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Purely Arithmetic PDEs Over a p-Adic Field: δ-Characters and δ-Modular Forms

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Abstract

A formalism of arithmetic partial differential equations (PDEs) is being developed in which one considers several arithmetic differentiations at one fixed prime. In this theory solutions can be defined in algebraically closed p-adic fields. As an application we show that for at least two arithmetic directions every elliptic curve possesses a non-zero arithmetic PDE Manin map of order 1; such maps do not exist in the arithmetic ODE case. Similarly, we construct and study "genuinely PDE" differential modular forms. As further applications we derive a Theorem of the kernel and a Reciprocity theorem for arithmetic PDE Manin maps and also a finiteness Diophantine result for modular parameterizations. We also prove structure results for the spaces of "PDE differential modular forms defined on the ordinary locus." We also produce a system of differential equations satisfied by our PDE modular forms based on Serre and Euler operators.

Keywords. Arithmetic differential equations, modular forms, quasi-canonical lifts, overconvergence

Mathematics Subject Classification (2020). Primary 11F32; Secondary 11F85, 11G07, 11G18

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Contents

Chapter 1

Introduction

Arithmetic analogues of ordinary differential equations were introduced in [6]. The role of functions of one variable t was played by elements of the completed valuation ring R of the maximal unramified extension of $\mathbb{Q}_p$. The role of the derivation $\frac{d}{dt}$ was played by the *p-derivation* $\delta_p = \delta : R \to R$ defined by

$$\delta a := \frac{\phi(a) - a^p}{p}, \quad a \in R,$$

where $\phi : R \to R$ is the unique Frobenius lift on R. One interpreted δ as an "arithmetic differentiation" with respect to the "arithmetic direction" p. This theory was successfully applied to a series of problems in Diophantine geometry [7–10, 13]. For example a special case of the main results of [13] states that for the modular curve $X := X_1(N)$ over R (with $N \geq 4$ coprime to p) and an elliptic curve E over R if $\Theta\colon X \to E$ is any surjective morphism then the intersection the CL-locus of X with the inverse along Θ of any finite rank subgroup of $E(R)$ must be finite. This is morally done by producing interesting homomorphisms $\psi\colon E(R) \to R$ and showing that $\mathrm{CL} \cap \Theta^{-1}(\mathrm{Ker}(\psi))$ is finite where CL is the CL-locus of canonical lift points (which are the analogues, in the local setting, of CM points). The functions ψ are arithmetic versions of Manin maps ("δ-characters") of order 2 while the theory of "δ-modular forms" provides functions that vanish on CL. The δ-characters, as well as the δ-modular forms, are basic examples of *arithmetic ODEs* of order $\leq r$ where the latter are defined as R-valued functions on sets of R-points of schemes which are locally given, in the Zariski topology, by restricted power series in the *p-derivatives* $\delta^i a_j$ of the affine coordinates a_j of the points with $i \leq r$. In particular, this theory only applies to unramified settings and concerns a single Frobenius lift.

In this memoir, we describe a significant enhancement of the foundational theory of arithmetic differential equations. On the one hand, for any prime element π in a Galois extension of $\mathbb{Q}_p$, we consider the ramified setting $R_\pi := R[\pi]$, where the elements of R_π are viewed as analogues of functions of several variables. Also, we will consider several Frobenius lifts $\phi_{\pi,1}, \ldots, \phi_{\pi,n}$ on R_π and their corresponding *π-derivations*, $\delta_{\pi,i} : R_\pi \to R_\pi$,

$$\delta_{\pi,i} a := \frac{\phi_{\pi,i}(a) - a^p}{\pi}, \quad a \in R_\pi, \quad i \in \{1, \ldots, n\},$$

leading to an arithmetic PDE theory. We will present a series of applications. As an example for $n = 2$ and E an elliptic curve over R_π, we produce a genuinely new homomorphism $\psi\colon E(R_\pi) \to R_\pi$ which is an order 1 arithmetic PDE analogue of a Manin map, for which we can prove the following finiteness theorem replacing CL with the locus of quasi-canonical lifts considered in [20]. For an open set $X \subset X_1(X)_{R_\pi}$, denote by $\mathrm{QCL}(X(R_\pi))$ the set of *quasi-canonical lift points* in $X(R_\pi)$ i.e., points corresponding to ordinary elliptic curves whose Serre–Tate parameter is a root of unity. We prove the following finiteness result (cf. Corollary 7.69).

Theorem 1.1. *Consider a surjective morphism of R_π-schemes $\Theta : X_1(N) \to E$ and denote by $\Theta_{R_\pi} : X_1(N)(R_\pi) \to E(R_\pi)$ the induced map. There exists an open set $X \subset X_1(N)$ with non-empty reduction mod π such that for all except finitely many cosets C of* $\mathrm{Ker}(\psi)$ *in $E(R_\pi)$ the set* $\mathrm{QCL}(X(R_\pi)) \cap \Theta_{R_\pi}^{-1}(C)$ *is finite.*

As a further application we will show that the map ψ above (and other maps similar to it) satisfy a remarkable "Reciprocity theorem" as follows. Let E_0 be an ordinary elliptic curve over $R_\pi/\pi R_\pi$. For every $\alpha, \beta \in R_\pi$ with absolute value less than $p^{-\frac{1}{p-1}}$ let E_α and E_β be the elliptic curves over R_π with reduction E_0 and with logarithms of the Serre–Tate parameters equal to α and β respectively. Furthermore, let $P_{\alpha,\beta} \in E_\beta(R_\pi)$ and $P_{\beta,\alpha} \in E_\alpha(R_\pi)$ be the points whose elliptic logarithms equal α and β, respectively. Finally, let ψ_α and ψ_β be the corresponding arithmetic PDE Manin maps of order 1 attached to E_α and E_β, respectively. The Reciprocity theorem for our arithmetic Manin maps referred to above is the following statement (cf. Theorem 7.44).

Theorem 1.2. *The following equality holds,*

$$\psi_\beta(P_{\alpha,\beta}) + \psi_\alpha(P_{\beta,\alpha}) = 0.$$

1.1 Background

The present work is essentially self-contained. However, for convenience, we explain its background in what follows.

As already mentioned above, a theory of arithmetic ordinary differential equations (ODEs) was initiated in [6] and had a series of Diophantine applications; cf. [6–8, 13]. In particular, in [6] arithmetic analogues of the classical Manin maps [27] were constructed and in [6, 8] arithmetic analogues of Manin's theorem of the kernel were proved. We recall that, for a function field F, the classical Manin maps are F-valued non-linear differential operators of order 2 defined on the set of F-rational points of an abelian F-variety. Similarly, the arithmetic Manin maps in [6] had order 2 and were defined on the set of points of an abelian variety over a p-adic field. Other basic ODEs were shown to have arithmetic analogues. This is the case for Schwarzian-type ODEs satisfied by classical modular forms, cf. [3, 9] and [10, Chapter 8] where a theory of differential modular forms was developed.

In the framework of [6, 10] the only solutions of arithmetic ODEs that were defined were "unramified solutions" i.e., solutions (with coordinates) in the completion R of the maximum unramified extension of $\mathbb{Z}_p$. Subsequently, the δ-overconvergence machinery in [12, 14] allowed one to define "ramified solutions" to the main arithmetic ODEs of the theory, i.e., solutions in the ring of integers R^{alg} of the algebraic closure K^{alg} of $K := R[1/p]$ and sometimes even in the ring of integers $\mathbb{C}_p^{\circ}$ of the complex p-adic field $\mathbb{C}_p$.

A theory of arithmetic PDEs with two "directions" one of which was arithmetic and the other geometric was then developed in [17, 18]. This theory combined an arithmetic differentiation δ_p in the "arithmetic direction p" with usual differentiation $\delta_q := \frac{d}{dq}$ with respect to a "geometric direction" defined by a variable q. The two operators δ_p and δ_q were viewed as acting on the power series ring $R[\![q]\!]$ and solutions were well defined (and extensively studied) in this ring. A somewhat surprising outcome of [15] was that, in this arithmetic PDE context, analogues of Manin maps exist that have order 1 (rather than 2) and interesting interactions were found between the order 2 ODE Manin maps (both arithmetic and geometric) and the newly discovered order 1 PDE Manin maps. In some sense the existence of order 1 Manin maps was an effect of the arithmetic direction p and the geometric direction q "conspiring" to create lower order Manin maps. In [18] a theory of differential modular forms in this setting was developed. This version of the theory was an "unramified theory" in the sense that solutions were defined in $R[\![q]\!]$ and did not make sense in $R^{\text{alg}}[\![q]\!]$.

It is reasonable instead to hope for a "purely arithmetic" PDE theory i.e., a PDE theory in which all the directions are "arithmetic." Along these lines a theory of arithmetic PDEs with $n \geq 2$ arithmetic directions was developed in [4, 16] in which n arithmetic differentiation operators were attached to n distinct prime integers. In this version of the theory, the solutions of arithmetic PDEs were only defined in number fields that were unramified at the primes in question. Arithmetic Manin maps were constructed in this context using a technique introduced in [16] called *analytic continuation between primes*. The order of the arithmetic Manin maps in this setting was $2n \geq 4$; hence, in some sense, the primes involved acted as if they obstructed each other in the process of creating Manin maps.

There is a basic version of the theory that is missing from the above series of approaches, namely a purely arithmetic PDE theory where the *several arithmetic differentiations* are all attached to *one* fixed prime p. For such a theory to be relevant one needs to make sense of "ramified solutions," i.e. of solutions in R^{alg}. It is the aim of the present work to systematically develop such a theory and provide new applications. We have already made clear in the introduction that tangible Diophantine applications will come out of this enhancement. However, there is even more. Additionally, certain ODE versions of the PDEs appearing in classical Riemannian geometry related to Chern and Levi-Civita connections have been developed (cf. [11]). The fundamental framework we describe here will have ramifications in that theory as well, and will be explored in upcoming work.

1.2 Framework of this memoir

Our starting point is the observation that in the ramified p-adic world one should envision not one but many arithmetic directions reflecting the fact that the absolute Galois group of $\mathbb{Q}_p$ does not have one but several (in fact one can take 4) topological generators (cf. [22]). These topological generators can be chosen to be *Frobenius automorphisms of* $\mathbb{Q}_p^{\text{alg}}$; cf. Definition 2.3. One can then develop the theory starting from an arbitrary finite collection $\phi_1, \dots, \phi_n$ of Frobenius automorphisms of K^{alg}. Remarkably this approach, combined with the δ-overconvergence technique in [12, 14], allows one to define solutions to our equations in R^{alg}. As an application we will again construct arithmetic Manin maps which (as in [17] but unlike in [16]) have order 1; so the various arithmetic differentiation operators at p conspire, again, to create lower order arithmetic Manin maps. On the other hand for $n = 2$ one can introduce a remarkable order 2 arithmetic PDE Manin map that can be viewed as the "Laplacian" of our context. This is very different from the order 4 arithmetic Laplacian in the context of [16]. An arithmetic PDE version of the theory of differential modular forms in [3, 9, 10] will also be developed in this memoir and a series of new "purely PDE" phenomena will be put forward.

We summarize our discussion above in the following table. Here N_{pr} below is the number of primes involved, N_{ari} is the number of arithmetic directions, and N_{geo} is the number of geometric directions.

Reference	N_{pr}	N_{ari}	N_{geo}	Ramified solutions defined
[6]	1	1	0	NO
[12]	1	1	0	YES
[17]	1	1	1	NO
[16]	n	n	0	NO
This work	1	n	0	YES

1.3 Terminology

In this memoir, unless otherwise stated, all rings will be commutative with identity. A morphism of Noetherian rings will be called *smooth* if it is of finite type and is 0-smooth in the sense of [28, page 193]. Throughout this memoir we fix an odd prime $p \in \mathbb{Z}$ and for any ring S and any Noetherian scheme X we denote by $\widehat{S}$ and $\widehat{X}$ the respective p-adic completions. The superscript "alg" will mean algebraic closure.

The superscript “ur” will mean maximum unramified extension. By an *elliptic curve* over a ring we mean an abelian scheme of relative dimension one. There are two contexts in which the word “ordinary” appears in this memoir: one as in “ordinary versus partial differential equation”; and the other as in “ordinary versus supersingular elliptic curve.” To avoid confusion we will always say “ODE” instead of “ordinary” in the first situation. Also, we will often use “ODE” and “PDE” as adjectives as in “ODE arithmetic Manin maps,” “PDE differential modular forms,” etc.

1.4 Main results

In what follows we explain some of our main results including the previously mentioned theorems in more context. For the precise definitions of our concepts we refer to the body of the memoir. For simplicity we assume, for the rest of this introduction, that the number of Frobenius automorphisms is $n = 2$. Some of the results below have variants that will be proved for arbitrary n.

Let Π be the set of all prime elements π in all finite Galois extensions of $\mathbb{Q}_p$. With $R = \widehat{\mathbb{Z}_p^{\mathrm{ur}}}$ and $K = R[1/p]$ and $\pi \in \Pi$ as above let $R_\pi := R[\pi]$ and $K_\pi := K(\pi)$. Recall from [6] that a *π-derivation* on a flat R_π-algebra A is a map $\delta_\pi : A \to A$ such that the map $\phi : A \to A$ defined by $\phi(x) = x^p + \pi\delta_\pi(x)$ is a ring homomorphism which is then referred to as a *π-Frobenius lift*. We fix a pair $\Phi = (\phi_1, \phi_2)$ of Frobenius automorphisms of K^{alg}; the automorphisms ϕ_1, ϕ_2 induce *π-derivations* on R_π. For any smooth scheme X over R_π we will define a sequence of p-adic formal schemes $J^r_{\pi,\Phi}(X)$ called the *partial π-jet spaces* of X. The ring of functions on $J^r_{\pi,\Phi}(X)$ will be referred to as the ring of (purely) *arithmetic PDEs* on X order $\leq r$ (cf. Definition 2.25). We will then define its subring of *totally δ-overconvergent* elements (cf. Definition 2.28). There is a natural action of ϕ_1, ϕ_2 on the colimit as $r \to \infty$ of these rings. Every arithmetic PDE f on X defines a map of sets $f_{R_\pi} : X(R_\pi) \to R_\pi$. If f is totally δ-overconvergent then the map f_{R_π} extends to a map of sets $f^{\mathrm{alg}} := f^{\mathrm{alg}}_{R_\pi} : X(R^{\mathrm{alg}}) \to K^{\mathrm{alg}}$ and the preimage of 0 under this map is the *set of solutions* in R^{alg} of the arithmetic PDE f.

Let E be an elliptic curve over R_π. We define a *partial δ_π-character* of order $\leq r$ on E to be an arithmetic PDE of order $\leq r$ which, viewed as a morphism $J^r_{\pi,\Phi}(E) \to \widehat{\mathbb{G}_a}$, is a group homomorphism; cf. Definition 3.1. Extending terminology from [6] “partial δ_π-characters” is the name for our “arithmetic Manin maps” in our PDE setting. To each E and every basis ω for the 1-forms on E we will attach two families of elements in R_π called (primary, respectively secondary) *arithmetic Kodaira–Spencer classes*; cf. Definitions 5.5 and 5.15. Finally, to each partial δ_π-character of E we will attach a *Picard–Fuchs symbol* which is a formal K_π-linear combination of non-commutative monomials in ϕ_1, ϕ_2; cf. Definition 3.7. The arithmetic Kodaira–Spencer classes appear then as coefficients of the symbols of certain distinguished partial δ_π-characters. Among the primary Kodaira–Spencer classes a special role will be played by elements denoted by $f_1, f_2 \in R_\pi$. One of our main results will be the following (cf. Corollaries 5.14, 3.9 and Proposition 3.13). This can be viewed as a simultaneous generalization of the main results in [6] and [12].

Theorem 1.3. *Let E be an elliptic curve over R_π.*

(1) *If $f_1 \neq 0$ or $f_2 \neq 0$ then the R_π-module of partial δ_π-characters of order $\leq r$ has rank equal to $2^{r+1} - 3$.*

(2) *If $f_1 = f_2 = 0$ then the R_π-module of partial δ_π-characters of order $\leq r$ has rank equal to $2^{r+1} - 2$.*

(3) *Every partial δ_π-character ψ is totally δ-overconvergent and the induced group homomorphism $\psi^{\text{alg}} : E(K^{\text{alg}}) \to K^{\text{alg}}$ can be extended to a continuous homomorphism $\psi^{\mathbb{C}_p} : E(\mathbb{C}_p) \to \mathbb{C}_p$. If ϕ_1, ϕ_2 are monomially independent then ψ is uniquely determined by ψ^{alg}.*

The homomorphisms ψ^{alg} are not given by algebraic (or even by analytic) functions in the coordinates but rather by analytic (in fact rigid analytic) functions in the coordinates and their various "$\delta_{\pi'}$-derivatives" for various π''s dividing π. The recipe for defining these maps involves the notion of total δ-overconvergence which is analogous to the one in [12] and will be explained in the body of the memoir. Note that since E is projective over R_π we have $E(K^{\text{alg}}) = E(R^{\text{alg}})$; however, it is an important feature of the theory that the images of the maps ψ^{alg} are not contained in R^{alg}.

For the case of order ≤ 2 we have more precise results. Consider the subset $\mathbb{M}_2^{2,+} := \{1, 2, 11, 22, 12, 21\}$ of non-empty words of length ≤ 2 in the free monoid with identity generated by the set $\{1, 2\}$. There are 6 primary Kodaira–Spencer classes of order ≤ 2,

$$f_\mu, \quad \mu \in \mathbb{M}_2^{2,+}. \tag{1.1}$$

The classes f_1, f_2, f_{11}, f_{22} come from the ODE theory [10]. On the other hand the classes f_{12}, f_{21} are "genuinely PDE" (not "reducible to ODEs"). The secondary Kodaira–Spencer classes will be denoted by

$$f_{\mu,\nu}, \quad \mu, \nu \in \mathbb{M}_2^{2,+}, \quad \mu \neq \nu. \tag{1.2}$$

They satisfy $f_{\mu,\nu} + f_{\nu,\mu} = 0$. The classes $f_{11,1}, f_{22,2}$ come from the ODE theory while the others classes $f_{\mu,\nu}$ are, again, "genuinely PDE". By the theory in [10] the secondary classes $f_{ii,i}$, $i \in \{1, 2\}$, are known to be expressible in terms of the primary ones f_i as $f_{ii,i} = p\phi_i f_i$; cf. Remark 7.17. Note that if E has ordinary reduction then $f_i = 0$ for some i if and only if $f_\mu = f_{\mu,\nu} = 0$ for all μ and ν, if and only if "the" Serre–Tate parameter of E is a root of unity; cf. Proposition 7.39. Finally, note (cf. Theorem 5.33) that there exist π, ϕ_1, ϕ_2 and a pair (E, ω) over R_π such that E has ordinary reduction and all classes (1.1) and (1.2) attached to (E, ω) are non-zero.

In case $\pi = p$ we can be more specific. Indeed, for all (E, ω) over R we have $f_1 = f_2$, $f_{11} = f_{22}$ and $f_{1,2} = f_{11,22} = 0$. (In [3] and [10] f_i was denoted by f^1 and f_{ii} was denoted by f^2.) In this case we have that $f_i = 0$ if and only if E has ordinary reduction and is a canonical lift of its reduction; cf. Remark 5.9. Also, if E comes from a curve $E_{\mathbb{Z}_p}$ over $\mathbb{Z}_p$ and has ordinary reduction but is not a canonical lift then $f_{ii} = a_p f_i \neq 0$ where $a_p \in \mathbb{Z}$ is the trace of Frobenius on the reduction mod p of $E_{\mathbb{Z}_p}$; cf. Remark 5.27.

Going back to the general situation when π is arbitrary we let $N(\pi)$ be the smallest integer $N \in \mathbb{Z}$ such that for all integers $n \geq 1$ we have $\pi^n/n \in p^{-N}\mathbb{Z}_p$; in particular $N(p) = -1$. If $\mu = i \in \{1, 2\}$ we set $\phi_\mu = \phi_i$ while for $\mu = ij$ with $i, j \in \{1, 2\}$ we set $\phi_\mu = \phi_i \phi_j$. We also set $\tilde{f}_\mu = p^{N(\pi)+1} f_\mu$. We will prove (see Corollaries 5.23 and 5.24) the following summary result, which also may be viewed as a generalization of the main results of [6].

Theorem 1.4. *Assume in Theorem 1.3 that $f_1 f_2 \neq 0$. The following hold:*

(1) *For all $\mu, \nu \in \mathbb{M}_2^{2,+}$ there is a unique δ_π-character $\psi_{\mu,\nu}$ with Picard–Fuchs symbol $\tilde{f}_\nu \phi_\mu - \tilde{f}_\mu \phi_\nu + f_{\mu,\nu}$.*

(2) *A basis modulo torsion of the R_π-module of partial δ_π-characters of order ≤ 1 consists of $\psi_{1,2}$.*

(3) *A basis modulo torsion of the R_π-module of partial δ_π-characters of order ≤ 2 consists of the elements $\psi_{1,2}$, $\phi_1 \psi_{1,2}$, $\phi_2 \psi_{1,2}$, $\psi_{11,1}$, $\psi_{22,2}$.*

Here and in the following by a *basis modulo torsion* of an R_π-module M we mean a family of elements in M inducing a basis of the K_π-linear space $M \otimes_{R_\pi} K_\pi$.

One is tempted to view $\psi_{11,22}$ as the "Laplacian" equation in our context while $\psi_{12,21}$ reflects, in some sense, the non-commutation of ϕ_1 and ϕ_2 and can be viewed as a "Poisson bracket operator." In case $f_1 = f_2 = 0$ a result similar to Theorem 1.4 will be proved; cf. Corollary 5.14.

The main flavor of our results above is "global on E". However, by looking at the completion of E at the origin, one obtains in particular the following integrality statement; cf. Corollary 5.19.

Theorem 1.5. *Let E be an elliptic curve over R_π with logarithm $\sum_{N=1}^\infty \frac{b_N}{N} T^N$, $b_N \in R_\pi$. Let $\mu, \nu \in \mathbb{M}_2^{2,+}$ and let $r, s \in \{1, 2\}$ be the lengths of the words μ, ν, respectively. Assume $r \geq s$. Then the following relations hold for all $N \geq 1$:*

$$\tilde{f}_\nu \frac{\phi_\mu(b_N)}{N} - \tilde{f}_\mu \frac{\phi_\nu(b_{p^{r-s}N})}{p^{r-s}N} + f_{\mu,\nu} \frac{b_{p^r N}}{p^r N} \in pR_\pi. \tag{1.3}$$

This integrality statement can be viewed as an analogue (for several "conjugates" of an elliptic curve) of the integrality statement of Atkin and Swinnerton-Dyer for a given elliptic curve [1, 34].

The next step in the theory will be to extend some of the theory of δ-modular forms [10] to the PDE case by defining *partial δ-modular forms* (whose weights are $\mathbb{Z}$-linear combinations of non-commutative monomials in ϕ_1, ϕ_2) and *isogeny covariance* for such forms; cf. Chapter 7. We will also attach *symbols* to isogeny covariant partial δ-modular forms for weights of degree -2; these symbols are, again, K-linear combinations of non-commutative monomials in ϕ_1, ϕ_2.

To state our main result we need to consider the standard modular curve $Y_1(N) = X_1(N) \setminus \{\text{cusps}\}$ over R_π (with $N \geq 4$ coprime to p) and the natural $\mathbb{G}_m$-bundle $B = B_1(N)$ over the $Y_1(N)$; so B classifies pairs consisting of an elliptic curve with $\Gamma_1(N)$-structure and a basis for the 1-forms. Let B_{ord} be the preimage in B of the ordinary locus in $Y_1(N)$. We will show (cf. Theorems 7.11, 7.13, 7.18, 7.19, 7.34, Proposition 7.38 and Corollary 7.30) the following characterization of these forms.

Theorem 1.6. *The following hold:*

(1) *The classes f_μ and $f_{\mu,\nu}$ are induced by isogeny covariant partial δ-modular forms, denoted by f_μ^{jet} and $f_{\mu,\nu}^{\text{jet}}$, of weight $-1-\phi_\mu$ and $-\phi_\mu - \phi_\nu$, respectively.*

(2) *There exists $c \in \mathbb{Z}_p^\times$ such that for every distinct words $\mu, \nu \in \mathbb{M}_2^{2,+}$ of lengths $r, s \in \{1, 2\}$, respectively, the symbols of f_μ^{jet} and $f_{\mu,\nu}^{\text{jet}}$ are equal to $c(\phi_\mu - p^r)$ and $c(p^s\phi_\mu - p^r\phi_\nu)$, respectively.*

(3) *The forms f_μ^{jet} and $f_{\mu,\nu}^{\text{jet}}$ naturally induce totally overconvergent arithmetic* PDE*s on B and the induced maps $B(R^{\text{alg}}) \to K^{\text{alg}}$ restricted to $B_{\text{ord}}(R^{\text{alg}})$ extend to continuous maps $B_{\text{ord}}(\mathbb{C}_p^\circ) \to \mathbb{C}_p$.*

(4) *The form $f_{1,2}^{\text{jet}}$ is a basis modulo torsion of the module of isogeny covariant partial δ-modular forms of order ≤ 1 and weight $-\phi_1 - \phi_2$.*

The forms $f_\mu^{\text{jet}}, f_{\mu,\nu}^{\text{jet}}$ in the theorem satisfy a series of cubic and quadratic relations (cf. Theorems 7.18 and 7.19). We will use these relations to determine the *Serre–Tate expansions* of the forms involved (cf. Theorem 7.28) which is what, in particular, leads to the determination of the corresponding symbols in part 2 of the theorem above. As a consequence of these computations we will derive some explicit formulae for the values of δ-characters in terms of Serre–Tate parameters. These formulae will exhibit a rather unexpected antisymmetry property that translates into a *Reciprocity theorem* similar to Theorem 1.2 and valid for arbitrary δ-characters $\psi_{\mu,\nu}$ where $\mu, \nu \in \mathbb{M}_2^{2,+}$; cf. Theorem 7.44 for details and a precise statement. The critical map ψ in Theorems 1.1 and 1.2 is the following. For $\pi \in \Pi$ and an elliptic curve E over R_π we recall our δ_π-character $\psi_{1,2}$. The map ψ in Theorems 1.1 and 1.2 is the induced group homomorphism

$$\psi_{R_\pi} := (\psi_{1,2})_{R_\pi} : E(R_\pi) \to R_\pi.$$

We also note that the proof of Theorem 1.1 utilizes a version of a classic theorem of Strassman, see Lemma 7.68. It would be immediate to conclude a uniform version of Theorem 1.1 from a uniform version of Lemma 7.68.

In addition, our considerations above will lead to an explicit description of the kernel of $\psi^{\text{alg}}_{\mu,\nu,\beta}$ in terms of β. This result can be viewed as an arithmetic PDE *Theorem of the kernel* analogue of Manin's theorem of the kernel [27] and extending the arithmetic ODE results in [6, Theorems A and B] and [8, Theorem 1.6]. This utilizes an interesting pairing defined as follows.

For μ, ν of lengths r and s respectively define the $\mathbb{Q}_p$-bilinear map

$$\langle\ ,\ \rangle_{\mu,\nu} : K^{\text{alg}} \times K^{\text{alg}} \to K^{\text{alg}}$$

by the formula

$$\langle \alpha, \beta\rangle_{\mu,\nu} = \beta^{\phi_\nu}\alpha^{\phi_\mu} - \beta^{\phi_\mu}\alpha^{\phi_\nu} + p^s(\alpha\beta^{\phi_\mu} - \beta\alpha^{\phi_\mu}) + p^r(\beta\alpha^{\phi_\nu} - \alpha\beta^{\phi_\nu}).$$

The version of the Theorem of the kernel (cf. Theorem 7.42) is as follows.

Theorem 1.7. *We have a natural group isomorphism*

$$\text{Ker}(\psi^{\text{alg}}_{\mu,\nu,\beta}) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \{\alpha \in K^{\text{alg}} \mid \langle \alpha, \beta\rangle_{\mu,\nu} = 0\}.$$

Note that for E_β ordinary with β not a root of unity we have that $\text{Ker}(\psi^{\text{alg}}_{\mu,\nu,\beta})$ (which always contains the torsion group of $E_\beta(R^{\text{alg}})$) does not reduce to the torsion group.

Another application of our theory of δ-modular forms is the construction, for every weight w, of a δ*-period map*

$$\mathfrak{p}_w : Y_1(N)(R^{\text{alg}})^{\text{ss}}_w \to \mathbb{P}^{N_w}(R^{\text{alg}})$$

where $Y_1(N)(R^{\text{alg}})^{\text{ss}}_w \subset Y_1(N)(R^{\text{alg}})$ is a natural set of *semistable* points; cf. Definition 7.35. The terminology is motivated by the following analogy with geometric invariant theory. Group actions are replaced, in our setting, with the action of Hecke correspondences and the "components" of our δ-period maps are given by isogeny covariant δ-modular forms which should be viewed as analogues of invariant sections of line bundles in geometric invariant theory. As we shall see the δ-period maps are rather non-trivial already for w of order 2 and degree 4; cf. Example 7.37. On the other hand isogeny covariance will imply the following result (cf. Theorem 7.36).

Theorem 1.8. *The δ-period maps $\mathfrak{p}_w$ are constant on prime to p isogeny classes.*

Next, as in [3, 9, 10], we will construct certain 'crystalline forms' f^{crys}_μ, $f^{\text{crys}}_{\mu,\nu}$ and prove they are proportional to the forms f^{jet}_μ, $f^{\text{jet}}_{\mu,\nu}$; cf. Corollary 7.52. In addition, we will consider δ*-modular forms on the ordinary locus*. (Such forms were called "ordinary" in [10, Chapter 8] but here we will avoid this term so that no confusion arises with its use in the ODE/PDE distinction.) Then using a crystalline construction as in loc.cit. we will completely determine the structure of the spaces of isogeny covariant δ-modular forms on the ordinary locus for the weights of degree 0 and -2; cf. Corollary 7.58.

1.5 Leitfaden

In Chapter 2, we begin by discussing Frobenius lifts and Frobenius automorphisms of K^{alg} after which we introduce partial δ_π-jet spaces which are a PDE analogue of the ODE π-jet spaces in [6]. In Chapter 3, we introduce and study δ_π-characters of group schemes as well as their Picard–Fuchs symbols. Chapter 4, is devoted to analyzing these concepts for the multiplicative group $\mathbb{G}_m$. Chapter 5, does a similar analysis for elliptic curves. Here we introduce and study the arithmetic Kodaira–Spencer classes f_μ, $f_{\mu,\nu}$ and the δ_π-characters $\psi_{\mu,\nu}$. All the above discussion is made in the context of an arbitrary number n of Frobenius automorphisms and an arbitrary order r. We next specialize our discussion of elliptic curves to the case $n = r = 2$, and we derive a series of quadratic and cubic relations satisfied by the arithmetic Kodaira–Spencer classes. Chapter 6, summarily explains how all the above theory can be developed in a "relative setting;" this is necessary for Chapter 7 where we introduce partial δ-modular forms which are a PDE version of the ODE concept introduced in [9]. The relative arithmetic Kodaira–Spencer classes define such forms. We then introduce and compute the Serre–Tate expansions of these forms, we construct our δ-period maps, and we derive the Theorem of the kernel and the Reciprocity theorem for arithmetic Manin maps. We continue by discussing the crystalline side of the story and forms on the ordinary locus, and we present a construction of finite covers defined by δ-modular forms (cf. Theorem 7.64) which is then used to prove our main Diophantine application to modular parameterizations (cf. Corollary 7.69). We end our Chapter 7 by introducing a PDE version of the ODE δ-Serre operators in [3,10]; these PDE δ-Serre operators lead to genuine (not arithmetic) PDEs satisfied by our arithmetic PDEs and can be viewed as Pfaffian systems of equations on the arithmetic jet spaces. The memoir ends with an Appendix where we briefly discuss a more general theoretical framework in which commutation relations and inversion of Frobenius lifts are "built into" our jet spaces. We will provide there some simple computations illustrating the complexity of this more general framework.

Chapter 2

Purely arithmetic PDEs

2.1 Frobenius automorphisms

We start with the following standard definition.

Definition 2.1. By a *Frobenius lift* for an A-algebra $\varphi : A \to B$ we understand a ring homomorphism $\phi : A \to B$ such that the induced homomorphism $\overline{\phi} : A/pA \to B/pB$ equals the composition of the induced homomorphism $\overline{\varphi} : A/pA \to B/pB$ with the p-power Frobenius on A/pA. If $B = A$ and $\varphi = 1_A$ we say that ϕ is a *Frobenius lift* on A.

For every, not necessarily algebraic, field extension $F \subset L$ we denote by $\mathfrak{G}(L/F)$ the group of all field automorphisms of L that are the identity on F. For every field L we denote by $\mathfrak{G}_L$ the absolute Galois group $\mathfrak{G}(L^{\text{alg}}/L)$, where L^{alg} is an algebraic closure of L.

We recall the main setting in [12]. Consider the field of p-adic numbers with absolute value $|\ |$ normalized by $|p| = p^{-1}$. Let $\mathbb{Q}_p^{\text{alg}}$ be an algebraic closure of $\mathbb{Q}_p$, let $\mathbb{Q}_p^{\text{ur}}$ be the maximum unramified extension of $\mathbb{Q}_p$ inside $\mathbb{Q}_p^{\text{alg}}$, let K be the metric completion of $\mathbb{Q}_p^{\text{ur}}$ and let K^{alg} be the algebraic closure of K in the metric completion $\mathbb{C}_p$ of $\mathbb{Q}_p^{\text{alg}}$. We still denote by $|\ |$ the induced absolute value on all of these fields. We denote by $\mathbb{Z}_p^{\text{ur}}, \mathbb{Z}_p^{\text{alg}}, R, R^{\text{alg}}, \mathbb{C}_p^{\circ}$ the valuation rings of $\mathbb{Q}_p^{\text{ur}}, \mathbb{Q}_p^{\text{alg}}, K, K^{\text{alg}}, \mathbb{C}_p$, respectively. In particular, $R := \widehat{\mathbb{Z}_p^{\text{ur}}}$. We set $k := R/pR$; so $k \simeq \mathbb{F}_p^{\text{alg}}$.

Remark 2.2. The natural ring homomorphism

$$\mathbb{Q}_p^{\text{alg}} \otimes_{\mathbb{Q}_p^{\text{ur}}} K \to K^{\text{alg}} \tag{2.1}$$

is an isomorphism. Indeed, this map is surjective because by Krasner's lemma, we have $K^{\text{alg}} := K\mathbb{Q}_p^{\text{alg}}$; cf. [5, Proposition 5, page 149]. To check that the map (2.1) is injective write and $\mathbb{Q}_p^{\text{alg}} = \bigcup F_i$ with $F_i/\mathbb{Q}_p$ finite and let $F_i^0 \subset F_i$ be the maximum unramified extension of $\mathbb{Q}_p$ contained in F_i; so F_i/F_i^0 is totally ramified and $\mathbb{Q}_p^{\text{ur}} = \bigcup F_i^0$. It is enough to check that $F_i \otimes_{F_i^0} K \to K^{\text{alg}}$ is injective for all i. To check this note that F_i/F_i^0 is generated by a root of an Eisenstein polynomial f_i in $F_i^0[x]$; but every such polynomial is an Eisenstein polynomial in $K[x]$ and so $F_i \otimes_{F_i^0} K = K[x]/(f_i)$ is a field, therefore it injects into K^{alg}.

Definition 2.3. Let L be a subfield of $\mathbb{C}_p$ containing $\mathbb{Q}_p$. A *Frobenius automorphism* of L is a continuous automorphism $\phi \in \mathfrak{G}(L/\mathbb{Q}_p)$ such that ϕ induces the p-power Frobenius on the residue field of the valuation ring of L. We denote by $\mathfrak{F}^{(1)}(L/\mathbb{Q}_p)$ the set of Frobenius automorphisms of L.

More generally the theory of the present memoir can be developed based on the set $\mathfrak{F}^{(s)}(L/\mathbb{Q}_p)$ of all continuous automorphisms $\phi \in \mathfrak{G}(L/\mathbb{Q}_p)$ such that ϕ induces the p^s-power Frobenius on the residue field of the valuation ring of L where s is some fixed positive integer; for simplicity we will not consider this more general situation in what follows.

Note that if ϕ is a Frobenius automorphism of K^{alg} then ϕ sends R into R, induces the Frobenius lift on R, and induces an automorphism of R^{alg} (which is however not a Frobenius lift on R^{alg} in the sense of Definition 2.1). Conversely, every automorphism $\phi \in \mathfrak{G}(K^{\text{alg}}/\mathbb{Q}_p)$ extending the Frobenius lift on R is a Frobenius automorphism of K^{alg}. Indeed, for every finite Galois extension L_0 of $\mathbb{Q}_p$, the field $L := L_0K$ is sent onto itself by ϕ and the absolute values $|\ |$ and $|\phi(\ )|$ on L have the same restriction to K, hence must coincide; cf. [25, page 32]; in particular ϕ is continuous and induces the p-power Frobenius on k.

The set $\mathfrak{F}^{(1)}(K^{\text{alg}}/\mathbb{Q}_p)$ is a principal homogeneous space for the absolute Galois group $\mathfrak{G}_K$ under the action given by $(\gamma, \phi) \mapsto \gamma\phi$ for $\phi \in \mathfrak{F}^{(1)}(K^{\text{alg}}/\mathbb{Q}_p)$ and $\gamma \in \mathfrak{G}_K$. On the other hand, by the fact that the homomorphism (2.1) is an isomorphism we immediately get that the restriction homomorphism $\mathfrak{G}_K \to \mathfrak{G}_{\mathbb{Q}_p^{\text{ur}}}$ an isomorphism of topological groups and the restriction map $\mathfrak{F}^{(1)}(K^{\text{alg}}/\mathbb{Q}_p) \to \mathfrak{F}^{(1)}(\mathbb{Q}_p^{\text{alg}}/\mathbb{Q}_p)$ is a bijection. Note that $\mathfrak{F}^{(1)}(\mathbb{Q}_p^{\text{alg}}/\mathbb{Q}_p)$ has a purely (topological) group characterization as a subset of $\mathfrak{G}_{\mathbb{Q}_p}$; cf. [31, Lemma 12.1.8, page 665]. The elements of $\mathfrak{F}^{(1)}(\mathbb{Q}_p^{\text{alg}}/\mathbb{Q}_p)$ are referred to in loc.cit. as *Frobenius lifts* but adopting that terminology here would conflict with our Definition 2.1.

By the way, the absolute Galois group $\mathfrak{G}_{\mathbb{Q}_p}$ is known to have 4 topological generators one of which is in $\mathfrak{F}^{(1)}(\mathbb{Q}_p^{\text{alg}}/\mathbb{Q}_p)$; the relations among these topological generators are also known, cf. [22] or [30, Theorem 7.5.10, page 360]. We say that a subset of a topological group is a set of *topological generators* if the subgroup generated by this set is dense in the group. One can easily see, by the way, that one can find a set of 4 topological generators of $\mathfrak{G}_{\mathbb{Q}_p}$ that is contained in $\mathfrak{F}^{(1)}(\mathbb{Q}_p^{\text{alg}}/\mathbb{Q}_p)$. We will not use this observation in what follows. What we will be interested in is the monoid (rather than the group) generated by our Frobenius automorphisms, as explained in the next subsection.

2.2 Monomial independence

In what follows monoids will not necessarily be commutative. Let $\mathbb{M}_n$ be the free (non-commutative) monoid with identity generated by the set $\{1, \ldots, n\}$,

$$\mathbb{M}_n := \{0\} \cup \{i_1 \ldots i_s \mid l \in \mathbb{N},\ i_1, \ldots, i_s \in \{1, \ldots, n\}\};$$

its elements will be referred to as *words*, the *length* $|\mu|$ of a word $\mu := i_1 \ldots i_s$ is defined by $|\mu| = s$, 0 is called the *empty word* and its length is defined by $|0| = 0$.

Multiplication is given by concatenation $(\mu, \nu) \mapsto \mu\nu$ and 0 is the identity element. For all $r \in \mathbb{N} \cup \{0\}$ let $\mathbb{M}_n^r$ be the set of all elements in $\mathbb{M}_n$ of length $\leq r$. Set $\mathbb{M}_n^+ := \mathbb{M}_n \setminus \{0\}$ and $\mathbb{M}_n^{r,+} := \mathbb{M}_n^r \setminus \{0\}$.

Definition 2.4. A family of distinct elements $\phi_1, \ldots, \phi_n$ in a monoid $\mathfrak{G}$ with identity 1 is called *monomially independent* if the monoid homomorphism

$$\mathbb{M}_n \to \mathfrak{G}, \quad \mu = i_1 \ldots i_l \mapsto \phi_\mu := \phi_{i_1} \ldots \phi_{i_l}, \quad \text{for } l \in \mathbb{N}, \quad \text{and } 0 \mapsto 1$$

is injective.

Remark 2.5. Note that in our notation above we have the formula $\phi_\mu \phi_\nu = \phi_{\mu\nu}$ for $\mu, \nu \in \mathbb{M}_n$. Note also that if $\mathfrak{G}$ is a group and $\phi_1, \ldots, \phi_n \in \mathfrak{G}$ are monomially independent in $\mathfrak{G}$ then the subgroup of $\mathfrak{G}$ generated by $\phi_1, \ldots, \phi_n$ is not necessarily freely generated by $\phi_1, \ldots, \phi_n$; an example that naturally occurs in our context is given in Remark 2.9.

The following lemma follows trivially from the well known "algebraic independence of field automorphisms" but, for convenience, we provide a proof.

Lemma 2.6. *Let L be a field of characteristic zero and let $\phi_1, \ldots, \phi_n$ be monomially independent elements in $\mathfrak{G}(L/\mathbb{Q})$. Let $F = F(\ldots, x_\mu, \ldots)$ be a polynomial with L-coefficients in the variables x_μ with $\mu \in \mathbb{M}_n$ and consider the function $f : L \to L$ defined by*

$$f(\lambda) = F(\ldots, \phi_\mu(\lambda), \ldots), \quad \lambda \in L.$$

Let A be a subring of L and assume $f(\lambda) = 0$ for all $\lambda \in A$. Then $F = 0$.

