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On Einstein's Effective Viscosity Formula

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Abstract

In his PhD thesis, Einstein derived an explicit first-order expansion for the effective viscosity of a Stokes fluid with a suspension of small rigid particles at low density. His formal derivation relied on two implicit assumptions: (i) there is a scale separation between the size of the particles and the observation scale; and (ii) at first order, dilute particles do not interact with one another. In mathematical terms, the first assumption amounts to the validity of a homogenization result defining the effective viscosity tensor, which is now well understood. Next, the second assumption allowed Einstein to approximate this effective viscosity at low density by considering particles as being isolated. The rigorous justification is, in fact, quite subtle as the effective viscosity is a nonlinear nonlocal function of the ensemble of particles and as hydrodynamic interactions have borderline integrability.

In the present memoir, we establish Einstein's effective viscosity formula in the most general setting. In addition, we pursue the low-density expansion to arbitrary order in form of a cluster expansion, where the summation of hydrodynamic interactions crucially requires suitable renormalizations. In particular, we justify a celebrated result by Batchelor and Green on the second-order correction and we explicitly describe all higher-order renormalizations for the first time. In some specific settings, we further address the summability of the whole cluster expansion. Our approach relies on a combination of combinatorial arguments, variational analysis, elliptic regularity, probability theory, and diagrammatic integration methods.

Keywords. Einsteins's formula, effective viscosity, dilute expansion, homogenization, point processes, stochastic analysis, elliptic regularity, diagrammatic integration

Mathematics Subject Classification (2020). Primary 76T20; Secondary 35R60, 76M50, 35Q35, 76D03, 76D07, 60G55

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Chapter 1

General overview

1.1 Historical context

At the dawn of the 20th century, the debate was still raging on the existence of atoms, and Einstein's PhD thesis "A New Determination of Molecular Dimensions" [22] aimed to support the atomic theory. This was the second of his five celebrated 1905 contributions and still constitutes his most cited work. The main part was devoted to the hydrodynamic derivation of a formula for the effective viscosity of a fluid with a dilute suspension of rigid particles: the so-called Einstein formula in fluid mechanics, which is the focus of the present memoir. In the same work, Einstein also derived a relation between the diffusion constant for suspended particles and their mobility: the so-called Einstein relation in kinetic theory. He then applied these two relations to sugar dissolved in water: using available empirical data, he deduced an estimate of the Avogadro number and of the size of sugar molecules (after eliminating a calculation error [23]). We refer to [61] for an inspiring account of this seminal work. As discussed by Perrin in his extensive report [58] at the first Solvay conference in 1911 in Brussels, these discoveries were confirmed by further experiments and shown to agree with other methods to determine the Avogadro number, which sealed the triumph of the atomic theory.

We briefly describe Einstein's argument to estimate the effective viscosity of a dilute suspension. Viscosity of a fluid is usually measured by shear-flow experiments: a cylindrical vessel is filled with the fluid, a rotating spindle is immersed in it, and one measures the torque needed to make it rotate at constant angular speed. Assume now that the fluid contains a suspension of small rigid spherical particles and consider their influence on the measured viscosity. As particles are rigid, they act as obstacles and hinder the fluid flow, thus effectively increasing the measured viscosity. A first challenging question concerns the dynamics of the particles: do they reach a statistical steady state? If this is the case and if one indeed measures a constant-in-time effective viscosity, then the latter depends on the steady state, hence possibly on the speed of the spindle itself, which corresponds to possible non-Newtonian behaviors [31, Section 7]. Einstein's main idea in [22] was that, in the low-density regime, for spherical particles, the first-order effective change in viscosity should only depend on the volume fraction of the particles and not on their distribution. In particular, this universality would relegate non-Newtonian effects to higher-order corrections. More precisely, in 3D, given a fluid with isotropic viscosity Id and given suspended spherical particles with small volume fraction $\varphi \ll 1$, Einstein's formula for the effective

viscosity takes the form

$$\bar{\mathbf{B}} = \mathrm{Id}\left(1 + \frac{5}{2}\varphi + o(\varphi)\right). \tag{1.1}$$

Heuristically, the argument is as follows: at low density, particles are scarce and typically well separated, hence their interactions are negligible to leading order. The first-order effect on the viscosity should thus be proportional to the volume fraction and correspond to the energy dissipation of a single isolated particle in the fluid. The latter can be computed explicitly for spherical particles and leads to the celebrated $\frac{5}{2}$ factor in (1.1); we refer to Section 2.6 below, where this classical calculation is reproduced.

This type of low-density expansion was not new in the physics community at the time: it was very much in line with other work on the micromechanics of heterogeneous media in the late 19th century. Einstein's formula is indeed comparable to the Clausius–Mossotti formula for the effective dielectric constant [10, 51, 52], to Maxwell's formula for the effective conductivity in electrostatics [49], or to the Lorentz–Lorenz formula for the effective refractive index in optics [46, 47]; we refer to [48] for an account of the historical context.

Einstein's formula triggered a lot of long-lasting activity in fluid mechanics: the large-scale rheology of suspensions was soon considered as a topic in its own right [24, 41, 42]. Various works have aimed at understanding to what extent Einstein's formula is robust and accurate. Robustness has been addressed in particular by establishing corresponding formulas for particles of different shapes, as e.g. the explicit formulas by Jeffery [40] for suspensions of ellipsoids (see also [35, 44]). Accuracy is a more subtle issue and essentially amounts to capturing the next-order term in the low-density expansion. While particle interactions are neglected at first order, the next-order correction consists of including the effects of pairwise interactions. Due to their long-range nature, the sum of pairwise contributions is not summable and some renormalization is therefore needed. This was first achieved by Batchelor and Green [7], and we refer to [1, 34, 56] for other formal renormalization ideas. A related, yet different, topic concerns the sedimentation of suspended particles under gravity and the computation of their effective settling speed, which happens to require a similar renormalization: the above-mentioned contribution by Batchelor and Green [7] was indeed inspired by Batchelor's work [6] on sedimentation. Interestingly, the renormalization of higher-order corrections to the effective viscosity had remained an open problem in the physics community.

We also refer to [2, 55, 63] for the asymptotic analysis of the effective viscosity for dilute *periodic* arrays of suspended particles and, in a more mathematical spirit, we mention the pioneering work by Sánchez-Palencia et al. [45, 60] using formal two-scale expansions for locally periodic suspensions.

1.2 Mathematical reformulation and objectives

As described above, Einstein's formal derivation of (1.1) in [22] relies on the following two implicit hypotheses:

(E1) *Scale separation.* There is a scale separation between the "microscopic" particle size and the "macroscopic" observation scale. Therefore, the suspension behaves on the observation scale like an "effective" fluid with some effective viscosity tensor $\overline{\mathbf{B}}$ that can then be measured by shear-flow experiments.

(E2) *Particle interactions are negligible.* In the low-density regime, particles are typically well separated and therefore, to leading order, they do not interact and can be treated as being isolated.

We briefly discuss the validity of these two working hypotheses and then turn to describing the literature and our objectives in the present memoir.

1.2.1 Einstein's hypothesis (E1): Scale separation

This first hypothesis concerns the definition of a notion of effective viscosity for suspensions when the particle size $O(\varepsilon)$ is much smaller than the observation scale $O(1)$. Consider a shear-flow experiment to measure the viscosity, say using a rotational viscometer. Let D denote the fluid domain in this device and let $\{x_{\varepsilon,n}^t\}_n \subset D$ stand for positions of suspended particles at time t, which evolve over time in the fluid flow. If inertia is neglected, the dynamics is greatly simplified: given particle positions at a given time, the fluid velocity satisfies steady Stokes equations, which determine instantaneous particle velocities. In this context, the emergence of an effective viscosity can be split into two parts:

— *Steady-state microstructure.* As the measured effective viscosity is expected not to depend on time, it implicitly requires particle positions to reach a statistical steady state in the long run. Focussing on a portion of the fluid in the bulk, we may consider without much loss of generality that the statistical ensemble is stationary (henceforth, "stationarity" stands for statistical spatial homogeneity). In other words, the point set $\{x_{\varepsilon,n}^t\}_n$ can be approximately replaced by a random point set $\{\varepsilon x_n : \varepsilon x_n \in D\}$ that is the ε-rescaling of some stationary random point process $\mathcal{P} = \{x_n\}_n$. The law of this steady state may depend itself on the prescribed shear flow in the viscometer, which leads to possible non-Newtonian effects [31, Section 7].

— *Steady homogenization problem.* Given a statistical ensemble of particle positions, under an ergodicity assumption, the steady Stokes equations for the fluid velocity are expected to homogenize on the macroscopic observation scale, in the sense that it can be replaced by effective steady Stokes equations with some effective viscosity tensor $\overline{\mathbf{B}}$, which amounts to an averaged effect of suspended particles.

While the rigorous analysis of the steady-state flow-induced microstructure remains a fully open problem at this time, the steady homogenization problem, in contrast, has been extensively studied under various assumptions in our recent series of articles [13, 18, 19, 21] and is by now very well understood. Given a statistical ensemble of particle positions, this provides a rigorous definition of the effective viscosity together with a homogenization result. More precisely, considering the system at the particle scale, we denote by $\mathcal{I} = \bigcup_n I_n$ the random ensemble of particles (not necessarily spherical), centered at the points of a point process $\mathcal{P} = \{x_n\}_n$, say in the d-dimensional Euclidean space $\mathbb{R}^d$ for generality. The effective viscosity tensor $\bar{\mathbf{B}}$ is defined as a quadratic form on the set $\mathbb{M}^{\mathrm{sym}}_0 \subset \mathbb{R}^{d\times d}$ of trace-free symmetric matrices,

$$E : \bar{\mathbf{B}}E := \mathbb{E}\big[\big|\mathrm{D}(\psi_E) + E\big|^2\big] = |E|^2 + \mathbb{E}\big[\big|\mathrm{D}(\psi_E)\big|^2\big], \quad E \in \mathbb{M}^{\mathrm{sym}}_0, \tag{1.2}$$

where $\mathrm{D}(\psi_E)$ is the unique stationary symmetric gradient solution, with bounded second moment and vanishing expectation, of the corrector problem

$$\begin{cases} -\triangle\psi_E + \nabla\Sigma_E = 0, & \text{in } \mathbb{R}^d \setminus \mathcal{I}, \\ \mathrm{div}(\psi_E) = 0, & \text{in } \mathbb{R}^d \setminus \mathcal{I}, \\ \mathrm{D}(\psi_E + Ex) = 0, & \text{in } \mathcal{I}, \\ \int_{\partial I_n} \sigma_E \nu = 0, & \forall n, \\ \int_{\partial I_n} \Theta(x - x_n)\cdot\sigma_E\nu = 0, & \forall n,\ \forall \Theta \in \mathbb{M}^{\mathrm{skew}}, \end{cases} \tag{1.3}$$

in terms of the associated Cauchy stress tensor

$$\sigma_E := \sigma(\psi_E + Ex, \Sigma_E) := 2\mathrm{D}(\psi_E + Ex) - \Sigma_E\,\mathrm{Id}, \tag{1.4}$$

where $\mathbb{M}^{\mathrm{skew}} \subset \mathbb{R}^{d\times d}$ is the set of skew-symmetric matrices. Throughout this work, we assume for simplicity that the plain fluid has isotropic viscosity Id. The so-called corrector ψ_E can be viewed as the correction of the linear straining flow

$$x \mapsto Ex$$

in presence of rigid suspended particles $\{I_n\}_n$. The last two boundary conditions in (1.3) correspond to the balance of forces and torques on each particle. Note that, if $\mathcal{I}$ contains an unbounded chain of touching particles, then the rigidity constraint

$$\mathrm{D}(\psi_E + Ex)|_{\mathcal{I}} = 0$$

entails that the field ψ_E would grow linearly along this chain, which would prevent $\mathrm{D}(\psi_E)$ from having vanishing expectation: it shows that this corrector problem can only be well posed provided that some suitable non-clustering assumption is made. Different sets of sufficient assumptions are recalled in Section 2.1 below and we refer to our previous work [13, 18, 19, 21] for a detailed account.

1.2.2 Einstein's hypothesis (E2): Interactions are negligible

As it appears from (1.3), the corrector ψ_E depends nonlocally and nonlinearly on the set $\mathcal{I}$ of particles via boundary conditions: this corresponds to the multibody nature of hydrodynamic interactions. Einstein's second hypothesis can be reinterpreted as claiming that ψ_E can be approximated around each inclusion I_n by the unique decaying solution $\psi_E^{\{n\}}$ of the single-particle corrector problem

$$\begin{cases} -\triangle \psi_E^{\{n\}} + \nabla \Sigma_E^{\{n\}} = 0, & \text{in } \mathbb{R}^d \setminus I_n, \\ \operatorname{div}\big(\psi_E^{\{n\}}\big) = 0, & \text{in } \mathbb{R}^d \setminus I_n, \\ \mathrm{D}\big(\psi_E^{\{n\}} + Ex\big) = 0, & \text{in } I_n, \\ \int_{\partial I_n} \sigma\big(\psi_E^{\{n\}} + Ex, \Sigma_E^{\{n\}}\big)\nu = 0, & \\ \int_{\partial I_n} \Theta(x - x_n) \cdot \sigma\big(\psi_E^{\{n\}} + Ex, \Sigma_E^{\{n\}}\big)\nu = 0, & \forall \Theta \in \mathbb{M}^{\mathrm{skew}}. \end{cases} \tag{1.5}$$

This amounts to neglecting the effect of other particles on ψ_E around I_n, thus precisely neglecting the multibody nature of the problem. To give a more precise statement, consider the Voronoi tessellation $\{V_n\}_n$ associated with the set of particles $\{I_n\}_n$, that is,

$$V_n := \big\{x : \operatorname{dist}(x, I_n) < \inf_{m:m\neq n} \operatorname{dist}(x, I_m)\big\}.$$

The relevant approximation of ψ_E then takes the form

$$\mathrm{D}(\psi_E) \approx \Psi_E^{\mathrm{Einstein}} := \sum_n \mathrm{D}\big(\psi_E^{\{n\}}\big)\mathbb{1}_{V_n}. \tag{1.6}$$

Inserting this into the definition (1.2) of the effective viscosity yields, after straightforward calculations,

$$\begin{aligned} E : \bar{\mathbf{B}}E &= |E|^2 + \mathbb{E}\big[\big|\mathrm{D}(\psi_E)\big|^2\big] \\ &\approx |E|^2 + \mathbb{E}\big[\big|\Psi_E^{\mathrm{Einstein}}\big|^2\big] \\ &= |E|^2 + \sum_n \mathbb{E}\left[\frac{\mathbb{1}_{0\in I_n}}{|I_n|}\int_{V_n} \big|\mathrm{D}\big(\psi_E^{\{n\}}\big)\big|^2\right], \end{aligned}$$

and thus, replacing single-particle energies $\int_{V_n} |\mathrm{D}(\psi_E^{\{n\}})|^2$ in Voronoi cells by corresponding whole-space energies,

$$E : \bar{\mathbf{B}}E \approx |E|^2 + \sum_n \mathbb{E}\left[\frac{\mathbb{1}_{0\in I_n}}{|I_n|}\int_{\mathbb{R}^d} \big|\mathrm{D}\big(\psi_E^{\{n\}}\big)\big|^2\right]. \tag{1.7}$$

In case of spherical particles, single-particle problems can be solved explicitly and we get

$$E : \bar{\mathbf{B}}E = |E|^2\left(1 + \frac{d+2}{2}\varphi + o(\varphi)\right), \tag{1.8}$$

in terms of the particle volume fraction

$$\varphi := \varphi(\mathcal{I}) := \lim_{R\uparrow\infty} R^{-d}|\mathcal{I} \cap RQ|, \tag{1.9}$$

where $Q := [-\frac{1}{2}, \frac{1}{2}]^d$ stands for the unit cube. We refer to Section 2.6 for the detailed computation, and we note that in 3D we recover Einstein's celebrated $\frac{5}{2}$ factor, cf. (1.1). Corrections to Einstein's formula are naturally obtained by further taking into account particle interactions. As we shall see, in the low-density regime, this is naturally captured in form of a cluster expansion: the next-order correction, known in the physics literature as the Batchelor–Green correction [7], involves the sum of two-particle contributions, and so on.

1.2.3 Objectives

In this memoir, we focus on the rigorous analysis of Einstein's hypothesis (E2): we start from the relevant notion of effective viscosity (1.2) as defined by homogenization theory and we study its asymptotic behavior at low density, aiming to justify Einstein's formula (1.8) and to describe all higher-order corrections.

The early works [32, 45, 60] focussed on Einstein's formula for locally periodic dilute arrays of particles. It was extended in [33, 54] to the dilute disordered setting under the simplifying assumption that the minimal interparticle distance is large enough (that is, $\ell(\mathcal{P}) \gg 1$ with the notation (1.13) below). The next-order Batchelor–Green correction was captured in [27, 29] in the same setting. The uniform separation assumption is particularly convenient as it allows to exploit the reflection method and rigorously neglect many-particle interactions, e.g. [36–39, 54], but it is physically quite restrictive and unsatisfactory. More recently, it was replaced in [28] by some weaker non-concentration condition in the proof of Einstein's formula, however still requiring some control on the minimal interparticle distance. In this context, we shall address the following two main points:

— We shall justify Einstein's formula under the weakest assumptions under which homogenization is known to hold, in particular covering the case of the general subcritical percolation condition in [21]. At the same time, we aim at optimal error estimates: the error $o(\varphi)$ in (1.1) was often claimed to be $O(\varphi^2)$ in the physics literature, but we shall see that it actually strongly depends on the structure of the random ensemble of particles.

— We shall describe higher-order corrections to Einstein's formula in form of a cluster expansion. Due to the long-range nature of hydrodynamic interactions, renormalizations are needed to make sense of cluster contributions. In the physics literature, formal renormalizations were actually still missing beyond the second-order Batchelor–Green correction. On the rigorous side, even the justification of the latter was restricted to some specific regimes [26, 27, 29].

In terms of techniques, previous results on the topic mostly relied on a deterministic analysis, more precisely on various forms of the reflection method. In the present memoir, we rather take inspiration from our work [15] on the Clausius–Mossotti conductivity formula based on the triad consisting of: (1) finite-volume approximation; (2) cluster expansion; (3) uniform $\ell^1 - \ell^2$ energy estimates. Substantially refining on this analysis, we go far beyond [15] (and beyond [26], which also builds on the cluster expansion of [15]) as we manage to cover general dilute regimes (beyond the case of explicit dilution by random deletion in [15]). We further describe explicitly for the first time the renormalization of cluster coefficients to all orders.

1.3 Cluster expansion formalism

While Einstein's formula (1.8) is obtained by considering dilute particles as being isolated, next-order corrections amount to taking into account many-particle interactions and the multibody structure of the corrector field ψ_E. At low density, particles are scarce, hence have weak interactions, and one might want to consider contributions of finite subsets of particles only. As in [15], taking inspiration from statistical mechanics, see e.g. [62, Chapter 19], this is naturally expressed by means of cluster expansions, which provide natural asymptotic series at low density. We recall the formalism, discuss the accuracy of cluster expansions, and describe the key difficulty to apply it to the effective viscosity problem: the long-range nature of hydrodynamic interactions.

1.3.1 Cluster expansions of multibody quantities

We recall the cluster expansion formalism in the compact form that we introduced in [15]. As particles are indexed by natural numbers, we denote by $P(\mathbb{N})$ the set of subsets of the index set $\mathbb{N}$ and we consider the space $M(\mathbb{N})$ of set functions from $P(\mathbb{N})$ to a given vector space V. Starting from the corrector problem (1.3), for any index subset $H \in P(\mathbb{N})$, we may consider[1] the associated corrector ψ_E^H obtained by replacing the full set $\mathcal{I}$ of particles by its corresponding subset $\mathcal{I}^H := \bigcup_{n\in H} I_n$ in the corrector equation (1.3). The map

$$\psi_E^\# : H \mapsto \psi_E^H$$

is then viewed as an element of $M(\mathbb{N})$, where $\psi_E^\varnothing \equiv 0$ and where $\psi_E^{\mathbb{N}} \equiv \psi_E$ is the original corrector defined in (1.3).

[1] The corrector problem (1.3) is, in fact, not well posed in general for a given deterministic infinite subset H of particles. In the sequel, we shall rather consider finite-volume approximations of the corrector problem, for which well-posedness is always trivial. We skip this detail at the level of the present discussion.

In this setting, for all $n \in \mathbb{N}$, we introduce a *difference operator* $\delta^{\{n\}} : M(\mathbb{N}) \to M(\mathbb{N})$, defined for all $\Phi \in M(\mathbb{N})$ by

$$\delta^{\{n\}}\Phi^H := \delta^{\{n\}}\Phi^{H\cup\#} := \Phi^{H\cup\{n\}} - \Phi^H, \quad H \subset \mathbb{N},$$

which provides a natural measure of the sensitivity of Φ with respect to the index n (it plays the role of a discrete derivative). Note that for all $n \neq m$,

$$(\delta^{\{n\}})^2 = -\delta^{\{n\}}, \quad \delta^{\{n\}}\delta^{\{m\}} = \delta^{\{m\}}\delta^{\{n\}}.$$

For any finite $F \subset \mathbb{N}$, we also define the higher-order difference operator

$$\delta^F := \prod_{n\in F} \delta^{\{n\}},$$

which acts as follows: for all $\Phi \in M(\mathbb{N})$,

$$\delta^F\Phi^H = \sum_{G\subset F} (-1)^{|F\setminus G|}\Phi^{G\cup H}, \quad H \subset \mathbb{N}. \tag{1.10}$$

We take the natural convention $\delta^\varnothing\Phi^H := \Phi^H$. These difference operators are the building blocks to construct the so-called *cluster expansions*, e.g. [62, Chapter 19]: to order k, the cluster expansion of $\Phi \in M(\mathbb{N})$ takes the form

$$\Phi^{\mathbb{N}} \sim \Phi^\varnothing + \sum_n \delta^{\{n\}}\Phi^\varnothing + \frac{1}{2!}\sum_{n_1,n_2}^{\neq} \delta^{\{n_1,n_2\}}\Phi^\varnothing + \cdots + \frac{1}{k!}\sum_{n_1,\ldots,n_k}^{\neq} \delta^{\{n_1,\ldots,n_k\}}\Phi^\varnothing,$$

where we use the shorthand notation $\sum^{\neq}_{n_1,\ldots,n_j}$ for sums over j-tuples $(n_1,\ldots,n_j)$ of distinct indices. This can be rewritten in the more compact form

$$\Phi^{\mathbb{N}} \sim \sum_{j=0}^{k}\sum_{\sharp F=j} \delta^F\Phi^\varnothing, \tag{1.11}$$

where $\sum_{\sharp F=j}$ stands for the sum over all sets F of j distinct indices. This expansion is particularly relevant in the low-density regime when particles are very scarce: the 0th-order term corresponds to the situation without any particle, the 1st-order term corresponds to contributions of isolated particles, the 2nd-order term to contributions of pairs of particles, etc. Formally, it can be viewed as a Taylor expansion at $\varnothing$ with respect to the difference operator δ, where under suitable assumptions higher-order terms will be shown to be indeed of higher order at low density. Note that, if $\Phi \in M(\mathbb{N})$ only depends on indices in a finite subset $K \subset \mathbb{N}$ in the sense that $\Phi^H = \Phi^{H\cap K}$ for all $H \subset \mathbb{N}$, then the expansion (1.11) is always a finite sum and is actually *equal* to $\Phi^{\mathbb{N}}$ provided that $k \geq \sharp K$.

1.3.2 Multi-point intensities

The general estimation of the terms in the cluster expansion (1.11) naturally leads to the notion of multi-point intensities, which appear as refined measures of diluteness and seem new to the literature. Given an ergodic stationary point process $\mathcal{P} = \{x_n\}_n$, we start by recalling the standard notion of *intensity* of the point process (or *one-point intensity* in our terminology below),

$$\lambda(\mathcal{P}) := \lambda_1(\mathcal{P}) := \mathbb{E}\big[\sharp(\mathcal{P} \cap Q)\big].$$

By the ergodic theorem, we have almost surely

$$\lambda(\mathcal{P}) = \lim_{R\uparrow\infty} R^{-d}\sharp\{n : x_n \in RQ\}. \tag{1.12}$$

In particular, provided that random shapes satisfy $|I_n^\circ| \simeq 1$ almost surely for all n, this relates to the particle volume fraction (1.9) via

$$\varphi(\mathcal{I}) \simeq \lambda(\mathcal{P}),$$

so that the low-density regime $\varphi(\mathcal{I}) \ll 1$ is equivalently characterized by the condition $\lambda(\mathcal{P}) \ll 1$. Yet, as we consider nonlinear functions of the point process (like the effective viscosity $\overline{\mathbf{B}}$), this linear notion of diluteness is not strong enough and we need to introduce refined notions of "multi-point intensities".

For that purpose, we start by introducing a notation for the *minimal distance* of the point process $\mathcal{P}$,

$$\ell := \ell(\mathcal{P}) := \inf_{n\neq m} |x_n - x_m|_\infty, \tag{1.13}$$

which is almost surely a deterministic characteristic length of $\mathcal{P}$. The point process is called hardcore if $\ell(\mathcal{P}) > 0$, which is the case of all the processes considered in this memoir, cf. (H_ρ) below. For all $j \geq 1$, provided $\ell = \ell(\mathcal{P}) > 0$, we then define the *$j$-point intensity*

$$\lambda_j(\mathcal{P}) := \sup_{z_1,\ldots,z_j} \mathbb{E}\left[\sum_{n_1,\ldots,n_j}^{\neq} \ell^{-d}\mathbb{1}_{Q_\ell(z_1)}(x_{n_1})\cdots\ell^{-d}\mathbb{1}_{Q_\ell(z_j)}(x_{n_j})\right], \tag{1.14}$$

where $Q_r(z) := z + rQ$ stands for the cube of sidelength r centered at z. Note that, by definition (1.13), each cube $Q_\ell(z)$ contains at most one point of $\mathcal{P}$. This definition corresponds to the maximum expected number of j-tuples of points of $\mathcal{P}$ that lie in the ℓ-neighborhood of an element of $(\mathbb{R}^d)^j$, properly normalized by ℓ. Alternatively, recalling that the *j-point density* f_j associated with $\mathcal{P}$ is the nonnegative function defined by the following relation,

$$\mathbb{E}\left[\sum_{n_1,\ldots,n_j}^{\neq} \zeta(x_{n_1},\ldots,x_{n_j})\right] = \int_{(\mathbb{R}^d)^j} \zeta f_j \quad \text{for all } \zeta \in C_c^\infty\big((\mathbb{R}^d)^j\big), \tag{1.15}$$

the definition (1.14) of j-point intensity can be reformulated as

$$\lambda_j(\mathcal{P}) = \sup_{z_1,\dots,z_j} \fint_{Q_\ell(z_1)\times\cdots\times Q_\ell(z_j)} f_j. \tag{1.16}$$

In the case $\ell(\mathcal{P}) = 0$, this definition is naturally extended to $\lambda_j(\mathcal{P}) = \|f_j\|_{\mathrm{L}^\infty((\mathbb{R}^d)^j)}$ for completeness. In view of upcoming arguments, it is convenient to further introduce the following quantities,

$$\underline{\lambda}_j(\mathcal{P}) := \min_{\sum_i j_i = j} \prod_i \lambda_{j_i}(\mathcal{P}) \le \overline{\lambda}_j(\mathcal{P}) := \max_{\sum_i j_i = j} \prod_i \lambda_{j_i}(\mathcal{P}). \tag{1.17}$$

For a Poisson point process, these quantities are, in fact, all equivalent since independence yields $\lambda_j(\mathcal{P}) = \lambda(\mathcal{P})^j$ for all $j \ge 1$, hence $\underline{\lambda}_j(\mathcal{P}) = \overline{\lambda}_j(\mathcal{P}) = \lambda(\mathcal{P})^j$. For a hardcore Poisson point process, we similarly find $\lambda_j(\mathcal{P}) \simeq_j \lambda(\mathcal{P})^j$. In other words, the one-point intensity $\lambda(\mathcal{P})$ is enough to fully describe low-density regimes in those cases. However, multi-point intensities are nontrivial in general: for any $\beta \in [0,1]$, one can construct examples of point processes with $\lambda_2(\mathcal{P}) \simeq \lambda(\mathcal{P})^{1+\beta}$ (see last paragraph of Section 5.1). For instance, given $e \in \mathbb{R}^d$, the point process

$$\mathcal{P}_e := \mathcal{P} \cup (\mathcal{P} + e)$$

consists of pairs of points $\{x_n, x_n + e\}$ and thus satisfies $\lambda_2(\mathcal{P}_e) \simeq \ell(\mathcal{P}_e)^{-d}\lambda(\mathcal{P}_e)$, hence $\lambda_2(\mathcal{P}_e) \simeq \lambda(\mathcal{P}_e)$ provided that $\mathcal{P}_e$ is hardcore. This indicates that heuristically the condition $\lambda_2(\mathcal{P}) \ll \lambda(\mathcal{P})$ can be understood as the scarcity of clusters in $\mathcal{P}$, and more generally the condition $\lambda_{k+1}(\mathcal{P}) \ll \lambda_k(\mathcal{P})$ as the scarcity of k-point clusters. The following lemma states some general properties.

Lemma 1.1 (Multi-point intensities). *Let $\mathcal{P} = \{x_n\}_n$ be an ergodic stationary random point process.*

(i) *For all $j \ge 1$, we have*

$$\lambda_{j+1}(\mathcal{P}) \le \ell(\mathcal{P})^{-d}\lambda_j(\mathcal{P}).$$

(ii) *If $\mathcal{P}$ is strongly mixing, then for all $j \ge 1$ we have*

$$\lambda(\mathcal{P})^j = \underline{\lambda}_j(\mathcal{P}) \le \overline{\lambda}_j(\mathcal{P}) = \lambda_j(\mathcal{P}).$$

(The same holds for all $j \le n$ under the mixing assumption (Mix^n_ω) introduced in Section 4.3 provided the rate ω decays at infinity.)

(iii) *Given $j \ge 2$ and $\theta \in [0,1]$, for any nonnegative function $\phi \in C^\infty_c((\mathbb{R}^d)^j)$ that satisfies*

$$\begin{cases} \max_i |z_i - z_i'|_\infty \le \theta\ell(\mathcal{P}) \\ \min_{j\neq i} |z_i - z_j|_\infty \ge \ell(\mathcal{P}) \end{cases} \implies \phi(z_1,\dots,z_j) \le C\phi(z_1',\dots,z_j'),$$

we have

$$\int_{(\mathbb{R}^d)^k} \phi f_k \le C\theta^{-dk}\lambda_k(\mathcal{P}) \int_{(\mathbb{R}^d)^k} \phi.$$

Proof. As each cube $Q_\ell(z)$ contains at most one point of the point process $\mathcal{P}$, we find that $\sum_n \ell^{-d} \mathbb{1}_{Q_\ell(z)}(x_n) \le \ell^{-d}$, so that item (i) readily follows from definition (1.14).

We turn to the proof of (ii). Given $j \ge 1$, for any partition $0 = k_1 < k_2 < \cdots < k_l = j$, setting $j_i := k_{i+1} - k_i$, the strong mixing of the point process implies

$$⨍_{Q_\ell(z_1)\times\cdots\times Q_\ell(z_j)} f_j - \prod_{i=1}^{l} \Big(⨍_{Q_\ell(z_{k_i+1})\times\cdots\times Q_\ell(z_{k_{i+1}})} f_{j_i} \Big) \to 0,$$

as $\min_{i\neq i'} \operatorname{dist}(Z_i, Z_{i'}) \to \infty$, where we have set for shortness

$$Z_i := \{z_{k_i+1}, \ldots, z_{k_{i+1}}\}.$$

In view of (1.16), using stationarity, this proves the estimate $\lambda_j(\mathcal{P}) \ge \prod_{i=1}^l \lambda_{j_i}(\mathcal{P})$, from which the claim (ii) easily follows.

Finally, item (iii) is a direct consequence of definition (1.16) of multi-point intensities, further using that the j-point density satisfies $f_j(x_1, \ldots, x_j) = 0$ whenever there are some indices $i \neq i'$ with $|x_i - x_{i'}| < \ell(\mathcal{P})$. ∎

1.3.3 Scaling of cluster expansions

With the above definitions, we may now determine the scaling of the terms in the cluster expansion (1.11) and show the relevance of multi-point intensities in this context. For that purpose, by way of illustration, we place ourselves in the elementary setting of short-range interactions, which will serve as a guideline in the sequel. More precisely, consider a set function $\Phi : P(\mathbb{N}) \to \mathbb{R}$ of the form

$$\Phi^H := \mathbb{E}\Big[g\Big(\sum_{n\in H} h(x_n) \Big) \Big], \tag{1.18}$$

for some $h : \mathbb{R}^d \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that

(a) h is *short-range*, in the sense that $\int_{\mathbb{R}^d} (\sup_{B(z)} |h|)\, dz < \infty$;

(b) g is *smooth*, in the sense that $g \in C_b^\infty(\mathbb{R})$,

and set $\Phi^\varnothing = g(0)$. The cluster expansion of $\Phi^{\mathbb{N}}$, cf. (1.11), then takes the form

$$\Phi^{\mathbb{N}} \sim \sum_{j=0}^{\infty} \frac{1}{j!} \bar{\Phi}^j, \quad \text{where } \bar{\Phi}^j := j! \sum_{\sharp F = j} \delta^F \Phi^\varnothing. \tag{1.19}$$

Although cluster coefficients $\{\bar{\Phi}^j\}_j$ are defined by infinite series, these series are always summable in this short-range setting and we show that they are naturally estimated by multi-point intensities. In particular, the second-order coefficient $\bar{\Phi}^2$ is of

order $O(\lambda_2(\mathcal{P}))$, which contradicts in general the bound $O(\lambda(\mathcal{P})^2) = O(\varphi^2)$ that one could have naively expected. Our main goal in this memoir is precisely to establish corresponding expansions and estimates for the effective viscosity (1.2).

Lemma 1.2 (Cluster expansions in the short-range setting). *Let $\mathcal{P} = \{x_n\}_n$ be an ergodic stationary point process on $\mathbb{R}^d$ with $0 < \ell(\mathcal{P}) \lesssim 1$, let Φ be a set function of the form (1.18) satisfying the short-range and smoothness assumptions* (a) and (b) *above, and let $\{\bar\Phi^j\}_j$ be the associated cluster coefficients (1.19). Then we have for all $k \geq 1$,*

$$\left| \Phi^{\mathbb{N}} - \sum_{j=0}^{k} \frac{1}{j!} \bar\Phi^j \right| \lesssim_{k,g,h} \lambda_{k+1}(\mathcal{P}), \quad |\bar\Phi^k| \lesssim_{k,g,h} \lambda_k(\mathcal{P}), \tag{1.20}$$

in terms of multi-point intensities $\{\lambda_j(\mathcal{P})\}_j$, cf. (1.14).

Proof. Given a sequence $Y := \{y_n\}_n \subset \mathbb{R}$ (that will be chosen as $y_n = h(x_n)$ below), define a set function $\Psi_Y : P(\mathbb{N}) \to \mathbb{R}$ by

$$\Psi_Y^H := g\Big(\sum_{n \in H} y_n \Big), \quad H \subset \mathbb{N}.$$

By definition of difference operators, cf. (1.10), we find, in the spirit of Taylor's remainder formulas,

$$\delta^{\{n_1,\ldots,n_k\}} \Psi_Y^{\varnothing} = \int_0^{y_{n_1}} \cdots \int_0^{y_{n_k}} g^{(k)}(t_1 + \cdots + t_k)\, dt_1 \cdots dt_k,$$

$$\Psi_Y^{\mathbb{N}} - \sum_{j=0}^{k} \sum_{\sharp F = j} \delta^F \Psi_Y^{\varnothing}$$

$$= \sum_{n_1 < \cdots < n_{k+1}} \int_0^{y_{n_1}} \cdots \int_0^{y_{n_{k+1}}} g^{(k+1)}\Big(t_1 + \cdots + t_{k+1} + \sum_{n < n_1} y_n \Big)\, dt_1 \cdots dt_{k+1}.$$

These identities yield in particular

$$|\delta^{\{n_1,\ldots,n_k\}} \Psi_Y^{\varnothing}| \leq \|g^{(k)}\|_{\mathrm{L}^\infty(\mathbb{R})} \prod_{j=1}^{k} |y_{n_j}|,$$

$$\left| \Psi_Y^{\mathbb{N}} - \sum_{j=0}^{k} \sum_{\sharp F = j} \delta^F \Psi_Y^{\varnothing} \right| \leq \|g^{(k+1)}\|_{\mathrm{L}^\infty(\mathbb{R})} \sum_{\sharp F = k+1} \prod_{n \in F} |y_n|.$$

Setting $Y := \{h(x_n)\}_n$, noting that (1.18) reads $\Phi^H = \mathbb{E}[\Psi_Y^H]$, inserting the definition (1.19) of cluster coefficients, and recalling the definition (1.15) of multi-point

density functions, this yields

$$|\bar\Phi^k| \le \|g^{(k)}\|_{\mathrm{L}^\infty(\mathbb{R})} \int_{(\mathbb{R}^d)^k} |h|^{\otimes k} f_k,$$
$$\left|\Phi^{\mathbb{N}} - \sum_{j=0}^{k} \frac{1}{j!}\bar\Phi^j\right| \le \frac{1}{(k+1)!}\|g^{(k+1)}\|_{\mathrm{L}^\infty(\mathbb{R})} \int_{(\mathbb{R}^d)^{k+1}} |h|^{\otimes(k+1)} f_{k+1}.$$

By definition (1.16) of multi-point intensities, the conclusion follows. ∎

1.3.4 Effective viscosity: long-range issues and renormalization

We now apply the above cluster expansion formalism to the effective viscosity (1.2). For a finite subset $H \subset \mathbb{N}$, recall the notation ψ_E^H for the solution of the corrector problem (1.3) with the set $\mathcal{I}$ of particles replaced by its subset $\mathcal{I}^H = \bigcup_{n\in H} I_n$ (this corrector problem is trivially well posed in $\dot H^1(\mathbb{R}^d)^d$ when H is finite). We then define a symmetric linear map $\bar{\mathbf{B}}^H$ on $\mathbb{M}_0^{\mathrm{sym}}$ by

$$E : \bar{\mathbf{B}}^H E := \mathbb{E}\big[\big|\mathrm{D}(\psi_E^H)(0) + E\big|^2\big], \quad E \in \mathbb{M}_0^{\mathrm{sym}}.$$

(This indeed makes sense for a given measurable enumeration of the point process.) In these terms, the formal cluster expansion of the effective viscosity (1.2) takes the form

$$\bar{\mathbf{B}} \sim \sum_{j=0}^{\infty} \frac{1}{j!}\bar{\mathbf{B}}^j, \quad \text{where } \bar{\mathbf{B}}^j := j! \sum_{\sharp F = j} \delta^F \bar{\mathbf{B}}^\varnothing. \tag{1.21}$$

Note that $\bar{\mathbf{B}}^0 = \bar{\mathbf{B}}^\varnothing = \mathrm{Id}$ is the plain fluid viscosity. In contrast with the short-range setting of Lemma 1.2 above, however, series defining cluster coefficients $\{\bar{\mathbf{B}}^j\}_{j\ge1}$ are not summable due to the long-range nature of hydrodynamic interactions. Indeed, the first coefficient $\bar{\mathbf{B}}^1$ takes the form

$$\begin{aligned} E : \bar{\mathbf{B}}^1 E &= \sum_n E : \delta^{\{n\}}\bar{\mathbf{B}}^\varnothing E \\ &= \sum_n \mathbb{E}\big[\big|\mathrm{D}(\psi_E^{\{n\}})(0)\big|^2 + 2E : \mathrm{D}(\psi_E^{\{n\}})(0)\big]. \end{aligned} \tag{1.22}$$

As $\psi_E^{\{n\}}$ satisfies the single-particle problem (1.5), it is easily checked to have borderline decay $|\mathrm{D}(\psi_E^{\{n\}})(x)| \simeq \langle x - x_n\rangle^{-d}$, which entails that the above series is not absolutely convergent,

$$\sum_n \mathbb{E}\big[\big|\mathrm{D}(\psi_E^{\{n\}})(0)\big|\big] = \infty.$$

The same borderline divergence is observed for all cluster coefficients $\{\bar{\mathbf{B}}^j\}_{j\ge1}$. In order to make sense of them, suitable renormalization procedures are thus required and constitute the major difficulty of the problem.

To first order, the needed renormalization happens to be trivial: by definition of the intensity of the point process, identity (1.22) can be equivalently rewritten as follows (say, in case of deterministic particle shapes),

$$E : \bar{\mathbf{B}}^1 E = \lambda(\mathcal{P}) \int_{\mathbb{R}^d} \left(\left| \mathrm{D}(\psi_E^\circ) \right|^2 + 2E : \mathrm{D}(\psi_E^\circ) \right),$$

where ψ_E° stands for the solution of the single-particle problem (1.5) with a particle centered at the origin. Here, we observe that in any finite-volume approximation the linear term $\int_{\mathbb{R}^d} E : \mathrm{D}(\psi_E^\circ)$ would be given a vanishing value as the integral of a gradient. Removing this linear term, we are left with the following summable integral,

$$E : \bar{\mathbf{B}}^1 E = \lambda(\mathcal{P}) \int_{\mathbb{R}^d} \left| \mathrm{D}(\psi_E^\circ) \right|^2, \tag{1.23}$$

which coincides with Einstein's formula (1.7). In contrast, higher-order renormalizations are not obtained by such simple cancellations. In the physics literature, the difficulty was recognized by Batchelor and Green [7], who managed to provide a heuristic renormalization for the second-order term $\bar{\mathbf{B}}^2$. The systematic renormalization of higher-order terms is more involved and has remained an open problem so far even on the heuristic level in physics. The present memoir is precisely devoted to the systematic treatment of this difficulty: we provide suitable renormalizations of cluster coefficients and in turn justify the expansion (1.21) to all orders. In the end, we prove essentially the same estimates on the cluster expansion as in the short-range setting (1.20), up to (sharp) logarithmic corrections that are persisting manifestations of the long-range nature of interactions, cf. (1.26) below.

1.4 Main results

This section is devoted to a brief, informal account of the main results of this memoir, with precise references to the relevant chapters. We refer to the conclusion in Chapter 5 for a detailed recap of all our results. We start with the main assumptions on the ensemble of rigid particles.

1.4.1 Main assumptions

Given an underlying probability space $(\Omega, \mathbb{P})$, let $\mathcal{P} = \{x_n\}_n$ be a random point process on $\mathbb{R}^d$, consider an associated collection of random shapes $\{I_n^\circ\}_n$, where each I_n° is a random simply connected open subset of the unit ball B, centered at the origin in the sense of $\int_{I_n^\circ} y \, dy = 0$, and then define the corresponding inclusions

$$I_n := x_n + I_n^\circ.$$

Note that random shapes are not required to be independent of the point process $\mathcal{P}$. We then consider the random set

$$\mathcal{I} := \bigcup_n I_n,$$

which we assume to satisfy the following conditions. Note that the disjointness and ρ-regularity conditions below entail that the point process $\mathcal{P}$ is hardcore with $\ell(\mathcal{P}) \gtrsim \rho$, cf. (1.13).

