}}(F)$ is isomorphic to $\Lambda[\frac{1}{p}]^{m_F} \oplus X'_F$ for a torsion $\Lambda[\frac{1}{p}]$-module X'_F. Put $X_F := X'_F \oplus \mathcal{G}_?(k)^\vee$ and decompose

$X_F = \bigoplus_{\mathfrak{P}} \Lambda[\frac{1}{p}]/\mathfrak{P}^{e_F(\mathfrak{P})}$ for maximal ideals $\mathfrak{P}$ of $\Lambda[\frac{1}{p}]$. Put $\mathrm{Char}_{\Lambda[\frac{1}{p}]}(X_F) = \prod_{\mathfrak{P}} \mathfrak{P}^{e_F(\mathfrak{P})}$. If $P \in \mathrm{Spec}(\Lambda)(\overline{\mathbb{Q}}_p)$ is prime to $\mathrm{Char}_{\Lambda[\frac{1}{p}]}(X_F)$,

$$\mathbb{J}_{\mathbb{T}}(F)/\alpha(\mathbb{J}_{\mathbb{T}}(F)) \cong (\Lambda/P)^{m_F} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

In other words, the Λ_P/P-dimension of $\mathbb{J}_{\mathbb{T}}(F)/\alpha(\mathbb{J}_{\mathbb{T}}(F))$ is independent of P for most of P. We formulate this fact for $F = k$ as follows:

THEOREM 7.1. *Let the notation be as above. Write W for the $\mathfrak{p}$-adic integer ring of k and Q for the quotient field of Λ. Then the Q-vector space $\check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}} \otimes_\Lambda Q$ has dimension equal to $g = \mathrm{rank}_{\mathbb{Z}_p} W \cdot \mathrm{rank}_\Lambda \mathbb{T}$.*

Proof. We use the notation introduced in Corollary 6.7. Pick $\alpha \in \Omega$, and let $A \subset J_r[\alpha]$ be the identity connected component. Define $J_r \twoheadrightarrow B$ to be the dual quotient of $A \hookrightarrow J_r$. By the control Theorem 6.6 (2), we have $\mathbb{J}_{\mathbb{T}}(k)/\alpha\mathbb{J}_{\mathbb{T}}(k) \cong \check{A}(k)^*_{\mathrm{ord},\mathbb{T}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ for all $\alpha \in \Omega$. Moreover, we have $\dim_{\kappa(P)} \mathbb{J}_{\mathbb{T}}(k)/\alpha\mathbb{J}_{\mathbb{T}}(k) = m_k$ outside a finite set $S \subset \Omega$. The set S is made of prime factors in Ω of $\mathrm{Char}_{\Lambda[\frac{1}{p}]}(X_k^*)$. Note that $m_k = \mathrm{rank}_{\Lambda[\frac{1}{p}]} \mathbb{J}_{\mathbb{T}}(k) = \dim_Q \check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}} \otimes_\Lambda Q$; so, we compute $\mathrm{rank}_{\Lambda[\frac{1}{p}]} \mathbb{J}_{\mathbb{T}}(k)$.

By [M55] or [T66], we have $\widehat{A}(k) \cong W^{\dim A} \times \Delta$ for a finite p-abelian group Δ. Regarding $A(k)$ as a p-adic Lie group, we have a logarithm map $\log : A(k) \to Lie(A_{/k})$. For a ring R, write $h(R)$ (resp. $h_r(R)$) for the scalar extension to R of

$$\mathbb{Z}[T(n)|n = 1, 2, \dots] \subset \mathrm{End}(A_{/\mathbb{Q}}) \cong \mathrm{End}(B_{/\mathbb{Q}})$$
$$(\text{resp. } \mathbb{Z}[T(n)|n = 1, 2, \dots] \subset \mathrm{End}(J_{r/\mathbb{Q}})).$$

The Lie algebra $Lie(A_{/\mathbb{Q}_p})$ is the dual of $\Omega_{B/\mathbb{Q}_p}$.
Note that $\Omega_{J_r/\mathbb{Q}} \cong \Omega_{X_r/\mathbb{Q}}$ (e.g., [GME, Theorem 4.1.7]). By q-expansion at the infinity cusp, we have an embedding $i : \Omega_{X_r/\mathbb{Q}} \hookrightarrow \mathbb{Q}[[q]]$ sending ω to $i(\omega)\frac{dq}{q}$. Writing $i(\omega) = \sum_{n=1}^\infty a(n, \omega)q^n$, we have $a(m, \omega|T(n)) = \sum_{0<d|(m,n),(d,Np)=1} d \cdot a(\frac{mn}{d^2}, \omega|\langle d\rangle)$ for the diamond operator $\langle d\rangle$ associated to $d \in (\mathbb{Z}/Np^r\mathbb{Z})^\times$. From this, the pairing $\langle\cdot,\cdot\rangle : h_r(\mathbb{Q}) \times \Omega_{X_r/\mathbb{Q}} \to \mathbb{Q}$ given by $\langle H, \omega\rangle = a(1, \omega|H)$ is non-degenerate (see [GME, §3.2.6]). Thus we have

$$\Omega_{J_r/k} \cong \mathrm{Hom}_k(h_r(k), k) \ \text{ and } \ \Omega_{A/k} \cong \mathrm{Hom}_k(h(k), k) \ \text{ as modules over } h_r(k),$$

since $h(k)$ is naturally a quotient of $h_r(k)$ and $B = J_r/\alpha(J_r)$ for $(\alpha) = \mathrm{Ker}(h_r(\mathbb{Z}_p) \twoheadrightarrow h(\mathbb{Z}_p))$ in $h_r(\mathbb{Z}_p)$. By the duality between $Lie(A_{/k})$ and $\Omega_{A/k}$, we have

$$Lie(A_{/k}) \cong h(k) \ \text{ as an } h(k)\text{-module}.$$

This leads to an isomorphism of h-modules:

$$\widehat{A}(k)^{\mathrm{ord}}_{\mathbb{T}} \otimes_{\mathbb{Z}_p} k \xrightarrow[\sim]{\log} Lie(A_{/k})_{\mathbb{T}} \cong (\mathbb{T}/(\alpha)\mathbb{T}) \otimes_{\mathbb{Z}_p} k$$

as $\mathbb{T}/(\alpha)\mathbb{T}$ is canonically isomorphic to a ring direct summand $h(\mathbb{Z}_p)^{\mathrm{ord}}_{\mathbb{T}}$ of $h(\mathbb{Z}_p)^{\mathrm{ord}}$ as $\mathbb{Z}_p$-algebras by the control theorem (cf. [GME, §3.2.6]). Thus

$$\mathrm{rank}_W \widehat{A}(k)^{\mathrm{ord}}_{\mathbb{T}} = [k : \mathbb{Q}_p] \, \mathrm{rank}_{\Lambda/(\alpha)} \mathbb{T}/(\alpha)\mathbb{T} = [k : \mathbb{Q}_p] \, \mathrm{rank}_\Lambda \mathbb{T}.$$

This proves the desired assertion, as $[k:\mathbb{Q}_p]=\operatorname{rank}_{\mathbb{Z}_p} W$. $\square$

We have a natural Λ-linear map

$$\check{J}_\infty(\kappa)^{\mathrm{ord}}_{\mathbb{T}} \xrightarrow{\iota} \check{J}_\infty(k)^{\mathrm{ord}}_{\mathbb{T}} \text{ and } \check{J}_\infty(k)^*_{\mathrm{ord},\,\mathbb{T}}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p \xrightarrow{\iota^*} \check{J}_\infty(\kappa)^*_{\mathrm{ord},\,\mathbb{T}}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p.$$

We would like to study their kernel and cokernel.

Take a reduced irreducible component $\mathrm{Spec}(\mathbb{I})\subset\mathrm{Spec}(\mathbb{T})$. Let $\widetilde{\mathbb{I}}$ be the normalization of $\mathbb{I}$, and write $Q(\mathbb{I})$ for the quotient field of $\mathbb{I}$. Then $\mathbb{J}_F := \check{J}_\infty(F)^*_{\mathrm{ord},\widetilde{\mathbb{I}}}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ is a $\widetilde{\mathbb{I}}[\frac{1}{p}]$-module of finite type for $F=\kappa,k$. Note that $\widetilde{\mathbb{I}}[\frac{1}{p}]$ is a Dedekind domain. This we can decompose $\mathbb{J}_F = L_F\oplus X_F$ for a locally free $\widetilde{\mathbb{I}}[\frac{1}{p}]$-module L_F of finite constant rank and a torsion module X_F isomorphic to $\bigoplus_{\mathfrak{P}}\widetilde{\mathbb{I}}[\frac{1}{p}]/\mathfrak{P}^{e_F(\mathfrak{P})}$ for finitely many maximal ideals $\mathfrak{P}$ of $\widetilde{\mathbb{I}}[\frac{1}{p}]$. We put $\mathrm{Char}_{\mathbb{I}}(X_F)=\prod_{\mathfrak{P}}\mathfrak{P}^{e_F(\mathfrak{P})}$.

For an abelian variety A over κ, write $\overline{A(\kappa)}\subset\widehat{A}(k)$ for the p-adic closure of the image of $A(\kappa)$ in $\widehat{A}(k)$. Pick an arithmetic point $P\in\mathrm{Spec}(\mathbf{h})(\overline{\mathbb{Q}}_p)$ of weight 2. Suppose that the abelian variety A_P is realized in J_r and satisfies the condition (A). By Theorem 6.6 (2), the natural map

$$\mathbb{J}_F/P\mathbb{J}_F \to \widehat{A}_P(F)^*_{\mathrm{ord},\,\widetilde{\mathbb{I}}}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$$

is an isomorphism. Thus as long as $P\nmid \mathrm{Char}(X_k)\cdot\mathrm{Char}(X_\kappa)$, we have a surjective linear map

$$\widehat{A}_P(k)^*_{\mathrm{ord},\,\widetilde{\mathbb{I}}}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p \twoheadrightarrow \overline{A_P(\kappa)}^*_{\mathrm{ord},\,\widetilde{\mathbb{I}}}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p\cong \iota_P^*(\mathbb{J}_k/P\mathbb{J}_k) \tag{7.1}$$

$\mathbb{Q}_p$-dual to the inclusion

$$\overline{A_P(\kappa)}^{\mathrm{ord}}_{\widetilde{\mathbb{I}}}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p\subset\widehat{A}_P(k))^{\mathrm{ord}}_{\widetilde{\mathbb{I}}}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p,$$

where $\iota_P^*=\iota^*\otimes\mathrm{id}:\mathbb{J}_k\otimes_{\mathbf{h}}\mathbf{h}/P=\mathbb{J}_k/P\mathbb{J}_k\to\widehat{A}_P(\kappa)_{\mathrm{ord},\,\widetilde{\mathbb{I}}}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ induced by ι^*. Put

$$r_k(F;\mathbb{I}) := \dim_{Q(\mathbb{I})}\mathbb{J}_F\otimes_{\widetilde{\mathbb{I}}[\frac{1}{p}]}Q(\mathbb{I}) = \operatorname{rank}_{\widetilde{\mathbb{I}}[\frac{1}{p}]}\mathbb{J}_F$$

for the quotient field $Q(\mathbb{I})$ of $\widetilde{\mathbb{I}}$.

We now assume

(a) Taking $r=r(P)$ and A_r to be A_P, the condition (A) holds for A_P for almost all arithmetic points $P\in\mathrm{Spec}(\mathbb{I})$ of weight 2.

By Proposition 5.1 (P2), the condition (A) holds for "all" arithmetic points $P\in\mathrm{Spec}(\mathbb{I})$ of weight 2 if $\mathbb{T}=\mathbb{I}$ and p is unramified in $\mathbb{T}/P$ for one arithmetic point $P\in\mathrm{Spec}(\mathbb{T})(\overline{\mathbb{Q}}_p)$. Indeed, as shown in [F02, Theorem 3.1], $\mathbb{T}$ is regular under this assumption (and the regularity guarantees the validity of (A) by Proposition 5.1 (P2)).

Pick a base arithmetic point $P_0\in\mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$ of weight 2. The point P_0 gives rise to $f=f_{P_0}\in S_2(\Gamma_1(Np^{r+1}))$ with $B_0=B_{P_0}$ and $A_0=A_{P_0}$ satisfying $f|T(n)=P_0(T(n))f$ for all $n>0$. By Theorem 6.6 (2), we have for $F=k,\kappa$,

(ct) $\mathbb{J}_F/P_0\mathbb{J}_F$ is isomorphic to the $\mathbb{Q}_p$-dual of $\widehat{A}_0(F)^{\mathrm{ord}}_{\widetilde{\mathbb{I}}}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$.

Choosing P_0 outside $\mathrm{Char}_{\mathbb{I}}(X_\kappa)\cdot\mathrm{Char}_{\mathbb{I}}(X_k)$, we may assume the following condition for $F = k, \kappa$:

(dim) $\dim_{\mathbb{Q}_p(f)} \mathbb{J}_F/P_0\mathbb{J}_F = r_k(F;\mathbb{I})$.

Here $\mathbb{Q}_p(f)$ is the quotient field of $\mathbb{I}/P_0\mathbb{I}$ and is generated by $P_0(T(n))$ for all n over $\mathbb{Q}_p$.

Since $A_0(\kappa)\otimes_{\mathbb{Z}}\mathbb{Q}$ is a $\mathbb{Q}(f)$ vector space, if $A_0(\kappa)\otimes\mathbb{Q}\neq 0$, we have $\dim_{\mathbb{Q}(f)} A_0(\kappa)\otimes\mathbb{Q} > 0$, which implies that $\overline{A_0(\kappa)}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p\neq 0$. Suppose

$$k = \mathbb{Q}_p \text{ and } (\widehat{A}_0(\kappa)^{\mathrm{ord}}_{\widetilde{\mathbb{I}}}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)\neq 0.$$

Then $(\overline{A_0(\kappa)}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)^{\mathrm{ord}}_{\widetilde{\mathbb{I}}}$ is a finite dimensional vector subspace over $\mathbb{Q}_p$ of $\widehat{A}_0(k)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ stable under $T(n)$ for all n. Let us identify $P_0(T(n))\in\overline{\mathbb{Q}}_p$ with a system of eigenvalues of $T(n)$ occurring on $\widehat{A}_0(\kappa)^{\mathrm{ord}}_{\widetilde{\mathbb{I}}}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$. Then $(\overline{A_0(\kappa)}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)^{\mathrm{ord}}_{\widetilde{\mathbb{I}}}$ and $\widehat{A}_0(k)^{\mathrm{ord}}_{\widetilde{\mathbb{I}}}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ are $\mathbb{Q}_p(f)$-vector spaces. Thus we conclude

$$0<\dim_{\mathbb{Q}_p(f)}(\overline{A_0(\kappa)}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)^{\mathrm{ord}}_{\widetilde{\mathbb{I}}}\leq\dim_{\mathbb{Q}_p(f)}(\widehat{A}_0(k)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)^{\mathrm{ord}}_{\widetilde{\mathbb{I}}}=1,$$

which implies

$$0<\dim_{\mathbb{Q}_p(f)}(\overline{A_0(\kappa)}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)^{\mathrm{ord}}_{\widetilde{\mathbb{I}}}=\dim_{\mathbb{Q}_p(f)}\widehat{A}_0(k)^{\mathrm{ord}}_{\widetilde{\mathbb{I}}}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p=1.$$

By (7.1), we get

$$\begin{aligned}&\dim_{\mathbb{Q}_p}\iota^*_{P_0}(\mathbb{J}_\kappa/P_0\mathbb{J}_\kappa)\\&\qquad=\dim_{\mathbb{Q}_p}(\overline{A_0(\kappa)}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)=\dim_{\mathbb{Q}_p}(\widehat{A}_0(\mathbb{Q}_p)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)=\dim_{\mathbb{Q}_p}\mathbb{Q}_p(f).\end{aligned}$$

In other words, by Theorem 6.6 (2), the kernel of the map ι^*: $K := \mathrm{Ker}(\iota^* : \mathbb{J}_k\to\mathbb{J}_\kappa)$ for $k=\mathbb{Q}_p$ is a torsion $\widetilde{\mathbb{I}}[\frac{1}{p}]$-module. Now we move weight 2 arithmetic points $P\in\mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)\subset\mathrm{Spec}(\mathbb{T})(\overline{\mathbb{Q}}_p)$. Then $K_P = K/PK$ covers surjectively

$$\mathrm{Ker}(\iota^*_P : \mathbb{J}_k/P\mathbb{J}_k\to\mathbb{J}_\kappa/P\mathbb{J}_\kappa).$$

By $\widetilde{\mathbb{I}}[\frac{1}{p}]$-torsion property of K, $K/PK = 0$ for almost all points in $\mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$, and we get

COROLLARY 7.2. *Let the notation and the assumption be as above. Suppose the condition* (a)*,* (dim)*, $k=\kappa_{\mathfrak{p}}=\mathbb{Q}_p$ and*

$$\dim_{\mathbb{Q}(f)} A_0(\kappa) > 0.$$

Then except for finitely many arithmetic points of $\mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$ *weight* 2*, we have* $\dim_{\mathbb{Q}(f_P)} A_P(\kappa) > 0$ *and*

$$\dim_{\mathbb{Q}_p(f_P)}(\overline{A_P(\kappa)}^{\mathrm{ord}}\otimes_{\mathbb{I}/P}\mathbb{Q}_p(f_P))=\dim_{\mathbb{Q}_p}\mathbb{Q}_p(f_P).$$

For general abelian variety $A_{/\mathbb{Q}}$, an estimate of $\dim_{\mathbb{Q}_p}\overline{A(\mathbb{Q})}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ relative to $\dim_{\mathbb{Q}} A(\mathbb{Q})\otimes_{\mathbb{Z}}\mathbb{Q}$ and a conjecture is given in [W11]. Here we studied the dimension over a family and showed its co-ordinary (or ordinary) part stays maximal for most of members of the family if one is maximal.

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Motivic Equivalence and Similarity of Quadratic Forms

To Sasha Merkurjev on the occasion of his 60th birthday

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Abstract. A result by Vishik states that given two anisotropic quadratic forms of the same dimension over a field of characteristic not 2, the Chow motives of the two associated projective quadrics are isomorphic iff both forms have the same Witt indices over all field extensions, in which case the two forms are called motivically equivalent. Izhboldin has shown that if the dimension is odd, then motivic equivalence implies similarity of the forms. This also holds for even dimension ≤ 6, but Izhboldin also showed that this generally fails in all even dimensions ≥ 8 except possibly in dimension 12. The aim of this paper is to show that motivic equivalence does imply similarity for fields over which quadratic forms can be classified by their classical invariants provided that in the case of formally real such fields the space of orderings has some nice properties. Examples show that some of the required properties for the field cannot be weakened.

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1. Introduction

Throughout this note, we will consider only fields of characteristic not 2. By a form over F we will mean a finite dimensional nondegenerate quadratic form over F, and by a quadric over F a smooth projective quadric $X_\varphi = \{\varphi = 0\}$ for some form φ over F.

An important theme in the theory of quadratic forms is the study of forms in terms of geometric properties of their associated quadrics. Suppose, for example, that for two given forms φ and ψ over F one has that the motives $M(X_\varphi)$ and $M(X_\psi)$ are isomorphic in the category of Chow motives, in which case we call φ and ψ *motivically equivalent* and we write $\varphi \overset{\text{mot}}{\sim} \psi$. Does this already imply that the quadrics are isomorphic as projective varieties? The converse is of course trivially true. It is well known that the quadrics X_φ and X_ψ are isomorphic iff φ and ψ are similar (see, e.g. [18, Th. 2.2]), i.e. there exists $c \in F^\times = F \setminus \{0\}$ with $\varphi \cong c\psi$ in which case we write $\varphi \overset{\text{sim}}{\sim} \psi$. The above question then reads as follows: Let φ and ψ be forms of the same dimension over F. Does $\varphi \overset{\text{mot}}{\sim} \psi$ imply $\varphi \overset{\text{sim}}{\sim} \psi$?

In fact, Izhboldin has shown that the answer is yes if $\dim \varphi$ is odd ([14, Cor. 2.9]) or even and at most 6 ([14, Prop. 3.1]), and that there are counterexamples in every even dimension ≥ 8 except possibly 12 over suitably chosen fields ([15, Th. 0.1]). To our knowledge, it seems to be still open if such counterexamples exist in dimension 12.

The purpose of the present note is to give criteria for fields that guarantee that motivic equivalence implies similarity in all dimensions. We show that it holds for fields over which forms of a given dimension can be classified by their classical invariants determinant, Clifford invariant and signatures provided that in the case of formally real fields the space of orderings satisfies a certain property called *effective diagonalization* ED (which will be defined below). We show furthermore that there are counterexamples once the condition ED is only slightly weakened.

Rather than working with motives of quadrics, we will use an alternative criterion for motivic equivalence due to Vishik [24, Th. 1.4.1] (see also Vishik [25, Th. 4.18] or Karpenko [16, § 5]). If we denote the Witt index of a form φ by $i_W(\varphi)$, this important criterion reads as follows.

Vishik's Criterion 1.1. *Let φ and ψ be forms over F with $\dim \varphi = \dim \psi$. Then $\varphi \overset{\text{mot}}{\sim} \psi$ if and only if $i_W(\varphi_E) = i_W(\psi_E)$ for every field extension E/F.*

Let us remark that while Vishik formulated his criterion in terms of integral Chow motives, it still holds for Chow motives with $\mathbb{Z}/2\mathbb{Z}$ coefficients, see [8].

The proofs of our results will concern mainly formally real fields (in the sequel we will call such fields *real* for short). For nonreal fields, the results are still valid but can often be shown in a much quicker and simpler fashion. The real case will involve various arguments concerning the space of orderings X_F of a real field and the signatures $\mathrm{sgn}_P(\varphi)$ of a form φ over F with respect to an ordering $P \in X_F$.

Consider the Witt ring WF and the torsion ideal W_tF (we have $WF = W_tF$ iff F in nonreal). By Pfister's local-global principle (see, e.g., [20, Ch. VIII, Th. 3.2]), a form φ is torsion iff $\mathrm{sgn}_P(\varphi) = 0$ for all $P \in X_F$. We call a form totally indefinite if $|\,\mathrm{sgn}_P(\varphi)| < \dim\varphi$ for all $P \in X_F$. Also, we will use the fact that the Witt ring only contains 2-primary torsion.

Let IF be the fundamental ideal in WF generated by even-dimensional forms in F and let $I^nF = (IF)^n$. We define $I_t^nF = I^nF \cap W_tF$. A real field F is said to satisfy *effective diagonalization* (ED) if any form φ over F has a diagonalization $\langle a_1, \ldots, a_n\rangle$ such that for all $1 \le i < n$ and for all $P \in X_F$ one has $a_i <_P 0 \Longrightarrow a_{i+1} <_P 0$ (see [26] or [23]). Recall that the u-invariant and the Hasse number $\tilde{u}$ are defined as follows:

$$\begin{aligned} u(F) &= \sup\{\dim\varphi \,|\, \varphi \text{ is anisotropic and } \varphi \in W_tF\} \\ \tilde{u}(F) &= \sup\{\dim\varphi \,|\, \varphi \text{ is anisotropic and totally indefinite}\} \end{aligned}$$

For nonreal F, we thus have $u(F) = \tilde{u}(F)$. It is also well known that these invariants cannot take the values $3, 5, 7$ (see [5, Ths. F–G] for the more involved case $\tilde{u}$ for real fields).

Our main result reads as follows.

MAIN THEOREM 1.2. *Let F be an ED-field and let φ, ψ be anisotropic forms over F of the same dimension. If $\varphi \overset{\text{mot}}{\sim} \psi$ then there exists $x \in F^\times$ such that $\varphi \perp -x\psi \in I_t^3F$.*

COROLLARY 1.3. *Let F be an ED-field with $I_t^3F = 0$ and let φ, ψ be anisotropic forms over F of the same dimension. Then $\varphi \overset{\text{mot}}{\sim} \psi$ if and only if $\varphi \overset{\text{sim}}{\sim} \psi$.*

Recall that fields with $I_t^3F = 0$ are exactly those fields over which quadratic forms can be classified by their classical invariants dimension, (signed) determinant, Clifford invariant and signatures, see [4].

Now fields with finite $\tilde{u}$ are always ED (see, e.g., [7, Th. 2.5]). By the Arason-Pfister Hauptsatz (see, e.g., [20, Ch. X, 5.1]) we thus get

COROLLARY 1.4. *Let F be a field with $\tilde{u}(F) \le 6$ and let φ, ψ be anisotropic forms over F of the same dimension. Then $\varphi \overset{\text{mot}}{\sim} \psi$ if and only if $\varphi \overset{\text{sim}}{\sim} \psi$.*

This corollary applies to global fields for which $\tilde{u} = 4$ (this follows from the well known Hasse-Minkowski theorem) and fields of transcendence degree one over a real closed field for which $\tilde{u} = 2$ (see, e.g., [5, Th. I]). However, for each $k \in \{2n \,|\, n \in \mathbb{N}\} \cup \{\infty\}$ there exist ED-fields F (in fact, fields F with a unique ordering) with $\tilde{u}(F) = k$ and $I_t^3F = 0$ (see [13, Th. 2.7] or [11, Th. 3.1]) to which Corollary 1.3 can still be applied.

In § 2, we investigate how determinants and Clifford invariants behave under motivic equivalence. The third section does the same for signatures and there we also prove the main theorem by putting all this together. In § 4, we give a few examples that show that under weakening some of the imposed conditions, one cannot expect any longer that motivic equivalence implies similarity.

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2. Comparing determinants and Clifford invariants

We will freely use without reference various basic facts from the algebraic theory of quadratic forms in characteristic $\neq 2$. All such facts and any unexplained terminology can be found in the books [20] or [3]. If φ is a form defined on an F-vector space V, we put $D_F(\varphi) = \{\varphi(x) \mid x \in V\} \cap F^\times$. We use the convention $\langle\langle a_1, \ldots, a_n \rangle\rangle$ to denote the n-fold Pfister form $\langle 1, -a_1\rangle \otimes \ldots \otimes \langle 1, -a_n\rangle$. A form φ over a field F is called a Pfister neighbor if there exists a Pfister form π over F and some $a \in F^\times$ such that $a\varphi$ is a subform of π (i.e. there exists another form ψ over F with $a\varphi \perp \psi \cong \pi$) and $2 \dim \varphi > \dim \pi$. Since such a Pfister form π is known to be either anisotropic or hyperbolic, it follows that a Pfister neighbor φ of π is anisotropic iff π is anisotropic. We call two forms φ and ψ over F half-neighbors if there exist an integer $n \geq 0$, $a, b \in F^\times$ and an $(n+1)$-fold Pfister form π such that $\dim \varphi = \dim \psi = 2^n$ and $a\varphi \perp -b\psi \cong \pi$. Now in this situation, if E is any field extension of F over which φ or ψ is isotropic then π_E is hyperbolic and thus $a\varphi_E \cong b\psi_E$ and it readily follows that $\varphi \overset{\text{mot}}{\sim} \psi$. Thus, a good way to construct examples of nonsimilar motivically equivalent forms is to find nonsimilar half-neighbors, see § 4. The function field $F(\varphi)$ of a form φ is defined to be the function field of the associated quadric $F(X_\varphi)$ (we put $F(\varphi) = F$ if $\dim \varphi = 1$ or φ a hyperbolic plane).

In the sequel, we state some definitions and facts concerning generic splitting of quadratic forms. We refer to Knebusch's original paper [17] on that topic for details.

Let φ be a form over F. The generic splitting tower of φ is constructed inductively as follows. Let $F = F_0$ and $\varphi_0 = \varphi_{\text{an}}$ be its anisotropic part over F. Suppose that for $i \geq 0$ we have constructed the field extension F_i/F. Consider the anisotropic form $\varphi_i \cong (\varphi_{F_i})_{\text{an}}$. If $\dim \varphi_i \geq 2$ we put $F_{i+1} = F_i(\varphi_i)$ and $\varphi_{i+1} \cong (\varphi_{F_{i+1}})_{\text{an}}$. Note that if $\dim \varphi_i \geq 2$, we have $2i_W(\varphi_{F_i}) = \dim \varphi - \dim \varphi_i < 2i_W(\varphi_{F_{i+1}})$ or, equivalently, $\dim \varphi_i > \dim \varphi_{i+1}$. The smallest h such that $\dim \varphi_h \leq 1$ is called the height of φ. The generic splitting tower of φ is then given by

$$F = F_0 \subset F_1 \subset \ldots \subset F_{h-1} \subset F_h \ .$$

F_{h-1} is called the leading field of φ. It is known that

$$\mathfrak{S}_a(\varphi) := \{i_W(\varphi_E) \mid E/F \text{ field extension}\} = \{i_W(\varphi_{F_i}) \mid 0 \leq i \leq h\} \ .$$

We call $\mathfrak{S}_a(\varphi)$ the absolute splitting pattern of φ. In the literature, it has often proved to be of advantage to consider instead the relative splitting pattern $\mathfrak{S}_r(\varphi)$ defined as follows. If $\mathfrak{S}_a(\varphi) = \{i_\ell = i_W(\varphi_{F_\ell}) \mid 0 \leq \ell \leq h\}$, then put $j_m = i_m - i_{m-1}$, $1 \leq m \leq h$, the increase of the Witt index at the m-th step in the splitting tower. Then $\mathfrak{S}_r(\varphi) = (j_1, \ldots, j_h)$ as an ordered sequence, but we won't need this here.

The degree $\deg(\varphi)$ is defined as follows. If the dimension of φ is odd, then $\deg(\varphi) = 0$. If φ is hyperbolic one defines $\deg(\varphi) = \infty$. So suppose φ is not hyperbolic and $\dim\varphi$ is even. Then the anisotropic form φ_{h-1} over F_{h-1} becomes hyperbolic over its own function field $F_h = F_{h-1}(\varphi_{h-1})$ and is thus similar to an n-fold Pfister form for some $n \geq 1$. We then define $\deg(\varphi) = n$. Now the above implies that if φ is not hyperbolic then

$$2^{\deg(\varphi)} = \min\{\dim(\varphi_E)_{\mathrm{an}} \mid E/F \text{ is a field extension with } \varphi_E \text{ not hyperbolic}\}\ ,$$

and it follows that if $\dim(\varphi_E)_{\mathrm{an}} = 2^{\deg(\varphi)}$, then $(\varphi_E)_{\mathrm{an}}$ is similar to an n-fold Pfister form over E. An important and deep theorem which we will also use states that $I^nF = \{\varphi \in WF \mid \deg\varphi \geq n\}$, see [22, Th. 4.3].
While part (i) of the following lemma is rather trivial, part (ii) is a bit less so and seems to be due to Izhboldin (see [16, Remark 2.7]) but to our knowledge a proof was not yet in the literature, so we included one for the reader's convenience.

Lemma 2.1. *Let φ and ψ be anisotropic forms over F with $\varphi \overset{\mathrm{mot}}{\sim} \psi$. Then*

(i) $\deg(\varphi) = \deg(\psi)$*;*
(ii) *For every $a \in F^\times$ we have* $\deg(\varphi \perp -a\psi) > \deg(\varphi)$.

Proof. Part (i) follows immediately from the definition of degree and Vishik's criterion for motivic equivalence.
Let now $\deg(\varphi) = \deg(\psi) = n$. Part (ii) is trivial for $n = 0$, so assume $n \geq 1$. If $\varphi \perp -a\psi$ is hyperbolic there is nothing to show. So assume $\tau \cong (\varphi \perp -a\psi)_{\mathrm{an}} \neq 0$. By the degree characterization of I^nF, we have $\tau \in I^nF$ and hence $\deg(\tau) \geq n$. Suppose $\deg(\tau) = n$. Let E/F be the leading field of φ. By what was said preceding the lemma, $(\varphi_E)_{\mathrm{an}}$ and $(\psi_E)_{\mathrm{an}}$ are anisotropic n-fold Pfister forms which are clearly motivically equivalent and thus similar (this follows readily from, e.g., [20, Ch. X, Cor. 4.9]). Hence, there exist an n-fold Pfister form π over E and $x, y \in E^\times$ such that in WE, $\varphi_E = x\pi$, $\psi_E = y\pi$. Thus, $\tau_E = \langle x, -ay\rangle \otimes \pi \in I^{n+1}E$ and therefore $\deg(\tau) = n < n+1 \leq \deg(\tau_E)$. But this implies $\deg(\varphi) \leq n - 2$ by [1, Satz 19], a contradiction. □

The signed determinant of a form φ over F will be denoted by $d(\varphi)$. For a diagonalization $\varphi \cong \langle a_1, \ldots, a_n\rangle$ we have $d(\varphi) = (-1)^{n(n-1)/2}\prod_{i=1}^n a_i \in F^\times/F^{\times 2}$ and the map $\varphi \mapsto d(\varphi)$ induces an isomorphism $IF/I^2F \to F^\times/F^{\times 2}$. The Clifford invariant $c(\varphi)$ of φ is defined as follows. The Clifford algebra $C(\varphi)$ is a central simple algebra over F if $\dim\varphi$ is even, and its even part $C_0(\varphi)$ is central simple if $\dim\varphi$ is odd. In both cases, these algebras are Brauer-equivalent to a tensor product of quaternion algebras and thus their classes lie in the 2-torsion part $\mathrm{Br}_2(F)$ of the Brauer group of F. One defines

$$c(\varphi) = \begin{cases} [C(\varphi)] \in \mathrm{Br}_2(F) & \text{if } \dim\varphi \text{ even} \\ [C_0(\varphi)] \in \mathrm{Br}_2(F) & \text{if } \dim\varphi \text{ odd} \end{cases}$$

By Merkurjev's theorem [21], c induces an isomorphism $I^2F/I^3F \to \mathrm{Br}_2(F)$.

Corollary 2.2. *Let* φ *and* ψ *be forms over* F *of even dimension* $\dim\varphi = \dim\psi$. *Let* $d = d(\varphi) \in F^\times/F^{\times 2}$ *and* $K = F$ *if* $d = 1$ *and* $K = F(\sqrt{d})$ *if* $d \neq 1$. *If* $\varphi \overset{\text{mot}}{\sim} \psi$ *then* $d = d(\varphi) = d(\psi)$ *and* $c(\varphi_K) = c(\psi_K)$.

Proof. We have $\varphi, \psi \in IF$ and also $\varphi \perp -\psi \in I^2F$ and thus $\varphi \equiv \psi \bmod I^2F$ since $\varphi \overset{\text{mot}}{\sim} \psi$ and by Lemma 2.1. The above isomorphism $IF/I^2F \cong F^\times/F^{\times 2}$ immediately implies $d(\varphi) = d(\psi)$.
Now over K we then have $\varphi_K, \psi_K \in I^2K$ since $d(\varphi_K) = d(\psi_K) = 1$. This time, Lemma 2.1 yields $\varphi_K \equiv \psi_K \bmod I^3K$ and by invoking Merkurjev's theorem we readily get $c(\varphi_K) = c(\psi_K)$. □

Corollary 2.3. *Let* φ *and* ψ *be forms over* F *of even dimension* $\dim\varphi = \dim\psi$. *Let* $d = d(\varphi) \in F^\times/F^{\times 2}$ *and suppose that* $\varphi \overset{\text{mot}}{\sim} \psi$.

(i) *There exists* $a \in F^\times$ *such that* $\varphi \perp -\psi \equiv \langle\langle a, d\rangle\rangle \bmod I^3F$.
(ii) *With* a *as in (i), if* $b \in F^\times$, *then* $\varphi \perp -b\psi \equiv \langle\langle ab, d\rangle\rangle \bmod I^3F$.

In particular, with a *as before, we have* $\varphi \perp -a\psi \in I^3F$.

Proof. (i) If $d = 1$ then Corollary 2.2 together with Merkurjev's theorem implies $\varphi, \psi \in I^2F$ and $\varphi \perp -\psi \equiv 0 \bmod I^3F$. The result follows since $\langle\langle a, d\rangle\rangle = \langle\langle a, 1\rangle\rangle = 0$ in WF for any $a \in F^\times$.
If $d \neq 1$, we still have $\varphi \perp -\psi \in I^2F$ since $d(\psi) = d$ and this time for $K = F(\sqrt{d})$ that $(\varphi \perp -\psi)_K \in I^3K$. Hence, the central simple F-algebra $C(\varphi \perp -\psi)$ splits over the quadratic extension K, so its index is at most 2 and it is well known that then there exists a quaternion algebra $(a, d)_F$ for some $a \in F^\times$ such that $C(\varphi \perp -\psi) \sim (a, d)_F$ in $\mathrm{Br}_2(F)$. Hence, it follows again readily from Merkurjev's theorem and the fact that $c(\langle\langle a, d\rangle\rangle) = [(a, d)_F]$ that we have $\varphi \perp -\psi \equiv \langle\langle a, d\rangle\rangle \bmod I^3F$.
(ii) We have $\varphi \perp -\psi, \psi \perp -b\psi \in I^2F$ and $-\psi \perp \psi = 0 \in WF$. Furthermore, by denoting the class of a quaternion algebra by its own symbol and using well known rules for manipulating Clifford invariants (see, e.g., [20, p. 118]), we get

$$\begin{aligned} c(\varphi \perp -b\psi) &= c(\varphi \perp -\psi \perp \psi \perp -b\psi) \\ &= c(\varphi \perp -\psi)c(\psi \perp -b\psi) \\ &= (a, d)_F c(\psi)c(-db\psi) \\ &= (a, d)_F c(\psi)c(\psi)(-db, d)_F \\ &= (ab, d)_F\ . \end{aligned}$$

We conclude as in (i) that now $\varphi \perp -b\psi \equiv \langle\langle ab, d\rangle\rangle \bmod I^3F$. □

3. Comparing signatures and proof of the Main Theorem

The following lemma compares signatures of motivically equivalent forms.

Lemma 3.1. *Let* φ *and* ψ *be forms of the same dimension over a real field* F. *If* $\varphi \overset{\text{mot}}{\sim} \psi$ *then* $|\operatorname{sgn}_P(\varphi)| = |\operatorname{sgn}_P(\psi)|$ *for all* $P \in X_F$.

Proof. We first note that if γ is any form of dimension ≥ 2 over any real field K and if $Q \in X_K$, then for $L = K(\gamma)$ we have that Q extends to an

ordering $Q' \in X_L$ iff γ is indefinite at Q, i.e. $\dim\gamma > |\operatorname{sgn}_Q(\gamma)|$ (see, e.g. [6, Th. 3.5]). In this case, we clearly have $\operatorname{sgn}_Q(\gamma) = \operatorname{sgn}_{Q'}(\gamma_L)$ which implies $\dim(\gamma_L)_{\mathrm{an}} \geq |\operatorname{sgn}_{Q'}(\gamma_L)| = |\operatorname{sgn}_Q(\gamma)|$.
Applied to φ, ψ and $P \in X_F$, it now follows readily that there exists an extension E/F with E in the generic splitting tower of φ such that P extends to $P' \in X_E$ and

$$\dim(\varphi_E)_{\mathrm{an}} = |\operatorname{sgn}_{P'}(\varphi_E)| = |\operatorname{sgn}_P(\varphi)| \ .$$

By motivic equivalence, we have $\dim(\varphi_E)_{\mathrm{an}} = \dim(\psi_E)_{\mathrm{an}}$ and hence

$$|\operatorname{sgn}_P(\varphi)| = \dim(\psi_E)_{\mathrm{an}} \geq |\operatorname{sgn}_{P'}\psi_E| = |\operatorname{sgn}_P\psi| \ .$$

By symmetry, we also have $|\operatorname{sgn}_P\psi| \geq |\operatorname{sgn}_P(\varphi)|$. □

Remark 3.2. The above proof also shows that $\frac{1}{2}(\dim\varphi - |\operatorname{sgn}_P(\varphi)|) \in \mathfrak{S}_a(\varphi)$, a fact that was already noticed in [9, Prop. 2.2].

We need a few properties regarding spaces of orderings of real fields. For more details regarding the following, we refer to [19], [7], [23]. Recall that the space of orderings X_F is a topological space whose topology has as sub-basis the so-called Harrison sets $H(a) = \{P \in X_F \mid a >_P 0\}$ for $a \in F^\times$. These are clopen sets, and F has the strong approximation property SAP if each clopen set is a Harrison set. F has the property S_1 if every binary torsion form represents a totally positive element. SAP and S_1 together are equivalent to ED, see [23, Th. 2].

Lemma 3.3. *Let F be a real SAP field and let φ and ψ be forms over F of the same dimension with $\varphi \overset{\mathrm{mot}}{\sim} \psi$. Then there exist $a, b \in F^\times$ such that $\operatorname{sgn}_P(a\varphi) = \operatorname{sgn}_P(b\psi) \geq 0$ for all $P \in X_F$.*

Proof. Let $U = \{P \in X_F \mid \operatorname{sgn}_P(\varphi) < 0\}$. Then $U \subset X_F$ is clopen and SAP implies that there exists $a \in F^\times$ with $U = H(-a)$. Then $\operatorname{sgn}_P(a\varphi) \geq 0$ for all $P \in X_F$. Similarly, there exists $b \in F^\times$ with $\operatorname{sgn}_P(b\psi) \geq 0$ for all $P \in X_F$. Since $a\varphi \overset{\mathrm{mot}}{\sim} \varphi \overset{\mathrm{mot}}{\sim} \psi \overset{\mathrm{mot}}{\sim} b\psi$, we have $\operatorname{sgn}_P(a\varphi) = \operatorname{sgn}_P(b\psi)$ for all $P \in X_F$ by Lemma 3.1. □

Let $\sum^\times F^2$ denote the set of nonzero sums of squares in F. If F is nonreal, then it is well known that $F^\times = \sum^\times F^2$.

Lemma 3.4. *Let F be a real S_1 field and let φ and ψ be forms over F of the same dimension with $\varphi \overset{\mathrm{mot}}{\sim} \psi$ and $\operatorname{sgn}_P(\varphi) = \operatorname{sgn}_P(\psi)$ for all $P \in X_F$. Then there exists $s \in \sum^\times F^2$ with $\varphi \perp -s\psi \in I_t^3 F$.*

Proof. Note first that the signatures don't change by scaling with an $s \in \sum^\times F^2$. Hence $\varphi \perp -s\psi$ has total signature zero for any such s and thus $\varphi \perp -s\psi \in W_t F$.
On the other hand, by Corollary 2.3, there exists $a \in F^\times$ with $\varphi \perp -\psi \equiv \langle\langle a, d\rangle\rangle \bmod I^3 F$ where $d = d(\varphi) = d(\psi) \in F^\times/F^{\times 2}$. Now if $P \in X_F$ and if π is an n-fold Pfister form over F, then $\operatorname{sgn}_P(\pi) \in \{0, 2^n\}$, hence, for $\tau \in I^n F$

we have $\mathrm{sgn}_P(\tau) \equiv 0 \bmod 2^n$. Now comparing signatures mod 8 immediately yields that $\langle\langle a, d\rangle\rangle \cong \langle 1, -a, -d, ad\rangle$ has total signature zero and is therefore torsion.

Consider the n-fold Pfister form $\sigma_n \cong 2^n \times \langle 1\rangle$. For n large enough, the $(n+2)$-fold Pfister form $\sigma_n \otimes \langle 1, -a, -d, ad\rangle$ will now be hyperbolic, so its Pfister neighbor $\sigma_n \otimes \langle 1, -d\rangle \perp \langle -a\rangle$ will be isotropic. It follows readily that there exist $u, v \in D_F(\sigma_n) \subseteq \sum^\times F^2$ with $\langle u, -a, -dv\rangle$ isotropic, so in particular, $au \in D_F(\langle 1, -duv\rangle)$. Since $uv \in \sum^\times F^2$, we can apply the characterization of S_1 in [12, Lemma 2.2(iii)] to find $t \in \sum^\times F^2$ such that $aut \in D_F(\langle 1, -d\rangle)$. But then $s := ut \in \sum^\times F^2$ and $\langle 1, -as, -d\rangle$ is isotropic. Therefore the Pfister form $\langle\langle as, d\rangle\rangle$ is hyperbolic, i.e. $\langle\langle as, d\rangle\rangle = 0$ in WF.

By the above and Corollary 2.3, we now have $\varphi \perp -s\psi \in W_tF \cap I^3F = I_t^3F$ as desired. □

Proof of Main Theorem 1.2. Let F be an ED-field and let φ, ψ be anisotropic forms over F of the same dimension n with $\varphi \overset{\text{mot}}{\sim} \psi$. We have to show that there exists $x \in F^\times$ such that $\varphi \perp -x\psi \in I_t^3F$.

The theorem is trivial for odd n by Izhboldin's result because it implies $\varphi \overset{\text{sim}}{\sim} \psi$. So we may assume that n is even.

If F is nonreal (in which case $I_t^3F = I^3F$ and ED is an empty condition), the result follows already from Corollary 2.3 with $x = b = a$.

So suppose that F is real. Now ED is equivalent to SAP plus S_1. Because of SAP, we may assume by Lemma 3.3 that, possibly after scaling, $\mathrm{sgn}_P(\varphi) = \mathrm{sgn}_P(\psi)$ for all $P \in X_F$. Since we also have S_1, we can apply Lemma 3.4 to conclude. □

4. EXAMPLES

The following two examples show that in Corollary 1.3 the condition $I_t^3F = 0$ does not suffice for motivic equivalence to imply similarity once the condition ED is only slightly weakened.

Example 4.1. Let $F = \mathbb{R}((x))((y))$ be the iterated power series field in two variables x, y over the reals. It is well known that $S = \{\pm 1, \pm x, \pm y, \pm xy\}$ is a set of representatives of $F^\times/F^{\times 2}$. Let $\tau_n \cong n \times \langle 1\rangle$ (where we allow the 0-dimensional form τ_0). Then Springer's theorem implies that up to isometry the anisotropic forms over F are exactly the forms of type

$$\epsilon_1\tau_{n_1} \perp \epsilon_2 x\tau_{n_2} \perp \epsilon_3 y\tau_{n_3} \perp \epsilon_4 xy\tau_{n_4}$$

with $\epsilon_i \in \{\pm 1\}$ and $n_i \geq 0$, and that the isometry type is uniquely determined by the four pairs (ϵ_i, n_i) (see, e.g., [20, Ch. VI, Cor. 1.6, Prop 1.9]).

Since $u(\mathbb{R}) = 0$, it also follows from the above that $u(F) = 0$, in particular $W_tF = I_t^3F = 0$. Now consider the anisotropic forms

$$\varphi \cong \langle 1, 1, 1, x, x, x, y, y\rangle \quad \text{and} \quad \psi \cong \langle 1, x, y, y, xy, xy, xy, xy\rangle\ .$$

We have $\varphi \perp \psi \cong \langle\langle -1,-1,-x,-y\rangle\rangle$, so φ and ψ are half-neighbors and thus $\varphi \overset{\text{mot}}{\sim} \psi$. However, one also readily sees that there is no $s \in S$ with $s\varphi \cong \psi$, hence $\varphi \overset{\text{sim}}{\not\sim} \psi$.

Of course, it is also well known that F lacks the property SAP and thus ED as, for example, the totally indefinite form $\langle 1, x, y, -xy\rangle$ is not weakly isotropic.

We can be more precise. Recall that the reduced stability index $\mathrm{st}(F)$ of a field F can be characterized as the least n such that $I^{n+1}F = 2I^nF \bmod W_tF$, and that SAP is equivalent to $\mathrm{st}(F) \le 1$ (see [2]).

For $F = \mathbb{R}((x))((y))$, we trivially have property S_1 since $W_tF = 0$, and one also readily sees that $\mathrm{st}(F) = 2$.

Now Corollary 1.3 applies to fields with $I_t^3F = 0$, S_1 and $\mathrm{st}(F) \le 1$, but the above shows that generally, it cannot be extended to fields satisfying $I_t^3F = 0$, S_1 and $\mathrm{st}(F) = 2$. □

In [7], the property S_1 has been generalized as follows. A field F is said to have property S_n for $n \ge 1$ if for every n-fold Pfister form $\pi \cong \langle 1\rangle \perp \pi'$ over F and every $a \in \sum^\times F^2$ there exists an $m \ge 1$ with

$$D_F(\langle 1, -a\rangle) \cap D_F(\underbrace{\langle 1, \dots, 1\rangle}_{m} \otimes \pi') \neq \emptyset\ .$$

Example 4.2. It is not difficult to construct real fields K with $|K^\times/K^{\times 2}| = 4$ and where the square classes are represented by $\{\pm 1, \pm 2\}$ (see, e.g., [20, Remark II.5.3]). Clearly, K is uniquely ordered and $u(K) = \tilde{u}(K) = 2$. Consider $F = K((t))$. Then $u(F) = 4$, so in particular $I_t^3F = 0$, F has two orderings (see, e.g., [20, Prop. VIII.4.11]) and thus is SAP. Furthermore, one readily checks that F has property S_2.

Now consider the anisotropic forms

$$\varphi \cong \langle 1,1,1,1,1,1\rangle \perp t\langle 1,2\rangle \quad \text{and} \quad \psi \cong \langle 1,1\rangle \perp t\langle 1,1,1,1,1,2\rangle\ .$$

Since $\langle 1,1\rangle \cong \langle 2,2\rangle$ we have $\varphi \perp \psi \cong \langle\langle -1,-1,-1,-t\rangle\rangle$. So φ and ψ are half-neighbors and hence $\varphi \overset{\text{mot}}{\sim} \psi$. On the other hand, since $2 \notin F^{\times 2}$, it follows readily that $\varphi \overset{\text{sim}}{\not\sim} \psi$.

Hence, in general, Corollary 1.3 cannot be extended to fields satisfying $I_t^3F = 0$, S_2 and SAP (i.e. $\mathrm{st}(F) \le 1$). □

Note that the two forms in the previous example also provide motivically equivalent nonsimilar forms over $\mathbb{Q}((t))$, a field that also satisfies S_2 and SAP. However, this would give a weaker counterexample in the sense that $I_t^4\mathbb{Q}((t)) = 0$ but $I_t^3\mathbb{Q}((t)) \neq 0$ as can be readily seen.

Example 4.3. If F is nonreal and $u(F) < 2^{n+1}$, then $(n+1)$-fold Pfister forms will always be hyperbolic over F and thus half-neighbors of dimension 2^n will always be similar. However, in [10, Cor. 3.6], it was shown that for any $n \ge 3$ there exist nonreal fields F with $u(F) = 2^{n+1}$ over which one can find nonsimilar half-neighbors of dimension 2^n. In fact, one can take any nonreal field E with $u(E) = 4$ and take $F = E((x_1))\dots((x_{n-1}))$. As a consequence, there exist nonreal fields F with $u(F) = 16$ and motivically equivalent nonsimilar forms of dimension 8. □

It should be noted that to our knowledge, all constructions of nonsimilar motivically equivalent forms over nonreal fields (e.g. in [15]) require the existence of anisotropic 4-fold Pfister forms, so for these fields one would have $I^4F \neq 0$ and in particular $u(F) \geq 16$. Thus, also in view of the above examples, we ask the following.

Question 4.4. Are there fields F with $u(F) < 16$ which in the real case also satisfy ED, such that there exist nonsimilar motivically equivalent forms over F?

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Birational Geometry and Localisation of Categories

With Appendices by Jean-Louis Colliot-Thélène and Ofer Gabber

To Alexander Merkurjev, with warmest wishes for his 60th birthday

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Abstract. We explore connections between places of function fields over a base field F and birational morphisms between smooth F-varieties. This is done by considering various categories of fractions involving function fields or varieties as objects, and constructing functors between these categories. The main result is that in the localised category $S_b^{-1}\mathbf{Sm}(F)$, where $\mathbf{Sm}(F)$ denotes the usual category of smooth varieties over F and S_b is the set of birational morphisms, the set of morphisms between two objects X and Y, with Y proper, is the set of R-equivalence classes $Y(F(X))/R$.

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Contents

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Introduction

Let Φ be a functor from the category of smooth proper varieties over a field F to the category of sets. We say that Φ is *birational* if it transforms birational morphisms into isomorphisms. In characteristic 0, examples of such functors are obtained by choosing a function field K/F and defining $\Phi_K(X) = X(K)/R$, the set of R-equivalence classes of K-rational points [5, Prop. 10]. One of the main results of this paper is that *any birational functor Φ is canonically a direct limit of functors of the form Φ_K.*

This follows from Theorem 1 below via the complement to Yoneda's lemma ([SGA4, Exp. I, Prop. 3.4 p. 19] or [28, Ch. III, §, Th. 1 p. 76]). Here is the philosophy which led to this result and others presented here:

Birational geometry over a field F is the study of function fields over F, viewed as generic points of algebraic varieties[2], or alternately the study of algebraic F-varieties "up to proper closed subsets". In this context, two ideas seem related:

- places between function fields;
- rational maps.

The main motivation of this paper has been to understand the precise relationship between them. We have done this by defining two rather different "birational categories" and comparing them.

The first idea gives the category **place** (objects: function fields; morphisms: F-places), that we like to call the *coarse birational category*. For the second idea, one has to be a little careful: the naïve attempt at taking as objects smooth varieties and as morphisms rational maps does not work because, as was pointed out to us by Hélène Esnault, one cannot compose rational maps in general. On the other hand, one can certainly start from the category **Sm** of smooth F-varieties and *localise* it (in the sense of Gabriel-Zisman [12]) with respect to the set S_b of birational morphisms. We like to call the resulting category $S_b^{-1}\mathbf{Sm}$ the *fine birational category*. By hindsight, the problem mentioned by Esnault can be understood as a problem of calculus of fractions of S_b in **Sm**.

In spite of the lack of calculus of fractions, the category $S_b^{-1}\mathbf{Sm}$ was studied in [21] and we were able to show that, under resolution of singularities, the natural functor $S_b^{-1}\mathbf{Sm}^{\text{prop}} \to S_b^{-1}\mathbf{Sm}$ is an equivalence of categories, where $\mathbf{Sm}^{\text{prop}}$ denotes the full subcategory of smooth proper varieties (*loc. cit.* , Prop. 8.5).

What was not done in [21] was the computation of Hom sets in $S_b^{-1}\mathbf{Sm}$. This is the first main result of this paper:

Theorem 1 (*cf.* Th. 6.6.3 and Cor. 6.6.4). *Let X, Y be two smooth F-varieties, with Y proper. Then,*

a) In $S_b^{-1}\mathbf{Sm}$, we have an isomorphism

$$\operatorname{Hom}(X, Y) \simeq Y(F(X))/R$$

[2] By convention all varieties are irreducible here, although not necessarily geometrically irreducible.

where the right hand side is the set of R-equivalence classes in the sense of Manin.
b) The natural functor

$$S_b^{-1}\mathbf{Sm}_*^{\mathrm{prop}} \to S_b^{-1}\mathbf{Sm}$$

is fully faithful. Here $\mathbf{Sm}_*^{\mathrm{prop}}$ *is the full subcategory of* $\mathbf{Sm}$ *with objects those smooth proper varieties whose function field has a cofinal set of smooth proper models (see Definition 4.2.1).*

For the link with the result mentioned at the beginning of the introduction, note that $\mathbf{Sm}_*^{\mathrm{prop}} = \mathbf{Sm}^{\mathrm{prop}}$ in characteristic 0, and any birational functor on smooth proper varieties factors uniquely through $S_b^{-1}\mathbf{Sm}^{\mathrm{prop}}$, by the universal property of the latter category.
Theorem 1 implies that $X \mapsto X(F)/R$ is a birational invariant of smooth proper varieties in any characteristic (Cor. 6.6.6), a fact which seemed to be known previously only in characteristic 0 [5, Prop. 10]. It also implies that one can define a composition law on classes of R-equivalence (for smooth proper varieties), a fact which is not at all obvious a priori.
The second main result is a comparison between the coarse and fine birational categories. Let **dv** be the subcategory of **place** whose objects are separably generated function fields and morphisms are generated by field extensions and places associated to “good” discrete valuation rings (Definition 6.1.1).

Theorem 2 (*cf.* Th. 6.5.2 and 6.7.1). *a) There is an equivalence of categories*

$$\overline{\Psi} : (\mathbf{dv} \,/\, \mathrm{h}')^{\mathrm{op}} \xrightarrow{\sim} S_b^{-1}\mathbf{Sm}$$

where $\mathbf{dv} \,/\, \mathrm{h}'$ *is the quotient category of* **dv** *by the equivalence relation generated by two elementary relations: homotopy of places (definition 6.4.1) and “having a common centre of codimension* 2 *on some smooth model”.*
b) If $\operatorname{char} F = 0$*, the natural functor* $\mathbf{dv} \,/ h' \to \mathbf{place} \,/\, \mathrm{h}''$ *is an equivalence of categories, where* h'' *is generated by homotopy of places and “having a common centre on some smooth model”.*

(See §1.2 for the notion of an equivalence relation on a category.)
Put together, Theorems 1 and 2 provide an answer to a question of Merkurjev: given a smooth proper variety X/F, give a purely birational description of the set $X(F)/R$. This answer is rather clumsy because the equivalence relation h' is not easy to handle; we hope to come back to this issue later.
Let us introduce the set S_r of *stable birational morphisms*: by definition, a morphism $s : X \to Y$ is in S_r if it is dominant and the function field extension $F(X)/F(Y)$ is purely transcendental. We wondered about the nature of the localisation functor $S_b^{-1}\mathbf{Sm} \to S_r^{-1}\mathbf{Sm}$ for a long time, until the answer was given us by Colliot-Thélène through a wonderfully simple geometric argument (see Appendix A):

Theorem 3 (*cf.* Th. 1.7.2). *The functor* $S_b^{-1}\mathbf{Sm} \to S_r^{-1}\mathbf{Sm}$ *is an equivalence of categories.*

This shows a striking difference between birational functors and numerical birational invariants, many of which are not stably birationally invariant (for example, plurigenera).
Theorems 1 and 2 are substantial improvements of our results in the first version of this paper [22], which were proven only in characteristic 0: even in characteristic 0, Theorem 2 is new with respect to [22]. Their proofs are intertwined in a way we shall describe now.
The first point is to relate the coarse and fine birational categories, as there is no obvious comparison functor between them. There are two essentially different approaches to this question. In the first one:

- We introduce (Definition 2.2.1) an "incidence category" $\mathbf{SmP}$, whose objects are smooth F-varieties and morphisms from X to Y are given by pairs (f, λ), where f is a morphism $X \to Y$, λ is a place $F(Y) \rightsquigarrow F(X)$ and f, λ are *compatible* in an obvious sense. This category maps to both $\mathbf{place}^{\mathrm{op}}$ and $\mathbf{Sm}$ by obvious forgetful functors. Replacing $\mathbf{Sm}$ by $\mathbf{SmP}$ turns out to have a strong rigidifying effect.
- We embed $\mathbf{place}^{\mathrm{op}}$ in the category of locally ringed spaces via the "Riemann-Zariski" variety attached to a function field.

In this way, we obtain a naturally commutative diagram

$$\begin{array}{ccccc} & & S_b^{-1}\mathbf{Sm}_*^{\mathrm{prop}}\mathbf{P} & & \\ & \swarrow^{\overline{\Phi}_1^*} & & \searrow^{\overline{\Phi}_2^*} & \\ \mathbf{place}_*^{\mathrm{op}} & & & & S_b^{-1}\mathbf{Sm}_*^{\mathrm{prop}} \\ & \searrow^{\bar{\Sigma}} & & \swarrow^{\bar{J}} & \\ & & S_b^{-1}\widehat{\mathbf{Sm}}_*^{\mathrm{prop}} & & \end{array}$$

where $\mathbf{place}_*$ denotes the full subcategory of $\mathbf{place}$ consisting of the function fields of varieties in $\mathbf{Sm}_*^{\mathrm{prop}}$ (compare Theorem 1). Then J is an *equivalence of categories*[3] and the induced functor

$$(*) \qquad \Psi_* : \mathbf{place}_*^{\mathrm{op}} \to S_b^{-1}\mathbf{Sm}_*^{\mathrm{prop}}$$

is *full and essentially surjective* (Theorems 4.2.3 and 4.2.4).
This is more or less where we were in the first version of this paper [22], except for the use of the categories $\mathbf{Sm}_*$ and $\mathbf{place}_*$ which allow us to state results in any characteristic; in [22], we also proved Theorem 1 when $\operatorname{char} F = 0$, using resolution of singularities and a complicated categorical method.[4]
The second approach is to construct a functor $\mathbf{dv}^{\mathrm{op}} \to S_b^{-1}\mathbf{Sm}$ directly. Here the new and decisive input is the recent paper of Asok and Morel [1], and especially the results of its §6: they got the insight that, working with discrete

[3]So is $\overline{\Phi}_1^*$.

[4]Another way to prove Theorem 1 in characteristic 0, which was our initial method, is to define a composition law on R-equivalence classes by brute force (still using resolution of singularities) and to proceed as in the proof of Proposition 6.4.3.

valuations of rank 1, all the resolution that is needed is "in codimension 2". We implement their method in §6 of the present paper, which leads to a rather simple proof of Theorems 1 and 2 in any characteristic. Another key input is a recent uniformisation theorem of Knaf and Kuhlmann [23].

Let us now describe the contents in more detail. We start by setting up notation in Section 1, which ends with Theorem 3. In Section 2, we introduce the incidence category **SmP** sitting in the larger category **VarP**, the forgetful functors **VarP** $\to$ **Var** and **VarP** $\to$ $\mathbf{place}^{\mathrm{op}}$, and prove elementary results on these functors (see Lemmas 2.3.2 and 2.3.4). In Section 3, we endow the abstract Riemann variety with the structure of a locally ringed space, and prove that it is a cofiltered inverse limit of proper models, viewed as schemes (Theorem 3.2.8): this ought to be well-known but we couldn't find a reference. We apply these results to construct in §4 the functor (*), using calculus of fractions. In section 5, we study calculus of fractions in greater generality; in particular, we obtain a partial calculus of fractions in $S_b^{-1}\mathbf{Sm}_*$ in Proposition 5.4.1.

In §6, we introduce a notion of homotopy on **place** and the subcategory **dv**. We then relate our approach to the work of Asok-Morel [1] to prove Theorems 1 and 2. We make the link between the first and second approaches in Theorem 6.7.1 = Theorem 2 b).

Section 7 discusses variants of Kollár's notion of rational chain connectedness (which goes back to Chow under the name of linear connectedness), recalls classical theorems of Murre, Chow and van der Waerden, states new theorems of Gabber including the one proven in Appendix B, and draws some consequences in Theorem 7.3.1. Section 8 discusses some applications, among which we like to mention the existence of a "universal birational quotient" of the fundamental group of a smooth variety admitting a smooth compactfication (§8.4). We finish with a few open questions in §8.8.

This paper grew out of the preprint [20], where some of its results were initially proven. We decided that the best was to separate the present results, which have little to do with motives, from the rest of that work. Let us end with a word on the relationship between $S_b^{-1}\mathbf{Sm}$ and the $\mathbf{A}^1$-homotopy category of schemes $\mathcal{H}$ of Morel-Voevodsky [32]. One of the main results of Asok and Morel in [1] is a proof of the following conjecture of Morel in the proper case (*loc. cit.* Th. 2.4.3):

Conjecture 1 ([31, p. 386]). *If X is a smooth variety, the natural map*

$$X(F) \to \mathrm{Hom}_{\mathcal{H}}(\mathrm{Spec}\, F, X)$$

is surjective and identifies the right hand side with the quotient of the set $X(F)$ by the equivalence relation generated by

$$(x \sim y) \iff \exists h : \mathbf{A}^1 \to X \mid h(0) = x \text{ and } h(1) = y.$$

(Note that this "$\mathbf{A}^1$-equivalence" coincides with R-equivalence if X is proper.) Their result can then be enriched as follows:

Theorem 4 ([4]). *The Yoneda embedding of* **Sm** *into the category of simplicial presheaves of sets on* **Sm** *induces a fully faithful functor*

$$S_b^{-1}\mathbf{Sm} \longrightarrow S_b^{-1}\mathcal{H}$$

where $S_b^{-1}\mathcal{H}$ is a suitable localisation of $\mathcal{H}$ with respect to birational morphisms.

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Conventions. F is the base field. "Variety" means irreducible separated F-scheme of finite type. All morphisms are F-morphisms. If X is a variety, η_X denotes its generic point.

1. Preliminaries and notation

In this section, we collect some basic material that will be used in the paper. This allows us to fix our notation.

1.1. Localisation of categories and calculus of fractions. We refer to Gabriel-Zisman [12, Chapter I] for the necessary background. Recall [12, I.1] that if $\mathcal{C}$ is a small category and S is a collection of morphisms in $\mathcal{C}$, there is a category $\mathcal{C}[S^{-1}]$ and a functor $\mathcal{C} \to \mathcal{C}[S^{-1}]$ which is universal among functors from $\mathcal{C}$ which invert the elements of S. When S satisfies *calculus of fractions* [12, I.2] the category $\mathcal{C}[S^{-1}]$ is equivalent to another one, denoted $S^{-1}\mathcal{C}$ by Gabriel and Zisman, in which the Hom sets are more explicit.
If $\mathcal{C}$ is only essentially small, one can construct a category verifying the same 2-universal property by starting from an equivalent small category, provided S contains the identities. All categories considered in this paper are subcategories of $\mathbf{Var}(F)$ (varieties over our base field F) or $\mathbf{place}(F)$ (finitely generated extensions of F, morphisms given by places), hence are essentially small.
We shall encounter situations where calculus of fractions is satisfied, as well as others where it is not. We shall take the practice to abuse notation and write $S^{-1}\mathcal{C}$ rather than $\mathcal{C}[S^{-1}]$ even when calculus of fractions is not verified.

1.1.1. Notation. If $(\mathcal{C}, S)$ is as above, we write $\langle S\rangle$ for the *saturation* of S: it is the set of morphisms s in $\mathcal{C}$ which become invertible in $S^{-1}\mathcal{C}$. We have $S^{-1}\mathcal{C} = \langle S\rangle^{-1}\mathcal{C}$ and $\langle S\rangle$ is maximal for this property.

Note the following easy lemma:

1.1.2. Lemma. *Let $T : \mathcal{C} \to \mathcal{D}$ be a full and essentially surjective functor. Let $S \in Ar(\mathcal{C})$ be a set of morphisms. Then the induced functor $\bar{T} : S^{-1}\mathcal{C} \to T(S)^{-1}\mathcal{D}$ is full and essentially surjective.*

Proof. Essential surjectivity is obvious. Given two objects $X, Y \in S^{-1}\mathcal{C}$, a morphism from $\bar{T}(X)$ to $\bar{T}(Y)$ is given by a zig-zag of morphisms of $\mathcal{D}$. By the essential surjectivity of T, lift all vertices of this zig-zag, then lift its edges thanks to the fullness of T. □

1.2. Equivalence relations.

1.2.1. Definition. Let $\mathcal{C}$ be a category. An *equivalence relation* on $\mathcal{C}$ consists, for all $X, Y \in \mathcal{C}$, of an equivalence relation $\sim_{X,Y} = \sim$ on $\mathcal{C}(X, Y)$ such that $f \sim g \Rightarrow fh \sim gh$ and $kf \sim kg$ whenever it makes sense.

In [28, p. 52], the above notion is called a 'congruence'. Given an equivalence relation $\sim$ on $\mathcal{C}$, we may form the *factor category* $\mathcal{C}/\sim$, with the same objects as $\mathcal{C}$ and such that $(\mathcal{C}/\sim)(X,Y) = \mathcal{C}(X,Y)/\sim$. This category and the projection functor $\mathcal{C} \to \mathcal{C}/\sim$ are universal for functors from $\mathcal{C}$ which equalise equivalent morphisms.

1.2.2. *Example.* Let $\mathcal{A}$ be an Ab-category (sets of morphisms are abelian groups and composition is bilinear). An *ideal* $\mathcal{I}$ in $\mathcal{A}$ is given by a subgroup $\mathcal{I}(X,Y) \subseteq \mathcal{A}(X,Y)$ for all $X, Y \in \mathcal{A}$ such that $\mathcal{I}\mathcal{A} \subseteq \mathcal{I}$ and $\mathcal{A}\mathcal{I} \subseteq \mathcal{I}$. Then the ideal $\mathcal{I}$ defines an equivalence relation on $\mathcal{A}$, compatible with the additive structure.

Let $\sim$ be an equivalence relation on the category $\mathcal{C}$. We have the collection $S_\sim = \{f \in \mathcal{C} \mid f \text{ is invertible in } \mathcal{C}/\sim\}$. The functor $\mathcal{C} \to \mathcal{C}/\sim$ factors into a functor $S_\sim^{-1}\mathcal{C} \to \mathcal{C}/\sim$. Conversely, let $S \subset \mathcal{C}$ be a set of morphisms. We have the equivalence relation $\sim_S$ on $\mathcal{C}$ such that $f \sim_S g$ if $f = g$ in $S^{-1}\mathcal{C}$, and the localisation functor $\mathcal{C} \to S^{-1}\mathcal{C}$ factors into $\mathcal{C}/\sim_S \to S^{-1}\mathcal{C}$. Neither of these two factorisations is an equivalence of categories in general; however, [15, Prop. 1.3.3] remarks that if $f \sim g$ implies $f = g$ in $S_\sim^{-1}\mathcal{C}$, then $S_\sim^{-1}\mathcal{C} \to \mathcal{C}/\sim$ is an isomorphism of categories.

1.2.3. *Exercise.* Let A be a commutative ring and $I \subseteq A$ an ideal.
a) Assume that the set of minimal primes of A that do not contain I is finite (*e.g.* that A is noetherian). Show that the following two conditions are equivalent:

- (i) There exists a multiplicative subset S of A such that $A/I \simeq S^{-1}A$ (compatibly with the maps $A \to A/I$ and $A \to S^{-1}A$).
- (ii) I is generated by an idempotent.

(Hint: show first that, without any hypothesis, (i) is equivalent to

- (iii) For any $a \in I$, there exists $b \in I$ such that $ab = a$.)

b) Give a counterexample to (i) ⇒ (ii) in the general case (hint: take $A = k^{\mathbf{N}}$, where k is a field).

1.3. Places, valuations and centres [40, Ch. VI], [2, Ch. 6]. Recall [2, Ch. 6, §2, Def. 3] that a *place* from a field K to a field L is a map $\lambda : K \cup \{\infty\} \to L \cup \{\infty\}$ such that $\lambda(1) = 1$ and λ preserves sum and product whenever they are defined. We shall usually denote places by screwdriver arrows:

$$\lambda : K \rightsquigarrow L.$$

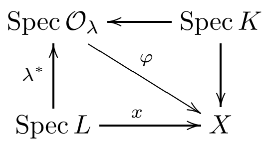
Then $\mathcal{O}_\lambda = \lambda^{-1}(L)$ is a valuation ring of K and $\lambda_{|\mathcal{O}_\lambda}$ factors as

$$\mathcal{O}_\lambda \twoheadrightarrow \kappa(\lambda) \hookrightarrow L$$

where $\kappa(\lambda)$ is the residue field of $\mathcal{O}_\lambda$. Conversely, the data of a valuation ring $\mathcal{O}$ of K with residue field κ and of a field homomorphism $\kappa \to L$ uniquely defines a place from K to L (*loc. cit.* , Prop. 2). It is easily checked that the composition of two places is a place.

1.3.1. *Caution.* Unlike Zariski-Samuel [40] and other authors [39, 23], we compose places in the same order as extensions of fields: so if $K \overset{\lambda}{\rightsquigarrow} L \overset{\mu}{\rightsquigarrow} M$ are two successive places, their composite is written $\mu\lambda$ in this paper. We hope this will not create confusion.

If K and L are extensions of F, we say that λ is an *F-place* if $\lambda_{|F} = Id$ and then write $F(\lambda)$ rather than $\kappa(\lambda)$.
In this situation, let X be an integral F-scheme of finite type with function field K. A point $x \in X$ is a *centre* of a valuation ring $\mathcal{O} \subset K$ if $\mathcal{O}$ dominates the local ring $\mathcal{O}_{X,x}$. If $\mathcal{O}$ has a centre on X, we sometimes say that $\mathcal{O}$ is *finite* on X. As a special case of the valuative criterion of separatedness (*resp.* of the valuative criterion of properness), x is unique (*resp.* and exists) for all $\mathcal{O}$ if and only if X is separated (*resp.* proper) [16, Ch. 2, Th. 4.3 and 4.7].
On the other hand, if $\lambda : K \rightsquigarrow L$ is an F-place, then a point $x \in X(L)$ is a *centre* of λ if there is a map $\varphi : \operatorname{Spec} \mathcal{O}_\lambda \to X$ letting the diagram



commute. Note that the image of the closed point by φ is then a centre of the valuation ring $\mathcal{O}_\lambda$ and that φ uniquely determines x.
In this paper, when X is separated we shall denote by $c_X(v) \in X$ the centre of a valuation v and by $c_X(\lambda) \in X(L)$ the centre of a place λ, and carefully distinguish between the two notions (one being a scheme-theoretic point and the other a rational point).
We have the following useful lemma from Vaquié [39, Prop. 2.4]; we reproduce its proof.

1.3.2. LEMMA. *Let $X \in$ **Var**, $K = F(X)$, v a valuation on K with residue field κ and $\bar v$ a valuation on κ. Let $v' = \bar v \circ v$ denote the composite valuation.*
a) If v' is finite on X, so is v.
b) Assume that v is finite on X, and let $Z \subset X$ be the closure of its centre (so that $F(Z) \subseteq \kappa$). Then v' is finite on X if and only if [the restriction to $F(Z)$ of] $\bar v$ is finite on Z, and then $c(\bar v) \in Z$ equals $c(v') \in X$.

Proof. We may assume that $X = \operatorname{Spec} A$ is an affine variety. Denoting respectively by $V, V', \bar{V}$ and $\mathfrak{m}, \mathfrak{m}', \bar{\mathfrak{m}}$ the valuation rings associated to $v, v', \bar{v}$ and their maximal ideals, we have $(0) \subset \mathfrak{m} \subset \mathfrak{m}' \subset V' \subset V \subset K$ and $\bar{\mathfrak{m}} \subset \bar{V} = V'/\mathfrak{m} \subset \bar{K} = V/\mathfrak{m}$.
a) v' is finite on X if and only if $A \subset V'$, which implies $A \subset V$.
b) The centres of the valuations v and v' on X are defined by the prime ideals $\mathfrak{p} = A \cap \mathfrak{m}$ and $\mathfrak{p}' = A \cap \mathfrak{m}'$ of A, and the centre of the valuation $\bar{v}$ on $Z = \operatorname{Spec} \bar{A}$, with $\bar{A} = A/\mathfrak{p}$ is defined by the prime ideal $\bar{\mathfrak{p}} = \bar{A} \cap \bar{\mathfrak{m}}$ of $\bar{A}$. Then the claim is a consequence of the equality $\bar{\mathfrak{p}} = \mathfrak{p}'/\mathfrak{p}$. □

1.4. Rational maps. Let X, Y be two F-schemes of finite type, with X integral and Y separated. Recall that a *rational map* from X to Y is a pair (U, f) where U is a dense open subset of X and $f : U \to Y$ is a morphism. Two rational maps (U, f) and (U', f') are equivalent if there exists a dense open subset U'' contained in U and U' such that $f_{|U''} = f'_{|U''}$. We denote by $\mathbf{Rat}(X, Y)$ the set of equivalence classes of rational maps, so that

$$\mathbf{Rat}(X, Y) = \varinjlim \operatorname{Map}_F(U, Y)$$

where the limit is taken over the open dense subsets of X. There is a largest open subset U of X on which a given rational map $f : X \dashrightarrow Y$ is defined [16, Ch. I, Ex. 4.2]. The (reduced) closed complement $X - U$ is called the *fundamental set* of f (notation: $\operatorname{Fund}(f)$). We say that f is *dominant* if $f(U)$ is dense in Y.
Similarly, let $f : X \to Y$ be a birational morphism. The complement of the largest open subset of X on which f is an isomorphism is called the *exceptional locus* of f and is denoted by $\operatorname{Exc}(f)$.
Note that the sets $\mathbf{Rat}(X, Y)$ only define a *precategory* (or diagram, or diagram scheme, or quiver) $\mathbf{Rat}(F)$, because rational maps cannot be composed in general. To clarify this, let $f : X \dashrightarrow Y$ and $g : Y \dashrightarrow Z$ be two rational maps, where X, Y, Z are varieties. We say that f and g are *composable* if $f(\eta_X) \notin \operatorname{Fund}(g)$, where η_X is the generic point of X. Then there exists an open subset $U \subseteq X$ such that f is defined on U and $f(U) \cap \operatorname{Fund}(g) = \emptyset$, and $g \circ f$ makes sense as a rational map. This happens in two important cases:

- f is dominant;
- g is a morphism.

This composition law is associative wherever it makes sense. In particular, we do have the category $\mathbf{Rat}_{\mathrm{dom}}(F)$ with objects F-varieties and morphisms dominant rational maps. Similarly, the category $\mathbf{Var}(F)$ of 1.7 acts on $\mathbf{Rat}(F)$ on the left.

1.4.1. Lemma ([21, Lemma 8.2]). *Let $f, g : X \to Y$ be two morphisms, with X integral and Y separated. Then $f = g$ if and only if $f(\eta_X) = g(\eta_X) =: y$ and f, g induce the same map $F(y) \to F(X)$ on the residue fields.* □

For X, Y as above, there is a well-defined map

$$\mathbf{Rat}(X,Y) \to Y(F(X)) \tag{1.1}$$
$$(U,f) \mapsto f_{|\eta_X}$$

where η_X is the generic point of X.

1.4.2. Lemma. *The map (1.1) is bijective.*

Proof. Surjectivity is clear, and injectivity follows from Lemma 1.4.1. □

1.5. The graph trick. We shall often use this well-known and basic device, which allows us to replace a rational map by a morphism.
Let U, Y be two F-varieties. Let $j : U \to X$ be an open immersion (X a variety) and $g : U \to Y$ a morphism. Consider the graph $\Gamma_g \subset U \times Y$. By the first projection, $\Gamma_g \xrightarrow{\sim} U$. Let $\bar{\Gamma}_g$ be the closure of Γ_g in $X \times Y$, viewed as a reduced scheme. Then the rational map $g : X \dashrightarrow Y$ has been replaced by $g' : \bar{\Gamma}_g \to Y$ (second projection) through the birational map $p : \bar{\Gamma}_g \to X$ (first projection). Clearly, if Y is proper then p is proper.

1.6. Structure theorems on varieties. Here we collect two well-known results, for future reference.

1.6.1. Theorem (Nagata [34]). *Any variety X can be embedded into a proper variety $\bar{X}$. We shall sometimes call $\bar{X}$ a* compactification *of X.*

1.6.2. Theorem (Hironaka [17]). *If* $\operatorname{char} F = 0$,
a) For any variety X there exists a projective birational morphism $f : \tilde{X} \to X$ with $\tilde{X}$ smooth. (Such a morphism is sometimes called a modification.) Moreover, f may be chosen such that it is an isomorphism away from the inverse image of the singular locus of X. In particular, any smooth variety X may be embedded as an open subset of a smooth proper variety (projective if X is quasi-projective).
b) For any proper birational morphism $p : Y \to X$ between smooth varieties, there exists a proper birational morphism $\tilde{p} : \tilde{Y} \to X$ which factors through p and is a composition of blow-ups with smooth centres.

In some places we shall assume characteristic 0 in order to use resolution of singularities. We shall specify this by putting an asterisk to the statement of the corresponding result (so, the asterisk will mean that the characteristic 0 assumption is due to the use of Theorem 1.6.2).

1.7. Some multiplicative systems. Let $\mathbf{Var}(F) = \mathbf{Var}$ be the category of *F-varieties*: objects are F-varieties (*i.e.* integral separated F-schemes of finite type) and morphisms are all F-morphisms. We write $\mathbf{Sm}(F) = \mathbf{Sm}$ for its full subcategory consisting of smooth varieties. As in [21], the superscripts ${}^{\mathrm{qp}}, {}^{\mathrm{prop}}, {}^{\mathrm{proj}}$ respectively mean quasi-projective, proper and projective.
As in [21], we shall use various collections of morphisms of $\mathbf{Var}$ that are to be inverted:

- *Birational morphisms* S_b: $s \in S_b$ if s is dominant and induces an isomorphism of function fields.
- *Stably birational morphisms* S_r: $s \in S_r$ if s is dominant and induces a purely transcendental extension of function fields.

In addition, we shall use the following subsets of S_b:

- S_o: open immersions
- S_b^p: proper birational morphisms

and of S_r:

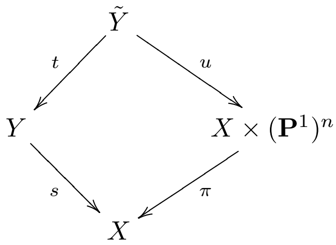
- S_r^p: proper stably birational morphisms
- S_h: the projections $pr_2 : X \times \mathbf{P}^1 \to X$.

We shall need the following lemma:

1.7.1. Lemma. *a) In* **Var** *and* **Sm**, *we have* $\langle S_b\rangle = \langle S_o\rangle$ *and* $\langle S_r\rangle = \langle S_b \cup S_h\rangle$ *(see Notation 1.1.1).*
b) We have $\langle S_r^p\rangle = \langle S_b^p \cup S_h\rangle$ *in* **Var**, **and also in* **Sm** *under resolution of singularities.*

Proof. a) The first equality is left to the reader. For the second one, given a morphism $s : Y \to X$ in S_r with $X, Y \in$ **Var** or **Sm**, it suffices to consider a commutative diagram

(1.2)



with $t, u \in S_o$, $\tilde{Y}$ a common open subset of Y and $X \times (\mathbf{P}^1)^n$.
b) For a morphism $s : Y \to X$ in S_r^p with $X, Y \in$ **Var**, we get a diagram (1.2), this time with $t, u \in S_b^p$ and $\tilde{Y}$ obtained by the graph trick. If $X, Y \in$ **Sm**, we use resolution to replace $\tilde{Y}$ by a smooth variety. □

Here is now the main result of this section.

1.7.2. Theorem. *In* **Sm**, *the sets* S_b *and* S_r *have the same saturation. *This is also true for* S_b^p *and* S_r^p *under resolution of singularities.*
In particular, the obvious functor $S_b^{-1}\mathbf{Sm} \to S_r^{-1}\mathbf{Sm}$ *is an equivalence of categories.*

Proof. Let us prove that S_h is contained in the saturation of S_b^p, hence in the saturation of S_b. Let Y be smooth variety, and let $f : Y \times \mathbf{P}^1 \to Y$ be the first projection. We have to show that f becomes invertible in $(S_b^p)^{-1}\mathbf{Sm}$. By Yoneda's lemma, it suffices to show that $F(f)$ is invertible for any (representable) functor $F : (S_b^p)^{-1}\mathbf{Sm}^{\mathrm{op}} \to \mathbf{Sets}$. This follows from taking the proof of Appendix A and "multiplying" it by Y.

To get Theorem 1.7.2, we now apply Lemma 1.7.1 a) and b). (Applying b) is where resolution of singularities is required.) □

1.7.3. *Remark.* Theorem 1.7.2 is also valid in **Var**, without resolution of singularities hypothesis (same proof). Recall however that the functor $S_b^{-1}\mathbf{Sm} \to S_b^{-1}\mathbf{Var}$ induced by the inclusion $\mathbf{Sm} \hookrightarrow \mathbf{Var}$ is far from being fully faithful [21, Rk. 8.11].

2. Places and morphisms

2.1. The category of places.

2.1.1. Definition. We denote by $\mathbf{place}(F) = \mathbf{place}$ the category with objects finitely generated extensions of F and morphisms F-places. We denote by $\mathbf{field}(F) = \mathbf{field}$ the subcategory of $\mathbf{place}(F)$ with the same objects, but in which morphisms are F-homomorphisms of fields. We shall sometimes call the latter *trivial places.*

2.1.2. *Remark.* If $\lambda : K \rightsquigarrow L$ is a morphism in **place**, then its residue field $F(\lambda)$ is finitely generated over F, as a subfield of the finitely generated field L. On the other hand, given a finitely generated extension K/F, there exist valuation rings of K/F with infinitely generated residue fields as soon as $\operatorname{trdeg}(K/F) > 1$, *cf.* [40, Ch. VI, §15, Ex. 4].

In this section, we relate the categories **place** and **Var**. We start with the main tool, which is the notion of compatibility between a place and a morphism.

2.2. A compatibility condition.

2.2.1. Definition. Let $X, Y \in \mathbf{Var}$, $f : X \dashrightarrow Y$ a rational map and $v : F(Y) \rightsquigarrow F(X)$ a place. We say that f and v are *compatible* if

- v is finite on Y (*i.e.* has a centre in Y).
- The corresponding diagram

$$\begin{array}{ccc} \eta_X & \xrightarrow{v^*} & \operatorname{Spec} \mathcal{O}_v \\ \downarrow & & \downarrow \\ U & \xrightarrow{f} & Y \end{array}$$

  commutes, where U is an open subset of X on which f is defined.

2.2.2. Proposition. *Let X, Y, v be as in Definition 2.2.1. Suppose that v is finite on Y, and let $y \in Y(F(X))$ be its centre. Then a rational map $f : X \dashrightarrow Y$ is compatible with v if and only if*

- $y = f(\eta_X)$ *and*

- *the diagram of fields*

$$\begin{array}{ccc} F(v) & & \\ \uparrow & \searrow^{v} & \\ F(y) & \xrightarrow{f^*} & F(X) \end{array}$$

commutes.

In particular, there is at most one such f.

Proof. Suppose v and f compatible. Then $y = f(\eta_X)$ because $v^*(\eta_X)$ is the closed point of $\operatorname{Spec} \mathcal{O}_v$. The commutativity of the diagram then follows from the one in Definition 2.2.1. Conversely, if f verifies the two conditions, then it is obviously compatible with v. The last assertion follows from Lemma 1.4.1. □

2.2.3. Corollary. *a) Let $Y \in$ **Var** and let $\mathcal{O}$ be a valuation ring of $F(Y)/F$ with residue field K and centre $y \in Y$. Assume that $F(y) \xrightarrow{\sim} K$. Then, for any rational map $f : X \dashrightarrow Y$ with X integral, such that $f(\eta_X) = y$, there exists a unique place $v : F(Y) \rightsquigarrow F(X)$ with valuation ring $\mathcal{O}$ which is compatible with f.*
b) If f is an immersion, the condition $F(y) \xrightarrow{\sim} K$ is also necessary for the existence of v.
c) In particular, let $f : X \dashrightarrow Y$ be a dominant rational map. Then f is compatible with the trivial place $F(Y) \hookrightarrow F(X)$, and this place is the only one with which f is compatible.

Proof. This follows immediately from Proposition 2.2.2. □

2.2.4. Proposition. *Let $f : X \to Y$, $g : Y \to Z$ be two morphisms of varieties. Let $v : F(Y) \rightsquigarrow F(X)$ and $w : F(Z) \rightsquigarrow F(Y)$ be two places. Suppose that f and v are compatible and that g and w are compatible. Then $g \circ f$ and $v \circ w$ are compatible.*

Proof. We first show that $v \circ w$ is finite on Z. By definition, the diagram

$$\begin{array}{ccc} \eta_Y & \xrightarrow{w^*} & \operatorname{Spec} \mathcal{O}_w \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathcal{O}_v & \longrightarrow & \operatorname{Spec} \mathcal{O}_{v \circ w} \end{array}$$

is cocartesian. Since the two compositions

$$\eta_Y \xrightarrow{w^*} \operatorname{Spec} \mathcal{O}_w \to Z$$

and

$$\eta_Y \to \operatorname{Spec} \mathcal{O}_v \to Y \xrightarrow{g} Z$$

coincide (by the compatibility of g and w), there is a unique induced (dominant) map $\operatorname{Spec} \mathcal{O}_{v\circ w} \to Z$. In the diagram

$$\begin{array}{ccccc}
\eta_X & \xrightarrow{v^*} & \operatorname{Spec} \mathcal{O}_v & \longrightarrow & \operatorname{Spec} \mathcal{O}_{v\circ w} \\
\downarrow & & \downarrow & & \downarrow \\
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z
\end{array}$$

the left square commutes by compatibility of f and v, and the right square commutes by construction. Therefore the big rectangle commutes, which means that $g \circ f$ and $v \circ w$ are compatible. □

2.3. The category VarP.

2.3.1. Definition. We denote by $\mathbf{VarP}(F) = \mathbf{VarP}$ the following category:

- Objects are F-varieties.
- Let $X, Y \in \mathbf{VarP}$. A morphism $\varphi \in \mathbf{VarP}(X,Y)$ is a pair (λ, f) with $f : X \to Y$ a morphism, $\lambda : F(Y) \rightsquigarrow F(X)$ a place and λ, f compatible.
- The composition of morphisms is given by Proposition 2.2.4.

If $\mathcal{C}$ is a full subcategory of **Var**, we also denote by $\mathcal{C}\mathbf{P}(F) = \mathcal{C}\mathbf{P}$ the full subcategory of **VarP** whose objects are in $\mathcal{C}$.

We now want to do an elementary study of the two forgetful functors appearing in the diagram below:

$$\begin{array}{ccc}
\mathbf{VarP} & \xrightarrow{\Phi_1} & \mathbf{place}^{\mathrm{op}} \\
{\scriptstyle \Phi_2}\downarrow & & \\
\mathbf{Var}\,. & &
\end{array} \tag{2.1}$$

Clearly, Φ_1 and Φ_2 are essentially surjective. Concerning Φ_2, we have the following partial result on its fullness:

2.3.2. Lemma. *Let $f : X \dashrightarrow Y$ be a rational map, with X integral and Y separated. Assume that $y = f(\eta_X)$ is a* regular *point (i.e. $A = \mathcal{O}_{Y,y}$ is regular). Then there is a place $v : F(Y) \rightsquigarrow F(X)$ compatible with f.*

Proof. By Corollary 2.2.3 a), it is sufficient to produce a valuation ring $\mathcal{O}$ containing A and with the same residue field as A.
The following construction is certainly classical. Let $\mathfrak{m}$ be the maximal ideal of A and let $(a_1, \dots, a_d)$ be a regular sequence generating $\mathfrak{m}$, with $d = \dim A = \operatorname{codim}_Y y$. For $0 \le i < j \le d+1$, let

$$A_{i,j} = (A/(a_j, \dots, a_d))_{\mathfrak{p}}$$

where $\mathfrak{p} = (a_{i+1}, \dots, a_{j-1})$ (for $i = 0$ we invert no a_k, and for $j = d+1$ we mod out no a_k). Then, for any (i,j), $A_{i,j}$ is a regular local ring of dimension

$j-i-1$. In particular, $F_i = A_{i,i+1}$ is the residue field of $A_{i,j}$ for any $j \geq i+1$. We have $A_{0,d+1} = A$ and there are obvious maps

$$\begin{aligned} A_{i,j} &\to A_{i+1,j} \quad \text{(injective)}\\ A_{i,j} &\to A_{i,j-1} \quad \text{(surjective)}. \end{aligned}$$

Consider the discrete valuation v_i associated to the discrete valuation ring $A_{i,i+2}$: it defines a place, still denoted by v_i, from F_{i+1} to F_i. The composition of these places is a place v from $F_d = F(Y)$ to $F_0 = F(y)$, whose valuation ring dominates A and whose residue field is clearly $F(y)$. □

2.3.3. *Remark.* In Lemma 2.3.2, the assumption that y is a regular point is necessary. Indeed, take for f a closed immersion. By [2, Ch. 6, §1, Th. 2], there exists a valuation ring $\mathcal{O}$ of $F(Y)$ which dominates $\mathcal{O}_{Y,y}$ and whose residue field κ is an algebraic extension of $F(y) = F(X)$. However we cannot choose $\mathcal{O}$ such that $\kappa = F(y)$ in general. The same counterexamples as in [21, Remark 8.11] apply (singular curves, the point $(0,0,\dots,0)$ on the affine cone $x_1^2 + x_2^2 + \cdots + x_n^2 = 0$ over $\mathbf{R}$ for $n \geq 3$).

Now concerning Φ_1, we have:

2.3.4. Lemma. *Let X, Y be two varieties and $\lambda : F(Y) \rightsquigarrow F(X)$ a place. Assume that λ is finite on Y. Then there exists a unique rational map $f : X \dashrightarrow Y$ compatible with λ.*

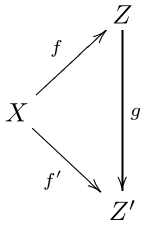
Proof. Let y be the centre of $\mathcal{O}_\lambda$ on Y and $V = \operatorname{Spec} R$ an affine neighbourhood of y, so that $R \subset \mathcal{O}_\lambda$, and let S be the image of R in $F(\lambda)$. Choose a finitely generated F-subalgebra T of $F(X)$ containing S, with quotient field $F(X)$. Then $X' = \operatorname{Spec} T$ is an affine model of $F(X)/F$. The composition $X' \to \operatorname{Spec} S \to V \to Y$ is then compatible with v. Its restriction to a common open subset U of X and X' defines the desired map f. The uniqueness of f follows from Proposition 2.2.2. □

2.3.5. *Remark.* Let Z be a third variety and $\mu : F(Z) \rightsquigarrow F(Y)$ be another place, finite on Z; let $g : Y \dashrightarrow Z$ be the rational map compatible with μ. If f and g are composable, then $g \circ f$ is compatible with $\lambda \circ \mu$: this follows easily from Proposition 2.2.4. However it may well happen that f and g are not composable. For example, assume Y smooth. Given μ, hence g (that we suppose not to be a morphism), choose $y \in \operatorname{Fund}(g)$ and find a λ with centre y, for example by the method in the proof of Lemma 2.3.2. Then the rational map f corresponding to λ has image contained in $\operatorname{Fund}(g)$.

We conclude this section with a useful lemma which shows that places rigidify the situation very much.

2.3.6. Lemma. *a) Let Z, Z' be two models of a function field L, with Z' separated, and v a valuation of L with centres z, z' respectively on Z and Z'.*

Assume that there is a birational morphism $g : Z \to Z'$. Then $g(z) = z'$.
b) Consider a diagram



with g a birational morphism. Let $K = F(X)$, $L = F(Z) = F(Z')$ and suppose given a place $v : L \rightsquigarrow K$ compatible both with f and f'. Then $f' = g \circ f$.

Proof. a) Let $f : \operatorname{Spec} \mathcal{O}_v \to Z$ be the dominant map determined by z. Then $f' = g \circ f$ is a dominant map $\operatorname{Spec} \mathcal{O}_v \to Z'$. By the valuative criterion of separatedness, it must correspond to z'. b) This follows from a) and Proposition 2.2.2. □

3. Places, valuations and the Riemann varieties

In this section, we give a second categorical relationship between the idea of places and that of algebraic varieties. This leads us to consider Zariski's "abstract Riemann surface of a field" as a locally ringed space. We start by giving the details of this theory, as we could not find it elaborated in the literature[5]. We remark however that the study of 'Riemann-Zariski spaces' has recently been revived by different authors independently (see [10], [36], [37], [39]).

3.1. Strict birational morphisms. It will be helpful to work here with the following notion of *strict birational morphisms*:

$$\underline{S}_b = \{s \in S_b \mid s \text{ induces an equality of function fields}\}$$

In fact, the difference between S_b and $\underline{S}_b$ is immaterial in view of the following

3.1.1. Lemma. *Any birational morphism of (separated) varieties is the composition of a strict birational morphism and an isomorphism.*

Proof. Let $s : X \to Y$ be a birational morphism. First assume X and Y affine, with $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. Let $K = F(X)$ and $L = F(Y)$, so that K is the quotient field of A and L is the quotient field of B. Let $s^* : L \xrightarrow{\sim} K$ be the isomorphism induced by s. Then $A' \xrightarrow{\sim} A = s^*(A)$, hence s may be factored as $X \xrightarrow{s'} X' \xrightarrow{u} Y$ with $X' = \operatorname{Spec} A'$, where s' is strict birational and u is an isomorphism. In the general case, we may patch the above construction (which is canonical) over an affine open cover (U_i) of Y and an affine open cover of X refining $(s^{-1}(U_i))$. □

[5] Except for a terse allusion in [17, 0.6, p. 146]: we thank Bernard Teissier for pointing out this reference.

3.2. The Riemann-Zariski variety as a locally ringed space.

3.2.1. Definition. We denote by $\mathcal{R}(F) = \mathcal{R}$ the full subcategory of the category of locally ringed spaces such that $(X, \mathcal{O}_X) \in \mathcal{R}$ if and only if $\mathcal{O}_X$ is a sheaf of local F-algebras.

(Here, we understand by "local ring" a commutative ring whose non-invertible elements form an ideal, but we don't require it to be Noetherian.)

3.2.2. Lemma. *Cofiltering inverse limits exist in $\mathcal{R}$. More precisely, if $(X_i, \mathcal{O}_{X_i})_{i\in I}$ is a cofiltering inverse system of objects of $\mathcal{R}$, its inverse limit is represented by $(X, \mathcal{O}_X)$ with $X = \varprojlim X_i$ and $\mathcal{O}_X = \varinjlim p_i^*\mathcal{O}_{X_i}$, where $p_i : X \to X_i$ is the natural projection.*

Sketch. Since a filtering direct limit of local rings for local homomorphisms is local, the object of the lemma belongs to $\mathcal{R}$ and we are left to show that it satisfies the universal property of inverse limits in $\mathcal{R}$. This is clear on the space level, while on the sheaf level it follows from the fact that inverse images of sheaves commute with direct limits. □

Recall from Zariski-Samuel [40, Ch. VI, §17] the *abstract Riemann surface* S_K of a function field K/F: as a set, it consists of all nontrivial valuations on K which are trivial on F. It is topologised by the following basis $\mathcal{E}$ of open sets: if R is a subring of K, finitely generated over F, $E(R) \in \mathcal{E}$ consists of all valuations v such that $\mathcal{O}_v \supseteq R$.
As has become common practice, we slightly modify this definition:

3.2.3. Definition. The *Riemann variety* Σ_K of K is the following ringed space:

- As a topological space, $\Sigma_K = S_K \cup \{\eta_K\}$ where η_K is the trivial valuation of K. (The topology is defined as for S_K.)
- The set of sections over $E(R)$ of the structural sheaf of Σ_K is the intersection $\bigcap_{v\in E(R)} \mathcal{O}_v$, *i.e.* the integral closure of R.

3.2.4. Lemma. *The stalk at $v \in \Sigma_K$ of the structure sheaf is $\mathcal{O}_v$. In particular, $\Sigma_K \in \mathcal{R}$.*

Proof. Let $x_1, \dots, x_n \in \mathcal{O}_v$. The subring $F[x_1, \dots, x_n]$ is finitely generated and contained in $\mathcal{O}_v$, thus $\mathcal{O}_v$ is the filtering direct limit of the R's such that $v \in E(R)$. □

Let R be a finitely generated F-subalgebra of K. We have a canonical morphism of locally ringed spaces $c_R : E(R) \to \operatorname{Spec} R$ defined as follows: on points we map $v \in E(R)$ to its centre $c_R(v)$ on $\operatorname{Spec} R$. On the sheaf level, the map is defined by the inclusions $\mathcal{O}_{X,c_X(v)} \subset \mathcal{O}_v$.
We now reformulate [40, p. 115 ff] in scheme-theoretic language. Let $X \in \mathbf{Var}$ be provided with a dominant morphism $\operatorname{Spec} K \to X$ such that the corresponding field homomorphism $F(X) \to K$ is an inclusion (as opposed to a monomorphism). We call such an X a *Zariski-Samuel model* of K; X is

a *model* of K if, moreover, $F(X) = K$. Note that Zariski-Samuel models of K form a cofiltering ordered set. Generalising $E(R)$, we may define $E(X) = \{v \in \Sigma_K \mid v \text{ is finite on } X\}$ for a Zariski-Samuel model of K; this is still an open subset of Σ_K, being the union of the $E(U_i)$, where (U_i) is some finite affine open cover of X. We still have a morphism of locally ringed spaces $c_X : E(X) \to X$ defined by glueing the affine ones. If X is proper, $E(X) = \Sigma_K$ by the valuative criterion of properness. Then:

3.2.5. Theorem (Zariski-Samuel). *The induced morphism of ringed spaces*

$$\Sigma_K \to \varprojlim X$$

where X runs through the proper Zariski-Samuel models of K, is an isomorphism in $\mathcal{R}$. The generic point η_K is dense in Σ_K.

Proof. Zariski and Samuel's theorem [40, th. VI.41 p. 122] says that the underlying morphism of topological spaces is a homeomorphism; thus, by Lemma 3.2.2, we only need to check that the structure sheaf of Σ_K is the direct limit of the pull-backs of those of the X. This amounts to showing that, for $v \in \Sigma_K$, $\mathcal{O}_v$ is the direct limit of the $\mathcal{O}_{X,c_X(v)}$.
We argue essentially as in [40, pp. 122–123] (or as in the proof of Lemma 3.2.4). Let $x \in \mathcal{O}_v$, and let X be the projective Zariski-Samuel model determined by $\{1, x\}$ as in *loc. cit.* , bottom p. 119, so that either $X \simeq \mathbf{P}^1_{F'}$ or $X = \operatorname{Spec} F'$ where F' is a finite extension of F contained in K. In both cases, $c = c_X(v)$ actually belongs to $\operatorname{Spec} F[x]$ and $x \in \mathcal{O}_{X,c} \subset \mathcal{O}_v$.
Finally, η_K is contained in every basic open set, therefore is dense in Σ_K. □

3.2.6. Definition. Let $\mathcal{C}$ be a full subcategory of **Var**. We denote by $\hat{\mathcal{C}}$ the full subcategory of $\mathcal{R}$ whose objects are cofiltered inverse limits of objects of $\mathcal{C}$ under morphisms of $\underline{S}_b$ (*cf.* §1.7). The natural inclusion $\mathcal{C} \subset \hat{\mathcal{C}}$ is denoted by J.

Note that, for any function field K/F, $\Sigma_K \in \widehat{\mathbf{Var}^{\rm prop}}$ by Theorem 3.2.5. Also, for any $X \in \widehat{\mathbf{Var}}$, the function field $F(X)$ is well-defined.

3.2.7. Lemma. *Let $X \in \widehat{\mathbf{Var}}$ and $K = F(X)$.*
a) For a finitely generated F-algebra $R \subset K$, the set

$$E_X(R) = \{x \in X \mid R \subset O_{X,x}\}$$

is an open subset of X. These open subsets form a basis for the topology of X.
b) The generic point $\eta_K \in X$ is dense in X, and X is quasi-compact.

Proof. a) If X is a variety, then $E_X(R)$ is open, being the set of definition of the rational map $X \dashrightarrow \operatorname{Spec} R$ induced by the inclusion $R \subset K$. In general, let $(X, \mathcal{O}_X) = \varprojlim_\alpha (X_\alpha, \mathcal{O}_{X_\alpha})$ with the X_α varieties and let $p_\alpha : X \to X_\alpha$ be the projection. Since R is finitely generated, we have

$$E_X(R) = \bigcup_\alpha p_\alpha^{-1}(E_{X_\alpha}(R))$$

which is open in X.
Let $x \in X$: using Lemma 3.2.2, we can find an α and an affine open $U \subset X_\alpha$ such that $x \in p_\alpha^{-1}(U)$. Writing $U = \operatorname{Spec} R$, we see that $x \in E_X(R)$, thus the $E_X(R)$ form a basis of the topology of X.
In b), the density follows from a) since clearly $\eta_K \in E_X(R)$ for every R. The space X is a limit of spectral spaces under spectral maps, and hence quasi-compact. Alternately, X is compact in the constructible topology as compactness is preserved under inverse limits, and hence quasi-compact in the weaker Zariski topology. □

We are grateful to M. Temkin for pointing out an error in our earlier proof of quasi-compactness and providing the proof of b) above.

3.2.8. Theorem. *Let $X = \varprojlim X_\alpha$, $Y = \varprojlim Y_\beta$ be two objects of $\widehat{\mathbf{Var}}$. Then we have a canonical isomorphism*

$$\widehat{\mathbf{Var}}(X,Y) \simeq \varprojlim_\beta \varinjlim_\alpha \mathbf{Var}(X_\alpha, Y_\beta).$$

Proof. Suppose first that Y is constant. We then have an obvious map

$$\varinjlim_\alpha \mathbf{Var}(X_\alpha, Y) \to \widehat{\mathbf{Var}}(X,Y).$$

Injectivity follows from Lemma 1.4.1. For surjectivity, let $f : X \to Y$ be a morphism. Let $y = f(\eta_K)$. Since η_K is dense in X by Lemma 3.2.7 b), $f(X) \subseteq \overline{\{y\}}$. This reduces us to the case where f is *dominant*.
Let $x \in X$ and $y = f(x)$. Pick an affine open neighbourhood $\operatorname{Spec} R$ of y in Y. Then $R \subset \mathcal{O}_{X,x}$, hence $R \subset \mathcal{O}_{X_\alpha, x_\alpha}$ for some α, where $x_\alpha = p_\alpha(x)$, $p_\alpha : X \to X_\alpha$ being the canonical projection. This shows that the rational map $f_\alpha : X_\alpha \dashrightarrow Y$ induced by restricting f to the generic point is defined at x_α for α large enough.
Let U_α be the set of definition of f_α. We have just shown that X is the increasing union of the open sets $p_\alpha^{-1}(U_\alpha)$. Since X is quasi-compact, this implies that $X = p_\alpha^{-1}(U_\alpha)$ for some α, *i.e.* that f factors through X_α for this value of α.
In general we have

$$\widehat{\mathbf{Var}}(X,Y) \xrightarrow{\sim} \varprojlim_\beta \widehat{\mathbf{Var}}(X, Y_\beta)$$

by the universal property of inverse limits, which completes the proof. □

3.2.9. *Remark.* Let $\mathrm{pro}_{\underline{S}_b} - \mathbf{Var}$ be the full subcategory of the category of pro-objects of $\mathbf{Var}$ consisting of the (X_α) in which the transition maps $X_\alpha \to X_\beta$ are strict birational morphisms. Then Theorem 3.2.8 may be reinterpreted as saying that the functor

$$\varprojlim : \mathrm{pro}_{\underline{S}_b} - \mathbf{Var} \to \widehat{\mathbf{Var}}$$

is an *equivalence of categories.*

3.3. Riemann varieties and places. We are going to study two functors

$$\mathrm{Spec} : \mathbf{field}^{\mathrm{op}} \to \widehat{\mathbf{Var}}$$
$$\Sigma : \mathbf{place}^{\mathrm{op}} \to \widehat{\mathbf{Var}}$$

and a natural transformation $\eta : \mathrm{Spec} \Rightarrow \Sigma \circ \iota$, where ι is the embedding $\mathbf{field}^{\mathrm{op}} \hookrightarrow \mathbf{place}^{\mathrm{op}}$.

The first functor is simply $K \mapsto \mathrm{Spec}\, K$. The second one maps K to the Riemann variety Σ_K. Let $\lambda : K \rightsquigarrow L$ be an F-place. We define $\lambda^* : \Sigma_L \to \Sigma_K$ as follows: if $w \in \Sigma_L$, we may consider the associated place $\tilde{w} : L \rightsquigarrow F(w)$; then $\lambda^* w$ is the valuation underlying $\tilde{w} \circ \lambda$.

Let $E(R)$ be a basic open subset of Σ_K. Then

$$(\lambda^*)^{-1}(E(R)) = \begin{cases} \emptyset & \text{if } R \not\subseteq \mathcal{O}_\lambda \\ E(\lambda(R)) & \text{if } R \subseteq \mathcal{O}_\lambda. \end{cases}$$

Moreover, if $R \subseteq \mathcal{O}_\lambda$, then λ maps $\mathcal{O}_{\lambda^* w}$ to $\mathcal{O}_w$ for any valuation $w \in (\lambda^*)^{-1} E(R)$. This shows that λ^* is continuous and defines a morphism of locally ringed spaces. We leave it to the reader to check that $(\mu \circ \lambda)^* = \lambda^* \circ \mu^*$.

Note that we have for any K a morphism of ringed spaces

$$\eta_K : \mathrm{Spec}\, K \to \Sigma_K \tag{3.1}$$

with image the trivial valuation of Σ_K (which is its generic point). This defines the natural transformation η we alluded to.

3.3.1. Proposition. *The functors* Spec *and* Σ *are fully faithful; moreover, for any* K, L, *the map*

$$\widehat{\mathbf{Var}}(\Sigma_L, \Sigma_K) \xrightarrow{\eta_L^*} \widehat{\mathbf{Var}}(\mathrm{Spec}\, L, \Sigma_K)$$

is bijective.

Proof. The case of Spec is obvious. For the rest, let $K, L \in \mathbf{place}(F)$ and consider the composition

$$\mathbf{place}(K, L) \xrightarrow{\Sigma} \widehat{\mathbf{Var}}(\Sigma_L, \Sigma_K) \xrightarrow{\eta_L^*} \widehat{\mathbf{Var}}(\mathrm{Spec}\, L, \Sigma_K).$$

It suffices to show that η_L^* is injective and $\eta_L^* \circ \Sigma$ is bijective.

Let $\psi_1, \psi_2 \in \widehat{\mathbf{Var}}(\Sigma_L, \Sigma_K)$ be such that $\eta_L^* \psi_1 = \eta_L^* \psi_2$. Pick a proper model X of K; by Theorem 3.2.8, $c_X \circ \psi_1$ and $c_X \circ \psi_2$ factor through morphisms $f_1, f_2 : Y \to X$ for some model Y of L. By Lemma 1.4.1, $f_1 = f_2$, hence $c_X \circ \psi_1 = c_X \circ \psi_2$ and finally $\psi_1 = \psi_2$ by Theorem 3.2.5. Thus η_L^* is injective.

On the other hand, let $\varphi \in \widehat{\mathbf{Var}}(\mathrm{Spec}\, L, \Sigma_K)$ and $v = \varphi(\mathrm{Spec}\, L)$: then φ induces a homomorphism $\mathcal{O}_v \to L$, hence a place $\lambda : K \rightsquigarrow L$ and clearly $\varphi = \eta_L^* \circ \Sigma(\lambda)$. This is the only place mapping to φ. This shows that the composition $\eta_L^* \circ \Sigma$ is bijective, which concludes the proof. □

4. Two equivalences of categories

In this section, we compare the localised categories $S_r^{-1}\,\mathbf{place}$ and a suitable version of $S_b^{-1}\mathbf{Sm}^{\mathrm{prop}}$ by using the techniques of the previous section. First, we prove in Theorem 4.2.3 that a suitable version of the functor Φ_1 of (2.1) becomes an equivalence of categories after we invert birational morphisms. Next, we construct a full and essentially surjective functor

$$\mathbf{place}_*^{\mathrm{op}} \to S_b^{-1}\mathbf{Sm}_*^{\mathrm{prop}}$$

in Corollary 4.2.4, where $\mathbf{Sm}_*^{\mathrm{prop}}$ is the full subcategory of $\mathbf{Sm}$ formed of smooth varieties having a cofinal system of smooth proper models, and $\mathbf{place}_* \subseteq \mathbf{place}$ is the full subcategory of their function fields.

4.1. The basic diagram.

We start from the commutative diagram of functors

(4.1)
$$\begin{array}{ccccc} & & \mathbf{VarP} & & \\ & \swarrow^{\Phi_1} & & \searrow^{\Phi_2} & \\ \mathbf{place}^{\mathrm{op}} & & & & \mathbf{Var} \\ & \searrow_{\Sigma} & & \swarrow_{J} & \\ & & \widehat{\mathbf{Var}} & & \end{array}$$

where Φ_1, Φ_2 are the two forgetful functors of (2.1). Note that Σ takes values in $\widehat{\mathbf{Var}}^{\mathrm{prop}}$, so this diagram restricts to a similar diagram where $\mathbf{Var}$ is replaced by $\mathbf{Var}^{\mathrm{prop}}$.

We can extend the birational morphisms S_b to the categories appearing in this diagram:

4.1.1. Definition (*cf.* Theorem 3.2.8). Let $X, Y \in \widehat{\mathbf{Var}}$, with $X = \varprojlim X_\alpha$, $Y = \varprojlim Y_\beta$. A morphism $s : X \to Y$ is *birational* if, for each β, the projection $X \xrightarrow{s} Y \to Y_\beta$ factors through a birational map $s_{\alpha,\beta} : X_\alpha \to Y_\beta$ for some α (this does not depend on the choice of α). We denote by $S_b \subset Ar(\widehat{\mathbf{Var}})$ the collection of these morphisms.

In $\mathbf{Var\,P}$, we write S_b for the set of morphisms of the form (u, f) where u is an isomorphism of function fields and f is a birational morphism. In $\mathbf{place}$, we take for S_b the set of isomorphisms.

4.2. Main results.

4.2.1. Definition. Let

- $\mathbf{place}_*$ be the full subcategory of $\mathbf{place}$ formed of function fields which have a cofinal system of smooth proper models.
- $\mathbf{Sm}_*^{\mathrm{prop}} \subseteq \mathbf{Sm}^{\mathrm{prop}}$ be the full subcategory of those X such that, for any $Y \in \mathbf{Var}^{\mathrm{prop}}$ birational to X, there exists $X' \in \mathbf{Sm}^{\mathrm{prop}}$ and a (proper) birational morphism $s : X' \to Y$.

Note that $\mathbf{Sm}_*^{\rm prop} = \mathbf{Sm}^{\rm prop}$ in characteristic 0 and that $X \in \mathbf{Sm}^{\rm prop} \Rightarrow X \in \mathbf{Sm}_*^{\rm prop}$ if $\dim X \leq 2$ in any characteristic. On the other hand, it is not clear whether $\mathbf{Sm}_*^{\rm prop}$ is closed under products, or even under product with $\mathbf{P}^1$.
The following lemma is clear:

4.2.2. LEMMA. *a) If $X, X' \in \mathbf{Sm}^{\rm prop}$ are birational, then $X \in \mathbf{Sm}_*^{\rm prop} \iff X' \in \mathbf{Sm}_*^{\rm prop}$.*
b) $K \in \mathbf{place}_* \iff K$ has a model in $\mathbf{Sm}_*^{\rm prop}$, and then any smooth proper model of K is in $\mathbf{Sm}_*^{\rm prop}$. □

If $X \in \mathbf{Sm}_*^{\rm prop}$, we have $F(X) \in \mathbf{place}_*$, hence with these definitions, (4.1) induces a commutative diagram of localised categories:

$$(4.2)\qquad \begin{array}{ccccc} & & S_b^{-1}\mathbf{Sm}_*^{\rm prop}\mathbf{P} & & \\ & \overline{\Phi}_1^* \swarrow & & \searrow \overline{\Phi}_2^* & \\ \mathbf{place}_*^{\rm op} & & & & S_b^{-1}\mathbf{Sm}_*^{\rm prop} \\ & \bar{\Sigma} \searrow & & \swarrow \bar{J} & \\ & & S_b^{-1}\widehat{\mathbf{Sm}_*^{\rm prop}} & & \end{array}$$

4.2.3. THEOREM. *In (4.2), $\bar{J}$ and $\overline{\Phi}_1^*$ are equivalences of categories.*

Composing $\bar{\Sigma}$ with a quasi-inverse of $\bar{J}$, we get a functor

$$(4.3)\qquad \Psi_* : \mathbf{place}_*^{\rm op} \to S_b^{-1}\mathbf{Sm}_*^{\rm prop}.$$

This functor is well-defined up to unique natural isomorphism, by the essential uniqueness of a quasi-inverse to $\bar{J}$.

4.2.4. THEOREM. *a) The functor Ψ_* is full and essentially surjective.*
b) Let $K, L \in \mathbf{place}_*$ and $\lambda, \mu \in \mathbf{place}_*(K, L)$. Suppose that λ and μ have the same centre on some model $X \in \mathbf{Sm}_*^{\rm prop}$ of K. Then $\Psi_*(\lambda) = \Psi_*(\mu)$.
c) Let $S_r \subset \mathbf{place}_*$ denote the set of field extensions $K \hookrightarrow K(t)$ such that $K \in \mathbf{place}_*$ and $K(t) \in \mathbf{place}_*$. Then the composition $\mathbf{place}_*^{\rm op} \xrightarrow{\Psi_*} S_b^{-1}\mathbf{Sm}_*^{\rm prop} \to S_b^{-1}\mathbf{Sm}$ factors through a (full) functor, still denoted by Ψ_*:

$$\Psi_* : S_r^{-1}\mathbf{place}_*^{\rm op} \to S_b^{-1}\mathbf{Sm}.$$

The proofs of Theorems 4.2.3 and 4.2.4 go in several steps, which are given in the next subsections.

4.3. PROOF OF THEOREM 4.2.3: THE CASE OF $\bar{J}$. We apply Proposition 5.10 b) of [21]. To lighten notation we drop the functor J. We have to check Conditions (b1), (b2) and (b3) of *loc. cit.* , namely:

(b1) *Given two maps $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ in $\mathbf{Sm}_*^{\rm prop}$ and a map $s : Z = \varprojlim Z_\alpha \to X$ in $S_b \subset \widehat{\mathbf{Sm}_*^{\rm prop}}$, $fs = gs \Rightarrow f = g$.* This is clear by Lemma 1.4.1, since by Theorem 3.2.8 s factors through some Z_α, with $Z_\alpha \to X$ birational.

(b2) *For any* $X = \varprojlim X_\alpha \in \widehat{\mathbf{Sm}^{\mathrm{prop}}_*}$, *there exists a birational morphism* $s : X \to X'$ *with* $X' \in \mathbf{Sm}^{\mathrm{prop}}_*$. It suffices to take $X' = X_\alpha$ for some α.

(b3) *Given a diagram*

$$\begin{array}{ccc} X_1 & & \\ {\scriptstyle s_1}\big\uparrow & & \\ X = \varprojlim X_\alpha & \xrightarrow{f} & Y \end{array}$$

with $X \in \widehat{\mathbf{Sm}^{\mathrm{prop}}_*}$, $X_1, Y \in \mathbf{Sm}^{\mathrm{prop}}_*$ *and* $s_1 \in S_b$, *there exists* $s_2 : X \to X_2$ *in* S_b, *with* $X_2 \in \mathbf{Sm}^{\mathrm{prop}}_*$, *covering both* s_1 *and* f. Again, it suffices to take $X_2 = X_\alpha$ for α large enough (use Theorem 3.2.8).

4.4. Calculus of fractions.

4.4.1. Proposition. *The category* $\mathbf{Sm}^{\mathrm{prop}}_*\mathbf{P}$ *admits a calculus of right fractions with respect to* S^p_b. *In particular, in* $(S^p_b)^{-1}\mathbf{Sm}^{\mathrm{prop}}_*\mathbf{P}$, *any morphism may be written in the form* fp^{-1} *with* $p \in S^p_b$. *The latter also holds in* $(S^p_b)^{-1}\mathbf{Sm}^{\mathrm{prop}}_*$.

Proof. Consider a diagram

$$\begin{array}{ccc} & & Y' \\ & & \big\downarrow{\scriptstyle s} \\ X & \xrightarrow{u} & Y \end{array} \tag{4.4}$$

in $\mathbf{Sm}^{\mathrm{prop}}_*\mathbf{P}$, with $s \in S^p_b$. Let $\lambda : F(Y) \rightsquigarrow F(X)$ be the place compatible with u which is implicit in the statement. By Proposition 2.2.2, λ has centre $z = u(\eta_X)$ on Y. Since s is proper, λ therefore has also a centre z' on Y'. By Lemma 2.3.6 a), $s(z') = z$. By Lemma 2.3.4, there exists a unique rational map $\varphi : X \dashrightarrow Y'$ compatible with λ, and $s \circ \varphi = u$ by Lemma 2.3.6 b). By the graph trick, we get a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{u'} & Y' \\ {\scriptstyle s'}\big\downarrow & & \big\downarrow{\scriptstyle s} \\ X & \xrightarrow{u} & Y \end{array} \tag{4.5}$$

in which $X' \subset X \times_Y Y'$ is the closure of the graph of φ, $s' \in S^p_b$ and u' is compatible with λ. Since $X \in \mathbf{Sm}^{\mathrm{prop}}_*$, we may birationally dominate X' by an $X'' \in \mathbf{Sm}^{\mathrm{prop}}_*$ by Lemma 4.2.2, hence replace X' by X'' in the diagram. Since Φ_1^* is full by Lemma 2.3.2, the same construction works in $\mathbf{Sm}^{\mathrm{prop}}_*$, hence the structure of morphisms in $(S^p_b)^{-1}\mathbf{Sm}^{\mathrm{prop}}_*\mathbf{P}$ and $(S^p_b)^{-1}\mathbf{Sm}^{\mathrm{prop}}_*$.

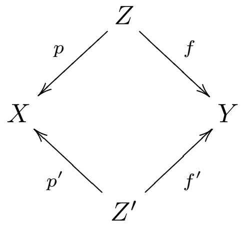
Let now

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{s} Y'$$

be a diagram in $\mathbf{Sm}^{\mathrm{prop}}_*\mathbf{P}$ with $s \in S^p_b$, such that $sf = sg$. By Corollary 2.2.3 c), the place underlying s is the identity. Hence the two places underlying f and g must be equal. But then $f = g$ by Proposition 2.2.2. □

4.4.2. PROPOSITION. *a) Consider a diagram in* $\mathbf{Sm}_*^{\mathrm{prop}}\mathbf{P}$

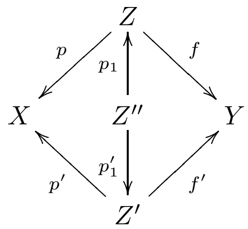
(4.6)



where $p, p' \in S_b^p$*. Let* $K = F(Z) = F(Z') = F(X)$*,* $L = F(Y)$ *and suppose given a place* $\lambda : L \rightsquigarrow K$ *compatible both with* f *and* f'*. Then* $(\lambda, fp^{-1}) = (\lambda, f'p'^{-1})$ *in* $(S_b^p)^{-1}\mathbf{Sm}_*^{\mathrm{prop}}\mathbf{P}$*.*
b) Consider a diagram (4.6) in $\mathbf{Sm}_*^{\mathrm{prop}}$*. Then* $fp^{-1} = f'p'^{-1}$ *in* $(S_b^p)^{-1}\mathbf{Sm}_*^{\mathrm{prop}}$ *if* (f,p) *and* (f',p') *define the same rational map from* X *to* Y*.*

Proof. a) By the graph trick, complete the diagram as follows:

(4.7)



with $p_1, p_1' \in S_b^p$ and $Z'' \in \mathbf{Sm}_*^{\mathrm{prop}}\mathbf{P}$. Since $X \in \mathbf{Sm}_*^{\mathrm{prop}}$, we may take Z'' in $\mathbf{Sm}_*^{\mathrm{prop}}$.Then we have

$$pp_1 = p'p_1', \quad fp_1 = f'p_1'$$

(the latter by Lemma 2.3.6 b)), hence the claim.
b) If (f,p) and (f',p') define the same rational map, then arguing as in a) we get a diagram (4.7) in $\mathbf{Sm}_*^{\mathrm{prop}}$, hence $fp^{-1} = f'p'^{-1}$ in $(S_b^p)^{-1}\mathbf{Sm}_*^{\mathrm{prop}}$. □

4.5. THE MORPHISM ASSOCIATED TO A RATIONAL MAP. Let $X, Y \in \mathbf{Sm}_*^{\mathrm{prop}}$, and let $\varphi : Y \dashrightarrow X$ be a rational map. By the graph trick, we may find $p : Y' \to Y$ proper birational and a morphism $f : Y' \to X$ such that φ is represented by (f,p); since $Y \in \mathbf{Sm}_*^{\mathrm{prop}}$, we may choose Y' in $\mathbf{Sm}_*^{\mathrm{prop}}$. Then $fp^{-1} \in (S_b^p)^{-1}\mathbf{Sm}_*^{\mathrm{prop}}$ does not depend on the choice of Y' by Proposition 4.4.2 b): we simply write it φ.

4.6. PROOF OF THEOREM 4.2.4. Let $K, L \in \mathbf{place}_*$ and $\lambda \in \mathbf{place}_*(K,L)$. Put $X = \Psi_*(K), Y = \Psi_*(L)$, so that X (resp. Y) is a smooth proper model of K (resp. L) in $\mathbf{Sm}_*$ (see 4.2.1). Since X is proper, λ is finite on X and by Lemma 2.3.4 there exists a unique rational map $\varphi : Y \dashrightarrow X$ compatible with λ, that we view as a morphism in $(S_b^p)^{-1}\mathbf{Sm}_*^{\mathrm{prop}}$ by §4.5.

4.6.1. LEMMA. *With the above notation, we have* $\Psi_*(\lambda) = \varphi$*.*

Proof. Consider the morphisms $(\lambda, f) \in \mathbf{Sm}_*^{\mathrm{prop}}\mathbf{P}(Y', X)$ and $(1_L, s) \in \mathbf{Sm}_*^{\mathrm{prop}}\mathbf{P}(Y', Y)$. In (4.2) Φ_1^* sends the first morphism to λ and the second one to 1_L, while Φ_2^* sends the first morphism to f and the second one to s. The conclusion now follows from the commutativity of (4.2) and the construction of Ψ_*. □

We can now prove Theorem 4.2.4:
a) The essential surjectivity of Ψ_* is tautological. Let now $X = \Psi_*(K), Y = \Psi_*(L)$ for some $K, L \in \mathbf{place}_*$ and let $\varphi \in (S_b^p)^{-1}\mathbf{Sm}_*^{\mathrm{prop}}(X, Y)$. By Proposition 4.4.1, we may write $\varphi = fs^{-1}$ where f, s are morphisms in $\mathbf{Sm}_*^{\mathrm{prop}}$ and $s \in S_b^p$. Let $\tilde{\varphi} : X \dashrightarrow Y$ be the corresponding rational map. By Lemma 2.3.2, f is compatible with some place λ and by Corollary 2.2.3 c), s is compatible with the corresponding isomorphism ι of function fields. Then $\tilde{\varphi}$ is compatible with $\iota^{-1}\lambda$, and $\Psi_*(\iota^{-1}\lambda) = \varphi$ by Lemma 4.6.1. This proves the fullness of Ψ_*. (One could also use Lemma 1.1.2.)
b) By Lemma 4.6.1, $\Psi_*(\lambda)$ and $\Psi_*(\mu)$ are given by the respective rational maps $f, g : \Psi_*(L) \dashrightarrow \Psi_*(K)$ compatible with λ, μ. By the definition of $\mathbf{Sm}_*^{\mathrm{prop}}$, we can find a model $X' \in \mathbf{Sm}_*^{\mathrm{prop}}$ of K and two birational morphisms $s : X' \to X$, $t : X' \to \Psi_*(K)$. The hypothesis and Lemma 2.3.4 imply that $st^{-1}f = st^{-1}g$, hence $f = g$ in $S_b^{-1}\mathbf{Sm}_*^{\mathrm{prop}}$.
c) The said composition sends morphisms in S_r to morphisms in S_r, hence induces a functor

$$S_r^{-1}\mathbf{place}_*^{\mathrm{op}} \to S_r^{-1}\mathbf{Sm}.$$

But $S_b^{-1}\mathbf{Sm} \xrightarrow{\sim} S_r^{-1}\mathbf{Sm}$ by Theorem 1.7.2.

4.7. Proof of Theorem 4.2.3: the case of $\overline{\Phi}_1^*$. Essential surjectivity is obvious by definition of $\mathbf{place}_*$. Let $X, Y \in \mathbf{Sm}_*^{\mathrm{prop}}\mathbf{P}$, and $K = \Phi_1^*(X), L = \Phi_1^*(Y)$. By Lemma 2.3.4, a place $\lambda : L \rightsquigarrow K$ is compatible with a (unique) rational map $\varphi : X \dashrightarrow Y$. Since $X \in \mathbf{Sm}_*^{\mathrm{prop}}$, we may write $\varphi = fs^{-1}$ with $f : X' \to Y$ for $X' \in \mathbf{Sm}_*^{\mathrm{prop}}$, and $s : X' \to X$ is a birational morphism. This shows the fullness of $\overline{\Phi}_1^*$.
We now prove the faithfulness of $\overline{\Phi}_1^*$. Let $(\lambda_1, \psi_1), (\lambda_2, \psi_2)$ be two morphisms from X to Y in $(S_b^p)^{-1}\mathbf{Sm}_*^{\mathrm{prop}}\mathbf{P}$ having the same image under $\overline{\Phi}_1^*$. By Proposition 4.4.1, we may write $\psi_i = f_i p_i^{-1}$ with f_i, p_i morphisms and $p_i \in S_b$. As they have the same image, it means that the places λ_1 and λ_2 from $F(Y)$ to $F(X)$ are equal. By Lemma 2.3.4, (f_1, p_1) and (f_2, p_2) define the same rational map $\varphi : X \dashrightarrow Y$. Therefore $\psi_1 = \psi_2$ by Proposition 4.4.2 b), and $(\lambda_1, \psi_1) = (\lambda_2, \psi_2)$.

4.8. Dominant rational maps. Recall from Subsection 1.4 the category $\mathbf{Rat}_{\mathrm{dom}}$ of dominant rational maps between F-varieties. Writing $\mathbf{Var}_{\mathrm{dom}}$ for the category of F-varieties and dominant maps, we have inclusions of categories

$$\mathbf{Var} \supset \mathbf{Var}_{\mathrm{dom}} \overset{\rho}{\hookrightarrow} \mathbf{Rat}_{\mathrm{dom}}\,. \tag{4.8}$$

Recall [16, Ch. I, Th. 4.4] that there is an anti-equivalence of categories

$$\begin{aligned} \mathbf{Rat}_{\mathrm{dom}} &\xrightarrow{\sim} \mathbf{field}^{\mathrm{op}} \\ X &\mapsto F(X). \end{aligned} \tag{4.9}$$

Actually this follows easily from Lemma 1.4.2. We want to revisit this theorem from the current point of view. For simplicity, we restrict to smooth varieties and separably generated extensions of F. Recall:

4.8.1. Lemma. *A function field K/F has a smooth model if and only if it is separably generated.*

Proof. Necessity: let p be the exponential characteristic of F. If X is a smooth model of K/F, then $X \otimes_F F^{1/p}$ is smooth over $F^{1/p}$ and irreducible, hence $K \otimes_F F^{1/p}$ is still a field. The conclusion then follows from Mac Lane's separability criterion [27, Chapter 8, §4]

Sufficiency: if K/F is separably generated, pick a separable transcendence basis $\{x_1, \dots, x_n\}$. Writing $F(x_1, \dots, x_n) = F(\mathbf{A}^n)$, we can find an affine model of finite type X of K/F with a dominant generically finite morphism $f : X \to \mathbf{A}^n$. By generic flatness [EGA4, 11.1.1], there is an open subset $U \subseteq \mathbf{A}^n$ such that $f^{-1}(U) \to U$ is flat. On the other hand, since $K/F(x_1, \dots, x_n)$ is separable, there is another open subset $V \subseteq \mathbf{A}^n$ such that $\Omega^1_{f^{-1}(V)/V} = 0$. Then $f^{-1}(U \cap V)$ is flat and unramified, hence étale, over $U \cap V$, hence is smooth over F since $U \cap V$ is smooth [EGA4, 17.3.3]. □

Instead of (4.1) and (4.2), consider now the commutative diagrams of functors

$$\begin{array}{ccccc} & & \mathbf{Sm}_{\mathrm{dom}}\mathbf{P} & & \\ & \swarrow^{\Phi_{1,\mathrm{dom}}} & & \searrow^{\Phi_{2,\mathrm{dom}}} & \\ \mathbf{field}_s^{\mathrm{op}} & & & & \mathbf{Sm}_{\mathrm{dom}} \\ & \searrow^{\mathrm{Spec}} & & \swarrow^{J_{\mathrm{dom}}} & \\ & & \widehat{\mathbf{Sm}_{\mathrm{dom}}} & & \end{array} \tag{4.10}$$

$$\begin{array}{ccccc} & & S_b^{-1}\mathbf{Sm}_{\mathrm{dom}}\mathbf{P} & & \\ & \swarrow^{\overline{\Phi}_{1,\mathrm{dom}}} & & \searrow^{\overline{\Phi}_{2,\mathrm{dom}}} & \\ \mathbf{field}_s^{\mathrm{op}} & & & & S_b^{-1}\mathbf{Sm}_{\mathrm{dom}}. \\ & \searrow^{\overline{\mathrm{Spec}}} & & \swarrow^{\bar{J}_{\mathrm{dom}}} & \\ & & S_b^{-1}\widehat{\mathbf{Sm}_{\mathrm{dom}}} & & \end{array}$$

Here, $\mathbf{field}_s \subseteq \mathbf{field}$ is the full subcategory of separably generated extensions, $\mathbf{Sm}_{\mathrm{dom}}\mathbf{P}$ is the subcategory of $\mathbf{VarP}$ given by varieties in $\mathbf{Sm}$ and morphisms are pairs (λ, f) where f is dominant (so that λ is an inclusion of function fields) and $\Phi_{1,\mathrm{dom}}$, $\Phi_{2,\mathrm{dom}}$ are the two forgetful functors of (2.1), restricted to $\mathbf{Sm}_{\mathrm{dom}}\mathbf{P}$. Similarly, J_{dom} is the analogue of J for $\mathbf{Sm}_{\mathrm{dom}}$. We extend the birational morphisms S_b as in Definition 4.1.1.

