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Conductive Homogeneity of Compact Metric Spaces and Construction of *p*-Energy

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Abstract

In the ordinary theory of Sobolev spaces on domains of $\mathbb{R}^n$, the p-energy is defined as the integral of $|\nabla f|^p$. In this paper, we try to construct a p-energy on compact metric spaces as a scaling limit of discrete p-energies on a series of graphs approximating the original space. In conclusion, we propose a notion called conductive homogeneity under which one can construct a reasonable p-energy if p is greater than the Ahlfors regular conformal dimension of the space. In particular, if $p = 2$, then we construct a local regular Dirichlet form and show that the heat kernel associated with the Dirichlet form satisfies upper and lower sub-Gaussian type heat kernel estimates. As examples of conductively homogeneous spaces, we present new classes of square-based self-similar sets and rationally ramified Sierpiński crosses, where no diffusions were constructed before.

In memory of the late Professors Robert S. Strichartz and Ka-Sing Lau,
who were my dearest friends and daring explorers of the frontiers

Keywords. Sobolev spaces, metric spaces, conductive homogeneity, p-energy, self-similar sets

Mathematics Subject Classification (2020). Primary 46E36; Secondary 28A80, 31C45, 31C25, 31E05, 30L10

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Contents

Chapter 1

Introduction

The main objective of this paper is to generalize the following elementary fact.

Let $I = [0, 1]$. Define

$$\mathcal{E}_p^n(f) = \sum_{i=1}^{2^n} \left| f\left(\frac{i-1}{2^n}\right) - f\left(\frac{i}{2^n}\right)\right|^p$$

for $n \geq 1$ and $f : I \to \mathbb{R}$. If f is smooth or more generally $f \in W^{1,p}(I)$, which is the $(1, p)$-Sobolev space, then

$$(2^{p-1})^n \mathcal{E}_p^n(f) \to \int_0^1 |\nabla f|^p dx$$

as $n \to \infty$, where ∇f is the derivative of f.

Our naive question is what is a counterpart of this in the case of metric spaces. More precisely, our general strategy of the study is:

(1) To fix an adequate sequence of discrete graphs $\{(T_n, E_n^*)\}_{n \geq 1}$, where T_n is a discrete approximation of the original metric space (X, d) and E_n^* is the collection of edges, i.e., pairs of points in T_n. For a function $f : T_n \to \mathbb{R}$, define

$$\mathcal{E}_p^n(f) = \frac{1}{2} \sum_{(x,y) \in E_n^*} |f(x) - f(y)|^p,$$

which is called the p-energy of the function f.

(2) To find a proper scaling constant σ such that the space of functions

$$\{f : X \to \mathbb{R} \mid \sigma^n \mathcal{E}_p^n(P_n f) \text{ is "convergent" as } n \to \infty\},$$

where $P_n f$ is a suitable discrete approximation of f, is rich enough to be a "Sobolev" space in some sense. From our perspective, we do not care about the existence of a derivative ∇f but pursue the convergence of $\sigma^n \mathcal{E}_p^n(P_n f)$.

Actually, in the case $p = 2$, this strategy was employed to construct Dirichlet forms inducing diffusion processes on self-similar sets like the Sierpiński gasket[1] and the Sierpiński carpet. (See Figure 1.4.) For the sake of simplicity, we confine ourselves to non-finitely ramified self-similar sets. (This excludes post critically finite

[1] In many papers, people use "Sierpinski" in place of "Sierpiński". Of course, originally "Sierpiński" is the correct one as a Polish family name but such a simplification often occurs when the subject becomes popular and a part of classics.

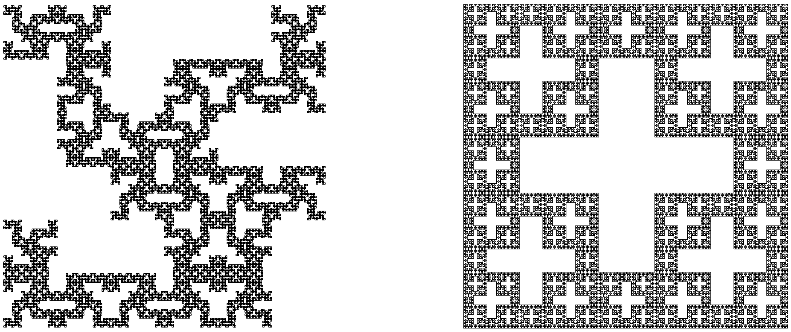


Figure 1.1. Square-based self-similar sets.

self-similar sets represented by the Sierpiński gasket.) Barlow and Bass constructed the Brownian motions on (generalized) Sierpiński carpets in [1–6] as scaling limits of the Brownian motions on regions approximating Sierpiński carpets. Later in [36], Kusuoka and Zhou employed the above strategy for $p = 2$ and directly constructed the Dirichlet form inducing the Brownian motion on the planar Sierpiński carpet. Note that all these works were done in the last century. Although more than 20 years have passed, no essential progress has been made on the construction of diffusion processes/Dirichlet forms on non-finitely ramified self-similar sets. In particular, no diffusion was constructed on square-based non-finitely ramified self-similar sets like those in Figure 1.1. The right-hand one is an example of rationally ramified Sierpiński crosses treated in Section 4.5. It has two different contraction ratios. The left-hand one is an example having no symmetry of the square. As a by-product of our results in this paper, we will construct non-trivial self-similar local regular Dirichlet forms on classes of square-based self-similar sets including those in Figure 1.1. See Sections 4.3, 4.4, and 4.5 for details.

From the viewpoint of construction of Sobolev spaces on metric spaces, there have already been established theories based on upper gradients, which correspond to local Lipschitz constants of Lipschitz functions. Compared with our strategy above, this direction is to seek a counterpart of ∇f instead of the convergence of $\sigma^n \mathcal{E}_p^n(P_n f)$ like us. The pioneering works of this theory are Hajłasz [22], Cheeger [15] and Shanmugalingam [40]. One can find a panoramic view of this theory in [23]. Recent studies by Kajino and Murugan in [26, 27], however, have suggested that they may not cover all the interesting cases. So far examples in question are higher-dimensional Sierpiński gaskets, the Vicsek set, and the planar Sierpiński carpet. What they have shown in [26, 27] is that the Brownian motions on those examples will not have the Gaussian heat kernel estimate under any time change by a pair (d, μ), where d is quasisymmetric to the Euclidean metric d_E and μ has the volume doubling property with respect to d_E. On the other hand, under the established theory, the heat kernel associated with a $(1, 2)$-Sobolev space satisfying a $(2, 2)$-Poincaré inequality should satisfy the Gaussian estimate due to the results in [21, 39, 42]. Thus, the Dirichlet forms associated with the Brownian motions on the above-mentioned self-similar sets can hardly be one of $(1, 2)$-Sobolev spaces based on upper gradients. Note that, in these cases, there exist plenty of rectifiable curves with respect to (the restriction of) the Euclidean metrics, which are even quasiconvex. Partly motivated by such a situation, we will try to provide an alternative theory of function spaces, which may be called Sobolev spaces or else, on metric spaces, and to construct natural diffusion processes at the same time.

Getting straight to the conclusion, we are going to propose a condition called p-conductive homogeneity and show that under this condition, the strategy consisting of (1) and (2) succeeds for $p > \dim_{AR}(K, d)$, where $\dim_{AR}(K, d)$ is the *Ahlfors regular conformal dimension* of a compact metric space (K, d). One can see a more precise and detailed exposition in what follows. The definition of the Ahlfors regular conformal dimension of (K, d) is

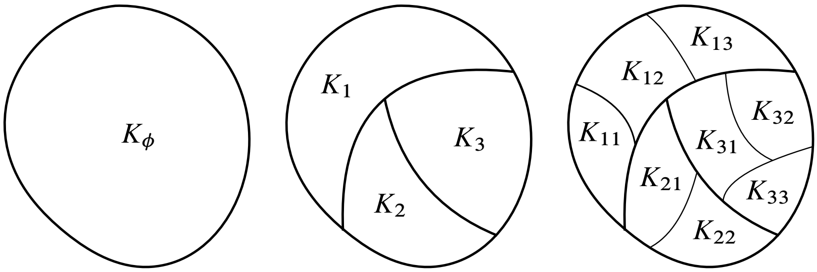
$$\begin{aligned}\dim_{AR}(K, d) = \inf\{\alpha \mid\ & \text{there exist a metric } \rho \text{ on } K \text{ which is} \\ & \text{quasisymmetric to } d \text{ and a Borel regular measure } \mu \\ & \text{which is } \alpha\text{-Ahlfors regular with respect to } \rho\},\end{aligned} \tag{1.1}$$

where the definition of Ahlfors regularity of a measure is given in (2.9). The notion of quasisymmetry was introduced in [43] as a certain generalization of quasiconformal maps of the complex plane. It is defined in the following way:

Definition 1.1. Let (X, d) be a metric space. A metric ρ on X is said to be *quasisymmetirc* to d if (X, ρ) gives the same topology as d and there exists a homeomorphism h from $[0, \infty)$ to itself satisfying $h(0) = 0$ and for any $t > 0$, $\rho(x, z) \leq h(t)\rho(x, y)$ whenever $d(x, z) < td(x, y)$.

In the direction of our study, Shimizu has done pioneering work for the case of the planar Sierpiński carpet, PSC for short, in the very recent paper [41]. Extending Kusuoka–Zhou's method, he has constructed a p-energy and the corresponding p-energy measure for $p > \dim_{AR}(\mathrm{PSC}, d_E)$, and done detailed analysis of those objects. In particular, he has shown that the collection of functions with finite p-energies is a Banach space that is reflexive and separable. His proof of reflexivity and separability can be easily extended to our general case as well. See Theorem 3.22 for details.

Our framework on metric spaces is the theory of partitions introduced in [34]. Let (K, d) be a compact metric space. We always suppose that (K, d) is connected in this paper. Roughly speaking, a partition of K is a sequence of successive divisions of K by some of its compact subsets. The idea is illustrated in Figure 1.2. Let $T_0 = \{\phi\}$



$T_0 = \{\phi\}$ $\quad$ $T_1 = \{1, 2, 3\}$ $\quad$ $S(1) = \{11, 12, 13\}$

$S(2) = \{21, 22\}$

$S(2) = \{31, 32, 33\}$

Figure 1.2. Partition.

and set $K_\phi = K$. Starting from K, we first divide K into a finite number of children K_w for $w \in T_1$, i.e.,

$$K = \bigcup_{w \in T_1} K_w.$$

The set T_1 is thought of as the collection of children of T_0 and denoted by $S(\phi)$. Then we repeat this process of division, i.e., each $w \in T_1$ has a collection of children, $S(w)$, such that

$$K_w = \bigcup_{v \in S(w)} K_v.$$

Define T_2 as the disjoint union of the $S(w)$'s for $w \in T_1$. So repeating this process inductively, we have $\{T_n\}_{n \geq 0}$ where each $w \in T_n$ has the collection of children $S(w) \subseteq T_{n+1}$. Set

$$T = \bigcup_{n \geq 0} T_n.$$

With several requirements described in Section 2.1, the family $\{K_w\}_{w \in T}$ is called a partition of K.

For each $n \geq 1$, T_n has a natural graph structure associated with a given partition $\{K_w\}_{w \in T}$. Namely, if

$$E_n^* = \{(u, v) \mid u, v \in T_n, K_u \cap K_v \neq \emptyset\},$$

then (T_n, E_n^*) is a connected graph, which is illustrated in Figure 1.3. To avoid technical complexity, we are going to explain our results under Assumption 2.15 hereafter in the introduction. In fact, if (K, d) is α-Ahlfors regular for some α and the metric d is 1-adapted in the sense of [34], then Assumption 2.15 holds. So our setting should be broad enough.

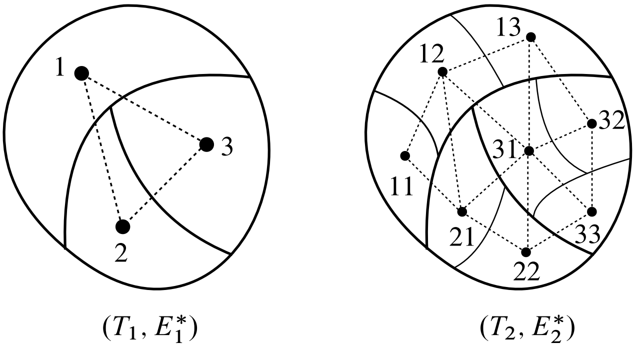


Figure 1.3. Graphs associated with a partition (dotted lines are vertices).

For $A \subseteq T_n$, we define the p-energy of a function on A by

$$\mathcal{E}_{p,A}^n(f) = \frac{1}{2} \sum_{\substack{u,v \in A \\ (u,v) \in E_n^*}} |f(u) - f(v)|^p.$$

To carry out our strategy, we introduce two key characteristic quantities: conductance and neighbor disparity constants. For $m \geq 0$, $A_1, A_2, A \subseteq T_n$ with $A_1, A_2 \subseteq A$ and $A_1 \cap A_2 = \emptyset$, define

$$\mathcal{E}_{p,m}(A_1, A_2, A) = \inf\big\{\mathcal{E}_{p,A}^{n+m}(f) \mid f\colon S^m(A) \to \mathbb{R}, f|_{S^m(A_1)} \equiv 1, f|_{S^m(A_2)} \equiv 0\big\},$$

where $S^m(A) \subseteq T_{n+m}$ is the collection of the descendants in the m-th generation from A. The quantity $\mathcal{E}_{p,m}(A_1, A_2, A)$ is called the *p-conductance* between A_1 and A_2 within A at the level m.

Remark. Attaching a resistor of resistance 1 to each edge $(u, v) \in E_{n+m}^*$, we may consider the graph (T_{n+m}, E_{n+m}^*) as an electric network. In this respect, the reciprocal of $\mathcal{E}_{2,m}(A_1, A_2, A)$ is the effective resistance between A_1 and A_2 within A and hence $\mathcal{E}_{2,m}(A_1, A_2, A)$ corresponds to the effective conductance. Such an analogy has been often used in the study of random walks. See [18] for a classical reference. In potential theory, the quantity $\mathcal{E}_{2,m}(A_1, A_2, A)$ is called "capacity" as well.

In particular, for $w \in T_n$, define

$$\mathcal{E}_{p,m}(w) = \mathcal{E}_{p,m}(\{w\}, \Gamma_1(w)^c, T_n),$$

where $\Gamma_1(w)$ is the collection of neighbors of w in T_n given by

$$\Gamma_1(w) = \{v \mid v \in T_n, (w, v) \in E_n^*\}.$$

The value $\mathcal{E}_{p,m}(w)$ represents the p-conductance between w and the complement of its neighborhood $\Gamma_1(w)$ in the m-th generation from w. In [34], it was shown that

$$\overline{\lim_{m\to\infty}} \Big(\sup_{w\in T} \mathcal{E}_{p,m}(w)^{\frac{1}{m}} \Big) < 1 \quad \text{if and only if} \quad p > \dim_{AR}(K, d). \tag{1.2}$$

The other one, the neighbor disparity constant, is defined as

$$\sigma_{p,m,n} = \sup_{(w,v)\in E_n^*} \Big(\sup_{f:S^m(w,v)\to\mathbb{R}} \frac{|(f)_{S^m(w)} - (f)_{S^m(v)}|^p}{\mathcal{E}^{n+m}_{p,S^m(w,v)}(f)} \Big),$$

where $S^m(w,v) = S^m(w) \cup S^m(v)$ and $(f)_{S^m(w)}$ is the average of f on $S^m(w)$ under a suitable measure μ. (This definition of the neighbor disparity constant is a simplified version for introductory purposes. The full version will be presented in Section 2.4.) For the case $p = 2$, this constant was introduced in [36]. The neighbor disparity constant controls the difference of means of a function on neighboring cells via the p-energy.

And now, p-conductive homogeneity, which is the principal notion of this paper, is defined as follows.

Definition 1.2. A metric space (K, d) is said to be *p-conductively homogeneous* if and only if there exists $c > 0$ such that

$$\sup_{w\in T} \mathcal{E}_{p,m}(w) \sup_{n\geq 1} \sigma_{p,m,n} \leq c$$

for any $m \geq 1$.

The above condition is essentially due to Kusuoka–Zhou [36] when $p = 2$. Cao and Qiu named this condition as condition (B) in [13], where they have constructed a diffusion process on so called unconstrained Sierpiński carpets by following the Kusuoka–Zhou's method.

At a glance, it does not quite look like "homogeneity". The following theorem, however, gives the legitimacy of the name.

Theorem 1.3 (Theorem 3.30). *A metric space (K, d) is p-conductively homogeneous if and only if there exist $\sigma > 0$ and $c_1, c_2 > 0$ such that*

$$c_1\sigma^{-m} \leq \mathcal{E}_{p,m}(w) \leq c_2\sigma^{-m}$$

for any $w \in T\backslash\{\phi\}$ and $m \geq 1$ and

$$c_1\sigma^{m} \leq \sigma_{p,m,n} \leq c_2\sigma^{m}$$

for any $m, n \geq 1$.

The next natural question is how the conductive homogeneity is related to the construction of a p-energy. The answer is the next theorem which follows by combining Theorems 3.5, 3.21, 3.23 and Lemma 3.34.

Theorem 1.4. *Suppose $p > \dim_{AR}(K,d)$ and (K,d) is p-conductively homogeneous. Let $C(K)$ be the collection of continuous functions on K. Define*

$$\mathcal{N}_p(f) = \Big(\sup_{m\geq 0} \sigma^m \mathcal{E}_p^m(P_m f)\Big)^{\frac{1}{p}}$$

for $f \in L^p(K,\mu)$, where

$$(P_m f)(w) = \frac{1}{\mu(K_w)} \int_{K_w} f(x)\mu(dx),$$

and

$$\mathcal{W}^p = \{f \mid f \in L^p(K,\mu), \mathcal{N}_p(f) < \infty\}.$$

Then

(1) *$\mathcal{N}_p(f) = 0$ if and only if f is constant on K.*

(2) *$\mathcal{N}_p$ is a semi-norm of $\mathcal{W}^p$.*

(3) *$(\mathcal{W}^p, \|\cdot\|_{p,\mu} + \mathcal{N}_p(\cdot))$ is a Banach space.*

(4) *$\mathcal{W}^p$ is a dense subset of $(C(K), \|\cdot\|_\infty)$.*

Moreover, there exists $\widehat{\mathcal{E}}_p \colon \mathcal{W}^p \to [0,\infty)$ such that $\widehat{\mathcal{E}}_p^{\frac{1}{p}}$ is a semi-norm of $\mathcal{W}^p$ which is equivalent to $\mathcal{N}_p(\cdot)$, $\widehat{\mathcal{E}}_p$ satisfies the Markov property and there exist $\tau > 0$ and $c_1, c_2 > 0$ such that

$$c_1 d(x,y)^\tau \leq \sup_{\substack{f\in\mathcal{W}^p\\ \widehat{\mathcal{E}}_p(f)\neq 0}} \frac{|f(x)-f(y)|^p}{\widehat{\mathcal{E}}_p(f)} \leq c_2 d(x,y)^\tau$$

for any $x, y \in K$. In particular, for $p = 2$, one can choose $(\widehat{\mathcal{E}}_2, \mathcal{W}^2)$ as a local regular Dirichlet form on $L^2(K,\mu)$.

Note that by (1.2), the condition $p > \dim_{AR}(K,d)$ implies $\sigma > 1$. An explicit description of the constant τ is given in Lemma 3.34. In addition, we show a sub-Gaussian type heat kernel estimate for the diffusion process induced by the Dirichlet form $(\widehat{\mathcal{E}}_2, \mathcal{W}^2)$ in Theorem 3.35. Moreover, if (K,d) is a self-similar set with rationally related contraction ratios, then a self-similar p-energy which is equivalent to $\mathcal{N}_p$ will be constructed in Section 4.1.

Another important question is how to show conductive homogeneity. The following theorem provides an equivalent and useful condition for this purpose.

Theorem 1.5 (Theorem 3.33). *Suppose that $p > \dim_{AR}(K, d)$. (K, d) is p-conductively homogeneous if and only if, for any $k \geq 1$, there exists $c(k) > 0$ such that*

$$\sup_{z \in T} \mathcal{E}_{p,m}(z) \leq c(k)\mathcal{E}_{p,m}(u, v, S^k(w)) \tag{1.3}$$

for any $m \geq 1$, $w \in T$ and $u, v \in S^k(w)$ with $u \neq v$.

The condition in the above theorem, (1.3), which is the same as (3.20) in Theorem 3.33, is a relative of the "knight move" condition in [36] described in the terminology of random walks, although the word "knight move" does not make sense from the appearance of (1.3) any longer. The original "knight move" in [1] was the name of an argument based on the symmetry of the Sierpiński carpet to show a probabilistic counterpart of (1.3). With certain symmetries of the space, it is possible to show (1.3) by the method of combinatorial modulus in [11]. Applying Theorem 1.5, we are going to show the conductive homogeneity for examples like those in Figure 1.1 in Sections 4.4 and 4.5.

Besides applications, Theorem 1.5 has a remarkable theoretical consequence; conductive homogeneity is determined only by conductance constants and is independent of the neighbor disparity constants if $p > \dim_{AR}(K, d)$. This is the reason conductive homogeneity is called "*conductive*".

The major methodological backgrounds of this paper are Kusuoka–Zhou's arguments in [36] and combinatorial moduli of path families on graphs introduced in [11]. On many occasions, we will extend Kusuoka–Zhou's results to compact metric spaces and to general values of p. On such occasions, we will put a reference to the original result by Kusuoka and Zhou right behind the number of a proposition or a lemma like Lemma 2.27 [36, Lemma 2.12]. Beyond Kusuoka–Zhou's arguments, the notion of combinatorial modulus will play a crucial role on several occasions. The most important one is in the proof of a sub-multiplicative inequality of conductance constants, Corollary 2.24. Moreover, by Lemma C.4, one can compare moduli of different graphs and this lemma is indispensable for showing (1.3) in Sections 4.3 and 4.5.

Regrettably, we do not have much for the case $p \leq \dim_{AR}(K, d)$. In Section 3.2, we will construct a function space $\mathcal{W}^p$ and a semi-norm $\widehat{\mathcal{E}}_p$ on $\mathcal{W}^p$ under p-conductive homogeneity for $p \in [1, \dim_{AR}(K, d)]$. In this case, however, $\mathcal{W}^p$ is given as a subspace of $L^p(K, \mu)$ and we do not know whether $\mathcal{W}^p \cap C(K)$ is dense in $(C(K), \|\cdot\|_\infty)$ or not. This is due to the lack of an elliptic Harnack principle of p-harmonic functions on the corresponding graphs. In the case $p = 2$, using the coupling method, Barlow and Bass conquered this difficulty for higher-dimensional Sierpiński carpets in [5, 6]. We have little idea what is an analytic counterpart of the coupling method at this moment. It is a big open problem for future work. In particular, it is interesting to know whether the following naive conjecture is true or not.

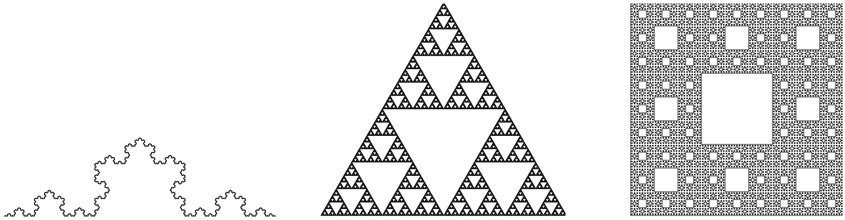


Figure 1.4. von Koch curve, Sierpiński gasket and Sierpiński carpet.

Conjecture. $\mathcal{W}^p \subseteq C(K)$ *if and only if* $p > \dim_{AR}(K, d)$.

Now we briefly explain what happens in the cases of familiar examples.

1. *Unit (hyper)cube* $[-1, 1]^n$: In this case, for any $p > n$,

$$\mathcal{W}^p = W^{1,p}([-1, 1]^n)$$

and there exists $c > 0$ such that

$$c\widehat{\mathcal{E}}_p(f) \leq \int_{[-1,1]^n} |\nabla f|^p dx \leq c^{-1}\widehat{\mathcal{E}}_p(f)$$

for any $f \in W^{1,p}([-1, 1]^n)$. See Example 4.31 for details. Even if $p \in [1, n]$, the above results should be true but we do not have any proof for now.

2. *von Koch curve (Figure 1.4)*: The von Koch curve does not contain any rectifiable curve, so that the approaches using upper gradients do not work from the beginning. However, our theory does not distinguish metric spaces which are snowflake equivalent, i.e., two metric spaces (X, d_X) and (Y, d_Y) are snowflake equivalent if there exist a homeomorphism $\varphi: X \to Y$, $c_1, c_2 > 0$ and $\alpha > 0$ such that

$$c_1 d_X(x_1, x_2)^\alpha \leq d_Y(\varphi(x_1), \varphi(x_2)) \leq c_2 d_X(x_1, x_2)^\alpha$$

for any $x_1, x_2 \in X$. Since the von Koch curve is snowflake equivalent to the unit interval $[0, 1]$, we see that $\mathcal{W}^p$ for the von Koch curve equals $W^{1,p}([0, 1])$ for any $p > 1$.

3. *Planar Sierpiński carpet (Figure 1.4)*: As is mentioned above, this is one of the original motivations of this paper and it is expected that our space $\mathcal{W}^p$ is quite different from what one may get from the upper gradient approaches. By Theorem 4.13, the planar Sierpiński carpet K is shown to be p-conductive homogeneous for any $p > \dim_{AR}(K, d_*)$, where d_* is the restriction of the Euclidean metric. Moreover, let

$$\alpha_H = \frac{\log 8}{\log 3} \quad \text{and} \quad \beta_p = \frac{\log 8\sigma}{\log 3},$$

where σ is the exponent appearing in Theorem 1.3. Then by [41, Theorem 2.19], we have a fractional Korevaar–Shoen type expression of $\mathcal{W}^p$ as follows:

$$\mathcal{W}^p = \left\{ f \mid f \in L^p(K,\mu), \overline{\lim_{r \downarrow 0}} \int_K \frac{1}{r^{\alpha_H}} \int_{B_{d_*}(x,r)} \frac{|f(x) - f(y)|^p}{r^{\beta_p}} \mu(dy)\mu(dx) < \infty \right\},$$

where μ is the normalized α_H-dimensional Hausdorff measure. Furthermore, it is shown in [41] that $\beta_p > p$. This fact implies that $\mathcal{W}^p$ should not coincide with any of the spaces obtained by approaches using upper gradients.

4. *Sierpiński gasket (Figure 1.4)*: Let K be the standard Sierpiński gasket and let d_* be the restriction of the Euclidean metric. Since K is one of nested fractals and

$$\dim_{AR}(K, d_*) = 1,$$

Theorem 4.50 yields that K is p-conductively homogeneous for any $p > 1$. Arguments analogous to those in [41, Section 5.3] give the same fractional Korevaar–Shoen type expression of $\mathcal{W}^p$ as the planar Sierpiński carpet. In this case,

$$\alpha_H = \frac{\log 3}{\log 2} \quad \text{and} \quad \beta_p = \frac{\log 3\sigma}{\log 2}.$$

We expect that $\beta_p > p$ for any $p > 1$. In fact, due to [8], we know

$$\beta_2 = \frac{\log 5}{\log 2} > 2.$$

Moreover, $\frac{\beta_p}{p}$ is monotonically decreasing by [34, Lemma 4.7.3]. So at least for $p \in (1, 2]$, $\beta_p > p$ and the space $\mathcal{W}^p$ does not seem to be obtained by the upper gradient approaches. However in this case, if we replace the Euclidean metric with the harmonic geodesic metric and the Hausdorff measure with the Kusuoka measure, then the heat kernel associated with the new pair of the metric and the measure has the Gaussian estimate. See [30] for details. Consequently, the Cheeger theory [15] is now in place for $\mathcal{W}^2$ at least. On the other hand, the replacement of the metric and the measure causes a change of the partition and, consequently, a change of the associated function space $\mathcal{W}^P$. So, we expect that $\mathcal{W}^p$ associated with the new pair may coincide with those obtained from the approaches based on upper gradients but we have no proof so far.

Before the conclusion of the introduction, we mention two related works. The first one is [10], where the authors constructed another type of "Sobolev spaces" $\dot{A}_p(X)$ on a compact metric space (Z, d) from its hyperbolic fillings X. The method is to construct a discretization Pf on X of $f \in L^1(Z)$, and to consider the weak ℓ^p-norm of the gradient of Pf. Their space $\dot{A}^p(Z)$ seems closely related to our space $\mathcal{W}^p$ but we merely know that $\mathcal{W}^p \subseteq \dot{A}_p(X)$ under suitable assumptions at this point.

The second one is [24], where the authors constructed a p-energy on Sierpiński gasket type self-similar sets by extending the notion of harmonic structures in the case of $p = 2$ for post critically finite self-similar sets. Their p-energy should be equivalent to ours, although they did not show the completeness of the domain of their p-energy. Despite the fact that their method can work only for finitely ramified self-similar sets even if $p = 2$, their work is the first pioneering study to construct a p-energy by renormalizing discrete counterparts.

The organization of this paper is as follows.

In Section 2.1, we review the basics of partitions of compact metric spaces and then give a framework of this paper including standing assumptions, Assumptions 2.6, 2.7, 2.10 and 2.12. In the end, we present Assumption 2.15, which is stronger than the combination of all the assumptions above but more concise.

In Section 2.2, we introduce the notion of conductance constant which is one of two principal quantities of this paper and we show the existence of a partition of unity associated with the conductance constant.

In Section 2.3, we introduce the notion of combinatorial moduli of path families on graphs and show a sub-multiplicative inequality for conductance constants using them.

In Section 2.4, we introduce the other principal quantity, the neighbor disparity constant and show its relation with the conductance constant and a sub-multiplicative inequality of them.

In Section 3.1, we construct our function space $\mathcal{W}^p$ and the p-energy $\widehat{\mathcal{E}}_p$ under Assumption 3.2 and show Theorem 1.4. At the same time, we propose a condition called p-conducive homogeneity and show that the condition $p > \dim_{AR}(K, d)$ and p-conductive homogeneity imply Assumption 3.2 in Section 3.3.

In Section 3.2, we see what we can do for $p \leq \dim_{AR}(K, d)$. In Section 3.3, we show Theorem 3.30 (= Theorem 1.3) and Theorem 3.33 (= Theorem 1.5). Moreover, in Theorem 3.35, we give a sub-Gaussian type heat kernel estimate for the diffusion process induced by the Dirichlet form $(\mathcal{E}, \mathcal{W}^2)$ given in Section 3.1.

In Section 4.1, we construct a self-similar p-energy for self-similar sets with rationally related contraction ratios. In Section 4.2, we give a sufficient condition for the conductive homogeneity for self-similar sets. Section 4.3 is devoted to a class of self-similar sets called subsystems of cubic tiling, for which conductive homogeneity is shown through Theorem 3.33. This class includes the Sierpiński carpets, the Menger curve, and the higher-dimensional hypercubes. In Section 4.4, we present examples of subsystems of cubic tiling having the conductive homogeneity. Also, Section 4.5 is devoted to showing conductive homogeneity of rationally ramified Sierpiński crosses.

In Sections 5.1, 5.2 and 5.3, we give a proof of Theorem 3.33. In Section 6.1, we show that conductance, Poincaré and neighbor disparity constants are uniformly bounded from below and above.

We will briefly discuss the modification of the graph structure in Section 6.2. Finally, in Section 6.3, we gather open problems and future directions of research. Appendices give basic facts used in this paper.

Chapter 2

Basic frameworks and key constants

2.1 Framework

In this section, we are going to make our framework of this paper clear. It is based on the notion of partitions of compact metric spaces parametrized by rooted trees, which was introduced in [34]. Roughly speaking, a partition is successive divisions of a given space like the binary division of the unit interval. See [34] for examples. Since this notion is relatively new and unfamiliar to most readers, we will give a minimal but detailed account of its definition.

To start with, we present the basics of graphs and trees.

Definition 2.1. Let T be a countable set and let $\mathcal{A}: T \times T \to \{0, 1\}$ which satisfies $\mathcal{A}(w, v) = \mathcal{A}(v, w)$ and $\mathcal{A}(w, w) = 0$ for any $w, v \in T$. We call the pair $(T, \mathcal{A})$ a (non-directed) *graph* with the *vertices* T and the *adjacency matrix* $\mathcal{A}$. An element $(u, v) \in T \times T$ is called an *edge* of $(T, \mathcal{A})$ if $\mathcal{A}(u, v) = 1$. We often identify the adjacency matrix $\mathcal{A}$ with the collection of edges $\{(u, v) \mid u, v \in T, \mathcal{A}(u, v) = 1\}$.

(1) A graph $(T, \mathcal{A})$ is called *locally finite* if $\#(\{v \mid \mathcal{A}(w, v) = 1\}) < \infty$ for any $w \in T$, where $\#(A)$ is the number of elements of a set A.

(2) For $w_0, \dots, w_n \in T$, $(w_0, w_1, \dots, w_n)$ is called a *path* between w_0 and w_n if $\mathcal{A}(w_i, w_{i+1}) = 1$ for any $i = 0, 1, \dots, n-1$. A path $(w_0, w_1, \dots, w_n)$ is called *simple* if $w_i \neq w_j$ for any i, j with $0 \leq i < j \leq n$ and $|i - j| < n$.

(3) $(T, \mathcal{A})$ is called a *tree* if there exists a unique simple path between w and v for any $w, v \in T$ with $w \neq v$. For a tree $(T, \mathcal{A})$, the unique simple path between two vertices w and v is called the *geodesic* between w and v and denoted by $\overline{wv}$. We write $u \in \overline{wv}$ if $\overline{wv} = (w_0, w_1, \dots, w_n)$ and $u = w_i$ for some i.

Next, we define fundamental notions on trees.

Definition 2.2. Let $(T, \mathcal{A})$ be a tree and let $\phi \in T$. The triple $(T, \mathcal{A}, \phi)$ is called a *rooted tree* with *root* (or *reference point*, see, e.g., [45]) ϕ.

(1) Define $\pi: T \to T$ by

$$\pi(w) = \begin{cases} w_{n-1} & \text{if } w \neq \phi \text{ and } \overline{\phi w} = (w_0, w_1, \dots, w_{n-1}, w_n), \\ \phi & \text{if } w = \phi \end{cases}$$

and, for $w \in T$, set

$$S(w) = \{v \mid \pi(v) = w\} \backslash \{w\}.$$

An element $v \in S(w)$ is thought of as a *child* of w. Moreover, for any $k \geq 1$, we define $S^k(w)$ inductively as

$$S^{k+1}(w) = \bigcup_{v \in S(w)} S^k(v),$$

which is the collection of descendants in the k-th generation from w.

(2) For $w \in T$ and $m \geq 0$, we define

$$|w| = \min\{n \mid n \geq 0,\ \pi^n(w) = \phi\} \quad \text{and} \quad T_m = \{w \mid w \in T,\ |w| = m\}.$$

(3) For any $w \in T$, define

$$T(w) = \{v \mid \text{there exists } n \geq 0 \text{ such that } \pi^n(v) = w\},$$

which is the collection of all the descendants of w.

(4) Define

$$\Sigma = \{(w(i))_{i \geq 0} \mid w(i) \in T_i \text{ and } w(i) = \pi(w(i+1)) \text{ for any } i \geq 0\}.$$

For $\omega = (\omega(i))_{i \geq 0} \in \Sigma$, set $[\omega]_m = \omega(m)$ for $m \geq 0$. An element $(w(i))_{i \geq 0} \in \Sigma$ is called a *geodesic ray* starting from ϕ in [45].

Remark. In [34], we have used $(T)_n$ and T_w in place of T_n and $T(w)$, respectively.

Throughout this paper, T is a countably infinite set and $(T, \mathcal{A})$ is a locally finite tree satisfying $\#(\{v \mid (w, v) \in \mathcal{A}\}) \geq 2$ for any $w \in T$.

Next, we define partitions.

Definition 2.3 (Partition). Let $(K, \mathcal{O})$ be a compact metrizable topological space having no isolated point, where $\mathcal{O}$ is the totality of open sets.

A collection of non-empty compact subsets $\{K_w\}_{w \in T}$ is called a *partition* of K parametrized by $(T, \mathcal{A}, \phi)$ if it satisfies the following conditions (P1) and (P2):

(P1) $K_\phi = K$ and for any $w \in T$, K_w has no isolated point and

$$K_w = \bigcup_{v \in S(w)} K_v.$$

(P2) For any geodesic ray $\omega \in \Sigma$, $\bigcap_{m \geq 0} K_{[\omega]_m}$ is a single point.

Originally in [34], we did not assume that K is connected to include spaces like the Cantor set. In this paper, however, we will only deal with connected spaces. In such cases, the assumption that K has no isolated point is always satisfied unless K is a single point.

As an illustrative example of partitions, we present the case of the unit square $[-1, 1]^2$ as a self-similar set. This is an example of the general construction of partitions associated with self-similar sets discussed in Section 4.1.

Example 2.4 (The unit square). Let $K = [-1, 1]^2$ and let $S = \{1, 2, 3, 4\}$. Set $p_1 = [-1, -1]$, $p_2 = [1, -1]$, $p_3 = [1, 1]$ and $p_4 = [-1, 1]$. For $i \in S$, define $f_i(x) = \frac{1}{2}(x - p_i) + p_i$ for any $x \in \mathbb{R}^2$. Then it is obvious that

$$K = \bigcup_{i \in S} f_i(K).$$

This is the expression of the unit square as the self-similar set with respect to the collection of contractions $\{f_i\}_{i \in S}$. Let

$$T_n = S^n = \{i_1 \dots i_n \mid i_j \in S \text{ for any } j = 1, \dots, n\}.$$

In particular, let $T_0 = \{\phi\}$. Moreover, define $T = \bigcup_{m \geq 0} T_m$ and define $\pi : T \to T$ by

$$\pi(i_1 \dots i_n i_{n+1}) = i_1 \dots i_n$$

for any $i_1 \dots i_n i_{n+1} \in T_{n+1}$ for $n \geq 1$ and $\pi(\phi) = \phi$. Define $\mathcal{A}(w, v)$ for $w, v \in T$ as $\mathcal{A}(w, v) = 1$ if $\pi(w) = v$ or $\pi(v) = w$ except for $(w, v) = (\phi, \phi)$. Then $(T, \mathcal{A}, \phi)$ is a rooted tree. For $w = w_1 \dots w_n \in T_n$, define

$$f_w = f_{w_1} \circ \cdots \circ f_{w_n} \quad \text{and} \quad K_w = f_w(K).$$

Then $\{K_w\}_{w \in T}$ is a partition of K parametrized by $(T, \mathcal{A}, \phi)$. See Figure 2.1.

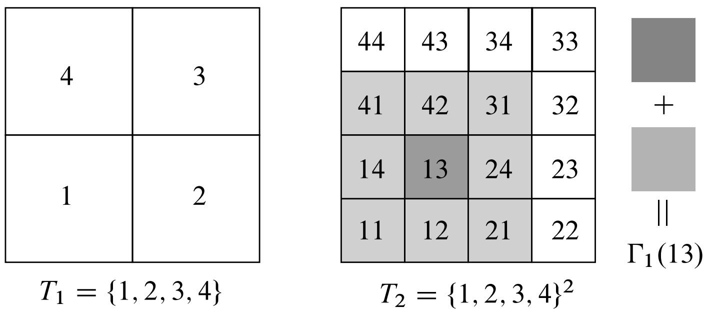


Figure 2.1. Partition of the unit square.

The following definition is a collection of notions concerning partitions.

Definition 2.5. Let $\{K_w\}_{w \in T}$ be a partition of K parametrized by $(T, \mathcal{A}, \phi)$.

(1) Define O_w and B_w for $w \in T$ by

$$O_w = K_w \setminus \Big(\bigcup_{v \in T_{|w|} \setminus \{w\}} K_v \Big), \quad B_w = K_w \cap \Big(\bigcup_{v \in T_{|w|} \setminus \{w\}} K_v \Big).$$

If $O_w \neq \emptyset$ for any $w \in T$, then the partition K is called *minimal*.

(2) For any $A \subseteq T_n$ and $w \in A$, define $\Gamma_M^A(w) \subseteq T_n$ as

$$\Gamma_M^A(w) = \{u \mid u \in A, \text{there exist } u(0), \dots, u(M) \in A \text{ such that } u(0) = w, u(M) = u \text{ and } K_{u(i)} \cap K_{u(i+1)} \neq \emptyset \text{ for any } i = 0, \dots, M-1\}.$$

For simplicity, for $w \in T_n$, we write $\Gamma_M(w) = \Gamma_M^{T_n}(w)$.

(3) $\{K_w\}_{w \in T}$ is called *uniformly finite* if

$$\sup_{w \in T} \#(\Gamma_1(w)) < +\infty.$$

If a partition is minimal, then O_w is actually the interior of K_w, and B_w is the topological boundary of K_w. See [34, Proposition 2.2.3] for details.

In the case of the unit square in Example 2.4, K_w is a square and O_w (resp. B_w) is the interior (resp. the boundary) of K_w. Therefore, it is minimal. Moreover,

$$\sup_{w \in T} \#(\Gamma_1(w)) \leq 8,$$

so that it is uniformly finite.

Now we give the first part of our framework in this paper.

As we declared partially before, through this paper, T is a countably infinite set, $\phi \in T$, $(T, \mathcal{A})$ is a locally finite tree satisfying $\#(\{w | (w, v) \in \mathcal{A}\}) \geq 2$ for any $w \in T$, $(K, \mathcal{O})$ is a compact connected metrizable space and $\{K_w\}_{w \in T}$ is a partition of K parametrized by $(T, \mathcal{A}, \phi)$.

Assumption 2.6. (1) *For any $w \in T$, K_w is connected.*

(2) *There exist M_* and $k_* \in \mathbb{N}$ such that*

$$\pi^{k_*}(\Gamma_{M_*+1}(w)) \subseteq \Gamma_{M_*}(\pi^{k_*}(w)) \tag{2.1}$$

for any $w \in T$.

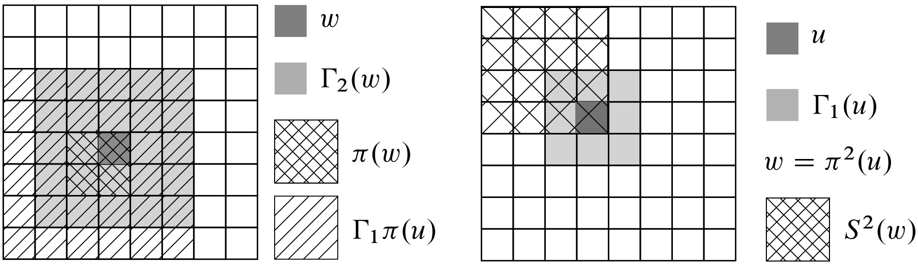
(3) *There exists $M_0 \geq M_*$ such that*

$$\Gamma_{M_*}(u) \cap S^k(w) \subseteq \Gamma_{M_0}^{S^k(w)}(u) \tag{2.2}$$

for any $w \in T$, $k \geq 1$ and $u \in S^k(w)$.

See Figure 2.2 for an illustrative exposition of Assumption 2.6 in the case of the unit square.

Remark. As is explicitly mentioned in Proposition 2.16, Assumption 2.6 (2) is always satisfied under mild additional assumptions.



(2.1): $k_* = 1,\ M_* = 1$ (2.2): $k = 2,\ M_* = M_0 = 1$

Figure 2.2. Assumption 2.6: the unit square.

Remark. If $M_* = 1$, then we have $\Gamma_{M_*}(w) \cap A = \Gamma^A_{M_*}(w)$ for any w and A. So in this case, by choosing $M_0 = M_* = 1$, Assumption 2.6 (3) is always satisfied.

Throughout this paper, we set

$$L_* = \sup_{w \in T} \#(\Gamma_1(w)). \tag{2.3}$$

Then, for any $m \in \mathbb{N}$,

$$\sup_{w \in T} \#(\Gamma_m(w)) \le (L_*)^m.$$

Under Assumption 2.6 (2), if the partition $\{K_w\}_{w \in T}$ is replaced by the partition $\{K_w\}_{w \in T^{(k_*)}}$, where $T^{(k_*)} = \bigcup_{i \ge 0} T_{ik_*}$, the constant k_* can be regarded as 1. So doing such a replacement, we will adopt the following assumption.

Assumption 2.7. *The constant k_* appearing in (2.1) is* 1.

For a given partition $\{K_w\}_{w \in T}$, we always associate the following graph structure E^*_n on T_n.

Proposition 2.8. *For $n \ge 0$, define*

$$E^*_n = \{(w, v) \mid w, v \in T_n,\ w \ne v,\ K_w \cap K_v \ne \emptyset\}.$$

Then (T_n, E^*_n) is a non-directed graph. Under Assumption 2.6, (T_n, E^*_n) is connected for any $n \ge 0$, and

$$\Gamma_1(w) = \{v \mid v \in T_n,\ (w, v) \in E^*_n\}$$

for any $n \ge 0$ and $w \in T_n$.

Remark. In [34], E^*_n is denoted by $J^h_{1,n}$.

Definition 2.9. For $w \in T_n$, define

$$\partial S^m(w) = \{v \mid v \in S^m(w),\ \text{there exists } v' \in T_{n+m} \text{ such that } (v, v') \in E^*_{n+m} \text{ and } \pi^m(v') \ne w\}.$$

The set $\partial S^m(w)$ is a kind of a boundary of $S^m(w)$. In fact, it is easy to see

$$\partial S^m(w) = \{v \mid v \in S^m(w),\ K_v \cap B_w \neq \emptyset\},$$

where B_w is the topological boundary of K_w as is mentioned above. So the next assumption means that the boundary is not the whole space.

Assumption 2.10. *There exists $m_0 \geq 1$ such that $S^m(w) \backslash \partial S^m(w) \neq \emptyset$ for any $w \in T$ and $m \geq m_0$.*

In Figure 2.3, we have an illustrative exposition of Assumption 2.10 in the case of the unit square.

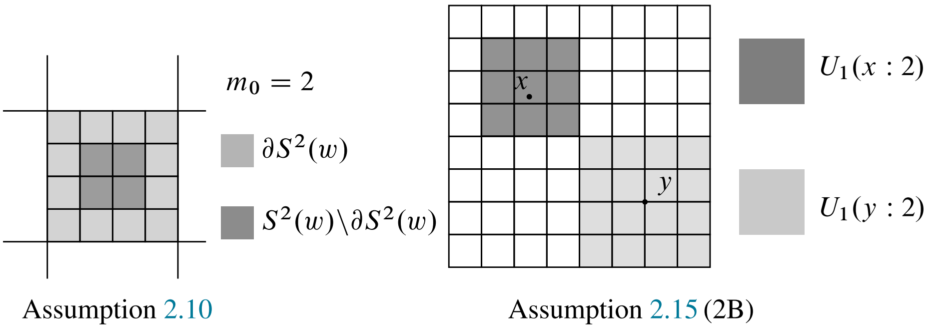


Figure 2.3. Assumptions 2.10 and 2.15 (2B); the unit square.

Definition 2.11. For $w \in T$, $M \geq 1$ and $k \geq 1$, define

$$B_{M,k}(w) = \left\{v \mid v \in S^k(w), \Gamma_{M-1}(v) \cap \partial S^k(w) \neq \emptyset\right\}.$$

Remark. $B_{1,k}(w) = \partial S^k(w)$.

The final assumption is an assumption on a measure μ on K.

Assumption 2.12. *The measure μ is a Borel regular probability measure on K satisfying*

$$\mu(K_w) = \sum_{v \in S(w)} \mu(K_v) \tag{2.4}$$

for any $w \in T$. There exists $\gamma \in (0, 1)$ such that

$$\mu(K_w) \geq \gamma\mu(K_{\pi(w)}) \tag{2.5}$$

for any $w \in T$. This property is called "super-exponential" in [34]. Moreover, there exists $\kappa > 0$ such that if $w, v \in T$, $|w| = |v|$ and $(w, v) \in E^*_{|w|}$, then

$$\mu(K_w) \leq \kappa\mu(K_v) \tag{2.6}$$

The above condition (2.6) corresponds to the gentleness of the measure μ introduced in [34]. Indeed, if μ has the volume doubling property, then this condition is satisfied. See Proposition 2.16 and its proof below for an exact statement.

Lemma 2.13. *Under Assumptions 2.6, 2.10 and 2.12,*

(1) *μ is exponential, i.e., μ satisfies (2.5) and there exist $m_1 \geq 1$ and $\gamma_1 \in (0, 1)$ such that $\mu(K_v) \leq \gamma_1 \mu(K_w)$ for any $w \in T$ and $v \in S^{m_1}(w)$.*

(2) $\sup_{w \in T} \#(S(w)) < \infty$.