Assumption ($\mathbf{H}_\rho$) – General conditions with parameter $\rho > 0$.

• Stationarity and ergodicity: *The point process $\mathcal{P} = \{x_n\}_n$ and the associated random set $\mathcal{I}$ are stationary and ergodic.*[2]

• Disjointness: *There holds $I_n \cap I_m = \varnothing$ almost surely for all $n \neq m$.*

• ρ-Regularity: *Random shapes $\{I_n^\circ\}_n$ almost surely satisfy interior and exterior ball conditions with radius ρ.*

Next, we define the effective viscosity tensor $\overline{\mathbf{B}}$ associated with the suspension $\mathcal{I}$ as the quadratic form on $\mathbb{M}^{\mathrm{sym}}_0$ given in (1.2). We emphasize that the corrector problem (1.3) only makes sense provided that all particles $\{I_n\}_n$ are well enough separated. If this separation is uniform, the pressure $\Sigma_E \mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}}$ can also be uniquely constructed as a stationary field with finite second moment and vanishing expectation, cf. [18, Proposition 2.1]. When particles are not well separated, the corrector problem should rather be considered via its variational formulation and the effective viscosity is then defined as the minimum value

$$\begin{aligned} E : \overline{\mathbf{B}} E = \inf \big\{ \mathbb{E}\big[\big|\mathrm{D}(\psi) + E\big|^2\big] : \psi \in \mathrm{L}^2\big(\Omega; H^1_{\mathrm{loc}}(\mathbb{R}^d)^d\big), \nabla\psi \text{ stationary}, \\ \mathrm{div}(\psi) = 0, \big(\mathrm{D}(\psi) + E\big)\big|_{\mathcal{I}} = 0, \mathbb{E}\big[\mathrm{D}(\psi)\big] = 0 \big\}. \end{aligned} \tag{1.24}$$

In general, nothing prevents this infimum from being infinite: as explained after (1.3) above, the issue originates from the possible existence of unbounded chains of touching particles. This will be excluded by means of further geometric assumptions, cf. ($\mathrm{H}^{\mathrm{unif}}_\rho$), ($\mathrm{H}^{\mathrm{mom}}_{\rho,\underline{\kappa}}$), or ($\mathrm{H}^{\mathrm{perc}}_{\rho,\kappa}$) below. Even if the infimum is finite, nothing ensures in general that $\overline{\mathbf{B}}$ defines the effective viscosity in the sense of homogenization theory: we view this as a separate question, which is extensively discussed in different settings in our previous work [13, 18, 19, 21] and will not be further discussed here. We are now in the position to describe our main results.

[2] More precisely, stationarity means that the laws of the translated point process $x + \mathcal{P}$ and of the translated random set $x + \mathcal{I}$ are independent of the shift $x \in \mathbb{R}^d$. Ergodicity then means that a measurable function of $\mathcal{P}$ or $\mathcal{I}$ is almost surely unchanged for $\mathcal{P}$ or $\mathcal{I}$ replaced by $x + \mathcal{P}$ or $x + \mathcal{I}$ for any $x \in \mathbb{R}^d$ only if it is almost surely constant. Note that shifts $x \in \mathbb{R}^d$ can be replaced by discrete shifts $x \in \mathbb{Z}^d$, and periodic point sets can be considered as a particular case, for which the expectation is replaced by the average over a period.

1.4.2 First-order expansion: Einstein's formula

Assuming that particles are uniformly separated by a positive distance, cf. ($\mathrm{H}_\rho^{\mathrm{unif}}$) below, we prove in the general ergodic stationary setting,

$$|\bar{\mathbf{B}} - \mathrm{Id} - \bar{\mathbf{B}}^1| \lesssim \lambda_2(\mathcal{P}) \log\left(2 + \frac{\lambda(\mathcal{P})}{\lambda_2(\mathcal{P})(\ell(\mathcal{P})+1)^d}\right), \quad |\bar{\mathbf{B}}^1| \simeq \lambda(\mathcal{P}), \tag{1.25}$$

where $\bar{\mathbf{B}}^1$ is given by the renormalized cluster formula (1.7) and takes the explicit form of Einstein's formula $\bar{\mathbf{B}}^1 = \frac{d+2}{2}\varphi\,\mathrm{Id}$ in case of spherical particles. This error estimate is new and optimal, and the stochastic assumption of mere ergodicity is minimal. In particular, we find that Einstein's formula provides a leading-order approximation, in the sense of $|\bar{\mathbf{B}} - \mathrm{Id} - \bar{\mathbf{B}}^1| \ll |\bar{\mathbf{B}}^1|$, provided that the condition $\lambda_2(\mathcal{P}) \ll \lambda(\mathcal{P})$ holds, which heuristically amounts to the scarcity of particle clusters and follows for instance from some local independence of $\mathcal{P}$.

Yet, the uniform separation assumption ($\mathrm{H}_\rho^{\mathrm{unif}}$) is not satisfactory from the physical point of view. At the price of weakening the error estimate (1.25), we may relax this assumption as we did for the homogenization result in [13,21]: either we assume moment bounds on the interparticle distance ($\mathrm{H}_{\rho,\kappa}^{\mathrm{mom}}$) (see also [28]), or we consider a subcritical percolation condition ($\mathrm{H}_{\rho,\kappa}^{\mathrm{perc}}$) (in which case particles are allowed to touch provided they do not cluster). We refer to Theorem 2.1 in Chapter 2 for a detailed statement.

1.4.3 Higher-order cluster corrections

For the higher-order analysis, we focus for simplicity on the setting of the uniform separation assumption ($\mathrm{H}_\rho^{\mathrm{unif}}$). Under a slight strengthening of ergodicity, the formal cluster expansion is well defined, up to suitable renormalization of cluster coefficients (1.21), and it essentially[3] satisfies for all $k \geq 1$,

$$\left|\bar{\mathbf{B}} - \sum_{j=0}^{k} \bar{\mathbf{B}}^j\right| \lesssim \lambda_{k+1}(\mathcal{P})\left|\log \lambda(\mathcal{P})\right|^k, \quad |\bar{\mathbf{B}}^k| \lesssim \lambda_k(\mathcal{P})\left|\log \lambda(\mathcal{P})\right|^{k-1}. \tag{1.26}$$

These estimates coincide remarkably with the corresponding result (1.20) in the short-range setting, to the exception of logarithmic corrections that are precisely the manifestations of the long-range nature of hydrodynamic interactions. The result is new for any $k \geq 2$ and logarithmic corrections are expected to be optimal (optimality is proved for $k = 2$, cf. Theorem 4.4). We also believe that the slightly strengthened ergodicity assumption is necessary for the result to hold. We refer to Theorem 5.2 in Chapter 5

[3] The correct estimate is in general slightly more complicated than what is stated here; cf. Theorem 5.2 in Chapter 5.

for a detailed statement. In particular, our analysis justifies the Batchelor–Green formula for the second-order term $\overline{\mathbf{B}}^2$, cf. Proposition 4.9 (see also Corollary 5.6), and we develop a systematic renormalization scheme for all higher-order cluster coefficients by means of diagrammatic expansions, cf. Section 4.4.

We emphasize that the above result (1.26) holds without any structural assumption on the dilution process (which we call the "model-free" setting). If we make the dilution more specific, considering for instance a random deletion procedure (as in [15]) or dilation, then the full cluster expansion can be further shown to define an absolutely converging series. We refer to Theorem 5.4 for a precise analyticity statement. All previous results on the second-order expansion [26, 27, 29] were, in fact, essentially restricted to such specific settings.

1.5 Roadmap to the main results

The rest of the memoir is divided into four chapters. Chapter 2 is dedicated to the proof of Einstein's formula. Chapter 3 studies the cluster expansion of finite-volume approximations $\{\overline{\mathbf{B}}_L\}_L$ of the effective viscosity $\overline{\mathbf{B}}$. In Chapter 4, we deal with the issue of systematic renormalization of cluster coefficients, which leads us to justifying the cluster expansion of $\overline{\mathbf{B}}$. Our different results are combined and summarized in Chapter 5. We briefly describe below our approach for each step.

1.5.1 Einstein's formula: first-order expansion—Chapter 2

We develop a new, purely variational approach to Einstein's formula (1.25); a short self-contained proof is given in Chapter 2. It amounts to constructing competitors for the variational problem (1.24) and to controlling their energy difference by means of elliptic regularity. The variational nature of the argument allows us to avoid uniform particle separation assumptions and to cover in particular the case of colliding particles under a general non-clustering assumption. It also allows to avoid the need for fine pressure estimates, which is a crucial point as such estimates would be problematic in case of colliding particles.

1.5.2 Cluster expansion of the effective viscosity—Chapter 3

While coefficients in the formal cluster expansion of the effective viscosity $\overline{\mathbf{B}}$ are given by infinite series that are not summable due to the long-range nature of hydrodynamic interactions, cf. Section 1.3.4, we start by considering finite-volume approximations $\{\overline{\mathbf{B}}_L\}_{L\geq 1}$ obtained by periodization of the variational problem (1.24). Chapter 3 provides a detailed analysis of the cluster expansion of $\overline{\mathbf{B}}_L$ for fixed L.

- First, we give explicit formulas for the coefficients $\{\overline{\mathbf{B}}_L^j\}_j$ of the cluster expansion, as well as an explicit estimate for the remainder $R_L^{k+1} := \overline{\mathbf{B}}_L - \sum_{j=0}^k \frac{1}{j!}\overline{\mathbf{B}}_L^j$, in terms of correctors associated with finite subsets of particles; see Theorem 3.1. The argument is essentially combinatorial. Note that the proof of remainder estimates further makes key use of the rigidity of the particles.
- Second, we prove that the cluster coefficients $\{\overline{\mathbf{B}}_L^j\}_j$ and the remainder R_L^{k+1} are bounded uniformly in L. The idea of the proof is as follows: if infinite-volume cluster formulas are given by infinite series that are not summable, they can in fact be viewed as complicated (non-explicit) combinations of Calderón–Zygmund kernels. As the effective viscosity is an L^2-based quantity, we may expect to estimate cluster formulas by means of suitable energy estimates, carefully avoiding taking absolute values of any Calderón–Zygmund kernel. Taking inspiration from our previous work [15], this is achieved by means of a hierarchy of so-called interpolating $\ell^1 - \ell^2$ energy estimates (also crucially used in [20,30]). As a corollary, uniform estimates allow us to define infinite-volume cluster coefficients in the limit $\{\overline{\mathbf{B}}^j\}_j := \lim_{L\uparrow\infty}\{\overline{\mathbf{B}}_L^j\}_j$. Yet, being based on energy arguments, these estimates do not display the desired dependence (1.26) on multi-point intensities $\{\lambda_j\}_j$.
- Third, we prove corresponding cluster estimates that have the same dependence on multi-point intensities as in the short-range setting, but display a logarithmic divergence in the large-volume limit. This is obtained by proceeding as for the short-range setting of Lemma 1.2, and the logarithmic divergence follows from estimating hydrodynamic interactions too roughly.

It remains to show that the dependence on multi-point intensities is actually kept in the large-volume limit (at the price of logarithmic corrections).

1.5.3 Renormalization of cluster formulas—Chapter 4

In order to prove the relevant infinite-volume cluster estimates (1.26), we need a better understanding of cluster formulas and of the underlying compensations that make them well defined in the large-volume limit. A first route proceeds by assuming an algebraic convergence rate for the finite-volume approximations $\{\overline{\mathbf{B}}_L\}_L$ of the effective viscosity: this is known to hold under quantitative α-mixing condition whose rate is then transmitted (suboptimally) to cluster coefficients $\{\overline{\mathbf{B}}_L^j\}_j$, which allows in turn to keep the desired dependence on multi-point intensities in the cluster estimates while removing the logarithmic divergence. This implicit renormalization argument is particularly robust (see also [15]), but it does not provide any understanding of underlying cancellations and leaves several questions open.

Next, further assuming for simplicity that particle shapes are independent of particle positions, we show that an explicit renormalization of cluster formulas can be

developed: taking advantage of several explicit cancellations, cluster formulas can be transformed into summable integral formulas. This renormalization is trivial for $\overline{\mathbf{B}}^1$, cf. (1.23), and the required cancellations are already more involved for $\overline{\mathbf{B}}^2$, as formally understood by Batchelor and Green [8]. At higher orders, renormalizations rely on a suitable diagrammatic decomposition of cluster formulas to make cancellations manifest. Finally, the direct analysis of renormalized formulas allows to recover the desired cluster estimates (1.26) and to show that logarithmic corrections in those bounds are actually optimal in general.

Notation

- For vector fields u, u' and matrix fields T, T', we set $(\nabla u)_{ij} = \nabla_j u_i$, $\mathrm{div}(T)_i = \nabla_j T_{ij}$, $T : T' = T_{ij} T'_{ij}$, $(u \otimes u')_{ij} = u_i u'_j$, where we systematically use Einstein's summation convention on repeated indices. We also denote by $(\mathrm{D}(u))_{ij} = \frac{1}{2}(\nabla_j u_i + \nabla_i u_j)$ the symmetrized gradient. For a velocity field u and associated pressure field P, we define the associated Cauchy stress tensor, cf. (1.4),

$$\sigma(u, P) := 2\mathrm{D}(u) - P\,\mathrm{Id}\,.$$

- We denote by $\mathbb{M}^{\mathrm{sym}}_0 \subset \mathbb{R}^{d\times d}$ the subset of symmetric trace-free matrices, and by $\mathbb{M}^{\mathrm{skew}}$ the subset of skew-symmetric matrices.
- We use the notation $\lesssim$ (resp. $\gtrsim$) for $\le C\times$ (resp. $\ge \frac{1}{C}\times$) with a constant C that depends only on the dimension d and on the parameters appearing in the different assumptions when applicable. Note that the value of the constant C is allowed to change from one line to another. We add subscripts to C, $\lesssim$, or $\gtrsim$ to indicate the dependence on other parameters. We write $a \simeq b$ when both $a \lesssim b$ and $a \gtrsim b$ hold. In addition, we write $\ll$ (resp. $\gg$) for $\le \frac{1}{C}\times$ (resp. $\ge C\times$) for some sufficiently large constant C.
- The ball centered at x of radius r in $\mathbb{R}^d$ is denoted by $B_r(x)$, and we set $B(x) = B_1(x)$, $B_r = B_r(0)$, and $B = B_1(0)$. We denote by $Q_r(x) = x + [-\frac{r}{2}, \frac{r}{2})^d$ the cube of sidelength r centered at x, and we set $Q(x) = Q_1(x)$, $Q_r = Q_r(0)$, and $Q = Q_1(0)$.
- For $x \in \mathbb{R}^d$, we denote by $|x|$ its Euclidean norm and by $|x|_\infty$ its supremum norm. We also set $\langle x\rangle = (1 + |x|^2)^{1/2}$, and similarly $\langle\nabla\rangle = (1 - \triangle)^{1/2}$.
- We use the shorthand notation $\sum^{\neq}_{n_1,\dots,n_j}$ for sums over j-tuples $(n_1, \dots, n_j)$ of distinct indices. We also use the notation $\sum_{\sharp F = j}$ for the sum over all subsets F of j distinct indices.

Chapter 2

Einstein's formula: First-order expansion

2.1 Main result

Assumption (H_ρ) first needs to be complemented with suitable geometric assumptions on the ensemble of particles to ensure that the effective viscosity (1.24) is finite. This can either be performed by means of conditions on interparticle distances,

$$\rho_n := \frac{1}{2} \min_{m:m\neq n} \operatorname{dist}(I_n, I_m), \tag{2.1}$$

or in terms of conditions on the size of clusters of close particles. This has been the subject of our recent series of articles [13,18,21], where the finiteness of the effective viscosity and the validity of a homogenization result are obtained under any of the following three types of assumptions:

— interparticle distances are uniformly bounded below, cf. [18];

— interparticle distances satisfy suitable reciprocal moment bounds, cf. [13, Theorem 1 for $d > 2$ and Remark 3.4 for $d = 2$];

— diameters of clusters of close particles satisfy suitable moment bounds in a subcritical percolation perspective, cf. [21].

These are formulated more precisely in Assumptions ($H_\rho^{\rm unif}$), ($H_{\rho,\kappa}^{\rm mom}$), and ($H_{\rho,\kappa}^{\rm perc}$) below, respectively.

Assumption ($\mathbf{H}_\rho^{\mathbf{unif}}$) – Uniform separation with parameter $\rho > 0$. *Particles are uniformly separated with minimal distance $\rho_n > \rho$, that is, we have almost surely*

$$(I_n + \rho B) \cap (I_m + \rho B) = \varnothing \quad \text{for all } n \neq m.$$

Assumption ($\mathbf{H}_{\rho,\kappa}^{\mathbf{mom}}$) – Moment condition with parameters $\rho > 0$, $\kappa > 1$.

• ρ-Uniform non-degeneracy of contact points: *Pairs of "ρ-close" particles can be "ρ-locally" included in pairs of disjoint spheres with "ρ-uniformly" bounded radius. For "ρ-close" particles, instead of (2.1), we then define ρ_n as (half of) the distance between locally covering spheres. For a more precise statement of this geometric condition, we refer to* [13, Assumption (H'_δ)]. *Note that this condition is trivially satisfied in case of spherical particles.*

• Reciprocal moment bound: *There exists $\mathcal{K}_\kappa < \infty$ such that*

$$\limsup_{R\uparrow\infty} \left(\frac{1}{\sharp\{n : I_n \subset Q_R\}} \sum_{n:I_n\subset Q_R} \mu(\rho_n)^\kappa \right)^{\frac{1}{\kappa}} \leq \mathcal{K}_\kappa,$$

in terms of

$$\mu(t) := \begin{cases} t^{-\frac{1}{2}(5-d)} & \text{if } d < 5, \\ \log(2+\frac{1}{t}) & \text{if } d = 5, \\ 1 & \text{if } d > 5. \end{cases} \tag{2.2}$$

Note that this condition is trivially satisfied for any $\kappa > 1$ in case $d > 5$.

Assumption ($\mathrm{H}^{\mathrm{perc}}_{\rho,\kappa}$) – Cluster condition with parameters $\rho > 0$, $\kappa > 1$. *Let $\{K_{q,\rho}\}_q$ be the family of connected components of the fattened set $\mathcal{I} + \rho B$, and consider the corresponding clusters*

$$J_{q,\rho} := \bigcup_{I_n \subset K_{q,\rho}} I_n.$$

Given $p_0 \gg 1$ large enough (related to the existence of correctors in [21, Proposition 2]*), there exists $\mathcal{K}_\kappa < \infty$ such that*

$$\limsup_{R\uparrow\infty} \left(\frac{1}{\sharp\{q : J_{q,\rho} \cap Q_R \neq \varnothing\}} \sum_{q: J_{q,\rho}\cap Q_R \neq \varnothing} \operatorname{diam}(J_{q,\rho})^{\kappa p_0 + d} \right)^{\frac{1}{\kappa}} \leq \mathcal{K}_\kappa. \tag{2.3}$$

Assumptions ($\mathrm{H}^{\mathrm{mom}}_{\rho,\kappa}$) and ($\mathrm{H}^{\mathrm{perc}}_{\rho,\kappa}$) are always weaker than ($\mathrm{H}^{\mathrm{unif}}_{\rho}$). While ($\mathrm{H}^{\mathrm{mom}}_{\rho,\kappa}$) only allows for particle contacts in dimension $d > 5$ and is in particular incompatible with the 3D steady-state behavior of the two-particle density as computed in [8], Assumption ($\mathrm{H}^{\mathrm{perc}}_{\rho,\kappa}$) is of a different nature and allows for particle contacts in any dimension, but implicitly requires some strong mixing condition to ensure the validity of moment bounds on cluster diameters, cf. [21]. In [13, 18, 21], we show that these assumptions ensure the finiteness of the effective viscosity (1.24) and the well-posedness of the corrector problem. In case of ($\mathrm{H}^{\mathrm{mom}}_{\rho,\kappa}$) or ($\mathrm{H}^{\mathrm{perc}}_{\rho,\kappa}$), the validity of the homogenization result requires further strengthened conditions.

The following theorem states the validity of Einstein's formula under each of the above assumptions. The proof is split between Sections 2.2, 2.3, 2.4, and 2.5 below.

Theorem 2.1 (Einstein's formula). *Under Assumption (H_ρ), provided that either ($\mathrm{H}^{\mathrm{unif}}_{\rho}$), ($\mathrm{H}^{\mathrm{mom}}_{\rho,\kappa}$), or ($\mathrm{H}^{\mathrm{perc}}_{\rho,\kappa}$) holds for some $\rho > 0$ and $\kappa > 1$, we have*

$$\begin{aligned} \left|\overline{\mathbf{B}} - (\mathrm{Id} + \overline{\mathbf{B}}^1)\right| &\lesssim_\rho \lambda_2(\mathcal{P}) \log\left(2 + \frac{\lambda(\mathcal{P})}{\lambda_2(\mathcal{P})\big(\ell(\mathcal{P})+1\big)^d}\right) \\ &+ \begin{cases} 0 & \text{in case of } (\mathrm{H}^{\mathrm{unif}}_{\rho}), \\ \mathcal{K}_\kappa \lambda_2(\mathcal{P})^{1-\frac{1}{\kappa}} \lambda(\mathcal{P})^{\frac{1}{\kappa}} & \text{in case of } (\mathrm{H}^{\mathrm{mom}}_{\rho,\kappa}) \text{ or } (\mathrm{H}^{\mathrm{perc}}_{\rho,\kappa}), \end{cases} \end{aligned} \tag{2.4}$$

where $\overline{\mathbf{B}}^1$ satisfies

$$|\overline{\mathbf{B}}^1| \simeq \lambda(\mathcal{P}),$$

and is defined for all $E \in \mathbb{M}_0^{\mathrm{sym}}$ *by*

$$E : \bar{\mathbf{B}}^1 E := \sum_n \mathbb{E}\bigg[\frac{\mathbb{1}_{0\in I_n}}{|I_n|}\int_{\mathbb{R}^d} \big|\mathrm{D}\big(\psi_E^{\{n\}}\big)\big|^2\bigg], \tag{2.5}$$

where $\psi_E^{\{n\}}$ *is the unique decaying solution of the single-particle problem* (1.5). *In particular, the estimate* $|\bar{\mathbf{B}} - (\mathrm{Id} + \bar{\mathbf{B}}^1)| = o(\lambda(\mathcal{P}))$ *holds provided the point process* $\mathcal{P}$ *satisfies* $\lambda_2(\mathcal{P}) = o(\lambda(\mathcal{P}))$.

As outlined in Section 2.2, our proof is variational and amounts to proving lower and upper bounds on $\bar{\mathbf{B}}$ that match with $\mathrm{Id} + \bar{\mathbf{B}}^1$ to the required accuracy. (For the case of the Clausius–Mossotti conductivity formula, we refer to [14], where we provide a PDE version of the present variational argument.) This new approach is very robust: it yields the first optimal error estimate and allows to cover the most general setting regarding particle separation assumptions. We briefly emphasize these two points:

— *Optimality:* In case of $(\mathrm{H}_\rho^{\mathrm{unif}})$, the error estimate (2.4) for Einstein's formula is new and sharp. As will be seen in Theorem 4.4, it indeed coincides with the general scaling of the next term $\bar{\mathbf{B}}^2$ in the cluster expansion: the logarithmic correction is related to the long-range nature of hydrodynamic interactions and cannot be avoided in general, thus contrasting with the short-range setting (1.20).[1] In case of $(\mathrm{H}_{\rho,\kappa}^{\mathrm{mom}})$ or $(\mathrm{H}_{\rho,\kappa}^{\mathrm{perc}})$, the error estimate (2.4) displays a further algebraic loss, which is also new and expected to be optimal in general. If for some exponent $\gamma \geq 1$ the moment bounds in $(\mathrm{H}_{\rho,\kappa}^{\mathrm{mom}})$ or $(\mathrm{H}_{\rho,\kappa}^{\mathrm{perc}})$ hold with constant $\mathcal{K}_\kappa \lesssim \kappa^\gamma$ for all $\kappa \geq 1$,[2] then the error estimate (2.4) could be upgraded to

$$\lambda_2(\mathcal{P}) \log^{1\vee\gamma}\bigg(2 + \frac{\lambda(\mathcal{P})}{\lambda_2(\mathcal{P})}\bigg)$$

after optimizing in κ.

— *Particle separation:* Most works on the topic [26–29, 33, 54] have focussed so far on the simplest setting of $(\mathrm{H}_\rho^{\mathrm{unif}})$ in case when diluteness further holds in the strong form of $\ell(\mathcal{P}) \gg 1$. The only exception is the recent independent work [28], where this last condition is relaxed and where the case of $(\mathrm{H}_{\rho,\kappa}^{\mathrm{mom}})$ is also covered. More generally, our approach allows to further cover essentially any situation for which the effective viscosity (1.24) can be proved to be finite. Applied to $(\mathrm{H}_{\rho,\kappa}^{\mathrm{perc}})$, it allows us to treat for the first time a 3D setting where particles are allowed to touch.

[1] In some special cases, however, for instance in the statistically isotropic setting, the logarithmic correction can be removed, cf. Theorem 4.4 (i).

[2] For $(\mathrm{H}_{\rho,\kappa}^{\mathrm{mom}})$, this would amount to having stretched exponential moment bounds. For $(\mathrm{H}_{\rho,\kappa}^{\mathrm{perc}})$, this holds for some point processes such as the random parking measure with γ large enough, cf. [21].

Next, we further simplify formula (2.5) for the first-order cluster coefficient $\bar{\mathbf{B}}^1$ in the case when particle shapes are independent: we recover the formula obtained in [33], as well as Einstein's explicit formula (1.8) in case of spherical particles. The proof is postponed to Section 2.6.

Proposition 2.2 (First-order coefficient). *On top of Assumption* (H_ρ)*, further assume*

(Indep) Independent shapes: *Random shapes $\{I_n^\circ\}_n$ are iid copies of a given random subset I°, independently of the point process $\mathcal{P}$.*

Then, the first-order coefficient $\bar{\mathbf{B}}^1$ defined in (2.5) *can be written as*

$$\bar{\mathbf{B}}^1 = \lambda(\mathcal{P})\hat{\mathbf{B}}^1, \quad E : \hat{\mathbf{B}}^1 E = \mathbb{E}\bigg[\int_{\mathbb{R}^d} \big|\mathrm{D}(\psi_E^\circ)\big|^2\bigg], \tag{2.6}$$

in terms of the unique decaying solution of the single-particle problem

$$\begin{cases} -\triangle\psi_E^\circ + \nabla\Sigma_E^\circ = 0, & \text{in } \mathbb{R}^d \setminus I^\circ, \\ \operatorname{div}(\psi_E^\circ) = 0, & \text{in } \mathbb{R}^d \setminus I^\circ, \\ \mathrm{D}(\psi_E^\circ) + E = 0, & \text{in } I^\circ, \\ \int_{\partial I^\circ} \sigma_E^\circ \nu = 0, & \\ \int_{\partial I^\circ} \Theta x \cdot \sigma_E^\circ \nu = 0, & \forall \Theta \in \mathbb{M}^{\mathrm{skew}}. \end{cases} \tag{2.7}$$

In case of spherical particles, $I_n = B(x_n, r_n)$, with iid random radii $\{r_n\}_n$, this reduces to Einstein's celebrated formula

$$\bar{\mathbf{B}}^1 = \frac{d+2}{2}\varphi\,\mathrm{Id}, \tag{2.8}$$

where the volume fraction is in this case $\varphi = \lambda(\mathcal{P})\mathbb{E}[|I_0|]$.

2.2 Variational approach

This section is devoted to setting up our variational approach to prove Theorem 2.1, which is partly inspired by the theory of optimal bounds in homogenization; see e.g. [50, Chapters 13 and 23]. The new main ingredients are the use of Voronoi tessellations and of elliptic regularity. Let $E \in \mathbb{M}_0^{\mathrm{sym}}$ be fixed. In the spirit of the heuristic approximation (1.6) for the corrector, we start by defining single-particle problems in the neighborhood of each particle. For a random set $\mathcal{I}$ satisfying (H_ρ), we define the associated Voronoi tessellation $\{V_n\}_n$ as follows,

$$V_n := \big\{x \in \mathbb{R}^d : \operatorname{dist}(x, I_n) < \operatorname{dist}(x, I_m)\ \forall m \neq n\big\}.$$

By definition, these Voronoi cells pave the whole space $\mathbb{R}^d$ and each V_n contains exactly one inclusion I_n. We then consider the single-particle problems in V_n, with

either homogeneous Dirichlet or Neumann boundary conditions on ∂V_n,

$$\mathcal{E}_{n,D} := \inf\Big\{ \int_{V_n} \big|\mathrm{D}(\psi)\big|^2 : \psi \in H^1_0(V_n)^d,\ \mathrm{div}(\psi) = 0,\ \big(\mathrm{D}(\psi) + E\big)\big|_{I_n} = 0\Big\}, \tag{2.9}$$

$$\mathcal{E}_{n,N} := \inf\Big\{ \int_{V_n} \big|\mathrm{D}(\psi)\big|^2 : \psi \in H^1(V_n)^d,\ \mathrm{div}(\psi) = 0,\ \big(\mathrm{D}(\psi) + E\big)\big|_{I_n} = 0\Big\}. \tag{2.10}$$

Provided that $\mathcal{E}_{n,D} < \infty$ (which is the case as soon as $I_n \Subset V_n$), the Dirichlet problem (2.9) is well posed and we denote by $\psi_{n,D}$ its unique minimizer. The Neumann problem (2.10), on the other hand, is always well posed and one has the deterministic uniform bound $\mathcal{E}_{n,N} \lesssim 1$. We denote by $\psi_{n,N}$ the corresponding minimizer, which is only defined up to a rigid motion and can be fixed for instance by choosing $\int_V \psi_{n,N} = 0$ and $\int_V \nabla\psi_{n,N} \in \mathbb{M}^{\mathrm{sym}}_0$. Next, we define the single-particle problem on the whole space via

$$\mathcal{E}_{n,\infty} := \inf\Big\{ \int_{\mathbb{R}^d} \big|\mathrm{D}(\psi)\big|^2 : \psi \in H^1(\mathbb{R}^d)^d,\ \mathrm{div}(\psi) = 0,\ \big(\mathrm{D}(\psi) + E\big)\big|_{I_n} = 0\Big\}. \tag{2.11}$$

Note that the unique minimizer $\psi_{n,\infty}$ of this variational problem coincides with the solution $\psi_E^{\{n\}}$ of (1.5). In case of ($\mathrm{H}^{\mathrm{perc}}_{\rho,\kappa}$), as we only control clusters of close particles, we naturally merge Voronoi cells that intersect the same cluster: more precisely, we consider the Voronoi cell associated with each cluster $J_{q,\rho}$,

$$W_q := \bigcup_{n : I_n \subset J_{q,\rho}} V_n,$$

and we then partition the whole space as

$$\mathbb{R}^d = \Big(\bigcup_{n\in\mathcal{S}} V_n\Big) \cup \Big(\bigcup_{q\in\mathcal{S}'} W_q\Big),$$

where $\mathcal{S} := \{n : \rho_n \geq \rho\}$ is the set of indices of well-separated particles and where $\mathcal{S}'$ is the set of indices q such that the cluster $J_{q,\rho}$ is made of at least two particles. For $n \in \mathcal{S}$ we shall consider the single-inclusion problems $\mathcal{E}_{n,D}, \mathcal{E}_{n,N}, \mathcal{E}_{n,\infty}$ as above, while for $q \in \mathcal{S}'$ it will suffice to consider the single-cluster problem with Dirichlet conditions,

$$\mathcal{F}_{q,D} := \inf\Big\{ \int_{W_q} \big|\mathrm{D}(\psi)\big|^2 : \psi \in H^1_0(W_q)^d,\ \mathrm{div}(\psi) = 0,\ \big(\mathrm{D}(\psi) + E\big)\big|_{J_{q,\rho}} = 0\Big\}. \tag{2.12}$$

The upcoming lemma shows that the error in the first-order expansion $\overline{\mathbf{B}} \sim \mathrm{Id} + \overline{\mathbf{B}}^1$ can be controlled using single-particle problems (as well as single-cluster problems in

case of ($\mathrm{H}^{\mathrm{perc}}_{\rho,\kappa}$)). This provides a drastic reduction of complexity since $\bar{\mathbf{B}}$ itself involves the corrector ψ_E with the full set of particles. The proof is postponed to Section 2.4 below.

Lemma 2.3. *Under the assumptions of Theorem 2.1, using the notation (2.9)–(2.12), we have*

$$\begin{aligned}
&\big|E:\big(\bar{\mathbf{B}}-(\mathrm{Id}+\bar{\mathbf{B}}^1)\big)E\big|\\
&\lesssim\begin{cases}
\mathbb{E}\Big[\sum_n \frac{\mathbb{1}_{0\in I_n}}{|I_n|}(\mathcal{E}_{n,D}-\mathcal{E}_{n,N})\Big] & \text{in case of } (\mathrm{H}^{\mathrm{unif}}_{\rho}) \text{ or } (\mathrm{H}^{\mathrm{mom}}_{\rho,\kappa}),\\
\mathbb{E}\Big[\sum_{n\in\mathcal{S}} \frac{\mathbb{1}_{0\in I_n}}{|I_n|}(\mathcal{E}_{n,D}-\mathcal{E}_{n,N})\Big] & \\
\quad+\mathbb{E}\Big[\sum_{q\in\mathcal{S}'} \frac{\mathbb{1}_{0\in J_{q,\rho}}}{|J_{q,\rho}|}\mathcal{F}_{q,D}\Big] & \text{in case of } (\mathrm{H}^{\mathrm{perc}}_{\rho,\kappa}),
\end{cases}
\end{aligned}\tag{2.13}$$

where $\bar{\mathbf{B}}^1$ is defined in (2.5).

It remains to control the right-hand side in the above preliminary error estimate (2.13), which amounts to comparing the single-particle problems with Dirichlet or Neumann boundary conditions on Voronoi cells. The proof is postponed to Section 2.5 below.

Lemma 2.4. *For all n, we have almost surely*

$$0\le\mathcal{E}_{n,D}-\mathcal{E}_{n,N}\lesssim\mu(\rho_n)\mathbb{1}_{\rho_n<1}+\rho_n^{-d}\mathbb{1}_{\rho_n\ge1},\tag{2.14}$$

where we recall that ρ_n stands for (half of) the interparticle distance, cf. (2.1), and that the weight μ is defined in (2.2). In addition, there is $p_0<\infty$ such that for all q we have almost surely

$$\mathcal{F}_{q,D}\lesssim\mathrm{diam}(J_{q,\rho})^{p_0}.\tag{2.15}$$

With these two lemmas at hand, combining the estimates, we may now quickly conclude the proof of Theorem 2.1.

Proof of Theorem 2.1. Combining Lemmas 2.3 and 2.4, we get

$$\begin{aligned}
&\big|\bar{\mathbf{B}}-(\mathrm{Id}+\bar{\mathbf{B}}^1)\big|\\
&\lesssim\begin{cases}
\mathbb{E}\Big[\sum_n \frac{\mathbb{1}_{0\in I_n}}{|I_n|}\big(\mu(\rho_n)\mathbb{1}_{\rho_n<1}+\rho_n^{-d}\mathbb{1}_{\rho_n\ge1}\big)\Big] & \text{in case of } (\mathrm{H}^{\mathrm{unif}}_{\rho}) \text{ or } (\mathrm{H}^{\mathrm{mom}}_{\rho,\kappa}),\\
\mathbb{E}\Big[\sum_n \frac{\mathbb{1}_{0\in I_n}}{|I_n|}\rho_n^{-d}\mathbb{1}_{\rho_n\ge\rho}\Big] & \\
\quad+\mathbb{E}\Big[\sum_{q\in\mathcal{S}'} \frac{\mathbb{1}_{0\in J_{q,\rho}}}{|J_{q,\rho}|}\mathrm{diam}(J_{q,\rho})^{p_0}\Big] & \text{in case of } (\mathrm{H}^{\mathrm{perc}}_{\rho,\kappa}),
\end{cases}
\end{aligned}\tag{2.16}$$

and it remains to estimate these expectations. We split the proof into two steps.

Step 1. Proof that, if $g \in \mathrm{L}^\infty(\mathbb{R}^+)$ is non-increasing with $g(r) \downarrow 0$ as $r \uparrow \infty$, then

$$\mathbb{E}\Big[\sum_n \frac{\mathbb{1}_{0\in I_n}}{|I_n|} g(\rho_n)\Big] \lesssim \lambda_2(\mathcal{P})\|g\|_{\mathrm{L}^\infty(\mathbb{R}^+)} + \int_{(\frac{1}{2}\ell-1)_+}^\infty |g'(r)|\big((\lambda_2(\mathcal{P})\langle r\rangle^d) \wedge \lambda(\mathcal{P})\big)\, dr. \tag{2.17}$$

To start with, we rewrite the left-hand side as

$$\mathbb{E}\Big[\sum_n \frac{\mathbb{1}_{0\in I_n}}{|I_n|} g(\rho_n)\Big] = \int_0^\infty g(r)\, d\Lambda(r), \tag{2.18}$$

where the positive measure Λ on $\mathbb{R}^+$ is defined by its distribution function

$$\Lambda\big([0,r]\big) := \mathbb{E}\Big[\sum_n \frac{\mathbb{1}_{0\in I_n}}{|I_n|}\mathbb{1}_{\rho_n\le r}\Big] = \mathbb{E}\Big[\sum_n \frac{\mathbb{1}_{0\in I_n}}{|I_n|}\mathbb{1}_{\exists m\neq n:\frac{1}{2}\operatorname{dist}(I_m,I_n)\le r}\Big]. \tag{2.19}$$

As $I_n \subset B(x_n)$ for all n, we can estimate the latter as

$$\Lambda\big([0,r]\big) \le \mathbb{E}\Big[\sum_n \mathbb{1}_{|x_n|\le 1}\mathbb{1}_{\exists m\neq n:|x_m-x_n|\le 2(r+1)}\Big].$$

Recalling that $\ell = \ell(\mathcal{P})$ is the minimal distance (1.13), we deduce that $\Lambda([0,r]) = 0$ for all $r \le \frac{1}{2}\ell - 1$. Moreover, we can bound, on the one hand,

$$\Lambda\big([0,r]\big) \le \mathbb{E}\Big[\sum_n \mathbb{1}_{|x_n|\le 1}\Big] = \lambda(\mathcal{P})|B|,$$

and on the other hand, in terms of the two-point density and intensity, for $r \ge \frac{1}{2}\ell - 1$,

$$\begin{aligned}\Lambda\big([0,r]\big) &\le \mathbb{E}\Big[\sum_{n\neq m} \mathbb{1}_{|x_n|\le 1}\mathbb{1}_{|x_m-x_n|\le 2(r+1)}\Big]\\ &= \iint_{B\times B_{2(r+1)}} f_2(x,x+y)\, dxdy\\ &= \big(2(r+1)\big)^{-d} \iint_{B_{2(r+1)}\times B_{2(r+1)}} f_2(x,x+y)\, dxdy\\ &\lesssim \lambda_2(\mathcal{P})\langle r\rangle^d.\end{aligned}$$

Combining these estimates yields

$$\Lambda\big([0,r]\big) \lesssim \big(\lambda_2(\mathcal{P})\langle r\rangle^d\big) \wedge \lambda(\mathcal{P}). \tag{2.20}$$

Under our assumptions on g, an integration by parts yields

$$\int_0^\infty g(r)\, d\Lambda(r) = -g(0)\Lambda\big(\{0\}\big) + \int_0^\infty |g'(r)|\Lambda\big([0,r]\big)\, dr,$$

and the conclusion follows in combination with (2.18) and (2.20).

Step 2. Conclusion. In case of ($\mathrm{H}^{\mathrm{unif}}_{\rho}$), as we have $\rho_n \geq \rho$ for all n, the contributions of $\rho_n < \rho$ can be removed in (2.16). Applying (2.17) with $g(r) = \langle r \rangle^{-d}$, we are then led to

$$\begin{aligned}&\big|\bar{\mathbf{B}} - (\mathrm{Id} + \bar{\mathbf{B}}^1)\big| \\ &\lesssim \mathbb{E}\bigg[\sum_n \frac{\mathbb{1}_{0\in I_n}}{|I_n|}\langle \rho_n\rangle^{-d}\bigg] \lesssim \lambda_2(\mathcal{P}) + \int_{(\frac{1}{2}\ell-1)_+}^{\infty} \langle r\rangle^{-d-1}\Big(\big(\lambda_2(\mathcal{P})\langle r\rangle^d\big) \wedge \lambda(\mathcal{P})\Big)\, dr,\end{aligned}$$

and the conclusion (2.4) follows after estimating this integral.

Next, in case of ($\mathrm{H}^{\mathrm{mom}}_{\rho,\kappa}$), repeating the same computation as above for the contributions of $\rho_n \geq 1$ in (2.16), and separating the contributions of $\rho_n \leq 1$, we find

$$\begin{aligned}&\big|\bar{\mathbf{B}} - (\mathrm{Id} + \bar{\mathbf{B}}^1)\big| \\ &\quad\lesssim \lambda_2(\mathcal{P}) \log\bigg(2 + \frac{\lambda(\mathcal{P})}{\lambda_2(\mathcal{P})\big(\ell(\mathcal{P})+1\big)^d}\bigg) + \mathbb{E}\bigg[\sum_n \frac{\mathbb{1}_{0\in I_n}}{|I_n|}\mu(\rho_n)\mathbb{1}_{\rho_n<1}\bigg],\end{aligned}$$

and it remains to estimate the last term. By Hölder's inequality, we can write for any $\kappa \geq 1$,

$$\mathbb{E}\bigg[\sum_n \frac{\mathbb{1}_{0\in I_n}}{|I_n|}\mu(\rho_n)\mathbb{1}_{\rho_n<1}\bigg] \leq \mathbb{E}\bigg[\sum_n \frac{\mathbb{1}_{0\in I_n}}{|I_n|}\mathbb{1}_{\rho_n<1}\bigg]^{1-\frac{1}{\kappa}} \mathbb{E}\bigg[\sum_n \frac{\mathbb{1}_{0\in I_n}}{|I_n|}\mu(\rho_n)^\kappa\bigg]^{\frac{1}{\kappa}}.$$

On the one hand, (2.20) yields

$$\mathbb{E}\bigg[\sum_n \frac{\mathbb{1}_{0\in I_n}}{|I_n|}\mathbb{1}_{\rho_n<1}\bigg] = \Lambda\big([0,1]\big) \lesssim \lambda_2(\mathcal{P}).$$

On the other hand, by the ergodic theorem, using (1.12) and the reciprocal moment condition in ($\mathrm{H}^{\mathrm{mom}}_{\rho,\kappa}$), we find

$$\begin{aligned}&\mathbb{E}\bigg[\sum_n \frac{\mathbb{1}_{0\in I_n}}{|I_n|}\,\mu(\rho_n)^\kappa\bigg] \\ &\quad= \lim_{R\uparrow\infty} R^{-d}\sum_n \frac{|I_n\cap Q_R|}{|I_n|}\,\mu(\rho_n)^\kappa \\ &\quad\leq \limsup_{R\uparrow\infty} \frac{\sharp\{n : I_n\cap Q_R\neq\varnothing\}}{R^d}\,\frac{1}{\sharp\{n : I_n\cap Q_R\neq\varnothing\}}\sum_{n:I_n\cap Q_R\neq\varnothing}\mu(\rho_n)^\kappa \\ &\quad\leq \lambda(\mathcal{P})(\mathcal{K}_\kappa)^\kappa,\end{aligned}$$

and the conclusion (2.4) follows.