Proof. By Artin's independence of characters, cf. [26, page 283], if $\mathfrak{A}$ is a monoid then every family of distinct monoid homomorphisms $\mathfrak{A} \to L^\times$ is L-linearly independent in the L-linear space of all maps $\mathfrak{A} \to L$. Let $\mathfrak{A} = A \setminus \{0\}$. Then by Artin's independence of characters it is enough to check that for distinct vectors $e := (e_\mu)_{\mu \in \mathbb{M}_n}$ with entries non-negative integers, almost all zero, the maps $f_e : A \to L$ defined by

$$f_e(\lambda) := \prod_\mu (\phi_\mu(\lambda))^{e_\mu}, \quad \lambda \in A$$

are distinct. Assume $f_e = f_{e'}$ and let us show that $e = e'$. For all integers $m \in \mathbb{Z}$ we have

$$\prod_\mu (m + \phi_\mu(\lambda))^{e_\mu} = \prod_\mu (m + \phi_\mu(\lambda))^{e'_\mu}, \quad \lambda \in A.$$

Since L has characteristic zero we have an equality

$$\prod_\mu (t + \phi_\mu(\lambda))^{e_\mu} = \prod_\mu (t + \phi_\mu(\lambda))^{e'_\mu}, \quad \lambda \in A$$

in the ring of polynomials $L[t]$. Looking at degrees in t we get $\sum_\mu e_\mu = \sum_\mu e'_\mu =: d$. Picking out the coefficient of t^{d-1} we get

$$\sum_\mu e_\mu \phi_\mu(\lambda) = \sum_\mu e'_\mu \phi_\mu(\lambda), \quad \lambda \in A.$$

By monomial independence of the ϕ_i's and, again, by Artin's independence of characters, the family $(\phi_\mu)_{\mu \in \mathbb{M}_n}$ is L-linearly independent in the L-linear space of maps $\mathfrak{A} \to L$ so, since L has characteristic zero, we conclude that $e_\mu = e'_\mu$ for all μ. ■

Example 2.7. In what follows we show that the set $\mathfrak{F}^{(1)}(K^{\text{alg}}/\mathbb{Q}_p)$ of Frobenius automorphisms of K^{alg} contains large subsets of monomially independent elements that remain monomially independent on "small" (abelian) extensions of K. We recall some standard constructions from Iwasawa theory; cf. [21]. Let $l \neq p$ be a prime. Consider sequences $\pi_m \in K^{\text{alg}}$ and $\zeta_{l^m} \in K$ with $m \geq 0$ such that

$$\pi_0 = p, \;\; \zeta_{l^0} = 1, \;\; \pi_{m+1}^l = \pi_m, \;\; \zeta_{l^{m+1}}^l = \zeta_{l^m}, \;\; m \geq 0.$$

Since the polynomial $x^{l^m} - p$ is Eisenstein over K and π_m is one of its roots we have that the field $K_{\pi_m} := K(\pi_m)$ generated by π_m is isomorphic to $K[x]/(x^{l^m} - p)$ and K_{π_m} is Galois over K with cyclic Galois group of order l^m generated by the automorphism τ_m satisfying $\tau_m \pi_m = \zeta_{l^m} \pi_m$. Define

$$K^{(l)} := \bigcup_{m \geq 0} K_{\pi_m}. \tag{2.2}$$

Clearly the automorphisms τ_m are compatible and yield an automorphism $\tau_{(l)} \in \mathfrak{G}(K^{(l)}/K)$. For all $\gamma \in \mathbb{Z}_l$ one defines $\tau_{(l)}^\gamma \in \mathfrak{G}(K^{(l)}/K)$ as follows: if $\gamma \equiv b_m$ mod l^m with $b_m \in \mathbb{Z}$ then one lets $\tau_{(l)}^\gamma$ to be $\tau_{(l)}^{b_m}$ on $K(\pi_m)$. Then the map $\mathbb{Z}_l \to \mathfrak{G}(K^{(l)}/K)$ given by $\gamma \mapsto \tau_{(l)}^\gamma$ is an isomorphism. On the other hand the fields K_{π_m} possess compatible automorphisms extending the Frobenius lift on R and fixing the π_m's; they induce an automorphism $\phi_{(l)}$ on $K^{(l)}$. One trivially checks that $\phi_{(l)}\tau_{(l)}$ and $\tau_{(l)}^p \phi_{(l)}$ coincide on all roots of unity in K (and hence on K) and also on all π_m's; so $\phi_{(l)}\tau_{(l)} = \tau_{(l)}^p \phi_{(l)}$ in $\mathfrak{G}(K^{(l)}/\mathbb{Q}_p)$. For each $\gamma \in \mathfrak{B}$ we set $\phi_{(l)}^{(\gamma)} := \tau_{(l)}^\gamma \phi_{(l)} \in \mathfrak{G}(K^{(l)}/\mathbb{Q}_p)$, and we let $\phi^{(\gamma)} \in \mathfrak{F}^{(1)}(K^{\text{alg}}/\mathbb{Q}_p)$ be an arbitrary extension of $\phi_{(l)}^{(\gamma)}$.

Proposition 2.8. *The following hold:*

(1) $\phi_{(l)}^{(0)}, \ldots, \phi_{(l)}^{(p-1)}$ *are monomially independent in* $\mathfrak{G}(K^{(l)}/\mathbb{Q}_p)$*. In particular,* $\phi^{(0)}, \ldots, \phi^{(p-1)}$ *are monomially independent in* $\mathfrak{G}(K^{\text{alg}}/\mathbb{Q}_p)$*.*

(2) *Let* $\gamma_1, \ldots, \gamma_n \in \mathbb{Z}_l$ *be* $\mathbb{Z}$*-linearly independent. Then* $\phi_{(l)}^{(\gamma_1)}, \ldots, \phi_{(l)}^{(\gamma_n)}$ *are monomially independent in* $\mathfrak{G}(K^{(l)}/\mathbb{Q}_p)$*; in particular* $\phi^{(\gamma_1)}, \ldots, \phi^{(\gamma_n)}$ *are monomially independent in* $\mathfrak{G}(K^{\text{alg}}/\mathbb{Q}_p)$*.*

Proof. We will prove Part 2. Part 1 is proved similarly. Write $\phi_{i,l} := \phi_{(l)}^{(\gamma_i)}$ for $i \in \{1, \dots, n\}$. Let $\mu = i_1 \dots i_s$ where $i_1, \dots, i_s \in \{1, \dots, n\}$ and similarly $\mu' = i'_1 \dots i'_{s'}$ where $i'_1, \dots, i'_{s'} \in \{1, \dots, n\}$. Assume

$$\phi_{i_1,l} \dots \phi_{i_s,l} = \phi_{i'_1,l} \dots \phi_{i'_{s'},l}$$

and let us prove that $\mu = \mu'$. We first note that for all integers $j \geq 0$ we have $\phi_{(l)} \tau_{(l)}^j = \tau_{(l)}^{pj} \phi_{(l)}$; this follows by induction on j. We conclude that $\phi_{(l)} \tau_{(l)}^{\gamma} = \tau_{(l)}^{p\gamma} \phi_{(l)}$ for all $\gamma \in \mathbb{Z}_l$; this equality holds because it holds on every K_{π_m}. Next note that for all integers $i \geq 1$ and for all $\gamma \in \mathbb{Z}_l$ we have $\phi_{(l)}^i \tau_{(l)}^{\gamma} = \tau_{(l)}^{p^i \gamma} \phi_{(l)}^i$; this follows by induction on i. Using the latter equalities we get

$$\phi_{i_1,l} \dots \phi_{i_s,l} = \tau_{(l)}^{\gamma_{i_1}} \phi_{(l)} \dots \tau_{(l)}^{\gamma_{i_s}} \phi_{(l)} = \tau_{(l)}^{\gamma_{i_1} + p\gamma_{i_2} + \dots + p^{s-1}\gamma_{i_s}} \phi_{(l)}^s$$

and similarly for μ', so we get

$$\tau_{(l)}^{\gamma_{i_1} + p\gamma_{i_2} + \dots + p^{s-1}\gamma_{i_s}} \phi_{(l)}^s = \tau_{(l)}^{\gamma_{i'_1} + p\gamma_{i'_2} + \dots + p^{s'-1}\gamma_{i_{s'}}} \phi_{(l)}^{s'}.$$

Since $\phi_{(l)}$ has infinite order on K we get $s = s'$. Since $\tau_{(l)}$ has infinite order on $K^{(l)}$ we get

$$\gamma_{i_1} + p\gamma_{i_2} + \dots + p^{s-1}\gamma_{i_s} = \gamma_{i'_1} + p\gamma_{i'_2} + \dots + p^{s-1}\gamma_{i'_s} \tag{2.3}$$

in F. We will be done if we prove the following.

Claim. *An equality of the form (2.3) implies that $i_j = i'_j$ for all $j \in \{1, \dots, s\}$.*

The claim can be proved by induction on s. The case $s = 1$ is trivial. The induction step follows if we show that the equality (2.3)implies that $i_1 = i'_1$. Assume $i_1 \neq i'_1$ and seek a contradiction. Recalling that $\gamma_1, \gamma_2, \dots, \gamma_n$ are $\mathbb{Z}$-linearly independent write the left-hand side of (2.3) as a sum $\sum_{i=1}^n c_i \gamma^i$ with $c_i \in \mathbb{Z}$ and write the right-hand side of (2.3) as a sum $\sum_{i=1}^n c'_i \gamma_i$ with $c'_i \in \mathbb{Z}$. So $c_i = c'_i$ for all i. Since $\gamma_{i_1} \neq \gamma_{i'_1}$ we get that $c_{i_1} \equiv 1 \bmod p$ while $c'_{i_1} \equiv 0 \bmod p$, a contradiction. This ends the proof of our claim and hence of our proposition. ∎

Remark 2.9. Note that, in spite of the fact that $s_1 := \phi_{(l)}^{(0)} = \phi_{(l)}$ and $s_2 := \phi_{(l)}^{(1)} = \tau_{(l)} \phi_{(l)}$ are monomially independent in $\mathfrak{G}(K^{(l)}/\mathbb{Q}_p)$ we have that the subgroup of $\mathfrak{G}(K^{(l)}/\mathbb{Q}_p)$ generated by s_1 and s_2 is not freely generated by s_1 and s_2; indeed we have the following relation:

$$s_1(s_2 s_1^{-1}) = (s_2 s_1^{-1})^p s_1.$$

2.3 π-Frobenius lifts

Throughout the memoir we denote by Π the set of all elements $\pi \in \mathbb{Q}_p^{\text{alg}}$ such that there exists a finite Galois extension $E/\mathbb{Q}_p$ with the property that π is a prime element in $\mathcal{O}_E$. Note that Π consists exactly of those elements $\pi \in \mathbb{Q}_p^{\text{alg}}$ which are roots of Eisenstein polynomials with coefficients in $\mathbb{Z}_p^{\text{ur}}$ and for which $\mathbb{Q}_p^{\text{ur}}(\pi)/\mathbb{Q}_p$ is Galois. We have $\mathbb{Q}_p^{\text{alg}} = \mathbb{Q}_p^{\text{ur}}(\Pi)$. For any $\pi \in \Pi$ write $K_\pi = K(\pi)$ and let $R_\pi = R[\pi]$ which equals the valuation ring of K_π. We write $\pi'|\pi$ if and only if $K_\pi \subset K_{\pi'}$. Note that $K^{\text{alg}} = K(\Pi)$. Clearly for $\pi \in \Pi$ the field K_π is mapped into itself by every Frobenius automorphism ϕ of K^{alg}. By continuity of ϕ we have an induced automorphism $\phi_\pi : R_\pi \to R_\pi$ (which we sometimes still denote by ϕ) inducing the p-power Frobenius on $R_\pi/\pi R_\pi = k$.

Remark 2.10. We take the opportunity to correct here a typo in [12]: in the definition of Π of Section 2.1 the exponent "ur" in the condition "$\mathbb{Q}_p^{\text{ur}}(\pi)/\mathbb{Q}_p$ is Galois" was inadvertently dropped.

More generally we will need the following.

Definition 2.11. Let A be an R_π-algebra. By a *π-Frobenius lift* for an A-algebra $\varphi : A \to B$ we understand a ring homomorphism $\phi : A \to B$ such that the induced homomorphism $\overline{\phi} : A/\pi A \to B/\pi B$ equals the composition of the induced homomorphism $\overline{\varphi} : A/\pi A \to B/\pi B$ with the p-power Frobenius on $A/\pi A$. If $B = A$ and $\varphi = 1_A$ we say that ϕ is a *π-Frobenius lift* on A.

In particular, for every Frobenius automorphism ϕ of K^{alg} and every $\pi \in \Pi$ the induced automorphism ϕ_π of R_π is a π-Frobenius lift.

2.4 Rings of symbols

Definition 2.12. Consider a family $\Phi := (\phi_1, \ldots, \phi_n)$, $\phi_i \in \mathfrak{F}^{(1)}(K^{\text{alg}}/\mathbb{Q}_p)$ of distinct Frobenius automorphisms and let $\pi \in \Pi$. Let $\mathbb{M}_\Phi$ be the free monoid with identity on the set Φ; so we have an isomorphism $\mathbb{M}_n \simeq \mathbb{M}_\Phi$, $i \mapsto \phi_i$. We define the *ring of symbols* $K_{\pi,\Phi}$ to be the free K_π-module with basis $\mathbb{M}_\Phi$ equipped with multiplication defined by

$$\phi_i \cdot \lambda = \phi_i(\lambda) \cdot \phi_i \tag{2.4}$$

for $\lambda \in K_\pi$, $i \in \{1, \ldots, n\}$. If in the above definition we replace K_π we obtain a ring $R_{\pi,\Phi}$.

So every element in $K_{\pi,\Phi}$ (respectively $R_{\pi,\Phi}$) can be uniquely written as

$$\sum_{\mu \in \mathbb{M}_n} \lambda_\mu \phi_\mu$$

with λ_μ in K_π (respectively in R_π). These rings have filtrations "by order" given by the subgroups:

$$K^r_{\pi,\Phi} := \left\{ \sum_{\mu \in \mathbb{M}^r_n} \lambda_\mu \phi_\mu \mid \lambda_\mu \in K_\pi \right\} \subset K_{\pi,\Phi},$$

$$R^r_{\pi,\Phi} := \left\{ \sum_{\mu \in \mathbb{M}^r_n} \lambda_\mu \phi_\mu \mid \lambda_\mu \in R_\pi \right\} \subset R_{\pi,\Phi}.$$

The ring $K_{\pi,\Phi}$ is a K_π-linear space with left multiplication by scalars but, of course, it is not a K_π-algebra. If $\mathrm{End}_{\mathrm{gr}}(K^{\mathrm{alg}})$ denotes the ring of all group endomorphisms of K^{alg} then we have a natural K_π-linear ring homomorphism

$$K_{\pi,\Phi} \to \mathrm{End}_{\mathrm{gr}}(K^{\mathrm{alg}}), \quad \theta \mapsto \theta^{\mathrm{alg}}. \tag{2.5}$$

Remark 2.13. Note that if $\phi_1, \dots, \phi_n \in \mathfrak{F}^{(1)}(K^{\mathrm{alg}}/\mathbb{Q}_p)$ are monomially independent in $\mathfrak{G}(K^{\mathrm{alg}}/\mathbb{Q}_p)$ then, by Lemma 2.6 (and in fact directly from Artin's "independence of characters") the natural ring homomorphism (2.5) is injective.

Remark 2.14. One can also consider the free ring $\mathbb{Z}_\Phi$ generated by Φ which we refer to as the ring of *integral symbols*; as an abelian group it is the free abelian group with basis $\mathbb{M}_\Phi$. So every element of this ring can uniquely be written as

$$w = \sum_{\mu \in \mathbb{M}_n} m_\mu \phi_\mu, \quad m_\mu \in \mathbb{Z}.$$

This ring has an order (with non-negative elements defined as those with non-negative coefficients) and has a filtration "by order" given by the subgroups $\mathbb{Z}^r_\Phi$ consisting of $\mathbb{Z}$-linear combinations of elements ϕ_μ with $\mu \in \mathbb{M}^r_n$. Then for all $\lambda \in R^\times_\pi$ and all $w \in \mathbb{Z}_\Phi$ we write

$$\lambda^w = \prod_{\mu \in \mathbb{M}_n} (\phi_\mu(\lambda))^{m_\mu} \in R^\times_\pi.$$

For every $w = \sum m_\mu \phi_\mu \in \mathbb{Z}_\Phi$ we define the *degree* of w to be $\deg(w) = \sum m_\mu$.

2.5 Partial π-jet spaces

For $\pi \in \Pi$ let $C_p(X, Y) \in \mathbb{Z}[X, Y]$ be the polynomial

$$C_p(X, Y) := \frac{X^p + Y^p - (X + Y)^p}{p}.$$

Following [6, 7, 23] a π-*derivation* from an R_π-algebra A into an A-algebra B is a map $\delta_\pi : A \to B$, $x \mapsto \delta_\pi x$, such that $\delta_\pi(1) = 0$ and

$$\delta_\pi(x + y) = \delta_\pi x + \delta_\pi y + \frac{p}{\pi} C_p(x, y),$$

$$\delta_\pi(xy) = x^p \cdot \delta_\pi y + y^p \cdot \delta_\pi x + \pi \cdot \delta_\pi x \cdot \delta_\pi y,$$

for all $x, y \in A$. Given a π-derivation as above and denoting by $\varphi : A \to B$ the structure map of the A-algebra B we always denote by $\phi_\pi : A \to B$ the map $\phi(x) = \varphi(x)^p + \pi\delta_\pi x$; then ϕ_π is a π-Frobenius lift. If π is a non-zero divisor in B then the above formula gives a bijection between the set of π-derivations from A to B and the set of π-Frobenius lifts from A to B.

Definition 2.15. By a *partial δ_π-ring* we understand an R_π-algebra A equipped with an n-tuple $(\delta_{\pi,1}, \dots, \delta_{\pi,n})$ of π-derivations $A \to A$. (We do *not* assume any “commutation relation” between them.)

Assume we are given a family $\Phi := (\phi_1, \dots, \phi_n) \in \mathfrak{F}^{(1)}(K^{\mathrm{alg}}/\mathbb{Q}_p)^n$ of distinct Frobenius automorphisms of K^{alg}. Note that for every $\pi \in \Pi$ we get an induced tuple $\Phi_\pi = (\phi_{\pi,1}, \dots, \phi_{\pi,n})$ of (not necessarily distinct) π-Frobenius lifts on R_π, called the restriction of Φ to R_π. We therefore get an induced tuple $(\delta_{\pi,1}, \dots, \delta_{\pi,n})$ of π-derivations on R_π and hence a structure of partial δ_π-ring on R_π.

Following the lead of [6] we need to consider the following generalization of the notion of partial δ_π-ring.

Definition 2.16. Define a category $\mathbf{Prol}^*_{\pi,\Phi}$ as follows. An object of this category is a countable family of p-adically complete R_π-algebras $S^* = (S^r)_{r\geq 0}$ equipped with the following data:

(1) R_π-algebra homomorphisms $\varphi\colon S^r \to S^{r+1}$;

(2) π-derivations $\delta_{\pi,j}\colon S^r \to S^{r+1}$ for $1 \leq j \leq n$.

We require that $\delta_{\pi,i}$ be compatible with the π-derivations on R_π and with φ, i.e., $\delta_{\pi,j} \circ \varphi = \varphi \circ \delta_{\pi,j}$. Morphisms are defined in a natural way. We denote by $\phi_{\pi,j} : S^r \to S^{r+1}$ the corresponding π-Frobenius lifts, defined by $\phi_{\pi,j}(x) = \varphi(x)^p + \pi\delta_{\pi,j}x$. Also, for all $\mu := i_1 \dots i_l \in \mathbb{M}_n$ and all $x \in S^r$ we set $\delta_{\pi,\mu}x := (\delta_{\pi,i_1} \circ \dots \circ \delta_{\pi,i_l})(x) \in S^{r+l}$ and $\phi_{\pi,\mu}x := (\phi_{\pi,i_1} \circ \dots \circ \phi_{\pi,i_l})(x) \in S^{r+l}$.

The objects of $\mathbf{Prol}^*_{\pi,\Phi}$ are called *prolongation sequences* (over R_π with respect to Φ or Φ_π). We sometimes identify elements $a \in S^r$ with the elements $\varphi(a) \in S^{r+1}$ if no confusion arises. We sometimes write $S^* = (S^r, \varphi, \delta_{\pi,1}, \dots, \delta_{\pi,n})$. We denote by $\mathbf{Prol}_{\pi,\Phi}$ the full subcategory of $\mathbf{Prol}^*_{\pi,\Phi}$ whose objects are the prolongation sequences (S^r) such that all S^r's are Noetherian and flat over R_π.

Remark 2.17. (1) If S is a p-adically complete partial δ_π-ring whose π-derivations are compatible with those on R_π then the sequence $S^* = (S^r)$ with $S^r = S$ has a natural structure of object of $\mathbf{Prol}^*_{\pi,\Phi}$ with φ the identity and obvious $\delta_{\pi,j}$. If in addition S is Noetherian and flat over R_π then S^* is an object of $\mathbf{Prol}_{\pi,\Phi}$. The initial object in $\mathbf{Prol}^*_{\pi,\Phi}$ (and also of $\mathbf{Prol}_{\pi,\Phi}$) is the sequence $R^*_\pi = (R^r_\pi)$ with $R^r_\pi := R_\pi$.

(2) If $S^* = (S^r, \varphi, \delta_{\pi,1}, \dots, \delta_{\pi,n})$ is an object of $\mathbf{Prol}^*_{\pi,\Phi}$ then the ring

$$\lim_{\substack{\to \\ \varphi}} S^r$$

has a natural structure of partial δ_π-ring.

Remark 2.18. For every $\pi'|\pi$ and every object S^* in $\mathbf{Prol}_{\pi,\Phi}$ the sequence

$$S^* \otimes_{R_\pi} R_{\pi'} := (S^r \otimes_{R_\pi} R_{\pi'})_{r\geq 0}$$

is naturally an object of $\mathbf{Prol}_{\pi',\Phi}$; cf. [12, Section 4.1].

Remark 2.19. For $\mu = i_1 \dots i_r \in \mathbb{M}^r_n \setminus \mathbb{M}^{r-1}_n$ we define the integral symbol:

$$w(\mu) := 1 + \phi_{i_1} + \phi_{i_1 i_2} + \phi_{i_1 i_2 i_3} + \cdots + \phi_{i_1 i_2 i_3 \dots i_{r-1}} \in \mathbb{Z}_\Phi.$$

For every object $S^* = (S^r)$ in $\mathbf{Prol}^*_{\pi,\Phi}$, every $r \geq 1$, every $\mu \in \mathbb{M}^r_n \setminus \mathbb{M}^{r-1}_n$ and every $a \in S^0$ there exists $a_\mu \in S^{r-1}$ such that

$$\phi_{\pi,\mu} a = \pi^{w(\mu)} \delta_{\pi,\mu} a + \varphi(a_\mu); \tag{2.6}$$

this is trivially proved by induction on r.

Definition 2.20. Consider two families of distinct Frobenius automorphisms $\Phi' := (\phi'_1, \dots, \phi'_{n'})$ and $\Phi'' := (\phi''_1, \dots, \phi''_{n''})$ of K^{alg}. Also let $\pi \in \Pi$. A map of sets

$$\epsilon : \{1, \dots, n'\} \to \{1, \dots, n''\} \tag{2.7}$$

is called a *selection map* (with respect to (Φ', Φ'', π)) if for all $j \in \{1, \dots, n'\}$ we have that $\phi_{\pi,j} = \phi_{\pi,\epsilon(j)}$. Consider next an object of $\mathbf{Prol}^*_{p,\Phi''}$,

$$S^* = (S^r, \varphi, \delta''_{\pi,1}, \dots, \delta''_{\pi,n}),$$

and let ϵ be a selection map as above. One defines the object S^*_ϵ in $\mathbf{Prol}^*_{p,\Phi'}$ by:

$$S^*_\epsilon := (S^r, \varphi, \delta''_{\pi,\epsilon(1)}, \dots, \delta''_{\pi,\epsilon(n')}).$$

This construction depends only on the restrictions Φ'_π and Φ''_π of Φ' and Φ'' to K_π.

Motivated by Proposition 2.8, introduce variables denoted by $\delta_{\pi,\mu} y_j$ for $\mu \in \mathbb{M}_n$, $\pi \in \Pi$, $j \in \{1, \dots, N\}$. Fix an integer N and consider the ring $R_\pi[y_1, \dots, y_N]$ and the rings

$$J^r_{\pi,\Phi}(R_\pi[y_1, \dots, y_N]) := R_\pi[\delta_{\pi,\mu} y_j \mid \mu \in \mathbb{M}^r_n, j \in \{1, \dots, N\}]\widehat{\ }. \tag{2.8}$$

The sequence $J^*_{\pi,\Phi}(R_\pi[y_1, \dots, y_N]) := (J^r_{\pi,\Phi}(R_\pi[y_1, \dots, y_N]))$ has a unique structure of object in $\mathbf{Prol}_{\pi,\Phi}$ such that $\delta_{\pi,i}\delta_{\pi,\mu} y := \delta_{\pi,i\mu} y$ for all $i = 1, \dots, n$. We have an induced evaluation map $F_{R_\pi} : R_\pi^N \to R_\pi$: for $(a_1, \dots, a_N) \in R_\pi^N$ we let $F_{R_\pi}(a_1, \dots, a_N) \in R_\pi$ be obtained from F by replacing the variables $\delta_{\pi,\mu} y_j$ with the elements $\delta_{\pi,\mu} a_j$. Note that the map

$$J^r_{\pi,\Phi}(R_\pi[y_1, \dots, y_N]) \to \mathrm{Fun}(R_\pi^N, R_\pi), \; F \mapsto F_{R_\pi} \tag{2.9}$$

is not injective in general, even if Φ is monomially independent. Here and in the following "Fun" stands for the set of set-theoretic maps. For instance if $\pi = p$ we have $(\delta_{p,i} y - \delta_{p,j} y)_R = 0$. This is in stark contrast with [12]. See, however, Remark 2.34.

Definition 2.21. For every R_π-algebra of finite type $A := R_\pi[y_1, \dots, y_N]/I$, we define

$$J^r_{\pi,\Phi}(A) := J^r_{\pi,\Phi}(R_\pi[y_1, \dots, y_N])/(\delta_{\pi,\mu} I \mid \mu \in \mathbb{M}^r_n).$$

This algebra is called the *partial π-jet algebra* of A of order r.

Note that $J^r_{\pi,\Phi}(A)$ is Noetherian and p-adically complete but generally not flat over R_π, even if $\pi = p$ and A is flat over R_π as one can see by taking $A = R[x]/(x^p)$. It is trivial to see that the sequence $J^*_{\pi,\Phi}(A) := (J^r_{\pi,\Phi}(A))$ has a natural structure of prolongation sequence, i.e., it is an object of $\mathbf{Prol}^*_{\pi,\Phi}$ (but, as just noted, it is not generally an object of $\mathbf{Prol}_{\pi,\Phi}$). Also note that $J^r_{\pi,\Phi}(A)$ depends only on r, π, A and on the restriction Φ_π of Φ to R_π.

Proposition 2.22. *If A is a smooth R_π-algebra, and $u\colon R_\pi[T_1, \dots, T_d] \to A$ is an étale morphism of R_π-algebras, then there is a (unique) isomorphism*

$$\widehat{A[\delta_{\pi,\mu} T_j \mid \mu \in \mathbb{M}^{r,+}_n, j \in \{1, \dots, d\}]} \cong J^r_{\pi,\Phi}(A)$$

sending $\delta_{\pi,\mu} T_j$ into $\delta_{\pi,\mu}(u(T_j))$ for all j and μ. In particular, $J^r_{\pi,\Phi}(A)$ is flat over R_π so the sequence $J^*_{\pi,\Phi}(A)$ is an object of $\mathbf{Prol}_{\pi,\Phi}$.

Proof. Similar to [10, Proposition 3.13]. ■

We have the following universal property.

Proposition 2.23. *Assume A is a finitely generated (respectively smooth) R_π-algebra. For every object T^* of* $\mathbf{Prol}^*_{\pi,\Phi}$ *(respectively in* $\mathbf{Prol}_{\pi,\Phi}$*) and every R_π-algebra map $u : A \to T^0$ there is a unique morphism $J^*_{\pi,\Phi}(A) \to T^*$ over S^* in* $\mathbf{Prol}^*_{\pi,\Phi}$ *(respectively in* $\mathbf{Prol}_{\pi,\Phi}$*) compatible with u.*

Proof. Similar to [10, Proposition 3.3]. ■

We next record the existence of "prolongations of derivations." Let S be a ring. Recall that by an *S-derivation* from an S-algebra A to an A-algebra B one understands an S-module endomorphism $A \to B$ satisfying the Leibniz rule.

Proposition 2.24. *Let A be a smooth R_π-algebra equipped with an R_π-derivation $D : A \to A$. Then for every $r \geq 1$ and every $\mu \in \mathbb{M}_n^r$ there exists a unique R_π-derivation $D_\mu : J^r_{\pi,\Phi}(A) \to J^r_{\pi,\Phi}(A)$ satisfying the following properties:*

(1) $D_\mu \phi_\mu a = p^r \cdot \phi_\mu Da$ *for all* $a \in A$*;*

(2) $D_\mu \phi_\nu a = 0$ *for all* $a \in A$ *and all* $\nu \in \mathbb{M}_n^r \setminus \{\mu\}$.

Proof. Similar to [10, Proposition 3.43]. We recall the argument. Uniqueness is clear. To prove existence let $u : S := R_\pi[T_1, \ldots, T_d] \to A$ be an étale map and let $a_i := DT_i \in A$. Then consider the derivation

$$\frac{p^r}{\pi^{w(\mu)}} \sum_{i=1}^{d} a_i^{\phi_\mu} \frac{\partial}{\partial \delta_{\pi,\mu} T_i} : J^r_{\pi,\Phi}(S) = R_\pi[\delta_{\pi,\nu} T \mid \nu \in \mathbb{M}_n^r] \to J^r_{\pi,\Phi}(A).$$

By Proposition 2.22 this derivation extends to a derivation $D_\mu : J^r_{\pi,\Phi}(A) \to J^r_{\pi,\Phi}(A)$. To check properties (1) and (2) it is enough to check them for $a = T_i$ because if (1) and (2) hold for two elements of $J^r_{\pi,\Phi}(S)$ then (1) and (2) hold for their sum and their product. But for $a = T_i$ the equalities (1) and (2) hold in view of formula (2.6). ∎

The jet construction can be globalized as follows.

Definition 2.25. For every smooth scheme X over R_π define the p-adic formal scheme

$$J^r_{\pi,\Phi}(X) = \bigcup \mathrm{Spf}(J^r_{\pi,\Phi}(\mathcal{O}(U_i))),$$

called the *partial π-jet space* of order r of X, where $X = \bigcup U_i$ is (any) affine open cover. The gluing involved in this definition is well defined because the formation of π-jet spaces is compatible with fractions; cf. Proposition 2.22. The elements of the ring $\mathcal{O}(J^r_{\pi,\Phi}(X))$, identified with morphisms of p-adic formal schemes $J^r_{\pi,\Phi}(X) \to \widehat{\mathbb{A}^1}$, are called (purely) *arithmetic PDEs* on X over R_π of order $\leq r$.

For all $\pi'|\pi$ we write $X_{\pi'} := X \otimes_{R_\pi} R_{\pi'}$. Clearly $J^0_{\pi',\Phi}(X_{\pi'}) = \widehat{X_{\pi'}}$. Note also that $J^r_{\pi',\Phi}(X_{\pi'})$ only depends on r, π', X and on the restriction Φ_π of Φ to R_π.

Proposition 2.26. *Assume A is a smooth R_π-algebra. For all $\pi''|\pi'|\pi$ there are natural homomorphisms*

$$\iota_{\pi'',\pi'} : J^r_{\pi'',\Phi}(A) \to J^r_{\pi',\Phi}(A) \otimes_{R_{\pi'}} R_{\pi''} \tag{2.10}$$

such that the homomorphism

$$\iota_{\pi'',\pi} : J^r_{\pi'',\Phi}(A) \to J^r_{\pi,\Phi}(A) \otimes_{R_\pi} R_{\pi''} \tag{2.11}$$

equals the composition

$$J^r_{\pi'',\Phi}(A) \xrightarrow{\iota_{\pi'',\pi'}} J^r_{\pi',\Phi}(A) \otimes_{R_{\pi'}} R_{\pi''} \xrightarrow{\iota_{\pi',\pi} \otimes 1} (J^r_{\pi,\Phi}(A) \otimes_{R_\pi} R_{\pi'}) \otimes_{R_{\pi'}} R_{\pi''}, \tag{2.12}$$

where the targets of the maps (2.11) *and* (2.12) *are naturally identified. Moreover, the homomorphisms* (2.10) *are injective.*

Proof. This follows similarly to [12, Proposition 4.1 (1)] and [14, Proposition 2.2]. The map $\iota_{\pi'',\pi'}$ is guaranteed by Proposition 2.23 as $(J^r_{\pi',\Phi}(A) \otimes_{R_{\pi'}} R_{\pi''})$ is naturally an object of $\mathbf{Prol}_{\pi'',\Phi}$; cf. Remark 2.18. The factorization (2.11) arises from naturality of base change. Finally, to address the injectivity of (2.10), pick an étale homomorphism $R_\pi[T_1,\dots,T_d] \to A$. Both the source and target of (2.10) then embed in the common ring

$$K_{\pi''}[\![\delta_{\pi'',\mu} T_j \mid \mu \in \mathbb{M}^r_n, j = 1,\dots,d]\!] \cong K_{\pi''}[\![\delta_{\pi',\mu} T_j \mid \mu \in \mathbb{M}^r_n, j = 1,\dots,d]\!]$$

recovering the natural base change (2.10) from which the injectivity is clear. ∎

Remark 2.27. For every smooth algebra A over R_π and every selection map ϵ with respect to (Φ', Φ'', π) we get (by the universality property of J^r) a natural morphism of prolongation sequences over R_π with respect to Φ', $J^*_{\pi,\Phi'}(A) \to J^*_{\pi,\Phi''}(A)_\epsilon$, cf. Definition 2.20 for the subscript notation. Hence for every smooth scheme X over R_π we get morphisms

$$J^r_{\pi,\Phi''}(X) \to J^r_{\pi,\Phi'}(X). \tag{2.13}$$

We shall be interested later in four special cases of this construction.

(1) Assume $\pi = p$, $\Phi' = \Phi''$, and $\epsilon : \{1,\dots,n\} \to \{1,\dots,n\}$ is a bijection. Then the above construction defines an action of the symmetric group Σ_n on $J^r_{\pi,\Phi}(X)$.

(2) Assume $n' = s$, $n'' = n$, $\Phi' = (\phi'_1,\dots,\phi'_s)$, $\Phi'' = \Phi = (\phi_1,\dots,\phi_n)$,

$$\phi'_1 = \phi_{i_1},\dots,\phi'_s = \phi_{i_s},\ 1 \le i_1 < i_2 < \cdots < i_s \le n,\ \epsilon(1) = i_1,\dots,\epsilon(s) = i_s.$$

Then we get a natural morphism (referred to as a *face* morphism)

$$J^r_{\pi,\Phi}(X) = J^r_{\pi,\phi_1,\dots,\phi_n}(X) \to J^r_{\pi,\phi_{i_1},\dots,\phi_{i_s}}(X).$$

(3) Assume $\pi = p$, $n' = n$, $n'' = 1$, $\Phi' = \Phi = (\phi_1,\dots,\phi_n)$, $\Phi'' = \{\phi\}$, and hence ϵ is the constant map. Then we get a natural morphism (referred to as the *degeneration* morphism):

$$J^r_{\pi,\phi}(X) \to J^r_{\pi,\Phi}(X).$$

(4) Assume $\pi = p$ and $\Phi = \{\phi_1,\dots,\phi_n\}$. Then one trivially checks that for all $i \in \{1,\dots,n\}$ the composition of the face and degeneration morphisms below is the identity:

$$\mathrm{id} : J^r_{p,\phi_i}(X) \to J^r_{p,\Phi}(X) \to J^r_{p,\phi_i}(X).$$

2.6 Total δ-overconvergence

The notion of δ-overconvergence was introduced in [14] and exploited in [12], cf. [12, Definition 2.5].

Definition 2.28. Assume A is a smooth R_π-algebra. An element $f_\pi \in J^r_{\pi,\Phi}(A)$ is called *totally δ-overconvergent* if it has the following property: for all $\pi'|\pi$ there exists an integer $N \geq 0$ such that $p^N f_\pi \otimes 1$ is in the image of the map

$$\iota_{\pi',\pi} : J^r_{\pi',\Phi}(A) \to J^r_{\pi,\Phi}(A) \otimes_{R_\pi} R_{\pi'}. \tag{2.14}$$

Let us denote by $J^r_{\pi,\Phi}(A)^\dagger$ the R-algebra of all totally δ-overconvergent elements in $J^r_{\pi,\Phi}(A)$. For every smooth scheme X/R_π an element (arithmetic PDE), $f \in \mathcal{O}(J^r_{\pi,\Phi}(X))$, will be called *totally δ-overconvergent* if for all affine open set $U \subset X$ (equivalently for every affine open set of a given affine open cover of X) the image of f in the ring $\mathcal{O}(J^r_{\pi,\Phi}(U)) = J^r_{\pi,\Phi}(\mathcal{O}(U))$ is totally δ-overconvergent. We denote by $\mathcal{O}(J^r_{\pi,\Phi}(X))^\dagger$ the ring of all totally δ-overconvergent elements of $\mathcal{O}(J^r_{\pi,\Phi}(X))$. A morphism $J^r_{\pi,\Phi}(X) \to \widehat{\mathbb{A}^1}$ will be called *totally δ-overconvergent* if the corresponding element in $\mathcal{O}(J^r_{\pi,\Phi}(X))$ is totally δ-overconvergent.

Remark 2.29. We caution the reader about the notation †. It is common for † superscripts to also denote overconvergence in a difference sense. Specifically, these superscripts are used extensively in the overconvergent Witt vectors or Monsky–Washnitzer algebras of rigid geometry. This memoir is written entirely in the formal setting. There are certainly overlaps between concepts used here and those in rigid geometry, however they remain for now in different realms. We hope this notation causes no confusion. To elucidate, all uses of † are in reference to δ-overconvergence.

Note that, again, the ring $\mathcal{O}(J^r_{\pi,\Phi}(X))^\dagger$ depends only on r, π, X and on the restriction Φ_π of Φ to R_π.

Using Proposition 2.22 one trivially checks the following two propositions.

Proposition 2.30. *For every smooth scheme X over R_π, every $r \geq 0$, and every map as in (2.7) the ring homomorphisms*

$$\mathcal{O}(J^r_{\pi,\Phi'}(X)) \to \mathcal{O}(J^r_{\pi,\Phi''}(X))$$

induced by the morphisms (2.13) induce ring homomorphisms

$$\mathcal{O}(J^r_{\pi,\Phi'}(X))^\dagger \to \mathcal{O}(J^r_{\pi,\Phi''}(X))^\dagger.$$

We will usually view the above ring homomorphisms as inclusions.

Proposition 2.31. *Assume that $u : \widehat{X} \to \widehat{Y}$ is a morphism between the p-adic completions of two smooth R_π-schemes and let $f : J^r_{\pi,\Phi}(Y) \to \widehat{\mathbb{A}^1}$ be a totally δ-overconvergent morphism. Then the composition*

$$J^r_{\pi,\Phi}(X) \xrightarrow{J^r(u)} J^r_{\pi,\Phi}(Y) \xrightarrow{f} \widehat{\mathbb{A}^1}$$

is totally δ-overconvergent, where $J^r(u)$ is the morphism induced by u via the universal property.

Similarly to [12] we make the following definition.

Definition 2.32. For every $f \in \mathcal{O}(J^r_{\pi,\Phi}(X))$ and every object $S^* = (S^r)$ in $\mathbf{Prol}_{\pi,\Phi}$ the universal property of π-jet spaces yields a map of sets

$$f_{S^*} : X(S^0) \to S^r. \tag{2.15}$$

On the other hand, if $f \in \mathcal{O}(J^r_{\pi,\Phi}(X))^\dagger$ then for every object $S^* = (S^r)$ in $\mathbf{Prol}_{\pi,\Phi}$ we can define the map of sets

$$f^{\text{alg}}_{S^*} : X(S^0 \otimes_{R_\pi} R^{\text{alg}}) \to S^r \otimes_{R_\pi} K^{\text{alg}} \tag{2.16}$$

as follows. We may assume $X = \operatorname{Spec} A$ is affine because the construction below allows gluing in the obvious sense. Let $P \in X(S^0 \otimes_{R_\pi} R^{\text{alg}})$. Choose $\pi'|\pi$ such that $P \in X(S^0 \otimes_{R_\pi} R_{\pi'})$ and choose $N \geq 1$ such that $p^N f \otimes 1 \in J^r_{\pi,\Phi}(A) \otimes_{R_\pi} R_{\pi'}$ is the image of some (necessarily unique) element $f_{\pi',N} \in J^r_{\pi',\Phi}(A)$ via the map (2.14). View P as a morphism $P : A \to S^0 \otimes_{R_\pi} R_{\pi'}$. By the universal property of π'-jet spaces we have an induced morphism $J^r(P) : J^r_{\pi',\Phi}(A) \to S^r \otimes_{R_\pi} R_{\pi'}$. Then we define

$$f^{\text{alg}}_{S^*}(P) = p^{-N}(J^r(P))(f_{\pi',N}) \in S^r \otimes_{R_\pi} K_{\pi'} \subset S^r \otimes_{R_\pi} K^{\text{alg}}.$$

The definition is independent of the choice of π' and N due to the injectivity part of Proposition 2.26. On the other hand $f^{\text{alg}}_{S^*}$ effectively depends on Φ (and not only on the restriction Φ_π on K_π). For $S^* = R^*_\pi$ we write $f_{R_\pi} := f_{R^*_\pi}$ and

$$f^{\text{alg}} := f^{\text{alg}}_{R_\pi} := f^{\text{alg}}_{R^*_\pi} : X(R^{\text{alg}}) \to K^{\text{alg}}. \tag{2.17}$$

Proposition 2.33. *Let $f \in \mathcal{O}(J^r_{\pi,\Phi}(X))$ and assume the map f_{S^*} is the zero map for every object S^* in $\mathbf{Prol}_{\pi,\Phi}$ with the property that S^r are integral domains and $\varphi : S^r \to S^{r+1}$ are injective. Then $f = 0$. In particular, if $f \in \mathcal{O}(J^r_{\pi,\Phi}(X))^\dagger$ and the map $f^{\text{alg}}_{S^*}$ is the zero map for every object S^* in $\mathbf{Prol}_{\pi,\Phi}$ as above, then $f = 0$.*

Proof. Take $S^* = (S^r)$, $S^r := \mathcal{O}(J^r_{\pi,\Phi}(U))$ for various affine open sets $U \subset X$; one gets that the image of f in $\mathcal{O}(J^r_{\pi,\Phi}(U))$ is 0, hence $f = 0$. ∎

Remark 2.34. Assume $\phi_1, \dots, \phi_n$ are monomially independent in $\mathfrak{G}(K^{\mathrm{alg}}/\mathbb{Q}_p)$. It would be interesting to know when/if the ring homomorphism

$$\mathcal{O}(J^r_{\pi,\Phi}(X))^\dagger \to \mathrm{Fun}(X(R^{\mathrm{alg}}), K^{\mathrm{alg}}), \quad f \mapsto f^{\mathrm{alg}} \tag{2.18}$$

is injective. For $n = 1$ this is true; cf. [12, proof of Proposition 4.4]. See also Proposition 3.13 and Proposition 7.38 for related results. Clearly, if we do not assume $\phi_1, \dots, \phi_n$ are monomially independent in $\mathfrak{G}(K^{\mathrm{alg}}/\mathbb{Q}_p)$ then (2.18) is not injective in general: to get an example take X the affine line, $n = 2$, and $\phi_1 = \phi_2$.

Chapter 3

Partial δ-characters

3.1 Definition and the additive group

We start with the following PDE definition extending the ODE case in [6].