Throughout this paper, we set

$$N_* = \sup_{w \in T} \#(S(w)). \tag{2.7}$$

Proof. (1) In fact, we set $m_1 = m_0$. For any w with $|w| \geq 1$ and $m \geq 0$, we see that $\partial S^m(w) \neq \emptyset$ because K is connected. Hence by Assumption 2.10, $\#(S^{m_0}(w)) \geq 2$ for any $w \in T$. Let $v \in S^{m_1}(w)$. Then there exists $u \in S^{m_1}(w)$ with $v \neq u$. By (2.5),

$$\mu(K_w) \geq \mu(K_v) + \mu(K_u) \geq \mu(K_v) + \gamma^{m_1} \mu(K_w),$$

so that $\mu(K_v) \leq (1 - \gamma^{m_1})\mu(K_w)$.

(2) $\mu(K_w) = \sum_{v \in S(w)} \mu(K_v) \geq \gamma \sum_{v \in S(w)} \mu(K_w) = \gamma \#(S(w)) \mu(K_w)$.

Hence $\#(S(w)) \leq \frac{1}{\gamma}$. ∎

Lemma 2.14. *Under Assumptions 2.6, 2.10 and 2.12,*

$$S^m(w) \backslash B_{M,m}(w) \neq \emptyset$$

for any $w \in T$, $M \geq 1$ and $m \geq M m_0$. Moreover,

$$\mu\Big(\bigcup_{v \in S^n(S^m(w) \backslash B_{M,m}(w))} K_v \Big) \geq \gamma^{m_0 M} \mu(K_w) \tag{2.8}$$

for any $w \in T$, $n \geq 0$ and $m \geq M m_0$.

Proof. By Assumption 2.10, we can inductively choose $v_i \in S^{i m_0}(w)$ for $i \geq 1$ such that $v_{i+1} \in S^{m_0}(v_i) \backslash \partial S^{m_0}(v_i)$ for any $i \geq 1$. At the same time, we see $v_i \notin B_{i, i m_0}(w)$. If $m_0 i < k \leq m_0(i+1)$, then $v \notin B_{i,k}(w)$ for $v = \pi^{m_0(i+1)-k}(v_{i+1})$. So the first part of the claim has been verified. Now if $v \in S^m(w) \backslash B_{M,m}(w)$, then

$$\mu\Big(\bigcup_{v \in S^n(S^m(w) \backslash B_{M,m}(w))} K_v \Big) \geq \mu(K_v) \geq \gamma^{m_0 M} \mu(K_w)$$

by Assumption 2.12. ∎

Until now, we have not considered any metric of $(K, \mathcal{O})$, which was merely assumed to be compact and metrizable. The introduction of a metric d on K having suitable properties enables us to integrate the above assumptions into the following one.

Assumption 2.15. *The metric space (K, d) is a compact connected metric space and* $\mathrm{diam}(K, d) = 1$, *where*

$$\mathrm{diam}(A, d) = \sup_{x,y \in A} d(x, y)$$

for a subset $A \subseteq B$. The partition $\{K_w\}_{w \in T}$ is minimal and uniformly finite.

(1) *For any $w \in T$, K_w is connected.*

(2) *There exist $M_* \geq 1$ and $r \in (0, 1)$ such that the following properties hold:*

(2A) *Define $h_r : T \to (0, 1]$ as $h_r(w) = r^{|w|}$. Then there exist $c_1, c_2 > 0$ such that*

$$c_1 h_r(w) \leq \mathrm{diam}(K_w, d) \leq c_2 h_r(w)$$

for any $w \in T$.

(2B) *For $x \in K$ and $n \geq 1$, define*

$$U_M(x : n) = \bigcup_{\substack{w \in T_n \\ x \in K_w}} \bigcup_{v \in \Gamma_M(w)} K_v.$$

(See Figure 2.3 for examples of $U_1(\cdot : 2)$ in the case of the unit square.) Then there exist $c_1, c_2 > 0$ such that

$$B_d(x, c_1 r^n) \subseteq U_{M_*}(x : n) \subseteq B_d(x, c_2 r^n)$$

for any $n \geq 1$ and $x \in K$, where $B_d(x, r) = \{y \mid d(x, y) < r\}$.

(2C) *There exist $c > 0$ such that, for any $n \geq 1$ and $w \in T_n$, there exists $x \in K_w$ such that*

$$K_w \supseteq B_d(x, cr^n).$$

(3) *μ is a Borel regular probability measure on K. Moreover, μ is exponential and has the volume doubling property with respect to the metric d. Furthermore, μ satisfies (2.4) for any $w \in T$.*

(4) *There exists M_0 such that (2.2) holds for any $w \in T$, $k \geq 1$ and $u \in S^k(w)$.*

(5) *For any $w \in T$, $\pi(\Gamma_{M_*+1}(w)) \subseteq \Gamma_{M_*}(\pi(w))$.*

Remark. In the terminology of [34], (2A) corresponds to the bi-Lipschitz equivalence of d and h_r, (2B) says that the metric d is M_*-adapted to h_r and (2C) together with (2B) yields d being thick. The combination of (2A), (2B) and (2C) is equivalent to that of (BF1) and (BF2) in [34, Section 4.3].

Remark. Modifying the original partition $\{K_w\}_{w\in T}$, we may always obtain Assumption 2.15 (5) from Assumption 2.15 (1), (2), (3), and (4). Namely, by Proposition 2.16, we have k_* satisfying (2.1) under Assumption 2.15 (1), (2), (3) and (4). So, replacing the original partition $\{K_w\}_{w\in T}$ with $\{K_w\}_{w\in T^{(k_*)}}$, we may suppose $k_* = 1$.

Proposition 2.16. *Assumption* 2.15 (1), (2), (3) *and* (4) *suffice Assumptions* 2.6, 2.10 *and* 2.12.

Proof. About Assumption 2.6, (1) and (3) are included in Assumption 2.15. Since d is M_*-adapted, [34, Proposition 4.4.4] shows the existence of k_* required in Assumption 2.6 (2). By (2C) and (2B), there exists $m_0 \geq 1$ such that

$$K_w \supseteq B_d(x, cr^n) \supseteq U_{M_*}(x : n + m_0)$$

for any $n \geq 1$ and $w \in T_n$, where the point $x \in K_w$ is chosen as in (2C). So if $v \in T_{n+m_0}$ and $x \in K_v$, then $K_v \subseteq B_d(x, cr^n)$ and hence $K_v \cap B_w = \emptyset$. Therefore, Assumption 2.10 is satisfied. Assumption 2.15 includes (2.4) and (2.5) follows from the fact that μ is exponential. Finally, (2.6) is a consequence of the volume doubling property by [34, Theorem 3.3.4]. ∎

Under Assumption 2.15, we may suppose further properties of the metric d and the measure μ. Namely, if $\alpha > \dim_{AR}(K, d)$, then by (1.1), there exist an α-Ahlfors regular metric d_* which is quasisymmetric to d and a Borel regular measure ν which is α-Ahlfors regular with respect to d_*, i.e., there exist $c_1, c_2 > 0$ such that

$$c_1 r^\alpha \leq \nu(B_{d_*}(x, r)) \leq c_2 r^\alpha \tag{2.9}$$

for any $x \in K$ and $r \in (0, 2\mathrm{diam}(K, d)]$. Replacing d and μ by d_* and ν, respectively, we may assume that d is α-Ahlfors regular. Note that if μ is α-Ahlfors regular with respect to d, then α is the Hausdorff dimension of (K, d).

2.2 Conductance constant

In this section, we introduce the conductance constant $\mathcal{E}_{M,p,m}(w, A)$ and show the existence of a partition of unity whose p-energies are estimated by conductance constants from above. In the next section, using the method of combinatorial modulus, we will establish a sub-multiplicative inequality of conductance constants.

Through this section, T is a countably infinite set, $\phi \in T$, $(T, \mathcal{A})$ is a locally finite tree satisfying $\#(\{w | (w, v) \in \mathcal{A}\}) \geq 2$ for any $w \in T$, $(K, \mathcal{O})$ is a compact connected metrizable space and $\{K_w\}_{w\in T}$ is a partition of K parametrized by $(T, \mathcal{A}, \phi)$. Moreover, hereafter in this paper, we always presume Assumptions 2.6, 2.7, 2.10 and 2.12.

To begin with, we define p-energies of functions on graphs (T_n, E_n^*) and the associated p-conductances between subsets.

Notation. Let A be a set. Set

$$\ell(A) = \{f \mid f\colon A \to \mathbb{R}\}. \tag{2.10}$$

Definition 2.17. (1) Let $A \subseteq T_n$. For $f \in \ell(A)$, define $\mathcal{E}^n_{p,A}(f)$ by

$$\mathcal{E}^n_{p,A}(f) = \frac{1}{2} \sum_{u,v \in A, (u,v) \in E_n^*} |f(u) - f(v)|^p.$$

In particular, if $A = T_n$, we define $\mathcal{E}^n_p(f) = \mathcal{E}^n_{p,T_n}(f)$ for $f \in \ell(T_n)$.

(2) Let $A \subseteq T_n$ and let $A_1, A_2 \subseteq A$. Define

$$\mathcal{E}_{p,m}(A_1, A_2, A) = \inf\big\{\mathcal{E}^{n+m}_{p,S^m(A)}(f) \mid f \in \ell(S^m(A)), f|_{S^m(A_1)} \equiv 1, \\ f|_{S^m(A_2)} \equiv 0\big\}.$$

(3) Let $A \subseteq T_n$. For $w \in A$, define

$$\mathcal{E}_{M,p,m}(w, A) = \mathcal{E}_{p,m}(\{w\}, A \backslash \Gamma^A_M(w), A),$$

which is called the *p-conductance constant* of w in A at level m.

For simplicity, we often denote a set consisting of a single point, $\{w\}$, by w. For example, if A_1 and A_2 are single points u and v respectively, we sometimes write $\mathcal{E}_{p,m}(u, v, A)$ instead of $\mathcal{E}_{p,m}(\{u\}, \{v\}, A)$.

Remark. As we have mentioned in the introduction, the quantity $\mathcal{E}_{M,p,m}(w, A)$ can be regarded as "p-capacity" from the viewpoint of the potential theory.

Lemma 2.18. *For any $w \in T$, $k \geq 0$ and $u \in S^k(w)$,*

$$\mathcal{E}_{M_0,p,m}(u, S^k(w)) \leq \mathcal{E}_{M_*,p,m}(u, T_{|w|+k}).$$

Proof. This follows from Assumption 2.6 (3). ■

Remark. In the case $M_* = 1$, we always have $\Gamma^A_1(w) = \Gamma_1(w) \cap A$. Hence even without (2.2),

$$\mathcal{E}_{1,p,m}(w, S^k(w)) \leq \mathcal{E}_{1,p,m}(w, T_{|w|+k})$$

for any $w \in T$, $k \geq 0$ and $u \in S^k(w)$.

The following lemma shows the existence of a partition of unity.

Lemma 2.19. *Let $p \geq 1$ and let $A \subseteq T_n$. For any $w \in A$, there exists $\varphi_w\colon S^m(A) \to [0,1]$ such that*

$$\sum_{w \in A} \varphi_w \equiv 1, \quad \varphi_w|_{S^m(w)} \geq (L_*)^{-M}, \quad \varphi_w|_{S^m(A) \setminus S^m(\Gamma_M^A(w))} \equiv 0$$

and

$$\mathcal{E}_{p,S^m(A)}^{n+m}(\varphi_w) \leq ((L_*)^{2M+1} + 1)^p \max_{w' \in \Gamma_{2M+1}^A(w)} \mathcal{E}_{M,p,m}(w', A).$$

Proof. Choose $h_w \in \ell(S^m(A))$ such that $h_w|_{S^m(w)} \equiv 1$, $h_w|_{S^m(A) \setminus S^m(\Gamma_M^A(w))} \equiv 0$, and $\mathcal{E}_{M,p,m}(w, A) = \mathcal{E}_{p,S^m(A)}^{n+m}(h_w)$. Define $h \in \ell(S^m(A))$ as

$$h(v) = \sum_{w \in A} h_w(v)$$

for any $v \in S^m(A)$. Note that $1 \leq h(v) \leq (L_*)^M$. Set

$$\varphi_w = \frac{h_w}{h} \quad \text{and} \quad E_{n+m}(w) = E_{n+m}^* \cap S^m(\Gamma_{M+1}^A(w))^2.$$

It follows that $\varphi_w(u) = \varphi_w(v) = 0$ for any $(u,v) \notin E_{n+m}(w)$. Let $(u,v) \in E_{n+m}(w)$. Then $h_w(v)(h_{w'}(v) - h_{w'}(u)) = 0$ for any $w' \notin \Gamma_{2M+1}^A(w)$. Hence

$$\begin{aligned}
|\varphi_w(u) - \varphi_w(v)| &= \left| \frac{1}{h(u)h(v)} (h(v)(h_w(u) - h_w(v)) + h_w(v)(h(v) - h(u))) \right| \\
&\leq |h_w(u) - h_w(v)| + \sum_{w' \in \Gamma_{2M+1}^A(w)} |h_{w'}(u) - h_{w'}(v)|.
\end{aligned}$$

Set $C = (L_*)^{2M+1} + 1$. Then the last inequality yields

$$\begin{aligned}
\mathcal{E}_p^{n+m}(\varphi_w) &= \frac{1}{2} \sum_{(u,v) \in E_{n+m}(w)} |\varphi_w(u) - \varphi_w(v)|^p \\
&\leq \frac{C^{p-1}}{2} \sum_{(u,v) \in E_{n+m}(w)} \Big(|h_w(u) - h_w(v)|^p \\
&\qquad + \sum_{w' \in \Gamma_{2M+1}^A(w)} |h_{w'}(u) - h_{w'}(v)|^p \Big) \\
&\leq C^{p-1} \Big(\mathcal{E}_{p,S^m(A)}^{n+m}(h_w) + \sum_{w' \in \Gamma_{2M+1}^A(w)} \mathcal{E}_{p,S^m(A)}^{n+m}(h_{w'}) \Big) \\
&\leq C^p \max_{w' \in \Gamma_{2M+1}^A(w)} \mathcal{E}_{M,p,m}(w', A).
\end{aligned}$$

■

In particular, in the case $A = T_n$, the associated partition of unity defined below will be used to show the regularity of the p-energy constructed in Section 3.1.

Definition 2.20. For $w \in T$, define $h^*_{M,w,m} \in \ell(T_{|w|+m})$ as the unique function h satisfying $h|_{S^m(w)} = 1$, $h|_{T_{|w|+m} \setminus S^m(\Gamma_M(w))} = 0$ and

$$\mathcal{E}_p^{|w|+m}(h) = \mathcal{E}_{M,p,m}(w, T_{|w|}).$$

Moreover, define $\varphi^*_{M,w,m} \in \ell(T_{|w|+m})$ by

$$\varphi^*_{M,w,m} = \frac{h^*_{M,w,m}}{\sum_{v \in T_{|w|}} h^*_{M,v,m}}.$$

By the proof of Lemma 2.19,

$$\mathcal{E}_p^{n+m}(\varphi^*_{M,w,m}) \leq ((L_*)^{2M+1} + 1)^p \max_{v \in T_n} \mathcal{E}_{M,p,m}(v, T_n)$$

for any $w \in T_n$.

2.3 Combinatorial modulus

Another principal tool of this paper is the notion of combinatorial modulus of a path family of a graph introduced in [11]. The general theory will be briefly reviewed in Appendix 6.3. In this section, we introduce the notion of the p-modulus of paths between two sets and show a sub-multiplicative inequality for them. As in the last section, T is a countably infinite set, $\phi \in T$, $(T, \mathcal{A})$ is a locally finite tree satisfying $\#(\{w | (w, v) \in \mathcal{A}\}) \geq 2$ for any $w \in T$, $(K, \mathcal{O})$ is a compact connected metrizable space and $\{K_w\}_{w \in T}$ is a partition of K parametrized by $(T, \mathcal{A}, \phi)$.

Definition 2.21. Let $M, m \in \mathbb{N}$.

(1) Define

$$E^*_{M,m} = \{(w, v) \mid w, v \in T_m, v \in \Gamma_M(w)\}.$$

Note that $E^*_m = E^*_{1,m}$. Moreover, define

$$\theta_m(w, v) = \min\{M \mid v \in \Gamma_M(w)\}$$

for $w, v \in T_m$. $\theta_m(w, v)$ is called the *graph distance* of the graph (T_m, E^*_m).

(2) Let $A \subseteq T_n$ and let $A_1, A_2 \subseteq A$. For $k \geq 0$, define

$$\begin{aligned} \mathcal{C}_m^{(M)}(A_1, A_2, A) = \{&(v(1), \ldots, v(l)) \mid v(i) \in S^m(A) \text{ for any } i = 1, \ldots, l, \\ &\text{there exist } v(0) \in S^m(A_1) \text{ and } v(l+1) \in S^m(A_2) \text{ such} \\ &\text{that } (v(i), v(i+1)) \in E^*_{M,n+m} \text{ for any } i = 0, \ldots, l\}, \end{aligned} \tag{2.11}$$

$$\begin{aligned} \mathcal{A}_m^{(M)}(A_1, A_2, A) = \Big\{&f \mid f : T_{n+m} \to [0, \infty), \textstyle\sum_{i=1}^l f(w(i)) \geq 1 \\ &\text{for any } (w(1), \ldots, w(l)) \in \mathcal{C}_m^{(M)}(A_1, A_2, A)\Big\} \end{aligned}$$

and

$$\mathcal{M}_{p,m}^{(M)}(A_1, A_2, A) = \inf_{f \in \mathcal{A}_m^{(M)}(A_1,A_2,A)} \sum_{u \in T_{n+m}} f(u)^p. \tag{2.12}$$

(3) For $w \in T_n$, define

$$\mathcal{C}_{N,m}^{(M)}(w) = \mathcal{C}_m^{(M)}(\{w\}, \Gamma_N(w)^c, T_n), \mathcal{A}_{N,m}^{(M)}(w) = \mathcal{A}_m^{(M)}(\{w\}, \Gamma_N(w)^c, T_n)$$

and

$$\mathcal{M}_{N,p,m}^{(M)}(w) = \mathcal{M}_{p,m}^{(M)}(\{w\}, \Gamma_N(w)^c, T_n).$$

The quantity $\mathcal{M}_{p,m}^{(M)}(A_1, A_2, A)$ is called the *p-modulus* of the family of paths between A_1 and A_2 inside A.

Remark. In (2.11) and (2.12), the domain of f is T_{n+m}. However, since we only use $f(u)$ for $u \in S^m(A)$ in (2.11) and the sum in (2.12) becomes smaller by setting $f(u) = 0$ for $u \in T_{n+m} \backslash S^m(A)$, we may think of the domain of f as $S^m(A)$.

As in the case of conductances, if A_1 and A_2 consist of single points u and v, respectively, then we write $\mathcal{C}_m^{(M)}(u, v, A)$, $\mathcal{A}_m^{(M)}(u, v, A)$ and $\mathcal{M}_{p,m}^{(M)}(u, v, A)$ instead of $\mathcal{C}_m^{(M)}(\{u\}, \{v\}, A)$, $\mathcal{A}_m^{(M)}(\{u\}, \{v\}, A)$ and $\mathcal{M}_{p,m}^{(M)}(\{u\}, \{v\}, A)$, respectively.

In accordance with [34, Proposition 4.8.4], the following simple relation between $\mathcal{E}_{p,m}(A_1, A_2, A)$ and $\mathcal{M}_{p,m}^{(1)}(A_1, A_2, A)$ holds. Hence to know $\mathcal{M}_{p,m}^{(1)}(A_1, A_2, A)$ is essentially to know $\mathcal{E}_{p,m}(A_1, A_2, A)$.

Lemma 2.22. *Let $A \subseteq T_n$ and let $A_1, A_2 \subseteq A$ with $A_1 \cap A_2 = \emptyset$. Then for any $m \geq 1$ and $p > 0$,*

$$\begin{aligned}\frac{1}{L_*}\mathcal{E}_{p,m}(A_1, A_2, A) &\leq \mathcal{M}_{p,m}^{(1)}(A_1, A_2, A) \\ &\leq 2\max\{1, (L_*)^{p-1}\}\mathcal{E}_{p,m}(A_1, A_2, A).\end{aligned} \tag{2.13}$$

The following theorem is the main result of this section.

Theorem 2.23 (Sub-multiplicative inequality). *Let $k_0, L, M \in \mathbb{N}$. Suppose that*

$$\pi^{k_0}(\Gamma_{L+1}(u)) \subseteq \Gamma_M(\pi^{k_0}(u))$$

for any $u \in T$. Then

$$\mathcal{M}_{M,p,k+l}^{(1)}(w) \leq c_{2.23}\mathcal{M}_{M,p,k}^{(1)}(w) \max_{v \in S^k(\Gamma_M(w))} \mathcal{M}_{L,p,l}^{(1)}(v)$$

for any $l \in \mathbb{N}$, $k \geq k_0$, $w \in T$ and $p > 0$, where $c_{2.23}$ depends only on p, L_* and L.

Remark. If $\pi^{k_0}(\Gamma_{L+1}(u)) \subseteq \Gamma_M(\pi^{k_0}(u))$, then $\pi^k(\Gamma_{L+1}(u)) \subseteq \Gamma_M(\pi^k(u))$ for any $k \geq k_0$.

Similar sub-multiplicative inequalities for moduli of curve families have been shown in [11, Proposition 3.6], [14, Lemma 3.8] and [34, Lemma 4.9.3].

By Assumption 2.7, the assumption $\pi^{k_0}(\Gamma_{L+1}(u)) \subseteq \Gamma_M(\pi^{k_0}(u))$ is satisfied with $M = L = M_*$ and $k_0 = 1$. This fact along with Lemma 2.22 shows the following sub-multiplicative inequality of conductance constants.

Corollary 2.24. *For any $n, k, l \geq 1$, $w \in T_n$ and $p \geq 1$.*

$$\mathcal{E}_{M_*,p,k+l}(w, T_n) \leq c_{2.24}\mathcal{E}_{M_*,p,k}(w, T_n) \max_{v \in S^k(\Gamma_M(w))} \mathcal{E}_{M_*,p,l}(v, T_{n+k}), \qquad (2.14)$$

where the constant $c_{2.24} = c_{2.24}(p, L_*, M_*)$ depends only on p, L_* and M_*.

The rest of this section is devoted to a proof of Theorem 2.23.

Lemma 2.25. *Let $A \subseteq T_n$ and let $A_1, A_2 \subseteq A$ with $A_1 \cap A_2 = \emptyset$. Assume that $\Gamma_M(u) \cap S^m(A)$ is connected for any $u \in S^m(A)$. Then*

$$\mathcal{M}^{(1)}_{p,m}(A_1, A_2, A) \leq \mathcal{M}^{(M)}_{p,m}(A_1, A_2, A) \leq (L_*)^{(p+1)M}\mathcal{M}^{(1)}_{p,m}(A_1, A_2, A).$$

Proof. By definition,

$$\mathcal{C}^{(M)}_m(A_1, A_2, A) \supseteq \mathcal{C}^{(1)}_m(A_1, A_2, A) \quad \text{and} \quad \mathcal{A}^{(M)}_m(A_1, A_2, A) \subseteq \mathcal{A}^{(1)}_m(A_1, A_2, A).$$

This shows

$$\mathcal{M}^{(1)}_{p,m}(A_1, A_2, A) \leq \mathcal{M}^{(M)}_{p,m}(A_1, A_2, A).$$

Define $H_u = \Gamma_M(u)$ for any $u \in T_{n+m}$. Then

$$\#(H_u) \leq (L_*)^M \quad \text{and} \quad \#(\{v \mid u \in H_v\}) \leq (L_*)^M.$$

Let $(u(1), \ldots, u(l)) \in \mathcal{C}^{(M)}_m(A_1, A_2, A)$. Then there exist $u(0) \in S^m(A_1) \cap \Gamma_M(u(1))$ and $u(l+1) \in S^m(A_2) \cap \Gamma_M(u(l))$. Since $u(0)$ and $u(1)$ is connected by a chain in $\Gamma_M(u(1))$ and $u(i)$ and $u(i+1)$ is connected by a chain for $i = 1, \ldots, l$ in $\Gamma_M(u(i))$, we have a chain belonging to $\mathcal{C}^{(1)}_m(A_1, A_2, A)$ and contained in $\bigcup_{i=1,\ldots,n} H_{u(i)}$. Thus Lemma C.4 shows

$$\mathcal{M}^{(M)}_{p,m}(A_1, A_2, A) \leq (L_*)^{(p+1)M}\mathcal{M}^{(1)}_{p,m}(A_1, A_2, A). \qquad \blacksquare$$

Proof of Theorem 2.23. Let $f \in \mathcal{A}^{(L+1)}_{M,k}(w)$ and let $g_v \in \mathcal{A}^{(1)}_{L,l}(v)$ for any $v \in T_{|w|+k}$. Define $h\colon T_{|w|+k+l} \to [0, \infty)$ by

$$h(u) = \max\left\{f(v)g_v(u) \mid v \in \Gamma_L(\pi^l(u)) \cap S^k(\Gamma_M(w))\right\}\chi_{S^{k+l}(\Gamma_M(w))}(u).$$

Claim 1. $h \in \mathcal{A}^{(1)}_{M,k+l}(w)$.

Proof. Let $(u(1), \ldots, u(m)) \in \mathcal{C}^{(1)}_{M,k+l}(w)$. There exist such $u(0) \in S^{k+l}(w)$ and $u(m+1) \in T_{|w|+k+l} \setminus S^{k+l}(\Gamma_M(w))$ that $u(0) \in \Gamma_1(u(1))$ and $u(m+1) \in \Gamma_1(u(m))$. Set $v(i) = \pi^l(u(i))$ for $i = 0, \ldots, m+1$. Let $v_*(0) = v(0)$ and let $i_0 = 0$. Define n_*, $v_*(n)$ and i_n for $i = 1, \ldots, n_*$ inductively as follows: If

$$\max\{j \mid i_n \le j \le m, v(j) \in \Gamma_L(v_*(n))\} = m,$$

then $n = n_*$. If

$$\max\{j \mid i_n \le j \le m, v(j) \in \Gamma_L(v_*(n))\} < m,$$

then define

$$i_{n+1} = \max\{j \mid i_n \le j \le m, v(j) \in \Gamma_L(v_*(n))\} + 1 \quad \text{and} \quad v_*(n+1) = v(i_{n+1}).$$

The fact that $\pi^k(\Gamma_{L+1}(v_*(0))) \subseteq \Gamma_M(\pi^k(v(0)))$ implies $n_* \ge 1$. Since $v(i_{n+1} - 1) \in \Gamma_L(v_*(n))$, we have $v_*(n+1) \in \Gamma_{L+1}(v_*(n))$. Hence

$$(v_*(1), \ldots, v_*(n_*)) \in \mathcal{C}^{(L+1)}_{M,k}(w).$$

Moreover, since $v_*(n-1) \notin \Gamma_L(v_*(n))$ for $n = 1, \ldots, n_*$, there exist j_n and m_n such that $i_{n-1} < j_n \le m_n < i_n$ and $(u(j_n), \ldots, u(m_n)) \in \mathcal{C}^{(1)}_{L,l}(v_*(n))$. Since $g_{v_*(n)} \in \mathcal{A}^{(1)}_{L,l}(v_*(n))$, we have

$$\sum_{i=j_n}^{m_n} h(u(i)) \ge \sum_{i=j_n}^{m_n} f(v_*(n)) g_{v_*(n)}(u(i)) \ge f(v_*(n)).$$

This and the fact that $(v_*(1), \ldots, v_*(n_*)) \in \mathcal{C}^{(L+1)}_{M,k}(w)$ yield

$$\sum_{i=1}^{m} h(u(i)) \ge \sum_{j=1}^{n_*} f(v_*(j)) \ge 1.$$

Thus Claim 1 has been verified. □

Set $C_0 = \max\{(L_*)^{L(p-1)}, 1\}$. Then by Lemma A.1, for $u \in S^{k+l}(\Gamma_M(w))$,

$$\begin{aligned} h(u)^p &\le \Bigg(\sum_{v \in \Gamma_L(\pi^l(u)) \cap S^k(\Gamma_M(w))} f(v) g_v(u) \Bigg)^p \\ &\le C_0 \sum_{v \in \Gamma_L(\pi^l(u)) \cap S^k(\Gamma_M(w))} f(v)^p g_v(u)^p. \end{aligned}$$

The above inequality and Claim 1 yield

$$\mathcal{M}^{(1)}_{M,p,k+l}(w) \le \sum_{u \in S^{k+l}(\Gamma_M(w))} h(u)^p \le C_0 \sum_{v \in S^k(\Gamma_M(w))} \sum_{u \in T_{|w|+k+l}} f(v)^p g_v(u)^p.$$

Taking infimum over $g_v \in \mathcal{A}^{(1)}_{L,l}(v)$ and $f \in \mathcal{A}^{(L+1)}_{M,k}(w)$, we have

$$\begin{aligned}\mathcal{M}^{(1)}_{M,p,k+l}(w) &\leq C \sum_{v \in S^k(\Gamma_M(w))} f(v)^p \mathcal{M}^{(1)}_{L,p,l}(v) \\ &\leq C_0 \sum_{v \in T_{|w|+k}} f(v)^p \max_{v \in S^k(\Gamma_M(w))} \mathcal{M}^{(1)}_{L,p,l}(v) \\ &\leq C_0 \mathcal{M}^{(L+1)}_{M,p,k}(w) \max_{v \in S^k(\Gamma_M(w))} \mathcal{M}^{(1)}_{L,p,l}(v).\end{aligned}$$

Finally, applying Lemma 2.25, we have the desired inequality. This completes the proof of Theorem 2.23. ■

2.4 Neighbor disparity constant

Another important constant in this paper is $\sigma_{p,m}(\cdot)$, which is called the neighbor disparity constant. The neighbor disparity constant controls the differences between means of a function on several cells via the p-energy of the function. For $p = 2$, $\sigma_{2,m}$ was introduced in [36] for the case of self-similar sets.

Notation. For $A \subseteq T_n$ and $f \in \ell(A)$, define

$$(f)_A = \frac{1}{\sum_{v \in A} \mu(K_w)} \sum_{v \in A} f(w)\mu(K_w).$$

Furthermore, set

$$E^*_n(A) = (A \times A) \cap E^*_n. \tag{2.15}$$

Definition 2.26. Let $A \subseteq T_n$.

(1) Define $P_{n,m}\colon \ell(S^m(A)) \to \ell(A)$ by

$$(P_{n,m} f)(w) = (f)_{S^m(w)}$$

for any $f \in \ell(S^m(A))$ and $w \in A$.

(2) For $m \geq 0$ and $p \geq 1$, define

$$\sigma_{p,m}(A) = \sup_{f \in \ell(S^m(A))} \frac{\mathcal{E}^n_{p,A}(P_{n,m} f)}{\mathcal{E}^{n+m}_{p,S^m(A)}(f)},$$

which is called the *p-neighbor disparity constant* of A at level m.

(3) Let $\{G_i\}_{i=1,\ldots,k}$ be a collection of subsets of T_n. The family $\{G_i\}_{i=1,\ldots,k}$ is called a *covering* of $(A, E^*_n(A))$ with *covering numbers* (N_T, N_E) if

$$A = \bigcup_{i=1}^{k} G_i, \quad \max_{x \in A} \#(\{i \mid x \in G_i\}) \leq N_T,$$

and for any $(u,v) \in E_n^*(A)$, there exist $l \le N_E$ and $\{w(1),\dots,w(l+1)\} \subseteq A$ such that $w(1) = u$, $w(l+1) = v$ and $(w(i), w(i+1)) \in \bigcup_{j=1,\dots,k} E_n^*(G_j)$ for any $i = 1,\dots,l$.

Remark. The neighbor disparity constant $\sigma_{p,m}(w,v)$ defined in the introduction is equal to $\sigma_{p,m}(A)$ with $A = \{w,v\}$.

One of the advantages of neighbor disparity constants is their compatibility with the integral projection $P_{n,m}$ from $\ell(T_{n+m})$ to $\ell(T_n)$ as follows.

Lemma 2.27 ([36, Lemma 2.12]). *Let A be a connected subset of T_n, let $m \ge 0$ and let $\{G_i\}_{i=1}^k$ be a covering of $(A, E_n^*(A))$ with covering numbers (N_T, N_E). Then*

$$\mathcal{E}_{p,A}^n(P_{n,m}f) \le c_{2.27} \max_{i=1,\dots,k} \sigma_{p,m}(G_i)\mathcal{E}_{p,S^m(A)}^{n+m}(f)$$

for any $f \in \ell(S^m(A))$, where $c_{2.27} = (L_*)^{N_E}(N_E)^{p-1}N_T$, and

$$\sigma_{p,m}(A) \le c_{2.27} \max_{i=1,\dots,k} \sigma_{p,m}(G_i).$$

In particular, if $A_1, A_2 \subseteq A$, then

$$\mathcal{E}_{p,0}(A_1,A_2,A) \le c_{2.27} \max_{i=1,\dots,k} \sigma_{p,m}(G_i)\mathcal{E}_{p,m}(A_1,A_2,A) \tag{2.16}$$

for any $m \ge 0$.

Proof. For $(w,v) \in E_n^*$, set

$$\begin{aligned} D_l(w,v) = \{(u_1,u_2) \mid\ & (u_1,u_2) \in E_n^*, \text{there exists } (w(1),\dots,w(l),w(l+1)) \\ & \text{such that } w(1) = u_1, w(l+1) = u_2 \\ & \text{and } (w(i),w(i+1)) = (w,v) \text{ for some } i = 1,\dots,l\}. \end{aligned}$$

If $(u_1,u_2) \in D_l(w,v)$, then $u_1 \in \Gamma_{l-1}(w)$ and $u_2 \in \Gamma_1(u_1)$. Hence

$$\#(D_l(w,v)) \le (L_*)^l.$$

Since $\{G_i\}_{i=1,\dots,k}$ is a covering of A with covering numbers (N_T, N_E), we have

$$\begin{aligned} \mathcal{E}_{p,A}^n(P_{n,m}f) &= \frac{1}{2}\sum_{(u_1,u_2)\in E_n^*(A)} |P_{n,m}(f)(u_1) - P_{n,m}(f)(u_2)|^p \\ &\le (N_E)^{p-1} \max_{(w,v)\in E_n^*} \#(D_{N_E}(w,v)) \sum_{i=1}^k \mathcal{E}_{p,G_i}^m(P_{n,m}f) \\ &\le (L_*)^{N_E}(N_E)^{p-1} \sum_{i=1}^k \sigma_{p,m}(G_i)\mathcal{E}_{p,S^m(G_i)}^{n+m}(f) \\ &\le c_{2.27} \max_{i=1,\dots,k} \sigma_{p,m}(G_i)\mathcal{E}_{p,S^m(A)}^{n+m}(f). \end{aligned}$$

Next, choose f such that $f|_{A_1} \equiv 1$, $f|_{A_0} \equiv 0$ and $\mathcal{E}_{p,m}(A_1, A_2, A) = \mathcal{E}^{n+m}_{p,S^m(A)}(f)$. Then

$$\mathcal{E}_{p,0}(A_1, A_2, A) \le \mathcal{E}^n_{p,A}(P_{n,m} f).$$

So we have (2.16). ∎

Lemma 2.28 ([36, Proposition 2.13 (3)]). *Let $p \ge 1$ and let $A \subseteq T_k$. If $\{B_i\}_{i=1,\ldots,l}$ is a covering of $(S^n(A), E^*_{k+n}(S^n(A)))$ with covering number (N_T, N_E), then*

$$\sigma_{p,n+m}(A) \le c_{2.27}\sigma_{p,n}(A) \max_{i=1,\ldots,l} \sigma_{p,m}(B_i).$$

Proof. By Lemma 2.27, for any $f \in \ell(T_{k+n+m})$,

$$\begin{aligned}\mathcal{E}^k_{p,A}(P_{k,n}(P_{k+n,m} f)) &\le \sigma_{p,n}(A)\mathcal{E}^{k+n}_{p,S^n(A)}(P_{k+n,m} f)\\ &\le \sigma_{p,n} c_{2.27}\sigma_{p,n}(A) \max_{i=1,\ldots,l} \sigma_{p,m}(B_i)\mathcal{E}^{k+n+m}_{p,S^{m+n}(A)}(f).\end{aligned}$$ ∎

Due to Theorem 3.33, we will see that if $p > \dim_{AR}(K, d)$, then it is enough to consider neighbor disparity constants for a family $\mathcal{J}_* = \{\{w, v\} | (w, v) \in \bigcup_{n\ge0} E^*_n\}$. As we will mention right after Example 2.30, however, allowing all the pairs in $\mathcal{J}_*$ might cause a trouble, so that we need the following notion of a covering system in general.

Definition 2.29. Let $\mathcal{J} \subseteq \bigcup_{n\ge0}\{A \mid A \subseteq T_n\}$. The collection $\mathcal{J}$ is called a *covering system* with *covering numbers* (N_T, N_E) if the following conditions are satisfied:

(1) $\sup_{A\in\mathcal{J}} \#(A) < \infty$.

(2) For any $w \in T$ and $m \ge 1$, there exists a finite subset $\mathcal{N} \subseteq \mathcal{J}$ such that $\mathcal{N}$ is a covering of $(S^m(w), E^*_{n+m}(S^m(w)))$ with covering numbers (N_T, N_E).

(3) For any $G \in \mathcal{J}$ and $m \ge 0$, if $G \subseteq T_n$, then there exists a finite subset $\mathcal{N} \subseteq \mathcal{J}$ such that $\mathcal{N}$ is a covering of $(S^m(G), E^*_{n+m}(S^m(G)))$ with covering numbers (N_T, N_E).

For a covering system $\mathcal{J}$, set

$$\sigma^{\mathcal{J}}_{p,m,n} = \max\{\sigma_{p,m}(A) \mid A \in \mathcal{J}, A \subseteq T_n\} \quad \text{and} \quad \sigma^{\mathcal{J}}_{p,m} = \sup_{n\ge0} \sigma^{\mathcal{J}}_{p,m,n}.$$

Remark. By (2.6), applying Theorem 6.10, we see that

$$0 < \sigma^{\mathcal{J}}_{p,m,n} < \infty \quad \text{and} \quad 0 < \sigma^{\mathcal{J}}_{p,m} < \infty.$$

Example 2.30. Define

$$\mathcal{J}_* = \{\{w, v\} \mid (w, v) \in E^*_n \text{ for some } n \ge 0\}.$$

Then $\mathcal{J}_*$ is a covering system with covering numbers $(L_*, 1)$.

If we allow all the pairs in $\mathcal{J}_*$, we may end up with the following situation.

Proposition 2.31. *Let $\mathcal{J}$ be a covering system and let $\{w, v\} \in \mathcal{J}$. Assume $K_w \cap K_v$ is a single point $\{x\}$, and for any $m \geq 0$, there exist $w' \in S^m(w)$ and $v' \in S^m(v)$ such that $\{w', v'\} = \{u \mid u \in T_{n+m}, x \in K_u\}$. Then*

$$\sigma^{\mathcal{J}}_{p,m,|w|} \geq 1 \quad \text{and} \quad \sigma^{\mathcal{J}}_{p,m} \geq 1$$

for any $p > 0$ and $m \geq 0$.

Proof. Set $n = |w|$. Let $f = \chi_{S^m(w)}$. Then $P_{n,m} f = \chi_{\{w\}}$. Hence

$$\mathcal{E}^{n+m}_{p,S^m(w)\cup S^m(v)}(f) = 1 \quad \text{and} \quad \mathcal{E}^{n}_{p,\{w,v\}}(P_{n,m} f) = 1,$$

so that $\sigma_{p,m}(\{w, v\}) \geq 1$. ■

As we will observe in the following sections, the consequence of the above proposition should be avoided if $p < \dim_{AR}(K, d)$ because we expect (but do not have a proof in general) that $\lim_{m\to 0} \sigma^{\mathcal{J}}_{p,m} = 0$ for $p < \dim_{AR}(K, d)$. For example, a suitable substitute of $\mathcal{J}_*$ for the unit square described in Example 2.4 is given as follows.

Example 2.32. Let K be the unit square $[-1, 1]^2$ treated in Example 2.4. Define

$$\mathcal{J}_\ell = \{\{w, v\} \mid \{w, v\} \in \mathcal{J}_*, K_w \cap K_v \text{ is a line segment}\},$$

where the subscript ℓ in $\mathcal{J}_\ell$ represents the word "line". Then $\mathcal{J}_\ell$ is a covering system with covering numbers $(4, 2)$. Note that no $\{w, v\} \in \mathcal{J}_\ell$ satisfies the assumption of Proposition 2.31.

Similar modification of $\mathcal{J}_*$ can be made in the case of subsystems of cubic tilings studied in Section 4.3 including the Sierpiński carpet. See (4.15) for details.

Now, we start to investigate the properties of the neighbor disparity constants of a fixed covering system.

The following lemma is a consequence of Lemma 2.27 connecting the conductance constants with the neighbor disparity constants.

Lemma 2.33. *Let $\mathcal{J}$ be a covering system with covering numbers (N_T, N_E). Let $p \geq 1$ and let $w \in T$. For any $k \geq 1$, $m, l \geq 0$ and $v \in S^k(w)$,*

$$\mathcal{E}_{M,p,m}(v, S^k(w)) \leq c_{2.27} \sigma^{\mathcal{J}}_{p,l,|w|+k+m} \mathcal{E}_{M,p,m+l}(v, S^k(w)). \tag{2.17}$$

In particular, there exists $c_{2.33}$, depending only on N_T, N_E, M, p and L_*, such that if $S^k(w) \neq \Gamma^{S^k(w)}_M(v)$, then

$$c_{2.33} \leq \sigma^{\mathcal{J}}_{p,l,|w|+k} \mathcal{E}_{M,p,l}(v, S^k(w)) \tag{2.18}$$

for any $n \geq 1$ and $l \geq 0$.

Proof. Let $A = S^{k+m}(w)$ and choose a covering $\mathcal{N} \subseteq \mathcal{J}$ of $S^{k+m}(w)$ with covering number (N_T, N_E). Then by (2.16),

$$\begin{aligned}&\mathcal{E}_{p,0}(S^m(v), S^m(\Gamma_M^{S^k(w)}(v)^c), S^{m+k}(w))\\&\qquad \le c_{2.27} \max_{G\in\mathcal{N}} \sigma_{p,l}(G)\mathcal{E}_{p,l}(S^m(v), S^m(\Gamma_M^{S^k(w)}(v)^c), S^{m+k}(w)).\end{aligned}$$

This implies (2.17). To obtain (2.18), letting $m = 0$ in (2.17), we have

$$\mathcal{E}_{M,p,0}(v, S^k(w)) \le c_{2.27}\sigma^{\mathcal{J}}_{p,l,|w|+k}\mathcal{E}_{M,p,l}(v, S^k(w)).$$

According to Theorem 6.3, $\underline{c}_{\mathcal{E}}(L_*, (L_*)^{M-1}, p) \le \mathcal{E}_{M,p,0}(v, S^k(w))$. This immediately implies (2.18). ■

Another important consequence of Lemma 2.27 is a sub-multiplicative inequality of neighbor disparity constants.

Lemma 2.34. *Let $\mathcal{J}$ be a covering system with covering numbers (N_T, N_E) and let $p \ge 1$. Then*

$$\sigma^{\mathcal{J}}_{p,n+m,k} \le c_{2.27}\sigma^{\mathcal{J}}_{p,n,k}\sigma^{\mathcal{J}}_{p,m,k+n}$$

for any $n, m, k \in \mathbb{N}$.

Proof. This is straightforward by Lemma 2.28. ■

In the rest of this section, we study an estimate of the difference $f(u) - f(v)$ for $f: T_n \to \mathbb{R}$ and $u, v \in T$ by means of the p-energy $\mathcal{E}_p^n(f)$ and neighbor disparity constants.

Lemma 2.35. *Let $\mathcal{J}$ be a covering system with covering numbers (N_T, N_E). Let $w \in T$ and let $m \ge 1$. For any $f \in \ell(S^m(w))$ and $u \in S(w)$,*

$$|(f)_{S^m(w)} - (f)_{S^{m-1}(u)}| \le N_*(\sigma^{\mathcal{J}}_{p,m-1,|w|+1})^{\frac{1}{p}}\mathcal{E}^{|w|+m}_{p,S^m(w)}(f)^{\frac{1}{p}}.$$

Proof. Let $\mathcal{N} \subseteq \mathcal{J}$ be a covering of $(S(w), E^*_{|w|+1}(S(w)))$ with covering numbers (N_T, N_E). For any $v \in S(w)$, there exist $v_1, v_2, \dots, v_k \in S(w)$ and $G_1, \dots, G_k \in \mathcal{N}$ such that $k \le N_*$, $v_1 = v$, $v_k = u$ and $(v_i, v_{i+1}) \in E^*_n(G_i)$ for any $i = 1, \dots, k-1$. Hence

$$\begin{aligned}&|(f)_{S^{m-1}(v)} - (f)_{S^{m-1}(u)}|\\&\quad\le \sum_{i=1}^{k-1}|(f)_{S^{m-1}(v_i)} - (f)_{S^{m-1}(v_{i+1})}| \le \sum_{i=1}^{k-1}\mathcal{E}^{|w|+1}_{p,G_i}(P_{|w|+1,m-1}f)^{\frac{1}{p}}\\&\quad\le (\sigma^{\mathcal{J}}_{p,m-1,|w|+1})^{\frac{1}{p}}\sum_{i=1}^{k-1}\mathcal{E}^{|w|+m}_{p,S^{m-1}(G_i)}(f)^{\frac{1}{p}} \le N_*(\sigma^{\mathcal{J}}_{p,m-1,|w|+1})^{\frac{1}{p}}\mathcal{E}^{|w|+m}_{p,S^m(w)}(f)^{\frac{1}{p}}.\end{aligned}$$

Combining this with

$$(f)_{S^m(w)} - (f)_{S^{m-1}(u)} = \frac{1}{\mu(w)} \sum_{v \in S(w)} ((f)_{S^{m-1}(v)} - (f)_{S^{m-1}(u)})\mu(v),$$

we obtain the desired inequality. ■

Lemma 2.36. *Suppose that $\mathcal{J}$ is a covering system with covering numbers (N_T, N_E). For any $u, v \in T_n$ and $f \in \ell(T_{n+m})$,*

$$|(f)_{S^m(u)} - (f)_{S^m(v)}| \leq N_E \theta_n(u, v)\big(\sigma^{\mathcal{J}}_{p,m,n} \mathcal{E}^{n+m}_p(f)\big)^{\frac{1}{p}}.$$

Proof. Suppose that $\mathcal{N} \subseteq \mathcal{J}$ is a covering of T_n with covering number (N_T, N_E). Set $N = \theta_n(u, v)$ and $g = P_{n,m} f$. There exists $(u(1), \ldots, u(N+1)) \subseteq T_n$ such that $u(1) = u, u(N+1) = v$ and $(u(i), u(i+1)) \in E^*_n$ for any $i = 1, \ldots, N$. For any i, there exist $G_{i,1}, \ldots, G_{i,N_E} \in \mathcal{H}$ and $(u(i,1), \ldots, u(i, N_E + 1))$ such that $u(i,1) = u(i), u(i, N_E + 1) = u(i+1)$ and $(u(i,j), u(i,j+1)) \in E^*_n(G_{i,j})$ for any $j = 1, \ldots, N_E$. Then,

$$\begin{aligned}
|g(u) - g(v)| &\leq \sum_{i=1}^{N} |g(u(i)) - g(u(i+1))| \\
&\leq \sum_{i=1}^{N} \Big((N_E)^{p-1} \sum_{j=1}^{N_E} |g(u(i,j)) - g(u(i,j+1))|^p\Big)^{\frac{1}{p}} \\
&\leq \sum_{i=1}^{N} \Big((N_E)^{p-1} \sum_{j=1}^{N_E} \mathcal{E}^n_{p,G_{i,j}}(P_{n,m} f)\Big)^{\frac{1}{p}} \\
&\leq \sum_{i=1}^{N} \Big((N_E)^{p-1} \sigma^{\mathcal{J}}_{p,m,n} \sum_{j=1}^{N_E} \mathcal{E}^n_{p,S^m(G_{i,j})}(f)\Big)^{\frac{1}{p}} \\
&\leq \sum_{i=1}^{N} \big((N_E)^{p-1} \sigma^{\mathcal{J}}_{p,m,n} N_E \mathcal{E}^{n+m}_{p,T_{n+m}}(f)\big)^{\frac{1}{p}} \\
&\leq N N_E \big(\sigma^{\mathcal{J}}_{p,m,n} \mathcal{E}^{n+m}_p(f)\big)^{\frac{1}{p}}.
\end{aligned}$$

■

Lemma 2.37. *Let $\mathcal{J}$ be a covering system with covering numbers (N_T, N_E). Let $n \geq m$. Then, for any $u, v \in T_n$ and $f \in \ell(T_n)$,*

$$\begin{aligned}
|f(u) - f(v)| \leq \Big(&N_E \theta_m(\pi^{n-m}(u), \pi^{n-m}(v))(\sigma^{\mathcal{J}}_{p,n-m,m})^{\frac{1}{p}} \\
&+ 2N_* \sum_{i=1}^{n-m} (\sigma^{\mathcal{J}}_{p,n-m-i,m+i})^{\frac{1}{p}}\Big) \mathcal{E}^n_p(f)^{\frac{1}{p}}.
\end{aligned} \tag{2.19}$$

Proof. Set $v(i) = \pi^{n-m-i}(v)$ for $i = 0, \ldots, n-m$. Then by Lemma 2.35,

$$|f(v) - (f)_{S^{n-m}(\pi^{n-m}(v))}| \leq \sum_{i=1}^{n-m} |(f)_{S^{n-m-i}(v(i))} - (f)_{S^{n-m-i+1}(v(i-1))}|$$
$$\leq N_* \sum_{i=1}^{n-m} (\sigma^{\mathcal{J}}_{p,n-m-i,m+i})^{\frac{1}{p}} \mathcal{E}^n_p(f)^{\frac{1}{p}}. \tag{2.20}$$

The same inequality holds if we replace v by u. Set $v' = \pi^{n-m}(v)$ and $u' = \pi^{n-m}(u)$. Applying Lemma 2.36, we obtain

$$|(f)_{S^{n-m}(u')} - (f)_{S^{n-m}(v')}| \leq N_E \theta_m(u', v')(\sigma^{\mathcal{J}}_{p,n-m,m})^{\frac{1}{p}} \mathcal{E}^n_p(f)^{\frac{1}{p}}. \tag{2.21}$$

By (2.20) and (2.21), we have (2.19). ∎

Chapter 3

Conductive homogeneity and its consequences

3.1 Construction of p-energy: $p > \dim_{AR}(K, d)$

In this section, we are going to construct a p-energy on K as a scaling limit of the discrete counterparts $\mathcal{E}_p^n$'s step by step under Assumption 3.2, which consists of the following two requirements:

(3.1) Neighbor disparity constants and (conductance constants)$^{-1}$ have the same asymptotic behavior.