Finally, in case of $(\mathrm{H}^{\mathrm{perc}}_{\rho,\kappa})$, repeating again the same computation for the contributions of $\rho_n \geq \rho$ in (2.16), we find

$$\begin{aligned}
&\big|\overline{\mathbf{B}} - (\mathrm{Id} + \overline{\mathbf{B}}^1)\big| \\
&\quad \lesssim \lambda_2(\mathcal{P}) \log\left(2 + \frac{\lambda(\mathcal{P})}{\lambda_2(\mathcal{P})\big(\ell(\mathcal{P})+1\big)^d}\right) + \mathbb{E}\bigg[\sum_{q\in\mathcal{S}'} \frac{\mathbb{1}_{0\in J_{q,\rho}}}{|J_{q,\rho}|}\operatorname{diam}(J_{q,\rho})^{p_0}\bigg],
\end{aligned}$$

and it remains to estimate the last term. By Hölder's inequality, we can write for any $\kappa \geq 1$,

$$\begin{aligned}
&\mathbb{E}\bigg[\sum_{q\in\mathcal{S}'} \frac{\mathbb{1}_{0\in J_{q,\rho}}}{|J_{q,\rho}|}\operatorname{diam}(J_{q,\rho})^{p_0}\bigg] \\
&\qquad \leq \mathbb{E}\bigg[\sum_{q\in\mathcal{S}'} \frac{\mathbb{1}_{0\in J_{q,\rho}}}{|J_{q,\rho}|}\bigg]^{1-\frac{1}{\kappa}} \mathbb{E}\bigg[\sum_{q} \frac{\mathbb{1}_{0\in J_{q,\rho}}}{|J_{q,\rho}|}\operatorname{diam}(J_{q,\rho})^{\kappa p_0}\bigg]^{\frac{1}{\kappa}}.
\end{aligned}$$

On the one hand, by definition of $\mathcal{S}'$ and by definition (2.19) of Λ, we get from (2.20),

$$\mathbb{E}\bigg[\sum_{q\in\mathcal{S}'} \frac{\mathbb{1}_{0\in J_{q,\rho}}}{|J_{q,\rho}|}\bigg] \leq \mathbb{E}\bigg[\sum_{n} \frac{\mathbb{1}_{0\in I_n}}{|I_n|}\mathbb{1}_{\rho_n\leq\rho}\bigg] = \Lambda\big([0,\rho]\big) \lesssim \lambda_2(\mathcal{P}).$$

On the other hand, by the ergodic theorem, using (1.12), the condition (2.3) in $(\mathrm{H}^{\mathrm{perc}}_{\rho,\kappa})$, and the fact that there are less clusters than particles, we find

$$\begin{aligned}
&\mathbb{E}\bigg[\sum_{q} \frac{\mathbb{1}_{0\in J_{q,\rho}}}{|J_{q,\rho}|}\operatorname{diam}(J_{q,\rho})^{\kappa p_0}\bigg] = \lim_{R\uparrow\infty} R^{-d}\sum_{q} \frac{|J_{q,\rho}\cap Q_R|}{|J_{q,\rho}|}\operatorname{diam}(J_{q,\rho})^{\kappa p_0} \\
&\leq \limsup_{R\uparrow\infty} \frac{\sharp\{q : J_{q,\rho}\cap Q_R \neq \varnothing\}}{R^d} \frac{1}{\sharp\{q : J_{q,\rho}\cap Q_R \neq \varnothing\}} \sum_{q:J_{q,\rho}\cap Q_R\neq\varnothing} \operatorname{diam}(J_{q,\rho})^{\kappa p_0} \\
&\leq \limsup_{R\uparrow\infty} \frac{\sharp\{n : I_n\cap Q_R \neq \varnothing\}}{R^d} \frac{1}{\sharp\{q : J_{q,\rho}\cap Q_R \neq \varnothing\}} \sum_{q:J_{q,\rho}\cap Q_R\neq\varnothing} \operatorname{diam}(J_{q,\rho})^{\kappa p_0} \\
&\leq \lambda(\mathcal{P})(\mathcal{K}_\kappa)^\kappa,
\end{aligned}$$

and the conclusion (2.4) follows. ∎

2.3 Preliminary lemmas

Before turning to the proof of Lemmas 2.3 and 2.4, which are key to Theorem 2.1 as explained above, we start with a few preliminary PDE and probabilistic lemmas. We first prove the following trace estimates at particle boundaries.

Lemma 2.5 (Trace estimates). *For all n, we have the following estimates.*

(i) *For any $\psi \in H^1(I_n)$, we have*

$$\inf_{\kappa\in\mathbb{R}^d,\,\Theta\in\mathbb{M}^{\mathrm{skew}}} \int_{\partial I_n} \big|\psi - \big(\kappa + \Theta(x - x_n)\big)\big|^2 \lesssim \int_{I_n} \big|\mathrm{D}(\psi)\big|^2.$$

(ii) *For any $\psi \in H^1(I_n + \rho B)$ satisfying the following relations, and for some $E \in \mathbb{M}^{\mathrm{sym}}_0$,*

$$\begin{cases} -\triangle\psi + \nabla\Sigma = 0, & \text{in } (I_n + \rho B) \setminus I_n, \\ \mathrm{div}(\psi) = 0, & \text{in } (I_n + \rho B) \setminus I_n, \\ \mathrm{D}(\psi) + E = 0, & \text{in } I_n, \end{cases}$$

we have

$$\inf_{c\in\mathbb{R}} \int_{\partial I_n} \big|\sigma(\psi, \Sigma) - c\,\mathrm{Id}\big|^2 \lesssim \int_{I_n+\rho B} \big|\mathrm{D}(\psi)\big|^2,$$

where we recall that multiplicative constants may implicitly depend on ρ.

Proof. We split the proof into two steps.

Step 1. Proof of (i). We appeal to a trace estimate in form of

$$\int_{\partial I_n} \big|\psi - \big(\kappa + \Theta(x - x_n)\big)\big|^2 \lesssim \int_{I_n} \big|\langle\nabla\rangle^{\frac12}\big(\psi - \big(\kappa + \Theta(x - x_n)\big)\big)\big|^2,$$

and the conclusion then follows from Poincaré's and Korn's inequalities.

Step 2. Proof of (ii). By definition of the Cauchy stress tensor, a trace estimate yields

$$\int_{\partial I_n} \big|\sigma(\psi, \Sigma) - c\,\mathrm{Id}\big|^2 \lesssim \int_{(I_n+\frac12\rho B)\setminus I_n} \big|\langle\nabla\rangle^{\frac12}\nabla\psi\big|^2 + \big|\langle\nabla\rangle^{\frac12}(\Sigma - c)\big|^2. \tag{2.21}$$

By the local regularity theory for the steady Stokes equation near a boundary, e.g. [25, Theorems IV.5.1–IV.5.3], we have for all $m \geq 0$, for all constants $\kappa \in \mathbb{R}^d$ and $c \in \mathbb{R}$,

$$\begin{aligned} &\|\nabla\psi\|_{H^{m+1}((I_n+\frac12\rho B)\setminus I_n)} + \|\Sigma - c\,\mathrm{Id}\|_{H^{m+1}((I_n+\frac12\rho B)\setminus I_n)} \\ &\quad\lesssim \|\psi|_{I_n} - \kappa\|_{H^{m+\frac32}(\partial I_n)} + \|\psi - \kappa\|_{H^1((I_n+\rho B)\setminus I_n)} + \|\Sigma - c\,\mathrm{Id}\|_{\mathrm{L}^2((I_n+\rho B)\setminus I_n)}. \end{aligned}$$

Choosing $c := \fint_{(I_n+\rho B)\setminus I_n} \Sigma$ and using a local pressure estimate for the steady Stokes equation, e.g. [19, Lemma 3.3], we find

$$\|\Sigma - c\,\mathrm{Id}\|_{\mathrm{L}^2((I_n+\rho B)\setminus I_n)} \lesssim \|\nabla\psi\|_{\mathrm{L}^2((I_n+\rho B)\setminus I_n)},$$

so that the above reduces to

$$\begin{aligned} &\|\nabla\psi\|_{H^{m+1}((I_n+\frac12\rho B)\setminus I_n)} + \|\Sigma - c\,\mathrm{Id}\|_{H^{m+1}((I_n+\frac12\rho B)\setminus I_n)} \\ &\quad\lesssim \|\psi|_{I_n} - \kappa\|_{H^{m+\frac32}(\partial I_n)} + \|\psi - \kappa\|_{H^1((I_n+\rho B)\setminus I_n)}. \end{aligned}$$

As ψ is affine in I_n, we have

$$\|\psi|_{I_n} - \kappa\|_{H^{m+\frac{3}{2}}(\partial I_n)} \lesssim \|\psi - \kappa\|_{H^{m+2}(I_n)} = \|\psi - \kappa\|_{H^1(I_n)},$$

and the above then becomes

$$\|\nabla\psi\|_{H^{m+1}((I_n+\frac{1}{2}\rho B)\setminus I_n)} + \|\Sigma - c\,\mathrm{Id}\|_{H^{m+1}((I_n+\frac{1}{2}\rho B)\setminus I_n)} \lesssim \|\psi - \kappa\|_{H^1(I_n+\rho B)}.$$

Further, choosing $\kappa := \fint_{I_n+\rho B} \psi$ and applying Poincaré's inequality, we deduce

$$\|\nabla\psi\|_{H^{m+1}((I_n+\frac{1}{2}\rho B)\setminus I_n)} + \|\Sigma - c\,\mathrm{Id}\|_{H^{m+1}((I_n+\frac{1}{2}\rho B)\setminus I_n)} \lesssim \|\nabla\psi\|_{\mathrm{L}^2(I_n+\rho B)}.$$

In particular, combined with (2.21), this leads us to

$$\inf_{c\in\mathbb{R}} \int_{\partial I_n} \big|\sigma(\psi,\Sigma) - c\,\mathrm{Id}\big|^2 \lesssim \int_{I_n+\rho B} |\nabla\psi|^2.$$

Noting that $\sigma(\psi,\Sigma)$ and the equations satisfied by (ψ,Σ) are unchanged if a rigid motion is added to ψ, the conclusion now follows from Korn's inequality. ∎

Next, we recall the following standard elliptic regularity estimate for solutions of the free steady Stokes equation.

Lemma 2.6 (Mean-value property). *Given $r > 0$, if (ψ,Σ) is a weak solution of the free Stokes equation in B_r,*

$$-\triangle\psi + \nabla\Sigma = 0, \quad \mathrm{div}(\psi) = 0, \quad \text{in } B_r,$$

then it satisfies

$$\big|\mathrm{D}(\psi)(0)\big|^2 \lesssim \fint_{B_r} \big|\mathrm{D}(\psi)\big|^2.$$

Proof. By scaling, it suffices to consider $r = 1$. For $m > \frac{d}{2}$, the Sobolev embedding yields

$$\big|\mathrm{D}(\psi)(0)\big| \lesssim \big\|\mathrm{D}(\psi)\big\|_{H^m(\frac{1}{2}B)}, \tag{2.22}$$

and it remains to estimate this Sobolev norm. By the local regularity theory for the steady Stokes equation, e.g. [25, Theorem IV.4.1], we find for all $\kappa \in \mathbb{R}^d$ and $c \in \mathbb{R}$,

$$\|\nabla\psi\|_{H^m(\frac{1}{2}B)} + \|\Sigma - c\|_{H^m(\frac{1}{2}B)} \lesssim \|\psi - \kappa\|_{H^1(B)} + \|\Sigma - c\|_{\mathrm{L}^2(B)}.$$

Choosing $c = \fint_B \Sigma$ and using a local pressure estimate for the steady Stokes equation, e.g. [19, Lemma 3.3], we find

$$\|\Sigma - c\|_{\mathrm{L}^2(B)} \lesssim \|\nabla\psi\|_{\mathrm{L}^2(B)}.$$

Inserting this into the above and applying Poincaré's inequality for the choice $\kappa = \fint_B \psi$, we deduce

$$\|\nabla\psi\|_{H^m(\frac{1}{2}B)} \lesssim \|\nabla\psi\|_{L^2(B)}.$$

For any $\Theta \in \mathbb{M}^{\mathrm{skew}}$, this entails

$$\big\|D(\psi)\big\|_{H^m(\frac{1}{2}B)} \le \big\|\nabla(\psi - \Theta x)\big\|_{H^m(\frac{1}{2}B)} \lesssim \big\|\nabla(\psi - \Theta x)\big\|_{L^2(B)},$$

hence, by Korn's inequality,

$$\big\|D(\psi)\big\|_{H^m(\frac{1}{2}B)} \lesssim \big\|D(\psi)\big\|_{L^2(B)}.$$

Inserting this into (2.22), the conclusion follows. ∎

Finally, the following lemma provides a useful property of Voronoi tessellations. Although it could be obtained as a direct consequence of Palm theory, we include a more elementary proof by means of an approximation argument.

Lemma 2.7 (Property of Voronoi tessellations). *Under Assumption* (H_ρ)*, for all stationary random fields* ζ *with* $\mathbb{E}[|\zeta|] < \infty$*, we have*

$$\mathbb{E}[\zeta] = \mathbb{E}\bigg[\sum_n \frac{\mathbb{1}_{0\in I_n}}{|I_n|}\int_{V_n}\zeta\bigg]. \tag{2.23}$$

In case of ($H^{\mathrm{perc}}_{\rho,\kappa}$)*, we can alternatively decompose*

$$\mathbb{E}[\zeta] = \mathbb{E}\bigg[\sum_{n\in\mathcal{S}} \frac{\mathbb{1}_{0\in I_n}}{|I_n|}\int_{V_n}\zeta + \sum_{q\in\mathcal{S}'} \frac{\mathbb{1}_{0\in J_{q,\rho}}}{|J_{q,\rho}|}\int_{W_q}\zeta\bigg]. \tag{2.24}$$

Remark 2.8. In [14], we refer to Lemma 2.7 above[3] in the following slightly different form: given a general stationary ergodic point process $\widetilde{\mathcal{P}} = \{\tilde{x}_n\}_n$, considering spherical inclusions $\widetilde{I}_n = B(\tilde{x}_n)$, denoting by $\{\widetilde{\rho}_n\}_n$ the associated interparticle distances, by $\{\widetilde{V}_n\}_n$ the associated Voronoi cells, and defining

$$\widetilde{\mathcal{S}} := \{n : \widetilde{\rho}_n \ge 1\}$$

(which can possibly be empty), we have for all stationary random fields ζ with $\mathbb{E}[|\zeta|] < \infty$,

$$\mathbb{E}[\zeta] = \mathbb{E}\bigg[\sum_{n\in\widetilde{\mathcal{S}}} \frac{\mathbb{1}_{0\in B(\tilde{x}_n)}}{|B|}\int_{\widetilde{V}_n}\zeta\bigg] + \mathbb{E}\big[\zeta\mathbb{1}_{\mathbb{R}^d\setminus\bigcup_{n\in\widetilde{\mathcal{S}}}\widetilde{V}_n}\big]. \tag{2.25}$$

Indeed, given that the restricted inclusion process $\{B(\tilde{x}_n) : n \in \widetilde{\mathcal{S}}\}$ satisfies (H_ρ),

[3] Though with the erroneous reference "Lemma 2.5".

denoting the associated Voronoi cells by $\{\widetilde{V}'_n : n \in \widetilde{\mathcal{S}}\}$, we can apply (2.23) in the form

$$\begin{aligned}\mathbb{E}[\zeta] &= \mathbb{E}\big[\zeta \mathbb{1}_{\bigcup_{m\in\widetilde{\mathcal{S}}} \widetilde{V}_m}\big] + \mathbb{E}\big[\zeta \mathbb{1}_{\mathbb{R}^d \setminus \bigcup_{m\in\widetilde{\mathcal{S}}} \widetilde{V}_m}\big] \\ &= \mathbb{E}\bigg[\sum_{n\in\widetilde{\mathcal{S}}} \frac{\mathbb{1}_{0\in B(\tilde{x}_n)}}{|B|} \int_{\widetilde{V}'_n} \zeta \mathbb{1}_{\bigcup_{m\in\widetilde{\mathcal{S}}} \widetilde{V}_m}\bigg] + \mathbb{E}\big[\zeta \mathbb{1}_{\mathbb{R}^d \setminus \bigcup_{m\in\widetilde{\mathcal{S}}} \widetilde{V}_m}\big],\end{aligned}$$

and identity (2.25) immediately follows since the trivial inclusion

$$\{\tilde{x}_n : n \in \widetilde{\mathcal{S}}\} \subset \widetilde{\mathcal{P}}$$

implies $\widetilde{V}_n \subset \widetilde{V}'_n$ for all n.

Proof. By the monotone convergence theorem, it is enough to prove the result for any bounded nonnegative random field $0 \le \zeta \le M$ with any fixed $M > 0$. Let such a ζ be fixed. We split the proof into two steps.

Step 1. Proof of (2.23) and (2.24) under the additional assumption that almost surely

$$\sup_n \operatorname{diam}(V_n) < \infty. \tag{2.26}$$

In that case, let $K \ge 1$ be such that $\operatorname{diam}(V_n) \le K$ almost surely for all n. We consider (2.23) and (2.24) separately, and split the proof into two further substeps.

Substep 1.1. Proof of (2.23) under assumption (2.26). By the ergodic theorem, we have almost surely

$$\mathbb{E}\bigg[\sum_n \frac{\mathbb{1}_{0\in I_n}}{|I_n|} \int_{V_n} \zeta\bigg] = \lim_{R\uparrow\infty} R^{-d} \sum_n \frac{|I_n \cap Q_R|}{|I_n|} \int_{V_n} \zeta.$$

As $\zeta \ge 0$ and as assumption (2.26) entails $V_n \subset B_K(x_n)$ for all n, we easily get the two-sided estimate

$$\int_{Q_{R-CK}} \zeta \le \sum_n \frac{|I_n \cap Q_R|}{|I_n|} \int_{V_n} \zeta \le \int_{Q_{R+CK}} \zeta,$$

and the claim (2.23) then follows from the ergodic theorem.

Substep 1.2. Proof of (2.24) under assumption (2.26). By the ergodic theorem, we have almost surely

$$\begin{aligned}&\mathbb{E}\bigg[\sum_{n\in\mathcal{S}} \frac{\mathbb{1}_{0\in I_n}}{|I_n|} \int_{V_n} \zeta + \sum_{q\in\mathcal{S}'} \frac{\mathbb{1}_{0\in J_{q,\rho}}}{|J_{q,\rho}|} \int_{W_q} \zeta\bigg] \\ &\quad = \lim_{R\uparrow\infty} R^{-d} \bigg(\sum_{n\in\mathcal{S}} \frac{|I_n \cap Q_R|}{|I_n|} \int_{V_n} \zeta + \sum_{q\in\mathcal{S}'} \frac{|J_{q,\rho} \cap Q_R|}{|J_{q,\rho}|} \int_{W_q} \zeta\bigg).\end{aligned}$$

As $0 \le \zeta \le M$ and as assumption (2.26) entails $V_n \subset B_K(x_n)$ for all n, we get the

two-sided estimate

$$\begin{aligned}\int_{Q_{R-CK}} \zeta - M \sum_{q: J_{q,\rho} \cap (Q_{R+1} \setminus Q_R) \neq \varnothing} |W_q| \\ \leq \sum_{n \in \mathcal{S}} \frac{|I_n \cap Q_R|}{|I_n|} \int_{V_n} \zeta + \sum_{q \in \mathcal{S}'} \frac{|J_{q,\rho} \cap Q_R|}{|J_{q,\rho}|} \int_{W_q} \zeta \\ \leq \int_{Q_{R+CK}} \zeta + M \sum_{q: J_{q,\rho} \cap (Q_{R+1} \setminus Q_R) \neq \varnothing} |W_q|.\end{aligned}$$

By the ergodic theorem, in order to prove (2.24), it thus remains to show almost surely

$$\lim_{R \uparrow \infty} R^{-d} \sum_{q: J_{q,\rho} \cap (Q_{R+1} \setminus Q_R) \neq \varnothing} |W_q| = 0,$$

which would follow provided that we show almost surely

$$\limsup_{R \uparrow \infty} R^{-d} \sum_{q: J_{q,\rho} \cap Q_R \neq \varnothing} |W_q| < \infty. \tag{2.27}$$

As $|W_q| \lesssim (\operatorname{diam}(J_{q,\rho}) + K)^d$, we can estimate

$$\begin{aligned}R^{-d} \sum_{q: J_{q,\rho} \cap Q_R \neq \varnothing} |W_q| \lesssim K^d \big(R^{-d} \sharp\{q : J_{q,\rho} \cap Q_R \neq \varnothing\} \big) \\ \times \bigg(\frac{1}{\sharp\{q : J_{q,\rho} \cap Q_R \neq \varnothing\}} \sum_{q: J_{q,\rho} \cap Q_R \neq \varnothing} \operatorname{diam}(J_{q,\rho})^d \bigg).\end{aligned}$$

To bound the first factor, we simply note that

$$\begin{aligned}R^{-d} \sharp\{q : J_{q,\rho} \cap Q_R \neq \varnothing\} \leq R^{-d} \sharp\{n : I_n \cap Q_R \neq \varnothing\} \\ \leq R^{-d} \sharp\{n : x_n \in Q_{R+1}\} \xrightarrow{R \uparrow \infty} \lambda(\mathcal{P}).\end{aligned}$$

Appealing to the condition (2.3) in $(\mathrm{H}^{\mathrm{perc}}_{\rho,\kappa})$ to estimate the second factor, the claim (2.27) follows.

Step 2. Relaxing assumption (2.26). It remains to consider the case when

$$\sup_n \operatorname{diam}(V_n) = \infty,$$

and we proceed by approximation. Consider a point process

$$\mathcal{P}' = \{x'_n\}_n$$

independent of $\mathcal{P}$ and of $\mathcal{I}$ such that almost surely

$$\min_{n\neq m} |x'_n - x'_m| \geq \frac{1}{2}, \qquad \min_{m:m\neq n} |x'_n - x'_m| \leq 1 \quad \text{for all } n.$$

For instance, $\mathcal{P}'$ can be chosen as the random parking process of parameter $\frac{1}{4}$, cf. [57]. Now, for any integer $\alpha \geq 1$, we define the "enriched" point process $\mathcal{P}_\alpha$ as follows,

$$\mathcal{P}_\alpha := \mathcal{P} \cup \big\{2^{2\alpha} x'_n : \operatorname{dist}(2^{2\alpha} x'_n, \mathcal{P}) \geq 2^{2\alpha+3}\big\},$$

as well as the corresponding random set

$$\mathcal{I}_\alpha := \mathcal{I} \cup \bigcup_{n:\operatorname{dist}(2^{2\alpha}x'_n,\mathcal{P})\geq 2^{2\alpha+3}} B(2^{2\alpha} x'_n).$$

Denote by $V_\alpha(x_n)$ the Voronoi cell associated with I_n in $\mathcal{I}_\alpha$, and by $V_\alpha(x'_n)$ the Voronoi cell associated with $B(2^{2\alpha}x'_n)$. By construction, it can be checked that for all n,

$$V_n \cap B_{2^{2\alpha+2}}(x_n) \subset V_\alpha(x_n) \subset V_n \cap B_{2^{2\alpha+4}}(x_n),$$

which entails that $V_\alpha(x_n) \uparrow V_n$ increasingly as $\alpha \uparrow \infty$ (over integers). In addition, $\mathcal{P}_\alpha$, $\mathcal{I}_\alpha$ satisfy (H_ρ), as well as (2.26) with Voronoi diameters bounded by $O(2^{2\alpha})$. They further satisfy ($\mathrm{H}^{\mathrm{unif}}_\rho$), ($\mathrm{H}^{\mathrm{mom}}_{\rho,\kappa}$), or ($\mathrm{H}^{\mathrm{perc}}_{\rho,\kappa}$) provided that $\mathcal{P}$, $\mathcal{I}$ satisfy the corresponding assumption. We focus on the case of ($\mathrm{H}^{\mathrm{unif}}_\rho$) or ($\mathrm{H}^{\mathrm{mom}}_{\rho,\kappa}$), while the case of ($\mathrm{H}^{\mathrm{perc}}_{\rho,\kappa}$) is analogous. Applying the result (2.23) of Step 1, by definition of $\mathcal{P}_\alpha$, we get

$$\begin{aligned}\mathbb{E}[\zeta] = \mathbb{E}\bigg[\sum_n \frac{\mathbb{1}_{0\in I_n}}{|I_n|}\int_{V_\alpha(x_n)} \zeta\bigg] \\ + \mathbb{E}\bigg[\sum_n \mathbb{1}_{\operatorname{dist}(2^{2\alpha}x'_n,\mathcal{P})\geq 2^{2\alpha+3}} \frac{\mathbb{1}_{0\in B(2^{2\alpha}x'_n)}}{|B|}\int_{V_\alpha(x'_n)} \zeta\bigg].\end{aligned}$$

As $0 \leq \zeta \leq M$, as Voronoi diameters are bounded by $C2^{2\alpha}$ almost surely, and using stationarity and the independence of $\mathcal{P}$, $\mathcal{I}$ and $\mathcal{P}'$, $\mathcal{I}'$, the second right-hand side term satisfies

$$\begin{aligned}0 &\leq \mathbb{E}\bigg[\sum_n \mathbb{1}_{\operatorname{dist}(2^{2\alpha}x'_n,\mathcal{P})\geq 2^{2\alpha+3}} \frac{\mathbb{1}_{0\in B(2^{2\alpha}x'_n)}}{|B|}\int_{V_\alpha(x'_n)} \zeta\bigg] \\ &\lesssim M 2^{2\alpha d}\,\mathbb{E}\bigg[\sum_n \mathbb{1}_{\operatorname{dist}(2^{2\alpha}x'_n,\mathcal{P})\geq 2^{2\alpha+3}} \frac{\mathbb{1}_{0\in B(2^{2\alpha}x'_n)}}{|B|}\bigg] \\ &= M 2^{2\alpha d}\,\mathbb{E}\bigg[\sum_n \frac{\mathbb{1}_{0\in B(2^{2\alpha}x'_n)}}{|B|}\bigg]\mathbb{P}\big[\operatorname{dist}(0,\mathcal{P}) \geq 2^{2\alpha+3}\big] \\ &= M\lambda(\mathcal{P}')\,\mathbb{P}\big[\operatorname{dist}(0,\mathcal{P}) \geq 2^{2\alpha+3}\big].\end{aligned}$$

Inserting this into the above, we deduce

$$\mathbb{E}\Big[\sum_n \frac{\mathbb{1}_{0\in I_n}}{|I_n|}\int_{V_\alpha(x_n)}\zeta\Big] \le \mathbb{E}[\zeta] \le \mathbb{E}\Big[\sum_n \frac{\mathbb{1}_{0\in I_n}}{|I_n|}\int_{V_\alpha(x_n)}\zeta\Big] + CM\lambda(\mathcal{P}')\,\mathbb{P}\big[\operatorname{dist}(0,\mathcal{P})\ge 2^{2\alpha+3}\big].$$

and the conclusion (2.23) follows from the monotone convergence theorem. ∎

2.4 Proof of Lemma 2.3

Without loss of generality, we can assume that $\mathcal{E}_{n,D}<\infty$ almost surely as otherwise the claimed estimate (2.13) would be trivial. The variational definition of the effective viscosity (1.24) can be written as

$$\begin{aligned} &E:\bar{\mathbf{B}}E\\ &=|E|^2+\inf\big\{\mathbb{E}\big[\big|\mathrm{D}(\psi)\big|^2\big]:\psi\in \mathrm{L}^2\big(\Omega;H^1_{\mathrm{loc}}(\mathbb{R}^d)^d\big),\ \nabla\psi \text{ stationary},\\ &\qquad \operatorname{div}(\psi)=0,\ \big(\mathrm{D}(\psi)+E\big)\big|_{\mathcal{I}}=0,\ \mathbb{E}\big[\mathrm{D}(\psi)\big]=0\big\}, \end{aligned} \tag{2.28}$$

and the definition (2.5) of $\bar{\mathbf{B}}^1$ as

$$E:\bar{\mathbf{B}}^1E=\mathbb{E}\Big[\sum_n \frac{\mathbb{1}_{0\in I_n}}{|I_n|}\mathcal{E}_{n,\infty}\Big]. \tag{2.29}$$

Note that an energy estimate for (2.11) using Bogovskii's construction yields the uniform bound $\mathcal{E}_{n,\infty}\lesssim|E|^2$. In order to prove (2.13), it remains to compare (2.28) to a superposition of the single-particle problems $\{\mathcal{E}_{n,\infty}\}_n$ and to recognize (2.29). We split the proof into three steps.

Step 1. Upper bound: proof that we have in case of $(\mathrm{H}^{\mathrm{unif}}_{\rho})$ or $(\mathrm{H}^{\mathrm{mom}}_{\rho,\kappa})$,

$$E:\bar{\mathbf{B}}E\le|E|^2+\mathbb{E}\Big[\sum_n \frac{\mathbb{1}_{0\in I_n}}{|I_n|}\mathcal{E}_{n,D}\Big], \tag{2.30}$$

and in case of $(\mathrm{H}^{\mathrm{perc}}_{\rho,\kappa})$,

$$E:\bar{\mathbf{B}}E\le|E|^2+\mathbb{E}\Big[\sum_{n\in\mathcal{S}} \frac{\mathbb{1}_{0\in I_n}}{|I_n|}\mathcal{E}_{n,D}+\sum_{q\in\mathcal{S}'} \frac{\mathbb{1}_{0\in J_{q,\rho}}}{|J_{q,\rho}|}\mathcal{F}_{q,D}\Big]. \tag{2.31}$$

We focus on (2.31), the proof of (2.30) being identical. We define almost surely

$$\psi_D:=\sum_{n\in\mathcal{S}}\psi_{n,D}+\sum_{q\in\mathcal{S}'}\psi_{q,D}\quad\in H^1_{\mathrm{loc}}(\mathbb{R}^d)^d,$$

where the summands $\psi_{n,D} \in H^1_0(V_n)^d$ and $\psi_{q,D} \in H^1_0(W_q)^d$ are implicitly extended by zero outside V_n and W_q, respectively. Properties of Dirichlet minimizers $\{\psi_{n,D}\}_n$ and $\{\psi_{q,D}\}_q$ ensure that $\nabla\psi_D$ is stationary and that it satisfies $\operatorname{div}(\psi_D) = 0$ and $(\mathrm{D}(\psi_D) + E)|_{\mathcal{I}} = 0$. Assume that $\mathrm{D}(\psi_D) \in \mathrm{L}^2(\Omega)^{d\times d}$ (for otherwise the claim is trivial by (2.24)). Then, appealing to (2.24), we find $\mathbb{E}[\mathrm{D}(\psi_D)] = 0$. We may then use ψ_D as a test function in the variational problem (2.28), to the effect of

$$E : \bar{\mathbf{B}}E \le |E|^2 + \mathbb{E}\big[\big|\mathrm{D}(\psi_D)\big|^2\big],$$

and the claim (2.31) now follows from (2.24).

Step 2. Lower bound: proof that

$$E : \bar{\mathbf{B}}E \ge |E|^2 + \mathbb{E}\bigg[\sum_n \frac{\mathbb{1}_{0\in I_n}}{|I_n|}\mathcal{E}_{n,N}\bigg]. \tag{2.32}$$

By (2.23), we can write

$$\begin{aligned} E : \bar{\mathbf{B}}E &= |E|^2 + \mathbb{E}\big[\big|\mathrm{D}(\psi_E)\big|^2\big] \\ &= |E|^2 + \mathbb{E}\bigg[\sum_n \frac{\mathbb{1}_{0\in I_n}}{|I_n|}\int_{V_n}\big|\mathrm{D}(\psi_E)\big|^2\bigg]. \end{aligned}$$

Using the corrector ψ_E as a test function for the Neumann single-particle problem (2.10), we find $\mathcal{E}_{n,N} \le \int_{V_n} |\mathrm{D}(\psi_E)|^2$ and the claim (2.32) follows.

Step 3. Conclusion. In view of (2.30) and (2.32), it remains to compare $\mathcal{E}_{n,D}$ and $\mathcal{E}_{n,N}$ to $\mathcal{E}_{n,\infty}$. On the one hand, since $\psi_{n,D}$ is an admissible test function for $\mathcal{E}_{n,\infty}$, we have

$$\mathcal{E}_{n,\infty} \le \int_{\mathbb{R}^d}\big|\mathrm{D}(\psi_{n,D})\big|^2 = \int_{V_n}\big|\mathrm{D}(\psi_{n,D})\big|^2 = \mathcal{E}_{n,D}.$$

On the other hand, since the restriction $\psi_{n,\infty}|_{V_n}$ is an admissible test function for $\mathcal{E}_{n,N}$, we have

$$\mathcal{E}_{n,N} \le \int_{V_n}\big|\mathrm{D}(\psi_{n,\infty})\big|^2 \le \mathcal{E}_{n,\infty}.$$

This yields

$$\mathcal{E}_{n,N} \le \mathcal{E}_{n,\infty} \le \mathcal{E}_{n,D},$$

or alternatively,

$$|\mathcal{E}_{n,N} - \mathcal{E}_{n,\infty}| + |\mathcal{E}_{n,D} - \mathcal{E}_{n,\infty}| = \mathcal{E}_{n,D} - \mathcal{E}_{n,N}.$$

Further, note that the minimality of Neumann problems entails

$$\sum_{n:I_n\subset J_{q,\rho}} \mathcal{E}_{n,N} \le \mathcal{F}_{q,D},$$

and thus

$$\mathbb{E}\Big[\sum_{n\notin\mathcal{S}}\frac{\mathbb{1}_{0\in I_n}}{|I_n|}\mathcal{E}_{n,N}\Big]=\mathbb{E}\Big[\sum_{q\in\mathcal{S}'}\frac{\mathbb{1}_{0\in J_{q,\rho}}}{|J_{q,\rho}|}\sum_{n:I_n\subset J_{q,\rho}}\mathcal{E}_{n,N}\Big]$$
$$\le\mathbb{E}\Big[\sum_{q\in\mathcal{S}'}\frac{\mathbb{1}_{0\in J_{q,\rho}}}{|J_{q,\rho}|}\mathcal{F}_{q,D}\Big].$$

Combining these observations with (2.30) and (2.32), the conclusion (2.13) follows.

2.5 Proof of Lemma 2.4

The bound (2.15) on $\mathcal{F}_{q,D}$ follows from Bogovskii's construction in form of [21, Lemma 4.2]. We turn to the proof of (2.14). By [13, Section 4.1], there exists $w_n\in W_0^{1,\infty}(V_n)^d$ that is an admissible test function for the Dirichlet problem $\mathcal{E}_{n,D}$ such that

$$\mathcal{E}_{n,D}\le\int_{V_n}\big|\mathrm{D}(w_n)\big|^2\lesssim\mu(\rho_n),$$

which entails

$$0\le\mathcal{E}_{n,D}-\mathcal{E}_{n,N}\le\mathcal{E}_{n,D}\lesssim\mu(\rho_n).$$

To prove (2.14), it remains to show that in the case $\rho_n\ge1$ we have

$$\mathcal{E}_{n,D}-\mathcal{E}_{n,N}\lesssim\rho_n^{-d}.\tag{2.33}$$

This amounts to investigating the role of the different boundary conditions on ∂V_n. We assume from now on that $\rho_n\ge1$ and, without loss of generality, $x_n=0$. We drop the index n to simplify notation and we set $r=\rho_n$ (to avoid confusion with the constant ρ in Assumption (H_ρ) and elsewhere). We split the argument into three steps.

Step 1. Proof that

$$\mathcal{E}_D-\mathcal{E}_N=\int_V\big|\mathrm{D}(\psi_D-\psi_N)\big|^2.\tag{2.34}$$

By the Euler–Lagrange equation for ψ_N in form of

$$\int_V\mathrm{D}(\psi_N):\mathrm{D}(\psi_D-\psi_N)=0,$$

we find

$$\int_V\big|\mathrm{D}(\psi_D)\big|^2-\int_V\big|\mathrm{D}(\psi_N)\big|^2=\int_V\mathrm{D}(\psi_D+\psi_N):\mathrm{D}(\psi_D-\psi_N)$$
$$=\int_V\big|\mathrm{D}(\psi_D-\psi_N)\big|^2,$$

that is, (2.34).

Step 2. Proof that

$$\mathcal{E}_D - \mathcal{E}_N \lesssim \left(\int_{I+\frac{1}{2}B} |\mathrm{D}(\psi_D - \psi_N)|^2 \right)^{\frac{1}{2}}. \tag{2.35}$$

As in (1.5), the Euler–Lagrange equation for ψ_D takes the following form, in terms of the associated pressure field Σ_D,

$$\begin{cases} -\triangle \psi_D + \nabla \Sigma_D = 0, & \text{in } V \setminus I, \\ \mathrm{div}(\psi_D) = 0, & \text{in } D \setminus I, \\ \psi_D = 0, & \text{on } \partial V, \\ \mathrm{D}(\psi_D + Ex) = 0, & \text{in } I, \\ \int_{\partial I} \sigma(\psi_D + Ex, \Sigma_D)\nu = 0, & \\ \int_{\partial I} \Theta x \cdot \sigma(\psi_D + Ex, \Sigma_D)\nu = 0, & \forall \Theta \in \mathbb{M}^{\mathrm{skew}}, \end{cases} \tag{2.36}$$

and similarly the equation for ψ_N is as follows, in terms of the associated pressure Σ_N,

$$\begin{cases} -\triangle \psi_N + \nabla \Sigma_N = 0, & \text{in } V \setminus I, \\ \mathrm{div}(\psi_N) = 0, & \text{in } V \setminus I, \\ \sigma(\psi_N, \Sigma_N)\nu = 0, & \text{on } \partial V, \\ \mathrm{D}(\psi_N + Ex) = 0, & \text{in } I, \\ \int_{\partial I} \sigma(\psi_N + Ex, \Sigma_N)\nu = 0, & \\ \int_{\partial I} \Theta x \cdot \sigma(\psi_N + Ex, \Sigma_N)\nu = 0, & \forall \Theta \in \mathbb{M}^{\mathrm{skew}}. \end{cases} \tag{2.37}$$

Testing the above equations for ψ_D and ψ_N, and using boundary conditions, we find the following energy identities,

$$\begin{aligned} 2\int_{V\setminus I} \left|\mathrm{D}(\psi_D)\right|^2 &= \int_{V\setminus I} \mathrm{div}\left(\sigma(\psi_D, \Sigma_D)\psi_D\right) = -\int_{\partial I} \psi_D \cdot \sigma(\psi_D, \Sigma_D)\nu, \\ 2\int_{V\setminus I} \left|\mathrm{D}(\psi_N)\right|^2 &= \int_{V\setminus I} \mathrm{div}\left(\sigma(\psi_N, \Sigma_N)\psi_N\right) = -\int_{\partial I} \psi_N \cdot \sigma(\psi_N, \Sigma_N)\nu, \end{aligned}$$

and thus, using boundary conditions for ψ_D, ψ_N on ∂I,

$$\begin{aligned} 2\int_{V\setminus I} \left|\mathrm{D}(\psi_D)\right|^2 &= \int_{\partial I} Ex \cdot \sigma(\psi_D, \Sigma_D)\nu, \\ 2\int_{V\setminus I} \left|\mathrm{D}(\psi_N)\right|^2 &= \int_{\partial I} Ex \cdot \sigma(\psi_N, \Sigma_N)\nu. \end{aligned}$$

Taking the difference and noticing $|\mathrm{D}(\psi_D)|^2 - |\mathrm{D}(\psi_N)|^2 = 0$ in I, we get

$$\mathcal{E}_D - \mathcal{E}_N = \frac{1}{2}\int_{\partial I} Ex \cdot \sigma(\psi_D - \psi_N, \Sigma_D - \Sigma_N)\nu.$$

Noting that the trace-free condition for E yields $\int_{\partial I} Ex \cdot \nu = 0$, we can add any constant to the pressure field $\Sigma_D - \Sigma_N$ in the right-hand side, and the claim (2.35) then follows from the trace estimate in Lemma 2.5 (ii).

Step 3. Mean-value property with rigid inclusion:

$$\int_{I+\frac{1}{2}B} \left|\mathrm{D}(\psi_D - \psi_N)\right|^2 \lesssim r^{-d} \int_V \left|\mathrm{D}(\psi_D - \psi_N)\right|^2. \tag{2.38}$$

If the difference $\psi := \psi_D - \psi_N$ satisfied the free steady Stokes equation in the whole domain V, then Lemma 2.6 would already yield

$$\int_{I+\frac{1}{2}B} \left|\mathrm{D}(\psi)\right|^2 \lesssim r^{-d} \int_V \left|\mathrm{D}(\psi)\right|^2,$$

that is indeed the claim (2.38). However, ψ is rigid in I and does not satisfy the free steady Stokes equation in the whole domain.[4] To overcome this issue, we shall compare ψ to a suitable proxy: we consider the solution $(\widetilde{\psi}, \widetilde{\Sigma})$ of the following auxiliary Dirichlet problem in V,

$$\begin{cases} -\triangle \widetilde{\psi} + \nabla \widetilde{\Sigma} = 0, & \text{in } V, \\ \mathrm{div}(\widetilde{\psi}) = 0, & \text{in } V, \\ \widetilde{\psi} = \psi, & \text{on } \partial V. \end{cases}$$

Testing this latter equation with $\widetilde{\psi}$ or with ψ, we find

$$2\int_V |\mathrm{D}(\widetilde{\psi})|^2 = \int_{\partial V} \widetilde{\psi} \cdot \widetilde{\sigma}\nu = \int_{\partial V} \psi \cdot \widetilde{\sigma}\nu = 2\int_V \mathrm{D}(\psi) : \mathrm{D}(\widetilde{\psi}),$$

and thus, by the Cauchy–Schwarz inequality,

$$\int_V \left|\mathrm{D}(\widetilde{\psi})\right|^2 \leq \int_V \left|\mathrm{D}(\psi)\right|^2. \tag{2.39}$$

Next, for the approximation error, we similarly compute

$$2\int_V \left|\mathrm{D}(\widetilde{\psi} - \psi)\right|^2 = -2\int_V \mathrm{D}(\widetilde{\psi} - \psi) : \mathrm{D}(\psi),$$

and thus, testing the equations (2.36) and (2.37) for $\psi = \psi_D - \psi_N$ with $\widetilde{\psi}$, and using boundary conditions,

$$2\int_V \left|\mathrm{D}(\widetilde{\psi} - \psi)\right|^2 = \int_{\partial I} (\widetilde{\psi} - \psi) \cdot \sigma(\psi, \Sigma)\nu = \int_{\partial I} \widetilde{\psi} \cdot \sigma(\psi, \Sigma)\nu.$$

[4] As shown in Lemma A.2 in Appendix A, the mean-value property can actually be extended in presence of rigid particles. Rather than appealing to this general result here, we provide a self-contained and more elementary approach in the present single-particle setting.