Definition 3.1. A *partial δ_π-character of order $\leq r$* of a commutative smooth group scheme G/R_π is a group homomorphism $J^r_{\pi,\Phi}(G) \to \widehat{\mathbb{G}_a}$ in the category of p-adic formal schemes. (So in particular a partial δ_π-character can be identified with an element of $\mathcal{O}(J^r_{\pi,\Phi}(G))$, i.e., with an arithmetic PDE of order $\leq r$.) We denote by

$$\mathbf{X}^r_{\pi,\Phi}(G) := \mathrm{Hom}(J^r_{\pi,\Phi}(G), \widehat{\mathbb{G}_a})$$

the R_π-module of partial δ_π-characters of G of order $\leq r$ which we identify with a submodule of $\mathcal{O}(J^r_{\pi,\Phi}(G))$. For $\pi'|\pi$ we set $\mathbf{X}_{\pi',\Phi}(G) := \mathbf{X}_{\pi',\Phi}(G_{\pi'})$, and we call the elements of the latter *partial $\delta_{\pi'}$-characters* of G. For $n = 1$ partial δ_π-characters will be referred to as *ODE δ_π-characters*. An element of $\mathbf{X}^r_{\pi,\Phi}(G)$ will be said to have order r if it is not in the image of the canonical (injective) map $\varphi : \mathbf{X}^{r-1}_{\pi,\Phi}(G) \to \mathbf{X}^r_{\pi,\Phi}(G)$ induced by φ. We also consider the naturally induced semilinear maps ϕ_i : $\mathbf{X}^{r-1}_{\pi,\Phi}(G) \to \mathbf{X}^r_{\pi,\Phi}(G)$ induced by ϕ_i.

Consider the R_π-module $\mathbf{X}^r_{\pi,\Phi}(G)^\dagger$ of totally δ-overconvergent partial δ_π-characters of G. So inside the ring $\mathcal{O}(J^r_{\pi,\Phi}(G))$ we have

$$\mathbf{X}^r_{\pi,\Phi}(G)^\dagger = \mathbf{X}^r_{\pi,\Phi}(G) \cap \mathcal{O}(J^r_{\pi,\Phi}(G))^\dagger.$$

Note that if $\psi \in \mathbf{X}^r_{\pi,\Phi}(G)$ and if $p^N\psi \otimes 1 \in \mathcal{O}(J^r_{\pi,\Phi}(G)) \otimes_{R_\pi} R_{\pi'}$ is the image of some (necessarily unique) $\psi_{\pi'} \in \mathcal{O}(J^r_{\pi',\Phi}(G))$ then $\psi_{\pi'} \in \mathbf{X}^r_{\pi',\Phi}(G)$. In particular, we have the following.

Lemma 3.2. *The image of the natural map*

$$\mathbf{X}^r_{\pi,\Phi}(G)^\dagger \to \mathrm{Fun}(G(R^{\mathrm{alg}}), K^{\mathrm{alg}}), \quad \psi \mapsto \psi^{\mathrm{alg}}$$

is contained in $\mathrm{Hom}_{\mathrm{gr}}(G(R^{\mathrm{alg}}), K^{\mathrm{alg}})$ *where* $\mathrm{Hom}_{\mathrm{gr}}$ *is the* Hom *in the category of abstract groups.*

We also record the following obvious lemma.

Lemma 3.3. *If an element ψ of $\mathbf{X}^r_{\pi,\Phi}(G)$ times a power of p belongs to $\mathbf{X}^r_{\pi,\Phi}(G)^\dagger$ then ψ itself belongs to $\mathbf{X}^r_{\pi,\Phi}(G)^\dagger$.*

We have the following description of partial δ_π-characters of the additive group $\mathbb{G}_a = \text{Spec } R_\pi[T]$. For $\mu = i_1 \dots i_s \in \mathbb{M}_n$ and $r \geq s$ recall that we write

$$\phi_{\pi,\mu} T := \phi_{\pi,i_1} \dots \phi_{\pi,i_s} T \in \widehat{R_\pi[\delta_{\pi,\nu} T \mid \nu \in \mathbb{M}_n^r]} = \mathcal{O}(J^r_{\pi,\Phi}(\mathbb{G}_a)).$$

Consider the embedding

$$\mathcal{O}(J^r_{\pi,\Phi}(\mathbb{G}_a)) \subset K_\pi[\![\delta_{\pi,\nu} T \mid \nu \in \mathbb{M}_n^r]\!] = K_\pi[\![\phi_{\pi,\nu} T \mid \nu \in \mathbb{M}_n^r]\!]$$

and consider the groups

$$K^r_{\pi,\Phi} T := \sum_{\mu \in \mathbb{M}_n^r} K_\pi \phi_{\pi,\mu} T \subset K_\pi[\![\phi_{\pi,\nu} T \mid \nu \in \mathbb{M}_n^r]\!],$$
$$R^r_{\pi,\Phi} T := \sum_{\mu \in \mathbb{M}_n^r} R_\pi \phi_{\pi,\mu} T \subset K^r_{\pi,\Phi} T.$$

These groups are naturally isomorphic to the groups of symbols $K^r_{\pi,\Phi}$ and $R^r_{\pi,\Phi}$, respectively.

Proposition 3.4. *The following equality holds,*

$$\mathbf{X}^r_{\pi,\Phi}(\mathbb{G}_a) = (K^r_{\pi,\Phi} T) \cap (R_\pi[\delta_{\pi,\nu} T \mid \nu \in \mathbb{M}_n^r]), \tag{3.1}$$

where the intersection is taken inside the ring $K_\pi[\![\phi_{\pi,\nu} T \mid \nu \in \mathbb{M}_n^r]\!]$. In particular

$$\mathbf{X}^r_{\pi,\Phi}(\mathbb{G}_a) = \mathbf{X}^r_{\pi,\Phi}(\mathbb{G}_a)^\dagger.$$

Proof. The inclusion $\supset$ in (3.1) is clear. To check the inclusion $\subset$ note that every element in $\mathbf{X}^r_{\pi,\Phi}(\mathbb{G}_a)$ defines an additive polynomial in the ring $K_\pi[\![\phi_{\pi,\nu} T \mid \nu \in \mathbb{M}_n^r]\!]$ hence, since K_π has characteristic 0, a linear polynomial. ∎

Lemma 3.5. *For $\psi := \sum_{\mu \in \mathbb{M}_n^r} \lambda_\mu \phi_\mu T$ in the group in (3.1) the following hold:*

(1) $\lambda_0 \in R_\pi$.

(2) *If $\psi \equiv 0 \bmod T$ in $K_\pi[\delta_{\pi,\nu} T \mid \nu \in \mathbb{M}_n^r]$ then $\lambda_\mu = 0$ for all $\mu \in \mathbb{M}_n^{r,+}$.*

(3) *If $n = 1$ then $\lambda_\mu \in R_\pi$ for all $\mu \in \mathbb{M}_n^r$. In other words the intersection in the right-hand side of (3.1) equals $R^r_{\pi,\Phi} T$.*

Proof. Part 1 follows by picking out the coefficient of T.

Part 2 follows by induction on the number of non-zero terms in ψ. For the induction step one orders $\mathbb{M}_n^r$ by letting all members of $\mathbb{M}_n^s \setminus \mathbb{M}_n^{s-1}$ be greater than all members of $\mathbb{M}_n^{s-1}$ for all $s \in \{1, \dots, r\}$ and by taking an arbitrary total order on each set $\mathbb{M}_n^s \setminus \mathbb{M}_n^{s-1}$. Then one picks out the coefficient of $\delta_{\pi,\mu} T$ in ψ where μ is the largest element in $\mathbb{M}_n^r$ with $\delta_{\pi,\mu} T$ appearing in ψ.

Part 3 follows again by induction on the number of non-zero terms in ψ. For the induction step one picks out the coefficient of T^{p^n} in ψ where n is the largest integer such that T^{p^n} appears in ψ. ∎

Remark 3.6. Assertion 3 in Lemma 3.5 fails for $n \geq 2$. For instance, for $\pi = p$, one immediately checks that

$$\frac{1}{p}\phi_1\phi_2 T - \frac{1}{p}\phi_2\phi_1 T \in \mathbf{X}^2_{p,\Phi}(\mathbb{G}_a) \setminus (R^r_\Phi T).$$

3.2 Picard–Fuchs symbol

Definition 3.7. Let G have relative dimension 1 over R_π and let ω be an invariant 1-form on G. By an *admissible coordinate* for G we mean an étale coordinate $T \in \mathcal{O}(U)$ on a neighborhood U of the origin of G generating the ideal of the origin of G in $\mathcal{O}(U)$. Let

$$\ell(T) = \ell_\omega(T) = \sum_{m=1}^{\infty} \frac{b_m}{m} T^m \in K_\pi[\![T]\!], \ \ b_m \in R_\pi,$$

be the logarithm of the formal group of G (with respect to T) normalized with respect to ω; so ℓ is the unique series in $K_\pi[\![T]\!]$ without constant term such that $d\ell = \omega$ in $K_\pi[\![T]\!]dT$. (We have $b_1 \in R_\pi^\times$.) Let $e(T) = e_\omega(T) \in K_\pi[\![T]\!]$ be the exponential normalized with respect to ω, i.e., the compositional inverse of $\ell(T)$. Then the series $e(pT)$ belongs to $p\widehat{R_\pi[T]}$ and so defines a morphism of groups in the category of p-adic formal schemes, $\mathcal{E} : \widehat{\mathbb{G}_a} \to \widehat{G}$. For every partial δ_π-character $\psi \in \mathbf{X}^r_{\pi,\Phi}(G)$ the composition

$$\theta(\psi) : J^r_{\pi,\Phi}(\mathbb{G}_a) \xrightarrow{\mathcal{E}^r} J^r_{\pi,\Phi}(G) \xrightarrow{\psi} \widehat{\mathbb{G}_a}$$

is a partial δ_π-character of $\mathbb{G}_a$ so, identified with an element of $\mathcal{O}(J^r_{\pi,\Phi}(\mathbb{G}_a))$, can be written (cf. Proposition 3.4 and Lemma 3.5, Part 1) as

$$\theta(\psi) = \sum_{\mu \in \mathbb{M}^r_n} \lambda_\mu \phi_{\pi,\mu} T \in \mathbf{X}^r_{\pi,\Phi}(\mathbb{G}_a) \subset K^r_{\pi,\Phi} T, \ \ \lambda_\mu \in K_\pi, \ \ \lambda_0 \in R_\pi. \qquad (3.2)$$

We define the *Picard–Fuchs symbol* (still denoted by $\theta(\psi)$) of ψ (with respect to T and ω) by

$$\theta(\psi) := \sum_{\mu \in \mathbb{M}^r_n} \lambda_\mu \phi_\mu \in K^r_{\pi,\Phi}.$$

The latter induces a $\mathbb{Q}_p$-linear map

$$\theta(\psi)^{\text{alg}} : K^{\text{alg}} \to K^{\text{alg}}.$$

Remark 3.8. (1) By our very definition, viewing ψ as an element of $R_\pi[\![\delta_{\pi,\mu} T \mid \mu \in \mathbb{M}^r_n]\!]$, we have the following equality in $K_\pi[\![\delta_{\pi,\mu} T \mid \mu \in \mathbb{M}^r_n]\!]$:

$$\psi = \frac{1}{p}\theta(\psi)\ell(T).$$

(2) For every ψ, writing $\theta(\psi) = \sum_\mu \lambda_\mu \phi_\mu$ we have that

$$\lambda_0 \in pR_\pi.$$

Indeed, by the equality in Part 1 we have that

$$\theta(\psi)\ell(T) \in pR_\pi[\![\delta_{\pi,\mu}T \mid \mu \in \mathbb{M}_n^r]\!]$$

and we are done by picking out the coefficient of T.

(3) The map

$$\theta : \mathbf{X}^r_{\pi,\Phi}(G) \to K^r_{\pi,\Phi}, \quad \psi \mapsto \theta(\psi) \tag{3.3}$$

is a group homomorphism. Moreover, for all μ we have

$$\theta(\phi_\mu \psi) = \phi_\mu \theta(\psi).$$

(4) If $\pi = p$ then the action of Σ_n on $J^r_{p,\Phi}(G)$ induces an action of Σ_n on $\mathbf{X}^r_{p,\Phi}(G)$. We also have an obvious action of Σ_n on K^r_Φ and the homomorphism (3.3) is Σ_n-equivariant.

In what follows let $\mathfrak{m} = \mathfrak{m}(R^{\text{alg}})$ be the maximal ideal of R^{alg}, let

$$G(\mathfrak{m}) := \text{Ker}(G(R^{\text{alg}}) \to G(k)),$$

and let $\ell^{\text{alg}} : G(\mathfrak{m}^{\text{alg}}) \to K^{\text{alg}}$ be the map induced by the logarithm series $\ell(T)$.

Corollary 3.9. *Let $\psi \in \mathbf{X}_{\pi,\Phi}(G)^\dagger$ be a totally δ-overconvergent δ_π-character and consider the homomorphism $\psi^{\text{alg}} : G(R^{\text{alg}}) \to K^{\text{alg}}$. The following hold:*

(1) *The restriction $\psi^{\mathfrak{m}}$ of ψ^{alg} to $G(\mathfrak{m})$ fits into a commutative diagram*

$$\begin{array}{ccc} G(\mathfrak{m}) & \xrightarrow{\ell^{\text{alg}}} & K^{\text{alg}} \\ {\scriptstyle \psi^{\mathfrak{m}}}\downarrow & & \downarrow{\scriptstyle \theta(\psi)^{\text{alg}}} \\ K^{\text{alg}} & = & K^{\text{alg}} \end{array} \tag{3.4}$$

(2) *The homomorphism ψ^{alg} can be extended to a (necessarily unique) continuous homomorphism $\psi^{\mathbb{C}_p} : G(\mathbb{C}_p^\circ) \to \mathbb{C}_p$.*

Proof. Part 1 follows directly from Remark 3.8, Part 1. To check Part 2 note that since ψ^{alg} is a homomorphism it is enough to check it can be extended by continuity on a ball in $G(\mathbb{C}_p^\circ)$ around the origin, cf. [12, Section 4.2]. This follows directly, exactly as in [12, proof of Proposition 6.8], from Part 1. ■

The following is a PDE version of the arithmetic ODE analogue (cf., [6, 8]) of Manin's Theorem of the kernel [27].

Corollary 3.10. *For every* $\psi \in \mathbf{X}_{\pi,\Phi}(G)^{\dagger}$ *there is a natural group isomorphism*

$$(\mathrm{Ker}(\psi^{\mathrm{alg}})) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathrm{Ker}(\theta(\psi)^{\mathrm{alg}}). \tag{3.5}$$

Proof. The exact sequence

$$0 \to G(\mathfrak{m}) \to G(R^{\mathrm{alg}}) \to G(k)$$

induces an exact sequence

$$0 \to \mathrm{Ker}(\psi^{\mathfrak{m}}) \to \mathrm{Ker}(\psi^{\mathrm{alg}}) \to G(k).$$

Since the group $G(k)$ is torsion we get an induced group isomorphism

$$(\mathrm{Ker}(\psi^{\mathfrak{m}})) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} (\mathrm{Ker}(\psi^{\mathrm{alg}})) \otimes_{\mathbb{Z}} \mathbb{Q}. \tag{3.6}$$

On the other hand recall that the map ℓ^{alg} in diagram (3.4) has a torsion kernel and cokernel; cf. [33, Proposition 3.2 and Theorem 6.4]. So tensoring the diagram (3.4) with $\mathbb{Q}$ the resulting upper horizontal map is an isomorphism. Taking the kernels of the resulting vertical maps we get an induced group isomorphism

$$(\mathrm{Ker}(\psi^{\mathfrak{m}})) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} \mathrm{Ker}(\theta(\psi)^{\mathrm{alg}}). \tag{3.7}$$

We are done by considering the composition of the map (3.7) with the inverse of the map (3.6). ∎

The following strengthened version of the above corollary is sometimes useful. Let L be a filtered union of complete subfields of K^{alg}, let $\mathcal{O}$ be the valuation ring of L, and let $\mathfrak{m}(\mathcal{O})$ be the maximal ideal of $\mathcal{O}$. Assume G comes via base change from a smooth group scheme $G_{\mathcal{O}}$ over $\mathcal{O}$ and write $G_{\mathcal{O}}(\mathcal{O}) =: G(\mathcal{O})$.

Corollary 3.11. *For every* $\psi \in \mathbf{X}_{\pi,\Phi}(G)^{\dagger}$ *the isomorphism in Corollary 3.10 induces an isomorphism*

$$(\mathrm{Ker}(\psi^{\mathrm{alg}}) \cap G(\mathcal{O})) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathrm{Ker}(\theta(\psi)^{\mathrm{alg}}) \cap L. \tag{3.8}$$

In particular, if $\mathrm{Ker}(\theta(\psi)^{\mathrm{alg}}) \cap L = 0$ *then*

$$\mathrm{Ker}(\psi^{\mathrm{alg}}) \cap G(\mathcal{O}) = G(\mathcal{O})_{\mathrm{tors}}.$$

Proof. It is enough to prove this for L complete. Let x be in the left-hand side of (3.8), hence in the left-hand side of (3.5). The image x' of x in the right-hand side of (3.5) is obtained as follows. One takes an integer $N \geq 1$ such that $Nx = P \otimes 1$ with $P \in \mathrm{Ker}(\psi^{\mathfrak{m}})$. Then $x' = \frac{1}{N}\ell^{\mathrm{alg}}(P)$. Since L is complete and ℓ has coefficients in $\mathcal{O}$ we get that ℓ^{alg} sends $G_{\mathcal{O}}(\mathfrak{m}(\mathcal{O}))$ into L, so we get that $x' \in L$ hence x' is in the right-hand side of (3.8). Conversely, let y' be in the right-hand side of (3.8). The image y of y' in the left-hand side of (3.5) is obtained as follows. There exists an integer $N \geq 1$ such that $Ny = \ell^{\text{alg}}(P) \otimes 1$ with $P \in G_{\mathcal{O}}(\mathfrak{m}(\mathcal{O}))$. By diagram (3.4) we have $P \in \text{Ker}(\psi^{\text{alg}})$. Then $y = P \otimes \frac{1}{N}$. So y is in the left-hand side of (3.5). ■

We end by providing an easy dimension evaluation. Define:

$$D(n,r) := \#\mathbb{M}_n^r = 1 + n + n^2 + \cdots + n^r. \tag{3.9}$$

The following proposition is trivial to check.

Proposition 3.12. *The map $\mathbf{X}^r_{\pi,\Phi}(G) \to K^r_{\pi,\Phi}$, $\psi \mapsto \theta(\psi)$ is injective. In particular*

$$\text{rank}_{R_\pi} \mathbf{X}^r_{\pi,\Phi}(G) \leq D(n,r). \tag{3.10}$$

3.3 Functions on points

The next proposition shows that, in the case of monomially independent Frobenius automorphisms, polynomial combinations of δ-characters are completely determined by their functions on points.

Proposition 3.13. *Assume $\phi_1, \dots, \phi_n$ are monomially independent in $\mathfrak{G}(K^{\text{alg}}/\mathbb{Q}_p)$ and denote by $R_\pi[\mathbf{X}^r_{\pi,\Phi}(G)^\dagger]$ the R_π-subalgebra of $\mathcal{O}(J^r_{\pi,\Phi}(G))$ generated by the elements of $\mathbf{X}^r_{\pi,\Phi}(G)^\dagger$. Then the R_π-algebra map*

$$R_\pi[\mathbf{X}^r_{\pi,\Phi}(G)^\dagger] \to \text{Fun}(G(R^{\text{alg}}), K^{\text{alg}}), \quad f \mapsto f^{\text{alg}}$$

is injective. In particular, the R_π-module homomorphism

$$\mathbf{X}^r_{\pi,\Phi}(G)^\dagger \to \text{Hom}_{\text{alg}}(G(R^{\text{alg}}), K^{\text{alg}}), \quad \psi \mapsto \psi^{\text{alg}}$$

is injective.

Proof. Let $\psi_1, \dots, \psi_N \in \mathbf{X}^r_{\pi,\Phi}(G)^\dagger$, let $F \in R_\pi[y_1, \dots, y_N]$ be a polynomial in the variables $y_1, \dots, y_N$, and let

$$f = F(\psi_1, \dots, \psi_N) \in \mathcal{O}(J^r_{\pi,\Phi}(G)).$$

Assume that the induced map $f^{\text{alg}} : G(R^{\text{alg}}) \to K^{\text{alg}}$ is the zero map. Then the composition of f^{alg} with the induced map $\mathcal{E}^{\text{alg}} : \mathbb{G}_a(R^{\text{alg}}) \to G(R^{\text{alg}})$ is the zero map. Write

$$\theta(\psi_i) = \sum_\mu \lambda_{i,\mu}\phi_\mu, \quad \lambda_{i,\mu} \in K_\mu.$$

Then for every $\lambda \in R^{\text{alg}}$ we have

$$0 = (f^{\text{alg}} \circ \mathcal{E}^{\text{alg}})(\lambda) = F\Big(\sum_\mu \lambda_{1,\mu}\phi_\mu(\lambda), \dots, \sum_\mu \lambda_{N,\mu}\phi_\mu(\lambda)\Big).$$

Let x_μ be variables indexed by $\mu \in \mathbb{M}_n^r$ and consider the polynomial

$$G(\dots, x_\mu, \dots) := F\Big(\sum_\mu \lambda_{1,\mu} x_\mu, \dots, \sum_\mu \lambda_{N,\mu} x_\mu\Big) \in R_\pi[\dots, x_\mu, \dots].$$

By Lemma 2.6 we get $G = 0$. But clearly f is obtained from G by replacing $x_\mu \mapsto \frac{1}{p}\phi_\mu \ell(T)$. So $f = 0$. ∎

Chapter 4

Multiplicative group

For $\pi \in \Pi$ define

$$N(\pi) := \min\left\{N \in \mathbb{Z} \;\middle|\; \frac{\pi^n}{n} \in \frac{1}{p^N}\mathbb{Z}_p \text{ for all } n \geq 1\right\}. \tag{4.1}$$

In particular, since p is odd we have $N(p) = -1$.

Denote by $\mathbb{G}_m$ the multiplicative group scheme Spec $R_\pi[x, x^{-1}]$ over R_π. Note that for each $i \in \{1, \ldots, n\}$ the series

$$p^{N(\pi)} \log\left(\frac{\phi_{\pi,i} x}{x^p}\right) := p^{N(\pi)} \log\left(1 + \pi \frac{\delta_{\pi,i} x}{x^p}\right) \tag{4.2}$$

(where log is the usual logarithm series) belongs to $R_\pi[x, x^{-1}, \delta_\pi x]\widehat{\ }$ and defines a totally δ-overconvergent δ_π-character $\psi_i \in \mathbf{X}^1_{\pi,\Phi}(\mathbb{G}_m)^\dagger$. Clearly the symbol of ψ_i is given by

$$\theta(\psi_i) = p^{N(\pi)+1}(\phi_i - p).$$

Theorem 4.1. *For all $r \geq 1$ we have*

$$\mathbf{X}^r_{\pi,\Phi}(\mathbb{G}_m)^\dagger = \mathbf{X}^r_{\pi,\Phi}(\mathbb{G}_m)$$

and a basis modulo torsion for this R_π-module is given by

$$\{\phi_\mu \psi_i \mid i \in \{1, \ldots, n\}, \mu \in \mathbb{M}_n^{r-1}\}. \tag{4.3}$$

Proof. By Proposition 3.12 the rank of $\mathbf{X}^r_{\pi,\Phi}(\mathbb{G}_m)$ is at most $1 + n + \cdots + n^r$. The symbols of the members of (4.3) are linearly independent so the $n + n^2 + \cdots + n^r$ elements of (4.3) are linearly independent. It is enough to check that $\mathbf{X}^r_{\pi,\Phi}(\mathbb{G}_m)$ does not have rank $1 + n + \cdots + n^r$; indeed this will make (4.3) a basis modulo torsion of $\mathbf{X}^r_{\pi,\Phi}(\mathbb{G}_m)$ and this will also force $\mathbf{X}^r_{\pi,\Phi}(\mathbb{G}_m)^\dagger$ and $\mathbf{X}^r_{\pi,\Phi}(\mathbb{G}_m)$ to have the same rank and hence to be equal by Lemma 3.3. Now if $\mathbf{X}^r_{\pi,\Phi}(\mathbb{G}_m)$ has rank $1 + n + \cdots + n^r$, by Proposition 3.12 the map $\mathbf{X}^r_{\pi,\Phi}(\mathbb{G}_m) \to K^r_{\pi,\Phi}$ tensored with K_π must be an isomorphism. So there must be an element $\psi \in \mathbf{X}^r_{\pi,\Phi}(\mathbb{G}_m)$ with symbol $\theta(\psi) =: c \in K_\pi$. Taking $T = x - 1$ we immediately get that the logarithm $\ell_{\mathbb{G}_m}(T) = \log(1 + T)$ of the formal group of $\mathbb{G}_m$ times a power of p belongs to $R_\pi[[T]]$ which is a contradiction. ∎

Remark 4.2. The moral of the above theorem is that the PDE story in the case of $\mathbb{G}_m$ can be reduced to the ODE story via face maps. As we shall see in the next section no such reduction is possible in the case of elliptic curves where 'genuinely PDE' δ-characters exist.

The next corollary is a strengthening of [12, Proposition 3.5].

Corollary 4.3. *Let* $\pi \in \Pi$ *and for* $i \in \{1, \ldots, n\}$ *let* ψ_i *be the* δ_π*-character in* (4.2). *Consider the induced homomorphism* $\psi_i^{\mathrm{alg}} : \mathbb{G}_m(R^{\mathrm{alg}}) \to K^{\mathrm{alg}}$. *Then*

$$\mathrm{Ker}(\phi_i^{\mathrm{alg}}) = \mathbb{G}_m(R^{\mathrm{alg}})_{\mathrm{tors}}. \tag{4.4}$$

Proof. This follows directly from Corollary 3.10 in view of the fact that the map

$$\theta(\psi_i)^{\mathrm{alg}} : \mathbb{Q}_p^{\mathrm{alg}} \to \mathbb{Q}_p^{\mathrm{alg}}, \quad \beta \mapsto \phi_i(\beta) - p\beta$$

is injective. ∎

Chapter 5

Elliptic curves

5.1 General case

Throughout this subsection E is an elliptic curve (abelian scheme of dimension 1) over R_π, and we fix an invertible 1-form ω on E. By a *formal group* over a ring we will always understand a formal group law (i.e, a tuple of elements in a formal power series ring). For every family $\Phi := (\phi_1, \dots, \phi_n)$ of Frobenius automorphisms of K^{alg} define

$$N^r_{\pi,\Phi} := \operatorname{Ker}(J^r_{\pi,\Phi}(E) \to \widehat{E_{R_\pi}}).$$

Consider an admissible coordinate T on E. Exactly as in [10, Proposition 4.45] $N^r_{\pi,\Phi}$ is a group object in the category of p-adic formal schemes over R_π whose underlying p-adic formal scheme can be identified with the p-adic completion of the affine space

$$(\mathbb{A}^{n+\cdots+n^r}_{R_\pi})\widehat{\ } = \operatorname{Spf} R_\pi[\delta_{\pi,\mu} T \mid \mu \in \mathbb{M}^{r,+}_n]\widehat{\ }$$

and whose group law is obtained as follows. One considers the formal group law $F(T_1, T_2) \in R_\pi[[T_1, T_2]]$ of E with respect to T and one considers the group law on $N^r_{\pi,\Phi}$ defined by the series $F_\mu := (\delta_{\pi,\mu} F)|_{T=0}$ for $\mu \in \mathbb{M}^r_n$ (which turn out to be restricted power series rather than just formal power series).

Proposition 5.1. *The R_π-module* $\operatorname{Hom}(N^r_{\pi,\Phi}, \widehat{\mathbb{G}_a})$ *has rank*

$$\operatorname{rank}_{R_\pi} \operatorname{Hom}(N^r_{\pi,\Phi}, \widehat{\mathbb{G}_a}) = D(n,r) - 1 = n + \cdots + n^r.$$

Proof. Denote by $(N^r_{\pi,\Phi})^{\text{for}}$ the formal group over R_π associated to $N^r_{\pi,\Phi}$ and to the variables $\delta_{\pi,\mu} T$, $\mu \in \mathbb{M}^{r,+}_n$, and denote by $(N^r_{\pi,\Phi})^{\text{for}}_{K_\pi}$ the induced formal group law over K_π which is isomorphic to a power of the additive formal group law $\mathbb{G}^{\text{for}}_{a/K_\pi}$ (since $(N^r_{\pi,\Phi})^{\text{for}}_{K_\pi}$ is commutative over a field of characteristic zero). We have natural injective maps of K_π-vector spaces

$$\operatorname{Hom}(N^r_{\pi,\Phi}, \widehat{\mathbb{G}_a}) \otimes_{R_\pi} K_\pi \to \operatorname{Hom}_{\text{for.gr.}}((N^r_{\pi,\Phi})^{\text{for}}_{K_\pi}, \mathbb{G}^{\text{for}}_{a,K_\pi}) \simeq K_\pi^{n+\cdots+n^r}.$$

On the other hand we will construct, in what follows, $D(n,r) - 1$ K_π-linearly independent elements in $\operatorname{Hom}(N^r_{\pi,\Phi}, \widehat{\mathbb{G}_a})$; from this our proposition follows. Recall the logarithm series $\ell(T) = \ell_\omega(T) \in K_\pi[[T]]$ normalized with respect to ω. Recalling the integer $N(\pi)$ in (4.1) we have that, for $\mu \in \mathbb{M}^{r,+}_n$, the series

$$L^\mu_{\pi,\Phi} := (\phi_\mu(\ell(T)))|_{T=0} \in K_\pi[[\delta_{\pi,\nu} T \mid \nu \in \mathbb{M}^{r,+}_n]] \qquad (5.1)$$

satisfies

$$\tilde{L}^{\mu}_{\pi,\Phi} := p^{N(\pi)} L^{\mu}_{\pi,\Phi} \in R_\pi[\delta_{\pi,\nu} T \mid \nu \in \mathbb{M}^{r,+}_n]\widehat{\ }. \tag{5.2}$$

It follows that

$$\tilde{L}^{\mu}_{\pi,\Phi} \in \mathrm{Hom}(N^r_{\pi,\Phi}, \widehat{\mathbb{G}_a}). \tag{5.3}$$

It is trivial to check that the elements $\tilde{L}^{\mu}_{\pi,\Phi}$ are K_π-linearly independent, which ends our proof. ■

Remark 5.2. Exactly as in [12, Proposition 4.6] we get that for every $\pi'|\pi$ and $\mu \in \mathbb{M}^r_n$ the element $p^{N(\pi')-N(\pi)}\tilde{L}^{\mu}_{\pi,\Phi} \otimes 1$ is the image of $\tilde{L}^{\mu}_{\pi',\Phi}$ via the homomorphism

$$R_{\pi'}[\delta_{\pi',\mu} T \mid \mu \in \mathbb{M}^{r,+}_n]\widehat{\ } \to R_\pi[\delta_{\pi,\nu} T \mid \nu \in \mathbb{M}^{r,+}_n]\widehat{\ } \otimes_{R_\pi} R_{\pi'}.$$

Remark 5.3. Assume $\pi = p$. As in [10, page 124] for all $i_1 \dots i_r \in \mathbb{M}^{r,+}_n$ we have

$$\phi_{i_1\dots i_r} T - T^{p^r} - p(\delta_{i_r} T)^{p^{r-1}} \in (pT, p^2) \subset R[\delta_\nu T \mid \nu \in \mathbb{M}^r_n].$$

Hence, following [10, Proposition 4.41] we get that

$$\tilde{L}^{i_1\dots i_r}_{\pi,\Phi} \equiv (\delta_{i_r} T)^{p^{r-1}} \bmod p \ \text{ in } \ R[\delta_{p,\mu} T \mid \mu \in \mathbb{M}^{r,+}_n]\widehat{\ }.$$

Remark 5.4. Consider the following standard cohomology sequence (cf. [10, page 191] for the case $n = 1$):

$$0 = \mathrm{Hom}(\widehat{E}, \widehat{\mathbb{G}_a}) \to \mathrm{Hom}(J^r_{\pi,\Phi}(E), \widehat{\mathbb{G}_a}) \to \mathrm{Hom}(N^r_{\pi,\Phi}, \widehat{\mathbb{G}_a}) \xrightarrow{\partial^r} H^1(\widehat{E}, \mathcal{O}) \tag{5.4}$$

and consider the isomorphism defined by Serre duality,

$$\langle -, \omega\rangle : H^1(\widehat{E}, \mathcal{O}) \to R_\pi.$$

It is useful to recall the explicit construction of the map ∂^r. By Proposition 2.31 there exists an affine open cover $E = \bigcup_i U_i$ and sections $s^r_i : \widehat{U_i} \to \mathrm{pr}_r^{-1}(U_i)$ of the projection $\mathrm{pr}_r : \mathrm{pr}_r^{-1}(U_i) \to \widehat{U_i}$ induced by the projection $\mathrm{pr}_r : J^r_{\pi,\Phi}(E) \to \widehat{E}$. Then for all $L \in \mathrm{Hom}(N^r_{\pi,\Phi}, \widehat{\mathbb{G}_a})$ the element $\partial^r(L)$ is defined as the cohomology class

$$[L \circ (s^r_i - s^r_j)] \in H^1(\widehat{E}, \mathcal{O})$$

of the cocycle

$$(L \circ (s^r_i - s^r_j))_{ij}, \quad L \circ (s^r_i - s^r_j) : \widehat{U_i} \cap \widehat{U_j} \to N^r_{\pi,\Phi} \to \widehat{\mathbb{G}_a}.$$

The definition above is independent of the choice of sections s^r_i; such a change would change the cocycle by a coboundary.

Following [10, page 194] and recalling the elements in (5.3), we introduce the following.

Definition 5.5. For $\mu \in \mathbb{M}_n^{r,+}$ we define the *primary arithmetic Kodaira–Spencer class* of E attached to μ by the formula

$$f_\mu := \langle \partial^r(\tilde{L}^\mu_{\pi,\Phi}), \omega\rangle \in R_\pi$$

and consider the vector

$$\mathrm{KS}^r_{\pi,\Phi}(E) = (f_\mu)_{\mu \in \mathbb{M}_n^{r,+}} \in R_\pi^{\mathbb{M}_n^{r,+}}.$$

Remark 5.6. A few comments are important at this point.

(1) The elements f_μ are easily seen to depend only on the pair (E, ω) and not on the choice of T. This follows from the easily-checked fact that our construction can be presented in a coordinate-free manner: instead of the rings $R_\pi[\![T]\!]$, $K_\pi[\![T]\!]$ one may consider the completion A of the local ring of E at the closed point of the identity section and the corresponding completion A_{K_π} for $E \otimes_{R_\pi} K_\pi$. Instead of

$$R_\pi[\![\delta_{\pi,\mu} T \mid \mu \in \mathbb{M}_n^r]\!],\ K_\pi[\![\delta_{\pi,\mu} T \mid \mu \in \mathbb{M}_n^r]\!],\ R_\pi[\![T]\!][\delta_{\pi,\mu} T \mid \mu \in \mathbb{M}_n^{r,+}]\widehat{\ },$$

one may consider certain new types of "π-jet algebras" A^r, $A^r_{K_\pi}$, $\tilde{A}^r$, attached to A respectively, satisfying certain new corresponding universal properties. We will not go here into defining these new types of π-jet algebras. Then $\ell \in A_{K_\pi}$ is defined by the condition that it "vanishes" at the ideal of the zero section and $d\ell = \omega$ in the completed module of Kähler differentials of A_{K_π}, $\phi_\mu \ell$ makes sense as an element of $A^r_{K_\pi}$ while $\phi_\mu \ell_{|T=0}$ makes sense as an element of $\tilde{A}^r$.

(2) We will write $f_\mu(E, \omega)$ instead of f_μ if we want to emphasize the dependence on (E, ω). With notation as in Remark 2.14, we have

$$f_\mu(E, \lambda\omega) = \lambda^{\phi_\mu + 1} f_\mu(E, \omega)$$

for all $\lambda \in R_\pi^\times$; this follows from the fact that if one replaces ω by $\lambda\omega$ in our construction then, since $\ell_{\lambda\omega}(T) = \lambda \ell_\omega(T)$, we have that $L^\mu_{\pi,\Phi}$ gets replaced by $\phi_\mu(\lambda) L^\mu_{\pi,\Phi}$.

(3) It is easy to see that the elements f_μ do not change if r changes, as long as $\mu \in \mathbb{M}_n^{r,+}$; this follows from the fact that changing r amounts to changing the defining cocycle in our construction by a coboundary which does not change the cohomology class. This justifies not including r in our notation for f_μ. In particular, if $\mathrm{KS}^r_{\pi,\Phi}(E) \neq 0$ for some $r \geq 1$ then $\mathrm{KS}^{r'}_{\pi,\Phi}(E) \neq 0$ for all $r' \geq r$.

(4) It is easy to check that the formation of the elements f_μ is compatible with face maps in the sense that for every $\mu' \in \mathbb{M}_{n'}^{r,+}$ we have, in the above notation:

$$f_{\mu'} = f_{\epsilon(\mu')}. \tag{5.5}$$

On the other hand note that in general

$$f_{i\mu} \neq \phi_i f_\mu.$$

(5) For every isogeny $u : E' \to E$ of degree d prime to p of elliptic curves over R_π and every invertible 1-form ω on E, setting $\omega' = u^*\omega$, we have

$$f_\mu(E', \omega') = d \cdot f_\mu(E, \omega).$$

The argument is entirely similar to the one in [10, page 264].

(6) Let us write $f_{\pi,\mu}$ instead of f_μ if we want to emphasize dependence on π. Then for all $\pi' | \pi$ we clearly have

$$f_{\pi',\mu}(E_{\pi'}, \omega_{\pi'}) = p^{N(\pi')-N(\pi)} f_{\pi,\mu}(E, \omega) \in R_{\pi'}$$

where $E_{\pi'} := E \otimes_{R_\pi} R_{\pi'}$ and $\omega_{\pi'}$ is the induced form.

A special role will be played later by the primary arithmetic Kodaira–Spencer classes $f_i, f_{ii}, f_{iii}, \ldots$ We will write

$$f_{i^r} := f_{i \ldots i} \quad \text{with } i \text{ repeated } r \text{ times.}$$

These classes are the images, via the corresponding face maps, of the corresponding classes obtained by replacing Φ by $\{\phi_i\}$ in all our constructions. Note that in [3] and [10] the forms f_{i^r} were denoted by f^r.

Proposition 5.7. *Assume E over R_π has ordinary reduction and assume $f_i = 0$ for some i. Then for all $\mu \in \mathbb{M}_n$ we have $f_\mu = 0$.*

This proposition cannot be proved at this point in the memoir but is an immediate consequence of Theorem 7.39 that will be stated and proved later.

Lemma 5.8. *Assume $\pi = p$. Then for all $\mu \in \mathbb{M}_n^r$ and all $\sigma \in \Sigma_n$ we have*

$$f_{\sigma\mu} = f_\mu.$$

Proof. Let $s_i^r : \widehat{U}_i \to J_{p,\Phi}^r(\mathrm{pr}_r^{-1}(U_i))$ be local sections of the projection $\mathrm{pr}_r : J_{p,\Phi}^r(E) \to \widehat{E}$ as in Remark 5.4, and consider the group automorphism over E,

$$\sigma : J_{p,\Phi}^r(E) \to J_{p,\Phi}^r(E)$$

induced by $\sigma \in \Sigma_n$. Consider the local sections $\sigma \circ s_i^r : \widehat{U}_i \to J_{p,\Phi}^r(\mathrm{pr}_r^{-1}(U_i))$. By the independence of f_μ on the choice of local sections we get

$$\begin{aligned} f_\mu &= \langle [\tilde{L}_{\pi,\Phi}^\mu \circ (s_i^r - s_j^r)], \omega \rangle \\ &= \langle \tilde{L}_{\pi,\Phi}^\mu \circ (\sigma \circ s_i^r - \sigma \circ s_j^r), \omega \rangle \\ &= \langle \tilde{L}_{\pi,\Phi}^\mu \circ \sigma \circ (s_i^r - s_j^r), \omega \rangle \\ &= \langle \tilde{L}_{\pi,\Phi}^{\sigma\mu} \circ (s_i^r - s_j^r), \omega \rangle \\ &= f_{\sigma\mu}. \end{aligned}$$

∎

Remark 5.9. Assume $\pi = p$ and fix an index i. By its very construction $f_i = 0$ if and only if E possesses a Frobenius lift (i.e., a scheme endomorphism reducing modulo p to the p-power Frobenius). Recall that if E has ordinary reduction then E has a Frobenius lift if and only if E is a canonical lift of its reduction [29, Appendix]. On the other hand recall from [10, Corollary 8.89] that if E has supersingular reduction then $f_i \neq 0$. We conclude that for an arbitrary E over $R = R_p$ we have $f_i = 0$ if and only if E has ordinary reduction and is a canonical lift of its reduction.

Consider the K_π-linear space

$$K^{r,+}_{\pi,\Phi} = \left\{ \sum_{\mu \in \mathbb{M}^{r,+}_n} \lambda_\mu \phi_\mu \,\middle|\, \lambda_\mu \in K_\pi \right\} \subset K^r_{\pi,\Phi}$$

and the projection

$$\rho : K^r_{\pi,\Phi} \to K^{r,+}_{\pi,\Phi}, \quad \rho\Bigg(\sum_{\mu \in \mathbb{M}^r_n} \lambda_\mu \phi_\mu \Bigg) = \sum_{\mu \in \mathbb{M}^{r,+}_n} \lambda_\mu \phi_\mu .$$

We may consider the K_π-linear space of relations among the primary arithmetic Kodaira–Spencer classes:

$$\mathrm{KS}^r_{\pi,\Phi}(E)^\perp := \left\{ \sum_{\mu \in \mathbb{M}^{r,+}_n} \lambda_\mu \phi_\mu \in K^{r,+}_{\pi,\Phi} \,\middle|\, \sum_{\mu \in \mathbb{M}^{r,+}_n} \lambda_\mu f_\mu = 0 \right\} \tag{5.6}$$

and its R_π-submodule of "integral elements,"

$$\mathrm{KS}^r_{\pi,\Phi}(E)^\perp_{\mathrm{int}} := \left\{ \sum_{\mu \in \mathbb{M}^{r,+}_n} \lambda_\mu \phi_\mu \in \mathrm{KS}^r_{\pi,\Phi}(E)^\perp \,\middle|\, \lambda_\mu \in R_\pi \right\}.$$

Finally, recall the *symbol homomorphism*

$$\theta : \mathbf{X}^r_{\pi,\Phi}(E) \to K^r_{\pi,\Phi}, \quad \psi \mapsto \theta(\psi).$$

Theorem 5.10. *The following claims hold.*

(1) *There exists an R_π-module homomorphism P as in (5.7) below such that the composition*

$$\mathrm{KS}^r_{\pi,\Phi}(E)^\perp_{\mathrm{int}} \xrightarrow{P} \mathbf{X}^r_{\pi,\Phi}(E)^\dagger \subset \mathbf{X}^r_{\pi,\Phi}(E) \xrightarrow{\theta} K^r_{\pi,\Phi} \xrightarrow{\rho} K^{r,+}_{\pi,\Phi} \tag{5.7}$$

is the multiplication by $p^{N(\pi)+1}$ map. So for $\pi = p$ the composition (5.7) is the inclusion $\mathrm{KS}^r_{\pi,\Phi}(E)^\perp_{\mathrm{int}} \subset K^{r,+}_{\pi,\Phi}$.