4.8.2. Theorem. *In the top diagram of* (4.10), $\Phi_{2,\mathrm{dom}}$ *is an isomorphism of categories. In the bottom diagram, all functors are equivalences of categories.*

Proof. The first claim follows from Corollary 2.2.3 c). In the right diagram, the proofs for $\bar{J}_{\mathrm{dom}}$ and $\overline{\Phi}_{1,\mathrm{dom}}$ are exactly parallel to those of Theorems 4.2.3 and 4.2.4 with a much simpler proof for the latter. As $\overline{\Phi}_{2,\mathrm{dom}}$ is an isomorphism of categories, the 4th functor $\overline{\operatorname{Spec}}$ is an equivalence of categories as well. □

In Theorem 4.8.2, we could replace $\mathbf{Sm}_{\mathrm{dom}}$ by $\mathbf{Var}_{\mathrm{dom}}$ or $\mathbf{Var}_{\mathrm{dom}}^{\mathrm{prop}}$ (proper varieties) and $\mathbf{field}_s$ by $\mathbf{field}$ (same proofs).[6] Since $\Phi_{2,\mathrm{dom}}$ is an isomorphism of categories in both cases, we directly get a naturally commutative diagram of categories and functors

$$\begin{array}{ccccc} & & S_b^{-1}\mathbf{Sm}_{\mathrm{dom}} & \xrightarrow{\sim} & \mathbf{field}_s^{\mathrm{op}} \\ & & \downarrow & & \downarrow \\ S_b^{-1}\mathbf{Var}_{\mathrm{dom}}^{\mathrm{prop}} & \xrightarrow{\sim} & S_b^{-1}\mathbf{Var}_{\mathrm{dom}} & \xrightarrow{\sim} & \mathbf{field}^{\mathrm{op}}. \end{array} \tag{4.11}$$

where the horizontal ones are equivalences.

To make the link with (4.9), note that the functor ρ of (4.8) sends a birational morphism to an isomorphism. Hence ρ induces functors

$$S_b^{-1}\mathbf{Var}_{\mathrm{dom}}^{\mathrm{prop}} \to S_b^{-1}\mathbf{Var}_{\mathrm{dom}} \to \mathbf{Rat}_{\mathrm{dom}} \tag{4.12}$$

whose composition with the second equivalence of (4.11) is (4.9).

4.8.3. Proposition. *Let* $S = S_o, S_b$ *or* S_b^p.
a) S *admits a calculus of right fractions within* $\mathbf{Var}_{\mathrm{dom}}$.
b) The functors in (4.12) *are equivalences of categories.*

Proof. a) For any pair (u, s) of morphisms as in Diagram (4.4), with $s \in S$ and u dominant, the pull-back of s by u exists and is in S. Moreover, if $sf = sg$ with f and g dominant and $s \in S$, then $f = g$.
b) This follows from (4.11) and (4.9). □

Taking a quasi-inverse of (4.11), we now get an equivalence of categories

$$\Psi_{\mathrm{dom}} : \mathbf{field}_s^{\mathrm{op}} \xrightarrow{\sim} S_b^{-1}\mathbf{Sm}_{\mathrm{dom}} \tag{4.13}$$

which will be used in Section 6.

4.8.4. *Remark.* The functor $(S_b^p)^{-1}\mathbf{Var}_{\mathrm{dom}} \to \mathbf{field}^{\mathrm{op}}$ is not full (hence is not an equivalence of categories). For example, let X be a proper variety and Y an affine open subset of X, and let K be their common function field. Then the identity map $K \to K$ is not in the image of the above functor. Indeed,

[6] We could also replace dominant morphisms by flat morphisms, as in [19].

if it were, then by calculus of fractions it would be represented by a map of the form fs^{-1} where $s : X' \to X$ is proper birational. But then X' would be proper and $f : X' \to Y$ should be constant, a contradiction.
It can be shown that the localisation functor

$$(S_b^p)^{-1}\,\mathbf{Var}_{\mathrm{dom}} \to S_b^{-1}\,\mathbf{Var}_{\mathrm{dom}}$$

has a (fully faithful) right adjoint given by

$$(S_b^p)^{-1}\,\mathbf{Var}_{\mathrm{dom}}^{\mathrm{prop}} \to (S_b^p)^{-1}\,\mathbf{Var}_{\mathrm{dom}}$$

via the equivalence $(S_b^p)^{-1}\,\mathbf{Var}_{\mathrm{dom}}^{\mathrm{prop}} \xrightarrow{\sim} S_b^{-1}\,\mathbf{Var}_{\mathrm{dom}}$ given by Proposition 4.8.3 b). The proof is similar to that of Theorem 5.3.1 (ii) below.

4.9. Recapitulation. We constructed a full and essentially surjective functor (Theorem 4.2.4)

$$\Psi_* : S_r^{-1}\,\mathbf{place}_*^{\mathrm{op}} \to S_b^{-1}\mathbf{Sm}$$

and an equivalence of categories (4.13)

$$\Psi_{\mathrm{dom}} = \bar{J}_{\mathrm{dom}}^{-1} \circ \overline{\mathrm{Spec}} : \mathbf{field}_s^{\mathrm{op}} \xrightarrow{\sim} S_b^{-1}\mathbf{Sm}_{\mathrm{dom}}.$$

Consider the natural functor

$$\theta : S_b^{-1}\mathbf{Sm}_*^{\mathrm{prop}} \to S_b^{-1}\mathbf{Sm}. \tag{4.14}$$

In characteristic zero, θ is an equivalence of categories by [21, Prop. 8.5], noting that in this case $\mathbf{Sm}_*^{\mathrm{prop}} = \mathbf{Sm}^{\mathrm{prop}}$ by Hironaka. Let ι be the inclusion $\mathbf{field}_s^{\mathrm{op}} \hookrightarrow \mathbf{place}_*^{\mathrm{op}}$. Then the natural transformation $\eta : \mathrm{Spec} \Rightarrow \Sigma$ of (3.1) provides the following naturally commutative diagram

$$\begin{array}{ccccc}
\mathbf{field}_s^{\mathrm{op}} & \xrightarrow[\sim]{\Psi_{\mathrm{dom}}} & S_b^{-1}\mathbf{Sm}_{\mathrm{dom}} & \longrightarrow & S_b^{-1}\mathbf{Sm} \\
\downarrow{\scriptstyle \iota} & & & \underset{\sim}{\overset{\theta}{\nearrow}} & \\
S_r^{-1}\,\mathbf{place}_*^{\mathrm{op}} & \xrightarrow{\Psi_*} & S_b^{-1}\mathbf{Sm}_*^{\mathrm{prop}}. & &
\end{array} \tag{4.15}$$

(Note that η induces a natural isomorphism $\bar{\eta} : \overline{\mathrm{Spec}} \overset{\sim}{\Rightarrow} \bar{\Sigma}$.)
We can replace $^{\mathrm{prop}}$ by $^{\mathrm{proj}}$ in all this story.
In characteristic p, we don't know if $\mathbf{field}_s \subset \mathbf{place}_*$: to get an analogue of (4.15) we would have to take the intersection of these categories. We shall do this in Section 6 in an enhanced way, using a new idea (Lemma 6.3.4 a)). As a byproduct, we shall get the full faithfulness of θ in any characteristic (Corollary 6.6.4)

5. Other classes of varieties

In this section we prove that, given a full subcategory $\mathcal{C}$ of $\mathbf{Var}$ satisfying certain hypotheses, the functor

$$S_b^{-1}\mathcal{C}\mathbf{P} \to \mathbf{place}^{\mathrm{op}}$$

induced by the functor Φ_1 of Diagram (4.1) is fully faithful.

5.1. The $_*$ construction. We generalise Definition 4.2.1 as follows:

5.1.1. Definition. Let $\mathcal{C}$ be a full subcategory of **Var**. We write $\mathcal{C}_*$ for the full subcategory of $\mathcal{C}$ with the following objects: $X \in \mathcal{C}_*$ if and only if, for any $Y \in \mathbf{Var}^{\mathrm{prop}}$ birational to X, there exists $X' \in \mathcal{C}$ and a proper birational morphism $s : X' \to Y$.

5.1.2. Lemma. *a) $\mathcal{C}_*$ is closed under birational equivalence.*
b) We have $\mathcal{C}_* = \mathcal{C}$ for the following categories: $\mathbf{Var}, \mathbf{Norm}$ *and* $\mathbf{Sm}, \mathbf{Sm}^{\mathrm{qp}}$ *if* $\operatorname{char} F = 0$.
c) We have $\mathcal{C}_* \cap \mathcal{C}^{\mathrm{prop}} = (\mathcal{C}^{\mathrm{prop}})_*$, where $\mathcal{C}^{\mathrm{prop}} := \mathbf{Var}^{\mathrm{prop}} \cap \mathcal{C}$.

Proof. a) is tautological. b) is trivial for **Var**, is true for **Norm** because normalisation is finite and birational in **Var**, and follows from Hironaka's resolution for **Sm**. Finally, c) is trivial. □

5.1.3. Lemma. *Suppose $\mathcal{C}$ verifies the following condition: given a diagram*

$$\begin{array}{ccc} X' & \xrightarrow{j} & \tilde{X} \\ {\scriptstyle p}\downarrow & & \\ X & & \end{array}$$

with $X, \tilde{X} \in \mathcal{C}_*$, $p \in S_b^p$, $j \in S_o$ and $\tilde{X}$ proper, we have $X' \in \mathcal{C}$. (This holds in the following special cases: $\mathcal{C} \subseteq \mathbf{Var}^{\mathrm{prop}}$, or $\mathcal{C}$ stable under open immersions.)
a) Let $X \in \mathcal{C}_*$. Then the following holds: for any $s : Y \to X$ with $Y \in \mathbf{Var}$ and $s \in S_b^p$, there exists $t : X' \to Y$ with $X' \in \mathcal{C}_*$ and $t \in S_b^p$.
b) Let $X, Y \in \mathcal{C}_*$ with Y proper, and let $\gamma : X \dashrightarrow Y$ be a rational map. Then there exists $X' \in \mathcal{C}_*$, $s : X' \to X$ in S_b^p and a morphism $f : X' \to Y$ such that $\gamma = fs^{-1}$.

Proof. a) By Nagata's theorem, choose a compactification $\bar{Y}$ of Y. By hypothesis, there exists $\bar{X}' \in \mathcal{C}$ and a proper birational morphism $t' : \bar{X}' \to \bar{Y}$. If $X' = t'^{-1}(Y)$, then $t : X' \to Y$ is a proper birational morphism. The hypothesis on $\mathcal{C}$ then implies that $X' \in \mathcal{C}$, hence $X' \in \mathcal{C}_*$ by Lemma 5.1.2 a).
b) Apply a) to the graph of γ, which is proper over X. □

5.2. Calculus of fractions.

5.2.1. Proposition. *Under the condition of Lemma 5.1.3, Propositions 4.4.1 and 4.4.2 remain valid for $\mathcal{C}_*\mathbf{P}$. In particular, any morphism in $(S_b^p)^{-1}\mathcal{C}_*\mathbf{P}$ or $(S_b^p)^{-1}\mathcal{C}_*$ is of the form fp^{-1}, with $f \in \mathcal{C}_*\mathbf{P}$ or $\mathcal{C}_*$ and $p \in S_b^p$.*

Proof. Indeed, the only fact that is used in the proofs of Propositions 4.4.1 and 4.4.2 is the conclusion of Lemma 5.1.3 a). □

To go further, we need:

5.2.2. Proposition. *In $(S_b^p)^{-1}\mathcal{C}_*\mathbf{P}$, S_o admits a calculus of left fractions. In particular (cf. Proposition 5.2.1), any morphism in $S_b^{-1}\mathcal{C}_*\mathbf{P}$ may be written as $j^{-1}fq^{-1}$, with $j \in S_o$ and $q \in S_b^p$.*

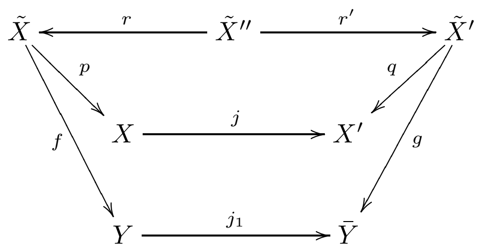
Proof. a) Consider a diagram in $(S_b^p)^{-1}\mathcal{C}_*\mathbf{P}$

$$\begin{array}{ccc} X & \xrightarrow{j} & X' \\ {\scriptstyle\varphi}\downarrow & & \\ Y & & \end{array}$$

with $j \in S_o$. By Proposition 5.2.1, we may write $\varphi = fp^{-1}$ with $p \in S_b^p$ and f a morphism of $\mathcal{C}_*\mathbf{P}$ (f, p originate from some common $\bar{X}$). We may embed Y as an open subset of a proper $\bar{Y}$. This gives us a rational map $X' \dashrightarrow \bar{Y}$. Using the graph trick, we may "resolve" this rational map into a morphism $g : \tilde{X}' \to \bar{Y}$, with $\tilde{X}' \in \mathbf{Var}$ provided with a proper birational morphism $q : \tilde{X}' \to X'$. Since $Y \in \mathcal{C}_*$, we may assume $\bar{X}' \in \mathcal{C}_*$. Let $\psi = gq^{-1} \in (S_b^p)^{-1}\mathcal{C}_*\mathbf{P}$. Then the diagram in $(S_b^p)^{-1}\mathcal{C}_*\mathbf{P}$

$$\begin{array}{ccc} X & \xrightarrow{j} & X' \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ Y & \xrightarrow{j_1} & \bar{Y} \end{array}$$

commutes because the following bigger diagram commutes in $\mathcal{C}_*\mathbf{P}$:



thanks to Lemma 2.3.6, for suitable $\tilde{X}'' \in \mathcal{C}_*$ and $r, r' \in S_b^p$.
b) Consider a diagram

$$X' \xrightarrow{j} X \overset{f}{\underset{g}{\rightrightarrows}} Y$$

in $(S_b^p)^{-1}\mathcal{C}_*\mathbf{P}$, where $j \in S_o$ and $fj = gj$. By Proposition 5.2.1, we may write $f = \tilde{f}p^{-1}$ and $g = \tilde{g}p^{-1}$, where $\tilde{f}, \tilde{g}$ are morphisms in $\mathcal{C}_*\mathbf{P}$ and $p : \tilde{X} \to X$ is in S_b^p. Let U be a common open subset to X' and $\tilde{X}$: then the equality $fj = gj$ implies that the restrictions of $\tilde{f}$ and $\tilde{g}$ to U coincide as morphisms of $(S_b^p)^{-1}\mathcal{C}_*\mathbf{P}$. Hence the places underlying $\tilde{f}$ and $\tilde{g}$ are equal, which implies that $\tilde{f} = \tilde{g}$ (Proposition 2.2.2), and thus $f = g$. □

5.2.3. *Remark.* S_o does not admit a calculus of *right* fractions, even in $(S_b^p)^{-1}\,\mathbf{VarP}$. Indeed, consider a diagram in $(S_b^p)^{-1}\,\mathbf{VarP}$

$$\begin{array}{ccc} & & Y' \\ & & \big\downarrow{\scriptstyle j} \\ X & \xrightarrow{f} & Y \end{array}$$

where $j \in S_o$ and, for simplicity, f comes from $\mathbf{VarP}$. Suppose that we can complete this diagram into a commutative diagram in $(S_b^p)^{-1}\,\mathbf{VarP}$

$$\begin{array}{ccc} \tilde{X}' & & \\ {\scriptstyle p}\big\downarrow & \searrow^{g} & \\ X' & \xrightarrow{\varphi} & Y' \\ {\scriptstyle j'}\big\downarrow & & \big\downarrow{\scriptstyle j} \\ X & \xrightarrow{f} & Y \end{array}$$

with $p \in S_b^p$ and g comes from $\mathbf{VarP}$. By Proposition 2.2.2 the localisation functor $\mathbf{VarP} \to (S_b^p)^{-1}\,\mathbf{VarP}$ is faithful, so the diagram (without φ) must already commute in $\mathbf{VarP}$. If $f(X) \cap Y' = \emptyset$, this is impossible.

5.3. Generalising Theorem 4.2.3.

5.3.1. Theorem. *Let $\mathcal{C}$ be a full subcategory of* $\mathbf{Var}$*. In diagram* (4.1)*,*
a) J induces an equivalence of categories $S_b^{-1}\mathcal{C} \to S_b^{-1}\hat{\mathcal{C}}$.
b) Suppose that $\mathcal{C}$ verifies the condition of Lemma 5.1.3. Consider the string of functors

$$(S_b^p)^{-1}\mathcal{C}_*^{\mathrm{prop}}\mathbf{P} \xrightarrow{S} (S_b^p)^{-1}\mathcal{C}_*\mathbf{P} \xrightarrow{T} S_b^{-1}\mathcal{C}_*\mathbf{P} \xrightarrow{\overline{\Phi_1^*}} \mathbf{place}^{\mathrm{op}}.$$

where S and T are the obvious ones and $\overline{\Phi_1^*}$ is induced by Φ_1. Then

(i) *S is fully faithful and T is faithful.*
(ii) *For any $X \in (S_b^p)^{-1}\mathcal{C}_*P$ and $Y \in (S_b^p)^{-1}\mathcal{C}_*^{\mathrm{prop}}\mathbf{P}$, the map*

$$T : \mathrm{Hom}(X, S(Y)) \to \mathrm{Hom}(T(X), TS(Y)) \tag{5.1}$$

is an isomorphism.

(iii) *TS is an equivalence of categories.*
(iv) *$\overline{\Phi_1^*}$ is fully faithful.*

Proof. a) It is exactly the same proof as for the case of $\bar{J}$ in Theorem 4.2.3.
b) In 4 steps:
A) We run through the proof of Theorem 4.2.3 given in §4.7 for $\overline{\Phi_1^*}$ in the case $\mathcal{C} = \mathbf{Sm}^{\mathrm{prop}}$. In view of Proposition 5.2.1, the proof of faithfulness for $\overline{\Phi_1^*}T$ goes through verbatim. The proof of fullness for $\overline{\Phi_1^*}TS$ also goes through (note that in *loc. cit.* , we need Y to be proper in order for λ to be finite on it). It follows that S is fully faithful and T is faithful.

B) By A), (5.1) is injective. Let $\varphi \in \operatorname{Hom}(T(X), TS(Y))$. By Proposition 5.2.2, $\varphi = j^{-1} f p^{-1}$ with $j \in S_o$ and $p \in S_b^p$. Since Y is proper, j is necessarily an isomorphism, which shows the surjectivity of (5.1). This proves (ii).
C) It follows from A) and B) that TS is fully faithful. Essential surjectivity follows from Lemma 5.1.2 a) and c) plus Nagata's theorem. This proves (iii).
D) We come to the proof of (iv). Since $\overline{\Phi_1^*}TS$ is faithful (see A)) and TS is an equivalence, $\overline{\Phi_1^*}$ is faithful. To show that it is full, let $X, Y \in \mathcal{C}_*\mathbf{P}$ and $\lambda : F(Y) \rightsquigarrow F(X)$ a place. Let $Y \to \bar{Y}$ be a compactification of Y. By Definition 5.1.1, we may choose $\bar{Y}' \xrightarrow{s} \bar{Y}$ with $s \in S_b^p$ and $\bar{Y}' \in \mathcal{C}_*^{\mathrm{prop}}$. Then λ is finite over $\bar{Y}'$. By Lemma 2.3.4, there is a rational map $f : X \dashrightarrow \bar{Y}'$ compatible with λ. Applying Lemma 5.1.3 b) to the rational maps $X \dashrightarrow \bar{Y}'$ and $Y \dashrightarrow \bar{Y}'$, we find a diagram in $\mathcal{C}_*$

$$\begin{array}{ccccc} X' & \xrightarrow{f} & \bar{Y}' & \xleftarrow{t'} & Y' \\ \downarrow{\scriptstyle t} & & & & \downarrow{\scriptstyle s'} \\ X & & & & Y \end{array}$$

with $t, s' \in S_b^p$ (and $t' \in S_b$). Then $\varphi = s' t'^{-1} f t^{-1} : X \to Y$ is such that $\overline{\Phi_1^*}(\varphi) = \lambda$. □

5.3.2. Corollary. *The localisation functor T has a right adjoint, given explicitly by $(TS)^{-1} \circ S$.* □

Consider now the commutative diagram of functors:
(5.2)

$$\begin{array}{ccccccc} (S_b^p)^{-1}\mathcal{C}_*^{\mathrm{prop}}\mathbf{P} & \xrightarrow{S} & (S_b^p)^{-1}\mathcal{C}_*\mathbf{P} & \xrightarrow{T} & S_b^{-1}\mathcal{C}_*\mathbf{P} & \xrightarrow{\overline{\Phi_1^*}} & \mathbf{place}^{\mathrm{op}} \\ \downarrow & & \downarrow & & \downarrow & & \| \\ (S_b^p)^{-1}\,\mathbf{Var}^{\mathrm{prop}}\,\mathbf{P} & \xrightarrow{S} & (S_b^p)^{-1}\,\mathbf{Var}\mathbf{P} & \xrightarrow{T} & S_b^{-1}\,\mathbf{Var}\mathbf{P} & \xrightarrow{\overline{\Phi_1^*}} & \mathbf{place}^{\mathrm{op}}. \end{array}$$

5.3.3. Corollary. *All vertical functors in* (5.2) *are fully faithful.*

Proof. For the first and third vertical functors, this is a byproduct of Theorem 5.3.1. The middle one is faithful by the faithfulness of T and $\overline{\Phi_1^*}$ in Theorem 5.3.1. For fullness, let $X, Y \in (S_b^p)^{-1}\mathcal{C}_*\mathbf{P}$ and $\varphi : X \to Y$ be a morphism in $(S_b^p)^{-1}\,\mathbf{Var}\mathbf{P}$. By Proposition 5.2.1, we may write $\varphi = fp^{-1}$, with $p : \tilde{X} \to X$ proper birational. By Lemma 5.1.3 a), we may find $p' : \tilde{X}' \to \tilde{X}$ proper birational with $\tilde{X}' \in \mathcal{C}_*$, and replace fp^{-1} by $fp'(pp')^{-1}$. □

5.3.4. *Remarks.* 1) Take $\mathcal{C} = \mathbf{Var}$ in Theorem 5.3.1 and let $X, Y \in (S_b^p)^{-1}\,\mathbf{Var}\mathbf{P}$. Then the image of $\operatorname{Hom}(X, Y)$ in $\operatorname{Hom}(\overline{\Phi_1^*}T(Y), \overline{\Phi_1^*}T(X))$ via $\overline{\Phi_1^*}T$ is contained in the set of places which are finite on Y. If X and Y are proper, then the image is all of $\operatorname{Hom}(\overline{\Phi_1^*}T(Y), \overline{\Phi_1^*}T(X))$. On the other hand, if X is proper and Y is affine, then for any map $\varphi = fp^{-1} : X \to Y$, the

source X' of p is proper hence $f(X')$ is a closed point of Y, so that the image is contained in the set of places from $F(Y)$ to $F(X)$ whose centre on Y is a closed point (and one sees easily that this inclusion is an equality). In general, the description of this image seems to depend heavily on the geometric nature of X and Y.
2) For "usual" subcategories $\mathcal{C} \subseteq \mathbf{Var}$, the functors Φ_1^*, $\Phi_1^* T$ and $\Phi_1^* TS$ of Theorem 5.3.1 b) are essentially surjective (hence so are those in Corollary 5.3.3): this is true for $\mathcal{C} = \mathbf{Var}$ or $\mathbf{Norm}$ (any function field has a normal proper model), and for $\mathcal{C} = \mathbf{Sm}$ in characteristic 0. For $\mathcal{C} = \mathbf{Sm}$ in positive characteristic, the essential image of these functors is the category $\mathbf{place}_*^{\mathrm{op}}$ of Definition 4.2.1.

5.4. Localising $\mathcal{C}_*$. In Theorem 5.3.1, we generalised Theorem 4.2.3 which was used to construct the functor Ψ_* of (4.3). A striking upshot is Corollary 5.3.3. What happens if we study $S_b^{-1}\mathcal{C}_*$ instead of $S_b^{-1}\mathcal{C}_*\mathbf{P}$?
This was done previously in [21, §8], by completely different methods. The two main points were:

- In characteristic 0, we have the following equivalences of categories:

$$S_b^{-1}\mathbf{Sm}^{\mathrm{proj}} \simeq S_b^{-1}\mathbf{Sm}^{\mathrm{prop}} \simeq S_b^{-1}\mathbf{Sm}^{\mathrm{qp}} \simeq S_b^{-1}\mathbf{Sm} \tag{5.3}$$

  induced by the obvious inclusion functors [21, Prop. 8.5].
- Working with varieties that are not smooth or at least regular leads to pathologies: for example, the functor $S_b^{-1}\mathbf{Sm} \to S_b^{-1}\mathbf{Var}$ is neither full nor faithful [21, Rk. 8.11]. This contrasts starkly with Corollary 5.3.3. The issue is closely related to the regularity condition appearing in Lemma 2.3.2; it is dodged in [21, Prop. 8.6] by restricting to those morphisms that send smooth locus into smooth locus.

Using the methods of [21], one can show that the functor

$$(S_b^p)^{-1}\mathcal{C}_*^{\mathrm{prop}} = S_b^{-1}\mathcal{C}_*^{\mathrm{prop}} \to S_b^{-1}\mathcal{C}_* \tag{5.4}$$

is an equivalence of categories for any $\mathcal{C} \subseteq \mathbf{Var}$ satisfying the condition of Lemma 5.1.3. For this, one should use [21, Th. 5.14] under a form similar to that given in [21, Prop. 5.10]. One can then deduce from Corollary 5.3.2 that the localisation functor

$$(S_b^p)^{-1}\mathcal{C}_* \xrightarrow{T} S_b^{-1}\mathcal{C}_*$$

has a right adjoint given (up to the equivalence (5.4)) by $(S_b^p)^{-1}\mathcal{C}_*^{\mathrm{prop}} \xrightarrow{S} (S_b^p)^{-1}\mathcal{C}_*$ (in particular, S is fully faithful): indeed, the unit and counit of the adjunction in Corollary 5.3.2 map by the essentially surjective forgetful functors

$$S_b^{-1}\mathcal{C}_*\mathbf{P} \to S_b^{-1}\mathcal{C}_*, \quad \text{etc.} \tag{5.5}$$

to natural transformations which keep enjoying the identities of an adjunction. Note however that (5.5) is not full unless $\mathcal{C} \subseteq \mathbf{Sm}$ (see Lemma 1.1.2 and Lemma 2.2.2 for this case).
For $\mathcal{C} = \mathbf{Sm}$ or $\mathbf{Sm}^{\mathrm{qp}}$, the equivalence (5.4) extends a version of (5.3) to positive characteristic. We won't give a detailed proof however, because it would be

tedious and we shall obtain a better result later (Corollary 6.6.4) by a different method.
The proofs given in [21] do not use any calculus of fractions. In fact, S_b^p *does not admit any calculus of fractions within* **Var**, contrary to the case of **VarP** (*cf.* Proposition 4.4.1). This is shown by the same examples as in Remark 2.3.3. If we restrict to $\mathbf{Sm}_*$, we can use Proposition 4.4.1 and Lemma 2.3.2 to prove a helpful part of calculus of fractions:

5.4.1. Proposition. *a) Let $s : Y \to X$ be in S_b^p, with X smooth. Then s is an envelope [9]: for any extension K/F, the map $Y(K) \to X(K)$ is surjective.*
b) The multiplicative set S_b^p verifies the second axiom of calculus of right fractions within $\mathbf{Sm}_*$.
c) Any morphism in $S_b^{-1}\mathbf{Sm}_*$ may be represented as $j^{-1}fp^{-1}$, where $j \in S_o$ and $p \in S_b^p$.

Proof. a) Base-changing to K, it suffices to deal with $K = F$. Let $x \in X(F)$. By lemma 2.3.2, there is a place λ of $F(X)$ with centre x and residue field F. The valuative criterion for properness implies that λ has a centre y on Y; then $s(y) = x$ by Lemma 2.3.6 and $F(y) \subseteq F(\lambda) = F$.
b) We consider a diagram (4.4) in $\mathbf{Sm}_*$, with $s \in S_b^p$. By a), $z = u(\eta_X)$ has a preimage $z' \in Y'$ with same residue field. Let $Z = \overline{\{z\}}$ and $Z' = \overline{\{z'\}}$: the map $Z' \to Z$ is birational. Since the map $\bar{u} : X \to Z$ factoring u is dominant, we get by Theorem 4.8.3 b) a commutative diagram like (4.5), with s' proper birational. By Lemma 5.1.3 a), we may then replace X' by an object of $\mathbf{Sm}_*$.
c) As that of Proposition 5.2.2. □

5.4.2. *Remark.* On the other hand, S_b^p is far from verifying the third axiom of calculus of right fractions within $\mathbf{Sm}_*$. Indeed, let $s : Y \to X$ be a proper birational morphism that contracts some closed subvariety $i : Z \subset Y$ to a point. Then, given any two morphisms $f, g : Y' \rightrightarrows Z$, we have $sif = sig$. But if $ift = igt$ for some $t \in S_b^p$, then $if = ig$ (hence $f = g$) since t is dominant.

6. Homotopy of places and R-equivalence

In this section, we do several things. In Subsection 6.1 we prove elementary results on divisorial valuations with separably generated residue fields. In Subsection 6.2 we introduce a subcategory **dv** of **place**, where morphisms are generated by field inclusions and places given by discrete valuation rings. We relate it in Subsection 6.3 with a construction of Asok-Morel [1] to define a functor

$$\Psi : S_r^{-1}\mathbf{dv} \to S_b^{-1}\mathbf{Sm}$$

extending the functor Ψ_{dom} of (4.13). This functor is compatible with the functor Ψ_* of Theorem 4.2.4. We then show in Proposition 6.4.3 that the localisation **place** $\to S_r^{-1}$**place** is also a quotient by a certain equivalence relation h; although remarkable, this fact is elementary.

Next, we reformulate a result of Asok-Morel to enlarge the equivalence relation h to another, h′, so that the functor Ψ factors through an equivalence of categories

$$\mathbf{dv} / \mathrm{h}' \xrightarrow{\sim} S_b^{-1}\mathbf{Sm}.$$

Finally, we use another result of Asok-Morel to compute some Hom sets in $S_b^{-1}\mathbf{Sm}$ as R-equivalence classes: in the first version of this paper, we had proven this only in characteristic 0 by much more complicated arguments.

6.1. Good dvr's.

6.1.1. Definition. A discrete valuation ring (dvr) R containing F is *good* if its quotient field K and its residue field E are finitely and separably generated over F, with $\operatorname{trdeg}(E/F) = \operatorname{trdeg}(K/F) - 1$.

6.1.2. Lemma. *A dvr R containing F is good if and only if there exist a smooth F-variety X and a smooth divisor $D \subset X$ such that $R \simeq \mathcal{O}_{X,D}$.*

Proof. Sufficiency is clear by Lemma 4.8.1. Let us show necessity. The condition on the transcendence degrees means that R is *divisorial* = a "prime divisor" in the terminology of [40]. By *loc. cit.* , Ch. VI, Th. 31, there exists then a model X of K/F such that $R = \mathcal{O}_{X,x}$ for some point x of codimension 1. (In particular, granting the finite generation of K, that of E is automatic.) Furthermore, the separable generation of E yields a short exact sequence

$$0 \to \mathfrak{m}/\mathfrak{m}^2 \to \Omega^1_{R/F} \otimes_R E \to \Omega^1_{E/F} \to 0$$

where $\mathfrak{m}$ is the maximal ideal of R (see Exercise 8.1 (a) of [16, Ch. II]). Therefore $\dim_E \Omega^1_{R/F} \otimes_R E = \operatorname{trdeg}(K/F) = \dim_K \Omega^1_{R/F} \otimes_R K$, thus $\Omega^1_{R/F}$ is free of rank $\operatorname{trdeg}(K/F)$ and x is a smooth point of X. Shrinking X around x, we may assume that it is smooth; if $D = \overline{\{x\}}$, it is generically smooth by Lemma 4.8.1, hence we may assume D is smooth up to shrinking X further. □

6.1.3. Lemma. *Let R be a good dvr containing F, with quotient field K and residue field E, and let K_0/F be a subextension of K/F. Then $R \cap K_0$ is either K_0 or a good dvr.*

Proof. By Mac Lane's criterion, K_0 is separably generated, and the same applies to the residue field $E_0 \subseteq E$ of $R \cap K_0$ if the latter is a dvr. □

6.2. The category **dv**.

6.2.1. Definition. Let K/F and L/F be two separably generated extensions. We denote by $\mathbf{dv}(K, L)$ the set of morphisms in $\mathbf{place}(K, L)$ of the form

$$K \rightsquigarrow K_1 \rightsquigarrow \dots \rightsquigarrow K_n \hookrightarrow L \tag{6.1}$$

where for each i, the place $K_i \rightsquigarrow K_{i+1}$ corresponds to a good dvr with quotient field K_i and residue field K_{i+1}. (Compare [40, Ch. VI, §3].)

6.2.2. Lemma. *In* $\mathbf{dv}(K,L)$*, the decomposition of a morphism in the form* (6.1) *is unique. The collection of the* $\mathbf{dv}(K,L)$*'s defines a subcategory* $\mathbf{dv}\subset\mathbf{place}$*, with objects the separably generated function fields.*

Proof. Uniqueness follows from [40, p. 10]. To show that $Ar(\mathbf{dv})$ is closed under composition, we immediately reduce to the case of a composition

$$K\overset{i}{\hookrightarrow}L\overset{\lambda}{\rightsquigarrow}L_1 \tag{6.2}$$

where (L,L_1) correspond to a good dvr R. Then the claim follows from applying Lemma 6.1.3 to the commutative diagram in **place**

$$\begin{array}{ccc} L & \xrightarrow{\lambda} & L_1\\ {\scriptstyle i}\uparrow & & {\scriptstyle i_1}\uparrow\\ K & \xrightarrow{\lambda_1} & K_1 \end{array} \tag{6.3}$$

where K_1 is the residue field of $R\cap K$ if this is a dvr, and $K_1=K$ otherwise (and then λ_1 is a trivial place). □

We shall need the following variant of a theorem of Knaf and Kuhlmann [23, Th. 1.1] (compare *loc. cit.* , pp. 834/835):

6.2.3. Theorem. *Let* $\lambda:K\rightsquigarrow L$ *be a morphism in* $\mathbf{dv}$*. Then* λ *is finite over a smooth model of* K*. Moreover, let* $K'\subseteq K$ *be a subextension of* K*, and let* Z *be a model of* K' *on which* $\lambda_{|K'}$ *has a centre* z*. Then there is a smooth model* X *of* K *on which* λ *has a centre of codimension* n*, the rank of* λ*, and a morphism* $X\to Z$ *inducing the extension* K/K'*.*

Proof. This actually follows from [23, Th. 1.1][7]: let U be an open affine neighbourhood of z and let $E:=\{y_1,\dots,y_r\}$ be a set of generators of the F-algebra $\mathcal{O}_Z(U)$ (ring of sections). Then by [23, Th. 1.1], there exists a model X_0 of K/F such that:

- λ is centred at a smooth point x of X_0,
- $\dim\mathcal{O}_{X_0,x}=n=\dim\mathcal{O}_\lambda$,
- E is contained in the maximal ideal of $\mathcal{O}_{X_0,x}$.

Hence $\mathcal{O}_Z(U)\subseteq\mathcal{O}_{X_0}(X)$ for some open affine neighbourhood X of x, which yields a morphism $X\to U$ that maps x to z. □

6.3. Relationship with the work of Asok and Morel. In [1, §6], Asok and Morel prove closely related results: let us translate them in the present setting.

Let us write $\mathcal{C}^\vee$ for the category of presheaves of sets on a category $\mathcal{C}$. In [1], the authors denote the category $(S_r^{-1}\mathbf{Sm})^\vee$ by $Shv_F^{h\mathbf{A}^1}$. Similarly, they write $\mathcal{F}_F^r-\mathbf{Set}$ for the category consisting of objects of $(\mathbf{field}_s^{\mathrm{op}})^\vee$ provided with

[7]We thank Hagen Knaf for his help in this proof.

"specialisation maps" for good dvrs. In [1, Th. 6.1.7], they construct a full embedding

$$Shv_F^{h\mathbf{A}^1} \to \mathcal{F}_F^r - \mathbf{Set} \tag{6.4}$$

(evaluate presheaves on function fields), and show that its essential image consists of those functors $\mathcal{S} \in \mathcal{F}_F^r - \mathbf{Set}$ satisfying a list of axioms (A1) – (A4) (ibid., Defn. 6.1.6).

The proof of Lemma 6.2.2 above shows that Conditions (A1) and (A2) mean that $\mathcal{S}$ defines a functor $\mathbf{dv}^{\mathrm{op}} \to \mathbf{Set}$, and Condition (A4) means that $\mathcal{S}$ factors through $S_r^{-1}\,\mathbf{dv}^{\mathrm{op}}$. In other words, they essentially[8] construct a functor

$$(S_r^{-1}\mathbf{Sm})^\vee \to (S_r^{-1}\,\mathbf{dv}^{\mathrm{op}})^\vee.$$

We now check that this functor is induced by a functor

$$\Psi : S_r^{-1}\,\mathbf{dv}^{\mathrm{op}} \to S_b^{-1}\mathbf{Sm}. \tag{6.5}$$

For this, we need a lemma:

6.3.1. Lemma. *Let $\mathbf{Sm}^{\mathrm{ess}}$ be the category of irreducible separated smooth F-schemes essentially of finite type. Then the full embedding $\mathbf{Sm} \hookrightarrow \mathbf{Sm}^{\mathrm{ess}}$ induces an equivalence of categories*

$$S_b^{-1}\mathbf{Sm} \xrightarrow{\sim} S_b^{-1}\mathbf{Sm}^{\mathrm{ess}}.$$

Proof. We use again the techniques of [21], to which we refer the reader: actually the first part of the proof of [21, Prop. 8.4] works with a minimal change. Namely, with notation as in *loc. cit.* , there are 3 conditions (b1) – (b3) to check:

- (b1) *Given $f, g : X \to Y$ in $\mathbf{Sm}$ and $s : Z \to X$ in $\mathbf{Sm}^{\mathrm{ess}}$ with $s \in S_b$, $fs = gs \Rightarrow f = g$*: this follows from Lemma 1.4.1 (birational morphisms are dominant).
- (b2) follows from the fact that any essentially smooth scheme may be embedded in a smooth scheme of finite type by an "essentially open immersion".
- (b3) We are given $i : X \to \bar{X}$ and $j : X \to Y$ where $X \in \mathbf{Sm}^{ess}$, $\bar{X}, Y \in \mathbf{Sm}$ and $i \in S_b$; we must factor i and j through $X \xrightarrow{s} U$ with U in $\mathbf{Sm}$ and s, $U \to \bar{X}$ in S_b. We take for U the smooth locus of the closure of the diagonal image of X in $\bar{X} \times Y$.

□

To define Ψ, it is now sufficient to construct it as a functor $\Psi : S_r^{-1}\,\mathbf{dv}^{\mathrm{op}} \to S_b^{-1}\mathbf{Sm}^{\mathrm{ess}}$. We first construct Ψ on $\mathbf{dv}^{\mathrm{op}}$ by extending the functor Ψ_{dom} of (4.13) from $\mathbf{field}_s^{\mathrm{op}}$ to $\mathbf{dv}^{\mathrm{op}}$. For this, we repeat the construction given on [1, p. 2041]: if $K \in \mathbf{dv}$ and $\mathcal{O}$ is a good dvr with quotient field K and residue field E, then the morphism $\operatorname{Spec} K \to \operatorname{Spec} \mathcal{O}$ is an isomorphism in $S_b^{-1}\mathbf{Sm}^{\mathrm{ess}}$, hence the quotient map $\mathcal{O} \to E$ induces a morphism $\operatorname{Spec} E \to \operatorname{Spec} K$.

[8] Essentially because Condition (A1) of [1, §6] only requires a commutation of diagrams coming from (6.3) when the ramification index is 1.

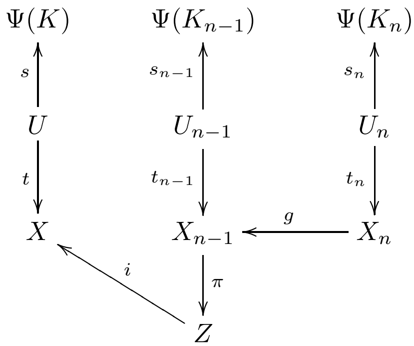
By Lemma 6.2.2, any morphism in **dv** has a unique expression in the form (6.1), which extends the definition of Ψ to all morphisms. To show that Ψ is a functor, it now suffices to check that it converts any diagram (6.3) into a commutative diagram, which is obvious by going through its construction. Finally, Ψ factors through $S_r^{-1}\,\mathbf{dv}^{\mathrm{op}}$ thanks to Theorem 1.7.2. It is now clear that the dual of Ψ gives back the Asok-Morel functor (6.4).
As in §4.5, we associate to a rational map f between smooth varieties a morphism in $S_b^{-1}\mathbf{Sm}$, still denoted by f. We need the following analogue of Lemma 4.6.1:

6.3.2. PROPOSITION. *Let $\lambda : K \rightsquigarrow L$ be a morphism in* **dv**. *Then, for any smooth model X of K on which λ is finite, we have $\Psi(\lambda) = st^{-1}f$, where $f : \Psi(L) \dashrightarrow X$ is the corresponding rational map and $s : U \hookrightarrow \Psi(K)$, $t : U \hookrightarrow X$ are open immersions of a common open subset U.*

Proof. We proceed by induction on the length n of a chain (6.1): If $n = 0$ the claim is trivial and if $n = 1$ it is true by construction. If $n > 1$, break λ as

$$K \overset{\lambda_1}{\rightsquigarrow} K_{n-1} \overset{\lambda_2}{\rightsquigarrow} K_n \hookrightarrow L$$

where λ_1 has rank $n-1$ and λ_2 has rank 1. We now apply Lemma 1.3.2: since λ is finite on X, so is λ_1, and if we write Z for the closure of $c_X(\lambda_1)$, then $z = c_X(\lambda) = c_Z(\lambda_2)$. If $n = 0$ the claim is trivial and if $n = 1$ it is true by construction. If $n > 1$, Theorem 6.2.3 provides us with $\pi : X_{n-1} \to Z$, X_{n-1} smooth with function field K_{n-1} on which λ_2 has a centre of codimension 1. Then we have a diagram



where i is the closed immersion $Z \hookrightarrow X$, $s, s_{n-1}, s_n, t, t_{n-1}$ are open immersions and g is the closed immersion of a smooth divisor obtained by applying Lemma 6.1.2 after possibliy shrinking X_{n-1}. Thus (gt_n, s_n) represents the rational map given by the centre of λ_2 on X_{n-1}. The rational map corresponding to λ_1 is represented by (f_{n-1}, s_{n-1}) with

$$f_{n-1} = i\pi t_{n-1}$$

and the one corresponding to $\lambda_2\lambda_1$ is represented by (f_n, s_n) with

$$f_n = i\pi g t_n$$

because this is compatible with $\lambda_2\lambda_1$ by Proposition 2.2.4 (also use the uniqueness in Lemma 2.3.4).
By induction and definition, we have

$$\Psi(\lambda_1) = st^{-1}f_{n-1}s_{n-1}^{-1}, \quad \Psi(\lambda_2) = s_{n-1}t_{n-1}^{-1}gt_ns_n^{-1}$$

so we have to show that

$$st^{-1}f_{n-1}s_{n-1}^{-1}s_{n-1}t_{n-1}^{-1}gt_ns_n^{-1} = st^{-1}f_ns_n^{-1}$$

or

$$f_{n-1}t_{n-1}^{-1}gt_n = f_n = i\pi gt_n$$

which is true because $f_{n-1} = i\pi t_{n-1}$. This concludes the proof. □

6.3.3. *Remark.* In this proof, there is no codimension condition on $c_X(\lambda)$. So Theorem 6.2.3 is used twice in a weak form: once, implicitly, to ensure the existence of X. Then a second time, to deal with Z. But here λ_2 is a discrete valuation of rank 1, so this special case can perhaps already be obtained by examining the proof of [40, Th. 31] (which may have been a source of inspiration for [23].)

6.3.4. Lemma. *a) Let* $\mathbf{dv}_*$ *be the full subcategory of* $\mathbf{dv}$ *whose objects are in* $\mathbf{place}_*$*. Then the diagram of functors*

$$\begin{array}{ccc}
S_r^{-1}\,\mathbf{place}_*^{\mathrm{op}} & \xrightarrow{\Psi_*} & S_b^{-1}\mathbf{Sm}_*^{\mathrm{prop}} \\
\uparrow & & \\
S_r^{-1}\,\mathbf{dv}_*^{\mathrm{op}} & & \Big\downarrow \\
\downarrow & & \\
S_r^{-1}\,\mathbf{dv}^{\mathrm{op}} & \xrightarrow{\Psi} & S_b^{-1}\mathbf{Sm}
\end{array}$$

is naturally commutative.
b) Let $K, L \in \mathbf{dv}$ *and* $\lambda, \mu \in \mathbf{dv}(K, L)$ *with the same residue field* $K' \subseteq L$*. Suppose that* λ *and* μ *have a common centre on some smooth model of* K*. Then* $\Psi(\lambda) = \Psi(\mu)$*.*

Proof. a) Same argument as in §4.9, using the natural transformation Spec $\Rightarrow \Sigma$ of (3.1). b) follows from Proposition 6.3.2 (compare proof of Theorem 4.2.4 b) in §4.6). □

6.4. Homotopy of places.

6.4.1. Definition. Let $K, L \in \mathbf{place}$. Two places $\lambda_0, \lambda_1 : K \rightsquigarrow L$ are *elementarily homotopic* if there exists a place $\mu : K \rightsquigarrow L(t)$ such that $s_i \circ \mu = \lambda_i, i = 0, 1$, where $s_i : L(t) \rightsquigarrow L$ denotes the place corresponding to specialisation at i.

The property of two places being elementarily homotopic is preserved under composition on the right. Indeed if λ_0 and λ_1 are elementarily homotopic and if $\mu : M \rightsquigarrow K$ is another place, then obviously so are $\lambda_0 \circ \mu$ and $\lambda_1 \circ \mu$. If on the other hand $\tau : L \rightsquigarrow M$ is another place, then $\tau \circ \lambda_0$ and $\tau \circ \lambda_1$ are not in general elementarily homotopic (we are indebted to Gabber for pointing this out), as one can see for example from the uniqueness of factorisation of places [40, p. 10].
Consider the equivalence relation h generated by elementary homotopy (*cf.* Definition 1.2.1). So h is the coarsest equivalence relation on morphisms in **place** which is compatible with left and right composition and such that two elementarily homotopic places are equivalent with respect to h.

6.4.2. Definition (*cf.* Def. 1.2.1). We denote by **place** /h the factor category of **place** by the homotopy relation h.

Thus the objects of **place** /h are function fields, while the set of morphisms consists of equivalence classes of homotopic places between the function fields. There is an obvious full surjective functor

$$\Pi : \mathbf{place} \to \mathbf{place}/\mathrm{h}.$$

The following proposition provides a more elementary description of S_r^{-1} **place** and of the localisation functor.

6.4.3. Proposition. *There is a unique isomorphism of categories*

$$\mathbf{place}/\mathrm{h} \to S_r^{-1}\,\mathbf{place}$$

which makes the diagram of categories and functors

$$\begin{array}{ccc} & \mathbf{place} & \\ {\scriptstyle \Pi}\swarrow & & \searrow{\scriptstyle S_r^{-1}} \\ \mathbf{place}/\mathrm{h} & \xrightarrow{\sim} & S_r^{-1}\,\mathbf{place} \end{array}$$

commutative. In particular, the localisation functor S_r^{-1} is full and its fibres are the equivalence classes for h. *These results remain true when restricted to the subcategory* **dv**.

Proof. [9] We first note that any two homotopic places become equal in S_r^{-1} **place**. Clearly it suffices to prove this when they are elementarily homotopic. But then s_0 and s_1 are left inverses of the natural inclusion $i : L \to L(t)$, which becomes an isomorphism in S_r^{-1} **place**. Thus s_0 and s_1 become equal in S_r^{-1} **place**. So the localisation functor **place** $\to S_r^{-1}$ **place** canonically factors through Π into a functor **place** /h $\longrightarrow$ S_r^{-1} **place**.
On the other hand we claim that, with the above notation, $i \circ s_0 : L(t) \rightsquigarrow L(t)$ is homotopic to $1_{L(t)}$ in **place**. Indeed they are elementarily homotopic via the trivial place $L(t) \rightsquigarrow L(t, s)$ that is the identity on L and maps t to st. Hence

[9] See also [15, Remark 1.3.4] for a closely related statement.

the projection functor Π factors as $S_r^{-1}\,\mathbf{place} \to \mathbf{place}\,/\mathrm{h}$, and it is plain that this functor is inverse to the previous one.
The claim concerning **dv** is clear since the above proof only used good dvr's. □

6.5. Another equivalence of categories. In this subsection, we study the "fibres" of the functor Ψ of (6.5) in the light of the last condition of [1, §6], (A3). Using Proposition 6.4.3, we may view Ψ as a functor

$$\Psi : (\mathbf{dv}\,/\,\mathrm{h})^{\mathrm{op}} \to S_b^{-1}\mathbf{Sm}.$$

Condition (A3) of [1, §6] for a functor $\mathcal{S} \in \mathcal{F}_F^r - \mathbf{Set}$ requires that for any $X \in \mathbf{Sm}$ with function field K, for any $z \in X^{(2)}$ (with separably generated residue field) and for any $y_1, y_2 \in X^{(1)}$ both specialising to z, the compositions

$$\mathcal{S}(K) \to \mathcal{S}(F(y_i)) \to \mathcal{S}(z), \quad i = 1, 2$$

are equal. We can interpret this condition in the present context by introducing the equivalence relation h_{AM} in **dv** generated by h and the following relation $\equiv$:

Given $K, L \in \mathbf{dv}$ and two places $\lambda_1, \lambda_2 : K \rightsquigarrow L$ of the form

$$\text{(6.6)} \qquad \begin{array}{l} K \xrightarrow{\mu_1} K_1 \xrightarrow{\nu_1} L \\ K \xrightarrow{\mu_2} K_2 \xrightarrow{\nu_2} L \end{array}$$

where $\mu_1, \nu_1, \mu_2, \nu_2$ stem from good dvr's, $\lambda_1 \equiv \lambda_2$ if λ_1 and λ_2 have a common centre with residue field L on some smooth model of K.

By Yoneda's lemma, [1, Th. 6.1.7] then yields an equivalence of categories

$$\text{(6.7)} \qquad (\mathbf{dv}\,/\,\mathrm{h}_{AM})^{\mathrm{op}} \xrightarrow{\sim} S_b^{-1}\mathbf{Sm}.$$

Here we implicitly used Lemma 6.3.4 b) and Theorem 6.2.3 to see that the functor $(\mathbf{dv}\,/\,\mathrm{h})^{\mathrm{op}} \to S_b^{-1}\mathbf{Sm}$ factors through h_{AM}, as well as the following lemma:

6.5.1. Lemma. *Let $\psi : \mathcal{C} \to \mathcal{D}$ be a functor such that the induced functor $\psi^* : \mathcal{D}^\vee \to \mathcal{C}^\vee$ is an equivalence of categories. Then ψ is fully faithful, hence an equivalence of categories if it is essentially surjective.*

(Note that the essential surjectivity of (6.7) is obvious.)

Proof. By [SGA4, I.5.3], ψ^* has a left adjoint $\psi_!$ which commutes naturally with ψ via the Yoneda embeddings $y_\mathcal{C}, y_\mathcal{D}$. Since ψ^* is an equivalence of categories, so is $\psi_!$; the conclusion then follows from the full faithfulness of $y_\mathcal{C}$ and $y_\mathcal{D}$. □

We now slightly refine the equivalence (6.7):

6.5.2. Theorem. *a) The functor* Ψ *induces an equivalence of categories:*

$$\overline{\Psi} : (\mathbf{dv} \,/\, \mathrm{h}')^{\mathrm{op}} \xrightarrow{\sim} S_b^{-1}\mathbf{Sm}$$

where h' *is the equivalence relation generated by* h *and the relation* (6.6) *restricted to the tuples* $(\mu_1, \nu_1, \mu_2, \nu_2)$ *such that* ν_2 *is of the form* $s_0 : L(t) \rightsquigarrow L$ *(specialisation at* 0*). In particular,* Ψ *is full.*
b) Any morphism of $\mathbf{dv} \,/\, \mathrm{h}'$ *may be written in the form* $\iota^{-1} f$ *for* f *a morphism of the form* (6.2) *and* ι *a rational extension of function fields.*

Proof. a) Let us show that $\mathrm{h}' = \mathrm{h}_{AM}$. Starting from K, λ_1 and λ_2 as above, we get a smooth model X of K and $z, y_1, y_2 \in X$ with z of codimension 2, such that μ_i is specialisation to y_i and ν_i is specialisation from y_i to z. Shrinking, we may assume that the closures Z, Y_1, Y_2 of z, y_1, y_2 are smooth. Let $X' = \mathrm{Bl}_Z(X)$ be the blow-up of X at Z and let Y_1', Y_2' be the proper transforms of Y_1 and Y_2 in X'. The exceptional divisor P is a projective line over Z and $Z_i = P \cap Y_i'$ maps isomorphically to Z for $i = 1, 2$. We then get new places

$$\begin{aligned} \lambda_1' &: K \xrightarrow{\mu'} M \xrightarrow{\nu_1'} L \\ \lambda_2' &: K \xrightarrow{\mu'} M \xrightarrow{\nu_2'} L \end{aligned} \tag{6.8}$$

where $M = F(P)$, $L = F(Z)$ and $\lambda_i' \equiv \lambda_i$.
In $\mathbf{dv} \,/\, \mathrm{h} \simeq S_r^{-1}\,\mathbf{dv}$, the morphisms ν_1' and ν_2' are inverse to the rational extension $L \hookrightarrow L(t) \simeq M$, hence are equal, which concludes the proof that $\mathrm{h}' = \mathrm{h}_{AM}$.
The fullness of Ψ now follows from the obvious fullness of $\mathbf{dv} \to \mathbf{dv} \,/\, \mathrm{h}'$.
The argument in the proof of a) shows in particular that any composition $\nu \circ \mu$ of two good dvr's is equal in $\mathbf{dv} \,/\, \mathrm{h}'$ to such a composition in which ν is inverse to a purely transcendental extension of function fields: b) follows from this by induction on the number of dvr's appearing in a decomposition (6.1). □

6.5.3. *Remarks.* 1) Via $\overline{\Psi}$, Theorem 6.5.2 yields a structural result for morphisms in $S_b^{-1}\mathbf{Sm}$, closely related to Proposition 5.4.1 c) but weaker. See however Theorem 6.6.3 below.
2) We don't know any example of an object in $\mathcal{F}_F^r - \mathbf{Set}$ which verifies (A1), (A2) and (A4) but not (A3): it would be interesting to exhibit one.

6.6. R-equivalence. Recall the following definition of Manin:

6.6.1. Definition. a) Two rational points x_0, x_1 of a (separated) F-scheme X of finite type are *directly* R*-equivalent* if there is a rational map $f : \mathbf{P}^1 \dashrightarrow X$ defined at 0 and 1 and such that $f(0) = x_0$, $f(1) = x_1$.
b) R-equivalence on $X(F)$ is the equivalence relation generated by direct R-equivalence.

Recall that, for any X, Y, we have an isomorphism

$$(X \times Y)(F)/R \xrightarrow{\sim} X(F)/R \times Y(F)/R. \tag{6.9}$$

The proof is easy.

If X is proper, any rational map as in Definition 6.6.1 a) extends to a morphism; the notion of R-equivalence is therefore the same as Asok-Morel's notion of $\mathbf{A}^1$-equivalence in [1]. Another of their results is then, in the above language:

6.6.2. Theorem ([1, Th. 6.2.1]). *Let X be a proper F-scheme. Then the rule*

$$Y \mapsto X(F(Y))/R$$

defines a presheaf of sets $\Upsilon(X) \in (S_b^{-1}\mathbf{Sm})^\vee$.

Note that $X \mapsto \Upsilon(X)$ is obviously functorial.
The main point is that R-equivalence classes on X specialise well with respect to good discrete valuations. Such a result was originally indicated by Kollár [25, p. 1] for smooth proper schemes over a discrete valuation ring R, and proven by Madore [29, Prop. 3.1] for projective schemes over R. Asok and Morel's proof uses Lipman's resolution of 2-dimensional schemes as well as a strong factorisation result of Lichtenbaum; as hinted by Colliot-Thélène, it actually suffices to use the more elementary results of Šafarevič [35, Lect. 4, Theorem p. 33].
Let X be proper and smooth. Its generic point $\eta_X \in X(F(X))$ defines by Yoneda's lemma a morphism of presheaves

$$\eta(X) : y(X) \to \Upsilon(X) \tag{6.10}$$

where $y(X) \in (S_b^{-1}\mathbf{Sm})^\vee$ is the presheaf of sets represented by X; $\eta : X \mapsto \eta(X)$ is clearly a morphism of functors.

6.6.3. Theorem. *η is an isomorphism of functors. Explicitly: for $Y \in \mathbf{Sm}$, $\eta(X)$ induces an isomorphism*

$$S_b^{-1}\mathbf{Sm}(Y, X) \xrightarrow{\sim} X(F(Y))/R. \tag{6.11}$$

Proof. Since $K \mapsto X(K)$ is a functor on $\mathbf{dv}^{\mathrm{op}}$ (compare [1, Lemma 6.2.3]), we have a commutative diagram for any $Y \in \mathbf{Sm}$:

$$\begin{array}{ccc}
\mathbf{dv}^{\mathrm{op}}(F(Y), F(X)) & \xrightarrow{\tilde\eta} & X(F(Y)) \\
{\scriptstyle\Psi}\downarrow & \overset{\varepsilon}{\swarrow} & \downarrow{\scriptstyle\pi} \\
S_b^{-1}\mathbf{Sm}(Y, X) & \xrightarrow{\eta} & X(F(Y))/R.
\end{array} \tag{6.12}$$

Here $\tilde\eta$ is obtained from η_X by Yoneda's lemma in the same way as (6.10), Ψ is (obtained from) the functor of (6.5), π is the natural projection and ε associates to a rational map its class in $S_b^{-1}\mathbf{Sm}(Y, X)$ (see comment just before Proposition 6.3.2). Here the commutativity of the top triangle follows from Proposition 6.3.2. The surjectivity of π shows the surjectivity of η. Note further that Ψ is surjective by Theorem 6.5.2 a). This shows that ε is also surjective.
To conclude, it suffices to show that ε factors through π, thus yielding an inverse to η. If $x_0, x_1 \in X(F(Y))$ are directly R-equivalent, up to shrinking Y

we have a representing commutative diagram

$$\begin{array}{ccc} Y & \overset{x_0}{\underset{x_1}{\rightrightarrows}} & X \\ {\scriptstyle s_0}\downarrow\downarrow{\scriptstyle s_1} & \nearrow{\scriptstyle h} & \\ \mathbf{P}^1_Y & & \end{array}$$

with s_0, s_1 the inclusions of 0 and 1. But if we view $X(F(Y))$ and $S_b^{-1}\mathbf{Sm}(Y,X)$ as functors of $F(Y) \in \mathbf{dv}^{\mathrm{op}}$ (the second one via Ψ), then ε is checked to be a natural transformation: indeed, this is easy in the case of an inclusion of function fields and follows from the properness of X in the case of a good dvr. Hence we get $\varepsilon(x_0) = \varepsilon(x_1)$ since $S_b^{-1}\mathbf{Sm}(Y,X) \xrightarrow{\sim} S_b^{-1}\mathbf{Sm}(\mathbf{P}^1_Y, X)$ by Theorem 1.7.2. □

6.6.4. Corollary. *The functor $\theta : S_b^{-1}\mathbf{Sm}_*^{\mathrm{prop}} \to S_b^{-1}\mathbf{Sm}$ of (4.14) is fully faithful.*

Proof. For $X, Y \in \mathbf{Sm}_*^{\mathrm{prop}}$, we have a commutative diagram similar to (6.12) replacing $\mathbf{dv}$ by $\mathbf{place}_*$ and $\mathbf{Sm}$ by $\mathbf{Sm}_*^{\mathrm{prop}}$. The map η_* corresponding to η is obtained from (6.12) by composition, while the map $\tilde\eta_*$ corresponding to $\tilde\eta$ exists because $K \mapsto X(K)$ is a functor on $\mathbf{place}^{\mathrm{op}}$ by the valuative criterion of properness. Further, the map corresponding to ε is well-defined thanks to Proposition 4.4.2 b) and the top triangle commutes thanks to Lemma 4.6.1. The natural map from this diagram to (6.12) yields a commutative diagram thanks to Lemma 6.3.4. Moreover, the map corresponding to Ψ is surjective thanks to Theorem 4.2.4 a). The same reasoning as above then shows that η_* is bijective: we just have to replace "up to shrinking Y" by "up to replacing Y by a birationally equivalent smooth projective variety", using the graph trick and the definition of $\mathbf{Sm}_*^{\mathrm{prop}}$. The graph trick can also be used to reduce the verification that ε is natural to the case where the rational maps involved are in fact morphisms. Hence the conclusion. □

6.6.5. *Remark.* One could replace $\mathbf{Sm}_*^{\mathrm{prop}}$ by $\mathbf{Sm}_*^{\mathrm{proj}}$ in Corollary 6.6.4, thus getting a full embedding $S_b^{-1}\mathbf{Sm}_*^{\mathrm{proj}} \hookrightarrow S_b^{-1}\mathbf{Sm}_*^{\mathrm{prop}}$.

The following corollary generalises [5, Prop. 10] to any characteristic:

6.6.6. Corollary. *Let $s : Y \dashrightarrow X$ be a rational map, with $X, Y \in \mathbf{Sm}^{\mathrm{prop}}$, Then s induces an map $s_* : Y(K)/R \to X(K)/R$ for any $K \in \mathbf{dv}$. Moreover, s_* is a bijection for any $K \in \mathbf{dv}$ if and only if the morphism $\tilde s$ associated to s in $S_b^{-1}\mathbf{Sm}$ (see comment just before Proposition 6.3.2) is an isomorphism.*
In particular, $s_* : Y(K)/R \xrightarrow{\sim} X(K)/R$ for any $K \in \mathbf{dv}$ when s is dominant and the field extension $F(Y)/F(X)$ is rational.

Proof. The morphism $\tilde s$ induces a morphism $S_b^{-1}(U,Y) \to S_b^{-1}(U,X)$ for any $U \in \mathbf{Sm}$, hence the first claim follows from Theorem 6.6.3. "If" is obvious, and "only if" follows from Yoneda's lemma. Finally, Theorem 1.7.2 implies that $\tilde s$ is an isomorphism under the last hypothesis on s, hence the conclusion. □

See Theorem 7.3.1 for a further generalisation.

6.7. Coronidis loco. Let us go back to the diagram in Lemma 6.3.4 a). Let h'_* be the equivalence relation on $\mathbf{dv}_*$ defined exactly as h' on $\mathbf{dv}$ (using objects of $\mathbf{dv}_*$ instead of objects of $\mathbf{dv}$). On the other hand, let h'' be the equivalence relation on $\mathbf{place}_*$ generated by h and

> *For $\lambda, \mu : K \rightsquigarrow L$, $\lambda \sim \mu$ if λ and μ have a common centre on some model $X \in \mathbf{Sm}^{\mathrm{prop}}_*$ of K.*

Clearly, the restriction of h'' to $\mathbf{dv}_*$ is coarser than h'; hence, using Theorem 4.2.4 b) and Proposition 6.4.3, we get an induced naturally commutative diagram:

$$\begin{array}{ccc}
(\mathbf{place}_* / \mathrm{h}'')^{\mathrm{op}} & \xrightarrow{\overline{\Psi}_*} & S_b^{-1}\mathbf{Sm}^{\mathrm{prop}}_* \\
\uparrow{\scriptstyle a} & & \\
(\mathbf{dv}_* / \mathrm{h}'_*)^{\mathrm{op}} & & \downarrow{\scriptstyle \theta} \\
\downarrow{\scriptstyle b} & & \\
(\mathbf{dv} / \mathrm{h}')^{\mathrm{op}} & \xrightarrow{\overline{\Psi}} & S_b^{-1}\mathbf{Sm}.
\end{array}$$

In this diagram, $\overline{\Psi}_*$ is full and essentially surjective by Theorem 4.2.4 a), $\overline{\Psi}$ is an equivalence of categories by Theorem 6.5.2 a) and θ is fully faithful by Corollary 6.6.4. Moreover, a is full by Lemma 2.3.4 and the proof of Lemma 2.3.2, and essentially surjective by defnition. All this implies:

6.7.1. *Theorem. *If* $\operatorname{char} k = 0$, *all functors in the above diagram are equivalences of categories.*

Proof. If $\operatorname{char} k = 0$, $\mathbf{dv}_* = \mathbf{dv}$ hence b is the identity functor. In view of the above remarks, the diagram then shows that a is faithful, hence an equivalence of categories. It follows that $\overline{\Psi}_*$ is also an equivalence of categories. Finally θ is essentially surjective, which completes the proof. □

As an application, we get a generalisation of the specialisation theorem to arbitrary places (already obtained in [20, Cor. 7.1.2]):

6.7.2. *Corollary. *Suppose* $\operatorname{char} F = 0$. *Let* $X \in \mathbf{Var}^{\mathrm{prop}}$, $K, L \in \mathbf{place}_*$ *and* $\lambda : K \rightsquigarrow L$ *be a place. Then* λ *induces a map*

$$\lambda_* : X(K)/R \to X(L)/R.$$

If $\mu : L \rightsquigarrow M$ *is another place, with* $M \in \mathbf{place}_*$, *then* $(\mu\lambda)_* = \mu_*\lambda_*$.

Proof. By Theorem 6.6.2, $K \mapsto X(K)/R$ defines a presheaf on $(\mathbf{dv}/h')^{\mathrm{op}}$, which extends to a presheaf on $(\mathbf{place}_* / \mathrm{h}'')^{\mathrm{op}}$ by Theorem 6.7.1. □

7. Linear connectedness of exceptional loci

7.1. Linear connectedness. We have the following definition of Chow [3]:

7.1.1. Definition. A (separated) F-scheme X of finite type is *linearly connected* if any two points of X (over a universal domain) may be joined by a chain of rational curves.

Linear connectedness is closely related to the notion of rational chain-connectedness of Kollár et al., for which we refer to [7, p. 99, Def. 4.21]. In fact:

7.1.2. Proposition. *The following conditions are equivalent:*

(i) *X is linearly connected.*
(ii) *For any algebraically closed extension K/F, $X(K)/R$ is reduced to a point.*
If X is a proper F-variety, these conditions are equivalent to:
(iii) *X is rationally chain-connected.*

Proof. (ii) $\Rightarrow$ (i) is obvious by definition (take for K a universal domain). For the converse, let $x_0, x_1 \in X(K)$. Then x_0 and x_1 are defined over some finitely generated subextension E/F. By assumption, there exists a universal domain $\Omega \supset E$ such that x_0 and x_1 are R-equivalent in $X(\Omega)$. Then the algebraic closure $\bar{E}$ of E embeds into Ω and K. If x_0 and x_1 are R-equivalent in $X(\bar{E})$, so are they in $X(K)$; this reduces us to the case where $K \subseteq \Omega$.
Let $\gamma_1, \dots, \gamma_n : \mathbf{P}^1_\Omega \dashrightarrow X_\Omega$ be a chain of rational curves linking x_0 and x_1 over Ω. Pick a finitely generated extension L of K over which all the γ_i are defined. We may write $L = K(U)$ for some K-variety U. Then the γ_i define rational maps $\tilde{\gamma}_i : U \times \mathbf{P}^1 \dashrightarrow X$. Since each γ_i is defined at 0 and 1 with $\gamma_i(1) = \gamma_{i+1}(0)$, we may if needed shrink U so that the domains of definition of all the $\tilde{\gamma}_i$ contain $U \times \{0\}$ and $U \times \{1\}$. Moreover, these restrictions coincide in the same style as above, since they do at the generic point of U. Pick a rational point $u \in U(K)$: then the fibres of the $\tilde{\gamma}_i$ at u are rational curves defined over K that link x_0 to x_1.
A rationally chain connected F-scheme is a proper variety by definition; then (i) $\iff$ (iii) if F is uncountable by [7, p. 100, Remark 4.22 (2)]. On the other hand, the property of linear connectedness is clearly invariant under algebraically closed extensions, and the same holds for rational chain-connectedness by [7, p. 100, Remark 4.22 (3)]. Thus (i) $\iff$ (iii) holds in general. □

We shall discuss the well-known relationship with rationally connected varieties in §8.5.
Proposition 7.1.2 suggests the following definition:

7.1.3. Definition. A separated F-scheme X of finite type is *strongly linearly connected* if $X(K)/R = *$ for any separable extension K/F.

7.2. Theorems of Murre, Chow, van der Waerden and Gabber. We start with the following not so well-known but nevertheless basic theorem of Murre [33], which was later rediscovered by Chow and van der Waerden [3, 38].

7.2.1. Theorem (Murre, Chow, van der Waerden). *Let $f : X \to Y$ be a projective birational morphism of F-varieties and $y \in Y$ be a smooth rational point. Then the fibre $f^{-1}(y)$ is linearly connected. In particular, by Proposition 7.1.2, $f^{-1}(y)(K)/R$ is reduced to a point for any algebraically closed extension K/F.*

For the sake of completeness, we give the general statement of Chow, which does not require a base field:

7.2.2. Theorem (Chow). *Let A be a regular local ring and $f : X \to \operatorname{Spec} A$ be a projective birational morphism. Let s be the closed point of $\operatorname{Spec} A$ and F its residue field. Then the special fibre $f^{-1}(s)$ is linearly connected (over F).*

Gabber has recently refined these theorems:

7.2.3. Theorem (Gabber). *Let A, X, f, s, F be as in Theorem 7.2.2, but assume only that f is proper. Let X_{reg} be the regular locus of X and $f^{-1}(s)^{\mathrm{reg}} = f^{-1}(s) \cap X_{\mathrm{reg}}$, which is known to be open in $f^{-1}(s)$. Then, for any extension K/F, any two points of $f^{-1}(s)^{\mathrm{reg}}(K)$ become R-equivalent in $f^{-1}(s)(K)$. In particular, if X is regular, then $f^{-1}(s)$ is strongly linearly connected.*

See Appendix B for a proof of Theorem 7.2.3.

7.2.4. Theorem (Gabber [11]). *If F is a field, X is a regular irreducible F-scheme of finite type and K/F a field extension, then the map*

$$\varprojlim X'(K)/R \to X(K)/R$$

has a section, which is contravariant in X and covariant in K. The limit is over the proper birational $X' \to X$.

7.3. Applications. The following theorem extends part of Corollary 6.6.6 to a relative setting:

7.3.1. Theorem. *a) Let $s : Y \to X$ be in S_b^p, with X, Y regular. Then the induced map $Y(K)/R \to X(K)/R$ is bijective for any field extension K/F. If K is algebraically closed, the hypothesis "Y regular" is not necessary.*
b) Let $f : Y \dashrightarrow Z$ be a rational map with Y regular and Z proper. Then there is an induced map $f_* : Y(K)/R \to Z(K)/R$, which depends functorially on K/F.

Proof. a) As in the proof of Proposition 5.4.1 a), it suffices to deal with $K = F$. By this proposition, we have to show injectivity.
We assume that $s \in S_b^p$. Let $y_0, y_1 \in Y(F)$. Suppose that $s(y_0)$ and $s(y_1)$ are R-equivalent. We want to show that y_0 and y_1 are then R-equivalent. By definition, $s(y_0)$ and $s(y_1)$ are connected by a chain of direct R-equivalences. Applying Proposition 5.4.1 a), the intermediate rational points lift to $Y(F)$. This reduces us to the case where $s(y_0)$ and $s(y_1)$ are directly R-equivalent.

Let $\gamma : \mathbf{P}^1 \dashrightarrow X$ be a rational map defined at 0 and 1 such that $\gamma(i) = s(y_i)$. Applying Proposition 5.4.1 a) with $K = F(t)$, we get that γ lifts to a rational map $\tilde\gamma : \mathbf{P}^1 \dashrightarrow Y$. Since s is proper, $\tilde\gamma$ is still defined at 0 and 1. Let $y_i' = \tilde\gamma(i) \in Y(F)$: then $y_i, y_i' \in s^{-1}(s(y_i))$. If F is algebraically closed, they are R-equivalent by Theorem 7.2.1, thus y_0 and y_1 are R-equivalent. If F is arbitrary but Y is regular, then we appeal to Theorem 7.2.3.
b) By the usual graph trick, as Z is proper, we can resolve f to get a morphism

$$Y \xleftarrow{\;p\;} \tilde Y \xrightarrow{\;\tilde f\;} Z$$

such that p is a proper birational morphism. By Theorem 7.2.4, the map $p_* : \tilde Y(K)/R \to Y(K)/R$ has a section which is "natural" in p (*i.e.* when we take a finer p, the two sections are compatible). The statement follows. □

7.3.2. *Remark.* Concerning Theorem 7.2.3, Fakhruddin pointed out that $f^{-1}(s)$ is in general not strongly linearly connected, while Gabber pointed out that $f^{-1}(s)^{\mathrm{reg}}(F)$ may be empty even if X is normal, when F is not algebraically closed. Here is Gabber's example: in dimension 2, blow-up the maximal ideal of A and then a non-rational point of the special fiber, then contract the proper transform of the special fiber. Gabber also gave examples covering Fakhruddin's remark: suppose $\dim A = 2$ and start from $X_0 =$ the blow-up of $\operatorname{Spec} A$ at s. Using [8], one can "pinch" X_0 so as to convert a non-rational closed point of the special fibre into a rational point. The special fibre of the resulting $X \to \operatorname{Spec} A$ is then a singular quotient of $\mathbf{P}^1_F$, with two R-equivalence classes. He also gave a normal example [11].

8. Examples, applications and open questions

In this section, we put together some concrete applications of the above results and list some open questions.

8.1. Composition of R-equivalence classes. As a by-product of Theorem 6.6.3, one gets for three smooth proper varieties X, Y, Z over a field of characteristic 0 a composition law

$$Y(F(X))/R \times Z(F(Y))/R \to Z(F(X))/R \tag{8.1}$$

which is by no means obvious. As a corollary, we have:

8.1.1. Corollary. *Let X be a smooth proper variety with function field K. Then $X(K)/R$ has a structure of a monoid with η_X as the identity element.* □

8.2. R-equivalence and birational functors. Here is a more concrete reformulation of part of Theorem 6.6.3 and Corollary 6.6.4:

8.2.1. Corollary. *Let*

$$P : \mathbf{Sm} \to \mathcal{A}$$

be a functor to some category $\mathcal{A}$. Suppose that P is a birational functor. Then if X, Y are two smooth varieties with X proper, any class $x \in X(F(Y))/R$ induces a morphism $x_* : P(Y) \to P(X)$. This assignment is compatible with the composition of R-equivalence classes from (8.1).
In particular, for two morphisms $f, g : Y \to X$, $P(f) = P(g)$ as soon as $f(\eta_Y)$ and $g(\eta_Y)$ are R-equivalent.
The same statement holds for a birational functor $P : \mathbf{Sm}_*^{\mathrm{prop}} \to \mathcal{A}$, with $X, Y \in \mathbf{Sm}_*^{\mathrm{prop}}$.

Theorem 6.6.3 further says that R-equivalence is "universal" among birational functors.

8.3. Algebraic groups and R-equivalence. As a special case of Corollary 8.1.1, we consider a connected algebraic group G defined over F. Recall that for any extension K/F, the set $G(K)/R$ is in fact a group. Let $\bar{G}$ denote a smooth compactification of G over F (we assume that there is one). It is known (P. Gille, [13]) that the natural map $G(F)/R \to \bar{G}(F)/R$ is an isomorphism if F has characteristic zero and G is reductive.
Let K denote the function field $F(G)$. By the above corollary, there is a composition law $\circ$ on $\bar{G}(K)/R$. On the other hand, the multiplication morphism

$$m : G \times G \to G$$

considered as a rational map on $\bar{G} \times \bar{G}$ induces a product map (Theorem 7.3.1)

$$\bar{G}(K)/R \times \bar{G}(K)/R \to \bar{G}(K)/R$$

which we denote by $(g, h) \mapsto g \cdot h$; this is clearly compatible with the corresponding product map on $G(K)/R$ obtained using the multiplication homomorphism on G. Thus we have two composition laws on $\bar{G}(K)/R$.
The following lemma is a formal consequence of Yoneda's lemma:

8.3.1. Lemma. *Let $g_1, g_2, h \in \bar{G}(K)/R$. Then we have $(g_1 \cdot g_2) \circ h = (g_1 \circ h) \cdot (g_2 \circ h)$.* □

In particular, let us take $G = SL_{1,A}$, where A is a central simple algebra over F. It is then known that $G(K)/R \simeq SK_1(A_K)$ for any function field K. If $\operatorname{char} F = 0$, we may use Gille's theorem and find that, for $K = F(G)$, $SK_1(A_K)$ admits a second composition law with unit element the generic element, which is distributive on the right with respect to the multiplication law. However, it is not distributive on the left in general:
Note that the natural map $\mathrm{Hom}(\operatorname{Spec} F, \bar{G}) = \bar{G}(F)/R \to \bar{G}(K)/R = \mathrm{Hom}(\bar{G}, \bar{G})$ is split injective, a retraction being induced by the unit section $\operatorname{Spec} F \to G \to \bar{G}$. Now let $g \in G(F)$; for any $\varphi \in G(K) = \mathbf{Rat}(G, G)$, we

clearly have $[g]\circ[\varphi]=[g]$. In particular, $[g]\circ([\varphi]\cdot[\varphi'])\neq([g]\circ[\varphi])\cdot([g]\circ[\varphi'])$ unless $[g]=1$. (This argument works for any group object in a category with finite products.)

8.4. Kan extensions and Π_1. Let as before $\mathbf{Sm}_*$ denote the full subcategory of $\mathbf{Sm}$ given by those smooth varieties which admit a cofinal system of smooth proper compactifications: then the functor θ of Corollary 6.6.4 induces an equivalence of categories $S_b^{-1}\mathbf{Sm}_*^{\text{prop}}\xrightarrow{\sim}S_b^{-1}\mathbf{Sm}_*$. Suppose we are given a functor $F:\mathbf{Sm}_*\to\mathcal{C}$ whose restriction to $\mathbf{Sm}_*^{\text{prop}}$ is birational. We then get an induced functor $\bar F: S_b^{-1}\mathbf{Sm}_*\to\mathcal{C}$ plus a natural transformation

$$\rho_X: F(X)\to\bar F(X)$$

for any $X\in\mathbf{Sm}_*$.
To construct $\bar F$, we set

$$\bar F(X)=\varprojlim_{\bar X}F(\bar X)$$

where the limit is on the category of open immersions $j:X\hookrightarrow\bar X$ with $\bar X\in\mathbf{Sm}_*^{\text{prop}}$: this is an inverse limit of isomorphisms, hence makes sense without any hypothesis on $\mathcal{C}$ and may be computed by taking any representative $\bar X$. To construct ρ_X, an open immersion $j:X\hookrightarrow\bar X$ as above yields a map $F(X)\xrightarrow{F(j)}F(\bar X)\simeq\bar F(X)$, and one checks that this does not depend on the choice of j. This is an instance of a *left Kan extension* [28, Ch. X, §3], compare [21, §3] and [18, lemme 6.5].
We may apply this to $F=\Pi_1$, the fundamental groupoid[10] (here $\mathcal{C}$ is the category of groupoids): the required property is [SGA1, Exp. X, Cor. 3.4]. As an extra feature, we get that the universal transformation ρ is an epimorphism, because $\Pi_1(U)\twoheadrightarrow\Pi_1(X)$ if $U\hookrightarrow X$ is an open immersion of smooth schemes. Thus, $\Pi_1(X)$ has a "universal birational quotient" which is natural in X.
As another application, we get that for $X\in\mathbf{Sm}_*^{\text{prop}}$, the "section map" (subject to a famous conjecture of Grothendieck when X is a curve)

$$X(F)\to\operatorname{Hom}_{\Pi_1(\operatorname{Spec}F)}(\Pi_1(\operatorname{Spec}F),\Pi_1(X)) \tag{8.2}$$

factors through R-equivalence. On the other hand, if X is projective and Y is a smooth hyperplane section, then $\Pi_1(Y)\xrightarrow{\sim}\Pi_1(X)$ as long as $\dim X>2$ by [SGA2, Exp. XII, Cor. 3.5]; so there are more morphisms to invert if one wishes to study (8.2) for $\dim X>1$ by the present methods.

8.5. Strongly linearly connected smooth proper varieties. One natural question that arises is the following: characterise morphisms $f:X\to Y$ between smooth proper varieties which become invertible in the category $S_b^{-1}\mathbf{Sm}$. Here we shall study this question only in the simplest case, where $Y=\operatorname{Spec}F$.

8.5.1. Theorem. *a) Let X be a smooth proper variety over F. Consider the following conditions:*

[10] Rather than fundamental group, to avoid the choices of base points.

(1) *$p : X \to \operatorname{Spec} F$ is an isomorphism in $S_b^{-1}\mathbf{Sm}$.*
(2) *p is an isomorphism in $S_r^{-1}\mathbf{Sm}$.*
(3) *For any separable extension E/F, $X(E)/R$ has one element (i.e. X is strongly linearly connected according to Definition 7.1.3).*
(4) *Same, for E/F of finite type.*
(5) *$X(F) \neq \emptyset$ and $X(K)/R$ has one element for $K = F(X)$.*
(6) *$X(F) \neq \emptyset$ and, given $x_0 \in X(F)$, there exists a chain of rational curves $(f_i : \mathbf{P}^1_K \to X_K)_{i=1}^n$ such that $f_1(0) = \eta_X$, $f_{i+1}(0) = f_i(1)$ and $f_n(1) = x_0$. Here $K = F(X)$ and η_X is the generic point of X.*
(7) *Same as (6), but with $n = 1$.*

Then $(1) \iff (2) \iff (3) \iff (4) \iff (5) \iff (6) \Leftarrow (7)$.
b) If $\operatorname{char} F = 0$, *$X$ satisfies Conditions* $(1) - (6)$ *and is projective, it is rationally connected.*

Proof. a) $(1) \Rightarrow (2)$ is trivial and the converse follows from Theorem 1.7.2. Thanks to Theorem 6.6.3, $(2) \iff (4)$ is an easy consequence of the Yoneda lemma. The implications $(3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6) \Leftarrow (7)$ are trivial and $(4) \Rightarrow (3)$ is easy by a direct limit argument. To see $(6) \Rightarrow (1)$, note that by Theorem 6.6.3 (6) implies that $1_X = x_0 \circ p$ in $S_b^{-1}\mathbf{Sm}(X, X)$, hence p is an isomorphism.
b) This follows from Proposition 7.1.2 plus the famous theorem of Kollár-Miyaoka-Mori [24, Th. 3.10], [7, p. 107, Cor. 4.28]. □

8.5.2. *Remark.* The example of an anisotropic conic shows that, in (5), the assumption $X(F) \neq \emptyset$ does not follow from the next one.

8.5.3. *Question.* In the situation of Theorem 8.5.1 b), does X verify condition (7)? We give a partial result in this direction in Proposition 8.6.2 below. (The reader may consult the first version of this paper for a non-conclusive attempt to answer this question in general.)

8.6. Retract-rational varieties. Recall that, following Saltman, X (smooth but not necessarily proper) is *retract-rational* if it contains an open subset U such that U is a retract of an open subset of $\mathbf{A}^n$. When F is infinite, this includes the case where there exists Y such that $X \times Y$ is rational, as in [5, Ex. A. pp. 222/223].
We have a similar notion for function fields:

8.6.1. Definition. A function field K/F is *retract-rational* if there exists an integer $n \geq 0$ and two places $\lambda : K \rightsquigarrow F(t_1, \dots, t_n)$, $\mu : F(t_1, \dots, t_n) \rightsquigarrow K$ such that $\mu\lambda = 1_K$.

Note that this forces λ to be a trivial place (*i.e.* an inclusion of fields). Using Lemma 2.3.2, we easily see that X is retract-rational if and only if $F(X)$ is retract-rational.

8.6.2. Proposition. *If X is a retract-rational smooth variety, then $X \xrightarrow{\sim} \operatorname{Spec} F$ in $S_b^{-1}\mathbf{Sm}$. If moreover X is proper and F is infinite, then X verifies Condition (7) of Theorem 8.5.1 for a Zariski dense set of points x_0.*

Proof. The first statement is obvious by Yoneda's lemma. Let us prove the second: by hypothesis, there exist open subsets $U \subseteq X$ and $V \subseteq \mathbf{A}^n$ and morphisms $f : U \to V$ and $g : V \to U$ such that $gf = 1_U$. This already shows that $U(F)$ is Zariski-dense in X. Let now $x_0 \in U(F)$, and let $K = F(X)$. Consider the straight line $\gamma : \mathbf{A}^1_K \to \mathbf{A}^n_K$ such that $\gamma(0) = f(x_0)$ and $\gamma(1) = f(\eta_X)$. Then $g \circ \gamma$ links x_0 to η_X, as desired. □

8.6.3. Corollary. *We have the following implications for a smooth proper variety X over a field F of characteristic 0: retract-rational ⇒ strongly linearly connected ⇒ rationally connected.*

Proof. The first implication follows from Theorem 8.5.1 and Proposition 8.6.2; the second implication follows from the theorem of Kollár-Miyaoka-Mori already quoted. □

8.6.4. *Remark.* In characteristic 0, if X is a smooth compactification of a torus, then it verifies Conditions (1) – (6) of Theorem 8.5.1 if and only if it is retract-rational, by [6, Prop. 7.4] (*i.e.* the first implication in the previous corollary is an equivalence for such X). This may also be true by replacing "torus" by "connected reductive group": at least it is so in many special cases, see [14, Th. 7.2 and Cor. 5.10].

8.7. S_r-local objects. Recall:

8.7.1. Definition. Let $\mathcal{C}$ be a category and S a family of morphisms of $\mathcal{C}$. An object $X \in \mathcal{C}$ is *local* relatively to S or S-*local* (*left closed* in the terminology of [12, Ch. 1, Def. 4.1 p. 19]) if, for any $s : Y \to Z$ in S, the map

$$\mathcal{C}(Z, X) \xrightarrow{s^*} \mathcal{C}(Y, X)$$

is bijective.

In this rather disappointing subsection, we show that there are not enough of these objects. They are the exact opposite of rationally connected varieties.

8.7.2. Definition. A proper F-variety X is *nonrational* if it does not carry any nonconstant rational curve (over the algebraic closure of F), or equivalently if the map

$$X(\bar{F}) \to X(\bar{F}(t))$$

is bijective.

8.7.3. Lemma. *a) Nonrationality is stable by product and by passing to closed subvarieties.*
b) Curves of genus > 0 and torsors under abelian varieties are nonrational.
c) Nonrational smooth projective varieties are minimal in the sense that their canonical bundle is nef.

Proof. a) and b) are obvious; c) follows from the Miyaoka-Mori theorem ([30], see also [26, Th. 1.13] or [7, Th. 3.6]). □

On the other hand, an anisotropic conic is not a nonrational variety. This is also true for some minimal models in dimension 2, even when F is algebraically closed.
Smooth nonrational varieties are the local objects of **Sm** with respect to S_r in the sense of Definition 8.7.1:

8.7.4. Lemma. *a) A proper variety X is nonrational if and only if, for any morphism $f : Y \to Z$ between smooth varieties such that $f \in S_r$, the map*

$$f^* : Map(Z, X) \to Map(Y, X)$$

is bijective.
b) A smooth proper nonrational variety X is stably minimal in the following sense: any morphism in S_r with source X is an isomorphism.

Proof. a) Necessity is clear (take $f : \mathbf{P}^1 \to \operatorname{Spec} F$). For sufficiency, f^* is clearly injective since f is dominant, and we have to show surjectivity. We may assume F algebraically closed. Let U be a common open subset to Y and $Z \times \mathbf{P}^n$ for suitable n. Let $\psi : Y \to X$. By [26, Cor. 1.5] or [7, Cor. 1.44], $\psi_{|U}$ extends to a morphism φ on $Z \times \mathbf{P}^n$. But for any closed point $z \in Z$, $\varphi(\{z\} \times \mathbf{P}^1)$ is a point, where $\mathbf{P}^1$ is any line of $\mathbf{P}^n$. Therefore $\varphi(\{z\} \times \mathbf{P}^n)$ is a point, which implies that φ factors through the first projection.
b) immediately follows from a). □

8.7.5. Lemma. *If X is nonrational, it remains nonrational over any extension K/F.*

Proof. It is a variant of the previous one: we may assume that F is algebraically closed and that K/F is finitely generated. Let $f : \mathbf{P}^1_K \to X_K$. Spread f to a U-morphism $\tilde{f} : U \times \mathbf{P}^1 \to U \times X$ and compose with the second projection. Any closed point $u \in U$ defines a map $f_u : \mathbf{P}^1 \to X$, which is constant, hence $p_2 \circ \tilde{f}$ factors through the first projection, which implies that f is constant. □

8.8. Open questions. We finish by listing a few problems that are not answered in this paper.

(1) Compute Hom sets in $S_b^{-1}\,\mathbf{Var}$. In [21, Rk. 8.11], it is shown that the functor $S_b^{-1}\mathbf{Sm} \to S_b^{-1}\,\mathbf{Var}$ is neither full nor faithful and that the Hom sets are in fact completely different.
(2) Compute Hom sets in $(S_b^p)^{-1}\mathbf{Sm}$.
(3) Let $d_{\le n}\mathbf{Sm}$ be the full subcategory of **Sm** consisting of smooth varieties of dimension $\le n$. Is the induced functor $S_b^{-1}d_{\le n}\mathbf{Sm} \to S_b^{-1}\mathbf{Sm}$ fully faithful?
(4) Give a categorical interpretation of rationally connected varieties.
(5) Finally one should develop additional functoriality: products and internal Homs, change of base field.

Appendix A. Invariance birationnelle et invariance homotopique

par Jean-Louis Colliot-Thélène
14 septembre 2006.

Soit k un corps. Soit F un foncteur contravariant de la catégorie des k-schémas vers la catégorie des ensembles. Si sur les morphismes k-birationnels de surfaces projectives, lisses et géométriquement connexes ce foncteur induit des bijections, alors l'application $F(k) \to F(\mathbf{P}^1_k)$ est une bijection.

Démonstration. Toutes les variétés considérées sont des k-variétés. On écrit $F(k)$ pour $F(\mathrm{Spec}(k))$. Soit W l'éclaté de $\mathbf{P}^1 \times \mathbf{P}^1$ en un k-point M. Les transformés propres des deux génératrices L_1 et L_2 passant par M sont deux courbes exceptionnelles de première espèce $E_1 \simeq \mathbf{P}^1$ et $E_2 \simeq \mathbf{P}^1$ qui ne se rencontrent pas. On peut donc les contracter simultanément, la surface que l'on obtient est le plan projectif $\mathbf{P}^2$. Notons M_1 et M_2 les k-points de $\mathbf{P}^2$ sur lesquels les courbes E_1 et E_2 se contractent.
On réalise facilement cette construction de manière concrète. Dans $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^2$ avec coordonnées multihomogènes $(u, v; w, z; X, Y, T)$ on prend pour W la surface définie par l'idéal $(uT - vX, wT - zY)$, et on considère les deux projections $W \to \mathbf{P}^1 \times \mathbf{P}^1$ et $W \to \mathbf{P}^2$.
On a un diagramme commutatif de morphismes

$$\begin{array}{ccc} E_1 & \longrightarrow & W \\ \wr\downarrow & & \downarrow \\ L_1 & \longrightarrow & \mathbf{P}^1 \times \mathbf{P}^1. \end{array}$$

Le composé de l'inclusion $L_1 \hookrightarrow \mathbf{P}^1 \times \mathbf{P}^1$ et d'une des deux projections $\mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1$ est un isomorphisme. Par fonctorialité, la restriction $F(\mathbf{P}^1 \times \mathbf{P}^1) \to F(L_1)$ est donc surjective. Par fonctorialité, le diagramme ci-dessus implique alors que la restriction $F(W) \to F(E_1)$ est surjective.
Considérons maintenant la projection $W \to \mathbf{P}^2$. On a ici le diagramme commutatif de morphismes

$$\begin{array}{ccc} E_1 & \longrightarrow & W \\ \downarrow & & \downarrow \\ M_1 & \longrightarrow & \mathbf{P}^2. \end{array}$$

Par l'hypothèse d'invariance birationnelle, on a la bijection $F(\mathbf{P}^2) \xrightarrow{\sim} F(W)$. Donc la flèche composée $F(\mathbf{P}^2) \to F(W) \to F(E_1)$ est surjective. Mais par le diagramme commutatif ci-dessus la flèche composée se factorise aussi comme $F(\mathbf{P}^2) \to F(M_1) \to F(E_1)$. Ainsi $F(M_1) \to F(E_1)$, c'est-à-dire $F(k) \to F(\mathbf{P}^1)$, est surjectif. L'injectivité de $F(k) \to F(\mathbf{P}^1)$ résulte de la fonctorialité et de la considération d'un k-point sur $\mathbf{P}^1$.

Appendix B. A letter from O. Gabber

June 12, 2007

Dear Kahn,

I discuss a proof of

B.0.1. Theorem. *Let A be a regular local ring with residue field k, $X' \to X = Spec(A)$ a proper birational morphism, X'_{reg} the regular locus of X', X'_s the special fiber of X', $X'_{reg,s} = X'_s \cap X'_{reg}$, which is known to be open in X'_s, F a field extension of k, then any two points of $X'_{reg,s}(F)$ are R-equivalent in $X'_s(F)$.*

The proof I tried to sketch by joining centers of divisorial valuations has a gap in the imperfect residue field case. It is easier to adapt the proof by deformation of local arcs.
(1) If $Y' \to Y$ is proper surjective map between separated k-schemes of finite type whose fibers are projective spaces then for every F/k, $Y'(F)/R \to Y(F)/R$ is bijective. In particular the theorem holds if X' is obtained from X by a sequence of blow-ups with regular centers.
(2) If A is a regular local ring of dimension > 1 with maximal ideal $\mathfrak{m}$, U an open non empty in Spec(A), then there is $f \in \mathfrak{m} - \mathfrak{m}^2$ s.t. the generic point of $V(f)$ is in U.
This is because U omits only a finite number of height 1 primes and there are infinitely many possibilities for $V(f)$, e.g. $V(x - y^i)$ where x, y is a part of a regular system of parameters.
Inductively we get that there is $P \in U$ s.t. A/P is regular 1-dimensional.
(3) If A is a regular local ring and P, P' different prime ideals with A/P and A/P' regular one dimensional, then there is a prime ideal $Q \subset P \cap P'$ with A/Q regular 2-dimensional.
Indeed let $x_1, \dots, x_n$ be a minimal system of generators of P; their images in A/P' generate a principal ideal; we may assume this ideal is generated by the image of x_1, and then we can substract some multiples of x_1 from $x_2, \dots, x_n$ so that the images of $x_2, \dots, x_n$ are 0; take $Q = (x_2, \dots, x_n)$.
To prove the theorem we may assume F is a finitely generated extension of k, so F is a finite extension of a purely transcendental extension k' of k. We replace A by the local ring at the generic point of the special fiber of an affine space over A that has residue field k'. So we reduce to F/k finite. Let x, y be F-points of X'_s centered at closed points a, b at which X' is regular. Let U be dense open of X above which $X' \to X$ is an isomorphism. Let $X'(a), X'(b)$ be the local schemes (Spec of the local rings at a and b). There are regular one dimensional closed subschemes

$$C \subset X'(a), C' \subset X'(b)$$

whose generic points map to U.
By EGA 0_{III} 10.3 there are finite flat $D \to C$, $D' \to C'$ which are $Spec(F)$ over the closed points of C, C'. Then D, D' are Spec's of DVRs essentially

of finite type over A (localization of finite type A-algebras). We form the pushout of $D \leftarrow Spec(F) \to D'$, which is Spec of a fibered product ring, which by some algebraic exercise is still an A-algebra essentially of finite type. The pushout can be embedded as a closed subscheme in Spec of a local ring of an affine space over A and then by (3) in some Y a 2-dimensional local regular A-scheme essentially of finite type. Now D, D' are subschemes of Y. We have a rational map $Y \to X'$ defined on the inverse image of U and in particular at the generic points of D and D'. By e.g. Theorem 26.1 in Lipman's paper on rational singularities (Publ. IHES 36) there is $Y' \to Y$ obtained as a succession of blow-ups at closed points s.t. the rational map gives a morphism $Y' \to X'$. Then x, y are images of F-points of Y' (closed points of the proper transforms of D, D'),and by (1) any two F-points of the special fiber of $Y' \to Y$ are R-equivalent.

Sincerely,

Ofer Gabber

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K-Theory as an Eilenberg-Mac Lane Spectrum

To Sasha Merkurjev, on the occasion of his 60th birthday

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Abstract. For an additive Waldhausen category linear over a ring k, the corresponding K-theory spectrum is a module spectrum over the K-theory spectrum of k. Thus if k is a finite field of characteristic p, then after localization at p, we obtain an Eilenberg-Mac Lane spectrum – in other words, a chain complex. We propose an elementary and direct construction of this chain complex that behaves well in families and uses only methods of homological algebra (in particular, the notions of a ring spectrum and a module spectrum are not used).

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Contents

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Introduction.

Various homology and cohomology theories in algebra or algebraic geometry usually take as input a ring A or an algebraic variety X, and produce as output a certain chain complex; the homology groups of this chain complex are by definition the homology or cohomology groups of A or X. Higher algebraic K-groups are very different in this respect – by definition, the groups $K_\bullet(A)$ are homotopy groups of a certain spectrum $\mathcal{K}(A)$. Were it possible to represent $K_\bullet(A)$ as homology groups of a chain complex, one would be able to study it by means of the well-developed and powerful machinery of homological algebra. However, this is not possible: the spectrum $\mathcal{K}(A)$ is almost never a spectrum of the Eilenberg-Mac Lane type.

If one wishes to turn $\mathcal{K}(A)$ into an Eilenberg-Mac Lane spectrum, one needs to complete it or to localize it in a certain set of primes. The cheapest way to do it is of course to localize in *all* primes – rationally, every spectrum is an Eilenberg-Mac Lane spectrum, and the difference between spectra and complexes disappears. The groups $K_\bullet(A) \otimes \mathbb{Q}$ are then the primitive elements in the homology groups $H_\bullet(BGL_\infty(A), \mathbb{Q})$, and this allows for some computations using homological methods. In particular, $K_\bullet(A) \otimes \mathbb{Q}$ has been computed by Borel when A is a number field, and the relative K-groups $K_\bullet(A, I) \otimes \mathbb{Q}$ of a $\mathbb{Q}$-algebra A with respect to a nilpotent ideal $I \subset A$ have been computed in full generality by Goodwillie [Go].

However, there is at least one other situation when $\mathcal{K}(A)$ becomes an Eilenberg-Mac Lane spectrum after localization. Namely, if A is a finite field k of characteristic p, then by a famous result of Quillen [Q], the localization $\mathcal{K}(A)_{(p)}$ of the spectrum $\mathcal{K}(A)$ at p is the Eilenberg-Mac Lane spectrum $H(\mathbb{Z}_{(p)})$ corresponding to the ring $\mathbb{Z}_{(p)}$. Moreover, if A is an algebra over k, then $\mathcal{K}(A)$ is a module spectum over $\mathcal{K}(k)$ by a result of Gillet [Gi]. Then $\mathcal{K}(A)_{(p)}$ is a module spectrum over $H(\mathbb{Z}_{(p)})$, thus an Eilenberg-Mac Lane spectrum corresponding in the standard way ([Sh, Theorem 1.1]) to a chain complex $K_\bullet(A)_{(p)}$ of $\mathbb{Z}_{(p)}$-modules. More generally, if we have a k-linear exact or Waldhausen category $\mathcal{C}$, the p-localization $\mathcal{K}(\mathcal{C})_{(p)}$ of the K-theory spectrum $\mathcal{K}(\mathcal{C})$ is also an Eilenberg-Mac Lane spectrum corresponding to a chain complex $K_\bullet(\mathcal{C})_{(p)}$. Moreover, if we have a nilpotent ideal $I \subset A$ in a k-algebra A, then the relative K-theory spectrum $\mathcal{K}(A, I)$ is automatically p-local. Thus $\mathcal{K}(A, I) \cong \mathcal{K}(A, I)_{(p)}$ is an Eilenberg-Mac Lane spectrum "as is", without further modifications.

Unfortunately, unlike in the rational case, the construction of the chain complex $K_\bullet(\mathcal{C})_{(p)}$ is very indirect and uncanonical, so it does not help much in practical computations. One clear deficiency is insufficient functoriality of the construction that makes it difficult to study its behaviour in families. Namely, a convenient axiomatization of the notion of a family of categories indexed by a small category $\mathcal{C}$ is the notion of a cofibered category $\mathcal{C}'/\mathcal{C}$ introduced in [Gr]. This is basicaly a functor $\pi : \mathcal{C}' \to \mathcal{C}$ satisfying some conditions; the conditions insure that for every morphism $f : c \to c'$ in $\mathcal{C}'$, one has a natural transition functor $f_! : \pi^{-1}(c) \to \pi^{-1}(c')$ between fibers of the cofibration π. Cofibration also behave nicely with respect to pullbacks – for any cofibered category $\mathcal{C}'/\mathcal{C}$ and any functor $\gamma : \mathcal{C}_1 \to \mathcal{C}$, we have the induced cofibration $\gamma^*\mathcal{C}' \to \mathcal{C}_1$. Within the context of algebraic K-theory, one would like to start with a cofibration $\pi : \mathcal{C}' \to \mathcal{C}$ whose fibers $\pi^{-1}(c)$, $c \in \mathcal{C}$, are k-linear additive categories, or maybe k-linear exact or Waldhausen categories, and one would like to pack the individual complexes $K_\bullet(\pi^{-1}(c))_{(p)}$ into a single object

$$K(\mathcal{C}'/\mathcal{C})_{(p)} \in \mathcal{D}(\mathcal{C}, \mathbb{Z}_{(p)})$$

in the derived category $\mathcal{D}(\mathcal{C}, \mathbb{Z}_{(p)})$ of the category of functors from $\mathcal{C}$ to $\mathbb{Z}_{(p)}$-modules. One would also like this construction to be functorial with respect to pullbacks, so that for any functor $\gamma : \mathcal{C}_1 \to \mathcal{C}$, one has a natural isomorphism

$$\gamma^* K(\mathcal{C}'/\mathcal{C})_{(p)} \cong K(\gamma^*\mathcal{C}'/\mathcal{C}_1)_{(p)}.$$

In order to achieve this by the usual methods, one has to construct the chain complex $K_\bullet(\mathcal{C})_{(p)}$ in such a way that it is exactly functorial in $\mathcal{C}$. This is probably possible but extremely painful.

The goal of this paper, then, is to present an alternative very simple construction of the objects $K(\mathcal{C}'/\mathcal{C})_{(p)} \in \mathcal{D}(\mathcal{C}, \mathbb{Z}_{(p)})$ that only uses direct homological methods, without any need to even introduce the notion of a ring spectrum. The only thing we need to set up the construction is a commutative ring k and a localization R of the ring $\mathbb{Z}$ in a set of primes S such that $K_i(k) \otimes R = 0$ for $i \geq 1$, and $K_0(k) \otimes R \cong R$. Starting from these data, we produce a family of objects $K^R(\mathcal{C}'/\mathcal{C}) \in \mathcal{D}(\mathcal{C}, R)$ with the properties listed above, and such that if $\mathcal{C}$ is the point category pt, then $K^R(\mathcal{C}'/\mathsf{pt})$ is naturally identified with the K-theory spectrum $\mathcal{K}(\mathcal{C}')$ localized in S.

Although the only example we have in mind is k a finite field of characteristic p, $R = \mathbb{Z}_{(p)}$, we formulate things in bigger generality to emphasize the essential ingredients of the construction. We do not need any information on how the isomorphism $K_0(k) \otimes R \cong R$ comes about, nor on why the higher K-groups vanish. As our entry point to algebraic K-theory, we use the formalism of Waldhausen categories, since it is the most general one available. However, were one to wish to use, for example, Quillen's Q-construction, everything would work with minimal modifications.

Essentially, our approach is modeled on the approach to Topological Hochschild Homology pioneered by M. Jibladze and T. Pirashvili [JP]. The construction itself is quite elementary. The underlying idea is also rather transparent and would work in much larger generality, but at the cost of much more technology to make things precise. Thus we have decided to present both the idea and its implementation but to keep them separate. In Section 1, we present the general idea of the construction, without making any mathematical statements precise enough to be proved. The rest of the paper is completely indepedent of Section 1. A rather long Section 2 contains the list of preliminaries; everything is elementary and well-known, but we need to recall these things to set up the notation and make the paper self-contained. A short explanation of what is needed and why is contained in the end of Section 1. Then Section 3 gives the exact statement of our main result, Theorem 3.4, and Section 4 contains its proof.

Acknowledgements. It is a pleasure and an honor to dedicate the paper to Sasha Merkurjev, as a birthday present. This is my first attempt to prove anything in algebraic K-theory, a subject I have always regarded with a lot of respect and a bit of trepidation, and if the resulting paper amuses him, I will be very happy. I am grateful to the referee for a thoughtful report and many useful suggestions.

1 Heuristics.

Assume given a commutative ring R, and let $M(R)$ be the category of finitely generated free R-modules. It will be useful to interpret $M(R)$ as the category of matrices: objects are finite sets S, morphisms from S to S' are R-valued matrices of size $S \times S'$.

Every R-module M defines a R-linear additive functor $\widetilde{M}$ from $M(R)$ to the category of R-modules by setting

$$\widetilde{M}(M_1) = \operatorname{Hom}_R(M_1^*, M) \tag{1.1}$$

for any $M_1 \in M(R)$, where we denote by $M_1^* = \operatorname{Hom}_R(M_1, R)$ the dual R-modules. This gives an equivalence of categories between the category R-mod of R-modules, and the category of R-linear additive functors from $M(R)$ to R-mod.

Let us now make the following observation. If we forget the R-module structure on M and treat it as a set, we of course lose information. However, if we do it pointwise with the functor $\widetilde{M}$, we can still recover the original R-module M. Namely, denote by $\operatorname{Fun}(M(R), R)$ the category of all functors from $M(R)$ to R-mod, without any additivity or linearity conditions, and consider the functor $R\text{-mod} \to \operatorname{Fun}(M(R), R)$ that sends M to $\widetilde{M}$. Then it has a left-adjoint functor

$$\mathsf{Add}_R : \operatorname{Fun}(M(R), R) \to R\text{-mod},$$

and for any $M \in R$-mod, we have

$$M \cong \mathsf{Add}_R(R[\widetilde{M}]), \tag{1.2}$$

where $R[\widetilde{M}] \in \mathrm{Fun}(M(R), R)$ sends $M_1 \in M(R)$ to the free R-module $R[\widetilde{M}(M_1)]$ generated by $\widetilde{M}(M_1)$. Indeed, by adjunction, Add_R commutes with colimits, so it suffices to check (1.2) for a finitely generated free R-module M; but then $R[\widetilde{M}]$ is a representable functor, and (1.2) follows from the Yoneda Lemma.

The functor Add_R also has a version with coefficients. If we have an R-algebra R', then for any R'-module M, the functor $\widetilde{M}$ defined by (1.1) is naturally a functor from $M(R)$ to R'-mod. Then by adjunction, we can define the functor

$$\mathsf{Add}_{R,R'} : \mathrm{Fun}(M(R), R') \to R'\text{-mod},$$

and we have an isomorphism

$$\mathsf{Add}_{R,R'}(R'[\widetilde{M}]) \cong M \otimes_R R' \tag{1.3}$$

for any flat R-module M.

What we want to do now is to obtain a homotopical version of the construction above. We thus replace sets with topological spaces. An abelian group structure on a set becomes an infinite loop space structure on a topological space; this is conveniently encoded by a special Γ-space of G. Segal [Se]. Abelian groups become connective spectra. Rings should become ring spectra. As far as I know, Segal machine does not extend directly to ring spectra – to describe ring spectra, one has to use more complicated machinery such as "functors with smash products", or an elaboration on them, ring objects in the category of symmetric spectra of [HSS]. However, in practice, if we are given a connective spectrum $\mathcal{X}$ represented by an infinite loop space X, then a ring spectrum structure on $\mathcal{X}$ gives rise to a multiplication map $\mu : X \times X \to X$, and in ideal situation, this is sufficiently associative and distributive to define a matrix category $\mathsf{Mat}(X)$ analogous to $M(R)$. This should be a small category enriched over topological spaces. Its objects are finite sets S, and the space of morphisms from S to S' is the space $X^{S \times S'}$ of X-valued matrices of size $S \times S'$, with compositions induced by the multiplication map $\mu : X \times X \to X$.

Ideal situations seem to be rare (the only example that comes to mind readily is a simplicial ring treated as an Eilenberg-MacLane ring spectrum). However, one might relax the conditions slightly. Namely, in practice, infinite loop spaces and special Γ-spaces often appear as geometric realizations of monoidal categories. The simplest example of this is the sphere spectrum $\mathcal{S}$. One starts with the groupoid $\overline{\Gamma}$ of finite sets and isomorphisms between them, one treats it as a monoidal category with respect to the disjoint union operation, and one produces a special Γ-space with underlying topological space $|\overline{\Gamma}|$, the geometric realization of the nerve of the category $\overline{\Gamma}$. Then by Barratt-Quillen Theorem, up to a stable homotopy equivalence, the corresponding spectrum is exactly $\mathcal{S}$. The sphere spectrum is of course a ring spectrum, and the multiplication operation μ also has a categorical origin: it is induced by the cartesian product functor $\overline{\Gamma} \times \overline{\Gamma} \to \overline{\Gamma}$. This functor is not associative or commutative on the nose, but it is associative and commutative up to canonical isomorphisms. The hypothetical matrix category $\mathsf{Mat}(|\overline{\Gamma}|)$ is then easily constructed as the geometric realization $|\mathcal{Q}\Gamma|$ of a strictification of a 2-category $\mathcal{Q}\Gamma$ whose objects are finite sets S, and whose category $\mathcal{Q}\Gamma(S,S')$ of morphisms from S to S' is the groupoid $\overline{\Gamma}^{S\times S'}$. Equivalently, $\mathcal{Q}\Gamma(S,S')$ is the category of diagrams

$$S \xleftarrow{\;l\;} \widetilde{S} \xrightarrow{\;r\;} S' \tag{1.4}$$

of finite sets, and isomorphisms between these diagrams. Compositions are obtained by taking pullbacks.

Any spectrum is canonically a module spectrum over $\mathcal{S}$. So, in line with the additivization yoga described above, we expect to be able to start with a connective spectrum $\mathcal{X}$ corresponding to an infinite loop space X, produce a functor $X_\bullet$ from $|\mathcal{Q}\Gamma|$ to topological spaces sending a finite set S to X^S, and then recover the infinite loop space structure on X from the functor $X_\bullet$.

This is exactly what happens – and in fact, we do not need the whole 2-category $\mathcal{Q}\Gamma$, it suffices to restrict our attention to the subcategory in $\mathcal{Q}\Gamma$ spanned by diagrams (1.4) with injective map l. Since such diagrams have no non-trivial automorphisms, this subcategory is actually a 1-category. It is equivalent to the category Γ_+ of pointed finite sets. Then restricting $X_\bullet$ to Γ_+ produced a functor from Γ_+ to topological spaces, that is, precisely a Γ-space in the sense of Segal. This Γ-space is automatically special, and one recovers the infinite loop space structure on X by applying the Segal machine.

It is also instructive to do the versions with coefficients, with R being the sphere spectrum, and R' being the Eilenberg-Mac Lane ring spectrum $H(A)$ corresponding to a ring A. Then module spectra over $H(A)$ are just complexes of A-modules, forming the derived category $\mathcal{D}(A)$ of the category A-mod, and functors from Γ_+ to $H(A)$-module spectra are complexes in the category $\operatorname{Fun}(\Gamma_+,A)$ of functors from Γ_+ to A-mod, forming the derived category $\mathcal{D}(\Gamma_+,A)$ of the abelian category $\operatorname{Fun}(\Gamma_+,A)$. One has a tautological functor from A-mod to $\operatorname{Fun}(\Gamma_+,A)$ sending an A-module M to $\widetilde{M}\in\operatorname{Fun}(\Gamma_+,A)$ given by $\widetilde{M}(S)=M[\overline{S}]$, where $\overline{S}\subset S$ is the complement to the distinguished element $o\in S$. This has a left-adjoint functor

$$\mathsf{Add}:\operatorname{Fun}(\Gamma_+,A)\to A\text{-mod},$$

with its derived functor $L^\bullet\mathsf{Add}:\mathcal{D}(\Gamma_+,A)\to\mathcal{D}(A)$. The role of the free A-module $A[S]$ generated by a set S is played by the singular chain complex $C_\bullet(X,A)$ of a topological space, and we expect to start with a special Γ-space $X_+:\Gamma_+\to\operatorname{Top}$, and obtain an analog of (1.3), namely, an isomorphism

$$L^\bullet\mathsf{Add}(C_\bullet(X_+,A))\cong H_\bullet(\mathcal{X},A),$$

where $H_\bullet(\mathcal{X},A)$ are the homology groups of the spectum $\mathcal{X}$ corresponding to X_+ with coefficients in A (that is, homotopy groups of the product $\mathcal{X}\wedge A$).

Such an isomorphism indeed exists; we recall a precise statement below in Lemma 4.1.
Moreover, we can be more faithful to the original construction and avoid restricting to $\Gamma_+ \subset \mathcal{Q}\Gamma$. This entails a technical difficulty, since one has to explain what is a functor from the 2-category $\mathcal{Q}\Gamma$ to complexes of A-modules, and define the corresponding derived category $\mathcal{D}(\mathcal{Q}\Gamma, A)$. It can be done in several equivalent ways, see e.g. [Ka2, Section 3.1], and by [Ka2, Lemma 3.4(i)], the answer remains the same – we still recover the homology groups $H_\bullet(\mathcal{X}, A)$.

Now, the point of the present paper is the following. The K-theory spectrum $\mathcal{K}(k)$ of a commutative ring k also comes from a monoidal category, namely, the groupoid $\mathsf{Iso}(k)$ of finitely generated projective k-modules and isomorphisms between them. Moreover, the ring structure on $\mathcal{K}(k)$ also has categorical origin – it comes from the tensor product functor $\mathsf{Iso}(k) \times \mathsf{Iso}(k) \to \mathsf{Iso}(k)$. And if we have some k-linear Waldhausen category $\mathcal{C}$, then the infinite loop space corresponding to the K-theory spectrum $\mathcal{K}(\mathcal{C})$ is the realization of the nerve of a category $S\mathcal{C}$ on which $\mathsf{Iso}(k)$ acts. Therefore one can construct a 2-category $\mathsf{Mat}(k)$ of matrices over $\mathsf{Iso}(k)$, and $\mathcal{C}$ defines a 2-functor $\mathsf{Vect}(S\mathcal{C}) : \mathsf{Mat}(k) \to$ Cat to the 2-category Cat of small categories. At this point, we can forget all about ring spectra and module spectra, define an additivization functor

$$\mathsf{Add} : \mathcal{D}(\mathsf{Mat}(k), R) \to \mathcal{D}(R),$$

and use an analog of (1.3) to recover if not $\mathcal{K}(\mathcal{C})$ then at least $\mathcal{K}(\mathcal{C}) \wedge_{\mathcal{K}(k)} H(R)$, where $H(R)$ is the Eilenberg-Mac Lane spectrum corresponding to R. This is good enough: if R is the localization of $\mathbb{Z}$ in a set of primes S such that $\mathcal{K}(k)$ localized in S is $H(R)$, then $\mathcal{K}(\mathcal{C}) \wedge_{\mathcal{K}(k)} H(R)$ is the localization of $\mathcal{K}(\mathcal{C})$ in S.
The implementation of the idea sketched above requires some preliminaries. Here is a list. Subsection 2.1 discusses functor categories, their derived categories and the like; it is there mostly to fix notation. Subsection 2.2 recalls the basics of the Grothendieck construction of [Gr]. Subsection 2.3 contains some related homological facts. Subsection 2.4 recalls some standard facts about simplicial sets and nerves of 2-categories. Subsection 2.5 discusses 2-categories and their nerves. Subsection 2.6 constructs the derived category $\mathcal{D}(\mathcal{C}, R)$ of functors from a small 2-category $\mathcal{C}$ to the category of modules over a ring R; this material is slightly non-standard, and we have even included one statement with a proof. We use an approach based on nerves, since it is cleaner and does not require any strictification of 2-categories. Then we introduce the 2-categories we will need: Subsection 2.7 is concerned with the 2-category $\mathcal{Q}\Gamma$ and its subcategory $\Gamma_+ \subset \mathcal{Q}\Gamma$, while Subsection 2.8 explains the matrix 2-categories $\mathsf{Mat}(k)$ and the 2-functors $\mathsf{Vect}(\mathcal{C})$. Finally, Subsection 2.9 explains how the matrix and vector categories are defined in families (that is, in the relative setting, with respect to a cofibration in the sense of [Gr]).
Having finished with the preliminaries, we turn to our results. Section 3 contains a brief recollection on K-theory, and then the statement of the main result, Theorem 3.4. Since we do not introduce ring spectra, we cannot really

state that we prove a spectral analog of (1.3). Instead, we construct directly a map $\mathcal{K}(\mathcal{C}) \to \mathcal{K}$ to a certain Eilenberg-Mac Lane spectrum $\mathcal{K}$, and we prove that the map becomes an isomorphism after the appropriate localization. The actual proof is contained in Section 4.

2 Preliminaries.

2.1 Homology of small categories. For any two objects $c, c' \in \mathcal{C}$ in a category $\mathcal{C}$, we will denote by $\mathcal{C}(c, c')$ the set of maps from c to c'. For any category $\mathcal{C}$, we will denote by $\mathcal{C}^o$ the opposite category, so that $\mathcal{C}(c, c') = \mathcal{C}^o(c', c)$, $c, c' \in \mathcal{C}$. For any functor $\pi : \mathcal{C}_1 \to \mathcal{C}_2$, we denote by $\pi^o : \mathcal{C}_1^o \to \mathcal{C}_2^o$ the induced functor between the opposite categories.
For any small category $\mathcal{C}$ and ring R, we will denote by $\operatorname{Fun}(\mathcal{C}, R)$ the abelian category of functors from $\mathcal{C}$ to the category R-mod of left R-modules, and we will denote by $\mathcal{D}(\mathcal{C}, R)$ its derived category. The triangulated category $\mathcal{D}(\mathcal{C}, R)$ has a standard t-structure in the sense of [BBD] whose heart is $\operatorname{Fun}(\mathcal{C}, R)$. For any object $c \in \mathcal{C}$, we will denote by $R_c \in \operatorname{Fun}(\mathcal{C}, R)$ the representable functor given by

$$R_c(c') = R[\mathcal{C}(c, c')], \tag{2.1}$$

where for any set S, we denote by $R[S]$ the free R-module spanned by S. Every object $E \in \mathcal{D}(\mathcal{C}, R)$ defines a functor $\mathcal{D}(E) : \mathcal{C} \to \mathcal{D}(R)$ from $\mathcal{C}$ to the derived category $\mathcal{D}(R)$ of the category R-mod, and by adjunction, we have a quasiisomorphism

$$\mathcal{D}(E)(c) \cong \operatorname{RHom}^\bullet(R_c, E) \tag{2.2}$$

for any object $c \in \mathcal{C}$ (we will abuse notation by writing $E(c)$ instead of $\mathcal{D}(E)(c)$). Any functor $\gamma : \mathcal{C} \to \mathcal{C}'$ between small categories induces an exact pullback functor $\gamma^* : \operatorname{Fun}(\mathcal{C}', R) \to \operatorname{Fun}(\mathcal{C}, R)$ and its adjoints, the left and right Kan extension functors $\gamma_!, \gamma_* : \operatorname{Fun}(\mathcal{C}, R) \to \operatorname{Fun}(\mathcal{C}', R)$. The derived functors $L^\bullet\gamma_!, R^\bullet\gamma_* : \mathcal{D}(\mathcal{C}, R) \to \mathcal{D}(C', R)$ are left resp. right-adjoint to the pullback functor $\gamma^* : \mathcal{D}(C', R) \to \mathcal{D}(\mathcal{C}, R)$. The homology resp. cohomology of a small category $\mathcal{C}$ with coefficients in a functor $E \in \operatorname{Fun}(\mathcal{C}, R)$ are given by

$$H_i(\mathcal{C}, E) = L^i\tau_!E, \qquad H^i(\mathcal{C}, E) = R^i\tau_*E, \qquad i \geq 0,$$

where $\tau : \mathcal{C} \to \mathsf{pt}$ is the tautological projection to the point category pt.
Assume that the ring R is commutative. Then for any $E \in \operatorname{Fun}(\mathcal{C}, R)$, $T \in \operatorname{Fun}(\mathcal{C}^o, R)$, the *tensor product* $E \otimes_{\mathcal{C}} T$ is the cokernel of the natural map

$$\bigoplus_{f : c \to c'} E(c) \otimes_R T(c') \xrightarrow{E(f) \otimes \mathsf{id} - \mathsf{id} \otimes T(f)} \bigoplus_{c \in \mathcal{C}} E(c) \otimes_R T(c).$$

Sending E to $E \otimes_{\mathcal{C}} T$ gives a right-exact functor from $\operatorname{Fun}(\mathcal{C}, R)$ to R-mod; we denote its derived functors by $\operatorname{Tor}_i^{\mathcal{C}}(E, T)$, $i \geq 1$, and we denote by $E \overset{\mathsf{L}}{\otimes} T$ the derived tensor product. If $T(c)$ is a free R-module for any $c \in \mathcal{C}$, then $- \otimes_{\mathcal{C}} T$ is left-adjoint to an exact functor $\mathcal{H}om(T,-) : R\text{-mod} \to \operatorname{Fun}(\mathcal{C}, R)$ given by

$$\mathcal{H}om(T,E)(c) = \operatorname{Hom}(T(c), E), \qquad c \in \mathcal{C}, E \in R\text{-mod}. \tag{2.3}$$

Being exact, $\mathcal{H}om(T,-)$ induces a functor from $\mathcal{D}(R)$ to $\mathcal{D}(\mathcal{C}, R)$; this functor is right-adjoint to the derived tensor product functor $- \overset{\mathsf{L}}{\otimes}_{\mathcal{C}} T$. For example, if $T = R$ is the constant functor with value R, then we have

$$H_{\bullet}(\mathcal{C}, E) \cong \operatorname{Tor}_{\bullet}^{\mathcal{C}}(E, R)$$

for any $E \in \operatorname{Fun}(\mathcal{C}, R)$.

2.2 Grothendieck construction. A morphism $f : c \to c'$ in a category $\mathcal{C}'$ is called *cartesian* with respect to a functor $\pi : \mathcal{C}' \to \mathcal{C}$ if any morphism $f_1 : c_1 \to c'$ in $\mathcal{C}'$ such that $\pi(f) = \pi(f_1)$ factors uniquely as $f_1 = f \circ g$ for some $g : c_1 \to c$. A functor $\pi : \mathcal{C}' \to \mathcal{C}$ is a *prefibration* if for any morphism $f : c \to c'$ in $\mathcal{C}$ and object $c'_1 \in \mathcal{C}'$ with $\pi(c'_1) = c'$, there exists a cartesian map $f_1 : c_1 \to c'_1$ in $\mathcal{C}'$ with $\pi(f_1) = f$. A prefibration is a *fibration* if the composition of two cartesian maps is cartesian. A functor $F : \mathcal{C}' \to \mathcal{C}''$ between two fibrations $\mathcal{C}', \mathcal{C}''/\mathcal{C}$ is *cartesian* if it commutes with projections to $\mathcal{C}$ and sends cartesian maps to cartesian maps. For any fibration $\mathcal{C}' \to \mathcal{C}$, a subcategory $\mathcal{C}'_0 \subset \mathcal{C}'$ is a *subfibration* if the induced functor $\mathcal{C}'_0 \to \mathcal{C}$ is a fibration, and the embedding functor $\mathcal{C}'_0 \to \mathcal{C}'$ is cartesian over $\mathcal{C}$.
A fibration $\pi : \mathcal{C}' \to \mathcal{C}$ is called *discrete* if its fibers $\pi_c = \pi^{-1}(c)$, $c \in \mathcal{C}$ are discrete categories. For example, for any $c \in \mathcal{C}$, let $\mathcal{C}/c$ be the category of objects $c' \in \mathcal{C}$ equipped with a map $c' \to c$. Then the forgetful functor $\varphi : \mathcal{C}/c \to \mathcal{C}$ sending $c' \to c$ to c' is a discrete fibration, with fibers $\varphi_{c'} = \mathcal{C}(c', c)$, $c' \in \mathcal{C}$.
For any functor $F : \mathcal{C}^o \to \operatorname{Cat}$ to the category Cat of small categories, let $\operatorname{Tot}(F)$ be the category of pairs $\langle c, s\rangle$ of an object $c \in \mathcal{C}$ and an object $s \in F(c)$, with morphisms from $\langle c, s\rangle$ to $\langle c', s'\rangle$ given by a morphism $f : c \to c'$ and a morphism $s \to F(f)(s')$. Then the forgetful functor $\pi : \operatorname{Tot}(F) \to \mathcal{C}$ is a fibration, with fibers $\pi_c \cong F(c)$, $c \in \mathcal{C}$. If F is a functor to sets, so that for any $c \in \mathcal{C}$, $F(c)$ is a discrete category, then the fibration π is discrete.
Conversely, for any fibration $\pi : \mathcal{C}' \to \mathcal{C}$ with of small categories, and any object $c \in \mathcal{C}$, let $\operatorname{Gr}(\pi)(c)$ be the category of cartesian functors $\mathcal{C}/c \to \mathcal{C}'$. Then $\operatorname{Gr}(\pi)(c)$ is contravariantly functorial in c and gives a functor $\operatorname{Gr}(\pi) : \mathcal{C}^o \to \operatorname{Cat}$. The two constructions are inverse, in the sense that we have a natural cartesian equivalence $\operatorname{Tot}(\operatorname{Gr}(\pi)) \cong \mathcal{C}'$ for any fibration $\pi' : \mathcal{C}' \to \mathcal{C}$, and a natural pointwise equivalence $F \to \operatorname{Gr}(\operatorname{Tot}(F))$ for any $F : \mathcal{C}^o \to \operatorname{Cat}$. In particular, we have equivalences

$$\pi_c \cong \operatorname{Gr}(\pi)(c), \qquad c \in \mathcal{C}.$$

These equivalences of categories are not isomorphisms, so that the fibers π_c themselves do not form a functor from $\mathcal{C}^o$ to Cat – they only form a pseudofunctor in the sense of [Gr] (we do have a transition functor $f^* : \pi_{c'} \to \pi_c$

for any morphism $f : c \to c'$ in $\mathcal{C}$, but this is compatible with compositions only up to a canonical isomorphism). Nevertheless, for all practical purposes, a fibered category over $\mathcal{C}$ is a convenient axiomatization of the notion of a family of categories contravariantly indexed by $\mathcal{C}$.
For any fibration $\pi : \mathcal{C}' \to \mathcal{C}$ of small categories, and any functor $\gamma : \mathcal{C}_1 \to \mathcal{C}$ from a small category $\mathcal{C}_1$, we can define a category $\gamma^*\mathcal{C}'$ and a functor $\pi_1 : \gamma^*\mathcal{C}' \to \mathcal{C}_1$ by taking the cartesian square

$$\begin{array}{ccc} \gamma^*\mathcal{C}' & \xrightarrow{\gamma'} & \mathcal{C}' \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathcal{C}_1 & \xrightarrow{\gamma} & \mathcal{C} \end{array} \tag{2.4}$$

in Cat. Then π_1 is also a fibration, called the *induced fibration*. The corresponding pseudofunctor $\mathrm{Gr}(\pi_1) : \mathcal{C}_1^o \to \mathrm{Cat}$ is the composition of the functor γ and $\mathrm{Gr}(\pi)$.
For covariantly indexed families, one uses the dual notion of a cofibration: a morphism f is cocartesian with respect to a functor π if it is cartesian with respect to π^o, a functor π is a cofibration if π^o is a fibration, a functor $F : \mathcal{C}' \to \mathcal{C}''$ between two cofibrations is cocartesian if F^o is cartesian, and a subcategory $\mathcal{C}'_0 \subset \mathcal{C}'$ is a subcofibration if $(\mathcal{C}'_0)^o \subset (\mathcal{C}')^o$ is a subfibration. The Grothendieck construction associates cofibrations over $\mathcal{C}$ to functors from $\mathcal{C}$ to Cat. We have the same notion of an induced cofibration. Functors to Sets $\subset$ Cat correspond to discrete cofibrations; the simplest example of such is the projection

$$\rho_c : c\backslash\mathcal{C} \to \mathcal{C} \tag{2.5}$$

for some object $c \in \mathcal{C}$, where $c\backslash\mathcal{C} = (\mathcal{C}^o/c)^o$ is the category of objects $c' \in \mathcal{C}$ equipped with a map $c \to c'$.