As the boundary conditions for ψ allow to add any rigid motion to $\widetilde{\psi}$ in the right-hand side, and as the incompressibility of $\widetilde{\psi}$ in form of $\int_{\partial I} \widetilde{\psi} \cdot \nu = 0$ allows to add any constant to the pressure field Σ, the trace estimates in Lemma 2.5 lead us to

$$\int_V \left|\mathrm{D}(\widetilde{\psi} - \psi)\right|^2 \lesssim \left(\int_I \left|\mathrm{D}(\widetilde{\psi})\right|^2 \right)^{\frac{1}{2}} \left(\int_{I+\frac{1}{2}B} \left|\mathrm{D}(\psi)\right|^2 \right)^{\frac{1}{2}},$$

and thus, decomposing $\psi = \widetilde{\psi} - (\widetilde{\psi} - \psi)$ in the last factor,

$$\int_V \left|\mathrm{D}(\widetilde{\psi} - \psi)\right|^2 \lesssim \int_{I+\frac{1}{2}B} \left|\mathrm{D}(\widetilde{\psi})\right|^2.$$

We then deduce

$$\int_{I+\frac{1}{2}B} \left|\mathrm{D}(\psi)\right|^2 \lesssim \int_{I+\frac{1}{2}B} \left|\mathrm{D}(\widetilde{\psi})\right|^2 + \int_V \left|\mathrm{D}(\widetilde{\psi} - \psi)\right|^2 \lesssim \int_{I+\frac{1}{2}B} \left|\mathrm{D}(\widetilde{\psi})\right|^2.$$

As $\widetilde{\psi}$ satisfies the free steady Stokes equation in V and as we have $|I + \frac{1}{2}B| \lesssim 1$ and $\mathrm{dist}(I + \frac{1}{2}B, \partial V) \geq \frac{r}{2}$, we may now appeal to Lemma 2.6, to the effect of

$$\int_{I+\frac{1}{2}B} \left|\mathrm{D}(\widetilde{\psi})\right|^2 \lesssim r^{-d} \int_V \left|\mathrm{D}(\widetilde{\psi})\right|^2.$$

The above then becomes

$$\int_{I+\frac{1}{2}B} \left|\mathrm{D}(\psi)\right|^2 \lesssim r^{-d} \int_V \left|\mathrm{D}(\widetilde{\psi})\right|^2,$$

and the claim (2.38) for $\psi = \psi_D - \psi_N$ follows in combination with (2.39).

Step 4. Conclusion. Combining (2.35) and (2.38), we get

$$\mathcal{E}_D - \mathcal{E}_N \lesssim r^{-\frac{d}{2}} \left(\int_V \left|\mathrm{D}(\psi_D - \psi_N)\right|^2 \right)^{\frac{1}{2}},$$

and thus, by (2.34),

$$\mathcal{E}_D - \mathcal{E}_N \lesssim r^{-\frac{d}{2}} (\mathcal{E}_D - \mathcal{E}_N)^{\frac{1}{2}},$$

which precisely proves the conclusion (2.33). ∎

2.6 Explicit form of Einstein's formula

This section is devoted to the proof of Proposition 2.2. Under Assumption (Indep), the definition (2.5) of $\overline{\mathbf{B}}^1$ becomes

$$E : \overline{\mathbf{B}}^1 E = \lambda(\mathcal{P}) \mathbb{E}\left[\int_{\mathbb{R}^d} \left|\mathrm{D}(\psi_E^\circ)\right|^2 \right],$$

that is (2.6), in terms of the unique decaying solution ψ_E° of the single-particle problem (2.7). It remains to prove Einstein's formula (2.8) for spherical particles, $I_n = B(x_n, r_n)$, with iid random radii $\{r_n\}_n$. By scaling, the above becomes

$$E : \bar{\mathbf{B}}^1 E = \lambda(\mathcal{P}) \mathbb{E}\big[(r_n)^d\big] \int_{\mathbb{R}^d} \big|\mathrm{D}(\widetilde{\psi}_E^\circ)\big|^2,$$

in terms of the unique decaying solution $\widetilde{\psi}_E^\circ$ of the rescaled problem

$$\begin{cases} -\triangle \widetilde{\psi}_E^\circ + \nabla \widetilde{\Sigma}_E^\circ = 0, & \text{in } \mathbb{R}^d \setminus B, \\ \operatorname{div}(\widetilde{\psi}_E^\circ) = 0, & \text{in } \mathbb{R}^d \setminus B, \\ \mathrm{D}(\widetilde{\psi}_E^\circ + Ex) = 0, & \text{in } B, \\ \int_{\partial B} \sigma(\widetilde{\psi}_E^\circ + Ex, \widetilde{\Sigma}_E^\circ)\nu = 0, & \\ \int_{\partial B} \Theta x \cdot \sigma(\widetilde{\psi}_E^\circ + Ex, \widetilde{\Sigma}_E^\circ)\nu = 0, & \forall \Theta \in \mathbb{M}^{\mathrm{skew}}. \end{cases}$$

Alternatively, using the energy identity for this equation,

$$E : \bar{\mathbf{B}}^1 E = \frac{1}{2} \lambda(\mathcal{P}) \, \mathbb{E}\big[(r_n)^d\big] \int_{\partial B} Ex \cdot \sigma(\widetilde{\psi}_E^\circ + Ex, \widetilde{\Sigma}_E^\circ)\nu. \tag{2.40}$$

As is well known, e.g. [31, Section 2.1.3], $\widetilde{\psi}_E^\circ$ coincides with the unique solution of

$$\begin{cases} -\triangle \widetilde{\psi}_E^\circ + \nabla \widetilde{\Sigma}_E^\circ = 0, & \text{in } \mathbb{R}^d \setminus B, \\ \operatorname{div}(\widetilde{\psi}_E^\circ) = 0, & \text{in } \mathbb{R}^d \setminus B, \\ \widetilde{\psi}_E^\circ = -Ex, & \text{on } \partial B, \end{cases}$$

and is explicitly given by the following formulas for $|x| \geq 1$,

$$\widetilde{\psi}_E^\circ(x) := -\frac{d+2}{2} \frac{(x \cdot Ex)x}{|x|^{d+2}} - \frac{1}{2}\left(2\frac{Ex}{|x|^{d+2}} - (d+2)\frac{(x \cdot Ex)x}{|x|^{d+4}}\right),$$
$$\widetilde{\Sigma}_E^\circ(x) := -(d+2)\frac{x \cdot Ex}{|x|^{d+2}}.$$

Inserting this into (2.40), a direct computation yields

$$E : \bar{\mathbf{B}}^1 E = \frac{d+2}{2} \lambda(\mathcal{P}) \, \mathbb{E}\big[(r_n)^d\big] |B| |E|^2,$$

that is, Einstein's formula (2.8). ∎

Chapter 3

Cluster expansion of the effective viscosity

This chapter is devoted to the higher-order cluster expansion of the effective viscosity $\overline{\mathbf{B}}$: starting from finite-volume approximations, we establish cluster formulas and prove uniform estimates in the large-volume limit. These results are mainly inspired by our previous work [15] on the Clausius–Mossotti conductivity formula, where we introduced the triad consisting of: (1) finite-volume approximation; (2) cluster expansion; (3) uniform $\ell^1 - \ell^2$ energy estimates. We further refine the analysis of [15], in particular improving on error estimates, and properly estimating cluster coefficients in case of uniform particle separation $\ell(\mathcal{P}) \gg 1$. Compared to [15], there are also some new twists due to the rigidity of the inclusions. Henceforth, in the rest of this memoir, we shall assume that particles are uniformly separated in the sense of Assumptions (H_ρ) and ($\mathrm{H}_\rho^{\mathrm{unif}}$).

3.1 Finite-volume approximations

In order to make sense of cluster expansions and avoid diverging series, we start by defining finite-volume approximations of the effective viscosity, obtained by a periodization procedure, which will in turn provide an implicit renormalization of cluster coefficients in the large-volume limit. More precisely, we define a restriction $\mathcal{P}_L$ on Q_L of the point process $\mathcal{P}$ via

$$\mathcal{P}_L := \{x_n : n \in N_L\}, \quad N_L := \{n : x_n \in Q_{L,\rho}\}, \quad Q_{L,\rho} := Q_{L-2(\ell(\mathcal{P})\vee(1+\rho))}$$

and we consider the corresponding random set

$$\mathcal{I}_L := \bigcup_{n\in N_L} I_n, \quad I_n = x_n + I_n^\circ. \tag{3.1}$$

For convenience, we choose an enumeration $\mathcal{P}_L = \{x_{n,L}\}_n$ and set $I_{n,L} = x_{n,L} + I_{n,L}^\circ$. Under Assumptions ($\mathrm{H}_\rho$) and ($\mathrm{H}_\rho^{\mathrm{unif}}$), we have the following:

- *Regularity and separation:* For all L, the periodized random set $\mathcal{I}_L + L\mathbb{Z}^d$ satisfies the ρ-regularity and uniform separation assumptions in (H_ρ) and ($\mathrm{H}_\rho^{\mathrm{unif}}$). Moreover, the periodized point process $\mathcal{P}_L + L\mathbb{Z}^d$ satisfies

$$\ell(\mathcal{P}_L + L\mathbb{Z}^d) \geq \ell(\mathcal{P}) \gtrsim 1.$$

- *Stabilization:* For all L, there holds $\mathcal{P}_L|_{Q_{L,\rho}} = \mathcal{P}|_{Q_{L,\rho}}$.

We then define the following finite-volume approximation of the effective viscosity $\overline{\mathbf{B}}$,

$$E : \overline{\mathbf{B}}_L E := \mathbb{E}\Big[\fint_{Q_L} \big|\mathrm{D}(\psi_{E;L}) + E\big|^2 \Big], \tag{3.2}$$

where $\psi_{E;L} \in \mathrm{L}^2(\Omega; H^1_{\mathrm{per}}(Q_L)^d)$ is almost surely the unique solution in $H^1_{\mathrm{per}}(Q_L)^d$, with vanishing average $\int_{Q_L} \psi_{E;L} = 0$, of the periodized version of the corrector problem (1.3),

$$\begin{cases} -\triangle \psi_{E;L} + \nabla \Sigma_{E;L} = 0, & \text{in } Q_L \setminus \mathcal{I}_L, \\ \mathrm{div}(\psi_{E;L}) = 0, & \text{in } Q_L \setminus \mathcal{I}_L, \\ \mathrm{D}(\psi_{E;L} + Ex) = 0, & \text{in } \mathcal{I}_L, \\ \int_{\partial I_{n;L}} \sigma_{E;L} \nu = 0, & \forall n, \\ \int_{\partial I_{n;L}} \Theta(x - x_{n;L}) \cdot \sigma_{E;L} \nu = 0, & \forall n,\ \forall \Theta \in \mathbb{M}^{\mathrm{skew}}, \end{cases} \tag{3.3}$$

where we use the shorthand notation $\sigma_{E;L} := \sigma(\psi_{E;L} + Ex, \Sigma_{E;L})$ for the Cauchy stress tensor. As a corollary of [18, Theorem 1],[1] in view of the above stabilization property, this finite-volume approximation (3.2) is consistent in the sense of

$$\lim_{L \uparrow \infty} \overline{\mathbf{B}}_L = \overline{\mathbf{B}}. \tag{3.4}$$

As opposed to $\overline{\mathbf{B}}$, we emphasize that the approximation $\overline{\mathbf{B}}_L$ depends only on the finite number of inclusions $\{I_{n,L}\}_n$. Indeed, by (H_ρ), the number of inclusions in Q_L has almost surely a deterministic upper bound CL^d. The associated cluster expansion is therefore well defined.

3.2 Main results

We start with the cluster expansion of the finite-volume approximation $\overline{\mathbf{B}}_L$, establishing suitable formulas for cluster coefficients and for the remainder. This is analogous to formulas obtained in our previous work on the conductivity problem [15]. While the formula (3.9) for the remainder naturally involves the original corrector with the whole set $\mathcal{P}_L$ of particles, we emphasize that the bound (3.10) only involves correctors associated with finite numbers of inclusions (uniformly in L): this is key to the optimal estimates obtained in the sequel and constitutes the first twist w.r.t. [15]. Indeed, this control is based on the rigidity of the inclusions and is therefore not available in the generality considered for the conductivity problem in [15]; it was first observed at second order by Gérard-Varet in [26]. The proof is displayed in Section 3.4.

[1] This requires to replace Dirichlet boundary conditions in [18] by periodic conditions, as is standard in homogenization theory.

Theorem 3.1 (Finite-volume cluster expansion). *Under Assumptions* (H_ρ) *and* ($\mathrm{H}_\rho^{\mathrm{unif}}$)*, finite-volume approximations of the effective viscosity can be expanded for all* L *and* $k \geq 1$*,*

$$\bar{\mathbf{B}}_L = \mathrm{Id} + \sum_{j=1}^{k} \frac{1}{j!} \bar{\mathbf{B}}_L^j + R_L^{k+1}, \tag{3.5}$$

where the coefficients and remainders are defined as follows:

• *The coefficients* $\{\bar{\mathbf{B}}_L^j\}_j$ *are given by cluster formulas, cf.* (1.21)*,*

$$E : \bar{\mathbf{B}}_L^j E := j! \sum_{\sharp F = j} \mathbb{E}\Big[\fint_{Q_L} \delta^F \big(\big| \mathrm{D}(\psi_{E;L}^{\varnothing}) + E \big|^2 \big) \Big], \tag{3.6}$$

which can be alternatively expressed as

$$\begin{aligned} E : \bar{\mathbf{B}}_L^j E &= \frac{1}{2} j!\, L^{-d} \sum_{\sharp F = j} \sum_{n \in F} \mathbb{E}\Big[\int_{\partial I_{n,L}} E(x - x_{n,L}) \cdot \delta^{F \setminus \{n\}} \sigma_{E;L}^{\{n\}} \nu \Big] && (3.7) \\ &= \frac{1}{2} j!\, L^{-d} \sum_{\sharp F = j} \sum_{n \in F} \mathbb{E}\Big[\int_{\partial I_{n,L}} \delta^{F \setminus \{n\}} \big(\psi_{E;L}^{\varnothing} + E(x - x_{n,L}) \big) \cdot \sigma_{E;L}^{F} \nu \Big], && (3.8) \end{aligned}$$

where, for any $H \subset \mathbb{N}$*,* $\psi_{E;L}^H$ *stands for the solution of the periodized corrector problem* (3.3) *with inclusion set* $\mathcal{I}_L$ *replaced by* $\mathcal{I}_L^H := \bigcup_{n \in H} I_{n,L}$*, where we use the shorthand notation* $\sigma_{E;L}^H := \sigma(\psi_{E;L}^H + Ex, \Sigma_{E;L}^H)$ *for the Cauchy stress tensor, and where we recall the notation of Section* 1.3.1 *for cluster difference operators.*

• *The remainder* R_L^{k+1} *can be represented as*

$$E : R_L^{k+1} E = \frac{1}{2} L^{-d} \sum_{\sharp F = k+1} \sum_{n \in F} \mathbb{E}\Big[\int_{\partial I_{n,L}} \delta^{F \setminus \{n\}} \big(\psi_{E;L}^{\varnothing} + E(x - x_{n,L}) \big) \cdot \sigma_{E;L} \nu \Big], \tag{3.9}$$

and is estimated as follows,

$$\begin{aligned} |E : R_L^{k+1} E| \lesssim \mathbb{E}\Big[L^{-d} \sum_n \int_{I_{n,L}} \Big| \sum_{\substack{\sharp F = k \\ n \notin F}} \mathrm{D}(\delta^F \psi_{E;L}^{\varnothing}) \Big|^2 \Big] \\ + \sum_{j=1}^{k} \mathbb{E}\Big[L^{-d} \sum_n \Big(\int_{I_{n,L}} \Big| \sum_{\substack{\sharp F = k \\ n \notin F}} \mathrm{D}(\delta^F \psi_{E;L}^{\varnothing}) \Big|^2 \Big)^{\frac{1}{2}} \\ \times \Big(\int_{I_{n,L} + \rho B} \Big| \sum_{\substack{\sharp F = j-1 \\ n \notin F}} \mathrm{D}\big(\delta^F \big(\psi_{E;L}^{\{n\}} + E(x - x_{n,L}) \big) \big) \Big|^2 \Big)^{\frac{1}{2}} \Big]. \end{aligned} \tag{3.10}$$

In view of the short-range setting (1.20), we expect

$$\bar{\mathbf{B}}_L^j = O\big(\lambda_j(\mathcal{P})\big)$$

and we aim to prove uniform-in-L estimates that would allow us to pass to the large-volume limit and to recover a dilute expansion for the original effective viscosity $\bar{\mathbf{B}}$. This is partially achieved in the upcoming theorem, which states fine estimates on cluster coefficients and on the remainder. However, note that we cannot directly obtain uniform-in-L estimates with the desired scalings $O(\lambda_j(\mathcal{P}))$. Instead, the result below is twofold:

— *Uniform estimates:* In (i), we state uniform-in-L estimates, which further display the optimal scaling w.r.t. the minimal distance $\ell = \ell(\mathcal{P})$, but fail to capture the general expected dependence on multi-point intensities $\{\lambda_j(\mathcal{P})\}_j$.

— *Non-uniform estimates:* In (ii), we state non-uniform estimates, which display a logarithmic divergence in the large-volume limit $L \uparrow \infty$, but have the merit of capturing the correct dependence on multi-point intensities.

Uniform estimates in (i) allow to deduce the convergence of cluster coefficients $\{\bar{\mathbf{B}}_L^j\}_j$ in the large-volume limit $L \uparrow \infty$, cf. (3.13) below: this actually *defines* infinite-volume cluster coefficients in a meaningful way, providing an implicit renormalization of diverging series and answering the question raised in Section 1.3.4. As they display the optimal dependence on the minimal distance $\ell = \ell(\mathcal{P})$, these estimates already yield the desired infinite-volume cluster expansion in the large-separation regime $\ell \gg 1$ with $\lambda_j(\mathcal{P})$ replaced by $(\ell^{-d})^j$, which is optimal in some cases (see dilation setting in Theorem 5.4 and Remark 5.5). To treat the general model-free dilute setting, however, uniform estimates need to be further derived with the correct dependence on multi-point intensities: this requires to overcome logarithmic divergences in non-uniform estimates in (ii), which is the subject of Chapter 4. The proof of the present result is split between Sections 3.5, 3.6, 3.7, and 3.8.

Theorem 3.2 (Cluster estimates and large-volume limit). *Under Assumptions* (H_ρ) *and* ($\mathrm{H}_\rho^{\mathrm{unif}}$)*, the coefficients and the remainder of the finite-volume cluster expansion in Theorem* 3.1 *satisfy the following two classes of estimates.*

(i) Uniform estimates: *For all L and $k, j \geq 1$,*

$$\begin{aligned} |\bar{\mathbf{B}}_L^j| &\leq j!(C\ell^{-d})^j, \\ |R_L^{k+1}| &\leq (C\ell^{-d})^{k+1}. \end{aligned} \tag{3.11}$$

(ii) Non-uniform estimates: *For all L and $k, j \geq 1$,*

$$\begin{aligned} |\bar{\mathbf{B}}_L^j| &\lesssim_j \lambda_j(\mathcal{P})\,(\log L)^{j-1}, \\ |R_L^{k+1}| &\lesssim_k \sum_{l=k}^{2k} \lambda_{l+1}(\mathcal{P})(\log L)^l. \end{aligned} \tag{3.12}$$

In particular, as a consequence of (i), for all $k, j \geq 1$, the following large-volume limits are well defined,

$$\bar{\mathbf{B}}^j := \lim_{L\uparrow\infty} \bar{\mathbf{B}}_L^j, \quad R^{k+1} := \lim_{L\uparrow\infty} R_L^{k+1}, \tag{3.13}$$

so that the cluster expansion (3.5) becomes, for all $k \geq 1$,

$$\left| \bar{\mathbf{B}} - \left(\mathrm{Id} + \sum_{j=1}^{k} \frac{1}{j!} \bar{\mathbf{B}}^j \right) \right| \leq |R^{k+1}| \leq (C\ell^{-d})^{k+1}.$$

3.3 Preliminary lemmas

Henceforth, we fix E with $|E| = 1$ and we skip the associated subscript for notational convenience. Before turning to the proof of Theorems 3.1 and 3.2, we state a series of preliminary lemmas. We start with the following useful reformulation of the corrector equation (1.3), where the rigidity constraint is viewed as generating a source term concentrated at particle boundaries in steady Stokes equations.

Lemma 3.3 (Reformulation of the corrector equation). *For all $H \subset \mathbb{N}$ we have in Q_L,*

$$-\triangle \psi_L^H + \nabla(\Sigma_L^H \mathbb{1}_{Q_L \setminus \mathcal{I}_L^H}) = -\sum_{n\in H} \delta_{\partial I_{n,L}} \sigma_L^H \nu, \tag{3.14}$$

where $\delta_{\partial I_{n,L}}$ stands for the Hausdorff measure on the boundary of $I_{n,L}$.[2]

Proof. For any test function

$$\phi_L \in C^\infty_{\mathrm{per}}(Q_L)^d,$$

recalling that ψ_L^H is divergence-free in Q_L and that it satisfies $\mathrm{D}(\psi_L^H + Ex) = 0$ in $\mathcal{I}_L^H$, we find

$$\begin{aligned}
&\int_{Q_L} \nabla\phi_L : \nabla\psi_L^H - \int_{Q_L\setminus\mathcal{I}_L^H} \Sigma_L^H \,\mathrm{div}(\phi_L) \\
&= 2\int_{Q_L} \nabla\phi_L : \mathrm{D}(\psi_L^H) - \int_{Q_L\setminus\mathcal{I}_L^H} \Sigma_L^H \,\mathrm{div}(\phi_L) \\
&= 2\int_{Q_L} \nabla\phi_L : \mathrm{D}(\psi_L^H + Ex) - \int_{Q_L\setminus\mathcal{I}_L^H} \Sigma_L^H \,\mathrm{div}(\phi_L) \\
&= \int_{Q_L\setminus\mathcal{I}_L^H} \nabla\phi_L : \sigma(\psi_L^H + Ex, \Sigma_L^H).
\end{aligned}$$

[2] More precisely, we define $\int_{Q_L} \phi_L \, \delta_{\partial I_{n,L}} := \int_{\partial I_{n,L}} \phi_L$ for any test function $\phi_L \in C^\infty_{\mathrm{per}}(Q_L)$.

Since the steady Stokes equation for ψ_L^H reads $\operatorname{div}(\sigma(\psi_L^H + Ex, \Sigma_L^H)) = 0$ in $Q_L \setminus \mathcal{I}_L^H$, we deduce by integration by parts,

$$\int_{Q_L} \nabla\phi_L : \nabla\psi_L^H - \int_{Q_L\setminus\mathcal{I}_L^H} \Sigma_L^H \operatorname{div}(\phi_L) = -\sum_{n\in H} \int_{\partial I_{n,L}} \phi_L \cdot \sigma(\psi_L^H + Ex, \Sigma_L^H)\nu.$$

By the arbitrariness of ϕ_L, this proves (3.14) with $\sigma_L^H = \sigma(\psi_L^H + Ex, \Sigma_L^H)$. ∎

Next, the following result provides corresponding Stokes equations for corrector differences, which will be used abundantly in the sequel.

Lemma 3.4 (Equations for corrector differences). *For all disjoint subsets $F, H \subset \mathbb{N}$ with F finite, we have in Q_L,*

$$\begin{aligned} &-\triangle\delta^F\psi_L^H + \nabla\delta^F\big(\Sigma_L^H \mathbb{1}_{Q_L\setminus\mathcal{I}_L^H}\big) \\ &\quad = -\sum_{n\in H} \delta_{\partial I_{n,L}}\delta^F\sigma_L^H\nu - \sum_{n\in F} \delta_{\partial I_{n,L}}\delta^{F\setminus\{n\}}\sigma_L^{H\cup\{n\}}\nu. \end{aligned} \tag{3.15}$$

Proof. The starting point is equation (3.14) satisfied by $\psi_L^{S\cup H}$,

$$-\triangle\psi_L^{S\cup H} + \nabla\big(\Sigma_L^{S\cup H}\mathbb{1}_{Q_L\setminus\mathcal{I}_L^{S\cup H}}\big) = -\sum_{n\in S\cup H} \delta_{\partial I_{n,L}}\sigma_L^{S\cup H}\nu.$$

Using the definition (1.10) of the difference operator, we deduce

$$-\triangle\delta^F\psi_L^H + \nabla\delta^F\big(\Sigma_L^H\mathbb{1}_{Q_L\setminus\mathcal{I}_L^H}\big) = -\sum_{S\subset F}(-1)^{|F\setminus S|}\sum_{n\in S\cup H}\delta_{\partial I_{n,L}}\sigma_L^{S\cup H}\nu,$$

and it remains to reformulate the right-hand side. For that purpose, we decompose

$$\begin{aligned} &-\triangle\delta^F\psi_L^H + \nabla\delta^F\big(\Sigma_L^H\mathbb{1}_{Q_L\setminus\mathcal{I}_L^H}\big) \\ &\quad = -\sum_{n\in H}\delta_{\partial I_{n,L}}\sum_{S\subset F}(-1)^{|F\setminus S|}\sigma_L^{S\cup H}\nu - \sum_{n\in F}\delta_{\partial I_{n,L}}\sum_{S\subset F}\mathbb{1}_{n\in S}(-1)^{|F\setminus S|}\sigma_L^{S\cup H}\nu. \end{aligned}$$

Changing summation variables and recognizing the definition (1.10) of the difference operator, the conclusion follows. ∎

We now state and prove trace estimates, which constitute an upgraded version of Lemma 2.5. We shall repeatedly appeal to these estimates to control force terms at particle boundaries, which appear in our formulation (3.15) of equations for corrector differences.

Lemma 3.5 (Trace estimates). *Under Assumptions (H_ρ) and ($\mathrm{H}_\rho^{\mathrm{unif}}$), for all families $\mathcal{F}$ of finite subsets of $\mathbb{N}$, for all $H \subset \mathbb{N}$ and $n \in \mathbb{N}$ with $n \notin \bigcup_{F\in\mathcal{F}} F$, we have*

$$\inf_{\kappa\in\mathbb{R}^d,\Theta\in\mathbb{M}^{\mathrm{skew}}} \int_{\partial I_{n,L}} \Big|\sum_{F\in\mathcal{F}}\delta^F\psi_L^H - \big(\kappa + \Theta(x - x_{n,L})\big)\Big|^2 \lesssim \int_{I_{n,L}} \Big|\sum_{F\in\mathcal{F}} \mathrm{D}(\delta^F\psi_L^H)\Big|^2,$$

and

$$\inf_{c\in\mathbb{R}} \int_{\partial I_{n,L}} \Big| \sum_{F\in\mathcal{F}} \delta^F \sigma_L^H - c\,\mathrm{Id} \Big|^2 \lesssim \int_{I_{n,L}+\rho B} \Big| \sum_{F\in\mathcal{F}} \mathrm{D}\big(\delta^F(\psi_L^H + Ex)\big)\Big|^2.$$

Proof. We split the proof into three steps. We set for abbreviation

$$\psi_L^{\mathcal{F},H} := \sum_{F\in\mathcal{F}} \delta^F \psi_L^H, \quad \Sigma_L^{\mathcal{F},H} := \sum_{F\in\mathcal{F}} \delta^F \Sigma_L^H, \quad \sigma_L^{\mathcal{F},H} := \sum_{F\in\mathcal{F}} \delta^F \sigma_L^H.$$

We also use the shorthand notation $\overline{\psi}_L^H := \psi_L^H + Ex$ and $\overline{\psi}_L^{\mathcal{F},H} := \sum_{F\in\mathcal{F}} \delta^F \overline{\psi}_L^H$. This last expression is equal to $\sum_{F\in\mathcal{F}} \delta^F \psi_L^H + Ex$ if $\varnothing \in \mathcal{F}$, and to $\sum_{F\in\mathcal{F}} \delta^F \psi_L^H$ otherwise.

Step 1. Proof of the first estimate on $\psi_L^{\mathcal{F},H}$. We appeal to a trace estimate in form of

$$\int_{\partial I_{n,L}} \big|\psi_L^{\mathcal{F},H} - \big(\kappa + \Theta(x - x_{n,L})\big)\big|^2 \lesssim \int_{I_{n,L}} \big|\langle\nabla\rangle^{\frac12}\big(\psi_L^{\mathcal{F},H} - \big(\kappa + \Theta(x-x_{n,L})\big)\big)\big|^2,$$

and the conclusion then follows from Poincaré's and Korn's inequalities.

Step 2. Proof of the second estimate on $\sigma_L^{\mathcal{F},H}$ in the case $n \notin H$. As $\sigma_L^{\mathcal{F},H} = \sigma(\overline{\psi}_L^{\mathcal{F},H}, \Sigma_L^{\mathcal{F},H})$, a trace estimate yields

$$\int_{\partial I_{n,L}} |\sigma_L^{\mathcal{F},H} - c\,\mathrm{Id}\,|^2 \lesssim \int_{(I_{n,L}+\frac12\rho B)\setminus I_{n,L}} \big|\langle\nabla\rangle^{\frac12}\nabla\overline{\psi}_L^{\mathcal{F},H}\big|^2 + \big|\langle\nabla\rangle^{\frac12}(\Sigma_L^{\mathcal{F},H} - c)\big|^2. \tag{3.16}$$

Given $n \notin H$, as the uniform separation assumption in ($\mathrm{H}_\rho^{\mathrm{unif}}$) ensures that no other particle intersects $I_{n,L} + \rho B$, we note that $(\overline{\psi}_L^{\mathcal{F},H}, \Sigma_L^{\mathcal{F},H})$ satisfies

$$-\triangle\overline{\psi}_L^{\mathcal{F},H} + \nabla\Sigma_L^{\mathcal{F},H} = 0, \quad \text{in } I_{n,L} + \rho B. \tag{3.17}$$

By the local regularity theory for steady Stokes equations, e.g. [25, Theorem IV.4.1], we deduce for all $m \geq 0$, for all constants $\kappa \in \mathbb{R}^d$ and $c \in \mathbb{R}$,

$$\begin{aligned}&\|\nabla\overline{\psi}_L^{\mathcal{F},H}\|_{H^{m+1}(I_{n,L}+\frac12\rho B)} + \|\Sigma_L^{\mathcal{F},H} - c\|_{H^{m+1}(I_{n,L}+\frac12\rho B)}\\ &\lesssim \|\overline{\psi}_L^{\mathcal{F},H} - \kappa\|_{H^1(I_{n,L}+\rho B)} + \|\Sigma_L^{\mathcal{F},H} - c\|_{\mathrm{L}^2(I_{n,L}+\rho B)}.\end{aligned}$$

Choosing $c := \fint_{I_{n,L}+\rho B} \Sigma_L^{\mathcal{F},H}$ and using a local pressure estimate for the steady Stokes equation, e.g. [19, Lemma 3.3], we find

$$\|\Sigma_L^{\mathcal{F},H} - c\|_{\mathrm{L}^2(I_{n,L}+\rho B)} \lesssim \|\nabla\overline{\psi}_L^{\mathcal{F},H}\|_{\mathrm{L}^2(I_{n,L}+\rho B)},$$

so that the above reduces to

$$\begin{aligned}&\|\nabla\overline{\psi}_L^{\mathcal{F},H}\|_{H^{m+1}(I_{n,L}+\frac12\rho B)} + \|\Sigma_L^{\mathcal{F},H} - c\|_{H^{m+1}(I_{n,L}+\frac12\rho B)}\\ &\lesssim \|\overline{\psi}_L^{\mathcal{F},H} - \kappa\|_{H^1(I_{n,L}+\rho B)}.\end{aligned}$$

Further choosing $\kappa := \fint_{I_{n,L}+\rho B} \bar{\psi}_L^{\mathcal{F},H}$ and applying Poincaré's inequality, we conclude

$$\begin{aligned}\|\nabla\bar{\psi}_L^{\mathcal{F},H}\|_{H^{m+1}(I_{n,L}+\frac{1}{2}\rho B)} + \|\Sigma_L^{\mathcal{F},H} - c\|_{H^{m+1}(I_{n,L}+\frac{1}{2}\rho B)}\\ \lesssim \|\nabla\bar{\psi}_L^{\mathcal{F},H}\|_{\mathrm{L}^2(I_{n,L}+\rho B)}.\end{aligned}$$

In particular, combining this with (3.16) and noting that the Cauchy stress tensor $\sigma_L^{\mathcal{F},H}$ is unchanged if we add a rigid motion to $\bar{\psi}_L^{\mathcal{F},H}$, the conclusion follows from Korn's inequality.

Step 3. Proof of the second estimate on $\sigma_L^{\mathcal{F},H}$ in the case $n \in H$. The starting point is again (3.16). Now, given $n \in H$, we note that $(\bar{\psi}_L^{\mathcal{F},H}, \Sigma_L^{\mathcal{F},H})$ satisfies, instead of (3.17),

$$-\triangle\bar{\psi}_L^{\mathcal{F},H} + \nabla\Sigma_L^{\mathcal{F},H} = 0, \quad \text{in } (I_{n,L}+\rho B)\setminus I_{n,L},$$

and $\bar{\psi}_L^{\mathcal{F},H}$ is affine in $I_{n,L}$. By the local regularity theory for the steady Stokes equation near a boundary, e.g. [25, Theorem IV.5.1–5.3], we obtain for all $m \geq 0$, for all constants $\kappa \in \mathbb{R}^d$ and $c \in \mathbb{R}$,

$$\begin{aligned}\|\nabla\bar{\psi}_L^{\mathcal{F},H}\|_{H^{m+1}((I_{n,L}+\frac{1}{2}\rho B)\setminus I_{n,L})} + \|\Sigma_L^{\mathcal{F},H} - c\|_{H^{m+1}((I_{n,L}+\frac{1}{2}\rho B)\setminus I_{n,L})}\\ \lesssim \|\bar{\psi}_L^{\mathcal{F},H}|_{I_{n,L}} - \kappa\|_{H^{m+\frac{3}{2}}(\partial I_{n,L})} + \|\bar{\psi}_L^{\mathcal{F},H} - \kappa\|_{H^1((I_{n,L}+\rho B)\setminus I_{n,L})}\\ + \|\Sigma_L^{\mathcal{F},H} - c\|_{\mathrm{L}^2((I_{n,L}+\rho B)\setminus I_{n,L})}.\end{aligned}$$

Choosing $c := \fint_{(I_{n,L}+\rho B)\setminus I_{n,L}} \Sigma_L^{\mathcal{F},H}$ and using a local pressure estimate for the steady Stokes equation, e.g. [19, Lemma 3.3], we find

$$\|\Sigma_L^{\mathcal{F},H} - c\|_{\mathrm{L}^2((I_{n,L}+\rho B)\setminus I_{n,L})} \lesssim \|\nabla\bar{\psi}_L^{\mathcal{F},H}\|_{\mathrm{L}^2((I_{n,L}+\rho B)\setminus I_{n,L})},$$

so that the above reduces to

$$\begin{aligned}\|\nabla\bar{\psi}_L^{\mathcal{F},H}\|_{H^{m+1}((I_{n,L}+\frac{1}{2}\rho B)\setminus I_{n,L})} + \|\Sigma_L^{\mathcal{F},H} - c\|_{H^{m+1}((I_{n,L}+\frac{1}{2}\rho B)\setminus I_{n,L})}\\ \lesssim \|\bar{\psi}_L^{\mathcal{F},H}|_{I_{n,L}} - \kappa\|_{H^{m+\frac{3}{2}}(\partial I_{n,L})} + \|\bar{\psi}_L^{\mathcal{F},H} - \kappa\|_{H^1((I_{n,L}+\rho B)\setminus I_{n,L})}.\end{aligned}$$

As $\bar{\psi}_L^{\mathcal{F},H}$ is affine in $I_{n,L}$, we have

$$\|\bar{\psi}_L^{\mathcal{F},H}|_{I_{n,L}} - \kappa\|_{H^{m+\frac{3}{2}}(\partial I_{n,L})} \lesssim \|\bar{\psi}_L^{\mathcal{F},H} - \kappa\|_{H^{m+2}(I_{n,L})} = \|\bar{\psi}_L^{\mathcal{F},H} - \kappa\|_{H^1(I_{n,L})},$$

and the above then becomes

$$\begin{aligned}\|\nabla\bar{\psi}_L^{\mathcal{F},H}\|_{H^{m+1}((I_{n,L}+\frac{1}{2}\rho B)\setminus I_{n,L})} + \|\Sigma_L^{\mathcal{F},H} - c\|_{H^{m+1}((I_{n,L}+\frac{1}{2}\rho B)\setminus I_{n,L})}\\ \lesssim \|\bar{\psi}_L^{\mathcal{F},H} - \kappa\|_{H^1(I_{n,L}+\rho B)}.\end{aligned}$$

Further choosing $\kappa := \fint_{I_{n,L}+\rho B} \bar{\psi}_L^{\mathcal{F},H}$ and applying Poincaré's inequality, we deduce

$$\begin{aligned}&\|\nabla\bar{\psi}_L^{\mathcal{F},H}\|_{H^{m+1}((I_{n,L}+\frac{1}{2}\rho B)\setminus I_{n,L})} + \|\Sigma_L^{\mathcal{F},H} - c\|_{H^{m+1}((I_{n,L}+\frac{1}{2}\rho B)\setminus I_{n,L})}\\ &\quad\lesssim \|\nabla\bar{\psi}_L^{\mathcal{F},H}\|_{\mathrm{L}^2(I_{n,L}+\rho B)}.\end{aligned}$$

In particular, combined with (3.16), this yields the conclusion as in Step 2. ∎

3.4 Cluster formulas

This section is devoted to the proof of Theorem 3.1. We start by establishing the validity of the expansion (3.5) with coefficients given by formula (3.8) and with the explicit remainder (3.9). The proof is similar to its counterpart for the conductivity problem in our previous work [15].

Lemma 3.6 (Finite-volume cluster expansion). *Under Assumptions* (H_ρ) *and* ($\mathrm{H}_\rho^{\mathrm{unif}}$), *finite-volume approximations of the effective viscosity can be expanded for all* L *and* $k \geq 1$ *as*

$$\bar{\mathbf{B}}_L = \mathrm{Id} + \sum_{j=1}^{k} \frac{1}{j!}\bar{\mathbf{B}}_L^j + R_L^{k+1}, \tag{3.18}$$

where the coefficients $\{\bar{\mathbf{B}}_L^j\}_j$ *and the remainder* R_L^{k+1} *are given by formulas* (3.8) *and* (3.9), *respectively.*

Proof. Given $E \in \mathbb{M}_0^{\mathrm{sym}}$ with $|E| = 1$, we recall that we drop the corresponding subscripts in the notation. We split the proof into three steps.

Step 1. General strategy. The starting point is formula (3.2) for the finite-volume approximation of the effective viscosity,

$$E : \bar{\mathbf{B}}_L E = 1 + \mathbb{E}\Big[\fint_{Q_L} \big|\mathrm{D}(\psi_L)\big|^2\Big].$$

The energy identity for the corrector equation (3.3) takes the form

$$2\int_{Q_L} \big|\mathrm{D}(\psi_L)\big|^2 = \sum_n \int_{\partial I_{n,L}} E(x - x_{n,L}) \cdot \sigma_L \nu, \tag{3.19}$$

and thus, further decomposing $\sigma_L = \sigma_L^{\{n\}} + (\sigma_L - \sigma_L^{\{n\}})$, we obtain

$$\begin{aligned}\mathbb{E}\Big[2\int_{Q_L} \big|\mathrm{D}(\psi_L)\big|^2\Big] &= \sum_n \mathbb{E}\Big[\int_{\partial I_{n,L}} E(x - x_{n,L}) \cdot \sigma_L^{\{n\}}\nu\Big]\\ &\quad + \sum_n \mathbb{E}\Big[\int_{\partial I_{n,L}} E(x - x_{n,L}) \cdot (\sigma_L - \sigma_L^{\{n\}})\nu\Big].\end{aligned}$$

In addition, we shall prove below that for all $k \geq 1$,

$$\sum_{\sharp F=k}\sum_{n\in F}\mathbb{E}\bigg[\int_{\partial I_{n,L}}\delta^{F\setminus\{n\}}\big(\psi_L^{\varnothing}+E(x-x_{n,L})\big)\cdot(\sigma_L-\sigma_L^F)\nu\bigg]$$
$$=\sum_{\sharp F=k+1}\sum_{n\in F}\mathbb{E}\bigg[\int_{\partial I_{n,L}}\delta^{F\setminus\{n\}}\psi_L^{\varnothing}\cdot\sigma_L\nu\bigg]. \tag{3.20}$$

We note that (3.19) already proves the claim (3.18) for $k=0$. Next, we proceed by induction: if (3.18) holds for some $k\geq 0$, formulas (3.8) and (3.9) for R_L^{k+1}, $\bar{\mathbf{B}}_L^{k+1}$ allow us to decompose

$$E:R_L^{k+1}E=\frac{1}{(k+1)!}E:\bar{\mathbf{B}}_L^{k+1}E$$
$$+\frac{1}{2}L^{-d}\sum_{\sharp F=k+1}\sum_{n\in F}\mathbb{E}\bigg[\int_{\partial I_{n,L}}\delta^{F\setminus\{n\}}\big(\psi_L^{\varnothing}+E(x-x_{n,L})\big)\cdot(\sigma_L-\sigma_L^F)\nu\bigg].$$

Inserting identity (3.20), noting that for $\sharp F=k+2$ there holds

$$\delta^{F\setminus\{n\}}\psi_L^{\varnothing}=\delta^{F\setminus\{n\}}\big(\psi_L^{\varnothing}+E(x-x_{n,L})\big),$$

and recognizing formula (3.9) for R_L^{k+2}, we deduce

$$R_L^{k+1}=\frac{1}{(k+1)!}\bar{\mathbf{B}}_L^{k+1}+R_L^{k+2},$$

hence the claim (3.18) follows with k replaced by $k+1$. It remains to prove (3.20), which we do in the next two steps.