(2) *The map $\rho \circ \theta$ is injective. In particular, if $\theta(\psi) \in K_\pi$ for some $\psi \in \mathbf{X}^r_{\pi,\Phi}(E)$ then $\psi = 0$ and hence $\theta(\psi) = 0$.*

Proof. To prove Part 1 note that if

$$\Lambda := \sum_{\mu \in \mathbb{M}_n^{r,+}} \lambda_\mu \phi_\mu \in \mathrm{KS}^r_{\pi,\Phi}(E)^{\perp}_{\mathrm{int}}$$

and if

$$L_\Lambda := \sum_{\mu \in \mathbb{M}_n^{r,+}} \lambda_\mu \tilde{L}^\mu_{\pi,\Phi} \in \mathrm{Hom}(N^r_{\pi,\Phi}, \widehat{\mathbb{G}_a})$$

then $\partial(L_\Lambda) = 0$ so L_Λ is the restriction of a unique element

$$P(\Lambda) \in \mathrm{Hom}(J^r_{\pi,\Phi}(E), \widehat{\mathbb{G}_a}) = \mathbf{X}^r_{\pi,\Phi}(E). \tag{5.8}$$

Clearly $\Lambda \mapsto P(\Lambda)$ is an R_π-linear map. By an argument entirely similar to the one in the proof of [12, Theorem 6.1] and using Remark 5.2 above it follows that $P(\Lambda)$ is totally δ-overconvergent: $P(\Lambda) \in \mathbf{X}^r_{\pi,\Phi}(E)^{\dagger}$. By an argument entirely similar to the one in the proof of [10, Proposition 7.20] one gets that

$$\theta(P(\Lambda))T \equiv p^{N(\pi)+1}\Big(\sum_{\mu \in \mathbb{M}_n^{r,+}} \lambda_\mu \phi_\mu T\Big) \bmod T$$

in the ring $K_\pi[\delta_{\pi,\mu} T \mid \mu \in \mathbb{M}^r_n]$. By Lemma 3.5, Part 2, we have

$$\theta(P(\Lambda))T = p^{N(\pi)+1}\Big(\sum_{\mu \in \mathbb{M}_n^{r,+}} \lambda_\mu \phi_\mu T\Big) + \lambda_0 T$$

for some $\lambda_0 \in R_\pi$. Hence

$$\rho(\theta(P(\Lambda))) = p^{N(\pi)+1}\Big(\sum_{\mu \in \mathbb{M}_n^{r,+}} \lambda_\mu \phi_\mu\Big)$$

and Part 1 follows.

Part 2 follows from the observation that if $\theta(\psi) \in K_\pi$ for some $\psi \in \mathbf{X}^r_{\pi,\Phi}(E)$ then by Remark 3.8, Part 1 it easily follows that

$$\psi \in \mathcal{O}(J^2_{\pi,\Phi}(E)) \cap K_\pi[\![T]\!] = \mathcal{O}(\widehat{E})$$

and hence ψ defines a homomorphism $\widehat{E} \to \widehat{\mathbb{G}_a}$; but the only such homomorphism is the zero homomorphism. ∎

Remark 5.11. Note that P in Theorem 5.10 is automatically injective. For all $\Lambda \in \mathrm{KS}^r_{\pi,\Phi}(E)^{\perp}_{\mathrm{int}}$ we write $\psi_\Lambda := P(\Lambda)$; hence, by Remark 3.8, Part 2, we have

$$\theta(\psi_\Lambda) = p^{N(\pi)+1}\Lambda + \lambda_0(\Lambda)$$

for some $\lambda_0(\Lambda) \in pR_\pi$. Clearly the map

$$\mathrm{KS}^r_{\pi,\Phi}(E)^\perp_{\mathrm{int}} \to pR_\pi, \quad \Lambda \mapsto \lambda_0(\Lambda)$$

is an R_π-module homomorphism.

Remark 5.12. The map P in Theorem 5.10 is compatible with the face maps (2.7) in an obvious sense.

Remark 5.13. If $f_i = 0$ for some i then $\phi_i \in \mathrm{KS}^r_{\pi,\Phi}(E)^\perp_{\mathrm{int}}$ hence

$$\psi_i := \psi_{\phi_i} \in \mathbf{X}^1_{\pi,\Phi}(E)^\dagger.$$

Moreover, the symbol of ψ_i is given by

$$\theta(\psi_i) = p^{N(\pi)+1}\phi_i + \lambda_0(\phi_i).$$

Corollary 5.14. *The following claims hold.*

(1) *If* $\mathrm{KS}^r_{\pi,\Phi}(E) \neq 0$ *(in particular if* $\mathrm{KS}^1_{\pi,\Phi}(E) \neq 0$*), then we have* $\mathbf{X}^r_{\pi,\Phi}(E) = \mathbf{X}^r_{\pi,\Phi}(E)^\dagger$ *and the rank of this* R_π*-module equals* $D(n,r) - 2$.

(2) *If* $\mathrm{KS}^1_{\pi,\Phi}(E) = 0$ *(equivalently, if* $f_i = 0$ *for all* $i \in \{1,\dots,n\}$*) then the equality* $\mathbf{X}^r_{\pi,\Phi}(E) = \mathbf{X}^r_{\pi,\Phi}(E)^\dagger$ *holds, the rank of this* R_π*-module equals* $D(n,r) - 1$*, and a basis modulo torsion for this* R_π*-module is given by*

$$\{\phi_\mu \psi_i \mid \mu \in \mathbb{M}^{r,+}_n,\ i \in \{1,\dots,n\}\}. \tag{5.9}$$

(3) *The cokernel of the injective homomorphism* P *in Theorem* 5.10 *is a torsion* R_π*-module.*

Proof. If $\mathrm{KS}^r_{\pi,\Phi}(E) \neq 0$ then the module $\mathrm{KS}^r_{\pi,\Phi}(E)^\perp_{\mathrm{int}}$ has rank $D(n,r) - 2$. Since P in Theorem 5.10 is injective the module $\mathbf{X}^r_{\pi,\Phi}(E)^\dagger$ has rank at least $D(n,r) - 2$. On the other hand by Proposition 5.1 and by the exact sequence (5.4) the module $\mathbf{X}^r_{\pi,\Phi}(E)$ has rank at most $D(n,r) - 2$. So the modules $\mathbf{X}^r_{\pi,\Phi}(E)$ and $\mathbf{X}^r_{\pi,\Phi}(E)^\dagger$ have the same rank $D(n,r) - 2$ and hence they are equal by Lemma 3.3. This proves Part 1.

Assume now $\mathrm{KS}^1_{\pi,\Phi}(E) = 0$. The subset (5.9) of $\mathbf{X}^r_{\pi,\Phi}(E)^\dagger$ is linearly independent (because so is the set of symbols of its elements). It follows that $\mathbf{X}^r_{\pi,\Phi}(E)^\dagger$ has rank at least $D(n,r) - 1$. On the other hand by Proposition 5.1 and the sequence (5.4) the module $\mathbf{X}^r_{\pi,\Phi}(E)$ has rank at most $D(n,r) - 1$. So the modules $\mathbf{X}^r_{\pi,\Phi}(E)$ and $\mathbf{X}^r_{\pi,\Phi}(E)^\dagger$ have the same rank $D(n,r) - 1$ and hence they are equal by Lemma 3.3, with basis modulo torsion given by (5.9). This proves Part 2.

Part 3 follows from the fact that the source and target of P have the same rank. ■

As an application to Theorem 5.10 we construct a series of special δ_π-characters of E as follows. Let $\mu, \nu \in \mathbb{M}_n^{r,+}$ be distinct and let ω be an invertible 1-form on E. Recalling the integers $N(\pi)$ in (4.1) set

$$\tilde{f}_\mu := p^{N(\pi)+1} f_\mu, \quad \mu \in \mathbb{M}_n. \tag{5.10}$$

In particular, if $\pi = p$ then $\tilde{f}_\mu = f_\mu$.

Note that

$$f_\nu \phi_\mu - f_\mu \phi_\nu \in \mathrm{KS}^r_{\pi,\Phi}(E)^\perp_{\mathrm{int}}$$

so we may consider the partial δ_π-character

$$\psi_{\mu,\nu} := \psi_{f_\nu \phi_\mu - f_\mu \phi_\nu} \in \mathbf{X}^r_{\pi,\Phi}(E)^\dagger. \tag{5.11}$$

By Theorem 5.10 we have

$$\theta(\psi_{\mu,\nu}) = \tilde{f}_\nu \phi_\mu - \tilde{f}_\mu \phi_\nu + f_{\mu,\nu} \tag{5.12}$$

for some $f_{\mu,\nu} \in pR_\pi$.

Definition 5.15. The above element $f_{\mu,\nu} \in pR_\pi$ is called the *secondary arithmetic Kodaira–Spencer class* attached to μ and ν.

Remark 5.16. Note that $\psi_{\mu,\nu}$ and $f_{\mu,\nu}$ do not change if r changes which justifies r not being included in the notation. Note also that $\psi_{\mu,\nu}$ and $f_{\mu,\nu}$ effectively depend on (E and) ω and if we want to emphasize this dependence we denote them by $\psi_{\mu,\nu}(E,\omega)$ and $f_{\mu,\nu}(E,\omega)$, respectively. Similarly, we write $\tilde{f}_\mu(E,\omega)$ in place of $\tilde{f}_\mu$. Then for all $\lambda \in R_\pi^\times$ we have (using the notation in Remark 2.14):

$$f_{\mu,\nu}(E, \lambda\omega) = \lambda^{\phi_\mu + \phi_\nu} f_{\mu,\nu}(E,\omega).$$

Indeed, by Remark 3.8, Part 1, and Remark 5.6, Part 1, we have the following equalities

$$\begin{aligned}
\psi_{\mu,\nu}(E,\omega) =& \frac{1}{p}(\tilde{f}_\nu(E,\omega)\phi_\mu - \tilde{f}_\mu(E,\omega)\phi_\nu + f_{\mu,\nu}(E,\omega))\ell_\omega(T),\\
\psi_{\mu,\nu}(E,\lambda\omega) =& \frac{1}{p}(\tilde{f}_\nu(E,\lambda\omega)\phi_\mu - \tilde{f}_\mu(E,\lambda\omega)\phi_\nu + f_{\mu,\nu}(E,\lambda\omega))\ell_{\lambda\omega}(T)\\
=& \frac{1}{p}(\lambda^{\phi_\nu+1}\tilde{f}_\nu(E,\omega)\phi_\mu - \lambda^{\phi_\mu+1}\tilde{f}_\mu(E,\omega)\phi_\nu\\
&+ f_{\mu,\nu}(E,\lambda\omega))(\lambda\ell_\omega(T))\\
=& \frac{1}{p}(\lambda^{\phi_\nu+\phi_\mu+1}\tilde{f}_\nu(E,\omega)\phi_\mu - \lambda^{\phi_\mu+\phi_\nu+1}\tilde{f}_\mu(E,\omega)\phi_\nu\\
&+ \lambda f_{\mu,\nu}(E,\lambda\omega))\ell_\omega(T).
\end{aligned}$$

We get

$$\begin{aligned}\psi^* :=& \lambda^{\phi_\mu+\phi_\nu+1}\psi_{\mu,\nu}(E,\omega) - \psi_{\mu,\nu}(E,\lambda\omega)\\ =& \frac{1}{p}(\lambda^{\phi_\mu+\phi_\nu+1} f_{\mu,\nu}(E,\omega) - \lambda f_{\mu,\nu}(E,\lambda\omega))\ell_\omega(T).\end{aligned}$$

Hence

$$\theta(\psi^*) = \frac{1}{p}(\lambda^{\phi_\mu+\phi_\nu+1} f_{\mu,\nu}(E,\omega) - \lambda f_{\mu,\nu}(E,\lambda\omega)) \in K_\pi.$$

By Theorem 5.10, Part 2, $\theta(\psi^*) = 0$ which ends the proof.

Remark 5.17. For all distinct μ, ν we have

$$f_{\mu,\nu} + f_{\nu,\mu} = 0. \tag{5.13}$$

Indeed, switching μ and ν in (5.12) we get

$$\theta(\psi_{\nu,\mu}) = \tilde{f}_\mu\phi_\nu - \tilde{f}_\nu\phi_\mu + f_{\nu,\mu}. \tag{5.14}$$

Adding (5.12) and (5.14) we may conclude by Theorem 5.10, Part 2.

Remark 5.18. Fix in what follows the elliptic curve E over R_π and an invertible 1-form ω. Write, as before, $\ell(T) = \ell_\omega(T) = \sum_{m=1}^\infty \frac{b_m}{m}T^m$, $b_m \in R_\pi$. Let $\mu, \nu \in \mathbb{M}_n$ be distinct of lengths $r \geq s$, respectively. By Remark 3.8, Part 1, we have that

$$\psi_{\mu,\nu} = \frac{1}{p}(\tilde{f}_\nu\phi_\mu - \tilde{f}_\mu\phi_\nu + f_{\mu,\nu})\ell(T) \in R_\pi[\![\delta_{\pi,\eta}T \mid \eta \in \mathbb{M}_n^r]\!]. \tag{5.15}$$

On the other hand we can write

$$\phi_{\pi,\mu}T = T^{p^r} + G_\mu, \quad \phi_{\pi,\nu}T = T^{p^s} + G_\nu$$

with G_μ, G_ν in the ideal I_r of $R_\pi[\delta_{\pi,\eta}T \mid \eta \in \mathbb{M}_n^r]$ generated by the set

$$\{\delta_{\pi,\eta}T \mid \eta \in \mathbb{M}_n^{r,+}\}.$$

A direct computation shows

$$\begin{aligned}p\psi_{\mu,\nu} =& \tilde{f}_\nu\Big(\sum_m \frac{\phi_\mu(b_m)}{m}(T^{p^r} + G_\mu)^m\Big)\\ &- \tilde{f}_\mu\Big(\sum_m \frac{\phi_\nu(b_m)}{m}(T^{p^s} + G_\nu)^m\Big)\\ &+ f_{\mu,\nu}\Big(\sum_m \frac{b_m}{m}T^m\Big).\end{aligned}$$

Reducing the above equality modulo the ideal I_r we get

$$\tilde{f}_\nu\Big(\sum_m \frac{\phi_\mu(b_m)}{m} T^{p^r m}\Big) - \tilde{f}_\mu\Big(\sum_m \frac{\phi_\nu(b_m)}{m} T^{p^s m}\Big) + f_{\mu,\nu}\Big(\sum_m \frac{b_m}{m} T^m\Big) \in pR_\pi[\![T]\!].$$

For all integers $N \geq 1$, picking out the coefficients of $T^{p^r N}$, we get the following analogue of the integrality conditions of Atkin and Swinnerton-Dyer [1, 34].

Corollary 5.19. *For all integers $N \geq 1$*

$$\tilde{f}_\nu \frac{\phi_\mu(b_N)}{N} - \tilde{f}_\mu \frac{\phi_\nu(b_{p^{r-s}N})}{p^{r-s}N} + f_{\mu,\nu} \frac{b_{p^r N}}{p^r N} \in pR_\pi. \tag{5.16}$$

Remark 5.20. For every isogeny $u : E' \to E$ of degree d prime to p of elliptic curves over R_π and every invertible 1-form ω on E, setting $\omega' = u^*\omega$, we have

$$f_{\mu,\nu}(E', \omega') = d \cdot f_{\mu,\nu}(E, \omega).$$

Indeed, we may identify two admissible coordinates for E and E' (call this parameter T) in which case we identify the images of ω' and ω in $R_\pi[\![T]\!]dT$, and we identify the two series ℓ_ω and $\ell_{\omega'}$ in $R_\pi[\![T]\!]$. As in 5.15 we consider the partial δ_π-characters of E and E' respectively:

$$\psi := \psi_{\mu,\nu} = \frac{1}{p}(\tilde{f}_\nu(E,\omega)\phi_\mu - \tilde{f}_\mu(E,\omega)\phi_\nu + f_{\mu,\nu}(E,\omega))\ell(T), \tag{5.17}$$

$$\psi' := \psi'_{\mu,\nu} = \frac{1}{p}(\tilde{f}_\nu(E',\omega')\phi_\mu - \tilde{f}_\mu(E',\omega')\phi_\nu + f_{\mu,\nu}(E',\omega'))\ell(T). \tag{5.18}$$

Identifying ψ with its image in the space of δ_π-characters of E' and using Remark 5.6, Part 5, we get that

$$\psi' - d \cdot \psi = (f_{\mu,\nu}(E',\omega') - d \cdot f_{\mu,\nu}(E,\omega))\ell_\omega(T).$$

Hence

$$\theta(\psi' - d \cdot \psi) = f_{\mu,\nu}(E',\omega') - d \cdot f_{\mu,\nu}(E,\omega) \in R_\pi.$$

By Theorem 5.10, Part 2, $\theta(\psi' - d \cdot \psi) = 0$ which ends the proof.

Remark 5.21. Let us write $f_{\pi,\mu,\nu}$ and $\psi_{\pi,\mu,\nu}$ instead of $f_{\mu,\nu}$ and $\psi_{\mu,\nu}$ if we want to emphasize dependence on π. Then for all $\pi'|\pi$ we have

$$\psi_{\pi',\mu,\nu} = p^{2N(\pi')-2N(\pi)}\psi_{\pi,\mu,\nu} \in \mathbf{X}^r_{\pi',\Phi}(E), \tag{5.19}$$

$$f_{\pi',\mu,\nu} = p^{2N(\pi')-2N(\pi)} f_{\pi,\mu,\nu} \in R_{\pi'}. \tag{5.20}$$

This follows from Remark 5.6, Part 6 by an argument similar to that in Remark 5.20.

5.2 The case $n = r = 2$

We continue to consider an elliptic curve E over R_π and a 1-form ω. Consider, in what follows, $\Phi = (\phi_1, \phi_2)$. We consider in this subsection the arithmetic Kodaira–Spencer classes of order ≤ 2, and we derive some basic quadratic and cubic relations among them that will play a key role in the next section.

Specializing the construction in the previous section to our case we may consider the partial δ_π-character

$$\psi_{1,2} = \psi_{f_2\phi_1 - f_1\phi_2} \in \mathbf{X}^1_{\pi,\phi_1,\phi_2}(E)^\dagger.$$

Remark 5.22. If $f_1 f_2 \neq 0$ then $\psi_{1,2}$ is a "genuinely partial" object (not expressible in terms of ODE objects via face maps); indeed, in this case, by Theorem 5.10, we have $\mathbf{X}^1_{\pi,\phi_1}(E) = \mathbf{X}^1_{\pi,\phi_2}(E) = 0$. On the other hand ψ^1_{12} can be viewed as an analogue of the transport equation in [17].

By Theorem 5.10 we have

$$\theta(\psi_{1,2}) = \tilde{f}_2\phi_1 - \tilde{f}_1\phi_2 + f_{1,2}.$$

By Remark 5.16 the dependence of $f_{1,2}$ on ω is as follows:

$$f_{1,2}(E, \lambda\omega) = \lambda^{\phi_1+\phi_2} f_{1,2}(E, \omega).$$

Next, for $i \in \{1, 2\}$, we may consider the partial δ_π-characters (induced via face maps by the ODE arithmetic Manin maps in [6]):

$$\psi_{ii,i} := \psi_{\tilde{f}_i\phi_i^2 - \tilde{f}_{ii}\phi_i} \in \mathbf{X}^2_{\pi,\phi_1,\phi_2}(E)^\dagger.$$

By Theorem 5.10 we have

$$\theta(\psi_{ii,i}) = \tilde{f}_i\phi_i^2 - \tilde{f}_{ii}\phi_i + f_{ii,i}.$$

By Remark 5.16 the dependence of $f_{ii,i}$ on ω is as follows:

$$f_{ii,i}(E, \lambda\omega) = \lambda^{\phi_i+\phi_i^2} f_{ii,i}(E, \omega).$$

Finally, we may consider the partial δ_π-character

$$\psi_{11,22} := \psi_{f_{22}\phi_1^2 - f_{11}\phi_2^2} \in \mathbf{X}^2_{\pi,\phi_1,\phi_2}(E)^\dagger.$$

By Theorem 5.10 we have

$$\theta(\psi_{11,22}) = \tilde{f}_{22}\phi_1^2 - \tilde{f}_{11}\phi_2^2 + f_{11,22}.$$

By Remark 5.16 the dependence of $f_{11,22}$ on ω is as follows:

$$f_{11,22}(E,\lambda\omega) = \lambda^{\phi_1^2+\phi_2^2} f_{11,22}(E,\omega).$$

One has the following 6 elements in the module $\mathbf{X}^2_{\pi,\phi_1,\phi_2}(E)$,

$$\psi_{1,2},\ \phi_1\psi_{1,2},\ \phi_2\psi_{1,2},\ \psi_{11,1},\ \psi_{22,2},\ \psi_{11,22}. \tag{5.21}$$

So if $f_1 \neq 0$ or $f_2 \neq 0$, since $\mathbf{X}^2_{\pi,\phi_1,\phi_2}(E)$ has rank $2+2^2-1=5$ (cf. Theorem 5.10), it follows that there must be a non-trivial R_π-linear relation among these 6 elements:

$$\lambda_1\psi_{1,2} + \lambda_2\phi_1\psi_{1,2} + \lambda_3\phi_2\psi_{1,2} + \lambda_4\psi_{11,1} + \lambda_5\psi_{22,2} + \lambda_6\psi_{11,22} = 0, \tag{5.22}$$

for some $\lambda_1,\dots,\lambda_6 \in R_\pi$, not all zero. The existence of such a relation implies the vanishing of all 6×6 minors of the 6×7 matrix Γ of the coefficients of the Picard–Fuchs symbols of the elements in (5.21) with respect to the basis

$$\phi_1^2,\ \phi_2^2,\ \phi_1\phi_2,\ \phi_2\phi_1,\ \phi_1,\phi_2, 1 \tag{5.23}$$

of K^2_{π,ϕ_1,ϕ_2}. One can compute this matrix explicitly. Indeed, denote by $\theta_1,\dots,\theta_6$ the Picard–Fuchs symbols of the elements in (5.21), let $e_1,\dots,e_7$ be the elements in (5.21) and let $\Gamma=(\gamma_{ij})$ be the 6×7 matrix defined by the equalities

$$\theta_i = \sum_{j=1}^{7} \gamma_{ij} e_j,\quad i=1,\dots,6.$$

We have the following matrix

$$\Gamma = \begin{pmatrix} 0 & 0 & 0 & 0 & \tilde{f}_2 & -\tilde{f}_1 & f_{1,2} \\ \tilde{f}_2^{\phi_1} & 0 & -\tilde{f}_1^{\phi_1} & 0 & f_{1,2}^{\phi_1} & 0 & 0 \\ 0 & -\tilde{f}_1^{\phi_2} & 0 & \tilde{f}_2^{\phi_2} & 0 & f_{1,2}^{\phi_2} & 0 \\ \tilde{f}_1 & 0 & 0 & 0 & -\tilde{f}_{11} & 0 & f_{11,1} \\ 0 & \tilde{f}_2 & 0 & 0 & 0 & -\tilde{f}_{22} & f_{22,2} \\ \tilde{f}_{22} & -\tilde{f}_{11} & 0 & 0 & 0 & 0 & f_{11,22} \end{pmatrix}.$$

The upper left 5×5 minor of the matrix Γ is non-zero if $f_1 f_2 \neq 0$. In particular, the following corollary is proved.

Corollary 5.23. *If $f_1 f_2 \neq 0$ then the first 5 elements in (5.21) are R_π-linearly independent and hence they form a basis up to torsion of $\mathbf{X}^2_{\pi,\Phi}(E)$.*

On the other hand the linear combination of the rows of Γ with coefficients $\lambda_1, \dots, \lambda_6$ is 0 from which we get the following corollary.

Corollary 5.24. *If $f_1 f_2 \neq 0$ then in (5.22) we have $\lambda_2 = \lambda_3 = 0$.*

Assume $f_1 f_2 \neq 0$ and denote by $\tilde{\Gamma}$ the 4×5 matrix obtained from Γ by removing the 3rd and 4th columns as well as the 2nd and the 3rd rows. The rows of $\tilde{\Gamma}$ are then linearly dependent, so we get that all 4×4 minors of the matrix $\tilde{\Gamma}$ vanish. The vanishing of the minor obtained by removing the fifth column of $\tilde{\Gamma}$ is tautologically trivial, so it does not yield any information. The vanishings of the rest of the minors of $\tilde{\Gamma}$ is equivalent to one cubic relation (5.24) given in the following corollary.

Corollary 5.25. *If $f_1 f_2 \neq 0$ then the following relation holds in R_π:*

$$f_{11} f_{22} f_{1,2} + f_2 f_{22} f_{11,1} - f_{11} f_1 f_{22,2} - f_1 f_2 f_{11,22} = 0. \tag{5.24}$$

Proposition 5.26. *Assume $\pi = p$. Then the following equalities hold in R:*

(1) $f_1 = f_2$, $f_{12} = f_{21}$.

(2) $f_{11,1} = f_{22,2}$.

(3) $f_{1,2} = f_{11,22} = 0$.

Proof. Part 1 follows from Lemma 5.8. Part 2 follows from the compatibility with face maps. In order to check Part 3 consider the compatible actions of $\Sigma_2 = \{e, \sigma\}$ on $\mathbf{X}^1_{p,\Phi}(E)$ and $K^1_{p,\Phi}$. We have

$$\theta(\sigma\psi_{1,2}) = \sigma(\theta(\psi_{1,2})) = \sigma(f_1\phi_1 - f_1\phi_2 + f_{1,2}) = f_1\phi_2 - f_1\phi_1 + f_{1,2}.$$

Hence

$$\theta(\psi_{1,2} + \sigma\psi_{1,2}) = 2 f_{1,2} \in K_\pi.$$

By Theorem 5.10, Part 2, it follows that $f_{1,2} = 0$. The equality $f_{11,22} = 0$ follows similarly. ∎

Remark 5.27. Assume that E comes from a curve $E_{\mathbb{Z}_p}$ over $\mathbb{Z}_p$ and denote by a_p the trace of Frobenius on $E_{\mathbb{Z}_p} \otimes \mathbb{F}_p$. Also fix an index i. It follows from [8, Theorem 1.10], that if E is not a canonical lift of an ordinary elliptic curve then

$$f_{ii} = a_p f_i, \quad f_{ii,i} = p f_i.$$

Recall that, if in addition $p \geq 5$, then $a_p = 0$ if and only if E has supersingular reduction.

We continue by considering the partial δ_π-character

$$\psi_{12,1} := \psi_{f_1\phi_1\phi_2 - f_{12}\phi_1}.$$

Its symbol is

$$\theta(\psi_{12,1}) = \tilde{f}_1\phi_1\phi_2 - \tilde{f}_{12}\phi_1 + f_{12,1}.$$

This symbol must be a linear combination of the symbols of

$$\psi_{1,2},\ \phi_1\psi_{1,2},\ \phi_2\psi_{1,2},\ \psi_{11,1},\ \psi_{22,2}.$$

Let Γ' be the matrix obtained by replacing the last row in Γ by the row

$$[0\ 0\ \tilde{f}_1\ 0\ -\tilde{f}_{12}\ 0\ f_{12,1}].$$

Then the determinants of the matrices obtained from Γ' by deleting the 5th and the 7th columns respectively must be 0. The vanishing of these determinants yields the following result.

Lemma 5.28. *If $f_1 f_2 \neq 0$ then the following relations hold in R_π:*

$$f_{12,1} f_1^{\phi_1} - f_{11,1} f_2^{\phi_1} = 0, \tag{5.25}$$

$$\tilde{f}_{12} \tilde{f}_1^{\phi_1} - \tilde{f}_{11} \tilde{f}_2^{\phi_1} - \tilde{f}_1 f_{1,2}^{\phi_1} = 0. \tag{5.26}$$

Similarly, by looking at the partial δ_π-character

$$\psi_{21,2} := \psi_{f_2\phi_2\phi_1 - f_{21}\phi_2}$$

we get the following lemma.

Lemma 5.29. *If $f_1 f_2 \neq 0$ then the following relations hold in R_π:*

$$f_{21,2} f_2^{\phi_2} - f_{22,2} f_1^{\phi_2} = 0, \tag{5.27}$$

$$\tilde{f}_{21} \tilde{f}_2^{\phi_2} - \tilde{f}_{22} \tilde{f}_1^{\phi_2} - \tilde{f}_2 f_{2,1}^{\phi_2} = 0. \tag{5.28}$$

Next consider the partial δ_π-character

$$\psi_{12,21} := \psi_{f_{21}\phi_1\phi_2 - f_{12}\phi_2\phi_1}.$$

Its symbol is

$$\theta(\psi_{12,21}) = \tilde{f}_{21}\phi_1\phi_2 - \tilde{f}_{12}\phi_2\phi_1 + f_{12,21}.$$

Set

$$\psi := \tilde{f}_1 \tilde{f}_2 \psi_{12,21} - \tilde{f}_2 \tilde{f}_{21} \psi_{12,1} + \tilde{f}_1 \tilde{f}_{12} \psi_{21,2} - \tilde{f}_{12} \tilde{f}_{21} \psi_{1,2}.$$

One trivially checks the following identity

$$\theta(\psi) = f_{12,21} \tilde{f}_1 \tilde{f}_2 - \tilde{f}_2 \tilde{f}_{21} f_{12,1} + \tilde{f}_1 \tilde{f}_{12} f_{21,2} - \tilde{f}_{12} \tilde{f}_{21} f_{1,2}.$$

By Theorem 5.10, Part 1 we get the following.

Lemma 5.30. *If $f_1 f_2 \neq 0$ then the following relation holds in R_π:*

$$f_{12,21} f_1 f_2 - f_2 f_{21} f_{12,1} + f_1 f_{12} f_{21,2} - f_{12} f_{21} f_{1,2} = 0. \tag{5.29}$$

Similarly consider the partial δ_π-characters

$$\begin{gathered}\tilde{f}_1 \psi_{11,2} - \tilde{f}_2 \psi_{11,1} - \tilde{f}_{11} \psi_{12},\\ \tilde{f}_1 \psi_{11,12} - \tilde{f}_{12} \psi_{11,1} + \tilde{f}_{11} \psi_{12,1},\\ \tilde{f}_1 \psi_{12,2} - \tilde{f}_2 \psi_{12,1} - \tilde{f}_{12} \psi_{1,2},\\ \tilde{f}_2 \psi_{11,21} - \tilde{f}_{21} \psi_{11,2} + \tilde{f}_{11} \psi_{21,2}.\end{gathered}$$

The symbols of these partial δ_π-characters are equal to the expressions in the left-hand sides of the equalities in the Lemma 5.31 below. By Theorem 5.10, Part 1, since these symbols are in R_π they must vanish. So we have the following lemma.

Lemma 5.31. *If $f_1 f_2 \neq 0$ then the following relations hold in R_π:*

$$f_1 f_{11,2} - f_2 f_{11,1} - f_{11} f_{1,2} = 0, \tag{5.30}$$

$$f_1 f_{11,12} - f_{12} f_{11,1} + f_{11} f_{12,1} = 0, \tag{5.31}$$

$$f_1 f_{12,2} - f_2 f_{12,1} - f_{12} f_{1,2} = 0, \tag{5.32}$$

$$f_2 f_{11,21} - f_{21} f_{11,2} + f_{11} f_{21,2} = 0. \tag{5.33}$$

Moreover, the relations obtained from the above relations by switching the indices 1 *and* 2 *also hold.*

Remark 5.32. One can ask if one can "extend" equations (5.24), (5.25), (5.26), (5.27), (5.28), (5.30), (5.31), (5.32), (5.33) by continuity so that these remain true without the condition $f_1 f_2 \neq 0$. We claim this is the case as an immediate consequence of Theorems 7.18, 7.19 and the formulae (7.6) and (7.7) to be stated and proved later. By the way we have the following result; this will be proved after the proof of Proposition 7.38.

Theorem 5.33. *Assume ϕ_1, ϕ_2 are monomially independent in $\mathfrak{G}(K^{\text{alg}}/\mathbb{Q}_p)$. Then there exist $\pi \in \Pi$ and a pair (E, ω) over R_π such that E has ordinary reduction and all classes f_μ, $f_{\mu,\nu}$ with $\mu, \nu \in \mathbb{M}_2^{2,+}$, $\mu \neq \nu$, attached to (E, ω) are non-zero.*

Chapter 6

The relative theory

The theory developed so far over R_π should be viewed as an "absolute" theory and has a "relative" version in which the δ_π-prolongation sequence R_π^* is replaced by an arbitrary object S^* in $\mathbf{Prol}_{\pi,\Phi}^*$. This relative version of the theory is crucial for developing the formalism of *partial δ_π-modular forms* in the next section. In the present section we present a quick discussion of this relative version of the theory.

Again we consider the variables $\delta_{\pi,\mu} y_j$ for $\mu \in \mathbb{M}_n$, $\pi \in \Pi$, $j \in \{1, \dots, N\}$. Let $S^* = (S^r)$ be an object in $\mathbf{Prol}_{\pi,\Phi}$. Fix a positive integer N and consider the ring $S^0[y_1, \dots, y_N]$ and the rings

$$J^r_{\pi,\Phi}(S^0[y_1,\dots,y_N]/S^*) := S^r[\delta_{\pi,\mu} y_j \mid \mu \in \mathbb{M}^r_m, j = 1,\dots,N]\widehat{\ }.$$

The sequence $J^*_{\pi,\Phi}(S^0[y_1,\dots,y_N]/S^*) := (J^r_{\pi,\Phi}(S^0[y_1,\dots,y_N]/S^*))$ has, again, a unique structure of an object in $\mathbf{Prol}^*_{\pi,\Phi}$ such that $\delta_{\pi,i}\delta_{\pi,\mu} y := \delta_{\pi,i\mu} y$ for all $i = 1,\dots,n$; if S^* is an object of $\mathbf{Prol}_{\pi,\Phi}$ then $J^*_{\pi,\Phi}(S^0[y_1,\dots,y_N]/S^*)$ is also an object of $\mathbf{Prol}_{\pi,\Phi}$.

For every object S^* in $\mathbf{Prol}^*_{\pi,\Phi}$ and every S^0-algebra of finite type written as $A := S^0[y_1,\dots,y_N]/I$, we define the ring

$$J^r_{\pi,\Phi}(A/S^*) := J^r_\pi(S^0[y_1,\dots,y_N]/S^*)/(\delta_{\pi,\mu} I \mid m \in \mathbb{M}^r_m).$$

If $S^* = R^*_\pi$ then $J^r_{\pi,\Phi}(A/S^*)$ coincides with the previously defined ring $J^r_{\pi,\Phi}(A)$.

If S^* is an object of $\mathbf{Prol}_{\pi,\Phi}$, A is a smooth S^0-algebra, and $u\colon S^0[T_1,\dots,T_d] \to A$ is an étale morphism of S^0-algebras, then, again, there is a (unique) isomorphism

$$(A \otimes_{S^0} S^r)[\delta_{\pi,\mu} T_j \mid \mu \in \mathbb{M}^{r,+}_m, j = 1,\dots,d]\widehat{\ } \cong J^r_{\pi,\Phi}(A/S^*)$$

sending $\delta_{\pi,\mu} T_j$ into $\delta_{\pi,\mu}(u(T_j))$ for all j and μ. In particular, $J^r_{\pi,\Phi}(A/S^*)$ is Noetherian and flat over R_π so the sequence $J^*_{\pi,\Phi}(A/S^*)$ is an object of $\mathbf{Prol}_{\pi,\Phi}$.

As in Proposition 2.23 we have the following universal property. Assume S^* is an object of $\mathbf{Prol}_{\pi,\Phi}$ and A is a smooth S^0-algebra. For every object T^* of $\mathbf{Prol}_{\pi,\Phi}$ and every S^0-algebra map $u : A \to T^0$ and any morphism $S^* \to T^*$ in $\mathbf{Prol}_{\pi,\Phi}$ there is a unique morphism $J^*_{\pi,\Phi}(A/S^*) \to T^*$ over S^* in $\mathbf{Prol}_{\pi,\Phi}$ compatible with u. (A similar result holds for $\mathbf{Prol}^*_{\pi,\Phi}$.)

As in Definition 2.25 for every object S^* in $\mathbf{Prol}_{\pi,\Phi}$ and every smooth scheme X over S^0 we define the *relative partial π-jet space*

$$J^r_{\pi,\Phi}(X/S^*) = \bigcup \mathrm{Spf}(J^r_{\pi,\Phi}(\mathcal{O}(U_i))/S^*),$$

where $X = \bigcup_i U_i$ is (any) affine open cover. If $S^* = R^*_\pi$, $J^r_{\pi,\Phi}(X/S^*)$ coincides with the previously defined formal scheme $J^r_{\pi,\Phi}(X)$.

Let S^* be an object in $\mathbf{Prol}_{\pi,\Phi}$ and G a commutative smooth group scheme over S^0. We define a *relative partial δ_π-character* of order $\leq r$ of G over S^* to be a group homomorphism $J^r_{\pi,\Phi}(G/S^*) \to \widehat{\mathbb{G}_{a,S^r}}$ in the category of p-adic formal schemes. (Here G_{a,S^r} is, of course, the additive group scheme over S^r.) We denote by $\mathbf{X}^r_{\pi,\Phi}(G/S^*) = \mathrm{Hom}(J^r_{\pi,\Phi}(G), \widehat{\mathbb{G}_{a,S^r}})$ the S^r-module of relative partial δ_π-characters of G of order $\leq r$.

For a family $\Phi := (\phi_1, \ldots, \phi_n)$, $\phi_i \in \mathfrak{F}^{(1)}(K^{\mathrm{alg}}/\mathbb{Q}_p)$ of distinct Frobenius automorphisms and for an object S^* in $\mathbf{Prol}_{\pi,\Phi}$ we define the S^r-modules of *symbols* $S^r_{\pi,\Phi}$ to be the free S^r-module with basis $\{\phi_\mu \in \mathbb{M}_\Phi \mid \mu \in \mathbb{M}^r_n\}$. We consider the rings $S^r \otimes \mathbb{Q} = S^r \otimes_{R_\pi} K_\pi$ and the $S^r \otimes \mathbb{Q}$-modules $S^r_{\pi,\Phi} \otimes \mathbb{Q}$; they play the roles, in this relative setting, of K_π and $K^r_{\pi,\Phi}$, respectively.

As in Proposition 3.4 we have

$$\mathbf{X}^r_{\pi,\Phi}(\mathbb{G}_a/S^*) = ((S^r \otimes \mathbb{Q})T) \cap (S^r[\delta_{\pi,\nu}T \mid \nu \in \mathbb{M}^r_n])$$

where the intersection is taken inside the ring $(S^r \otimes \mathbb{Q})[[\phi_{\pi,\nu}T \mid \nu \in \mathbb{M}^r_n]]$.

Let S^* be an object in $\mathbf{Prol}_{\pi,\Phi}$, let G have relative dimension 1 over S^0 and assume we are given an invariant 1-form ω on G/S^0 and an *admissible coordinate* T (defined in the obvious corresponding way) on G/S^0. (Note that ω and T may not exist in general, but they exist locally on $\mathrm{Spec}(S^0)$ in the Zariski topology.) Then, as in Definition 3.7, one can attach to every $\psi \in \mathbf{X}^r_{\pi,\Phi}(G/S^*)$ a *Picard–Fuchs symbol* $\theta(\psi) \in S^r_{\pi,\Phi} \otimes \mathbb{Q}$.

The various results about δ_π-characters obtained in the previous sections have (obvious) relative analogues (over objects S^* in $\mathbf{Prol}_{\pi,\Phi}$ instead of over R^*_π) that are proved using essentially identical arguments. We note, however, that the relative analogues over S^* of Corollary 5.14 and of the results in Section 5.2 need the hypothesis that the rings S^r be integral domains; indeed for Corollary 5.14 we need the concepts of rank and torsion of an S^r-module to be well defined while for the analysis in Section 5.2 we need the fact that linear dependence in torsion-free S^r-modules is expressed via vanishings of corresponding determinants. Rather than stating these relative analogues here we will use them freely in what follows with appropriate references to the corresponding "absolute" results in the previous sections.

Chapter 7

Partial δ-modular forms

7.1 Basic definitions

We start with the standard extension of [9] or [10, Section 8.4.1] to our setting of partial differential equations. In this section, $\pi \in \Pi$. Continue to set $\Phi = (\phi_1, \dots, \phi_n)$ a fixed choice of Frobenius automorphisms of K^{alg}. Consider the category of triples $(E/S^0, \omega, S^*)$ where S^* is an object in $\mathbf{Prol}_{\pi,\Phi}$, E/S^0 is an elliptic curve, and $\omega \in H^0(E, \Omega_{E/S_0})$ is a basis. A morphism in this category is defined as a map of tuples $(E/S^0, \omega, S^*) \to (E'/T^0, \omega', T^*)$ consisting of a map of prolongation sequences $T^* \to S^*$ and a compatible map $E/S^0 \to E'/T^0$ of curves pulling back ω' to ω.

Definition 7.1. A *partial δ_π-modular function of order at most* $r \geq 0$ is a rule f assigning to each object $(E/S^0, \omega, S^*)$ an element $f(E/S^0, \omega, S^*) \in S^r$, depending only on the isomorphism class of $(E/S^0, \omega, S^*)$, such that f commutes with base change of the prolongation sequence. Specifically, if $u \colon S^* \to T^*$ is a map of prolongation sequences, then $f((E \times_{S^0} T^0)/T^0, u^*\omega, T^0) = u^r f(E/S^0, \omega, S^*)$. We denote by $M^r_{\pi,\Phi}$ the set of all partial δ_π-modular functions of order at most $r \geq 0$; this set has an obvious structure of ring. An element of $M^r_{\pi,\Phi}$ is said to have order r if it is not in the image of the canonical map $M^{r-1}_{\pi,\Phi} \to M^r_{\pi,\Phi}$. The latter map is, by the way, injective as can be seen from the next remark.

Remark 7.2. Consider two variables a_4 and a_6, let $\Delta := 4a_4^3 + 27a_6^2$, and let us consider the R_π-algebra $M_\pi := R_\pi[a_4, a_6, \Delta^{-1}]$ and the affine scheme $B(1) := \text{Spec}(M_\pi)$. (The scheme $B(1)$ has a natural $\mathbb{G}_m$-action and a morphism to the "j-line" $Y(1)$ which is however not a $\mathbb{G}_m$-bundle; later we will consider level $\Gamma_1(N)$-structures, the modular curves $Y_1(N)$, and the corresponding $\mathbb{G}_m$-bundles $B_1(N)$.) For all R_π-algebras S the set $B(1)(S)$ of S-points of $B(1)$ is in a natural bijection with the set of pairs (E, ω) consisting of an elliptic curve E/S and a basis ω for the 1-forms on E/S. Then, as in [9], we have an identification of R_π-algebras

$$M^r_{\pi,\Phi} \simeq J^r_{\pi,\Phi}(M_\pi) = \mathcal{O}(J^r_{\pi,\Phi}(B(1))) \simeq \widehat{R_\pi[\delta_\mu a_4, \delta_\mu a_6, \Delta^{-1} \mid \mu \in \mathbb{M}^r_n]}.$$

In particular, $M^0_{\pi,\Phi} \simeq \widehat{M_\pi}$.