2.3 Base change. Assume given a cofibration $\pi : \mathcal{C}' \to \mathcal{C}$ of small categories and a functor $\gamma : \mathcal{C}_1 \to \mathcal{C}$, and consider the cartesian square (2.4). Then the isomorphism $\gamma^{'*} \circ \pi^* \cong \pi_1^* \circ \gamma^*$ induces by adjunction a base change map

$$L^\bullet \pi_{1!} \circ \gamma^{'*} \to \gamma^* \circ L^\bullet \pi_!.$$

This map is an isomorphism (for a proof see e.g. [Ka1]). In particular, for any object $c \in \mathcal{C}$, any ring R, and any $E \in \mathrm{Fun}(\mathcal{C}', R)$, we have a natural identification

$$L^\bullet \pi_! E(c) \cong H_\bullet(\pi_c, E|_c), \tag{2.6}$$

where $E|_c \in \mathrm{Fun}(\pi_c, R)$ is the restriction to the fiber $\pi_c \subset \mathcal{C}'$. If the cofibration π is discrete, then this shows that $L^i \pi_! E = 0$ for $i \geq 1$, and

$$\pi_! E(c) = \bigoplus_{c' \in \pi_c} E(c').$$

For example, for the discrete cofibration ρ_c of (2.5) and the constant functor $R \in \operatorname{Fun}(c\backslash\mathcal{C}, R)$, we obtain an identification

$$R_c \cong \rho_{c!}R \cong L^{\bullet}\rho_{c!}R, \tag{2.7}$$

where $R_c \in \operatorname{Fun}(\mathcal{C}, R)$ is the representable functor (2.1). For fibrations, we have exactly the same statements with left Kan extensions replaced by right Kan extensions, and sums replaced by products.

Moreover, assume that R is commutative, and assume given an object $T \in \operatorname{Fun}((\mathcal{C}')^o, R)$ that inverts all maps f in $\mathcal{C}'$ cocartesian with respect to π – that is, $T(f)$ is invertible for any such map. Then we can define the relative tensor product functor $- \otimes_\pi T : \operatorname{Fun}(\mathcal{C}', R) \to \operatorname{Fun}(\mathcal{C}, R)$ by setting

$$(E \otimes_\pi T)(c) = E|_c \otimes_{\pi_c} T|_c$$

for any $E \in \operatorname{Fun}(\mathcal{C}', R)$. This has individual derived functors $\operatorname{Tor}^{\pi}_{\bullet}(-, T)$ and the total derived functor $- \overset{\mathsf{L}}{\otimes}_\pi T$. For any $c \in \mathcal{C}$, we have

$$(E \overset{\mathsf{L}}{\otimes}_\pi T)(c) \cong E|_c \overset{\mathsf{L}}{\otimes}_{\pi_c} T|_c. \tag{2.8}$$

If $T(c)$ is a free R-module for any $c \in \mathcal{C}'$, then we also have the relative version

$$\mathcal{H}om_\pi(T, -) : \operatorname{Fun}(\mathcal{C}, R) \to \operatorname{Fun}(\mathcal{C}', R)$$

of the functor (2.3); it is exact and right-adjoint to $- \otimes_\pi T$, resp. $- \overset{\mathsf{L}}{\otimes}_\pi T$. In the case $T = R$, we have $E \overset{\mathsf{L}}{\otimes}_\pi R \cong L^{\bullet}\pi_! E$, and the isomorphism (2.8) is the isomorphism (2.6).

2.4 Simplicial objects. As usual, we denote by Δ the category of finite non-empty totally ordered sets, a.k.a. finite non-empty ordinals, and somewhat unusually, we denote by $[n] \in \Delta$ the set with n elements, $n \geq 1$. A simplicial object in a category $\mathcal{C}$ is a functor from Δ^o to $\mathcal{C}$; these form a category denoted $\Delta^o\mathcal{C}$. For any ring R and $E \in \operatorname{Fun}(\Delta^o, R) = \Delta^o R$-mod, we denote by $C_\bullet(E)$ the normalized chain complex of the simplicial R-module E. The homology of the complex $C_\bullet(E)$ is canonically identified with the homology $H_\bullet(\Delta^o, E)$ of the category Δ^o with coefficients in E. Even stronger, sending E to $C_\bullet(E)$ gives the *Dold-Kan equivalence*

$$\mathsf{N} : \operatorname{Fun}(\Delta^o, R) \to C_{\geq 0}(R)$$

between the category $\operatorname{Fun}(\Delta^o, R)$ and the category $C_{\geq 0}(R)$ of complexes of R-modules concentrated in non-negative homological degrees. The inverse equivalence is given by the denormalization functor $\mathsf{D} : C_{\geq 0}(R) \to \operatorname{Fun}(\Delta^o, R)$ right-adjoint to N.

For any simplicial set X, its homology $H_\bullet(X, R)$ with coefficients in a ring R is the homology of the chain complex

$$C_\bullet(X, R) = C_\bullet(R[X]),$$

where $R[X] \in \operatorname{Fun}(\Delta^o, R)$ is given by $R[X]([n]) = R[X([n])]$, $[n] \in \Delta$. By adjunction, for any simplicial set X and any complex $E_\bullet \in C_{\geq 0}(R)$, a map $C_\bullet(X,R) \to E_\bullet$ gives rise to a map of simplicial sets

$$X \longrightarrow R[X] \longrightarrow \mathsf{D}(E_\bullet), \tag{2.9}$$

where we treat simplicial R-modules $R[X]$ and $\mathsf{D}(E_\bullet)$ as simplicial sets. Conversely, every map of simplicial sets $X \to \mathsf{D}(E_\bullet)$ gives rise to a map $C_\bullet(X,R) \to E_\bullet$. In particular, if we take $X = \mathsf{D}(E_\bullet)$, we obtain the *assembly map*

$$C_\bullet(\mathsf{D}(E_\bullet), R) \to E_\bullet. \tag{2.10}$$

The constructions are mutually inverse: every map of complexes of R-modules $C_\bullet(X,R) \to E_\bullet$ decomposes as

$$C_\bullet(X,R) \longrightarrow C_\bullet(\mathsf{D}(E_\bullet),R) \longrightarrow E_\bullet, \tag{2.11}$$

where the first map is induced by the tautological map (2.9), and the second map is the assembly map (2.10).

Applying the Grothendieck construction to a simplicial set X, we obtain a category $\operatorname{Tot}(X)$ with a discrete fibration $\pi : \operatorname{Tot}(X) \to \Delta$. We then have a canonical identification

$$H_\bullet(\operatorname{Tot}(X)^o, R) \cong H_\bullet(\Delta^o, \pi_! R) \cong H_\bullet(\Delta^o, R[X]), \tag{2.12}$$

so that $H_\bullet(X,R)$ is naturally identified with the homology of the small category $\operatorname{Tot}(X)^o$ with coefficients in the constant functor R.

The *nerve* of a small category $\mathcal{C}$ is the simplicial set $N(\mathcal{C}) \in \Delta^o$ Sets such that for any $[n] \in \Delta$, $N(\mathcal{C})([n])$ is the set of functors from the ordinal $[n]$ to $\mathcal{C}$. Explicitly, elements in $N(\mathcal{C})([n])$ are diagrams

$$c_1 \longrightarrow \dots \longrightarrow c_n \tag{2.13}$$

in $\mathcal{C}$. We denote by $\mathcal{N}(\mathcal{C}) = \operatorname{Tot}(N(\mathcal{C}))$ the corresponding fibered category over Δ. Then by definition, objects of $\mathcal{N}(\mathcal{C})$ are diagrams (2.13), and sending such a diagram to c_n gives a functor

$$q : \mathcal{N}(\mathcal{C}) \to \mathcal{C}. \tag{2.14}$$

Say that a map $f : [n] \to [m]$ in Δ is *special* if it identifies $[n]$ with a terminal segment of the ordinal $[m]$. For any fibration $\pi : \mathcal{C}' \to \Delta$, say that a map f in $\mathcal{C}'$ is *special* if it is cartesian with respect to π and $\pi(f)$ is special in Δ, and say that a functor $F : \mathcal{C}' \to \mathcal{E}$ to some category E is *special* if it $F(f)$ is invertible for any special map f in $\mathcal{C}'$. Then the functor q of (2.14) is special, and any special functor factors uniquely through q. In particular, $\operatorname{Fun}(\mathcal{C}, R)$ is naturally identified the full subcategory in $\operatorname{Fun}(\mathcal{N}(\mathcal{C}), R)$ spanned by special functors. Moreover, on the level of derived categories, say that $E \in \mathcal{D}(\mathcal{C}', R)$

is *special* if so is $\mathcal{D}(E) : \mathcal{C}' \to \mathcal{D}(R)$, and denote by $\mathcal{D}_{sp}(\mathcal{C}', R) \subset \mathcal{D}(\mathcal{C}', R)$ the full subcategory spanned by special objects. Then the pullback functor

$$q^* : \mathcal{D}(\mathcal{C}, R) \to \mathcal{D}(\mathcal{N}(\mathcal{C}), R) \tag{2.15}$$

induces an equivalence between $\mathcal{D}(\mathcal{C}, R)$ and $\mathcal{D}_{sp}(\mathcal{N}(\mathcal{C}), R)$. In particular, we have a natural isomorphism

$$H_\bullet(\mathcal{C}, R) \cong H_\bullet(\mathcal{N}(\mathcal{C}), R), \tag{2.16}$$

and by (2.12), the right-hand side is also canonically identified with the homology $H_\bullet(N(\mathcal{C}), R)$ of the simplicial set $N(\mathcal{C})$.

The *geometric realization functor* $X \mapsto |X|$ is a functor from $\Delta^o\,\mathrm{Sets}$ to the category Top of topological spaces. For any simplicial set X and any ring R, the homology $H_\bullet(X, R)$ is naturally identified with the homology $H_\bullet(|X|, R)$ of its realization, and the isomorphism (2.16) can also be deduced from the following geometric fact: for any simplicial set X, we have a natural homotopy equivalence

$$|N(\mathrm{Tot}(X))| \cong |X|.$$

Even stronger, the geometric realization functor extends to a functor from $\Delta^o\,\mathrm{Top}$ to Top, and for any small category $\mathcal{C}$ equipped with a fibration $\pi : \mathcal{C} \to \Delta$, we have a natural homotopy equivalence

$$|N(\mathcal{C})| \cong ||N(\mathrm{Gr}(\pi))||, \tag{2.17}$$

where $N(\mathrm{Gr}(\pi)) : \Delta^o \to \Delta^o\,\mathrm{Sets}$ is the natural bisimplicial set corresponding to π, and $||-||$ in the right-hand side stands for the geometric realization of its pointwise geometric realization. Geometric realization commutes with products by the well-known Milnor Theorem, so that in particular, (2.17) implies that for any self-product $\mathcal{C} \times_\Delta \cdots \times_\Delta \mathcal{C}$, we have a natural homotopy equivalence

$$|N(\mathcal{C} \times_\Delta \cdots \times_\Delta \mathcal{C})| \cong |N(\mathcal{C})| \times \cdots \times |N(\mathcal{C})|. \tag{2.18}$$

2.5 2-categories. We will also need to work with 2-categories, and for this, the language of nerves is very convenient, since the nerve of a 2-category can be converted into a 1-category by the Grothendieck construction.

Namely, recall that a 2-category[2] $\mathcal{C}$ is given by a class of objects $c \in \mathcal{C}$, a collection of morphism categories $\mathcal{C}(c, c')$, $c, c' \in \mathcal{C}$, a collection of identity objects $\mathsf{id}_c \in \mathcal{C}(c, c)$ for any $c \in \mathcal{C}$, and a collection of composition functors

$$m_{c,c',c''} : \mathcal{C}(c, c') \times \mathcal{C}(c', c'') \to \mathcal{C}(c, c''), \qquad c, c', c'' \in \mathcal{C} \tag{2.19}$$

equiped with associativity and unitality isomorphisms, subject to standard higher contraints (see [Be]). A 1-category is then a 2-category $\mathcal{C}$ with discrete

[2] We use "2-category" to mean "weak 2-category" a.k.a. "bicategory"; we avoid current usage that seems to reserve "2-category" for "strict 2-category", a rather unnatural notion with no clear conceptual meaning.

$\mathcal{C}(c,c')$, $c,c' \in \mathcal{C}$. For any 2-category $\mathcal{C}$ and any $[n] \in \Delta$, one can consider the category

$$N(\mathcal{C})_n = \coprod_{c_1,\dots,c_n \in \mathcal{C}} \mathcal{C}(c_1,c_2) \times \dots \times \mathcal{C}(c_{n-1},c_n).$$

If $\mathcal{C}$ is a small 1-category, then $N(\mathcal{C})_n = N(\mathcal{C})([n])$ is the value of the nerve $N(\mathcal{C}) \in \Delta^o\,\mathrm{Sets}$ at $[n] \in \Delta$, and the structure maps of the functor $N(\mathcal{C}) : \Delta^o \to \mathrm{Sets}$ are induced by the composition and unity maps in $\mathcal{C}$. In the general case, the composition and unity functors turn $N(\mathcal{C})$ into a pseudofunctor from Δ^o to Cat. We let

$$\mathcal{N}(\mathcal{C}) = \mathrm{Tot}(N(\mathcal{C}))$$

be the corresponding fibered category over Δ, and call it the *nerve* of the 2-category $\mathcal{C}$.

The associativity and unitality isomorphisms in $\mathcal{C}$ give rise to the compatibility isomorphisms of the pseudofunctor $N(\mathcal{C})$, so that they are encoded by the fibration $\mathcal{N}(\mathcal{C}) \to \Delta$. One can in fact use this to give an alternative definition of a 2-category, see e.g. [Ka3], but we will not need this. However, it is useful to note what happens to functors. A *2-functor* $F : \mathcal{C} \to \mathcal{C}'$ between 2-categories $\mathcal{C}$, $\mathcal{C}'$ is given by a map F between their classes of objects, a collection of functors

$$F(c,c') : \mathcal{C}(c,c') \to \mathcal{C}'(F(c),F(c')), \qquad c,c' \in \mathcal{C}, \tag{2.20}$$

and a collection of isomorphisms $F(c,c)(\mathsf{id}_c) \cong \mathsf{id}_{F(c)}$, $c \in \mathcal{C}$, and

$$m_{F(c),F(c'),F(c'')} \circ (F(c,c') \times F(c',c'')) \cong F(c,c'') \circ m_{c,c',c''}, \qquad c,c',c'' \in \mathcal{C},$$

again subject to standard higher constraints. Such a 2-functor gives rise to a functor $\mathcal{N}(F) : \mathcal{N}(\mathcal{C}) \to \mathcal{N}(\mathcal{C}')$ cartesian over Δ, and the correspondence between 2-functors and cartesian functors is one-to-one.

The category Cat is a 2-category in a natural way, and the Grothendieck construction generalizes directly to 2-functors from a 2-category $\mathcal{C}$ to Cat. Namely, say that a cofibration $\pi : \mathcal{C}' \to \mathcal{N}(\mathcal{C})$ is *special* if for any special morphism $f : c \to c'$ in $\mathcal{N}(\mathcal{C})$, the transition functor $f_! : \pi_c \to \pi_{c'}$ is an equivalence. Then 2-functors $F : \mathcal{C} \to \mathrm{Cat}$ correspond to special cofibrations $\mathrm{Tot}(F) \to \mathcal{N}(\mathcal{C})$, and the correspondence is again one-to-one. If $\mathcal{C}$ is actually a 1-category, then a 2-functor $F : \mathcal{C} \to \mathrm{Cat}$ is exactly the same thing as a pseudofunctor $\overline{F} : \mathcal{C} \to \mathrm{Cat}$ in the sense of the usual Grothendieck construction, and we have $\mathrm{Tot}(F) \cong q^* \mathrm{Tot}(\overline{F})$, where q is the functor of (2.14) (one easily checks that every special cofibration over $\mathcal{N}(\mathcal{C})$ arises in this way).

The simplest example of a 2-functor from a 2-category $\mathcal{C}$ to Cat is the functor $\mathcal{C}(c,-)$ represented by an object $c \in \mathcal{C}$. We denote the corresponding special cofibration by

$$\widetilde{\rho}_c : \mathcal{N}(c\backslash\mathcal{C}) \to \mathcal{N}(\mathcal{C}). \tag{2.21}$$

If $\mathcal{C}$ is a 1-category, then $\widetilde{\rho}_c = q^*\rho_c$, where ρ_c is the discrete cofibration (2.5)

2.6 Homology of 2-categories. To define the derived category of functors from a small 2-category $\mathcal{C}$ to complexes of modules over a ring R, we use its nerve $\mathcal{N}(\mathcal{C})$, with its fibration $\pi : \mathcal{N}(\mathcal{C}) \to \Delta$ and the associated notion of a special map and a special object.

Definition 2.1. For any ring R and small 2-category $\mathcal{C}$, the *derived category of functors from $\mathcal{C}$ to R-mod* is given by

$$\mathcal{D}(\mathcal{C}, R) = \mathcal{D}_{sp}(\mathcal{N}(\mathcal{C}), R).$$

Recall that if $\mathcal{C}$ is a 1-category, then $\mathcal{D}_{sp}(\mathcal{N}(\mathcal{C}), R)$ is identified with $\mathcal{D}(\mathcal{C}, R)$ by the functor q^* of (2.15), so that the notation is consistent. Since the truncation functors with respect to the standard t-structure on $\mathcal{D}(\mathcal{N}(\mathcal{C}), R)$ send special objects to special objects, this standard t-structure induces a t-structure on $\mathcal{D}(\mathcal{C}, R) \subset \mathcal{D}(\mathcal{N}(\mathcal{C}), R)$ that we also call standard. We denote its heart by $\operatorname{Fun}(\mathcal{C}, R) \subset \mathcal{D}(\mathcal{C}, R)$; it is equivalent to the category of special functors from $\mathcal{N}(\mathcal{C})$ to R-mod. If $\mathcal{C}$ is a 1-category, every special functor factors uniquely through q of (2.14), so that the notation is still consistent.

Lemma 2.2. *For any 2-category $\mathcal{C}$, the embedding $\mathcal{D}(\mathcal{C}, R) \subset \mathcal{D}(\mathcal{N}(\mathcal{C}), R)$ admits a left and a right-adjoint functors $L^{sp}, R^{sp} : \mathcal{D}(\mathcal{N}(\mathcal{C}), R) \to \mathcal{D}(\mathcal{C}, R)$. For any object $c \in \mathcal{C}$ with the correspoding object $n(c) \in N(\mathcal{C})_1 \subset \mathcal{N}(\mathcal{C})$, we have*

$$L^{sp} R_{n(c)} \cong L^{\bullet} \widetilde{\rho}_{c!} R,$$

where $\widetilde{\rho}_c$ is the special cofibration (2.21)*, and R in the right-hand side is the constant functor.*

Proof. Say that a map f in $\mathcal{D}(\mathcal{N}(\mathcal{C}))$ is *co-special* if $\pi(f) : [n] \to [n']$ sends the initial object of the ordinal $[n]$ to the initial object of the ordinal $[n']$. Then as in the proof of [Ka2, Lemma 4.8], it is elementary to check that special and co-special maps in $\mathcal{N}(C)$ form a complementary pair in the sense of [Ka2, Definition 4.3], and then the adjoint functor L^{sp} is provided by [Ka2, Lemma 4.6]. Moreover, $L^{sp} \circ L^{sp} \cong L^{sp}$, and L^{sp} is an idempotent comonad on $\mathcal{D}(\mathcal{N}(\mathcal{C}), R)$, with algebras over this comonad being exactly the objects of $\mathcal{D}(\mathcal{C}, R)$. Moreover, by construction of [Ka2, Lemma 4.6], $L^{sp} : \mathcal{D}(\mathcal{N}(C), R) \to \mathcal{D}(\mathcal{N}(\mathcal{C}), R)$ has a right-adjoint functor $R^{sp} : \mathcal{D}(\mathcal{N}(\mathcal{C}), R) \to \mathcal{D}(\mathcal{N}(\mathcal{C}), R)$. By adjunction, R^{sp} is an idempotent monad, algebras over this monad are objects in $\mathcal{D}(\mathcal{C}, R)$, and R^{sp} factors through the desired functor $\mathcal{D}(\mathcal{N}(\mathcal{C}, R)) \to \mathcal{D}(\mathcal{C}, R)$ right-adjoint to the embedding $\mathcal{D}(\mathcal{C}, R) \subset \mathcal{D}(\mathcal{N}(\mathcal{C}), R)$. Finally, the last claim immediately follows by the same argument as in the proof of [Ka2, Theorem 4.2]. □

For any 2-functor $F : \mathcal{C} \to \mathcal{C}'$ between small 2-categories, the corresponding functor $\mathcal{N}(F)$ sends special maps to special maps, so that we have a pullback functor

$$F^* = \mathcal{N}(F)^* : \mathcal{D}(\mathcal{C}', R) \to \mathcal{D}(\mathcal{C}, R).$$

By Lemma 2.2, F^* has a left and a right-adjoint functor $F_!$, F_*, given by

$$F_! = L^{sp} \circ L^{\bullet}\mathcal{N}(F)_!, \qquad F_* = R^{sp} \circ R^{\bullet}\mathcal{N}(F)_*.$$

For any object $c \in \mathcal{C}$, we denote

$$R_c = L^{sp} R_{n(c)} \cong L^{\bullet}\widetilde{\rho}_{c!}R \in \mathcal{D}(\mathcal{C}, R). \tag{2.22}$$

If $\mathcal{C}$ is a 1-category, then this is consistent with (2.1) by (2.7). In the general case, by base change, we have a natural identification

$$R_c(c') \cong H_{\bullet}(\mathcal{C}(c,c'), R) \tag{2.23}$$

for any $c' \in \mathcal{C}$, an analog of (2.1). Moreover, by adjunction, we have a natural isomorphism

$$E(c) \cong \operatorname{Hom}(R_c, E) \tag{2.24}$$

for any $E \in \mathcal{D}(\mathcal{C}, R)$, a generalization of (2.2).

2.7 Finite sets. The first example of a 2-category that we will need is the following. Denote by Γ the category of finite sets. Then objects of the 2-category $\mathcal{Q}\Gamma$ are finite sets $S \in \Gamma$, and for any two $S_1, S_2 \in \Gamma$, the category $\mathcal{Q}\Gamma(S_1, S_2)$ is the groupoid of diagrams

$$S_1 \xleftarrow{\ l\ } S \xrightarrow{\ r\ } S_2 \tag{2.25}$$

in Γ and isomorphisms between them. The composition functors (2.19) are obtained by taking fibered products.

We can also define a smaller 2-category $\Gamma_+ \subset \mathcal{Q}\Gamma$ by keeping the same objects and requiring that $\Gamma_+(S_1, S_2)$ consists of diagrams (2.25) with injective map l. Then since such diagrams have no non-trivial automorphisms, Γ_+ is actually a 1-category. It is equivalent to the category of finite pointed sets. The equivalence sends a set S with a disntiguished element $o \in S$ to the complement $\overline{S} = S \setminus \{o\}$, and a map $f : S \to S'$ goes to the diagram

$$\overline{S} \xleftarrow{\ i\ } f^{-1}(\overline{S}') \xrightarrow{\ f\ } \overline{S}',$$

where $i : f^{-1}(\overline{S}') \to \overline{S}$ is the natural embedding. For any $n \geq 0$, we denote by $[n]_+ \in \Gamma_+$ the set with n non-distinguished elements (and one distinguished element o). In particular, $[0]_+ = \{o\}$ is the set with the single element o.

To construct 2-functors from $\mathcal{Q}\Gamma$ to Cat, recall that for any category $\mathcal{C}$, the *wreath product* $\mathcal{C} \wr \Gamma$ is the category of pairs $\langle S, \{c_s\}\rangle$ of a set $S \in \Gamma$ and a collection of objects $c_s \in \mathcal{C}$ indexed by elements $s \in S$. Morphisms from $\langle S, \{c_s\}\rangle$ to $\langle S', \{c'_s\}\rangle$ are given by a morphism $f : S \to S'$ and a collection of morphisms $c_s \to c'_{f(s)}$, $s \in S$. Then the forgetful functor $\rho : \mathcal{C} \wr \Gamma \to \Gamma$ is a fibration whose fiber over $S \in \Gamma$ is the product $\mathcal{C}^S$ of copies of the category

$\mathcal{C}$ numbered by elements $s \in S$, and whose transition functor $f^* : \mathcal{C}^{S_2} \to \mathcal{C}^{S_1}$ associated to a map $f : S_1 \to S_2$ is the natural pullback functor.
Assume that the category $\mathcal{C}$ has finite coproducts (including the coproduct of an empty collection of objects, namely, the initial object $0 \in \mathcal{C}$). Then all the transition functors f^* of the fibration ρ have left-adjoint functors $f_!$, so that ρ is also a cofibration. Moreover, for any diagram (2.25) in Γ, we have a natural functor

$$r_! \circ l^* : \mathcal{C}^{S_1} \to \mathcal{C}^{S_2}. \tag{2.26}$$

This defines a 2-functor $\mathsf{Vect}(\mathcal{C}) : \mathcal{Q}\Gamma \to \mathrm{Cat}$ – for any finite set $S \in \Gamma$, we let $\mathsf{Vect}(\mathcal{C})(S) = \mathcal{C}^S$, and for any $S_1, S_2 \in \Gamma$, the functor $\mathsf{Vect}(\mathcal{C})(S_1, S_2)$ of (2.20) sends a diagram (2.25) to the functor induced by (2.26). Moreover, for any subcategory $w(\mathcal{C}) \subset \mathcal{C}$ with the same objects as $\mathcal{C}$ and containing all isomorphisms, the collection of subcategories $\mathsf{Vect}(w(\mathcal{C}))(S) = w(\mathcal{C})^S \subset \mathcal{C}^S$ defines a subfunctor $\mathsf{Vect}(w(\mathcal{C})) \subset \mathsf{Vect}(\mathcal{C})$.
Restricting the 2-functor $\mathsf{Vect}(\mathcal{C})$ to the subcategory $\Gamma_+ \subset \mathcal{Q}\Gamma$ and applying the Grothendieck construction, we obtain a cofibration over Γ_+ that we denote by $\rho_+ : (\mathcal{C} \wr \Gamma)_+ \to \Gamma_+$. For any subcategory $w(\mathcal{C})$ with the same objects an containg all isomorphisms, we can do the same procedure with the subfunctor $\mathsf{Vect}(w(\mathcal{C})) \subset \mathsf{Vect}(\mathcal{C})$; this gives a subcofibration $(w(\mathcal{C}) \wr \Gamma)_+ \subset (\mathcal{C} \wr \Gamma)_+$, and in particular, ρ_+ restricts to a cofibration

$$\rho_+ : (w(\mathcal{C}) \wr \Gamma)_+ \to \Gamma_+. \tag{2.27}$$

Explicitly, the fiber of the cofibration ρ_+ over a pointed set $S \in \Gamma_+$ is identified with $w(\mathcal{C})^{\overline{S}}$, where $\overline{S} \subset S$ is the complement to the distiguished element. The transition functor corresponding to a pointed map $f : S \to S'$ sends a collection $\{c_s\} \in w(\mathcal{C})^{\overline{S}}$, $s \in \overline{S}$ to the collection $c'_{s'}$, $s' \in \overline{S}'$ given by

$$c'_{s'} = \bigoplus_{s \in f^{-1}(s')} c_s, \tag{2.28}$$

where $\oplus$ stands for the coproduct in the category $\mathcal{C}$.

2.8 Matrices and vectors. Now more generally, assume that we are given a small category $\mathcal{C}_0$ with finite coproducts and initial object, and moreover, $\mathcal{C}_0$ is a unital monoidal category, with a unit object $1 \in \mathcal{C}_0$ and the tensor product functor $- \otimes -$ that preserves finite coproducts in each variable. Then we can define a 2-category $\mathsf{Mat}(\mathcal{C}_0)$ in the following way:

(i) objects of $\mathsf{Mat}(\mathcal{C}_0)$ are finite sets $S \in \Gamma$,

(ii) for any $S_1, S_2 \in \Gamma$, $\mathsf{Mat}(\mathcal{C}_0)(S_1, S_2) \subset \mathcal{C}^{S_1 \times S_2}$ is the groupoid of isomorphisms of the category $\mathcal{C}^{S_1 \times S_2}$,

(iii) for any $S \in \Gamma$, $\mathsf{id}_S \in \mathsf{Mat}(\mathcal{C}_0)(S, S)$ is given by $\mathsf{id}_S = \delta_!(p^*(1))$, where $p : S \to \mathsf{pt}$ is the projection to the point, and $\delta : S \to S \times S$ is the diagonal embedding, and

(iv) for any $S_1, S_2, S_2 \in \Gamma$, the composition functor m_{S_1,S_2,S_3} of (2.19) is given by

$$m_{S_1,S_2,S_3} = p_{2!} \circ \delta_2^*,$$

where $p_2 : S_1 \times S_2 \times S_3 \to S_1 \times S_3$ is the product $p_2 = \mathsf{id} \times p \times \mathsf{id}$, and analogously, $\delta_2 = \mathsf{id} \times \delta \times \mathsf{id}$.

In other words, objects in $\mathsf{Mat}(\mathcal{C}_0)(S_1, S_2)$ are matrices of objects in $\mathcal{C}$ indexed by $S_1 \times S_2$, and the identity object and the composition functors are induces by those of $\mathcal{C}$ by the usual matrix multiplication rules. The associativity and unitality isomorphisms are also induced by those of $\mathcal{C}_0$. It is straightforward to check that this indeed defines a 2-category; to simplify notation, we denote its nerve by

$$\mathcal{M}at(\mathcal{C}_0) = \mathcal{N}(\mathsf{Mat}(\mathcal{C}_0)).$$

Moreover, assume given another small category $\mathcal{C}$ with finite coproducts, and assume that $\mathcal{C}$ is a unital right module category over the unital monoidal category $\mathcal{C}_0$ – that is, we have the action functor

$$- \otimes - : \mathcal{C} \times \mathcal{C}_0 \to \mathcal{C}, \tag{2.29}$$

preserving finite coproducts in each variable and equipped with the relevant unitality and asociativity isomorphism. Then we can define a 2-functor $\mathsf{Vect}(\mathcal{C}, \mathcal{C}_0)$ from $\mathsf{Mat}(\mathcal{C}_0)$ to Cat that sends $S \in \Gamma$ to $\mathcal{C}^S$, and sends an object $M \in \mathsf{Mat}(\mathcal{C}_0)(S_1, S_2)$ to the functor $\mathcal{C}^{S_1} \to \mathcal{C}^{S_2}$ induced by (2.29) via the usual rule of matrix action on vectors. We denote the corresponding special cofibration over $\mathcal{M}at(\mathcal{C}_0)$ by $\mathcal{V}ect(\mathcal{C}, \mathcal{C}_0)$. Moreover, given a subcategory $w(\mathcal{C}) \subset \mathcal{C}$ with the same objects and containing all the isomorphisms, we obtain a subfunctor $\mathsf{Vect}(w(\mathcal{C}), \mathcal{C}_0) \subset \mathsf{Vect}(\mathcal{C}, \mathcal{C}_0)$ given by

$$\mathsf{Vect}(w(\mathcal{C}), \mathcal{C}_0)(S) = w(\mathcal{C})^S \subset \mathcal{C}^S = \mathsf{Vect}(\mathcal{C}, \mathcal{C}_0)(S).$$

We denote the corresponding subcofibration by

$$\mathcal{V}ect(w(\mathcal{C}), \mathcal{C}_0) \subset \mathcal{V}ect(\mathcal{C}, \mathcal{C}_0).$$

If we take $\mathcal{C}_0 = \Gamma$, and let $- \otimes -$ be the cartesian product, then $\mathsf{Mat}(\mathcal{C}_0)$ is exactly the category $\mathcal{Q}\Gamma$ of Subsection 2.7. Moreover, any category $\mathcal{C}$ that has finite coproducts is automatically a module category over Γ with respect to the action functor

$$c \otimes S = \bigoplus_{s \in S} c, \qquad c \in \mathcal{C}, S \in \Gamma,$$

and we have $\mathsf{Vect}(\mathcal{C}, \Gamma) = \mathsf{Vect}(\mathcal{C})$, $\mathsf{Vect}(w(\mathcal{C}), \Gamma) = \mathsf{Vect}(w(\mathcal{C}))$. This example is universal in the following sense: for any associative unital category $\mathcal{C}_0$ with finite coproducts, we have a unique coproduct-preserving tensor functor $\Gamma \to \mathcal{C}_0$, namely $S \mapsto 1 \otimes S$, so that we have a canonical 2-functor

$$e : \mathcal{Q}\Gamma \to \mathsf{Mat}(\mathcal{C}_0). \tag{2.30}$$

For any $\mathcal{C}_0$-module category $\mathcal{C}$ with finite coproducts, we have a natural equivalence $e \circ \mathsf{Vect}(\mathcal{C}, \mathcal{C}_0) \cong \mathsf{Vect}(\mathcal{C})$, and similarly for $w(\mathcal{C})$.

2.9 The relative setting. Finally, let us observe that the 2-functors $\mathsf{Vect}(\mathcal{C}, \mathcal{C}_0)$, $\mathsf{Vect}(w(\mathcal{C}), \mathcal{C}_0)$ can also be defined in the relative situation. Namely, assume given a cofibration $\pi : \mathcal{C} \to \mathcal{C}'$ whose fibers π_c, $c \in \mathcal{C}'$ have finite coproducts. Moreover, assume that $\mathcal{C}$ is a module category over $\mathcal{C}_0$, and the action functor (2.29) commutes with projections to $\mathcal{C}'$, thus induces $\mathcal{C}_0$-module category structures on the fibers π_c of the cofibration π. Furthermore, assume that the induced action functors on the fibers π_c preserve finite coproducts in each variable. Then we can define a natural 2-functor $\mathsf{Vect}(\mathcal{C}/\mathcal{C}', \mathcal{C}_0) : \mathsf{Mat}(\mathcal{C}_0) \to \mathrm{Cat}$ by setting

$$\mathsf{Vect}(\mathcal{C}/\mathcal{C}', \mathcal{C}_0)(S) = \mathcal{C} \times_{\mathcal{C}'} \cdots \times_{\mathcal{C}'} \mathcal{C} \tag{2.31}$$

where the terms in the product in the right-hand side are numbered by elements of the finite set S. As in the absolute situation, the categories $\mathsf{Mat}(\mathcal{C}_0)(S_1, S_2)$ act by the vector multiplication rule. We denote by

$$\mathcal{V}ect(\mathcal{C}/\mathcal{C}', \mathcal{C}_0) \to \mathcal{M}at(\mathcal{C}_0)$$

the special cofibration corresponding to the 2-functor $\mathsf{Vect}(\mathcal{C}/\mathcal{C}', \mathcal{C}_0)$, and we observe that the cofibration π induces a natural cofibration

$$\mathcal{V}ect(\mathcal{C}/\mathcal{C}', \mathcal{C}_0) \to \mathcal{C} \tag{2.32}$$

whose fiber over $c \in \mathcal{C}$ is naturally identified with $\mathcal{V}ect(\pi_c, \mathcal{C}_0)$. Moreover, if we have a subcategory $w(\mathcal{C}) \subset \mathcal{C}$ with the same objects that contains all the isomorphisms, and $w(\mathcal{C}) \subset \mathcal{C}$ is a subcofibration, then we can let

$$\mathsf{Vect}(w(\mathcal{C})/\mathcal{C}', \mathcal{C}_0)(S) = w(\mathcal{C}) \times_{\mathcal{C}'} \cdots \times_{\mathcal{C}'} w(\mathcal{C}) \subset \mathsf{Vect}(\mathcal{C}/\mathcal{C}', \mathcal{C}_0)(S)$$

for any finite set $S \in \Gamma$, and this gives a subfunctor $\mathsf{Vect}(w(\mathcal{C})/\mathcal{C}', \mathcal{C}_0) \subset \mathsf{Vect}(\mathcal{C}/\mathcal{C}', \mathcal{C}_0)$ and a subcofibration $\mathcal{V}ect(w(\mathcal{C})/\mathcal{C}', \mathcal{C}_0) \subset \mathcal{V}ect(\mathcal{C}/\mathcal{C}', \mathcal{C}_0)$. The cofibration (2.32) then induces a cofibration

$$\mathcal{V}ect(w(\mathcal{C})/\mathcal{C}', \mathcal{C}_0) \to \mathcal{C} \tag{2.33}$$

whose fibers are identified with $\mathcal{V}ect(w(\pi_c), \mathcal{C}_0)$, $c \in \mathcal{C}$. As in the absolute case, in the case $\mathcal{C}_0 = \Gamma$, we simplify notation by setting $\mathcal{V}ect(w(\mathcal{C})/\mathcal{C}') = \mathcal{V}ect(w(\mathcal{C})/\mathcal{C}', \Gamma)$, and we denote by

$$((w(\mathcal{C})/\mathcal{C}') \wr \Gamma)_+ \to \Gamma_+ \tag{2.34}$$

the induced cofibration over $\Gamma_+ \subset \mathcal{Q}\Gamma$.

Analogously, if $\pi : \mathcal{C} \to \mathcal{C}'$ is a fibration, then the same constructions go through, except that $w(\mathcal{C}) \subset \mathcal{C}$ has to be a subfibration, and the functors (2.32), (2.33) are also fibrations, with the same identification of the fibers.

3 Statements.

3.1 Generalities on K-theory. To fix notations and terminology, let us summarize very briefly the definitions of algebraic K-theory groups.

First assume given a ring k, let $k\text{-mod}^{fp} \subset k\text{-mod}$ be the category of finitely generated projective k-modules, and let $\mathsf{Iso}(k) \subset k\text{-mod}^{fp}$ be the groupoid of finitely generated projective k-modules and their isomorphisms. Explicitly, we have

$$\mathsf{Iso}(k) \cong \coprod_{P \in k\text{-mod}^{fp}} [\mathsf{pt}/\operatorname{Aut}(P)],$$

where the sum is over all isomorphism classes of finitely generated projective k-modules, $\operatorname{Aut}(P)$ is the automorphism group of the module P, and for any group G, $[\mathsf{pt}/G]$ stands for the groupoid with one object with automorphism group G. The category $k\text{-mod}^{fp}$ is additive. In particular, it has finite coproducts. Since $\mathsf{Iso}(k) \subset k\text{-mod}^{fp}$ contains all objects and all the isomorphisms, we have the cofibration

$$\rho_+ : (\mathsf{Iso}(k) \wr \Gamma)_+ \to \Gamma_+$$

of (2.27). Its fiber $(\rho_+)_{[1]_+}$ over the set $[1]_+ \in \Gamma_+$ is $\mathsf{Iso}(k)$, and the fiber $(\rho_+)_S$ over a general $S \in \Gamma_+$ is the product $\mathsf{Iso}(k)^{\overline{S}}$. Applying the Grothendieck construction and taking the geometric realization of the nerve, we obtain a functor

$$|N(\operatorname{Gr}(\rho_+))| : \Gamma_+ \to \operatorname{Top}$$

from Γ_+ to the category Top of topological spaces, or in other terminology, a Γ-space. Then (2.28) immediately shows that this Γ-space is special in the sense of the Segal machine [Se], thus gives rise to a spectrum $\mathcal{K}(k)$. The algebraic K-groups $K_\bullet(k) = \pi_\bullet\mathcal{K}(k)$ are by definition the homotopy groups of this spectrum.

For a more general K-theory setup, assume given a small category $\mathcal{C}$ with the subcategories $c(\mathcal{C}), w(\mathcal{C}) \subset \mathcal{C}$ of cofibrations and weak equivalences, and assume that $\langle \mathcal{C}, c(\mathcal{C}), w(\mathcal{C})\rangle$ is a Waldhausen category. In particular, $\mathcal{C}$ has finite coproducts and the initial object $0 \in \mathcal{C}$. Then one lets $E\mathcal{C}$ be the category of pairs $\langle [n], \varphi\rangle$ of an object $[n] \in \Delta$ and a functor $\varphi : [n] \to \mathcal{C}$, with morphisms from $\langle [n], \varphi\rangle$ to $\langle [n'], \varphi'\rangle$ given by a pair $\langle f, \alpha\rangle$ of a map $f : [n] \to [n']$ and a map $\alpha : \varphi' \circ f \to \varphi$. Further, one lets $\widetilde{S\mathcal{C}} \subset E\mathcal{C}$ be the full subcategory spanned by pairs $\langle [n], \varphi\rangle$ such that φ factors through $c(\mathcal{C}) \subset \mathcal{C}$ and sends the initial object $o \in [n]$ to $0 \in \mathcal{C}$. The forgetful functor $s : \widetilde{S\mathcal{C}} \to \Delta$ sending $\langle [n], \varphi\rangle$ to $[n]$ is a fibration; explicitly, its fiber over $[n] \in \Delta$ is the category of diagrams (2.13) such that all the maps are cofibrations, and $c_1 = 0$. Finally, one says that a map f in $\widetilde{S\mathcal{C}}$ is *admissible* if in its canonical factorization $f = g \circ f'$ with $s(f) = s(f')$ and f' cartesian with respect to s, the morphism g pointwise lies in $w(\mathcal{C}) \subset \mathcal{C}$. Then by definition, $S\mathcal{C} \subset \widetilde{S\mathcal{C}}$ is the subcategory with the same objects and admissible maps between them. This is again a fibered category over Δ, with the fibration $S\mathcal{C} \to \Delta$ induced by the forgetful functor s. The

K-groups $K_\bullet(\mathcal{C})$ are given by

$$K_i(\mathcal{C}) = \pi_{i+1}(|N(S\mathcal{C})|), \qquad i \geq 0.$$

Moreover, since $\mathcal{C}$ has finite coproducts, the fibers of the fibration $\widetilde{S\mathcal{C}} \to \Delta$ also have finite coproducts, and since $S\mathcal{C} \subset \widetilde{S\mathcal{C}}$ contains all objects and all isomorphisms, we can form the cofibration

$$\rho_+ : ((S\mathcal{C}/\Delta) \wr \Gamma)_+ \to \Gamma_+ \tag{3.1}$$

of (2.34). Its fibers are the self-products $S\mathcal{C} \times_\Delta \cdots \times_\Delta S\mathcal{C}$. Then by (2.18),

$$|N(\mathrm{Gr}(\rho_+))| : \Gamma_+ \to \mathrm{Top}$$

is a special Γ-space, so that $|N(S\mathcal{C})|$ has a natural infinite loop space structure and defines a connective spectrum. The K-theory spectrum $\mathcal{K}(\mathcal{C})$ is given by $\mathcal{K}(\mathcal{C}) = \Omega|N(S\mathcal{C})|$.

Remark 3.1. Our definition of the category $S\mathcal{C}$ differs from the usual one in that the fibers of the fibration s are opposite to what one gets in the usual definition. This is harmless since passing to the opposite category does not change the homotopy type of the nerve, and this allows for a more succint definition.

The main reason we have reproduced the S-construction instead of using it as a black box is the following observation: the construction works just as well in the relative setting. Namely, let us say that a *family of Waldhausen categories* indexed by a category $\mathcal{C}'$ is a category $\mathcal{C}$ equipped with a cofibration $\pi : \mathcal{C} \to \mathcal{C}'$ with small fibers, and two subcofibrations $c(\mathcal{C}), w(\mathcal{C}) \subset \mathcal{C}$ such that for any $c \in \mathcal{C}'$, the subcategories

$$c(\pi_c) = c(\mathcal{C}) \cap \pi_c \subset \pi_c, \qquad w(\pi_c) = w(\mathcal{C}) \cap \pi_c \subset \pi_c$$

in the fiber π_c of the cofibration π turn it into a Waldhausen category. Then given such a family, one defines the category $E\mathcal{C}$ exactly as in the absolute case, and one lets $\widetilde{S(\mathcal{C}/\mathcal{C}')} \subset E\mathcal{C}$ be the full subcategory spanned by $\widetilde{S\pi_c} \subset E\pi_c \subset E\mathcal{C}$, $c \in \mathcal{C}'$. Further, one observes that the forgetful functor $s : \widetilde{S(\mathcal{C}/\mathcal{C}')} \to \Delta$ is a fibration, and as in the absolute case, one let $S(\mathcal{C}/\mathcal{C}') \subset \widetilde{S(\mathcal{C}/\mathcal{C}')}$ be the subcategory spanned by maps f in whose canonical factorization $f = g \circ f'$ with $s(f) = s(f')$ and f' cartesian with respect to s, the morphism g pointwise lies in $w(\mathcal{C}) \subset \mathcal{C}$. One then checks easily that the cofibration π induces a cofibration

$$S(\mathcal{C}/\mathcal{C}') \to \mathcal{C}'$$

whose fiber over $c \in \mathcal{C}'$ is naturally identified with $S\pi_c$. This cofibration is obviously functorial in $\mathcal{C}'$: for any functor $\gamma : \mathcal{C}'' \to \mathcal{C}'$ with the induced cofibration $\gamma^*\mathcal{C} \to \mathcal{C}''$, we have $S(\gamma^*\mathcal{C}/\mathcal{C}'') \cong \gamma^* S(\mathcal{C}/\mathcal{C}')$.

3.2 The setup and the statement. Now assume given a commutative ring k, so that $k\text{-mod}^{fp}$ is a monoidal category, and a Waldhausen category $\mathcal{C}$ that is additive and k-linear, so that $\mathcal{C}$ is a module category over $k\text{-mod}^{fp}$. Then all the fibers of the fibration $S\mathcal{C} \to \Delta$ are also module categories over $k\text{-mod}^{fp}$. To simplify notation, denote

$$\mathcal{M}at(k) = \mathcal{M}at(k\text{-mod}^{fp}), \quad \mathbb{K}(\mathcal{C}, k) = \mathcal{V}ect(S\mathcal{C}/\Delta, k\text{-mod}^{fp}).$$

More generally, assume given a family $\pi : \mathcal{C} \to \mathcal{C}'$ of Waldhausen categories, and assume that all the fibers of the cofibration π are additive and k-linear, and transition functors are additive k-linear functors. Then $\mathcal{C}$ is a $k\text{-mod}^{fp}$-module category over $\mathcal{C}$, and we can form the cofibration

$$\mathbb{K}(\mathcal{C}/\mathcal{C}', k) = \mathcal{V}ect(S(\mathcal{C}/\mathcal{C}')/\Delta, k\text{-mod}^{fp}) \to \mathcal{C}' \times \mathcal{M}at(k).$$

Denote by

$$\widetilde{\pi} : \mathbb{K}(\mathcal{C}/\mathcal{C}', k) \to \mathcal{C}', \qquad \varphi : \mathbb{K}(\mathcal{C}/\mathcal{C}', k) \to \mathcal{M}at(k) \tag{3.2}$$

its compositions with the projections to $\mathcal{C}'$ resp. $\mathcal{M}at(k)$. Then the fiber of the cofibration $\widetilde{\pi}$ over $c \in \mathcal{C}'$ is naturally idenitified with the category $\mathbb{K}(\pi_c, k)$.

Definition 3.2. Let R be the localization of $\mathbb{Z}$ in a set of primes. A commutative ring k is *R-adapted* if $K_i(k) \otimes R = 0$ for $i \geq 1$, and $K_0(k) \otimes R \cong R$ as a ring.

Example 3.3. Let k be a finite field of characteristic $\mathsf{char}(k) = p$, and let $R = \mathbb{Z}_{(p)}$ be the localization of $\mathbb{Z}$ in the prime ideal $p\mathbb{Z} \subset \mathbb{Z}$. Then k is R-adapted by the famous theorem of Quillen [Q].

Assume given an R-adapted commutative ring k. Any additive map $K_0(k) \to R$ induces a map of spectra

$$\mathcal{K}(k) \to H(R), \tag{3.3}$$

where $H(R)$ is the Eilenberg-MacLane spectrum corresponding to R, so that fixing an isomorphism $K_0(k) \otimes R \cong R$ fixes a map (3.3). Do this, and for any $P \in k\text{-mod}^{fp}$, denote by $\mathsf{rk}(P) \in R$ the image of its class $[P] \in K_0(k) \otimes R$ under the isomorphism we have fixed. Let $M(R)$ be the category of free finitely generated R-modules, and let $T \in \operatorname{Fun}(M(R)^o, R)$ be the functor sending a free R-module M to $M^* = \operatorname{Hom}_R(M, R)$. Equivalently, objects in $M(R)$ are finite sets S, and morphisms from S_1 to S_2 are elements in the set $R[S_1 \times S_2]$. In this description, sending $P \in k\text{-mod}^{fp}$ to $\mathsf{rk}(P)$ defines a 2-functor $rk : \mathsf{Mat}(k) \to M(R)$. By abuse of notation, we denote

$$\mathsf{rk} = q \circ \mathcal{N}(rk) : \mathcal{M}at(k) \to \mathcal{N}(M(R)) \to M(R).$$

Since the projection φ of (3.2) obviously inverts all maps cocartesian with respect to the cofibration $\widetilde{\pi}$, the pullback $\varphi^{o*}\,\mathsf{rk}^{o*}\,T \in \operatorname{Fun}(\mathbb{K}(\mathcal{C}/\mathcal{C}', k), R)$ also inverts all such maps. Therefore we are in the situation of Subsection 2.3, and we have a well-defined object

$$K^R_{\bullet}(\mathcal{C}/\mathcal{C}', k) = \mathbb{Z} \overset{\mathsf{L}}{\otimes}_{\widetilde{\pi}} \varphi^{o*}\, \mathsf{rk}^{o*}\, T \in \mathcal{D}(\mathcal{C}', R), \tag{3.4}$$

where $\mathbb{Z}$ on the left-hand side of the product is the constant functor with value $\mathbb{Z}$. If $\mathcal{C}' = \mathsf{pt}$ is the point category, we simplify notation by letting $K^R_{\bullet}(\mathcal{C}, k) = K^R_{\bullet}(\mathcal{C}/\mathsf{pt}, k)$. The object $K^R_{\bullet}(\mathcal{C}/\mathcal{C}', k)$ is clearly functorial in $\mathcal{C}'$: for any functor $\gamma : \mathcal{C}'' \to \mathcal{C}'$, we have a natural isomorphism

$$\gamma^* K^R_{\bullet}(\mathcal{C}/\mathcal{C}', k) \cong K^R_{\bullet}(\gamma^*\mathcal{C}/\mathcal{C}'', k).$$

In particular, the value of $K^R_{\bullet}(\mathcal{C}/\mathcal{C}', k)$ at an object $c \in \mathcal{C}'$ is naturally identified with $K^R_{\bullet}(\pi_c, k)$. Here is, then, the main result of the paper.

Theorem 3.4. *Assume given a k-linear additive small Waldhausen category $\mathcal{C}$, and a ring R that is k-adapted in the sense of Definition 3.2, and let $\mathcal{K}^R(\mathcal{C}, k)$ be the Eilenberg-Mac Lane spectum associated to the complex $K^R_{\bullet}(\mathcal{C}, k)$ of* (3.4)*. Then there exists a natural map of spectra*

$$\nu : \mathcal{K}(\mathcal{C}) \to \mathcal{K}^R(\mathcal{C}, k)$$

that induces an isomorphism of homology with coefficients in R.

Here a "spectrum" is understood as an object of the stable homotopy category without choosing any particular model for it. In practice, what we produce in proving Theorem 3.4 is two special Γ-spaces in the sense of the Segal machine representing the source and the target of our map ν, and we produce ν as a map of Γ-spaces. Note that our complex $\mathcal{K}^R_{\bullet}(\mathcal{C}, k)$ is concentrated in non-negative homological degrees. For such a complex, the simplest way to construct the corresponding Eilenberg-Mac Lane spectrum is to apply the Dold-Kan equivalence, and take the realization of the resulting simplicial abelian group — it is then trivially a special Γ-space. This is exactly what we do. As usual, we define "homology with coefficients in R" of a spectrum X by

$$H_{\bullet}(X, R) = \pi_{\bullet}(X \wedge H(R)).$$

If R is the localization of $\mathbb{Z}$ in the set of primes S, then by the standard spectral sequence argument, Theorem 3.4 implies that ν becomes a homotopy equivalence after localizing at the same set of primes S.

4 Proofs.

4.1 Additive functors. Before we prove Theorem 3.4, we need a couple of technical facts on the categories $\mathcal{D}(\mathsf{Mat}(k), R)$, $\mathcal{D}(M(R), R)$. Recall that we have a natural 2-functor $e : \mathcal{Q}\Gamma \to \mathsf{Mat}(k)$ of (2.30). Composing it with the natural embedding $\Gamma_+ \to \mathcal{Q}\Gamma$, we obtain a 2-functor

$$i : \Gamma_+ \to \mathsf{Mat}(k).$$

Composing further with the 2-functor $rk : \mathsf{Mat}(k) \to M(R)$, we obtain a functor

$$\bar{i} : \Gamma_+ \to M(R).$$

Explicitly, $\bar{i}$ sends a finite pointed set S to its reduced span

$$\bar{i}(S) = \overline{R[S]} = R[S]/R \cdot \{o\},$$

where $o \in S$ is the distinguished element. The object $T \in \mathrm{Fun}(M(R)^o, R)$ gives by pullback objects $\mathsf{rk}^{o*}T \in \mathrm{Fun}(\mathcal{M}at(k)^o, R)$, $\bar{i}^{o*}T \in \mathrm{Fun}(\Gamma_+^o, R)$. For any $E \in \mathcal{D}(\Gamma_+, R)$, denote

$$H_\bullet^\Gamma(E) = \mathrm{Tor}_\bullet^{\Gamma_+}(E, \bar{i}^*T). \tag{4.1}$$

Say that an object $E \in \mathcal{D}(\Gamma_+, R)$ is *pointed* if $E([0]_+) = 0$, where $[0]_+ = \{o\} \in \Gamma_+$ is the pointed set consisting of the distinguished element.

Lemma 4.1. (i) *For any two pointed objects $E_1, E_2 \in \mathcal{D}(\Gamma_+, R)$, we have $H_\bullet^\Gamma(E_1 \overset{\mathsf{L}}{\otimes} E_2) = 0$.*

(ii) *Assume given a spectrum X represented by a Γ-space $|X| : \Gamma_+ \to \mathrm{Top}$ special in the sense of Segal, and let $C_\bullet(|X|, R) \in \mathcal{D}(\Gamma_+, R)$ be the object obtained by taking pointwise the singular chain homology complex $C_\bullet(-, R)$. Then there exists a natural identification*

$$H_\bullet^\Gamma(C_\bullet(|X|, R)) \cong H_\bullet(X, R).$$

Proof. Although both claims are due to T. Pirashvili, in this form, (i) is [Ka4, Lemma 2.3], and its corollary (ii) is [Ka4, Theorem 3.2]. □

The category Γ_+ has coproducts – for any $S, S' \in \Gamma_+$, their coproduct $S \vee S' \in \Gamma_+$ is the disjoint union of S and S' with distinguished elements glued together. The embedding $S \to S \vee S'$ admits a canonical retraction $p : S \vee S' \to S$ identical on S and sending the rest to the distiguished element, and similarly, $S' \to S \vee S'$ has a canonical retraction $p' : S \vee S' \to S'$.

Definition 4.2. An object $E \in \mathcal{D}(\Gamma_+, R)$ is *additive* if for any $S, S' \in \Gamma_+$, the natural map

$$E(S \vee S') \to E(S) \oplus E(S') \tag{4.2}$$

induced by the retractions p, p' is an isomorphism. An object E in the category $\mathcal{D}(\mathsf{Mat}(k), R)$ resp. $\mathcal{D}(M(R), R)$ is *additive* if so is i^*E resp. $\bar{i}^*E$.

We denote by $\mathcal{D}_{add}(\Gamma_+, R)$, $\mathcal{D}_{add}(\mathsf{Mat}(k), R)$, $\mathcal{D}_{add}(M(R), R)$ the full subcategories in $\mathcal{D}(\Gamma_+, R)$, $\mathcal{D}(\mathsf{Mat}(k), R)$, $\mathcal{D}(M(R), R)$ spanned by additive objects. In fact, $\mathcal{D}_{add}(\Gamma_+, R)$ is easily seen to be equivalent to $\mathcal{D}(R)$. Indeed, $[0]_+ \in \Gamma_+$ is a retract of $[1]_+ \in \Gamma_+$, so that we have a canonical direct sum decomposition

$$R_1 \cong t \oplus R_0$$

for a certain $t \in \operatorname{Fun}(\Gamma_+, R)$, where to simplify notation, we denote $R_n = R_{[n]_+} \in \operatorname{Fun}(\Gamma_+, R)$, $n \geq 0$. Then for any pointed $E \in \mathcal{D}(\Gamma_+, R)$, the adjunction map induces a map

$$t \otimes M \to E, \tag{4.3}$$

where $M = E([1])_+ \in \mathcal{D}(R)$. Any additive object is automatically pointed, and the map (4.3) is an isomorphism if and only if E is additive. We actually have $t \otimes M \cong \mathcal{H}om(\overline{i}^{o*} T, M) \cong \overline{i}^* \mathcal{H}om(T, M)$, so that the equivalence $\mathcal{D}(R) \cong \mathcal{D}_{add}(\Gamma_+, R)$ is realized by the functor

$$\overline{i}^* \circ \mathcal{H}om(T, -) : \mathcal{D}(R) \xrightarrow{\sim} \mathcal{D}_{add}(\Gamma_+, R) \subset \mathcal{D}(\Gamma_+, R).$$

4.2 Adjunctions. By definition, $\overline{i}^*$ and i^* preserve additivity – namely, $\overline{i}^*$ sends $\mathcal{D}_{add}(M(R), R) \subset \mathcal{D}(M(R), R)$ into $\mathcal{D}_{add}(\Gamma_+, R) \subset \mathcal{D}(\Gamma_+, R)$, and i^* sends $\mathcal{D}_{add}(\mathsf{Mat}(k), R) \subset \mathcal{D}(M(R), R)$ into $\mathcal{D}_{add}(\Gamma_+, R) \subset \mathcal{D}(\Gamma_+, R)$. It turns out that their adjoint functors $R^{\bullet}\overline{i}_*$, i_* also preserve additivity.

Lemma 4.3. (i) *For any additive $\overline{E} \in \mathcal{D}(\Gamma_+, R)$, the objects $R^{\bullet}\overline{i}_*\overline{E} \in \mathcal{D}(M(R), R)$ and $i_*\overline{E} \in \mathcal{D}(\mathsf{Mat}(k), R)$ are additive.*

(ii) *For any additive $E \in \operatorname{Fun}(\mathsf{Mat}(k), R) \subset \mathcal{D}(\mathsf{Mat}(k), R)$, the adjunction unit map $E \to i_* i^* E$ is an isomorphism in homological degree 0 with respect to the standard t-structure.*

Proof. For the first claim, let $E = R^{\bullet}\overline{i}_*\overline{E}$, and note that we may assume that $\overline{E} = \overline{i}^* \mathcal{H}om(T, M)$ for some $M \in D(R)$. Then by adjunction, for any finite set S, we have

$$\begin{aligned} E(\overline{i}(S)) \cong \operatorname{Hom}(R_{\overline{i}(S)}, E) = \operatorname{Hom}(R_{\overline{i}(S)}, R^{\bullet}\overline{i}_*\overline{E}) \cong \\ \cong \operatorname{Hom}(\overline{i}^* R_{\overline{i}(S)}, \overline{E}) \cong \operatorname{Hom}(H^{\Gamma}_{\bullet}(\overline{i}^* R_{\overline{i}(S)}), M), \end{aligned}$$

where R_S is the representable functor (2.1), and $H^{\Gamma}_{\bullet}(-)$ is as in (4.1). Thus to to check that (4.2) is an isomorphism, we need to check that the natural map

$$H^{\Gamma}_{\bullet}(\overline{i}^* R_{\overline{i}(S)}) \oplus H^{\Gamma}_{\bullet}(\overline{i}^* R_{\overline{i}(S')}) \to H^{\Gamma}_{\bullet}(R_{\overline{i}(S \vee S')})$$

induced by the projections p, p' is an isomorphism. For any $S, S_1 \in \Gamma_+$, we have

$$\overline{i}^* R_{\overline{i}(S)}(S_1) \cong R[\overline{S} \times \overline{S}_1]. \tag{4.4}$$

In particular, $\overline{i}^* R_{\overline{i}(S)}([0]_+) \cong R$ indepedently of S, and the tautological projection $S \to [0]_+$ induces a functorial map

$$t : \overline{i}^* R_{\overline{i}([0]_+)} \to \overline{i}^* R_{\overline{i}(S)} \cong R$$

in $\operatorname{Fun}(\Gamma_+, R)$ identical after evaluation at $[0]_+ \in \Gamma_+$. Moreover, we have

$$\overline{i}^* R_{\overline{i}(S \vee S')} \cong \overline{i}^* R_{\overline{i}(S)} \otimes \overline{i}^* R_{\overline{i}(S')}, \tag{4.5}$$

and under these identifications, the projections p, p' induce maps $\mathsf{id} \otimes t$ resp. $t \otimes \mathsf{id}$. Then to finish the proof, in suffices to invoke Lemma 4.1 (i).
For the object $i_*\overline{E}$, the argument is the same, but we need to replace the representable functors $R_{\overline{i}(S)}$, $R_{\overline{i}(S')}$, $R_{\overline{i}(S \vee S')}$ by their 2-category versions of (2.22), and (4.4) becomes the isomorphism

$$i^* R_{i(S)} \cong H_\bullet(\mathsf{Iso}(k)^{\overline{S} \times \overline{S_1}}, R)$$

provided by (2.23). The corresponding version of (4.5) then follows from the Künneth formula.
For the second claim, note that since we have already proved that $i_*i^*\overline{E}$ is additive, it suffices to prove that the natural map

$$E([1]_+) \to i_*i^*E([1]_+)$$

is an isomorphism in homological degree 0. Again by Lemma 4.1 (ii) and adjunction, this amount to checking that the natural map

$$H_0(\mathcal{K}(k), R) \to R$$

induced by the rank map rk is an isomorphism. This follows from Definition 3.2 and Hurewicz Theorem. □

By definition, the functor rk^* also sends additive objects to additive objects, but here the situation is even better.

Lemma 4.4. *The functor $\mathsf{rk}_* : \mathcal{D}(\mathsf{Mat}(k), R) \to \mathcal{D}(M(R), R)$ sends additive objects to additive objects, and rk^*, rk_* induce mutually inverse equivalences between $\mathcal{D}_{add}(\mathsf{Mat}(k), R)$ and $\mathcal{D}_{add}(M(R), R)$.*

Proof. Assume for a moment that we know that for any additive $E \in \mathcal{D}(\mathsf{Mat}(k), R)$, $\mathsf{rk}_* E$ is additive, and the adjunction counit map $\mathsf{rk}^* \mathsf{rk}_* E \to E$ is an isomorphism. Then for any additive $E \in \mathcal{D}_{add}(M(R), R)$, the cone of the adjunction unit map $E \to \mathsf{rk}_* \mathsf{rk}^* E$ is annihilated by rk^*. Since the functor rk^* is obviously conservative, $E \to \mathsf{rk}_* \mathsf{rk}^* E$ then must be an isomorphism, and this would prove the claim.
It remains to prove that for any $E \in \mathcal{D}_{add}(\mathsf{Mat}(k), R)$, $\mathsf{rk}_* E$ is additive, and the map $\mathsf{rk}^* \mathsf{rk}_* E \to E$ is an isomorphism. Note that we have

$$E \cong \lim_{\stackrel{\leftarrow}{n}}{}^\bullet \tau_{\geq -n} E,$$

where $\tau_{\geq -n}E$ is the truncation with respect to the standard t-structure. If E is additive, then all its truncations are additive, and by adjunction, rk_* commutes with derived inverse limits. Moreover, since derived inverse limit commutes with finite sums, it preserves the additivity condition. Thus it suffices to prove the statement under assumption that E is bounded from below with respect to the standard t-structure. Moreover, it suffices to prove it separately in each homological degree n.

Since rk^* is obviously exact with respect to the standard t-structure, rk_* is right-exact by adjunction, and the statement is trivially true for $E \in \mathcal{D}^{\geq n+1}(\mathsf{Mat}(k), R)$. Therefore by induction, we may assume that the statement is proved for $E \in \mathcal{D}^{\geq m+1}_{add}(\mathsf{Mat}(k), R)$ for some m, and we need to prove it for $E \in \mathcal{D}^{\geq m}_{\underline{add}}(\mathsf{Mat}(k), R)$. Let $\overline{E} = i^*E$. Since E is additive, $\overline{E}$ is also additive, so that $i_*\overline{E}$ is additive by Lemma 4.3 (i). The functor i_* is also right-exact with respect to the standard t-structures by adjunction, and by Lemma 4.3 (ii), the cone of the adjunction map

$$E \to i_*i^*E = i_*\overline{E}$$

lies in $\mathcal{D}^{\geq m+1}_{add}(\mathsf{Mat}(k), R)$. Therefore it suffices to prove the statement for $i_*\overline{E}$ instead of E. Since $\mathsf{rk}_*\, i_*\overline{E} \cong R^\bullet \bar{i}_*\overline{E}$ is additive by Lemma 4.3 (i), it suffices to prove that the adjunction map

$$\mathsf{rk}^*\, \bar{i}_*\overline{E} \cong \mathsf{rk}^*\, \mathsf{rk}_*\, i_*\overline{E} \to i_*\overline{E}$$

is an isomorphism. Moreover, since both sides are additive, it suffices to prove it after evaluating at $i([1]_+)$. We may assume that $\overline{E} = \mathcal{H}om(\bar{i}^*T, M)$ for some $M \in \mathcal{D}(R)$, so that by adjunction, this is equivalent to proving that the natural map

$$H^\Gamma_\bullet(i^*R_{i([1]_+)}) \to H^\Gamma_\bullet(\bar{i}^*R_{\bar{i}([1]_+)})$$

is an isomorphism. But as in the proof of Lemma 4.3, this map is the map

$$H^\Gamma_\bullet(C_\bullet(\mathsf{Iso}(k)^{\overline{S}}, R)) \to H^\Gamma_\bullet(R[\overline{S}])$$

induced by the functor rk, and by Lemma 4.1 (ii), it is identified with the map of homology

$$H_\bullet(\mathcal{K}(k), R) \to H_\bullet(H(R), R)$$

induced by the map of spectra (3.3). This map is an isomorphism by Definition 3.2. $\square$

4.3 Proof of the theorem. We can now prove Theorem 3.4. We begin by constructing the map. To simplify notation, let $K = K^R_\bullet(\mathcal{C}, k) \in \mathcal{D}(R)$, and let

$$E = L^\bullet \pi_{2!}R \in \mathcal{D}(\mathsf{Mat}(k), \mathbb{Z}) \subset \mathcal{D}(\mathcal{M}at(k), \mathbb{Z}).$$

Then by the projection formula, we have a natural quasiisomorphism

$$K \cong E \overset{\mathsf{L}}{\otimes}_{\mathcal{M}at(k)} \mathsf{rk}^{o*}\, T,$$

so that by adjunction, we obtain a natural map

$$v : E \to \mathcal{H}om(\mathsf{rk}^{o*}\, T, K). \tag{4.6}$$

Restricting with respect to the 2-functor $i : \Gamma_+ \to \mathsf{Mat}(k)$, we obtain a map

$$\overline{v} : \overline{E} \to i^* \mathcal{H}om(\mathsf{rk}^{o*} T, K) \cong \mathcal{H}om(\overline{i}^{o*} T, K), \tag{4.7}$$

where we denote $\overline{E} = i^* E$. Now note that over $i(\mathcal{N}(\Gamma_+)) \subset \mathcal{M}at(k)$, the cofibration $\varphi : \mathbb{K}(\mathcal{C}, k) \to \mathcal{M}at(k)$ restricts to the special cofibration corresponding to the cofibration ρ_+ of (3.1). Therefore by base change, we have $\overline{E} \cong L^\bullet \rho_{+!} R$. Then to compute $\overline{E}$, we can apply the Grothendieck construction to the cofibration ρ_+ and use base change; this shows that $\overline{E} \in \mathcal{D}(\Gamma_+, R)$ can be represented by the homology complex

$$E_\bullet = C_\bullet(N(\mathrm{Gr}(\rho_+)), R).$$

Choose a complex $\overline{K}_\bullet$ representing $\mathcal{H}om(\overline{i}^* T, K) \in \mathcal{D}(\Gamma_+, R)$ in such a way that the map $\overline{v}$ of (4.7) is represented by a map of complexes

$$\overline{v}_\bullet : E_\bullet \to \overline{K}_\bullet.$$

Replacing $\overline{K}_\bullet$ with its truncation if necessary, we may assume that it is concentrated in non-negative homological degrees. Applying the Dold-Kan equivalence pointwise, we obtain a functor $\mathsf{D}(\overline{K}_\bullet)$ from Γ_+ to simplicial abelian groups. We can treat it as a functor to simplicial sets, and take pointwise the tautological map (2.9); this results in a map

$$\overline{\nu} : N(\mathrm{Gr}(\rho_+)) \to \mathsf{D}(\overline{K}_\bullet) \tag{4.8}$$

of functors from Γ_+ to simplicial sets. Taking pointwise geometric realization, we obtain a map of Γ-spaces, hence of spectra. By definition, the Γ-space $|N(\mathrm{Gr}(\rho_+))|$ corresponds to the spectrum $\mathcal{K}(\mathcal{C})$. Since $\overline{K}_\bullet$ represents the additive object $\overline{i}^* \mathcal{H}om(T, K) \in \mathcal{D}(\Gamma_+, R)$, the isomorphisms (4.2) induce weak equivalences of simplicial sets

$$\mathsf{D}(\overline{K}_\bullet)(S \vee S') \cong \mathsf{D}(\overline{K}_\bullet)(S) \times \mathsf{D}(\overline{K}_\bullet)(S'),$$

so that the Γ-space $|\,\mathsf{D}(\overline{K}_\bullet)|$ is special. It gives the Eilenberg-MacLane spectrum $\mathcal{K}$ corresponding to $K \cong \overline{K}_\bullet([1]_+) \in \mathcal{D}(R)$. Thus the map of spectra induced by $\overline{\nu}$ of (4.8) reads as

$$\mathcal{K}(\mathcal{C}) \to \mathcal{K}. \tag{4.9}$$

This is our map.

To prove the theorem, we need to show that the map $\overline{\nu}$ induces an isomorphism on homology with coefficients in R. Let $\overline{S} \in \mathcal{D}(\Gamma_+, R)$ be the object represented by the chain complex $C_\bullet(\mathsf{D}(\overline{K}_\bullet), R)$. Then by Lemma 4.1 (ii), it suffices to prove that the map

$$H^\Gamma_\bullet(\overline{E}) \to H^\Gamma_\bullet(\overline{S}) \tag{4.10}$$

induced by (4.8) is an isomorphism. Moreover, note that we can apply the procedure above to the map v of (4.6) instead of its restriction $\overline{v}$ of (4.7). This results in a map of functors

$$N(\mathrm{Gr}(\varphi)) \to \mathsf{D}(K_\bullet),$$

where $K_\bullet$ is a certain complex representing $\mathcal{H}om(\mathsf{rk}^* T, K) \in \mathcal{D}(\mathsf{Mat}(k), R)$. If we denote by $S \in \mathcal{D}(\mathsf{Mat}(k), R)$ the object represented by $C_\bullet(\mathsf{D}(K_\bullet), R)$ and let

$$\nu : E \to S \tag{4.11}$$

be the map induced by the map v, then we have $S_0 \cong i^* S$, $i^*\nu$ is the map induced by $\overline{\nu}$ of (4.8), and (4.10) becomes the map

$$H^\Gamma_\bullet(i^*\nu) : H^\Gamma_\bullet(i^*E) \to H^\Gamma_\bullet(i^*S).$$

By adjunction and Lemma 4.3 (i), it then suffices to prove that for any additive $N \in \mathcal{D}(\mathsf{Mat}(k), R)$, the map

$$\mathrm{Hom}(S, N) \to \mathrm{Hom}(E, N)$$

induced by the map $\nu : E \to S$ is an isomorphism. By Lemma 4.4, we may assume that $N \cong \mathsf{rk}^* \widetilde{N}$ for some additive $\widetilde{N} \in \mathcal{D}(M(R), R)$, and by induction on degree, we may further assume that $\widetilde{N}$ lies in a single homological degree. But since R is a localization of $\mathbb{Z}$, any additive functor from $M(R)$ to R-modules is R-linear, thus of the form $\mathcal{H}om(T, M)$ for some R-module M. Thus we may assume $\widetilde{N} = \mathcal{H}om(T, M)$ for some $M \in \mathcal{D}(R)$. Again by adjunction, it then suffices to prove that the map

$$E \overset{\mathsf{L}}{\otimes}_{\mathcal{M}at(k)} \mathsf{rk}^{o*} T \to S \overset{\mathsf{L}}{\otimes}_{\mathcal{M}at(k)} \mathsf{rk}^{o*} T$$

induced by the map ν of (4.11) is an isomorphism. But the adjunction map v of (4.6) has the decomposition (2.11) that reads as

$$E \xrightarrow{\quad\nu\quad} S \xrightarrow{\quad\kappa\quad} \mathcal{H}om(\mathsf{rk}^{o*} T, K),$$

where κ is the assembly map (2.10) for the complex $K_\bullet$. Thus to finish the proof, it suffices to check the following.

Lemma 4.5. *For any object $K \in \mathcal{D}(R)$ represented by a complex $K_\bullet$ of flat R-modules concentrated in non-negative homological degrees, denote by $\widetilde{S} \in \mathcal{D}(M(R), R)$ the object represented by the complex $C_\bullet(\mathsf{D}(\mathcal{H}om(T, K_\bullet)), R)$, let $S = \mathsf{rk}^* \widetilde{S}$, and let*

$$\mathsf{rk}^* \kappa : S \to \mathsf{rk}^* \mathcal{H}om(T, K) \cong \mathcal{H}om(\mathsf{rk}^{o*} T, K)$$

be the pullback of the assembly map $\kappa : \widetilde{S} \to \mathcal{H}om(T, K)$. Then the map

$$S \overset{\mathsf{L}}{\otimes}_{\mathcal{M}at(k)} \mathsf{rk}^{o*} T \to K$$

adjoint to $\mathsf{rk}^* \kappa$ is an isomorphism in $\mathcal{D}(R)$.

Proof. For any $M \in R$-mod, we can consider the functor $\mathcal{H}om(T, M)$ as a functor from $M(R)$ to sets, and we have the assembly map

$$R[\mathcal{H}om(T, M)] \to \mathcal{H}om(T, M). \tag{4.12}$$

If M is finitely generated and free, then by definition, we have

$$\begin{aligned} R[\mathcal{H}om(T, M)](M_1) = R[\mathcal{H}om(T, M)(M_1)] &= R[\operatorname{Hom}(M_1^*, M)] \\ &\cong R[\operatorname{Hom}(M^*, M_1)] \end{aligned}$$

for any $M_1 \in M(R)$, so that $R[\mathcal{H}om(T, M)] \cong R_{M^*}$ is a representable functor. Therefore $\operatorname{Tor}_i^{M(R)}(R[\mathcal{H}om(T, M)], T)$ vanishes for $i \geq 1$, and the map

$$R[\mathcal{H}om(T, M)] \overset{\mathsf{L}}{\otimes}_{M(R)} T \cong R[\mathcal{H}om(T, M)] \otimes_{M(R)} T \to M$$

adjoint to the assembly map (4.12) is an isomorphism. Since $- \overset{\mathsf{L}}{\otimes} -$ commutes with filtered direct limits, the same is true for an R-module M that is only flat, not necessarily finitely generated or free.

Moreover, consider the product $\Delta^o \times M(R)$, with the projections $\tau : \Delta^o \times M(R) \to M(R)$, $\tau' : \Delta^o \times M(R) \to \Delta^o$. Then for any simplicial pointwise flat R-module $M \in \operatorname{Fun}(\Delta^o, R)$, the map

$$a : R[\mathcal{H}om(\tau^* T, M)] \overset{\mathsf{L}}{\otimes}_{\tau'} \tau^* T \to M \tag{4.13}$$

adjoint to the assembly map $R[\mathcal{H}om(\tau^* T, M)] \to \mathcal{H}om(\tau^* T, M)$ is also an isomorphism. Apply this to $M = \mathsf{D}(K_\bullet)$, and note that we have

$$K \cong L^\bullet \tau_! M, \qquad \widetilde{S} \cong L^\bullet \tau_! R[\mathcal{H}om(\tau^* T, M)],$$

and the map $\widetilde{S} \overset{\mathsf{L}}{\otimes}_{M(R)} T \to K$ adjoint to the assembly map κ is exactly $L^\bullet \tau_!(a)$, where a is the map (4.13). Therefore it is also an isomorphism.

To finish the proof, it remains to show that the natural map

$$\widetilde{S} \overset{\mathsf{L}}{\otimes}_{M(R)} T \to \mathsf{rk}^* \widetilde{S} \overset{\mathsf{L}}{\otimes}_{\mathsf{Mat}(k)} \mathsf{rk}^{o*} T = S \overset{\mathsf{L}}{\otimes}_{\mathsf{Mat}(k)} \mathsf{rk}^{o*} T$$

is an isomorphism. By adjunction, it suffices to show that the natural map

$$\operatorname{Hom}(\widetilde{S}, E) \to \operatorname{Hom}(\widetilde{S}, \mathsf{rk}_* \mathsf{rk}^* E) \cong \operatorname{Hom}(S, \mathsf{rk}^* E)$$

is an isomorphism for any additive $E \in \mathcal{D}(M(R), R)$, and this immediately follows from Lemma 4.4. $\square$

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Minimal Canonical Dimensions of Quadratic Forms

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Abstract. Canonical dimension of a smooth complete connected variety is the minimal dimension of image of its rational endomorphism. The i-th canonical dimension of a non-degenerate quadratic form is the canonical dimension of its i-th orthogonal grassmannian. The maximum of a canonical dimension for quadratic forms of a fixed dimension is known to be equal to the dimension of the corresponding grassmannian. This article is about the minima of the canonical dimensions of an anisotropic quadratic form. We conjecture that they equal the canonical dimensions of an excellent anisotropic quadratic form of the same dimension and we prove it in a wide range of cases.

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Contents

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1. Introduction

The *canonical dimension* $\mathrm{cd}(X)$ of a smooth complete connected algebraic variety X over a field F is the minimum of dimension of the image of a rational map $X \dashrightarrow X$. This integer depends only on the class of field extensions L/F with $X(L) \neq \emptyset$. We refer to [9] and [16] for interpretations and basic properties of $\mathrm{cd}(X)$. We will also use a 2-local version $\mathrm{cd}_2(X)$ of $\mathrm{cd}(X)$ called *canonical* 2*-dimension.*

All fields here are of characteristic $\neq 2$. (The questions we are discussing can be raised in characteristic 2 as well, but all results we get are for characteristic $\neq 2$ mainly because their proofs need the Steenrod operations on Chow groups modulo 2 which are not available in characteristic 2.)

Let φ be a non-degenerate quadratic form over a field F. (Our general reference for quadratic forms is [3].) For any integer i lying in the interval $[1,\ (\dim\varphi)/2]$, the i-th canonical dimension $\mathrm{cd}[i](\varphi)$ is defined as the canonical dimension of the orthogonal grassmannian of i-dimensional totally isotropic subspaces of φ (*i-grassmannian of φ* for short). A little care should be given to the case of $i = (\dim\varphi)/2$ because the corresponding i-grassmannian is not connected if the discriminant of φ is trivial. However, the (two) connected components it has are isomorphic to each other so that we can define the canonical dimension by taking any of them.

For arbitrary i and a given field extension L/F, the i-grassmannian of φ has an L-point if and only if the Witt index $\mathfrak{i}_0(\varphi_L)$ is at least i. Therefore, $\mathrm{cd}[i](\varphi)$ is an invariant of the class of field extensions L/F satisfying $\mathfrak{i}_0(\varphi_L) \geq i$.

Similarly, the i-th canonical 2-dimension $\mathrm{cd}_2[i](\varphi)$ is the canonical 2-dimension of the i-grassmannian. Since in general, canonical 2-dimension is a lower bound for canonical dimension, we have $\mathrm{cd}[i](\varphi) \geq \mathrm{cd}_2[i](\varphi)$ for any i. This is known to be equality for $i = 1$ (see Section 5) and no example when this inequality is not an equality (for some $i > 1$) is known.

The study of canonical dimensions of quadratic forms naturally commences with the question about the range of their possible values for anisotropic quadratic forms of a fixed dimension (over all fields or over all field extensions of a given field). It has been shown in [12] (see also [13]) that the evident upper bound on $\mathrm{cd}[i](\varphi)$ and $\mathrm{cd}_2[i](\varphi)$, given by the dimension of the i-grassmannian, is sharp. Here is a formula for this dimension:

$$i(i-1)/2 + i(\dim\varphi - 2i).$$

The question on the sharp upper bound being therefore closed, the present paper addresses the question about the sharp lower bound. Natural candidates are canonical dimensions of excellent quadratic forms. We do not really have a strong evidence supporting this, but we may, for instance, recall [3, Theorem 84.1] where the excellent forms appear in the answer to the question about the minimal height of quadratic forms.

For any $n \geq 1$ and any $i \in [1,\ n/2]$, we write $\mathrm{cd}[i](n)$ (resp., $\mathrm{cd}_2[i](n)$) for the i-th canonical (2-)dimension of an anisotropic excellent n-dimensional quadratic form over some field. Note that $\mathrm{cd}[i](n)$ depends only on i, n and coincides with $\mathrm{cd}_2[i](n)$ (see Section 2).

The following conjecture therefore gives a complete answer to the question about the sharp lower bound on canonical dimension and canonical 2-dimension of anisotropic quadratic forms:

Conjecture 1.1. *Let φ be an anisotropic quadratic form over a field F satisfying $\dim\varphi > 2i$ for some $i \geq 1$. Then $\mathrm{cd}_2[i](\varphi) \geq \mathrm{cd}[i](\dim\varphi)$.*

The reason of excluding the case $2i = \dim\varphi$ in the statement is that in this case $\mathrm{cd}_2[i](\varphi) = \mathrm{cd}_2[i-1](\varphi_E)$ and $\mathrm{cd}[i](\varphi) = \mathrm{cd}[i-1](\varphi_E)$, where E/F is the discriminant field extension of φ ($E = F$ if the discriminant of φ is trivial) and $i \geq 2$. So, understanding of $\mathrm{cd}_2[i](\varphi)$ and $\mathrm{cd}[i](\varphi)$ for $i < (\dim\varphi)/2$ would provide their understanding for $i = (\dim\varphi)/2$ and, on the other hand, using these relations it is easy to get counter-examples to the formula of Conjecture 1.1 with $i = (\dim\varphi)/2$ (see Section 9).

In this paper we prove Conjecture 1.1 for "small" values of i, namely, for i not exceeding the 2-nd absolute Witt index of φ (see Theorem 6.1) as well as for $i \leq 5$ (see Theorems 7.1, 10.1 and 11.1). Finally, we prove Conjecture 1.1 with arbitrary i for all quadratic forms of height ≤ 3 (see Theorem 8.2).

The proofs make use of a wide spectrum of modern results on quadratic forms and Chow motives (the question seems to be a good testing ground for them). However most of the results under use already became "classical" at least in the sense that they have been exposed in a book (in [3] in most of the cases). For instance, we are using only a part of Excellent Connections Theorem [20, Theorem 1.3], called Outer, which was available already before the whole result and is exposed in [3, Corollary 80.13].

The most recent (and certainly yet non-classical) tool is a kind of going down principle for Chow motives due to Charles De Clercq [2], used in the proofs of Theorem 3.2 and (in a slightly different situation) Theorem 8.2. Applications of some particular cases of this principle exist already in the literature (see, e.g., [4]). We are using it here (in the proof of Theorem 3.2) in a new situation (still not in its full generality but in the biggest generality which may occur in the case of projective homogeneous varieties). This principle generalizes [10, Proposition 4.6], this older result is not sufficient for our purposes here.

Those methods can certainly be used to prove a bit more of Conjecture 1.1, but it seems that something is missing for a complete solution.

One could expect that the case of maximal i should be more accessible because maximal orthogonal grassmannians are so well-understood (mainly due to results of [19] also exposed in [3, Chapter XVI]). Though in our approach we have to go through all values of i in order to get to the maximal one.
This paper is an extended version of [6].
For more introduction see §12.

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2. Excellent forms

Here we recall some standard facts about excellent forms needed to complete the statement of Conjecture 1.1. Proofs (along with a definition) can be found, e.g., in [3, §28].
Every positive integer n is uniquely representable in the form of an alternating sum of 2-powers:

$$n = 2^{p_0} - 2^{p_1} + 2^{p_2} - \cdots + (-1)^{r-1}2^{p_{r-1}} + (-1)^r 2^{p_r}$$

for some integers $r \geq 0$ and $p_0, p_1, \ldots, p_r$ satisfying $p_0 > p_1 > \cdots > p_{r-1} > p_r + 1 > 0$.
For any integer $i \in [1,\ n/2]$, we define an integer $\mathrm{cd}[i](n)$ as

$$\mathrm{cd}[i](n) := 2^{p_{s-1}-1} - 1,$$

where s is the minimal positive integer with

$$n - 2i \geq 2^{p_s} - 2^{p_{s+1}} + \cdots + (-1)^{r-s}2^{p_r}.$$

Note that $\mathrm{cd}[i](n) \geq \mathrm{cd}[i+1](n)$ (for any i, n such that both sides are defined).

Lemma 2.1. *For any field k and any positive integer n, there exists an n-dimensional anisotropic quadratic form φ over an appropriate extension field F/k such that*

$$\mathrm{cd}[i](\varphi) = \mathrm{cd}_2[i](\varphi) = \mathrm{cd}[i](n)$$

for any $i \in [1,\ n/2]$.

Proof. One may take as F a field extension of k generated by p_0 algebraically independent elements. (For $k \subset \mathbb{R}$ one may simply take $F = \mathbb{R}$.) Then there exists an anisotropic p_0-fold Pfister form over F and therefore an anisotropic excellent quadratic form φ of dimension n. (For $F = \mathbb{R}$, the unique up to isomorphism anisotropic n-dimensional quadratic form is excellent.) We claim that canonical dimensions of such φ are as required. Indeed, for $i \in [1,\ n/2]$ let s be the defined above integer. Then by [3, Theorem 28.3], there exists a p_{s-1}-fold Pfister form ρ over F such that for any field extension L/F the condition $\mathfrak{i}_0(\varphi_L) \geq i$ is equivalent to isotropy of ρ_L. It follows that $\mathrm{cd}_2[i](\varphi) = \mathrm{cd}[i](\varphi) = 2^{p_{s-1}-1} - 1$. □

3. Upper motives

By *motives* we always mean the Chow motives with coefficients in $\mathbb{F}_2 := \mathbb{Z}/2\mathbb{Z}$; we use related terminology and notation as in [3, Chapter XII]. In particular, $M(X)$ is the motive of a variety X; the motive $M(\operatorname{Spec} F)$ and all its shifts $M(\operatorname{Spec} F)(i)$, $i \in \mathbb{Z}$, are called Tate motives. If M is a motive over F, $\bar{M}$ is the corresponding motive over an algebraic closure of F.

Let φ be a non-degenerate quadratic form over a field F. For an integer i with $0 \leq i < \dim \varphi/2$, let $X_i = X_i(\varphi)$ be the i-grassmannian of φ. In particular, X_0 is the point and $X := X_1$ is the projective quadric of φ.

According to the general notion of upper motive, introduced in [14] and [11], the upper motive $U(X_i)$ of the variety X_i is the unique summand in the complete motivic decomposition of X with the property that $\bar{U}(X_i)$ contains a Tate summand with no shift (i.e., with the shift 0). According to the general criterion of isomorphism for upper motives, $U(X_i) \simeq U(X_j)$ if and only if

$$\mathfrak{i}_0(\varphi_L) \geq i \iff \mathfrak{i}_0(\varphi_L) \geq j$$

for any extension field L/F. This means that i and j are in the same semi-open interval $(\mathfrak{j}_{r-1},\ \mathfrak{j}_r]$ for some $r \geq 0$, where $\mathfrak{j}_r$ is the r-th absolute Witt index of φ and $\mathfrak{j}_{-1} := -\infty$.

According to the general [11, Theorem 1.1], applied to quadrics, any summand of the complete motivic decomposition of X is a shift of $U(X_i)$ for some i or – in the case of even-dimensional φ with non-trivial discriminant – $U(\operatorname{Spec} E)$, where E/F is the quadratic discriminant field extension. Shifts of $U((X_i)_E)$, which may a priori appear by [11, Theorem 1.1], aren't possible because for any $j \neq (\dim X)/2$ the motive $\bar{M}(X)$ contains at most one Tate summand with the shift j while $\bar{U}((X_i)_E)$ contains two Tate summands without shift and two Tate summands with the shift $\dim U((X_i)_E)$.

A more precise information can be derived from [18, §4] (see also [3, §73]): if a shift of $U(X_i)$ for some $i \in (\mathfrak{j}_{r-1},\ \mathfrak{j}_r]$ with $r \geq 1$ really appears in the decomposition (note that this is always the case for $r = 1$), then it appears precisely $\mathfrak{i}_r := \mathfrak{j}_r - \mathfrak{j}_{r-1}$ times and the shifting numbers are $\mathfrak{j}_{r-1}, \mathfrak{j}_{r-1}+1, \dots, \mathfrak{j}_r - 1$. A shift of $U(\operatorname{Spec} E)$ appears if and only if φ_E is hyperbolic in which case it appears only once and with the shifting number $(\dim X)/2$. Note that $U(X_i)$ for $i \leq \mathfrak{j}_0$ is just the motive of a point (= the Tate summand with no shift), it appears precisely $2\mathfrak{j}_0$ times and the shifting numbers are $0, \dots, \mathfrak{j}_0 - 1$ and $\dim X, \dots, \dim X - (\mathfrak{j}_0 - 1)$.

Given any i and setting $Y := X_i$, one can answer the question, whether a shift of $U(Y)$ does appear, in terms of canonical dimension. First of all we have

Theorem 3.1 ([9, Theorem 5.1]). $\operatorname{cd}_2(Y) = \dim U(Y)$.

The following result is new. It provides a criterion of appearance of $U(Y)$ and is proved with a help of the going down principle of [2].

Theorem 3.2. *Assume that $i \in (\mathfrak{j}_{r-1},\ \mathfrak{j}_r]$ for some $r \geq 1$ and set $T := X_{\mathfrak{j}_{r-1}}$, $Y := X_i$. A shift of $U(Y)$ appears in the complete motivic decomposition of X*

if and only if

$$\mathrm{cd}_2(Y) = \mathrm{cd}_2(Y_{F(T)}).$$

Remark 3.3 (cf. §5). $\mathrm{cd}_2(Y_{F(T)}) = \dim\varphi - 2\mathfrak{j}_{r-1} - \mathfrak{i}_r - 1$.

Remark 3.4. Note that $\mathrm{cd}_2(Y) \geq \mathrm{cd}_2(Y_{F(T)})$ in general, [16].

Remark 3.5. As already mentioned, for $i = \mathfrak{j}_1$, the $\mathfrak{i}_1$ shifts of $U(X_i)$ appear always.

Remark 3.6. Sufficient criteria of appearance given in [18, Theorems 4.15 and 4.17] are easily derived from Theorem 3.2.

Proof of Theorem 3.2. By Theorem 3.1, we may replace $\mathrm{cd}_2(Y)$ with $\dim U(Y)$ as well as $\mathrm{cd}_2(Y_{F(T)})$ with $\dim U(Y_{F(T)})$ in the statement.
If a shift of $U(Y)$ does appear, then $\dim U(Y) = \dim U(Y_{F(T)})$ by [18, §4] (see also [3, §73]). This proves one ("easy") direction of Theorem 3.2. Let us concentrate on the opposite direction.
Note that a shift of $U(Y_{F(T)})$ is a summand in $M(X_{F(T)})$ (see Remark 3.5). If $\dim U(Y) = \dim U(Y_{F(T)})$, then we conclude by [2, Theorem 1.1] that the same shift of $U(Y)$ is a summand in $M(X)$. □

4. Some tools

In this section we recall some results which appear most frequently in the proofs below.

4a. Outer excellent connections. The following statement is a part of [20, Theorem 1.3]. It is also proved in [3, Corollary 80.13].

Theorem 4.1 (Outer Excellent Connections). *Let X be the quadric of an anisotropic quadratic form of dimension $2^n + m$ with $n \geq 1$ and $m \in [1,\ 2^n]$. Let M be a summand of the complete motivic decomposition of X. If $\bar{M}$ contains a Tate summand with a shift $i < m$, then it also contains a Tate summand with the shift $2^n - 1 + i = \dim X - (m-1) + i$.*

Using Theorem 4.1, we will be able to see that no shift of $U(Y)$ is a summand of $M(X)$ for certain concrete X and Y as in Theorem 3.2. The latter theorem will then tell us that $\mathrm{cd}_2(Y) > \mathrm{cd}_2(Y_{F(T)})$ (see Remark 3.4). Afterwards, we usually get even a sharper lower bound on $\mathrm{cd}_2(Y)$ using the motivic decomposition described right below.

4b. A motivic decomposition. Let φ be a non-degenerate quadratic form over F of dimension n and let Y be the $\mathfrak{i}_0$-grassmannian of φ. A variety is called *anisotropic* if all its closed points are of even degree.

Lemma 4.2 ([7, Theorem 15.8 and Corollary 15.14] or [1]). *The motive of Y decomposes in a sum of shifts of motives of some anisotropic varieties plus*

$$\bigoplus_{i=0}^{\mathfrak{i}_0} M(\Gamma_i)\Big(i(i-1)/2 + i(n - 2\mathfrak{i}_0)\Big),$$

where Γ_i *is the* i*-grassmannian of an* $\mathfrak{i}_0$*-dimensional vector space (*Γ_0 *and* $\Gamma_{\mathfrak{i}_0}$ *are points,* Γ_1 *and* $\Gamma_{\mathfrak{i}_0-1}$ *– projective spaces).*

Corollary 4.3. *The motive of* Y *does not contain any Tate summand with a positive shift strictly below* $n - 2\mathfrak{i}_0$*.*

Proof. By preceding Lemma, the motive of Y decomposes in a sum of shifts of motives of certain varieties. Those summands of this motivic decomposition which are motives of *isotropic* varieties[2] (and therefore can contain Tate summands while the motives of anisotropic varieties cannot, see, e.g., [14, Lemma 2.21]) come with shifts $i(i-1)/2 + i(n-2\mathfrak{i}_0)$, $i \geq 0$. For $i = 0$ the shifting number is 0 and the corresponding variety is just the point. For $i \geq 1$ the shifting numbers are at least $n - 2\mathfrak{i}_0$. □

4c. Maximal orthogonal grassmannian. Let φ be a non-degenerate quadratic form of dimension $2n+1$ and let $Y = X_n(\varphi)$ be the maximal orthogonal grassmannian of φ. Let $e_i \in \mathrm{Ch}^i(\bar{Y})$, $i = 0, 1, \dots, e_{2^{n-1}+1}$, be the standard generators of the modulo 2 Chow ring $\mathrm{Ch}(\bar{Y})$ defined as in [3, §86]. We say that e_i is *rational* if it is in the image of the change of field homomorphism $\mathrm{Ch}^i(Y) \to \mathrm{Ch}^i(\bar{Y})$; otherwise is *irrational*. We recall [3, Theorem 90.3] stating that $\mathrm{cd}_2(Y)$ is equal to the sum of all j such that e_j is irrational.

4d. Values of first Witt index. By [3, Proposition 79.4 and Remark 79.5], the first Witt index $\mathfrak{i}_1$ of an anisotropic quadratic form of dimension $d \geq 2$ satisfies the relations

$$\mathfrak{i}_1 \equiv d \pmod{2^r} \quad \text{and} \quad 1 \leq \mathfrak{i}_1 \leq 2^r$$

for some integer $r \geq 0$ with $2^r < d$.

4e. Dimensions of forms in I^n. By [3, Proposition 82.1], dimension d of an anisotropic quadratic form in I^n (the n-th power of the fundamental ideal in the Witt ring of the base field), where $n \geq 1$, is either $\geq 2^{n+1}$ or equals $2^{n+1} - 2^i$ with $1 \leq i \leq n+1$. Actually, apart from the old Arason-Pfister Hauptsatz (saying that $d \notin (0,\ 2^n)$), we are only using the statement about the "first hole", saying that d is outside of the open interval $(2^n,\ 2^n + 2^{n-1})$ and proved earlier ([18, Theorem 6.4]).

5. Level 1

We explain here that Conjecture 1.1 is actually already known in "level 1", that is, for i not exceeding the first Witt index of φ.
It is well-known that $\mathrm{cd}[1](\varphi) = \mathrm{cd}_2[1](\varphi) \geq \mathrm{cd}[1](\dim\varphi)$ for any anisotropic φ. This is a consequence of the formula $\mathrm{cd}[1](\varphi) = \mathrm{cd}_2[1](\varphi) = \dim\varphi - \mathfrak{i}_1(\varphi) - 1$ ([3, Theorem 90.2]) and the fact that the first Witt index of an excellent form is maximal among the first Witt indexes of quadratic forms of a given dimension ([5, Corollary 1]).

[2] A variety is *isotropic* here if it has a closed point of odd degree.

As an immediate consequence, we get the following, formally more general statement – (a bit more than) the "level 1" case of Conjecture 1.1:

Proposition 5.1. *Let φ be an anisotropic quadratic form over F of height ≥ 1. For any $i \leq \mathfrak{i}_1(\varphi)$ one has* $\operatorname{cd}[i](\varphi) = \operatorname{cd}_2[i](\varphi) \geq \operatorname{cd}[i](\dim\varphi)$.

Proof. $\operatorname{cd}[i](\varphi) = \operatorname{cd}_2[i](\varphi) = \operatorname{cd}_2[1](\varphi) \geq \operatorname{cd}[1](\dim\varphi) \geq \operatorname{cd}[i](\dim\varphi)$. □

6. Level 2

In this Section we prove (a bit more than) the "level 2" case of Conjecture 1.1:

Theorem 6.1. *Let φ be an anisotropic quadratic form over F of height ≥ 2. For any positive integer $i \leq \mathfrak{i}_1(\varphi) + \mathfrak{i}_2(\varphi)$ one has* $\operatorname{cd}_2[i](\varphi) \geq \operatorname{cd}[i](\dim\varphi)$.

Corollary 6.2. *Let φ be an anisotropic quadratic form over F of dimension $\dim\varphi \geq 4$. Then* $\operatorname{cd}_2[2](\varphi) \geq \operatorname{cd}[2](\dim\varphi)$. □

Corollary 6.3. *Let φ be an anisotropic quadratic form over F of height ≤ 2. Then* $\operatorname{cd}_2[i](\varphi) \geq \operatorname{cd}[i](\dim\varphi)$ *for any $i \in [1, (\dim\varphi)/2]$.* □

Proof of Theorem 6.1. We write $\mathfrak{i}_1$ for $\mathfrak{i}_1(\varphi)$ and $\mathfrak{i}_2$ for $\mathfrak{i}_2(\varphi)$. By Proposition 5.1, we may assume that $i \in (\mathfrak{i}_1,\ \mathfrak{i}_1 + \mathfrak{i}_2]$.
Let us write $\dim\varphi = 2^n + m$ with $n \geq 1$ and $m \in [1,\ 2^n]$. In the case of $\mathfrak{i}_1 = m$ we have

$$\operatorname{cd}_2[i](\varphi) \geq \operatorname{cd}_2[i-m](\varphi_1) \geq \operatorname{cd}[i-m](\dim\varphi_1) = \operatorname{cd}[i](\dim\varphi),$$

where φ_1 is the 1-st anisotropic kernel of φ, [3, §25]. The first inequality here is a particular case of the general principle saying that $\operatorname{cd}_2(T_L) \leq \operatorname{cd}_2(T)$ for a variety T over F and a field extension L/F, [16]. The second inequality holds by Proposition 5.1.
Below we are assuming that $\mathfrak{i}_1 < m$ and we have to show that $\operatorname{cd}_2[i](\varphi) \geq 2^n - 1$. In the case of $\mathfrak{i}_1 < m/2$ we have

$$\operatorname{cd}_2[i](\varphi) \geq \operatorname{cd}_2[1](\varphi_1) \geq \operatorname{cd}[1](2^n + m - 2\mathfrak{i}_1(\varphi)) = 2^n - 1.$$

Below we are assuming that $m/2 \leq \mathfrak{i}_1 < m$. It follows by §4d that $\mathfrak{i}_1 = m/2$ (in particular, m is ≥ 2 and even). This implies that $\mathfrak{i}_2 \leq 2^{n-1}$.
If $\mathfrak{i}_1 + \mathfrak{i}_2 < m$, then $\mathfrak{i}_1 + \mathfrak{i}_2 \leq m - \mathfrak{i}_1$ by [17, Theorem 1.2] which is impossible with $\mathfrak{i}_1 = m/2$. Therefore $\mathfrak{i}_1 + \mathfrak{i}_2 \geq m$ and it follows by Theorem 4.1 that $U(Y)(\mathfrak{i}_1)$ is not a direct summand of the motive of X, where X is the quadric of φ and Y is the $(\mathfrak{j}_2 = \mathfrak{i}_1 + \mathfrak{i}_2)$-th grassmannian of φ.
Since $\operatorname{cd}_2[i](\varphi) = \operatorname{cd}_2(Y)$, all we need to show is $\operatorname{cd}_2(Y) \geq 2^n - 1$.
First of all we have $\operatorname{cd}_2(Y) > \operatorname{cd}_2(Y_{F(X)})$ by Theorem 3.2 and Remark 3.4.
Now we claim that the complete decomposition of $M(Y_{F(X)})$ does not contain a summand $U(Y_{F(X)})(j)$ with j inside of the open interval

$$(0,\ 2^n + m - 2(\mathfrak{i}_1 + \mathfrak{i}_2)).$$

Indeed, if $U(Y_{F(X)})(j)$ with some j is there, then $M(Y_{F(Y)})$ contains a Tate summand with the shift j. By Corollary 4.3 we necessarily have $j = 0$ or $j \geq 2^n + m - 2(\mathfrak{i}_1 + \mathfrak{i}_2)$, and the claim is proved.

By [9, Proposition 5.2], the complete decomposition of $U(Y)_{F(X)}$ ends with a summand $U(Y_{F(X)})(j)$ with some $j \geq 0$. (We say "ends" meaning that $\dim U(Y)_{F(X)} = \dim U(Y_{F(X)}) + j$.) By the first claim, $j \neq 0$. It follows by the second claim that $j \geq 2^n + m - 2(\mathfrak{i}_1 + \mathfrak{i}_2)$. Thus

$$\operatorname{cd}_2(Y) = \dim U(Y) = \dim U(Y)_{F(X)} = \dim U(Y_{F(X)}) + j = \operatorname{cd}_2[1](\varphi_1) + j = (2^n + m - \mathfrak{i}_1 - 1) + j \geq (2^n + m - \mathfrak{i}_1 - 1) + (2^n + m - 2(\mathfrak{i}_1 + \mathfrak{i}_2)) = 2^{n+1} + 2m - 3\mathfrak{i}_1 - 2\mathfrak{i}_2 - 1 = 2^{n+1} + m/2 - 2\mathfrak{i}_2 - 1 \geq 2^n.$$

The last inequality here holds because $\mathfrak{i}_2 \leq 2^{n-1}$ and $m \geq 2$ (see above). The very first equality holds by Theorem 3.1. □

7. Third canonical dimension

THEOREM 7.1. *For any positive integer $i \leq 3$ and any anisotropic quadratic form φ of dimension $\geq 2i$, one has $\operatorname{cd}_2[i](\varphi) \geq \operatorname{cd}[i](\dim \varphi)$.*

PROPOSITION 7.2. *In order to prove Theorem 7.1, one only needs to show that $\operatorname{cd}_2[3](\varphi) \geq 2^n - 1$ for φ satisfying $\dim \varphi = 2^n + 3$ ($n \geq 2$) and $\mathfrak{i}_1(\varphi) = \mathfrak{i}_2(\varphi) = 1$.*

Proof. We are reduced to the case of $i = 3$ and of φ of height ≥ 3 with $\mathfrak{i}_1(\varphi) = \mathfrak{i}_2(\varphi) = 1$ by Theorem 6.1.
So, we assume that $\dim \varphi \geq 6$. Having written $\dim \varphi = 2^n + m$ with $m \in [1,\ 2^n]$ (where $n \geq 2$), we get

$$\operatorname{cd}_2[3](\varphi) \geq \operatorname{cd}_2[2](\varphi_1) \geq \operatorname{cd}[2](2^n + m - 2) = \begin{cases} 2^n - 1 = \operatorname{cd}[3](\dim \varphi) & \text{provided that } m \geq 4; \\ 2^{n-1} - 1 \geq \operatorname{cd}[3](\dim \varphi) & \text{for } m = 1, 2 \text{ and} \\ 2^{n-1} - 1 < 2^n - 1 = \operatorname{cd}[3](\dim \varphi) & \text{for } m = 3. \end{cases}$$

So, the only problematic value of m is 3. □

Proof of Theorem 7.1. We are showing that $\operatorname{cd}_2[i](\varphi) \geq 2^n - 1$ for φ as in Proposition 7.2. Let X be the quadric of φ, T the 2-grassmannian of φ, and Y its $(2 + \mathfrak{i}_3)$-grassmannian, where $\mathfrak{i}_3 = \mathfrak{i}_3(\varphi)$ is the third Witt index of φ. We have to show that $\operatorname{cd}_2(Y) \geq 2^n - 1$.
We claim that $\operatorname{cd}_2(Y) > \operatorname{cd}_2(Y_{F(T)})$. We get the claim as a consequence of Theorem 3.2 because by Theorem 4.1, $U(Y)(2)$ is not a summand of $M(X)$.
By §4b, the complete motivic decomposition of $M(Y_{F(Y)})$ does not contain a Tate summand with a positive shift strictly below

$$\dim \varphi - 4 - 2\mathfrak{i}_3 = 2^n - 1 - 2\mathfrak{i}_3.$$

Since $\operatorname{cd}_2(Y_{F(T)}) = \dim \varphi - 4 - \mathfrak{i}_3 - 1 = 2^n - 2 - \mathfrak{i}_3$, it follows that

$$\operatorname{cd}_2(Y) \geq (2^n - 2 - \mathfrak{i}_3) + (2^n - 1 - 2\mathfrak{i}_3).$$

Therefore $\operatorname{cd}_2(Y) \geq 2^n - 1$ provided that $3\mathfrak{i}_3 \leq 2^n - 2$.
The integer $\mathfrak{i}_3$ is the first Witt index $\mathfrak{i}_1(\varphi_2)$ of the anisotropic quadratic form φ_2 (the 2-nd anisotropic kernel of φ) of dimension $2^n - 1$. It follows by §4d

that $\mathfrak{i}_3 = 2^{n-1} - 1$ or $\mathfrak{i}_3 \leq 2^{n-2} - 1$. In the second case we are done and we are considering the first case below.
The equality $\mathfrak{i}_3 = 2^{n-1} - 1$ we are assuming now means that φ is a $(2^n + 3)$-dimensional anisotropic quadratic form of height 3 with the splitting pattern $(\mathfrak{i}_1, \mathfrak{i}_2, \mathfrak{i}_3) = (1, 1, 2^{n-1} - 1)$. This is actually possible only for $n = 2$ and $n = 3$ (see [18, §7.2] for $n \leq 4$), but we will not use this fact because our argument will work for arbitrary n.
Note that the variety Y is now the maximal grassmannian of φ. Therefore $\mathrm{cd}_2(Y)$ can be computed as in §4c in terms of the generators $e_i \in \mathrm{Ch}^i(\bar{Y})$, $i = 0, 1, \dots, e_{2^{n-1}+1}$.
Note that φ_2 is a $(2^n - 1)$-dimensional form of height 1. So, φ_2 is similar to a 1-codimensional subform of an anisotropic n-fold Pfister form. It follows by [3, Example 88.10] that $e_{2^{n-1}-1}$ is irrational.
As can be easily deduced from [3, Corollary 88.6], the homomorphism $\mathrm{Ch}(Y) \to \mathrm{Ch}(Y_{F(T)})$ is surjective in codimensions $\leq 2^{n-1} - 1$. Consequently, if both $e_{2^{n-1}}$ and $e_{2^{n-1}+1}$ are rational, then $\mathrm{cd}_2(Y_{F(T)}) = \mathrm{cd}_2(Y)$ contradicting the proved above claim. So, at least one of these two standard generators is irrational and it follows that $\mathrm{cd}_2(Y) \geq (2^{n-1} - 1) + 2^{n-1} = 2^n - 1$. □

8. Height 3

We prove (a bit more than) Conjecture 1.1 for all forms φ of height ≤ 3 in this Section.
We recall the classification of splitting patterns of quadratic forms of height 2 first (for reader's convenience, we include a proof):

Theorem 8.1 ([21, Theorem 2]). *Let φ be a non-zero anisotropic quadratic form of height ≤ 2 over a field of characteristic $\neq 2$ with a non-excellent splitting pattern. Then*

(1) *either* $\dim \varphi = 2^{n+1}$ *and* $\mathfrak{i}_1(\varphi) = 2^{n-1} = \mathfrak{i}_2(\varphi)$ *for some* $n > 0$ *or*
(2) $\dim \varphi = 2^n + 2^{n-1}$, $\mathfrak{i}_1(\varphi) = 2^{n-2}$, *and* $\mathfrak{i}_2(\varphi) = 2^{n-1}$ *for some* $n > 1$.

Proof. By [3, Theorem 84.1], the height of φ is at least the height of an anisotropic excellent form of dimension $\dim \varphi$. Moreover, for odd $\dim \varphi$ this is an equality by [3, Remark 84.6]. It follows that either $\dim \varphi = 2^n$ for some $n \geq 0$, or $\dim \varphi = 2^m - 2^{n-1}$ for some $m > n > 1$, or $\dim \varphi = 2^m - 2^n + 1$ for some $m > n > 1$. To finish, it suffices to look at the possible values of $\mathfrak{i}_1(\varphi)$ satisfying the condition of §4d together with the condition that $\dim \varphi - 2\mathfrak{i}_1(\varphi)$ is 2^r or $2^{r+1} - 1$ for some $r \geq 1$. The latter condition comes from the classical [15, Theorem 5.8] giving the list of possible dimensions of height 1 anisotropic quadratic forms. □

Theorem 8.2. *Let φ be an anisotropic quadratic form over F of height ≤ 3. For any positive integer $i \leq (\dim \varphi)/2$ one has $\mathrm{cd}_2[i](\varphi) \geq \mathrm{cd}[i](\dim \varphi)$. In particular, Conjecture 1.1 holds for all φ of height ≤ 3.*

Proof. By Theorem 6.1, we only need to consider φ of precisely height 3. Let $n := v_2(\dim \varphi)$.

Even-dimensional φ. We assume that $n \geq 1$ here. We have to show that

$$\mathrm{cd}_2[m](\varphi) \geq 2^{n-1} - 1,$$

where $m = (\dim \varphi)/2$.
If $2^{n-1} | \, \mathfrak{i}_1$, then $2^n | \, \dim \varphi_1$ and we are done. Otherwise, by §4d, $\mathfrak{i}_1 = 2^r$ for some $0 \leq r \leq n-2$. Since φ_1 is of height 2, it follows by Theorem 8.1 that $\dim \varphi = 2^n$.
If $r = n-2$ then $\mathfrak{i}_1 = 2^{n-2}$ and $\mathfrak{i}_2 = \mathfrak{i}_3 = 2^{n-3}$. It follows by [3, Corollary 83.4] that $\dim \varphi - \mathfrak{i}_1$ is a 2-power which is false. Therefore $r \leq n-3$ and we have $\mathfrak{i}_2 = 2^{n-1} - 2^{r+1}$, $\mathfrak{i}_3 = 2^r$; or $r = n-3$ and $\mathfrak{i}_2 = 2^{n-3}$, $\mathfrak{i}_3 = 2^{n-2}$. In the first case, it follows by [18, Theorem 7.7] as well as by [3, Theorem 83.3] that $U(Y_{F(X)})(\mathfrak{i}_1 + \mathfrak{i}_2)$ is a summand of $M(X_{F(X)})$, where X is the projective quadric and Y the m-grassmannian of φ. On the other hand, $U(Y)(\mathfrak{i}_1 + \mathfrak{i}_2)$ is not a summand of $M(X)$ by Theorem 4.1. It follows by [2, Theorem 1.1] that $\mathrm{cd}_2(Y_{F(X)}) < \mathrm{cd}_2(Y)$. Therefore the standard generator of maximal codimension $e_{2^{n-1}-1} \in \mathrm{Ch}(\bar{Y})$ is irrational and it follows that $\mathrm{cd}_2(Y) \geq 2^{n-1} - 1$. So, $\mathrm{cd}_2[m](\varphi) \geq 2^{n-1} - 1$ as required.
In the second case, we simply have

$$\mathrm{cd}_2(Y) = \mathrm{cd}_2[2^{n-3}+1](\varphi_1) \geq \mathrm{cd}[2^{n-3}+1](2^{n-1}+2^{n-2}) = 2^{n-1} - 1.$$

Odd-dimensional φ. Here we assume that $n = 0$. By [3, Theorem 84.1 and Remark 84.6], the height of an anisotropic excellent quadratic form of dimension $\dim \varphi$ is 1 or 3. In the first case we have $\dim \varphi = 2^n - 1$ for some $n \geq 2$ and we need to show that $\mathrm{cd}_2[2^{n-1} - 1](\varphi) \geq 2^{n-1} - 1$.
By §4d, $\mathfrak{i}_1 = 2^r - 1$ for some $1 \leq r \leq n-1$. Moreover, $r \leq n-2$ because height of φ is 3. It follows that $\dim \varphi_1 = 2^n - 2^{r+1} + 1$. Since φ_1 is of height 2, it has an excellent splitting pattern by Theorem 8.1 so that we have $\mathfrak{i}_2 = 2^{n-1} - 2^{r+1} + 1$ and $\mathfrak{i}_3 = 2^r - 1$.
Note that $n \geq 3$ at this stage. If $n = 3$ then we are done by Theorem 7.1.
Assuming that $n \geq 4$, we claim that $U(Y_{F(X)})(\mathfrak{i}_1 + \mathfrak{i}_2)$ is a summand of $M(X_{F(X)})$, where X is the quadric and Y the maximal grassmannian of φ. For $r \leq n-3$, this is a consequence of the inequality $\mathfrak{i}_2 > \mathfrak{i}_3$ and [18, Theorem 7.7]. For the remaining case of $r = n-2$ we have $\mathfrak{i}_2 = 1$ and the above argument does not work. However, Theorem 4.1 ensures that the first shell of φ is connected with the third one. Since $\mathfrak{i}_1 = 2^r - 1 > \mathfrak{i}_2 = 1$, the first shell is not connected with the second one, and the claim follows.
Using the claim, we finish the proof of the current case the way we did it above for even-dimensional φ.
It remains to consider the case when the height of an anisotropic excellent quadratic form of dimension $\dim \varphi$ is 3. This means that $\dim \varphi = 2^{n_0} - 2^{n_1} + 2^{n_2} - 1$ for some integers $n_0 > n_1 > n_2 \geq 2$.
The first Witt index $\mathfrak{i}_1$ should satisfy §4d and in the same time be such that the height of the integer[3] $\dim \varphi_1 = \dim \varphi - 2\mathfrak{i}_1$ is 2. It follows that $\dim \varphi_1 =$

[3]As in [3, §84], by the *height of a positive integer* we mean the height of an anisotropic excellent quadratic form of dimension equal this integer.

$2^{n_1} - 2^{n_2} + 1$ or $\dim \varphi_1 = 2^{n_0} - 2^{n_1} + 1$. In both cases we have

$$\mathrm{cd}_2[m](\varphi) \geq \mathrm{cd}_2[m_1](\varphi_1) \geq \mathrm{cd}[m_1](\dim \varphi_1) \geq \mathrm{cd}[m](\dim \varphi),$$

where $m := (\dim \varphi - 1)/2$ and $m_1 := (\dim \varphi_1 - 1)/2$. □

9. "Counter-example" with maximal grassmannian

Surprisingly, we didn't exclude $i = (\dim \varphi)/2$ in any case of Conjecture 1.1 proved so far. So, let us produce a "counter-example" to the case $i = (\dim \varphi)/2$ of Conjecture 1.1. By Theorem 7.1, i should be at least 4 and therefore $\dim \varphi$ should be at least 8. We produce it in dimension 8.

Let us find a field F and quadratic forms q and ψ such that q is 4-dimensional of discriminant a, $q_{F(\sqrt{a})}$ is anisotropic, ψ is 4-dimensional and divisible by $\langle\langle a \rangle\rangle$, and, finally, $\varphi := q \perp \psi$ is anisotropic. For instance, taking $F := k(a, b, c, d, e)$ with any field k and variables a, b, c, d, e, we can take $\psi = \langle\langle a, b \rangle\rangle$ and $q = \langle c, d, e, acde \rangle$. Then

$$\mathrm{cd}[4](\varphi) = \mathrm{cd}_2[4](\varphi) = \mathrm{cd}[2](q_{F(\sqrt{a})}) = 1 < 3 = \mathrm{cd}[4](\dim \varphi).$$

10. Fourth canonical dimension

Theorem 10.1. *Conjecture 1.1 holds for $i = 4$.*

Proposition 10.2. *It suffices to prove Theorem 10.1 only for φ of dimension $2^n + 4$ ($n \geq 3$), of height at least 4, and of Witt indexes satisfying either $\mathfrak{i}_1 = \mathfrak{i}_2 = \mathfrak{i}_3 = 1$; or $\mathfrak{i}_1 = 1, \mathfrak{i}_2 = 2$; or $\mathfrak{i}_1 = 2, \mathfrak{i}_2 = 1$. More precisely, it suffices to prove that $\mathrm{cd}_2[4](\varphi) \geq 2^n - 1$ for such φ.*

Proof. Note that Conjecture 1.1 for $i = 4$ is only about quadratic forms φ of dimension ≥ 9. We may assume that $\mathfrak{i}_1 \leq 2$ (Theorem 6.1) and that the height of φ is at least 4 (Theorem 8.2). Moreover, we may assume that $\mathfrak{i}_1 + \mathfrak{i}_2 + \mathfrak{i}_3 = 3$ or $\mathfrak{i}_1 + \mathfrak{i}_2 = 3$ (Theorem 7.1). Therefore, we have either $\mathfrak{i}_1 = \mathfrak{i}_2 = \mathfrak{i}_3 = 1$; or $\mathfrak{i}_1 = 1, \mathfrak{i}_2 = 2$; or $\mathfrak{i}_1 = 2, \mathfrak{i}_2 = 1$.

Let us write $\dim \varphi = 2^n + m$ with $n \geq 3$ and $1 \leq m \leq 2^n$. Assuming that $\mathfrak{i}_1 = 1$, we have

$$\mathrm{cd}_2[4](\varphi) \geq \mathrm{cd}[3](2^n + m - 2) = 2^n - 1 = \mathrm{cd}[4](\dim \varphi)$$

for $m \geq 5$. On the other hand,

$$\mathrm{cd}_2[4](\varphi) \geq \mathrm{cd}[3](2^n + m - 2) = 2^{n-1} - 1 = \mathrm{cd}[4](\dim \varphi)$$

for $m \leq 3$. So, the only problematic value of m is 4.

Assuming that $\mathfrak{i}_1 = 2$, we have

$$\mathrm{cd}_2[4](\varphi) \geq \mathrm{cd}[2](2^n + m - 4) = 2^n - 1 = \mathrm{cd}[4](\dim \varphi)$$

for $m \geq 6$. On the other hand,

$$\mathrm{cd}_2[4](\varphi) \geq \mathrm{cd}[2](2^n + m - 4) = 2^{n-1} - 1 = \mathrm{cd}[4](\dim \varphi)$$

for $m \leq 3$. Moreover, since $\mathfrak{i}_1 = 2$, m is necessarily even (§4d). So, the only problematic value of m is again 4. □

Proof of Theorem 10.1. Let φ be a quadratic form as in Proposition 10.2. Let r be the integer $\in \{3,4\}$ such that $\mathfrak{i}_1+\cdots+\mathfrak{i}_{r-1}=3$ (more concretely, $r:=3$ if $\mathfrak{i}_1+\mathfrak{i}_2=3$, $r:=4$ if $\mathfrak{i}_1+\mathfrak{i}_2+\mathfrak{i}_3=3$). Let X be the quadric, T the 3-grassmannian, and Y the $(3+\mathfrak{i}_r)$-grassmannian of φ. Since $\operatorname{cd}_2[4](\varphi)=\operatorname{cd}_2(Y)$, it suffices to prove that $\operatorname{cd}_2(Y)\geq 2^n-1$.

By Theorem 4.1, the motive $U(Y)(3)$ is not a summand of $M(X)$. It follows by Theorem 3.2 that $\operatorname{cd}_2(Y)>\operatorname{cd}_2(Y_{F(T)})$.

Now, using §4b in the standard way, we get that

$$\operatorname{cd}_2(Y)\geq \operatorname{cd}_2(Y_{F(T)})+(\dim\varphi-2(\mathfrak{i}_1+\cdots+\mathfrak{i}_r))= \\ (2^n-3-\mathfrak{i}_r)+(2^n-2-2\mathfrak{i}_r)=2^{n+1}-5-3\mathfrak{i}_r.$$

So, the inequality $\operatorname{cd}_2(Y)\geq 2^n-1$ holds if $2^{n+1}-5-3\mathfrak{i}_r\geq 2^n-1$, or, equivalently, if

$$2^n\geq 3\mathfrak{i}_r+4. \tag{10.3}$$

Since the integer $\mathfrak{i}_r$ is the first Witt index of the quadratic form φ_{r-1} of dimension $\dim\varphi_{r-1}=\dim\varphi-6=2^n-2$, we have $\mathfrak{i}_r=2^{n-1}-2$ or $\mathfrak{i}_r\leq 2^{n-2}-2$ or $\mathfrak{i}_r=1$ (the last case is not included in the previous one if $n=3$). The inequality (10.3) does not hold only in the case of $\mathfrak{i}_r=2^{n-1}-2$ which we consider now.

Recall that now our anisotropic quadratic form φ is of dimension 2^n+4 ($n\geq 3$) and has the splitting pattern

$$\text{either } (1,1,1,2^{n-1}-2,1),\ \text{or } (1,2,2^{n-1}-2,1),\ \text{or } (2,1,2^{n-1}-2,1).$$

Let $d\in F^\times$ represents the discriminant of φ. We evidently have $\varphi_{F(\sqrt{d})}\in I^n$. It follows that the Clifford algebra $C(\varphi)$ is Brauer-equivalent to a quaternion algebra (c,d) with some $c\in F^\times$. Let $\psi:=\varphi\perp c\,\langle\!\langle d\rangle\!\rangle$. Then $\operatorname{disc}(\psi)$ is trivial and it follows by [3, Lemma 14.2] that the Clifford invariant of ψ is trivial as well, so that $\psi\in I^3$. Let us show that $\psi\in I^n$. We know this already for $n=3$. To show this for $n\geq 4$, it suffices to show that ψ_L is hyperbolic for any extension field L/F such that $\dim(\psi_L)_{\mathrm{an}}\leq 2^{n-1}$. Since $\dim\psi=2^n+6$, the condition on L ensures that $\mathfrak{i}_0(\psi_L)\geq 2^{n-2}+3$. Since φ is a subform in ψ of codimension 2, $\mathfrak{i}_0(\varphi_L)\geq 2^{n-2}+1$ which is ≥ 4 because $n\geq 4$. It follows that $\mathfrak{i}_0(\varphi_L)\geq 4$ and therefore $\geq 2^{n-1}+1$ so that $\dim(\varphi_L)_{\mathrm{an}}\leq 2$ and $\dim(\psi_L)_{\mathrm{an}}\leq 4$. Since the discriminant and the Clifford invariant of ψ_L are trivial, it follows that ψ_L is hyperbolic.

We have shown that $\psi\in I^n$. On the other hand, $2^n+2\leq\dim\psi_{\mathrm{an}}\leq 2^n+6$ so that for $n\geq 4$ we get a contradiction with §4e.

We proved that none of the above splitting patterns of φ is possible in the case of $n\geq 4$. It remains to consider the case of $n=3$, that is, of $\dim\varphi=12$. The splitting patterns of 12-dimensional anisotropic quadratic forms have been classified in [18, §7.3]. In particular, it has been shown there that only the first of our three splitting patterns is possible. For φ of this possible splitting pattern $(1,1,1,2,1)$, the above procedure provides us with an anisotropic quadratic form $\psi':=\psi_{\mathrm{an}}\in I^3$ of dimension 14 or 12 such that for any extension field

L/F the condition $\mathfrak{i}_0(\varphi_L) \geq 4$ holds if and only if $\mathfrak{i}_0(\psi'_L) \geq 4$ is hyperbolic. It follows that $\mathrm{cd}_2[4](\varphi) = \mathrm{cd}_2[4](\psi')$. Since the height of ψ' is ≤ 3, it follows by Theorem 8.2 that $\mathrm{cd}_2[4](\psi') \geq \mathrm{cd}[4](\dim \psi') = 7 = 2^n - 1$. □

11. Fifth canonical dimension

Theorem 11.1. *Conjecture 1.1 holds for $i = 5$.*

Proposition 11.2. *It suffices to prove Theorem 11.1 only for φ of height at least 4 and with $\mathfrak{i}_1 + \cdots + \mathfrak{i}_r = 4$ for some r, having one of the following types:*

(1) $\dim \varphi = 2^n + 5$ *($n \geq 3$) and* $\mathfrak{i}_1 = 1$;
(2) $\dim \varphi = 2^n + 6$ *($n \geq 3$) and* $\mathfrak{i}_1 = 2$;
(3) $\dim \varphi = 2^n + 7$ *($n \geq 3$) and* $\mathfrak{i}_1 = 3$.

More precisely, it suffices to prove that $\mathrm{cd}_2[4](\varphi) \geq 2^n - 1$ *for above* φ.

Proof. Note that Conjecture 1.1 for $i = 5$ is only about quadratic forms φ of dimension ≥ 11. We may assume that $\mathfrak{i}_1 \leq 3$ (Theorem 6.1) and that the height of φ is at least 4 (Theorem 8.2). Also we may assume that $\mathfrak{i}_1 + \cdots + \mathfrak{i}_r = 4$ for some r (Theorem 10.1).
Let us write $\dim \varphi = 2^n + m$ with $n \geq 3$ and $1 \leq m \leq 2^n$.
Assuming that $\mathfrak{i}_1 = 1$, we have

$$\mathrm{cd}_2[5](\varphi) \geq \mathrm{cd}[4](2^n + m - 2) = 2^n - 1 = \mathrm{cd}[5](\dim \varphi)$$

for $m \geq 6$. On the other hand,

$$\mathrm{cd}_2[5](\varphi) \geq \mathrm{cd}[4](2^n + m - 2) = 2^{n-1} - 1 = \mathrm{cd}[5](\dim \varphi)$$

for $m \leq 4$. So, the only problematic value of m is 5.
Assuming that $\mathfrak{i}_1 = 2$, we have

$$\mathrm{cd}_2[5](\varphi) \geq \mathrm{cd}[3](2^n + m - 4) = 2^n - 1 = \mathrm{cd}[5](\dim \varphi)$$

for $m \geq 7$. On the other hand,

$$\mathrm{cd}_2[5](\varphi) \geq \mathrm{cd}[3](2^n + m - 4) = 2^{n-1} - 1 = \mathrm{cd}[5](\dim \varphi)$$

for $m \leq 4$. Moreover, since $\mathfrak{i}_1 = 2$, m is necessarily even (§4d). So, the only problematic value of m is 6.
Finally, assuming that $\mathfrak{i}_1 = 3$, we have

$$\mathrm{cd}_2[5](\varphi) \geq \mathrm{cd}[2](2^n + m - 6) = 2^n - 1 = \mathrm{cd}[5](\dim \varphi)$$

for $m \geq 8$. On the other hand,

$$\mathrm{cd}_2[5](\varphi) \geq \mathrm{cd}[2](2^n + m - 6) = 2^{n-1} - 1 = \mathrm{cd}[5](\dim \varphi)$$

for $m \leq 4$. Moreover, since $\mathfrak{i}_1 = 3$, m is necessarily odd (§4d). So, the only problematic values of m are 5 and 7. Since 3 cannot be the first Witt index of an anisotropic quadratic form of dimension $2^n + 5$ (§4d again), the value 5 is not possible for m. □

Proof of Theorem 11.1. Let φ be a quadratic form as in Proposition 11.2. Let r be the integer such that $\mathfrak{i}_1+\cdots+\mathfrak{i}_{r-1}=4$. Let X be the quadric, T the 4-grassmannian, and Y the $(4+\mathfrak{i}_r)$-grassmannian of φ. Since $\mathrm{cd}_2[5](\varphi)=\mathrm{cd}_2(Y)$, it suffices to prove that $\mathrm{cd}_2(Y)\geq 2^n-1$.

By Theorem 4.1, the motive $U(Y)(4)$ is not a summand of $M(X)$. It follows by Theorem 3.2 that $\mathrm{cd}_2(Y)>\mathrm{cd}_2(Y_{F(T)})$.

Now, using §4b in the standard way, we get that

$$\mathrm{cd}_2(Y)\geq \mathrm{cd}_2(Y_{F(T)})+(\dim\varphi-2(\mathfrak{i}_1+\cdots+\mathfrak{i}_r))\geq \\ (2^n+m-9-\mathfrak{i}_r)+(2^n+m-8-2\mathfrak{i}_r)=2^{n+1}+2m-17-3\mathfrak{i}_r.$$

So, the inequality $\mathrm{cd}_2(Y)\geq 2^n-1$ holds if $2^{n+1}+2m-17-3\mathfrak{i}_r\geq 2^n-1$, or, equivalently, if

$$(11.3)\qquad 2^n\geq 3\mathfrak{i}_r+16-2m.$$

Since the integer $\mathfrak{i}_r$ is the first Witt index of the quadratic form φ_{r-1} of dimension 2^n+m-8, we have $\mathfrak{i}_r=2^{n-1}+m-8$ or $\mathfrak{i}_r\leq 2^{n-2}+m-8$. For $n=3$ and $m=6$, there is an additional case of $\mathfrak{i}_r=1$. The inequality 11.3 does not hold only in the case of $\mathfrak{i}_r=2^{n-1}+m-8$ which we consider now.

Let us start with the case of $m=5$. So, φ is of dimension 2^n+5 and has the splitting pattern $(\dots,2^{n-1}-3,1)$.

First we consider the case of $n=3$. In this case we have $\mathrm{cd}_2(Y_{F(T)})=3$, $\mathrm{cd}_2(Y)\geq 6$, and §4b tells us that in the complete decomposition of $M(Y_{F(Y)})$ there is only one Tate summand with the shift 3. On the other hand, if $\mathrm{cd}_2(Y)=6$, then $U(Y)_{F(T)}$ contains summands $U(Y_{F(T)})$ and $U(Y_{F(T)})(3)$ so that there are two Tate summands with the shift 3 in the complete decomposition of $M(Y_{F(Y)})$. It follows that $\mathrm{cd}_2(Y)\geq 7$ and we are done in the case of $n=3$ and $m=5$.

In the case of $n\geq 4$ and $m=5$, the splitting pattern of φ is impossible. Indeed, the anisotropic part of a (2^n+6)-dimensional quadratic form of trivial discriminant containing φ is in I^n and has dimension 2^n+6 or 2^n+4.

We go ahead to the case $m=7$. Now φ is of dimension 2^n+7 and has the splitting pattern $(3,1,2^{n-1}-1)$. This is only possible for $n=3$, but anyway, the height of φ is 3 so that we don't need to do anything more here.