(3.2) Conductance constants have exponential decay.

Under these assumptions, the p-energy $\widehat{\mathcal{E}}_p$ is constructed in Theorem 3.21. Furthermore, in the case $p = 2$, we construct a local regular Dirichlet form in Theorem 3.23.

The question when Assumption 3.2 is fulfilled will be addressed in Section 3.3.

As in the previous sections, we continue to suppose that Assumptions 2.6, 2.7, 2.10 and 2.12 hold. Moreover, throughout this section, we fix $p \geq 1$.

Definition 3.1. For $M \geq 1, m \geq 0$ and $n \geq 1$, define

$$\mathcal{E}_{M,p,m,n} = \max_{v \in T_n} \mathcal{E}_{M,p,m}(v, T_n).$$

Remark. Theorem 6.3 shows that $\mathcal{E}_{M,p,m,n}$ is finite.

Assumption 3.2. *Let $\mathcal{J}$ be a covering system. There exist $c_1, c_2 > 0$ and $\alpha \in (0, 1)$ such that*

$$c_1 \leq \mathcal{E}_{M_*,p,m,n}\sigma^{\mathcal{J}}_{p,m,n} \leq c_2 \tag{3.1}$$

and

$$\mathcal{E}_{M_*,p,m,n} \leq c_2\alpha^m \tag{3.2}$$

for any $m \geq 0, n \geq 1$.

Hereafter in this section, we fix a covering system $\mathcal{J}$ with covering numbers (N_T, N_E) and use $\sigma_{p,m,n}$ (resp. $\sigma_{p,m}$) in place of $\sigma^{\mathcal{J}}_{p,m,n}$ (resp. $\sigma^{\mathcal{J}}_{p,m}$) for simplicity of notations.

By [34, Theorems 4.7.6 and 4.9.1], we have the following characterization of (3.2) under Assumption 2.15.

Proposition 3.3. *Under Assumption 2.15,*

$$\varlimsup_{m\to\infty} (\mathcal{E}_{M_*,p,m})^{\frac{1}{m}} < 1 \quad \textit{if and only if} \quad p > \dim_{AR}(K, d).$$

In particular, (3.2) holds if and only if $p > \dim_{AR}(K, d)$.

Note that since K is assumed to be connected, we have $\dim_{AR}(K, d) \geq 1$, so that $p > 1$.

In the following definition, we introduce the principal notion of this paper called conductive homogeneity. Due to Theorem 3.5, conductive homogeneity yields (3.1).

Definition 3.4 (Conductive homogeneity). Define

$$\mathcal{E}_{M,p,m} = \sup_{w \in T, |w| \geq 1} \mathcal{E}_{M,p,m}(w, T_{|w|}).$$

A compact metric space K (with a partition $\{K_w\}_{w \in T}$ and a measure μ) is said to be *p-conductively homogeneous* if

$$\sup_{m \geq 0} \sigma_{p,m} \mathcal{E}_{M_*,p,m} < \infty. \tag{3.3}$$

Remark. As in the case of $\mathcal{E}_{M,p,m,n}$, $\mathcal{E}_{M,p,m}$ is always finite due to Theorem 6.3.

Remark. As we will see in Theorem 3.33, if $p > \dim_{AR}(K, d)$, then the conductive homogeneity is solely determined by the conductance constants. Consequently, it is independent of a choice of a covering system $\mathcal{J}$. So, in the case $p > \dim_{AR}(K, d)$, the covering system $\mathcal{J}_*$ is good enough in the end.

Theorem 3.5. *If K is p-conductively homogeneous, then* (3.1) *holds.*

A proof of Theorem 3.5 will be provided in Section 3.3.

Under conductive homogeneity, it will be shown in Theorem 3.30 that there exist $c_1, c_2 > 0$ and $\sigma > 0$ such that

$$c_1 \sigma^m \leq \sigma_{p,m,n} \leq c_2 \sigma^m \quad \text{and} \quad c_1 \sigma^{-m} \leq \mathcal{E}_{M_*,p,m}(v, T_n) \leq c_2 \sigma^{-m}$$

for any $m \geq 1$, $n \geq 0$ and $v \in T_n$. This is why we have given the name “homogeneity” to this notion.

Now we start to construct a p-energy under Assumption 3.2. An immediate consequence of Assumption 3.2 is the following multiplicative property of $\sigma_{p,m,n}$.

Lemma 3.6. *There exist $c_1, c_2 > 0$ such that*

$$c_1 \sigma_{p,m,n+k} \sigma_{p,n,k} \leq \sigma_{p,n+m,k} \leq c_2 \sigma_{p,m,n+k} \sigma_{p,n,k}$$

for any $k \geq 1$, and $m, n \geq 0$.

Proof. By (2.14), we have

$$\mathcal{E}_{M_*,p,n+m,k} \leq c \mathcal{E}_{M_*,p,m,n+k} \mathcal{E}_{M_*,p,n,k}.$$

This along with (3.1) shows

$$c_1 \sigma_{p,m,n+k} \sigma_{p,n,k} \leq \sigma_{p,n+m,k}.$$

The other half of the desired inequality follows from Lemma 2.34. ■

Next, we study some geometry associated with the partition $\{K_w\}_{w \in T}$.

Definition 3.7. Let $L \geq 1$. Define

$$n_L(x, y) = \max\{n \mid \text{there exist } w, v \in T_n \text{ such that } x \in K_w,\ y \in K_v \text{ and } v \in \Gamma_L(w)\}.$$

Furthermore, fix $r \in (0, 1)$ and define

$$\delta_L(x, y) = r^{n_L(x,y)}. \tag{3.4}$$

Recall that $h_r \colon T \to (0, 1]$ is given as $h_r(w) = r^{|w|}$. Since $\Lambda_s^{h_r} = T_n$ if $r^{n-1} > s \geq r^n$, where

$$\Lambda_s^{h_r} = \{w \mid w \in T,\ h_r(\pi(w)) > s \geq h_r(w)\},$$

δ_L is nothing but $\delta_L^{h_r}$ defined in [34, Definition 2.3.8].

By [34, Proposition 2.3.7] and the discussions in its proof, we have the following fact.

Proposition 3.8. *Suppose that d is a metric on K giving the original topology $\mathcal{O}$ of K. Let $L \geq 1$. There exists a monotonically non-decreasing function $\eta_L \colon [0, 1] \to [0, 1]$ satisfying $\lim_{t\downarrow} \eta_L(t) = 0$ and $\delta_L(x, y) \leq \eta_L(d(x, y))$ for any $x, y \in K$.*

Proof. Define

$$\Lambda_{s,0}^{h_r}(x) = \{v \mid v \in \Lambda_s^{h_r},\ x \in K_v\}, \quad U_0^{h_r}(x, s) = \bigcup_{v \in \Lambda_{s,0}^{h_r}(x)} K_v$$

and

$$U_1^{h_r}(x, s) = \bigcup_{y \in U_0^{h_r}(x,s)} U_0^{h_r}(y, s)$$

for $s \in (0, 1]$ and $x \in K$. First we show that for any $\varepsilon > 0$, there exists $\gamma_\varepsilon > 0$ such that $\delta_L(x, y) \leq \varepsilon$ whenever $d(x, y) \leq \gamma_\varepsilon$. If this is not the case, then there exist $\varepsilon_0 > 0$, $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ such that $d(x_n, y_n) \leq \frac{1}{n}$ and $\delta_L(x_n, y_n) > \varepsilon_0$. Since K is compact, choosing an adequate subsequence $\{n_k\}_{k\to\infty}$, we see that there exists $x \in K$ such that $x_{n_k} \to x$ and $y_{n_k} \to x$ for $k \to \infty$. By [34, Proposition 2.3.7], $U_0^{h_r}(x, \frac{\varepsilon_0}{2})$ is a neighborhood of x. Hence both x_{n_k} and y_{n_k} belong to $U_0^{h_r}(x, \frac{\varepsilon_0}{2})$ for sufficiently large k. So, there exist $w, v \in \Lambda_{\varepsilon_0/2,0}^{h_r}(x)$ such that $x_{n_k} \in K_w$ and $y_{n_k} \in K_v$. Since $x \in K_w \cap K_v$, we see that $y \in U_1^{h_r}(x, \frac{\varepsilon_0}{2})$, so that $\delta_L(x_{n_k}, y_{n_k}) \leq \frac{\varepsilon_0}{2}$. This contradicts the assumption that $\delta_L(x_n, y_n) \geq \varepsilon_0$. Thus our claim at the beginning of this proof is verified. Note that with a modification if necessary, we may assume that γ_ε is monotonically non-decreasing as a function of ε and $\lim_{\varepsilon\downarrow 0} \gamma_\varepsilon = 0$. Define

$$\eta_L(t) = \inf\{\varepsilon \mid \varepsilon > 0, t \leq \gamma_\varepsilon\}.$$

Now it is routine to see that η is the desired function. ∎

Let $T_n = \{w(1), \ldots, w(l)\}$, where $l = \#(T_n)$. Inductively we define $\widetilde{K}_w$ by

$$\widetilde{K}_{w(1)} = K_{w(1)} \quad \text{and} \quad \widetilde{K}_{w(k+1)} = K_{w(k+1)} \backslash \Big(\bigcup_{i=1,\ldots,k} \widetilde{K}_{w(i)} \Big).$$

Note that (2.4) implies that $\mu(B_w) = 0$ for any $w \in T_n$ and hence we have

$$\widetilde{K}_w \supseteq O_w \quad \text{and} \quad \mu(K_w \backslash \widetilde{K}_w) = 0$$

for any $w \in T_n$. The latter equality is due to (2.4). Now define $J_n : \ell(T_n) \to \mathbb{R}^K$ by

$$J_n f = \sum_{w \in T_n} f(w) \chi_{\widetilde{K}_w}. \tag{3.5}$$

Since $\widetilde{K}_w$ is a Borel set, $J_n f$ is μ-measurable for any $f \in \ell(T_n)$. The definitions of $\widetilde{K}_w$ and J_n depend on an enumeration of T_n but $J_n f$ stays the same in the μ-a.e. sense regardless of an enumeration.

Define

$$\widetilde{\mathcal{E}}_p^m(f) = \sigma_{p,m-1,1} \mathcal{E}_p^m(f). \tag{3.6}$$

The next lemma yields the control of the difference of values of $J_n f$ through $\widetilde{\mathcal{E}}_p^n(f)$.

Lemma 3.9. *Suppose that Assumption 3.2 holds. There exists $C > 0$ such that for any $n \geq 1$, $f \in \ell(T_n)$ and $x, y \in K$,*

$$|(J_n f)(x) - (J_n f)(y)| \leq C \alpha^{\frac{m}{p}} \widetilde{\mathcal{E}}_p^n(f)^{\frac{1}{p}},$$

where $m = \min\{n_L(x, y), n\}$.

Proof. Let $m = \min\{n_L(x, y), n\}$. Then there exist $w, w' \in T_m$, $v \in S^{n-m}(w)$ and $u \in S^{n-m}(w')$ such that $x \in K_v$, $y \in K_u$, $(J_n f)(x) = f(v)$, $(J_n f)(y) = f(u)$ and $w' \in \Gamma_{L+2}(w)$. By (2.19),

$$|f(u) - f(v)| \leq c \sum_{i=0}^{n-m} (\sigma_{p,n-m-i,m+i})^{\frac{1}{p}} \mathcal{E}_p^n(f)^{\frac{1}{p}}, \tag{3.7}$$

where $c = \max\{2(N_*)^2, N_E(L+2)\}$. Lemma 3.6 shows that

$$c_1 \sigma_{p,m+i-1,1} \sigma_{p,n-m-i,m+i} \leq \sigma_{p,n-1,1}.$$

Combining this with Assumption 3.2, we obtain

$$\sigma_{p,n-m-i,m+i} \leq c_3 \alpha^{m+i} \sigma_{p,n-1,1}.$$

Using (3.7), we see

$$|f(u) - f(v)| \leq c_4 \alpha^{\frac{m}{p}} \widetilde{\mathcal{E}}_p^n(f)^{\frac{1}{p}}. \qquad \blacksquare$$

By this lemma, the boundedness of $\widetilde{\mathcal{E}}_p^n(f_n)$ gives a kind of equicontinuity to the family $\{f_n\}_{n\geq 1}$ and hence an analogue of Arzelà–Ascoli theorem, which we present in Appendix 6.3, shows the existence of a uniform limit as follows.

Lemma 3.10. *Suppose that Assumption 3.2 holds. Define* $\tau = \frac{\log \alpha}{\log r}$. *Let* $f_n \in \ell(T_n)$ *for any* $n \geq 1$. *If*

$$\sup_{n\geq 1} \widetilde{\mathcal{E}}_p^n(f_n) < \infty \quad \text{and} \quad \sup_{n\geq 1} |(f_n)_{T_n}| < \infty,$$

then there exist a subsequence $\{n_k\}_{k\geq 1}$ *and* $f \in C(K)$ *such that* $\{J_{n_k} f_{n_k}\}$ *converges uniformly to* f *as* $k \to \infty$, $\widetilde{\mathcal{E}}_p^{n_k}(f_{n_k})$ *is convergent as* $k \to \infty$ *and*

$$|f(x) - f(y)|^p \leq C\eta_L(d(x, y))^\tau \lim_{k\to\infty} \widetilde{\mathcal{E}}_p^{n_k}(f_{n_k}),$$

where η_L *was introduced in Proposition 3.8.*

Proof. Set $C_* = \sup_{n\geq 1} \widetilde{\mathcal{E}}_p^n(f_n)$. By Lemma 3.9, if $n \geq n_L(x, y)$, then

$$|J_n f_n(x) - J_n f_n(y)| \leq C\alpha^{\frac{n_L(x,y)}{p}} (C_*)^{\frac{1}{p}} \leq C\eta_L(d(x, y))^{\frac{\tau}{p}} (C_*)^{\frac{1}{p}}. \tag{3.8}$$

In the case $n < n_L(x, y)$, there exist $w, w' \in T_n$ such that $x \in K_w$, $J_n f_n(x) = f(w)$, $y \in K_{w'}$, $J_n f_n(w') = f(w')$ and $w' \in \Gamma_{L+2}(w)$. So there exists an E_n^*-path $(w(0), \ldots, w(L + 2))$ satisfying

$$w(0) = w \quad \text{and} \quad w' = w(L + 2).$$

By Lemma A.1,

$$\begin{aligned} |f(w) - f(w')|^p &\leq (L + 2)^{p-1} \sum_{i=0}^{L+1} |f(w(i)) - f(w(i + 1))|^p \\ &\leq (L + 2)^{p-1} \mathcal{E}_p^n(f_n). \end{aligned}$$

On the other hand, since $\widetilde{\mathcal{E}}_p^n(f_n) \leq C_*$, Assumption 3.2 implies

$$\mathcal{E}_p^n(f_n) \leq (\sigma_{p,n-1,1})^{-1} C_* \leq c_2 \mathcal{E}_{M_*,p,n-1,1} C_* \leq (c_2)^2 \alpha^{n-1} C_*.$$

Thus we have

$$|J_n f_n(x) - J_n f_n(y)| \leq c\alpha^{\frac{n}{p}} (C_*)^{\frac{1}{p}}. \tag{3.9}$$

Making use of (3.8) and (3.9), we see that

$$|J_n f_n(x) - J_n f_n(y)| \leq C\eta_L(d(x, y))^{\frac{\tau}{p}} (C_*)^{\frac{1}{p}} + c\alpha^{\frac{n}{p}} (C_*)^{\frac{1}{p}}$$

for any $x, y \in K$. Applying Lemma D.1 with $X = K$, $Y = \mathbb{R}$, $u_i = J_i f_i$, we obtain the desired result. ∎

Definition 3.11. Define $P_n \colon L^1(K,\mu) \to \ell(T_n)$ by

$$(P_n f)(w) = \frac{1}{\mu(w)} \int_{K_w} f \, d\mu$$

for any $n, m \geq 1$. For $f \in \ell(T_k)$, we define

$$P_n f = P_n J_k f.$$

The next lemma is one of the keys to the construction of a p-energy. A counterpart of this fact has already been used in Kusuoka–Zhou's construction of Dirichlet forms on self-similar sets in [36].

Lemma 3.12. *Under Assumption 3.2, there exists $C > 0$ such that for any $n, m \geq 1$ and $f \in L^1(K,\mu) \cup (\bigcup_{k\geq 1} \ell(T_k))$,*

$$C\,\widetilde{\mathcal{E}}_p^n(P_n f) \leq \widetilde{\mathcal{E}}_p^{n+m}(P_{n+m} f). \tag{3.10}$$

In particular,

$$C \sup_{n\geq 0} \widetilde{\mathcal{E}}_p^n(P_n f) \leq \varliminf_{n\to\infty} \widetilde{\mathcal{E}}_p^n(P_n f) \leq \varlimsup_{n\to\infty} \widetilde{\mathcal{E}}_p^n(P_n f) \leq \sup_{n\geq 0} \widetilde{\mathcal{E}}_p^n(P_n f) \tag{3.11}$$

for any $f \in L^1(K,\mu)$.

Remark. This lemma holds without (3.2).

Proof. Note that $P_n f = P_{n,m}(P_{n+m} f)$. Let $\mathcal{N} \subseteq \mathcal{J}$ be a covering of (T_n, E_n^*) with covering numbers (N_T, N_E). By Lemma 2.27,

$$\mathcal{E}_p^n(P_n f) \leq c_{2.27}\sigma_{p,m,n}\mathcal{E}_p^{n+m}(P_{n+m} f).$$

Hence

$$\frac{1}{\sigma_{p,n-1,1}}\widetilde{\mathcal{E}}_p^n(P_n f) \leq c_{2.27}\frac{\sigma_{p,m,n}}{\sigma_{p,n+m-1,1}}\widetilde{\mathcal{E}}_p^{n+m}(P_{n+m} f).$$

By Lemma 3.6, we have (3.10). ∎

By virtue of the last lemma, we have a proper definition of the domain $\mathcal{W}^p$ of a p-energy given in Theorem 3.21 and its semi-norm $\mathcal{N}_p$.

Lemma 3.13. *Define*

$$\mathcal{W}^p = \big\{ f \mid f \in L^p(K,\mu), \sup_{n\geq 1} \widetilde{\mathcal{E}}_p^n(P_n f) < +\infty \big\},$$

and

$$\mathcal{N}_p(f) = \sup_{n\geq 1} \widetilde{\mathcal{E}}_p^n(P_n f)^{\frac{1}{p}}$$

for $f \in \mathcal{W}^p$. Then $\mathcal{W}^p$ is a normed linear space with norm $\|\cdot\|_{p,\mu} + \mathcal{N}_p(\cdot)$, where $\|\cdot\|_{p,\mu}$ is the L^p-norm. Moreover, for any $f \in \mathcal{W}^p$, there exists $f_* \in C(K)$ such that $f(x) = f_*(x)$ for μ-a.e. $x \in K$. In this way, $\mathcal{W}^p$ is regarded as a subset of $C(K)$ and

$$|f(x) - f(y)|^p \leq C\eta_L(d(x,y))^\tau \mathcal{N}_p(f)^p \tag{3.12}$$

for any $f \in \mathcal{W}^p$ and $x, y \in K$, where η_L was introduced in Proposition 3.8. In particular, $\mathcal{N}_p(f) = 0$ if and only if f is constant on K.

If no confusion may occur, we write $\|\cdot\|_p$ in place of $\|\cdot\|_{p,\mu}$ hereafter.

In fact, $(\mathcal{W}^p, \|\cdot\|_p + \mathcal{N}_p(\cdot))$ turns out to be a Banach space by Lemma 3.16.

Proof. Note that

$$\widetilde{\mathcal{E}}_p^n(f+g)^{\frac{1}{p}} \leq \widetilde{\mathcal{E}}_p^n(f)^{\frac{1}{p}} + \widetilde{\mathcal{E}}_p^n(g)^{\frac{1}{p}}$$

and so $\widetilde{\mathcal{E}}_p^n(\cdot)^{\frac{1}{p}}$ is a semi-norm. This implies that $\mathcal{N}_p(\cdot)$ is a semi-norm of $\mathcal{W}^p$.

For $f \in \mathcal{W}^p$, by Lemma 3.10, there exist $\{n_k\}_{k\geq 1}$ and $f_* \in C(K)$ such that

$$\|J_{n_k} P_{n_k} f - f_*\|_\infty \to 0$$

as $k \to \infty$ and

$$|f_*(x) - f_*(y)|^p \leq C\eta_L(d(x,y))^\tau \varlimsup_{n\to\infty} \mathcal{E}_p^n(P_n f).$$

Since $\int_{K_w} P_{n_k} f d\mu \to \int_{K_w} f_* \, d\mu$ as $k \to \infty$, it follows that $\int_{K_w} f d\mu = \int_{K_w} f_* \, d\mu$ for any $w \in T$. Hence $f = f_*$ for μ-a.e. $x \in K$. Thus we identify f_* with f and so $f \in C(K)$. Moreover, (3.12) holds for any $x, y \in K$. By (3.12), $\mathcal{N}_p(f) = 0$ if and only if f is constant on K. ∎

We now examine the properties of the normed space $(\mathcal{W}^p, \|\cdot\|_p + \mathcal{N}_p(\cdot))$. The intermediate goals are to show its completeness (Lemma 3.16) and that it is dense in $C(K)$ with respect to the supremum norm (Lemma 3.19).

Lemma 3.14. *Suppose that Assumption 3.2 holds. The identity map*

$$I : (\mathcal{W}^p, \|\cdot\|_p + \mathcal{N}_p(\cdot)) \to (C(K), \|\cdot\|_\infty)$$

is continuous.

Proof. Let $\{f_n\}_{n\geq 1}$ be a Cauchy sequence in $(\mathcal{W}^p, \|\cdot\|_p + \mathcal{N}_p(\cdot))$. Fix $x_0 \in K$ and set $g_n(x) = f_n(x) - f_n(x_0)$. Then

$$\begin{aligned}|g_n(x) - g_m(x)| &= |(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))| \\ &\leq C\eta_L(d(x,x_0))^{\frac{\tau}{p}} \mathcal{N}_p(f_n - f_m)\end{aligned}$$

for any $x \in K$ and $n, m \geq 1$. Thus $\{g_n\}_{n\geq 1}$ is a Cauchy sequence in $C(K)$ with the norm $\|\cdot\|_\infty$, so that there exists $g \in C(K)$ such that $\|g - g_n\|_\infty \to 0$ as $n \to \infty$. On the other hand, since $\{f_n\}_{n\geq 1}$ is a Cauchy sequence of $L^p(X,\mu)$, there exists $f \in L^p(X,\mu)$ such that $\|f_n - f\|_p \to 0$ as $n \to \infty$. Thus $f_n(x_0) = f_n - g_n$ converges as $n \to \infty$ in $L^p(K,\mu)$. Let c be its limit. Then $f = g + c$ in $L^p(K,\mu)$. Therefore, $f \in C(K)$ and $\|f_n - f\|_\infty \to 0$ as $n \to \infty$. ■

Define $\overline{\mathcal{W}}^p$ as the completion of $(\mathcal{W}^p, \|\cdot\|_p + \mathcal{N}_p(\cdot))$. Then the map I is extended to a continuous map from $\overline{\mathcal{W}}_p \to C(K)$, which is denoted by I as well for simplicity.

Lemma 3.15 (Closability). *Suppose that Assumption 3.2 holds. The extended map $I\colon \overline{\mathcal{W}}^p \to C(K)$ is injective. In particular, $\overline{\mathcal{W}}^p$ is identified with a subspace of $C(K)$.*

Proof. Let $\{f_n\}_{n\geq 1}$ be a Cauchy sequence in $(\mathcal{W}^p, \|\cdot\|_p + \mathcal{N}_p(\cdot))$. Suppose that $\lim_{n\to\infty} \|f_n\|_\infty = 0$. Note that

$$\widetilde{\mathcal{E}}_p^k(P_k f_n - P_k f_m) \leq \sup_{l\geq 1} \widetilde{\mathcal{E}}_p^l(P_l f_n - P_l f_m) = \mathcal{N}_p(f_n - f_m)^p$$

for any $k, n, m \geq 1$. Hence, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\widetilde{\mathcal{E}}_p^k(P_k f_n - P_k f_m) \leq \varepsilon$$

for any $n, m \geq N$ and $k \geq 1$. As $\|f_m\|_\infty \to 0$ as $m \to \infty$, we see that

$$\widetilde{\mathcal{E}}_p^k(P_k f_n) \leq \varepsilon$$

for any $n \geq N$ and $k \geq 1$ and hence $\mathcal{N}_p(f_n)^p \leq \varepsilon$ for any $n \geq N$. Thus, $\mathcal{N}_p(f_n) \to 0$ as $n \to \infty$, so that $f_n \to 0$ in $\mathcal{W}^p$ as $n \to \infty$. ■

Lemma 3.16. *Suppose that Assumption 3.2 holds. Then*

$$\overline{\mathcal{W}}^p = \mathcal{W}^p.$$

Proof. Let $\{f_n\}_{n\geq 1}$ be a Cauchy sequence of $\mathcal{W}^p$ and let f be its limit in $\overline{\mathcal{W}}^p$. It follows that $\|f - f_n\|_\infty \to 0$ as $n \to \infty$. Using the same argument as in the proof of Lemma 3.15, we see that for sufficiently large n,

$$C\,\widetilde{\mathcal{E}}_p^k(P_k f_n - P_k f) \leq \varepsilon$$

for any $k \geq 1$. Since

$$\widetilde{\mathcal{E}}_p^k(P_k f)^{\frac{1}{p}} \leq \widetilde{\mathcal{E}}_p^k(P_k f - P_k f_n)^{\frac{1}{p}} + \widetilde{\mathcal{E}}_p^k(P_k f_n)^{\frac{1}{p}},$$

it follows that $\sup_{k\geq 1} \widetilde{\mathcal{E}}_p^k(P_k f) < \infty$ and hence $f \in \mathcal{W}^p$. ■

Lemma 3.17. *Suppose that Assumption 3.2 holds.*

(1) *Let $\{n_k\}_{k \geq 1}$ be a monotonically increasing sequence of $\mathbb{N}$. Suppose that $f_{n_k} \in \ell(T_{n_k})$ for any $k \geq 1$, that $\sup_{k \geq 1} \widetilde{\mathcal{E}}_p^{n_k}(f_{n_k}) < \infty$ and that there exists $f \in C(K)$ such that $\|J_{n_k} f_{n_k} - f\|_\infty \to 0$ as $k \to \infty$. Then $f \in \mathcal{W}^p$.*

(2) *Let $f, g \in \mathcal{W}^p$. Then $f \cdot g \in \mathcal{W}^p$.*

Proof. (1) Set $C_1 = \sup_{k \geq 1} \widetilde{\mathcal{E}}_p^{n_k}(f_{n_k})$. By (3.10), if $n \leq n_l$, then

$$C\, \widetilde{\mathcal{E}}_p^{n}(P_n f_{n_l}) \leq \widetilde{\mathcal{E}}_p^{n_l}(f_{n_l}) \leq C_1.$$

Letting $l \to \infty$, we obtain

$$C\, \widetilde{\mathcal{E}}_p^{n}(P_n f) \leq C_1$$

for any $k \geq 1$. This implies $f \in \mathcal{W}^p$.

(2) For any $\varphi, \psi \in \ell(T_n)$,

$$\begin{aligned}
\mathcal{E}_p^n(\varphi \cdot \psi) &= \frac{1}{2} \sum_{(w,v) \in E_n^*} |\varphi(w)\psi(w) - \varphi(v)\psi(v)|^p \\
&\leq 2^{p-1} \frac{1}{2} \sum_{(w,v) \in E_n^*} \left(|\varphi(w)|^p |\psi(w) - \psi(v)|^p + |\varphi(w) - \varphi(v)|^p |\psi(v)|^p\right) \\
&\leq 2^{p-1} (\|\varphi\|_\infty^p \mathcal{E}_p^n(\varphi) + \|\psi\|_\infty^p \mathcal{E}_p^n(\psi)).
\end{aligned}$$

Hence if $h_n = P_n f \cdot P_n g$, then

$$\widetilde{\mathcal{E}}_p^n(h_n) \leq 2^{p-1}\big(\|f\|_\infty^p \widetilde{\mathcal{E}}_p^n(P_n f) + \|g\|_\infty^p \widetilde{\mathcal{E}}_p^n(P_n g)\big).$$

Since $f, g \in \mathcal{W}_p$, we see that $\sup_{n \geq 1} \widetilde{\mathcal{E}}_p^n(h_n) < \infty$. Moreover, $\|J_n h_n - fg\|_\infty \to 0$ as $n \to \infty$. Using (1), we conclude that $fg \in \mathcal{W}^p$. ■

Lemma 3.18. *Suppose that Assumption 3.2 holds. There exist a monotonically increasing sequence $\{m_j\}_{j \in \mathbb{N}}$ and $h^*_{M_*,w}, \varphi^*_{M_*,w} \in \mathcal{W}^p$ for $w \in T$ such that*

(a) *For any $w \in T$,*

$$\begin{aligned}
&\lim_{j \to \infty} \|J_{m_j} h^*_{M_*,w,m_j - |w|} - h^*_{M_*,w}\|_\infty \\
&\qquad = \lim_{j \to \infty} \|J_{m_j} \varphi^*_{M_*,w,m_j - |w|} - \varphi^*_{M_*,w}\|_\infty = 0,
\end{aligned}$$

where $h^*_{M_*,w,m}$ and $\varphi^*_{M_*,w,m}$ are defined in Definition 2.20. For negative values of m, we formally define $h^*_{M_*.w,k-|w|} = P_k h^*_{M_*,w,0}$ and $\varphi^*_{M_*,w,k-|w|} = P_k \varphi^*_{M_*,w,0}$ for $k = 0, 1, \ldots, |w|$.

(b) *$\{\widetilde{\mathcal{E}}_p^{m_j}(h^*_{M_*,w,m_j - |w|})\}_{j \geq 1}$ and $\{\widetilde{\mathcal{E}}_p^{m_j}(\varphi^*_{M_*,w,m_j - |w|})\}_{j \geq 1}$ converge as $j \to \infty$.*

(c) *Set* $U_M(w) = \bigcup_{v \in \Gamma_M(w)} K_w$. *For any* $w \in T$, $h^*_{M_*,w}\colon K \to [0,1]$ *and*

$$h_{M_*,w}(x) = \begin{cases} 1 & \text{if } x \in K_w, \\ 0 & \text{if } x \notin U_{M_*}(w). \end{cases}$$

(d) *For any* $w \in T$, $\varphi^*_{M_*,w}\colon K \to [0,1]$, $\mathrm{supp}(\varphi^*_{M_*,w}) \subseteq U_{M_*}(w)$, *and*

$$\varphi^*_{M_*,w}(x) \geq (L_*)^{-M_*}$$

for any $x \in K_w$. *Moreover, for any* $n \geq 1$,

$$\sum_{w \in T_n} \varphi^*_{M_*,w} \equiv 1.$$

(e) *For any* $w \in T$ *and* $x \in K$,

$$\varphi^*_{M_*,w}(x) = \frac{h^*_{M_*,w}(x)}{\sum_{v \in T_{|w|}} h^*_{M_*,v}(x)}.$$

Remark. The family $\{\varphi^*_{M_*,w}\}_{w \in T_n}$ is a partition of unity subordinate to the covering $\{U_{M_*}(w)\}_{w \in T_n}$.

Proof. For ease of notation, write $\varphi^*_{w,m} = \varphi^*_{M_*,w,m}$ and $h^*_{w,m} = h^*_{M_*,w,m}$. By Lemma 2.19, (3.1) and Lemma 3.6, we see that

$$\begin{aligned} \widetilde{\mathcal{E}}_p^{|w|+m}(\varphi^*_{w,m}) &\leq ((L_*)^{2M+1} + 1)^p \sigma_{p,|w|+m-1,1} \mathcal{E}_{M,p,m}(w, T_{|w|}) \\ &\leq C\sigma_{p,|w|+m-1,1}\sigma^{-1}_{p,m,|w|} \leq C'\sigma_{p,|w|-1,1} \end{aligned}$$

for any $w \in T$ and $m \geq 0$. Similarly,

$$\widetilde{\mathcal{E}}_p^{|w|+m}(h^*_{w,m}) \leq C'\sigma_{p,|w|-1,1}.$$

Hence Lemma 3.10 shows that, for each w, there exists $\{n_k\}_{k \to \infty}$ such that the sequence $\{J_{|w|+n_k} h^*_{w,n_k}\}_{k \geq 1}$ (resp. $\{J_{|w|+m_j}\varphi^*_{w,n_k}\}_{k \geq 1}$) converges uniformly as $k \to \infty$. Let h^*_w (resp. φ^*_w) be its limit. Lemma 3.17 (1) implies that $h^*_w \in \mathcal{W}^p$ and $\varphi^*_w \in \mathcal{W}^p$. By the diagonal argument, we choose $\{m_j\}_{j \geq 1}$ such that (a) and (b) hold. Statements (c), (d) and (e) are straightforward from the properties of $h^*_{w,m}$ and $\varphi^*_{w,m}$. ∎

Lemma 3.19. *Under Assumption 3.2,* $\mathcal{W}^p$ *is dense in* $(C(K), \|\cdot\|_\infty)$.

Proof. Choose $x_w \in K_w$ for each $w \in T$. For $f \in C(K)$, define

$$f_n = \sum_{w \in T_n} f(x_w)\varphi^*_{M_*,w}.$$

Then by Lemma 3.18, it follows that $\|f_n - f\|_\infty \to 0$ as $n \to \infty$. Hence $\mathcal{W}^p$ is dense in $C(K)$. ∎

Definition 3.20. For $f \in L^P(K, \mu)$, define $\overline{f}$ by

$$\overline{f}(x) = \begin{cases} 1 & \text{if } f(x) \geq 1, \\ f(x) & \text{if } 0 < f(x) < 1, \\ 0 & \text{if } f(x) \leq 0 \end{cases}$$

for $x \in K$.

Now we construct the p-energy $\widehat{\mathcal{E}}_p$ as a Γ-cluster point of $\widetilde{\mathcal{E}}_p^n(P_n \cdot)$. The use of Γ-convergence in the construction of Dirichlet forms on self-similar sets has been around for some time. See [13, 20] for example.

Theorem 3.21. *Suppose that Assumption 3.2 holds. Then there exist* $\widehat{\mathcal{E}}_p \colon \mathcal{W}^p \to [0, \infty)$ *and* $c > 0$ *such that*

(a) $(\widehat{\mathcal{E}}_p)^{\frac{1}{p}}$ *is a semi-norm on* $\mathcal{W}^p$ *and*

$$c\mathcal{N}_p(f) \leq \widehat{\mathcal{E}}_p(f)^{\frac{1}{p}} \leq \mathcal{N}_p(f) \tag{3.13}$$

for any $f \in \mathcal{W}^p$.

(b) *For any* $f \in \mathcal{W}^p$, $\overline{f} \in \mathcal{W}^p$ *and*

$$\widehat{\mathcal{E}}_p(\overline{f}) \leq \widehat{\mathcal{E}}_p(f).$$

(c) *For any* $f \in \mathcal{W}^p$,

$$|f(x) - f(y)|^p \leq c\eta_L(d(x, y))^\tau \widehat{\mathcal{E}}_p(f).$$

In particular, for $p = 2$, $(\widehat{\mathcal{E}}_2, \mathcal{W}^2)$ *is a regular Dirichlet form on* $L^2(K, \mu)$ *and the associated non-negative self-adjoint operator has compact resolvent.*

Property (b) in the above theorem is called the Markov property.

Theorem 3.22 (Shimizu [41]). *Suppose that Assumption 3.2 holds. Then the Banach space* $(\mathcal{W}^p, \|\cdot\|_p + \widehat{\mathcal{E}}_p(\cdot))$ *is reflexive and separable.*

Remark. In [41], the reflexivity and separability are shown in the case of the planar Sierpiński carpet. His method, however, can easily be extended to our general case and one has the above theorem.

Proof of Theorem 3.21. Define $\widehat{\mathcal{E}}_p^n \colon L^p(K, \mu) \to [0, \infty)$ by $\widehat{\mathcal{E}}_p^n(f) = \widetilde{\mathcal{E}}_p^n(P_n f)$ for $f \in L^p(K, \mu)$. Then by [12, Proposition 2.14], there exists a Γ-convergent subsequence $\{\widehat{\mathcal{E}}_p^{n_k}\}_{k \geq 1}$. Define $\widehat{\mathcal{E}}_p$ as its limit. Let $f \in \mathcal{W}^p$. Then

$$\widehat{\mathcal{E}}_p(f) \leq \varliminf_{k \to \infty} \widehat{\mathcal{E}}_p^{n_k}(f) \leq \sup_{n \geq 1} \widetilde{\mathcal{E}}_p^n(P_n f) = \mathcal{N}_p(f)^p.$$

Let $\{f_{n_k}\}_{k\geq 1}$ be a recovering sequence for f, i.e., $\|f - f_{n_k}\|_p \to 0$ as $k \to \infty$ and $\lim_{k\to\infty} \widehat{\mathcal{E}}_p^{n_k}(f_{n_k}) = \mathcal{E}_p(f)$. By (3.12), if $n_k \geq n$, then

$$C\,\widetilde{\mathcal{E}}_p^n(P_n f_{n_k}) \leq \widetilde{\mathcal{E}}_p^{n_k}(P_{n_k} f_{n_k}) = \widehat{\mathcal{E}}_p^{n_k}(f_{n_k}).$$

Letting $k \to \infty$, we obtain

$$C\,\widetilde{\mathcal{E}}_p^n(P_n f) \leq \widehat{\mathcal{E}}_p(f),$$

so that

$$C\,\mathcal{N}_p(f)^p \leq \widehat{\mathcal{E}}_p(f).$$

The semi-norm property of $\widehat{\mathcal{E}}_p(\cdot)^{\frac{1}{p}}$ is straightforward from basic properties of Γ-convergence.

Next we show that $\widehat{\mathcal{E}}_p(\overline{f}) \leq \widehat{\mathcal{E}}_p(f)$ for any $f \in \mathcal{W}^p$. Define

$$Q_n f = \sum_{w\in T_n} (P_n f)(w)\chi_{K_w}. \tag{3.14}$$

Then

$$\begin{aligned}
&\int_K |f(y) - Q_n f(y)|^p \mu(dy)\\
&\qquad \leq \sum_{w\in T_n} \int_{K_w} \Big(\frac{1}{\mu(w)} \int_{K_w} |f(y) - f(x)|\mu(dx)\Big)^p \mu(dy)\\
&\qquad \leq \sum_{w\in T_n} \frac{1}{\mu(w)} \int_{K_w\times K_w} |f(y) - f(x)|^p \mu(dx)\mu(dy).
\end{aligned}$$

This shows that if $f \in C(K)$, then $\|f - Q_n f\|_p \to 0$ as $n \to \infty$. Let $\{f_{n_k}\}_{k\geq 1}$ be a recovering sequence for f. Since

$$\begin{aligned}
\|\overline{f} - \overline{Q_n g}\|_p &\leq \|\overline{f} - \overline{Q_n f}\|_p + \|\overline{Q_n f} - \overline{Q_n g}\|_p\\
&\leq \|f - Q_n f\|_p + \|Q_n f - Q_n g\|_p\\
&\leq \|f - Q_n f\|_p + \|f - g\|_p,
\end{aligned}$$

it follows that $\|\overline{f} - \overline{Q_{n_k} f_{n_k}}\|_p \to 0$ as $n \to \infty$. Then

$$\begin{aligned}
\widehat{\mathcal{E}}_p(\overline{f}) &\leq \varliminf_{k\to\infty} \widehat{\mathcal{E}}_p^{n_k}(\overline{Q_{n_k} f_{n_k}}) = \varliminf_{k\to\infty} \widetilde{\mathcal{E}}_p^{n_k}(\overline{P_{n_k} f_{n_k}})\\
&\leq \varliminf_{k\to\infty} \widetilde{\mathcal{E}}_p^{n_k}(P_{n_k} f_{n_k}) = \lim_{k\to\infty} \widehat{\mathcal{E}}_p^{n_k}(f_{n_k}) = \widehat{\mathcal{E}}_p(f).
\end{aligned}$$

Finally for $p = 2$, since a Γ-limit of quadratic forms is a quadratic form, we see that $(\widehat{\mathcal{E}}_2, \mathcal{W}^2)$ is a regular Dirichlet form on $L^2(K,\mu)$. Since the inclusion map from $(\mathcal{W}^2, \|\cdot\|_2 + \mathcal{N}_p(\cdot))$ to $(C(K), \|\cdot\|_\infty)$ is a compact operator, by [17, Exercise 4.2], the non-negative self-adjoint operator associated with $(\mathcal{E}_2, \mathcal{W}^p)$ has compact resolvent. ∎

For the case $p = 2$, due to the above theorem, $\mathcal{W}^2$ is separable. Hence, we may replace Γ-convergence by point-wise convergence as seen in the following theorem. This enables us to obtain the local property of our Dirichlet form, which turns out to be a resistance form as well.

Theorem 3.23. *Suppose that Assumption 3.2 holds for $p = 2$. Then there exists a subsequence $\{m_k\}_{k\geq 1}$ such that $\{\widetilde{\mathcal{E}}_2^{m_k}(P_{m_k} f, P_{m_k} g)\}_{k\geq 1}$ converges as $k \to \infty$ for any $f, g \in \mathcal{W}^2$. Furthermore, define $\mathcal{E}(f, g)$ as its limit. Then $(\mathcal{E}, \mathcal{W}^2)$ is a local regular Dirichlet form on $L^2(K, \mu)$, and there exist $c_1, c_2, c_3 > 0$ such that*

$$c_1 \mathcal{N}_2(f) \leq \mathcal{E}(f, f)^{\frac{1}{2}} \leq c_2 \mathcal{N}_2(f) \tag{3.15}$$

and

$$|f(x) - f(y)|^2 \leq c_3 \eta_L(d(x, y))^\tau \mathcal{E}(f, f) \tag{3.16}$$

for any $f \in \mathcal{W}^2$ and $x, y \in K$. In particular, $(\mathcal{E}, \mathcal{W}^2)$ is a resistance form on K and the associated resistance metric R gives the original topology $\mathcal{O}$ of K.

Proof. Existence of $\{m_k\}_{k\geq 1}$: By Lemma 3.21, the non-negative self-adjoint operator H associated with the regular Dirichlet form $(\widehat{\mathcal{E}}_2, \mathcal{W}^2)$ has compact resolvent. Hence there exist a complete orthonormal basis $\{\varphi_i\}_{i\geq 1}$ of $L^2(K, \mu)$ and $\{\lambda_i\}_{i\geq 1} \subseteq [0, \infty)$ such that $H\varphi_i = \lambda_i \varphi_i$ and $\lambda_i \leq \lambda_{i+1}$ for any $i \geq 1$ and $\lim_{i\to\infty} \lambda_i = \infty$. Note that $\{\frac{\varphi_i}{\sqrt{1+\lambda_i}}\}_{i\geq 1}$ is a complete orthonormal system of $(\mathcal{W}^2, (\cdot,\cdot)_{2,\mu} + \widehat{\mathcal{E}}_p(\cdot,\cdot))$. Hence setting

$$\mathcal{F} = \{a_{i_1}\psi_{i_1} + \cdots + a_{i_m}\psi_{i_m} \mid m \geq 1,\ i_1, \ldots, i_m \geq 1,\ a_{i_1}, \ldots, a_{i_m} \in \mathbb{Q}\},$$

we see that $\mathcal{F}$ is a dense subset of $\mathcal{W}^p$. For any $f, g \in \mathcal{F}$, since

$$|\widetilde{\mathcal{E}}_2^n(P_n f, P_n g)| \leq \widetilde{\mathcal{E}}_2^n(P_n f)^{\frac{1}{2}} \widetilde{\mathcal{E}}_2^n(P_n g)^{\frac{1}{2}} \leq \mathcal{N}_2(f)\mathcal{N}_2(g),$$

some subsequence of $\{\widetilde{\mathcal{E}}_2^n(P_n f, P_n g)\}_{n\geq 1}$ is convergent. Since $\mathcal{F} \times \mathcal{F}$ is countable, the standard diagonal argument shows the existence of a subsequence $\{m_k\}_{k\geq 1}$ such that $\widetilde{\mathcal{E}}_2^{m_k}(P_{m_k} f, P_{m_k} g)$ converges as $k \to \infty$ for any $f, g \in \mathcal{F}$. Define $\mathcal{E}_2(f, g)$ as its limit. For $f, g \in \mathcal{W}^2$, choose $\{f_i\}_{i\geq 1} \subseteq \mathcal{F}$ and $\{g_i\}_{i\geq 1} \in \mathcal{F}$ such that $f_i \to f$ and $g_i \to g$ as $i \to \infty$ in $\mathcal{W}^2$. Write $\widetilde{\mathcal{E}}_k(u, v) = \widetilde{\mathcal{E}}_2^{m_k}(P_{m_k} u, P_{m_k} v)$ for ease of notation. Then

$$\begin{aligned}
|\widetilde{\mathcal{E}}_k(f, g) - \widetilde{\mathcal{E}}_l(f, g)| &\leq |\widetilde{\mathcal{E}}_k(f, g) - \widetilde{\mathcal{E}}_k(f_i, g)| + |\widetilde{\mathcal{E}}_k(f_i, g) - \widetilde{\mathcal{E}}_k(f_i, g_i)| \\
&\quad + |\widetilde{\mathcal{E}}_k(f_i . g_i) - \widetilde{\mathcal{E}}_l(f_i, g_i)| + |\widetilde{\mathcal{E}}_l(f_i, g_i) - \widetilde{\mathcal{E}}_l(f_i, g)| \\
&\quad + |\widetilde{\mathcal{E}}_l(f_i, g) - \widetilde{\mathcal{E}}_l(f, g)| \\
&\leq |\widetilde{\mathcal{E}}_k(f_i, g_i) - \widetilde{\mathcal{E}}_l(f_i, g_i)| + 2\mathcal{N}_2(f_i)\mathcal{N}_2(g - g_i) \\
&\quad + 2\mathcal{N}_2(f - f_i)\mathcal{N}_2(g).
\end{aligned}$$

This shows that $\{\widetilde{\mathcal{E}}_k(f,g)\}_{k\geq 1}$ is convergent as $k\to\infty$. The equivalence between $\mathcal{N}_2$ and $\mathcal{E}$, (3.15), is straightforward.

Strongly local property: Let $f,g\in\mathcal{W}^p$. Assume that there exists an open set $U\subseteq K$ such that $\mathrm{supp}(f)\subseteq U$ and $g|_U$ is a constant. Consequently, for sufficiently large k, $\widetilde{\mathcal{E}}_k(f,g)=0$, so that $\mathcal{E}(f,g)=0$.

Markov property: By (3.13) and (3.15),

$$0\leq\mathcal{E}(f,f)\leq\widehat{\mathcal{E}}_2(f,f)$$

for any $f\in\mathcal{W}^2$. Since $(\widehat{\mathcal{E}}_2,\mathcal{W}^2)$ is a regular Dirichlet form, by [16, Theorem 2.4.2], we see that $\mathcal{E}(f,g)=0$ whenever

$$f,g\in\mathcal{W}^2\quad\text{and}\quad f(x)g(x)=0$$

for μ-a.e. $x\in K$. Now by the same argument as in the proof of [7, Theorem 2.1], we have the Markov property.