Step 2. Proof that for all $\sharp F=k\geq 1$ and $G\subset F$,

$$\sum_{n\in F\setminus G}\int_{\partial I_{n,L}}\big(\psi_L^G+E(x-x_{n,L})\big)\cdot(\sigma_L-\sigma_L^F)\nu$$
$$=\sum_{n\notin F}\int_{\partial I_{n,L}}(\psi_L^F-\psi_L^G)\cdot\sigma_L\nu. \tag{3.21}$$

On the one hand, testing equation (3.14) for ψ_L^G with the difference $\psi_L-\psi_L^F$, and using the boundary conditions for ψ_L, ψ_L^F, ψ_L^G on $\partial I_{n,L}$ with $n\in G\subset F$, we find

$$\int_{Q_L}\nabla(\psi_L-\psi_L^F):\nabla\psi_L^G=-\sum_{n\in G}\int_{\partial I_{n,L}}(\psi_L-\psi_L^F)\cdot\sigma_L^G\nu=0. \tag{3.22}$$

On the other hand, equations (3.14) for ψ_L and ψ_L^F entail

$$-\triangle(\psi_L-\psi_L^F)+\nabla\big(\Sigma_L\mathbb{1}_{Q_L\setminus\mathcal{I}_L}-\Sigma_L^F\mathbb{1}_{Q_L\setminus\mathcal{I}_L^F}\big)$$
$$=-\sum_{n\notin F}\delta_{\partial I_{n,L}}\sigma_L\nu-\sum_{n\in F}\delta_{\partial I_{n,L}}(\sigma_L-\sigma_L^F)\nu,$$

and thus, testing with ψ_L^G and using the boundary conditions for ψ_L, ψ_L^F, ψ_L^G on $\partial I_{n,L}$ with $n \in G \subset F$,

$$\begin{aligned}
&\int_{Q_L} \nabla\psi_L^G : \nabla(\psi_L - \psi_L^F) \\
&\quad = -\sum_{n\notin F} \int_{\partial I_{n,L}} \psi_L^G \cdot \sigma_L \nu - \sum_{n\in F} \int_{\partial I_{n,L}} \psi_L^G \cdot (\sigma_L - \sigma_L^F)\nu \\
&\quad = -\sum_{n\notin F} \int_{\partial I_{n,L}} \psi_L^G \cdot \sigma_L \nu - \sum_{n\in F\setminus G} \int_{\partial I_{n,L}} \psi_L^G \cdot (\sigma_L - \sigma_L^F)\nu \\
&\qquad + \sum_{n\in G} \int_{\partial I_{n,L}} E(x - x_{n,L}) \cdot (\sigma_L - \sigma_L^F)\nu.
\end{aligned}$$

Combined with (3.22), this entails

$$\begin{aligned}
&\sum_{n\in F\setminus G} \int_{\partial I_{n,L}} \psi_L^G \cdot (\sigma_L - \sigma_L^F)\nu \\
&\quad = \sum_{n\in G} \int_{\partial I_{n,L}} E(x - x_{n,L}) \cdot (\sigma_L - \sigma_L^F)\nu - \sum_{n\notin F} \int_{\partial I_{n,L}} \psi_L^G \cdot \sigma_L \nu,
\end{aligned}$$

or alternatively,

$$\begin{aligned}
&\sum_{n\in F\setminus G} \int_{\partial I_{n,L}} \big(\psi_L^G + E(x - x_{n,L})\big) \cdot (\sigma_L - \sigma_L^F)\nu \\
&\quad = \sum_{n\in F} \int_{\partial I_{n,L}} E(x - x_{n,L}) \cdot (\sigma_L - \sigma_L^F)\nu - \sum_{n\notin F} \int_{\partial I_{n,L}} \psi_L^G \cdot \sigma_L \nu. \qquad (3.23)
\end{aligned}$$

For $G = F$, the left-hand side vanishes, hence

$$\sum_{n\in F} \int_{\partial I_{n,L}} E(x - x_{n,L}) \cdot (\sigma_L - \sigma_L^F)\nu = \sum_{n\notin F} \int_{\partial I_{n,L}} \psi_L^F \cdot \sigma_L \nu,$$

which allows us to reformulate (3.23) into (3.21).

Step 4. Proof of (3.20). Denote by $T_{k,L}$ the left-hand side of (3.20). By the definition (1.10) of the difference operator, we have

$$T_{k,L} = -\sum_{\sharp F = k} \sum_{n\in F} \sum_{G\subset F\setminus\{n\}} (-1)^{|F\setminus G|} \mathbb{E}\bigg[\int_{\partial I_{n,L}} \big(\psi_L^G + E(x - x_{n,L})\big) \cdot (\sigma_L - \sigma_L^F)\nu\bigg],$$

or alternatively, after changing summation variables,

$$T_{k,L} = -\sum_{\sharp F = k} \sum_{G\subset F} (-1)^{|F\setminus G|} \mathbb{E}\bigg[\sum_{n\in F\setminus G} \int_{\partial I_{n,L}} \big(\psi_L^G + E(x - x_{n,L})\big) \cdot (\sigma_L - \sigma_L^F)\nu\bigg].$$

We now appeal to (3.21), to the effect of

$$T_{k,L} = -\sum_{\sharp F=k}\sum_{G\subset F}(-1)^{|F\setminus G|}\mathbb{E}\Big[\sum_{n\notin F}\int_{\partial I_{n,L}}(\psi_L^F-\psi_L^G)\cdot\sigma_L\nu\Big].$$

Using that $\sum_{G\subset F}(-1)^{|F\setminus G|}=0$ for $F\neq\varnothing$ and recalling the definition (1.10) of the difference operator, this implies

$$\begin{aligned}T_{k,L} &= \sum_{\sharp F=k}\sum_{G\subset F}(-1)^{|F\setminus G|}\mathbb{E}\Big[\sum_{n\notin F}\int_{\partial I_{n,L}}\psi_L^G\cdot\sigma_L\nu\Big]\\ &= \sum_{\sharp F=k}\mathbb{E}\Big[\sum_{n\notin F}\int_{\partial I_{n,L}}\delta^F\psi_L^\varnothing\cdot\sigma_L\nu\Big],\end{aligned}$$

and the claim (3.20) follows after changing summation variables. ∎

In the above result, we have naturally come up with the definition (3.8) of cluster coefficients $\{\bar{\mathbf{B}}_L^j\}_j$. We now further establish the alternative formulas (3.6) and (3.7). Note that (3.6) coincides with the periodized version of the expected cluster formula (1.21).

Lemma 3.7 (Equivalent cluster formulas). *Under Assumptions* (H_ρ) *and* ($\mathrm{H}_\rho^{\mathrm{unif}}$), *for all* L *and* $j\geq 1$, *the finite-volume cluster coefficient* $\bar{\mathbf{B}}_L^j$ *defined by formula* (3.8) *is equivalently given by* (3.6) *and* (3.7).

Proof. We split the proof into two steps.

Step 1. Equivalence of (3.7) and (3.8). It suffices to prove that for all finite $F\subset\mathbb{N}$,

$$\begin{aligned}&\sum_{n\in F}\int_{\partial I_{n,L}}\delta^{F\setminus\{n\}}\big(\psi_L^\varnothing+E(x-x_{n,L})\big)\cdot\sigma_L^F\nu\\ &\quad=\sum_{n\in F}\int_{\partial I_{n,L}}E(x-x_{n,L})\cdot\delta^{F\setminus\{n\}}\sigma_L^{\{n\}}\nu.\end{aligned}\tag{3.24}$$

Decomposing $\delta^{F\setminus\{n\}}\psi_L^\varnothing=\delta^{F\setminus\{n\}}\psi_L^{\{n\}}-\delta^F\psi_L^\varnothing$ for $n\in F$ and using the boundary conditions, we find

$$\sum_{n\in F}\int_{\partial I_{n,L}}\delta^{F\setminus\{n\}}\big(\psi_L^\varnothing+E(x-x_n)\big)\cdot\sigma_L^F\nu=-\sum_{n\in F}\int_{\partial I_{n,L}}\delta^F\psi_L^\varnothing\cdot\sigma_L^F\nu.$$

Testing equation (3.14) for ψ_L^F with $\delta^F\psi_L^\varnothing$, this becomes

$$\sum_{n\in F}\int_{\partial I_{n,L}}\delta^{F\setminus\{n\}}\big(\psi_L^\varnothing+E(x-x_n)\big)\cdot\sigma_L^F\nu=\int_{Q_L}\nabla\delta^F\psi_L^\varnothing:\nabla\psi_L^F.$$

Now testing equation (3.15) for $\delta^F \psi_L^\varnothing$ with ψ_L^F, and using the boundary conditions, we deduce

$$\sum_{n\in F}\int_{\partial I_{n,L}} \delta^{F\setminus\{n\}}\big(\psi_L^\varnothing + E(x-x_n)\big)\cdot\sigma_L^F\nu = -\sum_{n\in F}\int_{\partial I_{n,L}} \psi_L^F\cdot\delta^{F\setminus\{n\}}\sigma_L^{\{n\}}\nu$$
$$= \sum_{n\in F}\int_{\partial I_{n,L}} E(x-x_{n,L})\cdot\delta^{F\setminus\{n\}}\sigma_L^{\{n\}}\nu,$$

that is, (3.24).

Step 2. Equivalence of (3.6) and (3.7). It suffices to prove for all finite $F\subset\mathbb{N}$,

$$\fint_{Q_L}\delta^F\big|\mathrm{D}(\psi_L^\varnothing)\big|^2 = \frac{1}{2}L^{-d}\sum_{n\in F}\int_{\partial I_{n,L}} E(x-x_{n,L})\cdot\delta^{F\setminus\{n\}}\sigma_L^{\{n\}}\nu. \tag{3.25}$$

Recalling the definition (1.10) of the difference operator, we can write

$$\fint_{Q_L}\delta^F|\mathrm{D}(\psi_L^\varnothing)|^2 = \sum_{G\subset F}(-1)^{|F\setminus G|}\fint_{Q_L}\big|\mathrm{D}(\psi_L^G)\big|^2,$$

which entails, in view of the energy identity for ψ_L^G, cf. (3.19),

$$\fint_{Q_L}\delta^F\big|\mathrm{D}(\psi_L^\varnothing)\big|^2 = \frac{1}{2}L^{-d}\sum_{G\subset F}\sum_{n\in G}(-1)^{|F\setminus G|}\int_{\partial I_{n,L}} E(x-x_{n,L})\cdot\sigma_L^G\nu.$$

After changing summation variables and using again the definition (1.10) of the difference operator, this yields the claim (3.25). ∎

To conclude the proof of Theorem 3.1, it remains to establish the control (3.10) of the remainder, which is inspired by a recent work of Gérard-Varet [26] and which we prove in the slightly refined form of (3.26) below. This extends the argument of [26] to all $k>2$.

Lemma 3.8 (Control of the remainder). *Under Assumptions* (H_ρ) *and* ($\mathrm{H}_\rho^{\mathrm{unif}}$), *for all* L *and* $j\geq 1$, *the remainder term defined in* (3.9) *can be estimated by*

$$|R_L^{k+1}| \leq \mathbb{E}\bigg[L^{-d}\sum_n\int_{I_{n,L}}\Big|\sum_{\substack{\sharp F=k\\ n\notin F}}\mathrm{D}(\delta^F\psi_L^\varnothing)\Big|^2\bigg]$$
$$+\sum_{j=1}^{k}\bigg|\mathbb{E}\bigg[L^{-d}\sum_n\int_{I_{n,L}}\Big(\sum_{\substack{\sharp F=k\\ n\notin F}}\mathrm{D}(\delta^F\psi_L^\varnothing)\Big):\Big(\sum_{\substack{\sharp F=j-1\\ n\notin F}}\mathrm{D}(\delta^F\widehat{\psi}_{n,L}^{\{n\}})\Big)\bigg]\bigg|, \tag{3.26}$$

where in view of (1.10) *we have defined, with a slight abuse of notation,*

$$\delta^F\widehat{\psi}_{n,L}^{\{n\}} := \sum_{G\subset F}(-1)^{|F\setminus G|}\widehat{\psi}_{n,L}^{G\cup\{n\}},$$

where for all $H \subset \mathbb{N}$ and $n \in H$ we denote by $\widehat{\psi}^H_{n,L}$ the solution of the following Neumann boundary value problem in the inclusion $I_{n,L}$ (unique up to a rigid motion),

$$\begin{cases} -\triangle \widehat{\psi}^H_{n,L} + \nabla \widehat{\Sigma}^H_{n,L} = 0, & \text{in } I_{n,L}, \\ \operatorname{div}\big(\widehat{\psi}^H_{n,L}\big) = 0, & \text{in } I_{n,L}, \\ \sigma\big(\widehat{\psi}^H_{n,L}, \widehat{\Sigma}^H_{n,L}\big)\nu = \sigma^H_L \nu, & \text{on } \partial I_{n,L}. \end{cases} \tag{3.27}$$

In particular, this yields the bound (3.10)*.*

Proof. We split the proof into two steps, first showing that (3.27) is well posed, and then proving the bound (3.26).

Step 1. Proof that the Neumann problem (3.27) is well posed for all $H \subset \mathbb{N}$ and $n \in H$, and that the solution satisfies

$$\int_{I_{n,L}} \big|\mathrm{D}(\widehat{\psi}^H_{n,L})\big|^2 \lesssim \int_{I_{n,L}+\rho B} \big|\mathrm{D}(\psi^H_L) + E\big|^2. \tag{3.28}$$

In addition, the proof yields similarly

$$\int_{I_{n,L}} \Big| \sum_{\substack{\sharp F = j-1 \\ n \notin F}} \mathrm{D}(\delta^F \widehat{\psi}^{\{n\}}_{n,L})\Big|^2 \lesssim \int_{I_{n,L}+\rho B} \Big| \sum_{\substack{\sharp F = j-1 \\ n \notin F}} \mathrm{D}\big(\delta^F (\psi^{\{n\}}_L + Ex)\big)\Big|^2.$$

This last estimate entails that the bound (3.10) follows from (3.26).

We turn to the proof of (3.28). The weak formulation of equation (3.27) reads for all $\phi \in H^1(I_{n,L})^d$ with $\operatorname{div}(\phi) = 0$,

$$2\int_{I_{n,L}} \mathrm{D}(\phi) : \mathrm{D}(\widehat{\psi}^H_{n,L}) = \int_{\partial I_{n,L}} \phi \cdot \sigma^H_L \nu. \tag{3.29}$$

Let us analyze the linear functional defining the right-hand side. Using the incompressibility of ϕ in form of $\int_{\partial I_{n,L}} \phi \cdot \nu = 0$, we can add any multiple of the identity matrix to σ^H_L. Further noting that the boundary conditions for ψ^H_L on $\partial I_{n,L}$ with $n \in H$ allow to subtract a rigid motion from the test function ϕ, we are led to

$$\begin{aligned} \Big| \int_{\partial I_{n,L}} \phi \cdot \sigma^H_L \nu \Big| \le & \left(\inf_{\kappa \in \mathbb{R}^d,\, \Theta \in \mathbb{M}^{\mathrm{skew}}} \int_{\partial I_{n,L}} \big|\phi - \big(\kappa + \Theta(x - x_{n,L})\big)\big|^2 \right)^{\frac{1}{2}} \\ & \times \left(\inf_{c \in \mathbb{R}} \int_{\partial I_{n,L}} |\sigma^H_L - c\,\mathrm{Id}\,|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Appealing to the trace estimates of Lemma 3.5, this becomes

$$\Big| \int_{\partial I_{n,L}} \phi \cdot \sigma^H_L \nu \Big| \lesssim \left(\int_{I_{n,L}} \big|\mathrm{D}(\phi)\big|^2 \right)^{\frac{1}{2}} \left(\int_{I_{n,L}+\rho B} \big|\mathrm{D}(\psi^H_L) + E\big|^2 \right)^{\frac{1}{2}}. \tag{3.30}$$

This proves that the right-hand side in the weak formulation (3.29) is a continuous linear functional with respect to $\mathrm{D}(\phi) \in \mathrm{L}^2(I_{n,L})^{d\times d}$. The Lax–Milgram theorem then ensures that equation (3.27) is well posed in the sense that it admits a unique solution $\mathrm{D}(\hat{\psi}^H_{n,L}) \in \mathrm{L}^2(I_{n,L})^{d\times d}$, and the a priori bound (3.28) follows.

Step 2. Proof of (3.26). Inserting the energy identity (3.19) and the formula (3.7) for the coefficients, the cluster expansion (3.5) yields the following formula for the remainder,

$$\begin{aligned} E : R_L^{k+1} E &= E : \bar{\mathbf{B}}_L E - 1 - \sum_{j=1}^{k} \frac{1}{j!} E : \bar{\mathbf{B}}_L^j E \\ &= \frac{1}{2} L^{-d} \mathbb{E}\Big[\sum_n \int_{\partial I_{n,L}} E(x - x_{n,L}) \cdot \sigma_L \nu \Big] \\ &\quad - \sum_{j=1}^{k} \frac{1}{2} L^{-d} \sum_{\sharp F = j} \sum_{n \in F} \mathbb{E}\Big[\int_{\partial I_{n,L}} E(x - x_{n,L}) \cdot \delta^{F\setminus\{n\}} \sigma_L^{\{n\}} \nu \Big], \end{aligned}$$

or equivalently, changing summation variables,

$$\begin{aligned} &E : R_L^{k+1} E \\ &\quad = \frac{1}{2} L^{-d} \mathbb{E}\Big[\sum_n \int_{\partial I_{n,L}} E(x - x_{n,L}) \cdot \Big(\sigma_L - \sum_{j=1}^{k} \sum_{\substack{\sharp F = j-1 \\ n \notin F}} \delta^F \sigma_L^{\{n\}} \Big) \nu \Big]. \end{aligned} \tag{3.31}$$

Consider the cluster expansion error

$$\Psi_L^k := \psi_L - \sum_{j=1}^{k} \sum_{\sharp F = j} \delta^F \psi_L^{\varnothing}, \tag{3.32}$$

$$\Xi_L^k := \Sigma_L \mathbb{1}_{Q_L \setminus \mathcal{I}_L} - \sum_{j=1}^{k} \sum_{\sharp F = j} \delta^F \big(\Sigma_L^{\varnothing} \mathbb{1}_{Q_L \setminus \mathcal{I}_L^{\varnothing}} \big),$$

and note that in view of (3.15) it satisfies the following equation in Q_L,

$$-\triangle \Psi_L^k + \nabla \Xi_L^k = -\sum_n \delta_{\partial I_{n,L}} \Big(\sigma_L - \sum_{j=1}^{k} \sum_{\substack{\sharp F = j-1 \\ n \notin F}} \delta^F \sigma_L^{\{n\}} \Big) \nu. \tag{3.33}$$

Testing this equation with ψ_L and using the boundary conditions, the identity (3.31) for the remainder becomes

$$E : R_L^{k+1} E = \mathbb{E}\Big[\fint_{Q_L} \mathrm{D}(\psi_L) : \mathrm{D}(\Psi_L^k) \Big].$$

Adding and subtracting $\sum_{j=1}^{k}\sum_{\sharp F=j}\delta^F\psi_L^{\varnothing}$ to ψ_L, we deduce by (3.32),

$$|E:R_L^{k+1}E|\le\mathbb{E}\Big[\fint_{Q_L}\big|\mathrm{D}(\Psi_L^k)\big|^2\Big]+\sum_{j=1}^{k}\Big|\mathbb{E}\Big[\fint_{Q_L}\mathrm{D}(\Psi_L^k):\sum_{\sharp F=j}\mathrm{D}(\delta^F\psi_L^{\varnothing})\Big]\Big|.$$

The conclusion (3.26) then follows from the estimate

$$\int_{Q_L}\big|\mathrm{D}(\Psi_L^k)\big|^2\lesssim\sum_n\int_{I_{n,L}}\Big|\sum_{\substack{\sharp F=j-1\\ n\notin F}}\mathrm{D}(\delta^F\psi_L^{\varnothing})\Big|^2,\tag{3.34}$$

and from the identity for all $1\le j\le k$

$$\begin{aligned}&\int_{Q_L}\mathrm{D}(\Psi_L^k):\sum_{\sharp F=j}\mathrm{D}(\delta^F\psi_L^{\varnothing})\\&\quad=\sum_n\int_{I_{n,L}}\Big(\sum_{\substack{\sharp F=j-1\\ n\notin F}}\mathrm{D}(\delta^F\psi_L^{\varnothing})\Big):\Big(\sum_{\substack{\sharp F=j-1\\ n\notin F}}\mathrm{D}(\delta^F\widehat{\psi}_{n,L}^{\{n\}})\Big),\end{aligned}\tag{3.35}$$

which we prove in the next two substeps, respectively.

Substep 2.1. Proof of (3.34). In view of (3.33), the cluster expansion error Ψ_L^k satisfies

$$-\triangle\Psi_L^k+\nabla\Xi_L^k=0,\quad \mathrm{div}(\Psi_L^k)=0,\quad \text{in } Q_L\setminus\mathcal{I}_L,$$

which entails

$$\begin{aligned}\int_{Q_L}\big|\mathrm{D}(\Psi_L^k)\big|^2&=\sum_n\int_{I_{n,L}}\big|\mathrm{D}(\Psi_L^k)\big|^2+\int_{Q_L\setminus\mathcal{I}_L}\big|\mathrm{D}(\Psi_L^k)\big|^2\\&=\sum_n\int_{I_{n,L}}\big|\mathrm{D}(\Psi_L^k)\big|^2-\frac12\sum_n\int_{\partial I_{n,L}}\Psi_L^k\cdot\sigma(\Psi_L^k,\Xi_L^k)\nu.\end{aligned}$$

Hence, using the boundary conditions and the incompressibility constraint to smuggle in arbitrary constants in the different factors, as in the proof of (3.30), and appealing to the trace estimates of Lemma 2.5, we find

$$\int_{Q_L}\big|\mathrm{D}(\Psi_L^k)\big|^2\lesssim\sum_n\int_{I_{n,L}}\big|\mathrm{D}(\Psi_L^k)\big|^2+\sum_n\Big(\int_{I_{n,L}}\big|\mathrm{D}(\Psi_L^k)\big|^2\Big)^{\frac12}\Big(\int_{I_{n,L}^+}\big|\mathrm{D}(\Psi_L^k)\big|^2\Big)^{\frac12},$$

from which we deduce by Young's inequality,[3]

$$\int_{Q_L}\big|\mathrm{D}(\Psi_L^k)\big|^2\lesssim\sum_n\int_{I_{n,L}}\big|\mathrm{D}(\Psi_L^k)\big|^2.\tag{3.36}$$

[3] As argued in [26], this estimate (3.36) can alternatively be deduced from minimizing properties of Stokes equations for Ψ_L^k in $Q_L\setminus\mathcal{I}_L$ with prescribed symmetric gradient in $\mathcal{I}_L$. We rather give a PDE argument that is more in line with the other arguments of this memoir.

Next, the definition of Ψ_L^k and the rigidity constraint for ψ_L in $I_{n,L}$ yield

$$\mathrm{D}(\Psi_L^k) = -E - \sum_{j=1}^{k} \sum_{\sharp F = j} \mathrm{D}(\delta^F \psi_L^{\varnothing}) \quad \text{in } I_{n,L}. \tag{3.37}$$

Distinguishing between the cases $n \in F$ and $n \notin F$, and noting that for $n \in F$ we can decompose

$$\delta^F \psi_L^{\varnothing} = \delta^{F\setminus\{n\}} \psi_L^{\{n\}} - \delta^{F\setminus\{n\}} \psi_L^{\varnothing},$$

we find

$$\sum_{\sharp F = j} \mathrm{D}(\delta^F \psi_L^{\varnothing}) = \sum_{\substack{\sharp F = j \\ n \notin F}} \mathrm{D}(\delta^F \psi_L^{\varnothing}) + \sum_{\substack{\sharp F = j-1 \\ n \notin F}} \mathrm{D}(\delta^F \psi_L^{\{n\}}) - \sum_{\substack{\sharp F = j-1 \\ n \notin F}} \mathrm{D}(\delta^F \psi_L^{\varnothing}),$$

and thus, in view of the rigidity constraint for $\delta^F \psi_L^{\{n\}}$ in $I_{n,L}$,

$$\sum_{\sharp F = j} \mathrm{D}(\delta^F \psi_L^{\varnothing}) = -E \mathbb{1}_{j=1} + \sum_{\substack{\sharp F = j \\ n \notin F}} \mathrm{D}(\delta^F \psi_L^{\varnothing}) - \sum_{\substack{\sharp F = j-1 \\ n \notin F}} \mathrm{D}(\delta^F \psi_L^{\varnothing}) \quad \text{in } I_{n,L}.$$

Inserting this into (3.37) and recognizing a telescoping sum, we deduce for all n,

$$\mathrm{D}(\Psi_L^k) = - \sum_{\substack{\sharp F = j-1 \\ n \notin F}} \mathrm{D}(\delta^F \psi_L^{\varnothing}) \quad \text{in } I_{n,L}. \tag{3.38}$$

Combined with (3.36), this yields the claim (3.34).

Substep 2.2. Proof of (3.35). Testing equation (3.15) for $\delta^F \psi_L^{\varnothing}$ with Ψ_L^k, and changing summation variables, we find

$$\begin{aligned} 2 \int_{Q_L} \mathrm{D}(\Psi_L^k) : \sum_{\sharp F = j} \mathrm{D}(\delta^F \psi_L^{\varnothing}) &= - \sum_{\sharp F = j} \sum_{n \in F} \int_{\partial I_{n,L}} \Psi_L^k \cdot \delta^{F\setminus\{n\}} \sigma_L^{\{n\}} \nu \\ &= - \sum_{n} \int_{\partial I_{n,L}} \Psi_L^k \cdot \sum_{\substack{\sharp F = j-1 \\ n \notin F}} \delta^F \sigma_L^{\{n\}} \nu. \end{aligned}$$

In view of equation (3.29) for $\mathrm{D}(\delta^F \widehat{\psi}_{n,L}^{\{n\}})$, this can be rewritten as

$$\int_{Q_L} \mathrm{D}(\Psi_L^k) : \sum_{\sharp F = j} \mathrm{D}(\delta^F \psi_L^{\varnothing}) = - \sum_{n} \int_{I_{n,L}} \mathrm{D}(\Psi_L^k) : \sum_{\substack{\sharp F = j-1 \\ n \notin F}} \mathrm{D}(\delta^F \widehat{\psi}_{n,L}^{\{n\}}).$$

Combined with (3.38), this yields the claim (3.35). ∎

3.5 Uniform $\ell^1 - \ell^2$ energy estimates

In order to prove uniform cluster estimates, cf. Theorem 3.2 (i), our main analytical achievement is the following hierarchy of interpolating $\ell^1 - \ell^2$ energy estimates for corrector differences, inspired by our previous work [15] on the conductivity problem (which also considers the case of "overlapping particles"; see [20,30] for refinements in that direction). More precisely, we consider the following quantities, for all $H \subset \mathbb{N}$, all L, and $k, j \geq 0$,

$$S_L^H(k,j) := \sum_{\sharp G = k} \fint_{Q_L} \Big| \sum_{\substack{\sharp F = j \\ F \cap G = \varnothing}} \mathrm{D}(\delta^{F \cup G} \psi_L^H) \Big|^2,$$

$$T_L^H(k,j) := L^{-d} \sum_{\sharp G = k} \sum_{n \notin G \cup H} \int_{I_{n,L} + \rho B} \Big| \sum_{\substack{\sharp F = j \\ F \cap (G \cup \{n\}) = \varnothing}} \mathrm{D}(\delta^{F \cup G} \psi_L^H) \Big|^2,$$

and we prove the following result. The novelty with respect to [15] is that we further identify here the optimal dependence on the minimal distance

$$\ell = \ell(\mathcal{P}) \gtrsim 1,$$

which appears to be surprisingly challenging and relies on a fine use of elliptic regularity via a duality argument.

Theorem 3.9 (Uniform $\ell^1 - \ell^2$ energy estimates). *Under Assumptions* (H_ρ) *and* ($\mathrm{H}_\rho^{\mathrm{unif}}$)*, we have for all $H \subset \mathbb{N}$, all L, and $k, j \geq 0$,*

$$S_L^H(k,j) \lesssim \begin{cases} \ell^{-d} & \text{if } k = j = 0, \\ (C\ell^{-d})^{2(k+j)-1} & \text{if } k, j \geq 0,\ k + j \geq 1, \end{cases}$$

$$T_L^H(k,j) \lesssim \begin{cases} \ell^{-2d} & \text{if } k = j = 0, \\ (C\ell^{-d})^{2(k+j)+1} & \text{if } k, j \geq 0,\ k + j \geq 1. \end{cases}$$

The proof is split into two parts in the following two subsections: to simplify the presentation, we first give a short proof in the spirit of [15] without keeping track of the ℓ-dependence, and then we establish the estimates in their stated optimal form.

3.5.1 Proof of Theorem 3.9 without ℓ-dependence

This section is devoted to the proof that for all $H \subset \mathbb{N}$, all L, and $k, j \geq 0$,

$$S_L^H(k,j) + T_L^H(k,j) \lesssim C^{k+j}. \tag{3.39}$$

For notational convenience, we set $S_L^H(k,j) = T_L^H(k,j) = 0$ for $j < 0$ or $k < 0$. We split the proof into three steps.

Step 1. Reduction to S_L^H: for all $H \subset \mathbb{N}$ and L, k, j,

$$T_L^H(k,j) \lesssim S_L^H(k,j) + S_L^H(k,j-1), \tag{3.40}$$

which entails in particular that it suffices to prove the bound (3.39) for S_L^H.

First note that for all maps f and all $n \notin G$ we have

$$\sum_{\substack{\sharp F = j \\ F \cap G = \varnothing}} f(F \cup G) = \sum_{\substack{\sharp F = j \\ F \cap (G \cup \{n\}) = \varnothing}} f(F \cup G) + \sum_{\substack{\sharp F = j-1 \\ F \cap (G \cup \{n\}) = \varnothing}} f\big(F \cup G \cup \{n\}\big). \tag{3.41}$$

Using this identity to decompose $T_L^H(j,k)$ and changing summation variables, we find

$$\begin{aligned} T_L^H(k,j) \lesssim\ & L^{-d} \sum_{\sharp G = k} \sum_{n \notin G \cup H} \int_{I_{n,L} + \rho B} \Big| \sum_{\substack{\sharp F = j \\ F \cap G = \varnothing}} \mathrm{D}(\delta^{F \cup G} \psi_L^H) \Big|^2 \\ & + L^{-d} \sum_{\sharp G = k+1} \sum_{n \in G \setminus H} \int_{I_{n,L} + \rho B} \Big| \sum_{\substack{\sharp F = j-1 \\ F \cap G = \varnothing}} \mathrm{D}(\delta^{F \cup G} \psi_L^H) \Big|^2, \end{aligned}$$

and thus, using the disjointness of the fattened inclusions $\{I_{n,L} + \rho B\}_n$ and recognizing the definition of S_L^H, the claim (3.40) follows.

Step 2. Energy estimate for correctors: for all $H \subset \mathbb{N}$,

$$S_L^H(0,0) \lesssim 1. \tag{3.42}$$

As in (3.19), the energy identity for the corrector equation (3.14) for ψ_L^H takes the form

$$2 \int_{Q_L} \big|\mathrm{D}(\psi_L^H)\big|^2 = \sum_{n \in H} \int_{\partial I_{n,L}} E(x - x_{n,L}) \cdot \sigma_L^H \nu. \tag{3.43}$$

Using the incompressibility constraint $\mathrm{tr}(E) = 0$ to add an arbitrary constant to the pressure in σ_L^H, as in the proof of (3.30), and then appealing to the trace estimates of Lemma 2.5 (ii), we obtain

$$\int_{Q_L} \big|\mathrm{D}(\psi_L^H)\big|^2 \lesssim \sum_{n \in H} \bigg(\int_{I_{n,L} + \rho B} \big|\mathrm{D}(\psi_L^H) + E\big|^2 \bigg)^{\frac{1}{2}}.$$

Since the fattened inclusions $\{I_{n,L} + \rho B\}_n$ are disjoint, the Cauchy–Schwarz inequality then yields, recalling the choice of the periodization (3.1),

$$\int_{Q_L} \big|\mathrm{D}(\psi_L^H)\big|^2 \lesssim \sharp\{n \in H : x_n \in Q_L\}. \tag{3.44}$$

As the right-hand side is bounded by CL^d, the claim (3.42) follows. For future reference, we also note that this bound entails, when taking the expectation,

$$\mathbb{E}\Big[\fint_{Q_L} \big|\mathrm{D}(\psi_L)\big|^2 \Big] \lesssim \lambda(\mathcal{P}). \tag{3.45}$$

Step 3. Key recurrence relation: for all $H \subset \mathbb{N}$ and $k, j \geq 0$,

$$\begin{aligned} S_L^H(k,j) \lesssim \mathbb{1}_{k+j\leq 1} + S_L^H(k+1,j-1) + S_L^H(k,j-1) \\ + S_L^H(k-1,j) + S_L^H(k,j-2) + S_L^H(k-1,j-1), \end{aligned} \tag{3.46}$$

which then leads to the conclusion (3.39) by a direct double induction argument.

Let a finite subset $G \subset \mathbb{N}$ be momentarily fixed. In view of (3.15), the following equation holds in Q_L, for any $F \subset \mathbb{N}$ with $F \cap G = \varnothing$,

$$\begin{aligned} -\triangle \delta^{F\cup G}\psi_L^H + \nabla \delta^{F\cup G}\big(\Sigma_L^H \mathbb{1}_{Q_L\setminus \mathcal{I}_L^H}\big) = &- \sum_{n\in H} \delta_{\partial I_{n,L}} \delta^{F\cup G}\sigma_L^H \nu \\ &- \sum_{n\in F\setminus H} \delta_{\partial I_{n,L}} \delta^{(F\setminus\{n\})\cup G}\sigma_L^{H\cup\{n\}}\nu \\ &- \sum_{n\in G\setminus H} \delta_{\partial I_{n,L}} \delta^{F\cup(G\setminus\{n\})}\sigma_L^{H\cup\{n\}}\nu. \end{aligned}$$

Hence, after summing over F and changing summation variables,

$$\begin{aligned} &-\triangle\Big(\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}} \delta^{F\cup G}\psi_L^H\Big) + \nabla\Big(\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}} \delta^{F\cup G}(\Sigma_L^H \mathbb{1}_{Q_L\setminus\mathcal{I}_L^H})\Big) \\ &= -\sum_{n\in H}\delta_{\partial I_{n,L}}\Big(\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}} \delta^{F\cup G}\sigma_L^H\nu\Big) - \sum_{n\notin G\cup H}\delta_{\partial I_{n,L}}\Big(\sum_{\substack{\sharp F=j-1\\ F\cap(G\cup\{n\})=\varnothing}} \delta^{F\cup G}\sigma_L^{H\cup\{n\}}\nu\Big) \\ &\quad - \sum_{n\in G\setminus H}\delta_{\partial I_{n,L}}\Big(\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}} \delta^{F\cup(G\setminus\{n\})}\sigma_L^{H\cup\{n\}}\nu\Big). \end{aligned}$$

Testing this equation with the solution $\sum_{\sharp F=j:F\cap G=\varnothing}\delta^{F\cup G}\psi_L^H$ itself, we obtain the energy identity

$$2\int_{Q_L}\Big|\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}} \mathrm{D}(\delta^{F\cup G}\psi_L^H)\Big|^2 = A_L^1(G,j) + A_L^2(G,j) + A_L^3(G,j), \tag{3.47}$$

in terms of

$$A_L^1(G,j) := -\sum_{n\in H}\int_{\partial I_{n,L}}\Big(\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}}\delta^{F\cup G}\psi_L^H\Big)\cdot\Big(\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}}\delta^{F\cup G}\sigma_L^H\nu\Big), \tag{3.48}$$

$$A_L^2(G,j) := -\sum_{n\notin G\cup H}\int_{\partial I_{n,L}}\Big(\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}}\delta^{F\cup G}\psi_L^H\Big)\cdot\Big(\sum_{\substack{\sharp F=j-1\\ F\cap(G\cup\{n\})=\varnothing}}\delta^{F\cup G}\sigma_L^{H\cup\{n\}}\nu\Big),$$

$$A_L^3(G,j) := -\sum_{n\in G\setminus H}\int_{\partial I_{n,L}}\Big(\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}}\delta^{F\cup G}\psi_L^H\Big)\cdot\Big(\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}}\delta^{F\cup(G\setminus\{n\})}\sigma_L^{H\cup\{n\}}\nu\Big).$$

We analyze these three contributions separately and we start with the first one. In view of the boundary conditions for $\delta^{F\cup G}\psi_L^H$ on $\partial I_{n,L}$ with $n\in H$, we can rewrite

$$\begin{aligned}A_L^1(G,j) &= \sum_{n\in H}\int_{\partial I_{n,L}}\Big(\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}}\delta^{F\cup G}\big(E(x-x_{n,L})\big)\Big)\cdot\Big(\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}}\delta^{F\cup G}\sigma_L^H\nu\Big)\\ &= \mathbb{1}_{G=\varnothing,j=0}\sum_{n\in H}\int_{\partial I_{n,L}}E(x-x_{n,L})\cdot\sigma_L^H\nu.\end{aligned}$$

Summing over $G\subset\mathbb{N}$ with $\sharp G=k$, and using the energy identity (3.43), we deduce

$$L^{-d}\sum_{\sharp G=k}A_L^1(G,j) = \mathbb{1}_{k=j=0}\,S_L^H(0,0). \tag{3.49}$$

We turn to the second term A_L^2 in (3.47). Using the boundary conditions and the incompressibility constraints to smuggle in arbitrary constants in the different factors, as in the proof of (3.30), and then appealing to the trace estimates of Lemma 3.5, we find

$$\begin{aligned}\big|A_L^2(G,j)\big| \lesssim \sum_{n\notin G\cup H}&\bigg(\int_{I_{n,L}}\Big|\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}}\mathrm{D}(\delta^{F\cup G}\psi_L^H)\Big|^2\bigg)^{\frac12}\\ &\times\bigg(\int_{I_{n,L}+\rho B}\Big|\sum_{\substack{\sharp F=j-1\\ F\cap(G\cup\{n\})=\varnothing}}\mathrm{D}\big(\delta^{F\cup G}(\psi_L^{H\cup\{n\}}+Ex)\big)\Big|^2\bigg)^{\frac12}.\end{aligned} \tag{3.50}$$

Decomposing the second factor via the following identity, for all $n\notin F\cup G\cup H$ and $F\cap G=\varnothing$,

$$\delta^{F\cup G}(\psi_L^{H\cup\{n\}}+Ex) = \mathbb{1}_{G=F=\varnothing}Ex + \delta^{F\cup G}\psi_L^H + \delta^{F\cup G\cup\{n\}}\psi_L^H,$$

summing over $G\subset\mathbb{N}$ with $\sharp G=k$, using the Cauchy–Schwarz inequality, and using the disjointness of the fattened inclusions $\{I_{n,L}+\rho B\}_n$, we get

$$\begin{aligned}&L^{-d}\sum_{\sharp G=k}\big|A_L^2(G,j)\big|\\ &\lesssim \big(S_L^H(k,j)\big)^{\frac12}\big(\mathbb{1}_{k=0,j=1}+T_L^H(k,j-1)+S_L^H(k+1,j-1)\big)^{\frac12}.\end{aligned} \tag{3.51}$$

We turn to the third contribution A_L^3 in (3.47). For $n \in G \setminus H$ and $F \cap G = \varnothing$, decomposing

$$\delta^{F\cup G}\psi_L^H = \delta^{F\cup(G\setminus\{n\})}\psi_L^{H\cup\{n\}} - \delta^{F\cup(G\setminus\{n\})}\psi_L^H,$$

and using the boundary conditions, we can rewrite

$$\begin{aligned} A^3_{L,\ell}(G,j) &= \mathbb{1}_{\sharp G=1,j=0}\sum_{n\in G\setminus H}\int_{\partial I_{n,L}} E(x-x_{n,L})\cdot\sigma_L^{H\cup\{n\}}\nu \\ &+ \sum_{n\in G\setminus H}\int_{\partial I_{n,L}}\Big(\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}}\delta^{F\cup(G\setminus\{n\})}\psi_L^H\Big)\cdot\Big(\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}}\delta^{F\cup(G\setminus\{n\})}\sigma_L^{H\cup\{n\}}\nu\Big). \end{aligned}$$

Using the boundary conditions and the incompressibility constraints to smuggle in arbitrary constants in the different factors, as in the proof of (3.30), and then appealing to the trace estimates of Lemma 3.5, we find

$$\begin{aligned} \big|A_L^3(G,j)\big| &\lesssim \mathbb{1}_{\sharp G=1,j=0}\sum_{n\in G\setminus H}\Big(\int_{I_{n,L}+\rho B}\big|\mathrm{D}(\psi_L^{H\cup\{n\}})+E\big|^2\Big)^{\frac12} \\ &+ \sum_{n\in G\setminus H}\Big(\int_{I_{n,L}}\Big|\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}}\mathrm{D}(\delta^{F\cup(G\setminus\{n\})}\psi_L^H)\Big|^2\Big)^{\frac12} \\ &\times\Big(\int_{I_{n,L}+\rho B}\Big|\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}}\mathrm{D}\big(\delta^{F\cup(G\setminus\{n\})}(\psi_L^{H\cup\{n\}}+Ex)\big)\Big|^2\Big)^{\frac12}. \end{aligned} \tag{3.52}$$

Decomposing the first right-hand side term and the last factor of the second term via the following identities, for all $n \in G \setminus H$ and $F \cap G = \varnothing$,

$$\begin{aligned} \psi_L^{H\cup\{n\}} &= \psi_L^H + \delta^{\{n\}}\psi_L^H, \\ \delta^{F\cup(G\setminus\{n\})}(\psi_L^{H\cup\{n\}}+Ex) &= \mathbb{1}_{\sharp G=1,\sharp F=0}Ex + \delta^{F\cup(G\setminus\{n\})}\psi_L^H \\ &\quad + \delta^{F\cup(G\setminus\{n\})}\delta^{\{n\}}\psi_L^H, \end{aligned} \tag{3.53}$$

summing over $G \subset \mathbb{N}$ with $\sharp G = k$, and using the Cauchy–Schwarz inequality and the disjointness of the fattened inclusions $\{I_{n,L}+\rho B\}_n$, this becomes

$$\begin{aligned} &L^{-d}\sum_{\sharp G=k}\big|A_L^3(G,j)\big| \\ &\lesssim \mathbb{1}_{k=1,j=0}\big(1+S_L^H(0,0)+S_L^H(1,0)\big)^{\frac12} \\ &\quad+\big(T_L^H(k-1,j)\big)^{\frac12}\big(\mathbb{1}_{k=1,j=0}+T_L^H(k-1,j)+S_L^H(k,j)\big)^{\frac12}. \end{aligned} \tag{3.54}$$

Inserting this into (3.47), together with (3.49) and (3.51), we obtain

$$\begin{aligned} S_L^H(k,j) \lesssim\ & \mathbb{1}_{k=0,j=0}\, S_L^H(0,0) + \mathbb{1}_{k=1,j=0}\big(1 + S_L^H(0,0) + S_L^H(1,0)\big)^{\frac12} \\ & + \big(S_L^H(k,j)\big)^{\frac12}\big(\mathbb{1}_{k=0,j=1} + T_L^H(k,j-1) + S_L^H(k+1,j-1)\big)^{\frac12} \\ & + \big(T_L^H(k-1,j)\big)^{\frac12}\big(\mathbb{1}_{k=1,j=0} + T_L^H(k-1,j) + S_L^H(k,j)\big)^{\frac12}. \end{aligned}$$

Using Young's inequality to absorb the occurrences of $S_L^H(k,j)$ in the right-hand side into the left-hand side, we are led to

$$\begin{aligned} S_L^H(k,j) \lesssim\ & \mathbb{1}_{k=0,j=0}\, S_L^H(0,0) + \mathbb{1}_{k=0,j=1} + \mathbb{1}_{k=1,j=0}\big(1 + S_L^H(0,0)\big)^{\frac12} \\ & + S_L^H(k+1,j-1) + T_L^H(k,j-1) + T_L^H(k-1,j), \end{aligned}$$

and the claim (3.46) now follows in combination with (3.40) and (3.42). ∎

3.5.2 Proof of Theorem 3.9 with optimal ℓ-dependence

It remains to refine the proof of the previous section to capture the optimal dependence on the minimal distance

$$\ell = \ell(\mathcal{P}) \gtrsim 1.$$

The proof involves a new intricate induction argument that combines both S_L^H and T_L^H, and the optimal scaling is then captured by a suitable application of elliptic regularity via a duality argument. By the result of the previous section, we may assume $\ell \gg 1$, in which case the uniform separation assumption in ($\mathrm{H}_\rho^{\mathrm{unif}}$) holds in the stronger form of

$$\frac12 \inf_{n\neq m} \operatorname{dist}(I_{n,L}, I_{m,L}) \geq \frac12\ell - 1 \geq \frac14\ell \geq \rho, \tag{3.55}$$

and the definition (3.1) of the periodization further ensures

$$\inf_n \operatorname{dist}(I_{n,L}, \partial Q_L) \geq \ell - 1 \geq \frac12\ell \geq \rho.$$

We split the proof into four steps.