The remaining value of m is 6 so that $\dim\varphi=2^n+6$ now. The splitting pattern of φ is either $(2,1,1,2^{n-1}-2,1)$ or $(2,2,2^{n-1}-2,1)$. Adding to φ an appropriate binary quadratic form of discriminant $\mathrm{disc}(\varphi)$, we get a (2^n+8)-dimensional quadratic form ψ lying in I^3 and therefore in I^n. The anisotropic part of ψ has dimension 2^n+8, 2^n+6 or 2^n+4 and it follows that n is 3 or 4. Note that for any field extension L/F, the condition $\mathfrak{i}_0(\varphi_L)\geq 5$ is equivalent to $\mathfrak{i}_0(\psi_L)\geq 5$ so that $\mathrm{cd}_2[5](\varphi)=\mathrm{cd}_2[5](\psi)$.

If $n=4$, then ψ is anisotropic (of dimension 24) and of height 2. Therefore we have $\mathrm{cd}_2[5](\psi)\geq \mathrm{cd}[5](24)=15$ and the case is closed.

If $n=3$, then the anisotropic part ψ' of ψ has dimension 12, 14, or 16. If $\dim\psi'=12$, then $\mathrm{cd}_2[5](\psi)=\mathrm{cd}_2[3](\psi')\geq \mathrm{cd}[3](12)=7$. If $\dim\psi'=14$, then $\mathrm{cd}_2[5](\psi)=\mathrm{cd}_2[4](\psi')\geq \mathrm{cd}[4](14)=7$. Finally, if $\dim\psi'=16$, i.e., if ψ

is anisotropic, then either the height of ψ is ≤ 3 or $\mathfrak{i}_1(\psi) = 1$. If the height is ≤ 3, then $\mathrm{cd}_2[5](\psi) \geq \mathrm{cd}[5](16) = 7$. If the first Witt index is 1, then $\mathrm{cd}_2[5](\psi) \geq \mathrm{cd}_2[4](\psi_1) \geq \mathrm{cd}[4](14) = 7$. □

COROLLARY 11.4. *Conjecture 1.1 holds in full for φ of dimension ≤ 13.*

Proof. We only need to consider $\mathrm{cd}_2[6](\varphi)$ for a 13-dimensional φ. But $\mathrm{cd}[6](13) = 1$ so that the statement to prove is trivial. □

REMARK 11.5. To prove Conjecture 1.1 for 14-dimensional φ, one "only" needs to check that $\mathrm{cd}_2[6](\varphi) \geq 7$.

12. Final comments

The material of this section has been added on the suggestion of the editors. The following proposition justifies appearance of excellent forms in the statement of Conjecture 1.1. It also answers a question raised by H. Bermudez during my talk at the International Conference on the Algebraic and Arithmetic Theory of Quadratic Forms (Puerto Natales, Patagonia, Chile) in December 2013.

PROPOSITION 12.1. *Let φ be an anisotropic quadratic form over F such that for any integer i with $1 \leq i < (\dim \varphi)/2$, the i-th canonical dimension of φ is minimal among the i-th canonical dimensions of anisotropic quadratic forms (over field extensions of F) of dimension* $\dim \varphi$.
Then

(1) *the higher Witt indexes of φ are excellent , i.e., φ has the same height and the same higher Witt indexes as any anisotropic excellent quadratic form of the same dimension;*
(2) $\mathrm{cd}[i](\varphi) = \mathrm{cd}[i](\dim \varphi)$, *i.e. Conjecture 1.1 holds for quadratic forms of dimension* $\dim \varphi$*;*
(3) *the quadric of φ has excellent* motivic decomposition type*;*
(4) *assuming an open* [8, Conjecture 1.8], *φ is excellent.*

The statement of (3) will be explained in the proof. Since we do not know if such φ exists (in arbitrary dimension), (2) does not prove Conjecture 1.1. If the i-th canonical dimension $\mathrm{cd}[i](\varphi)$ of a given anisotropic quadratic form φ is minimal for some value of i, it is not necessarily minimal for other values of i. For instance, for any $r \geq 2$ and any positive $m < 2^{r-1}$, we may find a field F and an m-dimensional quadratic form ψ over F such that the even Clifford algebra of ψ is a division algebra and ψ is a subform of an anisotropic r-fold Pfister form π. Then the i-th canonical dimension $\mathrm{cd}[i](\varphi)$ of the complement φ of ψ in π is minimal for $i = 1, \ldots, \mathfrak{j}_1(\varphi)$. For the remaining values of i however, $\mathrm{cd}[i](\varphi)$ coincides with $\mathrm{cd}[i - \mathfrak{j}_1(\varphi)](\psi)$ which is equal to $\dim X_{i-\mathfrak{j}_1(\varphi)}(\psi)$ by [13]. In particular, $\mathrm{cd}[i](\varphi)$ is not minimal in general because $\mathrm{cd}[i](\dim \varphi) = \mathrm{cd}[i - \mathfrak{j}_1(\varphi)](\dim \psi)$.

Proof of Proposition 12.1. We write $\mathfrak{j}_r$ for $\mathfrak{j}_r(\varphi)$. Since $\mathrm{cd}[1](\varphi)$ is minimal, the first Witt index of φ is excellent and (2) holds for i up to $\mathfrak{j}_1$ by the results listed in §5.

If we already know for some $r \geq 1$ that the first $r-1$ higher Witt indexes of φ are excellent and (2) holds for i up to $\mathfrak{j}_{r-1}$, the inequality $\mathrm{cd}[\mathfrak{j}_{r-1}+1](\varphi) \geq \mathrm{cd}[1](\varphi_{r-1})$ (which is an equality for φ replaced by an anisotropic excellent form of the same dimension) tells us that $\mathfrak{j}_r = \mathfrak{j}_1(\varphi_{r-1})$ is excellent and (2) holds for i up to $\mathfrak{j}_r$.

We proved (1) and (2) at this point. As a byproduct, we see that the above inequality is in fact an equality, which means by Theorem 3.2 that a shift (and therefore precisely $\mathfrak{j}_r - \mathfrak{j}_{r-1}$ shifts) of $U(X_{\mathfrak{j}_r})$ appear(s) in the complete motivic decomposition of the quadric. Having this for every r and counting the ranks of the motives over an algebraic closure, we see that each undecomposable summand of the motive of the quadric is binary, i.e. becomes over an algebraic closure a sum of two Tate motives. More precisely, every indecomposable summand looks over an algebraic closure precisely the same as the corresponding summand in the complete motivic decomposition of an anisotropic excellent quadric of the same dimension. This is what (3) means.

Finally, [8, Conjecture 1.8] produces Pfister forms out of the binary motives and allows one to show that φ is excellent. In more details, since $U(X_1)$ is binary, [8, Conjecture 1.8] implies that φ is a neighbor of a Pfister form π. By similar reason, the complement of φ in π is also a Pfister neighbor. Continuing this way, we eventually see that φ is excellent. □

Example 12.2. To visualize the statement of Conjecture 1.1, it is probably a good idea to draw the graph of the function $i \mapsto \mathrm{cd}[i](n)$ for some concrete value of n. For $n = 60 = 2^6 - 2^2$, the function is constantly $31 = 2^{6-1} - 1$ on the interval [1, 28] and takes the value $1 = 2^{2-1} - 1$ at 29. As for arbitrary n, it is piecewise constant (with values given by some powers of 2 minus 1) and decreasing. Conjecture 1.1 claims that for any 60-dimensional anisotropic φ, the graph of the function $i \mapsto \mathrm{cd}[i](\varphi)$ is over the graph just described. We know that it is under the parabola $i \mapsto \dim X_i(\varphi) = i(i-1)/2 + i(60-2i)$. In contrast with the above lower bound, this piece of the parabola (constituting the upper bound for $\mathrm{cd}[i](\varphi)$) is not monotone: it increases until 19 and decreases after 20.

One may view Conjecture 1.1 as an analogue of the Outer Excellent Connection Theorem for quadrics, where the quadrics are replaced by higher orthogonal grassmannian. Note that according to Theorem 3.1, Conjecture 1.1 is a statement about the structure of the Chow motives of higher orthogonal grassmannians. As such, it clearly affects our understanding of their Chow groups. Finally, orthogonal grassmannians constitute a special and important case of a flag variety under a semisimple algebraic group; Conjecture 1.1 is to consider in this general context.

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Triality and algebraic groups of type $^3D_4$

Dedicated to Sasha Merkurjev on the occasion of his 60th birthday,

in fond memory of the time spent writing the Book of Involutions

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Abstract. We determine which simple algebraic groups of type $^3D_4$ over arbitrary fields of characteristic different from 2 admit outer automorphisms of order 3, and classify these automorphisms up to conjugation. The criterion is formulated in terms of a representation of the group by automorphisms of a trialitarian algebra: outer automorphisms of order 3 exist if and only if the algebra is the endomorphism algebra of an induced cyclic composition; their conjugacy classes are in one-to-one correspondence with isomorphism classes of symmetric compositions from which the induced cyclic composition stems.

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1. Introduction

Let G_0 be an adjoint Chevalley group of type D_4 over a field F. Since the automorphism group of the Dynkin diagram of type D_4 is isomorphic to the symmetric group $\mathfrak{S}_3$, there is a split exact sequence of algebraic groups

$$1 \longrightarrow G_0 \xrightarrow{\text{Int}} \mathbf{Aut}(G_0) \xrightarrow{\pi} \mathfrak{S}_3 \longrightarrow 1. \tag{1}$$

Thus, $\mathbf{Aut}(G_0) \cong G_0 \rtimes \mathfrak{S}_3$; in particular G_0 admits outer automorphisms of order 3, which we call *trialitarian automorphisms*. Adjoint algebraic groups of type D_4 over F are classified by the Galois cohomology set $H^1(F, G_0 \rtimes \mathfrak{S}_3)$ and the map induced by π in cohomology

$$\pi_* : H^1(F, G_0 \rtimes \mathfrak{S}_3) \to H^1(F, \mathfrak{S}_3)$$

associates to any group G of type D_4 the isomorphism class of a cubic étale F-algebra L. The group G is said to be of type ${}^1\mathsf{D}_4$ if L is split, of type ${}^2\mathsf{D}_4$ if $L \cong F \times \Delta$ for some quadratic separable field extension Δ/F, of type ${}^3\mathsf{D}_4$ if L is a cyclic field extension of F and of type ${}^6\mathsf{D}_4$ if L is a non-cyclic field extension. An easy argument given in Proposition 4.2 below shows that groups of type ${}^2\mathsf{D}_4$ and ${}^6\mathsf{D}_4$ do not admit trialitarian automorphisms defined over the base field. Trialitarian automorphisms of groups of type ${}^1\mathsf{D}_4$ were classified in [3], and by a different method in [2]: the adjoint groups of type ${}^1\mathsf{D}_4$ that admit trialitarian automorphisms are the groups of proper projective similitudes of 3-fold Pfister quadratic spaces; their trialitarian automorphisms are shown in [3, Th. 5.8] to be in one-to-one correspondence with the symmetric composition structures on the quadratic space. In the present paper, we determine the simple groups of type ${}^3\mathsf{D}_4$ that admit trialitarian automorphisms, and we classify those automorphisms up to conjugation.

Our main tool is the notion of a trialitarian algebra, as introduced in [9, Ch. X]. Since these algebras are only defined in characteristic different from 2, we assume throughout (unless specifically mentioned) that the characteristic of the base field F is different from 2. In view of [9, Th. (44.8)], every adjoint simple group G of type D_4 can be represented as the automorphism group of a trialitarian algebra $T = (E, L, \sigma, \alpha)$. In the datum defining T, L is the cubic étale F-algebra given by the map π_* above, E is a central simple L-algebra with orthogonal involution σ, known as the *Allen invariant* of G (see [1]), and α is an isomorphism relating (E, σ) with its Clifford algebra $C(E, \sigma)$ (we refer to [9, §43] for details). We show in Proposition 4.2 that if G admits an outer automorphism of order 3 modulo inner automorphisms, then L is either split (i.e., isomorphic to $F \times F \times F$), or it is a cyclic field extension of F (so G is of type ${}^1\mathsf{D}_4$ or ${}^3\mathsf{D}_4$), and the Allen invariant E of G is a split central simple L-algebra. This implies that T has the special form $T = \operatorname{End}\Gamma$ for some cyclic composition Γ. We further show in Theorem 4.3 that if G carries a trialitarian automorphism, then the cyclic composition Γ is *induced*, which means that it is built from some symmetric composition over F, and we establish a one-to-one correspondence between trialitarian automorphisms of G up to conjugation and isomorphism classes of symmetric compositions over F from which Γ is built. Note that we only consider outer automorphisms of order 3, hence we do not investigate the weaker property considered by Garibaldi in [6], about the existence of outer automorphisms whose third power is inner. Nevertheless, our Theorem 4.3 has bearing on it, in view of a result recently announced by Garibaldi and Petersson [7], establishing the existence of outer automorphisms whose third power is inner for any group of type ${}^3\mathsf{D}_4$ with trivial Allen invariant. If Γ is a cyclic composition that is not induced (examples are given in Remark 2.1), the group of automorphisms of $\operatorname{End}\Gamma$ does not admit trialitarian automorphisms, but the Garibaldi–Petersson result shows that it has outer automorphisms whose third power is inner.

The notions of symmetric and cyclic compositions are recalled in §2. Trialitarian algebras are discussed in §3, which contains the most substantial part of the argument: we determine the trialitarian algebras that have semilinear automorphisms of order 3 (Theorem 3.1) and we classify these automorphisms up to conjugation (Theorem 3.5). The group-theoretic results follow easily in §4 by using the correspondence between groups of type D_4 and trialitarian algebras.

Notation is generally as in the Book of Involutions [9], which is our main reference. For an algebraic structure S defined over a field F, we let $\operatorname{Aut}(S)$ denote the group of automorphisms of S, and write $\mathbf{Aut}(S)$ for the corresponding group scheme over F.

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2. Cyclic and symmetric compositions

Cyclic compositions were introduced by Springer in his 1963 Göttingen lecture notes ([11], [12]) to get new descriptions of Albert algebras. We recall their definition from [12][1] and [9, §36.B], restricting to the case of dimension 8.

Let F be an arbitrary field (of any characteristic). A *cyclic composition* (of dimension 8) over F is a 5-tuple $\Gamma = (V, L, Q, \rho, *)$ consisting of

- a cubic étale F-algebra L;
- a free L-module V of rank 8;
- a quadratic form $Q\colon V \to L$ with nondegenerate polar bilinear form b_Q;
- an F-automorphism ρ of L of order 3;
- an F-bilinear map $*\colon V \times V \to V$ with the following properties: for all x, y, $z \in V$ and $\lambda \in L$,

$$(x\lambda) * y = (x * y)\rho(\lambda), \qquad x * (y\lambda) = (x * y)\rho^2(\lambda),$$

[1] A cyclic composition is called a *normal twisted composition* in [11] and [12].

$$Q(x*y)=\rho\big(Q(x)\big)\cdot\rho^2\big(Q(y)\big),$$
$$b_Q(x*y,z)=\rho\big(b_Q(y*z,x)\big)=\rho^2\big(b_Q(z*x,y)\big).$$

These properties imply the following (see [9, §36.B] or [12, Lemma 4.1.3]): for all $x,\ y\in V$,

$$(x*y)*x=y\rho^2\big(Q(x)\big)\quad\text{and}\quad x*(y*x)=y\rho\big(Q(x)\big). \tag{2}$$

Since the cubic étale F-algebra L has an automorphism of order 3, L is either a cyclic cubic field extension of F, and ρ is a generator of the Galois group, or we may identify L with $F\times F\times F$ and assume ρ permutes the components cyclically. We will almost exclusively restrict to the case where L is a field; see however Remark 2.3 below.

Let $\Gamma'=(V',L',Q',\rho',*')$ be also a cyclic composition over F. An *isotopy*[2] $\Gamma\to\Gamma'$ is defined to be a pair (ν,f) where $\nu\colon (L,\rho)\xrightarrow{\sim}(L',\rho')$ is an isomorphism of F-algebras with automorphisms (i.e., $\nu\circ\rho=\rho'\circ\nu$) and $f\colon V\xrightarrow{\sim}V'$ is a ν-semilinear isomorphism for which there exists $\mu\in L^\times$ such that

$$Q'\big(f(x)\big)=\nu\big(\rho(\mu)\rho^2(\mu)\cdot Q(x)\big)\quad\text{and}\quad f(x)*'f(y)=f(x*y)\nu(\mu)$$

for $x,\ y\in V$. The scalar μ is called the *multiplier* of the isotopy. Isotopies with multiplier 1 are *isomorphisms*. When the map ν is clear from the context, we write simply f for the pair (ν,f), and refer to f as a *ν-semilinear isotopy*.

Examples of cyclic compositions can be obtained by scalar extension from symmetric compositions over F, as we now show. Recall from [9, §34] that a *symmetric composition* (of dimension 8) over F is a triple $\Sigma=(S,n,\star)$ where (S,n) is an 8-dimensional F-quadratic space (with nondegenerate polar bilinear form b_n) and $\star\colon S\times S\to S$ is a bilinear map such that for all $x,\ y,\ z\in S$

$$n(x\star y)=n(x)n(y)\quad\text{and}\quad b_n(x\star y,z)=b_n(x,y\star z).$$

If $\Sigma'=(S',n',\star')$ is also a symmetric composition over F, an *isotopy* $\Sigma\to\Sigma'$ is a linear map $f\colon S\to S'$ for which there exists $\lambda\in F^\times$ (called the *multiplier*) such that

$$n'\big(f(x)\big)=\lambda^2n(x)\quad\text{and}\quad f(x)\star'f(y)=f(x\star y)\lambda\quad\text{for } x,\ y\in S.$$

Note that if $f\colon\Sigma\to\Sigma'$ is an isotopy with multiplier λ, then $\lambda^{-1}f\colon\Sigma\to\Sigma'$ is an isomorphism. Thus, symmetric compositions are isotopic if and only if they are isomorphic. For an explicit example of a symmetric composition, take a Cayley (octonion) algebra $(C,\cdot)$ with norm n and conjugation map $\overline{}$. Letting $x\star y=\overline{x}\cdot\overline{y}$ for $x,\ y\in C$ yields a symmetric composition $\widetilde{C}=(C,n,\star)$, which is called a *para-Cayley composition* (see [9, §34.A]).
Given a symmetric composition $\Sigma=(S,n,\star)$ and a cubic étale F-algebra L with an automorphism ρ of order 3, we define a cyclic composition $\Sigma\otimes(L,\rho)$ as follows:

$$\Sigma\otimes(L,\rho)=(S\otimes_F L,L,n_L,\rho,*)$$

[2] The term used in [9, p. 490] is *similarity*.

where n_L is the scalar extension of n to L and $*$ is defined by extending $\star$ linearly to $S \otimes_F L$ and then setting

$$x * y = (\mathrm{Id}_S \otimes \rho)(x) \star (\mathrm{Id}_S \otimes \rho^2)(y) \qquad \text{for } x,\, y \in S \otimes_F L.$$

(See [9, (36.11)].) Clearly, every isotopy $f\colon \Sigma \to \Sigma'$ of symmetric compositions extends to an isotopy of cyclic compositions $(\mathrm{Id}_L, f)\colon \Sigma \otimes (L,\rho) \to \Sigma' \otimes (L,\rho)$. Observe for later use that the map $\widehat{\rho} = \mathrm{Id}_S \otimes \rho \in \mathrm{End}_F(S \otimes_F L)$ defines a ρ-semilinear automorphism

$$(3) \qquad \widehat{\rho}\colon \Sigma \otimes (L,\rho) \xrightarrow{\sim} \Sigma \otimes (L,\rho)$$

such that $\widehat{\rho}^3 = \mathrm{Id}$.
We call a cyclic composition that is isotopic to $\Sigma \otimes (L,\rho)$ for some symmetric composition Σ *induced*. Cyclic compositions induced from para-Cayley symmetric compositions are called *reduced* in [12].

Remark 2.1. Induced cyclic compositions are not necessarily reduced. This can be shown by using the following cohomological argument. We assume for simplicity that the field F contains a primitive cube root of unity ω. There is a cohomological invariant $g_3(\Gamma) \in H^3(F, \mathbb{Z}/3\mathbb{Z})$ attached to any cyclic composition Γ. The cyclic composition Γ is reduced if and only if $g_3(\Gamma) = 0$ (we refer to [12, §8.3] or [9, §40] for details). We construct an induced cyclic composition Γ with $g_3(\Gamma) \neq 0$. Let $a,\, b \in F^\times$ and let $A(a,b)$ be the F-algebra with generators α, β and relations $\alpha^3 = a$, $\beta^3 = b$, $\beta\alpha = \omega\alpha\beta$. The algebra $A(a,b)$ is central simple of dimension 9 and the space A^0 of elements of $A(a,b)$ of reduced trace zero admits the structure of a symmetric composition $\Sigma(a,b) = (A^0, n, \star)$ (see [9, (34.19)]). Such symmetric compositions are called *Okubo symmetric compositions*. From the Elduque–Myung classification of symmetric compositions [5, p. 2487] (see also [9, (34.37)]), it follows that symmetric compositions are either para-Cayley or Okubo. Let $L = F(\gamma)$ with $\gamma^3 = c \in F^\times$ be a cubic cyclic field extension of F, and let ρ be the F-automorphism of L such that $\gamma \mapsto \omega\gamma$. We may then consider the induced cyclic composition $\Gamma(a,b,c) = \Sigma(a,b) \otimes (L,\rho)$. Its cohomological invariant $g_3\big(\Gamma(a,b,c)\big)$ can be computed by the construction in [12, §8.3]: Using ω, we identify the group μ_3 of cube roots of unity in F with $\mathbb{Z}/3\mathbb{Z}$, and for any $u \in F^\times$ we write $[u]$ for the cohomology class in $H^1(F, \mathbb{Z}/3\mathbb{Z})$ corresponding to the cube class $uF^{\times 3}$ under the isomorphism $F^\times/F^{\times 3} \cong H^1(F, \mu_3)$ arising from the Kummer exact sequence (see [9, p. 413]). Then $g_3\big(\Gamma(a,b,c)\big)$ is the cup-product $[a] \cup [b] \cup [c] \in H^3(F, \mathbb{Z}/3\mathbb{Z})$. Thus any cyclic composition $\Gamma(a,b,c)$ with $[a] \cup [b] \cup [c] \neq 0$ is induced but not reduced. Another cohomological argument can be used to show that there exist cyclic compositions that are not induced. We still assume that F contains a primitive cube root of unity ω. There is a further cohomological invariant of cyclic compositions $f_3(\Gamma) \in H^3(F, \mathbb{Z}/2\mathbb{Z})$ which is zero for any cyclic composition induced by an Okubo symmetric composition[3] and is given by the class in $H^3(F, \mathbb{Z}/2\mathbb{Z})$ of the 3-fold Pfister form which is the norm of $\widetilde{C}$ if Γ is induced

[3] The fact that F contains a primitive cubic root of unity is relevant for this claim.

from the para-Cayley $\widetilde{C}$ (see for example [9, §40]). Thus a cyclic composition Γ with $f_3(\Gamma) \neq 0$ and $g_3(\Gamma) \neq 0$ is not induced. Such examples can be given with the help of the Tits process used for constructing Albert algebras (see [9, §39 and §40]). However, for example, cyclic compositions over finite fields, p-adic fields or algebraic number fields are reduced, see [12, p. 108].

Examples 2.2. (i) Let $F = \mathbb{F}_q$ be the field with q elements, where q is odd and $q \equiv 1 \bmod 3$. Thus F contains a primitive cube root of unity and we are in the situation of Remark 2.1. Let $L = \mathbb{F}_{q^3}$ be the (unique, cyclic) cubic field extension of F, and let ρ be the Frobenius automorphism of L/F. Because $H^3(F, \mathbb{Z}/3\mathbb{Z}) = 0$, every cyclic composition over F is reduced; moreover every 3-fold Pfister form is hyperbolic, hence every Cayley algebra is split. Therefore, up to isomorphism there is a unique cyclic composition over F with cubic algebra (L, ρ), namely $\Gamma = \widetilde{C} \otimes (L, \rho)$ where $\widetilde{C}$ is the split para-Cayley symmetric composition. If Σ denotes the Okubo symmetric composition on 3×3 matrices of trace zero with entries in F, we thus have $\Gamma \cong \Sigma \otimes (L, \rho)$, which means that Γ is also induced by Σ. By the Elduque–Myung classification of symmetric compositions, every symmetric composition over F is isomorphic either to the Okubo composition Σ or to the split para-Cayley composition $\widetilde{C}$. Therefore, Γ is induced by exactly two symmetric compositions over F up to isomorphism.

(ii) Assume that F contains a primitive cube root of unity and that F carries an anisotropic 3-fold Pfister form n. Let C be the non-split Cayley algebra with norm n and let $\widetilde{C}$ be the associated para-Cayley algebra. For any cubic cyclic field extension (L, ρ) the norm n_L of the cyclic composition $\widetilde{C} \otimes (L, \rho)$ is anisotropic. Thus it follows from the Elduque–Myung classification that any symmetric composition Σ such that $\Sigma \otimes (L, \rho)$ is isotopic to $\widetilde{C} \otimes (L, \rho)$ must be isomorphic to $\widetilde{C}$.

(iii) Finally, we observe that the cyclic compositions of type $\Gamma(a, b, c)$, described in Remark 2.1, have invariant g_3 equal to zero if $c = a$. Since the f_3-invariant is also zero, they are all isotopic to the cyclic composition induced by the split para-Cayley algebra. Thus we can get (over suitable fields) examples of many mutually non-isomorphic symmetric compositions $\Sigma(a, b)$ that induce isomorphic cyclic compositions $\Gamma(a, b, c)$.

Of course, besides this construction of cyclic compositions by induction from symmetric compositions, we can also extend scalars of a cyclic composition: if $\Gamma = (V, L, Q, \rho, *)$ is a cyclic composition over F and K is any field extension of F, then $\Gamma_K = (V \otimes_F K, L \otimes_F K, Q_K, \rho \otimes \mathrm{Id}_K, *_K)$ is a cyclic composition over K.

Remark 2.3. Let $\Gamma = (V, L, Q, \rho, *)$ be an arbitrary cyclic composition over F with L a field. Write θ for ρ^2. We have an isomorphism of L-algebras

$$\nu\colon L \otimes_F L \xrightarrow{\sim} L \times L \times L \quad \text{given by} \quad \ell_1 \otimes \ell_2 \mapsto (\ell_1\ell_2, \rho(\ell_1)\ell_2, \theta(\ell_1)\ell_2).$$

Therefore, the extended cyclic composition Γ_L over L has a split cubic étale algebra. To give an explicit description of Γ_L, note first that under the isomorphism ν the automorphism $\rho\otimes\mathrm{Id}_L$ is identified with the map $\widetilde{\rho}$ defined by $\widetilde{\rho}(\ell_1,\ell_2,\ell_3)=(\ell_2,\ell_3,\ell_1)$. Consider the twisted L-vector spaces ${}^\rho V$, ${}^\theta V$ defined by

$$ {}^\rho V=\{{}^\rho x\mid x\in V\},\qquad {}^\theta V=\{{}^\theta x\mid x\in V\} $$

with the operations

$$ {}^\rho(x+y)={}^\rho x+{}^\rho y,\ {}^\theta(x+y)={}^\theta x+{}^\theta y,\text{ and } {}^\rho(x\lambda)=({}^\rho x)\rho(\lambda),\ {}^\theta(x\lambda)=({}^\theta x)\theta(\lambda) $$

for x, $y\in V$ and $\lambda\in L$. Define quadratic forms ${}^\rho Q\colon {}^\rho V\to L$ and ${}^\theta Q\colon {}^\theta V\to L$ by

$$ {}^\rho Q({}^\rho x)=\rho\big(Q(x)\big)\quad\text{and}\quad {}^\theta Q({}^\theta x)=\theta\big(Q(x)\big)\quad\text{for } x\in V, $$

and L-bilinear maps

$$ *_{\mathrm{Id}}\colon {}^\rho V\times{}^\theta V\to V,\quad *_\rho\colon {}^\theta V\times V\to{}^\rho V,\quad *_\theta\colon V\times{}^\rho V\to{}^\theta V $$

by

$$ {}^\rho x *_{\mathrm{Id}} {}^\theta y=x*y,\quad {}^\theta x *_\rho y={}^\rho(x*y),\quad x*_\theta {}^\rho y={}^\theta(x*y)\quad\text{for } x,\ y\in V. $$

We may then consider the quadratic form

$$ Q\times{}^\rho Q\times{}^\theta Q\colon V\times{}^\rho V\times{}^\theta V\to L\times L\times L $$

and the product $\diamond\colon (V\times{}^\rho V\times{}^\theta V)\times(V\times{}^\rho V\times{}^\theta V)\to(V\times{}^\rho V\times{}^\theta V)$ defined by

$$ (x,{}^\rho x,{}^\theta x)\diamond(y,{}^\rho y,{}^\theta y)=({}^\rho x*_{\mathrm{Id}}{}^\theta y,{}^\theta x*_\rho y,\ x*_\theta{}^\rho y). $$

Straightforward calculations show that the F-vector space isomorphism $f\colon V\otimes_F L\to V\times{}^\rho V\times{}^\theta V$ given by

$$ f(x\otimes\ell)=(x\ell,({}^\rho x)\ell,({}^\theta x)\ell)\qquad\text{for } x\in V\text{ and } \ell\in L $$

defines with ν an isomorphism of cyclic compositions

$$ \Gamma_L\xrightarrow{\sim}(V\times{}^\rho V\times{}^\theta V,\ L\times L\times L,\ Q\times{}^\rho Q\times{}^\theta Q,\ \widetilde{\rho},\ \diamond). $$

3. Trialitarian algebras

In this section, we assume that the characteristic of the base field F is different from 2. Trialitarian algebras are defined in [9, §43] as 4-tuples $T=(E,L,\sigma,\alpha)$ where L is a cubic étale F-algebra, (E,σ) is a central simple L-algebra of degree 8 with an orthogonal involution, and α is an isomorphism from the Clifford algebra $C(E,\sigma)$ to a certain twisted scalar extension of E. We just recall in detail the special case of trialitarian algebras of the form $\operatorname{End}\Gamma$ for Γ a cyclic composition, because this is the main case for the purposes of this paper.

Let $\Gamma = (V, L, Q, \rho, *)$ be a cyclic composition (of dimension 8) over F, with L a field, and let $\theta = \rho^2$. Let also σ_Q denote the orthogonal involution on $\operatorname{End}_L V$ adjoint to Q. We will use the product $*$ to see that the Clifford algebra $C(V, Q)$ is split and the even Clifford algebra $C_0(V, Q)$ decomposes into a direct product of two split central simple L-algebras of degree 8. Using the notation of Remark 2.3, to any $x \in V$ we associate L-linear maps

$$\ell_x\colon {}^{\rho}V \to {}^{\theta}V \quad \text{and} \quad r_x\colon {}^{\theta}V \to {}^{\rho}V$$

defined by

$$\ell_x({}^{\rho}y) = x *_{\theta} {}^{\rho}y = {}^{\theta}(x * y) \quad \text{and} \quad r_x({}^{\theta}z) = {}^{\theta}z *_{\rho} x = {}^{\rho}(z * x)$$

for y, $z \in V$. From (2) it follows that for $x \in V$ the L-linear map

$$\alpha_*(x) = \left(\begin{smallmatrix} 0 & r_x \\ \ell_x & 0 \end{smallmatrix}\right)\colon {}^{\rho}V \oplus {}^{\theta}V \to {}^{\rho}V \oplus {}^{\theta}V \quad \text{given by} \quad ({}^{\rho}y, {}^{\theta}z) \mapsto \big(r_x({}^{\theta}z), \ell_x({}^{\rho}y)\big)$$

satisfies $\alpha_*(x)^2 = Q(x)\,\mathrm{Id}$. Therefore, there is an induced L-algebra homomorphism

$$\alpha_*\colon C(V, Q) \to \operatorname{End}_L({}^{\rho}V \oplus {}^{\theta}V). \tag{4}$$

This homomorphism is injective because $C(V, Q)$ is a simple algebra, hence it is an isomorphism by dimension count. It restricts to an L-algebra isomorphism

$$\alpha_{*0}\colon C_0(V, Q) \xrightarrow{\sim} \operatorname{End}_L({}^{\rho}V) \times \operatorname{End}_L({}^{\theta}V),$$

see [9, (36.16)]. Note that we may identify $\operatorname{End}_L({}^{\rho}V)$ with the twisted algebra ${}^{\rho}(\operatorname{End}_L V)$ (where multiplication is defined by ${}^{\rho}f_1 \cdot {}^{\rho}f_2 = {}^{\rho}(f_1 \circ f_2)$) as follows: for $f \in \operatorname{End}_L V$, we identify ${}^{\rho}f$ with the map ${}^{\rho}V \to {}^{\rho}V$ such that ${}^{\rho}f({}^{\rho}x) = {}^{\rho}(f(x))$ for $x \in V$. On the other hand, let σ_Q be the orthogonal involution on $\operatorname{End}_L V$ adjoint to Q. The algebra $C_0(V, Q)$ is canonically isomorphic to the Clifford algebra $C(\operatorname{End}_L V, \sigma_Q)$ (see [9, (8.8)]), hence it depends only on $\operatorname{End}_L V$ and σ_Q. We may regard α_{*0} as an isomorphism of L-algebras

$$\alpha_{*0}\colon C(\operatorname{End}_L V, \sigma_Q) \xrightarrow{\sim} {}^{\rho}(\operatorname{End}_L V) \times {}^{\theta}(\operatorname{End}_L V).$$

Thus, α_{*0} depends only on $\operatorname{End}_L V$ and σ_Q. The trialitarian algebra $\operatorname{End}\Gamma$ is the 4-tuple

$$\operatorname{End}\Gamma = (\operatorname{End}_L V, L, \sigma_Q, \alpha_{*0}).$$

An *isomorphism* of trialitarian algebras $\operatorname{End}\Gamma \xrightarrow{\sim} \operatorname{End}\Gamma'$, for $\Gamma' = (V', L', Q', \rho', *')$ a cyclic composition, is defined to be an isomorphism of F-algebras with involution $\varphi\colon (\operatorname{End}_L V, \sigma_Q) \xrightarrow{\sim} (\operatorname{End}_{L'} V', \sigma_{Q'})$ subject to the following conditions:

(i) the restriction of φ to the center of $\operatorname{End}_L V$ is an isomorphism $\varphi|_L\colon (L, \rho) \xrightarrow{\sim} (L', \rho')$, and

(ii) the following diagram (where $\theta' = \rho'^2$) commutes:

$$\begin{array}{ccc} C(\operatorname{End}_L V, \sigma_Q) & \xrightarrow{\alpha_{*0}} & {}^{\rho}(\operatorname{End}_L V) \times {}^{\theta}(\operatorname{End}_L V) \\ \downarrow{\scriptstyle C(\varphi)} & & \downarrow{\scriptstyle {}^{\rho}\varphi \times {}^{\theta}\varphi} \\ C(\operatorname{End}_{L'} V', \sigma_{Q'}) & \xrightarrow{\alpha_{*'0}} & {}^{\rho'}(\operatorname{End}_{L'} V') \times {}^{\theta'}(\operatorname{End}_{L'} V') \end{array}$$

For example, it is straightforward to check that every isotopy $(\nu, f)\colon \Gamma \to \Gamma'$ induces an isomorphism $\operatorname{End}\Gamma \to \operatorname{End}\Gamma'$ mapping $g \in \operatorname{End}_L V$ to $f \circ g \circ f^{-1} \in \operatorname{End}_{L'} V'$. As part of the proof of the main theorem below, we show that every isomorphism $\operatorname{End}\Gamma \xrightarrow{\sim} \operatorname{End}\Gamma'$ is induced by an isotopy; see Lemma 3.4. (A cohomological proof that the trialitarian algebras $\operatorname{End}\Gamma$, $\operatorname{End}\Gamma'$ are isomorphic if and only if the cyclic compositions Γ, Γ' are isotopic is given in [9, (44.16)].)

We show that the trialitarian algebra $\operatorname{End}\Gamma$ admits a ρ-semilinear automorphism of order 3 if and only if Γ is induced. More precisely:

Theorem 3.1. *Let $\Gamma = (V, L, Q, \rho, *)$ be a cyclic composition over F, with L a field.*

(i) *If Σ is a symmetric composition over F and $f\colon \Sigma \otimes (L, \rho) \to \Gamma$ is an L-linear isotopy, then the automorphism $\tau_{(\Sigma,f)} = \operatorname{Int}(f \circ \widehat{\rho} \circ f^{-1})|_{\operatorname{End}_L V}$ of $\operatorname{End}\Gamma$, where $\widehat{\rho}$ is defined in (3), is such that $\tau^3_{(\Sigma,f)} = \operatorname{Id}$ and $\tau_{(\Sigma,f)}|_L = \rho$. The automorphism $\tau_{(\Sigma,f)}$ only depends, up to conjugation in $\operatorname{Aut}_F(\operatorname{End}\Gamma)$, on the isomorphism class of Σ.*

(ii) *If $\operatorname{End}\Gamma$ carries an F-automorphism τ such that $\tau|_L = \rho$ and $\tau^3 = \operatorname{Id}$, then Γ is induced. More precisely, there exists a symmetric composition Σ over F and an L-linear isotopy $f\colon \Sigma \otimes (L, \rho) \to \Gamma$ such that $\tau = \tau_{(\Sigma,f)}$.*

Proof. (i) It is clear that $\tau^3_{(\Sigma,f)} = \operatorname{Id}$ and $\tau_{(\Sigma,f)}|_L = \rho$. For the last claim, note that if $g\colon \Sigma \otimes (L, \rho) \to \Gamma$ is another L-linear isotopy, then $f \circ g^{-1}$ is an isotopy of Γ, hence $\operatorname{Int}(f \circ g^{-1})$ is an automorphism of $\operatorname{End}\Gamma$, and

$$\tau_{(\Sigma,f)} = \operatorname{Int}(f \circ g^{-1}) \circ \tau_{(\Sigma,g)} \circ \operatorname{Int}(f \circ g^{-1})^{-1}.$$

The proof of claim (ii) relies on three lemmas. Until the end of this section, we fix a cyclic composition $\Gamma = (V, L, Q, \rho, *)$, with L a field. We start with some general observations on ρ-semilinear automorphisms of $\operatorname{End}_L V$. For this, we consider the inclusions

$$L \hookrightarrow \operatorname{End}_L V \hookrightarrow \operatorname{End}_F V.$$

The field L is the center of $\operatorname{End}_L V$, hence every automorphism of $\operatorname{End}_L V$ restricts to an automorphism of L.

Lemma 3.2. *Let $\nu \in \{\operatorname{Id}_L, \rho, \theta\}$ be an arbitrary element in the Galois group $\operatorname{Gal}(L/F)$. For every F-linear automorphism φ of $\operatorname{End}_L V$ such that $\varphi|_L = \nu$, there exists an invertible transformation $u \in \operatorname{End}_F V$ such that $\varphi(f) = u \circ f \circ u^{-1}$ for all $f \in \operatorname{End}_L V$. The map u is uniquely determined up to a factor in $L^\times$;*

it is ν-semilinear, i.e., $u(x\lambda) = u(x)\nu(\lambda)$ for all $x \in V$ and $\lambda \in L$. Moreover, if $\varphi \circ \sigma_Q = \sigma_Q \circ \varphi$, then there exists $\mu \in L^\times$ such that

$$Q\bigl(u(x)\bigr) = \nu\bigl(\mu \cdot Q(x)\bigr) \qquad \textit{for all } x \in V.$$

Proof. The existence of u is a consequence of the Skolem–Noether theorem, since $\operatorname{End}_L V$ is a simple subalgebra of the simple algebra $\operatorname{End}_F V$: the automorphism φ extends to an inner automorphism $\operatorname{Int}(u)$ of $\operatorname{End}_F V$ for some invertible $u \in \operatorname{End}_F V$. Uniqueness of u up to a factor in $L^\times$ is clear because L is the centralizer of $\operatorname{End}_L V$ in $\operatorname{End}_F V$, and the ν-semilinearity of u follows from the equation $\varphi(f) = u \circ f \circ u^{-1}$ applied with f the scalar multiplication by an element in L.

Now, suppose φ commutes with σ_Q, hence for all $f \in \operatorname{End}_L V$

$$u \circ \sigma_Q(f) \circ u^{-1} = \sigma_Q(u \circ f \circ u^{-1}). \tag{5}$$

Let $\operatorname{Tr}_*(Q)$ denote the transfer of Q along the trace map $\operatorname{Tr}_{L/F}$, so $\operatorname{Tr}_*(Q)\colon V \to F$ is the quadratic form defined by $\operatorname{Tr}_*(Q)(x) = \operatorname{Tr}_{L/F}\bigl(Q(x)\bigr)$. The adjoint involution $\sigma_{\operatorname{Tr}_*(Q)}$ coincides on $\operatorname{End}_L V$ with σ_Q, hence from (5) it follows that $\sigma_{\operatorname{Tr}_*(Q)}(u)u$ centralizes $\operatorname{End}_L V$. Therefore, $\sigma_{\operatorname{Tr}_*(Q)}(u)u = \mu$ for some $\mu \in L^\times$. We then have $b_{\operatorname{Tr}_*(Q)}\bigl(u(x), u(y)\bigr) = b_{\operatorname{Tr}_*(Q)}(x, y\mu)$ for all x, $y \in V$, which means that

$$\operatorname{Tr}_{L/F}\bigl(b_Q(u(x), u(y))\bigr) = \operatorname{Tr}_{L/F}\bigl(\mu b_Q(x,y)\bigr). \tag{6}$$

Now, observe that since u is ν-semilinear, the map $c\colon V \times V \to L$ defined by $c(x,y) = \nu^{-1}\bigl(b_Q(u(x), u(y))\bigr)$ is L-bilinear. From (6), it follows that $c - \mu b_Q$ is a bilinear map on V that takes its values in the kernel of the trace map. But the value domain of an L-bilinear form is either L or $\{0\}$, and the trace map is not the zero map. Therefore, $c - \mu b_Q = 0$, which means that

$$\nu^{-1}\bigl(b_Q(u(x), u(y))\bigr) = \mu b_Q(x,y) \qquad \text{for all } x,\ y \in V,$$

hence $Q\bigl(u(x)\bigr) = \nu\bigl(\mu \cdot Q(x)\bigr)$ for all $x \in V$. □

Note that the arguments in the preceding proof apply to any quadratic space (V, Q) over L. By contrast, the next lemma uses the full cyclic composition structure: Let again $\nu \in \{\operatorname{Id}_L, \rho, \theta\}$. Given an invertible element $u \in \operatorname{End}_F V$ and $\mu \in L^\times$ such that for all $x \in V$ and $\lambda \in L$

$$u(x\lambda) = u(x)\nu(\lambda) \quad \text{and} \quad Q\bigl(u(x)\bigr) = \nu\bigl(\mu \cdot Q(x)\bigr),$$

we define an L-linear map $\beta_u\colon {}^\nu V \to \operatorname{End}_L({}^\rho V \oplus {}^\theta V)$ by

$$\beta_u({}^\nu x) = \begin{pmatrix} 0 & \nu(\mu)^{-1} r_{u(x)} \\ \ell_{u(x)} & 0 \end{pmatrix} \in \operatorname{End}_L({}^\rho V \oplus {}^\theta V) \quad \text{for } x \in V.$$

Then from (2) we get $\beta_u(x)^2 = \nu\bigl(Q(x)\bigr) = {}^\nu Q({}^\nu x)$. Therefore, the map β_u extends to an L-algebra homomorphism

$$\beta_u\colon C({}^\nu V, {}^\nu Q) \to \operatorname{End}_L({}^\rho V \oplus {}^\theta V).$$

Just like α_* in (4), the homomorphism β_u is an isomorphism. We also have an isomorphism of F-algebras $C({}^{\nu}\cdot)\colon C(V,Q)\to C({}^{\nu}V,{}^{\nu}Q)$ induced by the F-linear map $x\mapsto {}^{\nu}x$ for $x\in V$, so we may consider the F-automorphism ψ_u of $\operatorname{End}_L({}^{\rho}V\oplus{}^{\theta}V)$ that makes the following diagram commute:

$$\begin{array}{ccc} C(V,Q) & \xrightarrow{\alpha_*} & \operatorname{End}_L({}^{\rho}V\oplus{}^{\theta}V)\\ {\scriptstyle C({}^{\nu}\cdot)}\downarrow & & \downarrow{\scriptstyle\psi_u}\\ C({}^{\nu}V,{}^{\nu}Q) & \xrightarrow{\beta_u} & \operatorname{End}_L({}^{\rho}V\oplus{}^{\theta}V)\end{array}\tag{7}$$

LEMMA 3.3. *The F-algebra automorphism ψ_u restricts to an F-algebra automorphism ψ_{u0} of $\operatorname{End}_L({}^{\rho}V)\times\operatorname{End}_L({}^{\theta}V)$. The restriction of ψ_{u0} to the center $L\times L$ is either $\nu\times\nu$ or $(\nu\times\nu)\circ\varepsilon$ where ε is the switch map $(\ell_1,\ell_2)\mapsto(\ell_2,\ell_1)$. Moreover, if $\psi_{u0}|_{L\times L}=\nu\times\nu$, then there exist invertible ν-semilinear transformations u_1, $u_2\in\operatorname{End}_F V$ such that*

$$\psi_u(f)=\begin{pmatrix}{}^{\rho}u_1 & 0\\ 0 & {}^{\theta}u_2\end{pmatrix}\circ f\circ\begin{pmatrix}{}^{\rho}u_1^{-1} & 0\\ 0 & {}^{\theta}u_2^{-1}\end{pmatrix}\quad \textit{for all } f\in\operatorname{End}_L({}^{\rho}V\oplus{}^{\theta}V).$$

For any pair (u_1,u_2) satisfying this condition, we have

$$u_2(x*y)=u(x)*u_1(y)\ \textit{and}\ u_1(x*y)=\big(u_2(x)*u(y)\big)\theta\nu(\mu)^{-1}\ \textit{for all } x,\, y\in V.$$

Proof. The maps α_* and β_u are isomorphisms of graded L-algebras for the usual $(\mathbb{Z}/2\mathbb{Z})$-gradings of $C(V,Q)$ and $C({}^{\nu}V,{}^{\nu}Q)$, and for the "checker-board" grading of $\operatorname{End}_L({}^{\rho}V\oplus{}^{\theta}V)$ defined by

$$\operatorname{End}_L({}^{\rho}V\oplus{}^{\theta}V)_0=\operatorname{End}_L({}^{\rho}V)\times\operatorname{End}_L({}^{\theta}V)$$

and

$$\operatorname{End}_L({}^{\rho}V\oplus{}^{\theta}V)_1=\begin{pmatrix}0 & \operatorname{Hom}_L({}^{\theta}V,{}^{\rho}V)\\ \operatorname{Hom}_L({}^{\rho}V,{}^{\theta}V) & 0\end{pmatrix}.$$

Therefore, ψ_u also preserves the grading, and it restricts to an automorphism ψ_{u0} of the degree 0 component. Because the map $C({}^{\nu}\cdot)$ is ν-semilinear, the map ψ_u also is ν-semilinear, hence its restriction to the center of the degree 0 component is either $\nu\times\nu$ or $(\nu\times\nu)\circ\varepsilon$.

Suppose $\psi_{u0}|_{L\times L}=\nu\times\nu$. By Lemma 3.2 (applied with ${}^{\rho}V\oplus{}^{\theta}V$ instead of V), there exists an invertible ν-semilinear transformation $v\in\operatorname{End}_F({}^{\rho}V\oplus{}^{\theta}V)$ such that $\psi_u(f)=v\circ f\circ v^{-1}$ for all $f\in\operatorname{End}_F({}^{\rho}V\oplus{}^{\theta}V)$. Since ψ_{u0} fixes $\begin{pmatrix}\mathrm{Id}_{{}^{\rho}V} & 0\\ 0 & 0\end{pmatrix}$, the element v centralizes $\begin{pmatrix}\mathrm{Id}_{{}^{\rho}V} & 0\\ 0 & 0\end{pmatrix}$, hence $v=\begin{pmatrix}{}^{\rho}u_1 & 0\\ 0 & {}^{\theta}u_2\end{pmatrix}$ for some invertible u_1, $u_2\in\operatorname{End}_F V$. The transformations u_1 and u_2 are ν-semilinear because v is ν-semilinear. From the commutativity of (7) we have $v\circ\alpha_*(x)=\beta_u({}^{\nu}x)\circ v=\alpha_*\big(u(x)\big)\circ v$ for all $x\in V$. By the definition of α_*, it follows that

$$u_1(z*x)=\theta\nu^{-1}(\mu)\big(u_2(z)*u(x)\big)\ \text{and}\ u_2(x*y)=u(x)*u_1(y)\ \text{for all } y,\, z\in V.$$

□

LEMMA 3.4. *Let $\nu\in\{\mathrm{Id}_L,\rho,\theta\}$. For every F-linear automorphism φ of $\operatorname{End}\Gamma$ such that $\varphi|_L=\nu$, there exists an invertible transformation $u\in\operatorname{End}_F V$,* *uniquely determined up to a factor in* $L^\times$*, such that* $\varphi(f) = u \circ f \circ u^{-1}$ *for all* $f \in \operatorname{End}_L V$*. Every such* u *is a* ν*-semilinear isotopy* $\Gamma \to \Gamma$*.*

Proof. The existence of u, its uniqueness up to a factor in $L^\times$, and its ν-semilinearity, were established in Lemma 3.2. It only remains to show that u is an isotopy.

Since φ is an automorphism of $\operatorname{End}\Gamma$, it commutes with σ_Q, hence Lemma 3.2 yields $\mu \in L^\times$ such that $Q\bigl(u(x)\bigr) = \nu\bigl(\mu \cdot Q(x)\bigr)$ for all $x \in V$. We may therefore consider the maps β_u and ψ_u of Lemma 3.3. Now, recall from [9, (8.8)] that $C_0(V,Q) = C(\operatorname{End}_L V, \sigma_Q)$ by identifying $x \cdot y$ for x, $y \in V$ with the image in $C(\operatorname{End}_L V, \sigma_Q)$ of the linear transformation $x \otimes y$ defined by $z \mapsto x \cdot b_Q(y,z)$ for $z \in V$. We have

$$\varphi(x \otimes y) = u \circ (x \otimes y) \circ u^{-1} \colon z \mapsto u\bigl(x \cdot b_Q(y, u^{-1}(z))\bigr) \qquad \text{for } x,\ y,\ z \in V.$$

Since u is ν-semilinear and $Q\bigl(u(x)\bigr) = \nu\bigl(\mu \cdot Q(x)\bigr)$ for all $x \in V$, it follows that

$$u\bigl(x \cdot b_Q(y, u^{-1}(z))\bigr) = u(x) \cdot \nu\bigl(b_Q(y, u^{-1}(z))\bigr) = u(x) \cdot \nu(\mu)^{-1} b_Q(u(y), z).$$

Therefore, $\varphi(x \otimes y) = u(x) \otimes u(y)\nu(\mu)^{-1}$ for x, $y \in V$, hence the following diagram (where β_u and $C({}^\nu\cdot)$ are as in (7)) is commutative:

$$\begin{array}{ccc}
C_0(V,Q) & \xrightarrow{C({}^\nu\cdot)|_{C_0(V,Q)}} & C_0({}^\nu V, {}^\nu Q) \\
{\scriptstyle C(\varphi)}\big\downarrow & & \big\downarrow{\scriptstyle \beta_u|_{C_0({}^\nu V, {}^\nu Q)}} \\
C_0(V,Q) & \xrightarrow{\alpha_{*0}} & \operatorname{End}_L({}^\rho V) \times \operatorname{End}_L({}^\theta V)
\end{array}$$

On the other hand, the following diagram is commutative because φ is an automorphism of $\operatorname{End}\Gamma$:

$$\begin{array}{ccc}
C_0(V,Q) & \xrightarrow{\alpha_{*0}} & \operatorname{End}_L({}^\rho V) \times \operatorname{End}_L({}^\theta V) \\
{\scriptstyle C(\varphi)}\big\downarrow & & \big\downarrow{\scriptstyle {}^\rho\varphi \times {}^\theta\varphi} \\
C_0(V,Q) & \xrightarrow{\alpha_{*0}} & \operatorname{End}_L({}^\rho V) \times \operatorname{End}_L({}^\theta V)
\end{array}$$

Therefore, $\beta_u|_{C_0({}^\nu V, {}^\nu Q)} \circ C({}^\nu\cdot)|_{C_0(V,Q)} = ({}^\rho\varphi \times {}^\theta\varphi) \circ \alpha_{*0}$. By comparing with (7), we see that $\psi_{u0} = {}^\rho\varphi \times {}^\theta\varphi$, hence $\psi_{u0}|_{L\times L} = \nu \times \nu$. Lemma 3.3 then yields ν-semilinear transformations u_1, $u_2 \in \operatorname{End}_F V$ such that

$$\psi_u(f) = \begin{pmatrix} {}^\rho u_1 & 0 \\ 0 & {}^\theta u_2 \end{pmatrix} \circ f \circ \begin{pmatrix} {}^\rho u_1^{-1} & 0 \\ 0 & {}^\theta u_2^{-1} \end{pmatrix} \qquad \text{for all } f \in \operatorname{End}_L({}^\rho V \oplus {}^\theta V),$$

hence $\psi_{u0} = \operatorname{Int}({}^\rho u_1) \times \operatorname{Int}({}^\theta u_2)$. But we have $\psi_{u0} = {}^\rho\varphi \times {}^\theta\varphi = \operatorname{Int}({}^\rho u) \times \operatorname{Int}({}^\theta u)$. Therefore, multiplying (u_1, u_2) by a scalar in $L^\times$, we may assume $u = u_1$ and $u_2 = u\zeta$ for some $\zeta \in L^\times$. Lemma 3.3 then gives

$$u(x*y)\zeta = u(x) * u(y) \text{ and } u(x*y) = \bigl((\zeta u(x)) * u(y)\bigr)\theta\nu(\mu)^{-1} \text{ for all } x,\ y \in V.$$

The second equation implies that $u(x*y) = \bigl(u(x) * u(y)\bigr)\rho(\zeta)\theta\nu(\mu)^{-1}$. By comparing with the first equation, we get $\rho(\zeta)\theta\nu(\mu)^{-1} = \zeta^{-1}$, hence $\nu(\mu) = \rho(\zeta)\theta(\zeta)$. Therefore, (ν, u) is an isotopy $\Gamma \to \Gamma$ with multiplier $\nu^{-1}(\zeta)$. □

We start with the proof of claim (ii) of Theorem 3.1. Suppose τ is an F-automorphism of $\operatorname{End}\Gamma$ such that $\tau|_L = \rho$ and $\tau^3 = \mathrm{Id}$. By Lemma 3.4, we may find an invertible ρ-semilinear transformation $t \in \operatorname{End}_F V$ such that $\tau(f) = t \circ f \circ t^{-1}$ for all $f \in \operatorname{End}_L V$, and every such t is an isotopy of Γ. Since $\tau^3 = \mathrm{Id}$, it follows that t^3 lies in the centralizer of $\operatorname{End}_L V$ in $\operatorname{End}_F V$, which is L. Let $t^3 = \xi \in L^\times$. We have $\rho(\xi) = t\xi t^{-1} = \xi$, hence $\xi \in F^\times$. The F-subalgebra of $\operatorname{End}_F V$ generated by L and t is a crossed product (L, ρ, ξ); its centralizer is the F-subalgebra $(\operatorname{End}_L V)^\tau$ fixed under τ, and we have

$$\operatorname{End}_F V \cong (L, \rho, \xi) \otimes_F (\operatorname{End}_L V)^\tau.$$

Now, $\deg(L, \rho, \xi) = 3$ and $\deg(\operatorname{End}_L V)^\tau = 8$, hence (L, ρ, ξ) is split. Therefore $\xi = N_{L/F}(\eta)$ for some $\eta \in L^\times$. Substituting $\eta^{-1}t$ for t, we get $t^3 = \mathrm{Id}_V$, and t is still a ρ-linear isotopy of Γ. Let $\mu \in L^\times$ be the corresponding multiplier, so that for all x, $y \in V$

$$Q\big(t(x)\big) = \rho\big(\rho(\mu)\theta(\mu)Q(x)\big) \quad \text{and} \quad t(x) * t(y) = t(x * y)\rho(\mu). \tag{8}$$

From the second equation we deduce that $t^3(x) * t^3(y) = t^3(x * y)N_{L/F}(\mu)$ for all x, $y \in V$, hence $N_{L/F}(\mu) = 1$ because $t^3 = \mathrm{Id}_V$. By Hilbert's Theorem 90, we may find $\zeta \in L^\times$ such that $\mu = \zeta\theta(\zeta)^{-1}$. Define $Q' = \rho(\zeta)\theta(\zeta)Q$ and let $x *' y = (x * y)\zeta$ for x, $y \in V$. Then Id_V is an isotopy $\Gamma \to \Gamma' = (V, L, Q', \rho, *')$ with multiplier ζ, and (8) implies that

$$Q'\big(t(x)\big) = \rho\big(Q'(x)\big) \quad \text{and} \quad t(x) *' t(y) = t(x *' y) \quad \text{for all } x,\, y \in V.$$

Now, observe that because t is ρ-semilinear and $t^3 = \mathrm{Id}_V$, the Galois group of L/F acts by semilinear automorphisms on V, hence we have a Galois descent (see [9, (18.1)]): the fixed point set $S = \{x \in V \mid t(x) = x\}$ is an F-vector space such that $V = S \otimes_F L$. Moreover, since $Q'\big(t(x)\big) = \rho\big(Q'(x)\big)$ for all $x \in V$, the restriction of Q' to S is a quadratic form $n\colon S \to F$, and we have $Q' = n_L$. Also, because $t(x *' y) = t(x) *' t(y)$ for all x, $y \in V$, the product $*'$ restricts to a product $\star$ on S, and $\Sigma = (S, n, \star)$ is a symmetric composition because Γ' is a cyclic composition. The canonical map $f\colon S \otimes_F L \to V$ yields an isomorphism of cyclic compositions $f\colon \Sigma \otimes (L, \rho) \xrightarrow{\sim} \Gamma'$, hence also an isotopy $f\colon \Sigma \otimes (L, \rho) \to \Gamma$. We have $t = f \circ \widehat{\rho} \circ f^{-1}$, hence τ is conjugation by $f \circ \widehat{\rho} \circ f^{-1}$. □

Theorem 3.5. *The assignment $\Sigma \mapsto \tau_{(\Sigma,f)}$ induces a bijection between the isomorphism classes of symmetric compositions Σ for which there exists an L-linear isotopy $f\colon \Sigma \otimes (L, \rho) \to \Gamma$ and conjugacy classes in $\operatorname{Aut}_F(\operatorname{End}\Gamma)$ of automorphisms τ of $\operatorname{End}\Gamma$ such that $\tau^3 = \mathrm{Id}$ and $\tau|_L = \rho$.*

Proof. We already know by Theorem 3.1 that the map induced by $\Sigma \mapsto \tau_{(\Sigma,f)}$ is onto. Therefore, it suffices to show that if the automorphisms $\tau_{(\Sigma,f)}$ and $\tau_{(\Sigma',f')}$ associated to symmetric compositions Σ and Σ' are conjugate, then Σ and Σ' are isomorphic. Assume $\tau_{(\Sigma',f')} = \varphi \circ \tau_{(\Sigma,f)} \circ \varphi^{-1}$ for some $\varphi \in \operatorname{Aut}_F(\operatorname{End}\Gamma)$, and let $t = f \circ \widehat{\rho} \circ f^{-1}$, $t' = f' \circ \widehat{\rho} \circ f'^{-1} \in \operatorname{End}\Gamma$ be the ρ-semilinear transformations such that $\tau_{(\Sigma,f)} = \operatorname{Int}(t)|_{\operatorname{End}_L V}$ and $\tau_{(\Sigma',f')} =$

$\mathrm{Int}(t')|_{\mathrm{End}_L V}$. By Lemma 3.4 we may find an isotopy $(\nu, u)\colon \Gamma \to \Gamma$ such that $\varphi = \mathrm{Int}(u)|_{\mathrm{End}_L V}$. The equation $\tau_{(\Sigma',f')} = \varphi \circ \tau_{(\Sigma,f)} \circ \varphi^{-1}$ then yields $\mathrm{Int}(t')|_{\mathrm{End}_L V} = \mathrm{Int}(u \circ t \circ u^{-1})|_{\mathrm{End}_L V}$, hence there exists $\xi \in L^\times$ such that $u \circ t \circ u^{-1} = \xi t'$. Because $t^3 = {t'}^3 = \mathrm{Id}_V$, we have $N_{L/F}(\xi) = 1$, hence Hilbert's Theorem 90 yields $\eta \in L^\times$ such that $\xi = \rho(\eta)\eta^{-1}$. Then $\eta^{-1}u\colon \Gamma \to \Gamma$ is a ν-semilinear isotopy such that $(\eta^{-1}u) \circ t \circ (\eta^{-1}u)^{-1} = t'$, and we have a commutative diagram

$$\begin{array}{ccc} \Sigma \otimes (L,\rho) & \xrightarrow{{f'}^{-1}\circ(\eta^{-1}u)\circ f} & \Sigma' \otimes (L,\rho) \\ \Big\downarrow{\scriptstyle \widehat{\rho}} & & \Big\downarrow{\scriptstyle \widehat{\rho}} \\ \Sigma \otimes (L,\rho) & \xrightarrow{{f'}^{-1}\circ(\eta^{-1}u)\circ f} & \Sigma' \otimes (L,\rho) \end{array}$$

The restriction of ${f'}^{-1} \circ (\eta^{-1}u) \circ f$ to Σ is an isotopy of symmetric compositions $\Sigma \to \Sigma'$; a scalar multiple of this map is an isomorphism $\Sigma \xrightarrow{\sim} \Sigma'$. □

4. Trialitarian automorphisms of groups of type D_4

Let F be a field of characteristic different from 2. By [9, (44.8)], for every adjoint simple group G of type D_4 over F there is a trialitarian algebra $T = (E, L, \sigma, \alpha)$ such that G is isomorphic to $\mathbf{Aut}_L(T)$.

Proposition 4.1. *The natural map $\Phi : \mathbf{Aut}_F(T) \to \mathbf{Aut}(G)$ induced by conjugation is an isomorphism of group schemes.*

Proof. The group G is the connected component of $\mathbf{Aut}_F(T)$ by construction. By [4, Exp. XXIV, Th. 1.3], the group $\mathbf{Aut}(G)$ is a smooth algebraic group scheme, and the conjugation homomorphism Φ is a homomorphism of algebraic groups. Since G is adjoint semisimple the restriction of Φ to the connected component is an injective homomorphism $G \to \mathbf{Aut}(G)$, hence by [9, (22.2)] the differential $d\Phi$ is injective. On the other hand, since the correspondence between trialitarian algebras and adjoint simple groups of type D_4 is actually shown in [9, (44.8)] to be an equivalence of groupoids, over an algebraic closure F_{alg} the map $\Phi_{\mathrm{alg}}\colon \mathbf{Aut}_F(T)(F_{\mathrm{alg}}) \to \mathbf{Aut}(G)(F_{\mathrm{alg}})$ is an isomorphism. By [9, (22.5)] it follows that Φ is an isomorphism of group schemes. □

We thus have a commutative diagram with exact rows:

$$(9)\qquad \begin{array}{ccccccccc} 1 & \longrightarrow & \mathbf{Aut}_L(T) & \longrightarrow & \mathbf{Aut}_F(T) & \longrightarrow & \mathbf{Aut}_F(L) & \longrightarrow & 1 \\ & & \Big\| & & \Big\downarrow{\scriptstyle \Phi} & & \Big\| & & \\ 1 & \longrightarrow & G & \longrightarrow & \mathbf{Aut}(G) & \xrightarrow{\pi} & (\mathfrak{S}_3)_L & \longrightarrow & 1 \end{array}$$

where $(\mathfrak{S}_3)_L$ is a (non-constant) twisted form of the symmetric group $\mathfrak{S}_3$. Here $\mathbf{Aut}_F(L)$ is the group scheme given by $\mathbf{Aut}_F(L)(R) = \mathrm{Aut}_{R\text{-alg}}(L \otimes_F R)$ for any commutative F-algebra R. Thus, the type of the group G is related as follows to the type of L and to $\mathbf{Aut}_F(L)$:

(i) type $^1\mathsf{D}_4$: $L \cong F \times F \times F$ and $\mathbf{Aut}_F(L)(F) \cong \mathfrak{S}_3$;
(ii) type $^2\mathsf{D}_4$: $L \cong F \times \Delta$ (with Δ a quadratic field extension of F) and $\mathbf{Aut}_F(L)(F) \cong \mathfrak{S}_2$;
(iii) type $^3\mathsf{D}_4$: L a cyclic cubic field extension of F and $\mathbf{Aut}_F(L)(F) \cong \mathbb{Z}/3\mathbb{Z}$;
(iv) type $^6\mathsf{D}_4$: L a non-cyclic cubic field extension of F and $\mathbf{Aut}_F(L)(F) = 1$.

Proposition 4.2. *Let G be an adjoint simple group of type D_4 over F. If $\mathbf{Aut}(G)(F)$ contains an outer automorphism φ such that φ^3 is inner, then G is of type $^1\mathsf{D}_4$ or $^3\mathsf{D}_4$, and in the trialitarian algebra $T = (E, L, \sigma, \alpha)$ such that $G \cong \mathbf{Aut}_L(T)$, the central simple L-algebra E is split.*

Proof. The exactness of the bottom row of (9) implies the exactness of

$$1 \longrightarrow G(F) \longrightarrow \mathbf{Aut}(G)(F) \xrightarrow{\pi} (\mathfrak{S}_3)_L(F) \tag{10}$$

Since the image $\pi(\varphi) \in (\mathfrak{S}_3)_L(F)$ has order 3, $\mathbf{Aut}_F(L)(F)$ must be isomorphic to $\mathfrak{S}_3$ or to $\mathbb{Z}/3\mathbb{Z}$ and hence the cases $^2\mathsf{D}_4$ and $^6\mathsf{D}_4$ can be ruled out from the characterization of the various types above. Therefore the type of G is $^1\mathsf{D}_4$ or $^3\mathsf{D}_4$. If G is of type $^1\mathsf{D}_4$, then the algebra E is split by [6, Example 17] or by [2, Theorem 13.1]. If G is of type $^3\mathsf{D}_4$, then after scalar extension to L the group G_L has type $^1\mathsf{D}_4$, so $E \otimes_F L$ is split. Therefore, the Brauer class of E has 3-torsion since it is split by a cubic extension. But it also has 2-torsion since E carries an orthogonal involution, hence E is split. □

For the rest of this section, we focus on trialitarian automorphisms (i.e., outer automorphisms of order 3) of groups of type $^3\mathsf{D}_4$. Let G be an adjoint simple group of type $^3\mathsf{D}_4$ over F, and let L be its associated cyclic cubic field extension of F. Thus,

$$(\mathfrak{S}_3)_L(F) = \mathrm{Gal}(L/F) \cong \mathbb{Z}/3\mathbb{Z}.$$

If G carries a trialitarian automorphism φ defined over F, then the map $\pi\colon \mathbf{Aut}(G)(F) \to \mathrm{Gal}(L/F)$ is a split surjection, hence $\mathbf{Aut}(G)(F) \cong G(F) \rtimes (\mathbb{Z}/3\mathbb{Z})$. Therefore, it is easy to see that for any other trialitarian automorphism φ' of G defined over F, the elements φ and φ' are conjugate in $\mathbf{Aut}(G)(F)$ if and only if there exists $g \in G(F)$ such that $\varphi' = \mathrm{Int}(g) \circ \varphi \circ \mathrm{Int}(g)^{-1}$. When this occurs, we have $\pi(\varphi) = \pi(\varphi')$.

Theorem 4.3. (i) *Let G be an adjoint simple group of type $^3\mathsf{D}_4$ over F. The group G carries a trialitarian automorphism defined over F if and only if the trialitarian algebra $T = (E, L, \sigma, \alpha)$ (unique up to isomorphism) such that $G \cong \mathbf{Aut}_L(T)$ has the form $T \cong \operatorname{End}\Gamma$ for some induced cyclic composition Γ.*

(ii) *Let $G = \mathbf{Aut}_L(\operatorname{End}\Gamma)$ for some induced cyclic composition Γ. Every trialitarian automorphism φ of G has the form $\varphi = \mathrm{Int}(\tau)$ for some uniquely determined F-automorphism τ of $\operatorname{End}\Gamma$ such that $\tau^3 = \mathrm{Id}$ and $\tau|_L = \pi(\varphi)$. For a given nontrivial $\rho \in \mathrm{Gal}(L/F)$, the assignment $\Sigma \mapsto \mathrm{Int}(\tau_{(\Sigma,f)})$ defines a bijection between the isomorphism classes* *of symmetric compositions for which there exists an L-linear isotopy $f\colon \Sigma\otimes(L,\rho)\to\Gamma$ and conjugacy classes in $\mathbf{Aut}(G)(F)$ of trialitarian automorphisms φ of G such that $\pi(\varphi)=\rho$.*

Proof. Suppose first that φ is a trialitarian automorphism of G, and let $G = \mathbf{Aut}_L(T)$ for some trialitarian algebra $T=(E,L,\sigma,\alpha)$. Proposition 4.2 shows that the central simple L-algebra E is split, hence by [9, (44.16), (36.12)], we have $T = \operatorname{End}\Gamma$ for some cyclic composition $\Gamma=(V,L,Q,\rho,*)$ over F. Substituting φ^2 for φ if necessary, we may assume $\pi(\varphi)=\rho$. The preimage of φ under the isomorphism $\Phi_F\colon \mathbf{Aut}_F(T)(F)\xrightarrow{\sim}\mathbf{Aut}(G)(F)$ (from (9)) is an F-automorphism τ of T such that $\varphi=\operatorname{Int}(\tau)$, $\tau^3=\operatorname{Id}$, and $\tau|_L=\rho$. Since Φ_F is a bijection, τ is uniquely determined by φ. By Theorem 3.1(ii), the existence of τ implies that the cyclic composition Γ is induced.
Conversely, if Γ is induced, then by Theorem 3.1(i), the trialitarian algebra $\operatorname{End}\Gamma$ carries automorphisms τ such that $\tau^3=\operatorname{Id}$ and $\tau|_L\neq\operatorname{Id}_L$. For any such τ, conjugation by τ is a trialitarian automorphism of G.
The last statement in (ii) readily follows from Theorem 3.5 because trialitarian automorphisms $\operatorname{Int}(\tau)$, $\operatorname{Int}(\tau')$ are conjugate in $\mathbf{Aut}(G)(F)$ if and only if τ, τ' are conjugate in $\operatorname{Aut}_F(\operatorname{End}\Gamma)$. □

The following proposition shows that the algebraic subgroup of fixed points under a trialitarian automorphism of the form $\operatorname{Int}(\tau_{(\Sigma,f)})$ is isomorphic to $\mathbf{Aut}(\Sigma)$, hence in characteristic different from 2 and 3 it is a simple adjoint group of type G_2 or A_2, in view of the classification of symmetric compositions (see [3, §9]).

Proposition 4.4. *Let $G=\mathbf{Aut}_L\big(\operatorname{End}(\Sigma\otimes(L,\rho))\big)$ for some symmetric composition $\Sigma=(S,n,\star)$ over F and some cyclic cubic field extension L/F with nontrivial automorphism ρ. The subgroup of G fixed under the trialitarian automorphism $\operatorname{Int}(\widehat{\rho})$ is canonically isomorphic to $\mathbf{Aut}(\Sigma)$.*

Proof (Sketch). Mimicking the construction of the map α_* in (4), we may use the product $\star$ to construct an F-algebra isomorphism

$$\alpha_\star\colon C(S,n)\xrightarrow{\sim}\operatorname{End}_F(S\oplus S)$$

such that $\alpha_\star(x)(y,z)=(z\star x, x\star y)$ for x, y, $z\in S$. This isomorphism restricts to an isomorphism

$$\alpha_{\star 0}\colon C_0(S,n)\xrightarrow{\sim}(\operatorname{End}_F S)\times(\operatorname{End}_F S).$$

Let $\mathbf{Aut}(\operatorname{End}\Sigma)$ be the group scheme whose rational points are the F-algebra automorphisms φ of $(\operatorname{End}_F S,\sigma_n)$ that make the following diagram commute:

$$\begin{array}{ccc} C(\operatorname{End}_F S,\sigma_n) & \xrightarrow{\alpha_{\star 0}} & (\operatorname{End}_F S)\times(\operatorname{End}_F S)\\ {\scriptstyle C(\varphi)}\downarrow & & \downarrow{\scriptstyle \varphi\times\varphi}\\ C(\operatorname{End}_F S,\sigma_n) & \xrightarrow{\alpha_{\star 0}} & (\operatorname{End}_F S)\times(\operatorname{End}_F S)\end{array}$$

Arguing as in Lemma 3.4, one proves that every such automorphism has the form $\mathrm{Int}(u)$ for some isotopy u of Σ. But if u is an isotopy of Σ with multiplier μ, then $\mu^{-1}u$ is an automorphism of Σ. Therefore, mapping every automorphism u of Σ to $\mathrm{Int}(u)$ yields an isomorphism $\mathbf{Aut}(\Sigma) \xrightarrow{\sim} \mathbf{Aut}(\operatorname{End}\Sigma)$. The extension of scalars from F to L yields an isomorphism

$$\mathbf{PGL}(S) \xrightarrow{\sim} R_{L/F}\big(\mathbf{PGL}(S\otimes_F L)\big)^{\mathrm{Int}(\widehat{\rho})},$$

which carries the subgroup $\mathbf{Aut}(\operatorname{End}\Sigma)$ to $G^{\mathrm{Int}(\widehat{\rho})}$. □

To conclude, we briefly mention without proof the analogue of Theorem 4.3 for simply connected groups, which we could have considered instead of adjoint groups. (Among simple algebraic groups of type D_4, only adjoint and simply connected groups may admit trialitarian automorphisms.)

Theorem 4.5. (i) *For any cyclic composition $\Gamma = (V, L, Q, \rho, *)$ over F, with L a field, the group $\mathbf{Aut}_L(\Gamma)$ is simple simply connected of type $^3\mathsf{D}_4$, and there is an exact sequence of algebraic groups*

$$1 \longrightarrow \boldsymbol{\mu}_2^2 \longrightarrow \mathbf{Aut}_L(\Gamma) \xrightarrow{\mathrm{Int}} \mathbf{Aut}_L(\operatorname{End}\Gamma) \longrightarrow 1.$$

(ii) *A simple simply connected group of type $^3\mathsf{D}_4$ admits trialitarian automorphisms defined over F if and only if it is isomorphic to the automorphism group of an induced symmetric composition $\Gamma = (V, L, Q, \rho, *)$, with L a field. Conjugacy classes of trialitarian automorphisms of $\mathbf{Aut}_L(\Gamma)$ defined over F are in bijection with isomorphism classes of symmetric compositions Σ for which there is an isotopy $\Sigma \otimes (L, \rho) \to \Gamma$.*

Theorems 4.3 and 4.5 apply in particular to show that over a finite field of characteristic different from 2 and 3, every simple adjoint or simply connected group of type $^3\mathsf{D}_4$ admits trialitarian automorphisms. This follows because the Allen invariant is trivial and cyclic compositions are reduced, see [12, §4.8]. Note that the property holds without restriction on the characteristic (needed for the arguments in [12, §4.8]), and is a particular case of a more general result: every simple adjoint or simply connected linear algebraic group over a finite field is quasi-split by a theorem of Lang [10, Prop. 6.1], and therefore $\mathrm{Aut}(G)$ is a semidirect product, see [4, Exp. XXIV, 3.10] or [9, (31.4)].[4]

Examples 4.6. (i) Let $F = \mathbb{F}_q$ be the field with q elements, where q is odd and $q \equiv 1 \bmod 3$. As observed in Example 2.2(i), every symmetric composition over F is isomorphic either to the Okubo composition Σ or to the split para-Cayley composition $\widetilde{C}$, and (up to isomorphism) there is a unique cyclic composition $\Gamma \cong \widetilde{C} \otimes (L, \rho) \cong \Sigma \otimes (L, \rho)$ with cubic algebra (L, ρ). Therefore, the simply connected group $\mathbf{Aut}_L(\Gamma)$ and the adjoint group $\mathbf{Aut}_L(\operatorname{End}\Gamma)$ have exactly two conjugacy classes of trialitarian automorphisms defined over F. See also [8, (9.1)].

[4] We are indebted to Skip Garibaldi for this observation.

(ii) Example 2.2(ii) describes a cyclic composition induced by a unique (up to isomorphism) symmetric composition. Its automorphism group is a group of type ${}^3\mathsf{D}_4$ admitting a unique conjugacy class of trialitarian automorphisms.

(iii) In contrast to (i) and (ii) we get from Example 2.2(iii) examples of groups of type ${}^3\mathsf{D}_4$ with many conjugacy classes of trialitarian automorphisms.

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Quotients of MGL, Their Slices and Their Geometric Parts

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Abstract. Let $x_1, x_2, \ldots$ be a system of homogeneous polynomial generators for the Lazard ring $\mathbb{L}^* = MU^{2*}$ and let MGL_S denote Voevodsky's algebraic cobordism spectrum in the motivic stable homotopy category over a base-scheme S [Vo98]. Relying on Hopkins-Morel-Hoyois isomorphism [Hoy] of the 0th slice s_0MGL_S for Voevodsky's slice tower with $MGL_S/(x_1, x_2, \ldots)$ (after inverting all residue characteristics of S), Spitzweck [S10] computes the remaining slices of MGL_S as $s_nMGL_S = \Sigma^n_T H\mathbb{Z} \otimes \mathbb{L}^{-n}$ (again, after inverting all residue characteristics of S). We apply Spitzweck's method to compute the slices of a quotient spectrum $MGL_S/(\{x_i : i \in I\})$ for I an arbitrary subset of $\mathbb{N}$, as well as the mod p version $MGL_S/(\{p, x_i : i \in I\})$ and localizations with respect to a system of homogeneous elements in $\mathbb{Z}[\{x_j : j \notin I\}]$. In case $S = \operatorname{Spec} k$, k a field of characteristic zero, we apply this to show that for $\mathcal{E}$ a localization of a quotient of MGL as above, there is a natural isomorphism for the theory with support

$$\Omega_*(X) \otimes_{\mathbb{L}^{-*}} \mathcal{E}^{-2*,-*}(k) \to \mathcal{E}_X^{2m-2*,m-*}(M)$$

for X a closed subscheme of a smooth quasi-projective k-scheme M, $m = \dim_k M$.

To Sasha Merkurjev with warmest regards on his 60th birthday

Contents

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Introduction

This paper has a two-fold purpose. We consider Voevodsky's slice tower on the motivic stable homotopy category $\mathcal{SH}(S)$ over a base-scheme[2] S [Vo00]. For $\mathcal{E}$ in $\mathcal{SH}(S)$, we have the nth layer $s_n\mathcal{E}$ in the slice tower for $\mathcal{E}$. Let MGL denote Voevodsky's algebraic cobordism spectrum in $\mathcal{SH}(S)$ [Vo98] and let $x_1, x_2, \ldots$ be a system of homogeneous polynomial generators for the Lazard ring $\mathbb{L}_*$. Via the classifying map for the formal group law for MGL, we may consider x_i as an element of $MGL^{-2i,-i}(S)$, and thereby as a map $x_i : \Sigma^{2i,i}MGL \to MGL$, giving the quotient $MGL/(x_1, x_2, \ldots)$. Spitzweck [S10] shows how to build on the Hopkins-Morel-Hoyois isomorphism [Hoy]

$$MGL/(x_1, x_2, \ldots) \cong s_0 MGL$$

to compute all the slices $s_n MGL$ of MGL. Our first goal here is to extend Spitzweck's method to handle quotients of MGL by a subset of $\{x_1, x_2, \ldots\}$, as well as localizations with respect to a system of homogeneous elements in the ring generated by the remaining variables; we also consider quotients of such spectra by an integer. Some of these spectra are Landweber exact, and the slices are thus computable by the results of Spitzweck on the slices of Landweber exact spectra [S12], but many of these, such as the truncated Brown-Peterson spectra or Morava K-theory, are not.

The second goal is to extend results of [DL14, L09, L15], which consider the "geometric part" $X \mapsto \mathcal{E}^{2*,*}(X)$ of the bi-graded cohomology defined by an oriented weak commutative ring T-spectrum $\mathcal{E}$ and raise the question: is the classifying map

$$\mathcal{E}^*(k) \otimes_{\mathbb{L}^*} \Omega^* \to \mathcal{E}^*$$

an isomorphism of oriented cohomology theories, that is, is the theory $\mathcal{E}^*$ a theory of *rational type* in the sense of Vishik [Vi12]? Starting with the case $\mathcal{E} = MGL$, discussed in [L09], which immediately yields the Landweber exact case, we have answered this affirmatively for "slice effective" algebraic K-theory in [DL14], and extended to the case of slice-effective covers of a Landweber exact theory in [L15]. In this paper, we use our computation of the slices of a quotient of MGL to show that the classifying map is an isomorphism for the quotients and localizations of MGL described above.

The paper is organized as follows: in §1 and §2, we abstract Spitzweck's method from [S10] to a more general setting. In §1 we give a description of quotients in a suitable symmetric monoidal model category in terms of a certain homotopy colimit. In §2 we begin by recalling some basic facts and the slice tower and its construction. We then apply the results of §1 to the category of $\mathcal{R}$-modules for $\mathcal{R} \in \mathcal{SH}(S)$ a commutative ring T-spectrum (with some additional technical assumptions), developing a method for computing the slices of an $\mathcal{R}$-module $\mathcal{M}$, assuming that $\mathcal{R}$ and $\mathcal{M}$ are effective and that the 0th slice $s_0\mathcal{M}$ is of the form $\mathcal{M}/(\{x_i : i \in I\})$ for some collection $\{[x_i] \in \mathcal{R}^{-2d_i,-d_i}(S), d_i < 0\}$

[2] In this paper a "scheme" will mean a noetherian separated scheme of finite Krull dimension.

of elements in $\mathcal{R}$-cohomology of the base-scheme S; see theorem 2.3. We also discuss localizations of such $\mathcal{R}$-modules and the mod p case (corollary 2.4 and corollary 2.5). We discuss the associated slice spectral sequence for such $\mathcal{M}$ and its convergence properties in §3, and apply these results to our examples of interest: truncated Brown-Peterson spectra, Morava K-theory and connective Morava K-theory, as well as the Landweber exact examples, the Brown-Peterson spectra BP and the Johnson-Wilson spectra $E(n)$, in §4.

The remainder of the paper discusses the classifying map from algebraic cobordism Ω_* and proves our results on the rationality of certain theories. This is essentially taken from [L15], but we need to deal with a technical problem, namely, that it is not at present clear if the theories $[MGL/(\{x_i : i \in I\})]^{2*,*}$ have a multiplicative structure. For this reason, we extend the setting used in [L15] to theories that are modules over ring-valued theories. This extension is taken up in §5 and we apply this theory to quotients and localizations of MGL in §6.

We are grateful to the referee for suggesting a number of improvements to an earlier version of this paper, especially for pointing out to us how to use works of Spitzweck to extend many of our results to an arbitrary base-scheme.

1. Quotients and homotopy colimits in a model category

In this section we consider certain quotients in a model category and give a description of these quotients as a homotopy colimit (see proposition 1.9). This is an abstraction of the methods developed in [S10] for computing the slices of MGL.

In what follows, we use the term "fibrant replacement" of an object x in a model category $\mathcal{C}$ to mean a morphism $\alpha : x \to x^f$ in $\mathcal{C}$, where x^f is fibrant and α is a cofibration and a weak equivalence. A cofibrant replacement of x is similarly a morphism $\beta : x^c \to x$ in $\mathcal{C}$ with x^c cofibrant and β a fibration and a weak equivalence.

Let $(\mathcal{C}, \otimes, 1)$ be a closed symmetric monoidal simplicial pointed model category with cofibrant unit 1. We assume that 1 admits a fibrant replacement $\alpha : 1 \to \mathbf{1}$ such that $\mathbf{1}$ is a 1-algebra in $\mathcal{C}$, that is, there is an associative multiplication map $\mu_{\mathbf{1}} : \mathbf{1} \otimes \mathbf{1} \to \mathbf{1}$ such that $\mu_1 \circ (\alpha \otimes \mathrm{id})$ and $\mu_1 \circ (\mathrm{id} \otimes \alpha)$ are the respective multiplication isomorphisms $1 \otimes \mathbf{1} \to \mathbf{1}$, $\mathbf{1} \otimes 1 \to \mathbf{1}$. We assume in addition that the functor $K \mapsto 1 \otimes K$, giving part of the simplicial structure, is a symmetric monoidal left Quillen functor.

For a cofibrant object T in $\mathcal{C}$, the map $T \cong T \otimes 1 \xrightarrow{\mathrm{id} \otimes \alpha} T \otimes \mathbf{1}$ is a cofibration and weak equivalence. Indeed, the functor $T \otimes (-)$ preserves cofibrations, and also maps that are both a cofibration and a weak equivalence, whence the assertion.

Remark 1.1. We will be applying the results of this section to the following situation: $\mathcal{M}$ is a cofibrantly generated symmetric monoidal simplicial model category satisfying the monoid axiom [ScSh, definition 3.3]; ; we assume in addition that the functor $K \mapsto e \wedge K$, e the unit in $\mathcal{M}$, giving part of the simplicial structure, is a symmetric monoidal left Quillen functor. We fix in addition a commutative monoid $\mathcal{R}$ in $\mathcal{M}$, cofibrant in $\mathcal{M}$, and $\mathcal{C}$ is the category of $\mathcal{R}$-modules in $\mathcal{M}$, with model structure as in [ScSh, §4], that is, a map is a fibration or a weak equivalence in $\mathcal{C}$ if and only if it is so as a map in $\mathcal{M}$, and cofibrations are determined by the LLP with respect to acyclic fibrations. By [ScSh, theorem 4.1(3)], the category $\mathcal{R}$-Alg of monoids in $\mathcal{C}$ has the structure of a cofibrantly generated model category, with fibrations and weak equivalence those maps which become a fibration or weak equivalence in $\mathcal{M}$, and each cofibration in $\mathcal{R}$-Alg is a cofibration in $\mathcal{C}$. The unit 1 in $\mathcal{C}$ is just $\mathcal{R}$ and we may take $\alpha : 1 \to \mathbf{1}$ to be a fibrant replacement in $\mathcal{R}$-Alg.

Let $\{x_i : T_i \to \mathbf{1} \mid i \in I\}$ be a set of maps with cofibrant sources T_i. We assign each T_i an integer degree $d_i > 0$.
Let $\mathbf{1}/(x_i)$ be the homotopy cofiber (i.e., mapping cone) of the map $x_i : \mathbf{1}\otimes T_i \to \mathbf{1}$ and let $p_i : \mathbf{1} \to \mathbf{1}/(x_i)$ be the canonical map.
Let $A = \{i_1, \ldots, i_k\}$ be a finite subset of I and define $\mathbf{1}/(\{x_i : i \in A\})$ as

$$\mathbf{1}/(\{x_i : i \in A\}) := \mathbf{1}/(x_{i_1}) \otimes \ldots \otimes \mathbf{1}/(x_{i_k}).$$

Of course, the object $\mathbf{1}/(\{x_i : i \in A\})$ depends on a choice of ordering of the elements in A, but only up to a canonical symmetry isomorphism. We could for example fix the particular choice by fixing a total order on A and taking the product in the proper order. The canonical maps p_i, $i \in I$ composed with the map $1 \to \mathbf{1}$ give rise to the canonical map

$$p_I : 1 \to \mathbf{1}/(\{x_i : i \in A\})$$

defined as the composition

$$1 \xrightarrow{\mu^{-1}} 1^{\otimes k} \to \mathbf{1}^{\otimes k} \xrightarrow{p_{i_1}\otimes\ldots\otimes p_{i_k}} \mathbf{1}/(\{x_i : i \in A\}).$$

For finite subsets $A \subset B \subset I$, define the map

$$\rho_{A\subset B} : \mathbf{1}/(\{x_i : i \in A\}) \to \mathbf{1}/(\{x_i : i \in B\})$$

as the composition

$$\mathbf{1}/(\{x_i : i \in A\}) \xrightarrow{\mu^{-1}} \mathbf{1}/(\{x_i : i \in A\}) \otimes 1$$
$$\xrightarrow{\mathrm{id}\otimes p_{B\setminus A}} \mathbf{1}/(\{x_i : i \in A\}) \otimes \mathbf{1}/(\{x_i : i \in B\setminus A\}) \cong \mathbf{1}/(\{x_i : i \in B\}).$$

where the last isomorphism is again the symmetry isomorphism.
Because $\mathcal{C}$ is a symmetric monoidal category with unit 1, we have a well-defined functor from the category $\mathcal{P}_{\mathrm{fin}}(I)$ of finite subsets of I to $\mathcal{C}$:

$$\mathbf{1}/(-) : \mathcal{P}_{\mathrm{fin}}(I) \to \mathcal{C}$$

sending $A \subset I$ to $\mathbf{1}/(\{x_i : i \in A\})$ and sending each inclusion $A \subset B$ to $\rho_{A\subset B}$.

Definition 1.2. The object $\mathbf{1}/(\{x_i : i \in I\})$ of $\mathcal{C}$ is defined by

$$\mathbf{1}/(\{x_i\}) = \operatorname*{hocolim}_{A\in\mathcal{P}_{\mathrm{fin}}(I)} \mathbf{1}/(\{x_i : i \in A\}).$$

More generally, for $M \in \mathcal{C}$, we define $M/(\{x_i : i \in I\})$ as

$$M/(\{x_i : i \in I\}) := \mathbf{1}/(\{x_i : i \in I\}) \otimes QM,$$

where $QM \to M$ is a cofibrant replacement for M. In case the index set I is understood, we often write these simply as $\mathbf{1}/(\{x_i\})$ or $M/(\{x_i\})$.

Remark 1.3. 1. The object $\mathbf{1}/(x_i)$ is cofibrant and hence the objects $\mathbf{1}/(\{x_i : i \in A\})$ are cofibrant for all finite sets A. As a pointwise cofibrant diagram has cofibrant homotopy colimit [Hir03, corollary 14.8.1, example 18.3.6, corollary 18.4.3], $\mathbf{1}/(\{x_i : i \in I\})$ is cofibrant. Thus $M/(\{x_i : i \in I\}) := \mathbf{1}/(\{x_i : i \in I\}) \otimes QM$ is also cofibrant.
2. We often select a single cofibrant object T and take $T_i := T^{\otimes d_i}$ for certain integers $d_i > 0$. As T is cofibrant, so is $T^{\otimes d_i}$. In this case we set $\deg T = 1$, $\deg T^{\otimes d_i} = d_i$.

We let $[n]$ denote the set $\{0, \dots, n\}$ with the standard order and Δ the category with objects $[n]$, $n = 0, 1, \dots,$ and morphisms the order-preserving maps of sets. For a small category A and a functor $F : A \to \mathcal{C}$, we let $\operatorname{hocolim}_A F_*$ denote the standard simplicial object of $\mathcal{C}$ whose geometric realization is $\operatorname{hocolim}_A F$, that is

$$\operatorname*{hocolim}_A F_n = \bigvee_{\sigma:[n]\to A} F(\sigma(0)).$$

LEMMA 1.4. *Let $\{x_i : T_i \to \mathbf{1} : i \in I_1\}$, $\{x_i : T_i \to \mathbf{1} : i \in I_2\}$ be two sets of maps in $\mathcal{C}$, with cofibrant sources T_i, and with I_1, I_2 disjoint index sets. Then there is a canonical isomorphism*

$$\mathbf{1}/(\{x_i : i \in I_1 \amalg I_2\}) \cong \mathbf{1}/(\{x_i : i \in I_1\}) \otimes \mathbf{1}/(\{x_i : i \in I_2\}).$$

Proof. The category $\mathcal{P}_{\text{fin}}(I_1 \amalg I_2)$ is clearly equal to $\mathcal{P}_{\text{fin}}(I_1) \times \mathcal{P}_{\text{fin}}(I_2)$. For functors $F_i : \mathcal{A}_i \to \mathcal{C}$, $i = 1, 2$, $[\operatorname{hocolim}_{\mathcal{A}_1 \times \mathcal{A}_2} F_1 \otimes F_2]_*$ is the diagonal simplicial space associated to the bisimplicial space $(n, m) \mapsto [\operatorname{hocolim}_{\mathcal{A}_1} F_1]_n \otimes [\operatorname{hocolim}_{\mathcal{A}_2} F_2]_m$. Thus

$$\operatorname*{hocolim}_{\mathcal{A}_1 \times \mathcal{A}_2} F_1 \otimes F_2 \cong \operatorname*{hocolim}_{\mathcal{A}_2} [\operatorname*{hocolim}_{\mathcal{A}_1} F_1] \otimes F_2.$$

This gives us the isomorphism

$$\begin{aligned}
&\mathbf{1}/(\{x_i : i \in I_1 \amalg I_2\}) \\
&\quad = \operatorname*{hocolim}_{(A_1, A_2) \in \mathcal{P}_{\text{fin}}(I_1) \times \mathcal{P}_{\text{fin}}(I_2)} \mathbf{1}/(\{x_i : i \in A_1\}) \otimes \mathbf{1}/(\{x_i : i \in A_2\}) \\
&\quad \cong \operatorname*{hocolim}_{A_1 \in \mathcal{P}_{\text{fin}}(I_1)} \mathbf{1}/(\{x_i : i \in A_1\}) \otimes \operatorname*{hocolim}_{A_2 \in \mathcal{P}_{\text{fin}}(I_2)} \mathbf{1}/(\{x_i : i \in A_2\}) \\
&\quad = \mathbf{1}/(\{x_i : i \in I_1\}) \otimes \mathbf{1}/(\{x_i : i \in I_2\}).
\end{aligned}$$

□

Remark 1.5. Via this lemma, we have the isomorphism for all $M \in \mathcal{C}$,

$$M/(\{x_i : i \in I_1 \amalg I_2\}) \cong (M/(\{x_i : i \in I_1\})/(\{x_i : i \in I_2\}).$$

Let $\mathcal{I}$ be the category of formal monomials in $\{x_i\}$, that is, the category of maps $N : I \to \mathbb{N}$, $i \mapsto N_i$, such that $N_i = 0$ for all but finitely many $i \in I$, and with a unique map $N \to M$ if $N_i \geq M_i$ for all $i \in I$. As usual, the monomial in the x_i corresponding to a given N is $\prod_{i\in I} x_i^{N_i}$, written x^N. The index $N = 0$, corresponding to $x^0 = 1$, is the final object of $\mathcal{I}$.
Take an $i \in I$. For $m > k \geq 0$ integers, define the map

$$\times x_i^{m-k} : \mathbf{1} \otimes T_i^{\otimes m} \to \mathbf{1} \otimes T_i^{\otimes k}$$

as the composition

$$\mathbf{1}\otimes T_i^{\otimes m} = \mathbf{1}\otimes T_i^{\otimes m-k}\otimes T_i^{\otimes k} \xrightarrow{\mathrm{id}_{\mathbf{1}}\otimes x_i^{\otimes m-k}\otimes \mathrm{id}_{T_i^{\otimes k}}} \mathbf{1}^{\otimes m-k+1}\otimes T_i^{\otimes k} \xrightarrow{\mu\otimes\mathrm{id}} \mathbf{1}\otimes T_i^{\otimes k}.$$

In case $k = 0$, we use $\mathbf{1}$ instead of $\mathbf{1} \otimes 1$ for the target; we define $\times x^0$ to be the identity map. The associativity of the maps $\mu_{\mathbf{1}}$ shows that $\times x_i^{m-k} \circ \times x_i^{n-m} = \times x_i^{n-k}$, hence the maps $\times x_i^n$ all commute with each other.
Now suppose we have a monomial in the x_i; to simplify the notation, we write the indices occurring in the monomial as $\{1, \ldots, r\}$ rather than $\{i_1, \ldots, i_r\}$. This gives us the monomial $x^N := x_1^{N_1} \cdot \ldots \cdot x_r^{N_r}$. Define

$$T_*^N := \mathbf{1} \otimes T_1^{\otimes N_1} \otimes \ldots \otimes \mathbf{1} \otimes T_r^{\otimes N_r} \otimes \mathbf{1};$$

in case $N_i = 0$, we replace $\ldots\otimes\mathbf{1}\otimes 1\otimes\mathbf{1}\otimes T_{i+1}^{\otimes M_{i+1}}\otimes\ldots$ with $\ldots\otimes\mathbf{1}\otimes T_{i+1}^{\otimes M_{i+1}}\otimes\ldots$, and we set $T_*^0 := \mathbf{1}$.
Let $N \to M$ be a map in $\mathcal{I}$, that is $N_i \geq M_i \geq 0$ for all i. We again write the relevant index set as $\{1, \ldots, r\}$. Define the map

$$\times x^{N-M} : T_*^N \to T_*^M$$

as the composition

$$T_*^N \xrightarrow{\bigotimes_{j=1}^r \times x_j^{N_j-M_j}} \mathbf{1} \otimes T_1^{\otimes M_1} \otimes \ldots \otimes \mathbf{1} \otimes T_r^{\otimes M_r} \otimes \mathbf{1} \xrightarrow{\mu_M} T_*^{\otimes M};$$

the map μ_M is a composition of $\otimes$-product of multiplication maps $\mu_{\mathbf{1}} : \mathbf{1}\otimes\mathbf{1} \to \mathbf{1}$, with these occurring in those spots with $M_j = 0$. In case $N_i = M_i = 0$, we simply delete the term $\times x_i^0$ from the expression.
The fact that the maps $\mu_{\mathbf{1}}$ satisfy associativity yields the relation

$$\times x^{M-K} \circ \times x^{N-M} = \times x^{N-K}$$

and thus the maps $\times x^{N-M}$ all commute with each other.
Defining $\mathcal{D}_x(N) := T_*^N$ and $\mathcal{D}_x(N \to M) = \times x^{N-M}$ gives us the $\mathcal{I}$-diagram

$$\mathcal{D}_x : \mathcal{I} \to \mathcal{C}.$$

We consider the following full subcategories of $\mathcal{I}$. For a monomial M let $\mathcal{I}_{\geq M}$ denote the subcategory of monomials which are divisible by M, and for a positive integer n, recalling that we have assigned each T_i a positive integral degree d_i, let $\mathcal{I}_{\deg\geq n}$ denote the subcategory of monomials of degree at least n, where the degree of $N := (N_1, \ldots, N_k)$ is $N_1d_1 + \cdots + N_kd_k$. One defines similarly the full subcategories $\mathcal{I}_{>M}$ and $\mathcal{I}_{\deg>n}$.

Let $\mathcal{I}^\circ$ be the full subcategory of $\mathcal{I}$ of monomials $N \neq 0$ and $\mathcal{I}^\circ_{\leq 1} \subset \mathcal{I}^\circ$ be the full subcategory of monomials N for which $N_i \leq 1$ for all i. We have the corresponding subdiagrams $\mathcal{D}_x : \mathcal{I}^\circ \to \mathcal{C}$ and $\mathcal{D}_x : \mathcal{I}^\circ_{\leq 1} \to \mathcal{C}$ of $\mathcal{D}_x$. For $J \subset I$ a subset, we have the corresponding full subcategories $\mathcal{J} \subset \mathcal{I}$, $\mathcal{J}^\circ \subset \mathcal{I}^\circ$ and $\mathcal{J}^\circ_{\leq 1} \subset \mathcal{I}^\circ_{\leq 1}$ and corresponding subdiagrams $\mathcal{D}_x$. If the collection of maps x_i is understood, we write simply $\mathcal{D}$ for $\mathcal{D}_x$.
Let $F : A \to \mathcal{C}$ be a functor, a an object in $\mathcal{C}$, $c_a : A \to \mathcal{C}$ the constant functor with value a and $\varphi : F \to c_a$ a natural transformation. Then φ induces a canonical map $\tilde{\varphi} : \mathrm{hocolim}_A F \to a$ in $\mathcal{C}$. As in the proof of [S10, Proposition 4.4], let $C(A)$ be the category A with a final object $*$ adjoined and $C(F,\varphi) : C(A) \to \mathcal{C}$ the functor with value a on $*$, with restriction to A being F, and which sends the unique map $y \to *$ in $C(A)$, $y \in A$, to $\varphi(y)$. Let $[0,1]$ be the category with objects $0, 1$ and a unique non-identity morphism $0 \to 1$, and let $C(A)^\Gamma$ be the full subcategory of $C(A) \times [0,1]$ formed by removing the object $* \times 1$. We extend $C(F,\varphi)$ to a functor $C(F,\varphi)^\Gamma : C(A)^\Gamma \to \mathcal{C}$ by $C(F,\varphi)^\Gamma(y \times 1) = pt$, where pt is the initial/final object in $\mathcal{C}$.

LEMMA 1.6. *There is a natural isomorphism in $\mathcal{C}$*

$$\underset{C(A)^\Gamma}{\mathrm{hocolim}}\, C(F,\varphi)^\Gamma \cong \mathrm{hocofib}(\tilde{\varphi} : \underset{A}{\mathrm{hocolim}}\, F \to a).$$

Proof. For a category $\mathcal{A}$ we let $\mathcal{N}(\mathcal{A})$ denote the simplicial nerve of $\mathcal{A}$. We have an isomorphism of simplicial sets $\mathcal{N}(C(A)) \cong \mathrm{Cone}(\mathcal{N}(A), *)$, where $\mathrm{Cone}(\mathcal{N}(A), *)$ is the cone over $\mathcal{N}(A)$ with vertex $*$. Similarly, the full subcategory $A \times [0,1]$ of $C(A)^\Gamma$ has nerve isomorphic to $\mathcal{N}(A) \times \Delta[1]$. This gives an isomorphism of $\mathcal{N}(C(A)^\Gamma)$ with the push-out in the diagram

$$\begin{array}{ccc} \mathcal{N}(A) & \hookrightarrow & \mathrm{Cone}(\mathcal{N}(A), *) \\ {\scriptstyle \mathrm{id}\times\delta_0} \downarrow & & \\ \mathcal{N}(A) \times \Delta[1]. & & \end{array}$$

This in turn gives an isomorphism of the simplicial object $\mathrm{hocolim}_{C(A)^\Gamma} C(F,\varphi)^\Gamma_*$ with the pushout in the diagram

$$\begin{array}{ccc} \mathrm{hocolim}_A F & \hookrightarrow & C(\mathrm{hocolim}_A F, a) \\ \downarrow & & \\ C(\mathrm{hocolim}_A F, pt). & & \end{array}$$

This gives the desired isomorphism. ☐

LEMMA 1.7. *Let $J \subset K \subset I$ be finite subsets of I. Then the map*

$$\underset{\mathcal{J}^\circ_{\leq 1}}{\mathrm{hocolim}}\, \mathcal{D}_x \to \underset{\mathcal{K}^\circ_{\leq 1}}{\mathrm{hocolim}}\, \mathcal{D}_x$$

induced by the inclusion $J \subset K$ is a cofibration in $\mathcal{C}$.

Proof. We give the category of simplicial objects in $\mathcal{C}$, $\mathcal{C}^{\Delta^{\mathrm{op}}}$, the Reedy model structure, using the standard structure of a Reedy category on Δ^{op}. By [Hir03, theorem 19.7.2(1), definition 19.8.1(1)], it suffices to show that

$$\operatorname*{hocolim}_{\mathcal{J}^\circ_{\leq 1}} \mathcal{D}_* \to \operatorname*{hocolim}_{\mathcal{K}^\circ_{\leq 1}} \mathcal{D}_*$$

is a cofibration in $\mathcal{C}^{\Delta^{\mathrm{op}}}$, that is, for each n, the map

$$\varphi_n : \operatorname*{hocolim}_{\mathcal{J}^\circ_{\leq 1}} \mathcal{D}_n \amalg_{L^n \operatorname{hocolim}_{\mathcal{J}^\circ_{\leq 1}} \mathcal{D}_*} L^n \operatorname*{hocolim}_{\mathcal{K}^\circ_{\leq 1}} \mathcal{D}_* \to \operatorname*{hocolim}_{\mathcal{K}^\circ_{\leq 1}} \mathcal{D}_n$$

is a cofibration in $\mathcal{C}$, where L^n is the nth latching space.
We note that

$$\operatorname*{hocolim}_{\mathcal{J}^\circ_{\leq 1}} \mathcal{D}_n = \bigvee_{\sigma \in \mathcal{N}(\mathcal{J}^\circ_{\leq 1})_n} D(\sigma(0)),$$

where we view $\sigma \in \mathcal{N}(\mathcal{J}^\circ_{\leq 1})_n$ as a functor $\sigma : [n] \to \mathcal{J}^\circ_{\leq 1}$; we have a similar description of $\operatorname{hocolim}_{\mathcal{K}^\circ_{\leq 1}} \mathcal{D}_n$. The latching space is

$$L^n \operatorname*{hocolim}_{\mathcal{J}^\circ_{\leq 1}} \mathcal{D}_* = \bigvee_{\sigma \in \mathcal{N}(\mathcal{J}^\circ_{\leq 1})^{deg}_n} D(\sigma(0)),$$

where $\mathcal{N}(\mathcal{J}^\circ_{\leq 1})^{deg}_n$ is the subset of $\mathcal{N}(\mathcal{J}^\circ_{\leq 1})_n$ consisting of those σ which contain an identity morphism; $L^n \operatorname{hocolim}_{\mathcal{K}^\circ_{\leq 1}} \mathcal{D}_*$ has a similar description. The maps

$$L^n \operatorname*{hocolim}_{\mathcal{J}^\circ_{\leq 1}} \mathcal{D}_* \to \operatorname*{hocolim}_{\mathcal{J}^\circ_{\leq 1}} \mathcal{D}_n, L^n \operatorname*{hocolim}_{\mathcal{J}^\circ_{\leq 1}} \mathcal{D}_* \to L^n \operatorname*{hocolim}_{\mathcal{K}^\circ_{\leq 1}} \mathcal{D}*,$$
$$L^n \operatorname*{hocolim}_{\mathcal{K}^\circ_{\leq 1}} \mathcal{D}_* \to \operatorname*{hocolim}_{\mathcal{K}^\circ_{\leq 1}} \mathcal{D}_n, \operatorname*{hocolim}_{\mathcal{J}^\circ_{\leq 1}} \mathcal{D}_n \to \operatorname*{hocolim}_{\mathcal{K}^\circ_{\leq 1}} \mathcal{D}_n$$

are the unions of identity maps on $D(\sigma(0))$ over the respective inclusions of the index sets. As $\mathcal{N}(\mathcal{K}^\circ_{\leq 1})^{deg}_n \cap \mathcal{N}(\mathcal{J}^\circ_{\leq 1})_n = \mathcal{N}(\mathcal{J}^\circ_{\leq 1})^{deg}_n$, we have

$$\operatorname*{hocolim}_{\mathcal{J}^\circ_{\leq 1}} \mathcal{D}_n \amalg_{L^n \operatorname{hocolim}_{\mathcal{J}^\circ_{\leq 1}} \mathcal{D}_*} L^n \operatorname*{hocolim}_{\mathcal{K}^\circ_{\leq 1}} \mathcal{D}_* \cong \operatorname*{hocolim}_{\mathcal{J}^\circ_{\leq 1}} \mathcal{D}_n \bigvee C,$$

where

$$C = \bigvee_{\sigma \in \mathcal{N}(\mathcal{K}^\circ_{\leq 1})^{deg}_n \setminus \mathcal{N}(\mathcal{J}^\circ_{\leq 1})^{deg}_n} D(\sigma(0)),$$

and the map to $\operatorname{hocolim}_{\mathcal{K}^\circ_{\leq 1}} \mathcal{D}_n$ is the evident inclusion. As $D(N)$ is cofibrant for all N, this map is clearly a cofibration, completing the proof. $\square$

We have the n-cube $\square^n$, the category associated to the partially ordered set of subsets of $\{1, \ldots, n\}$, ordered under inclusion, and the punctured n-cube $\square^n_0$ of proper subsets. We have the two inclusion functors $i^+_n, i^-_n : \square^{n-1} \to \square^n$, $i^+_n(I) := I \cup \{n\}$, $i^-_n(I) = I$ and the natural transformation $\psi_n : i^-_n \to i^+_n$ given as the collection of inclusions $I \subset I \cup \{n\}$. The functor i^-_n induces the functor $i^-_{n0} : \square^{n-1} \to \square^n_0$.
For a functor $F : \square^n \to \mathcal{C}$, we have the iterated homotopy cofiber, $\mathrm{hocofib}_n F$, defined inductively as the homotopy cofiber of $\mathrm{hocofib}_{n-1}(F(\psi_n)) : \mathrm{hocofib}(F \circ i^-_n) \to \mathrm{hocofib}(F \circ i^+_n)$. Using this inductive construction, it is easy to define

a natural isomorphism $\mathrm{hocofib}_n F \cong \mathrm{hocolim}_{\square_0^{n+1}} \hat{F}$, where $\hat{F} \circ i^-_{n+10} = F$ and $\hat{F}(I) = pt$ if $n \in I$.
The following result, in the setting of modules over a model of MGL as a commutative $\mathbb{S}$-algebra, is proven in [S10, Lemma 4.3 and Proposition 4.4]. We give here a somewhat different proof in our context, which allows for a wider application.

Lemma 1.8. *Assume that I is countable. Then there is a canonical isomorphism in* $\mathbf{Ho}\,\mathcal{C}$

$$\mathbf{1}/(\{x_i \mid i \in I\}) \cong \mathrm{hocofib}[\operatorname*{hocolim}_{\mathcal{I}^\circ} \mathcal{D}_x \to \operatorname*{hocolim}_{\mathcal{I}} \mathcal{D}_x].$$

Proof. As 1 is the final object in $\mathcal{I}$, the collection of maps $\times x^N : T_*^N \to \mathbf{1}$ defines a weak equivalence $\pi : \mathrm{hocolim}_{\mathcal{I}} \mathcal{D}_x \to \mathbf{1}$. In addition, for each $N \in \mathcal{I}^\circ$, the comma category $N/\mathcal{I}^\circ_{\leq 1}$ has initial object the map $N \to \bar{N}$, where $\bar{N}_i = 1$ if $N_i > 0$, and $\bar{N}_i = 0$ otherwise. Thus $\mathcal{I}^\circ_{\leq 1}$ is homotopy right cofinal in $\mathcal{I}^\circ$ (see e.g. [Hir03, definition 19.6.1]). Since $\mathcal{D}_x$ is a diagram of cofibrant objects in $\mathcal{C}$, it follows from [Hir03, theorem 19.6.7] that the map $\mathrm{hocolim}_{\mathcal{I}^\circ_{\leq 1}} \mathcal{D}_x \to \mathrm{hocolim}_{\mathcal{I}^\circ} \mathcal{D}_x$ is a weak equivalence. This reduces us to identifying $\mathbf{1}/(\{x_i\})$ with the homotopy cofiber of $\pi^\circ_{\leq 1} : \mathrm{hocolim}_{\mathcal{I}^\circ_{\leq 1}} \mathcal{D}_x \to \mathbf{1}$, where $\pi^\circ_{\leq 1}$ is the composition of π with the natural map : $\mathrm{hocolim}_{\mathcal{I}^\circ_{\leq 1}} \mathcal{D}_x \to \mathrm{hocolim}_{\mathcal{I}} \mathcal{D}_x$.
Next, we reduce to the case of a finite set I. Take $I = \mathbb{N}$. Let $\mathcal{P}_{fin}(I)$ be the category of finite subsets of I, ordered by inclusion, consider the full subcategory $\mathcal{P}^O_{fin}(I)$ of $\mathcal{P}_{fin}(I)$ consisting of the subsets $I_n := \{1, \ldots, n\}$, $n = 1, 2 \ldots$, and let $\mathcal{I}^\circ_{n,\leq 1} \subset \mathcal{I}^\circ_{\leq 1}$ be the full subcategory with all indices in I_n. As $\mathcal{P}^O_{fin}(I)$ is cofinal in $\mathcal{P}_{fin}(I)$, we have

$$\operatorname*{colim}_n \operatorname*{hocolim}_{\mathcal{I}^\circ_{n,\leq 1}} \mathcal{D}_x \cong \operatorname*{hocolim}_{\mathcal{I}^\circ_{\leq 1}} \mathcal{D}_x.$$

Take $n \leq m$. By lemma 1.7 the the map $\mathrm{hocolim}_{\mathcal{I}^\circ_{n,\leq 1}} \mathcal{D}_x \to \mathrm{hocolim}_{\mathcal{I}^\circ_{m,\leq 1}} \mathcal{D}_x$ is a cofibration in $\mathcal{C}$. Thus, using the Reedy model structure on $\mathcal{C}^{\mathbb{N}}$ with $\mathbb{N}$ considered as a direct category, the $\mathbb{N}$-diagram in $\mathcal{C}$, $n \mapsto \mathrm{hocolim}_{\mathcal{I}^\circ_{n,\leq 1}} \mathcal{D}_x$, is a cofibrant object in $\mathcal{C}^{\mathbb{N}}$. As $\mathbb{N}$ is a direct category, the fibrations in $\mathcal{C}^{\mathbb{N}}$ are the pointwise ones, hence $\mathbb{N}$ has pointwise constants [Hir03, definition 15.10.1] and therefore [Hir03, theorem 19.9.1] the canonical map

$$\operatorname*{hocolim}_{n\in\mathbb{N}} \operatorname*{hocolim}_{\mathcal{I}^\circ_{n,\leq 1}} \mathcal{D}_x \to \operatorname*{colim}_{n\in\mathbb{N}} \operatorname*{hocolim}_{\mathcal{I}^\circ_{n,\leq 1}} \mathcal{D}_x$$

is a weak equivalence in $\mathcal{C}$. This gives us the weak equivalence in $\mathcal{C}$

$$\operatorname*{hocolim}_n \operatorname*{hocolim}_{\mathcal{I}^\circ_{n,\leq 1}} \mathcal{D}_x \to \operatorname*{hocolim}_{\mathcal{I}^\circ_{\leq 1}} \mathcal{D}_x.$$

Since $\mathbb{N}$ is contractible, the canonical map $\mathrm{hocolim}_{\mathbb{N}}\, \mathbf{1} \to \mathbf{1}$ is a weak equivalence in $\mathcal{C}$, giving us the weak equivalences

$$\begin{aligned}
&\mathrm{hocofib}[\operatorname*{hocolim}_{\mathcal{I}^\circ_{\leq 1}} \mathcal{D}_x \to \mathbf{1}] \\
&\qquad\qquad \sim \mathrm{hocofib}[\operatorname*{hocolim}_{n\in\mathbb{N}} \operatorname*{hocolim}_{\mathcal{I}^\circ_{n,\leq 1}} \mathcal{D}_x \to \operatorname*{hocolim}_{n\in\mathbb{N}} \mathbf{1}] \\
&\qquad\qquad\qquad\qquad \sim \operatorname*{hocolim}_{n\in\mathbb{N}}[\mathrm{hocofib}[\operatorname*{hocolim}_{\mathcal{I}^\circ_{n,\leq 1}} \mathcal{D}_x \to \mathbf{1}]].
\end{aligned}$$

Thus, we need only exhibit isomorphisms in $\mathbf{Ho}\,\mathcal{C}$

$$\rho_n : \mathrm{hocofib}[\operatorname*{hocolim}_{\mathcal{I}^\circ_{n,\leq 1}} \mathcal{D}_x \to \mathbf{1}] \to \mathbf{1}/(x_1, \ldots, x_n) := \mathbf{1}/(x_1) \otimes \ldots \otimes \mathbf{1}/(x_n),$$

which are natural in $n \in \mathbb{N}$.
By lemma 1.6 we have a natural isomorphism in $\mathcal{C}$,

$$\mathrm{hocofib}[\operatorname*{hocolim}_{\mathcal{I}^\circ_{n,\leq 1}} \mathcal{D}_x \to \mathbf{1}] \cong \operatorname*{hocolim}_{C(\mathcal{I}^\circ_{n,\leq 1})^\Gamma} C(\mathcal{D}_x, \pi)^\Gamma.$$

However, $\mathcal{I}^\circ_{n,\leq 1}$ is isomorphic to $\square^n_0$ by sending $N = (N_1, \ldots, N_n)$ to $I(N) := \{i \mid N_i = 0\}$. Similarly, $C(\mathcal{I}^\circ_{n,\leq 1})$ is isomorphic to $\square^n$, and $C(\mathcal{I}^\circ_{n,\leq 1})^\Gamma$ is thus isomorphic to $\square^{n+1}_0$. From our discussion above, we see that $\mathrm{hocolim}_{C(\mathcal{I}^\circ_{n,\leq 1})^\Gamma}\, C(\mathcal{D}_x, \pi)^\Gamma$ is isomorphic to $\mathrm{hocofib}_n C(\mathcal{D}_x, \pi)$, so we need only exhibit isomorphisms in $\mathbf{Ho}\,\mathcal{C}$

$$\rho_n : \mathrm{hocofib}_n C(\mathcal{D}_x, \pi) \to \mathbf{1}/(x_1) \otimes \ldots \otimes \mathbf{1}/(x_n)$$

which are natural in $n \in \mathbb{N}$.
We do this inductively as follows. To include the index n in the notation, we write $C(\mathcal{D}_x, \pi)_n$ for the functor $C(\mathcal{D}_x, \pi) : \square^n \to \mathcal{C}$. For $n = 1$, $\mathrm{hocofib}_1 C(\mathcal{D}_x, \pi)_1$ is the mapping cone of $\mu_1 \circ (\times x_1 \otimes \mathrm{id}) : \mathbf{1} \otimes T_1 \otimes \mathbf{1} \to \mathbf{1}$, which is isomorphic in $\mathbf{Ho}\,\mathcal{C}$ to the homotopy cofiber of $\times x_1 : \mathbf{1} \otimes T_1 \to \mathbf{1}$. As this latter homotopy cofiber is equal to $\mathbf{1}/(x_1)$, so we take $\rho_1 : \mathrm{hocofib}_1 C(\mathcal{D}_x, \pi)_1 \to \mathbf{1}/(x_1)$ to be this isomorphism. We note that $C(\mathcal{D}_x, \pi)_n \circ i^+_n = C(\mathcal{D}_x, \pi)_{n-1}$ and $C(\mathcal{D}_x, \pi)_n \circ i^-_n = C(\mathcal{D}_x, \pi)_{n-1} \otimes T_n \otimes \mathbf{1}$.
Define $C(\mathcal{D}_x, \pi)'_n$ by $C(\mathcal{D}_x, \pi)'_n \circ i^-_n = C(\mathcal{D}_x, \pi)_{n-1} \otimes \mathbf{1} \otimes T_n \otimes \mathbf{1}$, $C(\mathcal{D}_x, \pi)'_n \circ i^+_n = C(\mathcal{D}_x, \pi)_{n-1} \otimes \mathbf{1}$, with the natural transformation $C(\mathcal{D}_x, \pi)'_n \circ \psi_n$ given as

$$C(\mathcal{D}_x, \pi)_{n-1} \otimes \mathbf{1} \otimes T_n \otimes \mathbf{1} \xrightarrow{(\mathrm{id}\otimes\mu)\circ(\mathrm{id}\otimes\times x_n\otimes \mathrm{id}_{\mathbf{1}})} C(\mathcal{D}_x, \pi)_{n-1} \otimes \mathbf{1}.$$

The evident multiplication maps give a weak equivalence $C(\mathcal{D}_x, \pi)'_n \to C(\mathcal{D}_x, \pi)_n$, giving us the isomorphism in $\mathbf{Ho}\,\mathcal{C}$

$$\rho_n : \mathrm{hocofib}_n C(\mathcal{D}_x, \pi)_n \to \mathbf{1}/(x_1) \otimes \ldots \otimes \mathbf{1}/(x_n)$$

defined as the composition

$$\begin{aligned}
\mathrm{hocofib}_n C(\mathcal{D}_x,\pi)_n &\cong \mathrm{hocofib}_n C(\mathcal{D}_x,\pi)'_n \\
&\cong \mathrm{hocofib}(\mathrm{hocofib}_{n-1}(C(\mathcal{D}_x,\pi)_{n-1}\otimes \mathbf{1}\otimes T_n) \\
&\qquad \xrightarrow{\mathrm{hocofib}_{n-1}(\mathrm{id}\otimes\times x_n)} \mathrm{hocofib}_{n-1}(C(\mathcal{D}_x,\pi)_{n-1}\otimes \mathbf{1})) \\
&\cong \mathrm{hocofib}(\mathrm{hocofib}_{n-1}(C(\mathcal{D}_x,\pi)_{n-1})\otimes \mathbf{1}\otimes T_n \\
&\qquad \xrightarrow{\mathrm{id}\otimes\times x_n} \mathrm{hocofib}_{n-1}(C(\mathcal{D}_x,\pi)_{n-1})\otimes \mathbf{1}) \\
&\cong \mathrm{hocofib}_{n-1}(C(\mathcal{D}_x,\pi)_{n-1})\otimes \mathrm{hocofib}(\times x_n : \mathbf{1}\otimes T_n \to \mathbf{1}) \\
&= \mathrm{hocofib}_{n-1}(C(\mathcal{D}_x,\pi)_{n-1})\otimes \mathbf{1}/(x_n) \\
&\xrightarrow{\rho_{n-1}\otimes\mathrm{id}} \mathbf{1}/(x_1)\otimes\ldots\otimes\mathbf{1}/(x_{n-1})\otimes\mathbf{1}/(x_n).
\end{aligned}$$

Via the definition of $\mathrm{hocofib}_n$,

$$\begin{aligned}
\mathrm{hocofib}_n C(\mathcal{D}_x,\pi)_n = \mathrm{hocofib}[&\mathrm{hocofib}_{n-1}(C(\mathcal{D}_x,\pi)_n\circ i_n^-) \\
&\xrightarrow{\mathrm{hocofib}_{n-1}(C(\mathcal{D}_x,\pi)_{n-1}(\psi_n))} \mathrm{hocofib}_n(C(\mathcal{D}_x,\pi)_n\circ i_n^+]
\end{aligned}$$

and the identification $C(\mathcal{D}_x,\pi)_n\circ i_n^+ = C(\mathcal{D}_x,\pi)_{n-1}$, we have the canonical map $\mathrm{hocofib}_{n-1}(C(\mathcal{D}_x,\pi)_{n-1}) \to \mathrm{hocofib}_n(C(\mathcal{D}_x,\pi)_n)$. One easily sees that the diagram

$$\begin{array}{ccc}
\mathrm{hocofib}[\mathrm{hocolim}_{\mathcal{I}^\circ_{n-1,\leq 1}}\mathcal{D}_x\to\mathbf{1}] & \longrightarrow & \mathrm{hocofib}[\mathrm{hocolim}_{\mathcal{I}^\circ_{n,\leq 1}}\mathcal{D}_x\to\mathbf{1}] \\
\downarrow{\sim} & & \downarrow{\sim} \\
\mathrm{hocofib}_{n-1}(C(\mathcal{D}_x,\pi)_{n-1}) & \longrightarrow & \mathrm{hocofib}_n(C(\mathcal{D}_x,\pi)_n) \\
\downarrow{\rho_{n-1}} & & \downarrow{\rho_n} \\
\mathbf{1}/(x_1)\otimes\ldots\otimes\mathbf{1}/(x_{n-1}) & \xrightarrow[\rho_{\{1,\ldots,n-1\}\subset\{1,\ldots,n\}}]{} & \mathbf{1}/(x_1)\otimes\ldots\otimes\mathbf{1}/(x_n)
\end{array}$$

commutes in $\mathbf{Ho}\,\mathcal{C}$, giving the desired naturality in n. □

Now let M be an object in $\mathcal{C}$, let $QM\to M$ be a cofibrant replacement and form the $\mathcal{I}$-diagram $\mathcal{D}_x\otimes QM:\mathcal{I}\to\mathcal{C}$, $(\mathcal{D}_x\otimes QM)(N)=\mathcal{D}_x(N)\otimes QM$.

Proposition 1.9. *Assume that I is countable. Let M be an object in $\mathcal{C}$. Then there is a canonical isomorphism in* $\mathbf{Ho}\,\mathcal{C}$

$$M/(\{x_i\mid i\in I\})\cong \mathrm{hocofib}[\operatorname*{hocolim}_{\mathcal{I}^\circ}\mathcal{D}_x\otimes QM\to\operatorname*{hocolim}_{\mathcal{I}}\mathcal{D}_x\otimes QM].$$

Proof. This follows directly from lemma 1.8, noting the definition of $M/(\{x_i\mid i\in I\})$ as $[\mathbf{1}/(\{x_i\mid i\in I\})]\otimes QM$ and the canonical isomorphism

$$\begin{aligned}
\mathrm{hocofib}[&\operatorname*{hocolim}_{\mathcal{I}^\circ}\mathcal{D}_x\otimes QM\to\operatorname*{hocolim}_{\mathcal{I}}\mathcal{D}_x\otimes QM] \\
&\cong \mathrm{hocofib}[\operatorname*{hocolim}_{\mathcal{I}^\circ}\mathcal{D}_x\to\operatorname*{hocolim}_{\mathcal{I}}\mathcal{D}_x]\otimes QM.
\end{aligned}$$

□

Proposition 1.10. *Let $\mathcal{F} : \mathcal{I}_{\deg\geq n} \to \mathcal{C}$ be a diagram in a cofibrantly generated model category $\mathcal{C}$. Suppose for every monomial M of degree n the natural map* $\mathrm{hocolim}\, \mathcal{F}|_{\mathcal{I}_{>M}} \to \mathcal{F}(M)$ *is a weak equivalence. Then the natural map*

$$\mathrm{hocolim}\, \mathcal{F}|_{\deg\geq n+1} \to \mathrm{hocolim}\, \mathcal{F}$$

is a weak equivalence.

Proof. This is just [S10, lemma 4.5], with the following corrections: the statement of the lemma in *loc. cit.* has "$\mathrm{hocolim}\, \mathcal{F}|_{\mathcal{I}_{\geq M}} \to \mathcal{F}(M)$ is a weak equivalence" rather than the correct assumption "$\mathrm{hocolim}\, \mathcal{F}|_{\mathcal{I}_{>M}} \to \mathcal{F}(M)$ is a weak equivalence" and in the proof, one should replace the object $Q(M)$ with $\mathrm{colim}\, Q|_{I>M}$ rather than with $\mathrm{colim}\, Q|_{I\geq M}$. □

2. Slices of effective motivic module spectra

In this section we will describe the slices for modules for a commutative and effective ring T-spectrum $\mathcal{R}$, assuming certain additional conditions. We adapt the constructions used in describing slices of MGL in [S10].

Let us first recall from [Vo00] the definition of the slice tower in $\mathcal{SH}(S)$. We will use the standard model category $Mot := Mot(S)$ of symmetric T-spectra over S, $T := \mathbb{A}^1/\mathbb{A}^1 \setminus \{0\}$, with the motivic model structure as in [J00], for defining the triangulated tensor category $\mathcal{SH}(S) := \mathbf{Ho}\, Mot(S)$.

For an integer q, let $\Sigma^q_T \mathcal{SH}^{eff}(S)$ denote the localizing subcategory of $\mathcal{SH}(S)$ generated by $\mathcal{S}_q := \{\Sigma^q_T \Sigma^\infty_T X_+ \mid p \geq q, X \in \mathbf{Sm}/S\}$, that is, $\Sigma^q_T \mathcal{SH}^{eff}(S)$ is the smallest triangulated subcategory of $\mathcal{SH}(S)$ which contains $\mathcal{S}_q$ and is closed under direct sums and isomorphisms in $\mathcal{SH}(S)$. This gives a filtration on $\mathcal{SH}(S)$ by full localizing subcategories

$$\cdots \subset \Sigma^{q+1}_T \mathcal{SH}^{eff}(S) \subset \Sigma^q_T \mathcal{SH}^{eff}(S) \subset \Sigma^{q-1}_T \mathcal{SH}^{eff}(S) \subset \cdots \subset \mathcal{SH}(S).$$

The set $\mathcal{S}_q$ is a set of compact generators of $\Sigma^q_T \mathcal{SH}(S)$ and the set $\cup_q \mathcal{S}_q$ is similarly a set of compact generators for $\mathcal{SH}(S)$. By Neeman's triangulated version of Brown representability theorem [N97], the inclusion $i_q : \Sigma^q_T \mathcal{SH}^{eff}(S) \to \mathcal{SH}(S)$ has a right adjoint $r_q : \mathcal{SH}(S) \to \Sigma^q_T \mathcal{SH}^{eff}(S)$. We let $f_q := i_q \circ r_q$. The inclusion $\Sigma^{q+1}_T \mathcal{SH}^{eff}(S) \to \Sigma^q_T \mathcal{SH}^{eff}(S)$ induces a canonical natural transformation $f_{q+1} \to f_q$. Putting these together forms the slice tower

$$\cdots \to f_{q+1} \to f_q \to \cdots \to \mathrm{id}. \tag{2.1}$$

For each q there exists a triangulated functor $s_q : \mathcal{SH}(S) \to \mathcal{SH}(S)$ and a canonical and natural distinguished triangle

$$f_{q+1}(\mathcal{E}) \to f_q(\mathcal{E}) \to s_q(\mathcal{E}) \to \Sigma f_{q+1}(\mathcal{E})$$

in $\mathcal{SH}(S)$. In particular, $s_q(\mathcal{E})$ is in $\Sigma^q_T \mathcal{SH}^{eff}(S)$ for each $\mathcal{E} \in \mathcal{SH}(S)$.

Pelaez has given a lifting of the construction of the functors f_q to the model category level. For this, he starts with the model category Mot and forms for each n the right Bousfield localization of Mot with respect to the objects

$\Sigma^m_T F_n X_+$ with $m - n \geq q$ and $X \in \mathbf{Sm}/S$. Here $F_n X_+$ is the shifted T-suspension spectrum, that is, $\Sigma^{m-n}_T X_+$ in degree $m \geq n$, pt in degree $m < n$, and with identity bonding maps. Calling this Bousfield localization Mot_q, the functor r_q is given by taking a functorial cofibrant replacement in Mot_q. As the underlying categories are all the same, this gives liftings $\tilde{f}_q$ of f_q to endofunctors on Mot. The technical condition on Mot invoked by Pelaez is that of *cellularity* and right properness, which ensures that the right Bousfield localization exists; this follows from the work of Hirschhorn [Hir03]. Alternatively, one can use the fact that Mot is a *combinatorial* right proper model category, following work of J. Smith, detailed for example in [B10].

The combinatorial property passes to module categories, and so this approach will be useful here. The category Mot is a closed symmetric monoidal simplicial model category, with cofibrant unit the sphere (symmetric) spectrum $\mathbb{S}_S$ and product $\wedge$. Let $\mathcal{R}$ be a commutative monoid in Mot. We have the model category $\mathcal{C} := \mathcal{R}$-Mod of $\mathcal{R}$-modules, as constructed in [ScSh]. The fibrations and weak equivalences are the morphisms which are fibrations, resp. weak equivalences, after applying the forgetful functor to Mot; cofibrations are those maps having the left lifting property with respect to trivial fibrations. This makes $\mathcal{C}$ into a pointed closed symmetric monoidal simplicial model category; $\mathcal{C}$ is in addition cofibrantly generated and combinatorial. Assuming that $\mathcal{R}$ is a cofibrant object in Mot, the free $\mathcal{R}$-module functor, $\mathcal{E} \mapsto \mathcal{R} \wedge \mathcal{E}$, gives a left adjoint to the forgetful functor and gives rise to a Quillen adjunction. For details as to these facts and a general construction of this model category structure on module categories, we refer the reader to [ScSh]; another source is [Hov], especially theorem 1.3, proposition 1.9 and proposition 1.10.