Resistance form: Among the conditions for a resistance form in [32, Definition 3.1], (RF1), (RF2), (RF3), and (RF5) are immediate from what we have already shown. (RF4) is deduced from (3.16). In fact, (3.16) yields that

$$R(x,y)\leq c\eta_L(d(x,y))^{\tau}$$

for any $x,y\in K$. Assume that $R(x_n,x)\to 0$ as $n\to\infty$ and $\overline{\lim}_{n\to\infty}d(x,x_n)>0$. Note that the collection of

$$U_L^{h_r}(x,r^n)=\bigcup_{w\in T_n:x\in K_w}\Big(\bigcup_{v\in\Gamma_L(w)}K_v\Big)$$

for $n\geq 1$ is a fundamental system of neighborhoods of x by [34, Proposition 2.3.9]. Therefore, there exist $n\geq 1$ and $\{x_{m_k}\}_{k\geq 1}$ such that $x_{m_k}\notin U_L^{h_r}(x,r^n)$ for any $k\geq 1$. Choose $w\in T_n$ such that $x\in K_w$. Then x_{m_k} belongs to K_v for some $v\in\Gamma_L(w)^c$. So,

$$h^*_{L,w}(x)=1\quad\text{and}\quad h^*_{L,w}(x_{m_k})=0.$$

Hence

$$R(x_{m_k},x)\geq\frac{1}{\mathcal{E}(h^*_{L,w})}$$

for any $k\geq 1$. This contradicts the fact that $R(x,x_{m_k})\to 0$ as $k\to\infty$. Thus we have shown $d(x_n,x)\to 0$ as $n\to\infty$. Hence the topology induced by the resistance metric R is the same as the original topology $\mathcal{O}$. ■

3.2 Construction of p-energy: $p \leq \dim_{AR}(K, d)$

In this section, we will consider how much we can salvage the results in the previous section if $p \leq \dim_{AR}(K, d)$. Honestly, what we will have in this section is far from satisfactory mainly because we have no proof of the conjecture saying that $\mathcal{W}^p \cap C(K)$ is dense in $C(K)$ with respect to the supremum norm. In spite of this, we present what we have now for future study.

Throughout this section, we assume (3.1) and fix a covering system $\mathcal{J}$.

For $p < \dim_{AR}(K, d)$, a choice of a covering system really matters. As we have observed in Proposition 2.31, if $\{w, v\} \in \mathcal{J}$ and $K_w \cap K_v$ is a single point, then $\sigma^{\mathcal{J}}_{p,m,|w|} \geq 1$ for any $m \geq 1$. However, since we assume (3.1), this yields that $\mathcal{E}_{M_*,p,m,|w|} \leq c_2$ for any m, so that $\overline{\lim}_{m\to\infty}(\mathcal{E}_{M_*,p,m})^{\frac{1}{m}} \leq 1$. As long as

$$p \geq \dim_{AR}(K, d),$$

this inequality does not cause any inconsistency with Proposition 3.3. On the contrary, if $p < \dim_{AR}(K, d)$, then this seems troublesome. For example, in the case of the unit square, a direct calculation shows that $\overline{\lim}_{m\to\infty}(\mathcal{E}_{M_*,p,m})^{\frac{1}{m}} > 1$ for any $p < \dim_{AR}([-1, 1]^2) = 2$. A similar situation is expected in other cases including the Sierpiński carpet. So, for $p < \dim_{AR}(K, d)$, one should carefully choose $\mathcal{J}$ to avoid a pair sharing only a single point. In the case of the unit square, $\mathcal{J}_\ell$ given in Example 2.32 works for $p < 2$.

As in the previous section, we use $\sigma_{p,m}$ (resp. $\sigma_{p,m,n}$) in place of $\sigma^{\mathcal{J}}_{p,m}$ (reps. $\sigma^{\mathcal{J}}_{p,m,n}$).

Under (3.1), it is straightforward to see that Lemma 3.12 still holds. Replacing $(C(K), \|\cdot\|_\infty)$ by $(L^p(K,\mu), \|\cdot\|_p)$ in the statements and proofs of Lemmas 3.15 and 3.16, we have the following statement.

Lemma 3.24. *$\mathcal{W}^p$ is a Banach space with the norm $\|\cdot\|_p + \mathcal{N}_p(\cdot)$.*

Lemma 3.25. *Let $p > 1$. If $\{f_n\}_{n\geq 1}$ is a bounded sequence in the Banach space $\mathcal{W}^p$, then there exist $\{n_k\}_{k\geq 1}$ and $f \in \mathcal{W}^p$ such that f is the weak limit of $\{f_{n_k}\}_{k\geq 1}$ in $L^p(K,\mu)$,*

$$\|f\|_p \leq \sup_{n\geq 1} \|f_n\|_p \quad \textit{and} \quad \mathcal{N}_p(f) \leq \sup_{n\geq 1} \mathcal{N}_p(f_n).$$

Proof. Since $L^p(K,\mu)$ is reflexive, $\{f_n\}_{n\geq 1}$ contains a weakly convergent sub-sequence $\{f_{n_k}\}_{k\geq 1}$. (See [46, Section V.2].) Let $f \in L^p(K,\mu)$ be its weak limit. Since the map $f \to (P_m f)(w)$ is continuous, we see that $P_m f_{n_k} \to P_m f$ as $k \to \infty$ and hence

$$\widetilde{\mathcal{E}}^m_p(P_m f) = \lim_{k\to\infty} \widetilde{\mathcal{E}}^m_p(P_m f_{n_k}) \leq \sup_{k\geq 1} \mathcal{N}_p(f_{n_k})^{\frac{1}{p}}.$$ ■

Lemma 3.26. *Let $p > 1$. Suppose that $f_n \in \ell(T_n)$ for any $n \geq 1$ and that*

$$\sup_{n\geq 1} \|J_n f_n\|_p < \infty \quad \textit{and} \quad \sup_{n\geq 1} \widetilde{\mathcal{E}}_p^n(f_n) < \infty.$$

Then there exist a subsequence $\{n_k\}_{k\geq 1}$ and $f \in \mathcal{W}^p$ such that f is the weak limit of $\{J_{n_k} f_{n_k}\}_{k\geq 1}$ in $L^p(K,\mu)$ and

$$\|f\|_p \leq \sup_{n\geq 1} \|J_n f_n\|_p \quad \textit{and} \quad C\mathcal{N}_p(f)^p \leq \sup_{n\geq 1} \widetilde{\mathcal{E}}_p^n(f_n).$$

Proof. Since $L^p(K,\mu)$ is reflexive, $\{J_n f_n\}$ possesses a weak convergent sub-sequence $\{J_{n_k} f_{n_k}\}_{k\geq 1}$. (See [46, Section V.2].) Let $f \in L^p(K,\mu)$ be its weak limit. Lemma 3.12 shows that if $n_k \geq m$, then

$$C\widetilde{\mathcal{E}}_p^m(P_m J_{n_k} f_{n_k}) \leq \widetilde{\mathcal{E}}_p^{n_k}(P_{n_k} J_{n_k} f_{n_k}) = \widetilde{\mathcal{E}}_p^{n_k}(f_{n_k}) \leq \sup_{n\geq 1} \widetilde{\mathcal{E}}_p^n(f_n).$$

Letting $k \to \infty$, we see

$$C\widetilde{\mathcal{E}}_p^m(P_m f) \leq \sup_{n\geq 1} \widetilde{\mathcal{E}}_p^n(f_n)$$

for any $m \geq 1$. Thus $f \in \mathcal{W}^p$ and $C\mathcal{N}_p(f)^p \leq \sup_{n\geq 1} \widetilde{\mathcal{E}}_p^n(f_n)$. ∎

Using this lemma, we have a counterpart of Lemma 3.18 as follows.

Lemma 3.27. *There exist $\{h_w^*\}_{w\in T}$ and $\{\varphi_w^*\}_{w\in T} \subseteq \mathcal{W}^p$ such that*

(a) *Set $U_{M_*}(w) = \bigcup_{v\in\Gamma_{M_*}(w)} K_v$. For any $w \in T$, $h_w^*\colon K \to [0,1]$ and*

$$h_w^*(x) = \begin{cases} 1 & \textit{if } x \in K_w, \\ 0 & \textit{if } x \notin U_{M_*}(w). \end{cases}$$

(b) *For any $w \in T$, $\varphi_w^*\colon K \to [0,1]$, $\mathrm{supp}(\varphi_w^*) \subseteq U(w)$, and*

$$\varphi_w^*(x) \geq (L_*)^{-M_*}$$

for any $x \in K_w$. Moreover, for any $n \geq 1$,

$$\sum_{w\in T_n} \varphi_w^* \equiv 1.$$

(c) *For any $w \in T$ and $x \in K$,*

$$\varphi_w^*(x) = \frac{h_w^*(x)}{\sum_{v\in T_{|w|}} h_v^*(x)}.$$

By the above lemma, we have the next statement.

Lemma 3.28. *$\mathcal{W}^p$ is dense in $L^p(K, \mu)$.*

Finally, we have the following result on the construction of a p-energy.

Lemma 3.29. *There exist $\widehat{\mathcal{E}}_p \colon \mathcal{W}^p \to [0, \infty)$ and $c_1, c_2 > 0$ such that $\widehat{\mathcal{E}}_p^{\frac{1}{p}}$ is a semi-norm,*

$$c_1 \mathcal{N}_p(f)^p \leq \widehat{\mathcal{E}}_p(f) \leq c_2 \mathcal{N}_p(f)^p \quad \textit{and} \quad \widehat{\mathcal{E}}_p(\overline{f}) \leq \widehat{\mathcal{E}}_p(f)$$

for any $f \in \mathcal{W}^p$. In particular, for $p = 2$, $(\widehat{\mathcal{E}}_2, \mathcal{W}^2)$ is a Dirichlet form on $L^2(K, \mu)$.

3.3 Conductive homogeneity

In this section, we study the notion of conductive homogeneity, namely, its consequence and how one can show it.

Throughout this section, we suppose that Assumptions 2.6, 2.7, 2.10 and 2.12 hold. Moreover, we fix a covering system $\mathcal{J}$ with covering numbers (N_T, N_E). As in the previous sections, we omit $\mathcal{J}$ in the notations of $\sigma^{\mathcal{J}}_{p,m,n}$ and $\sigma^{\mathcal{J}}_{p,m}$ and use $\sigma_{p,m,n}$ and $\sigma_{p,m}$, respectively. In the end, we will see by Theorem 3.33 that the conductive homogeneity is solely determined by the conductance constants and a choice of $\mathcal{J}$ makes no difference.

The first theorem explains the reason why it is called “homogeneity”.

Theorem 3.30. *A metric space A is p-conductively homogeneous if and only if there exist $c_1, c_2 > 0$ and $\sigma > 0$ such that*

$$c_1 \sigma^{-m} \leq \mathcal{E}_{M_*,p,m}(v, T_n) \leq c_2 \sigma^{-m}, \tag{3.17}$$

and

$$c_1 \sigma^m \leq \sigma_{p,m,n} \leq c_2 \sigma^m$$

for any $m \geq 0$, $n \geq 1$ and $v \in T_n$.

An immediate corollary of this theorem is Theorem 3.5.

Corollary 3.31 (Theorem 3.5). *If K is p-conductively homogeneous, then* (3.1) *holds.*

Proof of Theorem 3.30. Assume that K is p-conductively homogeneous. Then by formula (2.18), there exists $c_1 > 0$ such that

$$c_1 \leq \sigma_{p,m} \mathcal{E}_{M_*,p,m}.$$

Also by Lemma 2.34, there exists $c_2 > 0$ such that

$$\sigma_{p,m+n} \leq c_2 \sigma_{p,m} \sigma_{p,n} \tag{3.18}$$

for any $n, m \geq 0$. Moreover, by (2.14), there exists $c_3 > 0$ such that

$$\mathcal{E}_{M_*,p,m+n} \leq c_3 \mathcal{E}_{M_*,p,m} \mathcal{E}_{M_*,p,n}$$

for any $n, m \geq 0$. These inequalities along with (3.3) shows that there exist $c_4, c_5 > 0$ such that

$$c_4 \sigma_{p,m} \sigma_{p,n} \leq \sigma_{p,m+n} \leq c_5 \sigma_{p,m} \sigma_{p,n} \quad \text{and} \quad c_4 \leq \sigma_{p,m} \mathcal{E}_{M_*,p,m} \leq c_5$$

for any $m, n \geq 0$. From these, there exist $c_6, c_7 > 0$ and $\sigma > 0$ such that

$$c_6 \sigma^m \leq \sigma_{p,m} \leq c_7 \sigma^m \quad \text{and} \quad c_6 \sigma^m \leq (\mathcal{E}_{M_*,p,m})^{-1} \leq c_7 \sigma^m$$

for any $m \geq 0$. Hence for any $w \in T$ and $n \geq 1$,

$$c_6 \sigma^m \leq (\mathcal{E}_{p,m})^{-1} \leq (\mathcal{E}_{M_*,p,m}(w, T_n))^{-1} \quad \text{and} \quad \sigma_{p,m,n} \leq c_7 \sigma^m.$$

Making use of (2.18), we see that there exists $c_8 > 0$ such that

$$c_6 \sigma^m \leq (\mathcal{E}_{M_*,p,m}(w, T_n))^{-1} \leq c_8 \sigma_{p,m,n} \leq c_8 c_7 \sigma^m$$

for any $m \geq 0$, $n \geq 1$ and $w \in T_n$.

The converse direction is straightforward. ■

Next, we show another consequence of conductive homogeneity. For simplicity, we set $\mathcal{E}_{p,m}(u, v, S^k(w)) = \mathcal{E}_{p,m}(\{u\}, \{v\}, S^k(w))$. (In other words, we deliberately confuse u with $\{u\}$.)

Lemma 3.32. *If K is p-conductively homogeneous, then there exists $c_{3.32} > 0$, depending only on p, L_*, N_*, M_*, k, N_T, N_E, such that*

$$\mathcal{E}_{M_*,p,m} \leq c_{3.32} \mathcal{E}_{p,m}(u, v, S^k(w))$$

for any $m \geq 0$, $w \in T$ and $u, v \in S^k(w)$ with $u \neq v$.

Proof. By (2.16), we see that

$$\mathcal{E}_{p,0}(u, v, S^k(w)) \leq c_{2.27} \sigma_{p,m} \mathcal{E}_{p,m}(u, v, S^k(w)).$$

Using Theorem 6.3, it follows that

$$\underline{c}_{\mathcal{E}}(L_*, (N_*)^k, p) \leq \mathcal{E}_{p,0}(u, v, S^k(w)) \leq c_{2.27} \sigma_{p,m} \mathcal{E}_{p,m}(u, v, S^k(w)).$$

Now Theorem 3.30 suffices. ■

When $p > \dim_{AR}(K, d)$, the converse direction of the above lemma is actually true.

Theorem 3.33. *Assume that there exist $c > 0$ and $\alpha \in (0,1)$ such that*

$$\mathcal{E}_{M_*,p,m} \leq c\alpha^m \tag{3.19}$$

for any $m \geq 0$. Then K is p-conductively homogeneous if and only if for any $k \geq 1$, there exists $c(k) > 0$ such that

$$\mathcal{E}_{M_*,p,m} \leq c(k)\mathcal{E}_{p,m}(u, v, S^k(w)) \tag{3.20}$$

for any $m \geq 0$, $w \in T$ and $u, v \in S^k(w)$ with $u \neq v$. In particular, under Assumption 2.15, if $p > \dim_{AR}(K, d)$, then whether K is p-conductively homogeneous or not is independent of neighbor disparity constants and hence a choice of a covering system $\mathcal{J}$.

The last part of the theorem justifies the name "*conductive*" homogeneity.

In fact, (3.19) is the same as (3.2). Recall that, by Proposition 3.3, (3.19) holds if and only if $p > \dim_{AR}(K, d)$ under Assumption 2.15.

As was mentioned in the introduction, (3.20) is an analytic relative of the "knight move" condition described in probabilistic terminologies in [36]. The name "knight move" originated from the epoch-making paper [1] where Barlow and Bass constructed the Brownian motion on the Sierpiński carpet.

The proof of the "only if" part of the above theorem is Lemma 3.32. A proof of the "if" part will be given in Chapter 5.

In the next chapter, we are going to give examples for which one can show p-conductive homogeneity by Theorem 3.33.

In the rest of this section, we study asymptotic behaviors of the heat kernel associated with the diffusion process induced by the Dirichlet form $(\mathcal{E}, \mathcal{W}^2)$ under Assumption 2.15. The next lemma shows that the associated resistance metric is bi-Lipschitz equivalent to a power of the original metric.

Lemma 3.34. *Suppose that Assumption 2.15 holds, $p > \dim_{AR}(K, d)$ and K is p-conductively homogeneous. Let σ be the same as in Theorem 3.30 and set $\tau_p = -\frac{\log \sigma}{\log r}$. Then there exist $c_1, c_2 > 0$ such that*

$$c_1 d(x,y)^{\tau_p} \leq \sup_{f \in \mathcal{W}^p, \widehat{\mathcal{E}}_p(f) \neq 0} \frac{|f(x) - f(y)|^p}{\widehat{\mathcal{E}}_p(f)} \leq c_2 d(x,y)^{\tau_p} \tag{3.21}$$

for any $x, y \in K$. In particular, if $2 > \dim_{AR}(K, d)$, then

$$c_1 d(x,y)^{\tau_2} \leq R(x,y) \leq c_2 d(x,y)^{\tau_2} \tag{3.22}$$

for any $x, y \in K$, where $R(x, y)$ is the resistance metric associated with the resistance form $(\mathcal{E}, \mathcal{W}^2)$.

Proof. Since $\mathcal{E}_p^m(h^*_{M_*,w,m-|w|}) = \mathcal{E}_{M_*,p,m-|w|}(w, T_{|w|})$, we have

$$c_1\sigma^{-m+|w|} \leq \mathcal{E}_p^m(h^*_{M_*,w,m-|w|}) \leq c_2\sigma^{-m+|w|}$$

by (3.17). This shows

$$c_1\sigma^{|w|} \leq \widehat{\mathcal{E}}_p(h^*_{M_*,w}) \leq c_2\sigma^{|w|}.$$

Note that d is M_*- adapted to h_r by Assumption 2.15. Hence by [34, (2.4.1)],

$$c_1 d(x,y) \leq \delta_{M_*}(x,y) \leq c_2 d(x,y) \tag{3.23}$$

for any $x, y \in K$. Choose $n = n_{M_*}(x,y)+1$. Let $w \in T_n$ satisfying $x \in K_w$. Since $n > n_{M_*}(x,y)$, it follows that if $v \in T_n$ and $y \in K_v$, then $v \notin \Gamma_{M_*}(w)$. Hence

$$h^*_{M_*,w}(x) = 1 \quad \text{and} \quad h^*_{M_*,w}(y) = 0.$$

Therefore (3.4) and (3.23) yield

$$\begin{aligned}\sup_{f \in \mathcal{W}^p, \widehat{\mathcal{E}}_p(f) \neq 0} \frac{|f(x)-f(y)|^p}{\widehat{\mathcal{E}}_p(f)} &\geq \frac{1}{\widehat{\mathcal{E}}_p(h^*_{M_*,w})} \\ &\geq c(\sigma_p)^{-n} \geq c' r^{n_{M_*}(x,y)\tau_p} \geq c'' d(x,y)^{\tau_p}.\end{aligned}$$

On the other hand in this case, $\eta_{M_*}(t) = t$ by (3.23). Hence Theorem 3.21 (c) implies the other side of the desired inequality. ∎

Due to the general theory of resistance forms in [32], once we have (3.22), it is straightforward to obtain asymptotic estimates of the heat kernel.

Theorem 3.35. *Suppose that Assumption 2.15 holds, $2 > \dim_{AR}(K,d)$ and K is 2-conductively homogeneous. Set $\tau_* = \tau_2$. Then there exists a jointly continuous hear kernel $p_\mu(t,x,y)$ on $(0,\infty)\times K \times K$ associated with the diffusion process induced by the local regular Dirichlet form $(\mathcal{E}, \mathcal{W}^2)$ on $L^2(K,\mu)$. Moreover,*

(1) *There exist $\beta \geq 2$, a metric ρ, which is quasisymmetric to d, and positive constants c_1, c_2, c_3, c_4 such that*

$$p_\mu(t,x,y) \leq \frac{c_1}{\mu(B_\rho(x,t^{\frac{1}{\beta}}))} \exp\left(-c_2\left(\frac{\rho(x,y)^\beta}{t}\right)^{\frac{1}{\beta-1}}\right) \tag{3.24}$$

for any $(t,x,y) \in (0,\infty)\times K \times K$ and

$$\frac{c_3}{\mu(B_\rho(x,t^{\frac{1}{\beta}}))} \leq p_\mu(t,x,y) \tag{3.25}$$

for any $y \in B_\rho(x, c_4 t^{\frac{1}{\beta}})$.

(2) *Suppose that μ is α_H-Ahlfors regular with respect to the metric d. Set*

$$\beta_* = \tau_* + \alpha_H.$$

Then $\beta_* \geq 2$ and there exist $c_7, c_8, c_9, c_{10} > 0$ such that

$$p_\mu(t, x, y) \leq c_6 t^{-\frac{\alpha_H}{\beta_*}} \exp\Big(-c_7\Big(\frac{d(x,y)^{\beta_*}}{t}\Big)^{\frac{1}{\beta_*-1}}\Big) \tag{3.26}$$

for any $(t, x, y) \in (0, \infty) \times K \times K$ and

$$c_9 t^{-\frac{\alpha_H}{\beta_*}} \leq p_\mu(t, x, y) \tag{3.27}$$

for any $y \in B_d(x, c_{10} t^{\frac{\alpha_H}{\beta_*}})$. In addition, suppose that d has the chain condition, i.e., for any $x, y \in K$ and $n \in \mathbb{N}$, there exist $x_0, \dots, x_n \in K$ such that $x_0 = x, x_n = y$ and

$$d(x_i, x_{i+1}) \leq \frac{Cd(x,y)}{n},$$

where the constant $C > 0$ is independent of x, y and n. Then there exist c_{11}, $c_{12} > 0$ such that

$$c_{11} t^{-\frac{\alpha_H}{\beta_*}} \exp\Big(-c_{12}\Big(\frac{d(x,y)^{\beta_*}}{t}\Big)^{\frac{1}{\beta_*-1}}\Big) \leq p_\mu(t, x, y). \tag{3.28}$$

The exponent α_H above is in fact the Hausdorff dimension of (K, d). The exponents β and β_* are called the walk dimensions.

Proof. We make use of [32, Theorems 15.10 and 15.11]. Since μ has the volume doubling property with respect to d, (3.22) shows that μ has the volume doubling property with respect to R as well. Since K is connected, (K, R) is uniformly perfect. Moreover, since $(\mathcal{E}, \mathcal{W}^2)$ has the local property, the annulus comparable condition (ACC) holds by [32, Proposition 7.6]. Thus, (C1) of [32, Theorem 15.11] is verified and so is (C3) of [32, Theorem 15.11]. Using [32, Theorem 15.11], we have (3.24). Consequently, by [32, Theorem 15.10], we see (3.25). Thus we have shown the first part of the statement. The fact that $\beta \geq 2$, which is beyond the reach of [32, Theorem 15.10], is due to [25]. See also [33, Theorem 22.2].

About the second part, assuming α_H-Ahlfors regularity, i.e., (2.9), we see that

$$h_d(x, s) = s^{\tau_* + \alpha_H} = s^{\beta_*},$$

where $h_d(x, s)$ is defined as

$$h_d(x, s) = \sup_{y \in B_d(x,s)} R(x, y) \cdot \mu(B_d(x, s)).$$

Hence following the flow of exposition of [32, Theorem 15.10], we have

$$g(s) = s^{\beta_*} \quad \text{and} \quad \Phi(s) = s^{\beta_* - 1},$$

where g and Φ appear in the statement of [32, Theorem 15.10]. Consequently, by [32, Theorem 15.10], we obtain (3.26), (3.27) and (3.28). The fact that $\beta_* \geq 2$ can be shown in the same way as we did for β above. ■

Chapter 4

Conductive homogeneity of self-similar sets

4.1 Self-similar sets and self-similarity of energy

In this section, we consider the case where K is a self-similar set with rationally related contraction ratios and construct self-similar energies under conductive homogeneity. Throughout this section, we fix a self-similar structure

$$\mathcal{L} = (K, S, \{f_s\}_{s \in S}).$$

The notion of the self-similar structure was introduced to give a purely topological description of self-similar sets. See [29, Section 1.3] for details.

Definition 4.1. Let K be a compact metrizable space, let S be a finite set, and let $\{f_s\}_{s \in S}$ be a family of continuous injective maps from K to itself.

(1) The triple $(K, S, \{f_s\}_{s \in S})$ is called a *self-similar structure* if there exists a continuous surjective map $\chi\colon S^{\mathbb{N}} \to K$ such that

$$\chi(s_1 s_2 \ldots) = f_{s_1}(\chi(s_2 s_3 \ldots))$$

for any $s_1 s_2 \ldots \in S^{\mathbb{N}}$, where $S^{\mathbb{N}}$ is equipped with the product topology.

(2) Define $W_* = \bigcup_{n \geq 0} S^n$, where $S^0 = \{\phi\}$. An element $(w_1, \ldots, w_n) \in S^n$ is denoted by $w_1 \ldots w_n$. For $w_1 \ldots w_n \in S^n$, set

$$f_w = f_{w_1} \circ \cdots \circ f_{w_n} \quad \text{and} \quad K_w = f_w(K).$$

In particular, f_ϕ is an identity map and $K_\phi = K$.

Hereafter in this section, $(K, S, \{f_s\}_{s \in S})$ is a self-similar structure.

By [29, Proposition 3.3], if $(K, S, \{f_s\}_{s \in S})$ is a self-similar structure, $\chi\colon S^{\mathbb{N}} \to K$ is uniquely given by

$$\{\chi(s_1 s_2 \ldots)\} = \bigcap_{m \geq 0} K_{s_1 \ldots s_m}$$

for any $s_1 s_2 \ldots \in S^{\mathbb{N}}$.

Typically, an example of self-similar structures is given by a self-similar set with respect to a family of contractions. Let (X, d) be a complete metric spaces and let $\{f_i\}_{i=1,\ldots,N}$ be a family of contractions of (X, d), i.e., $f_i\colon X \to X$ and

$$\sup_{x,y \in X, x \neq y} \frac{d(f_i(x), f_i(y))}{d(x, y)} < 1$$

for any $i \in \{1, \dots, N\}$. Then it is known that there exists a unique non-empty compact subset K of X satisfying

$$K = \bigcup_{i=1}^{N} f_i(K). \tag{4.1}$$

See [29, Theorem 1.1.4] for example. The set K is called a self-similar set with respect to $\{f_i\}_{i=1,\dots,N}$. By [29, Theorem 1.2.3], if $S = \{1, \dots, N\}$, then $(K, S, \{f_i\}_{i \in S})$ is a self-similar structure.

Definition 4.2. Let $r \in (0, 1)$ and let $j_s \in \mathbb{N}$ for $s \in S$.

(1) Define

$$j(w) = \sum_{i=1}^{m} j_{w_i} \quad \text{and} \quad g(w) = r^{j(w)} \tag{4.2}$$

for $w = w_1 \dots w_m \in S^m$. (In particular, $j(\phi) = 0$, $g(\phi) = 1$.) Define $\widetilde{\pi}(w_1 \dots w_m) = w_1 \dots w_{m-1}$ for $w = w_1 \dots w_m \in S^m$ and

$$\Lambda^g_{r^n} = \{w \mid w = w_1 \dots w_m \in W_*,\ g(\widetilde{\pi}(w)) > r^n \geq g(w)\}. \tag{4.3}$$

(2) Set

$$T_n = \{(n, w) \mid w \in \Lambda_{r^n}\}, \quad T = \bigcup_{n \geq 0} T_n$$

and define $\iota \colon T \to W^*$ as $\iota(n, w) = w$. Moreover, define

$$\mathcal{A} = \{((n, v), (n+1, w)) \mid n \geq 0,\ v = w \text{ or } v = \widetilde{\pi}(w)\}.$$

Note that $\Lambda^g_{r^n} \cap \Lambda^g_{r^{n+1}}$ can be non-empty. (See Section 4.5 for example.) Thus to distinguish $w \in \Lambda^g_{r^n}$ and $w \in \Lambda^g_{r^{n+1}}$, we have introduced T_n in the above definition.

The following proposition is straightforward.

Proposition 4.3. *The triple $(T, \mathcal{A}, \phi)$ is a rooted tree and $\{K_w\}_{w \in T}$ is a minimal partition of K parametrized by $(T, \mathcal{A}, \phi)$.*

In the rest of this section, we fix $\{j_s\}_{s \in S}$ and the associated partition $(T, \mathcal{A}, \phi)$. Furthermore, we presume the following assumption.

Assumption 4.4. *There exists a metric d on K giving the original topology of K and Assumption 2.15 holds with the metric d.*

If this assumption is satisfied, we say that $\{f_s\}_{s \in S}$ has *rationally related contraction ratios* $\{r^{j_s}\}_{s \in S}$.

In fact, under this assumption, in particular, by Assumption 2.15 (3), there exist $c_1, c_2 > 0$ such that

$$c_1 r^{j(w)} \leq \operatorname{diam}(K_w, d) \leq c_2 r^{j(w)} \tag{4.4}$$

for any $w \in T$. This enable us to regard the contraction ratio of f_s as r^{j_s}. This is why we say that contraction ratios of $\{f_s\}_{s\in S}$ are rationally related.

Combining (4.4) with Assumption 2.15 (2B), we obtain the following proposition.

Proposition 4.5. *Define α_H to be the unique number satisfying*

$$\sum_{s\in S} r^{j_s\alpha_H} = 1$$

and let μ be the self-similar measure on K with weight $\{r^{j_s\alpha_H}\}_{s\in S}$. Then μ is α_H-Ahlfors regular with respect to the metric d and α_H coincides with the Hausdorff dimension of (K, d).

Under our assumptions, let σ be the same constant as in Theorem 3.30. Note that even if we replace the definition of $\widetilde{\mathcal{E}}_p^m(u)$, (3.6), by

$$\widetilde{\mathcal{E}}_p^m(u) = \sigma^m \mathcal{E}_p^m(u), \tag{4.5}$$

all the arguments in Section 3.1 work and the results are unchanged. Our goal in this section is the next theorem.

Theorem 4.6. *Let $(K, S, \{f_s\}_{s\in S})$ be a self-similar structure and let $(T, \mathcal{A}, \phi)$ be given in Definition 4.2. Suppose that Assumption 4.4 is satisfied and that K is p-conductively homogeneous for some $p \in (\dim_{AR}(K, d), \infty)$.*

(1) *For any $w \in W_*$ and $f \in \mathcal{W}^p$,*

$$f \circ f_w \in \mathcal{W}^p.$$

(2) *There exists $\mathcal{E}_p \colon \mathcal{W}^p \to [0, \infty)$ satisfying*

(a) *$(\mathcal{E}_p)^{\frac{1}{p}}$ is a semi-norm on $\mathcal{W}^p$ and there exist $c_1, c_2 > 0$ such that*

$$c_1 \mathcal{N}_p(f) \le \mathcal{E}_p(f)^{\frac{1}{p}} \le c_2 \mathcal{N}_p(f)$$

and

$$c_1 d(x, y)^{\tau_p} \le \sup_{f\in\mathcal{W}^2, \mathcal{E}_p(f)\neq 0} \frac{|f(x) - f(y)|^p}{\mathcal{E}_p(f)} \le c_2 d(x, y)^{\tau_p}$$

for any $f \in \mathcal{W}^p$ and $x, y \in K$.

(b) *For any $f \in \mathcal{W}^p$, $\overline{f} \in \mathcal{W}^p$ and*

$$\mathcal{E}_p(\overline{f}) \le \mathcal{E}_p(f).$$

(c) *For any $f \in \mathcal{W}^p$,*

$$\mathcal{E}_p(f) = \sum_{s\in S} \sigma^{j_s} \mathcal{E}_p(f \circ f_s).$$

In particular, for $p = 2$, $(\mathcal{E}_2, \mathcal{W}^2)$ is a local regular Dirichlet form on $L^2(K, \mu)$.

Proof. Define

$$\mathcal{U} = \big\{A(\cdot) \mid A(\cdot) \text{ is a semi-norm on } \mathcal{W}^p, \text{ there exist } c_1, c_2 > 0 \text{ such that } c_1\mathcal{N}_p(f) \le A(f) \le c_2\mathcal{N}_p(f) \text{ for any } f \in \mathcal{W}^p\big\}.$$

For $A_1, A_2 \in \mathcal{U}$, we write $A_1 \le A_2$ if and only if $A_1(f) \le A_2(f)$ for any $f \in \mathcal{W}^p$. We give $\mathcal{U}$ the point-wise convergence topology, i.e., $\{A_n\}_{n\ge1} \subseteq \mathcal{U}$ is convergent to $A \in \mathcal{U}$ as $n \to \infty$ if and only if $A_n(f) \to A(f)$ as $n \to \infty$ for any $f \in \mathcal{W}^p$. Then due to the separability of $\mathcal{W}^p$ described in Theorem 3.22, $\mathcal{U}$ is an ordered topological cone in the sense of [28].

Let $w \in W_*$. For any $v = v_1 \dots v_k \in \Lambda_{r^{n-j(w)}}$, since

$$g(wv_1 \dots v_{k-1}) = g(w)g(v_1 \dots v_{k-1}) > g(w)r^{n-j(w)} = r^n \ge g(wv),$$

it follows that $wv \in \Lambda_{r^n}$. This shows that $\{(n, wv) | v \in \Lambda_{r^{n-j(w)}}\} \subseteq T_n$. In fact,

$$T_n = \bigcup_{w\in S^m} \{(n, wv) \mid v \in \Lambda_{r^{n-j(w)}}\},$$

which is a disjoint union. This yields

$$\sum_{w\in S^m} \mathcal{E}_p^{n-j(w)}(P_{n-j(w)}(f \circ f_w)) \le \mathcal{E}_p^n(P_n f)$$

for any $f \in L^p(K, \mu)$. Therefore,

$$\sum_{w\in S^m} \sigma^{j(w)} \widehat{\mathcal{E}}^{n-j(w)}(f \circ f_w) \le \widehat{\mathcal{E}}_p^n(f).$$

This inequality implies that $\sigma^{j(w)} \sup_{n\ge j(w)} \widehat{\mathcal{E}}^{n-j(w)}(f \circ f_w) \le \mathcal{N}_p(f)^p < \infty$ for any $f \in \mathcal{W}^p$, so that $f \circ f_w \in \mathcal{W}^p$. Thus we have verified the statement (1). Again by the above inequality,

$$\begin{aligned} c \sum_{w\in S^m} \sigma^{j(w)} \mathcal{N}_p(f \circ f_w)^p &\le \sum_{w\in S^m} \sigma^{j(w)} \varliminf_{n\to\infty} \widehat{\mathcal{E}}^{n-j(w)}(f \circ f_w) \\ &\le \sup_{n\ge0} \widehat{\mathcal{E}}_p^n(f) = \mathcal{N}_p(f)^p. \end{aligned} \tag{4.6}$$

Note that

$$\sum_{(n,v)\in T_n} \sigma^{j(v)} \widehat{\mathcal{E}}_p^{k-j(v)}(f \circ f_v) \le \sum_{w\in S^m} \sigma^{j(w)} \widehat{\mathcal{E}}^{n+k-j(w)}(f \circ f_w).$$

By (3.11), taking $\varliminf$ in the left-hand side and sup in the right-hand side, we see that

$$c \sum_{(n,v)\in T_n} \sigma^{j(v)} \mathcal{N}_p(f \circ f_v)^p \le \sum_{w\in S^m} \sigma^{j(w)} \mathcal{N}_p(f \circ f_w)^p. \tag{4.7}$$

On the other hand, for any $(n, v) \in T_n$ and $x \in K_v$, the self-similarity of μ and (3.21) show

$$
\begin{aligned}
|(P_n f)(v) - f(x)| &\le \int_K |f \circ f_v(y) - f \circ f_v(x_0)|\, \mu(dy) \\
&\le c \int_K d(x_0, y)^{\frac{\tau*}{p}} \mu(dy) \mathcal{N}_p(f \circ f_v) \le c' \mathcal{N}_p(f \circ f_v),
\end{aligned}
$$

where $x_0 = (f_v)^{-1}(x)$. Hence if $((n, v), (n, u)) \in E_n^*$, then

$$|(P_n f)(v) - (P_n f)(u)| \le c'(\mathcal{N}_p(f \circ f_v) + \mathcal{N}_p(f \circ f_w)).$$

This along with (4.7) yields

$$
\begin{aligned}
\widehat{\mathcal{E}}_p^n(f) &= \frac{\sigma^n}{2} \sum_{((n,v),(n,u)) \in E_n^*} |(P_n f)(v) - (P_n f)(u)|^p \\
&\le C \sum_{(n,v) \in T_n} \sigma^{j(w)} \mathcal{N}_p(f \circ f_v)^p \le C' \sum_{w \in S^m} \sigma^{j(w)} \mathcal{N}_p(f \circ f_w)^p.
\end{aligned}
$$

Taking sup in the right-hand side, we have

$$\mathcal{N}_p(f)^p \le C' \sum_{w \in S^m} \sigma^{j(w)} \mathcal{N}_p(f \circ f_w)^p. \tag{4.8}$$

Now for $A \in \mathcal{U}$, define $\mathcal{F}(A)$ by

$$\mathcal{F}(A)(f) = \Big(\sum_{s \in S} \sigma^{j_s} A(f \circ f_s)^p \Big)^{\frac{1}{p}}.$$

For any $A \in \mathcal{U}$, since $A \le c_2 \mathcal{N}_p$, (4.6) implies

$$\mathcal{F}(A) \le c_2 \mathcal{F}(\mathcal{N}_p) \le c' \mathcal{N}_p.$$

On the other hand, the fact $c_1 \mathcal{N}_p \le A$ and (4.8) yield

$$\mathcal{F}(A) \ge c_1 \mathcal{F}(\mathcal{N}_p) \ge c'' \mathcal{N}_p.$$

Thus $\mathcal{F}(A) \in \mathcal{U}$ and $\mathcal{F}\colon \mathcal{U} \to \mathcal{U}$. It is easy to see that $\mathcal{U}$ is continuous and

$$\mathcal{F}(A + B) \le \mathcal{F}(A) + \mathcal{F}(B).$$

Combining (4.6) and (4.8), we see that there exist $C_1, C_2 > 0$ such that

$$c_1 \mathcal{N}_p \le \mathcal{F}^j(\mathcal{N}_p) \le c_2 \mathcal{N}_p$$

for any $j \ge 1$. So, by [28, Theorem 1.5], there exists $\mathcal{E}_* \in \mathcal{U}$ such that $\mathcal{F}(\mathcal{E}_*) = \mathcal{E}_*$. Define

$$\mathcal{U}_M = \{A \mid A \in \mathcal{U}, A(\overline{f}) \le A(f) \text{ for any } f \in \mathcal{W}^p\}.$$

Then $\widehat{\mathcal{E}}_p \in \mathcal{U}_M$ and $\mathcal{U}_M$ is a closed subset of $\mathcal{U}$. Hence by [28, Corollary 1.6], we see there exists $\mathcal{E}' \in \mathcal{U}_M$ such that $\mathcal{F}(\mathcal{E}') = \mathcal{E}'$. Letting $\mathcal{E} = (\mathcal{E}')^p$, we have the desired $\mathcal{E}$. In the case $p = 2$, define

$$\mathcal{U}_{DF} = \{A \mid A \in \mathcal{U}, A \text{ satisfies the parallelogram law, the resulting quadratic form has both Markov and local property}\}.$$

Then $\mathcal{U}_{DF}$ is a closed subspace of $\mathcal{U}$ and Theorem 3.23 ensures that $\mathcal{U}_{DF} \neq \emptyset$. So again by [28, Corollary 1.6], we have the desired local regular Dirichlet form. ■

4.2 Conductive homogeneity of self-similar sets

In this section, we present a sufficient condition for conductive homogeneity of self-similar sets. The idea originated from [11], where the authors used symmetries of the spaces to show the combinatorial Loewner property of the Sierpiński carpet and the Menger curve, also known as the Menger sponge. Our sufficient condition, Theorem 4.8, will be used in Sections 4.3 and 4.6.

Throughout this section, we assume that $(K, S, \{f_s\}_{s\in S})$ is a self-similar structure and adopt the setting in Section 4.1, i.e., let $(T, \mathcal{A}, \phi)$ be given in Definition 4.2 and we suppose that Assumption 4.4 is satisfied. For simplicity, we also assume that $j_s = 1$ for any $s \in S$, so that $g(w) = r^{|w|}$ and $T_m = S^m$.

Definition 4.7. (1) For any $e = (w, v) \in \bigcup_{m\geq 1} E_m^*$, define

$$X(e) = (f_w)^{-1}(f_w(K) \cap f_v(K))$$

and $\varphi_e \colon X(e) \to X(e^r)$ by $\varphi_e = (f_v)^{-1} \circ f_w|_{X(e)}$, where $e^r = (v, w)$ for $e = (w, v)$. Furthermore, define

$$\mathcal{IT}(K, T) = \{(X(e), X(e^r), \varphi_e) \mid m \geq 1, e \in E_m^*\}.$$

An element of $\mathcal{IT}(K, T)$ is called an *intersection type* of (K, T).

(2) A homeomorphism $g\colon K \to K$ is said to be a *symmetry* of (K, T) if there exists $g^*\colon T \to T$ such that $|g^*(w)| = |w|$ and $g(K_w) = K_{g^*(w)}$ for any $w \in T$. Define $\mathcal{G}_{(K,T)}$ as the collection of symmetries of (K, T).

(3) For any $n \geq 0$, define $\psi_n\colon \bigcup_{m\geq 0} T_{n+m} \to T$ by $\psi_n(v) = u$ if $v \in T_{n+m}$ and $v = \pi^m(v)u$.

Remark. The notion of intersection types and the set $\mathcal{IT}(K, T)$ were introduced in [31].

Note that $\psi_n(T_{n+m}) = T_m$ and $(f_{\pi^m(v)})^{-1}(K_v) = K_{\psi_n(v)}$ for any $v \in T_{n+m}$.

Notation. For $A \subseteq T$, set

$$K(A) = \bigcup_{v \in A} K_v. \tag{4.9}$$

Theorem 4.8. *Suppose that there exist a finite subset $\mathcal{I} \subseteq \mathcal{IT}(K,T)$ and finite subgroups $\mathcal{G}_0$ and $\mathcal{G}_1$ of $\mathcal{G}_{(K,T)}$ satisfying the following properties:*

(a) $(T_m, E_m^{\mathcal{I}})$ *is connected for any $m \geq 1$, where*

$$E_m^{\mathcal{I}} = \{e \mid e \in E_m^*, (X(e), X(e^r), \varphi_e) \in \mathcal{I}\}.$$

(b) *For any $(X, Y, \varphi) \in \mathcal{I}$ and $x \in X$, there exists $g \in \mathcal{G}_0$ such that $g(x) = \varphi(x)$.*

(c) *For any $n \geq 1$, $w \in T_n$ and $\mathbf{p} \in \mathcal{C}_{M,m}^{(1)}(w)$, there exists $\mathcal{U}_{\mathbf{p}} \subseteq \bigcup_{g \in \mathcal{G}_1} g^*(\psi_n(\mathbf{p}))$ such that $K(\mathcal{U}_p)$ is connected and $g(K(\mathcal{U}_p)) \cap X \neq \emptyset$ and $g(K(\mathcal{U}_p)) \cap Y \neq \emptyset$ for any $(X, Y, \varphi) \in \mathcal{I}$ and $g \in \mathcal{G}_0$.*

Then for any $p \geq 1$, $n, k \geq 1$, $m \geq 1$, $u_*, v_* \in T_k$, and $w \in T_n$,

$$\mathcal{M}_{M,p,m}^{(1)}(w) \leq (L_*)^M \#(\mathcal{G}_1)^{p+1} \#(T_k)^p \mathcal{M}_{p,m}^{(1)}(u_*, v_*, T_k). \tag{4.10}$$

Furthermore, if Assumption 4.4 holds with $M_* = M$, then K is p-conductively homogeneous for any $p > \dim_{AR}(K, d)$.

Remark. Strictly, a path $\mathbf{p} = (w(1), \dots, w(k))$ of a graph is not a subset of vertices but a sequence of them. However, we use $\mathbf{p}$ to denote a subset $\{w(1), \dots, w(k)\}$ if no confusion may occur. For example, in the expression $\psi_n(\mathbf{p})$ above, we regard $\mathbf{p}$ as a subset of T_{n+m}.

Proof of Theorem 4.8. For $u \in S^m(\Gamma_M(w))$, define $H_u \subseteq T_{k+m}$ by

$$H_u = \{vg^*(\psi_n(u)) \mid g \in \mathcal{G}_1, v \in T_k\}.$$

Then we have that $\#(H_u) \leq \#(T_k)\#(\mathcal{G}_1)$ for any $u \in S^m(\Gamma_1(w))$ and $\#(\{u \mid v \in H_u\}) \leq \#(\Gamma_M(w))\#(\mathcal{G}_1)$ for any $v \in T_{k+m}$.

Now, since $(T_k, E_k^{\mathcal{I}})$ is connected, there exists $(w(0), w(1), \dots, w(l), w(l+1)) \in (T_k)^{l+2}$ such that $w(0) = u_*$, $w(l+1) = v_*$, $(w(i), w(i+1)) \in E_k^{\mathcal{I}}$ for any $i = 0, 1, \dots, l$. Set $e_i = (w(i), w(i+1))$. Then $(X(e_i), X((e_i)^r), \varphi_{e_i}) \in \mathcal{I}$.

Claim. *There exist $\mathcal{A}_i \subseteq T_m$, $x_i \in K$ and $g_i, h_i \in \mathcal{G}_0$ for $i = 1, 2, \dots, l$ such that*

(i) $\mathcal{A}_i = (h_i)^*(\mathcal{U}_{\mathbf{p}})$ *and* $K(\mathcal{A}_i) \cap X(e_i) \neq \emptyset$,

(ii) $x_i \in K(\mathcal{A}_i) \cap X(e_i)$ *and* $g_i(x_i) = \varphi_{e_i}(x_i)$,

(iii) $\mathcal{A}_{i+1} = (g_i)^*(\mathcal{A}_i)$.

Proof. For $i = 1$, let h_1 be the identity map. Then $\mathcal{A}_1 = \mathcal{U}_{\mathbf{p}}$. Since by (c) $K(\mathcal{A}_1) \cap X(e_1) \neq \emptyset$, we may choose $x_1 \in K(\mathcal{A}_1) \cap X(e_1)$. By (b), there exists $g_1 \in \mathcal{G}_0$ such that $g_1(x_1) = \varphi_{e_1}(x_1)$.

Assume that we have the desired objects for $i \in \{1, \dots, l-1\}$. Letting $h_{i+1} = g_i \circ h_i \in \mathcal{G}_0$ and $\mathcal{A}_{i+1} = (g_i)^*(\mathcal{A}_i)$, we obtain

$$\mathcal{A}_{i+1} = (g_i)^*(h_i)^*(\mathcal{U}_{\mathbf{p}}) = (h_{i+1})^*(\mathcal{U}_{\mathbf{p}}).$$

Using (c), we see that $K(\mathcal{A}_{i+1}) \cap X(e_{i+1}) \neq \emptyset$. Choose $x_{i+1} \in K(\mathcal{A}_{i+1}) \cap X(e_{i+1})$. By (b), there exists $g_{i+1} \in \mathcal{G}_0$ such that $g_{i+1}(x_{i+1}) = \varphi_{e_{i+1}}(x_{i+1})$.