Step 1. Energy estimate for correctors: for all $H \subset \mathbb{N}$,

$$S_L^H(0,0) = \fint_{Q_L} \big|\mathrm{D}(\psi_L^H)\big|^2 \lesssim \ell^{-d}, \tag{3.56}$$

$$T_L^H(0,0) = L^{-d} \sum_{n\notin H} \int_{I_{n,L}+\rho B} \big|\mathrm{D}(\psi_L^H)\big|^2 \lesssim \ell^{-2d}. \tag{3.57}$$

By the ℓ-separation property (3.55), the number of points of the process $\mathcal{P}_L$ in Q_L is bounded by $C(L/\ell)^d$, so that the first estimate (3.56) follows from (3.44). It remains to prove (3.57). For that purpose, first note that for $n \notin H$ the ℓ-separation property (3.55) entails that the following free steady Stokes equations hold in $I_{n,L} + \frac{1}{4}\ell B \subset Q_L \setminus \mathcal{I}_L^H$,

$$-\triangle \psi_L^H + \nabla \Sigma_L^H = 0, \quad \operatorname{div}(\psi_L^H) = 0, \quad \text{in } I_{n,L} + \frac{1}{4}\ell B.$$

Elliptic regularity in form of Lemma 2.6 then yields

$$\int_{I_{n,L}+\rho B} \left|\mathrm{D}(\psi_L^H)\right|^2 \lesssim \ell^{-d} \int_{I_{n,L}+\frac{1}{4}\ell B} \left|\mathrm{D}(\psi_L^H)\right|^2.$$

Summing this over $n \notin H$ and using the ℓ-separation property (3.55) in form of the disjointness of the fattened inclusions $\{I_{n,L} + \frac{1}{4}\ell B\}_n$, we deduce

$$\sum_{n\notin H} \int_{I_{n,L}+\rho B} \left|\mathrm{D}(\psi_L^H)\right|^2 \lesssim \ell^{-d} \int_{Q_L} \left|\mathrm{D}(\psi_L^H)\right|^2,$$

and the claim (3.57) now follows from (3.56).

Step 2. Recurrence relation for S_L^H: for all $H \subset \mathbb{N}$ and $k, j \geq 0$,

$$\begin{aligned} S_L^H(k,j) \lesssim\ & \mathbb{1}_{k+j\leq 1}\ell^{-d} + S_L^H(k+1,j-1) \\ & + T_L^H(k,j) + T_L^H(k,j-1) \\ & + T_L^H(k-1,j). \end{aligned} \tag{3.58}$$

This provides a refined version of the recurrence relation (3.46), which can indeed be recovered by appealing to (3.40) to bound T_L^H in terms of S_L^H. The present refined version will be combined with a recurrence relation for T_L^H in the next step.

Let $G \subset \mathbb{N}$ be momentarily fixed. As in the proof of (3.46), the starting point is identity (3.47), that is,

$$2\int_{Q_L} \Big| \sum_{\substack{\sharp F = j \\ F\cap G = \varnothing}} \mathrm{D}(\delta^{F\cup G}\psi_L^H)\Big|^2 = A_L^1(G,j) + A_L^2(G,j) + A_L^3(G,j), \tag{3.59}$$

where we recall that A_L^1, A_L^2, A_L^3 are defined in (3.48). We analyze these contributions separately. The first one satisfies (3.49), and thus, combined with the energy estimate (3.56),

$$L^{-d} \sum_{\sharp G = k} A_L^1(G,j) = \mathbb{1}_{k=j=0}\, S_L^H(0,0) \lesssim \mathbb{1}_{k=j=0}\ell^{-d}. \tag{3.60}$$

It remains to prove refined versions of (3.51) and (3.54) for A_L^2 and A_L^3, and we start with the contribution of A_L^2. The starting point is the trace estimate (3.50) used in the proof of (3.51), that is,

$$\begin{aligned}\big|A_L^2(G,j)\big| \lesssim \sum_{n\notin G\cup H}\bigg(\int_{I_{n,L}}\Big|\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}}\mathrm{D}(\delta^{F\cup G}\psi_L^H)\Big|^2\bigg)^{\frac12}\\ \times\bigg(\int_{I_{n,L}+\rho B}\Big|\sum_{\substack{\sharp F=j-1\\ F\cap(G\cup\{n\})=\varnothing}}\mathrm{D}\big(\delta^{F\cup G}(\psi_L^{H\cup\{n\}}+Ex)\big)\Big|^2\bigg)^{\frac12},\end{aligned}$$

which we shall now analyze more carefully. Using identity (3.41) to decompose the first factor, and decomposing the second factor via the following identity, for all $n\notin F\cup G\cup H$ and $F\cap G=\varnothing$,

$$\delta^{F\cup G}(\psi_L^{H\cup\{n\}}+Ex)=\mathbb{1}_{G=F=\varnothing}Ex+\delta^{F\cup G}\psi_L^H+\delta^{F\cup G\cup\{n\}}\psi_L^H,$$

we find

$$\begin{aligned}\big|A_L^2(G,j)\big| \lesssim \sum_{n\notin G\cup H}\bigg(\int_{I_{n,L}}\Big|\underbrace{\sum_{\substack{\sharp F=j\\ F\cap(G\cup\{n\})=\varnothing}}\mathrm{D}(\delta^{F\cup G}\psi_L^H)}_{\clubsuit}\Big|^2\\ +\underbrace{\Big|\sum_{\substack{\sharp F=j-1\\ F\cap(G\cup\{n\})=\varnothing}}\mathrm{D}(\delta^{F\cup G\cup\{n\}}\psi_L^H)\Big|^2}_{\diamondsuit}\bigg)^{\frac12}\\ \times\bigg(\mathbb{1}_{\sharp G=0,j=1}+\int_{I_{n,L}+\rho B}\Big|\underbrace{\sum_{\substack{\sharp F=j-1\\ F\cap(G\cup\{n\})=\varnothing}}\mathrm{D}(\delta^{F\cup G}\psi_L^H)}_{\spadesuit}\Big|^2\\ +\underbrace{\Big|\sum_{\substack{\sharp F=j-1\\ F\cap(G\cup\{n\})=\varnothing}}\mathrm{D}(\delta^{F\cup G\cup\{n\}}\psi_L^H)\Big|^2}_{\diamondsuit}\bigg)^{\frac12}.\end{aligned}$$

Summing over $G\subset\mathbb{N}$ with $\sharp G=k$, using Young's inequality, using the separation property in form of the disjointness of the fattened inclusions $\{I_{n,L}+\rho B\}_n$, using that the number of points of the process $\mathcal{P}_L$ in Q_L is bounded by $C(L/\ell)^d$, and

reorganizing the terms, we conclude

$$L^{-d}\sum_{\sharp G=k}\left|A_L^2(G,j)\right| \lesssim \ell^{-d}\mathbb{1}_{k=0,j=1} + S_L^H(k+1,j-1) + T_L^H(k,j) + T_L^H(k,j-1), \tag{3.61}$$

where the last three right-hand side terms come from $\diamondsuit$, $\clubsuit$, $\spadesuit$, respectively.

We turn to the contribution of A_L^3. The starting point is the trace estimate (3.52) used in the proof of (3.54). Further, using the decomposition (3.53), this estimate becomes

$$\begin{aligned}
&\left|A_L^3(G,j)\right| \\
&\lesssim \mathbb{1}_{\sharp G=1,j=0}\sum_{n\in G\setminus H}\left(1+\int_{I_{n,L}+\rho B}\left|\mathrm{D}(\psi_L^H)\right|^2+\left|\mathrm{D}(\delta^{\{n\}}\psi_L^H)\right|^2\right)^{\frac12} \\
&\quad+\sum_{n\in G\setminus H}\left(\int_{I_{n,L}}\Big|\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}}\mathrm{D}(\delta^{F\cup(G\setminus\{n\})}\psi_L^H)\Big|^2\right)^{\frac12} \\
&\quad\times\left(\int_{I_{n,L}+\rho B}\Big|\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}}\mathrm{D}(\delta^{F\cup(G\setminus\{n\})}\psi_L^H)\Big|^2+\Big|\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}}\mathrm{D}(\delta^{F\cup G}\psi_L^H)\Big|^2\right)^{\frac12}.
\end{aligned} \tag{3.62}$$

Summing the first right-hand side term over $G\subset\mathbb{N}$ with $\sharp G=1$, using the Cauchy–Schwarz inequality, recalling that the number of points of the process $\mathcal{P}_L$ in Q_L is bounded by $C(L/\ell)^d$, and appealing to the energy estimate (3.57), we find

$$\begin{aligned}
&\sum_{n\notin H}\left(1+\int_{I_{n,L}+\rho B}\left|\mathrm{D}(\psi_L^H)\right|^2+\left|\mathrm{D}(\delta^{\{n\}}\psi_L^H)\right|^2\right)^{\frac12} \\
&\lesssim L^{\frac{d}{2}}\ell^{-\frac{d}{2}}\left(L^d\ell^{-d}+\sum_{n\notin H}\int_{I_{n,L}+\rho B}\left|\mathrm{D}(\psi_L^H)\right|^2+\sum_{n\notin H}\int_{Q_L}\left|\mathrm{D}(\delta^{\{n\}}\psi_L^H)\right|^2\right)^{\frac12} \\
&\lesssim L^d\left(\ell^{-2d}+\ell^{-d}S_L^H(1,0)\right)^{\frac12}.
\end{aligned}$$

Now summing (3.62) over $G\subset\mathbb{N}$ with $\sharp G=k$, inserting the above estimate for the first right-hand side term, and using the Cauchy–Schwarz inequality, we find

$$\begin{aligned}
L^{-d}\sum_{\sharp G=k}\left|A_L^3(G,j)\right| &\lesssim \mathbb{1}_{k=1,j=0}\left(\ell^{-2d}+\ell^{-d}S_L^H(1,0)\right)^{\frac12} \\
&\quad+\left(T_L^H(k-1,j)\right)^{\frac12}\left(T_L^H(k-1,j)+S_L^H(k,j)\right)^{\frac12}.
\end{aligned}$$

Inserting this into (3.59), together with (3.60) and (3.61), we conclude

$$\begin{aligned} S_L^H(k,j) \lesssim\ & \mathbb{1}_{k=0,j\le 1}\ell^{-d} + \mathbb{1}_{k=1,j=0}\big(\ell^{-2d}+\ell^{-d}S_L^H(1,0)\big)^{\frac12} \\ & + S_L^H(k+1,j-1) + T_L^H(k,j) + T_L^H(k,j-1) \\ & + \big(T_L^H(k-1,j)\big)^{\frac12}\big(T_L^H(k-1,j)+S_L^H(k,j)\big)^{\frac12}. \end{aligned}$$

Using Young's inequality to absorb the occurrence of $S_L^H(k,j)$ in the right-hand side into the left-hand side, the claim (3.58) follows.

Step 3. Recurrence relation for T_L^H: for all $H\subset\mathbb{N}$ and $k,j\ge 0$,

$$\begin{aligned} T_L^H(k,j) \lesssim\ & \mathbb{1}_{k=j=0}\ell^{-2d} + \mathbb{1}_{k+j=1}\ell^{-3d} \\ & + \ell^{-2d}\big(T_L^H(k-1,j)+T_L^H(k,j-1)+T_L^H(k+1,j-2) \\ & + S_L^H(k,j)+S_L^H(k+1,j-1)+S_L^H(k+2,j-2)\big). \end{aligned} \tag{3.63}$$

Let $k,j\ge 0$ be fixed with $k+j\ge 1$ (the case $k=j=0$ already follows from (3.57)). For $G\subset\mathbb{N}$ and $n\notin G$, the ℓ-separation property (3.55) implies that the following free steady Stokes equations hold in $I_{n,L}+\frac14\ell B$,

$$\begin{aligned} & -\triangle\Big(\sum_{\substack{\sharp F=j\\ F\cap(G\cup\{n\})=\varnothing}}\delta^{F\cup G}\psi_L^H\Big)+\nabla\Big(\sum_{\substack{\sharp F=j\\ F\cap(G\cup\{n\})=\varnothing}}\delta^{F\cup G}(\Sigma_L^H\mathbb{1}_{Q_L\setminus\mathcal{I}_L^H})\Big)=0, \\ & \operatorname{div}\Big(\sum_{\substack{\sharp F=j\\ F\cap(G\cup\{n\})=\varnothing}}\delta^{F\cup G}\psi_L^H\Big)=0, \quad \text{in } I_{n,L}+\frac14\ell B, \end{aligned}$$

so that elliptic regularity in form of Lemma 2.6 yields

$$T_L^H(k,j)\lesssim L^{-d}\ell^{-d}\sum_{\sharp G=k}\sum_{n\notin G\cup H}\int_{I_{n,L}+\frac14\ell B}\Big|\sum_{\substack{\sharp F=j\\ F\cap(G\cup\{n\})=\varnothing}}\mathrm{D}(\delta^{F\cup G}\psi_L^H)\Big|^2. \tag{3.64}$$

In order to analyze the right-hand side, we shall appeal to elliptic regularity a second time, now via a duality argument. For that purpose, we use the following dual representation

$$\begin{aligned} & \sum_{\sharp G=k}\sum_{n\notin G\cup H}\int_{I_{n,L}+\frac14\ell B}\Big|\sum_{\substack{\sharp F=j\\ F\cap(G\cup\{n\})=\varnothing}}\mathrm{D}(\delta^{F\cup G}\psi_L^H)\Big|^2 \\ & \quad=\sup_{\alpha,h}\Big\{I(\alpha,h)^2:\sum_{\sharp G=k}\sum_{n\notin G\cup H}|\alpha_{n,G}|^2=1, \\ & \qquad \int_{Q_L}|h_{n,G}|^2=1,\ \operatorname{supp}h_{n,G}\subset I_{n,L}+\frac14\ell B,\ \forall n,G\Big\}, \end{aligned} \tag{3.65}$$

where for any $\alpha = \{\alpha_{n,G}\}_{n,G} \subset \mathbb{R}$ and $h = \{h_{n,G}\}_{n,G} \subset \mathrm{L}^2(Q_L)^{d\times d}_{\mathrm{sym}}$ we have set for abbreviation

$$I(\alpha,h) := \sum_{\sharp G=k} \sum_{n\notin G\cup H} \alpha_{n,G} \int_{Q_L} h_{n,G} : \Big(\sum_{\substack{\sharp F=j \\ F\cap(G\cup\{n\})=\varnothing}} \mathrm{D}(\delta^{F\cup G}\psi_L^H)\Big). \tag{3.66}$$

Let $\alpha = \{\alpha_{n,G}\}_{n,G} \subset \mathbb{R}$ and $h = \{h_{n,G}\}_{n,G} \subset \mathrm{L}^2(Q_L)^{d\times d}_{\mathrm{sym}}$ be momentarily fixed, satisfying the constraints in (3.65),

$$\sum_{\sharp G=k} \sum_{n\notin G\cup H} |\alpha_{n,G}|^2 = 1, \quad \int_{Q_L} |h_{n,G}|^2 = 1, \quad \operatorname{supp} h_{n,G} \subset I_{n,L} + \frac{1}{4}\ell B, \quad \forall n, G. \tag{3.67}$$

For $n \notin G \cup H$, consider the periodic solution $w_{h,n,G}$ of the following auxiliary steady Stokes problem,

$$\begin{cases} -\triangle w_{h,n,G} + \nabla P_{h,n,G} = \operatorname{div}(h_{n,G}), & \text{in } Q_L \setminus \mathcal{I}_L^H, \\ \operatorname{div}(w_{h,n,G}) = 0, & \text{in } Q_L \setminus \mathcal{I}_L^H, \\ \mathrm{D}(w_{h,n,G}) = 0, & \text{in } \mathcal{I}_L^H, \\ \int_{\partial I_{m,L}} \sigma(w_{h,n,G}, P_{h,n,G})\nu = 0, & \forall m \in H, \\ \int_{\partial I_{m,L}} \Theta(x-x_{m,L})\cdot\sigma(w_{h,n,G}, P_{h,n,G})\nu = 0, & \forall \Theta\in\mathbb{M}^{\mathrm{skew}},\ \forall m \in H. \end{cases} \tag{3.68}$$

Note that this problem is well posed since $h_{n,G}$ is supported in

$$I_{n,L} + \frac{1}{4}\ell B \subset Q_L \setminus \mathcal{I}_L^H.$$

The same argument as for (3.14) shows that $w_{h,n,G}$ satisfies in Q_L,

$$-\triangle w_{h,n,G} + \nabla\big(P_{h,n,G}\mathbb{1}_{Q_L\setminus\mathcal{I}_L^H}\big) = \operatorname{div}(h_{n,G}) - \sum_{m\in H} \delta_{\partial I_{m,L}}\sigma(w_{h,n,G}, P_{h,n,G})\nu,$$

and, appealing to (3.15) and changing summation variables, we also find in Q_L,

$$\begin{aligned} &-\triangle\Big(\sum_{\substack{\sharp F=j \\ F\cap(G\cup\{n\})=\varnothing}} \delta^{F\cup G}\psi_L^H\Big) + \nabla\Big(\sum_{\substack{\sharp F=j \\ F\cap(G\cup\{n\})=\varnothing}} \delta^{F\cup G}(\Sigma_L^H\mathbb{1}_{Q_L\setminus\mathcal{I}_L^H})\Big) \\ &\quad = -\sum_{m\in H} \delta_{\partial I_{m,L}}\Big(\sum_{\substack{\sharp F=j \\ F\cap(G\cup\{n\})=\varnothing}} \delta^{F\cup G}\sigma_L^H\nu\Big) \\ &\qquad -\sum_{m\in G\setminus H} \delta_{\partial I_{m,L}}\Big(\sum_{\substack{\sharp F=j \\ F\cap(G\cup\{n\})=\varnothing}} \delta^{F\cup(G\setminus\{m\})}\sigma_L^{H\cup\{m\}}\nu\Big) \\ &\qquad -\sum_{m\notin G\cup H\cup\{n\}} \delta_{\partial I_{m,L}}\Big(\sum_{\substack{\sharp F=j-1 \\ F\cap(G\cup\{n,m\})=\varnothing}} \delta^{F\cup G}\sigma_L^{H\cup\{m\}}\nu\Big). \end{aligned}$$

Testing the second of these two equations with the solution of the first one, and vice versa, and using the boundary conditions, we can reformulate $I(\alpha, h)$ in (3.66) as follows, provided $k + j \geq 1$,

$$\begin{aligned} I(\alpha, h) &= -\sum_{\sharp G = k} \sum_{n \notin G \cup H} 2\alpha_{n,G} \int_{Q_L} \mathrm{D}(w_{h,n,G}) : \Big(\sum_{\substack{\sharp F = j \\ F \cap (G \cup \{n\}) = \varnothing}} \mathrm{D}(\delta^{F \cup G} \psi_L^H) \Big) \\ &= I_1(\alpha, h) + I_2(\alpha, h), \end{aligned} \tag{3.69}$$

where we have set

$$\begin{aligned} &I_1(\alpha, h) \\ &:= \sum_{\sharp G = k} \sum_{n \notin G \cup H} \alpha_{n,G} \sum_{m \in G \setminus H} \int_{\partial I_{m,L}} w_{h,n,G} \cdot \Big(\sum_{\substack{\sharp F = j \\ F \cap (G \cup \{n\}) = \varnothing}} \delta^{F \cup (G \setminus \{m\})} \sigma_L^{H \cup \{m\}} \nu \Big), \end{aligned}$$

$$\begin{aligned} &I_2(\alpha, h) \\ &:= \sum_{\sharp G = k} \sum_{n \notin G \cup H} \alpha_{n,G} \sum_{m \notin G \cup H \cup \{n\}} \int_{\partial I_{m,L}} w_{h,n,G} \cdot \Big(\sum_{\substack{\sharp F = j-1 \\ F \cap (G \cup \{n,m\}) = \varnothing}} \delta^{F \cup G} \sigma_L^{H \cup \{m\}} \nu \Big). \end{aligned}$$

We only treat $I_1(\alpha, h)$ in detail since the argument for $I_2(\alpha, h)$ is similar. Appealing to identity (3.41), we can rewrite

$$\begin{aligned} &I_1(\alpha, h) \\ &= \sum_{\sharp G = k} \sum_{m \in G \setminus H} \int_{\partial I_{m,L}} \Big(\sum_{n \notin G \cup H} \alpha_{n,G} w_{h,n,G} \Big) \cdot \Big(\sum_{\substack{\sharp F = j \\ F \cap G = \varnothing}} \delta^{F \cup (G \setminus \{m\})} \sigma_L^{H \cup \{m\}} \nu \Big) \\ &- \sum_{\sharp G = k} \sum_{n \notin G \cup H} \alpha_{n,G} \sum_{m \in G \setminus H} \int_{\partial I_{m,L}} w_{h,n,G} \cdot \Big(\sum_{\substack{\sharp F = j-1 \\ F \cap (G \cup \{n\}) = \varnothing}} \delta^{F \cup (G \setminus \{m\}) \cup \{n\}} \sigma_L^{H \cup \{m\}} \nu \Big), \end{aligned}$$

or equivalently, after further changing summation variables in the second term,

$$\begin{aligned} I_1(\alpha, h) = &\sum_{\sharp G = k} \sum_{m \in G \setminus H} \int_{\partial I_{m,L}} \Big(\sum_{n \notin G \cup H} \alpha_{n,G} w_{h,n,G} \Big) \\ &\qquad \cdot \Big(\sum_{\substack{\sharp F = j \\ F \cap G = \varnothing}} \delta^{F \cup (G \setminus \{m\})} \sigma_L^{H \cup \{m\}} \nu \Big) \\ &- \sum_{\sharp G = k+1} \sum_{m \in G \setminus H} \int_{\partial I_{m,L}} \Big(\sum_{n \in G \setminus (H \cup \{m\})} \alpha_{n, G \setminus \{n\}} w_{h,n,G \setminus \{n\}} \Big) \\ &\qquad \cdot \Big(\sum_{\substack{\sharp F = j-1 \\ F \cap G = \varnothing}} \delta^{F \cup (G \setminus \{m\})} \sigma_L^{H \cup \{m\}} \nu \Big). \end{aligned}$$

Now using the boundary conditions and the incompressibility constraints to add arbitrary constants to the different factors, as in the proof of (3.30), and appealing to the trace estimates of Lemma 3.5, we are led to

$$\big|I_1(\alpha,h)\big| \lesssim I_{1,1}(\alpha,h) + I_{1,2}(\alpha,h), \tag{3.70}$$

where we have set

$$\begin{aligned}
I_{1,1}(\alpha,h) :=& \sum_{\sharp G=k}\sum_{m\in G\setminus H}\bigg(\int_{I_{m,L}}\Big|\sum_{n\notin G\cup H}\alpha_{n,G}\mathrm{D}(w_{h,n,G})\Big|^2\bigg)^{\frac12}\\
&\times\bigg(\int_{I_{m,L}+\rho B}\Big|\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}}\mathrm{D}\big(\delta^{F\cup(G\setminus\{m\})}(\psi_L^{H\cup\{m\}}+Ex)\big)\Big|^2\bigg)^{\frac12},\\
I_{1,2}(\alpha,h) :=& \sum_{\sharp G=k+1}\sum_{m\in G\setminus H}\bigg(\int_{I_{m,L}}\Big|\sum_{n\in G\setminus(H\cup\{m\})}\alpha_{n,G\setminus\{n\}}\mathrm{D}(w_{h,n,G\setminus\{n\}})\Big|^2\bigg)^{\frac12}\\
&\times\bigg(\int_{I_{m,L}+\rho B}\Big|\sum_{\substack{\sharp F=j-1\\ F\cap G=\varnothing}}\mathrm{D}\big(\delta^{F\cup(G\setminus\{m\})}(\psi_L^{H\cup\{m\}}+Ex)\big)\Big|^2\bigg)^{\frac12}.
\end{aligned}$$

We start by estimating $I_{1,1}(\alpha,h)$. Decomposing the second factor via the following identity, for all $m\in G\setminus H$ and $F\cap G=\varnothing$,

$$\delta^{F\cup(G\setminus\{m\})}(\psi_L^{H\cup\{m\}}+Ex) = \mathbb{1}_{\sharp G=1,\sharp F=0}Ex + \delta^{F\cup(G\setminus\{m\})}\psi_L^H + \delta^{F\cup G}\psi_L^H,$$

noting that the ℓ-separation property (3.55) entails that $\sum_{n\notin G\cup H}\alpha_{n,G}w_{h,n,G}$ satisfies the free steady Stokes equations in $I_{m,L}+\frac14\ell B$ for all $m\notin H$, and appealing to elliptic regularity in form of Lemma 2.6, we find

$$\begin{aligned}
I_{1,1}(\alpha,h) \lesssim& \sum_{\sharp G=k}\sum_{m\in G\setminus H}\bigg(\ell^{-d}\int_{I_{m,L}+\frac14\ell B}\Big|\sum_{n\notin G\cup H}\alpha_{n,G}\mathrm{D}(w_{h,n,G})\Big|^2\bigg)^{\frac12}\\
&\times\bigg(\mathbb{1}_{k=1,j=0}+\int_{I_{m,L}+\rho B}\Big|\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}}\mathrm{D}(\delta^{F\cup(G\setminus\{m\})}\psi_L^H)\Big|^2\\
&\qquad+\int_{I_{m,L}+\rho B}\Big|\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}}\mathrm{D}(\delta^{F\cup G}\psi_L^H)\Big|^2\bigg)^{\frac12}.
\end{aligned}\tag{3.71}$$

Next, the energy estimate for (3.68) yields

$$\sum_{\sharp G=k}\int_{Q_L}\Big|\sum_{n\notin G\cup H}\alpha_{n,G}\mathrm{D}(w_{h,n,G})\Big|^2 \lesssim \sum_{\sharp G=k}\int_{Q_L}\Big|\sum_{n\notin G\cup H}\alpha_{n,G}h_{n,G}\Big|^2,$$

and thus, using the constraints (3.67) on α, h, and noting that the ℓ-separation property (3.55) entails that the $h_{n,G}$'s have disjoint supports for different n's,

$$\sum_{\sharp G=k}\int_{Q_L}\Big|\sum_{n\notin G\cup H}\alpha_{n,G}\mathrm{D}(w_{h,n,G})\Big|^2\lesssim\sum_{\sharp G=k}\sum_{n\notin G\cup H}|\alpha_{n,G}|^2\int_{Q_L}|h_{n,G}|^2=1.$$

Inserting this into (3.71), using the Cauchy–Schwarz inequality, the ℓ-separation property (3.55) in form of the disjointness of the fattened inclusions $\{I_{m,L}+\frac{1}{4}\ell B\}_m$, using that the number of points of the process $\mathcal{P}_L$ in Q_L is bounded by $C(L/\ell)^d$, and changing summation variables, we deduce

$$L^{-d}I_{1,1}(\alpha,h)^2\lesssim\mathbb{1}_{k=1,j=0}\ell^{-2d}+\ell^{-d}\big(T_L^H(k-1,j)+S_L^H(k,j)\big).\tag{3.72}$$

We turn to a corresponding estimation for $I_{1,2}(\alpha,h)$. For that purpose, we first note that the disjointness of fattened inclusions $\{I_{m,L}+\frac{1}{4}\ell B\}_m$ allows us to decompose

$$\begin{aligned}&\sum_{\sharp G=k+1}\sum_{m\in G\setminus H}\int_{I_{m,L}+\frac{1}{4}\ell B}\Big|\sum_{n\in G\setminus(H\cup\{m\})}\alpha_{n,G\setminus\{n\}}\mathrm{D}(w_{h,n,G\setminus\{n\}})\Big|^2\\&\lesssim\sum_{\sharp G=k+1}\int_{Q_L}\Big|\sum_{n\in G\setminus H}\alpha_{n,G\setminus\{n\}}\mathrm{D}(w_{h,n,G\setminus\{n\}})\Big|^2\\&\quad+\sum_{\sharp G=k+1}\sum_{m\in G\setminus H}|\alpha_{m,G\setminus\{m\}}|^2\int_{Q_L}\big|\mathrm{D}(w_{h,m,G\setminus\{m\}})\big|^2,\end{aligned}$$

and the energy estimate for (3.68) then yields

$$\begin{aligned}&\sum_{\sharp G=k+1}\sum_{m\in G\setminus H}\int_{I_{m,L}+\frac{1}{4}\ell B}\Big|\sum_{n\in G\setminus(H\cup\{m\})}\alpha_{n,G\setminus\{n\}}\mathrm{D}(w_{h,n,G\setminus\{n\}})\Big|^2\\&\lesssim\sum_{\sharp G=k+1}\int_{Q_L}\Big|\sum_{n\in G\setminus H}\alpha_{n,G\setminus\{n\}}h_{n,G\setminus\{n\}}\Big|^2\\&\quad+\sum_{\sharp G=k+1}\sum_{m\in G\setminus H}|\alpha_{m,G\setminus\{m\}}|^2\int_{Q_L}|h_{m,G\setminus\{m\}}|^2,\end{aligned}$$

from which we deduce, using the constraints (3.67) on α, h and recalling that the $h_{n,G}$'s have disjoint supports for different n's,

$$\begin{aligned}&\sum_{\sharp G=k+1}\sum_{m\in G\setminus H}\int_{I_{m,L}+\frac{1}{4}\ell B}\Big|\sum_{n\in G\setminus(H\cup\{m\})}\alpha_{n,G\setminus\{n\}}\mathrm{D}(w_{h,n,G\setminus\{n\}})\Big|^2\\&\lesssim\sum_{\sharp G=k+1}\sum_{n\in G\setminus H}|\alpha_{n,G\setminus\{n\}}|^2=\sum_{\sharp G=k}\sum_{n\notin G\cup H}|\alpha_{n,G}|^2=1.\end{aligned}$$

With this estimate at hand, we may now repeat the same argument as for (3.72) and we obtain

$$L^{-d} I_{1,2}(\alpha,h)^2 \lesssim \mathbb{1}_{k=0,j=1}\ell^{-2d} + \ell^{-d}\big(T_L^H(k,j-1) + S_L^H(k+1,j-1)\big). \tag{3.73}$$

Likewise, the second term $I_2(\alpha,h)$ in (3.69) is easily estimated as follows,

$$L^{-d} I_2(\alpha,h)^2 \lesssim \mathbb{1}_{k=0,j=1}\ell^{-2d} + \ell^{-d}\big(T_L^H(k,j-1) + T_L^H(k+1,j-2) + S_L^H(k+1,j-1) + S_L^H(k+2,j-2)\big). \tag{3.74}$$

Combining these different estimates, that is, (3.70), (3.72), (3.73), and (3.74), inserting them into (3.65), and recalling (3.64), the claim (3.63) follows.

Step 4. Conclusion. By a direct double induction argument, starting with (3.57), the recurrence relation (3.63) entails, for all $H \subset \mathbb{N}$ and $k, j \geq 0$,

$$T_L^H(k,j) \lesssim \mathbb{1}_{k=j=0}\ell^{-2d} + \mathbb{1}_{k+j\geq 1}(C\ell^{-d})^{2(k+j)+1} + \sum_{l=0}^{k+j-1}(C\ell^{-d})^{2(l+1)}\sum_{i=0}^{2(l+1)} S_L^H(k+i-l,j-i). \tag{3.75}$$

Combined with the other recurrence relation (3.58), this yields

$$\begin{aligned} S_L^H(k,j) \lesssim{}& \mathbb{1}_{k=j=0}\ell^{-d} + \mathbb{1}_{k+j\geq 1}(C\ell^{-d})^{2(k+j)-1} + S_L^H(k+1,j-1) \\ &+ \sum_{l=0}^{k+j-1}(C\ell^{-d})^{2(l+1)}\sum_{i=0}^{2l+2} S_L^H(k+i-l,j-i) \\ &+ \sum_{l=0}^{k+j-2}(C\ell^{-d})^{2(l+1)}\sum_{i=0}^{2l+3} S_L^H(k+i-l-1,j-i). \end{aligned}$$

For $\ell \gg 1$, occurrences of $S_L^H(k,j)$ in the right-hand side can be absorbed into the left-hand side, and we are then left with

$$\begin{aligned} S_L^H(k,j) \lesssim{}& \mathbb{1}_{k=j=0}\ell^{-d} + \mathbb{1}_{k+j\geq 1}(C\ell^{-d})^{2(k+j)-1} + S_L^H(k+1,j-1) \\ &+ S_L^H(k+2,j-2) + \sum_{l=1}^{k+j-1}(C\ell^{-d})^{2(l+1)}\sum_{i=0}^{2l+2} S_L^H(k+i-l,j-i) \\ &+ \sum_{l=0}^{k+j-2}(C\ell^{-d})^{2(l+1)}\sum_{i=0}^{2l+3} S_L^H(k+i-l-1,j-i). \end{aligned}$$

By a double induction argument, this relation leads to the conclusion

$$S_L^H(k,j) \lesssim \begin{cases} \ell^{-d} & \text{if } k=j=0, \\ (C\ell^{-d})^{2(k+j)-1} & \text{if } k,j\geq 0,\ k+j\geq 1. \end{cases}$$

Combining this with (3.75) further yields

$$T_L^H(k,j) \lesssim \begin{cases} \ell^{-2d} & \text{if } k = j = 0, \\ (C\ell^{-d})^{2(k+j)+1} & \text{if } k, j \geq 0,\ k+j \geq 1. \end{cases}$$

Recalling that the case $\ell \simeq 1$ was already covered in (3.39), this finally concludes the proof of Theorem 3.9. ∎

3.6 Uniform cluster estimates

This section is devoted to the proof of Theorem 3.2 (i), based on the interpolating $\ell^1 - \ell^2$ energy estimates of Theorem 3.9. We focus on the bound (3.11) on the remainder R_L^{k+1}, while the corresponding bounds on cluster coefficients follow along the same lines. For $k \geq 1$, after changing summation variables, the definition (3.9) of the remainder can be written as

$$E : R_L^{k+1} E = \frac{1}{2} L^{-d} \sum_n \mathbb{E}\bigg[\int_{\partial I_{n,L}} \Big(\sum_{\substack{\sharp F = j-1 \\ n \notin F}} \delta^F \psi_L^{\varnothing} \Big) \cdot \sigma_L \nu \bigg].$$

Using the boundary conditions and the incompressibility constraint to smuggle in arbitrary constants in the different factors, as in the proof of (3.30), using the Cauchy–Schwarz inequality, and then appealing to the trace estimates of Lemma 3.5, we find

$$\begin{aligned} &|E : R_L^{k+1} E| \\ &\quad \lesssim L^{-d} \mathbb{E}\bigg[\sum_n \int_{I_{n,L}} \Big| \sum_{\substack{\sharp F = j-1 \\ n \notin F}} \mathrm{D}(\delta^F \psi_L^{\varnothing}) \Big|^2 \bigg]^{\frac{1}{2}} \mathbb{E}\bigg[\sum_n \int_{I_{n,L}+\rho B} \big| \mathrm{D}(\psi_L) + E \big|^2 \bigg]^{\frac{1}{2}}. \end{aligned}$$

Recalling the disjointness of the fattened inclusions $\{I_{n,L} + \rho B\}_n$, recognizing the definition of S_L and $T_L^{\varnothing}$, and using that in case $\ell \gg 1$ the ℓ-separation property (3.55) entails that the number of points of the process $\mathcal{P}_L$ in Q_L is bounded by $C(L/\ell)^d$, we are led to

$$|R_L^{k+1}| \lesssim \mathbb{E}\big[T_L^{\varnothing}(0,k)\big]^{\frac{1}{2}} \big(\ell^{-d} + \mathbb{E}\big[S_L(0,0)\big]\big)^{\frac{1}{2}},$$

and the conclusion (3.11) then follows from Theorem 3.9. ∎

3.7 Convergence of finite-volume approximations

This section is devoted to the proof of the convergence result (3.13) in Theorem 3.2. The idea is as follows: if $\{\overline{\mathbf{B}}_L^j\}_j$ could be viewed as derivatives of $\overline{\mathbf{B}}_L$ in some sense,

then the convergence of $\bar{\mathbf{B}}_L$ as $L \uparrow \infty$ and the uniform bounds on derivatives $\{\bar{\mathbf{B}}_L^j\}_j$ would ensure the convergence of the latter. We split the proof into two steps, first appealing to a probabilistic argument to view $\{\bar{\mathbf{B}}_L^j\}_j$ as true derivatives, and then concluding by means of standard real analysis.

Step 1. Dilution by random deletion. Taking inspiration from [53], given $p \in [0,1]$, we consider a sequence $\{b_n^{(p)}\}_n$ of iid Bernoulli variables, independent of $\mathcal{P}, \mathcal{I}$, with parameter

$$p = \mathbb{P}\big[b_n^{(p)} = 1\big],$$

and we define the corresponding decimated process

$$\mathcal{P}^{(p)} := \{x_n\}_{n\in N^{(p)}}, \quad \mathcal{I}^{(p)} := \bigcup_{n\in N^{(p)}} I_n, \quad N^{(p)} := \{n : b_n^{(p)} = 1\}. \tag{3.76}$$

Similarly, in the periodized setting (3.1), we set

$$\mathcal{P}_L^{(p)} := \{x_{n,L}\}_{n\in N^{(p)}}, \quad \mathcal{I}_L^{(p)} := \bigcup_{n\in N^{(p)}} I_{n,L}.$$

By definition, the decimated processes $\mathcal{P}^{(p)}, \mathcal{I}^{(p)}$ satisfy (H_ρ) and ($\mathrm{H}_\rho^{\mathrm{unif}}$) whenever $\mathcal{P}, \mathcal{I}$ do, and their periodized versions $\mathcal{P}_L^{(p)}, \mathcal{I}_L^{(p)}$ satisfy the same separation and stabilization properties as $\mathcal{P}_L, \mathcal{I}_L$ in Section 3.1. We use the notation $\bar{\mathbf{B}}^{(p)}$, $\bar{\mathbf{B}}_L^{(p)}$, $\{\bar{\mathbf{B}}_L^{(p),j}\}_j$, $\{R_L^{(p),k+1}\}_k$ for the effective viscosity, its periodized approximation, cluster coefficients, and cluster remainders associated with decimated processes $\mathcal{I}^{(p)}$, $\mathcal{I}_L^{(p)}$. As a corollary of [18, Theorem 1], as in (3.4), we have for all $p \in [0,1]$,

$$\lim_{L\uparrow\infty} \bar{\mathbf{B}}_L^{(p)} = \bar{\mathbf{B}}^{(p)}. \tag{3.77}$$

In the next two substeps, we shall further prove for all $k, j \geq 1$,

$$\bar{\mathbf{B}}_L^{(p),j} = p^j \bar{\mathbf{B}}_L^j, \tag{3.78}$$

$$|R_L^{(p),k+1}| \leq (Cp\ell^{-d})^{k+1}. \tag{3.79}$$

Combined with the cluster expansion (3.5), this yields for all L and $k \geq 1$,

$$\left|\bar{\mathbf{B}}_L^{(p)} - \left(\mathrm{Id} + \sum_{j=1}^{k} \frac{p^j}{j!} \bar{\mathbf{B}}_L^j\right)\right| \leq (Cp\ell^{-d})^{k+1}, \tag{3.80}$$

which entails that $\bar{\mathbf{B}}_L^j$ can be seen as the jth derivative of the map $p \mapsto \bar{\mathbf{B}}_L^{(p)}$ at $p = 0$. (Note that this estimate further shows that this map is real-analytic; we shall later come back to this observation as part of Theorem 5.4.)

Substep 1.1. Proof of (3.78). By definition of decimated processes, the cluster formula (3.6) for $\bar{\mathbf{B}}_L^{(p),j}$ can be written as

$$E : \bar{\mathbf{B}}_L^{(p),j} E = j! \sum_{\sharp F = j} \mathbb{E}\Big[\mathbb{1}_{F \subset N^{(p)}} \fint_{Q_L} \delta^F \big(\big| \mathrm{D}(\psi_L^{\varnothing}) + E \big|^2 \big) \Big].$$

As $N^{(p)}$ is independent of $\mathcal{I}$ and as

$$\mathbb{P}\big[F \subset N^{(p)} \big] = \mathbb{P}\big[b_n^{(p)} = 1, \ \forall n \in F \big] = p^{\sharp F},$$

we get

$$E : \bar{\mathbf{B}}_L^{(p),j} E = j! p^j \sum_{\sharp F = j} \mathbb{E}\Big[\fint_{Q_L} \delta^F \big(\big| \mathrm{D}(\psi_L^{\varnothing}) + E \big|^2 \big) \Big] = p^j \, E : \bar{\mathbf{B}}_L^{j} E,$$

that is, (3.78).