The model category $\mathcal{R}$-Mod inherits right properness from Mot. We may therefore form the right Bousfield localization $\mathcal{C}_q$ with respect to the free $\mathcal{R}$-modules $\mathcal{R} \wedge \Sigma^m_T F_n X_+$ with $m - n \geq q$ and $X \in \mathbf{Sm}/S$, and define the endofunctor $\tilde{f}^{\mathcal{R}}_q$ on $\mathcal{C}$ by taking a functorial cofibrant replacement in $\mathcal{C}_q$. By the adjunction, one sees that $\mathbf{Ho}\,\mathcal{C}_q$ is equivalent to the localizing subcategory of $\mathbf{Ho}\,\mathcal{C}$ (compactly) generated by $\{\mathcal{R} \wedge \Sigma^m_T F_n X_+ \mid m - n \geq q, X \in \mathbf{Sm}/S\}$. We denote this localizing subcategory by $\Sigma^q_T \mathbf{Ho}\,\mathcal{C}^{eff}$, or $\mathbf{Ho}\,\mathcal{C}^{eff}$ for $q = 0$. We call an object $\mathcal{M}$ of $\mathcal{C}$ *effective* if the image of $\mathcal{M}$ in $\mathbf{Ho}\,\mathcal{C}$ is in $\mathbf{Ho}\,\mathcal{C}^{eff}$, and denote the full subcategory of effective objects of $\mathcal{C}$ by $\mathcal{C}^{eff}$.

Just as above, Neeman's results give a right adjoint $r^{\mathcal{R}}_q$ to the inclusion $i^{\mathcal{R}}_q$: $\Sigma^q_T \mathcal{C}^{eff} \to \mathcal{C}$ and the composition $f^{\mathcal{R}}_q := i^{\mathcal{R}}_q \circ r^{\mathcal{R}}_q$ is represented by $\tilde{f}^{\mathcal{R}}_q$. One recovers the functors f_q and $\tilde{f}_q$ by taking $\mathcal{R} = \mathbb{S}_S$.

Lemma 2.1. *Let $\mathcal{R}$ be a cofibrant commutative monoid in Mot. The functors $f^{\mathcal{R}}_q : \mathbf{Ho}\,\mathcal{C} \to \mathbf{Ho}\,\mathcal{C}$ and their liftings $\tilde{f}^{\mathcal{R}}_q$ have the following properties.*

(1) *Each $f^{\mathcal{R}}_n$ is idempotent, i.e., $(f^{\mathcal{R}}_n)^2 = f^{\mathcal{R}}_n$.*
(2) *$f^{\mathcal{R}}_n \Sigma^1_T = \Sigma^1_T f^{\mathcal{R}}_{n-1}$ for $n \in \mathbb{Z}$.*
(3) *Each $\tilde{f}^{\mathcal{R}}_n$ commutes with homotopy colimits.*

(4) *Suppose that* $\mathcal{R}$ *is in* $\mathcal{SH}^{eff}(S)$. *Then the forgetful functor* $U : \mathbf{Ho}\,\mathcal{R}$-Mod $\to \mathcal{SH}(S)$ *induces an isomorphism* $U \circ f_q^{\mathcal{R}} \cong f_q \circ U$ *as well as an isomorphism* $U \circ s_q^{\mathcal{R}} \cong s_q \circ U$, *for all* $q \in \mathbb{Z}$.

Proof. (1) and (2) follow from universal property of triangulated functors $f_n^{\mathcal{R}}$. In case $\mathcal{R} = \mathbb{S}_S$, (3) is proved in [S10, Cor 4.6]; the proof for general $\mathcal{R}$ is the same. For (4), it suffices to prove the result for f_q and $f_q^{\mathcal{R}}$. Take $\mathcal{M} \in \mathcal{C}$. We check the universal property of $Uf_q^{\mathcal{R}}\mathcal{M} \to U\mathcal{M}$: Since $\mathcal{R}$ is in $\mathcal{SH}^{eff}(S)$ and the functor $- \wedge \mathcal{R}$ is compatible with homotopy cofiber sequences and direct sums, $-\wedge\mathcal{R}$ maps $\Sigma_T^q\mathcal{SH}^{eff}(S)$ into itself for each $q \in \mathbb{Z}$. As $U(\mathcal{R}\wedge\mathcal{E}) = \mathcal{R}\wedge\mathcal{E}$, it follows that $U(\Sigma_T^q\mathbf{Ho}\,\mathcal{R}\text{-Mod}^{eff}) \subset \Sigma_T^q\mathcal{SH}^{eff}(S)$ for each q. In particular, $U(f_q^{\mathcal{R}}(\mathcal{M}))$ is in $\Sigma_T^q\mathcal{SH}^{eff}(S)$. For $p \geq q$, $X \in \mathbf{Sm}/S$, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{SH}(S)}(\Sigma_T^p\Sigma_T^\infty X_+, U(f_q^{\mathcal{R}}(\mathcal{M}))) &\cong \mathrm{Hom}_{\mathbf{Ho}\,\mathcal{C}}(\mathcal{R}\wedge\Sigma_T^p\Sigma_T^\infty X_+, f_q^{\mathcal{R}}(\mathcal{M}))\\
&\cong \mathrm{Hom}_{\mathbf{Ho}\,\mathcal{C}}(\mathcal{R}\wedge\Sigma_T^p\Sigma_T^\infty X_+, \mathcal{M})\\
&\cong \mathrm{Hom}_{\mathcal{SH}(S)}(\Sigma_T^p\Sigma_T^\infty X_+, U(\mathcal{M})),
\end{aligned}$$

so the canonical map $U(f_q^{\mathcal{R}}(\mathcal{M})) \to f_q(U(\mathcal{M}))$ is therefore an isomorphism. □

From the adjunction $\mathrm{Hom}_{\mathcal{C}}(\mathcal{R},\mathcal{M}) \cong \mathrm{Hom}_{Mot}(\mathbb{S}_S,\mathcal{M})$ and the fact that $\mathbb{S}_S$ is a cofibrant object of Mot, we see that $\mathcal{R}$ is a cofibrant object of $\mathcal{C}$. Thus $\mathcal{C}$ is a closed symmetric monoidal simplicial model category with cofibrant unit $1 := \mathcal{R}$ and monoidal product $\otimes = \wedge_{\mathcal{R}}$. Similarly, $T_{\mathcal{R}} := \mathcal{R}\wedge T$ is a cofibrant object of $\mathcal{C}$. Abusing notation, we write $\Sigma_T(-)$ for the endofunctor $A \mapsto A\otimes T_{\mathcal{R}}$ of $\mathcal{C}$. The compatibility of the simplicial monoidal structure with monoidal structure of $\mathcal{C}$ follows directly from the construction of $\mathcal{C}$.
We recall that the category Mot satisfies the monoid axiom of Schwede-Shipley [ScSh, definition 3.3]; the reader can see for example the proof of [Hoy, lemma 4.2]. Following remark 1.1, there is a fibrant replacement $\mathcal{R} \to \mathbf{1}$ in $\mathcal{C}$ such that $\mathbf{1}$ is an $\mathcal{R}$-algebra; in particular, $\mathcal{R} \to \mathbf{1}$ is a cofibration and a weak equivalence in both $\mathcal{C}$ and in Mot, and $\mathbf{1}$ is fibrant in in both $\mathcal{C}$ and in Mot.
For each $\bar{x} \in \mathcal{R}^{-2d,-d}(S)$, we have the corresponding element $\bar{x} : T_{\mathcal{R}}^{\otimes d} \to \mathcal{R}$ in $\mathbf{Ho}\,\mathcal{C}$, which we may lift to a morphism $x : T_{\mathcal{R}}^{\otimes d} \to \mathbf{1}$ in $\mathcal{C}$. Thus, for a collection of elements $\{\bar{x}_i \in \mathcal{R}^{-2d_i,-d_i}(S) \mid i \in I\}$, we have the associated collection of maps in $\mathcal{C}$, $\{x_i : T_{\mathcal{R}}^{\otimes d_i} \to \mathbf{1} \mid i \in I\}$ and thereby the quotient object $\mathbf{1}/(\{x_i\})$ in $\mathcal{C}$. Similarly, for $\mathcal{M}$ an $\mathcal{R}$-module, we have the $\mathcal{R}$-module $\mathcal{M}/(\{x_i\})$, which is a cofibrant object in $\mathcal{C}$. We often write $\mathcal{R}/(\{x_i\})$ for $\mathbf{1}/(\{x_i\})$.

Lemma 2.2. *Suppose that* $\mathcal{R}$ *is in* $\mathcal{SH}^{eff}(S)$. *Then for any set*

$$\{\bar{x}_i \in \mathcal{R}^{-2d_i,-d_i}(S) \mid i \in I, d_i > 0\}$$

of elements of $\mathcal{R}$*-cohomology, the object* $\mathcal{R}/(\{x_i\})$ *is effective. If in addition* $\mathcal{M}$ *is an* $\mathcal{R}$*-module and is effective, then* $\mathcal{M}/(\{x_i\})$ *is effective.*

Proof. This follows from lemma 2.1 since $f_n^{\mathcal{R}}$ is a triangulated functor and $\mathcal{C}^{eff}$ is closed under homotopy colimits. □

Let A be an abelian group and SA the topological sphere spectrum with A-coefficients. For a T-spectrum $\mathcal{E}$ let us denote the spectrum $\mathcal{E} \wedge SA$ by $\mathcal{E} \otimes A$. Of course, if A is the free abelian group on a set S, then $\mathcal{E} \otimes A = \oplus_{s \in S} \mathcal{E}$.
Let $\{\bar{x}_i \in \mathcal{R}^{-2d_i,-d_i}(S) \mid i \in I, d_i > 0\}$ be a set of elements of $\mathcal{R}$-cohomology, with I countable. Suppose that $\mathcal{R}$ is cofibrant as an object in Mot and is in $\mathcal{SH}^{eff}(S)$. Let $\mathcal{M}$ be in $\mathcal{C}^{eff}$ and let $Q\mathcal{M} \to \mathcal{M}$ be a cofibrant replacement. By lemma 1.8, we have a homotopy cofiber sequence in $\mathcal{C}$,

$$\operatorname*{hocolim}_{\mathcal{I}^\circ} \mathcal{D}_x \otimes Q\mathcal{M} \to Q\mathcal{M} \to \mathcal{M}/(\{x_i\}).$$

Clearly $\operatorname{hocolim}_{\mathcal{I}^\circ} \mathcal{D}_x \otimes Q\mathcal{M}$ is in $\Sigma^1_T \mathbf{Ho}\,\mathcal{C}^{eff}$, hence the above sequence induces an isomorphism in $\mathbf{Ho}\,\mathcal{C}$

$$s_0^{\mathcal{R}}\mathcal{M} \xrightarrow{\sigma_{\mathcal{M}}} s_0^{\mathcal{R}}(\mathcal{M}/(\{x_i\})).$$

Composing the canonical map $\mathcal{M}/(\{x_i\}) \to s_0^{\mathcal{R}}(\mathcal{M}/(\{x_i\}))$ with $\sigma_{\mathcal{M}}^{-1}$ gives the canonical map

$$\pi^{\mathcal{R}}_{\mathcal{M}} : \mathcal{M}/(\{x_i\}) \to s_0^{\mathcal{R}}\mathcal{M}$$

in $\mathbf{Ho}\,\mathcal{C}$. Applying the forgetful functor gives the canonical map in $\mathcal{SH}(S)$

$$\pi_{\mathcal{M}} : U(\mathcal{M}/(\{x_i\})) \to U(s_0^{\mathcal{R}}\mathcal{M}) \cong s_0(U\mathcal{M}).$$

This equal to the canonical map $U(\mathcal{M}/(\{x_i\})) \to s_0(U(\mathcal{M}/(\{x_i\})))$ composed with the inverse of the isomorphism $s_0(U\mathcal{M}) \to s_0(U(\mathcal{M}/(\{x_i\})))$.

Theorem 2.3. *Let $\mathcal{R}$ be a commutative monoid in $Mot(S)$, cofibrant as an object in $Mot(S)$, such that $\mathcal{R}$ is in $\mathcal{SH}^{eff}(S)$. Let $X = \{\bar{x}_i \in \mathcal{R}^{-2d_i,-d_i}(S) \mid i \in I, d_i > 0\}$ be a countable set of elements of $\mathcal{R}$-cohomology. Let $\mathcal{M}$ be an $\mathcal{R}$-module in $\mathcal{C}^{eff}$ and suppose that the canonical map $\pi_{\mathcal{M}} : U(\mathcal{M}/(\{x_i\})) \to s_0(U\mathcal{M})$ is an isomorphism. Then for each $n \geq 0$, we have a canonical isomorphism in $\mathbf{Ho}\,\mathcal{C}$,*

$$s_n^{\mathcal{R}}\mathcal{M} \cong \Sigma^n_T s_0^{\mathcal{R}}\mathcal{M} \otimes \mathbb{Z}[X]_n,$$

where $\mathbb{Z}[X]_n$ is the abelian group of weighted-homogeneous degree n polynomials over $\mathbb{Z}$ in the variables $\{x_i, i \in I\}$, $\deg x_i = d_i$. Moreover, for each n, we have a canonical isomorphism in $\mathcal{SH}(S)$,

$$s_n U\mathcal{M} \cong \Sigma^n_T s_0 U\mathcal{M} \otimes \mathbb{Z}[X]_n.$$

Proof. Replacing $\mathcal{M}$ with a cofibrant model, we may assume that $\mathcal{M}$ is cofibrant in $\mathcal{C}$; as $\mathcal{R}$ is cofibrant in Mot, it follows that $U\mathcal{M}$ is cofibrant in Mot. Since $\pi_{\mathcal{M}} = U(\pi^{\mathcal{R}}_{\mathcal{M}})$, our assumption on $\pi_{\mathcal{M}}$ is the same as assuming that $\pi^{\mathcal{R}}_{\mathcal{M}}$ is an isomorphism in $\mathbf{Ho}\,\mathcal{C}$. By construction, $\pi^{\mathcal{R}}_{\mathcal{M}}$ extends to a map of distinguished triangles

$$\begin{array}{ccccccc}
(\operatorname{hocolim}_{\mathcal{I}^\circ} \mathcal{D}_x) \otimes \mathcal{M} & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{M}/(\{x_i\}) & \longrightarrow & \Sigma(\operatorname{hocolim}_{\mathcal{I}^\circ} \mathcal{D}_x) \otimes \mathcal{M} \\
\downarrow{\scriptstyle\alpha} & & \| & & \downarrow{\scriptstyle\pi^{\mathcal{R}}_{\mathcal{M}}} & & \downarrow{\scriptstyle\Sigma\alpha} \\
f_1^{\mathcal{R}}\mathcal{M} & \longrightarrow & \mathcal{M} & \longrightarrow & s_0^{\mathcal{R}}\mathcal{M} & \longrightarrow & \Sigma f_1^{\mathcal{R}}\mathcal{M},
\end{array}$$

and thus the map α is an isomorphism. We note that α is equal to the canonical map given by the universal property of $f_1^{\mathcal{R}}\mathcal{M} \to \mathcal{M}$.
We will now identify $f_n^{\mathcal{R}}\mathcal{M}$ in terms of the diagram $\mathcal{D}_x|_{\mathcal{I}_{\deg\geq n}} \otimes \mathcal{M}$, proving by induction on $n \geq 1$ that the canonical map $\mathrm{hocolim}\,\mathcal{D}_x \otimes \mathcal{M}|_{\deg\geq n} \to f_n^{\mathcal{R}}\mathcal{M}$ in $\mathbf{Ho}\,\mathcal{C}$ is an isomorphism.
As $\mathcal{I}^\circ = \mathcal{I}_{\deg\geq 1}$, the case $n = 1$ is settled. Assume the result for n. We claim that the diagram

$$\tilde{f}_{n+1}^{\mathcal{R}}[\mathcal{D}_x \otimes \mathcal{M}|_{\deg\geq n}] : \mathcal{I}_{\deg\geq n} \to \mathcal{C}$$

satisfies the hypotheses of proposition 1.10. That is, we need to verify that for every monomial M of degree n the natural map

$$\mathrm{hocolim}\,\tilde{f}_{n+1}^{\mathcal{R}}[\mathcal{D}_{>M} \otimes \mathcal{M}] \to \tilde{f}_{n+1}^{\mathcal{R}}[\mathcal{D}(M) \otimes \mathcal{M}]$$

is a weak equivalence in $\mathcal{C}$. This follows by the string of isomorphisms in $\mathbf{Ho}\,\mathcal{C}$

$$\begin{aligned}
\mathrm{hocolim}\,\tilde{f}_{n+1}^{\mathcal{R}}[\mathcal{D}_{>M} \otimes \mathcal{M}] &\cong \mathrm{hocolim}\,\tilde{f}_{n+1}^{\mathcal{R}}[\Sigma_T^n\mathcal{D}_{\deg\geq 1} \otimes \mathcal{M}]\\
&\cong \mathrm{hocolim}\,\Sigma_T^n\tilde{f}_1^{\mathcal{R}}[\mathcal{D}_{\deg\geq 1} \otimes \mathcal{M}]\\
&\cong \Sigma_T^n f_1^{\mathcal{R}}\,\mathrm{hocolim}[\mathcal{D}_{\deg\geq 1} \otimes \mathcal{M}]\\
&\cong \Sigma_T^n f_1^{\mathcal{R}} f_1^{\mathcal{R}}\mathcal{M}\\
&\cong \Sigma_T^n f_1^{\mathcal{R}}\mathcal{M}\\
&\cong f_{n+1}^{\mathcal{R}}\Sigma_T^n\mathcal{M}\\
&\cong f_{n+1}^{\mathcal{R}}[\mathcal{D}(M) \otimes \mathcal{M}].
\end{aligned}$$

Applying proposition 1.10 and our induction hypothesis gives us the string of isomorphisms in $\mathbf{Ho}\,\mathcal{C}$

$$\begin{aligned}
f_{n+1}^{\mathcal{R}}\mathcal{M} &\cong f_{n+1}^{\mathcal{R}} f_n^{\mathcal{R}}\mathcal{M} \cong f_{n+1}^{\mathcal{R}}\,\mathrm{hocolim}[\mathcal{D}_x \otimes \mathcal{M}|_{\deg\geq n}]\\
&\cong \mathrm{hocolim}\,\tilde{f}_{n+1}^{\mathcal{R}}[\mathcal{D}_x \otimes \mathcal{M}|_{\deg\geq n}] \cong \mathrm{hocolim}\,\tilde{f}_{n+1}^{\mathcal{R}}[\mathcal{D}_x \otimes \mathcal{M}|_{\deg\geq n+1}]\\
&\qquad\cong \mathrm{hocolim}\,\mathcal{D}_x \otimes \mathcal{M}|_{\deg\geq n+1},
\end{aligned}$$

the last isomorphism following from the fact that $\mathcal{D}_x(x^N) \otimes \mathcal{M}$ is in $\Sigma_T^{|N|}\mathcal{C}^{eff}$, and hence the canonical map $\tilde{f}_{n+1}^{\mathcal{R}}[\mathcal{D}_x \otimes \mathcal{M}] \to \mathcal{D}_x \otimes \mathcal{M}$ is an objectwise weak equivalence on $\mathcal{I}_{\deg\geq n+1}$.
For the slices s_n we have

$$\begin{aligned}
s_n^{\mathcal{R}}\mathcal{M} &:= \mathrm{hocofib}(\tilde{f}_{n+1}^{\mathcal{R}}\mathcal{M} \to \tilde{f}_n^{\mathcal{R}}\mathcal{M}) \cong \mathrm{hocofib}(\tilde{f}_{n+1}^{\mathcal{R}}\tilde{f}_n^{\mathcal{R}}\mathcal{M} \to \tilde{f}_n^{\mathcal{R}}\mathcal{M})\\
&\cong \mathrm{hocofib}(\mathrm{hocolim}\,\tilde{f}_{n+1}^{\mathcal{R}}[\mathcal{D}_{\deg\geq n} \otimes \mathcal{M}] \to \mathrm{hocolim}\,\mathcal{D}_{\deg\geq n} \otimes \mathcal{M})\\
&\cong \mathrm{hocolim}\,\mathrm{hocofib}(\tilde{f}_{n+1}^{\mathcal{R}}[\mathcal{D}_{\deg\geq n} \otimes \mathcal{M}] \to \mathcal{D}_{\deg\geq n} \otimes \mathcal{M}).
\end{aligned}$$

At a monomial of degree greater than n, the canonical map $\tilde{f}_{n+1}^{\mathcal{R}}[\mathcal{D}_{\deg\geq n} \otimes \mathcal{M}] \to \mathcal{D}_{\deg\geq n} \otimes \mathcal{M}$ is a weak equivalence, and at a monomial M of degree n

the homotopy cofiber is given by

$$\begin{aligned}\mathrm{hocofib}(\tilde{f}^{\mathcal{R}}_{n+1}[\mathcal{D}(M)\otimes\mathcal{M}]\to\mathcal{D}(M)\otimes\mathcal{M}) &= \mathrm{hocofib}((\tilde{f}^{\mathcal{R}}_{n+1}[\Sigma^n_T\mathcal{M}]\to\Sigma^n_T\mathcal{M})\\ &\cong \mathrm{hocofib}(\Sigma^n_T\tilde{f}^{\mathcal{R}}_1\mathcal{M}\to\Sigma^n_T\mathcal{M})\cong\Sigma^n_T s^{\mathcal{R}}_0\mathcal{M}\end{aligned}$$

Let $\tilde{s}^{\mathcal{R}}_0$ be the functor on $\mathcal{C}^{eff}$, $\mathcal{N}\mapsto\mathrm{hocofib}(\tilde{f}^{\mathcal{R}}_1\mathcal{N}\to\mathcal{N})$, and let $F_n\mathcal{M}:\mathcal{I}_{\deg\geq n}\to\mathcal{C}^{eff}$ be the diagram

$$F_n(M)=\begin{cases}pt & \text{for }\deg M>n\\ \Sigma^n_T\tilde{s}^{\mathcal{R}}_0\mathcal{M} & \text{for }\deg M=n.\end{cases}$$

We thus have a weak equivalence of pointwise cofibrant functors

$$\mathrm{hocofib}(\tilde{f}^{\mathcal{R}}_{n+1}[\mathcal{D}_{\deg\geq n}\otimes\mathcal{M}]\to\mathcal{D}_{\deg\geq n}\otimes\mathcal{M})\to F_n:\mathcal{I}_{\deg\geq n}\to\mathcal{C},$$

and therefore a weak equivalence on the homotopy colimits. As we have the evident isomorphism in $\mathbf{Ho}\,\mathcal{C}$

$$\operatorname*{hocolim}_{\mathcal{I}_{\deg\geq n}}F_n\cong\oplus_{M,\deg M=n}\Sigma^n_T s^{\mathcal{R}}_0\mathcal{M},$$

this gives us the desired isomorphism $s^{\mathcal{R}}_n\mathcal{M}\cong\Sigma^n_T s^{\mathcal{R}}_0\mathcal{M}\otimes\mathbb{Z}[X]_n$ in $\mathbf{Ho}\,\mathcal{C}$. Applying the forgetful functor and using lemma 2.1 gives the isomorphism $s_nU\mathcal{M}\cong\Sigma^n_T s_0U\mathcal{M}\otimes\mathbb{Z}[X]_n$ in $\mathcal{SH}(S)$. □

COROLLARY 2.4. *Let $\mathcal{R}$, X and $\mathcal{M}$ be as in theorem 2.3. Let $Z=\{z_j\in\mathbb{Z}[X]_{e_j}\}$ be a collection of homogeneous elements of $\mathbb{Z}[X]$, and let $\mathcal{M}[Z^{-1}]\in\mathcal{C}$ be the localization of $\mathcal{M}$ with respect to the collection of maps $\times z_j:\mathcal{M}\to\Sigma^{-e_j}_T\mathcal{M}$. Then there are natural isomorphisms*

$$s^{\mathcal{R}}_n\mathcal{M}[Z^{-1}]\cong\Sigma^n_T s^{\mathcal{R}}_0\mathcal{M}\otimes\mathbb{Z}[X][Z^{-1}]_n,$$

$$s_nU\mathcal{M}[Z^{-1}]\cong\Sigma^n_T s_0U\mathcal{M}\otimes\mathbb{Z}[X][Z^{-1}]_n.$$

Proof. Each map $\times z_j:\mathcal{M}\to\Sigma^{-e_j}_T\mathcal{M}$ induces the isomorphism $\times z_j:\mathcal{M}[Z^{-1}]\to\Sigma^{-e_j}_T\mathcal{M}[Z^{-1}]$ in $\mathbf{Ho}\,\mathcal{C}$, with inverse $\times z_j^{-1}:\Sigma^{-e_j}_T\mathcal{M}[Z^{-1}]\to\mathcal{M}[Z^{-1}]$. Applying $f^{\mathcal{R}}_q$ gives us the map in $\mathbf{Ho}\,\mathcal{C}$

$$\times z_j: f^{\mathcal{R}}_q\mathcal{M}\to f^{\mathcal{R}}_q\Sigma^{-e_j}_T\mathcal{M}\cong\Sigma^{-e_j}_T f^{\mathcal{R}}_{q+e_j}\mathcal{M}.$$

As $f^{\mathcal{R}}_{q+e_j}\mathcal{M}$ is in $\Sigma^{q+e_j}_T\mathbf{Ho}\,\mathcal{C}^{eff}$, both $\Sigma^{-e_j}_T f^{\mathcal{R}}_{q+e_j}\mathcal{M}$ and $f^{\mathcal{R}}_q\mathcal{M}$ are in $\Sigma^q_T\mathbf{Ho}\,\mathcal{C}^{eff}$. The composition

$$\Sigma^{-e_j}_T f^{\mathcal{R}}_{q+e_j}\mathcal{M}\to\Sigma^{-e_j}_T\mathcal{M}\xrightarrow{\times z_j^{-1}}\mathcal{M}[Z^{-1}]$$

gives via the universal property of $f^{\mathcal{R}}_q$ the map $\Sigma^{-e_j}_T f^{\mathcal{R}}_{q+e_j}\mathcal{M}\to f^{\mathcal{R}}_q\mathcal{M}[Z^{-1}]$. Setting $|N|=\sum_j N_je_j$, this extends to give a map of the system of monomial multiplications

$$\times z^{N-M}:\Sigma^{-|N|}_T f^{\mathcal{R}}_{q+|N|}\mathcal{M}\to\Sigma^{-|M|}_T f^{\mathcal{R}}_{q+|M|}\mathcal{M}$$

to $f_q^{\mathcal{R}}\mathcal{M}[Z^{-1}]$; the universal property of the truncation functors f_n and of localization shows that this system induces an isomorphism

$$\operatorname*{hocolim}_{N\in\mathcal{I}^{\mathrm{op}}}\Sigma_T^{-|N|}f^{\mathcal{R}}_{q+|N|}\mathcal{M}\cong f_q^{\mathcal{R}}\mathcal{M}[Z^{-1}]$$

in $\mathbf{Ho}\,\mathcal{C}$. As the slice functors s_q are exact and commute with hocolim, we have a similar collection of isomorphisms

$$\operatorname*{hocolim}_{N\in\mathcal{I}^{\mathrm{op}}}\Sigma_T^{-|N|}s^{\mathcal{R}}_{q+|N|}\mathcal{M}\cong s_q(\mathcal{M}[Z^{-1}]).$$

Theorem 2.3 gives us the natural isomorphisms

$$\Sigma_T^{-|N|}s^{\mathcal{R}}_{q+|N|}\mathcal{M}\cong \Sigma^q_Ts_0^{\mathcal{R}}\mathcal{M}\otimes\mathbb{Z}[X]_{q+|N|};$$

via this isomorphism, the map $\times z_j$ goes over to $\mathrm{id}_{\Sigma^q_Ts_0^{\mathcal{R}}\mathcal{M}}\otimes\times z_j$, which yields the result. □

Corollary 2.5. *Let $\mathcal{R}$, X and $\mathcal{M}$ be as in theorem 2.3. Let $Z=\{z_j\in\mathbb{Z}[X]_{e_j}\}$ be a collection of homogeneous elements of $\mathbb{Z}[X]$, and let $\mathcal{M}[Z^{-1}]\in\mathcal{C}$ be the localization of $\mathcal{M}$ with respect to the collection of maps $\times z_j:\mathcal{M}\to\Sigma_T^{-e_j}\mathcal{M}$. Let $m\geq 2$ be an integer. We let $\mathcal{M}[Z^{-1}]/m:=\mathrm{hocofib}\times m:\mathcal{M}[Z^{-1}]\to\mathcal{M}[Z^{-1}]$. Then there are natural isomorphisms*

$$s_n^{\mathcal{R}}\mathcal{M}[Z^{-1}]/m\cong\Sigma^n_Ts_0^{\mathcal{R}}\mathcal{M}/m\otimes\mathbb{Z}[X][Z^{-1}]_n,$$

$$s_nU\mathcal{M}[Z^{-1}]/m\cong\Sigma^n_Ts_0U\mathcal{M}/m\otimes\mathbb{Z}[X][Z^{-1}]_n.$$

This follows directly from corollary 2.4, noting that $s_n^{\mathcal{R}}$ and s_n are exact functors.

Remark 2.6. Let P be a multiplicatively closed subset of $\mathbb{Z}$. We may replace Mot with its localization $Mot[P^{-1}]$ with respect to P in theorem 2.3, corollary 2.4 and corollary 2.5, and obtain a corresponding description of $s_n^{\mathcal{R}}\mathcal{M}$ and $s_nU\mathcal{M}$ for a commutative monoid $\mathcal{R}$ in $Mot[P^{-1}]$ and an effective $\mathcal{R}$-module $\mathcal{M}$.
For $P=\mathbb{Z}\setminus\{p^n,n=1,2,\ldots\}$, we write $Mot\otimes\mathbb{Z}_{(p)}$ for $Mot[P^{-1}]$ and $\mathcal{SH}(S)\otimes\mathbb{Z}_{(p)}$ for $\mathbf{Ho}\,Mot\otimes\mathbb{Z}_{(p)}$.

3. The slice spectral sequence

The slice tower in $\mathcal{SH}(S)$ gives us the slice spectral sequence, for $\mathcal{E}\in\mathcal{SH}(S)$, $X\in\mathbf{Sm}/S$, $n\in\mathbb{Z}$,

$$E_2^{p,q}(n):=(s_{-q}(\mathcal{E}))^{p+q,n}(X)\Longrightarrow\mathcal{E}^{p+q,n}(X). \tag{3.1}$$

This spectral sequence is not always convergent, however, we do have a convergence criterion:

Lemma 3.1 ([L15, lemma 2.1]). *Suppose that $S=\mathrm{Spec}\,k$, k a perfect field. Take $\mathcal{E}\in\mathcal{SH}(S)$. Suppose that there is a non-decreasing function $f:\mathbb{Z}\to\mathbb{Z}$*

with $\lim_{n\to\infty} f(n) = \infty$*, such that* $\pi_{a+b,b}\mathcal{E} = 0$ *for* $a \leq f(b)$*. Then the for all* Y*, and all* $n \in \mathbb{Z}$*, the spectral sequence* (3.1) *is strongly convergent.*[3]

This yields our first convergence result. For $\mathcal{E} \in \mathcal{SH}(S)$, $Y \in \mathbf{Sm}/S$, $p, q, n \in \mathbb{Z}$, define

$$H^{p-q}(Y, \pi^{\mu}_{-q}(\mathcal{E})(n-q)) := \mathrm{Hom}_{\mathcal{SH}(S)}(\Sigma^{\infty}_T Y_+, \Sigma^{p+q,n} s_{-q}(\mathcal{E})).$$

Here $\Sigma^{a,b}$ is suspension with respect to the sphere $S^{a,b} \cong S^{a-b} \wedge \mathbb{G}_m^{\wedge b}$. This notation is justified by the case $S = \mathrm{Spec}\, k$, k a field of characteristic zero. In this case, there is for each q a canonically defined object $\pi^{\mu}_q(\mathcal{E})$ of Voevodsky's "big" triangulated category of motives $DM(k)$, and a canonical isomorphism

$$EM_{\mathbb{A}^1}(\pi^{\mu}_q(\mathcal{E})) \cong \Sigma^q_T s_q(\mathcal{E}),$$

where $EM_{\mathbb{A}^1} : DM(k) \to \mathcal{SH}(k)$ is the motivic Eilenberg-MacLane functor. The adjoint property of $EM_{\mathbb{A}^1}$ yields the isomorphism

$$\begin{aligned} H^{p-q}(Y, \pi^{\mu}_{-q}(\mathcal{E})(n-q)) &:= \mathrm{Hom}_{DM(k)}(M(Y), \pi^{\mu}_{-q}(\mathcal{E})(n-q)[p-q]) \\ &\cong \mathrm{Hom}_{\mathcal{SH}(S)}(\Sigma^{\infty}_T Y_+, \Sigma^{p+q,n} s_{-q}(\mathcal{E})). \end{aligned}$$

We refer the reader to [P11, RO08, Vo04] for details.

PROPOSITION 3.2. *Let* $\mathcal{R}$ *be a commutative monoid in* $Mot(S)$*, cofibrant as an object in* $Mot(S)$*, with* $\mathcal{R}$ *in* $\mathcal{SH}^{eff}(S)$*. Let* $X := \{\bar{x}_i \in \mathcal{R}^{-2d_i,-d_i}(S)\}$ *be a countable set of elements of* $\mathcal{R}$*-cohomology, with* $d_i > 0$*. Let* P *be a multiplicatively closed subset of* $\mathbb{Z}$ *and let* $\mathcal{M}$ *be an* $\mathcal{R}[P^{-1}]$*-module, with* $U\mathcal{M} \in \mathcal{SH}(S)^{eff}[P^{-1}]$*. Suppose that the canonical map*

$$U(\mathcal{M}/(\{x_i\})) \to s_0 U\mathcal{M}$$

is an isomorphism in $\mathcal{SH}(S)[P^{-1}]$*. Then*
1. The slice spectral sequence for $\mathcal{M}^{**}(Y)$ *has the following form:*

$$E_2^{p,q}(n) := H^{p-q}(Y, \pi^{\mu}_0(\mathcal{M})(n-q)) \otimes_{\mathbb{Z}} \mathbb{Z}[X]_{-q} \Longrightarrow \mathcal{M}^{p+q,n}(Y).$$

2. Suppose that $S = \mathrm{Spec}\, k$*,* k *a perfect field. Suppose further that there is an integer* a *such that* $\mathcal{M}^{2r+s,r}(Y) = 0$ *for all* $Y \in \mathbf{Sm}/S$*, all* $r \in \mathbb{Z}$ *and all* $s \geq a$*. Then the slice spectral sequence converges strongly for all* $Y \in \mathbf{Sm}/S$*,* $n \in \mathbb{Z}$*.*

Proof. The form of the slice spectral sequence follows directly from theorem 2.3, extended via remark 2.6 to the P-localized situation. The convergence statement follows directly from lemma 3.1, where one uses the function $f(r) = r - a$. □

We may extend the slice spectral sequence to the localizations $\mathcal{M}[Z^{-1}]$ as in corollary 2.4.

[3] As spectral sequence $\{E_r^{pq}\} \Rightarrow G^{p+q}$ *converges strongly* to G^* if for each n, the spectral sequence filtration F^*G^n on G^n is finite and exhaustive, there is an $r(n)$ such that for all p and all $r \geq r(n)$, all differentials entering and leaving $E_r^{p,n-p}$ are zero and the resulting maps $E_r^{p,n-p} \to E_\infty^{p,n-p} = \mathbf{Gr}^p_F G^n$ are all isomorphisms.

Proposition 3.3. *Let $\mathcal{R}$, X, P and $\mathcal{M}$ be as in proposition 3.3 and assume that all the hypotheses for (1) in that proposition hold. Let $Z = \{z_j \in \mathbb{Z}[X]_{e_j}\}$ be a collection of homogeneous elements of $\mathbb{Z}[X]$, and let $\mathcal{M}[Z^{-1}] \in \mathcal{C}$ be the localization of $\mathcal{M}$ with respect to the collection of maps $\times z_j : \mathcal{M} \to \Sigma_T^{-e_j}\mathcal{M}$. Then the slice spectral sequence for $\mathcal{M}[Z^{-1}]^{**}(Y)$ has the following form:*

$$E_2^{p,q}(n) := H^{p-q}(Y, \pi_0^\mu(\mathcal{M})(n-q)) \otimes_{\mathbb{Z}} \mathbb{Z}[X][Z^{-1}]_{-q} \Longrightarrow \mathcal{M}[Z^{-1}]^{p+q,n}(Y).$$

Suppose further that $S = \operatorname{Spec} k$, k a perfect field, and there is an integer a such that $\mathcal{M}^{2r+s,r}(Y) = 0$ for all $Y \in \mathbf{Sm}/S$ all $r \in \mathbb{Z}$ and all $s \geq a$. Then the slice spectral sequence converges strongly for all $Y \in \mathbf{Sm}/S$, $n \in \mathbb{Z}$.

The proof is same as for proposition 3.2, using corollary 2.4 to compute the slices of $\mathcal{M}[Z^{-1}]$.

Remark 3.4. Let $\mathcal{R}$ be a commutative monoid in Mot, with $\mathcal{R} \in \mathcal{SH}^{eff}(S)$. Suppose that there are elements $a_i \in \mathcal{R}^{2f_i,f_i}(S)$, $i = 1, 2, \ldots$, $f_i \leq 0$, so that $\mathcal{M}$ is the quotient module $\mathcal{R}/(\{a_i\})$. Suppose in addition that there is a constant c such that $\mathcal{R}^{2r+s,r}(Y) = 0$ for all $Y \in \mathbf{Sm}/S$, $r \in \mathbb{Z}$, $s \geq c$. Then $\mathcal{M}^{2r+s,r}(Y) = 0$ for all $Y \in \mathbf{Sm}/S$, $r \in \mathbb{Z}$, $s \geq c$. Indeed

$$\mathcal{M} := \operatorname*{hocolim}_n \mathcal{R}/(a_1, a_2, \ldots, a_n),$$

so it suffices to handle the case $\mathcal{M} = \mathcal{R}/(a_1, a_2, \ldots, a_n)$, for which we may use induction in n. Assuming the result for $\mathcal{N} := \mathcal{R}/(a_1, a_2, \ldots, a_{n-1})$, we have the long exact sequence ($f = f_n$)

$$\ldots \to \mathcal{N}^{p+2f,q+f}(Y) \xrightarrow{\times a_n} \mathcal{N}^{p,q}(Y) \to \mathcal{M}^{p,q}(Y) \to \mathcal{N}^{p+2f+1,q+f}(Y) \to \ldots .$$

Thus the assumption for $\mathcal{N}$ implies the result for $\mathcal{M}$ and the induction goes through.

4. Slices of quotients of MGL

The slices of a Landweber exact spectrum have been described by Spitzweck in [S12, S10], but a quotient of MGL or a localization of such is often not Landweber exact. We will apply the results of the previous section to describe the slices of the motivic truncated Brown-Peterson spectra $BP\langle n\rangle$, effective motivic Morava K-theory $k(n)$ and motivic Morava K-theory $K(n)$, as well as recovering the known computations for the Landweber examples [S12], such as the Brown-Peterson spectra BP and the Johnson-Wilson spectra $E(n)$.

Let MGL_p be the commutative monoid in $Mot \otimes \mathbb{Z}_{(p)}$ representing p-local algebraic cobordism, as constructed in [PPR, §2.1][4]. As noted in *loc. cit.*, MGL_p is a cofibrant object of $Mot \otimes \mathbb{Z}_{(p)}$. The motivic BP was first constructed by Vezzosi in [Ve01] as a direct summand of MGL_p by using Quillen's idempotent theorem. Here we construct BP and $BP\langle n\rangle$ as quotients of MGL_p; the effective Morava K-theory $k(n)$ is similarly a quotient of MGL_p/p. Our explicit

[4]This gives MGL as a symmetric spectrum, we take the image in the p-localized model structure to define MGL_p.

description of the slices allows us to describe the E_2-terms of slice spectral sequences for BP and $BP\langle n\rangle$.
The bigraded coefficient ring $\pi_{*,*}MGL_p(S)$ contains $\pi_{2*}MU \simeq \mathbb{L}_*$, localized at p, as a graded subring of the bi-degree $(2*,*)$ part, via the classifying map for the formal group law of MGL; see for example [Hoy, remark 6.3]. The ring $\mathbb{L}_{*p} := \mathbb{L}_* \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ is isomorphic to polynomial ring $\mathbb{Z}_{(p)}[x_1, x_2, \cdots]$ [A95, Part II, theorem 7.1], where the element x_i has degree $2i$ in π_*MU, degree $(2i, i)$ in $\pi_{*,*}MGL_p$ and degree i in $\mathbb{L}_*$.
The following result of Hopkins-Morel-Hoyois [Hoy] is crucial for the application of the general results of the previous sections to quotients of MGL and MGL_p.

Theorem 4.1 ([Hoy, theorem 7.12]). *Let p be a prime integer, S an essentially smooth scheme over a field of characteristic prime to p. Then the canonical maps $MGL_p/(\{x_i : i = 1, 2, \ldots\}) \to s_0MGL_p \to H\mathbb{Z}_{(p)}$ are isomorphisms in $\mathcal{SH}(S)$. In case $S = \operatorname{Spec} k$, k a perfect field of characteristic prime to p, the inclusion $\mathbb{L}_{*p} \subset \pi_{2*,*}MGL_p(S)$ is an equality.*

This has been extended by Spitzweck. He has constructed [S13] a motivic Eilenberg-MacLane spectrum $H\mathbb{Z}$ in $\mathbf{Spt}_{\mathbb{P}^1}(X)$ with a highly structured multiplication, for an arbitrary base-scheme X. For X smooth and of finite type over a Dedekind domain, $H\mathbb{Z}$ represents motivic cohomology defined as Bloch's higher Chow groups [Vo02]; this theory agrees with Voevodsky's motivic cohomology for smooth schemes of finite type over a perfect field. In addition, Spitzweck has extended theorem 4.1 to an arbitrary base-scheme.

Theorem 4.2 ([S13, theorem 11.3], [S14, corollary 6.6]). *Let p be a prime integer and let S be a scheme whose positive residue characteristics are all prime to p. Then the canonical maps $MGL_p/(\{x_i : i = 1, 2, \ldots\}) \to s_0MGL_p \to H\mathbb{Z}_{(p)}$ are isomorphisms in $\mathcal{SH}(S)$. In case $S = \operatorname{Spec} A$, A a Dedekind domain with all residue characteristics prime to p and with trivial class group, the inclusion $\mathbb{L}_{*p} \subset \pi_{2*,*}MGL_p(S)$ is an equality.*

We define a series of subsets of the set of generators $\{x_i \mid i = 1, 2 \ldots\}$,

$$\begin{aligned}
B_p^c &= \{x_i : i \neq p^k - 1, k \geq 1\},\\
B_p &= \{x_i : i = p^k - 1, k \geq 1\},\\
B\langle n\rangle_p^c &= \{x_i : i \neq p^k - 1, 1 \leq k \leq n\},\\
B\langle n\rangle_p &= \{x_i : i = p^k - 1, 1 \leq k \leq n\},\\
k\langle n\rangle_p &= \{x_{p^n-1}\}.
\end{aligned}$$

We also define

$$k\langle n\rangle_p^c = \{x_i : i \neq p^n - 1, \text{and } x_0 = p\} \subset \{p, x_i \mid i = 1, 2 \ldots\}.$$

Definition 4.3 (BP, $BP\langle n\rangle$ and $E(n)$). The Brown-Peterson spectrum BP is defined as

$$BP := MGL_p/(\{x_i \mid i \in B_p^c\}),$$

the truncated Brown-Peterson spectrum $BP\langle n\rangle$ is defined as

$$BP\langle n\rangle := MGL_p/(\{x_i \mid i \in B\langle n\rangle_p^c\})$$

and the Johnson-Wilson spectrum $E(n)$ is the localization

$$E(n) := BP\langle n\rangle[x_{p^n-1}^{-1}].$$

Definition 4.4 (Morava K-theories $k(n)$ and $K(n)$). *Effective Morava K-theory $k(n)$ is defined as*

$$k(n) := MGL_p/(\{x_i \mid i \in k\langle n\rangle_p^c\}) \cong BP\langle n\rangle/(x_{p-1}, \ldots, x_{p^{n-1}-1}, p).$$

Define Morava K-theory $K(n)$ to be the localization

$$K(n) := k(n)[x_{p^n-1}^{-1}].$$

The spectra $BP, BP\langle n\rangle$, $E(n), k(n)$ and $K(n)$ are MGL_p-modules. BP and $E(n)$ are Landweber exact. We let $\mathcal{C}$ denote the category of MGL_p-modules.

Lemma 4.5. *The MGL_p-module spectra $BP, BP\langle n\rangle$ and $k(n)$ are effective. BP and $E(n)$ have the structure of oriented weak commutative ring T-spectra in $\mathcal{SH}(S)$.*

Proof. The effectivity of these theories follows from lemma 2.2 and the fact that homotopy colimits of effective spectra are effective. The ring structure for BP and $E(n)$ follows from the Landweber exactness (see [NSO09]). □

We first discuss the effective theories $BP, BP\langle n\rangle$ and $k(n)$.

Proposition 4.6. *Let p be a prime and S a scheme with all residue characteristics prime to p. Then in $\mathcal{SH}(S)$:*
1. The zeroth slices of both BP and $BP\langle n\rangle$ are isomorphic to p-local motivic Eilenberg-MacLane spectrum $H\mathbb{Z}_{(p)}$, and the zeroth slice of $k(n)$ is isomorphic to $H\mathbb{Z}/p$.
2. The quotient maps from MGL_p induce isomorphisms

$$s_0BP \simeq (s_0MGL)_p \simeq s_0BP\langle n\rangle,$$
$$s_0k(n) \simeq (s_0MGL)_p/p.$$

3. The respective quotient maps from BP, $BP\langle n\rangle$ and $k(n)$ induce isomorphisms

$$BP/(\{x_i : x_i \in B_p\}) \simeq s_0BP,$$
$$BP\langle n\rangle/(\{x_i : x_i \in B\langle n\rangle_p\}) \simeq s_0BP\langle n\rangle,$$
$$k(n)/(x_{p^n-1}) \simeq s_0k(n).$$

Proof. By theorem 4.1 (in case S is essentially smooth over a field) or theorem 4.2 (for general S), the classifying map $MGL \to H\mathbb{Z}$ for motivic cohomology induces isomorphisms

$$MGL_p/(\{x_i : i = 1, 2, \ldots\}) \cong s_0MGL_p \cong H\mathbb{Z}_{(p)}$$

in $\mathcal{SH}(S) \otimes \mathbb{Z}_{(p)}$.

Now let $\mathcal{S} \subset \mathbb{N}$ be a subset and $\mathcal{S}^c$ its complement. By remark 1.5, we have an isomorphism

$$(MGL_p/(\{x_i : i \in \mathcal{S}^c\}))/(\{x_i : i \in \mathcal{S}\}) \cong MGL_p/(\{x_i : i \in \mathbb{N}\}).$$

Also, as x_i is a map $\Sigma^{2i,i} MGL_p \to MGL_p$, $i > 0$, the quotient map $MGL_p \to MGL_p/(\{x_i : i \in \mathcal{S}^c\})$ induces an isomorphism

$$s_0 MGL_p \to s_0[MGL_p/(\{x_i : i \in \mathcal{S}^c\})].$$

This gives us isomorphisms

$$(MGL_p/(\{x_i : i \in \mathcal{S}^c\}))/(\{x_i : i \in \mathcal{S}\}) \cong \\ \cong s_0[MGL_p/(\{x_i : i \in \mathcal{S}^c\})] \cong s_0 MGL_p,$$

with the first isomorphism induced by the quotient map

$$MGL_p/(\{x_i : i \in \mathcal{S}^c\} \to (MGL_p/(\{x_i : i \in \mathcal{S}^c\}))/(\{x_i : i \in \mathcal{S}\}).$$

Taking $\mathcal{S} = B_p, B\langle n\rangle_p, \{x_{p^n-1}\}$ proves the result for BP, $BP\langle n\rangle$ and $k(n)$, respectively. □

For motivic spectra $\mathcal{E} = BP,\ BP\langle n\rangle,\ k(n),\ E(n)$ and $K(n)$ defined in 4.3 and 4.4 let us denote the corresponding topological spectra by $\mathcal{E}^{top}$. The graded coefficient rings $\mathcal{E}^{top}_*$ of these topological spectra are

$$\mathcal{E}^{top}_* \simeq \left\{ \begin{array}{ll} \mathbb{Z}_p[v_1, v_2, \cdots] & \mathcal{E} = BP \\ \mathbb{Z}_p[v_1, v_2, \cdots, v_n] & \mathcal{E} = BP\langle n\rangle \\ \mathbb{Z}_p[v_1, v_2, \cdots, v_n, v_n^{-1}] & \mathcal{E} = E(n) \\ \mathbb{Z}/p[v_n] & \mathcal{E} = k(n) \\ \mathbb{Z}/p[v_n, v_n^{-1}] & \mathcal{E} = K(n) \end{array} \right\}$$

where $\deg v_n = 2(p^n - 1)$. The element v_n corresponds to the element $\bar{x}_n \in MGL^{2n,n}(k)$.

Corollary 4.7. *Let p be a prime integer and let S be a scheme whose positive residue characteristics are all prime to p. Then in $\mathcal{SH}(S)$, the slices of Brown-Peterson, Johnson-Wilson and Morava theories are given by*

$$s_i\mathcal{E} \simeq \left\{ \begin{array}{ll} \Sigma^i_T H_{\mathbb{Z}_p} \otimes \mathcal{E}^{top}_{2i} & \mathcal{E} = BP,\ BP\langle n\rangle \text{ and } E(n) \\ \Sigma^i_T H_{\mathbb{Z}/p} \otimes \mathcal{E}^{top}_{2i} & \mathcal{E} = k(n) \text{ and } K(n) \end{array} \right\}$$

where $\mathcal{E}^{top}_{2i}$ is degree $2i$ homogeneous component of coefficient ring of the corresponding topological theory.

Proof. The statement for BP and $BP\langle n\rangle$ follows from theorem 2.3, and remark 2.6. The case of $E(n)$ follows from corollary 2.4 and the cases of $k(n)$ and $K(n)$ follow from corollary 2.5. □

Theorem 4.8. *Let p be a prime integer and let S be a scheme whose positive residue characteristics are all prime to p. The slice spectral sequence for any of the spectra $\mathcal{E} = BP,\ BP\langle n\rangle,\ k(n),\ E(n)$ and $K(n)$ in $\mathcal{SH}(S)$ has the form*

$$\mathcal{E}^{p,q}_2(X, m) = H^{p-q}(X, \mathcal{Z}(m-q)) \otimes_{\mathbb{Z}} \mathcal{E}^{top}_{-2q} \Rightarrow \mathcal{E}^{p+q,m}(X),$$

where $\mathcal{Z} = \mathbb{Z}_p$ for $\mathcal{E} = BP$, $BP\langle n\rangle$ and $E(n)$, and $\mathcal{Z} = \mathbb{Z}/p$ for $\mathcal{E} = k(n)$ and $K(n)$. In case $S = \operatorname{Spec} k$ and k is perfect, these spectral sequences are all strongly convergent.

Proof. The form of the slice spectral sequence for $\mathcal{E}$ follows from corollary 4.7. The fact that the slice spectral sequences strongly converge for $S = \operatorname{Spec} k$, k perfect, follows from remark 3.4 and the fact that $MGL^{2r+s,r}(Y) = 0$ for all $Y \in \mathbf{Sm}/S$, $r \in \mathbb{Z}$ and $s \geq 1$. This in turn follows from the Hopkins-Morel-Hoyois spectral sequence

$$E_2^{p,q}(n) := H^{p-q}(Y, \mathbb{Z}(n-q)) \otimes \mathbb{L}_{-q} \Longrightarrow MGL^{p+q,n}(Y),$$

which is strongly convergent by [Hoy, theorem 8.12]. □

5. Modules for oriented theories

We will use the slice spectral sequence to compute the "geometric part" $\mathcal{E}^{2*,*}$ of a quotient spectrum $\mathcal{E} = MGL_p/(\{x_{i_j}\})$ in terms of algebraic cobordism, when working over a base field k of characteristic zero. As the quotient spectra are naturally MGL_p-modules but may not have a ring structure, we will need to extend the existing theory of oriented Borel-Moore homology and related structures to allow for modules over ring-based theories.

5.1. Oriented Borel-Moore homology. We first discuss the extension of oriented Borel-Moore homology. We use the notation and terminology of [LM09, §5]. Let $\mathbf{Sch}/k$ be the category of quasi-projective schemes over a field k and let $\mathbf{Sch}/k'$ denote the subcategory of projective morphisms in $\mathbf{Sch}/k$. Let $\mathbf{Ab}_*$ denote the category of graded abelian groups, $\mathbf{Ab}_{**}$ the category of bi-graded abelian groups.

Definition 5.1. Let A be an oriented Borel-Moore homology theory on $\mathbf{Sch}/k$ [LM09, definition 5.1.3]. An oriented A-module B is given by

(MD1) An additive functor $B_* : \mathbf{Sch}/k' \to \mathbf{Ab}_*$, $X \mapsto B_*(X)$.
(MD2) For each l. c. i. morphism $f : Y \to X$ in $\mathbf{Sch}/k$ of relative dimension d, a homomorphism of graded groups $f^* : B_*(X) \to B_{*+d}(Y)$.
(MD3) For each pair (X, Y) of objects in $\mathbf{Sch}/k$ a bilinear graded pairing

$$A_*(X) \otimes B_*(Y) \to B_*(X \times_k Y)$$
$$u \otimes v \mapsto u \times v$$

which is associative and unital with respect to the external products in the theory A.
These satisfy the conditions (BM1), (BM2), (PB) and (EH) of [LM09, definition 5.1.3]. In addition, these satisfy the following modification of (BM3).

(MBM3) Let $f : X' \to X$ and $g : Y' \to Y$ be morphisms in $\mathbf{Sch}/k$. If f and g are projective, then for $u' \in A_*(X')$, $v' \in B_*(Y')$, one has

$$(f \times g)_*(u' \times v') = f_*(u') \times g_*(v').$$

If f and g are l. c. i. morphisms, then for $u \in A_*(X)$, $v \in B_*(Y)$, one has

$$(f \times g)^*(u \times v) = f_*(u) \times g_*(v).$$

Let $f : A \to A'$ be a morphism of Borel-Moore homology theories, let B be an oriented A-module, B' an oriented A'-module. A morphism $g : B \to B'$ over f is a collection of homomorphisms of graded abelian groups $g_X : B_*(X) \to B'_*(X)$, $X \in \mathbf{Sch}/k$ such that the g_X are compatible with projective push-forward, l. c. i. pull-back and external products.

We do not require the analog of the axiom (CD) of [LM09, definition 5.1.3]; this axiom plays a role only in the proof of universality of Ω_*, whereas the universality of Ω for A-modules follows formally from the universality for Ω among oriented Borel-Moore homology theories (see proposition 5.3 below).

EXAMPLE 5.2. Let N_* be a graded module for the Lazard ring $\mathbb{L}_*$ and let A_* be an oriented Borel-Moore homology theory. Define $A_*^N(X) := A_*(X) \otimes_{\mathbb{L}_*} N_*$. Then with push-forward $f_*^N := f_*^A \otimes \mathrm{id}_{N_*}$, pull-back $f_N^* := f_A^* \otimes \mathrm{id}_{N_*}$, and product $u \times (v \otimes n) := (u \times v) \otimes n$, for $u \in A_*(X)$, $v \in A_*(Y)$, $n \in N_*$, A_*^N becomes an oriented A-module. Sending N_* to A_*^N gives a functor from graded $\mathbb{L}_*$-modules to oriented A-modules.
In case k has characteristic zero, we note that, for $A_* = \Omega_*$, we have a canonical isomorphism $\theta_{N_*} : \Omega_*^{N_*}(k) \cong N_*$, as the classifying map $\mathbb{L}_* \to \Omega_*(k)$ is an isomorphism [LM09, theorem 1.2.7].

Just as for a Borel-Moore homology theory, one can define operations of $A_*(Y)$ on $B_*(Z)$ via a morphism $f : Z \to Y$, assuming that Y is in $\mathbf{Sm}/k$: for $a \in A_*(Y)$, $b \in B_*(Z)$, define $a \cap_f b \in B_*(Z)$ by

$$a \cap_f b := (f, \mathrm{id}_Z)^*(a \times b),$$

where $(f, \mathrm{id}_Z) : Z \to Y \times_k Z$ is the (transpose of) the graph embedding. As Y is smooth over k, (f, id_Z) is an l. c. i. morphism, so the pullback $(f, \mathrm{id}_Z)^*$ is defined. Similarly, $B_*(Y)$ is an $A_*(Y)$-module via

$$a \cup_Y b := \delta_Y^*(a \times b).$$

These products satisfy the analog of the properties listed in [LM09, §5.1.4, proposition 5.2.1].

PROPOSITION 5.3. *Let A be an oriented Borel-Moore homology theory on $\mathbf{Sch}/k$ and let B be an oriented A-module. Let $\vartheta_A : \Omega_* \to A_*$ be the classifying map. There is a unique morphism $\theta_{A/B} : \Omega_*^{B_*(k)} \to B_*$ over ϑ_A such that $\theta_{A/B}(k) : \Omega_*^{B_*(k)}(k) \to B_*(k)$ is the canonical isomorphism $\theta_{B_*(k)}$.*

Proof. For $X \in \mathbf{Sch}/k$, $b \in B_*(k)$ and $u \in \Omega_*(X)$, we define $\theta_{A/B}(u \otimes b) := \vartheta_A(u) \times b \in B_*(X \times_k k) = B_*(X)$. It is easy to check that this defines a morphism over ϑ_A. Uniqueness follows easily from the fact that the product structure in A and Ω is unital. □

5.2. Oriented duality theories. Next, we discuss a theory of modules for an oriented duality theory (H, A). We use the notation and definitions from [L08]. In particular, we have the category $\mathbf{SP}$ of smooth pairs over k, with objects (M, X), $M \in \mathbf{Sm}/k$, $X \subset M$ a closed subset, and where a morphism $f : (M, X) \to (N, Y)$ is a morphism $f : M \to N$ in $\mathbf{Sm}/k$ such that $f^{-1}(Y) \subset X$.

Definition 5.4. Let A be a bi-graded oriented ring cohomology theory, in the sense of [L08, definition 1.5, remark 1.6]. An oriented A-module B is a bi-graded cohomology theory on $\mathbf{SP}$, satisfying the analog of [L08, definition 1.5], that is: for each pair of smooth pairs (M, X), (N, Y) there is a bi-graded homomorphism

$$\times : A^{**}_X(M) \otimes B^{**}_Y(N) \to B^{**}_{X\times Y}(M \times_k N)$$

satisfying

(1) *associativity*: $(a \times b) \times c = a \times (b \times c)$ for $a \in A^{**}_X(M)$, $b \in A^{**}_Y(N)$, $c \in B^{**}_Z(P)$.
(2) *unit*: $1 \times a = a$.
(3) *Leibniz rule*: Given smooth pairs (M, X), (M, X'), (N, Y) with $X \subset X'$ we have

$$\partial_{M\times N,X'\times N,X\times N}(a \times b) = \partial_{M,X',X}(a) \times b$$

for $a \in A^{**}_{X'\setminus X}(M \setminus X)$, $b \in B^{**}_Y(N)$. For a triple (N, Y', Y) with $Y \subset Y' \subset N$, $a \in A^{m,*}_{X'}(M)$, $b \in B^{**}_{Y'\setminus Y}(N \setminus Y)$ we have

$$\partial_{M\times N,M\times Y',M\times Y}(a \times b) = (-1)^m a \times \partial_{N,Y',Y}(b).$$

We write $a \cup b \in B_{X\cap Y}(M)$ for $\delta^*_M(a \times b)$, $a \in A^{**}_X(M)$, $b \in B^{**}_Y(M)$.
In addition, we assume that the "Thom classes theory" [P09, lemma 3.7.2] arising from the orientation on A induces an orientation on B in the following sense: Let (M, X) be a smooth pair and let $p : E \to M$ be a rank r vector bundle on M. Then the cup product with the Thom class $th(E) \in A^{2r,r}_M(E)$

$$B^{**}_X(M) \xrightarrow{p^*} B^{**}_{p^{-1}(X)}(E) \xrightarrow{th(E)\cup(-)} B^{2r+*,r+*}_X(E)$$

is an isomorphism.

We call an orientation on A that induces an orientation on B as above an orientation on (A, B), or just an orientation on B.
Given an orientation ω on A, one has 1st Chern classes in A for line bundles, where for $L \to M$ a line bundle over $M \in \mathbf{Sm}/k$ with zero section $s : M \to L$, one defines $c_1(L) \in A^{2,1}(X)$ as $s^*(th(L))$.
Let $\mathbf{SP}'$ be the category with the same objects (M, X) as in $\mathbf{SP}$, where a morphism $f : (M, X) \to (N, Y)$ is a projective morphism $f : M \to N$ such that $f(X) \subset Y$. One proceeds just as in [L08] to show that the orientation on B gives rise to an integration on B. To describe this more precisely, we first need to extend the notion of an *integration with support* [L08, definition 1.8] to the setting of bi-graded A-modules.

The discussion in [L08] is carried out in the setting of an ungraded cohomology theory; we modify this by introducing a bi-grading on the cohomology theory A as well as on the A-module B as above. An *integration with supports* for the pair (A, B) is defined by modifying the axioms of [L08, definition 1.8] as follows. We first discuss the modifications for A. The bi-grading is incorporated in that the pushforward map F_* associated to a morphisms $F : (M, X) \to (N, Y)$ in $\mathbf{SP}'$ has the form $F_* : A^{**}_X(M) \to A^{*-2d,*-d}_Y(N)$, where $d = \dim_k M - \dim_k N$. With this refinement, the remaining parts of definition 1.8 for A remain the same. For the module B, one requires as above that one has for each morphism $F : (M, X) \to (N, Y)$ in $\mathbf{SP}'$ a pushforward map $F_* : B^{**}_X(M) \to B^{*-2d,*-d}_Y(N)$. In addition, one modifies the multiplicative structure $f^*(-)\cup$ and $\cup$ for A in definition 1.8(2) of *loc. cit.* to bi-graded products

$$f^*(-)\cup : A^{**}_Z(M) \otimes B^{**}_Y(N) \to B^{**}_{Y\cap f^{-1}(Z)}(N)$$

and

$$\cup : A^{**}_Z(M) \otimes B^{**}_X(M) \to B_{X\cap Z}(M),$$

and, with these changes, we require that B satisfies the conditions of definition 1.8(2) of *loc. cit.* We call such a structure an *integration with supports on* (A, B).

Given an integration with supports on (A, B) and an orientation ω on (A, B) we say (as in [L08, definition 1.11]) that the integration with supports is subjected to ω if for each smooth pair (M, X) and each line bundle $p : L \to M$ with zero section $s : M \to L$, the compositions

$$A^{**}_X(M) \xrightarrow{s_*} A^{*-2,*-1}_{p^{-1}(X)}(L) \xrightarrow{s^*} A^{*-2,*-1}_X(M),$$

$$B^{**}_X(M) \xrightarrow{s_*} B^{*-2,*-1}_{p^{-1}(X)}(L) \xrightarrow{s^*} B^{*-2,*-1}_X(M)$$

are given by respective cup product with $c_1(L)$.

We have the analog of [L08, theorem 1.12] in the setting of oriented modules.

THEOREM 5.5. *Let A be a bi-graded ring cohomology theory with orientation ω and let B be an oriented A-module with orientation induced by ω. Then there is a unique integration with supports on (A, B) subjected to the orientation ω.*

The proof is exactly the same way as the proof of theorem 1.12 of *loc. cit.* We now extend the notion of an oriented duality theory to the setting of modules.

DEFINITION 5.6. Let (H, A) be an oriented duality theory, in the sense of [L08, definition 3.1]. An oriented (H, A)-module is a pair (J, B), where

(D1) $J : \mathbf{Sch}/k' \to \mathbf{Ab}_{**}$ is a functor.

(D2) B is an oriented A-module.

(D3) For each open immersion $j : U \to X$ there is a pullback map $j^* : J_{**}(X) \to J_{**}(U)$.

(D4) i. For each smooth pair (M, X) and each morphism $f : Y \to M$ in $\mathbf{Sch}/k$, there is a bi-graded cap product map

$$f^*(-)\cap : A_X(M) \otimes H(Y) \to H(f^{-1}(X)).$$

ii. For $X, Y \in \mathbf{Sch}/k$, there is a bi-graded external product

$$\times : H_{**}(X) \otimes J_{**}(Y) \to J_{**}(X \times Y).$$

(D5) For each smooth pair (M, X), there is a graded isomorphism

$$\beta_{M,X} : J_{**}(X) \to B_X^{2d-*,d-*}(M); \quad d = \dim_k M.$$

(D6) For each $X \in \mathbf{Sch}/k$ and each closed subset $Y \subset X$, there is a map

$$\partial_{X,Y} : J_{*+1,*}(X \setminus Y) \to J_{**}(Y).$$

These satisfy the evident analogs of properties (A1)-(A4) of [L08, definition 3.1], where we make the following changes: Let $d = \dim_k M$, $e = \dim_k N$. One replaces H with J_{**} throughout (except in (A3)(ii)), and

- in (A1) one replaces $A_Y(N)$, $A_X(M)$ with $B_Y^{2d-*,d-*}(N)$, $B_X^{2d-*,d-*}(M)$,
- in (A2) one replaces $A_Y(N)$, $A_X(M)$ with $B_Y^{2e-*,e-*}(N)$, $B_X^{2d-*,d-*}(M)$,
- in (A3)(i) one replaces $A_Y(M)$ with $B_Y^{2d-*,d-*}(M)$ and $A_{Y\cap f^{-1}(X)}(N)$ with $B_{Y\cap f^{-1}(X)}^{2e-*,e-*}(N)$,
- in (A3)(ii) one replaces $A_Y(M)$ with $B_Y^{2e-*,e-*}(N)$ and $A_{X\times Y}(M\times N)$ with $B_{X\times Y}^{2(d+e)-*,d+e-*}(M\times N)$, $H(X)$ with $H_{**}(X)$, $H(Y)$ with $J_{**}(Y)$ and $H(X\times Y)$ with $J_{**}(X\times Y)$,
- in (A4) one replaces $A_{X\setminus Y}(M\setminus Y)$ with $B_{X\setminus Y}^{2d-*,d-*}(M\setminus Y)$.

Remark 5.7. Let (H, A) be an oriented duality theory on $\mathbf{Sch}/k$, for k a field admitting resolution of singularities. By [L08, proposition 4.2] there is a unique natural transformation

$$\vartheta_H : \Omega_* \to H_{2*,*}$$

of functors $\mathbf{Sch}/k' \to \mathbf{Ab}_*$ compatible with all the structures available for $H_{2*,*}$ and, after restriction to $\mathbf{Sm}/k$ is just the classifying map $\Omega^* \to A^{2*,*}$ for the oriented cohomology theory $X \mapsto A^{2*,*}(X)$. We refer the reader to [L08, §4] for a complete description of the properties satisfied by ϑ_H.
Via ϑ_H and the ring homomorphism $\rho_\Omega : \mathbb{L}_* \to \Omega_*(k)$ classifying the formal group law for Ω_*, we have the ring homomorphism $\rho_H : \mathbb{L}_* \to H_{2*,*}(k)$. If (J, B) is an oriented (H, A)-module, then via the $H_{2*,*}(k)$-module structure on $J_{2*,*}(k)$, ρ_H makes $J_{2*,*}(k)$ a $\mathbb{L}_*$-module. We write J_* for the $\mathbb{L}_*$-module $J_{2*,*}(k)$.

Proposition 5.8. *Let k be a field admitting resolution of singularities. Let (H, A) be an oriented duality theory and (J, B) an oriented (H, A)-module. There is a unique natural transformation $\vartheta_{H/J} : \Omega_*^{J_*} \to J_{2*,*}$ from $\mathbf{Sch}/k' \to \mathbf{Ab}_*$ satisfying*

(1) *$\vartheta_{H/J}$ is compatible with pullback maps j^* for $j : U \to X$ an open immersion in $\mathbf{Sch}/k$.*
(2) *$\vartheta_{H/J}$ is compatible with fundamental classes.*
(3) *$\vartheta_{H/J}$ is compatible with external products.*

(4) *$\vartheta_{H/J}$ is compatible with the action of 1st Chern class operators.*
(5) *Identifying $\Omega_*^{J_*}(k)$ with $J_{2*,*}(k)$ via the product map $\Omega_*(k) \otimes_{\mathbb{L}_*} J_{2*,*}(k) \to J_{2*,*}(k)$, $\vartheta_{H/J}(k) : \Omega_*^{J_*}(k) \to J_{2*,*}$ is the identity map.*

Proof. For $X \in \mathbf{Sch}/k$, we define $\vartheta_{H/J}(X)$ by

$$\vartheta_{H/J}(u \otimes j) = \vartheta_H(u) \times j \in J_{2*,*}(X \times_k \operatorname{Spec} k) = J_{2*,*}(X),$$

for $u \otimes j \in \Omega_*^{J_*}(X) := \Omega_*(X) \otimes_{\mathbb{L}_*} J_{2*,*}(k)$. The properties (1)-(5) follow directly from the construction. As $\Omega_*(X)$ is generated by push-forwards of fundamental classes, the properties (2), (3) and (5) determine $\vartheta_{H/J}$ uniquely. □

Remark 5.9. Let k, (H, A) and (J, B) be as in proposition 5.8. Suppose that $J_* := J_{2*,*}$ has external products $\times_J$ and there is a unit element $1_J \in J_0(k)$ for these external products. Suppose further that these are compatible with the external products $H_*(X) \otimes J_*(Y) \to J_*(X \times_k Y)$, in the sense that

$$(h \times 1_J) \times_J b = h \times b \in J_*(X \times_k Y)$$

for $h \in H_*(X)$, $b \in J_*(Y)$, and that $1_H \times 1_J = 1_J$. Then $\vartheta_{H/J}$ is compatible with external products and is unital. This follows directly from our assumptions and the identity

$$\vartheta_{H/J}((u \otimes h) \times (u' \otimes j')) = \vartheta_H(u) \times \vartheta_{H/J}(u' \otimes (h \times j)).$$

5.3. MODULES FOR ORIENTED RING SPECTRA. We now discuss the oriented duality theory and oriented Borel-Moore homology associated to a module spectrum for an oriented weak commutative ring T-spectrum.

Let ph be the two-sided ideal of phantom maps in $\mathcal{SH}(S)$, where a phantom map is a map $f : \mathcal{E} \to \mathcal{F}$ such that $f \circ g = 0$ for each compact object $\mathcal{A}$ in $\mathcal{SH}(S)$ and each morphism $g : \mathcal{A} \to \mathcal{E}$. Let $\mathcal{E}$ be a weak commutative ring T-spectrum, that is, there are maps $\mu : \mathcal{E} \wedge \mathcal{E} \to \mathcal{E}$, $\eta : \mathbb{S}_S \to \mathcal{E}$ in $\mathcal{SH}(S)$ that satisfy the axioms for a monoid in $\mathcal{SH}(S)/\mathrm{ph}$. An $\mathcal{E}$-module is similarly an object $\mathcal{N} \in \mathcal{SH}(S)$ together with a multiplication map $\rho : \mathcal{E} \wedge \mathcal{N} \to \mathcal{E}$ in $\mathcal{SH}(S)$ that makes $\mathcal{N}$ into a unital $\mathcal{E}$-module in $\mathcal{SH}(S)/\mathrm{ph}$ (see for example [NSO09, §8], where a weak commutative ring T-spectrum is referred to as a T-spectrum $\mathcal{E}$ with a quasi-multiplication $\mu : \mathcal{E} \wedge \mathcal{E} \to \mathcal{E}$).

Suppose that $(\mathcal{E}, c)$ is an oriented weak commutative ring T-spectrum in $\mathcal{SH}(k)$, k a field admitting resolution of singularities. We have constructed in [L08, theorem 3.4] a bi-graded oriented duality theory $(\mathcal{E}'_{**}, \mathcal{E}^{**})$ by defining $\mathcal{E}'_{a,b}(X) := \mathcal{E}_X^{2m-a,m-b}(M)$, where $M \in \mathbf{Sm}/k$ is a chosen smooth quasi-projective scheme containing X as a closed subscheme and $m = \dim_k M$. Let $\mathcal{N}$ be an $\mathcal{E}$-module. For $E \to M$ a rank r vector bundle on $M \in \mathbf{Sm}/k$ and $X \subset M$ a closed subscheme, the Thom classes for $\mathcal{E}$ give rise to a Thom isomorphism $\mathcal{N}_X^{**}(M) \to \mathcal{N}_X^{2r+*,r+*}(E)$.

Using these Thom isomorphisms, the arguments used to construct the oriented duality theory $(\mathcal{E}'_{**}, \mathcal{E}^{**})$ go through without change to give $\mathcal{N}^{**}$ the structure of an oriented $\mathcal{E}^{**}$-module, and to define an oriented $(\mathcal{E}'_{**}, \mathcal{E}^{**})$-module $(\mathcal{N}'_{**}, \mathcal{N}^{**})$, with canonical isomorphisms $\mathcal{N}'_{a,b}(X) \cong \mathcal{N}_X^{2m-a,m-b}(M)$,

$m = \dim_k M$, and where the cap products are induced by the $\mathcal{E}$-modules structure on $\mathcal{N}$.

5.4. Geometrically Landweber exact modules.

Definition 5.10. Let $(\mathcal{E}, c)$ be a weak oriented ring T-spectrum and let $\mathcal{N}$ be an $\mathcal{E}$-module. The *geometric part* of $\mathcal{E}^{**}$ is the $(2*,*)$-part $\mathcal{E}^* := \mathcal{E}^{2*,*}$ of $\mathcal{E}^{**}$, the geometric part of $\mathcal{N}$ is the $\mathcal{E}^*$-module $\mathcal{N}^{2*,*}$, and the geometric part of $\mathcal{N}'$ is similarly given by $X \mapsto \mathcal{N}'_*(X) := \mathcal{N}'_{2*,*}(X)$. This gives us the $\mathbb{Z}$-graded oriented duality theory $(\mathcal{E}'_*, \mathcal{E}^*)$ and the oriented $(\mathcal{E}'_*, \mathcal{E}^*)$-module $(\mathcal{N}'_*, \mathcal{N}^*)$.

Let $(\mathcal{E}, c)$ be a weak oriented ring T-spectrum and let $\mathcal{N}$ be an $\mathcal{E}$-module. By proposition 5.8, we have a canonical natural transformation

$$\vartheta_{\mathcal{E}'/\mathcal{N}'} : \Omega_*^{\mathcal{N}'_*(k)} \to \mathcal{N}'_*$$

satisfying the compatibilities listed in that proposition.

We extend the definition of a geometrically Landweber exact weak commutative ring T -spectrum (see [L15, definition 3.7]) to the case of an $\mathcal{E}$-module:

Definition 5.11. Let $(\mathcal{E}, c)$ be a weak oriented ring T-spectrum and let $\mathcal{N}$ be an $\mathcal{E}$-module. We say that $\mathcal{N}$ is *geometrically Landweber exact* if for each point $\eta \in X \in \mathbf{Sm}/k$

i. The structure map $p_\eta : \eta \to \operatorname{Spec} k$ induces an isomorphism $p_\eta^* : \mathcal{N}^{2*,*}(k) \to \mathcal{N}^{2*,*}(\eta)$.

ii. The product map $\cup_\eta : \mathcal{E}^{1,1}(\eta) \otimes \mathcal{N}^{2*,*}(\eta) \to \mathcal{N}^{2*+1,*+1}(\eta)$ induces a surjection $k(\eta)^\times \otimes \mathcal{N}^{2*,*}(\eta) \to \mathcal{N}^{2*+1,*+1}(\eta)$.

Here we use the canonical natural transformation $t_{\mathcal{E}} : \mathbb{G}_m \to \mathcal{E}^{1,1}(-)$ defined in [L15, remark 1.5] to define the map $k(\eta)^\times \to \mathcal{E}^{1,1}(\eta)$ needed in (ii).

The following result generalizes [L15, theorem 6.2] from oriented weak commutative ring T-spectra to modules:

Theorem 5.12. *Let k be a field of characteristic zero, $\mathcal{N}$ an MGL-module in $\mathcal{SH}(k)$, $(\mathcal{N}'_{**}, \mathcal{N}^{**})$ the associated oriented (MGL'_{**}, MGL^{**})-module, and $\mathcal{N}'_*$ the geometric part of $\mathcal{N}'$. Suppose that $\mathcal{N}$ is geometrically Landweber exact. Then the classifying map*

$$\vartheta_{MGL'_*/\mathcal{N}'_*} : \Omega_*^{\mathcal{N}'_*(k)} \to \mathcal{N}'_*$$

is an isomorphism.

Remark 5.13. Let k be a field of characteristic zero, let $(\mathcal{E}, c)$ be an oriented weak commutative ring T-spectrum in $\mathcal{SH}(S)$, and let $\mathcal{N}$ be an $\mathcal{E}$-module. Via the classifying map $\varphi_{\mathcal{E},c} : MGL \to \mathcal{E}$, $\mathcal{N}$ becomes an MGL-module. In addition, the classifying map $\vartheta_{\mathcal{E}'} : \Omega_* \to \mathcal{E}'_*$ is induced from $\varphi_{\mathcal{E},c}$ and the classifying map $\vartheta_{MGL'_*/\mathcal{N}'_*}$ factors through the classifying map $\vartheta_{\mathcal{E}'_*/\mathcal{N}'_*} : \mathcal{E}_*'^{\mathcal{N}'_*(k)} \to \mathcal{N}'_*$ as

$$\vartheta_{MGL'_*/\mathcal{N}'_*} = \vartheta_{\mathcal{E}'_*/\mathcal{N}'_*} \circ (\varphi_{\mathcal{E},c} \otimes \mathrm{id}_{\mathcal{N}'_*(k)}).$$

Thus, theorem 5.12 applies to $\mathcal{E}$-modules for arbitrary $(\mathcal{E}, c)$. Moreover, if $(\mathcal{E}, c)$ is geometrically Landweber exact in the sense of [L15, definition 3.7], the map $\bar{\vartheta}_{\mathcal{E}'_*} : \Omega_*^{\mathcal{E}'_*(k)} \to \mathcal{E}'_*$ is an isomorphism ([L15, theorem 6.2]) hence the map $\vartheta_{\mathcal{E}'_*/\mathcal{N}'_*}$ is an isomorphism as well.

Proof of theorem 5.12. The proof of theorem 5.12 is essentially the same as the proof of [L15, theorem 6.2]. Indeed, just as in *loc. cit.*, one constructs a commutative diagram (see [L09, (6.4)])
(5.1)

$$\begin{array}{ccccccccc}
\oplus_{\eta\in X_{(d)}} k(\eta)^\times \otimes \mathcal{N}'_{*-d+1} & \xrightarrow{\mathrm{Div}_{\mathcal{N}}} & \Omega_*^{\mathcal{N}'_*(1)}(X) & \xrightarrow{i_*} & \Omega_*^{\mathcal{N}'_*}(X) & \xrightarrow{j^*} & \oplus_{\eta\in X_{(d)}} \Omega_*^{\mathcal{N}'_*}(\eta) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \bar{\vartheta}^{(1)}} & & \downarrow{\scriptstyle \bar{\vartheta}(X)} & & \downarrow{\scriptstyle \bar{\vartheta}} & & \\
\oplus_{\eta\in X_{(d)}} k(\eta)^\times \otimes \mathcal{N}'_{*-d+1} & \xrightarrow[div_{\mathcal{N}}]{} & \mathcal{N}'^{(1)}_{2*,*}(X) & \xrightarrow[i_*]{} & \mathcal{N}'_{2*,*}(X) & \xrightarrow[j^*]{} & \oplus_{\eta\in X_{(d)}} \mathcal{N}'_{2*,*}(\eta) & \longrightarrow & 0
\end{array}$$

where we write $\mathcal{N}'_*$ for $\mathcal{N}'_*(k)$, d is the maximum of $\dim_k X_i$ as X_i runs over the irreducible components of X, and $\mathcal{N}'^{(1)}_{2*,*}(X)$ is the colimit of $\mathcal{N}'_{2*,*}(W)$, as W runs over closed subschemes of X containing no dimension d generic point of X. A similarly defined colimit of the $\Omega_*^{\mathcal{N}'_*}(W)$ gives us $\Omega_*^{\mathcal{N}'_*(1)}(X)$. The maps $\bar{\vartheta}^{(1)}$, $\bar{\vartheta}(X)$ and $\bar{\vartheta}$ are all induced by the classifying map $\vartheta_{MGL'_*/\mathcal{N}'_*}$. The top row is a complex and the bottom row is exact; this latter fact follows from the surjectivity assumption in definition 5.11(ii). The map $\bar{\vartheta}$ is an isomorphism by part (i) of definition 5.11 and $\bar{\vartheta}^{(1)}$ is an isomorphism by induction on d. To show that $\bar{\vartheta}(X)$ is an isomorphism, it suffices to show that the identity map on $\oplus_\eta \mathcal{N}'_{*-d+1} \otimes k(\eta)^\times$ extends diagram (5.1) to a commutative diagram.
To see this, we note that the map $div_{\mathcal{N}}$ is defined by composing the boundary map

$$\partial : \oplus_{\eta\in X_{(d)}} \mathcal{N}'_{2*+1,*}(\eta) \to \mathcal{N}'^{(1)}_{2*,*}(X)$$

with the sum of the product maps $MGL'_{2d-1,d-1}(\eta) \otimes \mathcal{N}'_{*-d+1}(k) \to \mathcal{N}'_{2*+1,*}(\eta)$ and the canonical map $t_{MGL}(\eta) : k(\eta)^\times \to MGL^{1,1}(\eta) = MGL'_{2d-1,d-1}(\eta)$ (see [L09, remark 1.5]). For MGL', we have the similarly defined map

$$div_{MGL} : \oplus_{\eta\in X_{(d)}} k(\eta)^\times \otimes \mathbb{L}_{*-d+1} \to MGL'^{(1)}_{2*,*}(X),$$

after replacing $MGL'_{*-d+1}(k)$ with $\mathbb{L}_{*-d+1}$ via the classifying map $\mathbb{L}_* \to MGL'_*(k)$. We have as well the commutative diagram (see [L09, (5.4)])

$$\begin{array}{ccc}
\oplus_{\eta\in X_{(d)}} k(\eta)^\times \otimes \mathbb{L}_{*-d+1} & \xrightarrow{\mathrm{Div}} & \Omega_*^{(1)}(X) \\
\| & & \downarrow{\scriptstyle \vartheta^{(1)}_{MGL}} \\
\oplus_{\eta\in X_{(d)}} k(\eta)^\times \otimes \mathbb{L}_{*-d+1} & \xrightarrow[div_{MGL}]{} & MGL'^{(1)}_{2*,*}(X),
\end{array}$$

which after applying $-\otimes_{\mathbb{L}_*}\mathcal{N}'_*$ gives us the commutative diagram

$$
\begin{array}{ccc}
\oplus_{\eta\in X_{(d)}}k(\eta)^\times\otimes\mathcal{N}_{*-d+1} & \xrightarrow{\mathrm{Div}_{\mathcal{N}}} & \Omega_*^{\mathcal{N}'_*(1)}(X) \\
\| & & \downarrow{\scriptstyle \vartheta^{(1)}_{MGL}\otimes\mathrm{id}} \\
\oplus_{\eta\in X_{(d)}}k(\eta)^\times\otimes\mathcal{N}_{*-d+1} & \xrightarrow[\overline{div}_{MGL}]{} & MGL'^{(1)}_{2*,*}(X)\otimes_{\mathbb{L}_*}\mathcal{N}'_*.
\end{array} \tag{5.2}
$$

The Leibniz rule for ∂ gives us the commutative diagram

$$
\begin{array}{ccc}
\oplus_{\eta\in X_{(d)}}k(\eta)^\times\otimes\mathcal{N}_{*-d+1} & \xrightarrow{\overline{div}_{MGL}} & MGL'^{(1)}_{2*,*}(X)\otimes_{\mathbb{L}_*}\mathcal{N}'_* \\
\| & & \downarrow{\scriptstyle \cup} \\
\oplus_{\eta\in X_{(d)}}k(\eta)^\times\otimes\mathcal{N}'_{*-d+1} & \xrightarrow[div_{\mathcal{N}}]{} & \mathcal{N}'^{(1)}_{2*,*}(X);
\end{array} \tag{5.3}
$$

combining diagrams (5.2) and (5.3) yields the desired commutativity. □

6. Applications to quotients of MGL

We return to our discussion of quotients of MGL_p and their localizations. We select a system of polynomial generators for the Lazard ring, $\mathbb{L}_*\cong\mathbb{Z}[x_1,x_2,\ldots]$, deg $x_i=i$. Let $\mathcal{S}\subset\mathbb{N}$, $\mathcal{S}^c$ its complement and let $\mathbb{Z}[\mathcal{S}^c]$ denote the graded polynomial ring on the x_i, $i\in\mathcal{S}^c$, $\deg x_i=i$. Let $\mathcal{S}_0\subset\mathbb{Z}[\mathcal{S}^c]$ be a collection of homogeneous elements, $\mathcal{S}_0=\{z_j\in\mathbb{Z}[\mathcal{S}^c]_{e_j}\}$, and let $\mathbb{Z}[\mathcal{S}^c][\mathcal{S}_0^{-1}]$ denote the localization of $\mathbb{Z}[\mathcal{S}^c]$ with respect to $\mathcal{S}_0$.
We consider a quotient spectrum $MGL_p/(\mathcal{S}):=MGL_p/(\{x_i\mid i\in\mathcal{S}\})$ or an integral version $MGL/(\mathcal{S}):=MGL/(\{x_i\mid i\in\mathcal{S}\})$. We consider as well the localizations

$$MGL_p/(\mathcal{S})[\mathcal{S}_0^{-1}]:=MGL_p/(\mathcal{S})[\{z_j^{-1}\mid z_j\in\mathcal{S}_0\}],$$
$$MGL/(\mathcal{S})[\mathcal{S}_0^{-1}]:=MGL/(\mathcal{S})[\{z_j^{-1}\mid z_j\in\mathcal{S}_0\}].$$

and the mod p version

$$MGL/(\mathcal{S},p)[\mathcal{S}_0^{-1}]:=MGL_p/(\mathcal{S})[\mathcal{S}_0^{-1}]/p.$$

PROPOSITION 6.1. *Let p be a prime, and let $S=\operatorname{Spec}k$, k a perfect field with exponential characteristic prime to p. Let $\mathcal{S}$ be a subset of $\mathbb{N}$ and $\mathcal{S}_0$ a set of homogeneous elements of $\mathbb{Z}[\mathcal{S}^c]$. Then the spectra $MGL_p/(\mathcal{S})[\mathcal{S}_0^{-1}]$ and $MGL_p/(\mathcal{S},p)[\mathcal{S}_0^{-1}]$ are geometrically Landweber exact. In case $\operatorname{char}k=0$, $MGL/(\mathcal{S})[\mathcal{S}_0^{-1}]$ is geometrically Landweber exact.*

Proof. We discuss the cases $MGL_p/(\mathcal{S})[\mathcal{S}_0^{-1}]$ and $MGL_p/(\mathcal{S},p)[\mathcal{S}_0^{-1}]$; the case of $MGL/(\mathcal{S})[\mathcal{S}_0^{-1}]$ is exactly the same.
Let A be a finitely generated abelian group and let η be a point in some $X\in\mathbf{Sm}/k$. Then the motivic cohomology $H^*(\eta,A(*))$ satisfies

$$H^{2r}(\eta,A(r))=H^{2r+1}(\eta,A(r+1))=0$$

for $r \neq 0$,

$$H^0(\eta, A(0)) = A, \quad H^1(\eta, A(1)) = k(\eta)^\times \otimes_{\mathbb{Z}} A.$$

We consider the slice spectral sequences

$$E_2^{p,q}(n) := H^{p-q}(\eta, \mathbb{Z}(n-q)) \otimes \mathbb{Z}[\mathcal{S}^c][\mathcal{S}_0^{-1}]_{-q} \Rightarrow (MGL_p/(\mathcal{S})[\mathcal{S}_0^{-1}])^{p+q,n}(\eta)$$

and

$$E_2^{p,q}(n) := H^{p-q}(\eta, \mathbb{Z}/p(n-q)) \otimes \mathbb{Z}[\mathcal{S}^c][\mathcal{S}_0^{-1}]_{-q} \Rightarrow (MGL_p/(\mathcal{S},p)[\mathcal{S}_0^{-1}])^{p+q,n}(\eta)$$

given by proposition 3.3. As in the proof of theorem 4.8, $MGL_p^{2n+a,n}(\eta) = 0$ for $a > 0$ and $n \in \mathbb{Z}$, and thus by remark 3.4, the convergence hypotheses in proposition 3.3 are satisfied. Thus, these spectral sequences are strongly convergent. As discussed in the proof of [L15, proposition 3.8], the only non-zero E_2 term contributing to $(MGL_p/(\mathcal{S})[\mathcal{S}_0^{-1}])^{2n,n}(\eta)$ or to $(MGL_p/(\mathcal{S},p)[\mathcal{S}_0^{-1}])^{2n,n}(\eta)$ is $E_2^{n,n}(n)$, the only non-zero E_2 term contributing to $(MGL_p/(\mathcal{S})[\mathcal{S}_0^{-1}])^{2n-1,n}(\eta)$ or contributing to $(MGL_p/(\mathcal{S},p)[\mathcal{S}_0^{-1}])^{2n-1,n}(\eta)$ is $E_2^{n,n-1}(n)$, and all differentials entering or leaving these terms are zero.
This gives us isomorphisms

$$\begin{aligned}
&(MGL_p/(\mathcal{S})[\mathcal{S}_0^{-1}])^{2n,n}(\eta) \cong \mathbb{Z}_{(p)}[\mathcal{S}^c][\mathcal{S}_0^{-1}]_{-n}\\
&(MGL_p/(\mathcal{S},p)[\mathcal{S}_0^{-1}])^{2n,n}(\eta) \cong \mathbb{Z}/(p)[\mathcal{S}^c][\mathcal{S}_0^{-1}]_{-n}\\
&(MGL_p/(\mathcal{S})[\mathcal{S}_0^{-1}])^{2n-1,n}(\eta) \cong \mathbb{Z}_{(p)}[\mathcal{S}^c][\mathcal{S}_0^{-1}]_{1-n} \otimes k(\eta)^\times\\
&(MGL_p/(\mathcal{S},p)[\mathcal{S}_0^{-1}])^{2n-1,n}(\eta) \cong \mathbb{Z}/(p)[\mathcal{S}^c][\mathcal{S}_0^{-1}]_{1-n} \otimes k(\eta)^\times
\end{aligned}$$

from which it easily follows that $MGL_p/(\mathcal{S})[\mathcal{S}_0^{-1}]$ and $MGL_p/(\mathcal{S},p)[\mathcal{S}_0^{-1}]$ are geometrically Landweber exact. □

COROLLARY 6.2. *Let $S = \operatorname{Spec} k$, k a field of characteristic zero. Fix a prime p and let $\mathcal{N} = MGL/(\mathcal{S})[\mathcal{S}_0^{-1}]$, $MGL_p/(\mathcal{S})[\mathcal{S}_0^{-1}]$ or $MGL_p/(\mathcal{S},p)[\mathcal{S}_0^{-1}]$, let $(\mathcal{N}', \mathcal{N})$ be the associated (MGL', MGL)-module and $\mathcal{N}'_*$ the geometric part of $\mathcal{N}'_{**}$. Then the classifying map*

$$\vartheta_{\mathcal{N}'_*(k)} : \Omega_*^{\mathcal{N}'_*(k)} \to \mathcal{N}'_*$$

is an isomorphism of Ω_*-modules.

This follows directly from theorem 5.12 and proposition 6.1. As an immediate consequence, we have

COROLLARY 6.3. *Let $S = \operatorname{Spec} k$, k a field of characteristic zero. Fix a prime p and let $\mathcal{N} = BP$, $BP\langle n\rangle$, $E(n)$, $k(n)$ or $K(n)$, let $(\mathcal{N}', \mathcal{N})$ be the associated (MGL', MGL)-module and $\mathcal{N}'_*$ the geometric part of $\mathcal{N}'_{**}$. Then the classifying map*

$$\vartheta_{\mathcal{N}'_*(k)} : \Omega_*^{\mathcal{N}'_*(k)} \to \mathcal{N}'_*$$

is an isomorphism of Ω_*-modules. In case $\mathcal{N} = BP$ or $E(n)$, $\vartheta_{\mathcal{N}'_*(k)}$ is compatible with external products.

Remark 6.4. Suppose that the theory with supports $\mathcal{N}^{2*,*}$ has products and a unit, compatible with its $MGL^{2*,*}$-module structure. Then by remark 5.9, the classifying map $\vartheta_{\mathcal{N}'_*(k)}$ is also compatible with products.
In the case of a quotient $\mathcal{E}$ of MGL or MGL_p by a subset $\{x_i : i \in I\}$ of the set of polynomial generators, the vanishing of $MGL^{2r+s,r}(k)$ for $s > 0$ shows that $\mathcal{E}^{2*,*}(k) = MGL^{2*,*}(k)/(\{x_i : i \in I\})$, which has the evident ring structure induced by the natural $MGL^{2*,*}(k)$-module structure. Thus, the rational theory $\Omega_*^{\mathcal{E}_*(k)}$ has a canonical structure of an oriented Borel-Moore homology theory on **Sch**$/k$; the same holds for $\mathcal{E}$ a localization of this type of quotient. The fact that the classifying homomorphism $\vartheta_{\mathcal{E}} : \Omega_*^{\mathcal{E}'_*(k)} \to \mathcal{E}'_*$ is an isomorphism induces on $\mathcal{E}'_*$ the structure of an oriented Borel-Moore homology theory on **Sch**$/k$; it appears to be unknown if this arises from a multiplicative structure on the spectrum level.

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Essential Dimension of Separable Algebras Embedding in a Fixed Central Simple Algebra

To Alexander Merkurjev on His 60th Birthday

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Abstract. In this paper we fix a central simple F-algebra A of prime power degree and consider separable algebras over extensions K/F, which embed in A_K. We study the minimal number of independent parameters, called essential dimension, needed to define these separable algebras. In case the index of A does not exceed a certain bound, the task is equivalent to the problem of computing the essential dimension of the algebraic groups $(\mathbf{PGL}_d)^m \rtimes S_m$, which is extremely difficult in general. In the other case, however, we manage to compute the exact value of the essential dimension of the given class of separable algebras, except in one case for A of index 2, which we study in greater detail.

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1. Introduction

Central simple algebras over fields are at the core of non-commutative algebra. Their history is rooted in the middle of the 19th century, when W. Hamilton discovered the quaternions over the real numbers. In the early 20th century J. Wedderburn gave a classification of finite dimensional semisimple algebras by means of division rings and subsequently R. Brauer introduced the Brauer group of a field, which lead to diverse research in algebra and number theory. Moreover central simple algebras and the Brauer group arise naturally in Galois cohomology and are therefore central for the theory of algebraic groups over fields. We refer to [2, 1] for surveys on these topics, including discussion of open problems.

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Essential dimension is a more recent topic, introduced around 1995 by J. Buhler and Z. Reichstein [4] and in full generality by A. Merkurjev [3]. The essential dimension of a functor $\mathcal{F}\colon \mathrm{Fields}_F \to \mathrm{Sets}$ from the category of field extensions of a fixed base field F to the category of sets is defined as the least integer n, such that every object $a \in \mathcal{F}(K)$ over a field extension K/F is defined over a subextension K_0/F of transcendence degree at most n. Here $a \in \mathcal{F}(K)$ is said to be defined over K_0 if it lies in the image of the map $\mathcal{F}(K_0) \to \mathcal{F}(K)$ induced by the inclusion $K_0 \to K$. The functors $\mathcal{F}$ we are mostly interested in take a field extension K/F to the set of isomorphism classes of algebraic objects over K of some kind. The essential dimension of $\mathcal{F}$ is then roughly the number of independent paramters needed to define these objects.
The essential dimension of an algebraic group G over a field F is defined as the essential dimension of the Galois cohomology functor

$$H^1(-,G)\colon \mathrm{Fields}_F \to \mathrm{Sets}, \quad K \mapsto H^1(K,G).$$

It is denoted by $\mathrm{ed}(G)$ and measures the complexity of G-torsors up to isomorphism, and hence of isomorphism classes of certain objects such as central simple algebras (for projective linear groups), quadratic forms (for orthogonal groups), étale algebras (for symmetric groups) etc. See [21, 17] for recent surveys on the topic.

Two of the motivating problems in essential dimension are the computation of the essential dimension of the projective linear group $\mathbf{PGL}_d$ and the symmetric group S_n, since they provide insight to the structure of central simple algebras (of degree d) and étale algebras (of dimension n), respectively. The first problem goes back to C. Procesi [19], who asked for fields of definition of the universal division algebra and discovered, in modern terms, that $\mathrm{ed}(\mathbf{PGL}_d) \leq d^2$. This upper bound has been improved after the introduction of essential dimension, but it is still quadratic in d. See Remark 4.5 for details. A recent breakthrough has been made by A. Merkurjev [16] for a lower bound on $\mathrm{ed}(\mathbf{PGL}_d)$. Namely, if $d = p^a$ for some prime p different from $\mathrm{char}(F)$, he showed that $\mathrm{ed}(\mathbf{PGL}_d) \geq (a-1)p^a+1$. In fact he established this lower bound for the essential p-dimension of $\mathbf{PGL}_d$, denoted $\mathrm{ed}_p(\mathbf{PGL}_d)$, which measures complexity of degree d central simple algebras up to prime to p field extensions, and showed in particular that $\mathrm{ed}_p(\mathbf{PGL}_{p^2}) = p^2 + 1$ when $\mathrm{char}(F) \neq p$ [15]. For exponent $a \geq 3$ the problem of computing $\mathrm{ed}_p(\mathbf{PGL}_{p^a})$ is still wide open. Moreover even the value of $\mathrm{ed}(\mathbf{PGL}_p)$ is unknown for any prime $p \geq 5$ and related to the long-standing cyclicity-conjecture of degree p division algebras due to Albert.

The second problem is related to classical work of F. Klein, C. Hermite and F. Joubert on simplifying minimal polynomials of generators of separable field extensions (of degree $n = 5$ and 6) by means of Tschirnhaus-transformations, and was the main inspiration of [4]. In our language Hermite and Joubert showed that $\mathrm{ed}(S_5) \leq 2$ and $\mathrm{ed}(S_6) \leq 3$ (over a field F of characteristic zero), and Klein proved that $\mathrm{ed}(S_5) > 1$, hence $\mathrm{ed}(S_5) = 2$. The gap between the best lower bound (roughly $\frac{n}{2}$) and the best upper bound $n-3$ on $\mathrm{ed}(S_n)$ for $n \geq 5$

is still quite large in general. See [7], where it is also proven that $\mathrm{ed}(S_7) = 4$ in characteristic zero.

In this paper we study *separable* algebras B. A (finite-dimensional) algebra B over a field is called separable, if it is semisimple (i.e., its Jacobson radical is trivial) and remains semisimple over every field extension. This includes both the case of central simple algebras and étale algebras. We restrict our attention to those separable K-algebras which embed in $A_K = A \otimes_F K$ for a fixed central simple F-algebra A. Here F is our base field and K/F a field extension. This originates in my earlier paper [13], which covers the case where A is a division algebra. The aim in this paper is to prove results for lower index of A.

Throughout A is a central simple algebra over a field F and $B \subseteq A$ a separable subalgebra. The type of B in A is defined as the multiset $\theta_B = [(r_1, d_1), \dots, (r_m, d_m)]$ such that the algebra B and its centralizer $C = C_A(B)$ have the form

$$B_{\mathrm{sep}} \simeq M_{d_1}(F_{\mathrm{sep}}) \times \cdots \times M_{d_m}(F_{\mathrm{sep}}), \quad C_{\mathrm{sep}} \simeq M_{r_1}(F_{\mathrm{sep}}) \times \cdots \times M_{r_m}(F_{\mathrm{sep}})$$

over a separable closure F_{sep}. Note that central simple and étale subalgebras are those of type $\theta_B = [(d, r)]$ (with $d = \deg(B)$) and $\theta_B = [(1, r_1), \dots, (1, r_m)]$ (with $m = \dim(B)$), respectively. We will assume throughout that the type θ_B of B is constant, i.e. $\theta_B = [(d, r), \dots, (d, r)]$ (m-times) for some $r, d, m \geq 1$. This assumption is automatically satisfied if A is a division algebra. By [13, Lemma 4.2(a)] the product drm is the degree of A.

Denote by $\mathbf{Forms}(B)\colon \mathrm{Fields}_F \to \mathrm{Sets}$ the functor that takes a field extension K/F to the set of isomorphism classes of K-algebras B' which become isomorphic to B over a separable closure of K and by $\mathbf{Forms}_A^\theta(B)$ the subfunctor of $\mathbf{Forms}(B)$ formed by those isomorphism classes B' of forms of B which admit an embedding in A of type θ_B. We are interested in $\mathrm{ed}(\mathbf{Forms}_A^\theta(B))$. By [13, Lemma 4.6] we have a natural isomorphism

$$\mathbf{Forms}_A^\theta(B) \simeq H^1(-, G),$$

of functors $\mathrm{Fields}_F \to \mathrm{Sets}$, where G is the normalizer

$$G := N_{\mathbf{GL}_1(A)}(\mathbf{GL}_1(B)).$$

Our main result is the following theorem, which shows an interesting dichotomy between the case where the index of A exceeds the bound $\frac{r}{d}$ and when it does not. The case where A is a division algebra is [13, Theorem 4.10]. As there we get examples of algebraic groups, where $\mathrm{ed}(G)$ is determined explicitly, but $\mathrm{ed}(G_{\mathrm{alg}})$ is unknown. Here we see that the mystery starts exactly once $\mathrm{ind}(A) \leq \frac{r}{d}$.

Theorem 1.1. *Let $G = N_{GL_1(A)}(\mathbf{GL}_1(B))$ with A central simple and $B \subseteq A$ a separable subalgebra of type $\theta_B = [(d, r), \dots, (d, r)]$ (m-times). Suppose that $\deg(A) = drm$ is a power of a prime p and that $d \leq r$, so that $d|r$. Then exatly one of the following cases occurs:*

(a) $\operatorname{ind}(A) \le \frac{r}{d}$*: Then* $\mathbf{Forms}_A^\theta(B) = \mathbf{Forms}(B)$ *and the three functors* $H^1(-,G)$, $\mathbf{Forms}(B)$ *and* $H^1(-,(\mathbf{PGL}_d)^m \rtimes S_m)$ *are naturally isomorphic. In particular*

$$\operatorname{ed}(G) = \operatorname{ed}(\mathbf{Forms}(B)) = \operatorname{ed}((\mathbf{PGL}_d)^m \rtimes S_m).$$

(b) $\operatorname{ind}(A) > \frac{r}{d}$*: Then*

$$\begin{aligned}\operatorname{ed}(G) = \operatorname{ed}(\mathbf{Forms}_A^\theta(B)) &= \deg(A)\operatorname{ind}(A) - \dim(G),\\ &= drm\operatorname{ind}(A) - m(r^2+d^2-1).\end{aligned}$$

except possibly when $d = r > 1$ *and* $\operatorname{ind}(A) = 2$.

Note that the assumption $r \le d$ is harmless. Indeed since

$$N_{\mathbf{GL}_1(A)}(\mathbf{GL}_1(B)) \subseteq N_{\mathbf{GL}_1(A)}(\mathbf{GL}_1(C_A(B))) \subseteq N_{\mathbf{GL}_1(A)}(\mathbf{GL}_1(C_A(C_A(B))))$$

and $C_A(C_A(B)) = B$ by the double centralizer property of semisimple subalgebras [8, Theorem 4.10] we can always replace B by its centralizer (which amounts to switching r and d) without changing $\operatorname{ed}(G)$.

There is a big contrast between the two cases in Theorem 1.1. In case (a) the computation of $\operatorname{ed}(G) = \operatorname{ed}((\mathbf{PGL}_d)^m \rtimes S_m) = \operatorname{ed}(\mathbf{Forms}(B))$ is very hard in general. For instance when B is central simple (i.e., $m = 1$), we have $\operatorname{ed}(G) = \operatorname{ed}(\mathbf{PGL}_d)$ with $d = \deg(B)$, and in case B is étale (i.e., $d = 1$), $\operatorname{ed}(G) = \operatorname{ed}(S_m)$ where $m = \dim(B)$.
In contrast the above theorem gives the precise value of $\operatorname{ed}(G)$ in case (b) with only a small exception. The exception occurs when $d = r > 1$ and $\operatorname{ind}(A) = 2$, i.e., when $A \simeq M_{d/2}(Q)$ for a non-split quaternion F-algebra Q and B and the centralizer $C = C_A(B)$ become isomorphic to $(M_d(F_{\mathrm{sep}}))^m$ over F_{sep}. Note that we then automatically have $p = 2$, so $r = d$ and m are 2-primary. This special case will be treated separately. We will provide lower bounds and upper bounds on $\operatorname{ed}(G)$. When $m = 1$ the set $H^1(K,G)$ then classifies central simple K-algebras B' of degree d, whose tensor product with a fixed quaternion algebra over F is not division (see Example 4.1). In particular we will prove that $\operatorname{ed}(G)$ is either 2 or 3 when $r = d = 2$ and $m = 1$ (see Corollary 4.6).

The rest of the paper is structured as follows. In section 2 we study representations of $G = N_{GL_1(A)}(GL_1(B))$ with respect to generic freeness. This is used in section 3 to prove that $\operatorname{ed}(G)$ does not exceed the value suggested in Theorem 1.1(b). We will conclude the proof of the whole theorem in that section. It remains to study the case excluded from Theorem 1.1, where A has index 2 and $r = d > 1$. This is finally done in section 4.

2. Results on the Canonical Representation

The group $G = N_{\mathbf{GL}_1(A)}(\mathbf{GL}_1(B))$, as every subgroup of $\mathbf{GL}_1(A)$, has a canonical representation defined as follows:

Definition 2.1. Let H be a subgroup of $\mathbf{GL}_1(A)$ for a central simple algebra A. Let D be a division F-algebra representing the Brauer class of A. Fix an isomorphism $A \otimes_F D^{\mathrm{op}} \simeq \mathrm{End}(V)$ for an F-vector space V. We call the representation

$$H \hookrightarrow \mathbf{GL}_1(A) \hookrightarrow \mathbf{GL}_1(A \otimes_F D^{\mathrm{op}}) \simeq \mathbf{GL}(V)$$

the canonical representation of H, denoted $\rho^H_{\mathrm{can}}\colon H \to \mathbf{GL}(V)$.

Clearly ρ^H_{can} is faithful of dimension $\deg(A)\,\mathrm{ind}(A)$ and its equivalence class does not depend on the chosen isomorphism $A \otimes_F D^{\mathrm{op}} \simeq \mathrm{End}(V)$. Strictly speaking ρ^H_{can} depends on the embedding of H in $\mathbf{GL}_1(A)$. However it will always be clear from the context, which embedding is meant.

Recall that a representation $H \to \mathbf{GL}(W)$ of an algebraic group H over F in a F-vector space W is called *generically free*, if the affine space $\mathbb{A}(W)$ contains a non-empty H-invariant open subset U on which H acts freely, i.e., any $u \in U(F_{\mathrm{alg}})$ has trivial stabilizer in $H_{\mathrm{alg}} := H_{F_{\mathrm{alg}}}$. By stabilizer we will always mean the scheme-theoretic stabilizer (whose group of R-rational points for any commutative F_{alg}-algebra R is the subgroup of $H(R) = H_{\mathrm{alg}}(R)$ formed by those $h \in H(R)$ satisfying $hu = u$). Generic freeness of W can be tested over a separable or algebraic closure. In fact if $U \subseteq \mathbb{A}(W)_{F_{\mathrm{alg}}}$ is an H_{alg}-invariant nonempty open subset with free H_{alg}-action then the union of all $\mathrm{Gal}(F_{\mathrm{alg}}/F)$-translates of U descends to a nonempty H-invariant open subset with free H-action, see [23, Prop. 11.2.8].
Every generically free representation is faithful, but the converse need not be true. In particular, every generically free representation V of H has dimension $\dim(V) \geq \dim(H)$ and when $\mathrm{ed}(H) > 0$ this inequality is strict by [3, Proposition 4.11].
The main result of this section is the following Theorem:

Theorem 2.2. *Assume that d divides r. Then the canonical representation of $G = N_{\mathbf{GL}_1(A)}(\mathbf{GL}_1(B))$ is generically free if and only if the index of A satisfies*

$$\mathrm{ind}(A) \geq \begin{cases} 2, & \text{if } d = r = 1, m > 1, \\ 3, & \text{if } d = r > 1, \\ r, & \text{if } d = m = 1 \\ \frac{r}{d} + 1, & \text{if } d < r \text{ and } (d > 1 \text{ or } m > 1). \end{cases}$$

In order to prove Theorem 2.2 we start with a couple of intermediate results. We will need the notion of *stabilizer in general position*, abbreviated SGP. An SGP for an action of an algebraic group H (over a field F) on a geometrically irreducible F-variety X is a subgroup S of H with the property that there exists a non-empty open subscheme U of X such that all points $u \in U(F_{\mathrm{alg}})$ have (scheme-theoretic) stabilizers conjugate to $S_{\mathrm{alg}} = S_{F_{\mathrm{alg}}}$. We can always make such a subscheme U invariant under H as follows: Consider $U' := \bigcup_{h \in H(F_{\mathrm{alg}})} hU_{\mathrm{alg}}$, which is a nonempty H_{alg}-invariant open subscheme of X_{alg}. By construction the stabilizer of every $u \in U'(F_{\mathrm{alg}})$ is conjugate to S_{alg}.

Now $U'(F_{\mathrm{alg}})$ is also invariant under the action of the absolute Galois group of F. Therefore, by [23, Prop. 11.2.8] it descends to an H-invariant open subset of X with the same properties as U.

Clearly a representation of H is generically free, if and only if it the trivial subgroup of H is an SGP for that action. Moreover if H acts on X with kernel N, then S is an SGP for the H-action on X if and only if S contains N and S/N is an SGP for the (faithful) H/N-action on X.

The following lemma is well known for algebraically closed fields of characteristic 0. We adapt the proof of [18, Proposition 8] to our more general situation, when F is an arbitrary field.

Lemma 2.3. *Let H act on two geometrically irreducible F-varieties X and Y. Suppose that S_1 is an SGP for the H-action on X and S_2 is an SGP for the S_1-action on Y. Then S_2 is an SGP for the H-action on $X \times Y$.*

Proof. First by replacing X with a suitable non-empty H-invariant open subvariety we may assume that every $x \in X(F_{\mathrm{alg}})$ has stabilizer conjugate to $(S_1)_{\mathrm{alg}}$ in H_{alg}. Let U_Y be a non-empty S_1-invariant open subset of Y such that all $u \in U_Y(F_{\mathrm{alg}})$ have stabilizer conjugate to $(S_2)_{\mathrm{alg}}$ in $(S_1)_{\mathrm{alg}}$. Let $C = H \cdot (X^{S_1} \times U_Y^{S_2})$ denote the set-theoretical image of $(X^{S_1} \times U_Y^{S_2})$ in $X \times Y$ under the action map $H \times (X \times Y) \to X \times Y$. Endow the closure $Z := \bar{C}$ with the reduced scheme structure and consider the morphism $p_X \colon Z \to X$ of schemes given by the composition $Z \hookrightarrow X \times Y \overset{\pi_X}{\to} X$. The fiber of p_X over any $x \in X(F_{\mathrm{alg}})$ has dimension equal to $\dim Y$. In fact if $h_x \in H(F_{\mathrm{alg}})$ is such that $(H_{\mathrm{alg}})_x = h_x (S_1)_{\mathrm{alg}} h_x^{-1}$ then $p_X^{-1}(x)(F_{\mathrm{alg}})$ contains $\{x\} \times h_x U_Y(F_{\mathrm{alg}})$, as one easily checks. Therefore by the fiber dimension theorem $\dim Z = \dim X + \dim Y$ and it follows that C is dense in $X \times Y$. Since C is constructible (by Chevalley's Theorem) there exists a non-empty open subset $U \subset X \times Y$ contained in C. The stabilizer of every $u \in U(F_{\mathrm{alg}})$ is conjugate to $(S_2)_{\mathrm{alg}}$, since this is obviously true for elements of $(X^{S_1} \times U_Y^{S_2})(F_{\mathrm{alg}})$. Therefore S_2 is an SGP for the H-action on $X \times Y$. □

The following proposition will be the key step in order to establish the case of Theorem 2.2, where $m = 1$.

Proposition 2.4. *Let V be a vector space over a field F, whose dual we denote by V^*, and let*

$$H = \mathbf{GL}(V^*) \times \mathbf{GL}(V).$$

For any commutative F-algebra R and $\varphi \in \mathrm{End}(V_R)$ denote by $\varphi^* \in \mathrm{End}(V_R^*)$ the dual endomorphism (given by the formula $(\varphi^*(f))(v) = f(\varphi(v))$ for $v \in V_R$, $f \in V_R^* = \mathrm{Hom}_R(V_R, R)$).

(a) *The image $S \simeq \mathbf{GL}(V)$ of the homomorphism*

$$\mathbf{GL}(V) \to H,\ \varphi \mapsto ((\varphi^*)^{-1}, \varphi)$$

is an SGP for the natural H-action on $V^* \otimes_F V$.

(b) *Let E be a maximal étale subalgebra of* $\mathrm{End}(V)$*. Then the image* $T \simeq \mathbf{GL}_1(E)$ *of the homomorphism*

$$\mathbf{GL}_1(E) \to H,\ \varphi \mapsto ((\varphi^*)^{-1}, \varphi)$$

is an SGP for the natural H-action on $(V^* \otimes_F V)^{\oplus 2}$.

(c) *Let* $Z(H) \simeq \mathbf{G}_m \times \mathbf{G}_m$ *denote the center of H. The image of the homomorphism*

$$\mathbf{G}_m \to Z(H) \subseteq H, \quad \lambda \mapsto (\lambda^{-1}, \lambda)$$

is an SGP for the natural H-action on $(V^* \otimes_F V)^{\oplus 3}$.

(d) *Suppose* $V = V_1 \otimes_F V_2$ *and consider the subgroup*

$$H' = \mathbf{GL}(V_1^*) \times \mathbf{GL}(V)$$

of $H = \mathbf{GL}(V^*) \times \mathbf{GL}(V)$*. Let* $t = \dim(V_2)$*. Then the image S' of the homomorphism*

$$\mathbf{GL}(V_1) \to H',\ \varphi \mapsto ((\varphi^*)^{-1}, \varphi)$$

is an SGP for the natural H'-action on $(V_1^* \otimes_F V)^{\oplus t}$.

Moreover if $t > 1$, the image of the homomorphism

$$\mathbf{G}_m \to Z(H') \subseteq H', \quad \lambda \mapsto (\lambda^{-1}, \lambda)$$

is an SGP for the natural H'-action on $(V_1^* \otimes_F V)^{\oplus (t+1)}$.

Proof. (a) We use the canonical identification of $V^* \otimes_F V$ with the underlying F-vector space of the F-algebra $\mathrm{End}_F(V^*)$, where a pure tensor $f \otimes v$ corresponds to the endomorphism of V^* defined by $f' \mapsto f'(v)f$. The H-action on (the affine space associated with) $V^* \otimes_F V = \mathrm{End}_F(V^*)$ is then given by the formula

$$(\psi, \varphi) \cdot \rho = \psi \rho \varphi^*.$$

Let $U = \mathbf{GL}(V^*) \subseteq \mathbb{A}(\mathrm{End}(V^*))$, which is a non-empty and H-invariant open subset. The stabilizer of $\rho \in U(F_{\mathrm{alg}})$ in H_{alg} is given by the image of the homomorphism

$$\mathbf{GL}(V)_{\mathrm{alg}} \to H_{\mathrm{alg}}, \quad \varphi \mapsto (\rho(\varphi^*)^{-1}\rho^{-1}, \varphi)$$

which is a conjugate of S_{alg} over F_{alg}. This shows the claim.

(b) Let S be the subgroup of H from part (a). By Lemma 2.3 it suffices to show that T is an SGP for the S-action on $V^* \otimes_F V$. Let $U \subseteq \mathbb{A}(V^* \otimes_F V) = \mathbb{A}(\mathrm{End}(V^*))$ be as in part (a). Identify $(V^*)^*$ with V in the usual way, so that $\psi^* \in \mathrm{End}(V)$ for $\psi \in \mathrm{End}(V^*)$. For any $\rho \in U(F_{\mathrm{alg}})$ the stabilizer of ρ in S_{alg} is the image of the centralizer $C_{\mathbf{GL}(V)_{\mathrm{alg}}}(\rho^*)$ under the homomorphism $\mathbf{GL}(V)_{\mathrm{alg}} \to S_{\mathrm{alg}}, \varphi \mapsto ((\varphi^*)^{-1}, \varphi)$. When ρ^* is semisimple regular $C_{\mathbf{GL}(V)_{\mathrm{alg}}}(\rho^*)$ is a maximal torus of $\mathbf{GL}(V)_{\mathrm{alg}}$. Now the claim follows from the well known facts that all maximal tori of $\mathbf{GL}(V)_{\mathrm{alg}}$ are conjugate and the semisimple regular elements in $\mathbb{A}(\mathrm{End}(V^*))$ form a non-empty open subset.

(c) By part (b) $T \simeq \mathbf{GL}_1(E)$ is an SGP for the H-action on two copies of $V^* \otimes_F V$. The kernel of the T-action on $V^* \otimes_F V$ is the image of $\mathbf{G}_m$ in H and coincides with the SGP for this action, since T is a torus, see e.g. [12, Proposition 3.7(A)]. Now the claim follows with Lemma 2.3.

(d) Note that $(V_1^* \otimes_F V)^{\oplus t}$ is H-equivariantly isomorphic to $V_1^* \otimes_F V_2^* \otimes_F V \simeq V^* \otimes_F V$. Define the open subset $U \subseteq \mathbb{A}(V^* \otimes_F V)$ like in part (a). Then every $\rho \in U(F_{\text{alg}})$ has stabilizer in $(H')_{\text{alg}}$ given by the image of the homomorphism

$$\mathbf{GL}(V_1)_{\text{alg}} \to (H')_{\text{alg}}, \quad \alpha \mapsto ((\alpha^*)^{-1}, \rho^* \alpha (\rho^*)^{-1})$$

which is conjugate to $(S')_{\text{alg}}$ over F_{alg}. This shows the first claim.

As an S'-representation $V_1^* \otimes_F V$ is isomorphic to the t-fold direct sum of $W = \text{End}(V_1^*)$ where S' acts through the formula

$$((\varphi^*)^{-1}, \varphi) \cdot \rho = (\varphi^*)^{-1} \rho \varphi^*.$$

As in the proof of part (b) and (c) the S'-action on W has SGP isomorphic to $\mathbf{GL}_1(E')$ for a maximal étale subalgebra E' of $\mathbf{GL}(V_1)$ and the S'-action on $W^{\oplus 2}$ and, since $t > 1$, also on $W^{\oplus t} \simeq V_1^* \otimes_F V$, has as SGP the kernel of this action, which is the image of $\mathbf{G}_m$ in H' by the given homomorphism. Now the claim follows from Lemma 2.3.

□

The next lemma will allow a reduction to the case $m = 1$ in Theorem 2.2 when $d \neq 1$.

Lemma 2.5. *(a) Let $m \geq 1$. A representation of an algebraic group H on a vector space V of dimension $\dim(V) > \dim(H)$ is generically free if and only if the associated representation of the wreath product $H^m \rtimes S_m$ on $V^{\oplus m}$ is generically free.*

(b) Suppose A is split and $d \neq 1$. Then for any $t \geq 1$ generic freeness of $(\rho_{\text{can}}^G)^{\oplus t}$ depends only on r and d, not on m.

Proof. (a) If $H^m \rtimes S_m$ acts generically freely on $V^{\oplus m}$ then so does the subgroup H^m. Let

$$U \subseteq \mathbb{A}(V^{\oplus m}) = \underbrace{\mathbb{A}(V) \times \cdots \times \mathbb{A}(V)}_{m \text{ times}}$$

be a non-empty H^m-invariant open subset where H^m acts freely. Then the projection $\pi_1(U) \subseteq \mathbb{A}(V)$ is non-empty open and H-invariant with free H-action. Hence H acts generically freely on V.

Conversely suppose that H acts generically freely on V. Let $U_0 \subseteq \mathbb{A}(V)$ a friendly open subset, i.e., an H-invariant non-empty open subset such that there exists an H-torsor $\pi\colon U_0 \to Y$ for some irreducible F-scheme Y (which we will fix). Existence of U_0 is granted by a Theorem of P. Gabriel, see [3, Theorem 4.7] or [22, Exposé V, Théoréme 10.3.1]. Since $\dim(U_0) = \dim(V) > \dim(H)$ we have $\dim(Y) > 0$. Hence the open subset $Y^{(m)}$ of Y^m where the m coordinates are different, is nonempty open with free natural S_m-action on it. Now the inverse image of $Y^{(m)}$ in U_0^m under the morphism $\pi^m : U_0^m \to Y^m$ is $H^m \rtimes S_m$-invariant, nonempty and open with $H^m \rtimes S_m$ acting freely on it.

(b) Since the property of being generically free can be checked over an algebraic closure F_{alg} and $(\rho^G_{\mathrm{can}})_{F_{\mathrm{alg}}} = \rho^{G_{\mathrm{alg}}}_{\mathrm{can}}$ we may assume without loss of generality that F is algebraically closed. Let

$$H = (\mathbf{GL}(V_1) \times \mathbf{GL}(V_2))/\mathbf{G}_m,$$

where V_1 and V_2 are vector spaces of dimension $\dim(V_1) = d$, $\dim(V_2) = r$ and $\mathbf{G}_m$ is embedded in the center of $\mathbf{GL}(V_1) \times \mathbf{GL}(V_2)$ through $\lambda \mapsto (\lambda, \lambda^{-1})$. Then

$$G \simeq H^m \rtimes S_m.$$

In particular for $m = 1$ the two groups H and G are isomorphic. Moreover, in general, ρ^G_{can} is given by the obvious homomorphism

$$G \to \mathbf{GL}((V_1 \otimes_F V_2)^{\oplus m}).$$

In order to establish the claim, it suffices to show that the representation of H on $V := (V_1 \otimes_F V_2)^{\oplus t}$ is generically free if and only if the associated representation $(\rho^G_{\mathrm{can}})^{\oplus t}$ of G on $V^{\oplus m}$ is generically free. When $\dim(V) > \dim(H)$ the claim follows from part (a). On the other hand when $\dim(V) \le \dim(H)$ or equivalently $\dim(V^{\oplus m}) \le \dim G$ the two representations of G and H, respectively, are both not generically free, since otherwise the respective group would have essential dimension 0. This is both excluded by the assumption $d \neq 1$, since $B \simeq M_d(F)^m$ and $M_d(F)$ have nontrivial forms over some field extension K/F which embed in $A \otimes_F K \simeq M_{drm}(K)$. Correspondingly there is a non-trivial G-torsor (resp. H-torsor) over K. This torsor cannot be defined over any subfield of transcendence degree 0 over F, since F is algebraically closed.

□

The following lemma tells us how ρ^H_{can} looks over F_{sep}, for any subgroup H of $\mathbf{GL}_1(A)$.

Lemma 2.6. *Over F_{sep} the representation ρ^H_{can} decomposes as a direct sum of* $\mathrm{ind}(A)$ *copies of the canonical representation of $H_{\mathrm{sep}} = H_{F_{\mathrm{sep}}}$.*

Proof. Fix isomorphisms $A_{\mathrm{sep}} \xrightarrow{\sim} \mathrm{End}(V)$, $(D^{\mathrm{op}})_{\mathrm{sep}} \xrightarrow{\sim} \mathrm{End}(W)$ with F_{sep}-vector spaces V and W. Let $w_1, \dots, w_a$ be a basis of W, with $a = \dim(W) = \mathrm{ind}(A)$. Then $(\rho^H_{\mathrm{can}})_{F_{\mathrm{sep}}}$ is equivalent to the composition $H_{\mathrm{sep}} \hookrightarrow \mathbf{GL}(V) \hookrightarrow \mathbf{GL}(V \otimes_{F_{\mathrm{sep}}} W)$, whilst $\rho^{H_{\mathrm{sep}}}_{\mathrm{can}}$ is equivalent to the inclusion $H_{\mathrm{sep}} \hookrightarrow \mathbf{GL}(V)$. Since the subspaces $V \otimes_{F_{\mathrm{sep}}} F_{\mathrm{sep}} w_i$ of $V \otimes_{F_{\mathrm{sep}}} W$ are $\mathbf{GL}(V)$-invariant and $\mathbf{GL}(V)$-equivariantly (and therefore H_{sep}-equivariantly) isomorphic to V, the claim follows. □

We are now ready to prove our main result from this section.

Proof of Theorem 2.2. In view of Lemma 2.6 it suffices to show that the least integer $t \geq 1$ such that the t-fold direct sum of $\rho_{\mathrm{can}}^{G_{\mathrm{sep}}}$ is generically free, is given by the lower bound on the index in the statement of the theorem.

(a) Case $d = r = 1, m > 1$: Here B is a maximal étale subalgebra of A of dimension $\deg(A) = m > 1$. The canonical representation of G_{sep} is given by the natural action of $(\mathbf{G}_m)^m \rtimes S_m$ on $V = F^m$. Let $U \subseteq \mathbb{A}(V) = \mathbb{A}^m$ denote the open subset where all coordinates are non-zero. The group G_{sep} operates transitively on U. Therefore the stabilizer of any $u \in U(F_{\mathrm{alg}})$ is conjugate to the stabilizer of $(1, \ldots, 1)$ in G_{sep}, which is S_m. Therefore S_m is an SGP for the canonical representation of G_{sep}. Moreover S_m acts freely on the S_m-invariant open subset of U, where all coordinates are different. Thus the canonical representation of G_{sep} is not generically free, but two copies of it are, by Lemma 2.3.

(b) Case $d = r > 1$: We must show that two copies of the canonical representation of G_{sep} are not generically free, but three copies are. By Lemma 2.5, since $d > 1$, we may assume that $m = 1$. Let V be an F_{sep}-vector space of dimension $d = r$. Identify B_{sep} with $\mathrm{End}(V^*)$ and its centralizer in A_{sep} with $\mathrm{End}(V)$. This identifies G_{sep} with $(\mathbf{GL}(V^*) \times \mathbf{GL}(V))/\mathbf{G}_m$, where $\mathbf{G}_m$ is embedded in the center of $\mathbf{GL}(V^*) \times \mathbf{GL}(V)$ via $\lambda \mapsto (\lambda^{-1}, \lambda)$. Its canonical representation is given by the natural action on $V^* \otimes_F V$. By Proposition 2.4(b) the sum of two copies of this representation has an SGP in general position of the form $\mathbf{G}_m^d/\mathbf{G}_m$, hence it is not generically free. Moreover Proposition 2.4(c) shows that the sum of three copies of that representation is generically free.

(c) Case $d = m = 1$: Here $G = \mathbf{GL}_1(A)$ with A of degree $drm = r$. By dimension reasons we need at least r copies of the canonical representation of G_{sep} (whose dimension is r) in order to get a generically free representation. On the other hand r copies are clearly enough.

(d) Case $d < r$ and ($d > 1$ or $m > 1$):

First assume $d > 1$. This case is similar to case (b). We must show that $\frac{r}{d} + 1$ copies of the canonical representation of G_{sep} are generically free, but $\frac{r}{d}$ copies are not. By Lemma 2.5 we may assume that $m = 1$. Let V_1 and V_2 be F_{sep}-vector spaces of dimension d and $\frac{r}{d}$, respectively, and set $V = V_1 \otimes_{F_{\mathrm{sep}}} V_2$, which is of dimension r. Identify B_{sep} with $\mathrm{End}(V_1^*)$ and its centralizer in A_{sep} with $\mathrm{End}(V)$, so that $G_{\mathrm{sep}} = (\mathbf{GL}(V_1^*) \times \mathbf{GL}(V))/\mathbf{G}_m$. Its canonical representation is given by the natural action on $V_1^* \otimes_{F_{\mathrm{sep}}} V$. By Proposition 2.4(c) exactly $\dim(V_2) + 1 = \frac{r}{d} + 1$ copies of this representation are needed in order to get a generically free representation. This establishes the claim in case $d > 1$.

Now assume $d = 1 < r$ and $m > 1$. Here B is étale of dimension m with $1 < m < rm = \deg(A)$. Let V denote an r-dimensional

F_{sep}-vector space. Then $G_{\mathrm{sep}} \simeq (\mathbf{GL}(V))^m \rtimes S_m$ and its canonical representation is given by the natural action on $V^{\oplus m}$. We have $\dim G = r^2 m = r \cdot \dim(V^{\oplus m})$. Since G_{sep} is not connected it has $\mathrm{ed}(G_{\mathrm{sep}}) > 0$, see [11, Lemma 10.1], hence we need at least $r+1$ copies of $V^{\oplus m}$ in order to get a generically free representation. On the other hand the connected component $G^0_{\mathrm{sep}} \simeq (\mathbf{GL}(V))^m$ acts generically freely on r copies of $V^{\oplus m}$ and S_m acts generically freely on $V^{\oplus m}$, which implies that G_{sep} acts generically freely on $r+1$ copies of $V^{\oplus m}$. This concludes the proof.

□

3. Proof of Theorem 1.1

The purpose of this section consists in proving the results on $\mathrm{ed}(G)$ as formulated in our main theorem.

Proof of Theorem 1.1. (a) The inequality $\mathrm{ind}(A) \leq \frac{r}{d}$ implies that r is divisible by $d\,\mathrm{ind}(A)$, since $\mathrm{ind}\,A$, r and d are powers of p. In this case natural isomorphism between the functors of $H^1(-,G)$ and $\mathbf{Forms}(B)$ was established in [13, Remark 4.8]. In fact when r is divisible by $d\,\mathrm{ind}(A)$ every form B' of B over a field extension K/F can be embedded in $A \otimes_F K$ with type $[(d,r),\dots,(d,r)]$.

Now for every F-form B' of B the functors $\mathbf{Forms}(B)$ and $\mathbf{Forms}(B')$ are equivalent as functors to the category of sets. The split form of B over F is $M_d(F)^m$ and its automorphism group scheme is $(\mathbf{PGL}_d)^m \rtimes S_m$. This shows that $\mathbf{Forms}(B)$ is naturally isomorphic to the Galois cohomology functor $H^1(-,(\mathbf{PGL}_d)^m \rtimes S_m)$.