In what follows we discuss weights. In the case $\pi = p$ and $\Phi = \{\phi\}$, weights are taken to be elements of the polynomial ring $\mathbb{Z}[\phi]$ in the "variable" ϕ. In the partial differential setting considered here, we consider weights in the ring of integral symbols, $\mathbb{Z}_\Phi$. As before, if $w = \sum m_\mu \phi_\mu \in \mathbb{Z}_\Phi$, $m_\mu \in \mathbb{Z}$, and if S^* is an object of

Prol$_{\pi,\Phi}$ and $\lambda \in (S^0)^\times$, we write

$$\lambda^w = \prod_{\mu \in \mathbb{M}_n} (\phi_\mu(\lambda))^{m_\mu} \in (S^r)^\times.$$

We use a similar notation for $\lambda \in S^0$ in case all $\lambda_\mu \geq 0$. We have the formulae $\lambda^{w_1+w_2} = \lambda^{w_1}\lambda^{w_2}$ and $\lambda^{w_1 w_2} = (\lambda^{w_2})^{w_1}$.

Definition 7.3. A partial δ_π-modular function $f \in M^r_{\pi,\Phi}$ is called a *partial δ_π-modular form of weight* $w \in \mathbb{Z}^r_\Phi$ provided for all $(E/S^0, \omega, S^*)$ and for each $\lambda \in (S^0)^\times$ we have

$$f(E/S^0, \lambda\omega, S^*) = \lambda^{-w} f(E/S^0, \omega, S^*). \tag{7.1}$$

We denote by $M^r_{\pi,\Phi}(w)$ the R_π-module of partial δ_π-modular forms of weight w.

Remark 7.4. If t is a variable we may consider the ring

$$J^r_{\pi,\Phi}(M_\pi[t, t^{-1}]) \simeq R_\pi[\delta_\mu a_4, \delta_\mu a_6, \delta_\mu t, \Delta^{-1}, t^{-1} \mid \mu \in \mathbb{M}^r_n]\widehat{\ }. \tag{7.2}$$

Then for $f \in M^r_{\pi,\Phi}$ we have that $f \in M^r_{\pi,\Phi}(w)$ if and only if

$$f(\ldots, \delta_\mu(t^4 a_4), \ldots, \delta_\mu(t^6 a_6), \ldots) = t^w f(\ldots, \delta_\mu a_4, \ldots, \delta_\mu a_6, \ldots)$$

in the ring (7.2).

Remark 7.5. The direct sum $\bigoplus_{w \in \mathbb{Z}_\Phi} M^r_{\pi,\Phi}(w)$ has a natural structure of $\mathbb{Z}_\Phi$-graded R_π-algebra. Moreover, for every i and $f \in M^r_{\pi,\Phi}(w)$ we have a naturally defined form $f^{\phi_i} \in M^r_{\pi,\Phi}(\phi_i w)$. Consequently, we have natural *bracket* $\mathbb{Z}_p$-bilinear maps

$$\{\ ,\ \}_{\pi,i} : M^r_{\pi,\Phi}(w) \times M^r_{\pi,\Phi}(w) \to M^r_{\pi,\Phi}((\phi_i + p)w)$$

defined by

$$\{f, g\}_{\pi,i} := \frac{1}{\pi}(f^{\phi_i} g^p - g^{\phi_i} f^p) = g^p \delta_{\pi,i} f - f^p \delta_{\pi,i} g.$$

7.2 Jet construction

Examples of δ_π-modular forms are provided by primary and secondary arithmetic Kodaira–Spencer classes as we shall explain in what follows. We begin with primary classes where each f_μ for $\mu \in \mathbb{M}_n$ "comes from" a δ_π-modular function (which we denote by $f^{\text{jet}}_{\pi,\mu}$). Indeed, consider a prolongation sequence S^* over R, an elliptic curve E/S^0, and $\omega \in H^1(E, \Omega_{E/S^0})$ a basis. For fixed r and $\mu \in \mathbb{M}^r_n$, replicating the arguments in the construction of f_μ in Remark 5.2 (with jet spaces over R^* replaced by relative jet spaces over S^* as in Chapter 6) and setting η for the class of the corresponding $\partial^r(\tilde{L}^{\mu}_{\pi,\Phi})$ in Definition 5.5 we define $f^{\text{jet}}_{\pi,\mu}(E/S^0,\omega,S^*) = \langle \eta,\omega\rangle \in S^r$. In particular, using the notation in Remark 5.6, Part 2, for every (E,ω) over R_π we have

$$f_\mu(E,\omega) = f^{\text{jet}}_{\pi,\mu}(E/R_\pi,\omega,R^*_\pi). \tag{7.3}$$

Using the corresponding version over S^* of Remark 5.6, Part 2 we get the following.

Theorem 7.6. *The rule $f^{\text{jet}}_{\pi,\mu}$ defines a partial δ_π-modular form of weight $-1-\phi_\mu$.*

These forms are generalizations of the forms f_{jet} constructed in [9, Construction 4.1] or f^r_{jet} in [10, Section 8.4.2]. For $\pi = p$ we write

$$f^{\text{jet}}_{p,\mu} = f^{\text{jet}}_{\mu}.$$

Remark 7.7. Assume $\pi = p$ and let $E_{p-1} \in \mathbb{Z}_p[a_4,a_6] \subset M^0_{p,\Phi}$ be the polynomial that corresponds to the normalized Eisenstein series of weight $p-1$; we recall that the reduction mod p of E_{p-1} is the Hasse invariant. Then, exactly as in [10, Proposition 8.55] and using Remark 5.3, we get that for every $i_1\dots i_r \in \mathbb{M}^{r,+}_n$ we have

$$f^{\text{jet}}_{i_1\dots i_r} \equiv E^{1+p+\dots+p^{r-2}}_{p-1}\cdot (f^{\text{jet}}_{i_r})^{p^{r-1}} \mod p \ \text{ in } \ M^r_{p,\Phi}.$$

In particular, for all $\mu \in \mathbb{M}^{r,+}_n$ we have

$$f^{\text{jet}}_{\mu} \in M^r_{p,\Phi} \setminus pM^r_{p,\Phi}, \quad \text{hence} \quad f^{\text{jet}}_{\mu} \neq 0.$$

Next, for fixed r and $\mu,\nu \in \mathbb{M}^{r,+}_n$ distinct, replicating the arguments in the construction of $f_{\mu,\nu}$ in Definition 5.15 (with jet spaces over R^* replaced by relative jet spaces over S^* as in Chapter 6) we define $f^{\text{jet}}_{\pi,\mu,\nu}(E,\omega,S^*) \in S^r$. In particular, using the notation in Remark 5.16 for every E over R_π we have

$$f_{\mu,\nu}(E,\omega) = f^{\text{jet}}_{\pi,\mu,\nu}(E/R_\pi,\omega,R^*_\pi). \tag{7.4}$$

Using the corresponding version over S^* of Remarks 5.16 and 5.13, we get the following.

Theorem 7.8. *The rule $f^{\text{jet}}_{\pi,\mu,\nu}$, for $\mu,\nu \in \mathbb{M}^{r,+}_n$ distinct, defines a partial δ_π-modular form of weight $-\phi_\nu-\phi_\mu$. Moreover, $f^{\text{jet}}_{\pi,\mu,\nu} + f^{\text{jet}}_{\pi,\nu,\mu} = 0$.*

For $\pi = p$ we write

$$f^{\text{jet}}_{p,\mu,\nu} =: f^{\text{jet}}_{\mu,\nu}.$$

Our next goal is to show the above forms enjoy the special property of being "isogeny covariant" which we now define in our setting.

Definition 7.9. For every weight $w = \sum_{\mu \in \mathbb{M}_n} m_\mu \phi_\mu$, set $\deg(w) := \sum_{\mu \in \mathbb{M}_n} m_\mu$. Let f be a partial δ_π-modular form f of weight $w \in \mathbb{Z}_\Phi^r$ where $\deg(w)$ is even. We say f is *isogeny covariant* provided for every tuple $(E/S^0, \omega, S^*)$ and every isogeny of degree prime to p, $u\colon E' \to E$ over S^0, setting $\omega' = u^*\omega$, we have

$$f(E'/S^0, \omega', S^*) = [\deg(u)]^{-\deg(w)/2} f(E/S^0, \omega, S^*).$$

We denote by $I^r_{\pi,\Phi}(w)$ the R_π-module of isogeny covariant partial δ_π-modular forms of weight w.

Remark 7.10. The direct sum $\bigoplus_{w \in \mathbb{Z}_\Phi} I^r_{\pi,\Phi}(w)$ is a $\mathbb{Z}_\Phi$-graded R_π-subalgebra of $\bigoplus_{w \in \mathbb{Z}_\Phi} M^r_{\pi,\Phi}(w)$. For every $f \in I^r_{p,\Phi}(w)$ and every i we have $f^{\phi_i} \in I^r_{p,\Phi}(\phi_i w)$ and consequently the brackets in Remark 7.5 induce brackets

$$\{\ ,\ \}_{\pi,i} : I^r_{\pi,\Phi}(w) \times I^r_{\pi,\Phi}(w) \to I^r_{\pi,\Phi}((\phi_i + p)w).$$

Theorem 7.11. *The partial δ_π-modular forms $f^{\text{jet}}_{\pi,\mu}$ and $f^{\text{jet}}_{\pi,\mu,\nu}$ are isogeny covariant.*

Proof. This follows by adapting the arguments in Remark 5.6, Part 5 and Remark 5.20 with R_π^* replaced by an arbitrary prolongation sequence S^*. ∎

Exactly as in [12] the forms $f^{\text{jet}}_{\pi,\mu}$, $f^{\text{jet}}_{\pi,\mu,\nu}$ induce totally δ-overconvergent arithmetic PDEs on $B(1)$ and on certain natural bundles $B_1(N)$ over modular curves $Y_1(N)$. We explain this in what follows.

Definition 7.12. Consider the modular curve $Y_1(N) := X_1(N) \setminus \{\text{cusps}\}$ over R_π where $N \geq 4$, N coprime to p, and let L be the line bundle on $Y_1(N)$ equal to the direct image of $\Omega_{E_{\text{univ}}/Y_1(N)}$ where E_{univ} is the universal elliptic curve over $Y_1(N)$. Let $X \subset Y_1(N)$ be an affine open set, continue to denote by L the restriction of L to X, and consider the natural $\mathbb{G}_m$-bundle

$$B_1(N) := \operatorname{Spec}\Big(\bigoplus_{m \in \mathbb{Z}} L^m\Big) \to X.$$

The main example we have in mind is the case $X = Y_1(N)$.

Recall that if X is such that L is free over $X = \operatorname{Spec}(A)$ with basis ω then we have a natural identification

$$\mathcal{O}(J^r_{\pi,\Phi}(B_1(N))) \simeq J^r_{\pi,\Phi}(A)[x, x^{-1}, \delta_\mu x \mid \mu \in \mathbb{M}_n^{r,+}]\widehat{\ } \tag{7.5}$$

where x is a variable identified with the section ω. We define the R_π-module

$$M^r_{\pi,\Phi,X}(w) := \widehat{\mathcal{O}(X)} \cdot x^w.$$

If X is arbitrary and $X = \bigcup X_i$ is an open cover such that L is trivial on each X_i then we define $M^r_{\pi,\Phi,X}(w)$ to be the submodule of $\mathcal{O}(J^r_{\pi,\Phi}(B_1(N)))$ of all elements whose restriction to every $\mathcal{O}(J^r_{\pi,\Phi}(B_1(N)\times_X X_i))$ lies in $M^r_{\pi,\Phi,X_i}(w)$; this definition is independent of the covering considered.

Recall the scheme $B(1)$ defined in Remark 7.2. Exactly as in [12, Section 5.2] every partial δ_π-modular form

$$f = f^{B(1)} \in M^r_{\pi,\Phi}(w) \subset \mathcal{O}(J^r_{\pi,\Phi}(B(1)))$$

induces an element

$$f^{B_1(N)} \in M^r_{\pi,\Phi,X}(w) \subset \mathcal{O}(J^r_{\pi,\Phi}(B_1(N))).$$

We recall that for $X = \mathrm{Spec}(A)$ such that L has a basis x corresponding to a 1-form ω we define

$$f^{B_1(N)} := f(E_{\mathrm{univ}}, \omega, J^*_{\pi,\Phi}(A)) \cdot x^w;$$

for arbitrary X we glue the elements just defined. Exactly as in [12, Section 5.3] we have the following theorem.

Theorem 7.13. *Let B be either $B(1)$ or $B_1(N)$ with $N \geq 4$. For all $\mu, \nu \in \mathbb{M}^{r,+}_n$ the elements $(f^{\mathrm{jet}}_{\pi,\mu})^B, (f^{\mathrm{jet}}_{\pi,\mu,\nu})^B \in \mathcal{O}(J^r_{\pi,\Phi}(B))$ are totally δ-overconvergent. So there are induced maps*

$$((f^{\mathrm{jet}}_{\pi,\mu})^B)^{\mathrm{alg}}, ((f^{\mathrm{jet}}_{\pi,\mu,\nu})^B)^{\mathrm{alg}} : B(R^{\mathrm{alg}}) \to K^{\mathrm{alg}}.$$

Remark 7.14. For $\pi'|\pi$ and every point $P \in B(R_{\pi'})$, denoting by (E_P, ω_P) the corresponding elliptic curve over $R_{\pi'}$ equipped with the induced 1-form, we have that

$$((f^{\mathrm{jet}}_{\pi,\mu})^B)^{\mathrm{alg}}(P) = p^{-N(\pi')+N(\pi)} f_{\pi',\mu}(E_P/R_{\pi'}, \omega_P, R^*_{\pi'}), \tag{7.6}$$

$$((f^{\mathrm{jet}}_{\pi,\mu,\nu})^B)^{\mathrm{alg}}(P) = p^{-2N(\pi')+2N(\pi)} f_{\pi',\mu,\nu}(E_P/R_{\pi'}, \omega_P, R^*_{\pi'}); \tag{7.7}$$

cf. Remark 5.6, Part 6 and Remark 5.21.

Definition 7.15. For every selection map ϵ with respect to (Φ', Φ'', p) (cf. Definition 2.20) and every $f \in M^r_{p,\Phi'}$ we define $f_\epsilon \in M^r_{p,\Phi''}$ by letting

$$f_\epsilon(E/S^0, \omega, S^*) := f(E/S^0, \omega, S^*_\epsilon)$$

for every prolongation sequence S^*, every elliptic curve E/S^0 and every basis ω for the 1-forms on E. In particular, we get the following special cases:

(1) There is a natural action

$$\Sigma_n \times M^r_{p,\Phi} \to M^r_{p,\Phi}, \ \ (\sigma, f) \mapsto \sigma f := f_\sigma.$$

(2) For every $1 \leq i_1 < i_2 < \cdots < i_s \leq n$ there are natural *face* homomorphisms

$$M^r_{p,\phi_{i_1},\ldots,\phi_{i_s}} \to M^r_{p,\Phi}.$$

(3) For every $\phi \in \mathfrak{F}^{(1)}(K^{\mathrm{alg}}/\mathbb{Q}_p)$ there is a natural *degeneration* homomorphism

$$M^r_{p,\Phi} \to M^r_{p,\phi}.$$

(4) The composition of the face and degeneration maps below is the identity:

$$\mathrm{id} : M^r_{p,\phi_i} \to M^r_{p,\Phi} \to M^r_{p,\phi_i}.$$

(5) The face and degeneration maps induce maps between the corresponding modules $I^r_{p,\Phi}(w), I^r_{p,\phi_i}(w)$.

On the other hand by Remark 2.27 we have a natural action

$$\Sigma_n \times \mathcal{O}(J^r_{p,\Phi}(B_1(N))) \to \mathcal{O}(J^r_{p,\Phi}(B_1(N))), \quad (\sigma, u) \mapsto \sigma u.$$

The following is trivially checked.

Lemma 7.16. *The Σ_n-actions on $M^r_{p,\Phi} = \mathcal{O}(J^r_{p,\Phi}(B(1)))$ and $\mathcal{O}(J^r_{p,\Phi}(B_1(N)))$ are compatible in the sense that for every $\sigma \in \Sigma_n$ and every $f \in M^r_{p,\Phi}$ we have*

$$\sigma(f^{B_1(N)}) = (\sigma f)^{B_1(N)}.$$

7.3 The case $n = 2, \pi = p$

In this subsection we assume $n = 2, \pi = p$, and we may consider the forms

$$f^{\mathrm{jet}}_{\mu}, f^{\mathrm{jet}}_{\mu,\nu} \in M^2_{p,\phi_1,\phi_2}, \quad \mu, \nu \in \mathbb{M}^{2,+}_2, \quad \mu \neq \nu. \tag{7.8}$$

The forms (7.8) have weights $-\phi_\mu - 1$ and$-\phi_\mu - \phi_\nu$, respectively.

Remark 5.27 can be amplified as follows.

Remark 7.17. It follows from [10, Proposition 7.20, Corollary 8.84, and Remark 8.85] that for $i \in \{1, 2\}$ we have the following equality in M^2_{p,ϕ_1,ϕ_2}:

$$f^{\mathrm{jet}}_{ii,i} = p(f^{\mathrm{jet}}_i)^{\phi_i}. \tag{7.9}$$

It also follows from [10, Proposition 8.55] that $f^{\mathrm{jet}}_i, f^{\mathrm{jet}}_{ii}$ are non-zero in M^2_{p,ϕ_1,ϕ_2}; this also follows from (7.26) below. Hence by (7.9) it follows that $f^{\mathrm{jet}}_{ii,i}$ are non-zero in M^2_{p,ϕ_1,ϕ_2}. The fact that the rest of the forms f^{jet}_{μ} are non-zero was proved in Remark 7.7; the fact that the forms $f^{\mathrm{jet}}_{\mu,\nu}$ (for $\mu \neq \nu$) are non-zero is also true but more subtle and will be proved later; cf. Remark 7.31.

Theorem 7.18. *The following relation holds in M^2_{p,ϕ_1,ϕ_2}:*

$$f^{\mathrm{jet}}_{11} f^{\mathrm{jet}}_{22} f^{\mathrm{jet}}_{1,2} + f^{\mathrm{jet}}_{2} f^{\mathrm{jet}}_{22} f^{\mathrm{jet}}_{11,1} - f^{\mathrm{jet}}_{11} f^{\mathrm{jet}}_{1} f^{\mathrm{jet}}_{22,2} - f^{\mathrm{jet}}_{1} f^{\mathrm{jet}}_{2} f^{\mathrm{jet}}_{11,22} = 0. \tag{7.10}$$

Proof. In order to check our relation over a prolongation sequence S^* it is enough to check it after base change to a finite étale S^0-algebra. So we may assume E/S^0 has a $\Gamma_1(N)$-level structure with $(N, p) = 1$, $N \geq 4$. By functoriality it is then enough to check (7.1) for S^0 the ring of an affine open set of the modular curve $Y_1(N)$ over R_π and $S^r := J^r_{\pi,\Phi}(S^0)$. Note that f_1^{jet} and f_2^{jet} evaluated at the universal curve over S^0 are non-zero mod p in S^1; cf. [10, Lemma 4.4] (and also follows from Corollary 7.30 below). As in Corollary 5.25 the relation in our theorem holds with (S^r) replaced by (T^r) where $T^0 = S^0$ and $T^r := \widehat{(S^r_{f_1^{\text{jet}} f_2^{\text{jet}}})}$ for $r \geq 1$; this is because T^r are integral domains in which f_1^{jet}, f_2^{jet} are invertible. Note now that the homomorphisms $S^r \to T^r$ are injective; this follows because the homomorphisms $S^r/pS^r \to T^r/pT^r$ are injective as S^r/pS^r are integral domains and $f_1^{\text{jet}} f_2^{\text{jet}}$ is not zero in S^r/pS^r. We conclude that the relation in our theorem holds for (S^r). ∎

By an argument similar to the one in the proof of Theorem 7.18, using the corresponding version of Lemmas 5.28, 5.29, 5.30, 5.31 over an arbitrary prolongation sequence, we obtain the following.

Theorem 7.19. *The following relations hold in* M^2_{p,ϕ_1,ϕ_2}*:*

$$f_{12,1}^{\text{jet}}(f_1^{\text{jet}})^{\phi_1} - f_{11,1}^{\text{jet}}(f_2^{\text{jet}})^{\phi_1} = 0, \tag{7.11}$$

$$f_{12}^{\text{jet}}(f_1^{\text{jet}})^{\phi_1} - f_{11}^{\text{jet}}(f_2^{\text{jet}})^{\phi_1} - f_1^{\text{jet}}(f_{1,2}^{\text{jet}})^{\phi_1} = 0, \tag{7.12}$$

$$f_{12,21}^{\text{jet}} f_1^{\text{jet}} f_2^{\text{jet}} - f_2^{\text{jet}} f_{21}^{\text{jet}} f_{12,1}^{\text{jet}} + f_1^{\text{jet}} f_{12}^{\text{jet}} f_{21,2}^{\text{jet}} - f_{12}^{\text{jet}} f_{21}^{\text{jet}} f_{1,2}^{\text{jet}} = 0, \tag{7.13}$$

$$f_1^{\text{jet}} f_{11,2}^{\text{jet}} - f_2^{\text{jet}} f_{11,1}^{\text{jet}} - f_{11}^{\text{jet}} f_{1,2}^{\text{jet}} = 0, \tag{7.14}$$

$$f_1^{\text{jet}} f_{11,12}^{\text{jet}} - f_{12}^{\text{jet}} f_{11,1}^{\text{jet}} + f_{11}^{\text{jet}} f_{12,1}^{\text{jet}} = 0, \tag{7.15}$$

$$f_1^{\text{jet}} f_{12,2}^{\text{jet}} - f_2^{\text{jet}} f_{12,1}^{\text{jet}} - f_{12}^{\text{jet}} f_{1,2}^{\text{jet}} = 0, \tag{7.16}$$

$$f_2^{\text{jet}} f_{11,21}^{\text{jet}} - f_{21}^{\text{jet}} f_{11,2}^{\text{jet}} + f_{11}^{\text{jet}} f_{21,2}^{\text{jet}} = 0. \tag{7.17}$$

Moreover, the relations obtained from the above relations by switching the indices 1 *and* 2 *also hold.*

7.4 δ-Serre–Tate expansions

In this subsection we assume $\pi = p$ and n is arbitrary.

For the discussion of formal moduli in this paragraph we refer to [24]; we will use the notation in [10, Section 8.2]. Throughout our discussion we fix an ordinary elliptic curve E_0 over $k = R/pR$, a basis b of the Tate module of $T_p(E_0)$, and a basis $\check{b}$ of the Tate module of the dual $T_p(\check{E}_0)$. Let $S^0_{\text{for}} = R[\![T]\!]$, with T a variable, and consider the Serre–Tate universal deformation space (identified with $\text{Spf}(R[\![T]\!])$) of

E_0/k [24]. Let $E_{\rm for}/S^0_{\rm for}$ be the universal elliptic curve over $R[\![T]\!]$. For all Noetherian complete local rings $(A, \mathfrak{m}(A))$ with residue field k and every elliptic curve E/A lifting E_0/k we let $q(E) = q(E/A) \in 1 + \mathfrak{m}(A)$ be the *Serre–Tate parameter* of E, i.e., the value of the Serre–Tate pairing $q_{E/A} : T_p(E_0) \times T_p(\check{E}_0) \to 1 + \mathfrak{m}(A)$ at the pair $(b, \check{b})$. Then $q(E)$ is the image of $1 + T$ via the classifying map $R[\![T]\!] \to A$ corresponding to E/A. We denote by $\omega_{\rm for}$ the canonical 1-form on $E_{\rm for}$ attached to $\check{b}$; cf. [10, equation (8.67)] and the discussion before it.

Let now

$$S^r_{\rm for} := R[\![T]\!][\delta_{p,\mu} T \mid \mu \in \mathbb{M}^r_n]\widehat{\ }.$$

Clearly $S^*_{\rm for} = (S^r_{\rm for})$ is naturally an object of $\mathbf{Prol}_{p,\Phi}$. We define a ring homomorphism

$$\mathcal{E} = \mathcal{E}_{E_0,b,\check{b}} : M^r_{p,\Phi} \to S^r_{\rm for}$$

by attaching to every $f \in M^r_{p,\Phi}$ its δ-*Serre–Tate expansion* given by

$$\mathcal{E}(f) := f(E_{\rm for}/S^0_{\rm for}, \omega_{\rm for}, S^*_{\rm for}) \in S^r_{\rm for}.$$

Note that we have a natural action

$$\Sigma_n \times S^r_{\rm for} \to S^r_{\rm for}, \quad (\sigma, \delta_{p,i} T) \mapsto \delta_{p,\sigma(i)} T.$$

The following is trivially checked.

Lemma 7.20. *The δ-Serre–Tate expansion map $\mathcal{E}$ is compatible with the Σ_n-actions on $f \in M^r_{p,\Phi}$ and $S^r_{\rm for}$ in the sense that for every $\sigma \in \Sigma_n$ and every $f \in M^r_{p,\Phi}$ we have*

$$\sigma(\mathcal{E}(f)) = \mathcal{E}(\sigma f).$$

As in the arithmetic ODE case [9], one has the following *Serre–Tate expansion principle*.

Theorem 7.21. *For every $w \in \mathbb{Z}_\Phi$ the homomorphism*

$$\mathcal{E} : M^r_{p,\Phi}(w) \to S^r_{\rm for}, \quad f \mapsto \mathcal{E}(f)$$

is injective with torsion-free cokernel.

Proof. Let $f \in M^r_{p,\Phi}(w)$ be such that $\mathcal{E}(f) = 0$ (respectively $\mathcal{E}(f) \in pS^r_{\rm for}$); we want to show that for all S^* and E/S^0 we have $f(E/S^0, \omega, S^*) = 0$ (respectively $f(E/S^0, \omega, S^*) \in pS^r$). As in the proof of Theorem 7.18 it is enough to check this for S^0 the ring of an affine open set of the modular curve $Y_1(N)$ over R, E/S^0 the universal elliptic curve, and $S^r := J^r_{p,\Phi}(S^0)$. We conclude by the injectivity of the homomorphisms $S^r \to S^r_{\rm for}$ (respectively $S^r/pS^r \to S^r_{\rm for}/pS^r_{\rm for}$). ■

Corollary 7.22. *The ring $\bigoplus_{w \in \mathbb{Z}_\Phi} M^r_{p,\Phi}(w)$ is an integral domain.*

Remark 7.23. Write

$$f_{i^r}^{\text{jet}} = f_{i\dots i}^{\text{jet}} \quad \text{with } i \text{ repeated } r \text{ times.}$$

By [10, Propositions 8.22, 8.61, 8.84] plus the equality $\epsilon = 1$ in [10, page 236] we have

$$\mathcal{E}(f_{i^r}^{\text{jet}}) = c_r \Lambda_i^{r-1} \Psi_i, \tag{7.18}$$

where $c_r \in R^\times$,

$$\Lambda_i^{r-1} := \sum_{j=0}^{r-1} p^j \phi_i^{r-1-j},$$

and

$$\Psi_i := \frac{1}{p} \sum_{n \geq 1} (-1)^n \frac{p^n}{n} \Big(\frac{\delta_{p,i}(1+T)}{(1+T)^p} \Big)^n \in S^1_{\text{for}} = R[\![T]\!][\delta_{p,i} T]\widehat{\ }.$$

In fact, by the theory over $\mathbb{Z}_p$ in [2,9] (instead of over R as in [10]) one gets that

$$c_r \in \mathbb{Z}_p^\times. \tag{7.19}$$

Note that one has the following equality in $K[\![T, \delta_{p,i} T, \dots, \delta_{p,i}^r T]\!]$:

$$\mathcal{E}(f_{i^r}^{\text{jet}}) = c_r \frac{1}{p} (\phi_i^r - p^r) \log(1+T). \tag{7.20}$$

Now recall that the Serre–Tate expansion of E_{p-1} satisfies

$$\mathcal{E}(E_{p-1}) \equiv 1 \bmod p \ \text{ in } \ R[\![T]\!]; \tag{7.21}$$

cf., for instance, [10, Propositions 8.57 and 8.59]. (In loc. cit. $\overline{H}$ denotes the Hasse invariant which is the reduction mod p of E_{p-1}.) Taking $\mathcal{E}$ in the congruence

$$f_{ii}^{\text{jet}} \equiv E_{p-1} \cdot (f_i^{\text{jet}})^p \bmod p \ \text{ in } \ M^2_{p,\Phi},$$

cf. Remark 7.7, and using Fermat's little theorem we get

$$c_2 \Psi_i \equiv c_1 \Psi_i^p \bmod p \ \text{ in } \ R[\![T]\!]$$

hence

$$c_2 \equiv c_1 \bmod p \ \text{ in } \ \mathbb{Z}_p. \tag{7.22}$$

We claim that we have:

$$c_1 = c_2 = c_3 \tag{7.23}$$

and hence we set, in what follows,

$$c := c_1 = c_2 = c_3. \tag{7.24}$$

To check the equality (7.23) consider a prolongation sequence S^* with S^r integral domains and an arbitrary elliptic curve E/S^0 such that $f_1 \neq 0$. By the relative case of Corollary 5.14 (with $n = 1$) the S^3-module $\mathbf{X}^3_{p,\phi}(E/S^*)$ has rank ≤ 2 so the following δ-characters of E are S^3-linearly dependent:

$$\psi_{11,1},\ \phi_1\psi_{11,1},\ \psi_{111,1}.$$

Therefore the Picard–Fuchs symbols of these δ-characters,

$$f_1\phi^2 - f_{11}\phi + pf_1^{\phi},\ f_1^{\phi}\phi^3 - f_{11}^{\phi}\phi^2 + pf_1^{\phi^2}\phi,\ f_1\phi^3 - f_{111}\phi + f_{111,1}$$

are S^3-linearly dependent. We deduce that the determinant of the matrix

$$\begin{pmatrix} 0 & f_1 & -f_{11} \\ f_1^{\phi} & -f_{11}^{\phi} & pf_1^{\phi^2} \\ f_1 & 0 & -f_{111} \end{pmatrix}$$

vanishes, hence

$$pf_1^{\phi^2} f_1 - f_{11}^{\phi} f_{11} + f_1^{\phi} f_{111} = 0.$$

Since S^* was arbitrary (with S^r integral domains) we get

$$p(f_1^{\text{jet}})^{\phi^2} f_1^{\text{jet}} - (f_{11}^{\text{jet}})^{\phi} f_{11}^{\text{jet}} + (f_1^{\text{jet}})^{\phi} f_{111}^{\text{jet}} = 0.$$

Using the formula (7.18) we get

$$pc_1^2\Psi^{\phi^2}\Psi - c_2^2(\Psi^{\phi} + p\Psi)(\Psi^{\phi^2} + p\Psi^{\phi}) + c_1c_3\Psi^{\phi}(\Psi^{\phi^2} + p\Psi^{\phi} + p^2\Psi) = 0.$$

Identifying the coefficients we get $c_1^2 = c_2^2$ and $c_2^2 = c_3c_1$. So $c_3 = c_1$ and c_2 is either c_1 or $-c_1$. The equality $c_2 = -c_1$ together with the equality (7.22) leads to a congruence $c_1 \equiv -c_1 \mod p$ which is impossible for p odd. We conclude that $c_1 = c_2 = c_3$.

Theorem 7.24. *For every weight $w \in \mathbb{M}_n^r$ of degree* $\deg(w) = -2$ *and every $f \in I_{p,\Phi}^r(w)$ we have that $\mathcal{E}(f)$ is a K-linear combination of elements in the set*

$$\{\Psi_i^{\phi_\mu} \mid \mu \in \mathbb{M}_n^{r-1},\ i \in \{1, \dots, n\}\}.$$

Proof. The proof is entirely similar to that of the statement made in [10, paragraph after Proposition 8.30]. Here is a rough guide to the argument. By [10, Proposition 8.22] there exists a prime $l \neq p$ and an endomorphism $a := u_0 \in \text{End}(E_0) \subset \mathbb{Z}_p$ of degree l such that the quotient $\check{u}_0/u_0 \in \mathbb{Z}_p$ is not a root of unity. By standard Serre–Tate theory u_0 lifts to an isogeny of degree l between $E_{R[\![T]\!]}$ and the curve $E'_{R[\![T]\!]}$

obtained from $E_{R[\![T]\!]}$ by base change via the homomorphism $R[\![T]\!] \to R[\![T]\!]$ given by $T \mapsto (1+T)^a - 1$. This forces the series $F := \mathcal{E}(f)$ to satisfy the equation

$$F(\ldots, \delta_{p,\mu}((1+T)^a - 1), \ldots) = a \cdot F(\ldots, \delta_{p,\mu} T, \ldots); \tag{7.25}$$

cf. [10, Proposition 8.30]. We conclude exactly as in [10, Proposition 4.36] that F is a K-linear combination of series Ψ^{ϕ_μ}. ■

By Theorem 7.24 and the Serre–Tate expansion principle in Theorem 7.21 we get the following corollary.

Corollary 7.25. *For every $w \in \mathbb{Z}^r_\Phi$ of degree* $\deg(w) = -2$ *the R-module $I^r_{p,\Phi}(w)$ has rank at most*

$$D(n,r) - 1 = n + \cdots + n^r.$$

Definition 7.26. Let $w \in \mathbb{Z}^r_\Phi$ be a weight of degree $\deg(w) = -2$ and $f \in I^r_{p,\Phi}(w)$. Write

$$\mathcal{E}(f) = \sum_{i=1}^{n} \sum_{\mu \in \mathbb{M}_n^{r-1}} \lambda_{i,\mu} \Psi_i^{\phi_\mu}, \quad \lambda_{i,\mu} \in K_\pi;$$

cf. Theorem 7.24. Note that

$$\Psi_i^{\phi_\mu} = \frac{1}{p} \phi_i^\mu (\phi_i - p) \log(1+T).$$

Define the *symbol* $\theta(f) \in K_{\pi,\Phi}$ by

$$\theta(f) = \sum_{i=1}^{n} \sum_{\mu \in \mathbb{M}_n^{r-1}} \lambda_{i,\mu} \phi_\mu (\phi_i - p).$$

Hence the following holds in the ring S^r_{for}:

$$\mathcal{E}(f) = \frac{1}{p} \theta(f) \log(1+T).$$

Remark 7.27. For $n = 2$, using (7.18) and Remarks 7.17 and 7.23, we have:

$$\begin{aligned} \mathcal{E}(f_i^{\text{jet}}) &= c\Psi_i, \\ \mathcal{E}(f_{ii}^{\text{jet}}) &= c(\Psi_i^{\phi_i} + p\Psi_i), \\ \mathcal{E}(f_{ii,i}^{\text{jet}}) &= pc\Psi_i^{\phi_i}. \end{aligned} \tag{7.26}$$

In particular, the symbols of these forms are:

$$\begin{aligned} \theta(f_i^{\text{jet}}) &= c(\phi_i - p), \\ \theta(f_{ii}^{\text{jet}}) &= c(\phi_i^2 - p^2), \\ \theta(f_{ii,i}^{\text{jet}}) &= pc(\phi_i^2 - p^2\phi_i). \end{aligned} \tag{7.27}$$

For the rest of this subsection we assume that $n = 2$.

Theorem 7.28. *We have the following Serre–Tate expansions:*

$$
\begin{aligned}
\mathcal{E}(f^{\text{jet}}_{1,2}) &= pc(\Psi_1 - \Psi_2),\\
\mathcal{E}(f^{\text{jet}}_{11,22}) &= p^2c(\Psi_1^{\phi_1} + p\Psi_1 - \Psi_2^{\phi_2} - p\Psi_2),\\
\mathcal{E}(f^{\text{jet}}_{12,1}) &= pc\Psi_2^{\phi_1},\\
\mathcal{E}(f^{\text{jet}}_{12}) &= c(\Psi_2^{\phi_1} + p\Psi_1),\\
\mathcal{E}(f^{\text{jet}}_{12,21}) &= p^2c(\Psi_2^{\phi_1} + p\Psi_1 - \Psi_1^{\phi_2} - p\Psi_2),\\
\mathcal{E}(f^{\text{jet}}_{11,2}) &= pc(\Psi_1^{\phi_1} + p\Psi_1 - p\Psi_2),\\
\mathcal{E}(f^{\text{jet}}_{11,12}) &= p^2c(\Psi_1^{\phi_1} - \Psi_2^{\phi_1}),\\
\mathcal{E}(f^{\text{jet}}_{12,2}) &= pc(\Psi_2^{\phi_1} + p\Psi_1 - p\Psi_2),\\
\mathcal{E}(f^{\text{jet}}_{11,21}) &= p^2c(\Psi_1^{\phi_1} - \Psi_1^{\phi_2} + p\Psi_1 - p\Psi_2).
\end{aligned}
$$

Moreover, the relations obtained from the above relations by switching the indices 1 *and* 2 *also hold.*

Proof. We begin by proving the first 2 equalities.

Set $G_1 := \mathcal{E}(f^{\text{jet}}_{1,2})$, $G_2 := \mathcal{E}(f^{\text{jet}}_{11,22})$. Taking $\mathcal{E}$ in the cubic relation of Theorem 7.18 and using the formulae (7.26) we get

$$
\begin{aligned}
c^2(\Psi_1^{\phi_1} + p\Psi_1)(\Psi_2^{\phi_2} + p\Psi_2)G_1 + pc^3\Psi_2(\Psi_2^{\phi_2} + p\Psi_2)\Psi_1^{\phi_1}&\\
= pc^3(\Psi_1^{\phi_1} + p\Psi_1)\Psi_1\Psi_2^{\phi_2} + c^2\Psi_1\Psi_2G_2.&
\end{aligned}
\tag{7.28}
$$

By Theorem 7.24 we can write

$$
\begin{aligned}
G_1 &= \gamma_1\Psi_1 + \gamma_2\Psi_2\\
G_2 &= \sum_\mu \gamma'_\mu \Psi_1^{\phi_\mu} + \sum_\mu \gamma''_\mu \Psi_2^{\phi_\mu}
\end{aligned}
$$

with $\gamma_i, \gamma'_\mu, \gamma''_\mu \in K$. Plugging these expressions into the equation (7.28) and using the fact that the set

$$\{\Psi_i^{\phi_\mu} \mid i = 1, 2;\ \mu \in \mathbb{M}_2\}$$

is algebraically independent over K, we see that there is a unique tuple $(\gamma_i, \gamma'_\mu, \gamma''_\mu)$ satisfying the resulting equation, which leads to the desired formulae for $\mathcal{E}(f^{\text{jet}}_{1,2})$ and $\mathcal{E}(f^{\text{jet}}_{11,22})$. Taking $\mathcal{E}$ in (7.11), (7.12), (7.13), (7.14), (7.15), (7.16), (7.17) in this order we get the desired formulae for $\mathcal{E}(f^{\text{jet}}_{12,1})$, $\mathcal{E}(f^{\text{jet}}_{12})$, $\mathcal{E}(f^{\text{jet}}_{12,21})$, $\mathcal{E}(f^{\text{jet}}_{11,2})$, $\mathcal{E}(f^{\text{jet}}_{11,12})$, $\mathcal{E}(f^{\text{jet}}_{12,2})$, $\mathcal{E}(f^{\text{jet}}_{11,21})$, respectively. ∎

Corollary 7.29. *The following equalities hold:*

$$f^{\text{jet}}_{12,1} = p(f^{\text{jet}}_{2})^{\phi_1},$$
$$f^{\text{jet}}_{11,12} = p(f^{\text{jet}}_{1,2})^{\phi_1},$$
$$f^{\text{jet}}_{1,2} f^{\text{jet}}_{12,21} - p^2(f^{\text{jet}}_{2})^{\phi_1}(f^{\text{jet}}_{1})^{\phi_2} + f^{\text{jet}}_{21,1} f^{\text{jet}}_{12,2} = 0.$$

Moreover, the equalities obtained from the above equalities by switching the indices 1 *and* 2 *also hold.*

Proof. The forms $f^{\text{jet}}_{12,1}$ and $p(f^{\text{jet}}_{2})^{\phi_1}$ have the same weight and the same Serre–Tate expansion so by the Serre–Tate expansion principle they must be equal. The same argument holds for the other equalities. ∎

Corollary 7.30. *Let $\mu, \nu \in \mathbb{M}^{2,+}_2$, $\mu \neq \nu$, of length r, s respectively, with $r \geq s$.*

(1) *The following non-divisibility, respectively divisibility conditions hold:*

$$\begin{aligned} f^{\text{jet}}_{\mu} &\in I^r_{p,\phi_1,\phi_2}(-\phi_\mu - 1) \setminus pI^r_{p,\phi_1,\phi_2}(-\phi_\mu - 1), \\ f^{\text{jet}}_{\mu,\nu} &\in p^s I^r_{p,\phi_1,\phi_2}(-\phi_\mu - \phi_\nu). \end{aligned} \tag{7.29}$$

(2) *The symbols of f^{jet}_{μ}, $f^{\text{jet}}_{\mu,\nu}$ are given by*

$$\begin{aligned} \theta(f^{\text{jet}}_{\mu}) &= c(\phi_\mu - p^r), \\ \theta(f^{\text{jet}}_{\mu,\nu}) &= c(p^s\phi_\mu - p^r\phi_\nu). \end{aligned} \tag{7.30}$$

Proof. Part 1 follows from Theorem 7.28 plus the torsion-freeness part of the Serre–Tate expansion principle. (The first equation in Part 1 also follows from Remark 7.7.) Part 2 follows from a direct computation using our definitions. ∎

Remark 7.31. In particular, we have that the forms f^{jet}_{μ}, $f^{\text{jet}}_{\mu,\nu}$ for $\mu, \nu \in \mathbb{M}^{2,+}_2$, $\mu \neq \nu$ are non-zero in M^2_{p,ϕ_1,ϕ_2}. Note that all these forms, with the exception of the "ODE forms" f^{jet}_{i}, f^{jet}_{ii}, $f^{\text{jet}}_{ii,i}$, are "genuinely PDE" in the sense that they are not sums of products of ODE forms and their images by ϕ_1, ϕ_2 (as one can see by looking at weights).

Remark 7.32. We expect that Corollary 7.30 remains true for $\mathbb{M}^{2,+}_2$ replaced by $\mathbb{M}^{r,+}_n$ for n and r arbitrary. Our method of proof for $\mathbb{M}^{2,+}_2$ was based on "solving" a rather complicated system of quadratic and cubic equations satisfied by the arithmetic Kodaira–Spencer classes; extending this method to the case of $\mathbb{M}^{r,+}_n$ seems tedious. It would be interesting to find another approach to the proof of Corollary 7.30 that easily extends to arbitrary n and r.