Thus by induction, the claim has been proven. □

Now, by (c), $X(e_0) \cap K(\mathcal{A}_1) \neq \emptyset$ and $X((e_l)^r) \cap K(\mathcal{A}_l) \neq \emptyset$. This implies

$$f_{w(1)}(K(\mathcal{A}_1)) \cap K_{w(0)} \neq \emptyset \quad \text{and} \quad f_{w(l)}(K(\mathcal{A}_l)) \cap K_{w(l+1)} \neq \emptyset. \tag{4.11}$$

Next, (ii) yields $f_{w(i+1)}(g_i(x_i)) = f_{w(i)}(x_i)$. Since

$$g_i(x_i) \in K((g_i)^*(\mathcal{A}_i)) = K(\mathcal{A}_{i+1}),$$

we have

$$f_{w(i)}(K(\mathcal{A}_i)) \cap f_{w(i+1)}(K(\mathcal{A}_{i+1})) \neq \emptyset \tag{4.12}$$

for $i = 1, \dots, l$. Since $\mathcal{A}_i = (h_i)^*(\mathcal{U}_{\mathbf{p}}) \subseteq \bigcup_{g \in \mathcal{G}_1} g^*(\psi_n(\mathbf{p}))$, we see that

$$\bigcup_{i=1}^{l} w(i)\mathcal{A}_i \subseteq \bigcup_{u \in \mathbf{p}} H_u.$$

Note that $K(\bigcup_{i=1}^{l} w(i)\mathcal{A}_i) = \bigcup_{i=1}^{l} f_{w(i)}(\mathcal{A}_i)$. By formulas (4.12) and (4.11), we see that $K(\bigcup_{i=1}^{l} w(i)\mathcal{A}_i)$ is connected and intersects with $K_{w(0)}$ and $K_{w(l+1)}$. Therefore, there exists $\mathbf{p}_0 \in \mathcal{C}_m^{(1)}(u_*, v_*, T_k)$ included in $\bigcup_{i=1}^{l} w(i)\mathcal{A}_i \subseteq \bigcup_{u \in \mathbf{p}} H_u$. Consequently, Lemma C.4 shows (4.10). The conductive homogeneity follows from Lemma 2.22 and Theorem 3.33. ■

4.3 Subsystems of (hyper)cubic tiling

In this section, we present three classes of hypercube-based self-similar sets as examples of conductively homogeneous spaces. The first one given in Theorem 4.13 includes generalized Sierpiński carpets studied in the series of papers [1–6] by Barlow and Bass, the Menger curves (also known as the Menger sponge), and the hypercubes $[-1, 1]^L$ for $L \geq 1$. Unlike those examples, however, our examples also contain self-similar sets with fewer, or even no, symmetries of a hypercube. See Section 4.4, where we present explicit examples of self-similar sets belonging to the classes given in this section.

We start with basic notations on the hypercube $[-1, 1]^L$ and its symmetry group.

Definition 4.9. Let $L \in \mathbb{N}$ and let $C_*^L = [-1, 1]^L$. Moreover, let $\mathbb{B}_L$ be the L-dimensional *hyperoctahedral group*, that is,

$$\mathbb{B}_L = \{g \mid g \in O(L),\ g(C_*^L) = C_*^L\},$$

where $O(L)$ is the collection of orthogonal transformations of $\mathbb{R}^L$. For the case $L = 2$, $\mathbb{B}_2$ is often denoted by D_4 in a literature. Define

$$B_{j,i} = \{(x_1, \dots, x_L) \mid (x_1, \dots, x_L) \in [-1, 1]^L,\ x_j = i\}$$

for $j = \{1, \dots, L\}$ and $i \in \{-1, 0, 1\}$. Then the boundary of $[-1, 1]^L$ consists of $\{B_{j,i}\}_{j \in \{1,\dots,L\}, i \in \{1,-1\}}$. For $s = (s_1, \dots, s_L) \in \{1, \dots, N\}^L$, define

$$C_s^{L,N} = \prod_{i=1}^{L} \Big[\frac{2s_i - 2 - N}{N}, \frac{2s_i - N}{N}\Big],$$
$$c_s^{L,N} = \Big(\frac{2s_1 - 1 - N}{N}, \dots, \frac{2s_L - 1 - N}{N}\Big).$$

If no confusion may occur, we use C_*, C_s and c_s instead of C_*^L, $C_s^{L,N}$ and $c_s^{L,N}$ respectively hereafter.

In the course of this section, we are going to deal with particular elements of $\mathbb{B}_L$.

Definition 4.10. Define $R_j \in \mathbb{B}_L$ as the reflection in the hyperplane $B_{j,0}$ for $j \in \{1, \dots, L\}$. Furthermore, define $R^i_{j_1,j_2}$ as the reflection in the hyperplane

$$\mathcal{H}^i_{j_1,j_2} = \{(x_1, \dots, x_L) \mid x_{j_1} = i x_{j_2}\}$$

for $j_1, j_2 \in \{1, \dots, L\}$ with $j_1 \neq j_2$ and $i \in \{1, -1\}$.

In the next definition, we introduce key notions of this section.

Throughout this section, we fix $L \geq 1$ and $N \geq 2$.

Definition 4.11. (1) A self-similar structure $(K, S, \{f_s\}_{s \in S})$ is called a *subsystem of L-dimensional hypercubic tiling*, or a *subsystem of cubic tiling* for short, if $K \subseteq C_*$, $S \subseteq \{1, \dots, N\}^L$ and, for any $s \in S$, f_s is a restriction of a similitude from $\mathbb{R}^L$ to itself satisfying $f_s(C_*) = C_s$, i.e., there exists $\Phi_s \in \mathbb{B}_L$ such that

$$f_s(x) = \frac{1}{N}\Phi_s x + c_s$$

for any $x \in \mathbb{R}^L$. A subsystem of cubic tiling $(K, S, \{f_s\}_{s \in S})$ is called non-degenerate if $K \cap B_{j,i} \neq \emptyset$ for any $j \in \{1, \dots, L\}$ and $i \in \{1, -1\}$.

(2) A continuous map $\varphi\colon C_* \to C_*$ is called an *N-folding map* if and only if, for any $s \in \{1, \dots, N\}^L$, there exists $A_s \in \mathbb{B}_L$ such that

$$\varphi(x) = NA_s(x - c_s) \tag{4.13}$$

for any $x \in C_s$. If no confusion may occur, we omit N in the expression of an "N-folding" map and say a "folding map" for simplicity.

(3) Let $\mathcal{L} = (K, S, \{f_s\}_{s\in S})$ be a subsystem of cubic tiling. We use the framework of Section 4.1 to define $(T, \mathcal{A}, \phi)$ with $r = \frac{1}{N}$ and $j_s = 1$ for any $s \in S$. In this case, $T_n = S^n$ for any $n \geq 1$. Define a graph (T_n, E_n^ℓ) by

$$E_n^\ell = \{(w, v) \mid w, v \in T_n, w \neq v, f_w(C_*) \cap f_v(C_*) = f_w(B_{j,i}) \text{ for some } j \in \{1, \dots, L\} \text{ and } i \in \{1, -1\}\}.$$

The subsystem of cubic tiling $\mathcal{L}$ is said to be *strongly connected* if (T_n, E_n^ℓ) is connected for any $n \geq 1$.

(4) Let $\mathcal{L} = (K, S, \{f_s\}_{s\in S})$ be a subsystem of cubic tiling. $\mathcal{L}$ is called *locally symmetric* if $K_w \cup K_v$ is invariant under the reflection in the hyperplane including $f_w(C_*) \cap f_v(C_*)$ for any $n \geq 1$ and $(w, v) \in E_n^\ell$.

Remark. Let $\mathcal{L}$ be a subsystem of cubic tiling which is non-degenerate and locally symmetric. Then $E_n^\ell \subseteq E_n^*$ by the following arguments. Assume that $(w, v) \in E_n^\ell$. Set

$$\ell_{w,v} = f_w(C_*) \cap f_v(C_*). \tag{4.14}$$

By non-degeneracy, $K_w \cap \ell_{w,v} \neq \emptyset$ and by local symmetry, $K_w \cap \ell_{w,v} = K_v \cap \ell_{w,v} \neq \emptyset$. Hence $(w, v) \in E_n^*$. Note that even if $(w, v) \in T_n$ and $f_w(C_*) \cap f_v(C_*) \neq \emptyset$, it may happen that $K_w \cap K_v = \emptyset$.

Remark. Let $\mathcal{L}$ be a subsystem of cubic tiling which is non-degenerate, locally symmetric, and strongly connected. As in the case of the unit square in Example 2.32, define

$$\mathcal{J}_\ell = \Big\{\{w, v\} \;\Big|\; (w, v) \in \bigcup_{n\geq 0} E_n^\ell\Big\}. \tag{4.15}$$

For explicit examples in the next section except for the chipped Sierpiński carpet, $\mathcal{J}_\ell$ is a covering system and is a good substitute for $\mathcal{J}_*$ in the case $p < \dim_{AR}(K, d)$.

By properties of cubic tiling, it is easy to see that Assumption 2.15 holds. In summary, we have the next proposition. Recall that the edges of T_n are given not by E_n^ℓ but by E_n^* as it has always been in the previous sections.

Proposition 4.12. *Let $\mathcal{L} = (K, S, \{f_s\}_{s\in S})$ be a subsystem of cubic tiling. Then the family $\{K_w\}_{w\in T}$ is a partition of K parametrized by the tree $(T, \mathcal{A}, \phi)$. Let d_* be the restriction of the Euclidean metric on K and let μ be the self-similar measure satisfying $\mu(K_w) = (\#(S))^{-|w|}$ for any $w \in T$. Then Assumption 2.15 is satisfied with $d = d_*$, $r = \frac{1}{N}$, $M_* = 1$, $M_0 = 1$, $N_* = \#(S)$ and $L_* \leq 3^L - 1$. In this case, μ is α_H-Ahlfors regular with respect to d_*, where $\alpha_H = \frac{\log \#(S)}{\log N}$.*

The exponent α_H coincides with the Hausdorff dimension of (K, d_*). Note that $\#(S) \leq N^L$. Since $\#(S) = N^L$ implies $K = C_*$, we see that $\alpha_H < L$ unless $K = C_*$.

The following theorems are the main results of this section.

Theorem 4.13. *Let $\mathcal{L} = (K, S, \{f_s\}_{s\in S})$ be a subsystem of cubic tiling. Assume that $\mathcal{L}$ is non-degenerate, locally symmetric, and strongly connected. Moreover, suppose that the following condition* (SDR) *is satisfied:*

(SDR) *For any $j_1, j_2 \in \{1, \dots, L\}$ with $j_1 \neq j_2$, there exists $i \in \{1, -1\}$ such that $R^i_{j_1,j_2} \in \mathcal{G}_{(K,T)}$.*

Then K is p-conductively homogeneous for any $p > \dim_{AR}(K, d_*)$.

The name (SDR) represents "symmetric with respect to diagonal reflections" as $R^i_{j_1,j_2}$ is the reflection in the diagonal hyperplane $\mathcal{H}^i_{j_1,j_2}$. For generalized Sierpiński carpets, the Menger curve and the hypercube, it follows that $\mathcal{G}_{(K,T)} = \mathbb{B}_L$ and (SDR) is satisfied. However, $\mathcal{G}_{(K,T)}$ does not necessarily coincide with $\mathbb{B}_L$ to satisfy (SDR). For example, the group generated by $\{R^1_{j_1,j_2} \mid j_1, j_2 \in \{1, \dots, L\}, j_1 \neq j_2\}$ is (isomorphic to) the symmetric group of order L, $\mathcal{S}_L$, which is a proper subgroup of $\mathbb{B}_L$, and if $\mathcal{S}_L \subseteq \mathcal{G}_{(K,T)}$, then (SDR) is satisfied. See Example 4.30.

In the case $L = 2$, the advantage of being planar gives another two classes having conductive homogeneity.

Theorem 4.14. *Let $L = 2$ and let $\mathcal{L} = (K, S, \{f_s\}_{s\in S})$ be a subsystem of 2-dimensional cubic tiling. Assume that $\mathcal{L}$ is non-degenerate, locally symmetric, and strongly connected. Moreover, assume one of the following two conditions* (RS) *or* (NS)*:*

(RS) *$\Theta_{\frac{\pi}{2}} \in \mathcal{G}_{(K,T)}$, where $\Theta_{\frac{\pi}{2}}$ is the rotation by $\frac{\pi}{2}$ around* $(0, 0)$.

(NS) *For each $i, j \in \{1, \dots, N-1\}$, there exist $i_1, j_1 \in \{1, \dots, N\}$ such that*

$$\{(i, j_1), (i+1, j_1), (i_1, j), (i_1, j+1)\} \cap S = \emptyset.$$

Then K is p-conductively homogeneous for any $p > \dim_{AR}(K, d_*)$.

The expressions (RS) and (NS) represent "rotational symmetry" and "no symmetry", respectively.

At a glance at definitions, it may look difficult to verify the conditions like "non-degenerate", "strongly continuous", and "locally symmetric". In the course of the discussion, however, we will show useful criteria concerning only the first iteration $\{f_s(C_*)\}_{s\in S}$ to check those conditions.

Proofs of the above theorems will be given later in this section after necessary preparations. The main idea of the proof is to construct a family of paths required (c) of Theorem 4.8 by using local symmetry and an additional geometric condition (SDR), (RS), or (NS). Such an idea was used in [11] and can be traced back to the "knight move" argument by Barlow–Bass [1]. In those previous works, however, the full $\mathbb{B}_L$-

symmetry of the space was required but we find that weaker (or even no) symmetry is good enough under the presence of local symmetry.

Now we start to study the conditions “non-degenerate”, “strong continuous”, and “locally symmetric”. First, we study the nature of folding maps, which turns out to be closely related to the local symmetry.

Lemma 4.15. *Let $\varphi\colon C_* \to C_*$ be a folding map characterized as (4.13). Then for any $s, t \in \{1, \dots, N\}^L$,*

$$A_s = A_t R_j \quad \text{if } C_s \cap C_t = \frac{1}{N} B_{j,i} + c_s \text{ for some } i \in \{1, -1\}.$$

Proof. Assume that $C_s \cap C_t = \frac{1}{N} B_{j,i} + c_s$. Then $C_s \cap C_t = \frac{1}{N} B_{j,-i} + c_t$ as well and $x - c_t = R_j(x - c_s)$ for any $x \in C_s \cap C_t$. On the other hand, as φ is a folding map, we see that

$$NA_s(x - c_s) = NA_t(x - c_t)$$

for any $x \in C_s \cap C_t$. Hence $A_s(x - c_s) = A_t R_j(x - c_s)$ for any $x \in C_s \cap C_t$. This immediately implies $A_s = A_t R_j$. ∎

Note that $R_{j_1} R_{j_2} = R_{j_2} R_{j_1}$ for any $j_1, j_2 \in \{1, \dots, L\}$. So, by the above lemma, we can determine all the folding maps as follows.

Lemma 4.16. *Fix $s^* = (s_1^*, \dots, s_L^*) \in \{1, \dots, N\}^L$. For $A \in \mathbb{B}_L$, define $\varphi_{s^*,A}\colon C_* \to C_*$ by*

$$\varphi_{s^*,A}(x) = NA \prod_{j=1}^{L} (R_j)^{|s_j^* - s_j|}(x - c_{(i,j)}^N)$$

for any $x \in C_{(s_1,\dots,s_L)}$. Then $\varphi_{s_0,A}$ is a folding map. Moreover, $\{\varphi_{s^*,A} \mid A \in \mathbb{B}_L\}$ is the totality of folding maps for any $s^* \in \{1, \dots, N\}^L$.

Examples of folding maps in the case of $L = 2$ are given in Figure 4.1. In each example, $s^* = (1, 1)$ and $A = I$. The element of $\mathbb{B}_2$ in each square indicates the corresponding $A(R_1)^{|s_1 - s_1^*|}(R_2)^{|s_2 - s_2^*|}$.

Notation. Let $\mathcal{L} = (K, S, \{f_s\}_{s \in S})$ be a subsystem of cubic tiling. Set

$$K^{(m)} = \bigcup_{w \in T_m} f_w(C_*).$$

Due to the next lemma, one can easily determine the non-degeneracy of K by examining $K^{(1)}$.

Lemma 4.17. *Let $\mathcal{L} = (K, S, \{f_s\}_{s \in S})$ be a subsystem of cubic tiling. Then $\mathcal{L}$ is non-degenerate if and only if $K^{(1)} \cap B_{j,i} \neq \emptyset$ for any $j \in \{1, \dots, L\}$ and $i \in \{1, -1\}$.*

I	R_1	I
R_2	$-I$	R_2
I	R_1	I

$N = 3$

R_2	$-I$	R_2	$-I$
I	R_1	I	R_1
R_2	$-I$	R_2	$-I$
I	R_1	I	R_1

$N = 4$

Figure 4.1. Folding maps.

Proof. Since $K \subseteq K^{(1)}$, the "only if" part is obvious. Assume that $K^{(1)} \cap B_{j,i} \neq \emptyset$ for any $j \in \{1, \ldots, L\}$ and $i \in \{1, -1\}$. We are going to show that $K^{(k)} \cap B_{j,i} \neq \emptyset$ for any $j \in \{1, \ldots, L\}$, $i \in \{1, -1\}$, and $k \in \{1, \ldots, n\}$ by induction on n. Assume that the claim holds for n. Let $w \in T_n$ satisfying $f_w(C_*) \cap B_{j,i} \neq \emptyset$. Since

$$(f_w)^{-1}(f_w(C_*) \cap B_{j,i}) = B_{j_1,i_1}$$

for some $j_1 \in \{1, \ldots, L\}$ and $i_1 \in \{1, -1\}$, there exists $s \in T_1$ such that

$$f_s(C_*) \cap (f_w)^{-1}(f_w(C_*) \cap B_{j,i}) \neq \emptyset.$$

This implies that $f_{ws}(C_*) \cap B_{j,i} \neq \emptyset$. Thus we have shown the desired statement for $n + 1$. Now by induction,

$$K^{(k)} \cap B_{j,i} \neq \emptyset$$

for any $j \in \{1, \ldots, L\}$, $i \in \{1, -1\}$. Since $K^{(n)}$ is monotonically decreasing and $K = \bigcap_{n \geq 1} K^{(n)}$, it follows that $K \cap B_{j,i} \neq \emptyset$ for any $j \in \{1, \ldots, L\}$ and $i \in \{1, -1\}$. ∎

The locally symmetric property can also be determined by the first step of the iteration as follows.

Lemma 4.18. *Let $\mathcal{L} = (K, S, \{f_s\}_{s \in S})$ be a subsystem of cubic tiling. Then $\mathcal{L}$ is locally symmetric if and only if $K_s \cup K_t$ is invariant under the reflection in $\ell_{s,t}$ for any $(s, t) \in E_1^{\ell}$.*

Proof. The "only if" part is obvious. We show the following statement by induction on $n \geq 1$.

For any $k \in \{1, \ldots, n\}$ and $(w, v) \in E_k^{\ell}$, $K_w \cup K_v$ is invariant under the reflection in $\ell_{w,v}$.

The case $n = 1$ is exactly the assumption of the lemma. Suppose that the statement holds for n. Let $(w, v) \in E_{n+1}^{\ell}$. In the case $\pi^n(w) = \pi^n(v)$, let $s = \pi^n(w)$. Then $w = sw'$ and $v = sv'$ for some $w', v' \in T_n$. Since $f_w(C_*) = f_s(f_{w'}(C_*))$ and $f_v(C_*) =$

$f_s(f_{v'}(C_*))$, we see $\ell_{w',v'} \in E^\ell_n$. By induction hypothesis, $K_{w'} \cap K_{v'}$ is invariant under the reflection in $\ell_{w',v'}$. Applying f_s, we see that $K_w \cup K_v$ is invariant under the reflection in $\ell_{w,v}$. In the case $\pi^n(w) \neq \pi^n(v)$, let $s = \pi^n(w)$ and let $t = \pi^n(v)$. Since $\ell_{w,v} \subseteq \ell_{s,t} = f_s(B_{j,i})$ for some $j \in \{1, \dots, L\}$ and $i \in \{1, -1\}$, we obtain $(s,t) \in E^\ell_1$. So, $K_s \cup K_t$ is invariant under the reflection in $\ell_{s,t}$. Denoting this reflection by R, we see that R coincides with the reflection in $\ell_{w,v}$. Since $R(f_w(C_*)) = f_v(C_*)$, it follows that $R(K_w) = R(K_s \cap f_w(C_*)) = K_t \cap f_v(C_*) = K_v$. So we have verified the statement for $n + 1$. Thus by induction, we have the desired result. ■

Next, we consider the strong connectedness.

Lemma 4.19. *Let $\mathcal{L} = (K, S, \{f_s\}_{s\in S})$ be a locally symmetric subsystem of cubic tiling. If $\mathcal{L}$ is non-degenerate and (T_1, E^ℓ_1) is connected, then $\mathcal{L}$ is strongly connected.*

Proof. By the non-degeneracy, we see that $K^{(n)} \cap B_{j,i} \neq \emptyset$ for any $j \in \{1, \dots, L\}$ and $i \in \{1, -1\}$.

We are going to show that (T_k, E^ℓ_k) is connected for any $k \in \{1, \dots, n\}$ by induction on $n \geq 1$. Assume that $w, v \in T_{n+1}$. If $\pi^n(w) = \pi^n(v)$, then there exist $w', v' \in T_n$ such that $w = sw'$ and $v = sv'$, where $s = \pi^n(w)$. Since w' and v' are connected by an E^ℓ_n-path, w and v are connected by an E^ℓ_{n+1}-path. In the case $\pi^n(w) \neq \pi^n(v)$, let $s = \pi^n(w)$ and let $t = \pi^n(v)$. Then $w = sw'$ and $v = tv'$ for some $w', v' \in T_n$. Since (T_1, E^ℓ_1) is connected, there exists an E^ℓ_1-path $(s(0), \dots, s(m))$ such that $s(0) = s$, $s(m) = t$ and $(s(i), s(i+1)) \in E^\ell_1$ for any $i = 0, \dots, m-1$. For each $i = 0, \dots, m-1$, since $\bigcup_{w'\in T_n} f_{w'}(C_*) \cap B_{j,i} \neq \emptyset$ for any $j = \{1, \dots, L\}$ and $i \in \{1, -1\}$, there exists $u(i) \in T_n$ such that $f_{s(i)u(i)}(C_*) \cap \ell_{s(i),s(i+1)} \neq \emptyset$. Since $\mathcal{L}$ is locally symmetric, there exists $v(i) \in T_n$ such that $f_{s(i+1)v(i)}(C_*)$ is the image of $f_{s(i)u(i)}(C_*)$ by the reflection in $\ell_{s(i),s(i+1)}$. Define $v(-1) = w'$ and $u(m) = v'$. Then $w = s(0)v(-1)$ and $v = s(m)u(m)$. Since (T_n, E^ℓ_n) is connected, $v(i-1)$ and $u(i)$ are connected by an E^ℓ_n-path for any $i = 0, \dots, m-1$. Adding $s(i)$ at the top, we obtain an E^ℓ_{n+1}-path between $s(i)v(i-1)$ and $s(i)u(i)$. Combining all these E^ℓ_{n+1}-paths, we obtain an E^ℓ_{n+1}-path between w and v. Thus (T_{n+1}, E^ℓ_{n+1}) is connected. By induction, we see that $\mathcal{L}$ is strongly connected. ■

Lemma 4.20. *Let $\mathcal{L} = (K, S, \{f_s\}_{s\in S})$ be a subsystem of cubic tiling. Assume that $K \cap \mathrm{int}(C_*) \neq \emptyset$. For any $s \in \{1, \dots, N^m\}^L$, if $K \cap \mathrm{int}(C^{L,N^m}_s) \neq \emptyset$, then there exists $w \in T_m$ such that $f_w(C_*) = C^{L,N^m}_s$.*

Proof. Suppose that $f_w(C_*) \neq C^{L,N^m}_s$ for all $w \in T_m$. Then $f_w(C_*) \cap C^{L,N^m}_s$ is included in the boundary of C^{L,N^m}_s and hence $f_w(C_*) \cap \mathrm{int}(C^{L,N^m}_s) = \emptyset$. So,

$$K^{(m)} \cap \mathrm{int}(C^{L,N^m}_s) = \bigcup_{w\in T_m} \big(f_w(C_*) \cap \mathrm{int}(C^{L,N^m}_s)\big) = \emptyset.$$

Since $K \subseteq K^{(m)}$, it follows that $K \cap \mathrm{int}(C^{L,N^m}_s) = \emptyset$. ■

The following relation between a folding map and a subsystem of cubic tiling will be used to characterize local symmetry.

Lemma 4.21. *Let $\mathcal{L} = (K, S, \{f_s\}_{s\in S})$ be a subsystem of cubic tiling. Assume that $K \cap \mathrm{int}(C_*) \neq \emptyset$. Let φ be a folding map. Then the following four statements are equivalent:*

(a) $\varphi(K) = K$.

(b) $\varphi\circ f_s(K^{(m)}) = K^{(m)}$ *for any* $s \in S$ *and* $m \geq 0$.

(c) $\varphi\circ f_s(K) = K$ *for any* $s \in S$.

(d) $\varphi(K^{(m+1)}) = K^{(m)}$ *for any* $m \geq 0$.

Proof. (a) ⇒ (b): Let $s \in S$. Then $\varphi\circ f_s(K) \subseteq K$. For any $w \in T_m$, there exists $\tau = (\tau_1, \dots, \tau_L) \in \{1, \dots, N^m\}^L$ such that $\varphi\circ f_s(f_w(C_*)) = C_\tau^{L,N^m}$. Now

$$K \supseteq \varphi\circ f_s(f_w(K \cap \mathrm{int}(C_*))) = \varphi\circ f_s\circ f_w(K) \cap \mathrm{int}(C_\tau^{L,N^m}).$$

Since $K \cap \mathrm{int}(C_*) \neq \emptyset$, this implies $K \cap \mathrm{int}(C_\tau^{L,N^m}) \neq \emptyset$. Lemma 4.20 shows that $\varphi\circ f_s(f_w(C_*)) = C_\tau^{L,N^m} \subseteq K^{(m)}$, so that

$$\varphi\circ f_s(K^{(m)}) = \bigcup_{w\in T_m} \varphi\circ f_s(f_w(C_*)) \subseteq K^{(m)}.$$

As $\varphi\circ f_s \in \mathbb{B}_L$ preserves the Lebesgue measure of a set, we see $\varphi\circ f_s(K^{(m)}) = K^{(m)}$.

(b) ⇒ (c): Since $\bigcap_{m\geq 0} K^{(m)} = K$,

$$\varphi\circ f_s(K) = \varphi\circ f_s\Big(\bigcap_{m\geq 0} K^{(m)}\Big) = \bigcap_{m\geq 0} K^{(m)} = K.$$

(c) ⇒ (a): Since $K = \bigcup_{s\in S} f_s(K)$,

$$\varphi(K) = \varphi\Big(\bigcup_{s\in S} f_s(K)\Big) = K.$$

(b) ⇒ (d): Since $\bigcup_{s\in S} f_s(K^{(m)}) = K^{(m+1)}$,

$$\varphi(K^{(m+1)}) = \varphi\Big(\bigcup_{s\in S} f_s(K^{(m)})\Big) = K^{(m)}.$$

(d) ⇒ (a): Since $\bigcap_{m\geq 0} K^{(m)} = K$,

$$\varphi(K) = \varphi\Big(\bigcap_{m\geq 0} K^{(m+1)}\Big) = \bigcap_{m\geq 0} K^{(m)} = K.$$ ∎

The next theorem tells that a locally symmetric subsystem of cubic tiling is almost an inverse of a folding map.

Theorem 4.22. *Let $\mathcal{L} = (K, S, \{f_s\}_{s \in S})$ be a subsystem of cubic tiling.*

(1) *If $\mathcal{L}$ is strongly connected and locally symmetric, then there exists a folding map satisfying*
$$\varphi^n \circ f_w(K^{(m)}) = K^{(m)}$$
for any $n \geq 1$, $m \geq 0$ and $w \in T_n$. In particular,
$$\varphi^n(K^{(n+m)}) = K^{(m)}$$
for any $n \geq 1$, $m \geq 0$ and
$$\varphi^n(K) = K$$
for any $n \geq 1$. Furthermore, define $F_s \colon C_* \to C_s$ by $F_s = (\varphi|_{C_s})^{-1}$ for each $s \in S$. Then
$$K = \bigcup_{s \in S} F_s(K)$$
and $(K, S, \{F_s\}_{s \in S})$ is a self-similar structure.

(2) *Suppose that $K \cap \mathrm{int}(C_*) \neq \emptyset$. If there exists a folding map φ such that $\varphi(K) = K$, then $\mathcal{L}$ is locally symmetric.*

Proof. (1) Fix $s \in S$. Recall that there exists $\Phi_s \in \mathbb{B}_L$ such that
$$f_s(x) = \frac{1}{N}\Phi_s x + c_s$$
for any $x \in C_*$. Set $A_s = (\Phi_s)^{-1}$ and define $\varphi = \varphi_{s_0, A_s}$. Since $\varphi \circ f_s = I$, it follows that $\varphi^n \circ (f_s)^n = I$ for any $n \geq 1$. Thus letting
$$s_n = \underbrace{ss \cdots s}_{n\text{-times}},$$
we see that $\varphi^n \circ f_{s_n}(K) = K$. Choose $\tau = (\tau_1, \ldots, \tau_L) \in \{1, \ldots, N^n\}^L$ such that $C_\tau^{L,N^n} = f_{s_n}(C_*)$. Let $w \in T_n$. Choose $\xi = (\xi_1, \ldots, \xi_L) \in \{1, \ldots, N^n\}^L$ such that $C_\xi^{L,N^n} = f_w(C_*)$. Since $\mathcal{L}$ is strongly connected, there exists an E_n^ℓ-path $(w(0), \ldots, w(m))$ between s_n and w. Following this path and applying the reflections in $\ell_{w(i),w(i+1)}$, we see that
$$K_w - c_\xi^{L,N^n} = R(K_{s_n} - c_\tau^{L,N^n}),$$
where $R = \prod_{j=1}^L (R_j)^{|\tau_j - \xi_j|}$. Note that φ^n is an N^n-folding map. Hence, for any $\gamma \in \{1, \ldots, N^n\}^L$, there exists $A_\gamma \in \mathbb{B}_L$ such that
$$\varphi^n(x) = N^n A_\gamma(x - c_\gamma^{L,N^n})$$

for any $x \in C_\gamma^{L,N^n}$. Applying Lemma 4.16 to φ^n, we see that

$$\begin{aligned}\varphi^n \circ f_w(K) &= \varphi^n(K_w) = N^n A_\xi (K_w - c_\xi^{L,N^n}) \\ &= N^n A_\tau RR(K_{s_n} - c_\tau^{L,N^n}) = \varphi^n(K_{s_n}) = K.\end{aligned}$$

Hence

$$\varphi^n \circ f_w(K) = K$$

for any $n \geq 1$ and $w \in T_n$. Since $K \subseteq K^{(m)}$, it follows that $\varphi^n \circ f_w(K^{(m)}) \supseteq K$. Note that $\varphi^n \circ f_w(K^{(m)}) = \bigcup_{\gamma \in B} C_\gamma^{L,N^n}$ for some subset $B \subseteq \{1, \dots, N^n\}^L$ and $K^{(m)}$ is the minimal of such unions containing K. This shows $\varphi^n \circ f_w(K^{(m)}) \supseteq K^{(m)}$. Since $\varphi^n \circ f_w$ preserves the Lebesgue measure of a set, we conclude that

$$\varphi^n \circ f_w(K^{(m)}) = K^{(m)}.$$

Since $K^{(m+n)} = \bigcup_{w \in T_n} f_w(K^{(m)})$, we obtain $\varphi^n(K^{(n+m)}) = K^{(m)}$. Note that $K = \bigcup_{w \in T_n} f_w(K)$. Hence $\varphi^n(K) = K$. Moreover, if $\varphi(x) = NA_s(x - c_s)$ for $x \in C_s$, then by Lemma 4.21 (c), we have $K = NA_s(K_s - c_s)$. This implies

$$K_s = \frac{1}{N}(A_s)^{-1} K + c_s.$$

Hence letting $F_s(x) = \frac{1}{N}(A_s)^{-1}x + c_s$, we see $K = \bigcup_{s \in S} F_s(K)$.

(2) Suppose that $(s,t) \in E_1^\ell$. Then by Lemma 4.16, there exist $A_s \in \mathbb{B}_L$ and $j \in \{1, \dots, L\}$ such that

$$\varphi(x) = NA_s(x - c_s)$$

for any $x \in C_s$ and

$$\varphi(x) = NA_s R_j(x - c_t)$$

for any $x \in C_t$. Since $\varphi \circ f_s(K) = K$ and $\varphi \circ f_t(K) = K$ by Lemma 4.21, it follows that

$$K_s - c_s = \frac{1}{N}(A_s)^{-1} K \quad \text{and} \quad K_t - c_t = \frac{1}{N} R_j (A_s)^{-1} K.$$

Therefore,

$$R(K_s - c_s) = R\frac{1}{N}(A_s)^{-1} K = K_t - c_t,$$

so that $K_t \cup K_s$ is invariant under the reflection in $\ell_{s,t}$. Thus Lemma 4.18 shows that $\mathcal{L}$ is locally symmetric. ∎

By (2) of the above theorem, we immediately have the following sufficient condition for the local symmetry.

Corollary 4.23. *Let $S \subseteq \{1, \dots, N\}^L$. Assume that $B_{j,i} \cap (\bigcup_{s \in S} C_s) \neq \emptyset$ for any $j \in \{1, \dots, L\}$ and $i \in \{1, -1\}$. Let φ be an N-folding map. Define*

$$f_s = (\varphi|_{C_s})^{-1}$$

for any $s \in S$. Let K be the unique non-empty compact set satisfying

$$K = \bigcup_{s \in S} f_s(K).$$

Then, $\mathcal{L} = (K, S, \{f_s\}_{s \in S})$ is non-degenerate and locally symmetric.

Proof. Since $B_{i,i} \cap (\bigcup_{s \in S} C_s) \neq \emptyset$ for any $j \in \{1, \dots, L\}$ and $i \in \{1, -1\}$, Lemma 4.17 shows that $\mathcal{L}$ is non-degenerate and hence $K \cap \mathrm{int}(C_*) \neq \emptyset$. Moreover, it is immediate to see that $\varphi(K) = K$. Now Theorem 4.22 (2) suffices. ∎

Note that by Theorem 4.22 (1), if a subsystem of cubic tiling is locally symmetric and strongly connected, then it is given by a inverse of a folding map described in Corollary 4.23.

Now we are ready to give a proof of Theorem 4.13.

Proof of Theorem 4.13. By Theorem 4.22, we may assume that $\mathcal{L}$ is given by an inverse of a folding map described in Corollary 4.23 without loss of generality. Note that

$$(\varphi^m|_{f_w(C_*)})^{-1} = f_w \tag{4.16}$$

for any $m \geq 1$ and $w \in T_m$. For any $m \geq 1$ and $e = (w, v) \in E^\ell_m$, by (4.16),

$$\varphi^m|_{f_w(C_*) \cap f_v(C_*)} = (f_w)^{-1}|_{f_w(C_*) \cap f_v(C_*)} = (f_v)^{-1}|_{f_w(C_*) \cap f_v(C_*)}.$$

Hence $X(e) = X(e^r)$ and $\varphi_e = I$, where I is the identity map. Now let

$$\mathcal{I} = \left\{(X(e), X(e^r), \varphi_e) \;\middle|\; e \in \bigcup_{m \geq 1} E^\ell_m\right\},$$

and set $\mathcal{G}_0 = \{I\}$ and $\mathcal{G}_1 = \mathcal{G}_{(K,T)} \cap \mathbb{B}_L$. We are going to make use of Theorem 4.8. By the fact that $\mathcal{L}$ is strongly connected, we have (a) of Theorem 4.8. Since $\varphi_e = I$ for any $e \in \bigcup_{m \geq 1} E^\ell_m$, (b) of Theorem 4.8 is obvious.

Now it only remains to show (c) of Theorem 4.8. Let $w \in T_n$. Suppose that $f_w(C_*) = \prod_{i=1}^L [\alpha_i, \alpha_i + \frac{2}{N^n}]$. Then every path $\mathbf{p} \in \mathcal{C}^{(1)}_{1,m}(w)$ contains a path between hyperplanes

$$\{(x_1, \dots, x_L) \mid x_j = \alpha_j\} \quad \text{and} \quad \left\{(x_1, \dots, x_L) \;\middle|\; x_j = \alpha_j - \frac{2}{N^n}\right\}$$

or

$$\left\{(x_1, \dots, x_L) \;\middle|\; x_j = \alpha_j + \frac{2}{N^n}\right\} \quad \text{and} \quad \left\{(x_1, \dots, x_L) \;\middle|\; x_j = \alpha_j + \frac{4}{N^n}\right\}$$

for some $j \in \{1, \dots, L\}$. This implies that there exists $j_* \in \{1, \dots, L\}$ such that $\varphi^n(K(\mathbf{p})) \cap B_{j_*,i} \neq \emptyset$ for any $i \in \{1, -1\}$. Note that $\varphi^m(K(\mathbf{p})) = K(\psi_n(\mathbf{p}))$. Hence there exists a path $\mathbf{p}_{j_*} \subseteq \psi_n(\mathbf{p})$ between $B_{j_*,-1}$ and $B_{j_*,1}$. By (SDR), for any $j_1 \neq j_*$, there exists $i_* \in \{1, -1\}$ such that $R^{i_*}_{j_*,j_1} \in \mathcal{G}_{(K,T)}$. Set $\mathbf{p}_{j_1} = (R^{i_*}_{j_*,j_1})^*(\mathbf{p}_{j_*})$. Then $K(\mathbf{p}_{j_1}) \cap B_{j_1,i} \neq \emptyset$ for any $i \in \{1, -1\}$. Moreover, $K(\mathbf{p}_{j_*})$ and $K(\mathbf{p}_{j_1})$ intersects at $H^{i_*}_{j_*,j_1}$. Thus set $\mathbf{p}_* = \bigcup_{k=1}^{L} \mathbf{p}_k$. Then $\mathbf{p}_*$ is connected and $K(\mathbf{p}_*) \cap B_{k,i} \cap K \neq \emptyset$ for any $k \in \{1, \dots, L\}$ and $i \in \{1, -1\}$. Moreover, $\mathbf{p}_* \subseteq \bigcup_{g \in \mathcal{G}_{(K,T)} \cap \mathbb{B}_L} g^*(\psi_n(\mathbf{p}))$. Thus we have verified (c) of Theorem 4.8. ■

Proof of Theorem 4.14. The arguments are the same as in the proof of Theorem 4.13 except the deduction of (c) of Theorem 4.8.

In the case of (RS), to construct $\mathbf{p}_{j_1}$ from $\mathbf{p}_{j_*}$, we use $\Theta_{\frac{\pi}{2}}$ in place of $R^{i_*}_{j_*,j_1}$. Then the advantage of being planar yields $K(\mathbf{p}_{j_*}) \cap K(\mathbf{p}_j) \neq \emptyset$. The rest is the same as in the proof of Theorem 4.8.

Next, assume (NS). Let $w \in T_n$ and let $\mathbf{p} = (w(1), \dots, w(k)) \in \mathcal{C}^{(1)}_{M,m}(w)$ with $M = 4N - 3$. Note that

$$\#\big(\{\pi^m(w(1)), \dots, \pi^m(w(k))\}\big) \geq M.$$

We are going to show that

$$K(\psi_n(\mathbf{p})) \cap B_{j,i} \neq \emptyset \tag{4.17}$$

for any $j \in \{1, 2\}$ and $i \in \{1, -1\}$. Suppose $K(\psi_n(\mathbf{p})) \cap B_{1,1} = \emptyset$. As $\varphi^{-n}(B_{1,1})$ forms vertical lines at intervals of $\frac{2}{N^n}$, we see that $K(\mathbf{p})$ is contained in the interior of a vertical strip $\bigcup_{j=1,\dots,N^n} C^{2,N^n}_{(i_*,j)} \cup C^{2,N^n}_{(i_*+1,j)}$, which is denoted by Z_{i_*}, for some i_*. Let $C_1, \dots, C_l$ be the collection of connected components of

$$\Big(\bigcup_{w \in T_n} f_w(Q)\Big) \cap Z_{i_*}$$

and set

$$D_i = \{v \mid v \in T_n, f_v(C_*) \subseteq C_i\}$$

for $i = 1, \dots, l$. Then by (NS), we see that

$$\#(D_i) \leq 2(2N - 2).$$

Note that $\bigcup_{i=1}^{k} f_{\pi^m(w(i))}(C_*) \subset C_{i_*}$ for some i_*. Hence

$$4N - 4 \geq \#(D_{i_*}) \geq \#(\{\pi^m(w(i)) \mid i = 1, \dots, k\}) \geq M = 4N - 3.$$

This contradiction shows (4.17). Thus setting $\mathcal{U}_p = \psi_n(\mathbf{p})$, we have (c) of Theorem 4.8. ■

To conclude this section, we present a useful criterion to determine if $g \in \mathbb{B}_L$ is a symmetry of (K, T) or not.

Lemma 4.24. *Let $\mathcal{L} = (K, S, \{f_s\}_{s\in S})$ be a subsystem of cubic tiling. Assume that $\mathcal{L}$ is non-degenerate, locally symmetric and strongly connected. Let φ be the folding map satisfying Theorem 4.22 (1). Then for $g \in \mathbb{B}_L$, if there exists a map $g_*\colon S \to S$ such that, for any $s \in S$, $g(C_s) = C_{g_*(s)}$ and $A_{g_*(s)} g(A_s)^{-1} = g^k$ for some $k \geq 0$, then $g \in \mathcal{G}_{(K,T)}$.*

Recall that $A_s \in \mathbb{B}_2$ is given in Definition 4.11 (2).

Proof. We are going to show that $g(K^{(n)}) = K^{(n)}$ for any $n \geq 1$ by induction. For $n = 1$, since $g(C_s) = C_{g_*(s)}$, it follows $g(K^{(1)}) = K^{(1)}$. Next assume that

$$g(K^{(n)}) = K^{(n)}.$$

Then by Theorem 4.22, $\varphi\circ f_s(K^{(n)}) = K^{(n)}$, so that $A_s\Phi_s(K^{(n)}) = K^{(n)}$. Hence

$$f_s(K^{(n)}) = \frac{1}{N}(A_s)^{-1}(K^{(n)}) + c_s.$$

Set $t = g_*(s)$. Then

$$\begin{aligned} g(f_s(K^{(n)})) &= \frac{1}{N} g(A_s)^{-1}(K^{(n)}) + c_t = \frac{1}{N}(A_t)^{-1} A_t g(A_s)^{-1}(K^{(n)}) + c_t \\ &= \frac{1}{N}(A_t)^{-1} g^k(K^{(n)}) + c_t = f_t(K^{(n)}). \end{aligned}$$

Since $K^{(n+1)} = \bigcup_{s\in S} f_s(K^{(n)})$, this yields $g(K^{(n+1)}) = K^{(n+1)}$. Thus using induction, we see that $g(K^{(n)}) = K^{(n)}$ for any $n \geq 1$. Since $\bigcap_{n\geq 1} K^{(n)} = K$, we obtain $g(K) = K$. Now, since $g(K^{(n)}) = K^{(n)}$, it follows that, for any $w \in T_n$, there exists $v \in T_n$ such that $g(f_w(C_*)) = f_v(C_*)$. Set $v = g_*(w)$. Then $g_*\colon T_n \to T_n$. Since $g(f_w(C_*)) = f_{g_*(v)}(C_*)$ and $g(K_w) \subseteq K$, we see that

$$g(K_w) \subseteq g(f_w(C_*)) \cap K = f_{g_*(w)}(C_*) \cap K = K_{g_*(w)}.$$

Using g^{-1} in place of g in the arguments above, we obtain $g^{-1}(K_{g_*(w)}) \subseteq K_w$ as well. Thus we have shown $g(K_w) = K_{g_*(w)}$, so that $g \in \mathcal{G}_{(K,T)}$. ∎

4.4 Examples: subsystems of (hyper)cubic tiling

In this section, we present examples of subsystems of cubic tiling having conductive homogeneity.

We begin with planar examples where $\dim_{AR}(K, d_*) \leq \dim_H(K, d_*) < 2$, so that they are 2-conductively homogeneous and have self-similar local regular Dirichlet forms constructed in Theorem 4.6.

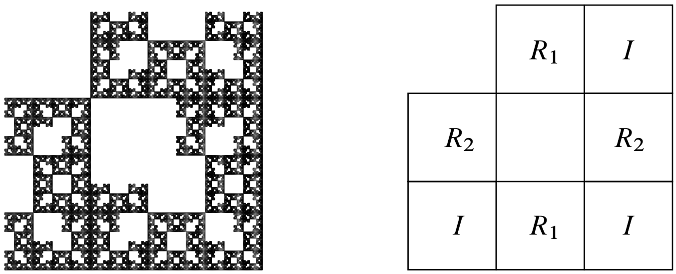


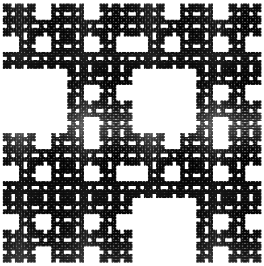
Figure 4.2. Chipped Sierpiński carpet.

Example 4.25 (Chipped Sierpiński carpet). Let $L = 2$ and let $N = 3$. Let S be the set of squares in the right figure of Figure 4.2 where one of R_1, R_2 or I is written. The corresponding f_s is given by

$$f_s(x) = \frac{1}{N}\Phi_s x + c_s^3,$$

where $\Phi_s \in \mathbb{B}_2$ is indicated in Figure 4.2. Note that if the upper-left square belonged to S as well, then K would be the Sierpiński carpet. Lemma 4.17 and Corollary 4.23 show that $\mathcal{L}$ is non-degenerate and locally symmetric, respectively. Then using Lemma 4.19, we see that $\mathcal{L}$ is strongly connected. Finally, Lemma 4.24 shows that $R_{1,2}^{-1} \in \mathcal{G}_{(K,T)}$, so that (SDR) is satisfied. Thus we have confirmed all the assumptions in Theorem 4.13. Note that $K \cap \partial C_*$ has two different ingredients, the line segment, and the Cantor set. The lack of rotational symmetry enables such a phenomenon. Another unique feature is the "countably ramified" property, that is, after removing a certain countable set, every remaining point becomes a connected component. Because of this property, $\mathcal{J}_\ell$ introduced in (4.15) is not a covering system. Furthermore, no matter how we choose a covering system $\mathcal{J} \subseteq \mathcal{J}_*$, we cannot avoid a pair $\{w, v\} \in \mathcal{J}$ where $K_w \cap K_v$ consists of a single point. It is our conjecture that $\dim_{AR}(K, d) = 1$ for the chipped Sierpiński gasket. In this example, since there are enough number of straight lines inside K, (K, d_*) has the chain condition and hence the heat kernel associated with $(\mathcal{E}, \mathcal{W}^2)$ satisfies (3.26) and (3.28).

Example 4.26. Let $L = 2$ and let $N = 4$. As in Example 4.25, S and $\{\Phi_s\}_{s \in S}$ are indicated in the right figure of Figure 4.3. It is easy to see that the corresponding self-similar structure is non-degenerate, locally symmetric, and strongly connected in the same way as Example 4.25. Moreover, Lemma 4.24 shows that $R_{1,2}^{1} \in \mathcal{G}_{(K,T)}$, so that (SDR) is satisfied. Thus we have confirmed all the assumptions of Theorem 4.13. Unlike the chipped Sierpiński carpet, this example is not "countably ramified". In this example, like the chipped Sierpiński carpet, K contains enough straight lines. This implies that (K, d_*) has the chain condition, so that the heat kernel associated with

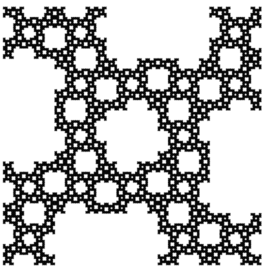


R_2	$-I$	R_2	$-I$
	R_1		R_1
R_2	$-I$	R_2	$-I$
I	R_1		R_1

Figure 4.3. Non-countably ramified example.

$(\mathcal{E}, \mathcal{W}^2)$ satisfies (3.26) and (3.28). In this example, $\mathcal{J}_\ell$ given by (4.15) is a covering system with covering numbers $(4, 2)$.

Example 4.27 (Moulin/Pinwheel). Let $L = 2$ and let $N = 5$. As in the above examples, S and $\{\Phi_s\}_{s \in S}$ are indicated in the right figure of Figure 4.4. The assumptions of Theorem 4.14 are verified in exactly the same way as before including (RS), i.e., $\Theta_{\frac{\pi}{2}} \in \mathcal{G}_{(K,T)}$. In this example, unlike previous ones, (K, d_*) does not have the chain condition and hence we have (3.26) and (3.27). In this example, $\mathcal{J}_\ell$ given by (4.15) is a covering system with covering numbers $(4, 2)$.



I	R_1			I
	$-I$	R_2	$-I$	R_2
	R_1		R_1	
R_2	$-I$	R_2	$-I$	
I			R_1	I

Figure 4.4. Moulin/Pinwheel.

The next two examples satisfy (NS) and have no $\mathbb{B}_2$-symmetry. Furthermore, $\mathcal{J}_\ell$ given by (4.15) is a covering system with covering numbers $(4, 2)$.

Example 4.28. Let $L = 2$ and let $N = 6$. As in the previous examples, S and $\{\Phi_s\}_{s \in S}$ are indicated in the right figure of Figure 4.5. In the same manner as before, we verify local symmetry, non-degeneracy and strongly connectedness. By the right figure of Figure 4.5, we verify (NS). We have $\#(S) = 23$, so that $\dim_H(K, d_*) = \frac{\log 23}{\log 6}$.