(b) Assume $\mathrm{ind}(A) > \frac{r}{d}$. For any algebraic group H over F we have the standard inequality

$$\mathrm{ed}(H) \leq \dim(\rho) - \dim(H)$$

for any generically free representation ρ of H, see [3, Proposition 4.11]. The canonical representation of the group $G = N_{\mathbf{GL}_1(A)}(\mathbf{GL}_1(B))$ has dimension $\deg(A)\,\mathrm{ind}(A)$. Therefore Theorem 2.2 yields the inequality

$$\mathrm{ed}(G) \leq \deg(A)\,\mathrm{ind}(A) - \dim(G) \tag{1}$$

in case

$$\mathrm{ind}(A) \geq \begin{cases} 2, & \text{if } d = r = 1, m > 1, \\ 3, & \text{if } d = r > 1, \\ r, & \text{if } d = m = 1 \\ \frac{r}{d} + 1, & \text{if } d < r \text{ and } (d > 1 \text{ or } m > 1). \end{cases}$$

Combining this with the assumption $\mathrm{ind}(A) > \frac{r}{d}$ shows that inequality (1) is always satisfied, except possibly when $d = r > 1$ and $\mathrm{ind}(A) = 2$.

Now we show the converse to inequality (1). We follow the approach given in [13]. Let $\mathbf{Aut}_F(A,B)$ denote the group scheme of automorphisms of B-preserving automorphisms of A. We have an exact sequence

$$1 \to \mathbf{G}_m \to G \overset{\mathrm{Int}}{\to} \mathbf{Aut}_F(A,B) \to 1,$$

where $\mathrm{Int}\colon G = N_{\mathbf{GL}_1(A)}(\mathbf{GL}_1(B)) \to \mathbf{Aut}_F(A,B)$ takes, for every commutative F-algebra R, the element $g \in G(R) \subseteq (A \otimes_F R)^\times$ to the inner automorphism of $A \otimes_F R$ given by conjugation by g. The connection map

$$H^1(K, \mathbf{Aut}_F(A,B)) \to H^2(K, \mathbf{G}_m) = \mathrm{Br}(K)$$

sends the isomorphism class of a K-form (A', B') of (A,B) to the Brauer class $[A'] - [A \otimes_F K] = [A' \otimes_F A^{\mathrm{op}}]$. Write $\deg(A) = p^s$. By [13, Lemma 2.3] there exists a field extension K/F and a central simple K-algebra A' of the form $A' = D_1 \otimes_K \cdots \otimes_K D_s$ for division K-algebras $D_1, \ldots, D_s$ of degree p, such that

$$\mathrm{ind}(A' \otimes_F A^{\mathrm{op}}) = p^s \,\mathrm{ind}(A) = \deg(A)\,\mathrm{ind}(A).$$

Write $d = p^a$, $r = p^b$, $m = p^c$, so that $a + b + c = s$. Choose a maximal étale K-subalgebra L_i of D_{a+i} for $i \in \{1, \ldots, c\}$. Then

$$B' := D_1 \otimes_K \cdots \otimes_K D_a \otimes_K L_1 \otimes_K \cdots \otimes_K L_c$$

is a separable K-subalgebra of A' of type $[(d,r), \ldots, (d,r)]$ (like B in A). This implies that (A', B') is a K-form of (A,B) by [13, Lemma 4.2(d)]. Therefore the maximal index of a Brauer class contained in the image of a connection map $H^1(K, \mathbf{Aut}_F(A,B)) \to \mathrm{Br}(K)$ for a field extension K/F is precisely $\deg(A)\,\mathrm{ind}(A)$. Now the inequality

$$\mathrm{ed}(G) \geq \deg(A)\,\mathrm{ind}(A) - \dim(G)$$

follows from [5, Corollary 4.2].

□

Remark 3.1. Theorem 1.1 holds with essential dimension replaced by essential p-dimension. For definition of $\mathrm{ed}_p(G)$ see [14] or [21]. In fact part (a) follows from the description of the Galois cohomology functor $H^1(-, G)$ like for essential dimension. Moreover we always have $\mathrm{ed}_p(G) \leq \mathrm{ed}(G)$ and the lower bounds given in part (b) are actually lower bounds on $\mathrm{ed}_p(G)$. This follows from the p-incompressibility of Severi-Brauer varieties of division algebras of p-power degree [9, Theorem 2.1] and [14, Theorem 4.6].

4. The Special Case

In this section we consider the case, which was not resolved by Theorem 1.1. Hence we assume throughout this section that

$$A = M_{2^n}(Q)$$

for some integer $n \geq 0$ and a non-split quaternion F-algebra Q, and $B \subseteq A$ is a separable subalgebra with

$$B_{\text{sep}} \simeq (M_{2^a}(F_{\text{sep}}))^{2^c} \simeq C_{\text{sep}},$$

where $C \subseteq A$ is the centralizer of B in A and a, c are integers with $a \geq 1, c \geq 0$. Note that the relation $drm = \deg(A)$ implies $2a + c = n + 1$. Recall that $G = N_{\mathbf{GL}_1(A)}(\mathbf{GL}_1(B))$.

Example 4.1. Suppose $m = 1$, which means that B is central simple of degree $d = 2^a$. Then the functor $H^1(-, G) \simeq \mathbf{Forms}_A^\theta(B)$ (with $\theta_B = [(d, d)]$) classifies central simple algebras B' of degree d over field extensions K/F such that $B' \otimes_F Q$ is not a division algebra. This is shown as follows: B' embeds in A_K if and only if $B' \otimes_F Q$ embeds in $A_K \otimes_F Q \simeq M_{2d^2}(K)$. If this is the case, the centralizer of $B' \otimes_F Q$ in $M_{2d^2}(K)$ has degree d and has opposite Brauer class to $B' \otimes_F Q$. Therefore the index of $B' \otimes_F Q$ divides d, i.e. $B' \otimes_F Q$ is not a division algebra. Conversely, if the index of $B' \otimes_F Q$ divides d, then the opposite algebra is Brauer equivalent to a degree d algebra, so $B' \otimes_F Q$ embeds in $M_{2d^2}(K)$.

Let L/F be a maximal separable subfield of Q (of dimension 2 over F). The algebra A splits over L. In particular we get the lower bound

$$\operatorname{ed}(\mathbf{Forms}(M_d(L)^m)) = \operatorname{ed}(\mathbf{Forms}(B_L)) = \operatorname{ed}(G_L) \leq \operatorname{ed}(G)$$

on $\operatorname{ed}(G)$ by Theorem 1.1(a) and [3, Proposition 1.5].
Moroever we have the upper bound

$$\begin{aligned}\operatorname{ed}(G) \leq 4\deg(A) - \dim(G) &= 4 \cdot 2^{2a+c} - 2^c((2^a)^2 + (2^a)^2 - 1)\\ &= 2^{2a+c+2} - 2^{2a+c+1} + 2^c\\ &= 2^c(2^{2a+1} + 1),\end{aligned}$$

since 2 copies of ρ^G_{can} are generically free by Theorem 2.2 and Lemma 2.6.
The main effort in this section will go into proving a better upper bound on $\operatorname{ed}(G)$.

For this purpose we will show that the canonical representation of the normalizer of a maximal torus (and even of some larger subgroup) of G is generically free. The following lemma reveals that this will improve the above upper bound on $\operatorname{ed}(G)$.

Lemma 4.2. *Let T be a maximal torus of G and H a subgroup of G containing the normalizer $N_G(T)$. Suppose that ρ^H_{can} is generically free. Then*

$$\begin{aligned}\operatorname{ed}(G) \leq \operatorname{ed}(H) &\leq 2\deg(A) - \dim H\\ &= 2^{c+2a+1} - \dim H.\end{aligned}$$

Proof. The connected component $G^0_{\text{alg}} \simeq ((\mathbf{GL}_{2^a} \times \mathbf{GL}_{2^a})/\mathbf{G}_m)^{2^c}$ of G_{alg} is reductive. Therefore the inclusion $\iota\colon N_G(T) \hookrightarrow G$ induces a surjection of functors

$$\iota_*\colon H^1(-, N_G(T)) \twoheadrightarrow H^1(-, G),$$

see e.g. [6, Corollary 5.3]. Note that G is supposed to be connected reductive there, but the proof goes through if only G^0 is reductive (over F_{alg}).
Since ι factors through H, the map ι_* factors through $H^1(-,H)$. By [3, Lemma 1.9] this proves the first inequality. The second inequality follows from $\dim(\rho^H_{\mathrm{can}}) = 2\deg(A)$ and [3, Proposition 4.11]. □

In order to make use of Lemma 4.2 we will need the following result:

Lemma 4.3. *Let R be a connected reductive algebraic group over F. Let T be a maximal torus of R and let $\mathcal{T}_R \simeq R/N_R(T)$ denote the variety of maximal tori in R. Assume that $R/Z(R)$ is simple, i.e., has no nontrivial normal subgroups. Then there exists a non-empty open subscheme U of $\mathcal{T}_R$ such that every maximal torus of R_{alg} contained in $U(F_{\mathrm{alg}})$ intersects $(N_R(T))_{\mathrm{alg}}$ exactly in $Z(R)_{\mathrm{alg}}$.*

Proof. First note that T contains $Z(R)$, since R is reductive. If T is central in R, then $T = Z(R)$, and the claim easily follows. Hence we may assume that T is non-central. We let R act on $\mathcal{T}_R$ through conjugation. The kernel of this action is a proper normal subgroup of R containing $Z(R)$. Hence it is equal to $Z(R)$. Therefore the kernel of the T-action on $\mathcal{T}_R$ obtained by restriction is also $Z(R)$. By [12, Proposition 3.7] the induced $T/Z(R)$-action on $\mathcal{T}_R$ is generically free. Hence there exists a non-empty open subscheme $\tilde{U}$ of $\mathcal{T}_R$ such that every $S \in \tilde{U}(F_{\mathrm{alg}})$ has stabilizer in T_{alg} equal to $Z(R)_{\mathrm{alg}}$, i.e. $N_{R_{\mathrm{alg}}}(S) \cap T_{\mathrm{alg}} = Z(R)_{\mathrm{alg}}$. For $a \in R(F_{\mathrm{alg}})$ with $S = aT_{\mathrm{alg}}a^{-1}$ this is equivalent to $(N_R(T))_{\mathrm{alg}} \cap (a^{-1}T_{\mathrm{alg}}a) = Z(R)_{\mathrm{alg}}$.
Denote by $\pi\colon R \to \mathcal{T}_R,\ a \mapsto aTa^{-1}$ the projection map and by $\iota\colon R \to R,\ a \mapsto a^{-1}$ the inversion map. Then $U := (\pi \circ \iota)(\pi^{-1}(\tilde{U}))$ has the desired property. □

Proposition 4.4. *With the standing assumptions $r = d = 2^a > 1$, $m = 2^c \geq 1$ and $\mathrm{ind}(A) = 2$:*

$$\begin{aligned}\mathrm{ed}(G) &\leq 2^{c+2a+1} - 2^c(2^{2a} + 2^a - 1)\\ &= 2^c(2^{2a} - 2^a + 1).\end{aligned}$$

Proof. We first consider the case $m = 1$ (i.e., $c = 0$): Let E be a maximal étale subalgebra of the centralizer $C = C_A(B)$ and let

$$H = (\mathbf{GL}_1(B) \times N_{\mathbf{GL}_1(C)}(\mathbf{GL}_1(E)))/\mathbf{G}_m \subseteq G.$$

We will show that ρ^H_{can} is generically free. Since $\dim(H) = 2^{2a} + 2^a - 1$ this would establish the claim in case $m = 1$ in view of Lemma 4.2. Over F_{alg} we may identify H_{alg} with $(\mathbf{GL}(V^*) \times N_{\mathbf{GL}(V)}(T))/\mathbf{G}_m$ where V is an F_{alg}-vector space of dimension $d = 2^a$ and T is a maximal torus of $\mathbf{GL}(V)$. Moreover ρ^H_{can} becomes a direct sum of two copies of the natural representation

$$H_{\mathrm{alg}} \to \mathbf{GL}(V^* \otimes_{F_{\mathrm{alg}}} V) = \mathbf{GL}(\mathrm{End}(V^*))$$

over F_{alg}. Hence it suffices to show that $\mathbf{G}_m$ is an SGP for the natural action of $H' := \mathbf{GL}(V^*) \times N_{\mathbf{GL}(V)}(T)$ on two copies of $W := \mathrm{End}(V^*)$. Identify $N := N_{\mathbf{GL}(V)}(T)$ with its image in H' under the map $\varphi \mapsto ((\varphi^*)^{-1}, \varphi)$. The proof of Proposition 2.4(b) shows that N is an SGP for the H' action on one copy of W. Moreover the stabilizer of any $\rho \in \mathrm{End}(V^*)$ in N is given by the intersection of N with the centralizer $C_{\mathbf{GL}(V)}(\rho^*)$. When ρ is semisimple regular, $C_{\mathbf{GL}(V)}(\rho^*)$ is a maximal torus of $\mathbf{GL}(V)$. It can be considered as a rational point of the variety of maximal tori $\mathcal{T}_{\mathbf{GL}(V)}$ of $\mathbf{GL}(V)$. By Lemma 4.3 there exists a non-empty open subscheme U of $\mathcal{T}_{\mathbf{GL}(V)}$ such that $N \cap S = \mathbf{G}_m$ for every $S \in U(F_{\mathrm{alg}})$. Let $\mathbf{GL}(V^*)^{\mathrm{ss,reg}} \subset \mathbb{A}(W)$ denote the open subset given by the regular semisimple elements. We have a morphism $\mathbf{GL}(V^*)^{\mathrm{ss,reg}} \to \mathcal{T}_{\mathbf{GL}(V)}$, sending a semisimple regular element ρ to the centralizer $C_{\mathbf{GL}(V)}(\rho^*)$. The preimage P of U in $\mathbf{GL}(V^*)^{\mathrm{ss,reg}}$ is a non-empty open subset of $\mathbb{A}(W)$ such that every $\rho \in P(F_{\mathrm{alg}})$ has stabilizer in N equal to $\mathbf{G}_m$. By Lemma 2.3 this implies the claim.

Now let $m = 2^c$ be arbitrary. Since the functor $H^1(-, G)\colon \mathrm{Fields}_F \to \mathrm{Sets}$ depends only on the type of B, we may replace B by any subalgebra of A of the same type as B without changing $\mathrm{ed}(G)$. As

$$A = M_{2^n}(Q) = M_m(B_0 \otimes_F C_0),$$

with $B_0 = M_{2^a}(F)$ and $C_0 = M_{2^{a-1}}(Q)$, we may take for B the m×m diagonal-matrices with entries in B_0. Its centralizer C is the set of m×m diagonal-matrices with entries in C_0. Therefore

$$G = (G_0)^m \rtimes S_m$$

where

$$G_0 = (\mathbf{GL}_1(B_0) \times \mathbf{GL}_1(C_0))/\mathbf{G}_m = N_{\mathbf{GL}_1(B_0 \otimes_F C_0)}(\mathbf{GL}_1(B_0))$$

has $\mathrm{ed}(G_0) \leq 2^{2a} - 2^a + 1$ by the case $m = 1$. By [13, Lemma 4.13] we have $\mathrm{ed}(G) \leq m\,\mathrm{ed}(G_0)$ and the claim follows. □

Remark 4.5. Consider the case $m = 1$. Since $\mathrm{ed}((\mathbf{PGL}_{2^a})_{\mathrm{sep}}) = \mathrm{ed}(G_{\mathrm{sep}}) \leq \mathrm{ed}(G)$ the upper bound

$$\mathrm{ed}(G) \leq 2^{2a} - 2^a + 1$$

should be compared with the best existing upper bound on the essential dimension of $(\mathbf{PGL}_{2^a})_{\mathrm{sep}}$, namely

$$\mathrm{ed}((\mathbf{PGL}_{2^a})_{\mathrm{sep}}) \leq 2^{2a} - 3 \cdot 2^a + 1$$

by [10, Proposition 1.6] (which assumes $\mathrm{char}(F) = 0$).

Corollary 4.6. *Suppose B is central simple (i.e., $m = 1$) and* $\mathrm{char}(F) \neq 2$. *Then*

$$\max\{2, (a-1)2^a + 1\} \leq \mathrm{ed}(G) \leq 2^{2a} - 2^a + 1.$$

In particular when B has degree 2 *we have*

$$\mathrm{ed}(G) \in \{2, 3\}$$

and when B has degree 4 *we have*

$$\mathrm{ed}(G) \in \{5, 6, \ldots, 13\}.$$

If B is central simple of degree 2 *and* $\mathrm{char}(F) = 2$ *we still have* $\mathrm{ed}(G) \in \{2, 3\}$.

Proof. The upper bound on $\mathrm{ed}(G)$ is contained in Proposition 4.4.
By Theorem 1.1(a) we have $\mathrm{ed}(\mathbf{Forms}(M_{2^a}(F_{\mathrm{sep}}))) = \mathrm{ed}(G_{\mathrm{sep}}) \leq \mathrm{ed}(G)$. Hence the lower bound

$$(a-1)2^a + 1 \leq \mathrm{ed}(G)$$

follows from [16, Theorem 6.1] (which assumes $\mathrm{char}(F) \neq 2$) and the lower bound $2 \leq \mathrm{ed}(G)$ follows from [20, Lemma 9.4(a)] (the paper assumes characteristic 0, but the proof works in arbitrary characteristic). □

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Divisible Abelian Groups are Brauer Groups

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2010 Mathematics Subject Classification: 16K50

It is well known that the Brauer group of a field is an abelian torsion group. Examples where the Brauer group of a field can be explicitly computed show that this group is close to being divisible. However, for a long time there was not a single known example of an abelian torsion group A such that $A \not\simeq \mathrm{Br}(F)$ for any field F. First examples of this type were constructed in [3], where it was shown that for $p = 2$ or 3 the p-component of the Brauer group of any field either is an elementary 2-group or contains a non-trivial divisible subgroup. In [1] Fein and Schacher conjectured that any abelian divisible torsion group is isomorphic to the Brauer group of some field. We will now give a proof of this conjecture.

Theorem. *For every abelian divisible torsion group A there exists a field F such that* $\mathrm{Br}(F) \simeq A$.

Proof. We will construct, inductively, a tower of fields $F_1 \subset F_2 \subset F_3 \subset \cdots$ and subgroups $A_i, B_i \subset \mathrm{Br}(F_i)$ satisfying the following conditions:

1. A is isomorphic to A_1.
2. $\mathrm{Br}(F_i) = A_i \oplus B_i$ $(i = 1, 2, \ldots)$.
3. The kernel of the natural homomorphism $\mathrm{Br}(F_i) \to \mathrm{Br}(F_{i+1})$ induced by the inclusion of fields $F_i \subset F_{i+1}$ is B_i. Moreover, this homomorphism restricts to an isomorphism between A_i and A_{i+1}.

[1] Translation given here with the kind permission of Uspekhi Mat. Nauk.

Let us begin with $i = 1$. By [2, Theorem 2] there exists a field F_1 such that A is isomorphic to some subgroup A_1 of $\mathrm{Br}(F_1)$. Since A_1 is divisible, it is a direct summand in $\mathrm{Br}(F_1)$, i.e., there exists a subgroup $B_1 \subset \mathrm{Br}(F_1)$ such that $\mathrm{Br}(F_1) = A_1 \oplus B_1$.

Now suppose we have constructed the fields $F_1 \subset F_2 \subset \cdots \subset F_n$ and subgroups $A_i, B_i \subset \mathrm{Br}(F_i)$ for $i = 1, \ldots, n$. By [2, Theorem 1] there exists a field F_{n+1} such that $F_n \subset F_{n+1}$, and the kernel of the homomorphism $\mathrm{Br}(F_n) \to \mathrm{Br}(F_{n+1})$ induced by this inclusion is B_n. Denote by A_{n+1} the image of A_n under this homomorphism, and by B_{n+1} any complement to A_{n+1} in $\mathrm{Br}(F_{n+1})$ (a complement to A_{n+1} exists because $A_{n+1} \simeq A$ is divisible). This completes the construction of the tower of fields $F_1 \subset F_2 \subset F_3 \subset \cdots$.

Now denote the union of the fields F_i $(i = 1, 2, \ldots)$ by F. Clearly

$$\mathrm{Br}(F) = \varinjlim \mathrm{Br}(F_i) = \varinjlim (A_i \oplus B_i) = \varinjlim A_i \simeq A\,,$$

as desired. □

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Editorial remarks

The above note is a translation of one presented at the October 4, 1983, meeting of the Leningrad Mathematical Society on the occasion of Merkurjev winning the Society's Young Mathematician Prize. It was originally published in Russian in 1985 and has not previously appeared in English.

The prehistory of the note was explained to us by Burt Fein: "Merkurjev made a tour of the US in the early 1980s and visited Oregon State. While he was here, Bill Jacob and I took him to the University of Oregon to give a seminar talk; we also took him to Cafe Zenon to sample their wonderful cream puffs. Over cream puffs I told him about the conjecture from [1] and asked him specifically about whether there was a field with Brauer group $\mathbb{Z}/3$. He solved it on the spot, first using K_2 and then coming up with a more traditional proof that it could not. I wrote up that proof and circulated it to the experts in the field under the title 'Merkurjev's Cream Puff Theorem'. That was the start of reference [3] and eventually to the note itself."

Zero Cycles on Singular Varieties and Their Desingularisations

To Sasha Merkurjev on the occasion of his 60th birthday

Matthew Morrow

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Abstract. We use pro cdh-descent of K-theory to study the relationship between the zero cycles on a singular variety X and those on its desingularisation X'. We prove many cases of a conjecture of S. Bloch and V. Srinivas, and relate the Chow groups of X to the Kerz–Saito Chow group with modulus of X' relative to its exceptional fibre.

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0 Introduction

Let $X' \to X$ be a desingularisation of a d-dimensional, integral variety over a field k, with exceptional fibre $E \hookrightarrow X$. Letting rE denote the r^{th} infinitesimal thickening of E, we denote by $F^dK_0(X', rE)$ the subgroup of the relative K-group $K_0(X', rE)$ generated by the cycle classes of closed points of $X' \setminus E$, for each $r \geq 1$. This inverse system

$$F^dK_0(X', E) \longleftarrow F^dK_0(X', 2E) \longleftarrow F^dK_0(X', 3E) \longleftarrow \cdots$$

was first studied by S. Bloch and V. Srinivas [16], in the case of normal surfaces, as a means of relating zero cycles on the singular variety X to zero cycles on the smooth variety X'. They conjectured [pg. 6, op. cit.] in 1985 that this inverse system would eventually stabilise, i.e., $F^dK_0(X', rE) \overset{\simeq}{\to} F^dK_0(X', (r-1)E)$ for $r \gg 1$, with stable value equal to $F^dK_0(X)$, the subgroup of $K_0(X)$ generated by cycle classes of smooth, closed points of X.

The Bloch–Srinivas conjecture was proved for normal surfaces by A. Krishna and Srinivas [9, Thm. 1.1], and later extended to higher dimensional, Cohen–Macaulay varieties with isolated singularities in characteristic zero by Krishna [6, Thm. 1.1] [7, Thm. 1.2]. The conjecture has not been previously verified in any case of non-isolated singularities, nor for any higher dimensional varieties in finite characteristic.

The primary goal of this paper is to prove the following cases of the Bloch–Srinivas conjecture for varieties which are regular in codimension one:

Theorem 0.1. *Let $\pi : X' \to X$ be a desingularisation of a d-dimensional, quasi-projective, integral variety X over an infinite, perfect field k which is assumed to have strong resolution of singularities. Let $E \hookrightarrow X$ be a closed embedding covering the exceptional fibre, and assume that* $\mathrm{codim}(X, \pi(E)) \geq 2$. *Then the associated Bloch–Srinivas conjecture is*

(i) true up to $(d-1)!$-torsion;

(ii) true if X is projective, $k = k^{alg}$, and $\mathrm{char}\, k = 0$*;*

(iii) true if X is projective, $k = k^{alg}$, and $d \leq$ $\mathrm{char}\, k \neq 0$*;*

(iv) true if X is affine and $k = k^{alg}$;

(v) true "up to a finite group" if $k = k^{alg}$ and X_{sing} is contained in an affine open of X;

(vi) true if $\pi(E)$ is finite;

(vii) true if the cycle class map $CH_0(X) \to F^d K_0(X)$ is an isomorphism.

The group $CH_0(X)$ appearing in part (vii) of Theorem 0.1 is the Levine–Weibel Chow group of zero cycles of the singular variety X [10, 12]; it will be reviewed in Section 1.1.

Part (iv) of the Theorem, combined with arguments of Krishna [7] and R. Murthy [15], has concrete applications to Chow groups of cones and to the structure of modules and ideals of graded algebras; see Theorem 1.17 and Corollaries 1.18 and 1.19.

This paper is intended partly to justify the author's pro cdh-descent theorem for K-theory [13]; indeed, the results of Theorem 0.1 are obtained in Section 1.2 as corollaries of the following general result, which itself is an immediate consequence of pro cdh-descent:

Theorem 0.2. *Let $\pi : X' \to X$ be a desingularisation of a d-dimensional, quasi-projective, integral variety over an infinite, perfect field k which is assumed to have strong resolution of singularities. Let $E \hookrightarrow X$ be a closed embedding covering the exceptional fibre. Then:*

(i) There exists a unique homomorphism $BS_r : F^d K_0(X', rE) \to F^d K_0(X)$ for $r \gg 1$ which is compatible with cycle classes of closed points.

(ii) *The associated Bloch–Srinivas conjecture is true if and only if the canonical map* $F^dK_0(X, rY) \to F^dK_0(X)$ *is an isomorphism for* $r \gg 1$*, where* $Y := \pi(E)_{red}$.

Section 2 concerns Chow groups of zero cycles with modulus. If X is a smooth, projective variety over a field k and D is an effective divisor on X, then the Chow group with modulus $CH_0(X; D)$ is defined to be the free abelian group on the closed points of $X \setminus D$, modulo rational equivalence coming from closed curves C which are not contained in $|D|$ and rational functions $f \in k(C)^\times$ which are $\equiv 1 \bmod D$. This Chow group is central in M. Kerz and S. Saito's [5] higher dimensional class field theory.
It is natural to formulate an analogue of the Bloch–Srinivas conjecture for the Chow groups with modulus given by successive thickenings of the exceptional fibre of a desingularisation. We will explain this further in Section 2, where we prove it in the following cases:

Theorem 0.3. *Let* $\pi : X' \to X$ *be a desingularisation of a d-dimensional, quasi-projective, integral variety over an algebraically closed field k which is assumed to have strong resolution of singularities. Let D be an effective Cartier divisor on X covering the exceptional fibre, and assume that* codim$(X, \pi(D)) \geq 2$.
Then the inverse system

$$CH_0(X'; D) \longleftarrow CH_0(X'; 2D) \longleftarrow CH_0(X'; 3D) \longleftarrow \cdots$$

eventually stabilises with stable value equal to $CH_0(X)$*, assuming that either*

(i) *X is projective and* $\operatorname{char} k = 0$*; or*

(ii) *X is projective and* $d \leq \operatorname{char} k \neq 0$*; or*

(iii) *X is affine.*

Whenever the assertions of Theorem 0.3 can be proved for a singular, projective variety X over a *finite* field (e.g., for surfaces, as we shall see in Remark 2.8), it has applications to the class field theory of X; in particular, it shows that there is a reciprocity isomorphism of finite groups $CH_0(X)^0 \xrightarrow{\simeq} \pi_1^{\mathrm{ab}}(X_{\mathrm{reg}})^0$. See Remark 2.7 for further details.
We prove Theorem 0.3 by reducing it to the analogous assertion in K-theory, which is precisely the Bloch–Srinivas conjecture, and then applying Theorem 0.1. This reduction is through the construction of a new cycle class homomorphism

$$CH_0(X; D) \longrightarrow F^dK_0(X, D),$$

which is valid for any effective Cartier divisor D on a smooth variety X. This also allows us to prove the following result, which appears related to a special case of a conjecture of Kerz and Saito [5, Qu. V]:

Theorem 0.4. *With notation and assumptions as in Theorem 0.3, the cycle class homomorphism*

$$CH_0(X';rD) \longrightarrow F^dK_0(X';rD)$$

is an isomorphism for $r \gg 1$.

Notation, conventions, etc.

A field k will be called *good* if and only if it is infinite, perfect, and has strong resolution of singularities, e.g., $\operatorname{char} k = 0$ suffices. A *k-variety* means simply a finite type k-scheme; further assumptions will be specified when required, and the reference to k with occasionally be omitted. Our conventions about "desingularisations" can be found at the start of Section 1.2.
A *curve* over k is a one-dimensional, integral k-variety. Given a closed point $x \in C_0$, there is an associated order function $\operatorname{ord}_x : k(X)^\times \to \mathbb{Z}$ characterised by the property that $\operatorname{ord}_x(t) = \operatorname{length}_{\mathcal{O}_{C,x}}(\mathcal{O}_{C,x}/t\mathcal{O}_{C,x})$ for any non-zero $t \in \mathcal{O}_{C,x}$; when C is smooth ord_x is the usual valuation associated to x.
An *effective divisor* D on X is by definition a closed subscheme whose defining sheaf of ideals $\mathcal{O}_X(-D)$ is an invertible $\mathcal{O}_X$-module, or, equivalently, is locally defined by a single non-zero-divisor; its associated support is denoted by $|D|$, but we write $X \setminus D$ in place of $X \setminus |D|$ for simplicity.
Given a closed embedding $Y = \operatorname{Spec} \mathcal{O}_X/\mathcal{I} \hookrightarrow X$, its r^{th} infinitesimal thickening is denoted by $rY = \operatorname{Spec} \mathcal{O}_X/\mathcal{I}^r$.
A *pro abelian group* $\{A_r\}_r$ is an inverse system of abelian groups, with morphisms given by the rule

$$\operatorname{Hom}_{\operatorname{Pro} Ab}(\{A_r\}_r, \{B_s\}_s) := \varprojlim_s \varinjlim_r \operatorname{Hom}_{Ab}(A_r, B_s).$$

The category of pro abelian groups is abelian; we refer to [1, App.] for more details.

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1 Zero cycles of desingularisations

In this section we prove cases of the Bloch–Srinivas conjecture relating zero cycles on a singular variety to those on its desingularisation.
There will be an important distinction between closed subsets $S \subseteq X$ and closed subschemes $Y \hookrightarrow X$; in an attempt to keep this clear we will use the

differentiating notation $\subseteq$ and $\hookrightarrow$ just indicated. Any closed subscheme $Y \hookrightarrow X$ has an associated support $|Y| \subseteq X$, though we will continue to write $X \setminus Y$ rather than $X \setminus |Y|$ for the associated open complement, and any closed subset $S \subseteq X$ has an associated reduced closed subscheme $S_{\mathrm{red}} \hookrightarrow X$. The singular locus of X is denoted by $X_{\mathrm{sing}} \subseteq X$.

1.1 Review of the Levine–Weibel Chow group

We begin by reviewing the Levine–Weibel Chow group of zero cycles [10, 12], restricting to the situation that the singularities of X are in codimension ≥ 2, since this is sufficient for our applications. Unless specified otherwise, k is an arbitrary field.

Definition 1.1. Let X be an integral k-variety which is regular in codimension one, and $S \subseteq X$ any closed subset containing X_{sing}. Then the associated *Levine–Weibel Chow group of zero cycles* is

$$CH_0(X;S) := \frac{\text{free abelian group on closed points of } X \setminus S}{\langle (f)_C \,:\, C \hookrightarrow X \text{ a curve not meeting } S, \text{ and } f \in k(C)^\times \rangle}$$

where $(f)_C := \sum_{x \in C_0} \mathrm{ord}_x(f)\, x$ as usual. In particular, $CH_0(X) := CH_0(X; X_{\mathrm{sing}})$.

Remark 1.2. Several remarks should be made:

(i) The group $CH_0(X;S)$ we have just defined can actually only reasonably be called the Levine–Weibel Chow group of zero cycles if we assume that $\mathrm{codim}(X,S) \geq 2$. But it is convenient to introduce the notation in slightly greater generality since it will be useful in Section 2.

(ii) An inclusion of closed subsets $S \subseteq S'$ of X, both containing X_{sing}, induces a canonical surjection $CH_0(X;S') \twoheadrightarrow CH_0(X;S)$. This surjection is an isomorphism if X is quasi-projective and S, S' have codimension ≥ 2, by a moving lemma [12, pg. 113].

(iii) Suppose that X is a smooth k-variety and that $S \subseteq X$ is a closed subset. Then there is a canonical surjection $CH_0(X;S) \twoheadrightarrow CH_0(X;\emptyset) = CH_0(X)$, which will be an isomorphism if S has codimension ≥ 2 and X is quasi-projective, by the aforementioned moving lemma.

(iv) Suppose that $X' \to X$ is a proper morphism which restricts to an isomorphism $X' \setminus S' \xrightarrow{\simeq} X \setminus S$ for some closed subsets $S \subseteq X$, $S' \subseteq X'$ containing the singular loci. Then the induced map $CH_0(X;S) \to CH_0(X';S')$ is an isomorphism. Indeed, both sides are generated by the closed points of $X' \setminus S' = X \setminus S$, and closed curves on X not meeting S correspond to closed curves on X' not meeting S'.

To review the relationship between $CH_0(X)$ and K-theory, we must first explain the cycle class map. Let X be a k-variety, and $i : Y \hookrightarrow X$ a fixed closed subscheme. If $j : C \hookrightarrow X$ is a closed subscheme with image disjoint from both $|Y|$ and X_{sing}, then j is of finite Tor dimension since it factors as $C \hookrightarrow X_{\mathrm{reg}} \to X$, and it is moreover proper; thus the pushforward map $j_* : K(C) \to K(X)$ on the K-theory spectra is well-defined. Moreover, the projection formula [19, Prop. 3.18] associated to the pullback diagram

$$\begin{array}{ccc} \emptyset & \longrightarrow & C \\ \big\downarrow & & \big\downarrow{\scriptstyle j} \\ Y & \underset{i}{\longrightarrow} & X \end{array}$$

shows that the composition $K(C) \xrightarrow{j_*} K(X) \xrightarrow{i^*} K(Y)$ is null-homotopic, and thus there is an induced pushforward $j_* : K(C) \to K(X,Y)$. The *cycle class* of C in $K_0(X,Y)$ is defined to be

$$[C] := j_*([\mathcal{O}_C]) \in K_0(X,Y).$$

Although this appears to depend a priori on a chosen null-homotopy, it was shown by K. Coombes [4] that the "obvious choices of homotopies" yield a class which is functorial with respect to both X and Y, and so we will follow Coombes' choices. A codimension filtration on $K_0(X,Y)$ is now defined by

$$F^pK_0(X,Y) := \langle [C] : C \hookrightarrow X \text{ an integral closed subscheme of } X \text{ of codim} \geq p \\ \text{disjoint from } |Y| \text{ and } X_{\mathrm{sing}} \rangle$$

In particular, $F^dK_0(X,Y)$ is the subgroup of $K_0(X,Y)$ generated by the cycle classes of smooth, closed points of $X \setminus Y$. The following is standard:

Lemma 1.3. *Let notation be as immediately above. If $j : C \hookrightarrow X$ is a closed embedding of a curve into X not meeting $|Y|$ or X_{sing}, and $f \in k(C)^\times$, then $\sum_{x \in C_0} \mathrm{ord}_x(f)[x] = 0$ in $K_0(X,Y)$.*

Proof. One has $\sum_{x \in C_0} \mathrm{ord}_x(f)[x] = j_*([\mathcal{O}_C] - [f\mathcal{O}_C]) = j_*(0) = 0$. □

Now suppose that X is a d-dimensional, integral k-variety which is regular in codimension one, let $Y \hookrightarrow X$ be a closed subscheme, and let $S \subseteq X$ be a closed subset containing both $|Y|$ and X_{sing}. It follows from Lemma 1.3 that the cycle class homomorphism

$$CH_0(X;S) \longrightarrow F^dK_0(X,Y), \quad x \longmapsto [x]$$

is well-defined. In particular, taking $S = X_{\mathrm{sing}}$ and $Y = \emptyset$ yields the cycle class homomorphism

$$[\ \ ] : CH_0(X) \longrightarrow F^dK_0(X),$$

which is evidently surjective. Moreover, as part of a general Riemann–Roch theory, M. Levine [11, 10] constructed a Chern class $ch_0 : F^d K_0(X) \to CH_0(X)$ such that the compositions $[\ \ ] \circ ch_0$ and $ch_0 \circ [\ \ ]$ are both multiplication by $(-1)^{d-1}(d-1)!$. In particular, $[\ \ ] : CH_0(X) \to F^d K_0(X)$ is an isomorphism if $d = 2$.
We complete our review of the Levine–Weibel Chow group of zero cycles by presenting the higher dimensional cases in which the cycle class homomorphism can be shown to be an isomorphism:

Theorem 1.4 (Barbieri Viale, Levine, Srinivas). *Let X be a d-dimensional, integral, quasi-projective variety over an algebraically closed field k which is regular in codimension one. Then the cycle class homomorphism $CH_0(X) \to F^d K_0(X)$ is*

(i) an isomorphism if X is projective and $\operatorname{char} k = 0$;

(ii) an isomorphism if X is projective and $d \leq \operatorname{char} k \neq 0$;

(iii) an isomorphism if X is affine and $\operatorname{char} k$ is arbitrary;

(iv) a surjection with finite kernel if X_{sing} is contained in an affine open subscheme of X and $\operatorname{char} k = 0$;

(v) a surjection with finite kernel if X_{sing} is contained in an affine open subscheme of X and $d \leq \operatorname{char} k \neq 0$;

Proof. Thanks to the existence of Levine's Chern class ch_0, it is enough to check that $CH_0(X)$ has no $(d-1)!$-torsion in cases (i)–(ii), that it has only a finite amount of $(d-1)!$-torsion in cases (iv)–(v), and that it has no torsion in case (iii).
Then (i) and (ii) are [10, Thm. 3.2], while (iv) and (v) are [2, Thm. A]. Finally, (iii) in characteristic zero (and when $d \leq \operatorname{char} k \neq 0$) is [10, Corol. 2.7], and so it remains only to deal with the following case: assuming that X is an integral, affine variety which is regular in codimension one, over an algebraically closed field of finite characteristic, we must show that $CH_0(X)$ is torsion-free. This is true for the normalisation $\widetilde{X}$ by [18], and so it remains only to check that $CH_0(X) \overset{\simeq}{\to} CH_0(\widetilde{X})$. But since X is assumed to be regular in codimension one, there are closed subsets $S \subseteq X$, $S' \subseteq \widetilde{X}$ (given by the conductor ideal, for example) of codimension ≥ 2, containing the singular loci, and such that the morphism $\widetilde{X} \to X$ restricts to an isomorphism $\widetilde{X} \setminus S' \overset{\simeq}{\to} X \setminus S$. Then, in the commutative diagram

$$\begin{array}{ccc} CH_0(\widetilde{X}; S') & \longrightarrow & CH_0(\widetilde{X}) \\ \uparrow & & \uparrow \\ CH_0(X; S) & \longrightarrow & CH_0(X) \end{array}$$

the horizontal arrows are isomorphisms by Remark 1.2(ii), while the left vertical arrow is an isomorphism by Remark 1.2(iv). Hence the right vertical arrow is an isomorphism, as required. ☐

1.2 The Bloch–Srinivas conjecture

Before we can carefully state the Bloch–Srinivas conjecture we must first fix some terminology concerning desingularisations. Given an integral variety X, a *desingularisation* is any proper, birational morphism $\pi : X' \to X$ where X' is smooth; in particular, we allow the desingularisation to change the smooth locus of X, though it is not clear if this is ever important in practice. There exists a smallest closed subset $S \subseteq X$ with the property that $X' \setminus \pi^{-1}(S) \stackrel{\simeq}{\to} X \setminus S$, and $\pi^{-1}(S)$ is known as the *exceptional set* of the resolution; setting $E := \pi^{-1}(S)_{\text{red}}$ yields the *exceptional fibre* $E \hookrightarrow X'$. Corollaries 1.10–1.15 will require that $\pi(|E|)$ has codimension ≥ 2 in X, which in particular implies that X is regular in codimension one.
If $X' \to X$ is a desingularisation of an integral variety X, with exceptional fibre $E \hookrightarrow X'$, then Bloch and Srinivas [16, pg. 6] made the following conjecture in 1985:

Conjecture 1.5 (Bloch–Srinivas). The inverse system

$$F^d K_0(X', E) \longleftarrow F^d K_0(X', 2E) \longleftarrow F^d K_0(X', 3E) \longleftarrow \cdots$$

stabilises, with stable value $F^d K_0(X)$.

Remark 1.6. To be precise, Bloch and Srinivas stated their conjecture in the case of a normal surface X over an algebraically closed field, assuming that the desingularisation did not alter the smooth locus of X. If Conjecture 1.5 is false because it has been formulated in excessive generality, it is the author's fault. In fact, we will consider Conjecture 1.5 in greater generality still, by replacing the exceptional fibre E by any reduced closed subscheme $E \hookrightarrow X'$ whose support contains the exceptional set (henceforth "covers the exceptional set").

We interpret part of the Bloch–Srinivas conjecture as an implicit statement that there exists a cycle class homomorphism

$$BS_r : F^d K_0(X', rE) \longrightarrow F^d K_0(X)$$

for $r \gg 1$ which is compatible with cycle classes of closed points $x \in X' \setminus E$, i.e., $BS_r([x]) = [x]$. Such a map BS_r is unique if it exists.
Our main technical theorem, which is an immediate consequence of the author's pro cdh-descent theorem for K-theory [13], proves the existence of the maps BS_r in full generality, and reduces the Bloch–Srinivas conjecture to the study of the K-theory of X:

Theorem 1.7. *Let X be a d-dimensional, integral variety over a good field k; let $\pi : X' \to X$ be a desingularisation, $E \hookrightarrow X'$ any reduced closed subscheme covering the exceptional set, and set $Y := \pi(|E|)_{red}$. Then:*

(i) *For $r \gg 1$, the canonical map $F^dK_0(X, rY) \to F^dK_0(X)$ factors through the surjection $F^dK_0(X, rY) \to F^dK_0(X', rE)$, i.e., there exists a commutative diagram*

$$\begin{array}{ccc} F^dK_0(X', rE) & \longrightarrow & F^dK_0(X') \\ \uparrow & \overset{\exists\, BS_r}{\searrow} & \uparrow \\ F^dK_0(X, rY) & \longrightarrow & F^dK_0(X) \end{array}$$

(ii) *The following are equivalent:*

(a) *The associated Bloch–Srinivas conjecture is true, i.e., BS_r is an isomorphism for $r \gg 1$.*

(b) *The canonical map $F^dK_0(X, rY) \to F^dK_0(X)$ is an isomorphism for $r \gg 1$.*

(c) *The canonical map $F^dK_0(X, rY) \to F^dK_0(X)$ is an isomorphism for all $r \geq 1$.*

Proof. There is an abstract blow-up square

$$\begin{array}{ccc} Y' & \longrightarrow & X' \\ \downarrow & & \downarrow \pi \\ Y & \longrightarrow & X \end{array}$$

where $Y' := X' \times_X Y$; note that Y' is a nilpotent thickening of E. By pro cdh-descent for K-theory [13, Thm. 0.1] (it is here that the field k is required to be good), the canonical homomorphism of pro abelian groups

$$\{K_0(X, rY)\}_r \longrightarrow \{K_0(X', rY')\}_r \cong \{K_0(X', rE)\}_r$$

is an isomorphism. Restricting to the codimension filtration we deduce that the homomorphism

$$\{F^dK_0(X, rY)\}_r \longrightarrow \{F^dK_0(X', rE)\}_r \qquad (\dagger)$$

is injective; but each map $F^dK_0(X, rY) \to F^dK_0(X', rE)$ is evidently surjective, since both sides are generated by the closed points of $X \setminus Y = X' \setminus E$. Thus (†) is an isomorphism.

By definition of an isomorphism of pro abelian groups, this implies that for any $s \geq 1$ there exists $r \geq s$ and a homomorphism $F^dK_0(X', rE') \to F^dK_0(X, sY)$ making the diagram commute:

$$\begin{array}{ccc} F^dK_0(X', rE) & & \\ \uparrow & \overset{\exists}{\searrow} & \\ F^dK_0(X, rY) & \longrightarrow & F^dK_0(X, sY) \end{array}$$

Note that the vertical and horizontal arrows are surjective, since the groups are generated by the closed points of $X \setminus Y = X' \setminus E$. This diagram shows that the canonical map $F^dK_0(X, rY) \to F^dK_0(X)$ factors through the surjection $F^dK_0(X, rY) \to F^dK_0(X', rE)$, proving (i).
This gives a commutative diagram

$$\begin{array}{ccccc} F^dK_0(X',rE) & & \xrightarrow{\quad BS_r \quad} & & \\ \uparrow & \searrow & & \searrow & \\ F^dK_0(X,rY) & \twoheadrightarrow & F^dK_0(X,sY) & \longrightarrow & F^dK_0(X) \end{array}$$

from which a simple diagram chase yields the following implications (valid for any $s \geq 1$ and $r \gg s$):

> $F^dK_0(X, rY) \to F^dK_0(X)$ is an isomorphism $\implies$ BS_r is an isomorphism.
> BS_r is an isomorphism $\implies$ $F^dK_0(X, sY) \to F^dK_0(X)$ is an isomorphism.

The equivalence of (a)–(c) follow, completing the proof. □

REMARK 1.8. Suppose that the desingularisation $X' \to X$ does not change the smooth locus of X and that E is equal to the exceptional fibre (this is probably the most important case of the conjecture). Then Theorem 1.7 states that the associated Bloch–Srinivas conjecture is true if and only if $F^dK_0(X, rY) \overset{\cong}{\to} F^dK_0(X)$ for $r \gg 1$, where $Y = (X_{\mathrm{sing}})_{\mathrm{red}}$.
In particular, under these additional hypotheses on X' and E we see that the Bloch–Srinivas conjecture depends only on X, and not on the chosen desingularisation. Even in the case of arbitrary desingularisations and general E covering the exceptional set, Theorem 1.7 shows that the associated Bloch–Srinivas conjecture depends only on X and $\pi(|E|)$.

REMARK 1.9. The proof of Theorem 1.7 also shows the following: the inverse system $F^dK_0(X', rE)$, $r \geq 1$, stabilises if and only if the inverse system $F^dK_0(X, rY)$, $r \geq 1$, stabilises, in which case the canonical map $F^dK_0(X, rY) \to F^dK_0(X', rE)$ is an isomorphism for $r \gg 1$.

The following corollary recovers all previously known cases of the Bloch–Srinivas conjecture (normal surfaces [9, Thm. 1.1]; Cohen–Macaulay varieties with isolated singularities in characteristic zero [6, Thm. 1.1] [7, Thm. 1.2]; note that in these cases one can use the reduction ideal trick of Weibel [20] to avoid assuming that k has resolution of singularities, c.f., Remark 2.8):

COROLLARY 1.10. *Let X be a d-dimensional, integral variety over a good field k; let $\pi : X' \to X$ be a desingularisation, and $E \hookrightarrow X'$ any reduced closed subscheme covering the exceptional set. Assume $\pi(|E|)$ is finite and $d \geq 2$. Then the associated Bloch–Srinivas conjecture is true.*

Proof. Set $Y := \pi(|E|)_{\mathrm{red}}$. According to Theorem 1.7, it is necessary and sufficient to show that the canonical map $F^dK_0(X, rY) \to F^dK_0(X)$ is an isomorphism for all $r \geq 1$. But this follows from [6, Lem. 3.1] since rY is zero dimensional. ☐

The next corollary proves the Bloch–Srinivas conjecture under the assumption that the cycle class homomorphism $CH_0(X) \to F^dK_0(X)$ is an isomorphism:

Corollary 1.11. *Let X be a d-dimensional, integral, quasi-projective variety over a good field k; let $\pi : X' \to X$ be a desingularisation, and $E \hookrightarrow X'$ any reduced closed subscheme covering the exceptional set. Assume* $\mathrm{codim}(X, \pi(|E|)) \geq 2$ *and that the cycle class map $CH_0(X) \to F^dK_0(X)$ is an isomorphism.*
Then the associated Bloch–Srinivas conjecture is true.

Proof. Set $Y = \pi(|E|)_{\mathrm{red}}$. According to Theorem 1.7, it is necessary and sufficient to show that the canonical map $F^dK_0(X, rY) \to F^dK_0(X)$ is an isomorphism for all $r \geq 1$. To prove this we consider the commutative diagram

$$\begin{array}{ccc} F^dK_0(X, rY) & \longrightarrow & F^dK_0(X) \\ \uparrow & & \uparrow \\ CH_0(X; |Y|) & \longrightarrow & CH_0(X) \end{array}$$

The right vertical arrow is an isomorphism by assumption, the bottom horizontal arrow is an isomorphism by Remark 1.2(ii), and the left vertical arrow is a surjection since the domain and codomain are generated by the closed points of $X \setminus Y$. It follows that the top horizontal arrow (and left vertical arrow – we will need this in the proof of Theorem 2.5) is an isomorphism, as desired. ☐

In particular, we have proved the Bloch–Srinivas conjecture for projective varieties over an algebraically closed field of characteristic zero which are regular in codimension one:

Corollary 1.12. *Let X be a d-dimensional, integral variety over an algebraically closed field k which has strong resolution of singularities; let $\pi : X' \to X$ be a desingularisation, and $E \hookrightarrow X'$ any reduced closed subscheme covering the exceptional set. Assume* $\mathrm{codim}(X, \pi(|E|)) \geq 2$ *and that one of the following is true:*

(i) X is projective and $\mathrm{char}\, k = 0$*; or*

(ii) X is projective and $d \leq \mathrm{char}\, k \neq 0$*; or*

(iii) X is affine and $\mathrm{char}\, k$ *is arbitrary.*

Then the associated Bloch–Srinivas conjecture is true.

Proof. This follows from Corollary 1.11 and the results of Levine and Srinivas recalled in Theorem 1.4. ☐

Remark 1.13. It seems plausible that some descent or base change technique should eliminate the requirement in Corollary 1.12 that k be algebraically closed.

We can also solve the Bloch–Srinivas conjecture up to $(d-1)!$-torsion whenever X is regular in codimension one:

Corollary 1.14. *Let X be a d-dimensional, integral, quasi-projective variety over a good field k; let $\pi : X' \to X$ be a desingularisation, and $E \hookrightarrow X'$ any reduced closed subscheme covering the exceptional set. Assume* $\operatorname{codim}(X, \pi(|E|)) \geq 2$.
Then the associated Bloch–Srinivas conjecture is true up to $(d-1)!$-torsion, i.e., the maps

$$BS_r : F^d K_0(X', rE) \otimes \mathbb{Z}[\tfrac{1}{(d-1)!}] \longrightarrow F^d K_0(X) \otimes \mathbb{Z}[\tfrac{1}{(d-1)!}]$$

are isomorphisms for $r \gg 1$.

Proof. Set $Y = \pi(|E|)_{\mathrm{red}}$. By a trivial modification of Theorem 1.7, it is necessary and sufficient to show that the canonical map $F^d K_0(X, rY) \to F^d K_0(X)$ is an isomorphism for all $r \geq 1$ after inverting $(d-1)!$. This follows exactly as in Corollary 1.11, since the cycle class map $CH_0(X) \to F^d K_0(X)$ is an isomorphism after inverting $(d-1)!$, thanks to the existence of Levine Chern class $ch_0 : F^d K_0(X) \to CH_0(X)$. □

The next result solves the Bloch–Srinivas conjecture up to a finite group when the singular locus X_{sing} has codimension ≥ 2 and is contained in an affine open of X. Note that the "obvious" cases in which this happens, namely when X_{sing} is finite or X itself is affine, are already largely covered by Corollaries 1.10 and 1.12(iii) respectively:

Corollary 1.15. *Let X be a d-dimensional, integral, quasi-projective variety over an algebraically closed field k which has strong resolution of singularities; let $\pi : X' \to X$ be a desingularisation, and $E \hookrightarrow X'$ any reduced closed subscheme covering the exceptional set. Assume* $\operatorname{codim}(X, \pi(|E|)) \geq 2$*, that X_{sing} is contained in an affine open of X, and moreover that $d \leq \operatorname{char} k$ if $\operatorname{char} k \neq 0$. Then the maps*

$$BS_r : F^d K_0(X', rE) \longrightarrow F^d K_0(X)$$

are surjective with finite kernel for $r \gg 1$, and the inverse system $F^d K_0(X', rE)$, $r \geq 1$, stabilises.

Proof. Set $Y = \pi(|E|)_{\mathrm{red}}$. We concatenate commutative diagrams we have already considered in Theorem 1.7 and Corollary 1.11:

$$\begin{array}{ccc}
F^dK_0(X', rE) & \xrightarrow{BS_r} & \\
\uparrow\!\!\uparrow & \searrow & \\
F^dK_0(X, rY) & \longrightarrow & F^dK_0(X) \\
\uparrow\!\!\uparrow & & \uparrow \\
CH_0(X; |Y|) & \xrightarrow{\cong} & CH_0(X)
\end{array}$$

The left vertical arrows are surjective since the groups are generated by the closed points of $X \setminus Y = X' \setminus E$; the bottom horizontal arrow is an isomorphism by Remark 1.2(ii); the right vertical arrow is surjective with finite kernel Λ by the result of Barbieri Viale recalled in Theorem 1.4.
A simple diagram chase shows that BS_r is surjective and that its kernel Λ_r is naturally a quotient of Λ. Since Λ is finite, this tower of quotients Λ_r must eventually stabilise, completing the proof. □

Remark 1.16. We finish our discussion of the Bloch–Srinivas conjecture with a remark about SK_1. Let $\pi : X' \to X$, E, Y, k be as in the statement of Theorem 1.7, and assume X is quasi-projective and $\operatorname{codim}(X, Y) \geq 2$.
The maps $F^dK_0(X, rY) \to F^dK_0(X)$ are surjective for all $r \geq 1$ (by Remark 1.2(ii) and existence of the cycle class maps); hence we may add

(b′) The canonical map $F^dK_0(X, rY) \to F^dK_0(X)$ is injective for $r \gg 1$.

to the list of equivalent conditions in Theorem 1.7(ii).
Next, it follows from [6, Lem. 3.1] that (b′) (hence the associated Bloch–Srinivas conjecture) would follow from showing that $\partial(SK_1(rY)) = 0$, where $\partial : K_1(rY) \to K_0(X, rY)$ is the boundary map and $SK_1(rY) := \operatorname{Ker}(K_1(rY) \twoheadrightarrow H^0(rY, \mathcal{O}_{rY}^\times))$; equivalently, it is enough to show that $SK_1(X) \to SK_1(rY)$ is surjective. Using the arguments of Theorem 1.7 it would even be enough to show, for each $r \gg 1$, that

$$\operatorname{Im}(SK_1(sY) \to SK_1(rY)) \subseteq \operatorname{Im}(SK_1(X) \to SK_1(rY))$$

for some $s \geq r$. It is not clear whether one should expect this to be true.

We finish the section with some consequence of the Bloch–Srinivas conjecture. The following result about Chow groups of cones was conjectured by Srinivas [17, §3] in 1987; it was proved by Krishna [7, Thm. 1.5] under the assumption that the cone X was normal and Cohen–Macaulay, and we will combine his argument with Theorem 1.7 to establish the result in general; due to the failure of Kodaira vanishing in finite characteristic we must restrict to characteristic zero:

Theorem 1.17. *Let $Y \hookrightarrow \mathbb{P}^N_k$ be a d-dimensional, smooth, projective variety over an algebraically closed field k of characteristic zero; assume $d > 0$ and $H^d(Y, \mathcal{O}_Y(1)) = 0$, and let X be the affine cone over Y. Then $CH_0(X) = 0$.*

Proof. We may resolve X, which has a unique singular point, to obtain X' which is a line bundle over Y, whose zero section is the exceptional fibre of the resolution $X' \to X$. By Corollary 1.10 or 1.12(iii), we know that $CH_0(X) \cong F^{d+1}K_0(X', rY)$ for $r \gg 1$; moreover, $CH_0(X')$ surjects onto $F^dK_0(X')$, and $CH_0(X') = 0$ since X' is a line bundle, so $F^dK_0(X') = 0$. So it is enough to show that the canonical map $F^{d+1}K_0(X', rY) \to F^{d+1}K_0(X')$ is an isomorphism. According to Krishna's proof of [7, Cor. 8.5], this would follows from knowing that:

(i) $H^d(X', \mathcal{K}_{d,X'}) \otimes k^\times \longrightarrow H^d(Y, \mathcal{K}_{d,Y}) \otimes k^\times$ is surjective; and

(ii) $H^d\left(rY, \frac{\Omega^d_{(rY,Y)}}{d\Omega^{d-1}_{(rY,Y)}}\right) = 0$ for $r \gg 1$.

Condition (i) is satisfied since the zero section $Y \hookrightarrow X'$ is split by the line bundle structure map $X' \to Y$. Condition (ii) is deduced from the Akizuki–Nakano vanishing theorem, as explained in Lem. 9.1 and the proof of Thm. 1.5 in [7]. □

Corollary 1.18. *Let Y, k be as in the previous theorem, and let A be its homogeneous coordinate ring. Then every projective module over A of rank at least d has a free direct summand of rank one.*

Proof. This follows from Theorem 1.17 using a result of R. Murthy [15, Cor. 3.9]. □

Corollary 1.19. *Let k be an algebraically closed field of characteristic zero, and $f \in k[\underline{t}] := k[t_0, \dots, t_d]$ a homogenous polynomial of degree at most $d+1$ which defines a smooth hypersurface in $\mathbb{P}^d_k$. Then every smooth closed point of* $\operatorname{Spec} k[\underline{t}]/\langle f\rangle$ *is a complete intersection.*
In other words, if $\mathfrak{m}$ is any maximal ideal of $k[\underline{t}]$ containing f other than the origin, then $\mathfrak{m} = \langle f, f_1, \dots, f_d\rangle$ for some $f_1, \dots, f_d \in k[\underline{t}]$.

Proof. This also follows from Theorem 1.17 thanks to Murthy [15, Thm. 4.4]. □

2 Chow groups with modulus

If X is a smooth variety over a field k, and D is an effective divisor on X, then the Chow group $CH_0(X; |D|)$ from Definition 1.1 may be a rather coarse invariant, as there may not be enough curves on X avoiding the codimension-one subset $|D|$. Of greater interest is $CH_0(X; D)$, the Chow group of zero cycles on X with modulus D, which we will define precisely in Definition 2.1; note

the notational difference, indicating that $CH_0(X;D)$ depends not only on the support of D, but on its schematic, and possibly non-reduced, structure.
According to the higher dimensional class field theory of M. Kerz and S. Saito, when k is finite and X is proper over k, the group $CH_0(X;D)$ classifies the abelian étale covers of $X \setminus D$ whose ramification is bounded by D; we refer the reader to [5] for details since we will not require any of their results.
We now turn to definitions, and refer again to [op. cit.] for a more detailed exposition. Let C be a smooth curve over a field k, and D an effective divisor on C; writing $D = \sum_{x \in |D|} m_x x$ as a Weil divisor, we let

$$k(C)^\times_D := \{f \in k(C)^\times : \mathrm{ord}_x(f-1) \geq m_x \text{ for all } x \in |D|\}$$

denote the rational functions on C which are $\equiv 1 \bmod D$. More generally, if X is a smooth variety over k and D is an effective divisor on X, then for any curve $C \hookrightarrow X$ which is not contained in $|D|$ we write

$$k(C)^\times_D := k(\widetilde{C})^\times_{\phi^* D},$$

where $\phi : \widetilde{C} \to C \hookrightarrow X$ is the resulting map from the normalisation $\widetilde{C}$ to X; evidently $k(C)^\times_D = k(C)^\times$ if C does not meet $|D|$.
The Chow group with modulus is defined as follows:

Definition 2.1. Let X be a smooth variety over k, and D an effective divisor on X. Then the associated *Chow group of zero cycles of X with modulus D* is

$$CH_0(X;D) := \frac{\text{free abelian group on closed points of } X \setminus D}{\langle (f)_C \ : \ C \hookrightarrow X \text{ a curve not contained in } |D|, \text{ and } f \in k(C)^\times_D \rangle}$$

where $(f)_C = \sum_{x \in C_0} \mathrm{ord}_x(f)\, x$.

If we were to define

$$k(C)^\times_{|D|} := \begin{cases} k(C)^\times & \text{if } C \text{ does not meet } |D|, \\ 1 & \text{if } C \text{ meets } |D|, \end{cases}$$

and repeat Definition 2.1 with $|D|$ in place of D, then the resulting group $CH_0(X;|D|)$ would coincide with that defined in Definition 1.1. Since $k(C)^\times_{|D|} \subseteq k(C)^\times_D$, we thus obtain a canonical surjection

$$CH_0(X;|D|) \twoheadrightarrow CH_0(X;D).$$

One sense in which $CH_0(X;D)$ is a more refined invariant than $CH_0(X;|D|)$ is that the cycle class homomorphism $CH_0(X;|D|) \to K_0(X,D)$ of Section 1.1 factors through $CH_0(X;D)$. There does not appear to be a proof of this important result in the literature, so we give one here, beginning with a much stronger result in the case of curves:

LEMMA 2.2. *Let C be a smooth curve over a field k, and D an effective divisor on C. Then the canonical map*

$$\text{free abelian group on closed points of } C \setminus D \longrightarrow K_0(C,D), \quad x \longmapsto [x]$$

induces an injective cycle class homomorphism

$$CH_0(C;D) \longrightarrow K_0(C,D),$$

which is an isomorphism if $D \neq 0$ (and has cokernel $= \mathbb{Z}$ if $D = 0$).

Proof. The Zariski descent spectral sequence for the K-theory of C relative to D degenerates to short exact sequences, since $\dim C = 1$, yielding in particular

$$0 \longrightarrow H^1(C, \mathcal{K}_{1,(C,D)}) \longrightarrow K_0(C,D) \longrightarrow H^0(C, \mathcal{K}_{0,(C,D)}) \longrightarrow 0.$$

Here $\mathcal{K}_{i,(C,D)}$ is by definition the Zariski sheafification on C of the presheaf $U \mapsto K_i(U, U \times_C D)$.
To describe these terms further we make some standard comments about the long exact sequence of sheaves

$$\mathcal{K}_{2,C} \to \mathcal{K}_{2,D} \to \mathcal{K}_{1,(C,D)} \to \mathcal{K}_{1,C} \to \mathcal{K}_{1,D} \to \mathcal{K}_{0,(C,D)} \to \mathcal{K}_{0,C} \to \mathcal{K}_{0,D}.$$

Firstly, $\mathcal{K}_{1,C} \cong \mathcal{O}_C^\times$ and $\mathcal{K}_{1,D} \cong \mathcal{O}_D^\times$, so the map $\mathcal{K}_{1,C} \to \mathcal{K}_{1,D}$ is surjective; moreover, the sheaves $\mathcal{K}_{2,C}$ and $\mathcal{K}_{2,D}$ are generated by symbols, and so the map $\mathcal{K}_{2,C} \to \mathcal{K}_{2,D}$ is also surjective. It follows that $\mathcal{K}_{1,(C,D)} \cong \operatorname{Ker}(\mathcal{O}_C^\times \to \mathcal{O}_D^\times) =: \mathcal{O}_{(C,D)}^\times$ and that $H^0(C, \mathcal{K}_{0,(C,D)}) = \operatorname{Ker}(H^0(C, \mathcal{K}_{0,C}) \to H^0(D, \mathcal{K}_{0,D}))$. Secondly, $\mathcal{K}_{0,C} \cong \mathbb{Z}$ via the rank map, and so $H^0(C, \mathcal{K}_{0,C}) \cong \mathbb{Z}$; similarly, $H^0(D, \mathcal{K}_{0,D}) \cong \bigoplus_{x \in |D|} \mathbb{Z}$ via the rank map. If $D \neq 0$, we deduce that the map $H^0(C, \mathcal{K}_{0,C}) \to H^0(D, \mathcal{K}_{0,D})$ is injective and so $H^0(C, \mathcal{K}_{0,(C,D)}) = 0$; while if $D = 0$ then evidently $H^0(X, \mathcal{K}_{0,(C,D)}) = H^0(X, \mathcal{K}_{0,C}) \cong \mathbb{Z}$.
In conclusion, it remains only to construct the cycle class isomorphism

$$CH_0(C;D) \xrightarrow{\simeq} H^1(C, \mathcal{O}_{(C,D)}^\times).$$

We will do this via a standard Gersten resolution.
Given an open subscheme $U \subseteq C$ containing $|D|$, let $j_U : U \to C$ denote the open inclusion. Then the canonical map $\mathcal{O}_{(C,D)}^\times \to j_{U*} j_U^* \mathcal{O}_{C,D}^\times$ fits into an exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_{(C,D)}^\times \longrightarrow j_{U*} j_U^* \mathcal{O}_{(C,D)}^\times \xrightarrow{(\mathrm{ord}_x)_x} \bigoplus_{x \in C \setminus U} i_{x*}\mathbb{Z} \longrightarrow 0,$$

where $i_{x*}\mathbb{Z}$ is a skyscraper sheaf at the closed point x. This remains exact after taking the filtered colimit over all open U containing $|D|$, yielding

$$0 \longrightarrow \mathcal{O}_{(C,D)}^\times \longrightarrow k(C)_D^\times \xrightarrow{(\mathrm{ord}_x)_x} \bigoplus_{x \in C_0 \setminus D} i_{x*}\mathbb{Z} \longrightarrow 0,$$

where $k(C)_D^\times$ denotes a constant sheaf by abuse of notation. This latter sequence is a flasque resolution of $\mathcal{O}_{(C,D)}^\times$, and using it to compute cohomology yields a natural isomorphism

$$\operatorname{coker}\big(k(C)_D^\times \xrightarrow{(\operatorname{ord}_x)_x} \bigoplus_{x\in C_0\setminus D}\mathbb{Z}\big) \xrightarrow{\simeq} H^1(C,\mathcal{O}_{(C,D)}^\times).$$

But the left side of this isomorphism is precisely $CH_0(C;D)$, thereby completing the proof. □

Proposition 2.3. *Let X be a smooth variety over a field k, and D an effective divisor on X. Then the canonical map*

$$\text{free abelian group on closed points of } X\setminus D \longrightarrow K_0(X,D), \quad x\longmapsto [x]$$

descends to a cycle class homomorphism

$$CH_0(X;D)\longrightarrow K_0(X,D).$$

Proof. We must show that if $C\hookrightarrow X$ is a curve not contained in $|D|$ and $f\in k(C)_D^\times$, then $\sum_{x\in C_0}\operatorname{ord}_x(f)[x]=0$ in $K_0(X,D)$. We will deduce this from Lemma 2.2 once we have verified a suitable pushforward formalism.
Let $\phi:\widetilde{C}\to C\hookrightarrow X$ be the resulting map from the normalisation $\widetilde{C}$ to X, and consider the following pullback square:

$$\begin{array}{ccc} \phi^*D & \xrightarrow{j'} & \widetilde{C} \\ {\scriptstyle\phi'}\downarrow & & \downarrow{\scriptstyle\phi} \\ D & \xrightarrow[j]{} & X \end{array}$$

We claim that ϕ and j are Tor-independent; that is, if y is a closed point of $\widetilde{C}$ such that $x:=\phi(y)$ lies in $|D|$, we must show that $\operatorname{Tor}^i_{\mathcal{O}_{X,x}}(\mathcal{O}_{D,x},\mathcal{O}_{\widetilde{C},y})=0$ for all $i>0$. But since D is an effective Cartier divisor, there exists a non-zero-divisor $t\in\mathcal{O}_{X,x}$ such that $\mathcal{O}_{D,x}=\mathcal{O}_{X,x}/t\mathcal{O}_{X,x}$; thus the only possible non-zero higher Tor is Tor^1, which equals the $\phi^*(t)$-torsion of $\mathcal{O}_{\widetilde{C},y}$; this could only be non-zero if $\phi^*(t)=0$ in $\mathcal{O}_{\widetilde{C},y}$, but this would contradict the condition that C does not lie in $|D|$. This proves the desired Tor-independence.
Moreover, ϕ is a finite morphism and X is assumed to be smooth, whence ϕ is proper and of finite Tor-dimension. Therefore the projection formula [19, Prop. 3.18] (or [4, Thm. 4.4]) states that the diagram

$$\begin{array}{ccc} K(\widetilde{C}) & \xrightarrow{j'^*} & K(\phi^*D) \\ {\scriptstyle\phi_*}\downarrow & & \downarrow{\scriptstyle\phi'_*} \\ K(X) & \xrightarrow[j^*]{} & K(D) \end{array}$$

is well-defined and commutes up to homotopy; so there is an induced pushforward map

$$\phi_* : K(\widetilde{C}, \phi^* D) \longrightarrow K(X, D),$$

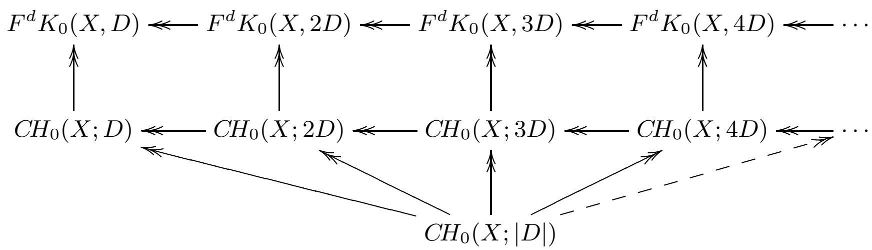
which by functoriality of pushforwards (as in Section 1.1 we must appeal to [4, §4–5] to know that the obvious choices of homotopies yield a functorial construction) satisfies $\phi_*[x] = [\phi(x)]$ for any $x \in \widetilde{C}_0$. Therefore

$$\begin{aligned}\sum_{x \in C_0} \mathrm{ord}_x(f)[x] &= \sum_{x \in \widetilde{C}_0} \mathrm{ord}_{\phi(x)}(f)[\phi(x)] \\ &= \phi_*\Big(\sum_{x \in \widetilde{C}_0} \mathrm{ord}_x(f)[x] \Big) \\ &= \phi_*(0) \\ &= 0,\end{aligned}$$

where $\sum_{x \in \widetilde{C}_0} \mathrm{ord}_x(f)[x] \in K_0(\widetilde{C}, \phi^* D)$ vanishes by Lemma 2.2. □

Remark 2.4. F. Binda [3] has independently proved Proposition 2.3, as well as constructing cycle class homomorphisms $CH_0(X; D; n) \to K_n(X, D)$ for the higher Chow groups with modulus.

Let X be a d-dimensional, smooth variety over k. Given effective divisors $D' \geq D$ with the same support, the inclusions $k(C)^\times_{D'} \subseteq k(C)^\times_D$ induce a canonical surjection $CH_0(X; D') \twoheadrightarrow CH_0(X; D)$. This applies in particular when $D' = rD$ is a thickening of D. Combining this observation with Proposition 2.3 we obtain a commutative diagram of inverse systems of Chow groups and relative K-groups (recall the definition of $F^d K_0$ from Section 1.1) in which all maps are surjective (since every group is generated by the closed points of $X \setminus D$):



There are two natural questions to consider concerning this diagram. Firstly, a question seemingly related to a conjecture of Kerz and Saito [5, Qu. V] is whether the cycle class homomorphism

$$\{CH_0(X; rD)\}_r \longrightarrow \{F^d K_0(X; rD)\}_r$$

is an isomorphism of pro abelian groups, perhaps at least ignoring $(d-1)!$-torsion.

Secondly, changing notation, now suppose that $X' \to X$ is a desingularisation of an integral variety X, whose exceptional fibre is an effective Cartier divisor D. Then, as a Chow-theoretic analogue of the Bloch–Srinivas conjecture, we ask whether the inverse system

$$CH_0(X'; D) \longleftarrow CH_0(X'; 2D) \longleftarrow CH_0(X'; 3D) \longleftarrow \cdots$$

eventually stabilises, with stable value most likely equal to the Levine–Weibel Chow group $CH_0(X)$ of X.

The following theorem simultaneously answers cases of these two questions, working under almost identical hypotheses to Corollary 1.11:

Theorem 2.5. *Let X be a d-dimensional, integral, quasi-projective variety over a good field k; let $\pi : X' \to X$ be a desingularisation, and D any effective Cartier divisor on X whose support contains the exceptional set. Assume* $\operatorname{codim}(X, \pi(|D|)) \geq 2$ *and that the cycle class map $CH_0(X) \to F^d K_0(X)$ is an isomorphism.*

Then $CH_0(X) \cong CH_0(X'; |D|)$, and the canonical maps

$$CH_0(X'; |D|) \longrightarrow CH_0(X'; rD) \longrightarrow F^d K_0(X'; rD)$$

are isomorphisms for $r \gg 1$.

Proof. Let $Y \hookrightarrow X$ be the reduced closed subscheme with support $\pi(|D|)$; this has codimension ≥ 2 and covers X_{sing}. Consider the following commutative diagram, which exists for any $r \gg 1$:

$$\begin{array}{ccccc} CH_0(X'; |D|) & \longrightarrow & CH_0(X'; rD) & \longrightarrow & F^d K_0(X'; rD) \\ \uparrow & & & & \downarrow{\scriptstyle BS_r} \\ CH_0(X; |Y|) & \longrightarrow & CH_0(X) & \longrightarrow & F^d K_0(X) \end{array}$$

The bottom right horizontal arrow is an isomorphism by assumption; the bottom left horizontal arrow is an isomorphism by Remark 1.2(ii); the left vertical arrow is an isomorphism by Remark 1.2(iv); the right vertical arrow is an isomorphism by Corollary 1.11. Since the two top horizontal arrows are surjective, it follows that they are isomorphisms. □

Corollary 2.6. *Let X be a d-dimensional, integral variety over an algebraically closed field k which has strong resolution of singularities; let $\pi : X' \to X$ be a desingularisation, and D any effective Cartier divisor on X whose support contains the exceptional set. Assume* $\operatorname{codim}(X, \pi(|D|)) \geq 2$ *and that one of the following is true:*

(i) X is projective and $\operatorname{char} k = 0$*; or*

(ii) X is projective and $d \leq \operatorname{char} k \neq 0$; or

(iii) X is affine.

Then $CH_0(X) \cong CH_0(X';|D|)$*, and the canonical maps*

$$CH_0(X';|D|) \longrightarrow CH_0(X';rD) \longrightarrow F^d K_0(X';rD)$$

are isomorphisms for $r \gg 1$*.*

Proof. This follows from Theorem 2.5 and the results of Levine and Srinivas recalled in Theorem 1.4. ☐

Remark 2.7 (Class field theory of singular varieties). In this remark we explain how the CH_0 isomorphism of Theorem 2.5 over a finite field $\mathbb{F}_q$ can be interpreted as part of an unramified class field theory for singular, projective varieties.

Let X be a projective variety over $\mathbb{F}_q$ which is regular in codimension one; suppose that a desingularisation $\pi : X' \to X$ exists, that D is an effective Cartier divisor on X whose support contains the exceptional set, and that $\operatorname{codim}(X, \pi(|D|)) \geq 2$. Write $U := X' \setminus D = X \setminus \pi(|D|)$.

The Kerz–Saito class group [5] of U is $C(U) := \varprojlim_r CH_0(X';rD)$, and their class field theory provides a reciprocity isomorphism $C(U)^0 \xrightarrow{\simeq} \pi_1^{\mathrm{ab}}(U)^0$, where the superscripts 0 denote degree-0 subgroups. Assuming that the conclusions of Theorem 2.5 are true in this setting, we deduce that $C(U) = CH_0(X';rD) \cong CH_0(X)$ for $r \gg 1$. Kerz–Saito prove moreover that each group $CH_0(X';rD)^0$ is finite.

In particular, this would prove finiteness of $CH_0(X)^0$, which is known in the smooth case thanks to the unramified class field theory of S. Bloch, K. Kato and Saito, et al. It would also yield a reciprocity isomorphism

$$CH_0(X)^0 \xrightarrow{\simeq} \pi_1^{\mathrm{ab}}(U)^0, \quad [x] \mapsto \mathrm{Frob}_x$$

However, since the canonical map $\pi_1^{\mathrm{ab}}(U) \to \pi_1^{\mathrm{ab}}(X)$ is surjective but generally not an isomorphism, we would obtain in general only a surjective reciprocity map

$$CH_0(X)^0 \longrightarrow \pi_1^{\mathrm{ab}}(X)^0,$$

indicating that the Levine–Weibel Chow group $CH_0(X)$ is not the correct class group for unramified class field theory of a singular variety.

Remark 2.8 (The case of surfaces). If X is an integral, projective surface over $\mathbb{F}_q$ which is regular in codimension one, then we have actually proved the observations of Remark 2.7 unconditionally: $CH_0(X)$ is isomorphic to the Kerz–Saito class group $C(X_{\mathrm{reg}})$, its degree-0 subgroup is finite, and there is a reciprocity isomorphism

$$CH_0(X)^0 \xrightarrow{\simeq} \pi_1^{\mathrm{ab}}(X_{\mathrm{reg}})^0$$

of finite groups. This was brought to the author's attention by [8], in which Krisha reproduced the argument while being unaware of the present paper.

To prove this we must only check that Theorem 2.5 is true for surfaces over finite fields. In fact, we will let X be a 2-dimensional, integral, quasi-projective variety over an *arbitrary* field k which is regular in codimension one. Then X admits a resolution of singularities $\pi : X' \to X$ with exceptional set equal to exactly $\pi^{-1}(X_{\mathrm{sing}})$; let $E := \pi^{-1}(X_{\mathrm{sing}})_{\mathrm{red}}$ and $Y := (X_{\mathrm{sing}})_{\mathrm{red}}$.
Then Theorem 1.7 is true for the data $X' \to X$, Y, E. Indeed, it is only necessary to establish the isomorphism (†) occurring in the proof, which may be broken into the two isomorphisms

$$\{F^d K_0(X, rY)\}_r \overset{\simeq}{\to} \{F^d K_0(\widetilde{X}, \widetilde{X} \times_X rY)\}_r \overset{\simeq}{\to} \{F^d K_0(X', rE)\}_r,$$

where $\widetilde{X} \to X$ denotes the normalisation of X. The second of these isomorphisms is due to Krishna and Srinivas [9, Thm. 1.1]; the first isomorphism follows from the isomorphism $\{K_0(X, rY)\}_r \overset{\simeq}{\to} \{K_0(\widetilde{X}, \widetilde{X} \times_X rY)\}_r$, which is a case of the author's pro-excision theorem [14, Corol. 0.4 & E.g. 2.5], and the obvious surjectivity just as in the proof of Theorem 1.7.
Now assume further (perhaps after blowing-up X' at finitely many points) that there is an effective divisor D on X' with support $\pi^{-1}(X_{\mathrm{sing}})$. Since the cycle class map $CH_0(X) \to F^d K_0(X)$ is automatically an isomorphism (as we remarked immediately before Theorem 1.4), it follows that the assertions of Theorem 2.5 are also true, as required: $CH_0(X) \cong CH_0(X'; |D|)$, and the canonical maps $CH_0(X'; |D|) \to CH_0(X'; rD) \to F^d K_0(X'; rD)$ are isomorphisms for $r \gg 1$.

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Wedderburn's Theorem for Regular Local Rings

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Abstract. Wedderburn's theorem is extended to Azumaya algebras over certain regular local rings.

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In [Pa] Ivan Panin proved the following theorem.

Theorem 1. *Let R be a regular local ring, K its field of fractions and (V, Φ) a quadratic space over R. Suppose R contains a field of characteristic zero. If $(V, \Phi) \otimes_R K$ is isotropic over K, then (V, Φ) is isotropic over R.*

The proof rests on a series of lemmas which can be summarized in a single one:

Lemma 2. *Let k be a field of characteristic zero, u a closed point of a smooth k-variety and $R = \mathcal{O}_{U,u}$ the local ring of U at u. Let further $\mathcal{X}$ be a projective R-scheme, smooth over R. Let K be the field of fractions of R and suppose that $\mathcal{X}$ has a K-point. Then, for every prime number p there exist an integral R-etale algebra S of degree prime to p and an S-point of $\mathcal{X}$.*

Proof. See [Pa], Lemma 3, Lemma 4 and proof of Theorem 1.

I want to show that the argument used for proving Theorem 1 also yields the following extension of Wedderburn's theorem to a large class of regular local rings.

Theorem 3. *Let R be a regular local ring, K its field of fractions and A an Azumaya algebra over R. Suppose R contains a field k of characteristic zero. If $A \otimes_R K$ is isomorphic to $M_n(D)$ where D is a central division algebra over K, then A is isomorphic to $M_n(\Delta)$ where Δ is a maximal (unramified) R-order of D. In other words, every class of the Brauer group of R is represented by an Azumaya algebra Δ such that $\Delta \otimes_R K$ is a division K-algebra.*

Proof. Let d^2 be the dimension of D over K. It suffices to show that A contains a right ideal I such that A/I is free of rank $(n^2 - n)d^2$ over R. In fact, since any A-module is projective over A if and only if it is projective over R, the projection $A \to A/I$ splits, I is a direct factor of the right A-module A, and $\Delta := End_A(I)$ is an Azumaya algebra equivalent to A. Clearly $\Delta \otimes_R K = D$ and by Morita theory

$$A = End_\Delta(Hom_A(I, A)) = M_n(\Delta).$$

In order to find a right ideal I of the right rank we consider the set $\mathcal{I}$ of all such ideals or, more precisely, we consider the functor $\mathcal{I}$ that associates to any R-algebra S the set of such ideals in $A \otimes_R S$.

Lemma 4. *$\mathcal{I}$ is a smooth closed subscheme of the Grassmannian scheme $\mathcal{G}$ consisting of all the free R-submodules of A which are direct factors of A and have rank nd^2.*

Proof. We denote by m the maximal ideal of R. To show that $\mathcal{I}$ is closed we first remark that A, as an R-module, is generated by the set A^* of all invertible elements of A. In fact for any $a \in A$ and any $\lambda \in k$ the reduced norm of $\lambda + a$ is a polynomial

$$P(\lambda) = \lambda^{nd} + c_1\lambda^{n-1} + \cdots + c_{nd}$$

whose coefficients are in R and only depend on a. Choosing λ in k^* such that $P(\lambda)$ is not 0 in R/m insures that $\lambda + a$ is invertible and allows to write $a = (\lambda + a) - \lambda$. So an R- submodule M of A is an ideal if $aM = M$ for every unit a. In other words, we must show that the set of fixed points of $\mathcal{G}$ under the action of A^* is closed. This is well-known.

The second point is the smoothness of $\mathcal{I}$. This means that for any R-algebra S and any ideal I of S, any S/I-point of $\mathcal{X}$ can be lifted to an S/I^2-point. But points correspond to right ideals generated by an idempotent and it is well-known that idempotents can be lifted.

Note that it suffices to treat the case when A is of prime power order in the Brauer group $Br(R)$ of R. In fact the class of A is a product of classes $[A_i]$ of order $p_i^{e_i}$ for some distinct primes $p_1, \ldots, p_r$. If each of them is represented by an order Δ_i in $D_i = \Delta_i \otimes_R K$ then A is Brauer equivalent to $\Delta_1 \otimes_R \cdots \otimes_R \Delta_r$ which is an order in $D = D_1 \otimes_K \cdots \otimes_K D_r$ and we know that D is a division algebra.

We now assume that R is of geometric type, in other words R is the local ring of a closed point u of a smooth k-variety. The general case then follows from this special case by a standard application of Dorin Popescu's theorem, saying that a regular ring containing a field is an inductive limit of smooth algebras. A self-contained proof of Popescu's theorem in the form needed here has been given by R. Swan [Sw]. For the original articles by Popescu see the references in [Sw].

Suppose now that A is of prime power exponent in $Br(R)$ and that the degree of D is p^e for some prime number p. Since $A \otimes_R K = M_n(D)$ the scheme

$\mathcal{I}$ has a K-point and according to Lemma 2 it also has an S-point, where S is an integral etale algebra whose degree d is prime to p. This means that $A \otimes_R S = M_n(B)$ for some maximal order B in $D \otimes_K L$, L being the field of fractions of S. Note that $D \otimes_K L$ remains a division algebra because the degree of L over K is prime to p. So the Brauer class $[A]_S$ of $A \otimes_R S$ in $Br(S)$ is represented by a degree p^e algebra. In [Ga] (see also [AdJ], Proposition 2.6.1) Gabber proved that any class $\alpha \in Br(R)$ which is represented by a degree m algebra when extended to a finite faithfully flat R-algebra S of degree d can be represented by an R-algebra of degree dm. We can thus find an Azumaya algebra A_1 of degree dp^e in the same class as A. On the other hand, we may also use Ferrand's [Fe] norm functor $N_{S/R}$ from S-algebras to R-algebras. Applying it to B we find that $N_{S/R}(B) = A_2$ is an Azumaya R-algebra equivalent to $A^{\otimes d}$ ([Fe], section 7.3), of degree p^{ed} ([Fe], Théorème 4.3.4). If the integer c is an inverse of d modulo p^e, the algebra $A_3 = A_2^{\otimes c}$ is Brauer equivalent to A and its degree is p^{cde}. Recall now that DeMeyer [DM] proved that every class in $Br(R)$ is represented by a unique "minimal" Azumaya algebra Δ with the property that every algebra in the same class is isomorphic to some matrix algebra over Δ. What is the degree m of this Δ in our case? We must have $A_1 \simeq M_{s_1}(\Delta)$ and $A_3 \simeq M_{s_3}(\Delta)$, hence $s_1 m = dp^e$ and $s_3 m = p^{cde}$. Since d is prime to p, this implies that m divides p^e and extending the scalars to K shows that $m = p^e$. The theorem is proved.

Easy and well-known examples (the simplest one being the usual quaternion algebra extended to $\mathbb{R}[x, y, z]/(x^2 + y^2 + z^2)$ localized at the origin) show that we cannot replace regularity by, say, normality.

Remark. As the referee pointed out, the proof of Theorem 3 could be extended to the case of a semi-local regular ring containing a field k of characteristic zero, although I do not see how to proceed if k has positive characteristic. Fortunately, since the time this article was written, new and stronger results have appeared. In [AB2] Benjamin Antieau and Ben Williams have generalized Theorem 3 to arbitrary semi-local regular rings. In [AB1] they have shown that Theorem 3 fails for arbitrary regular rings, in particular for certain smooth complex affine algebras of dimension 6.

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Rationally Isotropic Exceptional Projective Homogeneous Varieties Are Locally Isotropic

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Abstract. Assume that R is a regular local ring that contains an infinite field and whose field of fractions K has charactertistic $\neq 2$. Let X be an exceptional projective homogeneous scheme over R. We prove that in most cases the condition $X(K) \neq \emptyset$ implies $X(R) \neq \emptyset$.

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1. Introduction

The main result of the present article extends the main results of [Pa3] and [PP] to the case of exceptional groups. In the latter paper one can find historical remarks which might help the general reader. All the rings in the present paper are *commutative* and *Noetherian*. We prove the following theorem.

Theorem 1. *Let R be a regular local ring that contains an infinite field and whose field of fractions K has characteristic $\neq 2$. Let G be a split simple group of exceptional type (that is, E_6, E_7, E_8, F_4, or G_2), P be a parabolic subgroup of G, $[\xi]$ be a class from $\mathrm{H}^1(R, G)$, and $X = (G/P)_\xi$ be the corresponding homogeneous space over R. Assume that $P \neq P_7$, P_8, $P_{7,8}$ in case $G = E_8$, $P \neq P_7$ in case $G = E_7$, and $P \neq P_1$ in case $G = E_7^{ad}$. Then the condition $X(K) \neq \emptyset$ implies $X(R) \neq \emptyset$.*

The results of the present paper depend on the following yet unpublished results: [FP, Corollary of Theorem 1] and [Pa, Theorem 10.0.30].

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2. Purity of some H^1 functors

Let R be a commutative noetherian domain of finite Krull dimension with a fraction field F. We say that a functor $\mathcal{F}$ from the category of commutative R-algebras to the category of sets *satisfies purity* for R if we have

$$\mathrm{Im}\,[\mathcal{F}(R) \to \mathcal{F}(F)] = \bigcap_{\mathrm{ht}\,\mathfrak{p}=1} \mathrm{Im}\,[\mathcal{F}(R_{\mathfrak{p}}) \to \mathcal{F}(F)].$$

An element $a \in \mathcal{F}(F)$ *is called R-unramified* if it belongs to $\bigcap_{\mathrm{ht}\,\mathfrak{p}=1} \mathrm{Im}\,[\mathcal{F}(R_{\mathfrak{p}}) \to \mathcal{F}(F)]$. If $\mathfrak{p}$ is a height one prime ideal in R, the element a *is called $\mathfrak{p}$-unramified*, if it belongs to $\mathrm{Im}\,[\mathcal{F}(R_{\mathfrak{p}}) \to \mathcal{F}(F)]$.
If $\mathcal{H}$ is an étale group sheaf we write $\mathrm{H}^i(-,\mathcal{H})$ for $\mathrm{H}^i_{\text{ét}}(-,\mathcal{H})$ below through the text.
The following theorem is proven in the characteristic zero case [Pa2, Theorem 4.0.3]. We extend it here to reductive group schemes. Let R be a commutative noetherian ring. Recall that an R-group scheme G is called *reductive*, if it is affine and smooth as an R-scheme and if, moreover, for each algebraically closed field Ω and for each ring homomorphism $R \to \Omega$ the scalar extension G_Ω is a connected reductive algebraic group over Ω. This definition of a reductive R-group scheme coincides with [SGA, Exp. XIX, Definition 2.7].

Theorem 2. *Let R be the local ring of a closed point on a smooth scheme over an infinite field. Let G be a reductive R-group scheme. Let $i\colon Z \hookrightarrow G$ be a closed subgroup scheme of the center* $\mathrm{Cent}(G)$*. It is known that Z is of multiplicative type. Let $G' = G/Z$ be the factor group, $\pi\colon G \to G'$ be the projection.*
If the functor $\mathrm{H}^1(-,G')$ *satisfies purity for R, then the functor* $\mathrm{H}^1(-,G)$ *satisfies purity for R as well.*

It is known that π is surjective and strictly flat. Thus the exact sequence of R-group schemes

$$(*) \qquad \{1\} \to Z \xrightarrow{i} G \xrightarrow{\pi} G' \to \{1\}$$

induces an exact sequence of group sheaves in the fppf-topology.

Lemma 1. *Consider the category of R-algebras. The functor*

$$R' \mapsto \mathcal{F}(R') = \mathrm{H}^1_{\mathrm{fppf}}(R',Z)/\,\mathrm{Im}\,(\delta_{R'}),$$

where δ is the connecting homomorphism associated to sequence (*)*, satisfies purity for R.*

Proof. The lemma coincides with [Pa, Theorem 10.0.30]. □

Lemma 2. *The map*

$$\mathrm{H}^2_{\mathrm{fppf}}(R,Z) \to \mathrm{H}^2_{\mathrm{fppf}}(K,Z)$$

is injective.

Proof. See [C-TS, Theorem 4.3]. □

Proof of Theorem 2. Reproduce the diagram chase from the proof of [Pa2, Theorem 4.0.3]. For this purpose consider the commutative diagram

$$\begin{array}{ccccccccc}
\{1\} & \longrightarrow & \mathcal{F}(K) & \xrightarrow{\delta_K} & \mathrm{H}^1(K,G) & \xrightarrow{\pi_K} & \mathrm{H}^1(K,G') & \xrightarrow{\Delta_K} & \mathrm{H}^2_{\mathrm{fppf}}(K,Z) \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow{\scriptstyle\alpha} \\
\{1\} & \longrightarrow & \mathcal{F}(R) & \xrightarrow{\delta} & \mathrm{H}^1(R,G) & \xrightarrow{\pi} & \mathrm{H}^1(R,G') & \xrightarrow{\Delta} & \mathrm{H}^2_{\mathrm{fppf}}(R,Z)
\end{array}$$

Let $[\xi] \in \mathrm{H}^1(K,G)$ be an R-unramified class and let $[\bar{\xi}] = \pi_K([\xi])$. Clearly, $[\bar{\xi}] \in \mathrm{H}^1(K,G')$ is R-unramified. Thus there exists an element $[\bar{\xi}'] \in \mathrm{H}^1(R,G')$ such that $[\bar{\xi}']_K = [\bar{\xi}]$. The map α is injective by Lemma 2. One has $\Delta([\bar{\xi}']) = 0$, since $\Delta_K([\bar{\xi}]) = 0$. Thus there exists $[\xi'] \in \mathrm{H}^1(R,G)$ such that $\pi([\xi']) = [\bar{\xi}']$. Twisting G by ξ' we may assume that $[\bar{\xi}] = *$, so that $[\xi]$ comes from some $a \in \mathcal{F}(K)$.

Lemma 3. *The above constructed element $a \in \mathcal{F}(K)$ is R-unramified.*

Assume Lemma 3; we use it to complete the proof of Theorem 2. By Lemma 1 the functor $\mathcal{F}$ satisfies the purity for regular local rings containing the field k. Thus there exists an element $a' \in \mathcal{F}(R)$ with $a'_K = a$. It is clear that $[\delta(a')]_K = [\xi]$. It remains to prove Lemma 3. First we need a small variation of Nisnevich's theorem.

Lemma 4. *Let H be a reductive group scheme over a discrete valuation ring A. Let K be the fraction field of A. Then the map*

$$\mathrm{H}^1(A,H) \to \mathrm{H}^1(K,H)$$

is injective.

Proof. Let $[\xi_0], [\xi_1]$ be classes from $\mathrm{H}^1(A,H)$. Let $\mathcal{H}_0$ be a principal homogeneous H-bundle representing the class ξ_0. Let H_0 be the inner form of the group scheme H, corresponding to $\mathcal{H}_0$. Let $X = Spec(A)$. For each X-scheme S there is a well-known bijection $\phi_S \colon \mathrm{H}^1(S,H) \to \mathrm{H}^1(S,H_0)$ of non-pointed sets. That bijection takes the principal homogeneous H-bundle $\mathcal{H}_0 \times_X S$ to the trivial principal homogeneous H_0-bundle $H_0 \times_X S$. That bijection is functorial with respect to morphisms of X-schemes.
Assume that $[\xi_0]_K = [\xi_1]_K$. Then one has $* = \phi_K([\xi_0]_K) = \phi_K([\xi_1]_K) \in \mathrm{H}^1(K,H_0)$. The kernel of the map $\mathrm{H}^1(A,H_0) \to \mathrm{H}^1(K,H_0)$ is trivial by Nisnevich's theorem [Ni]. Thus $\phi_A([\xi]_1) = * = \phi_A([\xi]_0) \in \mathrm{H}^1(A,H_0)$. Whence $[\xi]_1 = [\xi]_0 \in \mathrm{H}^1(A,H)$. ☐

Now we go back to the proof of Lemma 3. Consider a height 1 prime ideal $\mathfrak{p}$ in R. Since $[\xi]$ is R-unramified there exists its lift up to an element $[\tilde{\xi}]$ in $\mathrm{H}^1(R_{\mathfrak{p}},G)$.
The map

$$\mathrm{H}^1(R_{\mathfrak{p}},G') \to \mathrm{H}^1(K,G')$$

is injective by Lemma 4. But

$$(\pi_{\mathfrak{p}}([\tilde{\xi}]))_K = \pi_K([\xi]) = *,$$

so $\pi_{\mathfrak{p}}[\tilde{\xi}] = *$. Therefore there exists a unique class $a_{\mathfrak{p}} \in \mathcal{F}(R_{\mathfrak{p}})$ such that $\delta(a_{\mathfrak{p}}) = [\tilde{\xi}] \in \mathrm{H}^1(R_{\mathfrak{p}}, G)$. So, $\delta_K(a_{\mathfrak{p},K}) = [\xi] \in \mathrm{H}^1(K, G)$ and finally $a = a_{\mathfrak{p},K}$. Lemma 3 is proven and Theorem 2 is proven as well. □

3. Purity of some H^1 functors, continued

Theorem 3. *Let R be such as in Theorem 1. The functor* $\mathrm{H}^1(-, \mathrm{PGL}_n)$ *satisfies purity for R.*

Proof. Let $[\xi] \in \mathrm{H}^1(K, \mathrm{PGL}_n)$ be an R-unramified element. Let $\delta\colon \mathrm{H}^1(-, \mathrm{PGL}_n) \to \mathrm{H}^2(-, \mathbb{G}_m)$ be the boundary map corresponding to the short exact sequence of étale group sheaves

$$1 \to \mathbb{G}_m \to \mathrm{GL}_n \to \mathrm{PGL}_n \to 1.$$

Let D_ξ be a central simple K-algebra of degree n corresponding ξ. If $D_\xi \cong M_l(D')$ for a skew-field D', then there exists $[\xi'] \in \mathrm{H}^1(K, \mathrm{PGL}_{n'})$ such that $D' = D_{\xi'}$. Then $\delta([\xi']) = [D'] = [D] = \delta(\xi)$. Replacing ξ by ξ', we may assume that $D := D_\xi$ is a central skew-field over K of degree n and the class $[D]$ is R-unramified. Since the functor $\mathrm{H}^2(-, \mathbb{G}_m)$ satisfies purity for R, there exists an Azumaya R-algebra A and an integer d such that $A_K = M_d(D)$.
There exists a projective left A-module P of finite rank such that each projective left A-module Q of finite rank is isomorphic to the left A-module P^m for an appropriative integer m (see [DeM, Cor.2]). In particular, two projective left A-modules of finite rank are isomorphic if they have the same rank as R-modules. One has an isomorphism $A \cong P^s$ of left A-modules for an integer s. Thus one has R-algebra isomorphisms $A \cong \mathrm{End}_A(P^s) \cong \mathrm{M}_s(\mathrm{End}_A(P))$. Set $B = \mathrm{End}_A(P)$. Observe, that $B_K = \mathrm{End}_{A_K}(P_K)$, since P is a finitely generated projective left A-module.
The class $[P_K]$ is a free generator of the group $K_0(A_K) = K_0(M_d(D)) \cong \mathbb{Z}$, since $[P]$ is a free generator of the group $K_0(A)$ and $K_0(A) = K_0(A_K)$. The P_K is a simple A_K-module, since $[P_K]$ is a free generator of $K_0(A_K)$. Thus $\mathrm{End}_{A_K}(P_K) = B_K$ is a skew-field.
We claim that the K-algebras B_K and D are isomorphic. In fact, $A_K = M_r(B_K)$ for an integer r, since P_K is a simple A_K-module. From the other side $A_K = M_d(D)$. As D, so B_K are skew-fields. Thus $r = d$ and D is isomorphic to B_K as K-algebras.
We claim further that B is an Azumaya R-algebra. That claim is local with respect to the étale topology on $\mathrm{Spec}(R)$. Thus it suffices to check the claim assuming that $\mathrm{Spec}(R)$ is strictly henselian local ring. In that case $A = M_l(R)$ and $P = (R^l)^m$ as an $M_l(R)$-module. Thus $B = \mathrm{End}_A(P) = M_m(R)$, which proves the claim.
Since B_K is isomorphic to D, one has $m = n$. So, B is an Azumaya R-algebra, and the K-algebra B_K is isomorphic to D. Let $[\zeta] \in \mathrm{H}^1(R, \mathrm{PGL}_n)$ be class

corresponding to B. Then $[\zeta]_K = [\xi]$, since $\delta([\zeta])_K = [B_K] = [D] = \delta([\xi]) \in \mathrm{H}^2(K, \mathbb{G}_m)$. □

We denote by Sim_n the group of similitudes of a *split* quadratic form of rank n and by Sim_n^+ its connected component. Recall that $\mathrm{H}^1(-, \mathrm{Sim}_n)$ classifies similarity classes of nondegenerate quadratic forms of rank n (see [KMRT, (29.15)]).

THEOREM 4. *Let R be such as in Theorem 1. The functor $\mathrm{H}^1(-, \mathrm{Sim}_n)$ satisfies purity for R.*

Proof. Let $[\xi] \in \mathrm{H}^1(K, \mathrm{Sim}_n)$ be an R-unramified element. Let φ be a quadratic form over K whose similarity class represents $[\xi]$. Diagonalizing φ we may assume that $\varphi = \sum_{i=1}^n f_i \cdot t_i^2$ for certain non-zero elements $f_1, f_2, \dots, f_n \in K$. For each i write f_i in the form $f_i = \frac{g_i}{h_i}$ with $g_i, h_i \in R$ and $h_i \neq 0$.
There are only finitely many height one prime ideals $\mathfrak{q}$ in R such that there exists $0 \le i \le n$ with f_i not in $R_\mathfrak{q}$. Let $\mathfrak{q}_1, \mathfrak{q}_2, \dots, \mathfrak{q}_s$ be all height one prime ideals in R with that property and let $\mathfrak{q}_i \neq \mathfrak{q}_j$ for $i \neq j$.
For all other height one prime ideals $\mathfrak{p}$ in R each f_i belongs to the group of units $R_\mathfrak{p}^\times$ of the ring $R_\mathfrak{p}$.
If $\mathfrak{p}$ is a height one prime ideal of R which is not from the list $\mathfrak{q}_1, \mathfrak{q}_2, \dots, \mathfrak{q}_s$, then $\varphi = \sum_{i=1}^n f_i \cdot t_i^2$ may be regarded as a quadratic space over $R_\mathfrak{p}$. We will write ${}_\mathfrak{p}\varphi$ for that quadratic space over $R_\mathfrak{p}$. Clearly, one has $({}_\mathfrak{p}\varphi) \otimes_{R_\mathfrak{p}} K = \varphi$ as quadratic spaces over K.
For each $j \in \{1, 2, \dots, s\}$ choose and fix a quadratic space ${}_j\varphi$ over $R_{\mathfrak{q}_j}$ and a non-zero element $\lambda_j \in K$ such that the quadratic spaces $({}_j\varphi) \otimes_{R_{\mathfrak{q}_j}} K$ and $\lambda_j \cdot \varphi$ are isomorphic over K. The ring R is factorial since it is regular and local. Thus for each $j \in \{1, 2, \dots, s\}$ we may choose an element $\pi_j \in R$ such that firstly π_j generates the only maximal ideal in $R_{\mathfrak{q}_j}$ and secondly π_j is an invertible element in $R_\mathfrak{n}$ for each height one prime ideal $\mathfrak{n}$ different from the ideal $\mathfrak{q}_j$.
Let $v_j \colon K^\times \to \mathbb{Z}$ be the discrete valuation of K corresponding to the prime ideal $\mathfrak{q}_j$. Set $\lambda = \prod_{i=1}^s \pi_j^{v_j(\lambda_j)}$ and

$$\varphi_{new} = \lambda \cdot \varphi.$$

Claim. The quadratic space φ_{new} is R-unramified. In fact, if a height one prime ideal $\mathfrak{p}$ is different from each of $\mathfrak{q}_j$'s, then $v_\mathfrak{p}(\lambda) = 0$. Thus, $\lambda \in R_\mathfrak{p}^\times$. In that case $\lambda \cdot ({}_\mathfrak{p}\varphi)$ is a quadratic space over $R_\mathfrak{p}$ and moreover one have isomorphisms of quadratic spaces $(\lambda \cdot ({}_\mathfrak{p}\varphi)) \otimes_{R_\mathfrak{p}} K = \lambda \cdot \varphi = \varphi_{new}$. If we take one of $\mathfrak{q}_j$'s, then $\frac{\lambda}{\lambda_j} \in R_{\mathfrak{q}_j}^\times$. Thus, $\frac{\lambda}{\lambda_j} \cdot ({}_j\varphi)$ is a quadratic space over $R_{\mathfrak{q}_j}$. Moreover, one has

$$\frac{\lambda}{\lambda_j} \cdot ({}_j\varphi) \otimes_{R_\mathfrak{q}} K = \frac{\lambda}{\lambda_j} \cdot \lambda_j \cdot \varphi = \varphi_{new}.$$

The Claim is proven.
By [PP, Corollary 3.1] there exists a quadratic space $\tilde{\varphi}$ over R such that the quadratic spaces $\tilde{\varphi} \otimes_R K$ and φ_{new} are isomorphic over K. This shows that the

similarity classes of the quadratic spaces $\tilde{\varphi}\otimes_R K$ and φ coincide. The theorem is proven. $\square$

Theorem 5. *Let R be such as in Theorem 1. The functor* $\mathrm{H}^1(-,\mathrm{Sim}_n^+)$ *satisfies purity for R.*

Proof. Consider an element $[\xi]\in\mathrm{H}^1(K,\mathrm{Sim}_n^+)$ such that for any $\mathfrak{p}$ of height 1 $[\xi]$ comes from $[\xi_{\mathfrak{p}}]\in\mathrm{H}^1(R_{\mathfrak{p}},\mathrm{Sim}_n^+)$. Then the image of $[\xi]$ in $\mathrm{H}^1(K,\mathrm{Sim}_n)$ by Theorem 4 comes from some $[\zeta]\in\mathrm{H}^1(R,\mathrm{Sim}_n)$. We have a short exact sequence

$$1\to\mathrm{Sim}_n^+\to\mathrm{Sim}_n\to\mu_2\to 1,$$

and $R^\times/(R^\times)^2$ injects into $K^\times/(K^\times)^2$. Thus the element $[\zeta]$ comes actually from some $[\zeta']\in\mathrm{H}^1(R,\mathrm{Sim}_n^+)$. It remains to show that the map

$$\mathrm{H}^1(K,\mathrm{Sim}_n^+)\to\mathrm{H}^1(K,\mathrm{Sim}_n)$$

is injective, or, by twisting, that the map

$$\mathrm{H}^1(K,\mathrm{Sim}^+(q))\to\mathrm{H}^1(K,\mathrm{Sim}(q))$$

has trivial kernel. The latter follows from the fact that the map

$$\mathrm{Sim}(q)(K)\to\mu_2(K)$$

is surjective (indeed, any reflection goes to $-1\in\mu_2(K)$). $\square$

4. Proof Theorem 1

Till the end of the proof of Lemma 9 we suppose that R is the local ring of a closed point on a smooth scheme over an infinite field. Let $[\xi]$ be a class from $\mathrm{H}^1(R,G)$, and $X=(G/P)_\xi$ be the corresponding homogeneous space. Denote by L a Levi subgroup of P.

Lemma 5. *Consider a parabolic subgroup P_1 in PGO_n^+, which is the stabilizer of an isotropic line. A Levi subgroup of P_1 is isomorphic to Sim_{n-2}^+.*

Proof. Is is clear from the matrix representation that a Levi subgroup of a parabolic subgroup P_1 in O_n^+ is isomorphic to $\mathrm{O}_{n-2}^+\times\mathbb{G}_m$. Now the homomorphism

$$\mathrm{O}_{n-2}^+\times\mathbb{G}_m\to\mathrm{Sim}_{n-2}^+$$

induced by the natural inclusions is surjective in the sense of groups schemes, and its kernel is μ_2. The claim follows. $\square$

Recall that a subset Ψ of a root system Φ is called *closed* if for any $\alpha,\beta\in\Psi$ such that $\alpha+\beta\in\Phi$ we have $\alpha+\beta\in\Psi$.

Lemma 6. *Let L modulo its center be isomorphic to PGO_{2m}^+ (resp., PGO_{2m+1}^+ or $\mathrm{PGO}_{2m}^+\times\mathrm{PGL}_2$). Denote by Φ the root system of G with respect to T, and by Ψ the root system of L with respect to T, where T is a maximal split torus in L. Assume that there is a root $\lambda\in\Phi$ such that the smallest closed set of roots Ψ' containing Ψ and $\pm\lambda$ is a root subsystem of type D_{m+1} (resp. B_{m+1} or $D_{m+1}+A_1$), and Ψ is the standard subsystem of type D_m (resp.*

B_m or $D_m + A_1$) therein. Then there is a surjective map $L \to \mathrm{Sim}^+_{2m}$ (resp., $L \to \mathrm{Sim}^+_{2m+1}$ or $L \to \mathrm{Sim}^+_{2m} \times \mathrm{PGL}_2$) whose kernel is a central closed subgroup scheme in L. In particular, the functor $\mathrm{H}^1(-, L)$ satisfies purity for R.

Proof. Consider the subgroup $H_{\Psi'}$ of G corresponding to Ψ' in the sense of [SGA, Exp. XXII, Definition 5.4.2]. Then $H_{\Psi'}$ is split reductive of type D_{m+1} (resp. B_{m+1} or $D_{m+1} + A_1$) by [SGA, Exp. XXII, Proposition 5.10.1], so it maps onto the split adjoint group of the same type. Under this map L maps onto a Levi subgroup of a parabolic subgroup P_1, which is isomorphic to Sim^+_{2m} (resp. Sim^+_{2m+1} or $\mathrm{Sim}^+_{2m} \times \mathrm{PGL}_2$) by Lemma 5. The purity claim follows from Theorem 5, Theorem 3 and Theorem 2. □

Lemma 7. *For any semi-local R-algebra S the map*

$$\mathrm{H}^1(S, L) \to \mathrm{H}^1(S, G)$$

is injective. Moreover, $X(S) \neq \emptyset$ if and only if $[\xi]_S$ comes from $\mathrm{H}^1(S, L)$.

Proof. See [SGA, Exp. XXVI, Cor. 5.10]. □

Lemma 8. *Assume that the functor $\mathrm{H}^1(-, L)$ satisfies purity for R. Then $X(K) \neq \emptyset$ implies $X(R) \neq \emptyset$.*

Proof. By Lemma 7 $[\xi]_K$ comes from some $[\zeta] \in \mathrm{H}^1(K, L)$, which is uniquely determined. Since X is smooth projective, for any prime ideal $\mathfrak{p}$ of height 1 we have $X(R_\mathfrak{p}) \neq \emptyset$. By Lemma 7 $\xi_{R_\mathfrak{p}}$ comes from some $[\zeta_\mathfrak{p}] \in \mathrm{H}^1(R_\mathfrak{p}, L)$. Now $[\zeta_\mathfrak{p}]_K = [\zeta]$, and so by the purity assumption there is $[\zeta'] \in \mathrm{H}^1(R, L)$ such that $[\zeta']_K = [\zeta]$.
Set $[\xi']$ to be the image of ζ' in $\mathrm{H}^1(R, G)$. We claim that $[\xi'] = [\xi]$. Indeed, by the construction $[\xi']_K = [\xi]_K$. It remains to recall that the map $\mathrm{H}^1(R, G) \to \mathrm{H}^1(K, G_K)$ is injective by [FP, Corollary of Theorem 1]. □

Lemma 9. *Let $Q \leq P$ be another parabolic subgroup, $Y = (G/Q)_\xi$. Assume that $X(K) \neq \emptyset$ implies $Y(K) \neq \emptyset$, and $Y(K) \neq \emptyset$ implies $Y(R) \neq \emptyset$. Then $X(K) \neq \emptyset$ implies $X(R) \neq \emptyset$.*

Proof. Indeed, there is a map $Y \to X$, so $Y(R) \neq \emptyset$ implies $X(R) \neq \emptyset$. □

Proof of Theorem 1. We first suppose that R is the local ring of a closed point on a smooth scheme over an infinite field. By Lemma 9 we may assume that P_K is a minimal parabolic subgroup of $(G_\xi)_K$. All possible types of such P_K are listed in [T, Table II]: the Dynkin diagram with circled vertices erased corresponds to the type of L. We show case by case that $\mathrm{H}^1(-, L)$ satisfies purity for R, hence we are in the situation of Lemma 8.
If $P = B$ is the Borel subgroup, obviously $\mathrm{H}^1(S, L) = \{*\}$ for any semi-local R-algebra S. In the case of index $E^9_{7,4}$ (resp. ${}^1E^{16}_{6,2}$) L modulo its center is isomorphic to $\mathrm{PGL}_2 \times \mathrm{PGL}_2 \times \mathrm{PGL}_2$ (resp. $\mathrm{PGL}_3 \times \mathrm{PGL}_3$), and we may apply Theorem 2 and Theorem 3. In the all other cases we provide an element $\lambda \in \mathrm{X}^*(T)$ such that the assumption of Lemma 6 holds ($\tilde{\alpha}$ stands for the maximal root, enumeration follows [B]). The indices $E^{78}_{7,1}$, $E^{133}_{8,1}$ and $E^{78}_{8,2}$ are

not in the list below since in those cases the L does not belong to one of the type D_m, B_m, $D_m \times A_1$. The index $E_{7,1}^{66}$ is not in the list below since in that case we need a weight λ which is not in the root lattice. So, the indices $E_{7,1}^{78}$, $E_{8,1}^{133}$, $E_{8,2}^{78}$ and $E_{7,1}^{66}$ are the exceptions in the statement of the Theorem.

Index	${}^1E_{6,2}^{28}$	$E_{7,1}^{48}$	$E_{7,2}^{31}$	$E_{7,3}^{28}$	$E_{8,1}^{91}$	$E_{8,2}^{66}$	$E_{8,4}^{28}$	$F_{4,1}^{21}$
λ	α_1	$-\tilde{\alpha}$	α_1	α_1	$-\tilde{\alpha}$	α_8	α_1	$-\tilde{\alpha}$

It remains to settle the case $P = P_1$ for $G = E_7^{sc}$. Denote by $\tilde{E}_7$ a Levi subgroup of a parabolic subgroup P_8 in E_8. Comparing the exact sequences

$$\mathrm{H}^1(R, E_7^{sc}) \to \mathrm{H}^1(R, E_7^{ad}) \to \mathrm{H}^2(R, \mu_2)$$

and

$$\mathrm{H}^1(R, \tilde{E}_7^{sc}) \to \mathrm{H}^1(R, E_7^{ad}) \to \mathrm{H}^2(R, \mathbb{G}_m)$$

and one sees that the image of $[\xi]$ in $\mathrm{H}^1(R, E_7^{ad})$ comes from some $[\zeta] \in \mathrm{H}^1(R, \tilde{E}_7)$. Let $\tilde{P}_1$ denote the corresponding parabolic subgroup in $\tilde{E}_7$; then we have $(E_7^{sc}/P_1)_\xi \simeq (\tilde{E}_7/\tilde{P}_1)_\zeta$.
We claim that $\mathrm{H}^1(-, \tilde{L})$ satisfies purity for R, where $\tilde{L}$ is a Levi subgroup of $\tilde{P}_1$. Indeed, consider a Levi subgroup G' of a parabolic subgroup P_1 inside E_8; then G' has type D_7 and $\tilde{L}$ is a Levi subgroup of a parabolic subgroup P_1 in G'. The rest of the proof goes exactly the same way as in Lemma 6.
Now suppose that R is a regular local ring containing an infinite field k. We first prove a general lemma. Let k' be an infinite field, X be a k'-smooth irreducible affine variety, Denote by $k'[X]$ the ring of regular functions on X and by $k'(X)$ the field of rational functions on X. Let $\mathfrak{p}$ be prime ideal in $k'[X]$, and let $\mathcal{O}_\mathfrak{p}$ be the corresponding local ring.

Lemma 10. *Theorem 1 holds for the local ring* $\mathcal{O}_\mathfrak{p}$.

Proof. Choose a maximal ideal $\mathfrak{m} \subset k'[X]$ containing $\mathfrak{p}$. One has inclusions of k'-algebras $\mathcal{O}_\mathfrak{m} \subset \mathcal{O}_\mathfrak{p} \subset k'(X)$. We already proved Theorem 1 for the ring $\mathcal{O}_\mathfrak{m}$. Thus Theorem 1 holds for the ring $\mathcal{O}_\mathfrak{p}$. □

The rest of the proof of Theorem 1 follows the arguments from [FP, page 5], which we reproduce here. Namely, let $\mathfrak{m}$ be the maximal ideal of R. Let k' be the algebraic closure of the prime field of R in k. Note that k' is perfect. It follows from Popescu's theorem ([P, Sw]) that R is a filtered inductive limit of smooth k'-algebras R_α. Modifying the inductive system R_α if necessary, we can assume that each R_α is integral. There are an index α, a 1-cocycle $\xi_\alpha \in Z^1(R_\alpha, G)$, and an element $f_\alpha \in R_\alpha$ such that $\xi = \varphi_\alpha(\xi_\alpha)$, f is the image of f_α under the homomorphism $\phi_\alpha : R_\alpha \to R$, the homogeneous space $X_\alpha := (G/H)_{\xi_\alpha}$ over R_α has a section over $(R_\alpha)_{f_\alpha}$.
If the field k' is infinite, then set $\mathfrak{p} = \phi_\alpha^{-1}(\mathfrak{m})$. The homomorphism ϕ_α induces a homomorphism of local rings $(R_\alpha)_\mathfrak{p} \to R$. By Lemma 10 one has $X_\alpha(R_\alpha) \neq \emptyset$, whence $X(R) \neq \emptyset$.

If the field k' is finite, then k contains an element t transcendental over k'. Thus R contains the subfield $k'(t)$ of rational functions in the variable t. So, if $R'_\alpha := R_\alpha \otimes_{k'} k'(t)$, then ϕ_α can be decomposed as follows $R_\alpha \xrightarrow{i_\alpha} R_\alpha \otimes_{k'} k'(t) = R'_\alpha \xrightarrow{\psi_\alpha} R$. Let $\xi' = i_\alpha(\xi_\alpha)$, $f'_\alpha = f_\alpha \otimes 1 \in R'_\alpha$, then the homogeneous space $X'_\alpha := (G/H)_{\xi'_\alpha}$ over R'_α has a section over $(R'_\alpha)_{f'_\alpha}$.
Let $\mathfrak{q} = \psi_\alpha^{-1}(\mathfrak{m})$. The ring R'_α is a $k'(t)$-smooth algebra over the infinite field $k'(t)$, and the homogeneous space $X'_\alpha := (G/H)_{\xi'_\alpha}$ over R'_α has a section over $(R'_\alpha)_{f'_\alpha}$. By Lemma 10 one has $X'_\alpha((R'_\alpha)_\mathfrak{q}) \neq \emptyset$. The homomorphism ψ_α can be factored as $R'_\alpha \to (R'_\alpha)_\mathfrak{q} \to R$. Thus $X(R) \neq \emptyset$. □

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Note on the Counterexamples for the Integral Tate Conjecture over Finite Fields

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Abstract. In this note we discuss some examples of non-torsion and non-algebraic cohomology classes for varieties over finite fields. The approach follows the construction of Atiyah-Hirzebruch and Totaro.

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1. Introduction

Let k be a finite field and let X be a smooth and projective variety over k. Let ℓ be a prime, $\ell \neq char(k)$. The Tate conjecture [20] predicts that the cycle class map

$$CH^i(X_{\bar{k}}) \otimes \mathbb{Q}_\ell \to \bigcup_H H^{2i}_{\acute{e}t}(X_{\bar{k}}, \mathbb{Q}_\ell(i))^H,$$

where the union is over all open subgroups H of $Gal(\bar{k}/k)$, is surjective.
In the integral version one is interested in the cokernel of the cycle class map

$$CH^i(X_{\bar{k}}) \otimes \mathbb{Z}_\ell \to \bigcup_H H^{2i}_{\acute{e}t}(X_{\bar{k}}, \mathbb{Z}_\ell(i))^H. \tag{1.1}$$

This map is not surjective in general: the counterexamples of Atiyah-Hirzebruch [1], revisited by Totaro [21], to the integral version of the Hodge conjecture, provide also counterexamples to the integral Tate conjecture [3]. More precisely, one constructs an ℓ-torsion class in $H^4_{\acute{e}t}(X_{\bar{k}}, \mathbb{Z}_\ell(2))$, which is not algebraic, for some smooth and projective variety X. However, one then wonders if there exists an example of a variety X over a finite field, such that the map

$$CH^i(X_{\bar{k}})\otimes\mathbb{Z}_\ell\to\bigcup_H H^{2i}_{\acute{e}t}(X_{\bar{k}},\mathbb{Z}_\ell(i))^H/torsion \tag{1.2}$$

is not surjective ([13, 3]). In the context of an integral version of the Hodge conjecture, Kollár [12] constructed such examples of curve classes. Over a finite field, Schoen [18] has proved that the map (1.2) is always surjective for curve classes, if the Tate conjecture holds for divisors on surfaces.

In this note we follow the approach of Atiyah-Hirzebruch and Totaro and we produce examples where the map (1.2) is not surjective for $\ell=2,3$ or 5.

Theorem 1.1. *Let ℓ be a prime from the following list: $\ell=2,3$ or 5. There exists a smooth and projective variety X over a finite field k, char$k\neq\ell$, such that the cycle class map*

$$CH^2(X_{\bar{k}})\otimes\mathbb{Z}_\ell\to\bigcup_H H^4_{\acute{e}t}(X_{\bar{k}},\mathbb{Z}_\ell(2))^H/torsion,$$

where the union is over all open subgroups H of $Gal(\bar{k}/k)$, is not surjective.

As in the examples of Atiyah-Hirzebruch and Totaro, our counterexamples are obtained as a projective approximation of the cohomology of classifying spaces of some simple simply connected groups, having ℓ-torsion in its cohomology. The non-algebraicity of a cohomology class is obtained by means of motivic cohomology operations: the operation Q_1 always vanishes on the algebraic classes and one establishes that it does not vanish on some class of degree 4. This is discussed in section 2. Next, in section 3 we investigate some properties of classifying spaces in our context and finally, following a suggestion of B. Totaro, we construct a projective variety approximating the cohomology of these spaces in small degrees in section 4.

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2. Motivic version of Atiyah-Hirzebruch arguments, revisited

2.1. Operations. Let k be a perfect field with $char(k)\neq\ell$ and let $\mathcal{H}.(k)$ be the motivic homotopy category of pointed k-spaces (see [15]). For $X\in\mathcal{H}.(k)$,

denote by $H^{*,*'}(X,\mathbb{Z}/\ell)$ the motivic cohomology groups with $\mathbb{Z}/\ell$-coefficients (*loc.cit.*). If X is a smooth variety over k (viewed as an object of $\mathcal{H}.(k)$), note that one has an isomorphism $CH^*(X)/\ell \xrightarrow{\sim} H^{2*,*}(X,\mathbb{Z}/\ell)$.
Voevodsky ([23], see also [17]) defined the reduced power operations P^i and the Milnor's operations Q_i on $H^{*,*'}(X,\mathbb{Z}/\ell)$:

$$P^i : H^{*,*'}(X,\mathbb{Z}/\ell) \to H^{*+2i(\ell-1),*'+i(\ell-1)}(X,\mathbb{Z}/\ell), i \geq 0$$

$$Q_i : H^{*,*'}(X,\mathbb{Z}/\ell) \to H^{*+2\ell^i-1,*'+(\ell^i-1)}(X,\mathbb{Z}/\ell), i \geq 0,$$

where $Q_0 = \beta$ is the Bockstein operation of degree $(1,0)$ induced from the short exact sequence $0 \to \mathbb{Z}/\ell \to \mathbb{Z}/\ell^2 \to \mathbb{Z}/\ell \to 0$.
One of the key ingredients for this construction is the following computation of the motivic cohomology of the classifying space $B_{ét}\mu_\ell \in \mathcal{H}.(k)$:

LEMMA 2.1. *([23, §6]) For each object $X \in \mathcal{H}.(k)$, the graded algebra $H^{*,*'}(X \times B_{ét}\mu_\ell, \mathbb{Z}/\ell)$ is generated over $H^{*,*'}(X,\mathbb{Z}/\ell)$ by elements x and y, $deg(x) = (1,1)$ and $deg(y) = (2,1)$, with $\beta(x) = y$ and* $x^2 = \begin{cases} 0 & \ell \text{ is odd} \\ \tau y + \rho x & \ell = 2 \end{cases}$
where τ is a generator of $H^{0,1}(Spec(k),\mathbb{Z}/2) \cong \mu_2$ and ρ is the class of (-1) in $H^{1,1}(Spec(k),\mathbb{Z}/2) \simeq k^*/(k^*)^2$.

For what follows, we assume that k contains a primitive ℓ^2-th root of unity ξ, so that $B_{ét}\mathbb{Z}/\ell \xrightarrow{\sim} B_{ét}\mu_\ell$ and $\beta(\tau) = \xi^\ell$ ($= \rho$ for $p = 2$) is zero in $k^*/(k^*)^\ell = H^{1,1}_{ét}(Spec(k);\mathbb{Z}/\ell)$.

We will need the following properties:

PROPOSITION 2.2. *Let $X \in \mathcal{H}.(k)$.*

(i) *$P^i(x) = 0$ for $i > m - n$ and $i \geq n$ and $x \in H^{m,n}(X,\mathbb{Z}/\ell)$;*
(ii) *$P^i(x) = x^\ell$ for $x \in H^{2i,i}(X,\mathbb{Z}/\ell)$;*
(iii) *if X is a smooth variety over k, the operation*

$$Q_i : CH^m(X)/\ell = H^{2m,m}(X,\mathbb{Z}/\ell) \to H^{2m+2\ell^i-1,m+(\ell^i-1)}(X,\mathbb{Z}/\ell)$$

is zero ;
(iv) *$Op.(\tau x) = \tau Op.(x)$ for $Op. = \beta, Q_i$ or P^i;*
(v) *$Q_i = [P^{\ell^{i-1}}, Q_{i-1}]$.*

Proof. See [23, §9]. For (iii) one uses that $H^{m,n}(X,\mathbb{Z}/\ell) = 0$ if $m - 2n > 0$ and X is a smooth variety over k, (iv) follows from the Cartan formula for the motivic cohomology.

2.2. Computations for $B_{ét}\mathbb{Z}/\ell$. The computations in this section are similar to [1, 21, 22].

Lemma 2.3. *In* $H^{*,*'}(B_{ét}\mathbb{Z}/\ell, \mathbb{Z}/\ell)$, *we have* $Q_i(x) = y^{\ell^i}$ *and* $Q_i(y) = 0$.

Proof. By definition $Q_0(x) = \beta(x) = y$. Using induction and Proposition 2.2, we compute

$$\begin{aligned} Q_i(x) = P^{\ell^{i-1}} Q_{i-1}(x) - Q_{i-1} P^{\ell^{i-1}}(x) &= P^{\ell^{i-1}} Q_{i-1}(x) \\ &= P^{\ell^{i-1}}(y^{\ell^{i-1}}) = y^{\ell^i}. \end{aligned}$$

Then $Q_1(y) = -Q_0 P^1(y) = -\beta(y^\ell) = 0$. For $i > 1$, using induction and Proposition 2.2 again, we conclude that $Q_i(y) = -Q_{i-1} P^{\ell^{i-1}}(y) = 0$.

□

Let $G = (\mathbb{Z}/\ell)^3$. As above, we view $B_{ét}G$ as an object of the category $\mathcal{H}.(k)$ and we assume that k contains a primitive ℓ^2-th root of unity. From Lemma 2.1, we have an isomorphism of modules over $H^{*,*'}(Spec(k), \mathbb{Z}/\ell)$:

$$H^{*,*'}(B_{ét}G, \mathbb{Z}/\ell) \cong H^{*,*'}(Spec(k), \mathbb{Z}/\ell)[y_1, y_2, y_3] \otimes \Lambda(x_1, x_2, x_3)$$

where $\Lambda(x_1, x_2, x_3)$ is isomorphic to the $\mathbb{Z}/\ell$-module generated by 1 and $x_{i_1} ... x_{i_s}$ for $i_1 < ... < i_s$, with relations $x_i x_j = -x_j x_i$ $(i \leq j)$, $\beta(x_i) = y_i$ and $x_i^2 = \tau y_i$ for $\ell = 2$.

Lemma 2.4. *Let* $x = x_1 x_2 x_3$ *in* $H^{3,3}(B_{ét}G, \mathbb{Z}/\ell)$. *Then*

$$Q_i Q_j Q_k(x) \neq 0 \in H^{2*,*}(B_{ét}G, \mathbb{Z}/\ell) \quad \textit{for } i < j < k.$$

Proof. Using Proposition 2.2(v) and Cartan formula for the operations on cup-products ([23] Proposition 9.7 and Proposition 13.4), we first get $Q_k(x) = y_1^{\ell^k} x_2 x_3 - y_2^{\ell^k} x_1 x_3 + y_3^{\ell^k} x_1 x_2$ and one then deduces

$$Q_i Q_j Q_k(x) = \sum_{\sigma \in S_3} \pm y_{\sigma(1)}^{\ell^k} y_{\sigma(2)}^{\ell^j} y_{\sigma(3)}^{\ell^i} \neq 0 \in \mathbb{Z}/\ell[y_1, y_2, y_3].$$

□

3. exceptional Lie groups

Let (G, ℓ) be a simple simply connected Lie group and a prime number from the following list:

$$(G, \ell) = \begin{cases} G_2, \ell = 2, \\ F_4, \ell = 3, \\ E_8, \ell = 5. \end{cases} \tag{3.1}$$

Then G is 2-connected and we have $H^3(G, \mathbb{Z}) \cong \mathbb{Z}$ for its (singular) cohomology group in degree 3. Hence BG, viewed as a topological space, is 3-connected and $H^4(BG, \mathbb{Z}) \cong \mathbb{Z}$ (see [14] for example). We write $x_4(G)$ for a generator of $H^4(BG, \mathbb{Z})$.

Given a field k with $char(k) \neq \ell$, let us denote by G_k the (split) reductive algebraic group over k corresponding to the Lie group G.
The Chow ring $CH^*(BG_k)$ has been defined by Totaro [22]. More precisely, one has

$$BG_k = \varinjlim(U/G_k), \tag{3.2}$$

where $U \subset W$ is an open set in a linear representation W of G_k, such that G_k acts freely on U. One can then identify $CH^i(BG_k)$ with the group $CH^i(U/G_k)$ if $\mathrm{codim}_W(W \setminus U) > i$, the group $CH^i(BG_k)$ is then independent of a choice of such U and W. Similarly, one can define the étale cohomology groups $H^i_{\acute{e}t}(BG_k, \mathbb{Z}_\ell(j))$ and the motivic cohomology groups $H^{*,*'}(BG_k, \mathbb{Z}/\ell)$ (see [8]), the latter coincide with the motivic cohomology groups of $B_{\acute{e}t}G$ as in [15] (cf. [8, Proposition 2.29 and Proposition 3.10]). We also have the cycle class map

$$cl : CH^*(BG_{\bar{k}}) \otimes \mathbb{Z}_\ell \to \bigcup_H H^{2*}_{\acute{e}t}(BG_{\bar{k}}, \mathbb{Z}_\ell(*))^H, \tag{3.3}$$

where the union is over all open subgroups H of $Gal(\bar{k}/k)$.
The following proposition is known.

Proposition 3.1. *Let (G, ℓ) be a group and a prime number from the list (3.1). Then*

(i) *the group G has a maximal elementary non toral subgroup of rank 3:*
$$i : A \simeq (\mathbb{Z}/\ell)^3 \subset G;$$
(ii) *$H^4(BG, \mathbb{Z}/\ell) \simeq \mathbb{Z}/\ell$, generated by the image x_4 of the generator $x_4(G)$ of $H^4(BG, \mathbb{Z}) \simeq \mathbb{Z}$;*
(iii) *$Q_1(i^*x_4) = Q_1Q_0(x_1x_2x_3)$, in the notations of Lemma 2.4. In particular, $Q_1(i^*x_4)$ is nonzero.*

Proof. For (i) see [5], for the computation of the cohomology groups with $\mathbb{Z}/\ell$-coefficients in (ii) see [14] VII 5.12; (iii) follows from [11] for $\ell = 2$ and [9, Proposition 3.2] for $\ell = 3, 5$ (see [10] as well). The class $Q_1(i^*x_4)$ is nonzero by Lemma 2.4 (see also [8, Théorème 4.1]). □

4. Algebraic approximation of BG

Write

$$BG_k = \varinjlim(U/G_k) \tag{4.1}$$

as in the previous section. Using proposition 3.1 and a specialization argument, we will first construct a quasi-projective algebraic variety X over a finite field k as a quotient $X = U/G_k$ (where $codim_W(W \setminus U)$ is big enough), such that the cycle class map (1.2) is not surjective for such X. However, if one is interested only in quasi-projective counterexamples for the surjectivity of the map (1.2), one can produce more naive examples, for instance as a complement of some smooth hypersurfaces in a projective space. Hence we are interested to find an approximation of Chow groups and the étale cohomology of $BG_{\bar{k}}$ as a smooth and projective variety. In the case when the group G is finite, this is done in [3, Théorème 2.1]. In this section we give such an approximation for the groups we consider here, this construction is suggested by B. Totaro. We will proceed in three steps. First, we construct a quasi-projective approximation in a family parametrized by $Spec\,\mathbb{Z}$. Then, for the geometric generic fibre we produce a projective approximation, by a topological argument. We finish the construction by specialization.