Remark 7.33. Let $\sigma \in \Sigma_2$ be the transposition (12). Then by Lemma 7.20 and by the δ-Serre–Tate expansion principle we have:

$$\sigma f^{\text{jet}}_{\mu} = f^{\text{jet}}_{\sigma\mu}, \quad \sigma f^{\text{jet}}_{\mu,\nu} = f^{\text{jet}}_{\sigma\mu,\sigma\nu}.$$

Theorem 7.34. *The form* $f_{1,2}^{\text{jet}}$ *is a basis modulo torsion of the* R*-module*

$$I_{p,\phi_1,\phi_2}^1(-\phi_1-\phi_2)$$

of isogeny covariant δ_p*-modular forms of order* 1 *and weight* $-\phi_1-\phi_2$.

Proof. Assume this R-module has rank ≥ 2 and seek a contradiction. Let f be an element of this module that is R-linearly independent of $\psi_{p,1,2}$. By Theorem 7.24 we have $\mathcal{E}(f)=\gamma_1\Psi_1+\gamma_2\Psi_2$ for some $\gamma_1,\gamma_2\in K$. By the Serre–Tate expansion principle (Theorem 7.21) $\mathcal{E}(f)$ is K-linearly independent of $\mathcal{E}(f_{1,2}^{\text{jet}})$. By Theorem 7.28 the latter equals $p\Psi_1-p\Psi_2$. Hence $\gamma_1+\gamma_2\neq 0$. Consider the degeneration morphism $d: M_{p,\phi_1,\phi_2}^1\to M_{p,\phi}^1$ where $\phi\in\mathfrak{F}^{(1)}(K^{\text{alg}}/\mathbb{Q}_p)$ and let Ψ be the series corresponding to f. Then $d(f)\in I_{p,\phi}^1(-2\phi)$. By [10, Theorem 8.83, Part 2] we have $I_{p,\phi}^1(-2\phi)=0$; hence $f=0$. But on the other hand $\mathcal{E}(f)=(\gamma_1+\gamma_2)\Psi\neq 0$, a contradiction. ∎

7.5 δ-period maps

In this subsection we revert to the case of an arbitrary value of n. For the next result let us consider a weight $w\in\mathbb{Z}_\Phi^r$ together with the set $P^r(w)$ of all polynomials $F(\ldots,y_{\eta,\mu},\ldots,y_{\eta,\mu,\nu},\ldots)$ with R-coefficients in the variables $y_{\eta,\mu}$, $y_{\eta,\mu,\nu}$, $\eta,\mu,\nu\in\mathbb{M}_n$, that are homogeneous of weight w when $y_{\eta,\mu}$, $y_{\eta,\mu,\nu}$ are given weights $-\phi_\eta-\phi_{\eta\mu}$ and $-\phi_{\eta\mu}-\phi_{\eta\nu}$, respectively. Moreover, we denote by $\text{KSI}_{p,\Phi}^r(w)$ the R-submodule of $M_{p,\Phi}^r$ of all elements f of the form

$$f=F(\ldots,\phi_\eta f_\mu^{\text{jet}},\ldots,\phi_\eta f_{\mu,\nu}^{\text{jet}},\ldots),\quad F\in P^r(w). \tag{7.31}$$

Clearly the elements of $\text{KSI}_{p,\Phi}^r(w)$ have weight w and are isogeny covariant, i.e.,

$$\text{KSI}_{p,\Phi}^r(w)\subset I_{p,\Phi}^r(w). \tag{7.32}$$

Alternatively $\text{KSI}_{p,\Phi}^r(w)$ is the R-span of all the products of the form

$$\prod_\mu (f_\mu^{\text{jet}})^{w_\mu}\cdot\prod_{\mu,\nu}(f_{\mu,\nu}^{\text{jet}})^{w_{\mu,\nu}} \tag{7.33}$$

where μ,ν run through $\mathbb{M}_n$, $w_\mu,w_{\mu,\nu}\in\mathbb{Z}_\Phi$ are ≥ 0 and

$$\sum_\mu w_\mu(1+\phi_\mu)+\sum_{\mu,\nu}w_{\mu,\nu}(\phi_\mu+\phi_\nu)=-w.$$

One should view the ring

$$\text{KSI}_{p,\Phi}^r:=\bigoplus_{w\in\mathbb{Z}_\Phi^r}\text{KSI}_{p,\Phi}^r(w)\subset\bigoplus_{w\in\mathbb{Z}_\Phi^r}I_{p,\Phi}^r(w)$$

as the "Φ-stable R-subalgebra generated by the arithmetic Kodaira–Spencer classes"; whence our notation. By the way we do not know if the inclusion (7.32) is an equality (or an equality modulo torsion).

Definition 7.35. Let $B = B_1(N)$, $N \geq 4$, $\pi = p$ and assume as before that the reduction mod p of $X \subset Y_1(N)$ is non-empty. Fix an order $r \geq 1$ and weight $w \in \mathbb{Z}^r_\Phi$ and consider a basis

$$f_{(0)}, \ldots, f_{(N_w)}$$

of the R-module $\mathrm{KSI}^r_{p,\Phi}(w)$ (so $N_w + 1$ is the rank of this module). Consider the map

$$\mathfrak{P}^B_w : B(R^{\mathrm{alg}}) \to \mathbb{A}^{N_w}(K^{\mathrm{alg}}) = (K^{\mathrm{alg}})^{N_w}$$

defined by

$$\mathfrak{P}^B_w(P) := (((f_{(0)})^B)^{\mathrm{alg}}(P), \ldots, ((f_{(N_w)})^B)^{\mathrm{alg}}(P))$$

and the induced map to the set of points of the projective space:

$$\mathfrak{p}^B_w : B(R^{\mathrm{alg}})^{\mathrm{ss}}_w := B(R^{\mathrm{alg}}) \setminus ((\mathfrak{P}^B_w)^{-1}(0)) \to \mathbb{P}^{N_w}(K^{\mathrm{alg}}) = \mathbb{P}^{N_w}(R^{\mathrm{alg}}).$$

The "ss" superscript stands for "semistable" (in analogy with geometric invariant theory; here instead of group actions we have an action of the Hecke correspondences and isogeny covariant forms are viewed as analogues of invariant sections of line bundles in geometric invariant theory). Assuming, for a moment, that the universal elliptic curve over X possesses a global invertible relative 1-form ω we get an induced section $\sigma : X \to B$ of the projection $B \to X$ and hence an induced map

$$\mathfrak{p}_w := \mathfrak{p}^X_w : X(R^{\mathrm{alg}})^{\mathrm{ss}}_w := \sigma^{-1}(B(R^{\mathrm{alg}})^{\mathrm{ss}}_w) \xrightarrow{\sigma} B(R^{\mathrm{alg}})^{\mathrm{ss}}_w \xrightarrow{\mathfrak{P}^B_w} \mathbb{P}^{N_w}(R^{\mathrm{alg}}). \quad (7.34)$$

The map (7.34) does not depend on the choice of ω (due to the fact that $f_{(i)}$ have the same weight) and hence this map is well defined for any X (not only for X such that an ω as above exists) and only depends on X and w (up to a projective transformation). The map (7.34) will be referred to as the δ-*period map* (of weight w) and the set $X(R^{\mathrm{alg}})^{\mathrm{ss}}_w$ will be referred to as the set of *semistable* points (relative to w).

Note that for $w, w' \in \mathbb{M}_n$ the composition

$$X(R^{\mathrm{alg}})^{\mathrm{ss}}_w \cap X(R^{\mathrm{alg}})^{\mathrm{ss}}_{w'} \xrightarrow{\mathfrak{p}_w \times \mathfrak{p}_{w'}} \mathbb{P}^{N_w}(R^{\mathrm{alg}}) \times \mathbb{P}^{N_{w'}}(R^{\mathrm{alg}}) \xrightarrow{\mathrm{Segre}} \mathbb{P}^{N_w N_{w'} + N_w + N_{w'}}(R^{\mathrm{alg}})$$

is obtained by composing the map

$$\mathfrak{p}_{w+w'} : X(R^{\mathrm{alg}})^{\mathrm{ss}}_w \to \mathbb{P}^{N_{w+w'}}(R^{\mathrm{alg}})$$

with a projection. And similarly $\mathfrak{p}_w$ is obtained from any of the maps $\mathfrak{p}_{\phi_i w}$ by composing with a projection followed by the ϕ_i map.

Theorem 7.36. *The δ-period map*

$$\mathfrak{p}_w : X(R^{\mathrm{alg}})^{\mathrm{ss}}_w \to \mathbb{P}^{N_w}(R^{\mathrm{alg}})$$

is constant on prime to p isogeny classes in the following sense: for every two points $P, Q \in X(R^{\mathrm{alg}})^{\mathrm{ss}}_w$ if there exists an isogeny of degree prime to p between the elliptic curves over R^{alg} corresponding to P and Q then

$$\mathfrak{p}_w(P) = \mathfrak{p}_w(Q).$$

Proof. By the density of prime to p ordinary isogeny classes [19] we may assume X is such that the universal elliptic curve over X possesses an invertible 1-form ω. Assume P and Q are as in the statement of the theorem and let $u : E_P \to E_Q$ be an isogeny of degree prime to p between the corresponding elliptic curves over R^{alg}. Let ω_P and ω_Q be the 1-forms on E_P and E_Q induced by ω, respectively. We may view both elliptic curves and the isogeny as being defined over some R_π. For each i let $f^X_{(i)}$ be the composition

$$f^X_{(i)} : J^r_{\pi,\Phi}(X) \stackrel{J^r(\sigma)}{\longrightarrow} J^r_{\pi,\Phi}(B) \stackrel{f^B_{(i)}}{\longrightarrow} \widehat{\mathbb{A}^1}.$$

Write for simplicity $f_{(i)}(P) := ((f_{(i)})^X)^{\mathrm{alg}}(P)$ for $P \in X(R^{\mathrm{alg}})$. By isogeny covariance, the weight condition, and the equalities (7.3), (7.4), (7.6), (7.7) we get that

$$f_{(i)}(P) = \lambda \cdot f_{(i)}(Q), \;\; i \in \{0, \dots, N_w\}$$

for some $\lambda \in R_\pi^\times$ depending on $E_P, E_Q, \omega_P, \omega_Q, u$ but not on i. This implies that $\mathfrak{p}_w(P) = \mathfrak{p}_w(Q)$. ∎

Example 7.37. Assume $n = r = 2$. Then one can explicitly describe the algebra $\mathrm{KSI}^2_{p,\Phi} \otimes_R K$ as follows. Consider the ring of polynomials

$$\mathcal{P} := K[\Psi_i, \Psi_i^{\phi_j} \mid i, j \in \{1, 2\}],$$

where $\Psi_i^{\phi_j}$ are viewed as variables and view this ring as graded by giving the variables the degree 1. We denote by $\mathcal{P}(i)$ the graded piece of degree $i \in \mathbb{Z}_{\geq 0}$, we set

$$t_0 := \frac{\Psi_2}{\Psi_1}, \;\; t_{ij} := \frac{\Psi_i^{\phi_j}}{\Psi_1}, \;\; i, j \in \{1, 2\},$$

and we consider the field of "homogeneous fractions of degree 0":

$$\mathcal{F} := \left\{ \frac{F}{G} \;\middle|\; F, G \in \mathcal{P}(i), \; i \geq 1, \; G \neq 0 \right\} = K(t_0, t_{11}, t_{12}, t_{21}, t_{22});$$

this is a field of rational functions over K in five variables. Consider new variables Λ^{ϕ_μ} for $\mu \in \mathbb{M}_2^2$ and the algebra of polynomials

$$\mathcal{P}[\Lambda^{\phi_\mu} \mid \mu \in \mathbb{M}_2^2].$$

Inside the latter consider the K-subalgebra $\mathcal{KSI}^2$ generated by all the elements of the form

$$\mathcal{E}(f_\mu^{\mathrm{jet}})\Lambda^{1+\phi_\mu}, \quad \mathcal{E}(f_{\mu,\nu}^{\mathrm{jet}})\Lambda^{\phi_\mu+\phi_\nu},$$

which have been explicitly computed in Remark 7.27 and Theorem 7.28. We then write

$$\mathcal{KSI}^2 = \bigoplus_{w \in \mathbb{Z}_\Phi^2} \mathcal{KSI}^2(w)\Lambda^w, \quad \mathcal{KSI}^2(w) \subset \mathcal{P}\left(\frac{\deg(w)}{2}\right)$$

(which makes sense since $\mathcal{KSI}^2(w) = 0$ for $\deg(w)$ odd). On the other hand, by the Serre–Tate expansion principle we get isomorphisms of K-vector spaces

$$\mathrm{KSI}_{p,\Phi}^2 \otimes_R K \simeq \mathcal{KSI}^2(w).$$

One may consider the subfield $\mathcal{F}_w \subset \mathcal{F}$ generated by the set

$$\left\{\frac{F}{G} \;\middle|\; F, G \in \mathcal{KSI}^2(w),\ m \geq 0,\ G \neq 0\right\}.$$

This field could be intuitively interpreted as the "field of rational functions on the image of the δ-period map $\mathfrak{p}_w$." If $\mathcal{KSI}^2(w) \neq 0$ and $\mathcal{KSI}^2(w') \neq 0$ then, clearly, $\mathcal{F}_w$ and $\mathcal{F}_{w'}$ are subfields of $\mathcal{F}_{w+w'}$.

As a special case of the above discussion let $w = -(1 + \phi_1 + \phi_2 + \phi_1^2)$. Then $\mathrm{KSI}_{p,\Phi}^r(w_1)$ is spanned by

$$f_1^{\mathrm{jet}} f_{11,2}^{\mathrm{jet}}, \quad f_2^{\mathrm{jet}} f_{11,1}^{\mathrm{jet}}, \quad f_{11}^{\mathrm{jet}} f_{1,2}^{\mathrm{jet}},$$

and has rank 2 with the "only relation" (7.14). The fact that this module has rank 2 follows by looking at the Serre–Tate expansions of the generators; cf. Theorem 7.28. Note that

$$\tau := \frac{\Psi_1(\Psi_1^{\phi_1} + p\Psi_1 - p\Psi_2)}{\Psi_2\Psi_1^{\phi_1}} = \frac{t_{11} + p - pt_0}{t_0 t_{11}} \in \mathcal{F}_w. \tag{7.35}$$

Similarly, let $w' = -(1 + \phi_1 + \phi_2 + \phi_1\phi_2)$. Then $\mathrm{KSI}_{p,\Phi}^r(w')$ is spanned by

$$f_1^{\mathrm{jet}} f_{12,2}^{\mathrm{jet}}, \quad f_2^{\mathrm{jet}} f_{12,1}^{\mathrm{jet}}, \quad f_{12}^{\mathrm{jet}} f_{1,2}^{\mathrm{jet}},$$

and has rank 2 with the "only relation" (7.16). Note that

$$\tau' := \frac{\Psi_1(\Psi_2^{\phi_1} + p\Psi_1 - p\Psi_2)}{\Psi_2\Psi_2^{\phi_1}} = \frac{t_{21} + p - pt_0}{t_0 t_{21}} \in \mathcal{F}_{w'}. \tag{7.36}$$

One has a similar discussion for the weights w'', w''' obtained by switching the indices 1 and 2 in the weights w, w', and we have corresponding elements

$$\tau'' := \frac{\Psi_2(\Psi_2^{\phi_2} + p\Psi_2 - p\Psi_1)}{\Psi_1\Psi_2^{\phi_2}} = \frac{t_0(t_{22} + pt_0 - p)}{t_{22}} \in \mathcal{F}_{w''}, \tag{7.37}$$

$$\tau''' := \frac{\Psi_2(\Psi_1^{\phi_2} + p\Psi_2 - p\Psi_1)}{\Psi_1\Psi_1^{\phi_2}} = \frac{t_0(t_{12} + pt_0 - p)}{t_{12}} \in \mathcal{F}_{w'''}. \tag{7.38}$$

It is trivial to check that

$$K(t_0, \tau, \tau', \tau'', \tau''') = K(t_0, t_{11}, t_{12}, t_{21}, t_{22}).$$

So in particular the field $K(\tau, \tau', \tau'', \tau''')$ is a field of rational functions in 4 variables. The union $\bigcup_w \mathcal{F}_w$ for all w's of order 2 is a subfield of $\mathcal{F}$, so it is finitely generated, hence equal to one of the fields $\mathcal{F}_{w_0}$. The field $\mathcal{F}_{w_0}$ has then transcendence degree 4 or 5 over K. It would be interesting to compute this field and in particular to compute its transcendence degree over K. The heuristic we are employing is that the order 2 partial p-jet space of $Y_1(N)$ has relative dimension $7 = |\mathbb{M}_2^2|$ over K and the field $\mathcal{F}_{w_0}$ (which has transcendence degree 4 or 5 over K) plays the role of "field of rational functions" of the quotient "of order 2" of $Y_1(N)$ by the action of the Hecke correspondences. The difference between the dimensions (which is either 3 or 2) should play the role of "dimension of the fibers" of the "order 2" projection from $Y_1(N)$ to this "quotient". All of this can be made rigorous in a "partial δ-geometry" which is a PDE analogue of the ODE δ-geometry in [10]; we will not pursue this in the present work. Suffices to say that, for every $P \in X(R^{\text{alg}})^{\text{ss}}_{w_0}$, we get in this way, a "large" system of arithmetic differential equations of order 2 satisfied by the points $P' \in X(R^{\text{alg}})^{\text{ss}}_{w_0}$ in the prime to p isogeny class of P. Roughly speaking these equations have the form

$$f_{(i)}(P)f_{(j)}(P') - f_{(j)}(P)f_{(i)}(P') = 0,$$

where we are using the notation in the proof of Theorem 7.36. Note that the analogous ODE δ-period maps of minimum order (implicit in [10, Section 8.6]) have order 3 rather than 2; this is an instance of the principle, already encountered in this memoir in the case of δ-characters, that replacing ODEs by PDEs reduces the order of the interesting arithmetic differential equations.

We end by discussing the maps on points on the ordinary locus.

Proposition 7.38. *Let $B = B_1(N)$, $N \geq 4$. Assume the reduction mod p of X is contained in the ordinary locus of the modular curve. Then the following hold:*

(1) *For all distinct $\mu, \nu \in \mathbb{M}_n$ and all $\eta \in \mathbb{M}_n$ the maps*

$$((\phi_\eta f_\mu^{\text{jet}})^B)^{\text{alg}}, ((\phi_\eta f_{\mu,\nu}^{\text{jet}})^B)^{\text{alg}} : B(R^{\text{alg}}) \to K^{\text{alg}}$$

(cf. Theorem 7.13) extend to continuous maps

$$((\phi_\eta f_\mu^{\text{jet}})^B)^{\mathbb{C}_p}, ((\phi_\eta f_{\mu,\nu}^{\text{jet}})^B)^{\mathbb{C}_p} : B(\mathbb{C}_p^\circ) \to \mathbb{C}_p.$$

(2) *Assume Φ is monomially independent in $\mathfrak{G}(K^{\text{alg}}/\mathbb{Q}_p)$. Then for every $w \in \mathbb{Z}_\Phi$ the R-module homomorphism*

$$\mathrm{KSI}^r_{p,\Phi}(w) \to \mathrm{Fun}(B(R^{\text{alg}}), K^{\text{alg}}), \quad f \mapsto (f^B)^{\text{alg}}$$

is injective.

Proof. Part 1 follows exactly as in [12, Proposition 5.17]. Here is a guide to the argument. Let $\mathrm{pr} : B \to X$ be the projection and let $P \in B(\mathbb{C}_p^\circ)$. Since $\mathrm{pr}(P) \in X(\mathbb{C}_p^\circ)$ corresponds to an elliptic curve with ordinary reduction E_0 we have at our disposal the δ-Serre–Tate expansion at E_0. Similar to [12, equation (5.29)] one gets that there exists a p-adic ball $\mathbb{B} \subset B(\mathbb{C}_p^\circ)$ containing P and a nowhere vanishing analytic map $u : \mathbb{B} \to \mathbb{C}_p$ with the following property: for all μ and ν, $((f_\mu^{\text{jet}})^B)^{\text{alg}}$ and $((f_{\mu,\nu}^{\text{jet}})^B)^{\text{alg}}$ may be expressed on $\mathbb{B} \cap B(R^{\text{alg}})$ as

$$((f_\mu^{\text{jet}})^B)^{\text{alg}}(P) = u(P)^{-1-\phi_\mu} \cdot \theta(f_\mu^{\text{jet}})^{\text{alg}}[\log(q(E_P))], \tag{7.39}$$

$$((f_{\mu,\nu}^{\text{jet}})^B)^{\text{alg}}(P) = u(P)^{-\phi_\mu-\phi_\nu} \cdot \theta(f_{\mu,\nu}^{\text{jet}})^{\text{alg}}[\log(q(E_P))], \tag{7.40}$$

where log is the usual logarithm defined on the open ball of $\mathbb{C}_p$ of radius 1, E_P is the elliptic curve corresponding to P, and $\theta(f_\mu^{\text{jet}})^{\text{alg}}$, $\theta(f_{\mu,\nu}^{\text{jet}})^{\text{alg}}$ are, as usual, the induced maps $K^{\text{alg}} \to K^{\text{alg}}$. Cf. loc. cit. for details on this representation. Extension by continuity then follows. A similar argument works for the above maps composed with ϕ_η.

By the way, if P corresponds to a pair (E_P, ω_P) then by the construction in loc. cit. we have

$$u(P) = 1 \text{ if } \omega_P \text{ is induced by } \omega_{\text{for}} \text{ via the classifying map.} \tag{7.41}$$

To check Part 2 let f be as in (7.31) and assume $(f^B)^{\text{alg}} = 0$. Then, by the homogeneity of F, the map $R^{\text{alg}} \to K^{\text{alg}}$ given by

$$\lambda \mapsto F(\ldots, \theta(\phi_\eta f_\mu^{\text{jet}})^{\text{alg}}(\lambda), \ldots, \theta(\phi_\eta f_{\mu,\nu}^{\text{jet}})^{\text{alg}}(\lambda), \ldots)$$

must vanish on the additive group $p^N R^{\text{alg}}$ for some N hence, again, by the homogeneity of F, this map vanishes on R^{alg}. Now we proceed as in the proof of Proposition 3.13. Indeed, write $\theta(\phi_\eta f_{\eta,\mu}) = \sum_\epsilon \lambda_{\eta,\mu,\epsilon}\phi_\epsilon$ and $\theta(\phi_\eta f_{\mu,\nu}) = \sum_\epsilon \lambda_{\eta,\mu,\nu,\epsilon}\phi_\epsilon$. Let x_ϵ be variables indexed by $\epsilon \in \mathbb{M}_n$. By Lemma 2.6 we get that the following polynomial vanishes:

$$F(\ldots, \sum_\epsilon \lambda_{\eta,\mu,\epsilon} x_\epsilon, \ldots, \sum_\epsilon \lambda_{\eta,\mu,\nu,\epsilon} x_\epsilon, \ldots) = 0. \tag{7.42}$$

But f is obtained from the left-hand side of (7.42) by replacing $x_\epsilon \mapsto \frac{1}{p}\phi_\epsilon T$. Hence $f = 0$. ∎

At this point we are ready to give the proof of Theorem 5.33.

Proof of Theorem 5.33. By Remark 7.31 the forms corresponding to the classes in Theorem 5.33 are all non-zero. Let f be the product of all these forms. By Corollary 7.22 $f \neq 0$. By Proposition 7.38, Part 2, for $B = B_1(N)$, $N \geq 4$, we get that $(f^B)^{\text{alg}} \neq 0$, so there exists $\pi \in \Pi$ and a point $P \in B(R_\pi)$ such that the pair (E_P, ω_P) over R_π corresponding to P satisfies $(f^B)^{\text{alg}}(P) \neq 0$. We conclude by (7.6) and (7.7). ∎

Here is a characterization of ordinary elliptic curves with vanishing arithmetic Kodaira–Spencer classes; it is an improvement (and generalization) of [12, Proposition 5.10].

Proposition 7.39. *Let E/R_π be an elliptic curve with ordinary reduction and ω a basis for its 1-forms. The following are equivalent:*

(1) $f_{\pi,i}(E,\omega) = 0$ *for some* $i \in \{1,\dots,n\}$.

(2) $f_{\pi,\mu}(E,\omega) = f_{\pi,\mu,\nu}(E,\omega) = 0$ *for all* $\mu, \nu \in \mathbb{M}_n$.

(3) *The Serre–Tate parameter $q(E)$ is a root of unity.*

Proof. Assume condition (1) holds. Let $P \in B(R_\pi)$ represent (E,ω). By formula (7.6) we have $((f_i^{\text{jet}})^B)^{\text{alg}}(P) = 0$ for $B = B_1(N)$, $N \geq 4$. By (7.39) (with $\mu = i$), since the map $\theta(f_i^{\text{jet}})^{\text{alg}} : R^{\text{alg}} \to K^{\text{alg}}$, $\beta \mapsto \phi_i(\beta) - p\beta$ is injective, it follows that $\log(q(E)) = 0$. Hence $q(E)$ is a root of unity, i.e. condition (3) holds. Similarly, condition (3) implies condition (2) due to (7.6) and (7.39), (7.40) (applied to arbitrary μ, ν). Finally, condition (2) trivially implies condition (1). ∎

7.6 Theorem of the kernel and Reciprocity theorem

Recall from the Introduction the following pairing.

Definition 7.40. Let $\mu, \nu \in \mathbb{M}_2^2$ have length $r, s \in \{1,2\}$, respectively. Define the $\mathbb{Q}_p$-bilinear map

$$\langle\ ,\ \rangle_{\mu,\nu} : K^{\text{alg}} \times K^{\text{alg}} \to K^{\text{alg}}$$

by the formula

$$\langle \alpha, \beta\rangle_{\mu,\nu} = \beta^{\phi_\nu}\alpha^{\phi_\mu} - \beta^{\phi_\mu}\alpha^{\phi_\nu} + p^s(\alpha\beta^{\phi_\mu} - \beta\alpha^{\phi_\mu}) + p^r(\beta\alpha^{\phi_\nu} - \alpha\beta^{\phi_\nu}). \quad (7.43)$$

Note that the above expression is antisymmetric in α, β:

$$\langle \alpha, \beta \rangle_{\mu,\nu} = -\langle \beta, \alpha \rangle_{\mu,\nu}$$

and also in μ, ν:

$$\langle \alpha, \beta \rangle_{\mu,\nu} = -\langle \alpha, \beta \rangle_{\nu,\mu}.$$

Consider in what follows the following data. Fix $\pi \in \Pi$ and an elliptic curve E over R_π with ordinary reduction E_0. Choose bases $b, \check{b}$ of $T_p(E_0), T_p(\check{E}_0)$ as in the beginning of Section 7.4. Now set $S^0_{\text{for}} = R[\![T]\!] \to R_\pi$ the classifying homomorphism given by the Serre–Tate theory. Choose the 1-form ω on E induced by the canonical form ω_{for} on the universal elliptic curve $E_{\text{for}}/S^0_{\text{for}}$ via the classifying homomorphism and set $q(E) \in R_\pi$ the Serre–Tate parameter of E, i.e., the image of $1 + T$ via the classifying homomorphism. Denote also by $\beta := \beta(E) := \log(q(E)) \in K_\pi$, where $\log\colon 1 + \pi R_\pi \to K_\pi$ is the usual logarithm. When $n = 2$, set $\Phi = \{\phi_1, \phi_2\}$, $\mu \neq \nu \in \mathbb{M}_2^{2,+}$ of length r, s respectively, with $r \geq s$. Consider $\psi_{\mu,\nu} := \psi_{\pi,\mu,\nu}(E, \omega) \in \mathbf{X}^2_{\pi,\Phi}(E)$ the δ_π-character attached to (E, ω) and $c \in \mathbb{Z}_p^\times$ the constant in (7.24). Recall this constant depends only on p.

Proposition 7.41. *The Picard–Fuchs symbol of $\psi_{\mu,\nu}$ is given by the following formula:*

$$\theta(\psi_{\mu,\nu}) = p^{N(\pi)+1} c[(\beta^{\phi_\nu} - p^s \beta)\phi_\mu - (\beta^{\phi_\mu} - p^r \beta)\phi_\nu + (p^s \beta^{\phi_\mu} - p^r \beta^{\phi_\nu})].$$

Proof. A direct computation using (in the following order) (5.12), (5.10), (7.6), (7.7), (7.39), (7.40), (7.41), (7.30). ■

In view of Definition 7.40, Proposition 7.41 yields then the formula

$$\theta(\psi_{\mu,\nu})^{\text{alg}}(\alpha) = p^{2N(\pi)+1} c \langle \alpha, \beta \rangle_{\mu,\nu}. \tag{7.44}$$

Hence

$$\operatorname{Ker}(\theta(\psi_{\mu,\nu})^{\text{alg}}) = \{\alpha \in K^{\text{alg}} \mid \langle \alpha, \beta \rangle_{\mu,\nu} = 0\}.$$

The above is a $\mathbb{Q}_p$-linear space; this space contains β which is non-zero if $q(E)$ is not a root of unity. So by Corollary 3.10 if $q(E)$ is not a root of unity then the group $\operatorname{Ker}(\psi^{\text{alg}}_{\mu,\nu})$ in *not torsion*. More generally by Corollary 3.10 we get the following.

Theorem 7.42. *(Theorem of the kernel) We have a natural group isomorphism*

$$\operatorname{Ker}(\psi^{\text{alg}}_{\mu,\nu}) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \{\alpha \in K^{\text{alg}} \mid \langle \alpha, \beta \rangle_{\mu,\nu} = 0\}.$$

Remark 7.43. In fact, in view of Corollary 3.11 a stronger result holds as follows. Let L be a filtered union of complete subfields of K^{alg} and let $\mathcal{O}$ be the valuation ring of L. Assume E comes via base change from an elliptic curve over $\mathcal{O}$. Then

$$(\operatorname{Ker}(\psi^{\text{alg}}_{\mu,\nu}) \cap E(\mathcal{O})) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \{\alpha \in L \mid \langle \alpha, \beta \rangle_{\mu,\nu} = 0\}. \tag{7.45}$$

In order to state our next result fix an ordinary elliptic curve E_0 over k and bases $b, \check{b}$ of $T_p(E_0), T_p(\check{E}_0)$ as in the beginning of Section 7.4. Let $\pi \in \Pi$ and let

$$\alpha, \beta \in K_\pi, \quad |\alpha|, |\beta| < p^{-\frac{1}{p-1}}. \tag{7.46}$$

By [33, Theorem 6.4] there exists $q \in 1 + \pi R_\pi$ such that $\log q = \beta$. Then the homomorphism $R[\![T]\!] \to R_\pi$, $T \mapsto q - 1$, defines an elliptic curve E_β over R_π with logarithm of the Serre–Tate parameter satisfying $\log(q(E_\beta)) = \beta$. Let $\ell^{\text{alg}}_{E_\beta} : E_\beta(\pi R_\pi) \to K_\pi$ be the logarithm of E_β and let ω_β be the 1-form on E_β induced from the canonical 1-form ω_{for} defined by $\check{b}$ on the universal deformation $E_{\text{for}}/R[\![T]\!]$ of E_0. Again, by [33, Theorem 6.4] there exists a point $P_{\alpha,\beta} \in E_\beta(\pi R_\pi)$ such that $\ell^{\text{alg}}_{E_\beta}(P_{\alpha,\beta}) = \alpha$. Since the roles of α and β can be interchanged we also have at our disposal an elliptic curve E_α over R_π, a 1-form ω_α on E_α, and a point $P_{\beta,\alpha} \in E_\alpha(\pi R_\pi)$. Let $\mu, \nu \in \mathbb{M}^2_n$ be distinct and let $\psi_{\mu,\nu,\beta}$ and $\psi_{\mu,\nu,\alpha}$ be the corresponding δ_π-characters attached to (E_β, ω_β) and $(E_\alpha, \omega_\alpha)$ over R_π, respectively. Then formula (7.44), the antisymmetry of $\langle\ ,\ \rangle_{\mu,\nu}$, and the commutative diagram (3.4) imply the following theorem.

Theorem 7.44. *(Reciprocity theorem) For every α, β as in (7.46) and every distinct $\mu, \nu \in \mathbb{M}^2_2$ we have*

$$\psi^{\text{alg}}_{\mu,\nu,\beta}(P_{\alpha,\beta}) = \psi^{\text{alg}}_{\nu,\mu,\alpha}(P_{\beta,\alpha}).$$

Remark 7.45. (1) The Reciprocity theorem works because of the antisymmetry of $\langle\ ,\ \rangle_{\mu,\nu}$. However, we feel that this antisymmetry comes as a surprise and should not be expected a priori; the only "explanation" we could give is the explicit computation of the constants involved in the expression of our bilinear map.

(2) Note that by antisymmetry in μ, ν we get

$$\psi^{\text{alg}}_{\mu,\nu,\alpha}(P_{\alpha,\alpha}) = 0. \tag{7.47}$$

In case $\pi = p$ and $\alpha \in pR$, (7.47) can also be derived as follows. Recall from [6, Section (4.1)] that if $\pi = p$ and $\alpha \in pR$ then the point $P_{\alpha,\alpha}$ belongs to the group $\bigcap_{m=1}^{\infty} p^m E(R)$ of infinitely p-divisible points of $E(R)$. On the other hand for every partial δ_p-character ψ of E_α the homomorphism $\psi^{\text{alg}} : E_\alpha(R^{\text{alg}}) \to K^{\text{alg}}$ sends $E_\alpha(R)$ into R. Since $\bigcap_{m=1}^{\infty} p^m R = 0$ we get $\psi^{\text{alg}}(P_{\alpha,\alpha}) = 0$. We expect that (7.47) can be derived along similar lines in the general case when π is arbitrary and α is arbitrary, satisfying $|\alpha| < p^{-\frac{1}{p-1}}$.

Example 7.46. Here is an illustration of the Theorem of the kernel; the example below can be easily generalized.

Let $\ell \leq p-1$ be a prime and consider the field $K^{(l)}$ in (2.2) and the notation of the paragraph containing that equation; in particular recall the elements π_m, ζ_{l^m}, and the automorphisms $\phi^{(\gamma)} \in \mathfrak{F}^{(1)}(K^{\text{alg}}/\mathbb{Q}_p)$. Set $\phi_1 := \phi^{(0)}$ and $\phi_2 := \phi^{(1)}$; hence ϕ_1

and ϕ_2 are Frobenius automorphisms whose restriction to K_{π_m} satisfy $\phi_1\pi_m = \pi_m$, $\phi_2\pi_m = \zeta_{l^m}\pi_m$. Let $\beta = \pi_1$ (hence $|\beta| < p^{-\frac{1}{p-1}}$), let $\mu, \nu \in \mathbb{M}_2^{2,+}$ be distinct of length 2 (a similar computation holds for length 1) and consider the $\mathbb{Q}_p$-bilinear map $\langle\ ,\ \rangle_{\mu,\nu} : K^{\mathrm{alg}} \times K^{\mathrm{alg}} \to K^{\mathrm{alg}}$ in (7.43).

Claim 1. For every $\alpha \in K^{(l)}$ we have $\langle \alpha, \beta\rangle_{\mu,\nu} = 0$ if and only if there exists $\lambda \in \mathbb{Q}_p$ such that $\alpha = \lambda\beta$.

The "if" part is clear. To check the "only if" part note that we may assume $\alpha \in R_{\pi_m}$ for some m, so we may write

$$\alpha = \sum_{i=0}^{l^m-1} \alpha_i \pi_m^i, \quad \alpha_i \in R.$$

Let ϕ be the restriction of ϕ_1, ϕ_2 to R. Picking out the coefficient of π_m^i in the right-hand side of the equality (7.43) we get from the equality $\langle \alpha, \beta\rangle_{\mu,\nu} = 0$ that

$$\alpha_i^{\phi}(\zeta_l - \zeta_{l^m}^i - p^2 + p^2\zeta_{l^m}^i) + \alpha_i(p^2 - p^2\zeta_l) = 0, \quad i \in \{0, \ldots, l^m - 1\}.$$

If i is such that $\alpha_i \neq 0$, since $|\phi(\alpha_i)| = |\alpha_i|$, it follows that p divides $\zeta_l - \zeta_{l^m}^i$ in R which forces $i = l^{m-1}$. This in turn implies $\phi(\alpha_{l^{m-1}}) = \alpha_{l^{m-1}}$, hence $\alpha_{l^{m-1}} \in \mathbb{Z}_p$ and our Claim 1 is proved.

Consider now the data $E_0, b, \check{b}$ in the paragraph before Theorem 7.44 and consider the elliptic curve E_β over R_{π_1} whose Serre–Tate parameter has logarithm equal to β. Consider, as in that paragraph, the partial δ_{π_1}-character $\psi_{11,22,\beta}$. Then by Claim 1 above and by the strengthening in Remark 7.43 of our Theorem of the kernel, the following claims hold.

Claim 2. The group $(\mathrm{Ker}(\psi^{\mathrm{alg}}_{\mu,\nu,\beta}) \cap E_\beta(K^{(l)})) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a one-dimensional $\mathbb{Q}_p$-linear space with basis any point $P_{\beta,\beta}$ whose elliptic logarithm is β.

On the other hand, if instead of the elliptic curve $E_\beta = E_{\pi_1}$ over R_{π_1} above we consider an elliptic curve E_γ over R with $\gamma \in pR$ then we get the following.

Claim 3. The group $(\mathrm{Ker}(\psi^{\mathrm{alg}}_{\mu,\nu,\gamma}) \cap E_\gamma(K)) \otimes_{\mathbb{Z}} \mathbb{Q}$ is naturally isomorphic to K, hence this group is an infinite-dimensional $\mathbb{Q}_p$-linear space.

Indeed, in this case, one has $\langle \alpha, \gamma\rangle_{\mu,\nu} = 0$ for every $\alpha \in K$.

The above is a "genuine PDE" example of an explicit computation for the kernel of a δ-character, in a tamely ramified situation. For the ODE case one has a complete description of the kernels of δ-characters if one restricts to the unramified situation; cf. [8, Introduction].

7.7 Crystalline construction

For background here we refer to [10, Section 8.4.3]. Let S^* be an object of $\mathbf{Prol}_{p,\Phi}$ and E/S^0 an elliptic curve. Let $H^1_{DR}(E/S^0)$ be the de Rham S^0-module. By crystalline theory for every $i \in \{1, \dots, n\}$ and every $r \geq 0$ we have ϕ_i-linear maps

$$H^1_{DR}(E/S^0) \otimes_{S^0} S^r \xrightarrow{\phi_i} H^1_{DR}(E/S^0) \otimes_{S^0} S^{r+1}.$$

Also, one has the de Rham pairing

$$\langle\ ,\ \rangle_{DR} : H^1_{DR}(E/S^0) \otimes_{S^0} S^r \times H^1_{DR}(E/S^0) \otimes_{S^0} S^r \to S^r.$$

Definition 7.47. For every basis ω of 1-forms on E/S^0 and every distinct $\mu, \nu \in \mathbb{M}_n^{r,+}$ we set

$$f^{\mathrm{crys}}_{\mu}(E/S^0, \omega, S^*) = \frac{1}{p}\langle \phi_\mu \omega, \omega\rangle_{DR} \in S^r,$$

$$f^{\mathrm{crys}}_{\mu,\nu}(E/S^0, \omega, S^*) = \frac{1}{p}\langle \phi_\mu \omega, \phi_\nu\omega\rangle_{DR} \in S^r.$$

It is trivial to check that the following proposition holds.

Proposition 7.48. *The rules f^{crys}_{μ} and $f^{\mathrm{crys}}_{\mu,\nu}$ define isogeny covariant partial δ_p-modular forms of order $\leq r$ and weights $-1-\phi_\mu$ and $-\phi_\mu-\phi_\nu$, respectively.*

On the other hand we have the next proposition.

Proposition 7.49. *Let $\mu, \nu \in \mathbb{M}_n^r$, $\mu \neq \nu$, of length r, s respectively, with $r \geq s$. The symbols of f^{crys}_{μ}, $f^{\mathrm{crys}}_{\mu,\nu}$ are given by*

$$\begin{aligned} \theta(f^{\mathrm{crys}}_{\mu}) &= \phi_\mu - p^r, \\ \theta(f^{\mathrm{crys}}_{\mu,\nu}) &= p^s\phi_\mu - p^r\phi_\nu. \end{aligned} \tag{7.48}$$

In particular, the Serre–Tate expansions $\mathcal{E}(f^{\mathrm{crys}}_{\mu})$ are R-linearly independent and not divisible by p in S^r_{for}.

Proof. This follows exactly as in the proof of [10, Proposition 8.61]. ∎

Remark 7.50. Following the lead of the ODE case we expect that the forms f^{crys}_{μ} and $f^{\mathrm{crys}}_{\mu,\nu}$ coincide up to a multiplicative constant in $\mathbb{Z}_p^\times$ with the forms f^{jet}_{μ} and $f^{\mathrm{jet}}_{\mu,\nu}$. Cf. also Remark 7.59 for more on this. In any case, by [10, Corollary 8.84] (adapted to the theory over $\mathbb{Z}_p$ instead of over R, as explained in Remark 7.23) we have the next corollary.

Corollary 7.51. *Assume $n = 1$. Then for all μ we have $f^{\mathrm{jet}}_{\mu} \in \mathbb{Z}_p^\times \cdot f^{\mathrm{crys}}_{\mu}$.*

For $n = 2$ and with $c \in \mathbb{Z}_p^\times$ as in (7.24) our theory yields the following result.

Corollary 7.52. *Let $\mu, \nu \in \mathbb{M}_2^{2,+}$. Then we have*

$$f_\mu^{\mathrm{jet}} = c \cdot f_\mu^{\mathrm{crys}},$$
$$f_{\mu,\nu}^{\mathrm{jet}} = c \cdot f_{\mu,\nu}^{\mathrm{crys}}.$$

Proof. By Corollary 7.30 and Proposition 7.49 the forms in the left-hand sides of our equations have the same Serre–Tate expansion as the forms in the right-hand sides, respectively. Since these forms have the same corresponding weights we conclude by the Serre–Tate expansion principle, cf. Theorem 7.21. ∎

7.8 Forms on the ordinary locus

Definition 7.53. A *δ_p-modular function of order $\leq r$ on the ordinary locus* is a rule f assigning to each object $(E/S^0, \omega, S^*)$ with E/S^0 ordinary an element $f(E/S^0, \omega, S^*)$ of the ring S^r, depending only on the isomorphism class of $(E/S^0, \omega, S^*)$, such that f commutes with base change of the prolongation sequence. We denote by $M_{p,\Phi,\mathrm{ord}}^r$ the set of all δ_p-modular function of order $\leq r$ on the ordinary locus; it has a structure of R-algebra.

In [10] such f's were called *ordinary*, but we want to avoid here the term *ordinary* so no confusion arises with the use of this word in relation to ODEs/PDEs.

Remark 7.54. The general theory developed in the preceding subsections can be developed in this context as follows.

(1) The set $M_{p,\Phi,\mathrm{ord}}^r$ has an obvious structure of ring. As in [9] we have a natural ring isomorphism

$$M_{p,\Phi,\mathrm{ord}}^r \simeq J_{\pi,\Phi}^r(M_{p,\Phi}[E_{p-1}^{-1}]) = R[\delta_\mu a_4, \delta_\mu a_6, \Delta^{-1}, E_{p-1}^{-1} \mid \mu \in \mathbb{M}_n^r]\widehat{\ },$$

where $E_{p-1} \in \mathbb{Z}_p[a_4, a_6]$ corresponds to the Eisenstein series of weight $p-1$.