Example 4.29. Let $L = 2$ and let $N = 7$. As in the previous examples, S and $\{\Phi_s\}_{s \in S}$ are indicated in the right figure of Figure 4.6. In the same manner as before, we verify

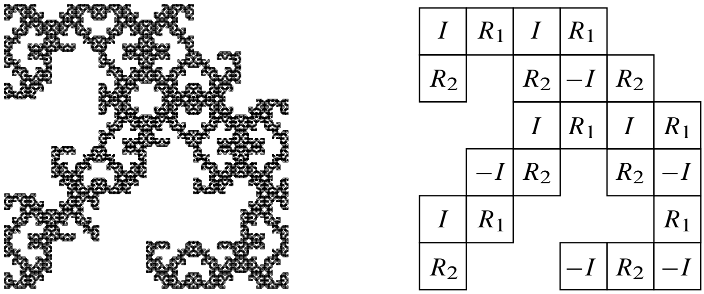


Figure 4.5. Non-symmetric example 1.

local symmetry, non-degeneracy and strongly connectedness. By the right figure of Figure 4.6, we verity (NS). In this example $\#(S) = 30$, so that $\dim_H(K, d_*) = \frac{\log 30}{\log 7}$. Note that

$$\dim_H(K \cap R_{2,1}) = \frac{\log 5}{\log 7} \quad \text{while} \;\; \dim_H(K \cap R_{2,-1}) = \frac{\log 4}{\log 7}.$$

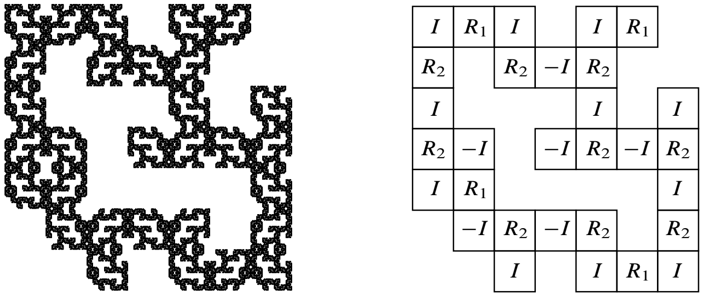


Figure 4.6. Non-symmetric example 2.

In the following examples, we may choose an arbitrary $L \geq 2$.

Example 4.30. Let $S = \{1, \ldots, N\}^L \backslash \{s_*\}$, where $s_* = (1, \ldots, 1)$. Also let $\varphi = \varphi_{s_*, I}$, i.e., φ is a folding map given by

$$\varphi(x) = N A_s (x - c_s)$$

for any $s = (s_1, \ldots, s_L) \in \{1, \ldots, N\}^L$ and $x \in Q_s$, where $A_s = \prod_{j=1}^{L} (R_j)^{|s_i - 1|}$. Note that $(A_s)^{-1} = A_s$. Define

$$f_s(x) = \frac{1}{N} A_s x + c_s$$

and let K be the unique non-empty compact set satisfying

$$K = \bigcup_{s \in S} f_s(K).$$

Then $\mathcal{L} = (K, S, \{f_s\}_{s \in S})$ is a self-similar structure. According to Corollary 4.23, $\mathcal{L}$ is non-degenerate and locally symmetric. Moreover, Lemma 4.19 shows that $\mathcal{L}$ is strongly connected. Additionally, using Lemma 4.24, we see that $\mathcal{G}_{(K,T)}$ is generated by $\{R^1_{j_1,j_2} \mid j_1, j_2 \in \{1, \dots, L\}, j_1 \neq j_2\}$ and it is isomorphic to the symmetric group of order L. Hence by Theorem 4.13, K is p-conductively homogeneous for any $p > \dim_{AR}(K, d_*)$. Note that $\mathcal{G}_{(K,T)}$ is a proper subgroup of $\mathbb{B}_L$ in this case. In this example, $\mathcal{J}_\ell$ given by (4.15) is a covering system with covering numbers $(2L, L)$.

Example 4.31 (Hypercube). Let $S = \{1, \dots, N\}^L$ and let $f_s(x) = \frac{1}{N}x + c_s$ for any $s \in S$ and $x \in [-1, 1]^L$. Set $K = [-1, 1]^L$. Then $(K, S, \{f_s\}_{s \in S})$ is a self-similar structure. Obviously, $\mathcal{L}$ is non-degenerate, strongly connected and locally symmetric. Moreover, $\mathcal{G}_{(K,T)} = \mathbb{B}_L$. In this case, $\mathcal{J}_\ell$ is a covering system with covering numbers $(2L, L)$. By Theorem 4.13, K is p-conductively homogeneous for any $p > L$. In fact, for any $p > L$, we see that $W^{1,p}(K) = \mathcal{W}^p$ and there exist $c > 0$ such that

$$c\mathcal{E}_p(f) \leq \int_K |\nabla f|^p dx \leq c^{-1}\mathcal{E}_p(f) \tag{4.18}$$

for any $f \in W^{1,p}(K)$, where $\mathcal{E}_p$ is the self-similar p-energy constructed in Section 4.1. The rest of this example is devoted to showing these facts. Let

$$A = \{w(1), w(2), w(3)\} \subseteq T_n.$$

Then $K_{w(1)}$, $K_{w(2)}$ and $K_{w(3)}$ are three consecutive cubes in x_1-direction, i.e.,

$$\begin{aligned} K_{w(1)} \cap K_{w(2)} &= f_{w(1)}(B_{1,1}) = f_{w(2)}(B_{1,-1}), \\ K_{w(2)} \cap K_{w(3)} &= f_{w(2)}(B_{1,1}) = f_{w(3)}(B_{1,-1}). \end{aligned}$$

Let $A_1 = \{w(1)\}$ and let $A_2 = \{w(3)\}$. Then, the function attaining the infimum in the definition of $\mathcal{E}_{p,m}(A_1, A_2, A)$ depends only on the first variable x_1 and is a piecewise linear function in the direction of x_1. Consequently, we see that

$$\mathcal{E}^\ell_{p,m}(A_1, A_2, A) \geq 2^{m(L-p)-1}.$$

On the other hand, the comparison of moduli shows

$$\mathcal{M}^{(1)}_{p,m}(A_1, A_2, A) \leq \mathcal{M}^{(1)}_{1,p,m}(w)$$

for any $w \in T$. Therefore, there exists $c_2 > 0$ such that

$$c_2 2^{m(L-p)} \leq \mathcal{E}_{1,p,m}(w, T_{|w|})$$

for any $m \geq 1$ and $w \in T$.

Now, for $f: K \to \mathbb{R}$, we define $\widetilde{f}_m: T_m \to T$ by $\widetilde{f}_m(w) = f(f_w(0))$. Then there exists $c > 0$ such that

$$2^{m(p-L)}\mathcal{E}_{p,T_m}(\widetilde{f}_m) \to c \int_K |\nabla f|^p dx \tag{4.19}$$

as $m \to \infty$ for any $f \in C^\infty(K)$. So there exists $c_3 > 0$ such that $\mathcal{E}_{1,p,m}(w, T_{|w|}) \leq c_3 2^{m(L-p)}$ for any $w \in T$. Thus the scaling exponent of σ appearing in (3.17) is 2^{L-p}. Combining this fact and arguments analogous to those in [41, Section 5.3], we have the following Korevaar–Shoen type expression of $\mathcal{W}^p$:

$$\mathcal{W}^p = \left\{ f \;\middle|\; f \in L^p(K, dx), \varlimsup_{r \downarrow 0} \int_K \frac{1}{r^L} \int_{B_{d_*}(x,r)} \frac{|f(x) - f(y)|^p}{r^p} dy dx < \infty \right\}.$$

This expressing enable us to identify $\mathcal{W}^p$ with $W^{1,p}(K)$. By (4.19), we see that (4.18) holds for any $f \in C^\infty(K)$. Since $C^\infty(K)$ is dense in $W^{1,p}(K)$, (4.18) holds for any $f \in \mathcal{W}^p$.

4.5 Rationally ramified Sierpiński crosses

In this section, we present another class of conductively homogeneous spaces called rationally ramified Sierpiński crosses. This example is a planar square-based self-similar set as those in the last section but the sizes of the squares constituting it are not one but two. See Figure 4.7. Consequently, although it has full $\mathbb{B}_2$-symmetry, we should make a little more complicated discussion than that of the previous section to show the conductive homogeneity.

The family of Sierpiński crosses was introduced in [31, Example 1.7.5].

Definition 4.32. Let $r_1, r_2 \in (0, 1)$ satisfying $2r_1 + r_2 = 1$ and $r_1 \geq r_2$. Let $p_1 = (-1, -1)$, $p_2 = (0, -1)$, $p_3 = (1, -1)$, $p_4 = (1, 0)$, $p_5 = (1, 1)$, $p_6 = (0, 1)$, $p_7 = (-1, 1)$ and $p_8 = (-1, 0)$. Set $S = \{1, \ldots, 8\}$. For $s \in S$, define $F_s: C_* \to C_*$ as

$$F_s(x) = \begin{cases} r_1(x - p_s) + p_s & \text{if } s \text{ is odd,} \\ r_2(x - p_s) + p_s & \text{if } s \text{ is even.} \end{cases}$$

The self-similar set K with respect to the family of contractions $\{F_s\}_{s \in S}$ is called the (r_1)-*Sierpiński cross*. Define

$$\ell_{\mathrm{L}} = \{-1\} \times [-1, 1], \quad \ell_{\mathrm{R}} = \{1\} \times [-1, 1],$$
$$\ell_{\mathrm{B}} = [-1, 1] \times \{-1\}, \quad \ell_{\mathrm{T}} = [-1, 1] \times \{1\},$$

where the symbols, L, R, B, and T correspond to left, right, bottom, and top, respectively.

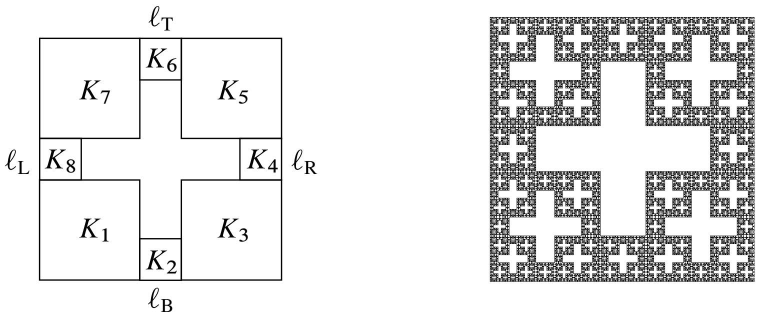


Figure 4.7. The ρ_*-Sierpiński cross: $\rho_* = \sqrt{2} - 1$.

In this section, we will show that if an (r_1)-Sierpiński cross K is rationally ramified, then it is p-conductively homogeneous for any $p > \dim_{AR}(K, d_*)$. Roughly speaking an (r_1)-Sierpiński cross is rationally ramified if $\bigcup_{v \in \Gamma_1(w)} K_v$, which represents the local geometry around $w \in T$, has finite types of variety up to the isometries when $w \in T$ varies. See [31] for the exact definition. In fact, in [31, Proposition 1.7.6], it is shown that an (r_1)-Sierpiński cross is rationally ramified if and only if $1 - r_1 = (r_1)^m$ for some $m \geq 2$. For simplicity of arguments, we confine ourselves to the case $m = 2$ hereafter in this section. The generalization to other values of m is a little complicated but the essential idea is the same.

In the case $m = 2$, the value of r_1 equals $\sqrt{2} - 1$. Set $\rho_* = \sqrt{2} - 1$. Our main object of study is now the ρ_*-Sierpiński cross. We take advantage of the framework of Section 4.1 with $r = \rho_*$ and

$$j_s = \begin{cases} 1 & \text{if } s \text{ is odd,} \\ 2 & \text{if } s \text{ is even} \end{cases}$$

to define $(T, \mathcal{A}, \phi)$ and the associated partition of K. In this case, $g(w)$ is the contraction ratio of the map $F_w = F_{w_1} \circ \cdots \circ F_{w_m}$ for $w = w_1 \ldots w_m \in S^m$. Note that $g(w) = (\rho_*)^n$ or $(\rho_*)^{n+1}$ for any $(n, w) \in T_n$. For example, $\Lambda^g_{\rho_*} = S$ and

$$\Lambda^g_{(\rho_*)^2} = \{1s, 3s, 5s, 7s \mid s \in S, s\colon \text{even}\} \cup \{1s, 3s, 5s, 7s \mid s \in S, s\colon \text{odd}\} \\ \cup \{2, 4, 6, 8\}.$$

Note that $g(1s) = (\rho_*)^3$ if s is even and $g(1s) = (\rho_*)^2$ if s is odd. Moreover, $\Lambda^g_{\rho_*} \cap \Lambda^g_{(\rho_*)^2} \neq \emptyset$ in this case. Let d_* be the restriction of the Euclidean metric to K. Let $h_{\rho_*}(n, w) = (\rho_*)^n$ for $(n, w) \in T_w$. It is straightforward to see that d_* is 1-adapted to the weight function h_{ρ_*}, i.e., Assumption 2.15 (2B) holds with $M_* = 1$.

For simplicity, to denote an element in T_n, we use w in place of (n, w) hereafter as long as no confusion may occur.

The Hausdorff dimension of (K, d_*) is given by the unique number α_H satisfying

$$4(\rho_*)^{2\alpha_H} + 4(\rho_*)^{\alpha_H} = 1.$$

Consequently, we see that

$$\alpha_H = 1 + \frac{\log 2}{\log(1+\sqrt{2})}.$$

Let μ be the self-similar measure with weight $(\mu_i)_{i \in S}$, where

$$\mu_i = \begin{cases} (\rho_*)^{\alpha_H} & \text{if } i \text{ is odd,} \\ (\rho_*)^{2\alpha_H} & \text{if } i \text{ is even.} \end{cases}$$

Then μ is the normalized α_H-dimensional Hausdorff measure and is α_H-Ahlfors regular with respect to d_*. After those observations, it is easy to see that Assumption 2.15 is satisfied with $M_* = M_0 = 1$, $N_* = 8$. Moreover, we see that $L_* \leq 8$.

The main result of this section is as follows.

Theorem 4.33. *For any $p > 0$, $n, m, k \geq 1$, $w \in T_n$ and $u, v \in T_k$,*

$$\mathcal{M}^{(1)}_{1,p,m}(w) \leq 8(24)^{p+1}\#(T_{k+1})^p \mathcal{M}^{(1)}_{p,m}(u, v, T_k).$$

An immediate consequence of the above theorem is the conductive homogeneity of the Sierpiński cross.

Corollary 4.34. *The ρ_*-Sierpiński cross K is p-conductively homogeneous for any $p > \dim_{AR}(K, d_*)$. Moreover, there exists a self-similar p-energy $\mathcal{E}_p$ on $\mathcal{W}^p$. In particular, there exists a local regular Dirichlet form $(\mathcal{E}, \mathcal{W}^2)$ on $L^2(K, \mu)$ whose associated heat kernel satisfies (3.26) and (3.28).*

Note that due to the two different values of j_s, the self-similarity of the p-energy $\mathcal{E}_p$ is given as

$$\mathcal{E}_p(f) = \sigma \sum_{s:\text{odd}} \mathcal{E}_p(f \circ F_s) + \sigma^2 \sum_{s:\text{even}} \mathcal{E}_p(f \circ F_s)$$

for any $f \in \mathcal{W}^p$.

Proof. By (2.13), it follows that

$$\mathcal{E}_{1,p,m}(w, T_n) \leq c_p \#(T_{k+1})^p \mathcal{E}_{p,m}(u, v, T_k)$$

for any $n, m, k \geq 1$, $w \in T_n$ and $u, v \in T_k$. Moreover, since $p > \dim_{AR}(K, d_*)$, there exist $c > 0$ and $\alpha \in (0, 1)$ such that

$$\mathcal{E}_{1,p,m} \leq c\alpha^m$$

for any $m \geq 1$. Thus we have obtained (3.19) and (3.20), so that K is p-conductively homogeneous by Theorem 3.33. In particular, since $\alpha_H < 2$, K is 2-conductively homogeneous and we have $(\mathcal{E}, \mathcal{W}^2)$. Since (K, d_*) has the chain condition, by Theorem 3.35, we have (3.26) and (3.28). ■

To show Theorem 4.33, we need to prepare several notions.

Definition 4.35. (1) Set

$$U = \{(2,13), (2,31), (4,35), (4,53), (6,57), (6,75), (8,17), (8,71)\}.$$

For $(i, jk) \in U$, define $R_{i,jk}: K_i \to K_{jk}$ as the reflection in the line segment $K_i \cap K_{jk}$. Moreover, define $R^*_{i,jk}(w)$ for $w \in T(i) \cup T(jk)$ as the unique $v \in T(i) \cup T(jk)$ satisfying $R_{i,jk}(K_w) = K_v$. $R^*_{i,jk}$ is a map from $T(i) \cup T(jk)$ to itself.

(2) For $g \in \mathbb{B}_2$, define $g^*: T \to T$ by

$$g^*(w) = v,$$

where v is the unique $v \in T$ with $g(K_w) = K_v$. Note that $g^*|_{T_n}: T_n \to T_n$.

(3) For $w \in T$, if $w \notin T(2) \cup T(4) \cup T(6) \cup T(8)$, then define

$$\mathcal{H}_w = \{g^*(v) \mid g \in \mathbb{B}_2\}.$$

Otherwise, if $w \in T(i)$ for $i = 2, 4, 6, 8$, then define

$$\mathcal{H}_w = \{g^*(v) \mid g \in \mathbb{B}_2\} \cup \{g_*(R^*_{i,jk}(v)) \mid g \in \mathbb{B}_2, (i, jk) \in U\}.$$

Note that $\#(\mathcal{H}_w) \leq 24$ for any $w \in T_n$.

By the construction of T_n, we see that $g(w) = (\rho_*)^n$ or $g(w) = (\rho_*)^{n+1}$ for any $w \in T_n^n$. In fact, we immediately obtain the following lemma.

Lemma 4.36. *Set $T_n^n = \{w \mid w \in T_n, g(w) = (\rho_*)^n\}$ and $T_n^{n+1} = \{w \mid w \in T_n, g(w) = (\rho_*)^{n+1}\}$. Then*

(1) *For any $w \in T_n^n$, $wv \in T_{n+m}$ if and only if $v \in T_m$.*

(2) *For any $w \in T_n^{n+1}$, $wv \in T_{n+m}$ if and only if $v \in T_{m-1}$.*

(3) *$w \in T_{n+1}^{n+1}$ if and only if $w \in T_n^{n+1}$ or $w = \tau j$ for some $\tau \in T_n^n$ and $j \in \{1, 3, 5, 7\}$.*

(4) *$w \in T_{n+1}^{n+2}$ if and only if $w = \tau j$ for some $\tau \in T_n^n$ and $j \in \{2, 4, 6, 8\}$.*

Definition 4.37. (1) Define $\psi^*_{n,m}: S^m(T_n^n) \to T_m$ by

$$\psi^*_{n,m}(wv) = v$$

for $w \in T_n^n$ and $v \in T_m$.

(2) For $w \in T$, define $\mathcal{H}_w^0 \subseteq T$ by

$$\mathcal{H}_w^0 = \begin{cases} \{w, R_{i,jk}^*(w)\} & \text{if } w \in T(jk) \text{ for some } (i, jk) \in U, \\ \{w\} & \text{otherwise.} \end{cases}$$

For $w \in T_{n+1}^{n+1}$ and $u \in T$, define

$$\mathcal{H}_{wu}^n = \begin{cases} \{\tau v \mid v \in \mathcal{H}_{ju}^0\} & \text{if } w = \tau j \text{ for some } \tau \in T_n^n \text{ and } j \in \{1, 3, 5, 7\}, \\ \{wu\} & \text{if } w \in T_n^{n+1}. \end{cases}$$

(3) Define

$$K_{\%} = \bigcup_{s \in S, K_s \cap \ell_{\%} \neq \emptyset} K_s \tag{4.20}$$

for $\% \in \{\mathrm{T}, \mathrm{B}, \mathrm{R}, \mathrm{L}\}$. For example, $K_{\mathrm{B}} = K_1 \cup K_2 \cup K_3$.

Note that if $w \in T_n$, then $\mathcal{H}_w^0 \in T_n$ and that if $w \in T_{n+1}^{n+1}$ and $u \in T_{m-1}$, then $\mathcal{H}_{wu}^n \subseteq T_{n+m}$.

Lemma 4.38. *Assume that there exists a path* $\mathbf{p} = (w(1), \dots, w(l))$ *of* T_{m-1} *contained in* K_{L} *such that* $K_{w(1)} \cap \ell_{\mathrm{B}} \neq \emptyset$, $K_{w(l)} \cap \ell_T \neq \emptyset$, *and* $\mathbf{p}$ *is* R_2^**-invariant. Set*

$$\mathcal{H}_u^* = \bigcup_{w \in T_{k+1}^{k+1}} \bigcup_{v \in \mathcal{H}_u} \mathcal{H}_{wv}^{k+1}$$

for $u \in T_{m-1}$. *Then for any* $u_1, u_2 \in T_k$, *there exists* $\mathbf{p}_0 \in \mathcal{C}_m^{(1)}(\{u_1\}, \{u_2\}, T_k)$ *such that*

$$\mathbf{p}_0 \subseteq \bigcup_{i=1}^{l} \mathcal{H}_{w(i)}^*. \tag{4.21}$$

Remark. Strictly, $\mathbf{p}_0$ is not a subset but a sequence of points. However, in (4.21), we use $\mathbf{p}_0$ to denote a subset consisting of the points in the sequence. We use such abuse of notations if no confusion may occur.

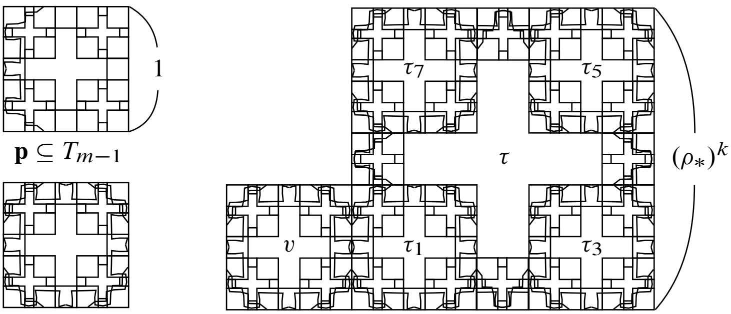
Proof. Set

$$Y = \mathbf{p} \cup \Theta_{\frac{\pi}{2}}^*(\mathbf{p}) \cup \Theta_{\pi}^*(\mathbf{p}) \cup \Theta_{\frac{3\pi}{2}}^*(\mathbf{p}).$$

Then $Y = g^*(Y)$ for any $g \in \mathbb{B}_2$. Let

$$\mathcal{H}^*(Y) = \bigcup_{w \in T_{k+1}^{k+1}} \bigcup_{v \in Y} \mathcal{H}_{wv}^{k+1}.$$

See Figure 4.8 for an illustration paths ψ and Y along with a part of $\mathcal{H}^*(Y)$. It follows that $K(\mathcal{H}^*(Y))$ is a connected set intersecting K_u for any $u \in T_k$. Therefore, we can choose a path $\mathbf{p}_0$ connecting K_{u_1} and K_{u_2} from $\mathcal{H}^*(Y)$. Since $\mathcal{H}^*(Y) \subseteq \bigcup_{i=1}^{l} \mathcal{H}_{w(i)}^*$, we have the desired statement. ■



$\mathbf{p} \subseteq T_{m-1}$ $\quad$ $Y \subseteq T_{m-1}$ $\quad$ $\tau \in T_k^k,\ v \in T_k^{k+1} \cap T_{k+1}^{k+1},\ \tau_1, \tau_3, \tau_5, \tau_7 \in T_{k+1}^{k+1}$

Figure 4.8. Paths ψ and Y, and a part of $\mathcal{H}^*(Y)$.

Proof of Theorem 4.33. Let $w \in T_n$ and let $u_1, u_2 \in T_k$. For any $\mathbf{p} \in \mathcal{C}_{1,m}^{(1)}(w)$, set

$$\mathcal{H}_{m-1}(\mathbf{p}) = \bigcup_{u \in \psi^*_{n+1,m-1}(\mathbf{p} \cap S^{m-1}(T_{n+1}^{n+1}))} \mathcal{H}_u.$$

Then $\mathcal{H}_{m-1}(\mathbf{p}) \subseteq T_{m-1}$ and $g^*(\mathcal{H}_{m-1}(\mathbf{p})) = \mathcal{H}_{m-1}(\mathbf{p})$ for any $g \in \mathbb{B}_2$.

Claim 1. *There exists a path* $\mathbf{p}^*$ *contained in* $\mathcal{H}_{m-1}(\mathbf{p})$ *such that one of the following four statements is true:*

(a) $K(\mathbf{p}^*) \cap \ell_{\mathrm{B}} \neq \emptyset$ *and* $K(\mathbf{p}^*) \cap K_{\mathrm{T}} \neq \emptyset$,

(b) $K(\mathbf{p}^*) \cap \ell_{\mathrm{T}} \neq \emptyset$ *and* $K(\mathbf{p}^*) \cap K_{\mathrm{B}} \neq \emptyset$,

(c) $K(\mathbf{p}^*) \cap \ell_{\mathrm{L}} \neq \emptyset$ *and* $K(\mathbf{p}^*) \cap K_{\mathrm{R}} \neq \emptyset$,

(d) $K(\mathbf{p}^*) \cap \ell_{\mathrm{R}} \neq \emptyset$ *and* $K(\mathbf{p}^*) \cap K_{\mathrm{L}} \neq \emptyset$.

Proof. Let $F_w(C_*) = [a, a+h] \times [b, b+h]$, where $h = (\rho_*)^n$ if $w \in T_n^n$ and $h = (\rho_*)^{n+1}$ if $w \in T_n^{n+1}$. Define

$$A_{w,\gamma} = [a-\gamma, a+h+\gamma] \times [b-\gamma, b+h+\gamma]$$

and $\widetilde{A}_w = K \cap (A_{w,(\rho_*)^{n+1}} \backslash A_{w,(\rho_*)^{n+2}})$. Two typical examples of $\widetilde{A}_w$ is illustrated in Figure 4.9. Since $K_{w(1)} \cap K_w \neq \emptyset$ and $K_{w(l)} \cap A_{w,(\rho_*)^{n+1}} = \emptyset$, a part of $\mathbf{p}$ contained in $\widetilde{A}_w$ connects

$$\{(a-(\rho_*)^{n+1}, y) \mid y \in [-1,1]\} \text{ and } \{(a-(\rho_*)^{n+2}, y) \mid y \in [-1,1]\},$$
$$\{(a+h, y+(\rho_*)^{n+2}) \mid y \in [-1,1]\} \text{ and } \{(a+h+(\rho_*)^{n+1}, y) \mid y \in [-1,1]\},$$
$$\{(x, b-(\rho_*)^{n+1}) \mid x \in [-1,1]\} \text{ and } \{(x, b-(\rho_*)^{n+2}) \mid x \in [-1,1]\},$$

or

$$\{(x, b+h+(\rho_*)^{n+2}) \mid x \in [-1,1]\} \text{ and } \{(x, b+h+(\rho_*)^{n+1}) \mid x \in [-1,1]\}.$$

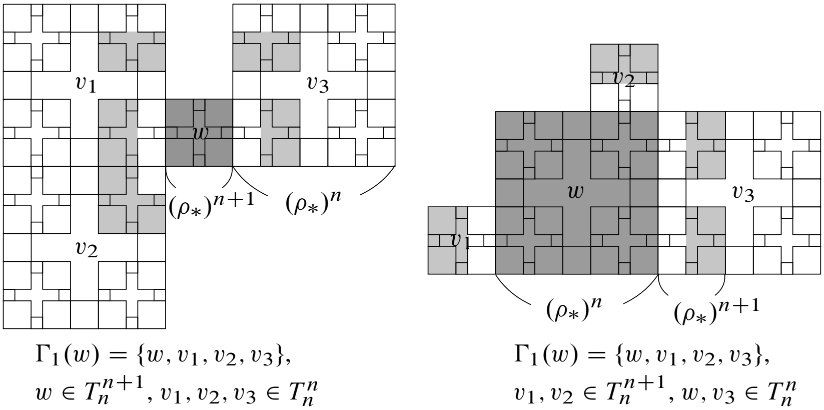


Figure 4.9. Two examples of $\widetilde{A}_w$ (dark grey regions are K_w, light grey regions are $\widetilde{A}_w$).

According to the four possibilities above, we have (a), (b), (c) or (d), where the exact correspondence depends on w. □

Hereafter we assume the first case (a) in Claim 1 in the course of discussion. Other cases may be treated exactly in the same manner. In the following claims, we are going to modify the initial path $\mathbf{p}^*$ step by step. This process of modification is illustrated in Figure 4.10.

Claim 2. *The union* $\mathbf{p}^* \cup R_2^*(\mathbf{p}^*)$ *contains an* R_2*-symmetric path*

$$\mathbf{p}_1 = (v(0), \dots, v(l_1))$$

between ℓ_B *and* ℓ_T*, i.e.,* $K_{v(0)} \cap \ell_\mathrm{B} \neq \emptyset$*,* $R_2^*(v(i)) = v(l_1 - i)$ *for* $i = 1, \dots, l_1$*.*

Proof. Let $p_* = (w(1), \dots, w(l))$. By (a), $K(\mathbf{p}^*)$ intersects with the line segment $[-1,1] \times \{0\}$. Set $i_* = \min\{i \mid w(i) \cap [-1,1] \times \{0\} \neq \emptyset\}$. Then connecting $(w(1), \dots, w(i_*))$ and its image by R_2^*, we obtain a desired path. □

Claim 3. *The union* $R_1^*(\mathbf{p}_1) \cup \mathbf{p}_1$ *contains an* R_2*-symmetric path* $\mathbf{p}_2$ *such that*

$$K(\mathbf{p}_2) \subseteq [-1,0] \times [-1,1].$$

Proof. If $\mathbf{p}_1$ or $R_1^*(\mathbf{p}_1)$ is contained in the left half of C_*, then choose $\mathbf{p}_1$ or $R_1^*(\mathbf{p}_1)$ accordingly as our path. Otherwise, applying R_1 to $K(\mathbf{p}_1) \cap [0,1] \times [-1,1]$, we obtain a desired path. □

Claim 4. *Set* $\mathcal{H}_{\mathbf{p}^*} = \bigcup_{u \in \mathbf{p}^*} \mathcal{H}_u$*. Then there exists an* R_2^**-symmetric path* $\mathbf{p}_3 \subseteq \mathcal{H}_{\mathbf{p}^*}$ *contained in* K_L *such that* $K(\mathbf{p}_3) \cap \ell_\mathrm{T} \neq \emptyset$ *and* $K(\mathbf{p}_3) \cap \ell_\mathrm{B} \neq \emptyset$*.*

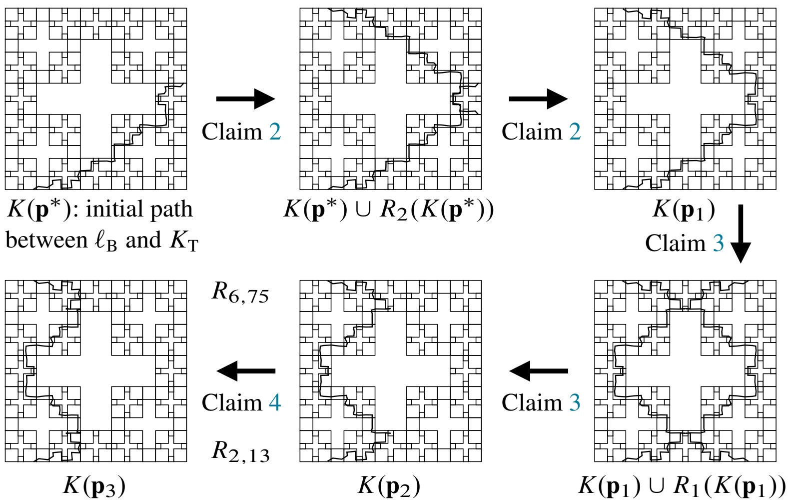


Figure 4.10. Modifications of a path.

Proof. If $K(\mathbf{p}_2) \subseteq K_L$, then we set $\mathbf{p}_2 = \mathbf{p}_3$. Otherwise, use $R^*_{2,13}$ (resp. $R^*_{6,75}$) to reflect the part $K(\mathbf{p}_2) \cap K_2$ (resp. $K(\mathbf{p}_2) \cap K_6$) into K_{13} (resp. K_{75}). Then we obtain a desired path. □

Now we have a path p_3 satisfying all the assumptions of Lemma 4.38. Applying Lemma 4.38 with $\mathbf{p} = \mathbf{p}_3$, we obtain a path $\mathbf{p}_0 \in \mathcal{C}^{(1)}_m(\{u_1\}, \{u_2\}, T_k)$. For $u \in S^m(\Gamma_1(w))$, define

$$H_u = \begin{cases} \bigcup_{v \in \mathcal{H}_{\psi^*_{n+1,m-1}(u)}} \mathcal{H}^*_v & \text{if } u \in S^{m-1}(T^{n+1}_{n+1}), \\ \emptyset & \text{otherwise.} \end{cases}$$

Then it follows that $\mathbf{p}_0 \subseteq \bigcup_{v \in \mathbf{p}} H_v$. Since $\#(\mathcal{H}_u) \leq 24$ and $\#(\Gamma_1(w)) \leq 8$,

$$\#(H_w) \leq 48\#(T_{k+1}) \quad \text{and} \quad \#(\{v \mid u \in H_v\}) \leq 24 \cdot 8.$$

So, Lemma C.4 suffices. ■

4.6 Nested fractals

In this section, we show conductive homogeneity of a class of self-similar sets, called strongly symmetric self-similar sets, that are highly symmetric and finitely ramified. This class is a natural extension of nested fractals introduced by Lindstrøm [37], where Brownian motions were constructed on them. In [29, Section 3.8], Lindstrøm's results were extended to strongly symmetric self-similar sets. Typical examples of strongly symmetric self-similar sets are the Sierpiński gasket, the pentakun ("Kun" means "Mr." in Japanese), and the snowflake, whose definitions are given below.

Let $\rho \in (0,1)$ and let S be a finite subset of $\mathbb{R}^L$ for some $L \in \mathbb{N}$. For each $q \in S$, let $f_q: \mathbb{R}^L \to \mathbb{R}^L$ be a ρ-similitude whose fixed point is q, i.e., there exists $U_q \in O(L)$ such that

$$f_q(x) = \rho U_q(x - q) + q$$

for any $x \in \mathbb{R}^L$. Let K be the self-similar set with respect to the family of contractions $\{f_q\}_{q \in S}$. Then the triple $(K, S, \{f_q\}_{q \in S})$ is a self-similar structure as is explained in Section 4.1. Throughout this section, we consider a self-similar structure $(K, S, \{f_q\}_{s \in S})$ given in this way.

Assumption 4.39. (1) *If $p, q \in S$ and $p \neq q$, then $p \notin f_q(K)$.*

(2) *There exists $U \subseteq S$ such that*

$$\bigcup_{\substack{q_1, q_2 \in S \\ q_1 \neq q_2}} f_{q_1}^{-1}(f_{q_1}(K) \cap f_{q_2}(K)) = U.$$

(3) *K is connected.*

For purposes of normalization, we assume $\sum_{q \in U} q = 0$ hereafter.

Proposition 4.40. *Under Assumption 4.39, $(K, S, \{f_q\}_{q \in S})$ is a post critically finite self-similar structure with*

$$V_0 = U. \tag{4.22}$$

Moreover, define $\{V_m\}_{m \geq 1}$ inductively by

$$V_{m+1} = \bigcup_{i \in S} f_i(V_m).$$

Then

$$V_m \subseteq V_{m+1} \tag{4.23}$$

for any $m \geq 0$.

The definitions of post critically finite (p.c.f. for short) self-similar structures and V_0 along with the proof of (4.22) is given in Appendix 6.3. Inclusion (4.23) is due to [29, Lemma 1.3.11].

For the self-similar structure $(K, S, \{f_q\}_{q \in S})$, we adopt the framework in Section 4.1 with $r = \rho$ and $j_q = 1$ for any $q \in S$. In this case,

$$T_m = S^m = \{w_1 \dots w_m \mid w_i \in S \text{ for any } i = 1, \dots, m\}.$$

Then we see that

$$V_0 = \bigcup_{e \in E_1^*} X(e),$$

where $X(e)$ is defined in Definition 4.7. Moreover, by [29, Proposition 1.3.5 (2)], it follows that

$$K_w \cap K_v = f_w(V_0) \cap f_v(V_0) \subseteq V_m \tag{4.24}$$

for any $w, v \in T_m$ with $w \neq v$. This implies that

$$V_0 = \bigcup_{(X,Y,\varphi) \in \mathcal{IT}(K,T)} X. \tag{4.25}$$

Let $\alpha_H = -\frac{\log N}{\log \rho}$. Note that $N\rho^{\alpha_H} = 1$. Let μ be the self-similar measure with weight $(\rho^{\alpha_H}, \ldots, \rho^{\alpha_H})$. Basic properties of μ are given in Appendix 6.3. Also, let d_* be the restriction of the Euclidean metric to K.

The following assumption is an equivalent condition of Assumption 2.15 (2B) when d is the (restriction of) Euclidean metric. Essentially, the same assumptions have been around from time to time for almost 30 years. See [35, Assumption 2.2] and [38, Assumption (P)]. The assumption is believed to be true for nested fractals but we have no proof so far. In [38], it was shown that this assumption is true if U_q is the same for any $q \in S$. In Appendix 6.3, this assumption is shown to be true if U_q is the identity map for any $q \in V_0$.

Assumption 4.41. *There exists $c > 0$ such that $d(K_w, K_v) \geq c\rho^{|w|}$ for any $n \geq 1$, and $(w, v) \in (T_n \times T_n)\backslash E_n^*$, where $d(A, B) = \inf_{x \in A, y \in B} |x - y|$ for subsets $A, B \subseteq \mathbb{R}^L$.*

Proposition 4.42. *Under Assumptions 4.39 and 4.41, Assumption 2.15 is satisfied with $d = d_*$, $r = \rho$, and $M_* = M_0 = 1$.*

The above proposition is proven in Appendix 6.3.

Definition 4.43. (1) Let $m_* = \#\{|x - y| \mid x, y \in V_0,\ x \neq y\}$, where $|x|$ is the Euclidean length of $x \in \mathbb{R}^L$. Define

$$l_0 = \min\{|x - y| \mid x, y \in V_0,\ x \neq y\}.$$

Moreover, define l_i for $i = 0, 1, \ldots, m_* - 1$ inductively by

$$l_{i+1} = \min\{|x - y| \mid x, y \in V_0,\ x \neq y,\ |x - y| > l_i\}.$$

(2) A sequence $(x_i)_{i=1,\ldots,k} \subseteq V_m$ is called an *m-walk* if there exists $w(i) \in T_m$ such that $x_i, x_{i+1} \in f_{w(i)}(V_0)$ for any $i = 1, \ldots, k-1$.

(3) A 0-walk $(x_i)_{i=1,\ldots,k}$ is called a *strict* 0-*walk* (between x_1 and x_k) if $|x_i - x_{i+1}| = l_0$ for any $i = 1, \ldots, k-1$.

(4) Define

$$\mathcal{G} = \{g \mid g \in O(L), g(V_0) = V_0 \text{ and there exists } g^*\colon T \to T \text{ such that } g(f_w(V_0)) = f_{g^*(w)}(V_0) \text{ for any } w \in T\}.$$

(5) For any $x, y \in \mathbb{R}^L$ with $x \neq y$, define

$$H_{xy} = \{z \mid z \in \mathbb{R}^L,\ |x - z| = |y - z|\}.$$

(H_{xy} is the hyperplane bisecting the line segment xy.) Also let $g_{xy}: \mathbb{R}^L \to \mathbb{R}^L$ be reflection in H_{xy}.

Definition 4.44. A self-similar structure $(K, S, \{f_q\}_{q \in S})$ is said to be *strongly symmetric* if Assumption 4.39 is satisfied and there exists a finite subgroup $\mathcal{G}_*$ of $\mathcal{G}$ such that the following properties hold:

(1) For any $x, y \in V_0$ with $x \neq y$, there exists a strict 0-walk between x and y.

(2) If $x, y, z \in V_0$ and $|x - y| = |x - z|$, then there exists $g \in \mathcal{G}_*$ such that $g(x) = x$ and $g(y) = z$.

(3) For any $i = 1, \dots, m_* - 2$, there exist x, y and $z \in V_0$ such that $|x - y| = l_i$, $|x - z| = l_{i+1}$ and $g_{yz} \in \mathcal{G}_*$.

(4) V_0 is $\mathcal{G}_*$-transitive, i.e., for any $x, y \in V_0$, there exists $g \in \mathcal{G}_*$ such that $g(x) = y$.

Remark. By Definition 4.44 (4), $|q_1| = |q_2|$ for any $q_1, q_2 \in V_0$.

Definition 4.45. A self-similar structure $(K, S, \{f_q\}_{q \in S})$ is called a *nested fractal* if Assumption 4.39 holds and $g_{xy} \in \mathcal{G}$ for any $x, y \in V_0$ with $x \neq y$.

By [29, Proposition 3.8.7], we have the following proposition.

Proposition 4.46. *A nested fractal is strongly symmetric.*

We give three examples of strongly symmetric self-similar sets. Note that Assumption 4.41 is satisfied for all three examples because of Lemma E.5. The first two are nested fractals.

Example 4.47 (Pentakun: Figure 4.11). Let $L = 2$ and let $S = \{p_1, \dots, p_5\}$ be a collection of vertices of a regular pentagon satisfying $\sum_{i=1}^{5} p_i = 0$ and let $\rho = \frac{3-\sqrt{5}}{2}$. Then the associated self-similar set K, called pentakun, is strongly symmetric. (See [29, Example 3.8.11].) In this case $\mathcal{G} = \mathcal{G}_* = D_5$, which is the group of symmetries of a regular pentagon, and $V_0 = \{p_1, \dots, p_5\}$.

Example 4.48 (Snowflake: Figure 4.12). Let $L = 2$ and let $\{p_1, \dots, p_6\}$ be a collection of vertices of a regular hexagon satisfying $\sum_{i=1}^{6} p_i = 0$ and let $S = \{p_1, \dots, p_7, 0\}$. Furthermore, let $\rho = \frac{1}{3}$. Then the associated self-similar set, called snowflake, is strongly symmetric. (See [29, Example 3.8.12].) In this case $\mathcal{G} = \mathcal{G}_* = D_6$, which is the group of symmetries of a regular hexagon and $V_0 = \{p_1, \dots, p_6\}$.

The last example is not a nested fractal.

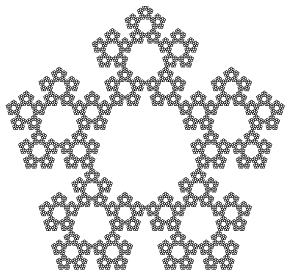


Figure 4.11. Pentakun.

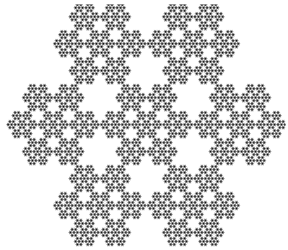


Figure 4.12. Snowflake.

Example 4.49. Let $L = 3$ and let

$$S = \{-1, 0, 1\}^3 \cup \left\{-\frac{1}{2}, \frac{1}{2}\right\}^3, \quad U = \{1, -1\}^3,$$

and $\rho = \frac{1}{5}$. Note that U is the collection of vertices of the cube $[-1, 1]^3$ and

$$f_q([-1, 1]^3) = \left[\frac{4q_1 - 1}{5}, \frac{4q_1 + 1}{5}\right] \times \left[\frac{4q_2 - 1}{5}, \frac{4q_2 + 1}{5}\right] \times \left[\frac{4q_3 - 1}{5}, \frac{4q_3 + 1}{5}\right]$$

for any $q = (q_1, q_2, q_3) \in S$. It is straightforward to see that the associated self-similar set is strongly symmetric with $V_0 = U$ and $\mathcal{G} = \mathcal{G}_* = \mathbb{B}_3$. This self-similar set is not a nested fractal because $g_{xy} \notin \mathcal{G}$ if $x = (-1, -1, -1)$ and $y = (1, 1, 1)$.

Using Theorem 4.8, we have the following theorem.

Theorem 4.50. *Suppose that $(K, S, \{f_i\}_{i \in S})$ is strongly symmetric and that Assumption 4.41 holds. Then (K, d_*) is p-conductively homogeneous for any $p >$ $\dim_{AR}(K, d_*)$.*

As for $\dim_{AR}(K, d_*)$, it was shown in [44] that $\dim_{AR}(K, d_*) = 1$ if (K, d_*) is the Sierpiński gasket. In general, we have the following fact.

Proposition 4.51. *Suppose that $(K, S, \{f_i\}_{i \in S})$ is strongly symmetric and that Assumption 4.41 holds. Then* $\dim_{AR}(K, d_*) < 2$.

Proof. For $m \geq 0$, define $\widetilde{E}_m = \{(f_w(x), f_w(y)) \mid w \in T_m, x, y \in V_0, x \neq y\}$. Then the sequence $\{(V_m, \widetilde{E}_m)\}_{m \geq 0}$ is a proper system of horizontal networks in the sense of [34, Definition 4.6.5]. Define

$$\begin{aligned}\mathcal{L}_s(V_0) = \{(D_{xy})_{x,y \in V_0} \mid\ & \text{there exists } (D_0, \ldots, D_{m_*-1}) \in [0, \infty)^{m_*} \\ & \text{such that } D_0 = 1,\ D_{xy} = D_i \text{ if } |x - y| = l_i, \\ & \text{and } \textstyle\sum_{y \in V_0} D_{xy} = 0 \text{ for any } x \in V_0\}.\end{aligned}$$

In particular, let $D^1 \in \mathcal{L}_s(V_0)$ satisfy $(D^1)_{xy} = 1$ for any $x, y \in V_0$ with $x \neq y$. For $D = (D_{xy})_{x,y \in V_0} \in \mathcal{L}_s(V_0)$, define

$$\mathcal{E}^D_{2,m}(h) = \frac{1}{2} \sum_{w \in T_m, x, y \in V_0} D_{xy}(h(f_w(x)) - h(f_w(y)))^2$$

for $h \in \ell(V_m)$ and

$$\mathcal{E}^D_{2,m,w} = \inf\{\mathcal{E}^D_{2,n+m}(h) \mid h \in \ell(V_{n+m}),\ h|_{V_{n+m} \cap K_w} = 1, \\ h|_{V_{n+m} \cap (\cup_{v \notin \Gamma_1(w)} K_v)} = 0\}$$

for any $w \in T_n$. Then by [29, Theorem 3.8.10 and Corollary 3.1.9], there exist $D_* \in \mathcal{L}_s(V_0)$ and $\sigma > 1$ such that $(D_*, (\sigma^{-1}, \dots, \sigma^{-1}))$ is a harmonic structure, that is, for any $h \in \ell(V_m)$,

$$\sigma^m \mathcal{E}^{D_*}_{2,m}(h) = \min\{\sigma^{m+1} \mathcal{E}^{D_*}_{2,m+1}(g) \mid g \in \ell(V_{m+1}), g|_{V_m} = h\}.$$

This implies that there exist $c_1, c_2 > 0$ and $k \geq 1$ such that

$$c_1 \sigma^{-m} \leq \sup_{w \in T \setminus T_k} \mathcal{E}^{D_*}_{2,m,w} \leq c_2 \sigma^{-m}.$$

On the other hand, there exist $c_3, c_4 > 0$ such that

$$c_3 \mathcal{E}^{D_*}_{2,m}(h) \leq \mathcal{E}^{D^1}_{2,m}(h) \leq c_4 \mathcal{E}^{D_*}_{2,m}(h)$$

for any $m \geq 0$ and $h \in \ell(V_m)$. Thus we see that $\sup_{w \in T} \mathcal{E}^{D_1}_{2,m,w} \leq C\sigma^{-m}$ for any $m \geq 0$. Therefore, by [34, Theorems 4.6.9 and 4.9.1], $\dim_{AR}(K, d_*) < 2$. ∎

The rest of this section is devoted to proving Theorem 4.50. We suppose that $(K, S, \{f_i\}_{i \in S})$ is strongly symmetric hereafter in this section. We have the following theorem by [29, Proposition 3.8.19],

Lemma 4.52. *If $(K, S, \{f_i\}_{i \in S})$ is strongly symmetric, then $g(K_w) = K_{g^*(w)}$ for any $g \in \mathcal{G}$ and $w \in T$. In particular, $\mathcal{G} \subseteq \mathcal{G}_{(K,T)}$.*

Lemma 4.53. *If $(K, S, \{f_i\}_{i \in S})$ is strongly symmetric, $x_1, x_2, y_1, y_2 \in V_0$ and $|x_1 - x_2| = |y_1 - y_2|$, then there exists $g \in \mathcal{G}_*$ such that $g(x_1) = y_1$ and $g(x_2) = y_2$.*

Proof. According to Definition 4.44 (4), there exists $g_1 \in \mathcal{G}_*$ such that $g_1(x_1) = y_1$. Let $g_1(x_2) = z$. Then $|y_1 - y_2| = |y_1 - z|$. Hence by Definition 4.44 (2), there exists $g_2 \in \mathcal{G}_*$ such that $g_2(y_1) = y_1$ and $g_2(z) = y_2$. Thus letting $g = g_2 \circ g_1$, we see that $g(x_1) = g_2(y_1) = y_1$ and $g(x_2) = g_2(z) = y_2$. ∎

Definition 4.54. A path $(w(1), \dots, w(k))$ of (T_m, E^*_m) is said to connect $x \in K$ and $y \in K$ if $x \in K_{w(1)}$ and $y \in K_{w(k)}$.