Substep 1.2. Proof of (3.79). Let $k \geq 1$. By definition of decimated processes, the remainder formula (3.9) for $R_L^{(p),k+1}$ can be written as

$$E : R_L^{(p),k+1} E = \frac{1}{2} L^{-d} \sum_{\sharp F = k+1} \sum_{n \in F} \mathbb{E}\Big[\mathbb{1}_{F \subset N^{(p)}} \int_{\partial I_{n,L}} \delta^{F \setminus \{n\}} \psi_L^{\varnothing} \cdot \sigma_L^{(p)} \nu \Big],$$

or equivalently, using the constraint $F \subset N^{(p)}$ to replace $\sigma_L^{(p)} = \sigma_L^{N^{(p)}}$ by $\sigma_L^{N^{(p)} \cup F}$,

$$E : R_L^{(p),k+1} E = \frac{1}{2} L^{-d} \sum_{\sharp F = k+1} \sum_{n \in F} \mathbb{E}\Big[\mathbb{1}_{F \subset N^{(p)}} \int_{\partial I_{n,L}} \delta^{F \setminus \{n\}} \psi_L^{\varnothing} \cdot \sigma_L^{N^{(p)} \cup F} \nu \Big].$$

In this expression, the integral

$$\int_{\partial I_{n,L}} \delta^{F \setminus \{n\}} \psi_L^{\varnothing} \cdot \sigma_L^{N^{(p)} \cup F} \nu$$

does not depend on the value of $\{b_n^{(p)}\}_{n \in F}$ and is thus independent of

$$\mathbb{1}_{F \subset N^{(p)}} = \prod_{n \in F} \mathbb{1}_{b_n^{(p)} = 1},$$

hence we are led to

$$E : R_L^{(p),k+1} E = \frac{1}{2} p^{k+1} L^{-d} \sum_{\sharp F = k+1} \sum_{n \in F} \mathbb{E}\Big[\int_{\partial I_{n,L}} \delta^{F \setminus \{n\}} \psi_L^{\varnothing} \cdot \sigma_L^{N^{(p)} \cup F} \nu \Big]. \tag{3.81}$$

It remains to estimate the right-hand side and deduce (3.79), which is easily done by adapting the proof of Theorem 3.2 (i) in Section 3.6. For that purpose, we first note that, for all $F \subset \mathbb{N}$, using that $\sum_{H' \subset H} (-1)^{|H'|} = 0$ if $H \neq \varnothing$, we have

$$\begin{aligned}\sum_{G \subset F} \delta^G \sigma_L^{(p)} &= \sum_{G \subset F} \sum_{G' \subset G} (-1)^{|G \setminus G'|} \sigma_L^{N^{(p)} \cup G'} \\ &= \sum_{G' \subset F} \Big(\sum_{G'' \subset F \setminus G'} (-1)^{|G''|} \Big) \sigma_L^{N^{(p)} \cup G'} \\ &= \sigma_L^{N^{(p)} \cup F},\end{aligned}$$

so that formula (3.81) can be decomposed as follows, after changing summation variables,

$$E : R_L^{(p),k+1} E = \frac{1}{2} p^{k+1} L^{-d} \sum_{\sharp F = k} \sum_{n \notin F} \sum_{G \subset F \cup \{n\}} \mathbb{E}\Big[\int_{\partial I_{n,L}} \delta^F \psi_L^{\varnothing} \cdot \delta^G \sigma_L^{(p)} \nu \Big].$$

Using the following identity, for all maps f and all $n \notin F$,

$$\sum_{G \subset F \cup \{n\}} f(G) = \sum_{G \subset F} f(G) + \sum_{G \subset F} f\big(G \cup \{n\}\big),$$

we deduce

$$\begin{aligned}&E : R_L^{(p),k+1} E \\ &\quad = \frac{1}{2} p^{k+1} L^{-d} \sum_{\sharp F = k} \sum_{G \subset F} \sum_{n \notin F} \mathbb{E}\Big[\int_{\partial I_{n,L}} \delta^F \psi_L^{\varnothing} \cdot \big(\delta^G \sigma_L^{(p)} + \delta^{G \cup \{n\}} \sigma_L^{(p)} \big) \nu \Big],\end{aligned}$$

or equivalently, further changing summation variables,

$$\begin{aligned}&E : R_L^{(p),k+1} E \\ &\quad = \frac{1}{2} p^{k+1} L^{-d} \sum_{j=0}^{k} \sum_{\sharp G = j} \sum_{n \notin G} \mathbb{E}\Big[\int_{\partial I_{n,L}} \Big(\sum_{\substack{\sharp F = k-j \\ F \cap (G \cup \{n\}) = \varnothing}} \delta^{F \cup G} \psi_L^{\varnothing} \Big) \\ &\qquad\qquad \cdot \Big(\delta^G \sigma_L^{(p)} + \delta^{G \cup \{n\}} \sigma_L^{(p)} \Big) \nu \Big].\end{aligned}$$

Using the boundary conditions for

$$\delta^G \sigma_L^{(p)} + \delta^{G \cup \{n\}} \sigma_L^{(p)} = \delta^G \sigma_L^{N^{(p)} \cup \{n\}}$$

and using the incompressibility constraint to smuggle in arbitrary constants in the different factors, as in the proof of (3.30), and then appealing to the trace estimates of

Lemma 3.5, we find

$$
\begin{aligned}
&|E : R_L^{(p),k+1} E| \\
&\quad \lesssim p^{k+1} L^{-d} \sum_{j=0}^{k} \sum_{\sharp G = j} \sum_{n \notin G} \mathbb{E}\Bigg[\int_{I_{n,L}} \Bigg| \sum_{\substack{\sharp F = k-j \\ F \cap (G \cup \{n\}) = \varnothing}} \mathrm{D}(\delta^{F \cup G} \psi_L^{\varnothing}) \Bigg|^2 \Bigg]^{\frac{1}{2}} \\
&\qquad \times \mathbb{E}\Bigg[\mathbb{1}_{j=0} + \int_{I_{n,L} + \rho B} \big| \mathrm{D}(\delta^{G} \psi_L^{(p)}) \big|^2 + \big| \mathrm{D}(\delta^{G \cup \{n\}} \psi_L^{(p)}) \big|^2 \Bigg]^{\frac{1}{2}}.
\end{aligned}
$$

Recalling the disjointness of fattened inclusions $\{I_{n,L} + \rho B\}_n$, recognizing the definition of $S_L^{(p)}$ and $T_L^{\varnothing}$, and using that in case $\ell \gg 1$ the ℓ-separation property (3.55) entails that the number of points of the process $\mathcal{P}_L$ in Q_L is bounded by $C(L/\ell)^d$, we deduce

$$
\begin{aligned}
&|E : R_L^{(p),k+1} E| \\
&\lesssim p^{k+1} \sum_{j=0}^{k} \mathbb{E}\big[T_L^{\varnothing}(j, k-j)\big]^{\frac{1}{2}} \big(\mathbb{1}_{j=0} \ell^{-d} + \mathbb{E}\big[S_L^{(p)}(j,0)\big] + \mathbb{E}\big[S_L^{(p)}(j+1,0)\big] \big)^{\frac{1}{2}}.
\end{aligned}
$$

Now appealing to Theorem 3.9, the claim (3.79) follows.

Step 2. Conclusion. While the uniform estimates of Theorem 3.2 (i) ensure that the sequence $\{\overline{\mathbf{B}}_L^j\}_{L \geq 1}$ converges as $L \uparrow \infty$ up to extraction of a subsequence, we shall use their interpretation as derivatives of the map $p \mapsto \overline{\mathbf{B}}_L^{(p)}$ at $p = 0$, together with some real analysis, to deduce the convergence of the full sequence. We argue by induction: given $k \geq 0$, we assume that the limits $\overline{\mathbf{B}}^j = \lim_{L \uparrow \infty} \overline{\mathbf{B}}_L^j$ exist for all $1 \leq j \leq k$, and we shall then prove that the limit

$$
\overline{\mathbf{B}}^{k+1} = \lim_{L \uparrow \infty} \overline{\mathbf{B}}_L^{k+1}
$$

also exists. As $\overline{\mathbf{B}}_L^{k+1}$ is bounded uniformly in L by Theorem 3.2 (i), it has a limit $\overline{\mathbf{C}}^{k+1}$ as $L \uparrow \infty$ up to extraction of a subsequence. Passing to the limit along this subsequence in (3.80), with k replaced by $k+1$, and using (3.77) and the induction assumptions, we get for all p,

$$
\left| \overline{\mathbf{B}}^{(p)} - \left(\mathrm{Id} + \sum_{j=1}^{k} \frac{p^j}{j!} \overline{\mathbf{B}}^j + \frac{p^{k+1}}{(k+1)!} \overline{\mathbf{C}}^{k+1} \right) \right| \leq (Cp)^{k+2},
$$

which proves that $\overline{\mathbf{C}}^{k+1}$ satisfies

$$
\overline{\mathbf{C}}^{k+1} = \lim_{p \downarrow 0} \frac{(k+1)!}{p^{k+1}} \left(\overline{\mathbf{B}}^{(p)} - \left(\mathrm{Id} + \sum_{j=1}^{k} \frac{p^j}{j!} \overline{\mathbf{B}}^j \right) \right),
$$

where in particular the limit exists. Since the right-hand side does not depend on the choice of the extracted subsequence, we deduce that the limit $\bar{\mathbf{C}}^{k+1}$ is uniquely defined, hence the limit $\bar{\mathbf{B}}^{k+1} := \bar{\mathbf{C}}^{k+1} = \lim_{L\uparrow\infty} \bar{\mathbf{B}}_L^{k+1}$ actually exists. By induction, this concludes the proof of the convergence result (3.13) in Theorem 3.2. ∎

3.8 Non-uniform cluster estimates

This section is devoted to the proof of Theorem 3.2 (ii). Taking inspiration from [12, Section 5.A], we proceed by a direct analysis of Green's representation formulas for corrector differences. More precisely, we introduce operators $\{\mathcal{J}^n_{L;H}\}_{n,H}$ that describe the fluid velocity generated by localized force dipoles in the presence of a finite number of rigid inclusions: these are viewed as Stokeslets for the problem with rigid inclusions and lead to a useful decomposition of corrector differences, cf. (3.82) below. The following lemma defines such operators and states their optimal decay properties, which are shown to coincide with the decay for the explicit Stokeslet associated with the problem in free space without rigid particles. This result is a particular case of Lemma A.1, the proof of which is postponed to Appendix A.

Lemma 3.10 (Decay of Stokeslets with rigid inclusions). *Let Assumptions* (H_ρ) *and* ($\mathrm{H}^{\mathrm{unif}}_\rho$) *hold, let $H \subset \mathbb{N}$ be finite and $n \notin H$, and let (ζ, P) satisfy the following Stokes problem in a neighborhood of $I_{n,L}$,*

$$\begin{cases} -\triangle\zeta + \nabla P = 0, & \text{in } (I_{n,L} + \rho B) \setminus I_{n,L}, \\ \operatorname{div}(\zeta) = 0, & \text{in } (I_{n,L} + \rho B) \setminus I_{n,L}, \\ \mathrm{D}(\zeta) = 0, & \text{in } I_{n,L}, \\ \int_{\partial I_{n,L}} \sigma(\zeta, P)\nu = 0, & \\ \int_{\partial I_{n,L}} \Theta(x - x_{n,L}) \cdot \sigma(\zeta, P)\nu = 0, & \forall \Theta \in \mathbb{M}^{\mathrm{skew}}. \end{cases}$$

Denote by $\mathcal{J}^n_{L;H}\zeta \in H^1_{\mathrm{per}}(Q_L)^d$ the solution of the following Stokes problem,

$$\begin{cases} -\triangle\mathcal{J}^n_{L;H}\zeta + \nabla\mathcal{Q}^n_{L;H}\zeta = -\delta_{\partial I_{n,L}}\sigma(\zeta, P)\nu, & \text{in } Q_L \setminus \mathcal{I}^H_L, \\ \operatorname{div}(\mathcal{J}^n_{L;H}\zeta) = 0, & \text{in } Q_L \setminus \mathcal{I}^H_L, \\ \mathrm{D}(\mathcal{J}^n_{L;H}\zeta) = 0, & \text{in } \mathcal{I}^H_L, \\ \int_{\partial I_{m,L}} \sigma(\mathcal{J}^n_{L;H}\zeta, \mathcal{Q}^n_{L;H}\zeta)\nu = 0, & \forall m \in H, \\ \int_{\partial I_{m,L}} \Theta(x - x_{m,L}) \cdot \sigma(\mathcal{J}^n_{L;H}\zeta, \mathcal{Q}^n_{L;H}\zeta)\nu = 0, & \forall m \in H,\ \forall \Theta \in \mathbb{M}^{\mathrm{skew}}. \end{cases}$$

Then, we have for all $z \in Q_L$,

$$\left(\int_{B(z)} \left|\mathrm{D}(\mathcal{J}^n_{L;H}\zeta)\right|^2 \right)^{\frac{1}{2}} \lesssim_{\sharp H} \langle (z - x_{n,L})_L \rangle^{-d} \left(\int_{I_{n,L}+\rho B} \left|\mathrm{D}(\zeta)\right|^2 \right)^{\frac{1}{2}}.$$

The above definition of operators $\{\mathcal{J}^n_{L;H}\}_{n,H}$ is motivated by the following observation: for all $F, H \subset \mathbb{N}$ with F finite and nonempty, equations (3.15) for corrector differences entail, in these terms,

$$\delta^F \psi^H_L = \sum_{n \in F \setminus H} \mathcal{J}^n_{L;H} \delta^{F\setminus\{n\}}(\psi^{H\cup\{n\}}_L + Ex). \tag{3.82}$$

Iterating this identity allows us to write $\delta^F \psi^H_L$ as a combination of iterations of $\mathcal{J}^n_{L;H}$'s, which are viewed as elementary single-particle contributions. With the above result at hand, we may now conclude with the proof of Theorem 3.2 (ii).

Proof of Theorem 3.2 (ii). We focus on the bound (3.12) on the remainder R^{k+1}_L, while the corresponding bound on cluster coefficients follows along the same lines. We split the proof into two steps.

Step 1. Estimation of corrector differences. For all finite $F, H \subset \mathbb{N}$ with F nonempty, and for all $n \in \mathbb{N}$, recalling the decomposition (3.82) for corrector differences, Lemma 3.10 yields

$$\begin{aligned}
&\bigg(\int_{I_{n,L}+\rho B} \big|\mathrm{D}(\delta^F \psi^H_L)\big|^2\bigg)^{\frac12} \\
&\quad\lesssim_{\sharp H} \sum_{m\in F\setminus H} \big\langle (x_{n,L} - x_{m,L})_L\big\rangle^{-d} \bigg(\int_{I_{m,L}+\rho B} \Big|\mathrm{D}\big(\delta^{F\setminus\{m\}}(\psi^{H\cup\{m\}}_L + Ex)\big)\Big|^2\bigg)^{\frac12}.
\end{aligned}$$

Iterating this bound, and recalling that the energy estimate (3.44) gives for all finite $G \subset \mathbb{N}$,

$$\int_{Q_L} \big|\mathrm{D}(\psi^G_L)\big|^2 \lesssim \sharp G,$$

we deduce for all n, setting $k := \sharp F \geq 1$,

$$\begin{aligned}
&\bigg(\int_{I_{n,L}+\rho B} \big|\mathrm{D}(\delta^F \psi^{\varnothing}_L)\big|^2 + \big|\mathrm{D}(\delta^F \psi^{\{n\}}_L)\big|^2\bigg)^{\frac12} \\
&\lesssim_k \sum_{n_1,\dots,n_k \in F}^{\neq} \big\langle (x_{n,L} - x_{n_1,L})_L\big\rangle^{-d} \big\langle (x_{n_1,L} - x_{n_2,L})_L\big\rangle^{-d} \cdots \big\langle (x_{n_{k-1},L} - x_{n_k,L})_L\big\rangle^{-d}.
\end{aligned} \tag{3.83}$$

Step 2. Conclusion. The starting point is the estimate (3.10) in Theorem 3.1 for the cluster remainder,

$$|E : R^{k+1}_L E| \lesssim A^\circ_k + \sum_{j=1}^k A_{j,k}, \tag{3.84}$$

in terms of

$$A_k^\circ := \mathbb{E}\Big[L^{-d}\sum_n \int_{I_{n,L}} \Big|\sum_{\substack{\sharp F=j-1\\ n\notin F}} \mathrm{D}(\delta^F \psi_L^{\varnothing})\Big|^2\Big],$$

$$A_{j,k} := \mathbb{E}\Big[L^{-d}\sum_n \Big(\int_{I_{n,L}} \Big|\sum_{\substack{\sharp F=j-1\\ n\notin F}} \mathrm{D}(\delta^F \psi_L^{\varnothing})\Big|^2\Big)^{\frac12} \tag{3.85}$$

$$\times \Big(\int_{I_{n,L}+\rho B} \Big|\sum_{\substack{\sharp F=j-1\\ n\notin F}} \mathrm{D}\big(\delta^F(\psi_L^{\{n\}}+Ex)\big)\Big|^2\Big)^{\frac12}\Big].$$

We shall prove for all $1 \le j \le k$,

$$A_k^\circ \lesssim_k \sum_{l=0}^{k} \lambda_{k+l+1}(\mathcal{P})(\log L)^{2l}, \tag{3.86}$$

$$A_{j,k} \lesssim_k \sum_{l=0}^{j-1} \lambda_{k+l+1}(\mathcal{P})(\log L)^{2l+k-j+1}. \tag{3.87}$$

Inserting this into (3.84), the conclusion (3.12) follows. We split the proof into two further substeps, separately proving (3.86) and (3.87).

Substep 2.1. Proof of (3.86). Let $k \ge 1$. The deterministic bound (3.83) yields

$$\sum_{\sharp F=k}\Big(\int_{I_{n,L}} \big|\mathrm{D}(\delta^F\psi_L^{\varnothing})\big|^2 + \big|\mathrm{D}(\delta^F\psi_L^{\{n\}})\big|^2\Big)^{\frac12}$$

$$\lesssim_k \sum_{n_1,\dots,n_k}^{\neq} D_L(x_{n,L}, x_{n_1,L},\dots,x_{n_k,L}), \tag{3.88}$$

where we have set

$$D_L(y_0,y_1,\dots,y_k) := \prod_{j=0}^{k-1} \big\langle (y_j - y_{j+1})_L\big\rangle^{-d}.$$

Inserting this in the definition (3.85) of A_k°, expanding the square, separating the different intersection patterns, and reformulating in terms of multi-points densities, cf. (1.15), we are led to

$$A_k^\circ \lesssim_k \sum_{l=0}^{k} L^{-d}\int_{(Q_L)^{k+l+1}} D_L(x,x_1,\dots,x_k) D_L(x,x_1,\dots,x_{k-l},y_1,\dots,y_l)$$

$$\times f_{k+l+1}(x,x_1,\dots,x_k,y_1,\dots,y_l)\,dx\,dx_1\cdots dx_k\,dy_1\cdots dy_l,$$

hence, in terms of multi-point intensities, appealing to Lemma 1.1 (iii),

$$A_k^\circ \lesssim_k \sum_{l=0}^{k} \lambda_{k+l+1}(\mathcal{P})\, L^{-d} \int_{(Q_L)^{k+l+1}} D_L(x, x_1, \dots, x_k) \\ \times D_L(x, x_1, \dots, x_{k-l}, y_1, \dots, y_l)\, dx\, dx_1 \cdots dx_k\, dy_1 \cdots dy_l.$$

First evaluating integrals over $x_{k-l+1}, \dots, x_k, y_1, \dots, y_l$, and noting that

$$\int_{Q_L} \langle (x-y)_L \rangle^{-d}\, dy \lesssim \log L,$$

we find

$$A_k^\circ \lesssim_k \sum_{l=0}^{k} \lambda_{k+l+1}(\mathcal{P}) (\log L)^{2l} L^{-d} \\ \times \int_{(Q_L)^{k-l+1}} D_L(x, x_1, \dots, x_{k-l})^2\, dx\, dx_1 \cdots dx_{k-l}.$$

Now evaluating the remaining integrals, noting that the square yields an integrable decay, the claim (3.86) follows.

Substep 2.2. Proof of (3.87). Let $k \geq j \geq 1$. Inserting (3.88) into the definition (3.85) of $A_{j,k}$, expanding the square, and separating the different intersection patterns, we now find

$$A_{j,k} \lesssim_k \sum_{l=0}^{j-1} L^{-d} \int_{(Q_L)^{k+l+1}} D_L(x, x_1, \dots, x_k) D_L(x, x_1, \dots, x_{j-l-1}, y_1, \dots, y_l) \\ \times f_{k+l+1}(x, x_1, \dots, x_k, y_1, \dots, y_l)\, dx\, dx_1 \cdots dx_k\, dy_1 \cdots dy_l,$$

where for notational convenience we define $D_L(x) := 1$. This integral can be evaluated exactly as in the proof of (3.86), and the claim (3.87) follows. ∎

Chapter 4

Renormalization of cluster formulas

This chapter is devoted to the proof of infinite-volume cluster estimates with optimal dependence on multi-point intensities $\{\lambda_j(\mathcal{P})\}_j$. It amounts to improving on the non-uniform cluster estimates (3.12) in Theorem 3.2, which captures the "short-range" dependence on multi-point intensities but displays a logarithmic divergence in the large-volume limit. This requires a better understanding of cluster formulas and of the underlying compensations that make them well defined in the large-volume limit.

4.1 Main results

We explore two different routes for the renormalization of infinite-volume cluster formulas, leading to two complementary results, cf. Theorems 4.1 and 4.3 below. We also discuss the optimality of our cluster estimates, cf. Theorem 4.4.

4.1.1 Implicit renormalization

Our first route relies on a slight algebraic quantification of the convergence of periodic approximations, cf. assumption (QPE) below: it implies a corresponding convergence rate for periodized cluster formulas, cf. (4.2) below, which in turn allows to remove the logarithmic divergence in the non-uniform cluster estimates of Theorem 3.2. This result is particularly general given that the quantitative periodization assumption (QPE) holds under a mere algebraic α-mixing condition for $\mathcal{I}$, cf. Remark 4.2 below. The obtained cluster estimates (4.1) differ from the canonical short-range setting of Lemma 1.2 by some logarithmic factors, which are expected to be optimal in general in link with the long-range nature of hydrodynamic interactions, cf. Theorem 4.4 below. The proof is displayed in Section 4.2.

Theorem 4.1 (Implicit renormalization of cluster formulas). *On top of Assumptions* (H_ρ) *and* ($\mathrm{H}_\rho^{\mathrm{unif}}$)*, let the following hold:*

(QPE) Quantitative periodization assumption: *There exist* $C, \gamma > 0$ *such that we have* $|\overline{\mathbf{B}}_L^{(p)} - \overline{\mathbf{B}}^{(p)}| \le C L^{-\gamma}$ *for all* $L \ge 1$ *and* $p \in [0,1]$*, where* $\overline{\mathbf{B}}_L^{(p)}$, $\overline{\mathbf{B}}^{(p)}$ *refer to the random deletion procedure introduced in Section 3.7, cf.* (3.76).

Then, we have the following estimates for the coefficients and the remainder of the infinite-volume cluster expansion defined by (3.13) *in Theorem 3.2: for all* $k, j \ge 1$*,*

$$|\overline{\mathbf{B}}^j| \lesssim_j \lambda_j(\mathcal{P}) \big|\log \lambda_j(\mathcal{P})\big|^{j-1}, \tag{4.1}$$

$$|R^{k+1}| \lesssim_k \sum_{l=k+1}^{2k+1} \lambda_l(\mathcal{P}) \big|\log \lambda_{k+1}(\mathcal{P})\big|^{l-1}.$$

In addition, the convergence result (3.13) *for finite-volume approximations can be quantified: for all* L *and* $k, j \geq 1$,

$$|\overline{\mathbf{B}}_L^j - \overline{\mathbf{B}}^j| \lesssim_j L^{-2^{-j}\gamma}, \quad |R_L^{k+1} - R^{k+1}| \lesssim_k L^{-2^{-k}\gamma}, \tag{4.2}$$

where γ *is the exponent in* (QPE).

Remark 4.2 (Quantitative periodization assumption). The validity of (QPE) can be shown to follow from a slight quantitative mixing condition for the inclusion process $\mathcal{I}$, such as the following:

(Mix) *Algebraic* α*-mixing condition:* There exist $C, \beta > 0$ such that for all Borel subsets $U, V \subset \mathbb{R}^d$ and all events $A \subset \sigma(\mathcal{I}|_U)$ and $B \in \sigma(\mathcal{I}|_V)$ we have

$$\big|\mathbb{P}[A \cap B] - \mathbb{P}[A]\mathbb{P}[B]\big| \leq C \operatorname{dist}(U, V)^{-\beta}. \tag{4.3}$$

More precisely, this condition (Mix) implies (QPE) for some $0 < \gamma \ll \beta$ (depending on β, d) and for all $0 \leq p \leq 1$ (since random deletion preserves (4.3)). This follows from by-now standard quantitative homogenization theory: we refer to Appendix B, where we adapt the techniques developed by Armstrong, Kuusi, Mourrat, and Smart in [3–5] to the present fluid context.

The above result provides optimal cluster estimates and its proof is extremely short, cf. Section 4.2. Yet, it has three main disadvantages, which call for a more detailed analysis.

— *No explicit renormalization:* While infinite-volume cluster formulas take the form of diverging series, cf. Section 1.3.4, cluster coefficients are defined as limits of finite-volume approximations, cf. (3.13). Using straightforward cancellations, we showed that the first-order cluster coefficient $\overline{\mathbf{B}}^1$ can be represented by a summable integral, cf. Proposition 2.2. A similar explicit renormalization was formally performed for the second-order coefficient $\overline{\mathbf{B}}^2$ by Batchelor and Green [7], based on more subtle cancellations. The implicit renormalization approach sheds no light on such questions. We aim to recover the Batchelor–Green renormalized formula for $\overline{\mathbf{B}}^2$ rigorously, as also discussed in [26, 27, 29], and to investigate how explicit renormalizations can be pursued to higher orders.

— *Mixing assumption:* In view of cluster formulas in Theorem 3.1, bounds on the cluster coefficient $\overline{\mathbf{B}}^j$ should only require assumptions on the j-point density. Likewise, in view of (3.10), bounds on the remainder R_L^{k+1} should only require assumptions on the $2k$-point density. Instead, assumptions (QPE) and (Mix) boldly involve the whole law of the inclusion process $\mathcal{I}$, which we aim to refine.

— *Convergence rates:* As the above approach builds on a convergence rate for periodic approximations of the effective viscosity $\overline{\mathbf{B}}$, cf. (QPE), it does not exploit the fact that cluster formulas only involve a finite number of particles at a time and are thus significantly simpler than $\overline{\mathbf{B}}$ itself. In particular, convergence rates for periodic approximations of cluster coefficients are not expected to be worse than for approximations of $\overline{\mathbf{B}}$ (on the contrary!), while the above result (4.2) displays an exponential degradation of the rates for higher-order coefficients.

4.1.2 Explicit renormalization

Our second route to renormalization of cluster formulas aims to remedy the above three issues and we proceed by an explicit analysis of cancellations. As in Proposition 2.2, we assume for convenience that particles have independent shapes, cf. (Indep), which makes cluster formulas somewhat simpler. While for $\overline{\mathbf{B}}^1$ and $\overline{\mathbf{B}}^2$ relatively simple cancellations are enough to turn cluster formulas into summable integrals, higher-order coefficients require a much deeper analysis: we are led to introducing a diagrammatic decomposition of corrector differences that allows to capture relevant cancellations. This fully resolves the higher-order renormalization question that was still open in the physics community. We refer in particular to Section 4.4 for an explicit display of renormalized formulas for $\overline{\mathbf{B}}^2$ and $\overline{\mathbf{B}}^3$, cf. Propositions 4.9 and 4.10: we recover the Batchelor–Green formula for $\overline{\mathbf{B}}^2$ and provide the first renormalized formula for $\overline{\mathbf{B}}^3$. Incidentally, these results only require assumptions on finite-order multi-point densities (instead of mixing assumptions) and Dini-type decay (instead of algebraic), which is beyond the reach of quantitative homogenization methods (and thus of the implicit renormalization approach). Renormalized formulas allow to recover the same cluster estimates (4.1) as obtained above via implicit renormalization and to further prove essentially optimal convergence rates for finite-volume approximations: the convergence rate (4.4) for $\overline{\mathbf{B}}^j$ below only degrades logarithmically when increasing j (as opposed to the exponential degradation in (4.2)), and it is always better (as it should) than the rate for approximations of the effective viscosity $\overline{\mathbf{B}}$ itself (cf. $\gamma \ll \beta$ in Remark 4.2). The proof is displayed in Section 4.4.

Theorem 4.3 (Explicit renormalization of cluster formulas). *On top of Assumptions* (H_ρ) *and* ($\mathrm{H}_\rho^{\mathrm{unif}}$)*, let the independence assumption* (Indep) *hold for particle shapes, as well as the following, for some rate function* $\omega \in C_b^\infty(\mathbb{R}^+)$*:*

(Mix_ω) α-Mixing condition with rate ω: *For all Borel subsets* $U, V \subset \mathbb{R}^d$ *and all events* $A \subset \sigma(\mathcal{I}|_U)$ *and* $B \in \sigma(\mathcal{I}|_V)$*, we have*

$$\big|\mathbb{P}[A \cap B] - \mathbb{P}[A]\mathbb{P}[B]\big| \le \omega\big(\mathrm{dist}(U, V)\big).$$

Then, the following hold.

(i) *For all $j \geq 2$, provided ω satisfies the Dini-type condition*

$$\int_1^\infty t^{-1}(\log t)^{j-2}\omega(t)\,dt < \infty,$$

the infinite-volume cluster coefficient $\bar{\mathbf{B}}^j$ can be described by means of summable integrals as detailed in Section 4.4.

(ii) *In case of an algebraic mixing rate $\omega(t) \leq Ct^{-\beta}$ for some $C, \beta > 0$, renormalized formulas lead to the same cluster estimates (4.1) for all $k, j \geq 1$. In addition, the following holds for finite-volume approximations: for all L and $j \geq 1$,*

$$|\bar{\mathbf{B}}_L^j - \bar{\mathbf{B}}^j| \lesssim_j \frac{(\log L)^{j-1}}{L^{\beta\wedge 1}}. \tag{4.4}$$

In addition, (Mix$_\omega$) can be replaced by a corresponding assumption on the j-point density for results on $\bar{\mathbf{B}}^j$, and on the $(2k+1)$-point density for results on R^{k+1}.

4.1.3 Optimality of cluster estimates

The following result states that logarithmic factors in cluster estimates (4.1) are optimal in general. These factors contrast with the prototypical short-range setting of Lemma 1.2: they are related to the long-range nature of hydrodynamic interactions and appear due to the lack of L^∞-boundedness of Calderón–Zygmund operators. We focus on the second-order coefficient $\bar{\mathbf{B}}^2$ for illustration, but, starting from renormalized formulas, the argument could be extended to higher orders as well. The proof is displayed in Section 4.5.

Theorem 4.4. *About the optimality of estimates on $\bar{\mathbf{B}}_2$, the following statements hold.*

(i) Isotropic setting: *On top of Assumptions (H$_\rho$), (H$_\rho^{\mathrm{unif}}$), and (Indep), assume that the 2-point correlation function $h_2(x,y) := f_2(x,y) - \lambda(\mathcal{P})^2$ satisfies the following decay assumption,*

$$\iint_{B(x)\times B(y)} |h_2| \leq \omega\big(|x-y|\big), \tag{4.5}$$

with some rate ω satisfying the Dini condition $\int_1^\infty t^{-1}\omega(t)\,dt < \infty$. If in addition the point process $\mathcal{P}$ is statistically isotropic, which entails that the correlation function is radial, then the following improved estimate holds,

$$|\bar{\mathbf{B}}^2| \lesssim \lambda_2(\mathcal{P}).$$

(ii) Optimality in the general setting: *There exists an inclusion process $\mathcal{I}$ that satisfies Assumptions (H$_\rho$), (H$_\rho^{\mathrm{unif}}$), (Indep), and (4.5), as well as the local independence condition $\lambda_2(\mathcal{P}) \simeq \lambda(\mathcal{P})^2 \ll \lambda(\mathcal{P})$, such that we have*

$$|\bar{\mathbf{B}}^2| \simeq \lambda_2(\mathcal{P})\big|\log \lambda_2(\mathcal{P})\big|.$$

4.2 Implicit renormalization of cluster formulas

This section is devoted to the short proof of Theorem 4.1, which we split into two steps. We start with the quantitative convergence result (4.2) for finite-volume approximations of cluster coefficients, which we obtain by quantifying the argument for the corresponding qualitative result (3.13) in Section 3.7. The claimed cluster estimates (4.1) then follow by optimization.

Step 1. Suboptimal convergence result: proof of (4.2). Starting from the cluster expansion (3.5) in Theorem 3.1, the triangle inequality yields for all $k \geq 0$,

$$|R_L^{k+1} - R^{k+1}| \leq |\bar{\mathbf{B}}_L - \bar{\mathbf{B}}| + \sum_{j=1}^{k} |\bar{\mathbf{B}}_L^j - \bar{\mathbf{B}}^j|,$$

so that the convergence rate for the remainder in (4.2) follows from Assumption (QPE) together with the convergence rate for cluster coefficients. It remains to prove the latter, that is, for all $j \geq 1$,

$$|\bar{\mathbf{B}}_L^j - \bar{\mathbf{B}}^j| \lesssim_j L^{-2^{-j}\gamma}. \tag{4.6}$$

For that purpose, we quantify the induction argument in the proof of the corresponding qualitative convergence result (3.13) in Section 3.7. Let $k \geq 0$ and assume that (4.6) holds for all $1 \leq j \leq k$. Taking the same notation as in Section 3.7 for the random deletion procedure, we recall the cluster expansion (3.80), for all L, p,

$$\left| \bar{\mathbf{B}}_L^{(p)} - \left(\mathrm{Id} + \sum_{j=1}^{k+1} \frac{p^j}{j!} \bar{\mathbf{B}}_L^j \right) \right| \leq (Cp)^{k+2}.$$

Hence, comparing to the corresponding estimate in the large-volume limit, we find

$$\left| (\bar{\mathbf{B}}_L^{(p)} - \bar{\mathbf{B}}^{(p)}) - \sum_{j=1}^{k+1} \frac{p^j}{j!} (\bar{\mathbf{B}}_L^j - \bar{\mathbf{B}}^j) \right| \leq (Cp)^{k+2}.$$

Isolating the difference $\bar{\mathbf{B}}_L^{k+1} - \bar{\mathbf{B}}^{k+1}$, and using Assumption (QPE) and the induction hypothesis to estimate other contributions, we deduce

$$\begin{aligned} |\bar{\mathbf{B}}_L^{k+1} - \bar{\mathbf{B}}^{k+1}| &\leq \frac{(k+1)!}{p^{k+1}} \left((Cp)^{k+2} + |\bar{\mathbf{B}}_L^{(p)} - \bar{\mathbf{B}}^{(p)}| + \sum_{j=1}^{k} \frac{p^j}{j!} |\bar{\mathbf{B}}_L^j - \bar{\mathbf{B}}^j| \right) \\ &\lesssim_k p + \sum_{j=0}^{k} p^{j-k-1} L^{-2^{-j}\gamma}. \end{aligned}$$

The choice $p = L^{-2^{-k-1}\gamma}$ then yields $|\bar{\mathbf{B}}_L^{k+1} - \bar{\mathbf{B}}^{k+1}| \lesssim_k L^{-2^{-k-1}\gamma}$, and the claim (4.6) follows by induction for all $j \geq 1$.

Step 2. Uniform cluster estimates: proof of (4.1). Combining the non-uniform estimates (3.12) of Theorem 3.2 with the suboptimal convergence result (4.2), we find for all $k \geq j \geq 1$,

$$|\bar{\mathbf{B}}^j| \lesssim_j L^{-2^{-j}\gamma} + \lambda_j(\mathcal{P})(\log L)^{j-1},$$
$$|R^{k+1}| \lesssim_j L^{-2^{-k}\gamma} + \sum_{l=k}^{2k} \lambda_{l+1}(\mathcal{P})(\log L)^l,$$

and the conclusion (4.1) follows from the choice $L^{-2^{-j}\gamma} = \lambda_j(\mathcal{P})$ or $L^{-2^{-k}\gamma} = \lambda_{k+1}(\mathcal{P})$, respectively. ∎

4.3 Preliminary to explicit renormalization

Before turning to the explicit renormalization of cluster formulas and to the proof of Theorem 4.3, we start with some preliminary definitions and technical tools: we define multi-point correlation functions, which provide a convenient framework to weaken the α-mixing condition, we revisit the decomposition (3.82) for corrector differences in terms of elementary single-particle contributions, and we state several crucial estimates on the latter.

4.3.1 Multi-point correlation functions

Multi-point correlation functions $\{h_j\}_j$ of the point process $\mathcal{P}$ can be defined inductively from the multi-point densities $\{f_j\}_j$, cf. (1.15), via the following relations:[1] for all $j \geq 1$,

$$f_j(x_1, \ldots, x_j) = \sum_{\pi} \prod_{H \in \pi} h_{\sharp H}(x_H), \tag{4.7}$$

where π runs over all partitions of the index set $\{1, \ldots, j\}$, where H runs over all cells of the partition π, and where for $H = \{i_1, \ldots, i_l\}$ we set

$$x_H := (x_{i_1}, \ldots, x_{i_l}).$$

For the first values of k, these relations read

$$f_1(z) = h_1(z) = \lambda(\mathcal{P}),$$
$$f_2(y,z) = \lambda(\mathcal{P})^2 + h_2(y,z),$$
$$f_3(x,y,z) = \lambda(\mathcal{P})^3 + \lambda(\mathcal{P})\big(h_2(x,y) + h_2(y,z) + h_2(z,x)\big) + h_3(x,y,z),$$

[1] Incidentally, these relations are known as Mayer's *cluster expansions*—although unrelated to the kind of cluster expansions otherwise studied in this work.

from which h_1, h_2, h_3 are easily extracted. More generally, note that the inductive definition (4.7) can be explicitly inverted: for all $j \geq 1$, we find

$$h_j(x_1, \ldots, x_j) := \sum_{\pi} (\sharp\pi - 1)!\,(-1)^{\sharp\pi - 1} \prod_{H \in \pi} f_{\sharp H}(x_H), \tag{4.8}$$

where π runs over all partitions of the index set $\{1, \ldots, j\}$ and where $\sharp\pi$ stands for the number of cells $H \in \pi$. The j-point correlation function h_j is thus a symmetric function on the product $(\mathbb{R}^d)^j$ and is a polynomial combination of multi-point densities $(f_i)_{i \leq j}$. The definition of multi-point intensities (1.16) then entails the following bounds on correlations, for all $j \geq 1$,

$$\sup_{z_1, \ldots, z_j} \fint_{Q_\ell(z_1) \times \cdots \times Q_\ell(z_j)} |h_j| \lesssim_j \bar{\lambda}_j(\mathcal{P}), \tag{4.9}$$

where we recall the notation (1.17). It is easily checked that the α-mixing assumption (Mix_ω) implies the decay of correlation functions in the following quantitative sense. Since we could not find any precise reference in the literature, we include a short proof below for completeness.

Lemma 4.5. *Assume that the point process $\mathcal{P}$ satisfies the α-mixing condition (Mix_ω) with a non-increasing rate function $\omega \in C_b^\infty(\mathbb{R}^+)$. Then, correlation functions satisfy for all $j \geq 2$ and $x_1, \ldots, x_j \in \mathbb{R}^d$,*

$$\int_{B(x_1) \times \cdots \times B(x_j)} |h_j| \leq C^j\, j!\, \min_{i \neq l} \omega\left(\left(\frac{1}{j}|x_i - x_l| - 2\right)_+\right). \tag{4.10}$$

In this view, it is natural to consider a "truncated" version of the α-mixing condition (Mix_ω) in form of the decay of a finite number of correlation functions only. This is the natural setting for cluster estimates.