(2) As in Definition 7.3 one defines what it means for an element $f \in M_{p,\Phi,\mathrm{ord}}^r$ to have *weight* $w \in \mathbb{Z}_\Phi^r$ by requiring that the condition in that definition be satisfied only for ordinary elliptic curves. We denote by $M_{p,\Phi,\mathrm{ord}}^r(w)$ the R-submodule of $M_{p,\Phi,\mathrm{ord}}^r$ of weight w.

(3) As in Definition 7.9, first one defines what it means for an element $f \in M_{p,\Phi,\mathrm{ord}}^r(w)$ to be *isogeny covariant* by requiring that the condition in that definition be satisfied only for ordinary elliptic curves. We denote by $I_{p,\Phi,\mathrm{ord}}^r(w)$ the submodule of all isogeny covariant elements of $M_{p,\Phi,\mathrm{ord}}^r(w)$. The direct sum $\bigoplus_{w \in \mathbb{Z}_\Phi} I_{p,\Phi,\mathrm{ord}}^r(w)$ is a Z_Φ-graded R-subalgebra of the R-algebra $\bigoplus_{w \in \mathbb{Z}_\Phi} M_{p,\Phi,\mathrm{ord}}^r(w)$. For every $f \in I_{p,\Phi,\mathrm{ord}}^r(w)$ and every i we have $f^{\phi_i} \in I_{p,\Phi,\mathrm{ord}}^r(\phi_i w)$.

(4) As in Theorem 7.21 for every $w \in \mathbb{Z}_\Phi$ there is a natural *Serre–Tate expansion* homomorphism $\mathcal{E} : M^r_{p,\Phi,\text{ord}}(w) \to S^r_{\text{for}}$, $f \mapsto \mathcal{E}(f)$, which is injective with torsion-free cokernel.

(5) As in Theorem 7.24 for every weight w of degree $\deg(w) = -2$ and every $f \in I^r_{p,\Phi,\text{ord}}(w)$ we have that $\mathcal{E}(f)$ is a K-linear combination of elements in the set

$$\{\Psi_i^{\phi_\mu} \mid \mu \in \mathbb{M}_n^{r-1},\ i \in \{1,\dots,n\}\}.$$

In particular, $I^r_{p,\Phi,\text{ord}}(w)$ has rank $\leq D(n,r) - 1$.

(6) For every weight w of degree $\deg(w) = 0$ and every $f \in I^r_{p,\Phi,\text{ord}}(w)$ we have that $\mathcal{E}(f) \in K$. (To prove this one proceeds as in the proof of Theorem 7.24 by noting that, in this case, the right-hand side of (7.25) reduces to $F(\dots,\delta_{p,\mu}T,\dots)$ which forces $F \in K$.) In particular, by the Serre–Tate expansion principle (4) above, the R-module $I^r_{p,\Phi,\text{ord}}(w)$ has rank 1.

(7) There are natural R-module homomorphisms $M^r_{p,\Phi}(w) \to M^r_{p,\Phi,\text{ord}}(w)$ and $I^r_{p,\Phi}(w) \to I^r_{p,\Phi,\text{ord}}(w)$ that are injective with torsion-free cokernel.

Recall the following result due to Barcau; cf. [3, Theorem 5.1, Corollary 5.1, Proposition 5.2] and [10, Theorem 8.83].

Theorem 7.55. *Assume $n = 1$, $\Phi = \{\phi\}$. There exist elements $f^\partial \in I^1_{p,\phi,\text{ord}}(\phi - 1)$ and $f_\partial \in I^1_{p,\phi,\text{ord}}(1 - \phi)$ such that*

(1) *f^∂ and f_∂ are bases modulo torsion for these R-modules, respectively;*

(2) *$f^\partial \cdot f_\partial = 1$ in $M^1_{p,\phi,\text{ord}}$;*

(3) *$\mathcal{E}(f^\partial) = \mathcal{E}(f_\partial) = 1$;*

(4) *$f^\partial \equiv E_{p-1}$ and $f_\partial \equiv E_{p-1}^{-1}$ mod p in $M^1_{p,\phi,\text{ord}}$;*

(5) *$I^1_{p,\phi}(\phi - 1) = I^1_{p,\phi}(1 - \phi) = 0$ hence $f^\partial, f_\partial \notin M^1_{p,\phi}$.*

Part 5 says intuitively that $f^\partial_\phi, f_\partial$ are "genuinely singular along the supersingular locus."

Proof. We recall the idea of the argument using references to [10]. For any triple $(E/S^0, \omega, S^*)$ with E/S^0 ordinary we define

$$f^\partial(E/S^0,\omega,S^*) := \frac{\langle \phi u, \omega\rangle_{DR}}{\phi(\langle u,\omega\rangle_{DR})} \in S^r$$

where $u \in H^1_{DR}(E/S^0)$ is a basis of the unit root subspace of $H^1_{DR}(E/S^0)$; cf. [10, page 269]. We also define

$$f_\partial(E/S^0,\omega,S^*) := \frac{\phi(\langle u,\omega\rangle_{DR})}{\langle \phi u, \omega\rangle_{DR}} \in S^r.$$

One readily checks that these formulae define elements of

$$I^1_{p,\phi,\mathrm{ord}}(\phi - 1),\ I^1_{p,\phi,\mathrm{ord}}(1 - \phi),$$

respectively. For the computation of their Serre–Tate expansion (Part 2 in the theorem) we refer to [10, Proposition 8.59]. Then these elements being non-zero are bases modulo torsion of the corresponding modules by Remark 7.54, Part 6, hence Part 1 of the theorem follows. Part 3 is obvious. Part 4 follows from [10, Theorem 8.83, Part 3]. ■

Definition 7.56. Let n be arbitrary and denote by f_i^{∂} and $f_{i,\partial}$ the images of f^{∂} and f_{∂} via the face maps

$$M^1_{p,\phi,\mathrm{ord}} \simeq M^1_{p,\phi_i,\mathrm{ord}} \to M^1_{p,\Phi,\mathrm{ord}}.$$

Corollary 7.57. *The following claims hold for every $i \in \{1, \ldots, n\}$:*

(1) *f_i^{∂} and $f_{i,\partial}$ are bases modulo torsion for the R-modules, $I^1_{p,\Phi,\mathrm{ord}}(\phi_i - 1)$ and $I^1_{p,\Phi,\mathrm{ord}}(1 - \phi_i)$, respectively;*

(2) *$f_i^{\partial} \cdot f_{i,\partial} = 1$ in $M^1_{p,\Phi,\mathrm{ord}}$;*

(3) *$\mathcal{E}(f_i^{\partial}) = \mathcal{E}(f_{i,\partial}) = 1$;*

(4) *$f_i^{\partial} \equiv E_{p-1}$ and $f_{i,\partial} \equiv E_{p-1}^{-1}$ mod p in $M^1_{p,\Phi,\mathrm{ord}}$;*

(5) *$f_i^{\partial}, f_{i,\partial} \notin M^1_{p,\Phi}$.*

Proof. Parts 1 to 4 follow from Parts 1 to 4 of Theorem 7.55. Part 5 follows from Part 5 of Theorem 7.55 by using the fact that the images of f_i^{∂} and $f_{i,\partial}$ via the degeneration map $M^1_{p,\Phi} \to M^1_{p,\phi_i}$ are f^{∂} and f_{∂}, respectively. ■

For the next result let us consider, for every $r \geq 1$, the unique group homomorphism

$$\{w \in \mathbb{Z}^r_{\Phi} \mid \deg(w) = 0\} \to (M^r_{p,\Phi,\mathrm{ord}})^{\times}, \quad w \mapsto f_{(w)}, \tag{7.49}$$

satisfying

$$f_{(\phi_{i_1\ldots i_s}-1)} := f^{\partial}_{i_1} \cdot (f^{\partial}_{i_2})^{\phi_{i_1}} \cdot (f^{\partial}_{i_3})^{\phi_{i_1 i_2}} \cdots (f^{\partial}_{i_s})^{\phi_{i_1 i_2 i_3 \ldots i_{s-1}}}, \quad s \in \{1, \ldots, r\}. \tag{7.50}$$

Note that, by Corollary 7.57, Part 4, we have the following congruences in $M^s_{p,\Phi,\mathrm{ord}}$:

$$f_{(\phi_{i_1\ldots i_s}-1)} \equiv E_{p-1}^{1+p+p^2+\cdots+p^{s-1}} \bmod p. \tag{7.51}$$

Corollary 7.58. *For every $r \geq 1$ the following claims hold.*

(1) *For every $w \in \mathbb{Z}^r_{\Phi}$ of degree $\deg(w) = 0$ the form $f_{(w)}$ is a basis of the R-module $I^r_{p,\Phi,\mathrm{ord}}(w)$.*

(2) *For every* $v \in \mathbb{Z}^r_\Phi$ *of degree* $\deg(v) = -2$ *the* R*-module* $I^r_{p,\Phi,\text{ord}}(v)$ *has rank* $D(n,r) - 1$ *and a basis modulo torsion is given by the set*

$$\{f_{(v+\phi_\mu+1)} f_\mu^{\text{crys}} \mid \mu \in \mathbb{M}_n^{r,+}\}.$$

Proof. To check Part 1 note that by Remark 7.54, Part 6, $I^r_{p,\Phi,\text{ord}}(w)$ has rank 1. We are done by noting that $f_{(w)}$ belongs to this module and is not divisible by p in this module.

To check Part 2 note that the forms $f_{(v+\phi_\mu+1)} f_\mu^{\text{crys}}$ have weight v hence belong to $I^r_{p,\Phi,\text{ord}}(v)$. Since the latter module has rank $\leq D(n,r) - 1$ (cf. Remark 7.54, Part 5) it is enough to check that the forms $f_{(v+\phi_\mu+1)} f_\mu^{\text{crys}}$ are R-linearly independent. For this it is enough to check that their Serre–Tate expansions are R-linearly independent. However, by Corollary 7.57, Part 3, we have

$$\mathcal{E}(f_{(v+\phi_\mu+1)} f_\mu^{\text{crys}}) = \mathcal{E}(f_\mu^{\text{crys}}),$$

and we may conclude by Proposition 7.49. ∎

Remark 7.59. Following the lead from the ODE case (cf. [3] or [10, Theorem 8.83, Part 2]) it is natural to ask if for all distinct $\mu, \nu \in \mathbb{M}^r_n$ we have

$$\text{rank}_R I^r_{p,\Phi}(-1 - \phi_\mu) = 1, \tag{7.52}$$

$$\text{rank}_R I^r_{p,\Phi}(-\phi_\mu - \phi_\nu) = 1, \tag{7.53}$$

$$I^r_{p,\Phi}(-2) = I^r_{p,\Phi}(-2\phi_\mu) = 0. \tag{7.54}$$

By loc. cit. the above equations hold if $n = 1$. For $n = 2$ the "simplest case" ($r = 1$, $\mu = 1$, $\nu = 2$) of (7.53) holds; cf. Theorem 7.34. Note that if the conditions (7.52) and (7.53) hold in general then there exist constants $\lambda_\mu \in \mathbb{Z}_p$ and $\lambda_{\mu,\nu} \in \mathbb{Q}_p$ such that

$$f_\mu^{\text{jet}} = \lambda_\mu \cdot f_\mu^{\text{crys}},$$
$$f_{\mu,\nu}^{\text{jet}} = \lambda_{\mu,\nu} \cdot f_{\mu,\nu}^{\text{crys}}.$$

Indeed, by Theorems 7.8 and 7.11 and by Proposition 7.48 the left-hand sides and the right-hand sides of the above equations belong to the same R-modules of rank one, respectively. On the other hand the forms f_μ^{crys} are not divisible by p while the forms $f_{\mu,\nu}^{\text{crys}}$ are non-zero (cf. Proposition 7.49). Moreover, again under the assumption that conditions (7.52) and (7.53) hold, since by Remark 7.7 the forms f_μ^{jet} are not divisible by p, we get $\lambda_\mu \in \mathbb{Z}_p^\times$. Nevertheless, even under the assumption that conditions (7.52) and (7.53) hold, we cannot conclude that $\lambda_{\mu,\nu} \neq 0$ (let alone that $\lambda_{\mu,\nu} \in \mathbb{Z}_p^\times$, as in Corollary 7.52). We recall that the proof of Corollary 7.52 involved "solving a system of quadratic and cubic equations" satisfied by the f^{jet} forms as in the proof of Theorem 7.28. So even if one can prove conditions (7.52) and (7.53) one still cannot go around solving our system of quadratic and cubic equations if one wants to prove the non-vanishing of the forms $f^{\text{jet}}_{\mu,\nu}$ for distinct μ, ν as in Corollary 7.52.

Finally, one may hope to prove conditions (7.52) and (7.53) along the lines of [3] or [10, Theorem 8.83]. However, even proving condition (7.52) for $n = 2, \mu = 1$ along these lines does not seem to work in an obvious way. Indeed, by Corollary 7.58 we have that $I^1_{p,\phi_1,\phi_2,\text{ord}}(-1-\phi_1)$ has a basis modulo torsion consisting of f_1 and $f_{1,\partial} f_2^{\partial} f_2$. So in order to prove that $I^1_{p,\phi_1,\phi_2}(-1-\phi_1)$ has rank 1 we need to show that the form $f_{1,\partial} f_2^{\partial} f_2 \in M^1_{p,\Phi,\text{ord}}$ does not belong to $M^1_{p,\Phi}$. The argument in loc. cit. for this type of statement was to show that the image of the corresponding form in $M^1_{p,\Phi,\text{ord}} \otimes_R k$ does not belong to $M^1_{p,\Phi} \otimes_R k$. But in our case the image of $f_{1,\partial} f_2^{\partial} f_2$ in $M^1_{p,\Phi,\text{ord}} \otimes_R k$ equals the image of f_2 which *does* belong to $M^1_{p,\Phi} \otimes_R k$.

On the other hand, as an application of the theory we get a whole series of identities between our forms f^{jet} and f^{∂}; here is an example.

Corollary 7.60. *The following formula holds in* $I^1_{p,\phi_1,\phi_2,\text{ord}}(-\phi_1-\phi_2)$*:*

$$f^{\text{jet}}_{1,2} = p(f^{\text{jet}}_1 f_{2,\partial} - f^{\text{jet}}_2 f_{1,\partial}).$$

Proof. The two sides of the formula have the same weight equal to $-\phi_1 - \phi_2$ and the same Serre–Tate expansions (cf. Remark 7.27 and Theorem 7.28). So they must be equal by the Serre–Tate expansion principle (Remark 7.54, Part 4). ∎

Remark 7.61. The formula in Corollary 7.60 is interesting in that $f_{1,\partial}$, $f_{2,\partial}$ in the right-hand side do not belong to $M^1_{p,\Phi}$ (they are "genuinely singular along the supersingular locus", cf. Corollary 7.57, Part 4) while the left-hand side does belong to $M^1_{p,\Phi}$; so, intuitively, the "singularities" in the right-hand side "cancel each other out." In view of Corollary 7.58 one has similar formulae (exhibiting similar "cancellations of singularities") for every $f \in I^r_{p,\phi_1,\phi_2}(w)$ with w of degree $\deg(w) = -2$.

Finally we address the total δ-overconvergence aspect, there by strengthening the results in [12].

Theorem 7.62. *Let $B = B_1(N)$ be the natural bundle over an open set $X \subset Y_1(N)$ of the modular curve $Y_1(N)$ over R, for $N \geq 4$, N coprime to p, and assume the reduction mod p of X is contained in the ordinary locus of the reduction mod p of $Y_1(N)$. Then the following hold:*

(1) *For every weight w of degree* $\deg(w) = 0$ *and every $f \in I^r_{p,\Phi,\text{ord}}(w)$ the element $f^B \in \mathcal{O}(J^r_{p,\Phi}(B))$ is totally δ-overconvergent.*

(2) *For every f as in (1) the map $(f^B)^{\text{alg}} : B(R^{\text{alg}}) \to K^{\text{alg}}$ extends to a continuous map $(f^B)^{\mathbb{C}_p} : B(\mathbb{C}_p^{\circ}) \to \mathbb{C}_p$.*

Proof. By Corollary 7.58 we may assume f is either $f_{i,\partial}$ or f_i^{∂}. Assume $f = f_i^{\partial}$; the other case is similar. Note that since f_i^{∂} is induced via a face map from the form f^{∂}

we are reduced to check Part 1 in case $n = 1$; but this was proved in [12, Corollary 5.12]. For Part 2 we proceed exactly as in the proof of Proposition 7.38; note that in our case here, since $\mathcal{E}(f_i^{\partial}) = 1$, we will simply have

$$((f_i^{\partial})^B)^{\text{alg}}(P) = u(P)^{\phi_i - 1} \tag{7.55}$$

for u as in that proof. ∎

Remark 7.63. The maps $(f^B)^{\mathbb{C}_p}$ in Part 2 of Theorem 7.62 satisfy a compatibility property with respect to isogenies at points with coordinates belonging to R because of the isogeny covariance of f^{∂}, f_{∂}; however we do not know if the maps $(f^B)^{\mathbb{C}_p}$ continue to satisfy a compatibility property with respect to isogenies at points with coordinates *not* belonging to R. For this to hold it would be sufficient to "naturally extend" the crystalline definition of f^{∂}, f_{∂} in the proof of Theorem 7.55 to the ramified case.

7.9 Finite covers defined by δ-modular forms

Throughout the discussion below fix an element $\pi \in \Pi$ and $\Phi = (\phi_1, \dots, \phi_n)$ with $n \geq 1$. We recall the forms $f_{\pi,i}^{\text{jet}}$ for $i \in \{1, \dots, n\}$; cf. Theorem 7.6. For an affine open set $X = \text{Spec}(A) \subset Y_1(N)$ with non-empty reduction mod π denote by E_X the corresponding universal elliptic curve over X. Assume there is a basis ω for the 1-forms on E_X/X (which can be achieved by shrinking X) and consider the unique elements

$$\check{f}_{\pi,i} \in J^1_{\pi,\phi_i}(A) \setminus \pi J^1_{\pi,\phi_i}(A)$$

such that there exist (necessarily unique) integers $n_i \geq 0$ with

$$\pi^{n_i} \cdot \check{f}_{\pi,i} = f_{\pi,i}^{\text{jet}}(E_X/X, \omega, J^*_{\pi,\phi_i}(A)) \in J^1_{\pi,\phi_i}(A).$$

We continue to denote by $\check{f}_{\pi,i}$ the images of these elements in $J^1_{\pi,\Phi}(A)$. Our main result here is the following theorem.

Theorem 7.64. *There exists an affine open set $X = \text{Spec}(A) \subset X_1(N)$ of the modular curve $X_1(N)$ over R_π, with non-empty reduction mod π, and a basis ω for the 1-forms on E_X/X such that the ring homomorphism*

$$\widehat{A} \to J^1_{\pi,\Phi}(A)/(\check{f}_{\pi,1}, \dots, \check{f}_{\pi,n})$$

is a finite algebra map.

If the map above is an isomorphism (which happens for instance if $\pi = p$ as one can easily see from the proof below) then one can view the arithmetic differential equations $\check{f}_{\pi,1},\dots,\check{f}_{\pi,n}$ as defining an 'arithmetic flow' on X; in the more general case when the map in the theorem is merely a finite algebra map one should view $\check{f}_{\pi,1},\dots,\check{f}_{\pi,n}$ as defining a structure slightly more general than that of an 'arithmetic flow.'

Proof. Since the source and the target of the map in the theorem are p-adically complete rings it is enough to show that there exists $X = \mathrm{Spec}(A)$ such that the map

$$A/\pi A \to (J^1_{\pi,\Phi}(A)/(\check{f}_{\pi,1},\dots,\check{f}_{\pi,n}))/(\pi)$$

is a finite algebra map. Start with an arbitrary X as in the paragraph before our theorem. Replacing X by an affine open set we may assume there is an étale map $R_\pi[y] \to A$, so we have identifications

$$J^1_{\pi,\phi_i}(A) = A[\delta_{\pi,i}y]\widehat{}, \quad J^1_{\pi,\Phi}(A) = A[\delta_{\pi,1}y,\dots,\delta_{\pi,n}y]\widehat{}.$$

The theorem will be proved if we show that for every $i \in \{1,\dots,n\}$ the image $\overline{f}_{\pi,i}$ of $\check{f}_{\pi,i}$ in the ring $(A/\pi A)[\delta_{\pi,i}y]$ is not contained in the ring $A/\pi A$. Note that by definition $\overline{f}_{\pi,i} \neq 0$ for all i. Assume for some i we have $\overline{f}_{\pi,i} \in A/\pi A$ and seek a contradiction. Consider the natural map

$$\mathcal{E}_\pi : J^1_{\pi,\phi_i}(A) \to R_\pi[\![T]\!][\delta_{\pi,i}T]\widehat{}$$

defined similarly to the Serre–Tate expansion map. Since the reduction mod π of this map,

$$\overline{\mathcal{E}_\pi} : J^1_{\pi,\phi_i}(A)/(\pi) = (A/\pi A)[\delta_{\pi,i}y] \to k[\![T]\!][\delta_{\pi,i}T]$$

is injective it follows that the image $\overline{E_\pi}(\overline{f}_{\pi,i}) \in k[\![T]\!][\delta_{\pi,i}T]$ of $\overline{f}_{\pi,i}$ is non-zero and is contained in $k[\![T]\!]$. Let z be a variable and consider the $k[\![T]\!]$-algebra isomorphism

$$\sigma : k[\![T]\!][z] \to k[\![T]\!][\delta_{\pi,i}T], \quad \sigma(z) = \frac{\delta_{\pi,i}(1+T)}{(1+T)^p}.$$

We have

$$0 \neq \sigma^{-1}(\overline{\mathcal{E}_\pi}(\overline{f}_{\pi,i})) \in k[\![T]\!] \subset k[\![T]\!][z]. \tag{7.56}$$

By the compatibility of $\mathcal{E}_\pi$ with the Serre–Tate expansion map and in view of Remark 7.27 it follows that there exists an integer $N \in \mathbb{Z}$ such that

$$\mathcal{E}(\check{f}_{\pi,i}) = u(T)^{-1-\phi_i} \cdot \pi^N \sum_{m\geq 1} (-1)^{m+1} \frac{\pi^m}{m} \left(\frac{\delta_{\pi,i}(1+T)}{(1+T)^p}\right)^m$$

for some invertible series $u(T) \in R_\pi[\![T]\!]^\times$. Letting $\lambda_m \in k$ be the image of

$$(-1)^{m+1}\frac{\pi^{N+n}}{m} \in R_\pi$$

we have that

$$\sigma^{-1}(\overline{\mathcal{E}_\pi}(\overline{f}_{\pi,i})) = u(T)^{-1-p} \cdot \sum_{m=1}^{d} \lambda_m z^m \in k[\![T]\!][z] \tag{7.57}$$

for some $d \geq 1$ which by (7.56) implies

$$0 \neq \sum_{m=1}^{d} \lambda_m z^m \in k[\![T]\!] \subset k[\![T]\!][z],$$

a contradiction. ∎

Remark 7.65. The integer N and the polynomial $S(z) = \sum_{i=1}^{d} \lambda_i z^i$ in the above proof can be computed explicitly. Indeed, let e be the ramification index of R_π over R. Then $-N$ is the minimum of the π-adic valuations of the numbers $\frac{\pi^m}{m}$ hence writing $m = p^\kappa s$ with $s \in \mathbb{Z} \setminus p\mathbb{Z}$ we have

$$-N = \min\{p^\kappa s - \kappa e \mid \kappa \geq 0,\ s \geq 1\} = \min\{p^\kappa - \kappa e \mid \kappa \geq 0\}.$$

On the other hand the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = p^x - ex$ has a unique minimum at

$$\theta := \frac{\log e - \log \log p}{\log p}.$$

So if $e < \log p$ then f is strictly increasing on $\mathbb{Z}_{\geq 0}$ which implies that $N = -1$ and $S(z) = z$. On the other hand if $e > \log p$ and $\kappa_0 \in \mathbb{Z}_{\geq 0}$ is such that $\theta \in [\kappa_0, \kappa_0 + 1]$ then the restriction of f to $\mathbb{Z}_{\geq 0}$ attains its minimum either at κ_0 or $\kappa_0 + 1$ or at both (the last case occurring if and only if $e = p^{\kappa_0+1} - p^{\kappa_0}$). Consequently we have that $S(z)$ is either $\lambda z^{p^{\kappa_0}}$ or $\lambda z^{p^{\kappa_0+1}}$ or $\lambda z^{p^{\kappa_0}} + \lambda' z^{p^{\kappa_0+1}}$ for some $\lambda, \lambda' \in k^\times$ (the last case occurring if and only if $e = p^{\kappa_0+1} - p^{\kappa_0}$).

7.10 Application to modular parameterizations

We present in what follows an application of Theorem 7.64 along the lines of [13, Theorem 1.3]. In this section, we prove Theorem 1.1 as well as the enhanced version of Strassman's theorem. We need some notation first.

We fix again $\pi \in \Pi$ and $n \geq 2$. Consider an elliptic curve E over R_π and a surjective morphism of R_π-schemes

$$\Theta : X_1(N) \to E$$

where $X_1(N)$ is the complete modular curve over R_π. In particular, one can take E to come from an elliptic curve over $\mathbb{Q}$ and Θ to be induced by a new form of weight 2 as in the Eichler–Shimura theory. We denote by

$$\Theta_{R_\pi} : X_1(N)(R_\pi) \to E(R_\pi)$$

the induced map on R_π-points. Similarly, for an open set $X = \mathrm{Spec}(A) \subset X_1(N)$ and a function $g \in \widehat{A}$ we denote by

$$g_{R_\pi} : X(R_\pi) \to R_\pi$$

the induced map on R_π-points. For such a g we consider the sets of zeros of g:

$$Z(g) = \{P \in X(R_\pi) \mid g_{R_\pi}(P) = 0\} \subset X(R_\pi).$$

Definition 7.66. A point $P \in X_1(N)(R_\pi)$ is called a *quasi-canonical lift* if the corresponding elliptic curve is ordinary with Serre–Tate parameter a root of unity. For an open set $X \subset X_1(N)$ we denote by $\mathrm{QCL}(X(R_\pi))$ the set of all points in $X(R_\pi)$ that are quasi-canonical lifts.

For every δ_π-character $\psi \in \mathbf{X}^1_{\pi,\Phi}(E)$ denote, as usual, by

$$\psi_{R_\pi} : E(R_\pi) \to R_\pi$$

the induced group homomorphism. For instance, if $n = 2$ one can take $\psi = \psi_{1,2}$ from Section 5.2. For $n \geq 3$ one can take ψ to be any R_π-linear combination of images of $\psi_{1,2}$ via the different face maps.

Theorem 7.67. *There exists an affine open set $X = \mathrm{Spec}(A) \subset X_1(N)$ with non-empty reduction mod π such that for every δ_π-character $\psi \in \mathbf{X}^1_{\pi,\Phi}(E)$ there exists a monic polynomial $G \in \widehat{A}[t]$ with the property that for every $P \in E(R_\pi)$ the following holds:*

$$\mathrm{QCL}(X(R_\pi)) \cap \Theta^{-1}_{R_\pi}(\mathrm{Ker}(\psi_{R_\pi}) + P) \subset Z(G(\psi_{R_\pi}(P))).$$

Proof. Take X as in Theorem 7.64. Then for every $\psi \in \mathbf{X}^1_{\pi,\Phi}(E)$ consider the composition

$$\Theta^\sharp : J^1_{\pi,\Phi}(X_1(N)) \xrightarrow{J^1(\Theta)} J^1_{\pi,\Phi}(E) \xrightarrow{\psi} \widehat{\mathbb{G}_{a,R_\pi}},$$

which we identify with an element (still denoted by)

$$\Theta^\sharp \in \mathcal{O}(J^1_{\pi,\Phi}(X_1(N))) \subset \mathcal{O}(J^1_{\pi,\Phi}(X)) = J^1_{\pi,\Phi}(A).$$

By Theorem 7.64 the image of $\Theta^\sharp$ in the ring

$$J^1_{\pi,\Phi}(A)/(\check{f}_{\pi,1}, \ldots, \check{f}_{\pi,n})$$

is integral over $\widehat{A}$ hence there exists a monic polynomial

$$G(t) = t^s + g_1 t^{s-1} + \cdots + g_s \in \widehat{A}[t], \quad g_1, \ldots, g_s \in \widehat{A}$$

and there exist $h_1, \dots, h_n \in J^1_{\pi,\Phi}(A)$ such that

$$(\Theta^\sharp)^s + g_1 \cdot (\Theta^\sharp)^{s-1} + \dots + g_s = h_1 \cdot \check{f}_{\pi,1} + \dots + h_n \cdot \check{f}_{\pi,n}$$

in the ring $J^1_{\pi,\Phi}(A)$. Let us denote by g_{i,R_π}, $\check{f}_{\pi,i,R_\pi}$, and h_{j,R_π}, respectively, the functions $X(R_\pi) \to R_\pi$ induced by g_i, $\check{f}_{\pi,i}$, and h_j. Then for all $Q \in X(R_\pi)$ we get an equality

$$(\psi_{R_\pi}(\Theta_{R_\pi}(Q)))^s + g_{1,R_\pi}(Q) \cdot (\psi_{R_\pi}(\Theta_{R_\pi}(Q)))^{s-1} + \dots + g_{s,R_\pi}(Q)$$
$$= h_{1,R_\pi}(Q) \cdot \check{f}_{\pi,1,R_\pi}(Q) + \dots + h_{n,R_\pi}(Q) \cdot \check{f}_{\pi,n,R_\pi}(Q).$$

By Proposition 7.39 we have

$$\check{f}_{\pi,i,R_\pi}(Q) = 0 \text{ for } i \in \{1, \dots, n\}, \quad Q \in \mathrm{QCL}(X(R_\pi)).$$

Hence for all $P \in E(R_\pi)$ and all $Q \in \mathrm{QCL}(X(R_\pi)) \cap \Theta^{-1}_{R_\pi}(\mathrm{Ker}(\psi_{R_\pi}) + P)$ we have the equality $\Theta_{R_\pi}(Q) = \psi_{R_\pi}(P)$ hence

$$(\psi_{R_\pi}(P))^s + g_{1,R_\pi}(Q) \cdot (\psi_{R_\pi}(P))^{s-1} + \dots + g_{s,R_\pi}(Q) = 0,$$

which implies $Q \in Z(G(\psi_{R_\pi}(P)))$. ■

We conclude with a finiteness result; cf. Corollary 7.69 below. We need the following variant of Strassman's theorem [32, page 306]. The classic case is that of the affine line over a not necessarily discrete valuation ring (DVR). We need here the case of an arbitrary smooth curve over a complete discrete valuation ring; the fact that the valuation ring is discrete greatly simplifies the proof.

Lemma 7.68 (Strassman's theorem for curves over a DVR). *Let V be a complete DVR with maximal ideal generated by $\pi \in V$. Fix X/V a smooth affine curve with connected closed fiber and let $\widehat{\mathcal{O}(X)}$ be the π-adic completion of $\mathcal{O}(X)$. Then every non-zero $g \in \widehat{\mathcal{O}(X)}$ has finitely many zeros in $\widehat{X}(V) = X(V)$.*

Proof. Set $A = \mathcal{O}(X)$ and for $g \in \widehat{A}$ denote by $Z(g)$ the set of zeros of g in $\widehat{X}(V) = X(V)$. Without loss of generality, assume g is not in $\pi\widehat{A}$ as π is a regular element. Note the usual bijection $Z(g) \cong \mathrm{Hom}_V(\widehat{A}/(g), V)$. For each map φ in this set denote by P_φ the kernel of φ and by M_φ the kernel of the composition

$$\overline{\varphi}\colon \widehat{A}/(g) \xrightarrow{\varphi} V \to k := V/\pi V.$$

Claim. *The map $\mathrm{Hom}_V(\widehat{A}/(g), V) \to \mathrm{Spec}(\widehat{A}/(g))$, $\varphi \mapsto P_\varphi$ is injective.*

Indeed, if $P_{\varphi_1} = P_{\varphi_2}$ and $\iota : R_\pi \to \widehat{A}/(g)$ is the natural map then for all $x \in \widehat{A}$ we have

$$\varphi_1(x - \iota(\varphi_1(x))) = \varphi_1(x) - \varphi_1(x) = 0$$

hence $x - \iota(\varphi_1(x)) \in P_{\varphi_1} = P_{\varphi_2}$ so

$$0 = \varphi_2(x - \iota(\varphi_1(x))) = \varphi_2(x) - \varphi_1(x)$$

and our claim is proved.

Similarly, we have that the map $\mathrm{Hom}_V(\widehat{A}/(g), k) \to \mathrm{Spec}(\widehat{A}/(g, \pi))$ is injective. Since $A/\pi A$ is regular and connected, it is an integral domain of dimension 1. It follows that $\widehat{A}/(g, \pi)$ is an Artin ring, and therefore it has a finite spectrum. In particular, the set $\mathrm{Hom}_V(\widehat{A}/(g), k)$ is finite.

Consider the natural map

$$\rho\colon \mathrm{Hom}_V(\widehat{A}/(g), V) \to \mathrm{Hom}_V(\widehat{A}/(g), k). \tag{7.58}$$

As the target of this map was shown to be finite the lemma follows by showing that the fibers of (7.58) are also finite. Fix a V-homomorphism $\varphi_0\colon \widehat{A}/(g) \to V$ and let $\varphi\colon \widehat{A}/(g) \to V$ be any V-homomorphism such that φ and φ_0 induce the same map $\overline{\varphi} = \overline{\varphi_0}\colon \widehat{A}/(g) \to k$ hence $M_\varphi = M_{\varphi_0}$. In particular, P_φ is contained in M_{φ_0}. But there are only finitely many prime ideals of $\widehat{A}/(g)$ contained in M_{φ_0} because the ring $(\widehat{A}/(g))_{M_{\varphi_0}}$ is local and Noetherian of dimension 1. We conclude by the claim above that there are only finitely many V-homomorphisms $\varphi : \widehat{A}/(g) \to V$ for which $\overline{\varphi} = \overline{\varphi_0}$ which proves the finiteness of the fibers of the map (7.58). ∎

Corollary 7.69. *There exists an open set $X \subset X_1(N)$ with non-empty reduction mod π such that for every δ_π-character $\psi \in \mathbf{X}^1_{\pi,\Phi}(E)$ there exists a finite set $\Sigma \subset R_\pi$ with the following property. For all $P \in E(R_\pi)$ if $\psi_\pi(P) \notin \Sigma$ then the set*

$$\mathrm{QCL}(X(R_\pi)) \cap \Theta^{-1}_{R_\pi}(\mathrm{Ker}(\psi_{R_\pi}) + P)$$

is finite.

Proof. Let G be the polynomial in Theorem 7.67. Then, in view of Lemma 7.68 applied to $V = R_\pi$, it is enough to take Σ the set of all roots of G. ∎

Remark 7.70. The above result is a ramified version of [13, Theorem 1.3] which dealt with the case $\pi = p$ and with the set of canonical lifts in $X(R)$. The group $\mathrm{Ker}(\psi_{R_\pi})$ in Corollary 7.69 contains the torsion of $E(R_\pi)$ but is generally bigger than the torsion. For $n = 2$ and $\psi = \psi_{1,2}$ this group was explicitly described in our arithmetic "Theorem of the kernel," cf. Theorem 7.42. As remarked before that theorem, for every E that is ordinary but not a quasi-canonical lift the group $\mathrm{Ker}(\psi_{R_\pi})$ is not torsion.

7.11 δ-Serre operators

We conclude now by applying, as in the ODE case, δ-Serre operators and δ-Euler operators to produce systems of differential equations satisfied by our partial δ-modular forms. In this subsection we assume n is arbitrary, $\pi = p$, and we consider an affine open subset $X \subset Y_1(N)$ with non-empty reduction mod p, where $N \geq 4$ is coprime to p. We let $B := B_1(N) = \mathrm{Spec}(\oplus_{m\in\mathbb{Z}} L^m) \to X$ be as in Definition 7.12. Recall from [10, Section 8.3.2 and Remark 3.58] the classic *Serre operator*

$$\partial\colon \bigoplus_{m\in\mathbb{Z}} L^m \to \bigoplus_{m\in\mathbb{Z}} L^m$$

(which has the property $\partial(L^{\otimes m}) \subset L^{\otimes m+2}$) and the *Euler operator*

$$\mathcal{D}\colon \bigoplus_{m\in\mathbb{Z}} L^m \to \bigoplus_{m\in\mathbb{Z}} L^m, \quad \mathcal{D}(\sum \alpha_m) := m\alpha_m, \quad \alpha_m \in L^{\otimes m}$$

(which has the property that $\mathcal{D}(L^{\otimes m}) \subset L^{\otimes m}$). The above operators induce R-derivations of the algebra $\mathcal{O}(B_1(N))$. Hence, if $\mu \in \mathbb{M}_n^r$ we may consider the induced derivations

$$\partial_\mu, \mathcal{D}_\mu\colon \mathcal{O}(J^r_{p,\Phi}(B_1(N))) \to \mathcal{O}(J^r_{p,\Phi}(B_1(N)));$$

cf. Proposition 2.24. Note that under the identification (7.5) we have $\mathcal{D} = x\frac{d}{dx}$ and hence, for $w = \sum a_\nu \phi_\nu$ and $f \in \mathcal{O}(J^r_{p,\Phi}(X))$ we have

$$\mathcal{D}_\mu(f \cdot x^w) = p^r \cdot a_\mu \cdot f \cdot x^w.$$

Proposition 7.71. *Assume we are given a derivation $D\colon \bigoplus_{m\in\mathbb{Z}} L^m \to \bigoplus_{m\in\mathbb{Z}} L^m$ so that there is $c \geq 0$ with $\partial(L^m) \subset L^{m+c}$ for all $m \geq 0$. Then, for all $\mu \in \mathbb{M}_n^r$ and all $w \in \mathbb{Z}^r_\Phi$, the induced derivation (cf. Proposition 2.24)*

$$D_\mu\colon \mathcal{O}(J^r_{p,\Phi}(B_1(N))) \to \mathcal{O}(J^r_{p,\Phi}(B_1(N)))$$

satisfies

$$D_\mu(M^r_{p,\Phi,X}(w)) \subset M^r_{p,\Phi,X}(w + c\phi_\mu).$$

Proof. Similar to [10, Proposition 3.56]. ∎

Recall from [10, Section 8.3.3] that if X has reduction mod p contained in the ordinary locus then we may consider the *Ramanujan form*

$$P \in L^2. \tag{7.59}$$

Then, as in [10, equation (8.94)], we consider the derivation

$$\partial^* := \partial + P\mathcal{D}\colon \bigoplus_{m\in\mathbb{Z}} L^m \to \bigoplus_{m\in\mathbb{Z}} L^m$$

which has the property that $\partial^*(L^m) \subset L^{m+2}$ for all $m \in \mathbb{Z}$.

On the other hand, as in [10, Section 8.3.5] we may consider the *canonical derivation*

$$\partial^{\text{can}} := (1+T)\frac{d}{dT} : R[\![T]\!] \to R[\![T]\!].$$

As in Proposition 2.24 one shows that for every $\mu \in \mathbb{M}_\mu^r$ there is a unique derivation

$$\partial_\mu^{\text{can}} : S_{\text{for}}^r \to S_{\text{for}}^r$$

satisfying

(1) $\partial_\mu^{\text{can}} \phi_\mu F = p^r \cdot \phi_\mu \partial^{\text{can}} F$ for all $F \in R[\![T]\!]$;

(2) $\partial_\mu^{\text{can}} \phi_\nu F = 0$ for all $F \in R[\![T]\!]$ and all $\nu \in \mathbb{M}_n^r \setminus \{\mu\}$.

It is given by

$$F \mapsto \partial_\mu^{\text{can}} F := (1+T^{\phi_\mu})\frac{\partial F}{\partial \delta_\mu T}.$$

Finally, recall the injective homomorphism $\mathcal{E} : M_{p,\Phi,\text{ord}}^r(w) \to S_{\text{for}}^r$ from Remark 7.54, Part 4. With the notation above we have the following.

Proposition 7.72. *For every $\mu \in \mathbb{M}_n^r$ and every $w \in \mathbb{Z}_\Phi^r$ we have an equality*

$$\mathcal{E} \circ \partial_\mu^* = \partial_\mu^{\text{can}} \circ \mathcal{E}$$

of maps $M_{\pi,\Phi,X}^r(w) \to S_{\text{for}}^r$. Equivalently, for $w = \sum a_\nu \phi_\nu$ and $f \in M_{p,\Phi,X}^r(w)$ we have

$$\mathcal{E}(\partial_\mu^* f) = \mathcal{E}(\partial_\mu f + a_\mu p^r P^{\phi_\mu} f) = (1+T^{\phi_\mu})\frac{\partial(\mathcal{E}(f))}{\partial \delta_\mu T}.$$

Proof. Similar to [10, Proposition 8.42]. The value of ϵ in loc. cit. is 1 in view of the comment after [10, equation (8.52)]. ∎

Remark 7.73. Note that for $\mu = i \in \{1, \ldots, n\}$ we easily compute

$$\partial_i^{\text{can}} \Psi_i = (1+T^{\phi_i})\frac{\partial \Psi_i}{\partial \delta_i T} = (1+T^{\phi_i})\frac{\partial}{\partial \delta_i T}\left\{\frac{1}{p}(\phi_i - p)\log(1+T)\right\} = 1. \quad (7.60)$$

The following corollary follows from the ODE case in [10, Proposition 8.64].

Corollary 7.74. *The following equality holds:*

$$\partial_i^* f_i^{\text{jet}} = \partial_i f_i^{\text{jet}} - pP^{\phi_i} f_i^{\text{jet}} = c f_i^{\partial}. \quad (7.61)$$

Proof. For convenience we repeat the argument. The form $\partial_i^* f_i^{\text{jet}}$ has weight $\phi_i - 1$ (cf. Proposition 7.71) and Serre–Tate expansion c (cf. Proposition 7.72 and Remarks 7.26 and 7.73) while the form $c f_i^{\partial}$ has the same weight and Serre–Tate expansion (cf. Theorem 7.55); hence the two forms must coincide by the Serre–Tate expansion principle. ∎

One can get new results along these lines in the PDE case; here is an example.

Corollary 7.75. *For $n = 2$ the following equalities hold:*

$$\partial_1^* f_{1,2}^{\text{jet}} = \partial_1 f_{1,2}^{\text{jet}} - pP^{\phi_1} f_{1,2}^{\text{jet}} = cpf_1^{\partial} f_{2,\partial},$$
$$\partial_2^* f_{1,2}^{\text{jet}} = \partial_2 f_{1,2}^{\text{jet}} - pP^{\phi_2} f_{1,2}^{\text{jet}} = -cpf_2^{\partial} f_{1,\partial},$$
$$(\partial_1 f_{1,2}^{\text{jet}} - pP^{\phi_1} f_{1,2}^{\text{jet}})(\partial_2 f_{1,2}^{\text{jet}} - pP^{\phi_2} f_{1,2}^{\text{jet}}) + c^2 p^2 = 0.$$

Proof. The form $\partial_1^* f_{1,2}^{\text{jet}}$ has weight $\phi_1 - \phi_2$ (cf. Proposition 7.71) and Serre–Tate expansion cp (cf. Proposition 7.72 and Remarks 7.26 and 7.73) while the other form $cpf_1^{\partial} f_{2,\partial}$ has the same weight and Serre–Tate expansion (cf. Theorem 7.55); hence the two forms must coincide by the Serre–Tate expansion principle which proves the first equality. The second equality is proved similarly. The third equality follows by multiplying the first two equalities. ∎

Appendix

Partial π-jet spaces with relations

Throughout this memoir, commutation relations between Frobenius lifts on particular fields K_π played no role; and similarly, no role was given to the inverses of our Frobenius lifts. In short, we assumed no dependence among the Frobenius lifts chosen. The aim of the Appendix is to briefly discuss a more general theoretical framework in which commutation relations and inversion of Frobenius lifts are "built into" our jet spaces. We will also provide some simple computations illustrating the complexity of this more general framework.