Lemma 4.55. *Let* $\mathbf{p}$ *be a path of* (T_m, E_m^*) *connecting* $x_1 \in V_0$ *and* $x_2 \in V_0$*. Suppose* $|x_1 - x_2| = l_i$ *for some* $i = 1, \dots, m_* - 1$*. Then there exist a path* $\mathbf{p}_1$ *of* (T_m, E_m^*)*,* $x \in V_0$ *and* $y \in V_0$ *such that* $\mathbf{p}_1$ *connects* x *and* y*,* $\mathbf{p}_1 \subseteq \bigcup_{g \in \mathcal{G}_*} g^*(\mathbf{p})$ *and* $|x - y| = l_{i-1}$*.*

Notation. For a path $\mathbf{p} = (w(1), \dots, w(k))$ and $g \in \mathcal{G}$, set

$$g^*(\mathbf{p}) = (g^*(w(1)), \dots, g^*(w(k))).$$

Remark. As was done before, we regard $\mathbf{p}_1$ and $g^*(\mathbf{p})$ as subsets of T_m in the above lemma. We are going to keep doing such an abuse of notation as long as no confusion may occur.

Proof. By Definition 4.44 (2), there exist $x, y, z \in V_0$ such that $|x - y| = l_{i-1}$, $|x - z| = l_i$ and $g_{yz} \in \mathcal{G}_*$. Also, Lemma 4.53 shows that there exists $h \in \mathcal{G}_*$ such that $h(x_1) = x$ and $h(x_2) = z$. Since $|x - y| < |x - z|$, x and z belong to different sides of H_{yz}. Hence the path $h^*(\mathbf{p})$ intersects with H_{yz}. Therefore, $h^*(\mathbf{p})$ and $(g_{yz})^* \circ h^*(\mathbf{p})$ has an intersection in H_{yz}. Since $(g_{yz})^* \circ h^*(\mathbf{p})$ connects $g_{yz}(x)$ and $y = g_{yz}(z)$, we can extract a path $\mathbf{p}_1$ from $h^*(\mathbf{p}) \cup (g_{yz})^* \circ h^*(\mathbf{p})$ connecting x and y, and included in $\bigcup_{g \in \mathcal{G}_*} g^*(\mathbf{p})$. Since $|x - y| = l_{i-1}$, $\mathbf{p}_1$ is a desired path. ∎

Lemma 4.56. *Let* $\mathbf{p}$ *be a path of* (T_m, E_m^*) *connecting two distinct points in* V_0*. Then for any* $x, y \in V_0$*, there exists a path* $\mathbf{p}'$ *of* (T_m, E_m^*) *connecting* x *and* y *such that* $\mathbf{p}' \subseteq \bigcup_{g \in \mathcal{G}_*} g^*(\mathbf{p})$*.*

Proof. Inductive use of Lemma 4.55 shows that there exists a path $\mathbf{p}_0$ of (T_m, E_m^*) connecting two distinct points z_1 and z_2 in V_0 such that $|z_1 - z_2| = l_0$ and $\mathbf{p}_0 \subseteq \bigcup_{g \in \mathcal{G}_*} g^*(\mathbf{p})$. By Definition 4.44 (1), there exists a strict 0-walk $(x_1, \dots, x_{j_0})$ satisfying $x_1 = x$ and $x_{j_0} = y$. By Lemma 4.53, for any $j = 1, \dots, j_0 - 1$, there exists $g_j \in \mathcal{G}_*$ such that $g_j(z_1) = x_j$ and $g_j(z_2) = x_{j+1}$. Concatenating $(g_1)^*(\mathbf{p}_0), \dots, (g_{j_0-2})^*(\mathbf{p}_0)$ and $(g_{j_0-1})^*(\mathbf{p}_0)$, we obtain a desired path connecting x and y. ∎

Proof of Theorem 4.50. We are going to use Theorem 4.8. Let $\mathcal{I} = \mathcal{IT}(K, T)$ and let $\mathcal{G}_0 = \mathcal{G}_1 = \mathcal{G}_*$. By (4.25) and the fact that $\mathcal{I} = \mathcal{IT}(K, T)$, we see that $E_m^{\mathcal{I}} = E_m^*$. Hence (a) of Theorem 4.8 is satisfied, and (b) is also satisfied due to the fact that $\mathcal{G}_*$ is transitive on V_0.

Let $w \in T_n$, let $u, v \in T_k$ and let $\mathbf{p} \in \mathcal{C}_{1,m}^{(1)}(w)$. Then $\mathbf{p}$ contains a path connecting two distinct points in $\bigcup_{w' \in T_n} f_{w'}(V_0)$. Thus $\psi_n(\mathbf{p})$ contains a path between two distinct points in V_0. By Lemma 4.56, for any $x, y \in V_0$, there exists a path $\mathbf{p}_{xy} \subseteq \bigcup_{g \in \mathcal{G}_*} g^*(\psi_n(\mathbf{p}))$ connecting x and y. Set $\mathcal{U}_{\mathbf{p}} = \bigcup_{x,y \in V_0} \mathbf{p}_{xy}$. Then since $K(\mathcal{U}_{\mathbf{p}}) \supseteq V_0$, it follows that $g(K(\mathcal{U}_{\varphi})) \supseteq V_0$ for any $g \in \mathcal{G}_*$. Moreover, $K(\mathcal{U}_{\mathbf{p}})$ is connected and $\mathcal{U}_{\mathbf{p}} \subseteq \bigcup_{g \in \mathcal{G}_*} g^*(\psi_n(\mathbf{p}))$. Thus we have verified (c) of Theorem 4.8. Now, Theorem 4.8 suffices. ∎

Chapter 5

Knight move implies conductive homogeneity

5.1 Conductance and Poincaré constants

From this section, we start preparations for a proof of Theorem 3.33. To begin with, we will introduce Poincaré constants and study a relationship between Poincaré and conductance constants in this section.

The next lemma concerns an extension of functions on T_n to those on T_{n+m} by means of the partition of unity $\{\varphi_w\}_{w \in T_n}$ given in Lemma 2.19.

Lemma 5.1 ([36, Lemma 2.8]). *Let $p \geq 1$, let $A \subseteq T_n$ and let $\{\varphi_w\}_{w \in A}$ be the partition of unity given in Lemma 2.19. Define $\widehat{I}_{A,m} \colon \ell(A) \to \ell(S^m(A))$ by*

$$(\widehat{I}_{A,m} f)(u) = \sum_{w \in A} f(w)\varphi_w(u).$$

Then

$$\mathcal{E}_{p,A}^{n+m}(\widehat{I}_{A,m} f) \leq c_{5.1}\bigl(\max_{w \in A} \mathcal{E}_{M,p,m}(w, A)\bigr)\mathcal{E}_{p,A}^{n}(f),$$

where the constant $c_{5.1} = c_{5.1}(p, L_*, M)$ depends only on p, L_* and M.

Proof. Let $(a_k(u, v))_{u,v \in T_k}$ be the adjacency matrix of (T_k, E_k^*). Set $\widetilde{f} = \widehat{I}_{A,m} f$. Then

$$\mathcal{E}_p^{n+m}(\widetilde{f}) = \frac{1}{2}\sum_{w \in A}\sum_{v \in S^m(w)}\sum_{u \in S^m(\Gamma_1^A(w))} a_{n+m}(u, v)|\widetilde{f}(u) - \widetilde{f}(v)|^p. \qquad (5.1)$$

Suppose $v \in S^m(w), u \in S^m(\Gamma_1^A(w))$ and $(u, v) \in E_{n+m}^*$. Then $\varphi_{w'}(u) = \varphi_{w'}(v) = 0$ for any $w' \notin \Gamma_{M+1}^A(w)$. Hence

$$\sum_{w' \in \Gamma_{M+1}^A(w)} \varphi_{w'}(u) = \sum_{w' \in \Gamma_{M+1}^A(w)} \varphi_{w'}(v) = 1.$$

Using this, we see

$$\begin{aligned}\widetilde{f}(u) - \widetilde{f}(v) &= \sum_{w' \in \Gamma_{M+1}^A(w)} f(w')(\varphi_{w'}(u) - \varphi_{w'}(v)) \\ &= \sum_{w' \in \Gamma_{M+1}^A(w)} (f(w') - f(w))(\varphi_{w'}(u) - \varphi_{w'}(v)).\end{aligned}$$

Let $q \geq 1$ be the conjugate of p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. Then by Lemma A.2,

$$
\begin{aligned}
|\widetilde{f}(u) - \widetilde{f}(v)|^p &\leq \sum_{w' \in \Gamma^A_{M+1}(w)} |f(w') - f(w)|^p \\
&\qquad \times \Bigl(\sum_{w' \in \Gamma^A_{M+1}(w)} |\varphi_{w'}(u) - \varphi_{w'}(v)|^q \Bigr)^{\frac{p}{q}} \\
&\leq C_1 \sum_{w' \in \Gamma^A_{M+1}(w)} |f(w') - f(w)|^p \\
&\qquad \times \sum_{w' \in \Gamma^A_{M+1}(w)} |\varphi_{w'}(u) - \varphi_{w'}(v)|^p,
\end{aligned}
$$

where $C_1 = \max\{1, (L_*)^{(M+1)(p-2)}\}$. If $w \in A$ and $w' \in \Gamma^A_{M+1}(w)$, then there exist $w(0), \ldots, w(M+1) \in A$ such that $w(0) = w$, $w(M+1) = w'$, $(w(j), w(j+1)) \in E^*_n$ for any $j = 0, \ldots, M$. Then

$$
|f(w') - f(w)|^p \leq (M+1)^{p-1} \sum_{j=0}^{M} |f(w(j)) - f(w(j+1))|^p.
$$

Since $\#(\Gamma^A_{M+1}(w)) \leq (L_*)^{M+1}$, it follows that

$$
\sum_{w' \in \Gamma^A_M(w)} |f(w') - f(w)|^p \leq C_2 \sum_{w', w'' \in \Gamma^A_M(w), (w', w'') \in E^*_n} |f(w') - f(w'')|^p,
$$

where $C_2 = (M+1)^{p-1}(L_*)^M$. On the other hand,

$$
\begin{aligned}
&\sum_{v \in S^m(w)} \sum_{u \in S^m(\Gamma^A_1(w))} a_{n+m}(u, v) \sum_{w' \in \Gamma^A_{M+1}(w)} |\varphi_{w'}(u) - \varphi_{w'}(v)|^p \\
&\quad \leq 2 \sum_{w' \in \Gamma^A_{M+1}(w)} \mathcal{E}^{n+m}_{p, S^m(A)}(\varphi_{w'}, \varphi_{w'}) \leq 2(L_*)^{M+1} \max_{w' \in A} \mathcal{E}^{n+m}_{p, S^m(A)}(\varphi_{w'}).
\end{aligned}
$$

Hence, by (5.1),

$$
\begin{aligned}
\mathcal{E}^{m+n}_{p, S^m(A)}(\widetilde{f}) &\leq C_1 C_2 (L_*)^{M+1} \max_{w \in A} \mathcal{E}^{n+m}_{p, S^m(A)}(\varphi_w) \\
&\qquad \times \sum_{w \in A} \Bigl(\sum_{w', w'' \in \Gamma^A_{M+1}(w), (w', w'') \in E^*_n} |f(w') - f(w'')|^p \Bigr) \\
&\leq C_1 C_2 (L_*)^{2(M+1)} \max_{w \in A} \mathcal{E}^{n+m}_{p, S^m(A)}(\varphi_w) \mathcal{E}^n_{p, A}(f).
\end{aligned}
$$

So, Lemma 2.19 suffices. ■

There is another simple way of extension of functions on T_n to those on T_{n+k}.

Lemma 5.2. *Let $p \geq 1$ and let $A \subseteq T_n$. Define $\widetilde{I}_{A,k}: \ell(A) \to \ell(S^k(A))$ by*

$$\widetilde{I}_{A,k} f = \sum_{w \in A} f(w) \chi_{S^k(w)}.$$

Then

$$\mathcal{E}^{n+k}_{p,S^k(A)}(\widetilde{I}_{A,k} f) \leq \max_{w \in A} \#(\partial S^k(w)) \mathcal{E}^n_{p,A}(f).$$

Proof. Let $\widehat{f} = \widetilde{I}_{A,k} f$. Then $\widehat{f}(u) = \widehat{f}(v)$ if $\pi^k(u) = \pi^k(v)$. So if $(u,v) \in E^*_{n+k}$ and $\widehat{f}(u) \neq \widehat{f}(v)$, then $(\pi^k(u), \pi^k(v)) \in E^*_n$. Fix $(w, w') \in E^*_n$. Then

$$\#\{(u,v) \mid (u,v) \in E_{n+k}, \pi^k(u) = w, \pi^k(v) = w'\} \leq \#(\partial S^k(w)).$$

This immediately implies the desired statement. ∎

Combining two previous extensions, we have the following estimate.

Lemma 5.3 ([36, Lemma 2.9]). *Let $p \geq 1$ and let $A \subseteq T_n$. Then, there exists $I_{A,k,m}$: $\ell(A) \to \ell(S^{k+m}(A))$ such that for any $f \in \ell(A)$,*

$$\mathcal{E}^{n+k+m}_{p,S^{k+m}(A)}(I_{A,k,m} f) \leq c_{5.3} \max_{w \in A} \#(\partial S^k(w)) \times \max_{v \in S^k(A)} \mathcal{E}_{M,p,m}(v, S^k(A)) \mathcal{E}^n_{p,A}(f), \tag{5.2}$$

where the constant $c_{5.3} = c_{5.3}(p, L_*, M)$ depends only on p, L_* and M, and

$$(I_{A,k,m} f)(u) = f(w) \tag{5.3}$$

for any $w \in A$ and $u \in S^m(S^k(w) \backslash B_{M,k}(w))$.

Proof. Define $I = \widehat{I}_{S^k(A),m} \circ \widetilde{I}_{A,k}$. Combining Lemmas 5.1 and 5.2, we immediately obtain (5.2). Let $u \in S^{m+k}(A)$. Set $v = \pi^m(u)$ and $w = \pi^k(w)$. If $\Gamma^{S^k(A)}_M(v) \subseteq S^k(w)$, then

$$\begin{aligned}(If)(u) &= \sum_{v' \in S^k(A)} f(\pi^k(v')) \varphi_{v'}(u) = \sum_{v' \in \Gamma^{S^k(A)}_M(v)} f(\pi^k(v')) \varphi_{v'}(u) \\ &= \sum_{v' \in \Gamma^{S^k(A)}_M(v)} f(w) \varphi_{v'}(u) = f(w).\end{aligned}$$

If $v \in S^k(w) \backslash B_{M,k}(w)$, then $\Gamma^{S^k(A)}_M(v) \subseteq \Gamma_M(v) \subseteq S^k(w)$. So the above equality suffices for (5.3). ∎

Next we introduce *p-Poincaré constants*. In fact, there are two kinds of Poincaré constants $\lambda_{p,m}(A)$ and $\widetilde{\lambda}_{p,m}(A)$ but they are almost the same in view of (5.4).

Definition 5.4. Define $\mu(w) = \mu(K_w)$ for $w \in T$. For $A \subseteq T_n$, define $\mu(A) = \sum_{w \in A} \mu(w)$ and $\mu_A \colon A \to [0, \infty)$ by

$$\mu_A(w) = \frac{\mu(w)}{\mu(A)}$$

for $w \in A$. For $f \in \ell(A)$, define

$$(f)_A = \sum_{u \in A} f(u)\mu_A(u)$$

and

$$\|f\|_{p,\mu_A} = \Big(\sum_{u \in A} |f(u)|^p \mu_A(u)\Big)^{\frac{1}{p}}.$$

Moreover, define

$$\lambda_{p,m}(A) = \sup_{f \in \ell(S^m(A))} \frac{\inf_{c \in \mathbb{R}}(\|f - c\chi_{S^m(A)}\|_{p,\mu_{S^m(A)}})^p}{\mathcal{E}^{n+m}_{p,S^m(A)}(f)}$$

and

$$\widetilde{\lambda}_{p,m}(A) = \sup_{f \in \ell(S^m(A))} \frac{(\|f - (f)_{S^m(A)}\|_{p,\mu_{S^m(A)}})^p}{\mathcal{E}^{n+m}_{p,S^m(A)}(f)}.$$

Remark. By Lemma B.2, it follows that

$$\Big(\frac{1}{2}\Big)^p \widetilde{\lambda}_{p,m}(A) \le \lambda_{p,m}(A) \le \widetilde{\lambda}_{p,m}(A). \tag{5.4}$$

Using the previous lemmas, we have a relation between Poincaré and conductance constants as follows.

Lemma 5.5 ([36, Proposition 2.10]). *Let $p \ge 1$ and let $A \subseteq T_n$. For any $m \ge 1$ and $k \ge Mm_0$,*

$$\max_{w \in A} \#(\partial S^k(w)) \max_{v \in S^k(A)} \mathcal{E}_{M,p,m}(v, S^k(A))\lambda_{p,k+m}(A) \ge c_{5.5}\lambda_{p,0}(A),$$

where the constant $c_{5.5} = c_{5.5}(\gamma, m_0, p, L_*, M)$ depends only on γ, m_0, p, L_* and M.

Proof. Choose $f_0 \in \ell(A)$ such that $\mathcal{E}^n_{p,A}(f_0) = 1$ and

$$\big(\min_{c \in \mathbb{R}} \|f_0 - c\chi_A\|_{p,\mu_A}\big)^p = \lambda_{p,0}(A).$$

Letting $f = I_{A,k,m} f_0$, by Lemma 5.3, we see that

$$\mathcal{E}^{n+k+m}_{p,S^{m+k}(A)}(f) \le c_{5.3} \max_{w\in A} \#(\partial S^k(w)) \max_{v\in S^k(A)} \mathcal{E}_{M,p,m}(v, S^k(A)). \tag{5.5}$$

On the other hand, by (5.3) and (2.8),

$$\begin{aligned}
&\frac{1}{\mu(A)} \sum_{v\in S^{k+m}(A)} |f(v) - c|^p \mu(v) \\
&\quad = \frac{1}{\mu(A)} \sum_{w\in A} \sum_{v\in S^m(S^k(w))} |f(v) - c|^p \mu(v) \\
&\quad \ge \frac{1}{\mu(A)} \sum_{w\in A} \sum_{v\in S^m(S^k(w)\setminus B_{M,k}(w))} |f_0(w) - c|^p \mu(v) \\
&\quad \ge \gamma^{m_0 M} \frac{1}{\mu(A)} \sum_{w\in A} |f_0(w) - c|^p \mu(w) \ge \gamma^{m_0 M} \lambda_{p,0}(A).
\end{aligned}$$

This and (5.5) yield the desired inequality. ∎

5.2 Relations of constants

In this section, we will establish relations between conductance, neighbor disparity, and Poincaré constants towards a proof of Theorem 3.33. As in the previous section we fix a covering system $\mathscr{J}$ with covering numbers (N_T, N_E) and we write $\sigma_{p,m}$ and $\sigma_{p,m,n}$ in place of $\sigma^{\mathscr{J}}_{p,m}$ and $\sigma^{\mathscr{J}}_{p,m,n}$, respectively.

Definition 5.6. For $w \in T$ and $n \ge 0$, define

$$\xi_n(w) = \max_{v\in S^n(w)} \frac{\mu(v)}{\mu(w)}.$$

First, we consider a relation between Poincaré and neighbor disparity constants.

Lemma 5.7 ([36, Proposition 2.13 (1)]). *Let $p \ge 1$. For any $w \in T$ and $n, m \ge 1$,*

$$\widetilde{\lambda}_{p,n+m}(w) \le 2^{p-1}\big(\xi_n(w) \max_{v\in S^n(w)} \widetilde{\lambda}_{p,m}(v) + L_* c_{2.27} \widetilde{\lambda}_{p,n}(w) \sigma_{p,m,n+|w|}\big).$$

Proof. By Theorem A.3, for any $f \in \ell(S^{n+m}(w))$,

$$\begin{aligned}
&\frac{1}{\mu(w)} \sum_{u\in S^{n+m}(w)} |f(u) - (f)_{S^{n+m}(w)}|^p \mu(v) \\
&\quad \le \frac{C_p}{\mu(w)} \sum_{v\in S^n(w)} \sum_{u\in S^m(v)} (|f(u) - (f)_{S^m(v)}|^p \\
&\qquad\qquad + |(f)_{S^m(v)} - (f)_{S^{n+m}(w)}|^p)\mu(u),
\end{aligned}$$

where $C_p = 2^{p-1}$ for $p \neq 2$ and $C_2 = 1$. Examining the first half of the above inequality, we obtain

$$\begin{aligned}\frac{1}{\mu(w)} \sum_{v \in S^n(w)} \sum_{u \in S^m(v)} &|f(u) - (f)_{S^m(v)}|^p \mu(u) \\ &\leq \sum_{v \in S^n(w)} \frac{\mu(v)}{\mu(w)} \widetilde{\lambda}_{p,m}(v) \mathcal{E}^{|w|+n+m}_{p,S^m(v)}(f) \leq \xi_n(w) \\ &\qquad \times \max_{v \in S^n(w)} \widetilde{\lambda}_{p,m}(v) \mathcal{E}^{|w|+n+m}_{p,S^{n+m}(w)}(f).\end{aligned}$$

For the other half, by Lemma 2.27,

$$\begin{aligned}\frac{1}{\mu(w)} \sum_{v \in S^n(w)} \sum_{u \in S^m(v)} &|(f)_{S^m(v)} - (f)_{S^{n+m}(w)}|^p \mu(u) \\ &= \sum_{v \in S^n(w)} \frac{\mu(v)}{\mu(w)} |(P_{n+|w|,m} f)(v) - (P_{n+|w|,m} f)_{S^n(w)}|^p \\ &\leq \widetilde{\lambda}_{p,n}(w) \mathcal{E}^{|w|+n}_{p,S^n(w)}(P_{n+|w|,m} f) \\ &\leq L_* \widetilde{\lambda}_{p,n}(w) c_{2.27} \sigma_{p,m,n+|w|} \mathcal{E}^{n+m+|w|}_{p,S^{n+m}(w)}(f).\end{aligned}$$

Combining all, we see

$$\begin{aligned}\widetilde{\lambda}_{p,n+m}(w) \leq C_p \Big(\xi_n(w) &\max_{v \in S^n(w)} \widetilde{\lambda}_{p,m}(v) \\ &+ L_* c_{2.27} \widetilde{\lambda}_{p,n}(w) \sigma_{p,m,n+|w|}(v, v')\Big).\end{aligned}$$ ■

Definition 5.8. Define

$$\overline{\lambda}_{p,m} = \sup_{w \in T} \widetilde{\lambda}_{p,m}(w).$$

By Theorem 6.7, $\overline{\lambda}_{p,m}$ is finite for any $m \geq 1$.

Making use of Lemma 5.7, we have the following inequality.

Lemma 5.9. *Define*

$$\xi_n = \sup_{w \in T} \xi_n(w).$$

Then

$$\overline{\lambda}_{p,n+m} \leq 2^{p-1} \big(\xi_n \overline{\lambda}_{p,m} + L_* c_{2.27} \overline{\lambda}_{p,n} \sigma_{p,m}\big) \tag{5.6}$$

for any $n, m \geq 1$.

Remark. By Lemma 2.13, μ is exponential, so that there exist $\xi \in (0, 1)$ and $c > 0$ such that

$$\xi_n \leq c \xi^n$$

for any $n \geq 1$.

Next, we examine the relationship between the conductance and Poincaré constants.

Lemma 5.10. *For any $w \in T$, $l, m \geq 1$ and $k \geq m_0 M_0$,*

$$\overline{D}_k \mathcal{E}_{M_*,p,m,|w|+k+l} \widetilde{\lambda}_{p,k+m+l}(w) \geq c_{5.10} \widetilde{\lambda}_{p,l}(w), \tag{5.7}$$

where $\overline{D}_k = \max_{v \in T \setminus \{\phi\}} \#(\partial S^k(v))$ and the constant $c_{5.10} = 2^{-p} c_{5.5}$ depends only on γ, m_0, p, L_* and M_0. In particular,

$$\overline{D}_k \mathcal{E}_{M_*,p,m} \overline{\lambda}_{p,k+m+l} \geq c_{5.10} \overline{\lambda}_{p,l} \tag{5.8}$$

Proof. Applying Lemma 5.5 with $M = M_0$ and $A = S^l(w)$, we obtain

$$\overline{D}_k \max_{v \in S^{k+l}(w)} \mathcal{E}_{M_0,p,m}(v, S^{k+l}(w)) \lambda_{p,k+m}(S^l(w)) \geq c_{5.5} \lambda_{p,0}(S^l(w)).$$

Lemma 2.18 shows

$$\mathcal{E}_{M_0,p,m}(v, S^{k+l}(w)) \leq \mathcal{E}_{M_*,p,m}(v, T_{|w|+k+l}) \leq \mathcal{E}_{M_*,p,m,|w|+k+l}.$$

Moreover, $\lambda_{p,k+m}(S^l(w)) = \lambda_{p,k+m+l}(w)$ and $\lambda_{p,0}(S^l(w)) = \lambda_{p,l}(w)$ by definition. So letting $c_{5.10} = 2^{-p} c_{5.5}$, we obtain (5.7). ∎

The next theorem is one of the main results of this section.

Theorem 5.11. *Assume that $p > 1$. If either*

$$\lim_{n \to \infty} \xi_n \mathcal{E}_{p,n-m_0 M_0} = 0 \tag{5.9}$$

or

$$\lim_{n \to \infty} \xi_n \overline{D}_{n-1} = 0, \tag{5.10}$$

then there exists $C > 0$ such that

$$\overline{\lambda}_{p,m} \leq C \sigma_{p,m}, \tag{5.11}$$

$$\overline{\lambda}_{p,m+n} \leq C \overline{\lambda}_{p,n} \sigma_{p,m} \tag{5.12}$$

and

$$(\mathcal{E}_{M_*,p,n})^{-1} \overline{\lambda}_{p,m} \leq C \overline{\lambda}_{p,m+n} \tag{5.13}$$

for any $n, m \geq 1$.

Remark. Inequalities (5.12) and (5.13) correspond to [36, (2.4)] and [36, (2.3)], respectively.

Unlike (5.9), (5.10) does not depend on p. So, once (5.10) holds, then we have (5.11), (5.12) and (5.13) for any $p > 1$. See Proposition 5.12 after the proof for more discussion on (5.10).

Proof. For ease of notation, we write $\overline{\lambda}_m = \overline{\lambda}_{p,m}$, $\sigma_m = \sigma_{p,m}$ and $\mathcal{E}_{M_*,p,m} = \mathcal{E}_m$. By (5.8), if $n > k \geq m_0 M_0$, then

$$\overline{D}_k \mathcal{E}_{n-k} \overline{\lambda}_{n+m} \geq c_{5.10} \overline{\lambda}_m. \tag{5.14}$$

This and (5.6) show

$$\overline{\lambda}_{n+m} \leq 2^{p-1}((c_{5.10})^{-1} \overline{D}_k \mathcal{E}_{n-k} \xi_n \overline{\lambda}_{n+m} + L_* c_{2.27} \overline{\lambda}_n \sigma_m). \tag{5.15}$$

Suppose that (5.9) holds. Let $k = m_0 M_0$. Then there exists n_0 such that, for any $n \geq n_0$,

$$2^{p-1}(c_{5.10})^{-1} \overline{D}_{m_0 M_0} \mathcal{E}_{n-m_0 M_0} \xi_n \leq \frac{1}{2}$$

and hence by (5.15),

$$\overline{\lambda}_{n+m} \leq 2^p L_* c_{2.27} \overline{\lambda}_n \sigma_m. \tag{5.16}$$

Next suppose that (5.10) holds. Then there exists n_0 such that, for any $n \geq n_0$,

$$2^{p-1}(c_{5.10})^{-1} \overline{D}_{n-1} \mathcal{E}_1 \xi_n \leq \frac{1}{2},$$

so that we have (5.16) as well. Thus we have seen that if either (5.9) or (5.10) holds, then there exists n_0 such that (5.16) holds for any $n \geq n_0$.

Now, let $n_* = \max\{m_0 M_0 + 1, n_0\}$. Then by (5.14) and (5.16),

$$c_{5.10}(\overline{D}_{m_0 M_0})^{-1} (\mathcal{E}_{p,n_*-m_0 M_0})^{-1} \overline{\lambda}_m \leq \overline{\lambda}_{n_*+m} \leq 2^p L_* c_{2.27} \overline{\lambda}_{n_*} \sigma_m$$

for any $m \geq 1$. This immediately implies (5.11). Using this and (3.18), we have

$$\overline{\lambda}_{m+n} \leq \sigma_{m+n} \leq C \sigma_m \sigma_n.$$

Therefore, for any $m \geq 1$ and $n \in \{1, \ldots, n_0\}$,

$$\frac{\overline{\lambda}_{m+n}}{\overline{\lambda}_n \sigma_{p,m}} \leq C \frac{\sigma_n}{\overline{\lambda}_n} \leq C \max_{n=1,\ldots,n_0} \frac{\sigma_n}{\overline{\lambda}_n}.$$

So we have verified (5.12) for any $n, m \geq 1$. Letting $k = m_0 M_0$ in (5.8) and using (5.12), we obtain (5.13) as well. ■

The following proposition gives a geometric sufficient condition for (5.10).

Proposition 5.12. *Suppose that Assumption 2.15 holds. Assume that μ is α_H-Ahlfors regular with respect to the metric d. If there exist $\tilde{\alpha} < \alpha_H$ and $c > 0$ such that*

$$\#(\partial S^m(w)) \leq c r^{-m\tilde{\alpha}}$$

for any $w \in T$ and $m \geq 0$, then (5.10) holds.

Under the assumptions of Proposition 5.12, $\alpha_H = \dim_H(K, d)$, which is the Hausdorff dimension of (K, d), while $\dim_H(B_w, d) \le \tilde{\alpha}$ for any $w \in T$. So, roughly speaking, Proposition 5.12 says that if

$$\dim_H(K, d) > \sup_{w \in T} \dim_H(B_w, d),$$

then (5.10) is satisfied. By this proposition, one can verify (5.10) for generalized Sierpiński carpets for example.

Proof. By [34, Theorem 3.1.21], there exist $c_1, c_2 > 0$ such that

$$c_1 r^{\alpha_H |w|} \le \mu(K_w) \le c_2 r^{\alpha_H |w|}$$

for any $w \in T$. Hence $\xi_n \le c r^{\alpha_H n}$, while $\overline{D}_n \le r^{-\tilde{\alpha} n}$. ■

To conclude this section, we present a lemma providing a control of the difference of a function on T_n through $\mathcal{E}_p^n(f)$ and the Poincaré constant.

Lemma 5.13. *For any $w \in T$, $n \ge m \ge 1$, $f \in \ell(S^n(w))$, and $u, v \in S^n(w)$, if $\pi^{n-m}(u) = \pi^{n-m}(v)$, then*

$$|f(u) - f(v)| \le 2\gamma^{-\frac{1}{p}} \mathcal{E}_{p,S^n(w)}^{n+|w|}(f)^{\frac{1}{p}} \sum_{i=1}^{n-m} (\overline{\lambda}_{p,i})^{\frac{1}{p}}.$$

Proof. Let $u \in S^n(w)$. Set

$$S_i(u) = S^i(\pi^i(u))$$

for $u \in S^n(w)$ and $i = 0, 1, \ldots, n$. By Lemma B.3 and (2.5), for any $k = 1, \ldots, n$,

$$\begin{aligned}
|f(u) - (f)_{S_k(u)}| &\le \sum_{i=1}^{k} |(f)_{S_{i-1}(u)} - (f)_{S_i(u)}| \\
&\le \sum_{i=1}^{k} \left(\frac{\mu(\pi^i(u))}{\mu(\pi^{i-1}(u))} \right)^{\frac{1}{p}} \left(\widetilde{\lambda}_{s,p,i}(\pi^i(u)) \mathcal{E}_{p,S_i(u)}^{n+w}(f) \right)^{\frac{1}{p}} \\
&\le \gamma^{-\frac{1}{p}} \mathcal{E}_{p,S^n(w)}^{n+|w|}(f)^{\frac{1}{p}} \sum_{i=1}^{k} \left(\widetilde{\lambda}_{p,i}(\pi^i(u)) \right)^{\frac{1}{p}}.
\end{aligned}$$

Hence

$$\begin{aligned}
|f(u) - f(v)| &\le |f(u) - (f)_{S_{n-m}(u)}| + |(f)_{S_{n-m}(v)} - f(v)| \\
&\le \gamma^{-\frac{1}{p}} \mathcal{E}_{p,S^n(w)}^{n+|w|}(f)^{\frac{1}{p}} \left(\sum_{i=1}^{n-m} \left((\widetilde{\lambda}_{p,i}(\pi^i(v)))^{\frac{1}{p}} + (\widetilde{\lambda}_{p,i}(\pi^i(w)))^{\frac{1}{p}} \right) \right).
\end{aligned}$$
■

5.3 Proof of Theorem 3.33

Finally, we are going to give a proof of the "if" part of Theorem 3.33. Recall that by (3.19), there exist $c > 0$ and $\alpha \in (0, 1)$ such that

$$\mathcal{E}_{M_*,p,m} \le c\alpha^m$$

for any $m \ge 0$. Then since $\xi_n \le 1$, (5.9) is satisfied and hence (5.11), (5.12) and (5.13) turn out to be true.

As in the previous sections, a set $\mathcal{J}$ is a covering system with covering numbers (N_T, N_E). Furthermore, recall that by the definition of covering systems,

$$\sup_{A\in\mathcal{J}} \#(A) < \infty.$$

We denote the above supremum by N_c.

Lemma 5.14. *Set $\rho = \alpha^{\frac{1}{p}}$. There exists $C > 0$ such that for any $w \in T$, $k, m \ge 1$ with $m \ge k$ and $f \in \ell(S^m(w))$, if $u, v \in S^m(w)$ and $\pi^{m-k}(u) = \pi^{m-k}(v)$, then*

$$|f(u) - f(v)| \le C\rho^k(\overline{\lambda}_{p,m})^{\frac{1}{p}}\mathcal{E}^{|w|+m}_{p,S^m(w)}(f)^{\frac{1}{p}}.$$

Proof. By (5.13),

$$\overline{\lambda}_{p,i} \le C\overline{\lambda}_{p,m}\mathcal{E}_{p,m-i} \le C\overline{\lambda}_{p,m}\rho^{p(m-i)}. \tag{5.17}$$

Using this and applying Lemma 5.13, we have

$$|f(u) - f(v)| \le C\mathcal{E}^{|w|+m}_{p,S^m(w)}(f)^{\frac{1}{p}}\sum_{i=1}^{m-k}(\overline{\lambda}_{p,i})^{\frac{1}{p}} C\mathcal{E}^{|w|+m}_{p,S^m(w)}(f)^{\frac{1}{p}}(\overline{\lambda}_{p,m})^{\frac{1}{p}}\sum_{i=k}^{m-1}\rho^i. \quad ∎$$

Lemma 5.15. *Set $\varepsilon = (N_c)^{-\frac{2}{p}}$. There exist $n_* \ge 1$ and $m_* \ge n_*$ such that if $m \ge m_*$, then there exist $w \in T$ and $f \in \ell(S^m(w))$ such that*

$$\min_{u\in S^{m-n_*}(y_1)} f(u) - \max_{u\in S^{m-n_*}(y_2)} f(u) \ge \frac{1}{8}\varepsilon$$

for some $y_1, y_2 \in S^{n_*}(w)$ and

$$\mathcal{E}^{|w|+m}_{p,S^m(w)}(f) \le \frac{2}{\sigma_{p,m}}.$$

Proof. Choose $A \in \mathcal{J}$ such that $\sigma_{p,m}(A) \ge \frac{1}{2}\sigma_{p,m}$. Suppose that $A \subseteq T_n$ and choose $f \in \ell(S^m(A))$ such that $\mathcal{E}^n_{p,A}(P_{n,m}f) = 1$ and

$$\mathcal{E}^{n+m}_{p,S^m(A)}(f) = \frac{1}{\sigma_{p,m}(A)}. \tag{5.18}$$

Claim 1. *There exists $c_1 > 0$, which is independent of m and A, such that if $u_1, u_2 \in S^m(A)$ and $(u_1, u_2) \in E^*_{n+m}$, then*

$$|f(u_1) - f(u_2)| \leq c_1 \rho^m. \tag{5.19}$$

Proof. By (5.11), (5.17) and (5.18), we have

$$|f(u_1) - f(u_2)|^p \leq \mathcal{E}^{n+m}_{p,S^m(A)}(f) = \frac{1}{\sigma_{p,m}(A)} \leq \frac{2}{\sigma_{p,m}} \leq \frac{C}{\overline{\lambda}_{p,m}} \leq C\rho^{pm}.$$

This proves the claim. □

Claim 2. *There exists $c_2 > 0$, which is independent of m and A, such that if $u_1, u_2 \in A$ and $\pi^{m-k}(u_1) = \pi^{m-k}(u_2)$ for some $k \in \{1, \ldots, m\}$, then $|f(u_1) - f(u_2)| \leq c_2 \rho^k$.*

Proof. It follows that $u_1, u_2 \in S^m(w)$ for some $w \in A$. Using Lemma 5.14, we obtain

$$\begin{aligned}|f(u_1) - f(u_2)| &\leq C\rho^k (\overline{\lambda}_{p,m})^{\frac{1}{p}} \mathcal{E}^{n+m}_{p,S^m(w)}(f)^{\frac{1}{p}} \\ &\leq C\rho^k (\overline{\lambda}_{p,m})^{\frac{1}{p}} \mathcal{E}^{n+m}_{p,S^m(A)}(f)^{\frac{1}{p}} \leq C\rho^k (\overline{\lambda}_{p,m})^{\frac{1}{p}} (\sigma_{p,m})^{-\frac{1}{p}}.\end{aligned}$$

Now (5.11) immediately shows the claim. □

Since $\#(A) \leq N_c$, it follows that $\#(E^*_n(A)) \leq (N_c)^2$. Therefore, the fact that $\mathcal{E}^n_{p,A}(P_{n,m} f) = 1$ shows that there exists $(w_1, w_2) \in E^*_n(A)$ such that

$$|(f)_{S^m(w_1)} - (f)_{S^m(w_2)}|^p \geq (N_c)^{-2} = \varepsilon^p.$$

Exchanging f by $-f$ if necessary, we may assume that

$$(f)_{S^m(w_1)} - (f)_{S^m(w_2)} \geq \varepsilon$$

without loss of generality. Define

$$\begin{aligned} n_* &= \inf\{n \mid n \in \mathbb{N}, \varepsilon \geq 16 c_2 \rho^n\}, \\ m_* &= \max\{n_*, \inf\{m \mid m \in \mathbb{N}, \varepsilon \geq 2 c_1 \rho^m\}\}. \end{aligned}$$

Hereafter, we assume that $m \geq m_*$.

Claim 3. *For $i = 1$ or 2, there exist $u_1, u_2 \in S^m(w_i)$ such that $u_2 \in \partial S^m(w_i)$ and*

$$|f(u_1) - f(u_2)| \geq \frac{1}{4}\varepsilon.$$

Proof. Choose $v_{11}, v_{12} \in S^m(w_1)$ and $v_{21}, v_{22} \in S^m(w_2)$ such that

$$f(v_{11}) \geq (f)_{S^m(w_1)}, \quad f(v_{22}) \leq (f)_{S^m(w_2)}, \quad (v_{12}, v_{21}) \in E^*_{|w_1|+m}.$$

Since

$$f(v_{11}) - f(v_{12}) + f(v_{12}) - f(v_{21}) + f(v_{21}) - f(v_{22}) = f(v_{11}) - f(v_{22}) \geq \varepsilon,$$

(5.19) shows that, for either $i = 1$ or 2,

$$|f(v_{i1}) - f(v_{i2})| \geq \frac{1}{2}(\varepsilon - c_1\rho^m) \geq \frac{1}{4}\varepsilon.$$

Letting $u_1 = v_{i1}$ and $u_2 = v_{i2}$, we have the claim. □

Let $w = w_i$ where i is chosen in Claim 3. Exchanging f by $-f$ if necessary, we see that there exists $u_1 \in S^m(w)$ and $u_2 \in \partial S^m(w)$ such that

$$f(u_1) - f(u_2) \geq \frac{1}{4}\varepsilon.$$

Set $y_i = \pi^{m-n_*}(u_i)$ for $i = 1, 2$. Note that $y_i \in S^{n_*}(w)$. By Claim 2,

$$\min_{u \in S^{m-n_*}(y_1)} f(u) - \max_{u \in S^{m-n_*}(y_2)} f(u) \geq \frac{1}{4}\varepsilon - 2c_2\rho^{n_*} \geq \frac{1}{8}\varepsilon.$$ ■

Proof of Theorem 3.33. Let $m \geq m_*$. Then there exist $w \in T$ and $f \in \ell(S^m(w))$ satisfying the conclusions of Lemma 5.15. Set $c_0 = \max_{u \in S^{m-n_*}(y_2)} f(u)$. Define

$$h(v) = \begin{cases} 1 & \text{if } 8(f(v) - c_0) \geq \varepsilon, \\ 8\varepsilon^{-1}(f(v) - c_0) & \text{if } 0 < 8(f(v) - c_0) < \varepsilon, \\ 0 & \text{if } 8(f(v) - c_0) < 0 \end{cases}$$

for any $v \in S^m(w)$. Then $h|_{S^{n_*}(y_1)} \equiv 1$, $h|_{S^{n_*}(y_2)} \equiv 0$ and

$$\mathcal{E}_{p,m-n_*}(y_1, y_2, S^{n_*}(w)) \leq \mathcal{E}^{|w|+m}_{p,S^m(w)}(h) \leq 8^p\varepsilon^{-p}\mathcal{E}^{|w|+m}_{p,S^m(w)}(f) \leq \frac{2^{3p+1}(N_c)^2}{\sigma_{p,m}}.$$

By (3.20),

$$\mathcal{E}_{M_*,p,m-n_*} \leq c(n_*)\mathcal{E}_{p,m-n_*}(y_1, y_2, S^{n_*}(w)) \leq \frac{c(n_*)2^{3p+1}(N_c)^2}{\sigma_{p,m}}.$$

Making use of the sub-multiplicative property of $\mathcal{E}_{M_*,p,n}$, we have

$$\mathcal{E}_{M_*,p,m} \leq C\mathcal{E}_{M_*,p,n_*}\mathcal{E}_{M_*,p,m-n_*}.$$

Finally, the last two inequalities show

$$\mathcal{E}_{M_*,p,m}\sigma_{p,m} \leq C\mathcal{E}_{M_*,p,n_*}c(n_*)2^{3p+1}(N_c)^2$$

for any $m \geq m_*$, where the right-hand side is independent of m. Thus K is p-conductively homogeneous. ■

Chapter 6

Miscellanea

6.1 Uniformity of constants

In this section, we study the uniformity of conductance, Poincaré and neighbor disparity constants with respect to the structure of graphs.

Definition 6.1. (1) A pair (V, E) is called a (non-directed) *graph* if and only if V is a countable set and $E \subseteq V \times V$ such that $(u, v) \in V$ if and only if $(v, u) \in V$. For a graph (V, E), V is called the *vertices* and E is called the *edges*.

(2) Let (V, E) and (V', E') be graphs. A bijective map $\iota\colon V \to V'$ is called an *isomorphism* between (V, E) and (V', E') if "$(w, v) \in E$" is equivalent to "$(\iota(w), \iota(v)) \in E'$" for any $u, v \in V$.

(3) Let (V, E) be a graph. For $p > 0$ and $f \in \ell(V)$, define $\mathcal{E}_p^{(V,E)}(f) \in [0, \infty]$ by

$$\mathcal{E}_p^{(V,E)}(f) = \frac{1}{2} \sum_{(u,v)\in E} |f(u) - f(v)|^p.$$

(4) Let (V, E) be a graph and let $A, B \subseteq V$ with $A \cap B = \emptyset$. Define

$$\mathcal{E}_p^{(V,E)}(A, B) = \inf\{\mathcal{E}_p^{(V,E)}(f) \mid f \in \ell(V), f|_A \equiv 1, f|_B \equiv 0\}.$$

In this section, we always identify isomorphic graphs.

First, we study the uniformity of conductance constants.

Definition 6.2. For $L, N \geq 1$, define

$$\begin{aligned}\mathcal{G}_{\mathcal{E}}(L, N) = \{(V, E) \mid\ & (V, E) \text{ is a connected graph, } V = \{\mathbf{t}, \mathbf{b}\} \cup V_*, \text{where}\\ & \text{the union is a disjoint union and } \mathbf{t} \neq \mathbf{b},\ 1 \leq \#(V_*) \leq LN,\\ & \#(\{v \mid v \in E, (w, v) \in E\}) \leq L \text{ for any } w \in V_*\}.\end{aligned}$$

Since $\mathcal{G}_{\mathcal{E}}(L, N)$ is a finite set up to graph isomorphisms, we have the following theorem.

Theorem 6.3. *For any $L, N \geq 1$ and $p > 0$,*

$$0 < \inf_{(V,E)\in\mathcal{G}_{\mathcal{E}}(L,N)} \mathcal{E}_p^{(V,E)}(\{\mathbf{t}\}, \{\mathbf{b}\}) \leq \sup_{(V,E)\in\mathcal{G}_{\mathcal{E}}(L,N)} \mathcal{E}_p^{(V,E)}(\{\mathbf{t}\}, \{\mathbf{b}\}) < \infty.$$

Definition 6.4. Define

$$\underline{c}_{\mathcal{E}}(L, N, p) = \inf_{(V,E)\in\mathcal{G}_{\mathcal{E}}(L,N)} \mathcal{E}_p^{(V,E)}(\{\mathbf{t}\}, \{\mathbf{b}\})$$

and

$$\overline{c}_{\mathcal{E}}(L, N, p) = \sup_{(V,E)\in\mathcal{G}_{\mathcal{E}}(L,N)} \mathcal{E}_p^{(V,E)}(\{\mathbf{t}\}, \{\mathbf{b}\}).$$

Next we consider Poincaré constants.

Definition 6.5. For $L \geq 1$ and $N \geq 2$, define

$$\mathcal{G}(L, N) = \{(V, E) \mid (V, E) \text{ is a connected graph, } 2 \leq \#(V) \leq N, \\ \#(\{v \mid v \in V, (w, v) \in E, \}) \leq L \text{ for any } w \in V\}.$$

For a connected graph (V, E), define

$$\mathcal{P}(V, E) = \Big\{\mu \mid \mu \in V \to [0, 1], \sum_{v\in V} \mu(v) = 1\Big\}.$$

For $\mu \in \mathcal{P}(V, E)$, define

$$(f)_\mu = \sum_{v\in V} f(v)\mu(v),$$

for $f \in \ell(V)$,

$$\widetilde{\lambda}_{p,\mu}^{(V,E)} = \sup_{f\in\ell(V)} \frac{\sum_{v\in V} |f - (f)_\mu|^p \mu(v)}{\mathcal{E}_p^{(V,E)}(f)}$$

for $p > 0$.