Let G be a compact Lie group as in (3.1). Let $\mathcal{G}$ be a split reductive group over $Spec\,\mathbb{Z}$ corresponding to G, such a group exists by [SGA3] XXV 1.3.

Lemma 4.1. *For any fixed integer $s \geq 0$ there exists a projective scheme $\mathcal{Y}/\mathrm{Spec}\,\mathbb{Z}$ and an open subscheme $\mathcal{W} \subset \mathcal{Y}$ such that*

(i) *$W \to \mathrm{Spec}\,\mathbb{Z}$ is smooth and the complement of $\mathcal{W}$ is of codimension at least s in each fiber of $\mathcal{Y} \to \mathrm{Spec}\,\mathbb{Z}$;*

(ii) *for any point $t \in \mathrm{Spec}\,\mathbb{Z}$ with residue field $\kappa(t)$ there is a natural map $\mathcal{W}_t \to B(\mathbb{G}_m \times \mathcal{G})_t$ inducing an isomorphism*

$$H^i_{ét}(\mathcal{W}_{\bar{t}}, \mathbb{Z}_\ell) \xrightarrow{\sim} H^i_{ét}(B(\mathbb{G}_m \times \mathcal{G})_{\bar{t}}, \mathbb{Z}_\ell) \text{ for } i \leq s, \ell \neq char\,\kappa(t). \tag{4.2}$$

Proof. Write $T = Spec\,\mathbb{Z}$, as it is an affine scheme of dimension 1, we can embed $\mathcal{G}$ as a closed subgroup of $\mathcal{H} = GL_{d,T}$ for some d (see [SGA3] VI_B 13.2). Moreover, it induces an embedding $\mathcal{G} \hookrightarrow PGL_{d,T}$, as the center of $\mathcal{G}$ is trivial for groups we consider here.

By a construction of [22, Remark 1.4] and [2, Lemme 9.2], there exists $n > 0$, a linear $\mathcal{H}$-representation $\mathcal{O}_T^{\oplus n}$ and an $\mathcal{H}$-invariant open subset $\mathcal{U} \subset \mathcal{O}_T^{\oplus n}$, which one can assume flat over T, such that the action of $\mathcal{H}$ is free on $\mathcal{U}$. Let $\mathcal{V}_N = \mathcal{O}_T^{\oplus Nn}$. Then the group $PGL_{n,T}$ acts on $\mathbb{P}(\mathcal{V}_N)$ and, taking N sufficiently large, one can assume that the action is free outside a subset S of high codimension (with respect to s).

By restriction, the group $\mathcal{G}$ acts on $\mathbb{P}(\mathcal{V}_N)$ as well, let $\mathcal{Y} = \mathbb{P}(\mathcal{V}_N)//\mathcal{G}$ be the GIT quotient for this action [16, 19]. The scheme $\mathcal{Y}$ is projective over T and we fix an embedding $\mathcal{Y} \subset \mathbb{P}^M_T$. Let

$$f : \mathcal{W} \to T \tag{4.3}$$

be the open set of $\mathcal{Y}$ corresponding to the quotient of the open set $\mathcal{U}$ as above where $\mathcal{G}_T$ acts freely. From the construction, one can assume that $\mathcal{W}$ has codimension at least s in $\mathcal{Y}$ in each fibre over T.

For any point $t \in T$ the fibre $\mathcal{W}_t$ is a smooth quasi-projective variety and if N is big enough, we have isomorphisms (cf. p. 263 in [22])

$$\mathcal{W}_t \cong (\mathbb{P}(\mathcal{V}_N) - S)_t/\mathcal{G}_t \cong ((\mathcal{V}_N - \{0\})/\mathbb{G}_m - S)_t)/\mathcal{G}_t \cong (\mathcal{V}_N - S')_t/(\mathbb{G}_m \times \mathcal{G})_t$$

where $S' = pr^{-1}S \cup \{0\}$ for the projection $pr : (\mathcal{V}_N - \{0\}) \to \mathbb{P}(\mathcal{V}_N)$. Hence we have isomorphisms

$$H^i_{ét}(\mathcal{W}_{\bar{t}}, \mathbb{Z}_\ell) \xrightarrow{\sim} H^i_{ét}(B(\mathbb{G}_m \times \mathcal{G})_{\bar{t}}, \mathbb{Z}_\ell) \text{ for } i \leq s, \ell \neq char\,\kappa(b),$$

induced by a natural map $\mathcal{W}_t \to B(\mathbb{G}_m \times \mathcal{G})_t$ from the presentation (4.1). □

Remark 4.2. More generally, in the statement above the map $\mathcal{W}_t \to B(\mathbb{G}_m \times \mathcal{G})_t$ induces an isomorphism $H^i_{ét}(\mathcal{W}_F, \mathbb{Z}_\ell) \xrightarrow{\sim} H^i_{ét}(B(\mathbb{G}_m \times \mathcal{G})_F, \mathbb{Z}_\ell)$, $i \leq s, \ell \neq char\, \kappa(t)$ for any F-point of T over t.

Lemma 4.3. *Let $Y \subset \mathbb{P}^M_{\mathbb{C}}$ be a projective variety over $\mathbb{C}$ and let $W \subset Y$ be a dense open in Y. Assume that W is smooth. Then for a general linear subspace L in $\mathbb{P}^M$ of codimension equal to $1 + dim(Y - W)$, the scheme $X = L \cap W$ is smooth and projective and the natural maps $H^i(W, \mathbb{Z}) \to H^i(X, \mathbb{Z})$ are isomorphisms for $i < dim\, X$.*

Proof. We apply a version of the Lefschetz hyperplane theorem for quasi-projective varieties, established by Hamm (as a special case of Theorem II.1.2 in [4]): for $V \subset \mathbb{P}^M$ a closed complex subvariety of dimension d, not necessarily smooth, $Z \subset V$ a closed subset, and H a hyperplane in $\mathbb{P}^M$, if $V - (Z \cup H)$ is local complete intersection (e.g. $V - Z$ is smooth) then

$$\pi_i((V - Z) \cap H) \to \pi_i(V - Z)$$

is an isomorphism for $i < d - 1$ and surjective for $i = d - 1$. In particular, $H^i(V - Z, \mathbb{Z}) \to H^i((V - Z) \cap H, \mathbb{Z})$ is an isomorphism for $i < d - 1$ and surjective for $i = d - 1$ by the Whitehead theorem.
Applying this statement to W and to successive intersections of W with linear forms defining L, we then deduce that $H^i(W, \mathbb{Z}) \to H^i(X, \mathbb{Z})$ is an isomorphism for $i < dim\, X$.

□

Proposition 4.4. *Let G be a compact Lie group as in (3.1).*
For all but finitely many primes p there exists a smooth and projective variety X_k over a finite field k with $char\, k = p$, an element $x_{4,\bar{k}} \in H^4_{ét}(B(\mathbb{G}_m \times G_{\bar{k}}), \mathbb{Z}_\ell(2))$, invariant under the action of $Gal(\bar{k}/k)$ and a map $\iota : X_k \to B(\mathbb{G}_m \times G_k)$ in the category $\mathcal{H}.(k)$ such that

- (i) *$\alpha_{\bar{k}} = \iota^* x_{4,\bar{k}}$ is a nonzero class in $H^4_{ét}(X_{\bar{k}}, \mathbb{Z}_\ell(2))/torsion$;*
- (ii) *the operation $Q_1(\bar{\alpha}_{\bar{k}})$ is nonzero, where we write $\bar{\alpha}_{\bar{k}}$ for the image of $\alpha_{\bar{k}}$ in $H^4_{ét}(X_{\bar{k}}, \mu_\ell^{\otimes 2})$.*

Proof. Let $\mathcal{W} \subset \mathcal{Y} \subset \mathbb{P}^M_{\mathbb{Z}}$ be as in Lemma 4.1 for $s \geq 4$.
Let $Y = \mathcal{Y}_{\mathbb{C}}$ and $W = \mathcal{W}_{\mathbb{C}}$ be the geometric generic fibres of $\mathcal{Y}$ and $\mathcal{W}$. Consider a general linear space L in $\mathbb{P}^M$ of codimension equal to $1 + dim(Y - W)$. We deduce from Lemma 4.3 above, that the variety $X := L \cap W$ is smooth and projective, and

$$H^i(X, R) \simeq H^i(B(\mathbb{G}_m \times G), R) \text{ for } i \leq s \text{ and } R = \mathbb{Z} \text{ or } \mathbb{Z}/n. \tag{4.4}$$

Hence $H^i_{ét}(X, \mathbb{Z}/n) \simeq H^i_{ét}(B(\mathbb{G}_m \times G), \mathbb{Z}/n), i \leq s$. In particular, by functoriality of the isomorphisms $H^i_{ét}(\cdot, \mathbb{Z}/n) \simeq H^i_{ét}(\cdot, \mu_n^{\otimes j}), i \leq s, j > 0$, for $\cdot = X$ and

$B(\mathbb{G}_m \times G)$, we get

$$H^i_{ét}(X, \mu_n^{\otimes j}) \simeq H^i_{ét}(B(\mathbb{G}_m \times G), \mu_n^{\otimes j}), i \leq s. \quad (4.5)$$

We can assume that we have an isomorphism as above for $i = 4$ and $i = 2\ell + 3$. Note that the cohomology of BG is a direct factor in the cohomology of $B(\mathbb{G}_m \times G)$ (cf. [8, Lemme 2.23]). Using Proposition 3.1, we then get an element $x_{4,\mathbb{C}}$ generating a direct factor isomorphic to $\mathbb{Z}_\ell$ in the cohomology group $H^4_{ét}(B(\mathbb{G}_m \times G), \mathbb{Z}_\ell(2))$. Denote $\alpha_{\mathbb{C}}$ its image in $H^4_{ét}(X, \mathbb{Z}_\ell(2))$.

We can now specialize the construction above to obtain the statement over a finite field. Note that one can assume that L is defined over $\mathbb{Q}$. One can then find an open $T' \subset \operatorname{Spec} \mathbb{Z}$ and a linear space $\mathcal{L} \subset \mathbb{P}^M_{T'}$ such that $\mathcal{L}_{\mathbb{C}} \simeq L$ and such that for any $t \in T'$ the fibre $\mathcal{X}_t$ of $\mathcal{X} = \mathcal{L} \cap \mathcal{T}$ is smooth. After passing to an étale cover T'' of T', one can assume that the inclusion $(\mathbb{Z}/\ell)^3 \subset G_{\mathbb{C}}$ from proposition 3.1 extends to an inclusion $i : \mathcal{A} = (\mathbb{Z}/\ell)^3_{T''} \hookrightarrow \mathcal{G}_{T''}$ (cf. [SGA3] XI.5.8).
Let $t \in T''$ and let $k = \kappa(t)$. As the schemes $\mathcal{X}_{T''}$, $\mathcal{W}_{T''}$ and $\mathcal{U}/\mathcal{A}$ are smooth over T'', we have the following commutative diagram, where the vertical maps are induced by the specialization maps (cf. [SGA4 1/2] Arcata V.3):

$$\begin{array}{ccccccc}
H^4_{ét}(X, \mathbb{Z}_\ell(2)) & \longleftarrow & H^4_{ét}(W, \mathbb{Z}_\ell(2)) & \longrightarrow & H^4_{ét}(\mathcal{U}_{\mathbb{C}}/(\mathbb{Z}/\ell)^3, \mathbb{Z}/\ell) & \xleftarrow{\simeq} & H^4_{ét}(B(\mathbb{Z}/\ell)^3_{\mathbb{C}}, \mathbb{Z}/\ell) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \simeq \\
H^4_{ét}(\mathcal{X}_{\bar{k}}, \mathbb{Z}_\ell(2)) & \longleftarrow & H^4_{ét}(\mathcal{W}_{\bar{k}}, \mathbb{Z}_\ell(2)) & \longrightarrow & H^4_{ét}(\mathcal{U}_{\bar{k}}/(\mathbb{Z}/\ell)^3, \mathbb{Z}/\ell) & \xleftarrow{\simeq} & H^4_{ét}(B(\mathbb{Z}/\ell)^3_{\bar{k}}, \mathbb{Z}/\ell)
\end{array}$$

The left vertical map is an isomorphism since $\mathcal{X}$ is proper, by a smooth-proper base change theorem. Hence we get a class $\alpha_{\bar{k}} \in H^4_{ét}(\mathcal{X}_{\bar{k}}, \mathbb{Z}_\ell(2))$, corresponding to $\alpha_{\mathbb{C}} \in H^4_{ét}(X, \mathbb{Z}_\ell(2))$. The map $H^4_{ét}(W, \mathbb{Z}_\ell(2)) \to H^4_{ét}(X, \mathbb{Z}_\ell(2))$ is an isomorphism by Lemma 4.3, so that $\alpha_{\bar{k}}$ comes from an element $x_{4,\bar{k}} \in H^4_{ét}(\mathcal{W}_{\bar{k}}, \mathbb{Z}_\ell(2))$. Let $\bar{\alpha}_{\mathbb{C}} \in H^4_{ét}(X, \mu_\ell^{\otimes 2})$ be the image of $\alpha_{\mathbb{C}}$ and let $\bar{\alpha}_{\bar{k}} \in H^4_{ét}(\mathcal{X}_{\bar{k}}, \mu_\ell^{\otimes 2})$ be the image of $\alpha_{\bar{k}}$. As the operation Q_1 commutes with the isomorphisms $H^i_{ét}(X, \mathbb{Z}/\ell) \to H^i_{ét}(X, \mu_\ell^{\otimes j})$, we get $Q_1(\bar{\alpha}_{\mathbb{C}}) \neq 0$ by proposition 3.1. The étale cohomology operation Q_1 also commutes with the specialization maps (cf. [7]), since these maps are obtained as composite of the natural maps $\phi \circ \psi^{-1}$ on the étale cohomology groups with torsion coefficients $H^i_{ét}(X_{\mathbb{C}}) \stackrel{\psi}{\leftarrow} H^i_{ét}(\mathcal{X}_S) \stackrel{\phi}{\rightarrow} H^i_{ét}(\mathcal{X}_{\bar{k}})$, where S is the strict henselization of T'' at t and ϕ is an isomorphism since $\mathcal{X}$ is smooth. Hence $Q_1(\bar{\alpha}_{\bar{k}})$ is nonzero as well. From the construction, the class $\alpha_{\bar{k}}$ generates a subgroup of $H^4_{ét}(\mathcal{X}_{\bar{k}}, \mathbb{Z}_\ell(2))$, which is a direct factor isomorphic to $\mathbb{Z}_\ell$, and is Galois-invariant. Letting $X_k = \mathcal{X}_k$ this finishes the proof of the proposition.

□

Remark 4.5. For the purpose of this note, the proposition above is enough. See also [6] for a a general statement on a projective approximation of the cohomology of classifying spaces.

Theorem 1.1 now follows from the proposition above:

Proof of theorem 1.1.
For k a finite field and X_k as in the proposition above, we find a nontrivial class $\alpha_{\bar{k}}$ in its cohomology in degree 4 modulo torsion, which is not annihilated by the operation Q_1. This class cannot be algebraic by proposition 2.2(iii). □

Remark 4.6. We can also adapt the arguments of [3, Théorème 2.1] to produce projective examples with higher torsion non-algebraic classes, while in *loc.cit.* one constructs ℓ-torsion classes. Let $G(n)$ be the finite group $G(\mathbb{F}_{\ell^n})$, so that we have

$$\varprojlim H^*_{\acute{e}t}(BG(n), \mathbb{Z}_\ell) = H^*_{\acute{e}t}(BG_{\bar{k}}, \mathbb{Z}_\ell).$$

Then, following the construction in *loc.cit.* one gets

> *For any $n > 0$, there exists a positive integer i_n and a Godeaux-Serre variety $X_{n,\bar{k}}$ for the finite group $G(i_n)$ such that*
> (1) *there is an element $x \in H^4_{\acute{e}t}(X_{n,\bar{k}}; \mathbb{Z}_\ell(2))$ generating $\mathbb{Z}/\ell^{n'}$ for some $n' \geq n$;*
> (2) *x is not in the image of the cycle class map (1.1).*

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AROUND THE ABHYANKAR–SATHAYE CONJECTURE

TO A. MERKURJEV ON HIS 60TH BIRTHDAY

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ABSTRACT. A "rational" version of the strengthened form of the Commuting Derivation Conjecture, in which the assumption of commutativity is dropped, is proved. A systematic method of constructing in any dimension greater than 3 the examples answering in the negative a question by M. El Kahoui is developed.

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1 INTRODUCTION

Throughout this paper k stands for an algebraically closed field of characteristic zero which serves as domain of definition for each of the algebraic varieties considered below.

Recall that an element c of the polynomial ring $k[x_1, \ldots, x_n]$ in variables $x_1, \ldots, x_n$ with coefficients in k is called a *coordinate* if there are the elements $t_1, \ldots, t_{n-1} \in k[x_1, \ldots, x_n]$ such that

$$k[c, t_1, \ldots, t_{n-1}] = k[x_1, \ldots, x_n] \tag{1}$$

(see, e.g., [vdEs 00]). Every coordinate is irreducible and, if $x_1, \ldots, x_n$ are the standard coordinate functions on the affine space $\mathbf{A}^n$, then the zero locus $\{c = 0\}$ of c in $\mathbf{A}^n$ is isomorphic to $\mathbf{A}^{n-1}$. The converse is claimed by the classical

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Abhyankar–Sathaye Conjecture. *If $f \in k[x_1, \dots, x_n]$ is an irreducible element whose zero locus in $\mathbf{A}^n$ is isomorphic to $\mathbf{A}^{n-1}$, then f is a coordinate.*

This conjecture is equivalent to the claim that every closed embedding $\iota\colon \mathbf{A}^{n-1} \hookrightarrow \mathbf{A}^n$ is *rectifiable*, i.e., there is an automorphism $\sigma \in \operatorname{Aut} \mathbf{A}^n$ such that $\sigma \circ \iota\colon \mathbf{A}^{n-1} \hookrightarrow \mathbf{A}^n$ is the standard embedding $(a_1, \dots, a_{n-1}) \mapsto (a_1, \dots, a_{n-1}, 0)$ (see [vdEs 00, Lemma 5.3.13]).
For $n = 2$ the Abhyankar–Sathaye conjecture is true (the Abhyankar–Moh–Suzuki theorem). For $n \geqslant 3$ it is still open, though there is a belief that in general it is false [vdEs 00, p. 103].
Exploration of this conjecture leads to the problem of constructing closed hypersurfaces in $\mathbf{A}^n$ isomorphic to $\mathbf{A}^{n-1}$, and irreducible polynomials in $k[x_1, \dots, x_n]$ whose zero loci in $\mathbf{A}^n$ are such hypersurfaces. The following two facts lead, in turn, to the idea of linking this problem with unipotent group actions:

(i) Every homogeneous space U/H, where U is a unipotent algebraic group and H its closed subgroup, is isomorphic to $\mathbf{A}^{\dim U/H}$ (see, e.g., [Gr 58, Prop. 2(ii)]).

(ii) All orbits of every morphic unipotent algebraic group action on a quasi-affine variety X are closed in X (see [Ro 61_2, Thm. 2]).

In view of (i) and (ii), every orbit of a morphic unipotent algebraic group action on $\mathbf{A}^n$ is the image of a closed embedding of some $\mathbf{A}^d$ in $\mathbf{A}^n$. In particular, orbits of dimension $n-1$ are the hypersurfaces of the sought-for type. Such actions, with a view of getting an approach to the Abhyankar–Sathaye conjecture, have been the object of study during the last decade, see [Ma 03], [EK 05], [DEM 08], [DEFM 11]. In particular, for commutative unipotent algebraic group actions, the following conjecture (whose formulation uses the equivalent language of locally nilpotent derivations, see [Fr 06]) has been put forward:

Commuting Derivations Conjecture ([Ma 03]). *Let D be a set of $n-1$ commuting locally nilpotent k-derivations of $k[x_1, \dots, x_n]$ linearly independent over $k[x_1, \dots, x_n]$. Then*

$$\{f \in k[x_1, \dots, x_n] \mid \partial(f) = 0 \textit{ for every derivation } \partial \in D\} = k[c], \tag{2}$$

where c is a coordinate in $k[x_1, \dots, x_n]$.

This conjecture is open for $n > 3$, proved in [Ma 03] for $n = 3$, and follows from Rentschler's theorem [Re 68] for $n = 2$. In [EK 05, Cor. 4.1] it is shown that it is equivalent to a weak version of the Abhyankar–Sathaye conjecture.
On the other hand, in [EK 05] the question is raised as to which extent $k[x_1, \dots, x_n]$ is characterized by (2). Namely, let $\mathcal{A}$ be a commutative associative unital k-algebra of transcendence degree $n > 0$ over k such that

(a) $\mathcal{A}$ is a unique factorization domain;

(b) there is a set D of $n-1$ commuting linearly independent over $\mathcal{A}$ locally nilpotent k-derivations of $\mathcal{A}$.

Consider the invariant algebra of D, i.e., the k-algebra

$$\mathcal{A}^D := \{a \in \mathcal{A} \mid \partial(a) = 0 \text{ for every } \partial \in D\}.$$

Question 1 ([EK 05, p. 449]). *Does the equality*

$$\mathcal{A}^D = k[c] \text{ for some element } c \in \mathcal{A} \tag{3}$$

imply the existence of elements $s_1, \ldots, s_{n-1} \in \mathcal{A}$ *and* $c_1, \ldots, c_{n-1} \in k[c]$ *such that* $\mathcal{A}$ *is the polynomial* k*-algebra* $k[c, s_1, \ldots, s_{n-1}]$ *and* $D = \{c_i \partial_{s_i}\}_{i=1}^{n-1}$?

Note that Equality (3) implies the transcendence of the element c over k, see, e.g., [Fr 06, p. 27, Principle 11(e)].
Question 1 is inspired by one of the main results of [EK 05], Theorem 3.1, claiming that for $n = 2$ the answer is affirmative. By [Mi 95, Thm. 2.6], Equality (3) holds and the answer to Question 1 is affirmative if Properties (a) and (b) hold, $\mathcal{A}$ is finitely generated over k, the multiplicative group $\mathcal{A}^\star$ of invertible elements of $\mathcal{A}$ coincides with $k^\star$, and $n = 2$.
The present paper contributes to the Commuting Derivation Conjecture and Question 1. In Section 2 a "rational" version of a strengthened form of the Commuting Derivation Conjecture is proved, in which the assumption of commutativity is dropped (see Theorem 2). Here "rational" means that the notion of "coordinate" is replaced by that of "rational coordinate" (see Definition 1 below). Geometrically, the latter means the existence of a birational (rather than biregular) automorphism of the ambient affine space that rectifies the corresponding hypersurface into the standard coordinate hyperplane. In Section 3, for every $n \geqslant 4$, a systematic method of constructing the pairs $(\mathcal{A}, D)$ is given, for which the answer to Question 1 is negative. Section 4 contains some remarks.

Notation, conventions, and some generalities

Below, as in [Bor 91], [Sp 98], "variety" means "algebraic variety" in the sense of Serre. The standard notation and conventions of [Bor 91], [Sp 98], and [PV 94] are used freely. In particular, the algebra of functions regular on a variety X is denoted by $k[X]$ (not by $\mathcal{O}(X)$ as in [DEFM 11], [DEM 08]).
Given an algebraic variety Z, below we denote the Zariski tangent space of Z at a point $z \in Z$ by $\mathrm{T}_{Z,z}$.
Let G be an algebraic group and let X be a variety. Given an action

$$\alpha \colon G \times X \to X \tag{4}$$

of G on X and the elements $g \in G$, $x \in X$, we denote $\alpha(g, x) \in X$ by $g \cdot x$. The G-orbit and the G-stabilizer of x are denoted resp. by $G \cdot x$ and G_x. If (4) is a morphism, then α is called a *regular* (or *morphic*) *action.* A regular action α is called *generically free* if there is a dense open subset U of X such that the G-stabilizer of every point of U is trivial.

Assume that X is irreducible. The map

$$\operatorname{Bir} X \to \operatorname{Aut}_k k(X), \qquad \varphi \mapsto (\varphi^*)^{-1}, \tag{5}$$

is a group isomorphism. We identify $\operatorname{Bir} X$ and $\operatorname{Aut}_k k(X)$ by means of (5) when we consider actions of the subgroups of $\operatorname{Bir} X$ by k-automorphisms of $k(X)$ and, conversely, actions of the subgroups of $\operatorname{Aut}_k k(X)$ by birational automorphisms of X.

Let $\theta\colon G \to \operatorname{Bir} X$ be an abstract group homomorphism. It determines an action of G on X by birational isomorphisms. If the partially defined map $G \times X \to X$, $(g, x) \mapsto \theta(g)(x)$ coincides on a dense open subset of $G \times X$ with a rational map $\varrho\colon G \times X \dashrightarrow X$, then ϱ is called a *rational action* of G on X.

By [Ro 56, Thm. 1], for every rational action ϱ there are a regular action of G on an irreducible variety Y, the open subsets X_0 and Y_0 of resp. X and Y (the subset Y_0 is not necessarily G-stable), and an isomorphism $Y_0 \to X_0$ such that the induced field isomorphism $k(X) = k(X_0) \to k(Y_0) = k(Y)$ is G-equivariant.

If ϱ is a rational action of G on X, then by

$$\pi_{G,X}\colon X \dashrightarrow X/\!\!/G$$

we denote a rational quotient of ϱ, i.e., $X/\!\!/G$ and $\pi_{G,X}$ are resp. a variety and a dominant rational map such that $\pi_{G,X}^*(k(X/\!\!/G)) = k(X)^G$ (see [PV 94, Sect. 2.4]). Depending on the situation we choose $X/\!\!/G$ as a suitable variety within the class of birationally isomorphic ones. A *rational section* for ϱ is a rational map $\sigma\colon X/\!\!/G \dashrightarrow X$ such that $\pi_{G,X} \circ \sigma = \mathrm{id}$.

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2 Rational coordinates

Since $\operatorname{char} k = 0$, equality (1) is equivalent to the property that the sequence $c, t_1, \ldots, t_{n-1}$ is a *coordinate system* on $\mathbf{A}^n$, i.e., that it separates points of $\mathbf{A}^n$. Thus to be a coordinate means to be an element of a coordinate system on $\mathbf{A}^n$. Considering separation of only points in general position in $\mathbf{A}^n$, we arrive to the following counterparts of these notions:

Definition 1. A sequence of rational functions $f_1, \ldots, f_n \in k(x_1, \ldots, x_n)$ is called a *rational coordinate system* on $\mathbf{A}^n$ if it separates points in general position in $\mathbf{A}^n$ or, equivalently,

$$k(f_1, \ldots, f_n) = k(x_1, \ldots, x_n).$$

If a rational function $f \in k(x_1, \ldots, x_n)$ is an element of a rational coordinate system on $\mathbf{A}^n$, then f is called a *rational coordinate.*

Theorem 1. *Let $\varrho\colon S\times X \dashrightarrow X$ be a rational action of a connected solvable affine algebraic group S on an irreducible algebraic variety X. Let*

$$\pi_{S,X}\colon X \dashrightarrow X/\!\!/S \tag{6}$$

be a rational quotient of this action. Then there are an integer $m\geqslant 0$ and a birational isomorphism $\varphi\colon X/\!\!/S\times \mathbf{A}^m \dashrightarrow X$ such that the following diagram is commutative

$$\begin{array}{ccc} X/\!\!/S\times \mathbf{A}^m & \overset{\varphi}{\dashrightarrow} & X \\ & {\scriptstyle \mathrm{pr}_1}\searrow \quad \swarrow{\scriptstyle \pi_{S,X}} & \\ & X/\!\!/S & \end{array}.$$

Proof. Replacing X by a birationally isomorphic variety, we may (and shall) assume that the action ϱ is regular. Put

$$m_{S,X} := \max_{x\in X} \dim S\cdot x. \tag{7}$$

First, consider the case

$$\dim S = 1. \tag{8}$$

In this case $m_{S,X}\leqslant 1$. If $m_{S,X}=0$, the action ϱ is trivial, hence $X/\!\!/S = X$, $\pi_{S,X} = \mathrm{id}$, and the claim is clear. Now let $m_{S,X}=1$. This means that S-stabilizers of points of a dense open subset are finite. In this case, we may assume that

$$\text{the action } \varrho \text{ is generically free.} \tag{9}$$

To prove this claim, recall (see, e.g., [Sp 98, Thm. 3.4.9]) that, given (8), we have $S=\mathbf{G}_a$ or $\mathbf{G}_m$. If $S=\mathbf{G}_a$, then the claim follows from the fact that, due to the assumption $\operatorname{char} k = 0$, there are no nontrivial finite subgroups in S. If $S=\mathbf{G}_m$, then S/F is isomorphic to S for any finite subgroup F, see, e.g., [Sp 98, 2.4.8(ii) and 6.3.6]. Therefore, taking as F the kernel of ϱ, we may assume that ϱ is faithful. As is well-known, since S is a torus, this, in turn, implies that ϱ is generically free, see, e.g., [Po 13, Lemma 2.4]. Thus (9) holds. Given (9), by [CTKPR 11, Thm. 2.13] we may replace X by an appropriate S-invariant open subset and assume that (6) is a torsor. Since S is a connected solvable affine algebraic group, by [Ro 56, Thm. 10] this torsor admits a rational section and therefore is trivial over an open subset of $X/\!\!/S$. As the group variety of S is birationally isomorphic to $\mathbf{A}^1$, this completes the proof of theorem in the case when (8) holds.

In the general case we argue by induction on $\dim S$. If $\dim S>0$, then solvability of S yields the existence of a closed connected normal subgroup N in S such that the (connected solvable affine) algebraic group $G:=S/N$ is one-dimensional. Put $Y:=X/\!\!/N$. By the inductive assumption, there are an integer

$r \geqslant 0$ and a birational isomorphism $\lambda\colon Y \times \mathbf{A}^r \dashrightarrow X$ such that the following diagram is commutative

$$\begin{array}{ccc} Y \times \mathbf{A}^r & \overset{\lambda}{\dashrightarrow} & X \\ & {\scriptstyle \mathrm{pr}_1} \searrow \quad \swarrow {\scriptstyle \pi_{N,X}} & \\ & Y & \end{array} . \tag{10}$$

Since $N \lhd S$ and $\pi_{N,X}^*(k(Y)) = k(X)^N$, the action ϱ induces a rational action of G on Y such that

$$Y/G = X/S, \tag{11}$$

$$\pi_{S,X} = \pi_{G,Y} \circ \pi_{N,X}. \tag{12}$$

Given (11) and using the proved validity of theorem for one-dimensional groups, we obtain that there are an integer $t \geqslant 0$ and a birational isomorphism $\gamma\colon X/S \times \mathbf{A}^t \dashrightarrow Y$ such that the following diagram is commutative

$$\begin{array}{ccc} X/S \times \mathbf{A}^t & \overset{\gamma}{\dashrightarrow} & Y \\ & {\scriptstyle \mathrm{pr}_1} \searrow \quad \swarrow {\scriptstyle \pi_{G,Y}} & \\ & X/S & \end{array} . \tag{13}$$

¿From (12) and diagrams (10), (13) we see that one can take $m = r + t$ and $\varphi = \lambda \circ (\gamma \times \mathrm{id}_{\mathbf{A}^r})$. This completes the proof. □

REMARK 1. The number m in the formulation of Theorem 1 is equal to the number $m_{S,X}$ given by (7).

COROLLARY. *In the notation of Theorem 1, there are the elements $f_1, \ldots, f_m$ of $k(X)$ such that*

(i) *$f_1, \ldots, f_m$ are algebraically independent over $k(X)^S$;*

(ii) *$k(X) = k(X)^S(f_1, \ldots, f_m)$.*

THEOREM 2. *Let a unipotent algebraic group U regularly act on $\mathbf{A}^n$. If*

$$\max_{a \in \mathbf{A}^n} \dim U \cdot a = n - 1, \tag{14}$$

then there is an irreducible polynomial $c \in k[\mathbf{A}^n]$ such that

(a) *$k[c] = k[\mathbf{A}^n]^U$;*

(b) *c is a rational coordinate.*

Proof. By Rosenlicht's theorem [Ro 56, Thm. 2] and the fiber dimension theorem, (14) implies that the transcendence degree of $k(\mathbf{A}^n)^U$ over k is 1 (cf. [PV 94, Sect. 2.3, Cor.]). Since U is unipotent, $k(\mathbf{A}^n)^U$ is the field of fractions of $k[\mathbf{A}^n]^U$, see [Ro 61_2, p. 220, Lemma]. By [Za 54] these properties imply that $k[\mathbf{A}^n]^U$ is a finitely generated k-algebra. Integral closedness of $k[\mathbf{A}^n]$ yields integral closedness of $k[\mathbf{A}^n]^U$, see, e.g., [PV 94, Thm. 3.16]. Thus $\mathbf{A}^n /\!\!/ U := \operatorname{Spec} k[\mathbf{A}^n]^U$ is an irreducible smooth affine algebraic curve. The curve $\mathbf{A}^n /\!\!/ U$ is unirational because $k(\mathbf{A}^n /\!\!/ U) = k(\mathbf{A}^n)^U$ is the subfield of the field of rational functions $k(\mathbf{A}^n)$. By Lüroth's theorem, from this we infer that this curve is rational. We then conclude that $\mathbf{A}^n /\!\!/ U$ is obtained from $\mathbf{P}^1$ by removing $s \geqslant 1$ points. Since $k[\mathbf{A}^n /\!\!/ U]^\star = k^\star$, we have $s = 1$, i.e., $\mathbf{A}^n /\!\!/ U = \mathbf{A}^1$, or, equivalently, $k[\mathbf{A}^n]^U = k[c]$ for an element $c \in k[\mathbf{A}^n]$. Since the group U is unipotent, it is connected (in view of $\operatorname{char} k = 0$) and admits no nontrivial algebraic homomorphisms $U \to \mathbf{G}_m$. This implies (see, e.g., [PV 94, Thm. 3.1]) that every nonconstant irreducible element of $k[\mathbf{A}^n]$ dividing c lies in $k[\mathbf{A}^n]^U$, which, in turn, easily implies irreducibility of c.
We now claim that c is a rational coordinate. Indeed, since $k(\mathbf{A}^n)^U$ is the field of fractions of $k[\mathbf{A}^n]^U$, we have $k(\mathbf{A}^n)^U = k(c)$. Hence by (14), Remark 1, and the Corollary of Theorem 1, there are elements $f_1, \ldots, f_{n-1} \in k(\mathbf{A}^n)$ such that $k(\mathbf{A}^n) = k(c, f_1, \ldots, f_{n-1})$. Whence the claim by Definition 1. □

3 Commuting derivations of unique factorization domains

First, we shall introduce the notation. Let G be a connected simply connected semisimple algebraic group. Fix a maximal torus T of G. Let X and $\mathrm{X}^\vee$ be, respectively, the character lattice and the cocharacter lattice of T in additive notation, and let $\langle\ ,\ \rangle\colon \mathrm{X} \times \mathrm{X}^\vee \to \mathbf{Z}$ be the natural pairing. The value of an element $\varphi \in \mathrm{X}$ at a point $t \in T$ denote by t^φ. Let Φ and $\Phi^+ \subset \mathrm{X}$ respectively be the root system of G with respect to T and the system of positive roots of Φ determined by a fixed Borel subgroup B of G containing T. Given a root $\alpha \in \Phi$, denote by $\alpha^\vee\colon \mathbf{G}_m \to T$ and U_α respectively the coroot and the one-dimensional unipotent root subgroup of G corresponding to α.
Let $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ be the system of simple roots of Φ_+ indexed as in [Bou 68]. If I is a subset of Δ, let Φ_I be the set of elements of Φ that are linear combinations of the roots in I. Denote by L_I be the subgroup of G generated by T and all the U_α's with $\alpha \in \Phi_I$. Let U_I (respectively, U_I^-) be the subgroup of G generated by all the U_α's with $\alpha \in \Phi^+ \setminus \Phi_I$ (respectively, $-\alpha \in \Phi^+ \setminus \Phi_I$). Then $P_I := L_I U_I$ and $P_I^- := L_I U_I^-$ are parabolic subgroups of G opposite to one another, U_I and U_I^- are the unipotent radicals of P_I and P_I^- respectively, L_I is a Levi subgroup of P_I and P_I^-, and

$$\dim U_I = \dim U_I^- = |\Phi^+ \setminus \Phi_I|, \tag{15}$$

$$\dim G = \dim L_I + 2 \dim U_I^-. \tag{16}$$

Every closed subgroup of G containing B is of the form P_I for some I. Every parabolic subgroup of G is conjugate to a unique P_I, called *standard* (with respect to T and B); see, e.g., [Sp 98, 8.4.3].

Let $\mathcal{D} \subset \mathrm{X}$ be the monoid of highest weights (with respect to T and B) of simple G-modules. Given a weight $\varpi \in \mathcal{D}$, let $E(\varpi)$ be a simple G-module with ϖ as the highest weight.

Denote by $\varpi_1, \ldots, \varpi_r$ the system of all indecomposable elements (i.e., *fundamental weights*) of $\mathcal{D}$ indexed in such a way that

$$\langle \varpi_i, \alpha_j^\vee \rangle = \delta_{ij}. \tag{17}$$

This system freely generates $\mathcal{D}$, i.e., for every weight $\varpi \in \mathcal{D}$ there are uniquely defined nonnegative integers $m_1, \ldots, m_r$ such that $\varpi = m_1\varpi_1 + \cdots + m_r\varpi_r$. By virtue of (17),

$$\langle \varpi, \alpha_i^\vee \rangle = m_i. \tag{18}$$

The integers (18) are called the *numerical labels of* ϖ. The "labeled" Dynkin diagram of $\alpha_1, \ldots, \alpha_r$, in which m_i is the label of the node α_i for every i, is called *the Dynkin diagram of* ϖ.

Given a nonzero $\varpi \in \mathcal{D}$, denote by $\mathbf{P}(E(\varpi))$ the projective space of all one-dimensional linear subspaces of $E(\varpi)$. The natural projection

$$\pi\colon E(\varpi) \setminus \{0\} \to \mathbf{P}(E(\varpi))$$

is G-equivariant with respect to the natural action of G on $\mathbf{P}(E(\varpi))$. The fixed point set of B in $\mathbf{P}(E(\varpi))$ is a single point $p(\varpi)$ and the G-orbit $\mathcal{O}(\varpi)$ of $p(\varpi)$ is the unique closed G-orbit in $\mathbf{P}(E(\varpi))$.

Consider in $E(\varpi)$ the affine cone $X(\varpi)$ over $\mathcal{O}(\varpi)$, i.e.,

$$X(\varpi) = \{0\} \sqcup \pi^{-1}(\mathcal{O}(\varpi)). \tag{19}$$

It is a G-stable irreducible closed subset of $E(\varpi)$. Let $\mathcal{A}(\varpi)$ be the coordinate algebra of $X(\varpi)$:

$$\mathcal{A}(\varpi) = k[X(\varpi)],$$

and let n be the transcendence degree of $\mathcal{A}(\varpi)$ over k, i.e.,

$$n = \dim X(\varpi). \tag{20}$$

Since every U_α is a one-dimensional unipotent group, its natural action on $X(\varpi)$ determines, as is well-known, an algebraic vector field $\mathcal{F}_\alpha$ on $X(\varpi)$, which, in turn, determines a locally nilpotent derivation ∂_α of $\mathcal{A}(\varpi)$; see, e.g., [Fr 06, 1.5]. Actually, ∂_α is induced by a locally nilpotent derivation of $k[E(\varpi)]$. Namely, as above, the natural action of U_α on $E(\varpi)$ determines a locally nilpotent derivation D_α of $k[E(\varpi)]$. Since the ideal $\mathcal{I}(\varpi)$ of $X(\varpi)$ in $k[E(\varpi)]$ is D_α-stable, D_α induces a locally nilpotent derivation of $\mathcal{A}(\varpi) = k[E(\varpi)]/\mathcal{I}(\varpi)$; the latter is ∂_α.

In the following theorem we have collected some facts we need. Some of them are probably folklore, but, for lack of references, we gave short elementary proofs in all cases.

Theorem 3. *For every nonzero weight $\varpi \in \mathcal{D}$, the following hold*:

(i) *The stabilizer $G_{p(\varpi)}$ of $p(\varpi)$ in G is $P_{I(\varpi)}$, where*

$$I(\varpi) = \{\alpha \in \Delta \mid \langle \varpi, \alpha^{\vee} \rangle = 0\}. \tag{21}$$

(ii) $\dim U^{-}_{I(\varpi)} = |\Phi^{+} \setminus \Phi_{I(\varpi)}| = n - 1.$

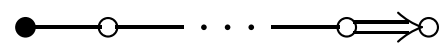
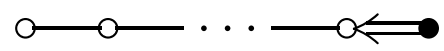
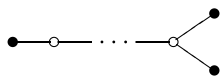
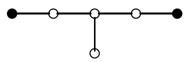
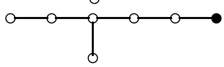
(iii) *The stabilizer of a point in general position for the natural action of $U^{-}_{I(\varpi)}$ on $X(\varpi)$ is trivial.*

(iv) *The set $\{\partial_{-\alpha} \mid \alpha \in \Phi^{+} \setminus \Phi_{I(\varpi)}\}$ of $n-1$ locally nilpotent derivations of the algebra $\mathcal{A}(\varpi)$ is linearly independent over $\mathcal{A}(\varpi)$.*

(v) *The following properties are equivalent*:

(C) $\{\partial_{-\alpha} \mid \alpha \in \Phi^{+} \setminus \Phi_{I(\varpi)}\}$ *is a set of commuting derivations. Equivalently, the unipotent group $U^{-}_{I(\varpi)}$ is commutative.*

(D) *In the Dynkin diagram of ϖ, every connected component S has at most one node with a nonzero label, and if such a node v exists, then S is not of type E_8, F_4, or G_2, and v is a black node of S colored as in the following table*:

type of S	colored S
A_l	
B_l	
C_l	
D_l	
E_6	
E_7	

(vi) $\mathcal{A}(\varpi)^{\star} = k^{\star}$.

(vii) *$\mathcal{A}(\varpi)$ is a unique factorization domain if and only if ϖ is a fundamental weight.*

(viii) *The following properties are equivalent*:

(s$_1$) *$X(\varpi)$ is singular*;

(s$_2$) $\dim E(\varpi) > n$;

(s$_3$) $X(\varpi) \neq E(\varpi)$.

The singular locus of every singular $X(\varpi)$ is the vertex 0.

Proof. (i): By the definition of $p(\varpi)$, the group B is contained in $G_{p(\varpi)}$. Hence

$$G_{p(\varpi)} = P_I \quad \text{for some } I. \tag{22}$$

In order to prove (i), fix a point $v \in \pi^{-1}(p(\varpi))$ and denote by G_v its stabilizer in G and by ℓ the line $\pi^{-1}(p(\varpi)) \cup \{0\}$ in $E(\varpi)$. We first show that the following properties of a root $\alpha \in \Delta$ are equivalent:

(a) $\alpha \in I$;

(b) $\langle \varpi, \alpha^\vee \rangle = 0$;

(c) the image of $\alpha^\vee$ is contained in G_v.

The definitions of $p(\varpi)$ and v imply that

$$t \cdot v = t^\varpi v \quad \text{for every element } t \in T, \tag{23}$$

and the definition of $\langle\ ,\ \rangle$ entails the equality

$$(\alpha^\vee(s))^\varpi = s^{\langle \varpi, \alpha^\vee \rangle} \quad \text{for every element } s \in \mathbf{G}_m. \tag{24}$$

Combining (23) and (24), we obtain the equivalence (b)$\Leftrightarrow$(c).
(a)$\Rightarrow$(c): By (22), the line ℓ is stable with respect to U_α. Being unipotent, the group U_α has no nontrivial characters and, therefore, no nontrivial one-dimensional modules. This proves that U_α is contained in G_v.
If (a) holds, then by (22) the line ℓ is stable with respect to $U_{-\alpha}$ as well. The same argument as for U_α then shows that $U_{-\alpha}$ is contained in G_v. Hence G_v contains the group S_α generated by U_α and $U_{-\alpha}$. But S_α contains the image of $\alpha^\vee$. This proves the implication (a)$\Rightarrow$(c).
(c)$\Rightarrow$(a): Assume that (c) holds. Since, as explained above, U_α is contained in G_v, the subgroup of S_α generated by U_α and the image of $\alpha^\vee$ is contained in G_v. This subgroup is a Borel subgroup of S_α. Therefore the S_α-orbit of v is a complete subvariety of $E(\varpi)$, i.e., a point. This means that S_α is contained in G_v. Therefore, $U_{-\alpha}$ is contained in G_v; whence (a) holds. This proves the implication (c)$\Rightarrow$(a).

Combining now (22) and (21) with the equivalence (a)$\Leftrightarrow$(c), we obtain the proof of Part (i).

(ii): Since $X(\varpi)$ is the affine cone over $\mathcal{O}(\varpi)$, we have

$$\dim X(\varpi) = \dim \mathcal{O}(\varpi) + 1. \tag{25}$$

On the other hand, (15), (16), and (i) entail

$$\dim \mathcal{O}(\varpi) = \dim U^-_{I_\varpi}. \tag{26}$$

Combining (25), (26), and (20), we obtain the proof of Part (ii).

(iii): Since $U_I^- \cap P_I = \{e\}$ for every I, the stabilizer of v for the natural action of $U_{I_\varpi}^-$ on $X(\varpi)$ is trivial because of (i). Hence $\dim X(\varpi)$ is the maximum of dimensions of $U_{I_\varpi}^-$-orbits in $X(\varpi)$. Since $U_{I_\varpi}^-$-orbits of points of a dense open subset of $X(\varpi)$ have maximal dimension, this means that the $U_{I_\varpi}^-$-stabilizer of a point in general position in $X(\varpi)$ is finite. But $U_{I_\varpi}^-$ has no nontrivial finite subgroups because it is a connected unipotent group and $\operatorname{char} k = 0$. This proves Part (iii).

(iv): Given a point $a \in X(\varpi)$, consider its $U_{I(\varpi)}^-$-orbit $U_{I(\varpi)}^- \cdot a$. By (iii), taking a suitable a, we may assume that

$$\dim \mathrm{T}_{U_{I(\varpi)}^- \cdot a, a} = \dim U_{I(\varpi)}^-. \tag{27}$$

Since $U_{I(\varpi)}^- = \prod_{\alpha \in \Phi^+ \setminus \Phi_{I(\varpi)}} U_{-\alpha}$ (the product being taken in any order), and $\operatorname{char} k = 0$,

$$\mathrm{T}_{U_{I(\varpi)}^- \cdot a, a} = \text{the linear span of } \{\mathcal{F}_{-\alpha}(a) \mid \alpha \in \Phi^+ \setminus \Phi_{I(\varpi)}\} \text{ over } k. \tag{28}$$

It follows from (27), (28), and (ii) that all the vectors $\mathcal{F}_{-\alpha}(a)$, where $\alpha \in \Phi^+ \setminus \Phi_{I(\varpi)}$ are linearly independent over k. Hence all the vector fields $\mathcal{F}_{-\alpha}$, where $\alpha \in \Phi^+ \setminus \Phi_{I(\varpi)}$, are linearly independent over $A(\varpi)$. This proves Part (iv).

(v): Since standard parabolic subgroups of G are products of standard parabolic subgroups of connected simple normal subgroups of G, the proof is reduced to the case, where G is simple. In this case (C)$\Leftrightarrow$(D) follows from (21) and the known classification of parabolic subgroups that have commutative unipotent radical (see, e.g., [RRS 92, Lemma 2.2 and Rem. 2.3]).

(vi): Since $\varpi \neq 0$, the action of T on $\pi^{-1}(p(\varpi))$ is nontrivial and, therefore, transitive. Since the restriction of π to $X(\varpi) \setminus \{0\}$ is a G-equivariant morphism onto the orbit $O(\varpi)$, this entails that

$$G \cdot v = X(\varpi) \setminus \{0\}. \tag{29}$$

By [PV 72, Thm. 2],

$$A(\varpi) \to k[G \cdot v], \quad f \mapsto f|_{G \cdot v}$$

is an isomorphism of k-algebras. On the other hand, the orbit map $G \to G \cdot v$ induces the embedding of $k[G \cdot v] \hookrightarrow k[G]$, and, being connected semisimple, G has no nontrivial characters, hence $k[G]^\star = k^\star$ by [Ro 61_1, Thm. 3]. This proves Part (vi).

(vii): This is proved, based on [Po 72, 74], in [PV 72, Thms. 4 and 5].

(viii): By virtue of (29), the singular locus of $X(\varpi)$ is either $\{0\}$ or empty; whence $X(\varpi)$ is singular if and only if $\dim \mathrm{T}_{X(\varpi),0} > n$. On the other hand, $\mathrm{T}_{E(\varpi),0} = E(\varpi)$ because $\mathrm{T}_{X(\varpi),0}$ is a submodule of the G-module $\mathrm{T}_{E(\varpi),0} = E(\varpi)$, which is simple. This implies (viii). $\square$

Thus, for every fundamental weight ϖ such that

— the property specified in Theorem 3(v)(D) holds;

— the variety $X(\varpi)$ is singular,

the answer to Question 1 for the pair $(\mathcal{A}, D)$, where

$$\begin{aligned}&\mathcal{A} := \mathcal{A}(\varpi),\\ &D := \{\partial_{-\alpha} \mid \alpha \in \Phi^+ \setminus \Phi_{I(\varpi)}\},\end{aligned}$$

is negative. There are examples of such pairs in any dimension $n \geqslant 4$.

Example 1. Let G be of type D_ℓ, $\ell \geqslant 3$, and $\varpi = \varpi_1$. Denote by V the underlying vector space of $E(\varpi)$ and by $\varphi_\varpi\colon G \to \mathrm{GL}(V)$ the homomorphism determining the G-module structure of $E(\varpi)$. Then $\dim V = 2\ell$ and $\varphi_\varpi(G)$ is the orthogonal group of a nondegenerate quadratic form f on V. There is a basis

$$e_1,\ e_2, \ldots, e_\ell, e_{-\ell}, e_{-\ell+1}, \ldots, e_{-1} \tag{30}$$

of V such that

$$f = x_{-1}x_1 + x_{-2}x_2 + \cdots + x_{-\ell}x_\ell,$$

where x_i is the ith coordinate function on V in basis (30). The variety $X(\varpi)$ coincides with that of all isotropic vectors of f,

$$X(\varpi) = \{v \in V \mid f(v) = 0\},$$

which, in turn, coincides with the closure of the G-orbit of e_1. Hence, if $\mathcal{P}_{2\ell}$ is the polynomial ring in 2ℓ variables $x_1,\ x_2, \ldots, x_\ell, x_{-\ell}, x_{-\ell+1}, \ldots, x_{-1}$ with coefficients in k (i.e., $\mathcal{P}_{2\ell} = k[E(\varpi)]$), then

$$\mathcal{A}(\varpi) = \mathcal{P}_{2\ell}/(f). \tag{31}$$

The k-algebra $\mathcal{A}(\varpi)$ is a unique factorization domain of transcendence degree $n := 2\ell - 1$ over k, and $\mathcal{A}(\varpi)^* = k^*$. The hypersurface of zeros of f in V is not smooth, hence $\mathcal{A}(\varpi)$ is not a polynomial ring over k.

Identifying every element of $\mathrm{GL}(V)$ with its matrix with respect to the basis (30), we may assume that $\mathrm{GL}(V) = \mathrm{GL}_{2\ell}$ and that the elements of $\varphi_\varpi(T)$ (resp. $\varphi_\varpi(B)$) are diagonal (resp. upper triangular) matrices (see, e.g., [Bou 75, Chap. VIII, §13, no. 4]). Using the explicit description of Φ, Δ, and U_α's available in this case (see loc.cit.), it is then not difficult to see that all the derivations $D_{-\alpha}$ of $\mathcal{P}_{2\ell}$, where $\alpha \in \Phi^+ \setminus \Phi_{I(\varpi)}$, are precisely the following $n-1$ commuting derivations D_j, $j = 2, 3, \ldots, \ell, -\ell, \ldots, -3, -2$, defined by the formula

$$\begin{aligned}&D_j(x_i) = \begin{cases} 0 & \text{for } i \neq j,\\ x_1 & \text{for } i = j\end{cases} \quad \text{if } i \neq -1,\\ &D_j(x_{-1}) = -x_{-j}.\end{aligned}$$

Let ∂_j be the locally nilpotent derivation of $\mathcal{A}(\varpi)$ induced (in view of $D_j(f) = 0$ and (31)) by D_j. Then $D := \{\partial_j\}$ is the set of $n-1$ commuting derivations that are linearly independent over $\mathcal{A}(\varpi)$; whence (3) holds (see, e.g., [Ma 03, Prop. 3.4], [DEFM 11, Lemma 1]). Thus in this case the answer to Question 1 is negative. □

Example 2. Let G be of type B_ℓ, $\ell \geqslant 2$, and $\varpi = \varpi_1$. An argument similar to that in Example 1 shows that if $\mathcal{P}_{2\ell+1}$ is the polynomial ring in $2\ell+1$ variables $x_1,\ x_2, \ldots, x_\ell, x_0, x_{-\ell}, x_{-\ell+1}, \ldots, x_{-1}$ with coefficients in k, then

$$\mathcal{A}(\varpi) = \mathcal{P}_{2\ell+1}/(h), \quad \text{where } h = x_0^2 + x_{-1}x_1 + x_{-2}x_2 + \cdots + x_{-\ell}x_\ell. \tag{32}$$

The k-algebra $\mathcal{A}(\varpi)$ is a unique factorization domain of transcendence degree $n := 2\ell$ over k, which is not a polynomial ring over k, and $\mathcal{A}(\varpi)^* = k^*$. The derivations $D_{-\alpha}$ of $\mathcal{P}_{2\ell+1}$, where $\alpha \in \Phi^+ \setminus \Phi_{I(\varpi)}$, are precisely the following $n-1$ commuting derivations D_j, $j = 2, 3, \ldots, \ell, 0, -\ell, \ldots, -3, -2$, defined by the formula

$$D_j(x_i) = \begin{cases} 0 & \text{for } i \neq j, \\ x_1 & \text{for } i = j \end{cases} \quad \text{if } i \neq -1,$$

$$D_j(x_{-1}) = \begin{cases} -x_{-j} & \text{for } j \neq 0, \\ 2x_0 & \text{for } j = 0. \end{cases}$$

Let ∂_j be the locally nilpotent derivation of $\mathcal{A}(\varpi)$ induced (in view of $D_j(h) = 0$ and (32)) by D_j. Then $D := \{\partial_j\}$ is the set of $n-1$ commuting derivations that are linearly independent over $\mathcal{A}(\varpi)$; whence (3) holds. Therefore, in this case the answer to Question 1 is negative as well. □

In Examples 1 and 2, the algebras $\mathcal{A}(\varpi)$ are hypersurfaces (quadratic cones). In the general case, they are quotient algebras of polynomial algebras modulo the ideals generated by finitely many quadratic forms. Namely, the G-module $\mathrm{S}^2(E(\varpi)^*)$ of quadratic forms on $E(\varpi)$ contains a unique submodule (the Cartan component) $C(\varpi)$ isomorphic to $E(2\varpi)^*$; whence there is a unique submodule $M(\varpi)$ such that $\mathrm{S}^2(E(\varpi)^*) = C(\varpi) \oplus M(\varpi)$. It is known that the ideal of $k[E(\varpi)]$ generated by $M(\varpi)$ is then the ideal of elements $k[E(\varpi)]$ vanishing on $X(\varpi)$ (see in [Br 85, Sect. 4.1, Thm.] the part concerning the ideal J). Therefore, $X(\varpi)$ is cut out in $E(\varpi)$ by

$$\frac{\dim E(\varpi)\big(\dim E(\varpi) + 1\big)}{2} - \dim E(2\varpi)$$

homogeneous quadrics (cf. also [Li 82]).

We note that a pair $(\mathcal{A}, D)$ with $\mathcal{A}$ of transcendence degree 3 over k, for which the answer to Question 1 is negative, exists as well: based on the famous theorem that the Koras–Russell threefold X is not isomorphic to $\mathbf{A}^3$ (see [Ma-Li 96]), in [EK 05] it is shown that one may take $\mathcal{A} = k[X]$.

4 Remarks

1. The same arguments as in the proof of Theorem 2 prove the following

Theorem 4. *Let X be an irreducible affine n-dimensional variety endowed with a regular action of a unipotent algebraic group U. Assume that*

(i) *X is unirational;*

(ii) *X is normal;*

(iii) *$k[X]^\star = k^\star$;*

(iv) $\max\limits_{x\in X} \dim U\cdot x = n-1$.

Then there is an irreducible element t of $k[X]$ and elements $f_1, \dots f_{n-1} \in k(X)$ such that

(a) *$k[X]^U = k[f]$;*

(b) *$k(X) = k(t, f_1, \dots, f_{n-1})$.*

In particular, X is rational.

2. Theorem 1 in [DEFM 11] (in which the notation $\mathcal{O}(X)$ is used in place of our $k[X]$) reads as follows:

Let U be an n-dimensional unipotent group acting faithfully on an affine n-dimensional variety X satisfying $\mathcal{O}(X)^\star = k^\star$. Then $X \cong \mathbf{A}^n$ if one of the following two conditions holds:

(a) *some $x \in X$ has trivial isotropy subgroup, or*

(b) $n = 2$, X is factorial, and U acts without fixed points.

The proof in [DEFM 11] shows that, in fact, X is also assumed to be irreducible. We remark that, actually, given (a), the assumption $\mathcal{O}(X)^\star = k^\star$ is superfluous and, changing the proof (see below), one may drop it. Moreover, in this case, more generally, affiness of X may be replaced by quasi-affiness, the assumption $\dim U = n$ may be dropped, and (a) may be replaced by the assumption

$$\dim U_x + \dim X = \dim U. \tag{33}$$

Proof. Indeed, (33) implies that $\dim U \cdot x = \dim X$. On the other hand, by [Ro 61_2, Thm. 2], unipotency of U implies that $U \cdot x$ is closed in X. Hence $U \cdot x = X$. Therefore, $X \cong U/U_x$, whence the claim by (i) in Introduction. □

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The Arason Invariant of Orthogonal Involutions of Degree 12 and 8, and Quaternionic Subgroups of the Brauer Group

To Sasha Merkurjev on his 60th birthday

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Abstract. Using the Rost invariant for torsors under Spin groups one may define an analogue of the Arason invariant for certain hermitian forms and orthogonal involutions. We calculate this invariant explicitly in various cases, and use it to associate to every orthogonal involution σ with trivial discriminant and trivial Clifford invariant over a central simple algebra A of even co-index an element $f_3(\sigma)$ in the subgroup $F^\times \cdot [A]$ of $H^3(F, \mathbb{Q}/\mathbb{Z}(2))$. This invariant $f_3(\sigma)$ is the double of any representative of the Arason invariant $e_3(\sigma) \in H^3(F, \mathbb{Q}/\mathbb{Z}(2))/F^\times \cdot [A]$; it vanishes when $\deg A \leq 10$ and also when there is a quadratic extension of F that simultaneously splits A and makes σ hyperbolic. The paper provides a detailed study of both invariants, with particular attention to the degree 12 case, and to the relation with the existence of a quadratic splitting field.

As a main tool we establish, when $\deg(A) = 12$, an additive decomposition of (A, σ) into three summands that are central simple algebras of degree 4 with orthogonal involutions with trivial discriminant, extending a well-known result of Pfister on quadratic forms of dimension 12 in I^3F. The Clifford components of the summands generate a subgroup U of the Brauer group of F, in which every element is represented by a quaternion algebra, except possibly the class of A. We show that the Arason invariant $e_3(\sigma)$, when defined, generates the homology of a complex of degree 3 Galois cohomology groups, attached to the subgroup U, which was introduced and studied by Peyre. In the final section, we use the results on degree 12 algebras to extend the definition of the Arason invariant to trialitarian triples in which all three algebras have index at most 2.

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1. Introduction

In quadratic form theory, the Arason invariant is a degree 3 Galois cohomology class with μ_2 coefficients attached to an even-dimensional quadratic form with trivial discriminant and trivial Clifford invariant. Originally defined by Arason in [1], it can also be described in terms of the Rost invariant of a split Spin group, as explained in [26, § 31.B]. It is not always possible to extend this invariant to the more general setting of orthogonal involutions, see [7, §3.4]. Nevertheless, one may use the Rost invariant of some possibly non-split Spin groups to define *relative* and *absolute* Arason invariants for some orthogonal involutions (see [41] or section 2 below for precise definitions). This was first noticed by Bayer-Fluckiger and Parimala in [6], where they use the Rost invariant to prove classification theorems for hermitian or skew-hermitian forms, leading to a proof of the so-called Hasse Principle conjecture II.

Below, we will focus on the absolute Arason invariant, which we will refer to simply as the Arason invariant. For orthogonal involutions, it was considered by Garibaldi, who uses the notation e_3^{hyp}, in [15], and by Berhuy in the index 2 case in [8]. In particular, the latter covers the case of central simple algebras of degree $2m$ with m odd, since such an algebra has index 1 or 2 when it is endowed with an orthogonal involution.

Based on the Rost invariant for the exceptional group E_8, Garibaldi also defined, for orthogonal involutions on degree 16 central simple algebras, another invariant related to the Arason invariant of quadratic forms, denoted by e_3^{16}. Bermudez and Ruozzi [9] extended this definition to all degrees divisible by 16. It follows from the proof of Corollary 10.11 in [15], and Remark 4.10 in Barry's paper [2], that these invariants do *not* coincide with what we call Arason invariant in this paper.

A systematic study of the relative and absolute Arason invariants for orthogonal involutions was recently initiated in [35], where the degree 8 case is studied in detail. In this paper, we continue with an investigation of absolute invariants in degree 12.

Let (A, σ) be a central simple algebra with orthogonal involution over a field F of characteristic different from 2. The Arason invariant $e_3(\sigma)$, when defined, belongs to the quotient

$$H^3(F, \mathbb{Q}/\mathbb{Z}(2))/F^\times \cdot [A],$$

where $F^\times \cdot [A]$ denotes the subgroup consisting of cup products $(\lambda) \cdot [A]$, for $\lambda \in F^\times$, $[A]$ the Brauer class of A. In § 2 below, we give a general formula for computing the Arason invariant of an algebra with involution admitting a rank 2 factor. It follows from this formula that the Arason invariant is not always represented by a cohomology class of order 2. This reflects the fact that the

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Dynkin index of a non-split Spin group, in large enough degree, is equal to 4. We define a new invariant $f_3(\sigma) \in H^3(F, \mu_2)$, attached to any orthogonal involution for which the Arason invariant is defined, and which vanishes if and only if the Arason invariant is represented by a cohomology class of order 2. This invariant is zero if the algebra is split, or of degree ≤ 10; starting in degree 12, we produce explicit examples where it is non-zero. This is an important motivation for studying the degree 12 case in detail.

The main results of the paper are given in sections 3 to 5. First, we prove that a degree 12 algebra with orthogonal involution (A, σ), having trivial discriminant and trivial Clifford invariant, admits a non-unique decomposition as a sum—in the sense of algebras with involution—of three degree 4 algebras with orthogonal involution of trivial discriminant. This can be seen as a refinement of the main result of [17], even though our proof in index 4 relies on the open-orbit argument of [17], see Remark 3.5. Using this additive decomposition, we associate to (A, σ) in a non-canonical way some subgroups of the Brauer group of F, which we call decomposition groups of (A, σ), see Definition 3.6. Such subgroups $U \subset \mathrm{Br}(F)$ are generated by (at most) three quaternion algebras; they were considered by Peyre in [32], where the homology of the following complex is studied:

$$F^\times \cdot U \to H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \to H^3(F_U, \mathbb{Q}/\mathbb{Z}(2)),$$

where $F^\times \cdot U$ denotes the subgroup generated by cup products $(\lambda) \cdot [B]$, for $\lambda \in F^\times$, $[B] \in U$, and F_U is the function field of the product of the Severi-Brauer varieties associated to the elements of U. Peyre's results are recalled in § 3.3.

In § 4, we restrict to those algebras with involution of degree 12 for which the Arason invariant is defined, and we prove $e_3(\sigma)$ detects isotropy of σ, and vanishes if and only if σ is hyperbolic. We then explore the relations between the decomposition groups and the values of the Arason invariant. Reversing the viewpoint, we also prove that the Arason invariant $e_3(\sigma)$ provides a generator of the homology of Peyre's complex, where (A, σ) is any algebra with involution admitting U as a decomposition group.

In § 5, we give a necessary and sufficient condition for the vanishing of $f_3(\sigma)$ in degree 12, in terms of decomposition groups of (A, σ). When there is a quadratic extension that splits A and makes σ hyperbolic, an easy corestriction argument shows that $f_3(\sigma) = 0$, see Proposition 2.5. We give in § 5.3 an explicit example to show that the converse does not hold. This also provides new examples of subgroups U for which the homology of Peyre's complex is non-trivial, which differ from Peyre's example in that the homology is generated by a Brauer class of order 2.

In the last section, we extend the definition of the Arason invariant in degree 8 to index 2 algebras with involution of trivial discriminant, and such that the two components of the Clifford algebra have index 2. In this case also, the algebra with involution has an additive decomposition, and the Arason invariant detects isotropy.

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Notation. Throughout this paper, we work over a base field F of characteristic different from 2. We use the notation $H^n(F, M) = H^n(\mathrm{Gal}(F_{\mathrm{sep}}/F), M)$ for any discrete torsion Galois module M. For every integer $n \geq 0$ we let

$$H^n(F) = H^n\big(F, \mathbb{Q}/\mathbb{Z}(n-1)\big),$$

(see [16, Appendix A, p. 151] for a precise definition). The cohomology classes we consider actually are in the 2-primary part of these groups, hence we shall not need the modified definition for the p-primary part when $\mathrm{char}(F) = p \neq 0$. For each integer $m \geq 0$ we let ${}_mH^n(F)$ denote the m-torsion subgroup of $H^n(F)$. Using the norm-residue isomorphism, one may check that

$${}_2H^n(F) = H^n(F, \mu_2) \qquad \text{and} \qquad {}_4H^3(F) = H^3(F, \mu_4^{\otimes 2}),$$

(see for instance [32, Remark 4.1]). In particular, ${}_2H^1(F) = F^\times/F^{\times 2}$. For every $a \in F^\times$ we let $(a) \in {}_2H^1(F)$ be the square class of a. For $a_1, \ldots, a_n \in F^\times$ we let $(a_1, \ldots, a_n) \in {}_2H^n(F)$ be the cup-product

$$(a_1, \ldots, a_n) = (a_1) \cdot \cdots \cdot (a_n).$$

We refer to [26] and to [28] for background information on central simple algebras with involution and on quadratic forms. However, we depart from the notation in [28] by letting $\langle\langle a_1, \ldots, a_n\rangle\rangle$ denote the n-fold Pfister form

$$\langle\langle a_1, \ldots, a_n\rangle\rangle = \langle 1, -a_1\rangle \cdot \cdots \cdot \langle 1, -a_n\rangle \qquad \text{for } a_1, \ldots, a_n \in F^\times.$$

Thus, the discriminant, the Clifford invariant and the Arason invariant, viewed as cohomological invariants e_1, e_2 and e_3, satisfy:

$$e_1\big(\langle\langle a_1\rangle\rangle\big) = (a_1), \quad e_2\big(\langle\langle a_1, a_2\rangle\rangle\big) = (a_1, a_2), \quad e_3\big(\langle\langle a_1, a_2, a_3\rangle\rangle\big) = (a_1, a_2, a_3).$$

For every central simple F-algebra A, we let $[A]$ be the Brauer class of A, which we identify to an element in $H^2(F)$. If L is a field extension of F, we let $A_L = A \otimes_F L$ be the L-algebra obtained from A by extending scalars.
Recall that the object function from the category Fields_F of field extensions of F to abelian groups defined by

$$L \mapsto \coprod_{n \geq 0} H^n(L)$$

is a cycle module over $\mathrm{Spec}F$ (see [37, Rem.1.11]). In particular, each group $H^n(L)$ is a module over the Milnor K-ring K_*L. The Brauer class $[A]$ of the algebra A generates a cycle submodule; we let M_A denote the quotient cycle module. Thus, for every field $L \supseteq F$, we have

$$M_A^n(L) = \begin{cases} H^n(L) & \text{if } n = 0 \text{ or } 1; \\ H^n(L)/\big(K_{n-2}L \cdot [A_L]\big) & \text{if } n \geq 2. \end{cases}$$

In particular, $M_A^2(L) = \mathrm{Br}(L)/\{0, [A_L]\}$ and $M_A^3(L) = H^3(L)/\big(L^\times \cdot [A_L]\big)$.

Let F_A denote the function field of the Severi–Brauer variety of A, which is a generic splitting field of A. Scalar extension from F to F_A yields group homomorphisms

$$M^2_A(F) \to M^2_A(F_A) = \mathrm{Br}(F_A) \qquad \text{and} \qquad M^3_A(F) \to M^3_A(F_A) = H^3(F_A).$$

The first map is injective by Amitsur's theorem, see [18, Th. 5.4.1]; the second one is injective if the Schur index of A divides 4 or if A is a division algebra that decomposes into a tensor product of three quaternion algebras, but it is not always injective (see [32], [23] and [24]).

2. Cohomological invariants of orthogonal forms and involutions

Most of this section recalls well-known facts on absolute and relative Arason invariants that will be used in the sequel of the paper. Since we will consider additive decompositions of algebras with involution, we need to state the results both for hermitian forms and for involutions. Some new results are also included. In Proposition 2.6 and Corollary 2.13, we give a general formula for the Arason invariant of an algebra with involution which has a rank 2 factor. In Definitions 2.4 and 2.15, we introduce a new invariant, called the f_3-invariant, which detects whether the Arason invariant is represented by a cohomology class of order 2. Finally, we state and prove in Proposition 2.7 a general formula for computing the f_3 invariant of a sum of hermitian forms, which is used in the proof of the main results of the paper.

Throughout this section, D is a central division algebra over an arbitrary field F of characteristic different from 2, and θ is an F-linear involution on D (i.e., an involution of the first kind). To any nondegenerate hermitian or skew-hermitian module (V, h) over (D, θ) we may associate the corresponding adjoint algebra with involution $\mathrm{Ad}_h = (\mathrm{End}_D V, \mathrm{ad}_h)$. Conversely, any central simple algebra A over F Brauer-equivalent to D and endowed with an F-linear involution σ can be represented as $(A, \sigma) \simeq \mathrm{Ad}_h$ for some nondegenerate hermitian or skew-hermitian module (V, h) over (D, θ). The hermitian or skew-hermitian module (V, h) is said to be a *hermitian module of orthogonal type* if the adjoint involution ad_h on $\mathrm{End}_D V$ is of orthogonal type. This occurs if and only if either h is hermitian and θ is of orthogonal type, or h is skew-hermitian and θ is of symplectic type, see [26, (4.2)]. Abusing terminology, we also say that h is a *hermitian form of orthogonal type* when (V, h) is a hermitian module of orthogonal type (even though h may actually be skew-hermitian if θ is symplectic).

2.1. Invariants of hermitian forms of orthogonal type. Let (V, h) be a hermitian module of orthogonal type over (D, θ); we call $r = \dim_D V$ the *relative rank* of h and $n = \deg \mathrm{End}_D V$ the *absolute rank* of h. These invariants are related by $n = r \deg D$. Cohomological invariants of h are defined in terms of invariants of the adjoint involution ad_h. Namely, if n is even, the *discriminant* of h, denoted $e_1(h) \in H^1(F, \mu_2)$, is the discriminant of ad_h;

the corresponding quadratic étale extension K/F is called the *discriminant extension.* If n is even and $e_1(h)$ is trivial, the *Clifford invariant* of h, denoted $e_2(h)$, is the class in $M^2_D(F)$ of any component of the Clifford algebra of ad_h.

Remark 2.1. It follows from the relations between the components of the Clifford algebra (see [26, (9.12)]) that the Clifford invariant is well-defined. However, since we do not assume n is divisible by 4, this invariant need not be represented by a cohomology class of order 2 in general.

Our definitions of rank and discriminant differ slightly from the definitions used by Bayer and Parimala in [5, §2], who call "rank" what we call the relative rank of h. The discriminant $d(h)$ of h in the sense of [5, §2.1] is related to $e_1(h)$ by

$$e_1(h) = d(h)\,\mathrm{disc}(\theta)^r,$$

where $\mathrm{disc}(\theta)$ is the discriminant of θ as defined in [26, §7], and $H^1(F,\mu_2)$ is identified with the group of square classes $F^\times/F^{\times 2}$. In particular, $e_1(h) = d(h)$ when h has even relative rank r. By [5, 2.1.3], the Clifford invariant $\mathcal{C}\ell(h)$ used by Bayer and Parimala coincides with our $e_2(h)$ when they are both defined, i.e., when h has even relative rank and trivial discriminant. Assume now that the hermitian form h has even relative rank, i.e., $\dim_D V$ is even. The vector space V then carries a hyperbolic hermitian form h_0 of orthogonal type, and the standard nonabelian Galois cohomology technique yields a canonical bijection between $H^1(F,\mathrm{O}(h_0))$ and the set of isomorphism classes of nondegenerate hermitian forms of orthogonal type on V, under which the trivial torsor corresponds to the isomorphism class of h_0, see [26, §29.D]. If $e_1(h)$ and $e_2(h)$ are trivial, the torsor corresponding to the isomorphism class of h has two different lifts to $H^1(F,\mathrm{O}^+(h_0))$, and one of these lifts can be further lifted to a torsor ξ in $H^1(F,\mathrm{Spin}(h_0))$. Bayer and Parimala consider the Rost invariant $R(\xi)\in H^3(F)$ and define in [6, §3.4, p. 664] an *Arason invariant* of h by the formula

$$e_3(h) = R(\xi) + F^\times\cdot[D] \in M^3_D(F).$$

This invariant satisfies the following properties:

Lemma 2.2 (Bayer–Parimala [6, Lemma 3.7, Corollary 3.9]). *Let h and h' be two hermitian forms of orthogonal type over (D,θ) with even relative rank, trivial discriminant, and trivial Clifford invariant.*

(i) *If h is hyperbolic, then $e_3(h) = 0$;*
(ii) *$e_3(h\perp h') = e_3(h) + e_3(h')$;*
(iii) *$e_3(\lambda h) = e_3(h)$ for any $\lambda\in F^\times$.*

In particular, it follows immediately that $e_3(h)$ is a well-defined invariant of the Witt class of h. Moreover, we have:

Corollary 2.3. *The Arason invariant e_3 has order* 2.

Proof. Indeed, for any h as above, we have $2e_3(h) = e_3(h)+e_3(h) = e_3(h)+e_3(-h) = e_3(h\perp(-h)) = 0$, since $h\perp(-h)$ is hyperbolic. □

Using the properties of the Arason invariant, we may define a new invariant as follows. Assume h is as above, a hermitian form of orthogonal type with even relative rank, trivial discriminant, and trivial Clifford invariant. Let c, $c' \in H^3(F)$ be two representatives of the Arason invariant $e_3(h)$. Since $c - c' \in F^\times \cdot [D]$, we have $2c = 2c' \in H^3(F)$, hence $2c$ depends only on h and not on the choice of the representative c of $e_3(h)$. Because of Corollary 2.3, the image of $2c$ in $M^3_D(F)$ vanishes, hence $2c \in F^\times \cdot [D]$. These observations lead to the following definition:

Definition 2.4. Given an arbitrary representative $c \in H^3(F)$ of the Arason invariant $e_3(h) \in M^3_D(F)$, we let $f_3(h) = 2c \in F^\times \cdot [D] \subset {}_2H^3(F)$.

Thus, the invariant $f_3(h)$ is well-defined; it vanishes if and only if the Arason invariant $e_3(h)$ is represented by a class of order at most 2, or equivalently, if every representative of $e_3(h)$ is a cohomology class of order at most 2. It is clear from the definition that the f_3 invariant is trivial when D is split. Another case where the f_3 invariant vanishes is the following:

Proposition 2.5. *If there exists a quadratic extension K/F such that D_K is split and h_K is hyperbolic, then $f_3(h) = 0$.*

Proof. Assume such a field K exists, and let $c \in H^3(F)$ be any representative of $e_3(h) \in M^3_D(F)$. Since h_K is hyperbolic, we have $e_3(h_K) = c_K = 0 \in M^3_D(K) = H^3(K)$. Hence, $\mathrm{cor}_{K/F}(c_K) = 2c = 0$, that is $f_3(h) = 0$. □

In §5, we will see that the f_3 invariant is trivial if the absolute rank is ≤ 10. In addition, we will prove that the converse of Proposition 2.5 does not hold, even in absolute rank 12.

Since the Dynkin index of the group $\mathrm{Spin}(\mathrm{Ad}_{h_0})$ divides 4, the Arason invariant $e_3(h)$ is represented by a cohomology class of order dividing 4. Moreover, there are examples where it is represented by a cohomology class of order equal to 4. Therefore, $f_3(h)$ is nonzero in general. Explicit examples can be constructed by means of Proposition 2.6 below, which yields the e_3 and f_3 invariants of hermitian forms with a rank 2 factor. (See also Corollary 2.19 for examples in the lowest possible degree, which is 12.)

2.2. Hermitian forms with a rank 2 factor. Consider a hermitian form which admits a decomposition as $\langle 1, -\lambda\rangle \otimes h$ for some $\lambda \in F^\times$ and some hermitian form h. In this case, we have the following explicit formulae for the Arason and the f_3-invariant, when they are defined:

Proposition 2.6. *Let h be a hermitian form of orthogonal type with even absolute rank n, and let K/F be the discriminant quadratic extension. For any $\mu \in K^\times$, the hermitian form $\langle 1, -N_{K/F}(\mu)\rangle h$ has even relative rank, trivial discriminant and trivial Clifford invariant. Moreover,*

$$e_3\big(\langle 1, -N_{K/F}(\mu)\rangle h\big) = \mathrm{cor}_{K/F}\big(\mu \cdot e_2(h_K)\big)$$

and

$$f_3\big(\langle 1, -N_{K/F}(\mu)\rangle h\big) = \begin{cases} 0 & \text{if } n \equiv 0 \bmod 4, \\ N_{K/F}(\mu)\cdot[D] & \text{if } n \equiv 2 \bmod 4. \end{cases}$$

In particular, if h has trivial discriminant, then for $\lambda \in F^\times$ we have

$$e_3\big(\langle 1, -\lambda\rangle h\big) = \lambda \cdot e_2(h)$$

and

$$f_3\big(\langle 1, -\lambda\rangle h\big) = \begin{cases} 0 & \text{if } n \equiv 0 \bmod 4, \\ \lambda\cdot[D] & \text{if } n \equiv 2 \bmod 4. \end{cases}$$

Proof. We first need to prove that the hermitian form $\langle 1, -N_{K/F}(\mu)\rangle h$ has trivial discriminant and trivial Clifford invariant. This can be checked after scalar extension to a generic splitting field of D, since the corresponding restriction maps ${}_2H^1(F) \to {}_2H^1(F_D)$ and $M_D^2(F) \to H^2(F_D)$ are injective. In the split case, the result follows from an easy computation for the discriminant, and from [28, Ch. V, §3] for the Clifford invariant. Alternatively, one may observe that the algebra with involution $\mathrm{Ad}_{\langle 1, -N_{K/F}(\mu)\rangle h}$ decomposes as $\mathrm{Ad}_{\langle 1, -N_{K/F}(\mu)\rangle} \otimes \mathrm{Ad}_h$, and apply [26, (7.3)(4)] and [39]. This computation also applies to the trivial discriminant case, where $\lambda = N_{F\times F/F}(\lambda, 1)$.

With this in hand, we may compute the Arason invariant by using the description of cohomological invariants of quasi-trivial tori given in [30]. Let us first assume h has trivial discriminant. Consider the multiplicative group scheme $\mathbb{G}_m$ as a functor from the category Fields_F to the category of abelian groups. For any field L containing F, consider the map

$$\varphi_L\colon \mathbb{G}_m(L) \to M_D^3(L) \quad \text{defined by} \quad \lambda \mapsto e_3\big(\langle 1, -\lambda\rangle h_L\big).$$

To see that φ_L is a group homomorphism, observe that in the Witt group of D_L we have for λ_1, $\lambda_2 \in L^\times$

$$\langle 1, -\lambda_1\lambda_2\rangle h_L = \langle 1, -\lambda_1\rangle h_L + \langle \lambda_1\rangle\langle 1, -\lambda_2\rangle h_L.$$

Therefore, Lemma 2.2 yields

$$e_3\big(\langle 1, -\lambda_1\lambda_2\rangle h_L\big) = e_3\big(\langle 1, -\lambda_1\rangle h_L\big) + e_3\big(\langle 1, -\lambda_2\rangle h_L\big).$$

The collection of maps φ_L defines a natural transformation of functors $\mathbb{G}_m \to M_D^3$, i.e., a degree 3 invariant of $\mathbb{G}_m$ with values in the cycle module M_D. By [29, Prop. 2.5], there is an element $u \in M_D^2(F)$ such that for any L and any $\lambda \in L^\times$

$$\varphi_L(\lambda) = \lambda \cdot u_L \qquad \text{in } M_D^3(L).$$

To complete the computation of $e_3\big(\langle 1, -\lambda\rangle h\big)$, it only remains to show that $u = e_2(h)$. Since the restriction map $M_D^2(F) \to M_D^2(F_D) = H^2(F_D)$ is injective, it suffices to show that $u_{F_D} = e_2(h)_{F_D}$. Now, since F_D is a splitting field for D, there exists a quadratic form q over F_D, with trivial discriminant, such that $(\mathrm{Ad}_h)_{F_D} \simeq \mathrm{Ad}_q$. Let t be an indeterminate over F_D. We have

$$\mathrm{Ad}_{\langle 1, -t\rangle} \otimes (\mathrm{Ad}_h)_{F_D(t)} \simeq \mathrm{Ad}_{\langle 1, -t\rangle} \otimes (\mathrm{Ad}_q)_{F_D(t)}$$

hence $e_3(\langle 1,-t\rangle h_{F_D(t)})$ is the Arason invariant of the quadratic form $\langle 1,-t\rangle q_{F_D(t)}$, which is $t\cdot e_2(q)=t\cdot e_2(h)_{F_D(t)}$. Therefore, we have

$$t\cdot u_{F_D(t)}=t\cdot e_2(h)_{F_D(t)}.$$

Taking the residue $\partial\colon H^3(F_D(t))\to H^2(F_D)$ for the t-adic valuation, we obtain $u_{F_D}=e_2(h)_{F_D}$, which completes the proof of the formula for $e_3\bigl(\langle 1,-\lambda\rangle h\bigr)$. To compute $f_3\bigl(\langle 1,-\lambda\rangle h\bigr)$, recall that $e_2(h)$ is represented by any of the two components C_+, C_- of the Clifford algebra of Ad_h. Therefore, $e_3\bigl(\langle 1,-\lambda\rangle h\bigr)$ is represented by $\lambda\cdot[C_+]$ or $\lambda\cdot[C_-]$, and

$$f_3\bigl(\langle 1,-\lambda\rangle h\bigr)=2(\lambda\cdot[C_+])=2(\lambda\cdot[C_-]).$$

By [26, (9.12)] we have

$$2[C_+]=2[C_-]=\begin{cases}0 & \text{if } n\equiv 0 \bmod 4,\\ [D] & \text{if } n\equiv 2 \bmod 4.\end{cases}$$

The formula for $f_3\bigl(\langle 1,-\lambda\rangle h\bigr)$ follows.

Assume now h has nontrivial discriminant. The proof in this case follows the same pattern. Let K/F be the discriminant field extension. We consider the group scheme $R_{K/F}(\mathbb{G}_m)$, which is the Weil transfer of the multiplicative group. For every field L containing F, the map

$$\mu\in R_{K/F}(\mathbb{G}_m)(L)=(L\otimes_F K)^\times\mapsto e_3(\langle 1,-N_{L\otimes K/L}(\mu)\rangle h_L)\in M^3_D(L)$$

defines a degree 3 invariant of the quasi-trivial torus $R_{K/F}(\mathbb{G}_m)$ with values in the cycle module M_D. By [30, Th. 1.1], there is an element $u\in M^2_D(K)$ such that for any field L containing F and any $\mu\in(L\otimes K)^\times$,

$$e_3(\langle 1,-N_{L\otimes K/L}(\mu)\rangle h_L)=\operatorname{cor}_{L\otimes K/L}(\mu\cdot u_{L\otimes K})\quad\text{in } M^3_D(L).$$

It remains to show that $u=e_2(h_K)$. To prove this, we consider the field $L=K(t)$, where t is an indeterminate. Since $e_1(h_{K(t)})=0$, the previous case applies. We thus get for any $\mu\in(K(t)\otimes_F K)^\times$

$$N_{K(t)\otimes K/K(t)}(\mu)\cdot e_2(h_{K(t)})=\operatorname{cor}_{K(t)\otimes K/K(t)}(\mu\cdot u_{K(t)\otimes K})\quad\text{in } M^3_D(K(t)).$$

Let ι be the nontrivial F-automorphism of K. The $K(t)$-algebra isomorphism $K(t)\otimes_F K\simeq K(t)\times K(t)$ mapping $\alpha\otimes\beta$ to $(\alpha\beta,\alpha\iota(\beta))$ yields an isomorphism $M^2_D(K(t)\otimes K)\simeq M^2_D(K(t))\times M^2_D(K(t))$ that carries $u_{K(t)\otimes K}$ to $(u_{K(t)},\iota(u)_{K(t)})$. Thus, for every $(\mu_1,\mu_2)\in K(t)^\times\times K(t)^\times$,

$$\mu_1\mu_2\cdot e_2(h_{K(t)})=\mu_1\cdot u_{K(t)}+\mu_2\cdot\iota(u)_{K(t)}\quad\text{in } M^3_D(K(t)).$$

In particular, if $\mu_1=t$ and $\mu_2=1$ we get $t\cdot e_2(h_{K(t)})=t\cdot u_{K(t)}$, hence taking the residue for the t-adic valuation yields $e_2(h_K)=u$, proving the formula for $e_3\bigl(\langle 1,-N_{K/F}(\mu)\rangle h\bigr)$.

To complete the proof, we compute $f_3\bigl(\langle 1,-N_{K/F}(\mu)\rangle h\bigr)$. Let C be the Clifford algebra of Ad_h, so $[C]$ represents $e_2(h_K)$ and

$$f_3\bigl(\langle 1,-N_{K/F}(\mu)\rangle h\bigr)=2\operatorname{cor}_{K/F}(\mu\cdot[C]).$$

By [26, (9.12)] we have

$$2[C] = \begin{cases} 0 & \text{if } n \equiv 0 \bmod 4, \\ [D_K] & \text{if } n \equiv 2 \bmod 4. \end{cases}$$

The formula for $f_3\big(\langle 1, -N_{K/F}(\mu)\rangle h\big)$ follows by the projection formula. □

2.3. Hermitian forms with an additive decomposition. We now present another approach for computing the f_3-invariant, which does not rely on the computation of the Arason invariant. This leads to an explicit formula in a more general situation, which will be used in the proof of Theorem 5.4:

Proposition 2.7. *Let (V_1, h_1), …, (V_m, h_m) be hermitian modules of orthogonal type and even absolute rank n_1, …, n_m over (D, θ), and let λ_1, …, $\lambda_m \in F^\times$. Let also $h = \langle 1, -\lambda_1\rangle h_1 \perp \ldots \perp \langle 1, -\lambda_m\rangle h_m$. If $\sum_{i=1}^m \lambda_i \cdot e_1(h_i) = 0$, then h has trivial Clifford invariant, and*

$$f_3(h) = \lambda_1^{n_1/2} \ldots \lambda_m^{n_m/2} \cdot [D].$$

To prove this proposition, we need some preliminary results. Let (V, h_0) be a hyperbolic module of orthogonal type over (D, θ). Recall from [26, (13.31)] the canonical map ("vector representation")

$$\chi\colon \mathrm{Spin}(h_0) \to \mathrm{O}^+(h_0).$$

Since proper isometries have reduced norm 1, we also have the inclusion

$$i\colon\ \mathrm{O}^+(h_0) \to \mathrm{SL}(V).$$

Lemma 2.8. *The following diagram, where R is the Rost invariant, is commutative:*

$$\begin{array}{ccc} H^1(F, \mathrm{Spin}(h_0)) & \xrightarrow{(i\circ\chi)_*} & H^1(F, \mathrm{SL}(V)) \\ {\scriptstyle R}\big\downarrow & & \big\downarrow{\scriptstyle R} \\ H^3(F) & \xrightarrow{\quad 2 \quad} & H^3(F) \end{array}$$

Proof. This lemma is just a restatement of the property that the Rost multiplier of the map $i \circ \chi$ is 2, see [16, Ex. 7.15, p. 124]. □

We next recall from [26, (29.27)] (see also [13]) the canonical description of the pointed set $H^1(F, \mathrm{O}^+(h_0))$. Define a functor $\mathrm{SSym}(h_0)$ from Fields_F to the category of pointed sets as follows: for any field L containing F, set

$$\mathrm{SSym}(h_0)(L) = \{(s, \lambda) \in \mathrm{GL}(V_L) \times L^\times \mid \mathrm{ad}_{h_0}(s) = s \text{ and } \mathrm{Nrd}(s) = \lambda^2\},$$

where the distinguished element is $(1, 1)$. Let F_s be a separable closure of F and let $\Gamma = \mathrm{Gal}(F_s/F)$ be the Galois group. We may identify $\mathrm{SSym}(h_0)(F_s)$ with the quotient $\mathrm{GL}(V_{F_s})/\,\mathrm{O}^+((h_0)_{F_s})$ by mapping a class $a \cdot \mathrm{O}^+((h_0)_{F_s})$ to $(a\,\mathrm{ad}_{h_0}(a), \mathrm{Nrd}(a))$ for $a \in \mathrm{GL}(V_{F_s})$. Therefore, we have an exact sequence of pointed Γ-sets

$$1 \to \mathrm{O}^+((h_0)_{F_s}) \to \mathrm{GL}(V_{F_s}) \to \mathrm{SSym}(h_0)(F_s) \to 1.$$

Since $H^1(F, \mathrm{GL}(V_{F_s})) = 1$ by Hilbert's Theorem 90, the induced exact sequence in Galois cohomology yields a canonical bijection between $H^1(F, \mathrm{O}^+(h_0))$ and the orbit set of $\mathrm{GL}(V)$ on $\mathrm{SSym}(h_0)(F)$. Abusing notation, we write simply $\mathrm{SSym}(h_0)$ for $\mathrm{SSym}(h_0)(F)$. The orbits of $\mathrm{GL}(V)$ on $\mathrm{SSym}(h_0)$ are the equivalence classes under the following relation:

$$(s,\lambda) \sim (s',\lambda') \quad \text{if } s' = as\,\mathrm{ad}_{h_0}(a) \text{ and } \lambda' = \lambda\,\mathrm{Nrd}(a) \text{ for some } a \in \mathrm{GL}(V).$$

Therefore, we may identify

$$H^1(F, \mathrm{O}^+(h_0)) = \mathrm{SSym}(h_0)/{\sim}.$$

Lemma 2.9. *The composition* $H^1(F, \mathrm{O}^+(h_0)) \xrightarrow{i_*} H^1(F, \mathrm{SL}(V)) \xrightarrow{R} H^3(F)$ *maps the equivalence class of* (s,λ) *to* $\lambda \cdot [A]$.

Proof. Let $\pi\colon \mathrm{SSym}(h_0)(F_s) \to F_s^\times$ be the projection $(s,\lambda) \mapsto \lambda$. We have a commutative diagram of pointed Γ-sets with exact rows:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{O}^+((h_0)_{F_s}) & \longrightarrow & \mathrm{GL}(V_{F_s}) & \longrightarrow & \mathrm{SSym}(h_0)(F_s) & \longrightarrow & 1 \\
 & & \Big\downarrow{\scriptstyle i} & & \Big\| & & \Big\downarrow{\scriptstyle \pi} & & \\
1 & \longrightarrow & \mathrm{SL}(V_{F_s}) & \longrightarrow & \mathrm{GL}(V_{F_s}) & \xrightarrow{\mathrm{Nrd}} & F_s^\times & \longrightarrow & 1
\end{array}$$

This diagram yields the following commutative square in cohomology:

$$\begin{array}{ccc}
\mathrm{SSym}(h_0) & \longrightarrow & H^1(F, \mathrm{O}^+(h_0)) \\
{\scriptstyle \pi}\Big\downarrow & & \Big\downarrow{\scriptstyle i_*} \\
F^\times & \longrightarrow & H^1(F, \mathrm{SL}(V))
\end{array}$$

On the other hand, the Rost invariant and the map $F^\times \to H^3(F)$ carrying λ to $\lambda \cdot [A]$ fit in the following commutative diagram (see [26, p. 437]):

$$\begin{array}{ccc}
F^\times & \longrightarrow & H^1(F, \mathrm{SL}(V)) \\
& \searrow \quad \swarrow{\scriptstyle R} & \\
& H^3(F) &
\end{array}$$

The lemma follows. □

For the next statement, let $\partial\colon H^1(F, \mathrm{O}^+(h_0)) \to {}_2H^2(F)$ be the connecting map in the cohomology exact sequence associated to

$$1 \to \mu_2 \to \mathrm{Spin}(h_0) \xrightarrow{\chi} \mathrm{O}^+(h_0) \to 1.$$

For any hermitian form h of orthogonal type on V, there exists a unique linear transformation $s \in \mathrm{GL}(V)$ such that $h(x,y) = h_0(x, s^{-1}(y))$ for all x, $y \in V$, hence $\mathrm{ad}_h = \mathrm{Int}(s) \circ \mathrm{ad}_{h_0}$ and $\mathrm{ad}_{h_0}(s) = s$. If the discriminant of h is trivial we have $\mathrm{Nrd}(s) \in F^{\times 2}$, hence there exists $\lambda \in F^\times$ such that $\lambda^2 = \mathrm{Nrd}(s)$, and we may consider (s,λ) and $(s,-\lambda) \in \mathrm{SSym}(h_0)$. By the main theorem of [13], $\partial(s,\lambda)$ and $\partial(s,-\lambda)$ are the Brauer classes of the two components of the Clifford algebra of $\mathrm{Ad}_{h_0\perp -h}$, so if the Clifford invariant of h is trivial we have

$$\{\partial(s,\lambda), \partial(s,-\lambda)\} = \{0, [D]\}.$$

Lemma 2.10. *With the notation above, we have $f_3(h) = \lambda \cdot [D]$ if $\partial(s,\lambda) = 0$.*

Proof. By definition of s, the torsor in $H^1(F, \mathrm{O}(h_0))$ corresponding to h lifts to $(s,\lambda) \in H^1(F, \mathrm{O}^+(h_0))$. If $\partial(s,\lambda) = 0$, then (s,λ) lifts to some $\xi \in H^1(F, \mathrm{Spin}(h_0))$, and by definition of the invariants e_3 and f_3 we have

$$e_3(h) = R(\xi) + F^\times \cdot [D] \in M_D^3(F) \qquad \text{and} \qquad f_3(h) = 2R(\xi) \in H^3(F).$$

Lemma 2.8 then yields $f_3(h) = R \circ (i \circ \chi)_*(\xi) = R \circ i_*(s,\lambda)$, and by Lemma 2.9 we have $R \circ i_*(s,\lambda) = \lambda \cdot [D]$. $\square$

In order to check the condition $\partial(s,\lambda) = 0$ in Lemma 2.10, the following observation is useful: Suppose (V_1, h_1) and (V_2, h_2) are hermitian modules of orthogonal type over (D,θ). The inclusions $V_i \hookrightarrow V_1 \perp V_2$ yield an F-algebra homomorphism $C(\mathrm{Ad}_{h_1}) \otimes_F C(\mathrm{Ad}_{h_2}) \to C(\mathrm{Ad}_{h_1\perp h_2})$, which induces a group homomorphism $\mathrm{Spin}(h_1) \times \mathrm{Spin}(h_2) \to \mathrm{Spin}(h_1 \perp h_2)$. This homomorphism fits into the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mu_2 \times \mu_2 & \longrightarrow & \mathrm{Spin}(h_1) \times \mathrm{Spin}(h_2) & \xrightarrow{\chi_1 \times \chi_2} & \mathrm{O}^+(h_1) \times \mathrm{O}^+(h_2) & \longrightarrow & 1 \\
 & & \Pi \big\downarrow & & \big\downarrow & & \big\downarrow \oplus & & \\
1 & \longrightarrow & \mu_2 & \longrightarrow & \mathrm{Spin}(h_1 \perp h_2) & \xrightarrow{\chi} & \mathrm{O}^+(h_1 \perp h_2) & \longrightarrow & 1
\end{array}$$

The left vertical map is the product, and the right vertical map carries (g_1, g_2) to $g_1 \oplus g_2$. The induced diagram in cohomology yields the commutative square

$$\begin{array}{ccc}
H^1(F, \mathrm{O}^+(h_1)) \times H^1(F, \mathrm{O}^+(h_2)) & \xrightarrow{\partial_1 \times \partial_2} & {}_2H^2(F) \times {}_2H^2(F) \\
\oplus \big\downarrow & & \big\downarrow \Pi \\
H^1(F, \mathrm{O}^+(h_1 \perp h_2)) & \xrightarrow{\partial} & {}_2H^2(F)
\end{array}$$

The following additivity property of the connecting maps ∂ follows: for $(s_1, \lambda_1) \in H^1(F, \mathrm{O}^+(h_1))$ and $(s_2, \lambda_2) \in H^1(F, \mathrm{O}^+(h_2))$,

$$\partial_1(s_1, \lambda_1) + \partial_2(s_2, \lambda_2) = \partial(s_1 \oplus s_2, \lambda_1 \lambda_2). \tag{1}$$

Proof of Proposition 2.7. Let $h_0 = \langle 1, -1\rangle h_1 \perp \ldots \perp \langle 1, -1\rangle h_m$, which is a hyperbolic form, and let $V = V_1^{\oplus 2} \oplus \cdots \oplus V_m^{\oplus 2}$ be the underlying vector space of h and h_0. The linear transformation $s \in \mathrm{GL}(V)$ such that $h(x,y) = h_0(x, s^{-1}(y))$ for all x, $y \in V$ is

$$s = 1 \oplus \lambda_1^{-1} \oplus 1 \oplus \lambda_2^{-1} \oplus \cdots \oplus 1 \oplus \lambda_m^{-1}.$$

By the additivity property (1), the connecting map

$$\partial\colon H^1(F, \mathrm{O}^+(h_0)) \to {}_2H^2(F)$$

satisfies

$$\partial(s, \lambda_1^{-n_1/2} \dots \lambda_m^{-n_m/2}) = \partial_1(\lambda_1^{-1}, \lambda_1^{-n_1/2}) + \dots + \partial_m(\lambda_m^{-1}, \lambda_m^{-n_m/2}).$$

A theorem of Bartels [4, p. 283] (see also [13]) yields $\partial_i(\lambda_i^{-1}, \lambda_i^{-n_i/2}) = \lambda_i^{-1} \cdot e_1(h_i)$ for all i. Therefore, if $\sum_{i=1}^m \lambda_i \cdot e_1(h_i) = 0$ we have $f_3(h) = \lambda_1^{n_1/2} \dots \lambda_m^{n_m/2} \cdot [D]$ by Lemma 2.10. □

2.4. Relative Arason invariant of hermitian forms of orthogonal type. By using the Rost invariant, one may also define a relative Arason invariant, in a broader context:

Definition 2.11. Let h_1 and h_2 be two hermitian forms of orthogonal type over (D, θ) such that their difference $h_1 + (-h_2)$ has even relative rank, trivial discriminant, and trivial Clifford invariant. Their relative Rost invariant is defined by

$$e_3(h_1/h_2) = e_3\big(h_1 \perp (-h_2)\big) \in M_D^3(F).$$

In particular, if both h_1 and h_2 have even relative rank, trivial discriminant, and trivial Clifford invariant, then $e_3(h_1/h_2) = e_3(h_1) + e_3(h_2) = e_3(h_1) - e_3(h_2)$.

Remark 2.12. Under the conditions of this definition, one may check that the involution ad_{h_2} corresponds to a torsor which can be lifted to a $\mathrm{Spin}(\mathrm{Ad}_{h_1})$ torsor (see [41, §3.5]). As explained in [6, Lemma 3.6], the relative Arason invariant $e_3(h_1/h_2)$ coincides with the class in $M_D^3(F)$ of the image of this torsor under the Rost invariant of $\mathrm{Spin}(\mathrm{Ad}_{h_1})$.

Combining the properties of the Arason invariant recalled in Lemma 2.2 and the computation of Proposition 2.6, we obtain:

Corollary 2.13. (i) *Let h be a hermitian form of orthogonal type with even absolute rank, and let K/F be the discriminant quadratic extension. For any $\mu \in K^\times$, the relative Arason invariant $e_3(\langle N_{K/F}(\mu)\rangle h/h)$ is well-defined, and*

$$e_3(\langle N_{K/F}(\mu)\rangle h/h) = \mathrm{cor}_{K/F}(\mu \cdot e_2(h_K)).$$

(ii) *Let h_1 and h_2 be two hermitian forms of orthogonal type with even absolute rank and trivial discriminant. We have*

$$e_3(h_1 \perp \langle\lambda\rangle h_2/h_1 \perp h_2) = e_3(\langle\lambda\rangle h_2/h_2) = \lambda \cdot e_2(h_2).$$

2.5. Arason and f_3 invariants of orthogonal involutions. Let (A, σ) be an algebra with orthogonal involution, Brauer-equivalent to the division algebra D over F. We pick an involution θ on D, so that (A, σ) can be represented as the adjoint $(A, \sigma) \simeq \mathrm{Ad}_h$ of some hermitian module (V, h) over (D, θ). The co-index of A, which is the dimension over D of the module V, is equal to the relative rank of h. If the form h has even relative rank, trivial discriminant, and trivial Clifford invariant, then its Arason invariant is well-defined. Moreover, by Lemma 2.2, we have $e_3(h) = e_3(\lambda h)$ for any $\lambda \in F^\times$, and, as explained in [6, Prop 3.8], $e_3(h)$ does not depend on the choice of θ. Therefore, we get a well-defined Arason invariant for the involution σ, provided the algebra A has even co-index, i.e. $\deg(A)/\operatorname{ind}(A) = \deg(A)/\deg(D) \in 2\mathbb{Z}$, and the involution σ has trivial discriminant and trivial Clifford invariant:

$$e_3(\sigma) = e_3(h) \in M_A^3(F) = M_D^3(F).$$

Remarks 2.14. (1) Under the assumptions above on (A,σ), one may also check that the algebra A carries a hyperbolic orthogonal involution σ_0, and the Arason invariant $e_3(\sigma)$ can be defined directly in terms of the Rost invariant of the group $\operatorname{Spin}(A,\sigma_0)$, see [41, §3.5].

(2) Similarly, we may also define a relative Arason invariant $e_3(\sigma_1/\sigma_2)$ if the involutions σ_1 and σ_2 both have trivial discriminant and trivial Clifford invariant. But we cannot relax those assumptions, as we did for hermitian forms. Indeed, if $e_2(h_2) = e_2(\operatorname{ad}_{h_2})$ is not trivial, then $e_3(\langle\lambda\rangle h_1/h_2)$ and $e_3(h_1/h_2)$ are generally different, as Corollary 2.13 shows.

In the setting above, we may also define an f_3-invariant by $f_3(\sigma) = f_3(h)$, or equivalently:

Definition 2.15. Let (A,σ) be an algebra with orthogonal involution. We assume A has even co-index, and σ has trivial discriminant and trivial Clifford invariant. We define $f_3(\sigma) \in F^\times \cdot [A] \subset {}_2H^3(F)$ by $f_3(\sigma) = 2c$, where c is any representative of the Arason invariant $e_3(\sigma) \in M_A^3(F)$.

Remark 2.16. (i) If A is split, then $F^\times \cdot [A] = \{0\}$, and $f_3(\operatorname{ad}_\varphi) = 0$ for all quadratic forms $\varphi \in I^3(F)$. This also follows from the fact that $e_3(\varphi) \in {}_2H^3(F)$.
(ii) Using the same process, one may define an invariant f_3^{16} from Garibaldi's invariant e_3^{16}, and from Bermudez-Ruozzi's generalization (see [15], [9]). This invariant has values in ${}_2H^3(F)$, but need not have values in $F^\times \cdot [A]$ in general.

Example 2.17. Let Q be a quaternion algebra over F, and consider the algebra with involution $(A,\sigma) = (Q,\rho) \otimes \operatorname{Ad}_\varphi$, where ρ is an orthogonal involution with discriminant $\delta \cdot F^{\times 2} \in F^\times/F^{\times 2}$, and φ is a even-dimensional quadratic form with trivial discriminant. We have $e_3(\sigma) = \delta \cdot e_2(\varphi) \mod F^\times \cdot [Q]$, and $f_3(\sigma) = 0$. Indeed, since the restriction map $M_Q^3(F) \to M_Q^3(F_Q) = H^3(F_Q)$ is injective, it is enough to check the formula in the split case, where it follows from a direct computation.

The computation in Proposition 2.6 can be again rephrased as follows:

Corollary 2.18. *Let (A,σ) be a central simple F-algebra of even degree n with orthogonal involution, and let K/F be the discriminant quadratic extension. For any $\mu \in K^\times$, the algebra with involution $\operatorname{Ad}_{\langle 1,-N_{K/F}(\mu)\rangle} \otimes (A,\sigma)$ has even co-index, trivial discriminant and trivial Clifford invariant. Its Arason invariant is given by*

$$e_3(\operatorname{ad}_{\langle 1,-N_{K/F}(\mu)\rangle} \otimes \sigma) = \operatorname{cor}_{K/F}\big(\mu \cdot e_2(\sigma_K)\big),$$

and

$$f_3(\mathrm{ad}_{\langle 1,-N_{K/F}(\mu)\rangle}\otimes\sigma)=\begin{cases}0 & \text{if } n\equiv 0 \bmod 4,\\ N_{K/F}(\mu)\cdot[A] & \text{if } n\equiv 2 \bmod 4.\end{cases}$$

In particular, if σ has trivial discriminant, we have for any $\lambda\in F^\times$

$$e_3(\mathrm{ad}_{\langle 1,-\lambda\rangle}\otimes\sigma)=\lambda\cdot e_2(\sigma)$$

and

$$f_3(\mathrm{ad}_{\langle 1,-\lambda\rangle}\otimes\sigma)=\begin{cases}0 & \text{if } n\equiv 0 \bmod 4,\\ \lambda\cdot[A] & \text{if } n\equiv 2 \bmod 4.\end{cases}$$

Hence, the formula given in [35, Th. 5.5] for algebras of degree 8 is actually valid in arbitrary degree.
With this in hand, one may easily check that the f_3 invariant is trivial up to degree 10. Indeed, since the co-index of the algebra is supposed to be even, the algebra is possibly non-split only when its degree is divisible by 4. In degree 4, any involution with trivial discriminant and Clifford invariant is hyperbolic, hence has trivial invariants. In degree 8, any involution with trivial discriminant and Clifford invariant admits a decomposition as in Corollary 2.18 by [35, Th. 5.5], hence its f_3 invariant is trivial. In degree 12, one may construct explicit examples of (A,σ) with $f_3(\sigma)\neq 0$ as follows. Suppose E is a central simple F-algebra of degree 4. Recall from [26, §10.B] that the second λ-power λ^2E is a central simple F-algebra of degree 6, which carries a canonical involution γ of orthogonal type with trivial discriminant, and which is Brauer-equivalent to $E\otimes_F E$.

Corollary 2.19. *Let E be a central simple F-algebra of degree and exponent 4. Pick an indeterminate t, and consider the algebra with involution*

$$(A,\sigma)=\mathrm{Ad}_{\langle 1,-t\rangle}\otimes(\lambda^2E,\gamma)_{F(t)}.$$

We have

$$f_3(\sigma)=t\cdot[A]\neq 0\in H^3(F(t)).$$

Proof. The formula $f_3(A,\sigma)=t\cdot[A]$ readily follows from Corollary 2.18. The algebra E has exponent 4, therefore $[A]=[E\otimes_F E]\neq 0$. Since t is an indeterminate, we get $t\cdot[A]\neq 0$. □

3. Additive decompositions in degree 12

In the next three sections, we concentrate on degree 12 algebras (A,σ) with orthogonal involution of trivial discriminant and trivial Clifford invariant. The main result of this section is Theorem 3.2, which generalizes a theorem of Pfister on 12-dimensional quadratic forms.

3.1. Additive decompositions. Given three algebras with involution, (A,σ), (A_1,σ_1) and (A_2,σ_2), we say that (A,σ) is a *direct sum* of (A_1,σ_1) and (A_2,σ_2), and we write

$$(A,\sigma)\in(A_1,\sigma_1)\boxplus(A_2,\sigma_2),$$

if there exist a division algebra with involution (D,θ) and hermitian modules (V_1,h_1) and (V_2,h_2) over (D,θ), which are both hermitian or both skew-hermitian, such that $(A_1,\sigma_1)=\mathrm{Ad}_{h_1}$, $(A_2,\sigma_2)=\mathrm{Ad}_{h_2}$ and $(A,\sigma)=\mathrm{Ad}_{h_1\perp h_2}$. In particular, this implies A, A_1 and A_2 are all three Brauer-equivalent to D, and the involutions σ, σ_1 and σ_2 are of the same type. This notion of direct sum for algebras with involution was introduced by Dejaiffe in [10]. As explained there, the algebra with involution (A,σ) is generally not uniquely determined by the data of the two summands (A_1,σ_1) and (A_2,σ_2). Indeed, multiplying the hermitian forms h_1 and h_2 by a scalar does not change the adjoint involutions, so the adjoint of $\lambda_1h_1\perp\lambda_2h_2$ also is a direct sum of (A_1,σ_1) and (A_2,σ_2) for any λ_1, $\lambda_2\in F^\times$. If one of the two summands, say $(A_1,\sigma_1)=(A_1,\mathrm{hyp})$ is hyperbolic, then all hermitian forms similar to h_1 actually are isomorphic to h_1. Hence in this case, there is a unique direct sum, and we may write

$$(A,\sigma)=(A_1,\mathrm{hyp})\boxplus(A_2,\sigma_2).$$

The cohomological invariants we consider, when defined, have the following additivity property:

Proposition 3.1. *Suppose σ, σ_1, σ_2 are orthogonal involutions such that $(A,\sigma)\in(A_1,\sigma_1)\boxplus(A_2,\sigma_2)$. We have:*

(i) $\deg A=\deg A_1+\deg A_2$.

(ii) *If* $\deg A_1\equiv\deg A_2\equiv 0 \bmod 2$, *then* $e_1(\sigma)=e_1(\sigma_1)+e_1(\sigma_2)$.

(iii) *If* $\deg A_1\equiv\deg A_2\equiv 0 \bmod 2$ *and* $e_1(\sigma_1)=e_1(\sigma_2)=0$, *then*

$$e_2(\sigma)=e_2(\sigma_1)+e_2(\sigma_2).$$

(iv) *If the co-indices of A_1 and A_2 are even and $e_i(\sigma_1)=e_i(\sigma_2)=0$ for $i=1$, 2, then*

$$e_3(\sigma)=e_3(\sigma_1)+e_3(\sigma_2)\ \textit{and}\ f_3(\sigma)=f_3(\sigma_1)+f_3(\sigma_2).$$

Proof. Assertion (i) is clear by definition, and (ii) was established by Dejaiffe [10, Prop. 2.3]. Assertion (iii) follows from [10, § 3.3] (see also the proof of the "Orthogonal Sum Lemma" in [14, §3]). To prove (iv), let D be the division algebra Brauer-equivalent to A, A_1, and A_2, and let θ be an F-linear involution on D. We may find hermitian forms of orthogonal type h_1, h_2 over (D,θ) such that $(A_i,\sigma_i)\simeq\mathrm{Ad}_{h_i}$ for $i=1$, 2, and $(A,\sigma)\simeq\mathrm{Ad}_{h_1\perp h_2}$. By Lemma 2.2(ii) we have

$$e_3(h_1\perp h_2)=e_3(h_1)+e_3(h_2).$$

By definition of the e_3-invariant of orthogonal involutions (see § 2.5), $e_3(\sigma)$ (resp. $e_3(\sigma_i)$ for $i=1$, 2) is represented by $e_3(h_1\perp h_2)$ (resp. $e_3(h_i)$), hence $e_3(\sigma)=e_3(\sigma_1)+e_3(\sigma_2)$. Likewise, the additivity of e_3 induces $f_3(\sigma)=f_3(\sigma_1)+f_3(\sigma_2)$, by definition of the f_3-invariant (see 2.15). □

By a result of Pfister [33, p.123-124], any 12-dimensional quadratic form φ in I^3F decomposes as $\varphi = \langle\alpha_1\rangle n_1 \perp \langle\alpha_2\rangle n_2 \perp \langle\alpha_3\rangle n_3$, where n_i is a 2-fold Pfister form and $\alpha_i \in F^\times$, for $1 \le i \le 3$. This can be rephrased as

$$\mathrm{Ad}_\varphi \in \mathrm{Ad}_{n_1} \boxplus \mathrm{Ad}_{n_2} \boxplus \mathrm{Ad}_{n_3},$$

where each summand Ad_{n_i} has degree 4 and discriminant 1. We now extend this result to the non-split case.

Theorem 3.2. *Let (A,σ) be a central simple F-algebra of degree 12 with orthogonal involution. Assume the discriminant and the Clifford invariant of σ are trivial. There is a central simple F-algebra A_0 of degree 4 and orthogonal involutions σ_1, σ_2, σ_3 of trivial discriminant on A_0 such that*

$$(A,\sigma) \in (A_0,\sigma_1) \boxplus (A_0,\sigma_2) \boxplus (A_0,\sigma_3).$$

Note that since $\deg A_0 = 4$ we have $e_2(\sigma_i) = 0$ if and only if σ_i is hyperbolic (see [41, Th. 3.10]); therefore, even when the index of A is 2 we *cannot* use Proposition 3.1(iv) to compute $e_3(\sigma)$ (unless each σ_i is hyperbolic).

Proof of Theorem 3.2. The index of A is a power of 2 since $2[A] = 0$ in $\mathrm{Br}(F)$, and it divides $\deg A = 12$, so $\operatorname{ind} A = 1$, 2 or 4. As we just pointed out, the index 1 case is Pfister's theorem. We consider separately the two remaining cases.

If $\operatorname{ind} A = 2$, we have $(A,\sigma) = \mathrm{Ad}_h$ for some skew-hermitian form h of relative rank 6 over a quaternion division algebra $(Q, \overline{})$ with its canonical involution. Let $q_1 \in Q$ be a nonzero pure quaternion represented by h, and write $h = \langle q_1\rangle \perp h'$. Over the quadratic extension $K_1 = F(q_1)$, the algebra Q splits and the form $\langle q_1\rangle$ becomes hyperbolic (because its discriminant becomes a square). Therefore, h_{K_1} and h'_{K_1} are Witt-equivalent, and $(\mathrm{ad}_{h'})_{K_1}$ is adjoint to a 10-dimensional form φ. The discriminant and Clifford invariant of σ are trivial, hence $\varphi \in I^3K_1$. Since there is no anisotropic 10-dimensional quadratic forms in I^3 (see [22, Th. 8.1.1]), it follows that h'_{K_1} is isotropic, hence by [34, Prop., p. 382], $h' = \langle -\lambda_1 q_1\rangle \perp k$ for some $\lambda_1 \in F^\times$ and some skew-hermitian form k of relative rank 4. We thus have

$$h = \langle q_1\rangle\langle 1, -\lambda_1\rangle \perp k,$$

and computation shows that $e_1\big(\langle q_1\rangle\langle 1,-\lambda_1\rangle\big) = 0$. Therefore, $e_1(k) = 0$, and $e_2(k) = e_2\big(\langle q_1\rangle\langle 1,-\lambda_1\rangle\big)$ because $e_2(h) = 0$. Now, let $q_2 \in Q$ be a nonzero pure quaternion represented by k, and let $K_2 = F(q_2)$, so $k = \langle q_2\rangle \perp k'$ for some skew-hermitian form k' of relative rank 3. The forms k_{K_2} and k'_{K_2} are Witt-equivalent, and $(\mathrm{ad}_{k'})_{K_2}$ is adjoint to a 6-dimensional form $\psi \in I^2K_2$, i.e., to an Albert form ψ. We have

$$e_2(\psi) = e_2(k')_{K_2} = e_2(k)_{K_2} = e_2\big(\langle q_1\rangle\langle 1,-\lambda_1\rangle\big)_{K_2},$$

hence the index of $e_2(\psi)$ is at most 2, and it follows that ψ is isotropic. Therefore, k'_{K_2} is isotropic, and $k' = \langle -\lambda_2 q_2\rangle \perp \ell$ for some $\lambda_2 \in F^\times$ and some skew-hermitian form ℓ of relative rank 2. Thus, we have

$$h = \langle q_1\rangle\langle 1, -\lambda_1\rangle \perp \langle q_2\rangle\langle 1, -\lambda_2\rangle \perp \ell.$$

Since $e_1\big(\langle q_1\rangle\langle 1,-\lambda_1\rangle\big) = e_1\big(\langle q_2\rangle\langle 1,-\lambda_2\rangle\big) = 0$ and $e_1(h) = 0$, we also have $e_1(\ell) = 0$. We thus obtain the required decomposition, with

$$A_0 = M_2(Q), \qquad \sigma_1 = \mathrm{ad}_{\langle q_1\rangle\langle 1,-\lambda_1\rangle}, \qquad \sigma_2 = \mathrm{ad}_{\langle q_2\rangle\langle 1,-\lambda_2\rangle}, \qquad \sigma_3 = \mathrm{ad}_\ell\,.$$

Suppose now $\operatorname{ind} A = 4$, and let D be the division algebra of degree 4 Brauer-equivalent to A. By [17, Th. 3.1], there exists a quadratic extension K of F such that $(A,\sigma)_K$ is hyperbolic. The co-index of A_K is therefore even, so the index of A_K is 2, hence we may identify K with a subfield of D. The following construction is inspired by the Parimala–Sridharan–Suresh exact sequence in Appendix 2 of [5]. We have $D = \widetilde{D} \oplus D'$, where $\widetilde{D}$ is the centralizer of K in D and, writing ι for the nontrivial F-automorphism of K,

$$D' = \{x \in D \mid xy = \iota(y)x \text{ for all } y \in K\}.$$

Let θ be an orthogonal involution on D that fixes K (such involutions exist by [26, (4.14)]). We may represent $(A,\sigma) = (\operatorname{End}_D V, \mathrm{ad}_h)$ for some hermitian form h of relative rank 3 over (D,θ). In view of the decomposition $D = \widetilde{D}\oplus D'$, we have for x, $y \in V$

$$h(x,y) = \widetilde{h}(x,y) + h'(x,y) \qquad \text{with } \widetilde{h}(x,y) \in \widetilde{D} \text{ and } h'(x,y) \in D'.$$

Since h is a hermitian form over (D,θ), it follows that $\widetilde{h}$ is a hermitian form on V viewed as a $\widetilde{D}$-vector space, with respect to the restriction of θ to $\widetilde{D}$. Clearly, $\operatorname{End}_D V \subset \operatorname{End}_{\widetilde{D}} V$. We may also embed K into $\operatorname{End}_{\widetilde{D}} V$ by identifying $\alpha \in K$ with the scalar multiplication $x \mapsto x\alpha$ for $x \in V$. Thus, we have a K-algebra homomorphism

$$(\operatorname{End}_D V) \otimes_F K \to \operatorname{End}_{\widetilde{D}} V.$$

This homomorphism is injective because the left side is a simple algebra, hence it is an isomorphism by dimension count. For $f \in \operatorname{End}_D V$ we have $\mathrm{ad}_h(f) = \mathrm{ad}_{\widetilde{h}}(f)$, so the isomorphism preserves the involution, and therefore $(\mathrm{Ad}_h)_K = \mathrm{Ad}_{\widetilde{h}}$. Since σ becomes hyperbolic over K, the form $\widetilde{h}$ is hyperbolic. Therefore, there is an h-orthogonal base of V consisting of $\widetilde{h}$-isotropic vectors, which yields a diagonalization

$$h = \langle a_1, a_2, a_3\rangle \qquad \text{with } a_1,\ a_2,\ a_3 \in D' \cap \operatorname{Sym}(\theta).$$

We thus have $(A,\sigma) \in (D,\sigma_1) \boxplus (D,\sigma_2) \boxplus (D,\sigma_3)$ with $\sigma_i = \operatorname{Int}(a_i^{-1}) \circ \theta$ for $i = 1$, 2, 3. To complete the proof, we show that the discriminant of each σ_i is trivial. Recall from [26, (7.2)] that the discriminant is the square class of any skew-symmetric unit. Let $\alpha \in K^\times$ be such that $\iota(\alpha) = -\alpha$. Since $a_i \in D'$ we have $\sigma_i(\alpha) = -\alpha$, so $\operatorname{disc}\sigma_i = \operatorname{Nrd}_D(\alpha) = N_{K/F}(\alpha)^2$. □

Recall that a central simple algebra of degree 4 with orthogonal involution (A_0,σ_0) of trivial discriminant decomposes as $(A_0,\sigma_0) \simeq (Q,\bar{\ }) \otimes (H,\bar{\ })$ where the quaternion algebras Q, H are the two components of the Clifford algebra

$C(A_0, \sigma_0)$ (see [26, (15.12)]). Therefore, Theorem 3.2 can be rephrased as follows:

Corollary 3.3. *Let (A, σ) be a central simple algebra of degree 12 with orthogonal involution of trivial discriminant and Clifford invariant. There exist quaternion F-algebras Q_i, H_i for $i = 1, 2, 3$ such that $[A] = [Q_i] + [H_i]$ for $i = 1, 2, 3$, $[H_1] + [H_2] + [H_3] = 0$, and*

$$(A, \sigma) \in \big((Q_1, \bar{\ }) \otimes (H_1, \bar{\ })\big) \boxplus \big((Q_2, \bar{\ }) \otimes (H_2, \bar{\ })\big) \boxplus \big((Q_3, \bar{\ }) \otimes (H_3, \bar{\ })\big).$$

Proof. Theorem 3.2 yields orthogonal involutions σ_1, σ_2, σ_3 of trivial discriminant on the central simple F-algebra A_0 of degree 4 Brauer-equivalent to A such that

$$(A, \sigma) \in (A_0, \sigma_1) \boxplus (A_0, \sigma_2) \boxplus (A_0, \sigma_3).$$

Each (A_0, σ_i) has a decomposition

$$(A_0, \sigma_i) \simeq (Q_i, \bar{\ }) \otimes (H_i, \bar{\ })$$

for some quaternion F-algebras Q_i, H_i such that

$$e_2(\sigma_i) = [Q_i] + \{0, [A]\} = [H_i] + \{0, [A]\} \in M_A^2(F).$$

From this decomposition, it follows that

$$[Q_i] + [H_i] = [A_0] = [A] \qquad \text{for } i = 1, 2, 3.$$

Moreover, we have $\sum_{i=1}^3 e_2(\sigma_i) = e_2(\sigma)$ by Proposition 3.1, and $e_2(\sigma) = 0$, so

$$\sum_{i=1}^{3} [Q_i] = \Big(\sum_{i=1}^{3} [H_i]\Big) + [A] \in \{0, [A]\} \subset {}_2H^3(F).$$

Therefore, interchanging Q_i and H_i if necessary, we may assume $\sum_{i=1}^3 [H_i] = 0$. □

Example 3.4. For any quaternion algebra Q with norm form n_Q, we have $\mathrm{Ad}_{n_Q} \simeq (Q, \bar{\ }) \otimes (Q, \bar{\ })$, see for instance [26, (11.1)]. Therefore, if A is split, and σ is adjoint to the 12-dimensional form $\varphi = \langle \alpha_1 \rangle n_1 \perp \langle \alpha_2 \rangle n_2 \perp \langle \alpha_3 \rangle n_3$, where n_i is the norm form of a quaternion algebra Q_i for $1 \leq i \leq 3$, then

$$(A, \sigma) \in \boxplus_{i=1}^3 (Q_i, \bar{\ }) \otimes (Q_i, \bar{\ }).$$

Conversely, any decomposition of a split $(A, \sigma) = \mathrm{Ad}_\varphi$ as in the corollary corresponds to a decomposition $\varphi = \langle \alpha_1 \rangle n_1 \perp \langle \alpha_2 \rangle n_2 \perp \langle \alpha_3 \rangle n_3$, where n_i is the norm form of $Q_i \simeq H_i$.

Remark 3.5. In [17], it is proved that any (A, σ) of degree 12 with trivial discriminant and trivial Clifford invariant can be described as a quadratic extension of some degree 6 central simple algebra with unitary involution (B, τ), with discriminant algebra Brauer-equivalent to A. This algebra (B, τ) can be described from the above additive decomposition as follows. Since $\sum_{i=1}^3 [H_i] = 0$, the algebras H_i have a common quadratic subfield K, see [28, Th. III.4.13]. All three products $(Q_i, \bar{\ }) \otimes (H_i, \bar{\ })$ are hyperbolic over K, so σ_K is hyperbolic. Moreover, as observed in [17, Ex. 1.3], the tensor product $(Q_i, \bar{\ }) \otimes (H_i, \bar{\ })$

is a quadratic extension of $(Q_i, \bar{\ }) \otimes (K, \bar{\ })$. Therefore, (A, σ) is a quadratic extension of some $(B, \tau) \in \boxplus_{i=0}^{3} (Q_i, \bar{\ }) \otimes (K, \bar{\ })$, and the discriminant algebra of (B, τ) is Brauer-equivalent to $[Q_1] + [Q_2] + [Q_3] = [Q]$. Note that in the case where $\operatorname{ind} A = 4$, we use in our proof the main result of [17], which guarantees the existence of a quadratic extension K such that $(A, \sigma)_K$ is hyperbolic. But for $\operatorname{ind} A \neq 4$, our proof is independent, and does not use the existence of an open orbit of a half-spin representation as in [17, p. 1220].

3.2. Decomposition groups of (A, σ). Until the end of this section, (A, σ) denotes a central simple F-algebra of degree 12 with an orthogonal involution of trivial discriminant and trivial Clifford algebra.

Definition 3.6. Given an additive decomposition as in Corollary 3.3

$$(A, \sigma) \in \boxplus_{i=1}^{3} \big((Q_i, \bar{\ }) \otimes_F (H_i, \bar{\ })\big) \qquad \text{with } \sum_{i=1}^{3} [H_i] = 0,$$

the subset

$$U = \{0, [A], [Q_1], [H_1], [Q_2], [H_2], [Q_3], [H_3]\} \subset {}_2\operatorname{Br}(F)$$

is called a *decomposition group* of (A, σ). It is indeed the subgroup of ${}_2\operatorname{Br}(F)$ generated by $[Q_1]$, $[Q_2]$, and $[Q_3]$, since $[A] = [Q_1] + [Q_2] + [Q_3]$ and $[H_i] = [A] + [Q_i]$ for $i = 1, 2, 3$.

As the following examples show, a given algebra with involution (A, σ) may admit several additive decompositions, corresponding to different decomposition groups, possibly not all of the same cardinality.

Example 3.7. Assume A is split. Since $[A] = 0$, we have $[H_i] = [Q_i]$ for all i. Hence all decomposition groups of (A, σ) have order dividing 4.
Consider three quaternion division algebras $[Q_1]$, $[Q_2]$ and $[Q_3]$ such that $[Q_1] + [Q_2] = [Q_3]$. By the "common slot lemma" [28, Th. III.4.13], there exist a, b_1, $b_2 \in F^\times$ such that $Q_i = (a, b_i)$ for $i = 1, 2$ and $Q_3 = (a, b_1 b_2)$. An easy computation then shows that the norm forms of Q_1, Q_2, Q_3, respectively denoted by n_1, n_2, n_3, satisfy $n_1 - n_2 = \langle b_2 \rangle n_3$ in the Witt group of F. Hence, extending scalars to a rational function fields in two variables over F, one may find scalars α_i for $1 \leq i \leq 3$ such that the form

$$\varphi = \langle \alpha_1 \rangle n_1 \perp \langle \alpha_2 \rangle n_2 \perp \langle \alpha_3 \rangle n_3$$

is either anisotropic, or isotropic and non-hyperbolic, or hyperbolic. By Example 3.4, in all three cases, $\{0, [Q_1], [Q_2], [Q_3]\}$ is a decomposition group of order 4 for the involution $\sigma = \operatorname{ad}_\varphi$.
On the other hand, the adjoint involution of an isotropic or a hyperbolic form also has smaller decomposition groups, as we now proceed to show. If the involution σ is isotropic, it is adjoint to a quadratic form φ which is Witt-equivalent to a 3-fold Pfister form π_3. Let Q be a quaternion algebra such that the norm form n_Q is a subform of π_3. There exists α_1, $\alpha_2 \in F^\times$ such that $\varphi = \langle \alpha_1, \alpha_2 \rangle \otimes n_Q \perp 2\mathbb{H}$, (where $\mathbb{H}$ denotes the hyperbolic plane) hence

$\{0,[Q]\}$ is a decomposition group of $\sigma = \mathrm{ad}\,\varphi$. If in addition σ, hence π_3, is hyperbolic, we may choose $[Q] = 0$.

Example 3.8. Assume now that $A = M_6(Q)$ has index 2. Since $0 \neq [Q] \in U$, all decomposition groups U have order 2, 4 or 8.
If σ is isotropic, then it is Witt-equivalent to a degree 8 algebra with involution $(M_4(Q), \sigma_0)$ that has trivial discriminant and trivial Clifford invariant, so

$$(A,\sigma) = (M_4(Q),\sigma_0) \boxplus (M_2(Q), \mathrm{hyp}) = (M_4(Q),\sigma_0) \boxplus \big((M_2(F),\bar{\ }) \otimes (Q,\bar{\ })\big).$$

Because $(M_4(Q), \sigma_0)$ has trivial discriminant and Clifford invariant, by [35, Th. 5.2] we may find λ, $\mu \in F^\times$ and an orthogonal involution ρ on Q such that

$$(M_4(Q),\sigma_0) \simeq \mathrm{Ad}_{\langle\langle \lambda,\mu\rangle\rangle} \otimes (Q,\rho).$$

Let Q_1 and H_1 be the two components of the Clifford algebra of $\mathrm{Ad}_{\langle\langle \mu\rangle\rangle} \otimes (Q,\rho)$. Then

$$\mathrm{Ad}_{\langle\langle \mu\rangle\rangle} \otimes (Q,\rho) \simeq (Q_1,\bar{\ }) \otimes (H_1,\bar{\ }).$$

Therefore,

$$\begin{aligned}\mathrm{Ad}_{\langle\langle \lambda,\mu\rangle\rangle} \otimes (Q,\rho) &\simeq \mathrm{Ad}_{\langle 1,-\lambda\rangle} \otimes (Q_1,\bar{\ }) \otimes (H_1,\bar{\ })\\ &\in \big((Q_1,\bar{\ }) \otimes (H_1,\bar{\ })\big) \boxplus \big((Q_1,\bar{\ }) \otimes (H_1,\bar{\ })\big),\end{aligned}$$

and finally

$$(A,\sigma) \in \big(Q_1,\bar{\ }) \otimes (H_1,\bar{\ })\big) \boxplus \big((Q_1,\bar{\ }) \otimes (H_1,\bar{\ })\big) \boxplus \big((M_2(F),\bar{\ }) \otimes (Q,\bar{\ })\big).$$

It follows that $\{0,[Q],[Q_1],[H_1]\}$ is a decomposition group for (A,σ).
If in addition σ is hyperbolic, we may choose $\mu = 1$, so that $\{[Q_1],[H_1]\} = \{0,[Q]\}$. Hence $\{0,[Q]\}$ is a decomposition group of (A,σ) in this case.