A.1 Main definitions

We now discuss technical structures on monoids which help to describe relationships among Frobenius lifts and their inverses. At the most basic, for every homomorphism $\rho : \mathbb{M} \to \mathbb{M}'$ of monoids with identity one defines its *kernel* $\mathbb{K}_\rho$ to be the set of all pairs $(\mu, \nu) \in \mathbb{M} \times \mathbb{M}$ such that $\rho(\mu) = \rho(\nu)$.

Fix in what follows $\pi \in \Pi$, R_π, K_π as in Subsection 2.3, and a family

$$\Phi = (\phi_1, \dots, \phi_n)$$

of distinct Frobenius elements in $\mathfrak{G}(K^{\text{alg}}/\mathbb{Q}_p)$. Furthermore, fix an integer n^* so that $0 \leq n^* \leq n$ and set

$$\Phi^* = (\phi_{n+1}, \dots, \phi_{n+n^*}) := (\phi_1^{-1}, \dots, \phi_{n^*}^{-1}).$$

By convention, for $n^* = 0$ we take $\Phi^* = \emptyset$. For all $i \in \{1, \dots, n^*\}$ we write $i^* = n + i$. From now on we set

$$\mathbb{M} := \mathbb{M}_{n+n^*}, \quad \mathbb{M}^r := \mathbb{M}^r_{n+n^*}.$$

We have a canonical monoid homomorphism

$$\text{can}_\pi : \mathbb{M} \to \mathfrak{G}(K_\pi/\mathbb{Q}_p)$$

defined by

$$\text{can}_\pi(i) = \phi_{\pi,i}, \quad i \in \{1, \dots, n + n^*\}.$$

Finally, fix a homomorphism of monoids with identity

$$\rho : \mathbb{M} \to \mathbb{M}'$$

into a monoid with identity $\mathbb{M}'$ such that $(ii^*, 0)$ and $(i^*i, 0)$ belong to the kernel $\mathbb{K}_\rho$ for all $i \in \{1, \dots, n^*\}$ and assume ρ is *compatible* with π in the sense that

$$\mathbb{K}_\rho \subset \mathbb{K}_{\mathrm{can}_\pi}.$$

We set

$$\mathbb{K}_\rho^r := (\mathbb{M}^r \times \mathbb{M}^r) \cap \mathbb{K}_\rho.$$

We will also fix a subset $\mathbb{M}_\rho \subset \mathbb{M}$ with following properties (such a set always exists):

(1) For all $\mu \in \mathbb{M}$ there exists a unique element $\overline{\mu} \in \mathbb{M}_\rho$ such that $\rho(\mu) = \rho(\overline{\mu})$.

(2) For all $\mu \in \mathbb{M}$ we have $|\mu| \geq |\overline{\mu}|$, where $|\ |$ denotes the length of a word.

Example A.1. We will consider later the following special cases:

(i) (Invertible case) Take $n = n^*$ and $\rho : \mathbb{M} = \mathbb{M}_{2n} \to G$ where G is a group. In case G is the free group on n generators and ρ sends $1, \dots, n$ into the generators one can take M_ρ to be the set of all words $\mu \in \mathbb{M}$ such that no sequence of 2 consecutive letters in μ is of the form ii^* or i^*i.

(ii) (Abelian case) Take $n^* = 0$ and

$$\rho : \mathbb{M} := \mathbb{M}_n \to \mathbb{M}_{\mathrm{ab}} := \mathbb{M}_{n,\mathrm{ab}} := \mathbb{Z}_{\geq 0}^n$$

the canonical homomorphism $\rho(i) = (0, \dots, 1, \dots, 0)$ with 1 on the i-th position. We assume that $\phi_1, \dots, \phi_n$ commute on K_π so ρ is compatible with π. In this case we can take $\mathbb{M}_\rho \subset \mathbb{M}$ to consist of all words of the form $1^{i_1} \dots n^{i_n}$. We will identify $\mathbb{M}_\rho$ with $\mathbb{M}_{\mathrm{ab}}$ via the bijection induced by ρ.

As π is fixed throughout, we will sometimes drop π from the notation $\delta_{\pi,i}, \phi_{\pi,i}$; this should not create confusion as the meaning of δ_i, ϕ_i will be clear every time from the context.

Definition A.2. Define the category $\mathbf{Prol} = \mathbf{Prol}_{\pi,\Phi,\Phi^*,\rho}$ as follows. An object of this category is a countable family of p-adically complete Noetherian flat R_π-algebras $S^* = (S^r)_{r\geq 0}$ equipped with the following data:

(1) R_π-algebra homomorphisms $\varphi \colon S^r \to S^{r+1}$;

(2) π-derivations $\delta_i \colon S^r \to S^{r+1}$ with attached Frobenius lifts mod π denoted by $\phi_i \colon S^r \to S^{r+1}$ for $1 \leq i \leq n$;

(3) ring homomorphisms $\phi_j \colon S^r \to S^{r+1}$ for $n + 1 \leq j \leq n + n^*$.

For all $i \in \{1, \dots, n + n^*\}$ we write

$$\epsilon_i := \begin{cases} \delta_i & \text{if } 1 \leq i \leq n, \\ \phi_i & \text{if } n+1 \leq i \leq n+n^*. \end{cases}$$

For $\mu := i_1 \dots i_l \in \mathbb{M}$ and $r \geq 0$ we write

$$\epsilon_\mu = \epsilon_{i_1} \dots \epsilon_{i_l} : S^r \to S^{r+l},$$
$$\phi_\mu = \phi_{i_1} \dots \phi_{i_l} : S^r \to S^{r+l}.$$

We require that ϕ_i on S^r be compatible with the ϕ_i on R_π for $i \in \{1, \dots, n + n^*\}$, and we have

$$\begin{aligned} \phi_i \circ \varphi &= \varphi \circ \phi_i && \text{for all } i \in \{1, \dots, n + n^*\}, \\ \phi_\mu &= \phi_\nu && \text{on } S^r \text{ for all } r \geq 0,\ (\mu, \nu) \in \mathbb{K}_\rho. \end{aligned}$$

Morphisms in this category are defined in the obvious way.

Note that since $(i i^*, 0), (i^* i, 0) \in \mathbb{K}_\rho^2$ we always have in the above definition that

$$\phi_i(\phi_{i^*}(x)) = \phi_{i^*}(\phi_i(x)) = x$$

for all $x \in S^r$ with $r \geq 0$ and all $i \in \{1, \dots, n^*\}$. So the homomorphisms ϕ_{i^*} play the role of "inverses" of ϕ_i for $i \in \{1, \dots, n^*\}$.

We next consider variables $\epsilon_\mu y_s$ for $\mu \in \mathbb{M}$ and $s \in \{1, \dots, N\}$ and consider the rings

$$J^r_{\pi,\Phi,\Phi^*}(R_\pi[N]) := R_\pi[\epsilon_\mu y_s \mid \mu \in \mathbb{M}^r,\ s \in \{1, \dots, N\}]\widehat{\ }.$$

We define ring homomorphisms

$$\phi_i : J^r_{\pi,\Phi,\Phi^*}(R_\pi[N]) \to J^{r+1}_{\pi,\Phi,\Phi^*}(R_\pi[N]),\ \ i \in \{1, \dots, n + n^*\},$$

extending ϕ_i on R_π by letting

$$\phi_i(\epsilon_\mu y_s) := \begin{cases} (\epsilon_\mu y)^p + \pi \epsilon_{i\mu} & \text{if } 1 \leq i \leq n, \\ \epsilon_{i\mu} y_s & \text{if } n \leq i \leq n + n^*. \end{cases}$$

We then have π-derivations

$$\delta_i : J^r_{\pi,\Phi,\Phi^*}(R_\pi[N]) \to J^{r+1}_{\pi,\Phi,\Phi^*}(R_\pi[N]),\ \ i \in \{1, \dots, n\},$$

with

$$\delta_i(\epsilon_\mu y_s) = \epsilon_{i\mu} y_s.$$

For each pair $(\mu, \nu) \in \mathbb{K}^r_\rho$ define the N-tuple

$$\Delta_{\mu,\nu} := \phi_\mu y - \phi_\nu y.$$

For an ideal I in a ring A we denote by $I : p^\infty$ the ideal of all $a \in A$ for which there exists an integer $m(a) \geq 0$ such that $p^{m(a)} a \in I$. If A is a $\mathbb{Z}_{(p)}$-algebra then

$$A/(I : p^\infty) \simeq (A/I)/\text{tors}.$$

Let $I \subset R_\pi[N]$ be an ideal. For $r \geq 1$ define

$$I_r := (\phi_\eta I, \Delta_{\mu,\nu} \mid \eta \in \mathbb{M}^r,\ (\mu,\nu) \in \mathbb{K}^r_\rho) : p^\infty \subset J^r_{\pi,\Phi,\Phi^*}(R_\pi[N]).$$

Definition A.3. For every finitely generated R_π-algebra $A := R_\pi[N]/I$ define the *π-ρ-jet algebra* of A by the formula

$$J^r_{\pi,\Phi,\Phi^*,\rho}(A) := J^r_{\pi,\Phi,\Phi^*}(R_\pi[N])/I_r.$$

If $m = 0$ (hence $\Phi^* = \emptyset$) we write $J^*_{\pi,\Phi,\emptyset,\rho}(A) =: J^*_{\pi,\Phi,\rho}(A)$.

Clearly the rings $J^r_{\pi,\Phi,\Phi^*,\rho}(A)$ are torsion free as groups, so they are flat over R_π. They are also Noetherian and p-adically complete. We claim they inherit π-derivations and ring homomorphisms from those on $J^r_{\pi,\Phi,\Phi^*}(R_\pi[N])$. This follows from the fact that for all $i \in \{1,\ldots,n+n^*\}$ we have that the components of the tuple $\phi_i(\Delta_{\mu,\nu})$ belong to the ideal $(\Delta_{\mu,\nu}, \Delta_{i\mu,i\nu})$ and hence

$$\delta_i I_r \subset I_r. \tag{A.1}$$

We claim $J^*_\rho(A) := J^*_{\pi,\Phi,\Phi^*,\rho}(A) := (J^r_{\pi,\Phi,\Phi^*,\rho}(A))$ is an object of **Prol**. Indeed, for all $(\mu,\nu) \in \mathbb{K}^r_\rho$ we have $\phi_\mu a_s = \phi_\nu a_s$ where a_s is the image of y_s. So $\phi_\mu a = \phi_\nu a$ for all $a \in A$. In particular,

$$\phi_\mu \phi_\eta a = \phi_{\mu\nu} a = \phi_{\nu\eta} a = \phi_\nu \phi_\eta a$$

for all $a \in A$ and all $\eta \in \mathbb{M}^r$. Using the fact that π is a non-zero divisor in $J^r_\rho(A)$ we get that $\phi_\mu \epsilon_\eta a = \phi_\nu \epsilon_\eta a$ for $a \in A$. By p-adic continuity we get that $\phi_\mu b = \phi_\nu b$ for all $b \in J^r_\rho(A)$.

Remark A.4. (1) The object $J^*_{\pi,\Phi,\Phi^*,\rho}(A)$ has the obvious universal property: for every object S^* in **Prol** and every R_π-algebra homomorphism $u : A \to S^0$ there is a unique morphism $J^*_{\pi,\Phi,\Phi^*,\rho}(A) \to S^*$ in **Prol** compatible with u.

(2) One has the following compatibility with fractions. For every object $S^* = (S^r)$ in **Prol** and every $f \in S^0 \setminus \pi S^0$ the sequence $(\widehat{S^r_f})$ has a natural structure of object in **Prol**. To check this it is enough to check that for every $r \geq 0$ and every $i \in \{1,\ldots,n+n^*\}$ we have that $\phi_i(f)$ is invertible in $\widehat{S^{r+1}_f}$. If $i \in \{1,\ldots,n\}$ the element $\phi_i(f) = f^p + \pi\delta_i f$ has inverse

$$\frac{1}{f^p}\Big(1 - \pi\frac{\delta_i f}{f^p} + \pi^2\Big(\frac{\delta_i f}{f^p}\Big)^2 + \cdots\Big).$$

If $i \in \{1,\ldots,n^*\}$ we have

$$f = \phi_{i^*}(\phi_i(f)) = \phi_{i^*}(f)^p + \phi_i^{-1}(\pi) \cdot \phi_{i^*}\delta_i f$$

hence the element $\phi_{i*}(f)$ has inverse

$$\frac{\phi_{i*}(f)^{p-1}}{f}\Big(1+\phi_i^{-1}(\pi)\cdot\frac{\phi_{i*}\delta_i f}{f}+\phi_i^{-1}(\pi)^2\cdot\Big(\frac{\phi_{i*}\delta_i f}{f}\Big)^2+\cdots\Big).$$

Using the above compatibility we get that for every finitely generated R_π-algebra A, $f\in A\setminus\pi A$, and every $r\geq 0$ one has natural isomorphisms

$$J^r_{\pi,\Phi,\rho}(A_f)\simeq((J^r_{\pi,\Phi,\rho}(A))_f)\widehat{\ }. \tag{A.2}$$

In view of (A.2) the functors $A\mapsto J^r_{\pi,\Phi,\Phi^*\rho}(A)$ can be globalized to give functors $X\mapsto J^r_{\pi,\Phi,\Phi^*,\rho}(X)$ from (not necessarily smooth) schemes of finite type over R_π to p-adic formal schemes. In case $\Phi^*=\emptyset$ we write $J^r_{\pi,\Phi,\Phi^*,\rho}(X)=:J^r_{\pi,\Phi,\rho}(X)$.

(3) If $n^*=0$ and ρ is injective, then for A smooth over R_π we have that the ring $J^r_{\pi,\Phi,\emptyset,\rho}(A)$ coincides with the ring $J^r_{\pi,\Phi}(A)$ previously defined in the body of the memoir; this is because the ring $J^r_{\pi,\Phi}(A)$ is, in this case, torsion free. So for X smooth over R_π we have that $J^r_{\pi,\Phi,\rho}(X)$ coincides with the formal scheme $J^r_{\pi,\Phi}(X)$ defined in the body of the memoir.

(4) For every $f\in J^r_{\pi,\Phi,\Phi^*,\rho}(A)$ and $X:=\mathrm{Spec}(A)$ we have an induced map $f_{R_\pi}:X(R_\pi)\to R_\pi$.

A.2 Structure over $\mathbb{Q}$

The π-ρ-jet algebras mod π have a complicated structure as we shall presently see. However, we have the following theorem about the behavior of these algebras over $\mathbb{Q}$. We need the following trivial fact.

Lemma A.5. *Let A be a flat R_π-algebra, P a prime ideal in A not containing π, $A':=A\otimes_{R_\pi}K_\pi$, and $P':=PA'$. Then $A_P\simeq A'_{P'}$.*

In what follows for a local ring B we denote by B^{for} the completion of B with respect to its maximal ideal. For a finitely generated R_π-algebra A we simply denote by $J^r_\rho(A)$ the algebra $J^r_{\pi,\Phi,\Phi^*,\rho}(A)$. For $u:A\to R_\pi$ an R_π-algebra map with kernel P we denote by P_r the kernel of the surjective homomorphism $J^r_\rho(A)\to R_\pi$ induced by u via the universal property of π-jet algebras; we refer to P_r as the r-th *prolongation* of P. We continue to denote by $P\in X(R_\pi)$ the point of $X:=\mathrm{Spec}(A)$ defined by $u:A\to R_\pi$.

Theorem A.6. *Let A be a finitely generated R_π-algebra and $A\to R_\pi$ be an R_π-algebra map with kernel P. Write $A=R_\pi[y]/I$ where y is an N-tuple of variables such that $P=(y)/I$ and let P_r be the prolongation of P. Then there is a canonical isomorphism*

$$(J^r_\rho(A)_{P_r})^{\mathrm{for}}\simeq\frac{K_\pi[\![\phi_{\overline{\mu}}y\mid\overline{\mu}\in\mathbb{M}^r_\rho]\!]}{(\phi_{\overline{\mu}}I\mid\overline{\mu}\in\mathbb{M}^r_\rho)}.$$

Moreover, if the image of some $F \in J^r_{\pi,\Phi,\Phi^*}(R_\pi[N])$ *in* $K_\pi[\![\phi_\mu y \mid \mu \in \mathbb{M}^r]\!]$ *belongs to the ideal* $(\Delta_{\mu,\nu} \mid (\mu,\nu) \in \mathbb{K}^r_\rho)$ *and if* f *is the image of* F *in* $J^r_\rho(A)$ *then* $f_{R_\pi}(P) = 0$.

Proof. In what follows (if not otherwise stated) η runs through $\mathbb{M}^r_n$ and (μ,ν) run through $\mathbb{K}^r_\rho$. In particular, $P_r = (\delta_\eta y)/I_r$. By Lemma A.5 we have

$$(J^r_\rho(A))_{(\delta_\eta y)} \simeq (J^r_\rho(A) \otimes_{R_\pi} K_\pi)_{(\delta_\eta y)}. \tag{A.3}$$

Note that

$$\begin{aligned} J^r_\rho(A) \otimes_{R_\pi} K_\pi &\simeq (\widehat{R_\pi[\delta_\eta y]} \otimes_{R_\pi} K_\pi)/((\delta_\eta I, \Delta_{\mu,\nu}) : p^\infty) \\ &\simeq (\widehat{R_\pi[\delta_\eta y]} \otimes_{R_\pi} K_\pi)/(\phi_\eta I, \Delta_{\mu,\nu}). \end{aligned} \tag{A.4}$$

Also, one easily checks that

$$((R_\pi[\delta_\eta y]\widehat{\ } \otimes_{R_\pi} K_\pi)_{(\delta_\eta y)})^{\text{for}} \simeq K_\pi[\![\delta_\eta y]\!] \simeq K_\pi[\![\phi_\eta y]\!]. \tag{A.5}$$

Finally note that since $I \subset (y)$, we have that

$$\phi_\mu f - \phi_\nu f \in (\Delta_{\mu,\nu}). \tag{A.6}$$

Using (A.5) and (A.6) we compute:

$$\begin{aligned} &((((\widehat{R_\pi[\delta_\eta y]}) \otimes_{R_\pi} K_\pi)/(\phi_\eta I, \Delta_{\mu,\nu}))_{(\delta_\eta y)})^{\text{for}} \\ &\quad\simeq (((\widehat{R_\pi[\delta_\eta y]}) \otimes_{R_\pi} K_\pi)_{(\delta_\eta y)})^{\text{for}}/(\phi_\eta I, \Delta_{\mu,\nu}) \\ &\quad\simeq K_\pi[\![\phi_\eta y]\!]/(\phi_\eta I, \Delta_{\mu,\nu}) \\ &\quad\simeq K_\pi[\![\phi_{\overline{\eta}} y : \overline{\eta} \in \mathbb{M}^r_\rho]\!]/(\phi_{\overline{\eta}} I \mid \overline{\eta} \in \mathbb{M}^r_\rho). \end{aligned} \tag{A.7}$$

We conclude the first assertion of the theorem by combining the isomorphisms (A.3), (A.4), (A.7). To check the second assertion of the theorem note that if the image of $F \in J^r_{\pi,\Phi,\Phi^*}(R_\pi[N])$ in $K_\pi[\![\phi_\mu y \mid \mu \in \mathbb{M}^r]\!]$ belongs to the ideal $(\Delta_{\mu,\nu} \mid (\mu,\nu) \in \mathbb{K}^r_\rho)$ then the image of F in $\frac{K_\pi[\![\phi_{\overline{\mu}} y \mid \overline{\mu} \in \mathbb{M}^r_\rho]\!]}{(\phi_{\overline{\mu}} I \mid \overline{\mu} \in \mathbb{M}^r_\rho)}$ is 0 hence the image of f in $J^r_\rho(A)_{P_r}$ is zero. So the image of f via the homomorphism $J^r_\rho(A)_{P_r} \to K_\pi$ is zero. Hence the image of f via the homomorphism $J^r_\rho(A) \to R_\pi$ is zero, hence $f_{R_\pi}(P) = 0$. ∎

Corollary A.7. *Under the assumptions of Theorem A.6 if* A *is smooth over* R_π *then the rings* $J^r_\rho(A)_{P_r}$ *are regular and the canonical homomorphisms*

$$J^r_\rho(A)_{P_r} \to J^{r+1}_\rho(A)_{P_{r+1}}$$

are injective.

Proof. If the R_π-algebra A is smooth the (y)-adic completion of $A \otimes_{R_\pi} K_\pi$ is isomorphic to $K_\pi[\![T]\!]$ for some tuple of variables T and hence

$$\frac{K_\pi[\![\phi_\mu y \mid \mu \in \mathbb{M}_\rho^r]\!]}{(\phi_\mu I \mid \mu \in \mathbb{M}_\rho^r)} \simeq K_\pi[\![\phi_\mu T \mid \mu \in \mathbb{M}_\rho^r]\!].$$

The latter power series ring is a regular local ring hence so is the ring $J_\rho^r(A)_{P_r}$. Hence the canonical homomorphisms in the theorem are injective because the corresponding homomorphisms between the power series rings are injective. ∎

Remark A.8. It would be interesting to know if for A smooth over R_π the rings $J_\rho^r(A)$ themselves are regular. If these rings even just domains then this would also imply that the homomorphisms

$$J_\rho^r(A) \to J_\rho^{r+1}(A)$$

are injective.

A.3 Invertible π-jets

Assume $n = n^*$ and $\rho : \mathbb{M} = \mathbb{M}_{2n} \to G$ with G a group; cf. Example A.1 (i). We refer to the algebra $J^r_{\pi,\Phi,\Phi^*,\rho}(A)$ as the *invertible π-jet algebra* of order r of A attached to ρ. The reduction modulo π of this algebra has a complicated structure as we shall illustrate in what follows.

First note that since $(ii^*, 0), (i^*i, 0) \in \mathbb{K}_\rho^2$ for $i \in \{1, \dots, n\}$ the following elements belong to the ideals I_r for all $r \geq 2$, $s \in \{1, \dots, N\}$, and $i \in \{1, \dots, n\}$:

$$\begin{aligned}\Delta_{ii^*,0,s} &= \phi_i\phi_{i^*}y_s - y_s = (\phi_{i^*}y_s)^p + \pi\delta_i\phi_{i^*}y_s - y_s,\\ \Delta_{i^*i,0,s} &= \phi_{i^*}\phi_i y_s - y_s = (\phi_{i^*}y_s)^p + \phi_i^{-1}\pi\cdot\phi_{i^*}\delta_i y_s - y_s.\end{aligned}$$

Take now $N = 1$, $n = 2$, $y = y_1$ (so we drop the index s), $I = 0$, and $\rho = \mathrm{can}_\pi : \mathbb{M} = \mathbb{M}_4 \to F_2$ the natural homomorphism to the free group on 2 generators. Consider the "linear relations:"

$$F_i := \delta_i\phi_{i^*}y - \frac{\phi_i^{-1}\pi}{\pi}\phi_{i^*}\delta_i y.$$

Then we have

$$\begin{aligned}J^2_{\pi,\Phi,\Phi^*,\mathrm{can}_\pi}(R_\pi[y]) &= J^2_{\pi,\Phi,\Phi^*}(R_\pi[y])/((\Delta_{13,0}, \Delta_{31,0}, \Delta_{24,0}, \Delta_{42,0}) : p^\infty)\\ &= (J^2_{\pi,\Phi,\Phi^*}(R_\pi[y])/(\Delta_{13,0}, \Delta_{24,0}, F_1, F_2))/\mathrm{tors}.\end{aligned}$$

A.4 Abelian π-jets

Assume $n^* = 0$ and $\rho : \mathbb{M} := \mathbb{M}_n \to \mathbb{M}_{\mathrm{ab}} := \mathbb{M}_{n,\mathrm{ab}} := \mathbb{Z}^n_{\geq 0}$ is the canonical homomorphism. We identify $\mathbb{M}_{\mathrm{ab}}$ with the subset $\mathbb{M}_\rho$ of $\mathbb{M}$ consisting of words of the form $1^{i_1} \dots n^{i_n}$; cf. Example A.1, (ii). We assume that $\phi_1, \dots, \phi_n$ commute on K_π so ρ is compatible with π. The algebra $J^r_{\pi,\Phi,\emptyset,\mathrm{ab}}(A) = J^r_{\pi,\Phi,\mathrm{ab}}(A)$ is referred to as the *abelian π-jet algebra* of order r. The reduction modulo π of this algebra also has a complicated structure and some comments on this will be made in what follows.

Assume for simplicity $N = 1$ $y = y_1$, $I = 0$. We begin with the following observation.

Lemma A.9. *For $\mu = i_1, \dots, i_r$ and variable y we have*

$$\phi_\mu(y) \equiv y^{p^r} + \pi(\delta_{i_1}\pi)(\delta_{i_2}\pi)^p \cdots (\delta_{i_{r-1}}(\pi)^{p^{r-2}})(\delta_{i_r} y)^{p^{r-1}} \mod \pi^2.$$

Proof. We proceed by induction on r. The case $r = 1$ holds by definition. For the inductive step note that, for any fixed Frobenius lift ϕ mod π with associated π-derivation δ, any $m \geq 0$ and any $F \in R_\pi[\delta_\mu y | \mu \in \mathbb{M}]$ we have

$$\phi(F^{p^m}) = (F^p + \pi\delta F)^{p^m} \equiv F^{p^{m+1}} + p^m \pi F^{p(p^m-1)}\delta F \mod \pi^2. \qquad \text{(A.8)}$$

Now assume $r \geq 2$ and set $\mu' = i_2 \dots i_r$ and $\phi = \phi_{i_1}$ so $\mu = i_1\mu'$. By induction

$$\phi_{\mu'}(y) \equiv y^{p^r} + \pi(\delta_{i_2}\pi)(\delta_{i_3}\pi)^p \cdots (\delta_{i_{r-1}}\pi)^{p^{r-3}}(\delta_{i_r} y)^{p^{r-2}} \mod \pi^2.$$

Repeatedly using (A.8) we have

$$\begin{aligned}
&\phi(\pi(\delta_{i_2}\pi)(\delta_{i_3}\pi)^p \cdots (\delta_{i_{r-1}}\pi)^{p^{r-3}}(\delta_{i_r} y)^{p^{r-2}}) \\
&\equiv (\pi^p + \pi\delta\pi)((\delta_{i_2}\pi)^p + \pi\delta\delta_{i_2}\pi) \cdots ((\delta_{i_r} y)^{p^{r-1}} + \pi p^{r-2}\delta\delta_{i_r} y(\delta_{i_r} y)^{p(p^{r-2}-1)}) \\
&\equiv (\pi\delta\pi)(\delta_{i_2}\pi)^p \cdots (\delta_{i_{r-1}}\pi)^{p^{r-2}}(\delta_{i_r} y)^{p^{r-1}} \mod \pi^2.
\end{aligned}$$

The result follows. ∎

For each $\mu = i_1 \dots i_r$ with $r \geq 2$ set

$$F_\mu := (\delta_{i_1}\pi)(\delta_{i_2}\pi)^p \cdots (\delta_{i_{r-1}}\pi)^{p^{r-2}}(\delta_{i_r} y)^{p^{r-1}}.$$

Note in particular that F_μ has order 1. Note also that if $(\mu, \nu) \in \mathbb{K}^r_\rho$ then μ and ν must have, in particular, the *same* length and the difference $F_\mu - F_\nu$ is a p^{r-1}-th power of a linear polynomial in the variables $\delta_i y$.

Proposition A.10. *One has a surjective ring homomorphism*

$$\frac{k[y, \delta_1 y, \dots, \delta_n y]}{(\{F_\mu - F_\nu : (\mu, \nu) \in \mathbb{K}^r_\rho\})}[\delta_\mu y : \mu \in \mathbb{M}^r \setminus \mathbb{M}^1] \to J^r_{\pi,\Phi,\mathrm{ab}}(R_\pi[y])/(\pi).$$

For $n = r = 2$ the above homomorphism is an isomorphism, i.e.,

$$J^2_{\pi,\Phi,\mathrm{ab}}(R_\pi[y])/(\pi) \simeq \frac{k[y,\delta_1 y,\delta_2 y]}{(\delta_1\pi\cdot(\delta_2 y)^p - \delta_2\pi\cdot(\delta_1 y)^p)}[\delta_1^2 y,\delta_1\delta_2 y,\delta_2\delta_1 y,\delta_2^2 y].$$

Proof. Recall that $J^r_{\pi,\Phi,\mathrm{ab}}(R_\pi[y])$ is obtained as $J^r_{\pi,\Phi,\Phi^*}(R_\pi[y])$ divided by the ideal $I_r = (\{\Delta_{\mu,\nu}\}) : p^\infty$, where the generators run over all $(\mu,\nu)\in\mathbb{K}^r_\rho$. From Lemma A.9, $\pi(F_\mu - F_\nu) \equiv \Delta_{\mu,\nu} \bmod \pi^2$, and so $F_\mu - F_\nu \in (I_r,\pi)$. Then the first part of the proposition follows. Assume now $n = r = 2$. Since π is a prime element in the ring $J^2_{\pi,\Phi}(R_\pi[y])$ and does not divide $F_{12,21}$ in this ring it follows that

$$I_2 := (\Delta_{12,21}) : p^\infty = (F_{12,21})$$

which implies the second part of the proposition. ∎

Remark A.11. In particular, for $n = r = 2$ the reduced ring

$$(J^2_{\pi,\Phi,\mathrm{ab}}(R_\pi[y])/(\pi))_{\mathrm{red}}$$

is isomorphic to a polynomial ring in 6 variables. However, note that, contrary to what one might expect, there is no equality (or even relation) in this ring between the images of the variables $\delta_1\delta_2 y$ and $\delta_2\delta_1 y$; instead we have an identification between the images of $(\delta_1\pi)^{1/p}\cdot\delta_2 y$ and $(\delta_2\pi)^{1/p}\cdot\delta_1 y$. As we see, a relation between the images of the variables $\delta_1\delta_2 y$ and $\delta_2\delta_1 y$ pops up in the ring $J^3_{\pi,\Phi,\mathrm{ab}}(R_\pi[y])/(\pi)$; cf. Proposition A.12.

To tackle the case $n = 2$ and $r = 3$ note that since $p \geq 3$,

$$\begin{aligned}\phi_1\phi_2 y &= \phi_1(y^p + \pi\delta_2 y)\\ &= (y^p + \pi\delta_1 y)^p + \phi_1(\pi)((\delta_2 y)^p + \pi\delta_1\delta_2 y)\\ &\equiv y^{p^2} + p\pi y^{p(p-1)}\delta_1 y + \pi\delta_1\pi\cdot(\delta_2 y)^p + \pi^2\delta_1\pi\cdot\delta_1\delta_2 y \mod \pi^3.\end{aligned}$$

hence

$$\check{\Delta}_{12,21} := \frac{1}{\pi}\Delta_{12,21} = \frac{\phi_1\phi_2 y - \phi_2\phi_1 y}{\pi} \equiv A + pB + \pi C \mod \pi^2$$

where

$$\begin{aligned}A &= \delta_1\pi\cdot(\delta_2 y)^p - \delta_2\pi\cdot(\delta_1 y)^p\\ B &= y^{p(p-1)}(\delta_1 y - \delta_2 y)\\ C &= \delta_1\pi\cdot\delta_1\delta_2 y - \delta_2\pi\cdot\delta_2\delta_1 y.\end{aligned}$$

Let $i \in \{1,2\}$. Using the fact that for any element z in a δ_π-ring we have

$$\delta_\pi z^p \equiv 0 \mod \pi$$

and assuming for simplicity that $R_\pi \neq R$ (so $p/\pi \in \pi R_\pi$) we get that

$$\begin{aligned} \delta_i(A) &\equiv \delta_i\delta_1\pi \cdot (\delta_2 y)^{p^2} - \delta_i\delta_2\pi \cdot (\delta_1 y)^{p^2} && \mod \pi, \\ \delta_i(pB) &\equiv 0 && \mod \pi, \\ \delta_i(\pi C) &\equiv \delta_i\pi \cdot C^p && \mod \pi. \end{aligned}$$

Hence

$$\begin{aligned} \delta_i(\check{\Delta}_{12,21}) \equiv& \delta_i(A + pB + \pi C) && \mod \pi \\ \equiv& \delta_i(A) + \delta_i(pB) + \delta_i(\pi C) && \mod \pi \\ \equiv& \delta_i\delta_1\pi \cdot (\delta_2 y)^{p^2} - \delta_i\delta_2\pi \cdot (\delta_1 y)^{p^2} \\ & + \delta_i\pi \cdot (\delta_1\pi \cdot \delta_1\delta_2 y - \delta_2\pi \cdot \delta_2\delta_1 y)^p && \mod \pi. \end{aligned}$$

On the other hand, by (A.1) and since $\check{\Delta}_{12,21} \in (\Delta_{12,21}) : p^\infty$ it follows that

$$\delta_i(\check{\Delta}_{12,21}) \in I_3 = (\Delta_{\mu,\nu} \mid (\mu,\nu) \in \mathbb{K}^3_\rho) : p^\infty.$$

In particular, we have proved the following.

Proposition A.12. *Assume that $R_\pi \neq R$. The image of the element*

$$\delta_i\pi \cdot (\delta_1\pi \cdot \delta_1\delta_2 y - \delta_2\pi \cdot \delta_2\delta_1 y)^p + \delta_i\delta_1\pi \cdot (\delta_2 y)^{p^2} - \delta_i\delta_2\pi \cdot (\delta_1 y)^{p^2} \in J^3_{\pi,\Phi}(R_\pi[y])$$

in the ring $J^3_{\pi,\Phi,\mathrm{ab}}(R_\pi[y])/(\pi)$ is 0.

Note that the image of the above element in $J^3_{\pi,\Phi}(R_\pi[y])/(\pi)$ is a p-th power. A similar (but slightly more complicated) formula holds in case $R_\pi = R$.

A.5 $\delta_{\pi\text{-}\rho}$-characters

We conclude with a discussion of characters for π-ρ-jets. Specifically, partial characters restrict to π-ρ-characters, but we will show this restriction map is not injective in the abelian case. It is a question whether the restriction map is surjective.

Definition A.13. Fix G a smooth commutative group scheme over R_π. The R_π-module

$$\mathbf{X}^r_{\pi,\Phi,\Phi^*,\rho}(G) := \mathrm{Hom}(J^r_{\pi,\Phi,\Phi^*,\rho}(G), \widehat{\mathbb{G}_a}).$$

will be called the *module of $\delta_{\pi\text{-}\rho}$-characters* of G of order $\leq r$. Let

$$\rho_{\mathrm{free}} : \mathbb{M}_{n+n^*} \to F_n$$

be the unique homomorphism of monoids with identity into the free group F_n on $\{1,\ldots,n\}$ that is the identity on $\mathbb{M}_n$ and sends i^* into i^{-1} for all $i \in \{1,\ldots,n\}$.

Note that $\mathbb{K}_{\rho_{\text{free}}} \subset \mathbb{K}_\rho$, so we have an induced closed immersion $J^r_{\pi,\Phi,\Phi^*,\rho}(G) \to J^r_{\pi,\Phi,\Phi^*,\rho_{\text{free}}}(G)$. The latter induces a restriction R_π-module homomorphism

$$\mathbf{X}^r_{\pi,\Phi,\Phi^*,\rho_{\text{free}}}(G) \xrightarrow{\text{res}} \mathbf{X}^r_{\pi,\Phi,\Phi^*\rho}(G).$$

For the further development of the theory it is important to understand the behavior of this canonical restriction. Note that in case $\Phi^* = \emptyset$ we have that $J^r_{\pi,\Phi,\Phi^*,\rho_{\text{free}}}(G)$ and $\mathbf{X}^r_{\pi,\Phi,\Phi^*,\rho_{\text{free}}}(G)$ identify with $J^r_{\pi,\Phi}(G)$ and $\mathbf{X}^r_{\pi,\Phi}(G)$ as defined in the body of the memoir.

Of particular interest are the *abelian δ_π-characters* of a commutative smooth group scheme G, defined as the $\delta_{\pi\text{-ab}}$-characters, i.e., the elements of the R_π-module

$$\mathbf{X}^r_{\pi,\Phi,\text{ab}}(G) := \text{Hom}(J^r_{\pi,\Phi,\text{ab}}(G), \widehat{\mathbb{G}_a}).$$

We therefore have a naturally induced restriction homomorphism

$$\mathbf{X}^r_{\pi,\Phi}(G) \to \mathbf{X}^r_{\pi,\Phi,\text{ab}}(G). \tag{A.9}$$

Define the K_π-module of *abelian symbols* $K^r_{\pi,\Phi,\text{ab}}$ as the free K_π-module with basis $\mathbb{M}^r_{\text{ab}}$ and define $R^r_{\pi,\Phi,\text{ab}}$ similarly. We would like to briefly look into the abelian δ_π-characters of $\mathbb{G}_a$ and $\mathbb{G}_m$.

Write $\mathbb{G}_a = \text{Spec } R_\pi[T]$. Recall from Corollary A.7 that we have an injective homomorphism

$$\mathcal{O}(J^r_{\pi,\Phi,\text{ab}}(\mathbb{G}_a))_{P_r} \to K_\pi[\![\phi_\mu T \colon \mu \in \mathbb{M}^r_{\text{ab}}]\!] \tag{A.10}$$

where $P_r = (\delta_\mu T \colon \mu \in \mathbb{M}^r)$. We may define

$$K^r_{\pi,\Phi,\text{ab}} T = \sum_{\mu \in \mathbb{M}^r_{\text{ab}}} K_\pi \phi_\mu T \subset K_\pi[\![\phi_\mu T \colon \mu \in \mathbb{M}^r_{\text{ab}}]\!]$$

and similarly for $R^r_{\pi,\Phi,\text{ab}} T$ which again are naturally isomorphic to the groups of abelian symbols. As in the non-abelian case, the image of $\mathbf{X}^r_{\pi,\Phi,\text{ab}}(\mathbb{G}_a)$ via the homomorphism (A.10) is contained in $K^r_{\pi,\Phi,\text{ab}} T$, so we get an induced homomorphism

$$\mathbf{X}^r_{\pi,\Phi,\text{ab}}(\mathbb{G}_a) \to K^r_{\pi,\Phi,\text{ab}} T, \ \ \psi \mapsto \psi(T). \tag{A.11}$$

If G has relative dimension 1 with invariant 1-form ω, the standard theory provides again a map of p-formal schemes $\mathcal{E} : \widehat{\mathbb{G}_a} \to \widehat{G}$. This again yields for each $\psi \in \mathbf{X}^r_{\pi,\Phi,\text{ab}}(G)$ a composition $\psi \circ \mathcal{E}^r \in \mathbf{X}^r_{\pi,\Phi,\text{ab}}(\mathbb{G}_a)$,

$$\psi \circ \mathcal{E}^r : J^r_{\pi,\Phi,\text{ab}}(\mathbb{G}_a) \xrightarrow{\mathcal{E}^r} J^r_{\pi,\Phi,\text{ab}}(G) \xrightarrow{\psi} \widehat{\mathbb{G}_a}.$$

We get an induced homomorphism

$$\mathbf{X}^r_{\pi,\Phi,\text{ab}}(G) \to K^r_{\pi,\Phi,\text{ab}} T, \ \ \psi \mapsto (\psi \circ \mathcal{E}^r)(T). \tag{A.12}$$

Writing $(\psi \circ \mathcal{E})(T) = \sum \lambda_\mu \phi_\mu T$ we define

$$\theta(\psi) := \sum \lambda_\mu \phi_\mu \in K^r_{\pi,\Phi,\mathrm{ab}}$$

to be the *abelian Picard–Fuchs symbol* of ψ.

Write now $\mathbb{G}_m = \operatorname{Spec} R_\pi[x, x^{-1}]$. Let $n = 2$ and note that $\phi_{21}(x)$ is invertible in the ring

$$J^2_{\pi,\Phi}(\mathbb{G}_m) = R_\pi[x, x^{-1}, \delta_1 x, \delta_2 x, \delta_1^2 x, \delta_1\delta_2 x, \delta_2\delta_1 x, \delta_2^2 x]\widehat{\ }.$$

We can now show the restriction map (A.9) can fail to be injective.

Proposition A.14. *Let N be the smallest integer so that $\psi := \frac{\pi^N}{p}\log\left(\frac{\phi_{12}(x)}{\phi_{21}(x)}\right)$ belongs to the ring $J^2_{\pi,\Phi}(\mathbb{G}_m)$. Then the restriction of $\psi \in \mathbf{X}^r_{\pi,\Phi}(\mathbb{G}_m)$ to $\mathbf{X}^r_{\pi,\Phi,\mathrm{ab}}(\mathbb{G}_m)$ vanishes.*

Proof. We can write

$$\frac{\phi_{12}(x)}{\phi_{21}(x)} = 1 + \left(\frac{\phi_{12}(x)}{\phi_{21}(x)} - 1\right) = 1 + \frac{\phi_{12}x - \phi_{21}x}{\phi_{21}x} = 1 + \pi\frac{\Delta_{12,21}(x)}{\phi_{21}x}$$

which yields

$$\frac{\pi^N}{p}\log\left(\frac{\phi_{12}(x)}{\phi_{21}(x)}\right) = \frac{\pi^N}{p}\sum(-1)^n\frac{\pi^n}{n}\left(\frac{\Delta_{12,21}(x)}{\phi_{21}x}\right)^n.$$

Clearly the latter series is in the ideal generated by $\Delta_{12,21}$ hence lies in the ideal I_2 and is therefore zero in $\mathcal{O}(J^2_{\pi,\Phi,\mathrm{ab}}(\mathbb{G}_m))$. ∎

Remark A.15. The previous discussion offers a glimpse into what the theory of abelian δ_π-characters should look like; the first non-trivial steps would have to tackle the case of elliptic curves which will not be pursued at this time.

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Purely Arithmetic PDEs Over a p-Adic Field: δ-Characters and δ-Modular Forms

A formalism of arithmetic partial differential equations (PDEs) is being developed in which one considers several arithmetic differentiations at one fixed prime. In this theory solutions can be defined in algebraically closed p-adic fields. As an application we show that for at least two arithmetic directions every elliptic curve possesses a non-zero arithmetic PDE Manin map of order 1; such maps do not exist in the arithmetic ODE case. Similarly, we construct and study "genuinely PDE" differential modular forms. As further applications we derive a Theorem of the kernel and a Reciprocity theorem for arithmetic PDE Manin maps and also a finiteness Diophantine result for modular parameterizations. We also prove structure results for the spaces of "PDE differential modular forms defined on the ordinary locus". We also produce a system of differential equations satisfied by our PDE modular forms based on Serre and Euler operators.

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