Lemma 6.6. *Let (V, E) be a connected finite graph. Then for any $p \geq 1$,*

$$0 < \inf_{\mu\in\mathcal{P}(V,E)} \widetilde{\lambda}_{p,\mu}^{(V,E)} \leq \sup_{\mu\in\mathcal{P}(V,E)} \widetilde{\lambda}_{p,\mu}^{(V,E)} < \infty.$$

Proof. Write $\mathcal{E}_p = \mathcal{E}_p^{(V,E)}$. For any $p \geq 1$,

$$|(f)_\mu| + \mathcal{E}_p(f)^{\frac{1}{p}}$$

is a norm on $\ell(V)$. Therefore, if

$$\mathcal{F}_\mu = \{f \mid f \in \ell(V), \mathcal{E}_p(f) = 1, (f)_\mu = 0\},$$

then $\mathcal{F}_\mu$ is a compact subset of $\ell(V)$. Fix $\mu_* \in \mathcal{P}(V, E)$ and set $\mathcal{F} = \mathcal{F}_{\mu_*}$. For any $f \in \ell(V)$ with $\mathcal{E}_p(f) \neq 0$, define

$$f_* = \mathcal{E}_p(f)^{-\frac{1}{p}}(f - (f)_{\mu_*}).$$

Then $f_* \in \mathcal{F}$ and

$$\frac{\sum_{v\in V} |f(v) - (f)_\mu|^p \mu(v)}{\mathcal{E}_p(f)} = \sum_{v\in V} |f_*(v) - (f_*)_\mu|^p \mu(v).$$

Hence letting

$$F(\mu, f_*) = \sum_{v \in V} |f_*(v) - (f_*)_\mu|^p \mu(v),$$

we see that

$$\widetilde{\lambda}^{(V,E)}_{p,\mu} = \sup_{f_* \in \mathcal{F}} F(\mu, f_*).$$

Since $\mathcal{P}(V, E) \times \mathcal{F}$ is compact and $F(\mu, f_*)$ is continuous on $\mathcal{P}(V, E) \times \mathcal{F}$, it follows that

$$\begin{aligned} 0 &< \inf_{\mu \in \mathcal{P}(V,E), f_* \in \mathcal{F}} F(\mu, f_*) \le \inf_{\mu \in \mathcal{P}} \widetilde{\lambda}^{(V,E)}_{p,\mu} \le \sup_{\mu \in \mathcal{P}(V,E)} \lambda^{(V,E)}_{p,\mu} \\ &< \sup_{\mu \in \mathcal{P}(V,E), f_* \in \mathcal{F}} F(\mu, f_*) < \infty. \end{aligned}$$ ■

Since $\mathcal{G}(L, N)$ is a finite set, the above lemma implies the following theorem.

Theorem 6.7. *For $p \ge 1$,*

$$0 < \inf_{(V,E) \in \mathcal{G}(L,N), \mu \in \mathcal{P}(V,E)} \widetilde{\lambda}^{(V,E)}_{p,\mu} \le \sup_{(V,E) \in \mathcal{G}(L,N), \mu \in \mathcal{P}(V,E)} \widetilde{\lambda}^{(V,E)}_{p,\mu} < \infty.$$

Definition 6.8. Define

$$\underline{c}_\lambda(p, L, N) = \inf_{(V,E) \in \mathcal{G}(L,N), \mu \in \mathcal{P}(V,E)} \widetilde{\lambda}^{(V,E)}_{p,\mu}$$

and

$$\overline{c}_\lambda(p, L, N) = \sup_{(V,E) \in \mathcal{G}(L,N), \mu \in \mathcal{P}(V,E)} \widetilde{\lambda}^{(V,E)}_{p,\mu}.$$

Finally, we study neighbor disparity constants.

Definition 6.9. Define

$$\begin{aligned} \mathcal{G}_\sigma(L, N_1, N_2) = \big\{ &(V, E_1, \{V_i\}_{i=1}^n, E_2) \mid (V, E_1) \in \mathcal{G}(L, N_1), \\ &(\{1, \ldots, n\}, E_2) \in \mathcal{G}(L, N_2), V_i \subseteq V \text{ and } V_i \neq \emptyset \\ &\text{for any } i = 1, \ldots, n, V = \textstyle\bigcup_{i=1}^n V_i, V_i \cap V_j = \emptyset \text{ if } i \neq j \big\}. \end{aligned}$$

Let (V, E) be a graph and let $\mu \in \mathcal{P}(V, E)$. For $U \subseteq V$ and $f \in \ell(V)$, define

$$\mu(U) = \sum_{v \in U} \mu(v)$$

and

$$(f)_{U,\mu} = \frac{1}{\mu(U)} \sum_{v \in U} f(v)\mu(v)$$

if $\mu(U) > 0$. For $G = (V, E_1, \{V_i\}_{i=1}^n, E_2) \in \mathcal{G}_\sigma(L, N_1, N_2)$, $\mu \in \mathcal{P}(V, E)$ and $p \geq 1$, define $P_{G,\mu}: \ell(V) \to \ell(\{1, \ldots, n\})$ and $\sigma_{p,\mu}(G)$ by

$$(P_{G,\mu} f)(i) = (f)_{V_i,\mu}$$

for $f \in \ell(V)$ and

$$\sigma_{p,\mu}(G) = \sup_{f \in \ell(V), \mathcal{E}_p^{(V,E)}(f) \neq 0} \frac{\mathcal{E}_p^{(\{1,\ldots,n\},E_2)}(P_{G,\mu} f)}{\mathcal{E}_p^{(V,E)}(f)}.$$

Moreover, define

$$\mathcal{P}(G, \kappa) = \{\mu \mid \mu \in \mathcal{P}(V, E),\ \mu(V_i) \geq \kappa\mu(V_j) \text{ for any } i, j \in \{1, \ldots, n\}\}$$

for $\kappa \in (0, 1]$.

Theorem 6.10. *For any $p \geq 1$, $L, N_1, N_2 \geq 1$ and $\kappa \in (0, 1]$,*

$$\begin{aligned} 0 &< \inf\{\sigma_{p,\mu}(G) \mid G \in \mathcal{G}_\sigma(L, N_1, N_2), \mu \in \mathcal{P}(G, \kappa)\} \\ &\leq \sup\{\sigma_{p,\mu}(G) \mid G \in \mathcal{G}_\sigma(L, N_1, N_2), \mu \in \mathcal{P}(G, \kappa)\} < \infty. \end{aligned}$$

Proof. First fix

$$G = (V, E_1, \{V_i\}_{i=1}^n, E_2) \in \mathcal{G}_\sigma(L, N_1, N_2)$$

and fix

$$\mu_* \in \mathcal{P}(G, \kappa).$$

Define $\mathcal{F}$ as in the proof of Lemma 6.6. For any $f \in \ell(V)$, setting

$$f_* = \mathcal{E}_p(f)^{-\frac{1}{p}} \times (f - (f)_{\mu_*}),$$

we see that $f_* \in \mathcal{F}$ and

$$\frac{|(f)_{V_1,\mu} - (f)_{V_2,\mu}|^p}{\mathcal{E}_p(f)} = |(f_*)_{V_1,\mu} - (f_*)_{V_2,\mu}|^p$$

for any $\mu \in \mathcal{P}(G, \kappa)$. Let $F: \mathcal{F} \times \mathcal{P}(G, \kappa) \to \mathbb{R}$ by

$$F(f, \mu) = |(f)_{V_1,\mu} - (f)_{V_2,\mu}|.$$

Since F is continuous and $\mathcal{F} \times \mathcal{P}(G, \kappa)$ is compact,

$$\begin{aligned} 0 &< \inf_{\mu \in \mathcal{P}(G,\kappa), f \in \mathcal{F}} F(f, \mu) \leq \inf_{\mu \in \mathcal{P}(G,\kappa)} \sigma_{p,\mu}(G) \leq \sup_{\mu \in \mathcal{P}(G,\kappa), f \in \mathcal{F}} F(f, \mu) \\ &= \sup_{\mu \in \mathcal{P}(G,\kappa)} \sigma_{p,\mu}(G) < \infty. \end{aligned}$$

Now the desired statement follows by the fact that $\mathcal{G}_\sigma(L, N)$ is a finite set up to graph isomorphisms. ∎

Definition 6.11. Define

$$\underline{c}_\sigma(L, N_1, N_2, \kappa) = \inf\{\sigma_{p,\mu}(G) \mid G \in \mathcal{G}_\sigma(L, N_1, N_2), \mu \in \mathcal{P}(G, \kappa)\},$$
$$\overline{c}_\sigma(L, N_1, N_2, \kappa) = \sup\{\sigma_{p,\mu}(G) \mid G \in \mathcal{G}_\sigma(L, N_1, N_2), \mu \in \mathcal{P}(G, \kappa)\}.$$

6.2 Modification of the structure of a graph

In the original work of Kusuoka–Zhou [36], they used a subgraph of (T_n, E_n^*) to define their version of $\mathcal{E}_2^m$ in the case of the Sierpiński carpet. Namely, in our terminology, their subgraph is

$$E_n^1 = \{(u, v) \mid (u, v) \in E_1^*, \dim_H (K_v \cap K_u) = 1\}$$

and their energy is

$$\mathcal{E}_p^{1,n}(f) = \frac{1}{2} \sum_{(u,v)\in E_n^1} |f(u) - f(v)|^p$$

for $f \in \ell(T_n)$. (They only consider the case $p = 2$.) Our theory in this paper works well if we replace our energy $\mathcal{E}_p^n$ with Kusuoka–Zhou's energy $\mathcal{E}_p^{1,n}$ because they are uniformly equivalent, i.e., there exist $c_1, c_2 > 0$ such that

$$c_2\mathcal{E}_p^n(f) \le \mathcal{E}_p^{1,n}(f) \le c_2\mathcal{E}_p^n(f)$$

for any $n \ge 1$ and $f \in \ell(T_n)$. More generally, if we replace our graph (T_n, E_n^*) with a subgraph (T_n, E_n) satisfying conditions (A) and (B) below, all the results in this paper remain true except for changes in the constants.

(A) $G_n = (T_n, E_n)$ is a connected graph for each n having the following properties:
  - (i) If $(w, v) \in E_n$, then $K_w \cap K_v \neq \emptyset$.
  - (ii) If $(w, v) \in E_n$ for $n \ge 1$, then $\pi(w) = \pi(v)$ or $(\pi(w), \pi(v)) \in E_{n-1}$.
  - (iii) If $(w, v) \in E_n$ for $n \ge 1$, then there exist $w_1 \in S(w)$ and $v_1 \in S(v)$ such that $(w_1, w_2) \in E_{n+1}$.
  - (iv) For any $n \ge 0$ and $w, v \in T_n$ with $K_w \cap K_v \neq \emptyset$, there exist $w(0), \ldots, w(k) \in \Gamma_1(w)$ satisfying $w(0) = w, w(k) = v$ and $(w(i), w(i+1)) \in E_n$ for any $i = 0, \ldots, k-1$.

(B) For any $w \in T$, the graphs $(S^n(w), E_{n+|w|}^{S^n(w)})$ associated with the partition $T(w)$ of K_w satisfies the counterparts of conditions (i), (ii), (iii) and (iv) of (A).

Naturally, the graph (T_n, E_n^*) satisfies (A) and (B).

6.3 Open problems

In the final section, we gather some of open problems and future directions of our research.

1. *Regularity of $\mathcal{W}^p$ for $p \in [1, \dim_{AR}(K,d)]$*: As we have already mentioned, it is not known whether or not $C(K) \cap W^p$ is dense in $\mathcal{W}^p$ for $p \in [1, \dim_{AR}(K,d)]$. The first step should be to establish an elliptic Harnack principle for p-harmonic functions on approximating graphs and/or the limiting object $(\mathcal{W}^p, \widehat{\mathcal{E}}_p(\cdot) + \|\cdot\|_{p,\mu})$. Even in the case of $p = 2$, this problem is open except for the case of generalized Sierpiński carpets. The conjecture

$$\mathcal{W}^p \subseteq C(K) \quad \text{if and only if} \quad p > \dim_{ARC}(K,d)$$

in the introduction is closely related to this problem as well.

2. *Construction of p-form and p-Laplacian*: In this paper, we have constructed a p-energy $\widehat{\mathcal{E}}_p(f)$ but not a p-form $\widehat{\mathcal{E}}_p(f,g)$. Let

$$\Phi_p(t) = \begin{cases} |t|^{p-2}t & \text{if } t \neq 0, \\ 0 & \text{if } t = 0. \end{cases}$$

On a graph $G = (V, E)$, if we define

$$\mathcal{E}_p(f,g) = -\sum_{x \in V} (\Delta_p f)(x) g(x)$$

for $f, g \in \ell(V)$, where Δ_p is the p-Laplacian defined by

$$(\Delta_p f)(x) = \sum_{y \in V, (x,y) \in E} \Phi_p(f(y) - f(x)),$$

then it follows that

$$\mathcal{E}_p(f) = \frac{1}{2} \sum_{(x,y) \in E} |f(x) - f(y)|^p = \mathcal{E}_p(f,f).$$

As a natural counterpart, we expect to have a p-form $\widehat{\mathcal{E}}_p(f,g)$ which is linear in g, satisfies

$$\widehat{\mathcal{E}}_p(f) = \widehat{\mathcal{E}}_p(f,f)$$

for any $f \in \mathcal{W}^p$, and has an expression such as

$$\mathcal{E}_p(f,g) = -\int_K (\Delta_p f)(x) g(x) \mu(dx).$$

3. *Existence of p-energy measure*: In the case $p = 2$, there is the notion of energy measures associated with a strongly local regular Dirichlet form $(\mathcal{E}, \mathcal{F})$, where $\mathcal{E}$ is the form and $\mathcal{F}$ is the domain. Roughly speaking, the energy measure μ_f associated with $f \in \mathcal{F}$ is a positive Radon measure satisfying

$$\int_X u(x)\, d\mu_f(dx) = 2\mathcal{E}(uf, f) - \mathcal{E}(f^2, u)$$

for any $u \in \mathcal{F} \cap C_0(X)$. See [19] for details. So, what is a counterpart of this in the case of $\widehat{\mathcal{E}}^p$? Is there any natural measure μ_f for $f \in \mathcal{W}^p$ such that

$$\int_K d\mu_f(dx) = \widehat{\mathcal{E}}_p(f)?$$

For $\mathbb{R}^n$, the answer is yes and

$$\mu_f = |\nabla f|^p dx.$$

For the planar Sierpiński carpet, this problem has already been studied in [41]. However, we know almost nothing beyond those examples.

4. *Fractional Korevaar–Shoen type expression*: As we have already mentioned, a fractional Korevaar–Shoen type expression of $\mathcal{W}^p$ has already shown in [41] in the case of the planar Sierpiński carpet. Namely, we have

$$\mathcal{W}^p = \left\{ f \;\middle|\; f \in L^p(K, \mu),\ \overline{\lim_{r \downarrow 0}} \int_K \frac{1}{r^{\alpha_H}} \int_{B_{d_*}(x,r)} \frac{|f(x) - f(y)|^p}{r^{\beta_p}}\, dxdy < \infty \right\},$$

and it is shown in [41] that $\beta_p > p$ for any $p > 1$. How about other cases? Suppose that Assumption 2.15 holds and μ is α_H-Ahlfors regular with respect to the metric d. Then we expect that

$$\beta_p = \alpha_H + \tau_p$$

and we know

$$\alpha_H + \tau_p \geq p$$

by [34, (4.6.14)]. Now our questions are:

- Do we have a fractional Korevaar–Shoen type expression as above?
- When does $\beta_p > p$ hold? (Apparently, if $K = [-1, 1]^L$, then $\beta_p = p$.)

A related question is: If $\beta_p = p$, then does $\mathcal{W}^p$ coincide with any of the Sobolev type spaces given by approaches using upper gradients?

5. *Without local symmetry*: In Sections 4.3, 4.4, 4.5 and 4.6, we have shown the conductive homogeneity of self-similar sets having local symmetry, which helped us to extend a path from one piece of K_w to neighbors by the reflection in its boundaries.

However, the local symmetry does not seem indispensable for having conductive homogeneity. Intuitively the essence should be the balance of conductances in different directions, for example, the vertical and the horizontal directions for square-based self-similar sets. Unfortunately, we have not had any example without local symmetry yet except for finitely ramified cases.

Appendices

A Basic inequalities

The next two lemmas can be deduced from the Hölder inequality.

Lemma A.1. *For $p \in (0, \infty)$,*

$$\Big|\sum_{i=1}^{n} a_i\Big|^p \leq \max\{1, n^{p-1}\} \sum_{i=1}^{n} |a_i|^p$$

for any $n \geq 1$ and $a_1, \dots, a_n \in \mathbb{R}$.

Lemma A.2. *Let $p, q \in [1, \infty]$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Then for any $n \in \mathbb{N}$ and $a_1, \dots, a_n \in \mathbb{R}$,*

$$\Big(\sum_{i=1}^{n} |a_i|^q\Big)^{\frac{1}{q}} \leq \max\left\{1, n^{\frac{p-2}{p}}\right\}\Big(\sum_{i=1}^{n} |a_i|^p\Big)^{\frac{1}{p}}.$$

The following fact implies the comparison of two types of Poincaré constants, $\lambda_{p,m}$ and $\widetilde{\lambda}_{p,m}$, as in (5.4).

Theorem A.3 ([9, Lemma 4.17]). *Let μ be a finite measure on a set X. Then for any $f \in L^p(X, \mu)$ and $c \in \mathbb{R}$,*

$$\|f - c\|_{p,\mu} \geq \frac{1}{2}\|f - (f)_\mu\|_{p,\mu},$$

where $\|\cdot\|_{p,\mu}$ is the L^p-norm with respect to μ and $(f)_\mu = \mu(X)^{-1}\int_X f d\mu$.

The following lemma is a discrete version of the above theorem.

Corollary A.4. *Let $(\mu_i)_{i=1,\dots,n} \in (0, 1)^n$ with $\sum_{i=1}^{n} \mu_i = 1$. Then*

$$\sum_{i=1}^{n} |x - a_i|^p \mu_i \geq \Big(\frac{1}{2}\Big)^p \sum_{i=1}^{n}\Big|\sum_{j=1}^{n} \mu_j a_j - a_i\Big|^p \mu_i$$

for any $x, a_1, \dots, a_n \in \mathbb{R}$.

B Basic facts on p-energy

Let $G = (V, E)$ be a finite graph. For $A \subseteq V$, set $E_A = \{(x, y) \mid x, y \in A, (w, y) \in E\}$ and $G_A = (A, E_A)$.

Definition B.1. Let $\mu\colon V \to (0,\infty)$ and let $A \subseteq V$. Define $\mathrm{supp}(\mu) = \{x \mid x \in V, \mu(x) > 0\}$. Let $p > 0$. For $u \in \ell(V)$, define

$$\mathcal{E}_p^G(u) = \frac{1}{2} \sum_{(x,y)\in E} |u(x) - u(y)|^p,$$

$$\|u\|_{p,\mu} = \Big(\sum_{x\in V} |u(x)|^p \mu(x)\Big)^{\frac{1}{p}},$$

$$(u)_\mu = \frac{1}{\sum_{y\in V}\mu(y)} \sum_{x\in V} \mu(x)u(x)$$

and

$$\lambda_{p,\mu}^G = \sup_{u\in\ell(V),u\neq 0} \frac{(\min_{c\in\mathbb{R}} \|u - c\chi_V\|_{p,\mu})^p}{\mathcal{E}_p^G(u)},$$

where $\chi_V \in \ell(V)$ is the characteristic function of the set V.

For $A \subseteq U$, set $\mathcal{E}_p^A = \mathcal{E}_p^{G_A}$ and $\lambda_{p,\mu}^A = \lambda_{p,\mu|_A}^{G_A}$.

Lemma B.2. *Define*

$$\widetilde{\lambda}_{p,\mu}^G = \sup_{u\in\ell(V),u\neq 0} \frac{(\|u - (u)_\mu \chi_V\|_{p,\mu})^p}{\mathcal{E}_p^G(u)}.$$

Then

$$\Big(\frac{1}{2}\Big)^p \widetilde{\lambda}_{p,\mu}^G \leq \lambda_{p,\mu}^G \leq \widetilde{\lambda}_{p,\mu}^G.$$

Proof. By Corollary A.4,

$$\begin{aligned}\sum_{x\in V} |u(x) - (u)_\mu|^p \mu(x) &\geq \min_{c\in\mathbb{R}} \sum_{x\in V} |u(x) - c|^p \mu(x) \\ &\geq \Big(\frac{1}{2}\Big)^p \sum_{x\in V} |u(x) - (u)_\mu|^p \mu(x).\end{aligned}$$ ■

Lemma B.3 ([36, Proposition 1.5 (2)]). *Let $p \in [1,\infty)$ and let $\mu\colon V \to (0,\infty)$. Assume that $A \subseteq B \subseteq V$. Then for any $u \in \ell(B)$,*

$$|(u)_A - (u)_B| \leq \frac{1}{\mu(A)^{\frac{1}{p}}} \big(\widetilde{\lambda}_{p,\mu}^B \mathcal{E}_p^B(u)\big)^{\frac{1}{p}}.$$

Proof. By the Hölder inequality,

$$|(u)_A - (u)_B| \leq \frac{1}{\mu(A)} \int_B \chi_A |u - (u)_B| d\mu \leq \frac{1}{\mu(A)^{\frac{1}{p}}} \Big(\int_B |u - (u)_B|^p d\mu\Big)^{\frac{1}{p}}.$$ ■

C Useful facts on combinatorial modulus

In this appendix, we have useful facts on combinatorial modulus. In particular, the last lemma, Lemma C.4, is a result on the comparison of moduli in two different graphs. This lemma plays a key role on several occasions in this paper.

Let V be a countable set and let $\mathcal{P}(V)$ be the power set of V. For $\rho\colon V \to [0,\infty)$ and $A \subseteq V$, define

$$L_\rho(A) = \sum_{x \in A} \rho(x).$$

For $\mathcal{U} \subseteq \mathcal{P}(V)$, define

$$\mathcal{A}(\mathcal{U}) = \{\rho \mid \rho\colon V \to [0,\infty),\ L_\rho(A) \geq 1 \text{ for any } A \in \mathcal{U}\}.$$

Moreover, for $\rho\colon V \to [0,\infty)$, define

$$M_p(\rho) = \sum_{x \in V} \rho(x)^p \quad \text{and} \quad \mathrm{Mod}_p(\mathcal{U}) = \inf_{\rho \in \mathcal{A}(\mathcal{U})} M_p(\rho).$$

Note that if $\mathcal{U} = \emptyset$, then $\mathcal{A}(\mathcal{U}) = [0,\infty)^V$ and $\mathrm{Mod}_p(\mathcal{U}) = 0$.

Lemma C.1. *Assume that $\mathcal{U}$ consists of finite sets. Then there exists $\rho_* \in \mathcal{A}(\mathcal{U})$ such that*

$$\mathrm{Mod}_p(\mathcal{U}) = M_p(\rho_*).$$

Proof. Choose $\{\rho_i\}_{i\geq 1} \subseteq \mathcal{A}(\mathcal{U})$ such that $M_p(\rho_i) \to \mathrm{Mod}_p(\mathcal{U})$ as $i \to \infty$. Since V is countable, there exists a subsequence $\{\rho_{n_j}\}_{j\geq 1}$ such that, for any $v \in V$, $\rho_{n_j}(v)$ is convergent as $j \to \infty$. Set $\rho_*(p) = \lim_{j\to\infty} \rho_{n_j}(p)$. For any $A \in \mathcal{U}$, since A is a finite set, it follows that $L_{\rho_*}(A) \geq 1$. Hence $\rho_* \in \mathcal{A}(\mathcal{U})$. For any $\varepsilon > 0$, there exists a finite set X_ε such that $\sum_{v \in X_\varepsilon} \rho_*(v)^p \geq M_p(\rho_*) - \varepsilon$. As

$$\mathrm{Mod}_p(\mathcal{U}) = \lim_{j\to\infty} M_p(\rho_{n_j}) \geq \lim_{j\to\infty} \sum_{v \in X_\varepsilon} \rho_{n_j}(v)^p,$$

we obtain $\mathrm{Mod}_p(\mathcal{U}) \geq M_p(\rho_*) - \varepsilon$ for any $\varepsilon > 0$. Hence $\mathrm{Mod}_p(\mathcal{U}) \geq M_p(\rho_*)$. On the other hand, since $\rho_* \in \mathcal{A}(\mathcal{U})$, we see $M_p(\rho_*) \geq \mathrm{Mod}_p(\mathcal{U})$. Therefore, $M_p(\rho_*) = \mathrm{Mod}_p(\mathcal{U})$. ∎

Lemma C.2. *Assume that $\mathcal{U}$ consists of finite sets. For $v \in V$, define $\mathcal{U}_v = \{A \mid A \in \mathcal{U}, v \in A\}$. Then*

$$\rho_*(v)^p \leq \mathrm{Mod}_p(\mathcal{U}_v)$$

for any $\rho_* \in \mathcal{A}(\mathcal{U})$ with $M_p(\rho_*) = \mathrm{Mod}_p(\mathcal{U})$. In particular, if $\mathcal{U}_v = \emptyset$, then

$$\rho_*(v) = 0.$$

Proof. Suppose that $\rho_* \in \mathcal{A}(\mathcal{U})$ and $M_p(\rho_*) = \mathrm{Mod}_p(\mathcal{U})$. Assume that $\mathcal{U}_v = \emptyset$ and $\rho_*(v) > 0$. Define ρ'_* by

$$\rho'_*(u) = \begin{cases} \rho_*(u) & \text{if } u \neq v, \\ 0 & \text{if } u = v. \end{cases}$$

Then $\rho'_* \in \mathcal{A}(\mathcal{U})$ and $M_p(\rho'_*) < M_p(\rho_*)$. This contradicts the fact that $M_p(\rho_*) = \mathrm{Mod}_p(\mathcal{U})$. Thus if $\mathcal{U}_v = \emptyset$, then $\rho_*(v) = 0$. Next assume that $\mathcal{U}_v \neq \emptyset$. Let $\rho_v \in \mathcal{A}(\mathcal{U}_v)$ with $M_p(\rho_v) = \mathrm{Mod}_p(\mathcal{U}_v)$. Note that such a ρ_v does exist by Lemma C.1. Define

$$\tilde{\rho}(u) = \begin{cases} \max\{\rho_*(u), \rho_v(u)\} & \text{if } u \neq v, \\ \rho_v(v) & \text{if } u = v. \end{cases}$$

Let $A \in \mathcal{U}$. If $v \notin A$, then $\tilde{\rho} \geq \rho_*$ on A, so that $\tilde{\rho} \in \mathcal{A}(A)$. If $v \in A$, then $\tilde{\rho} \geq \rho_v$ on A and hence $\tilde{\rho} \in \mathcal{A}(A)$. Thus we see that $\tilde{\rho} \in \mathcal{A}(\mathcal{U})$. Therefore,

$$\begin{aligned} \mathrm{Mod}_p(\mathcal{U}) \leq M_p(\tilde{\rho}) &\leq \sum_{u \neq v} \rho_*(u)^p + \sum_{u \in V} \rho_v(u)^p \\ &= \mathrm{Mod}_p(\mathcal{U}) - \rho_*(v)^p + \mathrm{Mod}_p(\mathcal{U}_v). \end{aligned}$$ ■

Define $\ell_+(V) = \{f \mid f : V \to [0, \infty)\}$.

Lemma C.3. *Let V_1 and V_2 be finite sets. Let $\mathcal{U}_i \subseteq \mathcal{P}(V_i)$ for $i = 1, 2$. If there exist maps $\xi : \mathcal{U}_2 \to \mathcal{U}_1$, $F : \ell_+(V_1) \to \ell_+(V_2)$ and constants $C_1, C_2 > 0$ such that*

$$C_1 L_{F(\rho)}(\gamma) \geq L_\rho(\xi(\gamma)) \quad \text{and} \quad M_p(F(\rho)) \leq C_2 M_p(\rho)$$

for any $\rho \in \ell_+(V_1)$ and $\gamma \in \mathcal{U}_2$, then

$$\mathrm{Mod}_p(\mathcal{U}_2) \leq (C_1)^p C_2 \mathrm{Mod}_p(\mathcal{U}_1)$$

for any $p > 0$.

Proof. Note that $C_1 F(\rho) \in \mathcal{A}(\mathcal{U}_2)$ for any $\rho \in \mathcal{A}(\mathcal{U}_1)$. Hence if $F'(\rho) = C_1 F(\rho)$, then

$$\begin{aligned} \mathrm{Mod}_p(\mathcal{U}_2) = \min_{\rho \in \mathcal{A}(\mathcal{U}_2)} M_p(\rho) &\leq \min_{\rho \in \mathcal{A}(\mathcal{U}_1)} M_p(F'(\rho)) \\ &= \leq (C_1)^P C_2 \min_{\rho \in \mathcal{A}(\mathcal{U}_1)} M_p(\rho) (C_1)^P C_2 \mathrm{Mod}_p(\mathcal{U}_1). \end{aligned}$$ ■

Lemma C.4. *Let V_1 and V_2 be countable sets and let $\mathcal{U}_i \subseteq \mathcal{P}(V_i)$ for $i = 1, 2$. Assume that $H_v \subseteq V_1$ and $\#(H_v) < \infty$ for any $v \in V_2$. Furthermore, assume that, for any $B \in \mathcal{U}_2$, there exists $A \in \mathcal{U}_1$ such that $A \subseteq \bigcup_{v \in B} H_v$. Then*

$$\mathrm{Mod}_p(\mathcal{U}_2) \leq \sup_{v \in V_2} \#(H_v)^p \sup_{u \in V_1} \#(\{v \mid v \in V_2, u \in H_v\}) \mathrm{Mod}_p(\mathcal{U}_1)$$

for any $p > 0$.

Proof. For $\rho\colon V_1 \to \mathbb{R}$, define

$$F(\rho)(v) = \max_{u \in H_v} \rho(u)$$

for any $v \in V_2$. Then $F\colon \ell_+(V_1) \to \ell_+(V_2)$ and

$$\begin{aligned} M_p(F(\rho)) &= \sum_{v \in V_2} \max_{u \in H_v} \rho(u)^p \leq \sum_{v \in V_2} \sum_{u \in H_v} \rho(u)^p \\ &\leq \sup_{u \in V_1} \#(\{v \mid v \in V_2, u \in H_v\}) M_p(\rho). \end{aligned}$$

On the other hand, for $B \in \mathcal{U}_2$, choose $\xi(B) \in \mathcal{U}_1$ such that $\xi(B) \subseteq \bigcup_{v \in B} H_v$. Then for any $\rho \in \ell_+(V_1)$ and $B \in \mathcal{U}_2$,

$$\begin{aligned} \sup_{u \in V_2} \#(H_u) L_{F(\rho)}(B) &\geq \sum_{u \in B} \#(H_u) F(\rho)(u) \geq \sum_{u \in B} \sum_{v \in H_u} \rho(v) \\ &= \sum_{v \in \bigcup_{u \in B} H_u} \#(\{u \mid v \in H_u\}) \rho(v) \\ &\geq \sum_{v \in \xi(B)} \rho(v) = L_\rho(\xi(B)). \end{aligned}$$

Hence by Lemma C.3, we have the desired conclusion. ∎

D An Arzelà–Ascoli theorem for discontinuous functions

The following lemma is a version of Arzelà–Ascoli theorem showing the existence of a uniformly convergent subsequence of a sequence of functions. The difference between the original version and the current one is that it can handle a sequence of discontinuous functions.

Lemma D.1 (Extension of Arzelà–Ascoli). *Let (X, d_X) be a totally bounded metric space and let (Y, d_Y) be a metric space. Let $u_i\colon X \to Y$ for any $i \geq 1$. Assume that there exist a monotonically increasing function $\eta\colon [0, \infty) \to [0, \infty)$ and a sequence $\{\delta_i\}_{i \geq 1} \in [0, \infty)$ such that $\eta(t) \to 0$ as $t \downarrow 0$, $\delta_i \to 0$ as $i \to \infty$ and*

$$d_Y(u_i(x_1), u_i(x_2)) \leq \eta(d_X(x_1, x_2)) + \delta_i \tag{D.1}$$

for any $i \geq 1$ and $x_1, x_2 \in X$. If $\overline{\bigcup_{i \geq 1} u_i(X)}$ is compact, then there exists a subsequence $\{u_{n_j}\}_{j \geq 1}$ such that $\{u_{n_j}\}_{j \geq 1}$ converges uniformly to a continuous function $u\colon X \to Y$ as $j \to \infty$ satisfying $d_Y(u(x_1), u(x_2)) \leq \eta(d_X(x_1, x_2))$ for any $x_1, x_2 \in X$.

Proof. Since X is totally bounded, there exists a countable subset $A \subseteq X$ which is dense in X and contains a finite τ-net A_τ of X for any $\tau > 0$. Let $K = \overline{\bigcup_{i \geq 1} u_i(X)}$.

Since K is compact and $\{u_i(x)\}_{i\geq 1} \subseteq K$ is bounded for any $x \in A$, there exists a subsequence $\{u_{m_k}(x)\}_{k\geq 1}$ converging as $k \to \infty$. By the standard diagonal argument, we may find a subsequence $\{u_{n_j}\}_{j\geq 1}$ such that $\{u_{n_j}(x)\}_{j\geq 1}$ converges as $j \to \infty$ for any $x \in A$. Set $v_j = u_{n_j}$ and $\alpha_j = \delta_{n_j}$. Define $v(x) = \lim_{j\to\infty} v(x)$ for any $x \in A$. By (D.1),

$$d_Y(v_j(x_1), v_j(x_2)) \leq \eta(d_X(x_1, x_2)) + \alpha_j$$

for any $x_1, x_2 \in A$. Letting $j \to \infty$, we see that

$$d_Y(v(x_1), v(x_2)) \leq \eta(d_X(x_1, x_2)) \tag{D.2}$$

for any $x_1, x_2 \in A$. Since A is dense in X, v is extended to a continuous function on X satisfying (D.2) for any $x_1, x_2 \in X$. Fix $\varepsilon > 0$. Choose $\tau > 0$ such that $\eta(\tau) < \frac{\varepsilon}{3}$. Since the τ-net A_τ is a finite set, there exists k_0 such that if $k \geq k_0$, then $\alpha_k < \frac{\varepsilon}{3}$ and $d_Y(v(z), v_k(z)) < \varepsilon$ for any $z \in A_\tau$. Let $x \in X$ and choose $z \in A_\tau$ such that $d_X(x, z) < \tau$. If $k \geq k_0$, then

$$\begin{aligned} d_Y(v_k(x), v(x)) &\leq d_Y(v_k(x), v_k(z)) + d_Y(v_k(z), v(z)) + d_Y(v(z), v(x)) \\ &\leq 2\eta(d_X(x, z)) + \alpha_k + d_Y(v_k(z), v(z)) < 2\varepsilon. \end{aligned}$$

Thus $\{v_j\}_{j\geq 1}$ converges uniformly to v as $j \to \infty$. ∎

E Geometric properties of strongly symmetric self-similar sets

In this appendix, we will give proofs of claims on topological and geometric properties of self-similar sets treated in Section 4.6. Namely, we will give proofs of Propositions 4.40 and 4.42. First, we recall the setting of Section 4.6. Let S be a finite subset of $\mathbb{R}^L$ and let $\rho \in (0, 1)$. Let $U_q \in O(L)$ for any $q \in S$. Define $f_q\colon \mathbb{R}^L \to \mathbb{R}^L$ by

$$f_q(x) = \rho U_q(x - q) + q$$

for $x \in \mathbb{R}^L$. Let K be the self-similar set with respect to $\{f_q\}_{q\in S}$, i.e., K is the unique non-empty compact set K satisfying

$$K = \bigcup_{q\in S} f_q(K).$$

The triple $(K, S, \{f_q\}_{q\in S})$ is know to be a self-similar structure defined in Definition 4.1 and the map $\chi\colon S^{\mathbb{N}} \to K$ is given by

$$\{\chi(q_1 q_2 \ldots)\} = \bigcap_{m\geq 0} f_{q_1\ldots q_m}(K)$$

as we have seen in Section 4.1.

Definition E.1. (1) Define $\widetilde{\sigma}\colon S^{\mathbb{N}} \to S^{\mathbb{N}}$ by

$$\widetilde{\sigma}(q_1 q_2 \dots) = q_2 q_3 \dots \quad \text{for } q_1 q_2 \dots \in S^{\mathbb{N}}.$$

(2) Define

$$C_K = \bigcup_{i \neq j \in S} K_i \cap K_j, \quad \mathcal{C} = \chi^{-1}(C_K), \quad \mathcal{P} = \bigcup_{k \geq 1} \widetilde{\sigma}^k(\mathcal{C}),$$

and $V_0 = \chi(\mathcal{P})$. The sets $\mathcal{C}$ and $\mathcal{P}$ are called the critical set and the post critical set of $(K, S, \{f_q\}_{q \in S})$, respectively. A self-similar structure $(K, S, \{f_q\}_{q \in S})$ is said to be post critically finite (p.c.f. for short) if $\mathcal{P}$ is a finite set.

By [29, Theorem 1.2.3], we have the following proposition.

Proposition E.2. *The map χ is continuous and surjective. Moreover,*

$$\chi(q_1 q_2 \dots) = f_{q_1}(\chi(\widetilde{\sigma}(q_1 q_2 \dots))) \tag{E.1}$$

for any $q_1 q_2 \dots \in S^{\mathbb{N}}$.

In this appendix, we suppose that Assumption 4.39 holds.

The next lemma gives a proof of Proposition 4.40.

Lemma E.3. *Under Assumption 4.39, we have*

(1) *For any $q \in S$, $\chi^{-1}(q) = \overline{q}$, where $\overline{q} = qqq \dots \in S^{\mathbb{N}}$.*

(2) *$\mathcal{P} = \{\overline{q} \mid q \in U\}$, where U is the set appearing in Assumption 4.39. In particular, the self-similar structure $(K, S, \{f_q\}_{q \in S})$ is post critically finite and $V_0 = U$.*

Proof. (1) Suppose $\chi(\tau_1 \tau_2 \dots) = q$. Then by (E.1),

$$q = \chi(\tau_1 \tau_2 \dots) = f_{\tau_1}(\chi(\tau_2 \tau_3 \dots)) \in K_{\tau_1}.$$

By Assumption 4.39 (1), it follows that $\tau_1 = q$. Since f_q is invertible, we see that $\chi(\tau_2 \tau_3 \dots) = q$. Using the same argument as above, we see that $\tau_2 = q$ as well. Thus we deduce that $\tau_k = q$ for any $k \in \mathbb{N}$ inductively.

(2) Suppose that $\chi(\tau_1 \tau_2 \dots) \in f_{\tau_1}(K) \cap f_q(K)$ for some $q \neq \tau_1$. By (E.1), it follows that $\chi(\tau_1 \tau_2 \dots) = f_{\tau_1}(\chi(\tau_2 \tau_3 \dots))$. Hence by Assumption 4.39 (2),

$$\chi(\tau_2 \tau_3 \dots) \in (f_{\tau_1})^{-1}(f_{\tau_1}(K) \cap f_q(K)) \subseteq U.$$

Thus $\tau_2 \tau_3 \dots = \overline{q'}$ for some $q' \in U$. Therefore, $\mathcal{P} \subseteq U$.

Conversely, again by Assumption 4.39 (2), for any $q \in U$, there exist $p_1, p_2 \in S$ with $p_1 \neq p_2$ such that $\chi(p_1 \overline{q}) \in f_{p_1}(K) \cap f_{p_2}(K)$. This shows that $p_1 \overline{q} \in \mathcal{C}$ and hence $\overline{q} \in \mathcal{P}$. ∎

In the next two lemmas, we are going to show a sufficient condition for Assumption 4.41.

Lemma E.4. *Suppose that Assumption 4.39 holds and that U_q is the identity map for any $q \in V_0$. Let $q = f_{p_1}(q_1) = f_{p_2}(q_2)$ for some $p_1, p_2 \in S$ with $p_1 \neq p_2$ and $q_1, q_2 \in V_0$. Then there exists $\gamma = \gamma(p_1, p_2, q_1, q_2) > 0$ such that*

$$d(\overline{K_{p_1} \backslash K_{p_1(q_1)^{m-1}}}, K_{p_2}) \geq \gamma \rho^m$$

for any $m \geq 1$, where $d(A, B) = \inf_{x \in A, y \in B} |x - y|$ and $(q)^k = \underbrace{q \dots q}_{k\text{-times}} \in T_k$.

In the following proof, we assume that

$$\#(f_{p_1}(K) \cap f_{p_2}(K)) \leq 1$$

to avoid a non-essential complication of arguments. Without this assumption, the lemma is still true with a technical modification of the proof.

Proof. Set $c_m = \inf\{d(K_w, K_v) \mid w, v \in T_m, K_w \cap K_v = \emptyset\}$. Define

$$X_m = \overline{K_{p_1} \backslash K_{p_1(q_1)^{m-1}}} \quad \text{and} \quad Y_m = \overline{K_{p_1 q_1} \backslash K_{p_1(q_1)^{m-1}}}$$

for $m \geq 1$. Then $X_m = Y_m \cup (\bigcup_{q \neq q_1} K_{p_1 q})$ and $K_{p_2} = K_{p_2 q_2} \cup (\bigcup_{q \neq q_2} K_{p_2 q})$. This implies that

$$d(X_m, K_{p_2}) \geq \min\{d(Y_m, K_{p_2 q_2}), c_2\}.$$

On the other hand, letting $f(x) = \rho(x - q) + q$, we see that

$$Y_m \cup K_{p_2 q_2} = f(X_{m-1} \cup K_{p_2}).$$

This yields $d(Y_m, K_{p_2 q_2}) = \rho d(X_{m-1}, K_{p_2})$. Consequently, we have

$$d(X_m, K_{p_2}) \geq \min\{\rho d(X_{m-1}, K_{p_2}), c_2\}.$$

Now inductive argument suffices. ∎

Lemma E.5. *Suppose that Assumption 4.39 holds and that U_q is the identity map for any $q \in V_0$. Then Assumption 4.41 holds.*

Remark. According to the notation in the proof of Lemma E.4, this lemma claims $c_m \geq c\rho^m$ for any $m \geq 1$.

Proof. Suppose that $w, v \in T_m$ and $K_w \cap K_v = \emptyset$. Let $w = w_1 \dots w_m$ and let $v = v_1 \dots v_m$. In the case $w_1 = w_2$,

$$d(K_w, K_v) = \rho d(K_{w_2 \dots w_m}, K_{v_2 \dots v_m}) \geq c_{m-1} \rho.$$

Otherwise, assume that $w_1 \neq v_1$. If $K_{w_1} \cap K_{v_1} = \emptyset$, then $d(K_w, K_v) \geq c_1$. So, the remaining possibility is that $w_1 \neq v_1$ and $K_{w_1} \cap K_{v_1} \neq \emptyset$. In this case, let $q = K_{w_1} \cap K_{v_1}$. Then $q = f_{w_1}(p_{j_1}) = f_{w_2}(p_{j_2})$ for some $j_1, j_2 \in \{1, \ldots, L\}$. By Lemma E.4, it follows that $d(K_w, K_v) \geq \overline{\gamma}\rho^m$, where $\overline{\gamma} = \min\{\gamma(p_1, p_2, q_1, q_2) \mid p_1, p_2 \in S, q_1, q_2 \in V_0, f_{p_2}(q_1) = f_{p_1}(q_2)\}$. Combining all the cases and using induction on m, we see that $c_m \geq \min\{c_1, \overline{\gamma}\}\rho^m$ for any $m \geq 1$. ■

Now we start showing Proposition 4.42, that is, Assumption 2.15 holds under Assumptions 4.39 and 4.41.

Lemma E.6. *Under Assumptions 4.39 and 4.41, Assumption 2.15 (2) holds with $r = \rho$, $M_* = 1$, and $d = d_*$, where d_* is the restriction of the Euclidean metric.*

Proof. (2A) is obvious. Set

$$\Gamma_{1,n}(x) = \bigcup_{\substack{w \in T_n \\ x \in K_w}} \Gamma_1(w)$$

for $x \in K$ and $n \geq 1$. Then for any $v \in T_n \backslash \Gamma_{1,n}(x)$, there exists $w \in T_n$ such that $x \in K_w$ and $K_w \cap K_v = \emptyset$. By Lemma E.5, we see that $d(K_w, x) \geq c\rho^n$ and hence $B_{d_*}(x, cr^n) \cap K_v = \emptyset$. Thus we have

$$B_{d_*}(x, c\rho^n) \subseteq U_1(x : n). \tag{E.2}$$

On the other hand, by (2A), there exists $c' > 0$ such that $\mathrm{diam}(K_w, d_*) \leq c'\rho^{|w|}$ for any $w \in T$. This implies

$$U_1(x : n) \subseteq B_{d_*}(x, 3c'\rho^n). \tag{E.3}$$

So we have (2B). Choose $x_0 \in K\backslash V_0$ and choose $m_0 \in \mathbb{N}$ such that $2\rho^{m_0} < d(x_0, V_0)$. Let $w \in T_n$ and let $u \in \Gamma_{1,m_0+n}(f_w(x_0))$. Suppose that $u \in T(v)$ for some $v \in T_n$ with $v \neq w$. Since $u \in \Gamma_{1,m_0+n}(f_w(x_0))$, there exists $u_0 \in T_{n+m_0}$ such that $f_w(x_0) \in K_{u_0}$ and $K_{u_0} \cap K_u \neq \emptyset$. Let $y \in K_u$. Since K is connected (and hence arcwise connected by [29, Theorem 1.6.2]), there exists a continuous curve $\zeta\colon [0, 1] \to K_{u_0} \cup K_u$ such that $\zeta(0) = f_w(x_0)$ and $\zeta(1) = y$. Note that $f_w(x_0) \in K_w$ and $y \in K_v$. By (4.24), the curve ζ intersects with $f_w(V_0)$. Therefore, $(K_u \cup K_{u_0}) \cap f_w(V_0) \neq \emptyset$. However, since $\mathrm{diam}(K_u, d_*) = \mathrm{diam}(K_{u_0}, d_*) = \rho^{m_0+n}$, it follows

$$d(f_w(x_0), K_u \cup K_{u_0}) \leq 2\rho^{m_0+n} < d(f_w(x_0), f_w(V_0)),$$

so that $(K_{u_0} \cup K_u) \cap f_w(V_0) = \emptyset$. This contradiction shows that $u \in T(w)$ and hence $U_1(f_w(x_0) : m_0 + n) \subseteq K_w$. By (E.2), we see that

$$B_{d_*}(f_w(x_0), c\rho^{m_0+n}) \subseteq U_1(f_w(x_0) : m_0 + n) \subseteq K_w.$$

This shows (2C). ■

Next set $\alpha_H = -\frac{\log \#(S)}{\log \rho}$. Note that $\rho^{\alpha_H} = \#(S)^{-1}$. Let μ be the self-similar measure on K with weight $(\rho^{\alpha_H}, \dots, \rho^{\alpha_H})$. By [31, Theorem 1.2.7], we see that $\mu(K_w) = \rho^{|w|}$ for any $w \in T$ and consequently $\mu(\{x\}) = 0$ for any $x \in K_w$. These facts show that μ satisfies Assumption 2.12. Moreover, we have the following proposition.

Proposition E.7. *Under Assumptions 4.39 and 4.41, there exist $c_1, c_2 > 0$ such that*

$$c_1 s^{\alpha_H} \leq \mu(B_{d_*}(x, s)) \leq c_1 s^{\alpha_H} \tag{E.4}$$

for any $s \in [0, 1]$. In particular, μ is α_H-Ahlfors regular with respect to d_* and the Hausdorff dimension of (K, d_*) equals α_H.

Proof. By (E.3), for any $x \in K$ and $n \geq 1$, if $w \in \Gamma_{1,n}(x)$, then

$$(\rho^n)^{\alpha_H} = \mu(K_w) \leq \mu(B_{d_*}(x, 3c'\rho^n)). \tag{E.5}$$

On the other hand, by [31, Proposition 1.6.11], there exists $J_* \in \mathbb{N}$ such that

$$\#(\Gamma_{1,n}(x)) \leq J_* \tag{E.6}$$

for any $x \in T$ and $n \geq 0$. (Note that $\Lambda^1_{\rho^n, x}$ defined in [31, Definition 1.3.3] equals $\Gamma_{1,n}(x)$.) Therefore by (E.2),

$$\mu(B_{d_*}(x, c\rho^n)) \leq \sum_{v \in \Gamma_{1,n}(x)} \mu(K_v) \leq J_*(\rho^n)^{\alpha_H}. \tag{E.7}$$

Combining (E.5) and (E.7), we obtain (E.4). ∎

The following proposition is immediately deduced from the previous propositions and lemmas. Note that $\Gamma_1(w) \subseteq \Gamma_{1,n}(x)$ for any $w \in T$ and $x \in K_w$. Hence by (E.6), we see that the partition $\{K_w\}_{w \in T}$ is uniformly finite.

Proposition E.8 (Proposition 4.42). *Under Assumptions 4.39 and 4.41, Assumption 2.15 holds with $r = \rho$, $d = d_*$ and $M_* = M_0 = 1$.*

The fact that $M_0 = 1$ is due to the second remark after Assumption 2.6.

F List of definitions and notations

Definitions

Notations

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Conductive Homogeneity of Compact Metric Spaces and Construction of p-Energy

In the ordinary theory of Sobolev spaces on domains of $\mathbb{R}^n$, the p-energy is defined as the integral of $|\nabla f|^p$. In this paper, we try to construct a p-energy on compact metric spaces as a scaling limit of discrete p-energies on a series of graphs approximating the original space. In conclusion, we propose a notion called conductive homogeneity under which one can construct a reasonable p-energy if p is greater than the Ahlfors regular conformal dimension of the space. In particular, if $p = 2$, then we construct a local regular Dirichlet form and show that the heat kernel associated with the Dirichlet form satisfies upper and lower sub-Gaussian type heat kernel estimates. As examples of conductively homogeneous spaces, we present new classes of square-based self-similar sets and rationally ramified Sierpiński crosses, where no diffusions were constructed before.

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