Example 3.9. If A has index 4, then all decomposition groups of (A,σ) have order 8. Indeed, since $[A] = [Q_i] + [H_i]$, the quaternion algebras Q_i and H_i all are division algebras. Therefore $[Q_i] \neq 0$ for $i = 1,2,3$. Moreover, we have $[Q_1] + [Q_2] + [Q_3] = [A]$. Since A has index 4, this guarantees $[Q_i] + [Q_j]$ is non zero if $i \neq j$. Therefore, $[Q_1],[Q_2],[Q_3]$ are $\mathbb{Z}/2$ linearly independent, and they do generate a group of order 8.
One may also check that the involution σ is anisotropic in this case. Indeed, $A \simeq M_3(D)$ for some degree 4 division algebra D, hence A does not carry any hyperbolic involution. Moreover, its isotropic involutions with trivial discriminant are Witt-equivalent to $(Q_1,\bar{\ }) \otimes (H_1,\bar{\ })$, for some quaternion division algebras Q_1 and H_1 such that $D \simeq Q_1 \otimes H_1$. Hence isotropic involutions on A with trivial discriminant have non trivial Clifford invariant

$$[Q_1] + \{0,[D]\} = [H_1] + \{0,[D]\} \neq 0 \in H^2(F)/\{0,[D]\}.$$

From these examples, we easily get the following characterization of isotropy and hyperbolicity:

Lemma 3.10. *Let (A,σ) be a degree 12 algebra with orthogonal involution with trivial discriminant and trivial Clifford invariant.*

(i) *The involution σ is isotropic if and only if it admits a decomposition group generated by $[A]$ and $[Q_1]$ for some quaternion algebra Q_1.*
(ii) *The involution σ is hyperbolic if and only if it admits $\{0, [A]\}$ as a decomposition group.*
(iii) *The algebra with involution (A, σ) is split and hyperbolic if and only if it admits $\{0\}$ as a decomposition group.*

Proof. Assertion (iii) is clear from the definition of a decomposition group, since $(M_2(F), \bar{\ }) \otimes (M_2(F), \bar{\ })$ is hyperbolic. For (i) and (ii), the direct implications immediately follow from the previous examples. To prove the converse, let us first assume (A, σ) admits $\{0, [A]\}$ as a decomposition group. Since this group has order 1 or 2, A cannot have index 4 by Example 3.9. Therefore, it is Brauer-equivalent to a quaternion algebra Q. Moreover, by definition,

$$(A, \sigma) \in \boxplus_{i=1}^{3} \big((Q, \bar{\ }) \otimes (M_2(F), \bar{\ })\big).$$

Since each summand is hyperbolic, this proves σ is hyperbolic.
Assume now that U is generated by $[A]$ and $[Q_1]$. The order of U then divides 4, hence by Example 3.9, the algebra A is Brauer-equivalent to some quaternion algebra Q. Thus $U = \{0, [Q], [Q_1], [H_1]\}$, with $M_2(Q) \simeq Q_1 \otimes H_1$. If $[Q_1] = 0$ or $[H_1] = 0$, then $U = \{0, [Q]\}$, and by the previous case, σ is hyperbolic. Assume now that both H_1 and Q_1 are non split. We get

$$(A, \sigma) \in \boxplus_{i=1}^{3} (Q_i, \bar{\ }) \otimes (H_i, \bar{\ }),$$

with for $i = 2, 3$ $\{[Q_i], [H_i]\}$ equal either to $\{0, [Q]\}$ or to $\{[Q_1], [H_1]\}$. Picking an arbitrary element in $\{[H_1], [Q_1]\}$ for $1 \le i \le 3$, we get three quaternion algebras whose sum is never 0. Therefore, since by Corollary 3.3 we have $[H_1] + [H_2] + [H_3] = 0$, at least one summand must be $(Q, \bar{\ }) \otimes (M_2(F), \bar{\ })$, and this proves σ is isotropic. □

Remark 3.11. Reversing the viewpoint, note that any subgroup $U \subset {}_2\operatorname{Br}(F)$ of order 8 in which all the nonzero elements except at most one have index 2 is the decomposition group of some central simple algebra of degree 12 with orthogonal involution of trivial discriminant and trivial Clifford invariant. If all the nonzero elements in U have index 2, pick a quaternion algebra D representing a nonzero element in U; otherwise, let D be the division algebra such that $[D] \in U$ and $\operatorname{ind} D > 2$. In each case, we may organize the other nonzero elements in U in pairs $[Q_i]$, $[H_i]$ such that $[D] = [Q_i] + [H_i]$ for $i = 1, 2, 3$, and $\sum_{i=1}^{3} [H_i] = 0$. Any algebra with involution (A, σ) in $\boxplus_{i=1}^{3} \big((Q_i, \bar{\ }) \otimes (H_i, \bar{\ })\big)$ has decomposition group U. Modifying the scalars in the direct sum leads to several nonisomorphic such (A, σ). Moreover, when all the nonzero elements in U have index 2, we may select for D various quaternion algebras, and thus obtain various (A, σ) that are not Brauer-equivalent. Similarly, any subgroup $U \subset {}_2\operatorname{Br}(F)$ of order 4 containing at most one element $[D]$ with $\operatorname{ind} D > 2$, and any subgroup $\{0, [Q]\}$ where Q is a quaternion algebra, is the decomposition group of some central simple algebra of degree 12 endowed with an isotropic orthogonal involution of trivial discriminant and trivial Clifford invariant.

The decomposition groups of (A, σ) are subgroups of the Brauer group generated by at most three quaternion algebras. Those subgroups were considered by Peyre in [32]. His results will prove useful to study degree 12 algebras with involution. For the reader's convenience, we recall them in the next section.

3.3. A complex of Peyre. Let F be an arbitrary field, and let $U \subset \operatorname{Br} F$ be a finite subgroup of the Brauer group of F. We let $F^\times \cdot U$ denote the subgroup of $H^3(F)$ generated by classes $\lambda \cdot \alpha$, with $\lambda \in F^\times$ and $\alpha \in U$; any element in $F^\times \cdot U$ can be written as $\sum_{i=1}^r \lambda_i \cdot \alpha_i$ for some $\lambda_i \in F^\times$, where $\alpha_1, \dots, \alpha_r$ is a generating set for the group U. Let F_U be the function field of the product of the Severi–Brauer varieties associated to elements of U. Clearly, F_U splits all the elements of U, hence the subgroup $F^\times \cdot U$ vanishes after scalar extension to F_U. Therefore, the following sequence is a complex, which was first introduced and studied by Peyre in [32, §4]:

$$F^\times \cdot U \to H^3(F) \to H^3(F_U).$$

We let $\mathcal{H}_U$ denote the corresponding homology group, that is

$$\mathcal{H}_U = \frac{\ker(H^3(F) \to H^3(F_U))}{F^\times \cdot U}.$$

We now return to our standing hypothesis that the characteristic of F is different from 2. Peyre considers in particular subgroups $U \subset \operatorname{Br} F$ generated by the Brauer classes of at most three quaternion algebras, and proves:

Theorem 3.12 (Peyre [32, Thm 5.1]). *If U is generated by the Brauer classes of two quaternion algebras, then $\mathcal{H}_U = 0$.*

In the next section, we need to consider only subgroups U such that all the elements of U are quaternion algebras; we call them *quaternionic subgroups* of the Brauer group. These subgroups have also been investigated by Sivatski [38]. We have:

Theorem 3.13 ([32, Prop. 6.1], [38, Cor. 11]). *If $U \subset \operatorname{Br} F$ is generated by the Brauer classes of three quaternion algebras, then $\mathcal{H}_U = 0$ or $\mathbb{Z}/2\mathbb{Z}$. Assume in addition U is quaternionic. Then the following conditions are equivalent:*

(a) *$\mathcal{H}_U = 0$;*
(b) *U is split by an extension of F of degree $2m$ for some odd m;*
(c) *U is split by a quadratic extension of F.*

The result that $\mathcal{H}_U = 0$ or $\mathbb{Z}/2\mathbb{Z}$ and the equivalence (a) $\iff$ (b) are due to Peyre [32, Prop. 6.1]. The equivalence (b) $\iff$ (c) was proved by Sivatski [38, Cor. 11].

We say that an extension K of F *splits* a subgroup $U \subset \operatorname{Br} F$ if it splits all the elements in U. If K splits a decomposition group of a central simple algebra with orthogonal involution (A, σ), then A_K is split because $[A] \in U$, and σ_K is hyperbolic by Lemma 3.10(iii) because (A_K, σ_K) has a trivial decomposition group. Therefore, Theorem 3.13 is relevant for the quadratic splitting of (A, σ), as we will see in § 5.2.

4. The Arason invariant and the homology of Peyre's complex

As in the previous section, (A,σ) is a degree 12 algebra with orthogonal involution of trivial discriminant and trivial Clifford invariant. From now on, we assume in addition that the Arason invariant $e_3(\sigma)$ is well defined. So the algebra A has even co-index, hence index 1 or 2. Under this assumption, any decomposition group of (A,σ) is quaternionic, that is consists only of Brauer classes of quaternion algebras. In this section, we relate the decomposition groups of (A,σ) with the values of the Arason invariant $e_3(\sigma)$. Reversing the viewpoint we then explain how one can use the Arason invariant to find explicit generators of the homology group $\mathcal{H}_U$ of Peyre's complex, for any quaternionic subgroup $U\subset \mathrm{Br}(F)$ of order dividing 8.

4.1. Arason invariant in degree 12. For orthogonal involutions on a degree 12 algebra, isotropy and hyperbolicity can be detected via the Arason invariant as follows:

Theorem 4.1. *Let (A,σ) be a degree 12 and index 1 or 2 algebra with orthogonal involution of trivial discriminant and trivial Clifford invariant.*

(i) *The involution σ is hyperbolic if and only if $e_3(\sigma)=0\in M^3_A(F)$.*

(ii) *The involution σ is isotropic if and only if*

$$e_3(\sigma)=e_3(\pi)+F^\times\cdot[A]\in M^3_A(F)$$

for some 3-fold Pfister form π, i.e. $f_3(\sigma)=0$ and $e_3(\sigma)$ is represented by a symbol.

Proof. Assume first that A is split, so that σ is adjoint to a 12-dimensional quadratic form φ, and $e_3(\sigma)=e_3(\varphi)\in {}_2H^3(F)$. Since the Arason invariant for quadratic forms has kernel the 4th power I^4F of the fundamental ideal of the Witt ring $W(F)$, the first equivalence follows from the Arason–Pfister Hauptsatz.

To prove (ii), note that there is no 10-dimensional anisotropic quadratic form in I^3F, see [28, Prop. XII.2.8]. So if φ is isotropic, then it has two hyperbolic planes, and it is Witt-equivalent to a multiple of some 3-fold Pfister form π. Hence, $e_3(\varphi)=e_3(\pi)$. Assume conversely that $e_3(\varphi)=e_3(\pi)$. By condition (i), φ becomes hyperbolic over the function field of π. Therefore, by [28, Th. X.4.11], the anisotropic kernel of φ is a multiple of π. In view of the dimensions, this implies $\varphi=\langle\alpha\rangle\pi+2\mathbb{H}$ for some $\alpha\in F^\times$. In particular, φ is isotropic.

Assume now $A=M_6(Q)$ for some quaternion division algebra Q. By a result of Dejaiffe [11] and of Parimala–Sridharan–Suresh [31, Prop 3.3], the involution σ is hyperbolic if and only if it is hyperbolic after scalar extension to a generic splitting field F_Q of the quaternion algebra Q. Since the restriction map $M^3_Q(F)\to H^3(F_Q)$ is injective, the split case gives the result in index 2. If (A,σ) is isotropic, it is Witt-equivalent to a degree 8 and index 2 algebra with involution. The explicit description of the Arason invariant in degree 8 given in [35, Th. 5.2] shows it is equal to $e_3(\pi) \mod F^\times\cdot[Q]\in M^3_Q(F)$ for some

3-fold Pfister form π. Assume conversely that $e_3(\sigma) = e_3(\pi) + F^\times \cdot [Q]$. After scalar extension to F_Q, the split case shows σ_{F_Q} is isotropic. By Parimala–Sridharan–Suresh [31, Cor. 3.4], this implies σ itself is isotropic. □

Remark 4.2. In the split isotropic case, the involution can be explicitly described from its Arason invariant: the proof of Theorem 4.1(ii) shows that σ is adjoint to $\pi + 2\mathbb{H}$ if $e_3(\sigma) = e_3(\pi) \in {}_2H^3(F)$. In index 2, we also get an explicit description of (A,σ) in the isotropic case. Indeed, we have

$$(A,\sigma) = (M_2(Q), \mathrm{hyp}) \boxplus (M_4(Q), \sigma_0)$$

for some orthogonal involution σ_0 with trivial discriminant and trivial Clifford invariant, and $e_3(\sigma) = e_3(\sigma_0)$. If (a,b,c) is a symbol representing $e_3(\sigma)$, then by [35, Th. 5.2] we may assume that one of the slots, say a, is such that $F(\sqrt{a})$ splits Q, hence Q carries an orthogonal involution ρ with discriminant a. Theorem 5.2 of [35] further shows that

$$(M_4(Q), \sigma_0) \simeq (Q,\rho) \otimes \mathrm{Ad}_{\langle\langle b,c\rangle\rangle}\,.$$

Under some additional condition, we also have the following classification result:

Proposition 4.3. *Let $A = M_6(Q)$ be a degree 12 algebra of index at most 2, and let σ and σ' be two orthogonal involutions with trivial discriminant and trivial Clifford invariant. We assume either A is split, or σ is isotropic. The involutions σ and σ' are isomorphic if and only if $e_3(\sigma) = e_3(\sigma')$.*

Proof. It is already known that two isomorphic involutions have the same Arason invariant, so we only need to prove the converse. Assume first that A has index 2, in which case we assume in addition that σ is isotropic. Since $e_3(\sigma) = e_3(\sigma')$, by Theorem 4.1, the involution σ' also is isotropic. The result then follows from the explicit description given in Remark 4.2, or equivalently from [35, Cor. 5.3(2)], which shows that the anisotropic parts of σ and σ' are isomorphic.

Assume now A is split, and σ and σ' are adjoint to φ and φ' respectively. We have $e_3(\varphi) = e_3(\varphi')$. If there exists a 3-fold Pfister form π such that $e_3(\varphi) = e_3(\varphi') = e_3(\pi)$, then φ and φ' are both similar to $\pi + 2\mathbb{H}$. Otherwise, they are anisotropic, and the result in this case follows by combining Pfister's theorem (see for instance [22, Th. 8.1.1]), which asserts that φ and φ' can be decomposed as tensor products of a 1-fold Pfister form and an Albert form, with Hoffmann's result [19, Corollary], which precisely says that two such forms are similar if and only if their difference is in I^4F.

□

4.2. Arason invariant and decomposition groups. Recall from Example 3.7 that the decomposition groups corresponding to additive decompositions of (A,σ) are quaternionic subgroups of order at most 4 when A is split. Hence, by Peyre's Theorem 3.12, the corresponding homology group is trivial, $\mathcal{H}_U = 0$. Using this, we have:

Proposition 4.4. *Let φ be a 12-dimensional quadratic form in I^3F, and let $U = \{0, [Q_1], [Q_2], [Q_3]\} \subset \mathrm{Br}(F)$ be a quaternionic subgroup of order at most 4. For $i = 1, 2, 3$, let n_i be the norm form of Q_i. The following are equivalent:*

(a) *There exists $\alpha_1, \alpha_2, \alpha_3 \in F^\times$ such that $\varphi = \langle\alpha_1\rangle n_1 \perp \langle\alpha_2\rangle n_2 \perp \langle\alpha_3\rangle n_3$;*
(b) *U is a decomposition group of Ad_φ;*
(c) *φ is hyperbolic over F_U;*
(d) *$e_3(\varphi) \in \ker\big(H^3(F) \to H^3(F_U)\big)$;*
(e) *$e_3(\varphi) \in F^\times \cdot U$.*

Proof. The equivalence between (a) and (b) follows from Example 3.4. Assume φ decomposes as in (a). Since the field F_U splits all three quaternion algebras Q_i, hence also their norm forms n_i, the form φ is hyperbolic over F_U, hence assertions (c) and (d) hold. By Peyre's result 3.12, we also get (e), and it only remains to prove that (e) implies (a).
Thus, assume now that $e_3(\varphi) \in F^\times \cdot U$. Since the subgroup U is generated by $[Q_1]$ and $[Q_2]$, there exists λ_1 and $\lambda_2 \in F^\times$ such that

$$e_3(\varphi) = (\lambda_1) \cdot [Q_1] + (\lambda_2) \cdot [Q_2].$$

The product $Q_1 \otimes Q_2 \otimes Q_3$ is split, so by the common slot lemma ([28, Th. III.4.13]), we may assume $Q_i = (a, b_i)_F$ for some a and $b_i \in F^\times$. A direct computation then shows that $n_1 - n_2 = \langle b_2\rangle n_3$. Hence the 12-dimensional quadratic form $\langle -\lambda_1\rangle n_1 + \langle\lambda_2\rangle n_2 + \langle b_2\rangle n_3$ is Witt-equivalent to $\langle 1, -\lambda_1\rangle n_1 + \langle -1\rangle\langle 1, -\lambda_2\rangle n_2$, which has the same Arason invariant as φ. By Proposition 4.3, this form is similar to φ, so that φ has an additive decomposition as required. □

Let us consider now the index 2 case. By Lemma 3.10, (A, σ) admits decomposition groups of order 4 if and only if it is isotropic. We prove:

Proposition 4.5. *Let $A = M_6(Q)$ be an algebra of index ≤ 2, and consider an orthogonal involution σ on A, with trivial discriminant and trivial Clifford invariant. Pick a subgroup $U = \{0, [Q], [Q_1], [H_1]\} \subset \mathrm{Br}(F)$ containing the class of Q. The following are equivalent:*

(a) *(A, σ) admits an additive decomposition of the following type:*

$$(A,\sigma) \in \big((Q_1, \bar{\ }) \otimes (H_1, \bar{\ })\big) \boxplus \big((Q_1, \bar{\ }) \otimes (H_1, \bar{\ })\big) \boxplus \big((M_2(F), \bar{\ }) \otimes (Q, \bar{\ })\big);$$

(b) *U is a decomposition group of (A, σ);*
(c) *σ is hyperbolic over F_U;*
(d) *$e_3(\sigma) \in \ker\big(M_A^3(F) \to H^3(F_U)\big)$;*
(e) *There exists $\alpha \in F^\times$ such that*

$$e_3(\sigma) = (\alpha) \cdot [Q_1] \mod F^\times \cdot [Q] \in M_Q^3(F).$$

Proof. The proof follows the same line as for the previous proposition. By the definition of decomposition groups, (a) implies (b). Conversely, if (b) holds, then (A, σ) has an additive decomposition with summands isomorphic to $(Q_1, \bar{\ }) \otimes (H_1, \bar{\ })$ or to $(M_2(F), \bar{\ }) \otimes (Q, \bar{\ })$. If Q_1 or H_1 is split, then the two kinds of summands are isomorphic, hence (a) holds. If Q_1 and H_1 are not

split, then the number of summands isomorphic to $(Q_1, ^-) \otimes (H_1, ^-)$ must be even because $e_2(\sigma) = 0$ (see Proposition 3.1), and it must be nonzero because U is the corresponding decomposition group. Therefore, (a) holds.
Now, assume (A, σ) satisfies (a). Since the field F_U splits Q, Q_1 and H_1, (c) holds. Assertion (d) follows since hyperbolic involutions have trivial Arason invariant. By Peyre's Proposition 3.12, we deduce assertion (e), and it only remains to prove that (e) implies (a). Hence, assume

$$e_3(\sigma) = (\alpha) \cdot [Q_1] \mod F^\times \cdot [Q] \in M^3_Q(F),$$

for some $\alpha \in F^\times$ and some quaternion algebra Q_1. By Theorem 4.1(ii), the involution σ is isotropic. Hence, in view of Proposition 4.3, it is enough to find an involution σ' satisfying (a) and having $e_3(\sigma') = (\alpha) \cdot [Q_1] \mod F^\times \cdot [Q]$. Since $Q \otimes Q_1 = H_1$ has index 2, the quaternion algebras Q and Q_1 have a common slot (see [28, Th. III.4.13]). Therefore, there exists a, b, $b_1 \in F^\times$ such that $Q = (a, b)$ and $Q_1 = (a, b_1)$. Let ρ be an orthogonal involution on Q with discriminant a, and let

$$(A, \sigma') = (M_2(Q), \mathrm{hyp}) \boxplus \big((Q, \rho) \otimes \mathrm{Ad}_{\langle\langle b_1, \alpha\rangle\rangle}\big).$$

One component of the Clifford algebra of $(Q, \rho) \otimes \mathrm{Ad}_{\langle\langle b_1\rangle\rangle}$ is given by the cup product of the discriminants of ρ and $\mathrm{ad}_{\langle\langle b_1\rangle\rangle}$, that is $(a, b_1) = Q_1$. By Remark 4.2, it follows that $e_3(\sigma') = e_3(\sigma)$, hence $(A, \sigma) \simeq (A, \sigma')$ by Proposition 4.3. Therefore (A, σ) satisfies (a) as required. □

4.3. Generators of the homology $\mathcal{H}_U$ of Peyre's complex. Let $A = M_6(Q)$ for some quaternion F-algebra Q, and let σ be an orthogonal involution on A with trivial discriminant and trivial Clifford invariant. Consider an additive decomposition of (A, σ) as in Theorem 3.2,

$$(A, \sigma) \in \boxplus_{i=1}^{3} \big((Q_i, ^-) \otimes (H_i, ^-)\big),$$

and let U be the corresponding decomposition group, which is a quaternionic subgroup of $\mathrm{Br}(F)$,

$$U = \{0, [Q], [Q_1], [H_1], [Q_2], [H_2], [Q_3], [H_3]\}.$$

Since $F^\times \cdot [Q] \subset F^\times \cdot U$, we may consider the canonical map

$$\overline{}^U : M^3_Q(F) \to H^3(F)/F^\times \cdot U.$$

As in §3.3, let F_U be the function field of the product of the Severi–Brauer varieties associated to elements of U. Since F_U splits U, Lemma 3.10 shows that A_{F_U} is split and σ_{F_U} is hyperbolic, hence $e_3(\sigma)_{F_U} = 0$. Therefore, $\overline{e_3(\sigma)}^U$ lies in the homology $\mathcal{H}_U$ of Peyre's complex.
As explained in Remark 3.11, for any quaternionic subgroup $U \subset \mathrm{Br}(F)$ of order dividing 8, we may find algebras with involution (A, σ) for which U is a decomposition group. The main result of this section is:

Theorem 4.6. *Let U be a quaternionic subgroup of $\operatorname{Br} F$ of order dividing 8. For any (A,σ) admitting U as a decomposition group, the class of the Arason invariant $\overline{e_3(\sigma)}^U$ is a generator of the homology group $\mathcal{H}_U$ of Peyre's complex.*

The main tool in the proof is the following proposition:

Proposition 4.7. *Let U be a quaternionic subgroup of $\operatorname{Br} F$ of order dividing 8, and pick an algebra $A = M_6(Q)$ with orthogonal involution σ, admitting U as a decomposition group.*

(i) *For all involutions σ' on A such that (A,σ') also admits U as a decomposition group, we have*

$$\overline{e_3(\sigma')}^U = \overline{e_3(\sigma)}^U.$$

(ii) *Conversely, for all $\xi \in M^3_A(F)$ such that $\overline{\xi}^U = \overline{e_3(\sigma)}^U$, there exists an involution σ' on A such that U is a decomposition group of (A,σ') and $e_3(\sigma') = \xi \mod F^\times \cdot [A]$.*

(iii) *There exists a hyperbolic involution σ' on A admitting U as a decomposition group if and only if $\overline{e_3(\sigma)}^U = 0$.*

Proof. (i) Since U is a decomposition group of (A,σ) and (A,σ'), we have

$$(A,\sigma) \text{ and } (A,\sigma') \in \boxplus_{i=1}^{3}\big((Q_i,\bar{\ }) \otimes (H_i,\bar{\ })\big).$$

Therefore, σ and σ' are adjoint to some skew-hermitian forms h and h' over $(Q,\bar{\ })$ satisfying

$$h = h_1 \perp h_2 \perp h_3 \text{ and } h' = \langle\alpha_1\rangle h_1 \perp \langle\alpha_2\rangle h_2 \perp \langle\alpha_3\rangle h_3,$$

for some h_i such that $\mathrm{ad}_{h_i} \simeq (Q_i,\bar{\ }) \otimes (H_i,\bar{\ })$, and some $\alpha_i \in F^\times$. Therefore,

$$e_3(\sigma) - e_3(\sigma') = e_3(\perp_{i=1}^3 \langle 1,-\alpha_i\rangle h_i).$$

Since h_i has discriminant 1, Proposition 2.6 applies to each summand and shows $\langle 1,-\alpha_i\rangle \otimes h_i$ has trivial discriminant and trivial Clifford invariant, and

$$e_3(\langle 1,-\alpha_i\rangle \otimes h_i) = \alpha_i \cdot e_2(h_i) = \alpha_i \cdot [Q_i] \in M^3_Q(F).$$

Therefore, $e_3(\sigma) - e_3(\sigma')$ is represented modulo $F^\times \cdot \{0,[Q]\}$ by $\sum_{i=1}^3 \alpha_i \cdot [Q_i]$. Since this element lies in $F^\times \cdot U$, we have $\overline{e_3(\sigma)}^U = \overline{e_3(\sigma')}^U$.
(ii) Consider a skew hermitian form h over $(Q,\bar{\ })$ such that $\sigma = \mathrm{ad}_h$, and a decomposition $h = h_1 \perp h_2 \perp h_3$ as in the proof of (i). Since $\overline{\xi}^U = \overline{e_3(\sigma)}^U$, the difference $e_3(\sigma) - \xi \in M^3_Q(F)$ is represented by a cohomology class of the form $\sum_{i=1}^3 \alpha_i \cdot [Q_i]$ for some $\alpha_i \in F^\times$. The computation in (i) shows that $e_3(\mathrm{ad}_{h'}) = \xi$ for $h' = \langle\alpha_1\rangle h_1 \perp \langle\alpha_2\rangle h_2 \perp \langle\alpha_3\rangle h_3$.
(iii) It follows from (ii) that $\overline{e_3(\sigma)}^U = 0$ if and only if there exists an involution σ' with decomposition group U and $e_3(\sigma') = 0$. Theorem 4.1(i) completes the proof by showing σ' is hyperbolic. □

With this in hand, we can now prove Theorem 4.6.

Proof of Theorem 4.6. Since $\mathcal{H}_U$ is either 0 or $\mathbb{Z}/2\mathbb{Z}$, in order to prove that $\overline{e_3(\sigma)}^U$ generates $\mathcal{H}_U$ it is enough to prove that $\mathcal{H}_U$ is trivial as soon as $\overline{e_3(\sigma)}^U = 0$. If U has order at most 4, then $\mathcal{H}_U$ is trivial by Theorem 3.12. Hence, let us assume U has order 8, and $\overline{e_3(\sigma)}^U = 0$. By Proposition 4.7, replacing σ by σ', we may assume σ is hyperbolic. Recall σ is adjoint to a skew-hermitian form h, which admits a decomposition $h = h_1 \perp h_2 \perp h_3$ with $\mathrm{Ad}_{h_i} = (Q_i, \bar{\ }) \otimes (H_i, \bar{\ })$. Since U has order 8, each summand h_i is anisotropic. The hyperbolicity of h says $h_1 \perp h_2 \simeq -h_3 \perp \mathbb{H}$ is isotropic. Therefore, there exists a pure quaternion q such that h_1 and h_2 represent q and $-q$ respectively. Over the quadratic extension $F(q)$ of F, the involutions ad_{h_1} and ad_{h_2} are isotropic. Since they are adjoint to 2-fold Pfister forms, they are hyperbolic. Hence $F(q)$ splits the Clifford algebra of h_1 and h_2, that is the quaternion algebras Q_1, Q_2. Since the Brauer classes of Q, Q_1 and Q_2 generate U, it follows that $F(q)$ is a quadratic splitting field of U. By Peyre's Theorem 3.13, we get $\mathcal{H}_U = 0$ as required. □

5. Quadratic splitting and the f_3 invariant

The f_3 invariant of an involution σ vanishes if the underlying algebra A is split, or of degree ≤ 10. We keep focusing on the case of degree 12 algebras, where we have explicit examples with $f_3(\sigma) \neq 0$, see Corollary 2.19. Thus, as in § 4, (A, σ) is a degree 12 algebra with orthogonal involution of trivial discriminant and trivial Clifford invariant for which the Arason and the f_3 invariants are defined. In particular, A has index at most 2.

Our first goal is to characterize the vanishing of $f_3(\sigma)$; this is done in Proposition 5.6 below. As pointed out in Proposition 2.5, $f_3(\sigma)$ vanishes if there exists a quadratic extension K/F over which (A, σ) is split and hyperbolic. Note that since A is Brauer-equivalent to a quaternion algebra, there exist quadratic extensions of the base field F over which A is split. Moreover, using the additive decompositions of Corollary 3.3, one may easily find quadratic extensions of the base field over which the involution is hyperbolic: it suffices to consider a common subfield of the quaternion algebras H_1, H_2, H_3, which exists by [28, Th. III.4.13] since $[H_1] + [H_2] + [H_3] = 0$. Yet, we give in Corollary 5.13 examples showing that the converse of Proposition 2.5 does not hold in degree 12: we may have $f_3(\sigma) = 0$ even when there is no quadratic extension that simultaneously splits A and makes σ hyperbolic.

First, we use quadratic forms to introduce an invariant of quaternionic subgroups of the Brauer group of F, which, as we next prove, coincides with the f_3-invariant of involutions admitting this subgroup as a decomposition group.

5.1. The invariants $f_3(U)$ and $f_3(\sigma)$.

To any quaternionic subgroup U of $\mathrm{Br}(F)$, we may associate in a natural way a quadratic form n_U by taking the sum of the norm forms n_H of the quaternion algebras H with Brauer class in U. We have:

Lemma 5.1. *Let U be a quaternionic subgroup of* $\operatorname{Br} F$ *generated by the Brauer classes of three quaternion algebras. The quadratic form $n_U = \sum_{[H]\in U} n_H$ satisfies $n_U \in I^3F$.*

Proof. Pick three generators $[Q_1]$, $[Q_2]$ and $[Q_3]$ of U, and let H_1, H_2, H_3, Q be quaternion algebras with Brauer classes $[H_1] = [Q_2] + [Q_3]$, $[H_2] = [Q_1] + [Q_3]$, $[H_3] = [Q_1] + [Q_2]$, and $[Q] = [Q_1] + [Q_2] + [Q_3]$. We have $[H_1] + [H_2] + [H_3] = 0$, and

$$U = \{0, [Q], [Q_1], [H_1], [Q_2], [H_2], [Q_3], [H_3]\}.$$

Since the difference $n_{Q_i} - n_{H_i}$ is Witt-equivalent to an Albert form of $Q_i \otimes H_i$, which is Brauer-equivalent to Q, there exists $\lambda_i \in F^\times$ such that in the Witt group of F, we have $n_{Q_i} - n_{H_i} = \langle \lambda_i \rangle n_Q \in WF$. Therefore,

$$n_U = \langle 1, \lambda_1, \lambda_2, \lambda_3 \rangle n_Q + \langle 1, 1 \rangle (n_{H_1} + n_{H_2} + n_{H_3}). \tag{2}$$

Since the right side is in I^3F, the lemma is proved. □

In view of Lemma 5.1, we may associate to U a cohomology class of degree 3 as follows:

Definition 5.2. For any quaternionic subgroup U generated by three elements, we let $f_3(U)$ be the Arason invariant of the quadratic form n_U:

$$f_3(U) = e_3(n_U) \in {}_2H^3(F).$$

We may easily compute $f_3(U)$ from formula (2): Since $[H_1] + [H_2] + [H_3] = 0$, we have $n_{H_1} + n_{H_2} + n_{H_3} \in I^3F$, hence $\langle 1, 1 \rangle (n_{H_1} + n_{H_2} + n_{H_3}) \in I^4F$ and therefore

$$f_3(U) = (\lambda_1\lambda_2\lambda_3) \cdot [Q]. \tag{3}$$

With this in hand, we get:

Proposition 5.3. *If $\mathcal{H}_U = 0$, then $f_3(U) = 0$.*

Proof. By Theorem 3.13, if $\mathcal{H}_U = 0$ then U admits a quadratic splitting field, i.e. the generators of U have a common quadratic subfield. So there exist a, b_1, b_2, and $b_3 \in F^\times$ such that $Q_i = (a, b_i)_F$ for $i = 1, 2, 3$. Thus, we have $H_1 = (a, b_2b_3)_F$ and

$$n_{Q_1} - n_{H_1} = \langle\langle a \rangle\rangle (\langle\langle b_1 \rangle\rangle - \langle\langle b_2b_3 \rangle\rangle) = \langle\langle a \rangle\rangle \langle -b_1, b_2b_3 \rangle = \langle -b_1 \rangle n_Q.$$

Similar formulas hold for $i = 2, 3$, and we get

$$f_3(U) = (-b_1b_2b_3) \cdot Q = (-b_1b_2b_3, a, b_1b_2b_3) = 0 \in {}_2H^3(F).$$

□

In [38], Sivatski asks about the converse[1], that is: if $f_3(U) = 0$, does the homology group $\mathcal{H}_U$ vanish, or equivalently by Peyre's Theorem 3.13, do the generators Q_1, Q_2, and Q_3 of the group U have a common quadratic subfield? Corollary 5.11 below shows that this is not the case.

[1]Sivatski's invariant has a different definition, but one may easily check the quadratic form he considers is equivalent to n_U modulo I^4F.

The relation between $f_3(U)$ and the f_3-invariant for involutions is given by the following:

Theorem 5.4. *Let (A,σ) be a central simple algebra of degree 12 and index ≤ 2, with orthogonal involution of trivial discriminant and trivial Clifford invariant. Let U be a quaternionic subgroup of the Brauer group, generated by three elements. If U is a decomposition group for (A,σ) then $f_3(\sigma) = f_3(U)$.*

Remark 5.5. (i) It follows that any two decomposition groups of a given algebra with involution have the same f_3-invariant, and any two algebras with involution having U as a decomposition group have the same f_3-invariant.
(ii) Let c be a generator of $\mathcal{H}_U$, and pick an arbitrary (A,σ) having U as a decomposition group. In view of Theorem 4.6, we have $e_3(\sigma) = c \mod F^\times \cdot U$. Hence $f_3(U) = f_3(\sigma) = 2c \in {}_2H^3(F)$. In particular, $f_3(U) = 0$ if and only if the homology group $\mathcal{H}_U$ is generated by a cohomology class of order 2.

Proof of Theorem 5.4. The result follows from the computation of $f_3(\sigma)$ in Proposition 2.7 and the computation of $f_3(U)$ in (3). We use the same notation as in Definition 3.6, and we let h_i be a rank 2 skew-hermitian form over $(Q, \bar{\ })$ such that

$$\mathrm{Ad}_{h_i} \simeq (Q_i, \bar{\ }) \otimes (H_i, \bar{\ }) \text{ and } \sigma = \mathrm{ad}_{h_1 \perp h_2 \perp h_3}.$$

For $i = 1, 2, 3$, let $q_i \in Q$ be a nonzero pure quaternion represented by h_i, and let $a_i = q_i^2 \in F^\times$. Let also $b_i \in F^\times$ be such that $Q = (a_i, b_i)_F$. Scalar extension to $F(q_i)$ makes h_i isotropic, hence hyperbolic since the discriminant of h_i is trivial. Therefore, we have $h_i \simeq \langle q_i \rangle \langle 1, -\lambda_i \rangle$ for some $\lambda_i \in F^\times$. The two components of the Clifford algebra of Ad_{h_i} are $(a_i, \lambda_i)_F$ and $(a_i, \lambda_i b_i)_F$, therefore

$$\{Q_i, H_i\} = \{(a_i, \lambda_i)_F, (a_i, \lambda_i b_i)_F\} \qquad \text{for } i = 1, 2, 3.$$

Since Q contains a pure quaternion which anticommutes with q_i and with square b_i, the form h_i is isomorphic to $\langle q_i \rangle \langle 1, -\lambda_i b_i \rangle$ for $i = 1, 2, 3$. Replacing some λ_i by $\lambda_i b_i$ if necessary, we may assume $H_i = (a_i, \lambda_i)_F$ for all i. Since $[H_1] + [H_2] + [H_3] = 0$, we get $\sum_{i=1}^3 (a_i, \lambda_i)_F = 0$. By Proposition 2.7 this implies $f_3(\sigma) = \lambda_1\lambda_2\lambda_3 \cdot [Q]$. On the other hand, since $n_{Q_i} - n_{H_i} = \langle\langle a_i, \lambda_i b_i \rangle\rangle - \langle\langle a_i, \lambda_i \rangle\rangle = \langle \lambda_i \rangle n_Q$, we have $f_3(U) = \lambda_1\lambda_2\lambda_2 \cdot [Q]$ by (3). □

5.2. Quadratic splitting, the f_3 invariant, and decomposition groups. By using Theorem 5.4 and Peyre's Theorem 3.13, we can now translate in terms of decomposition groups the two conditions we want to compare, as follows:

Proposition 5.6. *Let (A,σ) be a degree 12 and index ≤ 2 algebra with orthogonal involution of trivial discriminant and trivial Clifford invariant. The following conditions are equivalent:*

(a) $f_3(\sigma) = 0$*;*
(b) (A,σ) *has a decomposition group U with $f_3(U) = 0$;*
(c) $f_3(U) = 0$ *for all decomposition groups U of (A,σ).*

Likewise, the following conditions are equivalent:

(a') *there exists a quadratic extension K of F such that A_K is split and σ_K is hyperbolic;*

(b') *(A,σ) has a decomposition group U with $\mathcal{H}_U = 0$.*

Moreover, any of the conditions (a'), (b') *implies the equivalent conditions* (a), (b), (c).

Proof. The equivalence between conditions (a), (b), (c) follows directly from Theorem 5.4. Moreover, they can be deduced from (a'), (b') by Proposition 5.3 or Proposition 2.5. Hence, it only remains to prove that (a') and (b') are equivalent.

Assume first that (A,σ) has a decomposition group U with $\mathcal{H}_U = 0$. By Peyre's characterization of the vanishing of $\mathcal{H}_U$ for quaternionic groups, recalled in Theorem 3.13, U is split by a quadratic extension K of F. Hence, (A_K,σ_K) admits $\{0\}$ as a decomposition group. By Lemma 3.10, this implies (A_K,σ_K) is split and hyperbolic.

To prove the converse, let us assume there exists a quadratic field extension $K = F(d)$, with $d^2 = \delta \in F^\times$, such that A_K is split and σ_K is hyperbolic. If A is split, as explained in Example 3.7, all decomposition subgroups U of (A,σ) have order dividing 4, and therefore satisfy $\mathcal{H}_U = 0$ by Peyre's Theorem 3.12. Assume next $\operatorname{ind} A = 2$. Since A_K is split, we may identify $K = F(d)$ with a subfield of the quaternion division algebra Q Brauer-equivalent to A, and thus consider d as a pure quaternion in Q. Let h be a skew-hermitian form over $(Q, \bar{\ })$ such that $\sigma = \operatorname{ad}_h$. Since h_K is hyperbolic, it follows from [34, Prop., p. 382] that $h \simeq \langle d\rangle\varphi_0$ for some 6-dimensional quadratic form φ_0 over F. Decompose

$$\varphi_0 = \langle\alpha_1\rangle\langle 1,-\beta_1\rangle \perp \langle\alpha_2\rangle\langle 1,-\beta_2\rangle \perp \langle\alpha_3\rangle\langle 1,-\beta_3\rangle \qquad \text{for some } \alpha_i,\ \beta_i \in F^\times,$$

and let $Q_i = (\delta,\beta_i)_F$ be the quaternion F-algebra with norm $n_{Q_i} = \langle\langle \delta,\beta_i\rangle\rangle$ for $i = 1, 2, 3$. Computation shows that $e_2\big(\langle\alpha_i d\rangle\langle 1,-\beta_i\rangle\big)$ is represented by Q_i in $M_Q^2(F)$, hence (A,σ) decomposes as

$$(A,\sigma) \in \boxplus_{i=1}^{3} \operatorname{Ad}_{\langle\alpha_i d\rangle\langle 1,-\beta_i\rangle}\,.$$

So, the subgroup $U \subset \operatorname{Br} F$ generated by $[Q_1]$, $[Q_2]$ and $[Q_3]$ is a decomposition group for (A,σ). Again, U is split by K, hence $\mathcal{H}_U = 0$. □

5.3. Trivial f_3-invariant without quadratic splitting. We now construct an algebra with involution (A,σ), of degree 12 and index 2, such that $f_3(\sigma) = 0$, and yet, there is no quadratic extension K of F over which (A,σ) is both split and hyperbolic. In particular, by Peyre's Theorem 3.13, we have $\mathcal{H}_U \neq 0$ for all decomposition groups U of (A,σ). (See Remark 5.14 for an example where (A,σ) has a decomposition group U whith $\mathcal{H}_U \neq 0$ and another U' with $\mathcal{H}_{U'} = 0$.)

Remark 5.7. In his paper [32, §6.2], Peyre provides an example of a quaternionic subgroup $U \subset \operatorname{Br}(F)$ with $\mathcal{H}_U \neq 0$, but the way he proves $\mathcal{H}_U$ is nonzero is by describing an element $c \in H^3(F)$ which is not of order 2, hence does not belong to $F^\times \cdot U$, and yet is in the kernel of the restriction map $H^3(F) \to H^3(F_U)$. Thus, the group U in Peyre's example satisfies $f_3(U) \neq 0$ (see Remark 5.5). In this section, we construct an example of a different flavor, namely a subgroup U with $\mathcal{H}_U \neq 0$, but $f_3(U) = 0$. Hence, the homology group in this case is generated by a cohomology class which is of order 2, and in the kernel of $H^3(F) \to H^3(F_U)$, but does not belong to $F^\times \cdot U$.

Notation 5.8. Until the end of this section, k is a field (of characteristic different from 2), M is a triquadratic field extension of k (of degree 8) and K is a quadratic extension of k in M,

$$M = k(\sqrt{a}, \sqrt{b}, \sqrt{c}) \supset K = k(\sqrt{a}).$$

We let C be a central simple k-algebra of exponent 2 split by M and we write

$$[C] \in \mathrm{Dec}(M/k)$$

to express the property that there exist α, β, $\gamma \in k^\times$ such that

$$[C] = (a, \alpha)_k + (b, \beta)_k + (c, \gamma)_k.$$

The existence of algebras C as above such that $[C] \notin \mathrm{Dec}(M/k)$ is shown in [12, §5]. By contrast, it follows from a theorem of Albert that every central simple algebra of exponent 2 split by a biquadratic extension has a decomposition up to Brauer-equivalence into a tensor product of quaternion algebras adapted to the biquadratic extension (see [27, Prop. 5.2]), so (viewing M as $K(\sqrt{bc}, \sqrt{c})$) there exist x, $y \in K^\times$ such that

$$[C_K] = (bc, x)_K + (c, y)_K.$$

By multiplying x and y by squares in K, we may—and will—assume x, $y \notin k$. We have $\mathrm{cor}_{K/k}[C_K] = 2[C] = 0$, hence letting N denote the norm map from K to k, the previous equation leads, by the projection formula, to the following: $(bc, N(x))_k + (c, N(y))_k = 0$. We may then consider the following quaternion k-algebra:

$$H = (bc, N(x))_k = (c, N(y))_k. \tag{4}$$

Since $N(x)$, $N(y)$ are norms from $K = k(\sqrt{a})$ to k, we have $(a, N(x))_k = (a, N(y))_k = 0$, hence we may also write

$$H = (abc, N(x))_k = (ac, N(y))_k. \tag{5}$$

Let $B = (bc, x)_K \otimes_K (c, y)_K$ be the biquaternion algebra Brauer-equivalent to C_K, and let ψ be the Albert form of B over K defined by

$$\psi = \langle bc, x, -bcx, -c, -y, cy \rangle.$$

Let $s\colon K \to k$ be a nontrivial linear map such that $s(1) = 0$, and let $s_\star$ denote the corresponding Scharlau transfer. Using the properties of $s_\star$ (see for instance [28, p. 189, p. 198]), we can make the following computation in the Witt group $W(k)$:

$$\begin{aligned} s_\star(\psi) =& s_\star(\langle x, -bcx, -y, cy\rangle) = s_\star\big(\langle x\rangle\big)\langle\langle bc\rangle\rangle - s_\star\big(\langle y\rangle\big)\langle\langle c\rangle\rangle \\ =& \langle s(x)\rangle\langle\langle bc, N(x)\rangle\rangle - \langle s(y)\rangle\langle\langle c, N(y)\rangle\rangle. \end{aligned}$$

(Recall that we assume x, $y \notin k$, so $s(x)$, $s(y) \neq 0$.) In view of (4), the last equation yields

$$s_\star(\psi) = \langle s(x), -s(y)\rangle n_H,$$

where n_H is the norm form of H. Thus, $s_\star(\psi) \in I^3(k)$, and we may consider

$$e_3(s_\star(\psi)) = s(x)s(y)\cdot[H] \in {}_2H^3(k). \tag{6}$$

This class represents an invariant of B defined by Barry [2]. It is shown in [2, Prop. 4.4] that $e_3(s_\star(\psi)) \in N(K^\times)\cdot[C]$ if and only if the biquaternion algebra B has a descent to k, i.e., there exist quaternion k-algebras A_1, A_2 such that $B \simeq A_1 \otimes_k A_2 \otimes_k K$.

Finally, let t be an indeterminate over k, and let $F = k(t)$. Consider the subgroup $U \subset \mathrm{Br}(F)$ generated by the Brauer classes $(a,t)_F$, $(b,t)_F$ and $(c,t)_F + [H_F]$. In view of (4) and (5), one may easily check that U is a quaternionic subgroup of order 8:

$$\begin{aligned} U = \{0, (a,t)_F, (b,t)_F, (c, N(y)t)_F,& \\ (ab,t)_F, (ac, N(y)t)_F, (bc, N(x)t)_F, &(abc, N(x)t)_F\}. \end{aligned}$$

We set

$$\xi = t\cdot[C] + e_3(s_\star(\psi)) \in {}_2H^3(F).$$

This construction yields examples with trivial f_3 but with no quadratic splitting mentioned in the introduction to this section, as we proceed to show. First, we describe the group $\mathcal{H}_U$ and give a criterion for its vanishing:

Theorem 5.9. *Use the notation 5.8. Denote by* $\overline{\xi}^U \in H^3(F)/F^\times\cdot U$ *the image of* $\xi \in H^3(F)$*. We have*

$$\mathcal{H}_U = \{0, \overline{\xi}^U\} \qquad \textit{and} \qquad f_3(U) = 0.$$

Moreover, the following conditions are equivalent:

(a) $[C] \in \mathrm{Dec}(M/k)$*;*
(b) U *is split by some quadratic field extension* E/F*;*
(c) $\mathcal{H}_U = 0$*;*
(d) $\xi \in F^\times \cdot U$.

The core of the proof is the following technical lemma:

Lemma 5.10. *With the notation 5.8, every field extension of* F *that splits* U *also splits* ξ*.*

Proof. Let L be an extension of F that splits U. We consider two cases, depending on whether $a \in L^{\times 2}$ or $a \notin L^{\times 2}$. Suppose first $a \in L^{\times 2}$, so we may identify K with a subfield of L, hence x, $y \in L^\times$ and $[C_L] = (bc, x)_L + (c, y)_L$. Since L splits $(b,t)_F$, we have $(t, bc, x)_L = (t, c, x)_L$, hence $t \cdot [C_L] = xy \cdot (t, c)_L$. Since L also splits $(t, c)_F + [H_F]$, we have

$$t \cdot [C_L] = xy \cdot [H_L].$$

Comparing with (6), we see that it suffices to show $xy \cdot [H_L] = s(x)s(y) \cdot [H_L]$ to prove that L splits ξ.

Let ι be the nontrivial automorphism of K over k. Writing $x = x_0 + x_1\sqrt{a}$ and $y = y_0 + y_1\sqrt{a}$ with $x_i, y_i \in k$, we have

$$s(x)s(y) = x_1y_1s(\sqrt{a})^2 \text{ and } (x - \iota(x))(y - \iota(y)) = 4x_1y_1a.$$

Hence $s(x)s(y) \equiv (x - \iota(x))(y - \iota(y)) \bmod L^{\times 2}$. We also have

$$(x - \iota(x), N(x))_K = (x, N(x))_K \text{ because } (x^2 - N(x), N(x))_K = 0.$$

From the expression $H = (bc, N(x))_k$ it then follows that $x \cdot [H_K] = (x - \iota(x)) \cdot [H_K]$. Similarly, from $H = (c, N(y))_k$ we have $y \cdot [H_K] = (y - \iota(y)) \cdot [H_K]$, hence

$$xy \cdot [H_L] = s(x)s(y) \cdot [H_L].$$

Thus, we have proved L splits ξ under the additional hypothesis that $a \in L^{\times 2}$. For the rest of the proof of (i), assume $a \notin L^{\times 2}$. Let $L' = L(\sqrt{a}) = L \otimes_k K$, and write again $s\colon L' \to L$ for the L-linear extension of s to L' and $N\colon L' \to L$ for the norm map. If $t \in L^{\times 2}$, then $\xi_L = e_3(s_\star(\psi))_L$. Moreover, L splits H because it splits U. Therefore, (6) shows that L splits $e_3(s_\star(\psi))$. For the rest of the proof, we may thus also assume $t \notin L^{\times 2}$.

Since $(a, t)_L = 0$, we may find $z_0 \in L'$ such that $t = N(z_0)$. Because L splits $(b, t)_F$, we have $(b, N(z_0))_L = 0$, so $\operatorname{cor}_{L'/L}(b, z_0)_{L'} = 0$. It follows that $(b, z_0)_{L'}$ has an involution of the second kind, hence also a descent to L by a theorem of Albert (see [26, (2.22)]). We may choose a descent of the form $(b, z_0)_{L'} = (b, \zeta)_{L'}$ for some $\zeta \in L^\times$; see [40, (2.6)]. Let $z = z_0\zeta \in L'^\times$. We then have $(b, z)_{L'} = 0$, hence after taking the corestriction to L

$$(b, N(z))_L = 0.$$

We also have $t = N(z_0) \equiv N(z) \bmod L^{\times 2}$. Since L splits $[H_F] + (c, t)_F$, we have

$$H_L = (c, N(z))_L.$$

Since $H = (bc, N(x))_k = (c, N(y))_k$ by (4), it follows that

$$(bc, N(xz))_L = (c, N(yz))_L = 0.$$

If $s(xz) = 0$ (i.e., $xz \in L$), then $s_\star(\langle xz\rangle\langle\langle bc\rangle\rangle)$ is hyperbolic. If $s(xz) \neq 0$, computation yields

$$s_\star(\langle xz\rangle\langle\langle bc\rangle\rangle) = \langle s(xz)\rangle\langle\langle bc, N(xz)\rangle\rangle;$$

but since the quaternion algebra $(bc, N(xz))_L$ is split, the form $s_\star(\langle xz\rangle\langle\langle bc\rangle\rangle)$ is also hyperbolic in this case. Therefore, we may find $\lambda \in L^\times$ represented by $\langle xz\rangle\langle\langle bc\rangle\rangle$; we then have

$$(7)\qquad \langle xz\rangle\langle\langle bc\rangle\rangle = \langle\lambda\rangle\langle\langle bc\rangle\rangle, \qquad \text{hence also} \quad \langle x\rangle\langle\langle bc\rangle\rangle = \langle\lambda z\rangle\langle\langle bc\rangle\rangle.$$

Similarly, since the quaternion algebra $(c, N(yz))_L$ is split, the form $s_\star(\langle yz\rangle\langle\langle c\rangle\rangle)$ is hyperbolic, and we may find $\mu \in L^\times$ such that

$$(8)\qquad \langle yz\rangle\langle\langle c\rangle\rangle = \langle\mu\rangle\langle\langle c\rangle\rangle, \qquad \text{hence also} \quad \langle y\rangle\langle\langle c\rangle\rangle = \langle\mu z\rangle\langle\langle c\rangle\rangle.$$

As a result of (7) and (8), we have $\langle x, -bcx\rangle = \langle\lambda z, -\lambda zbc\rangle$ and $\langle -y, cy\rangle = \langle -\mu z, \mu zc\rangle$, hence we may rewrite ψ over L' as

$$\psi_{L'} = \langle bc, -c, \lambda z, -\lambda zbc, -\mu z, \mu zc\rangle.$$

Note that $z \notin L$ since $t \notin L^{\times 2}$, hence $s(z) \neq 0$. Using the last expression for $\psi_{L'}$ we may now compute

$$s_\star(\psi)_L = s_\star(\psi_{L'}) = s_\star(\langle z\rangle)\langle\lambda, -\lambda bc, -\mu, \mu c\rangle = \langle s(z)\rangle\langle\langle N(z)\rangle\rangle\langle\lambda, -\lambda bc, -\mu, \mu c\rangle.$$

Since $(bc, N(z))_L = (c, N(z))_L = H_L$, we have $\langle\langle N(z), bc\rangle\rangle = \langle\langle N(z), c\rangle\rangle = (n_H)_L$, hence $s_\star(\psi)_L = \langle s(z)\rangle\langle\lambda, -\mu\rangle(n_H)_L$, and therefore

$$(9)\qquad e_3(s_\star(\psi))_L = (\lambda\mu)\cdot[H_L].$$

On the other hand, we have $[C_K] = (bc, x)_K + (c, y)_K$, hence since $(b, z)_{L'} = 0$

$$[C_{L'}] = (bc, xz)_{L'} + (c, yz)_{L'}.$$

In view of (7) and (8), we may rewrite the right side as follows:

$$[C_{L'}] = (bc, \lambda)_{L'} + (c, \mu)_{L'}.$$

Therefore, $[C_L] + (bc, \lambda)_L + (c, y)_L$ is split by L'. We may then find $\nu \in L^\times$ such that

$$[C_L] = (bc, \lambda)_L + (c, \mu)_L + (a, \nu)_L.$$

Since L splits U, we have $(t, a)_L = (t, b)_L = 0$ and $(t, c)_L = H_L$. It follows that

$$(t)\cdot[C_L] = (t, c, \lambda\mu)_L = (\lambda\mu)\cdot[H_L].$$

By comparing with (9), we see that ξ vanishes over L. The proof of the lemma is thus complete. □

Proof of Theorem 5.9. Since $2\xi = 0$, the assertion $f_3(U) = 0$ follows from $\mathcal{H}_U = \{0, \overline{\xi}^U\}$, see Remark 5.5(ii). Moreover, the field F_U splits U. Therefore, by Lemma 5.10, we have $\xi \in \ker\big(H^3(F) \to H^3(F_U)\big)$, so that $\overline{\xi}^U \in \mathcal{H}_U$. Since we know from Theorem 3.13 that the order of $\mathcal{H}_U$ is at most 2, it suffices to show that $\overline{\xi}^U \neq 0$ when $\mathcal{H}_U \neq 0$ to establish $\mathcal{H}_U = \{0, \overline{\xi}^U\}$. Therefore, proving the equivalence of (a), (b), (c) and (d) completes the proof.

Let us first prove (a) ⇒ (b). Suppose $[C] = (a, \alpha)_k + (b, \beta)_k + (c, \gamma)_k$ for some $\alpha, \beta, \gamma \in k^\times$. Since $[C_K] = (bc, x)_K + (c, y)_K$, it follows that $(b, \beta)_K + (c, \gamma)_K = (bc, x)_K + (c, y)_K$, hence

$$(bc, \beta x)_K = (c, \beta\gamma y)_K.$$

By the common slot lemma [28, Th. III.4.13], we may find $z \in K^\times$ such that

$$(bc, \beta x)_K = (bc, z)_K = (c, z)_K = (c, \beta\gamma y)_K. \tag{10}$$

Let $E = F\big(\sqrt{N(z)t}\big)$, a quadratic extension of F. We claim that E splits U. First, observe that $N(z)t$ is represented by the form $\langle t, -at\rangle$, hence the quaternion algebra $(a,t)_F$ contains a pure quaternion with square $N(z)t$. Therefore, E splits $(a,t)_F$. Likewise, from (10) we see that $(b,z)_K = 0$, hence by taking the corestriction to k we have $(b, N(z))_k = 0$. Therefore, $N(z)t$ is represented by the form $\langle t, -bt\rangle$, and it follows that E splits $(b,t)_F$. Finally, by taking the corestriction of each side of the rightmost equation in (10), we obtain $(c, N(z))_k = (c, N(y))_k$, so $N(y)N(z)$ is represented by $\langle 1, -c\rangle$ and therefore $N(z)t$ is represented by $\langle N(y)t, -cN(y)t\rangle$. It follows that E splits the quaternion algebra $(c, N(y)t)_F$. We have thus shown that E splits three generators of U, hence E splits U.

The implication (b) $\Rightarrow$ (c) follows immediately from Peyre's Theorem 3.13. Moreover, (c) $\Rightarrow$ (d) is clear since $\overline{\xi}^U \in \mathcal{H}_U$. To complete the proof, we show (d) $\Rightarrow$ (a). Suppose there exist λ_1, λ_2, $\lambda_3 \in F^\times$ such that

$$\xi = (\lambda_1, a, t) + (\lambda_2, b, t) + (\lambda_3, c, N(y)t). \tag{11}$$

Let $\partial\colon H^i(F) \to H^{i-1}(k)$ be the residue map associated to the t-adic valuation, for $i = 2$, 3. Since $e_3(s_\star(\psi)) \in H^3(k)$ we have $\partial(e_3(s_\star(\psi))) = 0$, hence $\partial(\xi) = [C]$. Therefore, taking the image of each side of (11) under the residue map yields

$$[C] = a \cdot \partial(\lambda_1, t) + b \cdot \partial(\lambda_2, t) + c \cdot \partial(\lambda_3, N(y)t),$$

so that $[C] \in \operatorname{Dec}(M/k)$. □

As a corollary, we get:

Corollary 5.11. *Use the notation 5.8, and assume $[C] \notin \operatorname{Dec}(M/k)$. Then $U \subset \operatorname{Br}(F)$ is a quaternionic subgroup of order 8 such that $\sum_{[H]\in U} n_H \in I^4(F)$ (i.e., $f_3(U) = 0$), which is not split by any quadratic extension of F (i.e., $\mathcal{H}_U \neq 0$).*

To obtain an example of a central simple algebra with orthogonal involution of degree 12 without quadratic splitting, we need a more stringent condition on C:

Lemma 5.12. *With the notation 5.8, the following conditions are equivalent:*

(a) *there is a quadratic extension E of F that splits $(a,t)_F$ and ξ;*
(b) *the algebra C is Brauer-equivalent to a tensor product of quaternion k-algebras $A_1 \otimes_k A_2 \otimes_k A_3$ with A_3 split by K.*

Proof. (a) $\Rightarrow$ (b): Let $\widehat{F} = k((t))$ be the completion of F for the t-adic valuation. The field E does not embed in $\widehat{F}$ because $\widehat{F}$ does not split $(a,t)_F$. Therefore, E and $\widehat{F}$ are linearly disjoint over F and we may consider the field $\widehat{E} = E \otimes_F \widehat{F}$, which is a quadratic extension of $\widehat{F}$ that splits $(a,t)_{\widehat{F}}$ and $\xi_{\widehat{F}}$. Each square class in $\widehat{F}$ is represented by an element in $k^\times$ or an element of the

form ut with $u \in k^\times$, see [28, Cor. VI.1.3]. Therefore, we may assume that either $\widehat{E} = \widehat{F}(\sqrt{u})$ or $\widehat{E} = \widehat{F}(\sqrt{ut})$ for some $u \in k^\times$.
Suppose first $\widehat{E} = \widehat{F}(\sqrt{u})$ with $u \in k^\times$. Since the quaternion algebra $(a,t)_{\widehat{F}}$ is split by $\widehat{E}$, it must contain a pure quaternion with square u, hence u is represented by $\langle a, t, -at\rangle$ over $\widehat{F}$. Therefore, $u \equiv a \bmod k^{\times 2}$, and $\widehat{E} = K((t))$. From

$$\xi_{\widehat{E}} = t\cdot[C_{\widehat{E}}] + e_3(s_\star(\psi))_{\widehat{E}} = 0,$$

it follows by taking images under the residue map $H^3(\widehat{E}) \to H^2(K)$ associated to the t-adic valuation that $[C_K] = 0$. Then C is Brauer-equivalent to a quaternion algebra A_3 split by K, and (b) holds with A_1, A_2 split quaternion algebras.
Suppose next $\widehat{E} = \widehat{F}(\sqrt{ut})$ for some $u \in k^\times$. Since $\widehat{E}$ splits $(a,t)_{\widehat{F}}$, it follows as above that ut is represented by $\langle a, t, -at\rangle$ over $\widehat{F}$, hence u is represented by $\langle 1, -a\rangle$, which means that $u \in N(K^\times)$. Because ut is a square in $\widehat{E}$, we have $t\cdot[C_{\widehat{E}}] = u\cdot[C_{\widehat{E}}]$, hence the equation $\xi_{\widehat{E}} = 0$ yields

$$u\cdot[C_{\widehat{E}}] + e_3(s_\star(\psi))_{\widehat{E}} = \big(u\cdot[C] + e_3(s_\star(\psi))\big)_{\widehat{E}} = 0.$$

Since $\widehat{F} = k((t)) = k((ut))$ we have $\widehat{E} = k((\sqrt{ut}))$, hence the scalar extension map $H^3(k) \to H^3(\widehat{E})$ is injective. Therefore, the last equation yields

$$u\cdot[C] + e_3(s_\star(\psi)) = 0,$$

which shows that $e_3(s_\star(\psi)) \in N(K^\times)\cdot[C]$ because $u \in N(K^\times)$. By Barry's result [2, Prop. 4.4], it follows that the biquaternion algebra B has a descent to k: there exist quaternion k-algebras A_1, A_2 such that $B \simeq A_1 \otimes_k A_2 \otimes_k K$. Since C_K is Brauer-equivalent to B, it follows that $C \otimes A_1 \otimes_k A_2$ is split by K. It is therefore Brauer-equivalent to a quaternion algebra A_3 split by K, and C is Brauer-equivalent to $A_1 \otimes_k A_2 \otimes_k A_3$, proving (b).
(b) $\Rightarrow$ (a): Since B is Brauer-equivalent to C_K, condition (b) implies that $B \simeq A_1 \otimes_k A_2 \otimes_k K$. From Barry's result [2, Prop. 4.4], it follows that $e_3(s_\star(\psi)) = u\cdot[C]$ for some $u \in N(K^\times)$. Let $E = F(\sqrt{ut})$. Then $(a,t)_E \simeq (a,u)_E$, hence E splits $(a,t)_F$ because $u \in N(K^\times)$. Moreover, $\xi_E = \big(u\cdot[C] + e_3(s_\star(\psi))\big)_E$, so E also splits ξ. Therefore, (a) holds. □

Examples of algebras C for which condition (b) of Lemma 5.12 does not hold include indecomposable division algebras of degree 8 and exponent 2; other examples are given in [3]. Note that condition (b) is weaker than $[C] \in \mathrm{Dec}(M/k)$; it is in fact strictly weaker: see Remark 5.14.

Corollary 5.13. *Use the notation 5.8, and let $Q = (a,t)_F$. There exists an orthogonal involution ρ on $M_6(Q)$ with the following properties:*

(i) *ρ has trivial discriminant and trivial Clifford invariant;*
(ii) *$e_3(\rho) = \xi \mod F^\times\cdot[Q]$;*
(iii) *U is a decomposition group of ρ;*
(iv) *$f_3(\rho) = 0$.*

For any involution ρ satisfying (i) *and* (ii)*, there exists a quadratic extension of F over which Q is split and ρ is hyperbolic if and only if the equivalent conditions of Lemma 5.12 hold.*

Proof. By Remark 3.11, there is an orthogonal involution ρ on $M_6(Q)$ with trivial discriminant and trivial Clifford invariant, and with decomposition group U. By Theorems 4.6 and 5.4, $\overline{e_3(\rho)}^U$ generates $\mathcal{H}_U$, and $f_3(\rho) = f_3(U)$. Therefore, Theorem 5.9 yields $\overline{e_3(\rho)}^U = \overline{\xi}^U$ and $f_3(\rho) = 0$. By Proposition 4.7(ii), we may assume $e_3(\rho) = \xi \mod F^\times \cdot [Q]$. Thus, ρ satisfies conditions (i)–(iv).
Now, let ρ be any orthogonal involution on $M_6(Q)$ satisfying (i) and (ii). Because of (ii), condition (a) of Lemma 5.12 holds if and only if there is a quadratic extension E of F such that $[Q_E] = 0$ and $e_3(\rho)_E = 0$. By Theorem 4.1(i), the last equation holds if and only if ρ_E is hyperbolic. □

Suppose C does not satisfy condition (b) of Lemma 5.12 (e.g., C is an indecomposable division algebra of degree 8 and exponent 2 split by M). Then for any involution ρ on $M_6(Q)$ satisfying the properties (i)–(iv) of Corollary 5.13 we have $f_3(\rho) = 0$, and yet there is no quadratic extension of F over which Q is split and ρ is hyperbolic. From (b') ⇒ (a') in Proposition 5.6, it follows that $\mathcal{H}_{U'} \neq 0$ for every decomposition group U' of ρ.

Remark 5.14. By [40, Cor. 3.2], for any triquadratic extension M/k, any 2-torsion Brauer class in $\mathrm{Br}(k)$ split by M is represented modulo $\mathrm{Dec}(M/k)$ by a quaternion algebra. Therefore, if the triquadratic extension M/k is such that $\mathrm{Dec}(M/k)$ does not coincide with the subgroup of ${}_2\mathrm{Br}(k)$ split by M (see [12, §5] for examples of such extensions), we may find a quaternion k-algebra C split by M such that $[C] \notin \mathrm{Dec}(M/k)$. The algebra C obviously satisfies condition (b) of Lemma 5.12 (with A_2 and A_3 split), so for any involution ρ on $M_6(Q)$ satisfying the properties (i)–(iv) of Corollary 5.13 we may find a quadratic extension of F over which Q is split and ρ is hyperbolic. From (a') ⇒ (b') in Proposition 5.6, it follows that there exists a decomposition group U' of ρ such that $\mathcal{H}_{U'} = 0$. Yet, because $[C] \notin \mathrm{Dec}(M/k)$, the decomposition group U of ρ satisfies $\mathcal{H}_U \neq 0$ by Theorem 5.9.

6. Application to degree 8 algebras with involution

The Arason invariant in degree 8 was studied in [35] for orthogonal involutions with trivial discriminant and trivial Clifford algebra. In this section, we extend it to algebras of degree 8 and index 2, when the involution has trivial discriminant and the two components of the Clifford algebra also both have index 2. First, we prove an analogue of Theorem 3.2 on additive decompositions, for degree 8 algebras with orthogonal involution of trivial discriminant.

6.1. Additive decompositions in degree 8.

Let (A, σ) be a degree 8 algebra with orthogonal involution of trivial discriminant. We let $(C^+(A,\sigma), \sigma^+)$ and $(C^-(A,\sigma), \sigma^-)$ denote the two components of the Clifford algebra of (A,σ), endowed with the involutions induced by the canonical involution of the Clifford algebra. Both algebras have degree 8, both involutions have trivial discriminant, and by triality [26, (42.3)],

$$C(C^+(A,\sigma),\sigma^+) \simeq (C^-(A,\sigma),\sigma^-) \times (A,\sigma) \tag{12}$$

and

$$C(C^-(A,\sigma),\sigma^-) \simeq (A,\sigma) \times (C^+(A,\sigma),\sigma^+). \tag{13}$$

Assume (A,σ) decomposes as a sum $(A,\sigma) \in (A_1,\sigma_1) \boxplus (A_2,\sigma_2)$ of two degree 4 algebras with orthogonal involution of trivial discriminant. Each summand is a tensor product of two quaternion algebras with canonical involution, and we get

$$(A,\sigma) \in \big((Q_1,\bar{\ }) \otimes (Q_2,\bar{\ })\big) \boxplus \big((Q_3,\bar{\ }) \otimes (Q_4,\bar{\ })\big), \tag{14}$$

for some quaternion algebras Q_1, Q_2, Q_3 and Q_4 such that A is Brauer-equivalent to $Q_1 \otimes Q_2$ and $Q_3 \otimes Q_4$. By [36, Prop. 6.6], the two components of the Clifford algebra of (A,σ) then admit similar decompositions, namely, up to permutation of the two components, we have:

$$\big(C^+(A,\sigma),\sigma^+\big) \in \big((Q_1,\bar{\ }) \otimes (Q_3,\bar{\ })\big) \boxplus \big((Q_2,\bar{\ }) \otimes (Q_4,\bar{\ })\big), \tag{15}$$

and

$$\big(C^-(A,\sigma),\sigma^-\big) \in \big((Q_1,\bar{\ }) \otimes (Q_4,\bar{\ })\big) \boxplus \big((Q_2,\bar{\ }) \otimes (Q_3,\bar{\ })\big). \tag{16}$$

Mimicking the construction in §3, we associate to every decomposition of (A,σ) as above the subgroup W of the Brauer group of F generated by any three elements among the $[Q_i]$ for $1 \le i \le 4$. We call W a decomposition group of (A,σ). It consists of at most 8 elements, and can be described explicitly by

$$W = \{0, [A], [C^+(A,\sigma)], [C^-(A,\sigma)], [Q_1], [Q_2], [Q_3], [Q_4]\}.$$

In view of their additive decompositions, W also is a decomposition group of the two components $(C^+(A,\sigma),\sigma^+)$ and $(C^-(A,\sigma),\sigma^-)$ of the Clifford algebra. Note that, in contrast with the decomposition groups of algebras of degree 12 in Definition 3.6, the group W may contain three Brauer classes of index 4 instead of at most one. Nevertheless, it has similar properties; for instance, we prove:

Proposition 6.1. *Let (A,σ) be a central simple algebra of degree 8 with an orthogonal involution of trivial discriminant.*

(i) *Suppose (A,σ) has an additive decomposition as in (14), with decomposition group W. For any extension K/F which splits W, the algebra with involution (A_K,σ_K) is split and hyperbolic.*

(ii) *The converse holds for quadratic extensions: if (A,σ) is split and hyperbolic over a quadratic extension K of F, then (A,σ) has an additive decomposition with decomposition group split by K.*

Proof. (i) If a field K splits W, then it splits A, and moreover each summand in (14) is split and hyperbolic over K, therefore σ_K is hyperbolic.
(ii) To prove the converse, suppose $K = F(d)$ with $d^2 = \delta \in F^\times$, and assume A_K is split and σ_K is hyperbolic, hence $\operatorname{ind} A \le 2$. If A is split, we have as in

the proof of Proposition 5.6 $(A,\sigma) \simeq \mathrm{Ad}_\varphi$ with φ an 8-dimensional quadratic form multiple of $\langle 1,-\delta\rangle$. We may then find quaternion F-algebras Q_1, Q_2 split by K and scalars α_1, $\alpha_2 \in F^\times$ such that $\varphi \simeq \langle\alpha_1\rangle n_{Q_1} \perp \langle\alpha_2\rangle n_{Q_2}$. As in Example 3.4, we obtain a decomposition

$$(A,\sigma) \in \big((Q_1,\bar{\ }) \otimes (Q_1,\bar{\ })\big) \boxplus \big((Q_2,\bar{\ }) \otimes (Q_2,\bar{\ })\big).$$

The corresponding decomposition group is $\{0,[Q_1],[Q_2],[Q_1]+[Q_2]\}$; it is split by K.

If $\operatorname{ind} A = 2$, let Q be the quaternion division algebra Brauer-equivalent to A. Since K splits A, we may, again as in the proof of Proposition 5.6, identify K with a subfield of Q and find a skew-hermitian form h of the form $\langle d\rangle\langle 1,\alpha,\beta,\gamma\rangle$ (with α, β, $\gamma \in F^\times$) such that $(A,\sigma) \simeq \mathrm{Ad}_h$. Then

$$(A,\sigma) \in \mathrm{Ad}_{\langle d\rangle\langle 1,\alpha\rangle} \boxplus \mathrm{Ad}_{\langle d\rangle\langle \beta,\gamma\rangle}$$

is a decomposition in which each of the summands becomes hyperbolic over K. The corresponding decomposition group is therefore split by K. □

There exist quadratic forms φ of dimension 8 with trivial discriminant and Clifford algebra of index 4 that do not decompose into an orthogonal sum of two 4-dimensional quadratic forms of trivial discriminant, see [21, Cor. 16.8] or [20, Cor 6.2]. For such a form, neither Ad_φ nor the components of its Clifford algebra have additive decompositions as in (14). The next proposition shows, by contrast, that such a decomposition always exist if at least two among the algebras A, $C^+(A,\sigma)$ and $C^-(A,\sigma)$ have index ≤ 2.

Proposition 6.2. *Let (A,σ) be a central simple algebra of degree 8 with orthogonal involution of trivial discriminant. We assume at least two among the algebras A, $C^+(A,\sigma)$ and $C^-(A,\sigma)$ have index ≤ 2. Then all three algebras with involution (A,σ), $(C^+(A,\sigma),\sigma^+)$ and $(C^-(A,\sigma),\sigma^-)$ admit an additive decomposition as a sum of two degree 4 algebras with orthogonal involution of trivial discriminant as in (14).*

Proof. Assume two among $\operatorname{ind} A$, $\operatorname{ind} C^+(A,\sigma)$, $\operatorname{ind} C^-(A,\sigma)$ are 1 or 2. By triality, see (12) to (16) above, it is enough to prove that one of the three algebras with involution, say (A,σ) has an additive decomposition. Since A, $C^+(A,\sigma)$, $C^-(A,\sigma)$ are interchanged by triality, we may also assume $\operatorname{ind} A \leq 2$. If A is split, so $(A,\sigma) \simeq \mathrm{Ad}_\varphi$ for some 8-dimensional quadratic form φ with trivial discriminant and Clifford algebra of index at most 2, then (a) holds by a result of Knebusch [25, Ex. 9.12], which shows that φ is the product of a 2-dimensional quadratic form and a 4-dimensional quadratic form.

For the rest of the proof, assume $(A,\sigma) \simeq \mathrm{Ad}_h$ for some skew-hermitian form h over a quaternion division algebra $(Q,\bar{\ })$. Let $q \in Q$ be a nonzero quaternion represented by h, and let $h \simeq \langle q\rangle \perp h'$ for some skew-hermitian form h' of absolute rank 6. As we saw in the proof of Theorem 3.2, over the quadratic extension $K = F(q)$ the algebra Q splits and the form $\langle q\rangle$ becomes hyperbolic, hence h_K and h'_K are Witt-equivalent. In particular, it follows that $e_2((\mathrm{ad}_{h'})_K) = e_2((\mathrm{ad}_h)_K)$ has index at most 2. But $(\mathrm{ad}_{h'})_K = \mathrm{ad}_\psi$ for some

Albert form ψ over K, so ψ is isotropic. It follows by [34, Prop., p. 382] that h' represents some scalar multiple of q; thus $h \simeq \langle q\rangle\langle 1,-\lambda\rangle \perp h''$ for some $\lambda \in F^\times$ and some skew-hermitian form h'' of absolute rank 4. The discriminant of h'' must be trivial because h and $\langle q\rangle\langle 1,-\lambda\rangle$ have trivial discriminant, and we thus have the required decomposition for (A,σ). □

Remark 6.3. It follows that all trialitarian triples such that at least two of the algebras have index ≤ 2 have a description as in (14) to (16).

6.2. An extension of the Arason invariant in degree 8 and index 2. Throughout this section, (A,σ) is a central simple F-algebra of degree 8 and trivial discriminant. It is known that (A,σ) is a tensor product of quaternion algebras with involution if and only if $e_2(\sigma)=0$, see [26, (42.11)]. In this case, the Arason invariant $e_3(\sigma) \in M^3_A(F)$ is defined when A has index at most 4 (see §2.5) , and represented by an element of order 2 in $H^3(F)$, see [35]. Here, we extend the definition of the e_3 invariant under the following hypothesis:

$$\operatorname{ind} A = \operatorname{ind} C^+(A,\sigma) = \operatorname{ind} C^-(A,\sigma) = 2. \tag{17}$$

By Proposition 6.2, this condition implies that (A,σ) decomposes into a sum of two central simple algebras of degree 4 with involutions of trivial discriminant. Moreover, the associated decomposition group W is a quaternionic subgroup of $\operatorname{Br}(F)$. Let Q, Q^+, Q^- be the quaternion division algebras over F that are Brauer-equivalent to A, $C^+(A,\sigma)$, and $C^-(A,\sigma)$ respectively. From the Clifford algebra relations [26, (9.12)], we know $[Q^+]+[Q^-]=[Q]$. Therefore, the following is a subgroup of the Brauer group:

$$V = \{0,[Q],[Q^+],[Q^-]\} \subset \operatorname{Br}(F).$$

Condition (17) implies that $|V|=4$. Moreover, V also is a subgroup of every decomposition group of (A,σ).
To (A,σ), we may associate algebras of degree 12 with orthogonal involution with trivial discriminant and trivial Clifford invariant by considering any involution ρ of $M_6(Q)$ such that

$$(M_6(Q),\rho) \in (A,\sigma) \boxplus \big((Q^+,\bar{\ }) \otimes (Q^-,\bar{\ })\big). \tag{18}$$

Since the two components of the Clifford algebra of σ are Brauer-equivalent to Q^+ and Q^-, the involution ρ has trivial Clifford invariant. Therefore, we may consider its Arason invariant $e_3(\rho) \in M^3_Q(F)$. The following lemma compares the Arason invariant of two such involutions:

Lemma 6.4. *Let ρ and ρ' be two involutions of $M_6(Q)$ satisfying (18). There exists $\lambda \in F^\times$ such that*

$$e_3(\rho) - e_3(\rho') = (\lambda)\cdot[Q^+] = (\lambda)\cdot[Q^-] \in M^3_Q(F).$$

Moreover, $f_3(\rho) = f_3(\rho')$.

Proof. By definition of the direct orthogonal sum for algebras with involution, we may pick skew-hermitian forms h_1 and h_2 over $(Q,\bar{\ })$ such that $\sigma = \operatorname{ad}_{h_1}$,

$^{-}\otimes{}^{-} \simeq \mathrm{ad}_{h_2}$ and $\rho = \mathrm{ad}_{h_1\perp h_2}$. Moreover, there exists $\lambda \in F^\times$ such that $\rho' = \mathrm{ad}_{h_1\perp\langle\lambda\rangle h_2}$. Therefore, we have (see § 2.4 and 2.5):

$$e_3(\rho) - e_3(\rho') = e_3(\langle 1, -\lambda\rangle h_2) = (\lambda)\cdot[Q^+] \mod F^\times\cdot[Q],$$

by Proposition 2.6, since $e_2(h_2) = [Q^+] = [Q^-] \mod [Q]$. Moreover, if c and $c' \in H^3(F)$ are representatives of $e_3(\rho)$ and $e_3(\rho')$ respectively, then

$$c' - c \in (\lambda)\cdot[Q^+] + F^\times\cdot[Q].$$

So $2c = 2c'$, and this finishes the proof.

□

Let $F^\times \cdot V \subset H^3(F)$ be the subgroup consisting of the products $\lambda\cdot v$ with $\lambda \in F^\times$ and $v \in V = \{0, [Q], [Q^+], [Q^-]\}$. This subgroup contains $F^\times\cdot[Q]$, so we may consider the canonical map $\overline{}^V\colon M^3_Q(F) \to H^3(F)/F^\times\cdot V$. The previous lemma shows that the image $\overline{e_3(\rho)}^V$ of the Arason invariant of ρ does not depend on the choice of an involution ρ satisfying (18). This leads to the following:

Definition 6.5. With the notation above, we set

$$e_3(\sigma) = \overline{e_3(\rho)}^V \in H^3(F)/F^\times\cdot V \qquad \text{and} \qquad f_3(\sigma) = f_3(\rho) \in F^\times\cdot[Q] \subset H^3(F)$$

where ρ is any involution satisfying

$$(M_6(Q),\rho) \in (A,\sigma) \boxplus \big((Q^+,{}^-)\otimes(Q^-,{}^-)\big).$$

This definition functorially extends the definition of the Arason invariant. Indeed, if K is any extension of F that splits Q^+ or Q^- (or both), then the scalar extension map $\mathrm{Br}(F) \to \mathrm{Br}(K)$ carries V to $\{0, [Q]\}$, and any involution ρ as in (18) becomes Witt-equivalent to σ over K. Therefore, scalar extension carries $e_3(\sigma) \in H^3(F)/F^\times\cdot V$ defined above to $e_3(\sigma_K) \in M^3_Q(K)$ as defined in §2.5.

Example 6.6. Consider a central simple algebra $(M_6(Q),\rho)$ of degree 12 and index 2 with an orthogonal involution of trivial discriminant and trivial Clifford invariant. By Theorem 3.2 $(M_6(Q),\rho)$ admits additive decompositions

$$(M_6(Q),\rho) \in \boxplus_{i=1}^{3}\big((Q_i,{}^-)\otimes(H_i,{}^-)\big) \qquad \text{with} \sum_{i=1}^{3}[H_i] = 0,$$

so it contains symmetric idempotents e_1, e_2, e_3 such that

$$\big(e_iM_6(Q)e_i, \rho|_{e_iM_6(Q)e_i}\big) \simeq (Q_i,{}^-)\otimes(H_i,{}^-).$$

Consider the restriction of ρ to $(e_1+e_2)M_6(Q)(e_1+e_2)$; we thus obtain an algebra with involution $(M_4(Q),\sigma)$ such that

$$(M_6(Q),\rho) \in (M_4(Q),\sigma) \boxplus \big((Q_3,{}^-)\otimes(H_3,{}^-)\big)$$
$$\text{and} \quad (M_4(Q),\sigma) \in \boxplus_{i=1}^{2}\big((Q_i,{}^-)\otimes(H_i,{}^-)\big).$$

It is clear that the discriminant of σ is trivial. Since $e_2(\rho) = 0$, we have

$$e_2(\sigma) = e_2\big((Q_3, \bar{\ }) \otimes (H_3, \bar{\ })\big) = \{[Q_3], [H_3]\}.$$

Therefore, Condition (17) holds for $(M_4(Q), \sigma)$ if Q_3 and H_3 are not split. In that case, we have $V = \{0, [Q], [Q_3], [H_3]\}$ and, by definition,

$$e_3(\sigma) = \overline{e_3(\rho)}^V \in H^3(F)/F^\times \cdot V \quad \text{and} \quad f_3(\sigma) = f_3(\rho) \in F^\times \cdot [Q] \subset H^3(F).$$

The condition that Q_3 and H_3 are not split holds in particular when the decomposition group U generated by $[Q_1]$, $[Q_2]$, $[Q_3]$ has order 8.

Example 6.7. Take for $(M_6(Q), \rho)$ the algebra $\mathrm{Ad}_{\langle 1,-t\rangle} \otimes (\lambda^2 E, \gamma)_{F(t)}$ of Corollary 2.19, with E a division algebra of degree and exponent 4. (Note that $\lambda^2 E$ is Brauer-equivalent to $E \otimes E$, hence it has index 2.) Since $f_3(\rho) \neq 0$, every decomposition group of ρ has order 8; indeed, quaternionic subgroups $U \subset \mathrm{Br}(F)$ of order dividing 4 have $\mathcal{H}_U = 0$ by Theorem 3.12, hence trivial f_3 by Proposition 5.3. The construction in the previous example yields an algebra with involution $(M_4(Q), \sigma)$ of degree 8 satisfying Condition (17), with $f_3(\sigma) = t \cdot [Q] \neq 0$.

Example 6.8. Also, we may take for $(M_6(Q), \rho)$ the algebra with involution of Corollary 5.13, and obtain an algebra with involution $(M_4(Q), \sigma)$ of degree 8 satisfying Condition (17) such that (with the notation 5.8) $e_3(\sigma) = \overline{\xi}^V \in H^3(F)/F^\times V$. Since $\xi \notin F^\times \cdot U$, we have $e_3(\sigma) \neq 0$. Yet, we have $f_3(\sigma) = f_3(\rho) = 0$ by Corollary 5.13. Moreover, there is no quadratic extension K of F such that Q_K is split and σ_K is hyperbolic. Indeed, over such a field, $(M_6(Q), \rho)_K$ would be Witt-equivalent to an algebra of degree 4, hence it would be hyperbolic because $e_2(\rho) = 0$. Corollary 5.13 shows that such quadratic extensions K do not exist.

The next proposition shows that the e_3 invariant detects isotropy, for any central simple algebra with involution (A, σ) satisfying Condition (17). As in §6.1, we let σ^+ and σ^- denote the canonical involutions on $C^+(A, \sigma)$ and $C^-(A, \sigma)$.

Proposition 6.9. *Let (A, σ) be a central simple F-algebra of degree* 8 *with orthogonal involution of trivial discriminant satisfying* (17)*. With the notation above, we have $e_3(\sigma) = e_3(\sigma^+) = e_3(\sigma^-)$ and $f_3(\sigma) = f_3(\sigma^+) = f_3(\sigma^-)$. Moreover, the following conditions are equivalent:*

(a) $e_3(\sigma) = 0$;
(b) σ *is isotropic;*
(c) (A, σ) *is Witt-equivalent to* $(Q^+, \bar{\ }) \otimes (Q^-, \bar{\ })$.

Proof. As in §3.3, let F_V denote the function field of the product of the Severi–Brauer varieties associated to the elements of V. Extending scalars to F_V, we split Q and $e_2(\sigma)$, hence there is a 3-fold Pfister form π over F_V such that

$$\sigma_{F_V} \simeq \mathrm{ad}_\pi .$$

For Pfister forms, we have $\mathrm{ad}_\pi \simeq \mathrm{ad}_\pi^+ \simeq \mathrm{ad}_\pi^-$ (see [26, (42.2)]), hence $\sigma^+_{F_V} \simeq \sigma^-_{F_V} \simeq \sigma_{F_V}$, and therefore

$$e_3(\sigma)_{F_V} = e_3(\sigma^+)_{F_V} = e_3(\sigma^-)_{F_V} = e_3(\pi).$$

Since V is generated by the Brauer classes of two quaternion algebras, it follows from Theorem 3.12 that $F^\times \cdot V$ is the kernel of the scalar extension map $H^3(F) \to H^3(F_V)$, hence the preceding equations yield $e_3(\sigma) = e_3(\sigma^+) = e_3(\sigma^-)$. We then have $f_3(\sigma) = f_3(\sigma^+) = f_3(\sigma^-)$, since $f_3(\sigma)$ (resp. $f_3(\sigma^+)$, resp. $f_3(\sigma^-)$) is 2 times any representative of $e_3(\sigma)$ (resp. $e_3(\sigma^+)$, resp. $e_3(\sigma^-)$) in $H^3(F)$.
To complete the proof, we show that (a), (b), and (c) are equivalent. Clearly, (c) implies (b). The converse follows easily from [26, (15.12)] if A has index 2, and [26, (16.5)] if A is split. Moreover, in view of the definition of $e_3(\sigma)$, the equivalence between (a) and (c) follows from Proposition 4.5. □

As in §4, we may relate the e_3 invariant to the homology of the Peyre complex of any decomposition group, as follows:

Proposition 6.10. *Let (A,σ) be a central simple algebra of degree 8 with an orthogonal involution of trivial discriminant satisfying (17), and let W be a decomposition group of (A,σ). The image $\overline{e_3(\sigma)}^W$ of $e_3(\sigma) \in H^3(F)/F^\times \cdot V$ in $H^3(F)/F^\times \cdot W$ generates $\mathcal{H}_W$, and $f_3(\sigma) = f_3(W)$.*

Proof. As above, let Q be the quaternion division algebra Brauer-equivalent to A, so we may identify A with $M_4(Q)$. Let ρ be an involution on $M_6(Q)$ such that

$$(M_6(Q),\rho) \in (A,\sigma) \boxplus \big((Q^+,\bar{\ }) \otimes (Q^-,\bar{\ })\big).$$

By definition, we have $e_3(\sigma) = \overline{e_3(\rho)}^V$ and $f_3(\sigma) = f_3(\rho)$. Now, consider a decomposition of (A,σ) with decomposition group W (which necessarily contains V):

$$(A,\sigma) \in \big((C_1^+,\bar{\ }) \otimes (C_1^-,\bar{\ })\big) \boxplus \big((C_2^+,\bar{\ }) \otimes (C_2^-,\bar{\ })\big).$$

We have

$$(M_6(Q),\rho) \in \big((C_1^+,\bar{\ })\otimes(C_1^-,\bar{\ })\big) \boxplus \big((C_2^+,\bar{\ })\otimes(C_2^-,\bar{\ })\big) \boxplus \big((Q^+,\bar{\ })\otimes(Q^-,\bar{\ })\big),$$

which is a decomposition of $(M_6(Q),\rho)$ with decomposition group W. Therefore, Theorem 4.6 shows that $\overline{e_3(\rho)}^W$ generates $\mathcal{H}_W$ and $f_3(\rho) = f_3(W)$. The proposition follows because $f_3(\sigma) = f_3(\rho)$ and $\overline{e_3(\sigma)}^W = \overline{e_3(\rho)}^W$ since $V \subset W$. □

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Finite u Invariant and Bounds on Cohomology Symbol Lengths

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Abstract. In this note we answer a question of Parimala's, showing that fields with finite u invariant have bounds on the symbol lengths in their μ_2 cohomology in all degrees.

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Introduction

At the AIM Workshop on Period/Index problems in January 2011, Prof. Parimala asked whether fields of finite u invariant necessarily had bounded symbol length in their μ_2 cohomology. Parimala presented a proof that this was true in degrees one through three, using generic splitting constructions. In the subsequent breakout session on this problem, the first ideas of a proof were presented by myself, and I greatly benefited from the constructive comments by Prof. Parimala and Prof. Merkurjev. The note below is my write up of the argument. Similar results were simultaneously found by Daniel Krashen [K].

Much of the notation and definitions below are standard in quadratic form books (e.g. [Sc]), and we also assume familiarity with group and Galois cohomology and the Hochschild Serre spectral sequence (e.g. [NSW]). As notation which is perhaps not standard, for any field F let G_F be its absolute Galois group. That is, if F_s is the separable closure of F, then G_F is the Galois group of F_s/F. If R is a commutative domain, we let $q(R)$ be its field of fractions. Let me also add an extended discussion about Galois extensions which is also less standard ([Sa] p. 253). We list some conventions we use when talking about

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Galois extensions of rings. When we say S/R is a G Galois extension of rings, we include in this a fixed action of G on S. Similarly, if S/R and S'/R are G and G' Galois extensions respectively, and we write $S \subset S'$, we assume the inclusion preserves the Galois action. That is, there is a surjection $\rho : G' \to G$ such that the G' action on S' restricts to the G action on S via ρ.

Let S/R be an H Galois extension of commutative rings, where H is a subgroup of the finite group G. Then $\mathrm{Hom}_H(\mathbb{Z}[G], S)$ can be given the structure of an R algebra by pointwise operations in S, and has the coinduced G action. Together this defines $\mathrm{Ind}_H^G(S/R)$ which is a G Galois extension of R. Note that $\mathrm{Ind}_H^G(S/R)$ is of the form $S \oplus \ldots \oplus S$ as an R algebra. We will most often use this in the case $R = F$ is a field. Recall that any G Galois L/F (L is not necessarily a field) has the form $\mathrm{Ind}_H^G(K/F)$ where K/F is an H Galois extension of fields. In this case, suppose $K' \supset K \supset F$ is a tower of fields where K'/F is H' Galois and induces the surjection $\rho : H' \to H$ (we say $K' \supset K \supset F$ is Galois tower). Then there is a Galois tower of rings $\mathrm{Ind}_{H'}^{G'}(K') \supset L = \mathrm{Ind}_H^G(K) \supset F$ if and only if ρ extends to a surjection $\rho : G' \to G$ (as stated above, ρ is implicit in the inclusion $\mathrm{Ind}_H^G(K) \subset \mathrm{Ind}_{H'}^{G'}(K')$. Conversely, if $\mathrm{Ind}_{H'}^{G'}(K') \supset \mathrm{Ind}_H^G(K) \supset F$ is a Galois tower where K, K' are fields then there is an induced Galois tower $K' \supset K \supset F$. Finally, if S is a ring on which the group B acts we denote by S^B the subring of B fixed elements.

Underlying this work is the remarkable result of [OVV], based on the underlyng work of Voevodsky, that shows the maps $I_K^i/I_K^{i+1} \to H^i(K, \mu_2)$ are isomorphisms for all i. Since the paper [OVV] assumes all fields have characteristic 0 we also make this assumption, but really this work applies to any fields (characteristic not 2) where Milnor's Conjecture holds. We set μ_2 to be the group $\{1, -1\}$ of 2 roots of 1. Given a field F, then $H^i(F, \mu_2) = H^i(G_F, \mu_2)$ is the Galois cohomology group. If $a \in F^*$ then we abuse notation and write $a \in H^1(F, \mu_2)$ to be the character $\mathrm{Hom}(G_F, \mu_2) = H^1(F, \mu_2)$ determined by the field extension $F(a^{1/2})/F$. The cup product $a_1 \cup \ldots \cup a_i \in H^i(F, \mu_2)$ is called a SYMBOL and the symbol length of an element $\alpha \in H^i(F, \mu_2)$ is the least i such that α is a sum of i symbols. The canonical map $I_K^i/I_K^{i+1} \to H^i(K, \mu_2)$ of Milnor's Conjecture is determined by mapping the Pfister form (e.g., [Sc] p. 72) $\langle\langle a_1, \ldots, a_i \rangle\rangle \to -a_1 \cup \ldots \cup -a_i$ and this was shown in [OVV] to be an isomorphism. In particular, every element of $H^i(F, \mu_2)$ is a sum of symbols and so has a symbol length. Recall that the u invariant of a field is the integer or ∞, $u(F)$, such that any quadratic form over F of rank bigger than $u(F)$ is isotropic.

Suppose K/F is H Galois and $\beta \in H^i(H, \mu_2)$. Then there is a natural image of β in $H^i(F, \mu_2) = H^i(G_F, \mu_2)$ via inflation. In detail, a choice of embedding $K \subset F_s$ determines a surjection $\rho : G_F \to H$ and we inflate via ρ. This is well defined because a different choice of embedding changes ρ via conjugation by an element of H, and cohomology is invariant under conjugation. More generally, if $L = \mathrm{Ind}_H^G(K/F)$, K is a field, and $\beta \in H^i(G, \mu_2)$ then β has a natural image in Galois cohomology by first forming the restriction $\beta_H \in H^i(H, \mu_2)$ and then taking the image of β_H in $H^i(F, \mu_2) = H^i(G_F, \mu_2)$ via inflation as above. Of course, the subgroup $H \subset G$ is only determined by L up to conjugation, but again cohomology is invariant under conjugation so the map $H^i(G, \mu_2) \to H^i(F, \mu_2)$ is well defined. Note that this map commutes with inflation. That is, If $G' \to G$ is defined by a tower $F \subset L \subset L'$ of Galois extensions, and $L' = \mathrm{Ind}_{H'}^{G'}(K'/F)$ then we can choose H' such that we have the diagram:

$$\begin{array}{ccc} H' & \subset & G' \\ \downarrow & & \downarrow \\ H & \subset & G \end{array}$$

and the restriction of β to H inflated to H' is the same as the inflation of β to G' restricted to H'.

The result.

We say that a field F is u-bounded if there is an integer function $N(n)$, depending only on $u(F)$, such that $u(L) \leq N(n)$ for all extensions L/F of degree dividing n. Note that this is equivalent (e.g., [Sc] p. 104) to just saying that $u(F) < \infty$ but we phrase it in this way to emphasize that we are considering properties closed under finite extension. We say that F has bounded symbol length in degree d if there is an integer $M_d(n)$, depending only on $u(F)$, such that every element in $H^d(L, \mu_2)$ is a sum of $M_d(n)$ symbols for every L/F finite of degree dividing n. The point of this note is to show:

Theorem 1. *Every field with finite u invariant has bounded symbol length in degree i for all i.*

We prove this by induction and we note that every field has bounded symbol length of degree 1. Also, since the premise of this result is preserved by finite extensions, we may assume we have shown for such F that $M_j(n)$ exists for all $j < d$ and n, and show $M_d(1)$ exists. That is, we show that every element of $H^d(F, \mu_2)$ has bounded symbol length.

We remark that a partial converse of Theorem 1 is true, as was pointed out to us by Hoffmann and Garibaldi. Suppose F is non-real and $H^i(F, \mu_2)$ has bounded symbol length for all i. By [Ka] Proposition 1, F has finite 2 cohomological dimension. By [OVV], there is a d such that in the Witt ring of F, $I^d = 0$. Moreover, in each I^r/I^{r+1}, every element is represented by a Pfister form of bounded rank. If $[q]$ is the class of a quadratic form, then q is Witt equivalent to a sum of Pfister forms of bounded rank. In particular, if q is anisotropic, it has bounded rank.

The idea of the proof of Theorem 1 is the following. Since $u(F)$ is finite, we can write down a generic quadratic form which specializes to all anisotropic quadratic forms. We would like to modify this generic form so that it is generic and lies in the Witt Ring fundamental ideal I^i and specializes to all Witt classes of forms in that ideal. That generic form, defined over some $F' \supset F$, maps to $H^i(F', \mu_2)$ and there it is a sum of some number of symbols. By specializing, all elements of $H^i(F, \mu_2)$ are the sum of that many or fewer symbols, and we would be done.

At the moment, there is no construction for such a generic form. The difficulty can be illustrated as follows. Suppose $\alpha = \sum_{j=1}^m a_{1,j} \cup \ldots \cup a_{i,j}$ is an element of $H^i(F, \mu_2)$. Let $L = F(a_{k,l}^{1/2} |$ all $k, l)$ and let A be the Galois group of L/F. Then our form for α defines an element of $H^i(A, \mu_2)$. If this element maps to 0 in $H^i(G_F, \mu_2) = H^i(F, \mu_2)$, then there must be a finite extension $L' \supset L$ with L'/F Galois with group B and so an induced surjection $B \to A$ such that α maps to 0 in $H^i(B, \mu_2)$. However, we see no way to, in general, bound the size of B and if the size of B is not bounded there can be no generic way to force a cohomology class of α's form to be 0 because the B that works for such a generic construction would then give you a bound. However, we can construct a generic zeroing of α for a fixed B. We will call this generic with the limitation B (formal definition to follow).

Let us note that the argument of this paper does construct a finite set of generic elements of I^d, when $u(F) < \infty$. We make this explicit in:

Corollary 2. *Suppose $u(F) < \infty$ and $d \geq 1$. Then there are finitely many field extensions $F_i \supset F$ and α_i in $I(F_i)^d$ such that any $\alpha \in I(F)^d$ is the specialization of one of the α_i.*

To begin, we mention the following way of thinking about writing a cohomology element as a sum of symbols.

Lemma 3. *An element of $\alpha \in H^i(F, \mu_2)$ is a sum of symbols if and only if α is the image of $H^i(A, \mu_2)$ where $A = \mathrm{Gal}(L/F)$ is an elementary abelian 2 group. Moreover, that all α have bounded symbol length is equivalent to bounding the size of such A.*

Proof. The basic equivalence is immediate from (e.g., [E] p. 33), which says the well-known fact that any element of such an $H^i(A, \mu_2)$ is the sum of i degree monomials of elements in $H^1(A, \mu_2) = \mathrm{Hom}(A, \mu_2)$. Moreover, if $\alpha = \sum_{j=1}^M (a_{j,1} \cup \ldots \cup a_{j,i})$ then A can be taken of order dividing 2^{im}. Conversely, for A of order 2^M, $H^i(A, \mu_2)$ has a basis consisting of $\binom{M}{i}$ elements. □

In what follows A will always be an elementary abelian 2 group.

Let C be our ground field so all rings and fields will be C algebras. We suppose $\alpha \in H^i(F, \mu_2)$ is the image of $\beta \in H^i(A, \mu_2)$ where $A = \mathrm{Gal}(L/F)$ is as above. Further, suppose S/R is a A Galois extension with $q(R) = F$, R affine over C, and $L = S \otimes_R F$ as A Galois extensions. We call S/R, A and β a PRESENTATION of α. Since we are usually not interested in the specific rings S/R in a presentation, we define S'/R', A, β' EQUIVALENT to S/R, A, β if and only if $A = A'$, $\beta = \beta'$, and there are nonzero $r \in R$ and $r' \in R'$ such that $R(1/r) = R'(1/r')$ and $S(1/r)/R(1/r)$ and $S'(1/r')/R'(1/r')$ are isomorphic as A Galois extensions of $R(1/r)$. Obviously equivalent presentations have the same induced cohomology element. In discussing presentations up to equivalence, the ring extension S/R can often be surpressed and we can just say the presentation β, L/F where L/F is A Galois.

Presentations will be important to us because they allow specializations of cohomology classes as follows. In fact, we will be defining specializations of presentations. Let β, S/R, and A be a presentation of α. Suppose $\phi : R \to R_1 \subset F_1$ for a ring and field $R_1, F_1 \supset C$ with $q(R_1) = F_1$. If we set $S_1 = S \otimes_\phi R_1$ then β, S_1/R_1, and A is the specialization with respect to ϕ. This specialized presentation defines an $\alpha_1 \in H^i(F_1, \mu_2)$ which we can call a specialization of α.

Note that we have defined the notion of presentation without assuming $L = S \otimes_R F$ is a field. In fact, suppose $L = \mathrm{Ind}_{A_1}^A(L_1/F)$. Then there is some $0 \neq r \in R$ such that $S(1/r) = \mathrm{Ind}_{A_1}^A(S_1/R(1/r))$ and we can define (up to equivalence) β_1, $S_1/R(1/r)$, A_1 to be a restriction of the original presentation, and this restriction presents the same cohomology element.

Next we must talk about presentations that represent 0 and their so called limitations. Note that this is a key idea in the argument that follows. We will frequently be talking about β and L/F where β maps to $0 \in H^i(F, \mu_2)$ and so perhaps seem not to be important. In fact, we are very interested in why β maps to 0 and more specifically we will be bounding the reason why β maps to 0. To be precise, let $\beta \in H^i(A, \mu_2)$ be as above and suppose L/F is an A Galois extension. We say β and L/F PRESENTS 0 if the image of β in $H^i(F, \mu_2)$ is 0. We say that β and L/F has LIMITATION $B \to A$ (sometimes we write only B) if and only if there is a Galois extension L'/F with Galois group B such that L'/F contains L/F and induces $B \to A$ where β maps to 0 in $H^i(B, \mu_2)$. Suppose $L = \mathrm{Ind}_{A_1}^A(L_1/F)$, β restricts to $\beta_1 \in H^i(A_1, \mu_2)$, and β has limitation B realized by a B Galois extension L'/F. Write $L' = \mathrm{Ind}_{B_1}^B(K')$ where K' is a field. Then $K' \supset K \supset F$ is a Galois tower inducing a surjection $B_1 \to A_1$. We have the following commutative diagram:

$$\begin{array}{ccc} H^i(A, \mu_2) & \longrightarrow & H^i(B, \mu_2) \\ \downarrow & & \downarrow \\ H^i(A_1, \mu_2) & \longrightarrow & H^i(B_1, \mu_2) \end{array}$$

from which it follows that β_1 has limitation B_1. In particular, if β has some limitation B then β presents 0.

If β and L/F presents 0 and L is a field then there is a L'/F Galois extension containing L/F and an associated surjection of Galois groups $B \to A$ such that β presents 0 with limitation B. More generally, if $L = \mathrm{Ind}_{A_1}^A(L_1/F)$ with L_1 a field there is a Galois L_1'/F containing L_1/F and associated surjection $B_1 \to A_1$ such that β maps to 0 in $H^i(B_1, \mu_2)$.

For symmetry, though we do not need this fact, we observe that there is a Galois $L' \supset L \supset F$ and associated surjection $B \to A$ such that β maps to 0 in $H^i(B, \mu_2)$ and so β has limitation B. Write $A = A_1 \oplus A'$ and so $H^1(A, \mu_2) = H^1(A_1, \mu_2) \oplus H^1(A', \mu_2)$. We view any $\alpha \in H^1(A', \mu_2)$ as an element of $H^1(A, \mu_2)$ by setting $\alpha(A_1) = 1$. Write $\beta = \beta_1 + \beta'$ where all the symbols in β_1 have entries from $H^1(A_1, \mu_2)$, all the symbols in β' have have at least one entry from $H^1(A', \mu_2)$. Thus each symbol in β' has a subsymbol $\alpha \cup \alpha'$ which maps to $0 \in H^2(F, \mu_2)$ (since one of the α's is trivial). It follows that we can form $1 \to M \to B' \to A$ where each symbol of β' splits in B' and M is a direct sum of $\mathbb{Z}/2\mathbb{Z}$'s, one for each symbol. Form $B = B_1 \oplus B'$ and $L' = \mathrm{Ind}_{B_1}^B(L_1'/F)$. Note that the order of B is bounded in terms of the order of B_1, i, and the order of A.

Frequently when we specialize as above there will be many ways to do it and this is important. Given a presentation S/R, and $\beta \in H^i(A, \mu_2)$, we say β densely specializes to a presentation β_1 of $\alpha_1 \in H^i(F_1, \mu_2)$ if the following holds. For any $0 \neq r \in R$, there is a $\phi : R \to F_1$ such that $\phi(r) \neq 0$ and ϕ causes β to specialize to β_1 inducing the same presentation β_1. If R and R' are affine C algebras with $q(R) = F = q(R')$, then $R(1/r) = R'(1/r')$ for some $0 \neq r \in R$ and $0 \neq r' \in R'$ (e.g. [Sw] p. 152). Thus when β densely specializes to β_1 this is well defined up to equivalence.

Lemma 4. *Suppose $\beta \in H^i(A, \mu_2)$ and L/F, A are a presentation of 0 which densely specializes to β_1, L_1/F_1, A. Then if β has limitation $B \to A$ so does β_1 and in particular β_1 presents 0.*

Proof. Assume $L' \supset L \supset F$ is $B \to A$ Galois and β maps to 0 in $H^i(B, \mu_2)$. If S/R is A Galois and $q(R) = F$, there is a $0 \neq r \in R$ and a $S'/R(1/r)$ which is B Galois, contains $S(1/r)$, with $S' \otimes_{R(1/r)} F = L'$. Choose $\phi : R(1/r) \to F_1$ realizing the specialization and set $\tilde{L}_1' = S' \otimes_\phi F'$. Then $L_1' \supset L_1 \supset F_1$ is $B \to A$ Galois. □

Let $\alpha \in H^i(F, \mu_2)$ have presentation β, $\mathrm{Gal}(L/F)$. Assume $B \to A$ is a surjection of finite groups and β maps to 0 in $H^i(B, \mu_2)$. The following result is routine and we only include the proof for ease of the reader.

Proposition 5. *There is a field extension $F_B \supset F$ with the following properties (where we set $L_B = L \otimes_F F_B$).*

a) The extension L_B/F_B and β presents 0 with limitation B.

b) Suppose β, L/F densely specializes to β_1 and $A_1 = \mathrm{Gal}(K_1/F_1)$. If β_1 has limitation B, then the presentation β, L_B/F_B densely specializes to β_1.

Proof. All this really means is that we are constructing generically the B Galois extension extending L/F. To achieve this let V be the faithful B module $F[B]$. Let B act on the field of fractions $L(V)$ as follows. B acts on V as usual, and B acts on L via $B \to A$. Set $F_B = L(V)^B$. It is clear that $L(V)/F_B$ is B Galois. If N is the kernel of $B \to A$, then $L(V)^N = L_B = L \otimes_F F_B$ so $L(V) \supset L_B \supset F_B$ induces $B \to A$. This proves a).

Let S/R be A Galois such that $q(R) = F$. Then we can choose $t \in S[V]^B$ with the property that if $S_B = S[V](1/t)$ then S_B/R_B is B Galois with $R_B = (S_B)^B$. Suppose $0 \neq s \in S[V]^B$. It suffices to show that there is a $\phi : R_B \to F_1$ with $\phi(s) \neq 0$. By assumption there is a $\phi : R \to F_1$ specializing β to β_1. Set $L_1 = S \otimes_\phi F_1$ which is A Galois over F_1 and has the form $\mathrm{Ind}_{A_1}^A(K_1)$. It follows that ϕ extends to an A morphism $\phi : S \to L_1$. Since β_1 has limitation B, there is a B Galois extension $L_1'/F \supset L_1/F$ inducing $B \to A$. Since V has basis $\{x_g | g \in B\}$ with the obvious action, algebraic independence of Galois group elements (e.g. [BAI] p. 295) shows that we can define $\phi(x_g) = g(a) \in L_1'$ for some a such that $\phi(st) \neq 0$. Then ϕ extends to a B morphism and restricts to the needed ϕ on R_B. □

If β and $B \to A$ are as in Proposition 5, we say that F_B is the generic splitting field of β with limitation B.

Let's outline our argument a bit. We start with a generic quadratic form $\gamma = \sum_{i=1}^N a_i x_i^2$ (N is even) meaning that the ground field has the form $F_1 = C(a_1, \ldots, a_N)$ and the a_i are a transcendence base. Note that for any field $F \supset C$, this specializes to all Witt classes in the fundamental ideal I as long as $u(F) \leq N$. We want to write down a generic element in I^n with a fixed so called history as follows. Let F_2/F_1 be the extension defined by taking the square root of the determinant of γ. The extension γ_2 of γ to the Witt ring $W(F_2)$ is in $I_{F_2}^2$ and so defines an element $\alpha_2 \in H^2(F_2, \mu_2)$. We take F_3/F_2 to be a generic splitting field of α_2 and so the extension, $\gamma_3 \in W(F_3)$ is in $I_{F_3}^3$. So far there has been no limitations. However, if $\alpha_3 \in H^3(F_3, \mu_2)$ is the image of γ_3 then we can write γ_3 as a sum of Pfister forms and thereby write α_3 as a sum of symbols. Given that, we can choose a presentation β_3, $A_3 = \mathrm{Gal}(L_3/F_3)$ of α_3. For any $B_3 \to A_3$ that splits β_3, we form the generic splitting field of β_3 with limitation B_3 and call that F_4. We proceed by induction until the extension, $\gamma_n \in I^n_{F_n}$ is defined. The choice of presentations β_i and limitations B_i is the HISTORY of this construction.

Now given a u bounded field K every element $\alpha' \in H^i(K, \mu_2)$ is the image of a quadratic form γ' which is in I^i_K. We show that we can bound the order and hence number of the limitations which enforce this property of γ''s and hence write α' as the specialization of one of finitely many of the generic contructions of α_n (as above) as we vary the histories among finitely many choices of the B_i. This proves the result.

To make this argument more formal, if β, L/K is a presentation α then the ORDER of β is the order of the group $A = \mathrm{Gal}(L/K)$. Obviously the order of a presentation cannot increase under specialization. We say that a field K is LIMITATION BOUNDED in degree i if and only if for all d, all field extensions K'/K of degree dividing d, and all degree i presentations of zero, β over K' of order less than or equal to N, there is a $L(N, d)$ such that β has a limitation B of order less than or equal to $L(N, d)$. The above argument is an outline of the proof of:

THEOREM 6. *Suppose K is limitation bounded in all degrees $j < d$ and is also u bounded. Then all finite extensions of K have bounded symbol length in degree d and the bound is a function of the degree and the assumed u and limitation bounds.*

Proof. This is perhaps already clear except for the fact we are choosing presentations of zero. For simplicity we only treat K itself, the extension to the K'/K being clear. We prove by induction that every $\gamma' \in I^d_K$ is the specialization of some $\gamma_i \in I^d_{F_d}$ as above with only finitely many choices of histories. By induction there are finitely many histories such that γ' is the specialization some γ_{d-1}. For this γ_{d-1} there is a presentation β_{d-1} and thus a degree $d-1$ presentation of γ' we call β'_{d-1}. Since $\gamma' \in I^d_K$ it follows that β'_{d-1} is a presentation of zero and so there are only finitely many further choices of limitations B_d. We are done by Proposition 5. □

Given Theorem 6, we need to prove these u bounded fields are limitation bounded. This is an involved argument using the Hochschild–Serre spectral sequence. Note that we feel that the limitation bound we obtain is far from optimal. For this reason we will not be particularly explicit about the bound, as in the definition of "predictable" below. However, there is a group structure bound in our argument that seems interesting and so we will endeavor to prove it and make it explicit. In fact, let G be a finite group (for us usually abelian). A d-ABELIAN G GROUP is an extension $1 \to N \to G' \to G \to 1$ such that N contains G' normal subgroups $N = N(0) \supset N(1) \supset \ldots \supset N(d) = 1$ with $N(i)/N(i+1)$ abelian. Given any d abelian group we will use obvious modifications of the $N(i)$ notation above to denote the corresponding tower of groups.

Fix a field K with an absolute Galois group G_K. We say G' is a d-abelian G Galois group over K to mean G' is also an image of G_K (so $G' = \mathrm{Gal}(L(d)/K)$ for a field $L(d)$). Set $L(i) = L(d)^{N(i)}$ and $L = L(0)$. Whenever we talk about d abelian G Galois groups we will use the $L(i)$ notation, or obvious variants of it, to indicate the associated tower of fields.

In the course of the proof we will alter G' in several ways. In all cases we will want to construct Galois groups so we will usually construct these further groups via field theory.

If L'/L is abelian with L'/K Galois, then $L'(d)/K = L(d)L'/K$ is Galois with group G'' which is still a d-abelian G group. To see this, set $L(i)' = L(i)L'$ for $i > 0$ and $L'(0) = L(0)$. Then $L'(i)/L'(i-1)$ is abelian Galois with group a subgroup of $N(i-1)/N(i)$ for $i > 1$, but $L'(1)/L$ is abelian with Galois group a subgroup of $N_0/N_1 \oplus \mathrm{Gal}(L'/L)$. We call this EXPANDING the d-abelian G group G'.

Another construction we will need is the following. Suppose $K'(d')/K$ is Galois with d' abelian Galois group so $K'(d') \supset \ldots \supset K'(1) = L(1) \supset L(0) = K'(0) = L \supset K$ the point being here is that the beginning of the series of fields for K' coincides with the beginning for $L(d)$ (and $K'(i)/K'(i-1)$ is, of course, abelian Galois). Let d'' be the maximum of d and d', and set $K'(j) = K'(d')$ for $j > d'$ and similarly $L(j) = L(d)$ for $j > d$. Then if $L'(i) = L(i)K'(i)$ we have that $G'' = \mathrm{Gal}(L'(d'')/K)$ is a d''-abelian G group. Moreover, the first $\mathrm{Gal}(L'(1)/L'(0))$ is unchanged but the rest of the abelian series is larger. We call this REFINING the group G'. Note that expanding G' increases $\mathrm{Hom}(N_0, \mu_2)$ and so increases the cohomology cup products in $H^q(N_0, \mu_2)$. On the other hand, we will see that by refining our d abelian G groups we will introduce more relations among these cup products.

The above two constructions are special cases of the following. If $L(d)/K$ and $L_1(d')/K$ have d and d' abelian G Galois groups G' and G_1 respectively (with the same G Galois L/K) then the amalgamation $L(d)L'(d')/K$ has a d'' abelian G Galois group G_1' where d'' is the maximum of d and d'. We call G_1' the AMALGAMATION of G' and G_1.

Suppose $A' = \mathrm{Gal}(L'/L)$ is abelian and we have a d abelian A' Galois group over L, with associated field extensions $L'(d) \supset \ldots \supset L'(0) = L' \supset L$. Let L/K be G Galois as above. Then $L'(d)$ is not Galois over K, but if $L''(d)$ is the Galois closure of $L'(d)$ over K, then $L''(d)$ is the amalgamation of all the G conjugates of $L'(d)$ and so $\mathrm{Gal}(L''(d)/K)$ is a $d+1$ abelian G Galois group over K. We call this EXTENDING the d-abelian A' group to a $d+1$ abelian G group.

As another bit of terminology, if we have d and d_1 abelian G groups with a diagram

$$\begin{array}{ccccccccc} 1 & \to & N_1 & \to & G_1' & \to & G & \to & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \to & N & \to & G' & \to & G & \to & 1 \end{array}$$

where all vertical arrows are surjective then we say the d_1 abelian G group G_1' is a COVER of G' and if these maps are induced by field extensions we call it a Galois cover. Clearly, expanding, refining, amalgamating and extending are ways of constructing Galois covers.

When we expand or refine or extend a d-abelian G group G' we say that the size of the new group is PREDICTABLY bounded if the bound is only a function of G', the degrees of the cohomology groups involved, and previously proven symbol length bounds for field extensions of bounded degree. Note how unspecific this notion is. Any function of predictable bounds, or functions of predictable bounds and $|G|$ etc., also would be a predictable bound. For example, when we expand, or refine, or extend or amalgamate predictably bounded groups we get other ones.

Next we need some notation to help us navigate through the complexities of the Hochschild-Serre spectral sequence. We will employ this spectral sequence for sequences $1 \to \bar{N} \to \bar{G} \to G \to 1$ and $1 \to \bar{N}/\bar{N}' \to G' \to G \to 1$ where $\bar{N}/\bar{N}'$ is finite. Of course the natural map defines a morphism from the second spectral sequence to the first. Let me define notation in the first case as extension to the second is obvious.

In this spectral sequence, the $E_2^{p,q}$ term is $H^p(G, H^q(\bar{N}, \mu_2))$. The differential of this spectral sequence is d_r so $d_2 : H^p(G, H^q(\bar{N}, \mu_2)) \to H^{p+2}(G, H^{q-1}(\bar{N}, \mu_2))$. We wish to treat each $E_r^{p,q}$ as a subquotient of $H^p(G, H^q(\bar{N}, \mu_2))$ and so write

$$E_r^{p,q} = H^p(G, H^q(\bar{N}, \mu_2))_r^u / H^p(G, H^q(\bar{N}, \mu_2))_r^l.$$

Thus $H^p(G, H^q(\bar{N}, \mu_2))_2^u = H^p(G, H^q(\bar{N}, \mu_2))$ and $H^p(G, H^q(\bar{N}, \mu_2))_2^l = 0$. Moreover, the differentials d_r can be viewed as morphisms

$$d_r^{p,q} : H^p(G, H^q(\bar{N}, \mu_2))_r^u \to H^{p+r}(G, H^{q-r+1}(\bar{N}, \mu_2))/ H^{p+r}(G, H^{q-r+1}(\bar{N}, \mu_2))_r^l$$

and the kernel of $d_r^{p,q}$ is $H^p(G, H^q(\bar{N}, \mu_2))_{r+1}^u$ while the image of $d_r^{p,q}$ is

$$H^{p+r}(G, H^{q-r+1}(\bar{N}, \mu_2))_{r+1}^l / H^{p+r}(G, H^{q-r+1}(\bar{N}, \mu_2))_r^l.$$

Since all $d_r^{d,0}$ are 0, the kernel of $H^d(G, H^0(\bar{N}, \mu_2)) \to H^d(\bar{G}, \mu_2)$ is the union of all the $H^d(G, \mu_2)_r^l$ for all r.

In the arguments to follow we are going to use the above notions and work through the details of the Hochschild–Serre spectral sequence showing we can stay "predictably bounded" all the way. Since the overall argument is by induction, we will be able to assume the following at the degree d cohomology step:

(*) For all $q < d$ the following two facts hold. First, for all finite extensions of L/K of degree dividing n there is a symbol length bound, $M_q(n)$, for $H^q(L, \mu_2)$ only depending on n. Second, for any abelian extension L'/L of predictably bounded degree and Galois group G_1, and any element $\beta \in H^q(G_1, \mu_2)$ which presents 0, there is a predictably bounded $q-1$ abelian group G_1 Galois group over L written $1 \to N_1 \to G_1' \to G_1 \to 1$ such that β maps to 0 in $H^q(G_1', \mu_2)$.

Lemma 7. *Suppose $1 \to N \to G' \to G \to 1$ is a t-abelian G Galois group over K and $\gamma \in H^p(G, H^q(N, \mu_2))$ maps to 0 in $H^p(G, H^q(\bar{N}, \mu_2))$ where $q < d$. Assume (*). Let q' be the maximum of t and q. Then there is a predictably bounded q' abelian Galois cover of G' where γ maps to 0.*

Proof. The element γ is represented by a p-cocycle $c(g_1, \ldots, g_p) \in H^q(N, \mu_2)$ whose image in $H^q(\bar{N}, \mu_2))$ is the coboundary of a $p-1$ cochain $b(g_1, \ldots, g_{p-1})$. Since each $b(g_1, \ldots, g_{p-1})$ is a sum of $M_q([L : K])$ symbols, there is a 1-abelian G Galois group $1 \to \bar{N}/\bar{N}_1 \to G'' \to G \to 1$ such that the b's are the image of elements of $H^q(\bar{N}/\bar{N}_1, \mu_2)$ where $\bar{N}/\bar{N}_1$ is a predictably bounded elementary abelian 2 group. Expanding G' by this G'', and calling the result G' again, we have that the c's and b's are both in $H^p(G, H^q(N, \mu_2))$. There are $|G|^p$ relations in $H^q(N, \mu_2)$ that must be satisfied in order that the coboundary $\delta(b)$ equals c. Now let $G'' = N/N(1)$ be the abelian group. By assumption, we can iteratively refine the $t-1$ abelian G'' Galois group $1 \to N(1) \to N \to G'' \to 1$ to force these relations, and result is a $q'-1$ abelian G'' Galois group. Extending this to a q' abelian G Galois group we get the cover we need. Note that the size of the new group is again predictably bounded, though the bound is quite large. □

In a similar vein is:

Lemma 8. *Assume (*). Suppose $p + q < d$ and $\gamma \in H^p(G, H^q(\bar{N}, \mu_2))$. Then there is a q-abelian G Galois group $1 \to N \to G' \to G \to 1$ of predictably bounded order and an element $\gamma' \in H^p(G, H^q(N, \mu_2))$ which inflates to γ.*

Proof. The element γ is represented by a p cocycle $c(g_1, \ldots, g_p)$ consisting of less than or equal to $|G|^p$ elements of $H^q(\bar{N}, \mu_2)$. Each of these elements can be written as a sum of $M_q(|G|)$ symbols so that there are at most $|G|^p M(|G|)$ symbols and hence $|G|^p M_q(|G|)q$ elements of $\mathrm{Hom}(\bar{N}, \mu_2)$ are involved in writing all the $c(g_1, \ldots, g_p)$'s. Said differently, there is a $\bar{N}' \subset \bar{N}$ of index dividing $2^{|G|^p M_q(|G|)q}$ such that all these $c(g_1, \ldots, g_{d-r})$'s are in the image of

$H^q(\bar{N}/\bar{N}', \mu_2)$. Now $\bar{N}'$ is not normal in $\bar{G}$ so we take the intersection of the $|G|$ conjugates of $\bar{N}'$ to define $\bar{N}_1$ such that $\bar{N}_1$ is normal in $\bar{G}$ and all the $c(g_1, \ldots, g_{d-r})$'s come from $c_1(g_1, \ldots, g_p) \in H^q(\bar{N}/\bar{N}_1, \mu_2)$. Note that N/N_1 has order dividing $2^{|G|^{p+1}M(|G|)q}$. Now the c_1's do not form a q cocycle neccessarily, but their image in $H^q(\bar{N}, \mu_2)$ is a cocycle. Being a cocycle means that there are less than or equal to $|G|^{p+1}$ relations that must be satisfied. Since $q - 1 < d$, by assumption (*) there is a $q - 1$ abelian G' group where each of the cocycle relations become true after inflation (of predictably bounded size). By extending we get an q abelian G group for each needed cocycle relation, and we can refine all these together to get an q abelian G group $1 \to N \to G' \to G \to 1$ of predictably bounded size such that the c_1's inflate to an element $\gamma' \in H^p(G, H^q(N, \mu_2))$ which inflates to γ. □

The previous results do not suffice, as we need to show that we can achieve, element by element, the spectral sequence filtration in a predictably bounded cover. This is the next result.

Lemma 9. *Suppose $p + q < d$, and $1 \to N \to G' \to G \to 1$ is a t abelian G Galois group over K and $\gamma' \in H^p(G, H^q(N, \mu_2))$ maps via inflation to an element $\gamma \in H^p(G, H^q(\bar{N}, \mu_2))^u_s$. Let t' be the maximum of t and $q - 1$. Assume (*). Then there is a t' abelian Galois cover $1 \to N_1 \to G_1 \to G \to 1$ of predictably bounded size such that the inflation of γ' in $H^p(G, H^q(N_1, \mu_2))$ lies in $H^p(G, H^q(N_1, \mu_2))^u_s$.*

Proof. We prove this by induction on s. The statement is vacuous for $s = 2$ and by way of illustration the fact that γ' lies in $H^p(G, H^q(N_1, \mu_2))^u_3$ is equivalent to $d_2(\gamma') = 0 \in H^{p+2}(G, H^{q-1}(N_1, \mu_2))$ which (by Lemma 7) we can achieve after refining and thereby creating a predictably bounded t'-abelian G Galois cover where t' is the maximum of t and $q - 1$.

So assume the result for $s - 1$. We need to unpack the meaning when we say $\gamma \in H^p(G, H^q(\bar{N}, \mu_2))^u_s$. By definition, this is equivalent to $d_{s-1}(\gamma) \in H^{p+s-1}(G, H^{q-s+2}(\bar{N}, \mu_2))^l_{s-1}$ or $d_{s-1}(\gamma) - d_{s-2}(\gamma_1) \in H^{p+s-1}(G, H^{q-s+2}(\bar{N}, \mu_2))^l_{s-2}$ where $\gamma_1 \in H^{p+1}(G, H^{q-1}(\bar{N}, \mu_2))^u_{s-2}$. Proceeding by induction we have elements $\gamma_i \in H^{p+i}(G, H^{q-i}(\bar{N}, \mu_2))^u_{s-i-1}$ for bounded i such that $d_{s-1}(\gamma) = \sum_i d_{s-i-1}(\gamma_i) \in H^{p+s-1}(G, H^{q-s+2}(\bar{N}, \mu_2))$. By repeated use of Lemma 8 the γ_i are the image of $\gamma'_i \in H^{p+i}(G, H^{q-i}(N, \mu_2))$ for a $q - 1$ abelian G group G'. By induction we can assume the $\gamma'_i \in H^{p+i}(G, H^{q-i}(N, \mu_2))^u_{s-i-1}$ and one further refinement allows us to assert that $d_{s-1}(\gamma') = \sum_i d_{s-i-1}(\gamma'_i) \in H^{p+s-1}(G, H^{q-s+2}(N, \mu_2))^l_{s-1}$. □

Until now our spectral sequence notation has been unambiguous as to whether we are referring to the absolute sequence $1 \to \bar{N} \to \bar{G} \to G \to 1$ or some finite image of it, but we have to make such a distinction when we deal with $H^d(G, \mu_2)$

which appears in all these spectral sequences. Thus we will let $H^d(G,\mu_2)^l_r$ refer to the filtration induced by the absolute sequence and for the image sequence $1 \to \bar{N}/N' \to G' \to G \to 1$ we will use the notation $H^d_{G'}(G,\mu_2)^l_r$.

Proposition 10. *Assume (*). Suppose $\beta \in H^d(G,\mu_2)^l_{r+1}$ for some r. Then there is a $r-1$ abelian G Galois group $1 \to N \to G' \to G \to 1$ of predictably bounded order such that $\beta \in H^d_{G'}(G,\mu_2)^l_{r+1}$.*

Proof. There is a $\gamma \in H^{d-r}(G, H^{r-1}(\bar{N},\mu_2))^u_r$ which maps to β modulo $H^d(G,\mu_2)^l_r$. Let $\eta = d(\gamma) - \beta \in H^d(G,\mu_2)^l_r$. By induction on r there is a predictably bounded $r-2$ abelian G Galois group $1 \to N_1 \to G_1 \to G \to 1$ such that η is the inflation of $\eta' \in H^d_{G_1}(G,\mu_2)^l_r$. By Lemma 8 there is a predictably bounded $r-1$ abelian G Galois group $1 \to N \to G' \to G \to 1$ such that γ is the inflation of $\gamma' \in H^{d-r}(G, H^{r-1}(N,\mu_2))$. Since $\gamma \in H^{d-r}(G, H^{r-1}(\bar{N},\mu_2))^u_r$ we know from Lemma 9 that there is $r-1$ abelian G Galois group cover of G', which we also call G', such that $\gamma' \in H^{d-r}(G, H^{r-1}(N,\mu_2))^u_r$. We can amalgamate G_1 and G' to get a predictably bounded $r-1$ abelian G Galois group (we also call G') where $d_r(\gamma')$, β and η' are all defined and $\beta = d_r(\gamma') - \eta'$ so $\beta \in H^d_{G'}(G,\mu_2)^l_{r+1}$. □

Corollary 11. *Suppose $\beta \in H^d(G,\mu_2)$ maps to 0 in $H^d(\bar{G},\mu_2)$. Assume (*). Then there is a predictably bounded $d-1$ abelian G Galois group such that β maps to 0 in $H^d(G',\mu_2)$.*

Proof. In the spectral sequence the last nontrivial derivation with image $H^d(G,\mu_2)$ is $d_d : H^0(G, H^{d-1}(\bar{N},\mu_2))^u_d \to H^d(G,\mu_2)/H^d(G,\mu_2)^l_d$. That is, the ascending tower in $H^d(G,\mu_2)$ stabilizes at $H^d(G,\mu_2)^l_{d+1}$. The result follows from Proposition 10. □

Now we are in a position to prove Theorem 1, and we do it by noting it is a part of the following.

Theorem 12. *Let K have finite u invariant. Let K'/K be a any field extension of degree dividing n. For all degrees d, there is a symbol bound for $H^d(K',\mu_2)$ that only depends on n. Also, there is a limitation bound for K' that only depends on n. Moreover, given a presentation of zero $\beta \in H^d(G,\mu_2)$ and $G = \mathrm{Gal}(L'/K')$, this limitation bound is realized by $d-1$ abelian G Galois groups.*

Proof. As we have said all along, we prove this by induction and so we assume this statement for all degrees $j < d$. By Theorem 6 we have the symbol boundedness of K' in degree d. By Corollary 11, we have the limitation bound in degree d. □

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