(Mix_ω^n) *Mixing assumption with rate ω to order n:* Multi-point correlation functions satisfy for all $2 \leq j \leq n$ and $x_1, \ldots, x_j \in \mathbb{R}^d$,

$$\int_{B(x_1) \times \cdots \times B(x_j)} |h_j| \leq \min_{i \neq l} \omega\big(|x_i - x_l|\big).$$

Proof of Lemma 4.5. We argue by induction: given $j \geq 2$, we assume that the claimed decay estimate (4.10) is already known to hold for $h_2, \ldots, h_{j-1}$, and we prove that it also holds for h_j. Let $x_1, \ldots, x_j \in \mathbb{R}^d$ be fixed. The conclusion (4.10) is trivial when $\max_{i \neq l} \frac{1}{j}|x_i - x_l| \leq 2$, and we may thus assume $\max_{i \neq l} \frac{1}{j}|x_i - x_l| > 2$. Up to relabeling the points, we may further assume that there is $1 \leq j_* < j$ such that

$$\begin{aligned} |x_1 - x_j| &= \max_{i \neq l} |x_i - x_l|, \\ |x_i - x_l| &\geq \frac{1}{j}|x_1 - x_j| > 2 \quad \text{for all } 1 \leq i \leq j_* < l \leq j. \end{aligned} \tag{4.11}$$

(The latter condition is obtained by dividing the space between x_1 and x_j into j stripes of width $\frac{1}{j}|x_1 - x_j|$, by selecting the one that contains none of the points $\{x_i\}_{1<i<j}$, and by distinguishing the points on either side of this stripe.) Let

$$\phi \in C\big((\mathbb{R}^d)^{j_*}\big) \quad \text{and} \quad \phi' \in C\big((\mathbb{R}^d)^{j-j_*}\big)$$

be supported in $B(x_1) \times \cdots \times B(x_{j_*})$ and in $B(x_{j_*+1}) \times \cdots \times B(x_j)$, respectively, with $\|\phi\|_{\mathrm{L}^\infty((\mathbb{R}^d)^{j_*})} = \|\phi'\|_{\mathrm{L}^\infty((\mathbb{R}^d)^{j-j_*})} = 1$. Appealing to a standard covariance inequality, see e.g. [11, Lemma 1.2.3], the α-mixing condition (Mix$_\omega$) then yields

$$\begin{aligned}\Bigg|\mathrm{Cov}\Bigg[\sum_{n_1,\dots,n_{j_*}}^{\neq} \phi(x_{n_1},\dots,x_{n_{j_*}}); \sum_{n_{j_*+1},\dots,n_j}^{\neq} \phi'(x_{n_{j_*+1}},\dots,x_{n_j})\Bigg]\Bigg| \\ \le 4\,\omega\Bigg(\mathrm{dist}\Bigg(\bigcup_{i=1}^{j_*} B(x_i), \bigcup_{i=j_*+1}^{j} B(x_i)\Bigg)\Bigg) \le 4\,\omega\Bigg(\frac{1}{j}|x_1 - x_j| - 2\Bigg). \quad (4.12)\end{aligned}$$

Now we expand the covariance in terms of multi-point densities: by (4.11) and the support condition for ϕ, ϕ', we find that the product

$$\phi(x_{n_1},\dots,x_{n_{j_*}})\phi'(x_{n_{j_*+1}},\dots,x_{n_j})$$

vanishes whenever $n_i = n_l$ for some $1 \le i \le j_* < l \le j$, hence

$$\begin{aligned}&\mathrm{Cov}\Bigg[\sum_{n_1,\dots,n_{j_*}}^{\neq} \phi(x_{n_1},\dots,x_{n_{j_*}}); \sum_{n_{j_*+1},\dots,n_j}^{\neq} \phi'(x_{n_{j_*+1}},\dots,x_{n_j})\Bigg] \\ &= \int_{(\mathbb{R}^d)^j} (\phi \otimes \phi')\,(f_j - f_{j_*} \otimes f_{j-j_*}). \quad (4.13)\end{aligned}$$

Recalling the relation (4.7) for density functions in terms of correlations, we get

$$\begin{aligned}&(f_j - f_{j_*} \otimes f_{j-j_*})(z_1,\dots,z_j) \\ &\quad = \sum_{\pi} \mathbb{1}_{\exists H\in\pi: H\cap\{1,\dots,j_*\}\neq\varnothing\neq H\cap\{j_*+1,\dots,j\}} \prod_{H\in\pi} h_{\sharp H}(z_H).\end{aligned}$$

Combining this with (4.12) and (4.13), and isolating the contribution of the j-point correlation h_j (obtained for $\sharp\pi = 1$), we are led to

$$\begin{aligned}&\Bigg|\int_{(\mathbb{R}^d)^j} (\phi \otimes \phi')h_j\Bigg| \\ &\quad \le \sum_{\pi:\sharp\pi>1} \mathbb{1}_{\exists H\in\pi: H\cap\{1,\dots,j_*\}\neq\varnothing\neq H\cap\{j_*+1,\dots,j\}} \prod_{H\in\pi} \int_{B(x_H)} |h_{\sharp H}| \\ &\qquad + 4\,\omega\Bigg(\frac{1}{j}|x_1 - x_j| - 2\Bigg),\end{aligned}$$

where for $H = \{i_1, \dots, i_l\}$ we set $B(x_H) := B(x_{i_1}) \times \cdots \times B(x_{i_l})$. In view of (4.11), the induction hypothesis for $\{h_l\}_{l<j}$ entails

$$\begin{aligned}\Big| \int_{(\mathbb{R}^d)^j} (\phi \otimes \phi') h_j \Big| \\ \le \sum_{\ell=2}^{j} \sum_{i_1+\cdots+i_\ell=j} \binom{j}{i_1, \dots, i_\ell} \prod_{s=1}^{\ell} \Big(C^{i_s} i_s! \, \omega\Big(\frac{1}{j}|x_1 - x_j| - 2\Big)^{i_s} \Big) \\ + 4\,\omega\Big(\frac{1}{j}|x_1 - x_j| - 2\Big),\end{aligned}$$

from which we easily infer $| \int_{(\mathbb{R}^d)^j} (\phi \otimes \phi') h_j | \le C^j \, j! \, \omega(\frac{1}{j}|x_1 - x_j| - 2)$. By the arbitrariness of ϕ, ϕ' and of $x_1, \dots, x_j$, the conclusion (4.10) follows for h_j. ∎

4.3.2 Estimates on single-particle contributions

For notational simplicity, we henceforth assume that particles are spherical with unit radius, $I_n = B(x_n)$; the adaptation to the general case (Indep) with independent particle shapes is straightforward. As we shall see, the explicit renormalization of $\bar{\mathbf{B}}^j$ is particularly intricate for $j \ge 3$ since cancellations are not as apparent as they are for the first two orders: it will require us to decompose corrector differences into elementary single-particle contributions in the spirit of (3.82). We start by slightly changing the point of view for correctors, focussing on particle positions rather than on particle indices in the notation: given a set $Y \subset Q_L$ of "background" positions such that

$$\operatorname{dist}\big(B(y), B(y')\big) > 2\rho, \quad \operatorname{dist}\big(B(y), \partial Q_L\big) > \rho, \quad \text{for all } y, y' \in Y,\ y \neq y', \tag{4.14}$$

we denote by $\psi_L^Y \in H^1_{\mathrm{per}}(Q_L)^d$ the solution of the following periodic corrector problem, using the shorthand notation $\sigma_L^Y := \sigma(\psi_L^Y + Ex, \Sigma_L^Y)$,

$$\begin{cases} -\triangle \psi_L^Y + \nabla \Sigma_L^Y = 0, & \text{in } Q_L \setminus \bigcup_{y\in Y} B(y), \\ \operatorname{div}(\psi_L^Y) = 0, & \text{in } Q_L \setminus \bigcup_{y\in Y} B(y), \\ \mathrm{D}(\psi_L^Y + Ex) = 0, & \text{in } \bigcup_{y\in Y} B(y), \\ \int_{\partial B(y)} \sigma_L^Y \nu = 0, & \forall y \in Y, \\ \int_{\partial B(y)} \Theta(x-y) \cdot \sigma_L^Y \nu = 0, & \forall \Theta \in \mathbb{M}^{\mathrm{skew}},\ \forall y \in Y. \end{cases}$$

Next, similarly to (1.10), for any $z \in Q_L$ and any finite subset $Z \subset Q_L$, provided that the union set $\{z\} \cup Z \cup Y$ satisfies (4.14), we can define corrector differences

$$\delta^{\{z\}} \psi_L^Y := \psi_L^{\{z\}\cup Y} - \psi_L^Y, \quad \delta^Z \psi_L^Y := \sum_{W \subset Z} (-1)^{|Z\setminus W|} \psi_L^{W\cup Y}.$$

Compared with the notation that we use elsewhere in this memoir, this means for all index sets $F, H \subset \mathbb{N}$,

$$\psi_L^H \equiv \psi_L^{\{x_{n,L}\}_{n\in H}}, \quad \delta^F \psi_L^H \equiv \delta^{\{x_{n,L}\}_{n\in F}} \psi_L^{\{x_{n,L}\}_{n\in H}}.$$

For $Y = \{y_1, \ldots, y_m\}$ and $Z = \{z_1, \ldots, z_n\}$, we shall also write for convenience

$$\psi_L^{y_1,\ldots,y_m} := \psi_L^Y, \quad \delta^{z_1,\ldots,z_n} \psi_L^{y_1,\ldots,y_m} := \delta^Z \psi_L^Y. \tag{4.15}$$

Recall that Lemma 3.4 states that the corrector difference $\delta^Z \psi_L^Y$ satisfies

$$-\triangle \delta^Z \psi_L^Y + \nabla \delta^Z \Sigma_L^Y = -\sum_{z\in Z} \delta_{\partial B(z)} \delta^{Z\setminus\{z\}} \sigma_L^{Y\cup\{z\}} \nu \quad \text{in } Q_L \setminus \bigcup_{y\in Y} B(y), \tag{4.16}$$

together with the rigidity constraint $\mathrm{D}(\delta^Z \psi_L^Y) = 0$ in $\bigcup_{y\in Y} B(y)$ and with associated boundary conditions. In view of this equation, as in (3.82), we can decompose corrector differences into elementary single-particle contributions that we express in terms of operators $\{\mathcal{J}_{L;Y}^z\}_{z,Y}$ defined as follows: given a "tagged" position $z \in Q_L$, given a pair $(\zeta, P) \in H^1(B_{1+\rho}(z))^d \times \mathrm{L}^2(B_{1+\rho}(z) \setminus B(z))$ satisfying the following Stokes equations in a neighborhood of $B(z)$,

$$\begin{cases} -\triangle \zeta + \nabla P = 0, & \text{in } B_{1+\rho}(z) \setminus B(z), \\ \operatorname{div}(\zeta) = 0, & \text{in } B_{1+\rho}(z) \setminus B(z), \\ \mathrm{D}(\zeta) = 0, & \text{in } B(z), \\ \int_{\partial B(z)} \sigma(\zeta, P)\nu = 0, & \\ \int_{\partial B(z)} \Theta(x-z) \cdot \sigma(\zeta, P)\nu = 0, & \forall \Theta \in \mathbb{M}^{\mathrm{skew}}, \end{cases} \tag{4.17}$$

and given a finite subset $Y \subset Q_L$ of "background" positions such that $\{z\} \cup Y$ satisfies (4.14), we denote by $\mathcal{J}_{L;Y}^z \zeta \in H^1_{\mathrm{per}}(Q_L)^d$ the solution of the following Stokes problem,

$$\begin{cases} -\triangle \mathcal{J}_{L;Y}^z \zeta + \nabla \mathcal{Q}_{L;Y}^z \zeta = -\delta_{\partial B(z)} \sigma(\zeta, P)\nu, & \text{in } Q_L \setminus \bigcup_{y\in Y} B(y), \\ \operatorname{div}(\mathcal{J}_{L;Y}^z \zeta) = 0, & \text{in } Q_L \setminus \bigcup_{y\in Y} B(y), \\ \mathrm{D}(\mathcal{J}_{L;Y}^z \zeta) = 0, & \text{in } \bigcup_{y\in Y} B(y), \\ \int_{\partial B(y)} \sigma(\mathcal{J}_{L;Y}^z \zeta, \mathcal{Q}_{L;Y}^z \zeta)\nu = 0, & \forall y \in Y, \\ \int_{\partial B(y)} \Theta(x-y) \cdot \sigma(\mathcal{J}_{L;Y}^z \zeta, \mathcal{Q}_{L;Y}^z \zeta)\nu = 0, & \forall \Theta \in \mathbb{M}^{\mathrm{skew}},\ \forall y \in Y. \end{cases}$$

These operators $\{\mathcal{J}_{L;Y}^z\}_{z,Y}$ describe the fluid velocity generated by localized force dipoles in the presence of a finite number of rigid inclusions and are thus viewed as Stokeslets for the problem with rigid inclusions. In view of our upcoming analysis (see in particular cancellation properties in Lemma 4.6 below), we further need

to extend the definition of $\mathcal{J}^z_{L;Y}$ when the support $B(z)$ of the force dipole intersects rigid inclusions $\bigcup_{y\in Y} B(y)$ or the cell boundary ∂Q_L, which was excluded above by assuming that $\{z\}\cup Y$ satisfies (4.14). A convenient way to proceed is as follows: given $z\in Q_L$ and $Y\subset Q_L$ with only Y satisfying (4.14), we define $\mathcal{J}^z_{L;Y}\zeta\in H^1_{\rm per}(Q_L)^d$ as the solution of the following Stokes problem,

$$\begin{cases} -\triangle \mathcal{J}^z_{L;Y}\zeta+\nabla \mathcal{Q}^z_{L;Y}\zeta=-\delta_{\partial B^L(z)}\sigma(\zeta,P)\nu, & \text{in } Q_L\setminus\bigcup_{y\in Y}B(y),\\ \operatorname{div}(\mathcal{J}^z_{L;Y}\zeta)=0, & \text{in } Q_L\setminus\bigcup_{y\in Y}B(y),\\ \mathrm{D}(\mathcal{J}^z_{L;Y}\zeta)=0, & \text{in } \bigcup_{y\in Y\setminus Y_z}B(y),\\ \int_{\partial B(y)}\sigma(\mathcal{J}^z_{L;Y}\zeta,\mathcal{Q}^z_{L;Y}\zeta)\nu=0, & \forall y\in Y\setminus Y_z,\\ \int_{\partial B(y)}\Theta(x-y)\cdot\sigma(\mathcal{J}^z_{L;Y}\zeta,\mathcal{Q}^z_{L;Y}\zeta)\nu=0, & \forall\Theta\in\mathbb{M}^{\rm skew},\ \forall y\in Y\setminus Y_z,\\ \mathcal{J}^z_{L;Y}\zeta=V_z+\Theta_z(x-z), & \text{in } \bigcup_{y\in Y_z}B(y),\\ & \text{for some } V_z\in\mathbb{R}^d,\Theta_z\in\mathbb{M}^{\rm skew},\\ \sum_{y\in Y_z}\int_{\partial B(y)}\sigma(\mathcal{J}^z_{L;Y}\zeta,\mathcal{Q}^z_{L;Y}\zeta)\nu & \\ =\sum_{y\in Y_z}\int_{B(y)\cap\partial B^L(z)}\sigma(\zeta,P)\nu, & \\ \sum_{y\in Y_z}\int_{\partial B(y)}\Theta(x-z)\cdot\sigma(\mathcal{J}^z_{L;Y}\zeta,\mathcal{Q}^z_{L;Y}\zeta)\nu & \\ =\sum_{y\in Y_z}\int_{B(y)\cap\partial B^L(z)}\Theta(x-z)\cdot\sigma(\zeta,P)\nu, & \forall\Theta\in\mathbb{M}^{\rm skew}, \end{cases} \tag{4.18}$$

where $B^L(z):=(B(z)+L\mathbb{Z}^d)\cap Q_L$ stands for the periodization of the ball $B(z)$ in Q_L, where we have set $Y_z:=\{y\in Y: B(y)\cap B^L(z)\neq\varnothing\}$, and where we have implicitly extended (ζ,P) periodically to $B_{1+\rho}(z)+L\mathbb{Z}^d$. We emphasize that these equations are equivalent to the previous simpler ones when $\{z\}\cup Y$ satisfies (4.14) (hence $Y_z=\varnothing$). The solution $\mathcal{J}^z_{L;Y}\zeta$ is only defined up to a rigid motion in Q_L, which we fix by further choosing

$$\int_{Q_L}\mathcal{J}^z_{L;Y}\zeta=0,\quad \int_{Q_L}\nabla\mathcal{J}^z_{L;Y}\zeta\in\mathbb{M}^{\rm sym}_0.$$

Note that $\mathcal{J}^z_{L;Y}\zeta$ depends of course on the pair (ζ,P), not only on ζ, but we leave the pressure field implicit in the notation for convenience. We further define

$$\mathcal{J}^z_L\zeta:=\mathcal{J}^z_{L;\varnothing}\zeta,$$

for which the defining Stokes problem (4.18) reduces to

$$-\triangle\mathcal{J}^z_L\zeta+\nabla\mathcal{Q}^z_L\zeta=-\delta_{\partial B^L(z)}\sigma(\zeta,P)\nu,\quad \operatorname{div}(\mathcal{J}^z_L\zeta)=0,\quad \text{in } Q_L, \tag{4.19}$$

and we define $\mathcal{J}^z_Y\zeta$, $\mathcal{J}^z\zeta$ as the corresponding operators on whole space, that is, with $B^L(z)$ and Q_L replaced by $B(z)$ and $\mathbb{R}^d$, respectively, in (4.18) and (4.19).

In these terms, as in (3.82), given $Y, Z \subset Q_L$, provided that $Y \cup Z$ satisfies (4.14), equation (4.16) for corrector differences allows us to decompose

$$\delta^Z \psi_L^Y = \sum_{z \in Z} \mathcal{J}^z_{L;Y} \delta^{Z\setminus\{z\}} (\psi_L^{\{z\}\cup Y} + Ex). \tag{4.20}$$

The above definition (4.18) of $\mathcal{J}^z_{L;Y}$, with the particular choice of the extension to all $z \in Q_L$, is dictated by the following key observation. This constitutes the precise cancellation property that we shall repeatedly use for the explicit renormalization of cluster formulas.

Lemma 4.6 (Cancellation property). *For any $Y \subset Q_L$ satisfying (4.14), and for any function ζ satisfying (4.17) around $z = 0$, we have for $\zeta^z := \zeta(\cdot - z)$,*

$$\int_{Q_L} \big(\mathcal{J}^z_{L;Y} \zeta^z\big)\, dz = 0.$$

Proof. Integrating equations (4.18) for $\mathcal{J}^z_{L;Y}\zeta^z$ over z, noting that

$$\int_{Q_L} \Big(\sum_{y \in Y_z} \int_{B(y)\cap \partial B^L(z)} \sigma(\zeta^z, P^z)\nu \Big)\, dz = \sharp Y |B| \int_{\partial B} \sigma(\zeta, P)\nu = 0,$$

and similarly noting that

$$\begin{aligned}
&\int_{Q_L} \big(\delta_{\partial B^L(z)\setminus \bigcup_{y\in Y} B(y)} \sigma(\zeta^z, P^z)\nu\big)\, dz \\
&\quad = \int_{Q_L} \int_{\partial B^L(z)} \sigma(\zeta^z, P^z)\nu \, dz - \int_{Q_L} \Big(\sum_{y\in Y_z} \int_{B(y)\cap\partial B^L(z)} \sigma(\zeta^z, P^z)\nu \Big)\, dz \\
&\quad = \Big| Q_L \setminus \bigcup_{y\in Y} B(y) \Big| \int_{\partial B} \sigma(\zeta, P)\nu = 0,
\end{aligned}$$

the conclusion easily follows from the uniqueness of the solution to the Stokes problem (4.18). ∎

Next, we establish optimal decay estimates for these operators $\{\mathcal{J}^z_{L;Y}\}_{z,Y}$, which are shown to coincide with the decay for the explicit Stokeslets $\{\mathcal{J}^z_L\}_z$ associated with the problem in free space without rigid inclusions. This result corresponds to Lemma 3.10 and the proof is postponed to Appendix A in form of Lemma A.1.

Lemma 4.7 (Decay of Stokeslets with rigid inclusions). *Let $z \in Q_L$, let (ζ, P) satisfy (4.17) at z, and let $Y \subset Q_L$ satisfy (4.14). Then, we have for all $x \in Q_L$,*

$$\Big(\int_{B^L(x)} |\mathrm{D}(\mathcal{J}^z_{L;Y}\zeta)|^2 \Big)^{\frac12} \lesssim_{\sharp Y} \langle (x-z)_L \rangle^{-d} \Big(\int_{B_{1+\rho}(z)} \big|\mathrm{D}(\zeta)\big|^2 \Big)^{\frac12},$$

$$\Big(\int_{B(x)} \big|\mathrm{D}(\mathcal{J}^z_Y\zeta)\big|^2 \Big)^{\frac12} \lesssim_{\sharp Y} \langle x-z \rangle^{-d} \Big(\int_{B_{1+\rho}(z)} \big|\mathrm{D}(\zeta)\big|^2 \Big)^{\frac12}.$$

Finally, since we aim at finite-volume approximation error estimates, we need to quantify the difference $\mathcal{J}^z_{L;Y} - \mathcal{J}^z_Y$ between periodized and whole-space Stokeslets. The proof is postponed to Appendix A in form of Lemma A.3. We emphasize that the stated bounds are not optimal, but will be good enough for our purposes.

Lemma 4.8 (Periodization error). *Let $z \in Q_L$, let (ζ, P) satisfy (4.17) at z, and let $Y \subset Q_L$ such that $\{z\} \cup Y$ satisfies (4.14). Then, we have for all $x \in Q_L$,*

$$\begin{aligned}
&\Big(\int_{B^L_{1+\rho}(x)} \big|\mathrm{D}(\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_Y\zeta)\big|^2 \Big)^{\frac12} \\
&\quad \lesssim_{\sharp Y} \Big(\int_{B_{1+\rho}(z)} \big|\mathrm{D}(\zeta)\big|^2 \Big)^{\frac12} \\
&\qquad \times \Big(\mathbb{1}_{|x-z|>\frac{L}{4}} \big\langle (x-z)_L \big\rangle^{-d} + \mathbb{1}_{|x-z|\le\frac{L}{4}} \operatorname{dist}\big(Y \setminus \{x,z\}, \partial Q_L\big)^{-d} \Big),
\end{aligned}$$

where we set for notational convenience $\operatorname{dist}(\varnothing, \partial Q_L) := L$, *and where we denote by $B^L_r(z) = (B_r(z) + L\mathbb{Z}^d) \cap Q_L$ the periodization of the ball $B_r(z)$ in Q_L. In addition,*

$$\Big(\int_{B^L_{1+\rho}(x)} \big|\mathrm{D}(\psi^Y_L - \psi^Y)\big|^2 \Big)^{\frac12} \lesssim_{\sharp Y} \Big(\big\langle \operatorname{dist}(x, \partial Q_L) \big\rangle + \big\langle \operatorname{dist}\big(Y \setminus \{x\}, \partial Q_L\big) \big\rangle \Big)^{-d}.$$

4.4 Explicit renormalization of cluster formulas

This section is devoted to the proof of Theorem 4.3. We first describe the explicit renormalization of the second and third cluster coefficients $\overline{\mathbf{B}}^2$ and $\overline{\mathbf{B}}^3$, cf. Propositions 4.9 and 4.10 below, before turning to the general case, cf. Proposition 4.11. For notational simplicity, we assume that particles are spherical with unit radius, $I_n = B(x_n)$, but we emphasize that the general case follows along the same lines under the independence assumption (Indep). More precisely, it suffices to replace each occurrence of spherical particles below by iid random shapes and to further take the expectation with respect to the latter; we omit the detail.

4.4.1 Explicit renormalization of $\overline{\mathbf{B}}^2$: Batchelor–Green formula

We start with the analysis of $\overline{\mathbf{B}}^2$ and rigorously establish the so-called Batchelor–Green formula [7].

Proposition 4.9 (Batchelor–Green renormalization of $\overline{\mathbf{B}}^2$). *Let (H_ρ) and ($\mathrm{H}^{\mathrm{unif}}_\rho$) hold, and assume for simplicity that particles are spherical with unit radius, $I_n = B(x_n)$. Let also the mixing assumption (Mix^n_ω) hold to order $n = 2$ with some non-increasing*

rate $\omega \in C_b^\infty(\mathbb{R}^+)$ *satisfying the Dini condition* $\int_1^\infty \frac{1}{t}\,\omega(t)\,dt < \infty$, *as well as the doubling condition* $\omega(2t) \simeq \omega(t)$ *for all* $t \geq 0$. *Then, the infinite-volume second-order cluster coefficient* $\bar{\mathbf{B}}^2$ *defined in* (3.13) *can be expressed as follows,*

$$E : \bar{\mathbf{B}}^2 E = \int_{\mathbb{R}^d} \Big(\int_{\partial B} \psi^z \cdot \sigma^0 \nu \Big) h_2(0,z)\,dz + \int_{\mathbb{R}^d} \Big(\int_{\partial B} \psi^z \cdot \delta^z \sigma^0 \nu \Big) f_2(0,z)\,dz, \tag{4.21}$$

where both integrals are absolutely converging and where we use the notation (4.15). *In addition, the following estimates hold:*

(i) Uniform cluster estimate:

$$|\bar{\mathbf{B}}_L^2| \lesssim \lambda_2(\mathcal{P}) + \int_1^\infty \frac{1}{t}\big(\omega(t) \wedge \lambda_2(\mathcal{P})\big)\,dt,$$

hence, in case of an algebraic weight $\omega(t) \leq Ct^{-\beta}$ *for some* $C, \beta > 0$,

$$|\bar{\mathbf{B}}_L^2| \lesssim \lambda_2(\mathcal{P})\big|\log \lambda(\mathcal{P})\big|.$$

(ii) Periodization error estimate:

$$|\bar{\mathbf{B}}_L^2 - \bar{\mathbf{B}}^2| \lesssim \Big(\omega(L) + \frac{1}{L}\Big)\log L + \int_1^\infty \frac{1}{t+L}\,\omega(t)\,dt.$$

(iii) Uniform remainder estimate: *If* (Mix_ω^n) *further holds with* $n = 3$, *then*

$$|R_L^2| \lesssim \lambda_2(\mathcal{P}) + \int_1^\infty \frac{1}{t}\big(\omega(t) \wedge \lambda_2(\mathcal{P})\big)\,dt + \int_1^\infty \frac{\log t}{t}\big(\omega(t) \wedge \lambda_3(\mathcal{P})\big)\,dt,$$

hence, in case of an algebraic weight $\omega(t) \leq Ct^{-\beta}$ *for some* $C, \beta > 0$,

$$|R_L^2| \lesssim \lambda_2(\mathcal{P})\big|\log \lambda(\mathcal{P})\big| + \lambda_3(\mathcal{P})\big|\log \lambda(\mathcal{P})\big|^2.$$

Proof. We split the proof into four steps. Given $E \in \mathbb{M}_0^{\mathrm{sym}}$ with $|E| = 1$, for notational convenience, we write $\bar{\mathbf{B}}_L^2$, $\bar{\mathbf{B}}^2$, and R_L^2 for $E : \bar{\mathbf{B}}_L^2 E$, $E : \bar{\mathbf{B}}^2 E$, and $E : R_L^2 E$.

Step 1. Reformulation of $\bar{\mathbf{B}}_L^2$:

$$\begin{aligned}
\bar{\mathbf{B}}_L^2 = {} & L^{-d} \iint_{Q_{L,\rho}\times Q_{L,\rho}} \Big(\int_{\partial B(y)} \psi_L^z \cdot \sigma_L^y \nu \Big) h_2(y,z)\,dy dz \\
& + L^{-d} \iint_{Q_{L,\rho}\times Q_{L,\rho}} \Big(\int_{\partial B(y)} \psi_L^z \cdot \delta^z \sigma_L^y \nu \Big) f_2(y,z)\,dy dz \\
& - \lambda(\mathcal{P})^2 L^{-d} \iint_{Q_{L,\rho}\times (Q_L \setminus Q_{L,\rho})} \Big(\int_{\partial B(y)} \psi_L^z \cdot \sigma_L^y \nu \Big)\,dy dz,
\end{aligned} \tag{4.22}$$

where we recall the shorthand notation $Q_{L,\rho} = Q_{L-2(\ell\vee(1+\rho))}$, cf. (3.1).

By definition, cf. (3.8), the finite-volume approximation $\bar{\mathbf{B}}_L^2$ is given by

$$\bar{\mathbf{B}}_L^2 = L^{-d} \sum_{m \neq n} \mathbb{E}\bigg[\int_{\partial B(x_{n,L})} \psi_L^{\{m\}} \cdot \sigma_L^{\{m,n\}} \nu \bigg].$$

Decomposing

$$\sigma_L^{\{m,n\}} = \sigma_L^{\{n\}} + \delta^{\{m\}} \sigma_L^{\{n\}},$$

this turns into

$$\begin{aligned} \bar{\mathbf{B}}_L^2 = {} & L^{-d} \sum_{m \neq n} \mathbb{E}\bigg[\int_{\partial B(x_{n,L})} \psi_L^{\{m\}} \cdot \sigma_L^{\{n\}} \nu \bigg] \\ & + L^{-d} \sum_{m \neq n} \mathbb{E}\bigg[\int_{\partial B(x_{n,L})} \psi_L^{\{m\}} \cdot \delta^{\{m\}} \sigma_L^{\{n\}} \nu \bigg]. \end{aligned}$$

In terms of multi-point densities, cf. (1.15), recalling the choice of the finite-volume approximation with $\mathcal{P}_L = \{x_n : x_n \in Q_{L,\rho}\}$, cf. (3.1), and using the notation (4.15), we can rewrite

$$\begin{aligned} \bar{\mathbf{B}}_L^2 = {} & L^{-d} \iint_{Q_{L,\rho} \times Q_{L,\rho}} \bigg(\int_{\partial B(y)} \psi_L^z \cdot \sigma_L^y \nu \bigg) f_2(y,z)\, dy dz \\ & + L^{-d} \iint_{Q_{L,\rho} \times Q_{L,\rho}} \bigg(\int_{\partial B(y)} \psi_L^z \cdot \delta^z \sigma_L^y \nu \bigg) f_2(y,z)\, dy dz, \end{aligned} \tag{4.23}$$

and it remains to further analyze the first right-hand side term. For that purpose, we note that $\psi_L^z = \psi_L^0(\cdot - z)$ and $\sigma_L^y = \sigma_L^0(\cdot - y)$, so that

$$\int_{\partial B(y)} \psi_L^z \cdot \sigma_L^y \nu = \int_{\partial B} \psi_L^0(\cdot + y - z) \cdot \sigma_L^0 \nu.$$

Integrating over z, using the periodicity of ψ_L^0, and recalling that

$$\int_{\partial B} \sigma_L^0 \nu = 0,$$

we deduce

$$\int_{Q_L} \bigg(\int_{\partial B(y)} \psi_L^z \cdot \sigma_L^y \nu \bigg) dz = 0. \tag{4.24}$$

Decomposing

$$f_2(y,z) = \lambda(\mathcal{P})^2 + h_2(y,z)$$

in terms of the 2-point correlation function h_2, and then using this cancellation property (4.24) to reformulate the first right-hand side term in (4.23), the claim (4.22) follows.

Step 2. Uniform estimate: proof of (i). Using the boundary conditions and the incompressibility constraints to smuggle in arbitrary constants in the different factors, as in the proof of (3.30), and appealing to the trace estimates of Lemma 2.5, we find

$$\left|\int_{\partial B(y)} \psi_L^z \cdot \sigma_L^y \nu\right| \lesssim \left(\int_{B(y)} \left|\mathrm{D}(\psi_L^z)\right|^2\right)^{\frac{1}{2}} \left(\int_{B_{1+\rho}(y)} \left|\mathrm{D}(\psi_L^y)\right|^2\right)^{\frac{1}{2}},$$
$$\left|\int_{\partial B(y)} \psi_L^z \cdot \delta^z \sigma_L^y \nu\right| \lesssim \left(\int_{B(y)} \left|\mathrm{D}(\psi_L^z)\right|^2\right)^{\frac{1}{2}} \left(\int_{B_{1+\rho}(y)} \left|\mathrm{D}(\delta^z \psi_L^y)\right|^2\right)^{\frac{1}{2}}.$$

Hence, applying the decay estimates of Lemma 4.7 to $\psi_L^z = \mathcal{J}_L^z(\psi_L^z + Ex)$ and to $\delta^z \psi_L^y = \mathcal{J}_{L;y}^z(\psi_L^{y,z} + Ex)$, combined with the energy estimate (3.44), we get

$$\begin{aligned} \left|\int_{\partial B(y)} \psi_L^z \cdot \sigma_L^y \nu\right| &\lesssim \langle (y-z)_L \rangle^{-d}, \\ \left|\int_{\partial B(y)} \psi_L^z \cdot \delta^z \sigma_L^y \nu\right| &\lesssim \langle (y-z)_L \rangle^{-2d}. \end{aligned} \tag{4.25}$$

Formula (4.22) for $\bar{\mathbf{B}}_L^2$ can then be estimated as follows,

$$\begin{aligned} |\bar{\mathbf{B}}_L^2| \lesssim\ & L^{-d} \iint_{Q_L \times Q_L} \langle (y-z)_L \rangle^{-d} \left|h_2(y,z)\right| dydz \\ & + L^{-d} \iint_{Q_L \times Q_L} \langle (y-z)_L \rangle^{-2d} f_2(y,z)\, dydz \\ & + \lambda(\mathcal{P})^2 L^{-d} \iint_{Q_L \times (Q_L \setminus Q_{L,\rho})} \langle (y-z)_L \rangle^{-d} dydz. \end{aligned}$$

In terms of the two-point intensity, recalling that $\bar{\lambda}_2(\mathcal{P}) = \lambda_2(\mathcal{P})$ by Lemma 1.1 (ii) in view of (Mix_ω^n), we can estimate the 2-point correlation function as follows: appealing both to (4.9) and to the decay assumption (Mix_ω^n), and arguing as in Lemma 1.1 (iii), we find

$$\begin{aligned} & \iint_{Q_L \times Q_L} \langle (y-z)_L \rangle^{-d} \left|h_2(y,z)\right| dydz \\ & \quad \lesssim \iint_{Q_L \times Q_L} \langle (y-z)_L \rangle^{-d} \left(\omega(|y-z|) \wedge \lambda_2(\mathcal{P})\right) dydz. \end{aligned} \tag{4.26}$$

The above then becomes

$$\begin{aligned} |\bar{\mathbf{B}}_L^2| \lesssim\ & L^{-d} \iint_{Q_L \times Q_L} \langle (y-z)_L \rangle^{-d} \left(\omega(|y-z|) \wedge \lambda_2(\mathcal{P})\right) dydz \\ & + \lambda_2(\mathcal{P}) \Bigg(L^{-d} \iint_{Q_L \times Q_L} \langle (y-z)_L \rangle^{-2d} dydz \\ & \qquad + L^{-d} \iint_{Q_L \times (Q_L \setminus Q_{L,\rho})} \langle (y-z)_L \rangle^{-d} dydz \Bigg). \end{aligned}$$

As ω is non-increasing and as $|(y-z)_L| \le |y-z|$, the first right-hand side term is bounded by

$$\begin{aligned}
&L^{-d} \iint_{Q_L \times Q_L} \langle (y-z)_L \rangle^{-d} \big(\omega\big(|y-z|\big) \wedge \lambda_2(\mathcal{P})\big)\, dy dz \\
&\quad \le L^{-d} \iint_{Q_L \times Q_L} \langle (y-z)_L \rangle^{-d} \big(\omega\big(\big|(y-z)_L\big|\big) \wedge \lambda_2(\mathcal{P})\big)\, dy dz \\
&\quad \lesssim \int_{Q_L} \langle z \rangle^{-d} \big(\omega\big(|z|\big) \wedge \lambda_2(\mathcal{P})\big)\, dz \\
&\quad \lesssim \int_1^\infty \frac{1}{t} \big(\omega(t) \wedge \lambda_2(\mathcal{P})\big)\, dt,
\end{aligned}$$

and the conclusion (i) follows after similarly estimating the other terms.

Step 3. Convergence result: proof of (ii). Comparing identities (4.21) and (4.22), we have

$$|\overline{\mathbf{B}}_L^2 - \overline{\mathbf{B}}^2| \le A_L^1 + A_L^2 + A_L^3, \tag{4.27}$$

where we have set for abbreviation

$$\begin{aligned}
A_L^1 &:= \bigg| L^{-d} \iint_{Q_{L,\rho} \times Q_{L,\rho}} \bigg(\int_{\partial B(y)} \psi_L^z \cdot \sigma_L^y \nu \bigg) h_2(y,z)\, dy dz \\
&\qquad - \int_{\mathbb{R}^d} \bigg(\int_{\partial B} \psi^z \cdot \sigma^0 \nu \bigg) h_2(0,z)\, dz \bigg|, \\
A_L^2 &:= \bigg| L^{-d} \iint_{Q_{L,\rho} \times Q_{L,\rho}} \bigg(\int_{\partial B(y)} \psi_L^z \cdot \delta^z \sigma_L^y \nu \bigg) f_2(y,z)\, dy dz \\
&\qquad - \int_{\mathbb{R}^d} \bigg(\int_{\partial B} \psi^z \cdot \delta^z \sigma^0 \nu \bigg) f_2(0,z)\, dz \bigg|, \\
A_L^3 &:= \lambda(\mathcal{P})^2 L^{-d} \iint_{Q_{L,\rho} \times (Q_L \setminus Q_{L,\rho})} \bigg| \int_{\partial B(y)} \psi_L^z \cdot \sigma_L^y \nu \bigg|\, dy dz.
\end{aligned}$$

We estimate these three contributions separately and we start with A_L^1. Noting that stationarity yields $h_2(y,z) = h_2(0, z-y)$, and using that $\psi^z = \psi^{z-y}(\cdot - y)$ and $\sigma^y = \sigma^0(\cdot - y)$, we can write

$$\begin{aligned}
&L^{-d} \iint_{Q_{L,\rho} \times Q_{L,\rho}} \bigg(\int_{\partial B(y)} \psi^z \cdot \sigma^y \nu \bigg) h_2(y,z)\, dy dz \\
&\quad = L^{-d} \iint_{Q_{L,\rho} \times Q_{L,\rho}} \bigg(\int_{\partial B} \psi^{z-y} \cdot \sigma^0 \nu \bigg) h_2(0, z-y)\, dy dz \\
&\quad = \int_{\mathbb{R}^d} L^{-d} \big| Q_{L,\rho} \cap (Q_{L,\rho} + z) \big| \bigg(\int_{\partial B} \psi^z \cdot \sigma^0 \nu \bigg) h_2(0,z)\, dz,
\end{aligned}$$

and thus, setting for abbreviation $\gamma^2_{L,\rho}(z) := L^{-d}|Q_{L,\rho} \cap (Q_{L,\rho} + z)|$, we get by the triangle inequality,

$$A^1_L \lesssim \int_{\mathbb{R}^d} \big(1 - \gamma^2_{L,\rho}(z)\big) \left| \int_{\partial B} \psi^z \cdot \sigma^0 \nu \right| \big|h_2(0,z)\big|\, dz + L^{-d} \iint_{Q_L \times Q_L} \left(\left| \int_{\partial B(y)} (\psi^z_L - \psi^z) \cdot \sigma^y_L \nu \right| + \left| \int_{\partial B(y)} \psi^z \cdot (\sigma^y_L - \sigma^y)\nu \right| \right) \times |h_2(y,z)|\, dy dz. \tag{4.28}$$

Appealing to the trace estimates of Lemma 3.5, decomposing

$$\begin{aligned} \psi^z_L - \psi^z &= \mathcal{J}^z_L(\psi^z_L + Ex) - \mathcal{J}^z(\psi^z + Ex) \\ &= \mathcal{J}^z(\psi^z_L - \psi^z) + (\mathcal{J}^z_L - \mathcal{J}^z)(\psi^z_L + Ex), \end{aligned}$$

using the decay estimates of Lemma 4.7, the periodization error estimates of Lemma 4.8, and the energy estimate (3.44), we find

$$\begin{aligned} &\left| \int_{\partial B(y)} (\psi^z_L - \psi^z) \cdot \sigma^y_L \nu \right| \\ &\lesssim \left(\int_{B(y)} \big|\mathrm{D}(\psi^z_L - \psi^z)\big|^2 \right)^{\frac12} \left(\int_{Q_L} \big|\mathrm{D}(\psi^y_L + Ex)\big|^2 \right)^{\frac12} \\ &\lesssim \left(\int_{B(y)} \big|\mathrm{D}\big(\mathcal{J}^z(\psi^z_L - \psi^z)\big)\big|^2 + \big|\mathrm{D}\big((\mathcal{J}^z_L - \mathcal{J}^z)(\psi^z_L + Ex)\big)\big|^2 \right)^{\frac12} \\ &\lesssim \langle y - z\rangle^{-d} \left(\int_{B_{1+\rho}(z)} \big|\mathrm{D}(\psi^z_L - \psi^z)\big|^2 \right)^{\frac12} \\ &\quad + \left(\mathbb{1}_{|y-z|>\frac{L}{4}} \big\langle (y-z)_L \big\rangle^{-d} + \mathbb{1}_{|y-z|\le\frac{L}{4}} L^{-d} \right) \left(\int_{Q_L} \big|\mathrm{D}(\psi^z_L + Ex)\big|^2 \right)^{\frac12} \\ &\lesssim \mathbb{1}_{|y-z|>\frac{L}{4}} \big\langle (y-z)_L \big\rangle^{-d} + \mathbb{1}_{|y-z|\le\frac{L}{4}} L^{-d}, \end{aligned}$$

and similarly,

$$\left| \int_{\partial B} \psi^z \cdot \sigma^0 \nu \right| \lesssim \langle z\rangle^{-d},$$
$$\left| \int_{\partial B(y)} \psi^z \cdot (\sigma^y_L - \sigma^y)\nu \right| \lesssim L^{-d} \langle y - z\rangle^{-d}.$$

Inserting these estimates into (4.28), we get

$$A^1_L \lesssim \int_{\mathbb{R}^d} \big(1 - \gamma^2_{L,\rho}(z)\big) \langle z\rangle^{-d} \big|h_2(0,z)\big|\, dz + L^{-d} \iint_{Q_L \times Q_L} \big(\mathbb{1}_{|y-z|>\frac{L}{4}} \big\langle (y-z)_L \big\rangle^{-d} + \mathbb{1}_{|y-z|\le\frac{L}{4}} L^{-d} \big) \big|h_2(y,z)\big|\, dy dz.$$

Using the decay assumption (Mix$^n_\omega$) for h_2, noting that

$$1-\gamma^2_{L,\rho}(z)=1-L^{-d}\big|Q_{L,\rho}\cap(Q_{L,\rho}+z)\big|\lesssim\frac{|z|}{L}\wedge 1, \tag{4.29}$$

and using that $\int_{Q_L}\langle y\rangle^{-d}dy\lesssim\log L$, we conclude after straightforward simplifications,

$$\begin{aligned}A^1_L&\lesssim\int_{\mathbb{R}^d}\left(\frac{|z|}{L}\wedge 1\right)\langle z\rangle^{-d}\omega\big(|z|\big)\,dz+\omega(L)\log L+L^{-d}\int_{Q_{2L}}\omega\big(|z|\big)\,dz\\&\lesssim\omega(L)\log L+\int_0^\infty\frac{1}{t+L}\,\omega(t)\,dt.\end{aligned} \tag{4.30}$$

We turn to the estimation of the second term A^2_L in (4.27). By stationarity, as above, we find

$$\begin{aligned}A^2_L&\lesssim\int_{\mathbb{R}^d}\big(1-\gamma^2_{L,\rho}(z)\big)\left|\int_{\partial B}\psi^z\cdot\delta^z\sigma^0\nu\right|f_2(0,z)\,dz\\&+L^{-d}\iint_{Q_L\times Q_L}\left(\left|\int_{\partial B(y)}(\psi^z_L-\psi^z)\cdot\delta^z\sigma^y_L\nu\right|+\left|\int_{\partial B(y)}\psi^z\cdot(\delta^z\sigma^y_L-\delta^z\sigma^y)\nu\right|\right)\\&\qquad\times f_2(y,z)\,dy\,dz.\end{aligned}$$

Recalling $\psi^z_L-\psi^z=\mathcal{J}^z(\psi^z_L-\psi^z)+(\mathcal{J}^z_L-\mathcal{J}^z)(\psi^z_L+Ex)$, further decomposing

$$\begin{aligned}\delta^z\psi^y_L-\delta^z\psi^y&=\mathcal{J}^z_{L;y}(\psi^{y,z}_L+Ex)-\mathcal{J}^z_y(\psi^{y,z}+Ex)\\&=\mathcal{J}^z_y(\psi^{y,z}_L-\psi^{y,z})+(\mathcal{J}^z_{L;y}-\mathcal{J}^z_y)(\psi^{y,z}_L+Ex),\end{aligned}$$

and using the trace estimates of Lemma 3.5, the decay estimates of Lemma 4.7, the periodization error estimates of Lemma 4.8, and the energy estimate (3.44), we find

$$\begin{aligned}\left|\int_{\partial B}\psi^z\cdot\delta^z\sigma^0\nu\right|&\lesssim\langle z\rangle^{-2d},\\\left|\int_{\partial B(y)}(\psi^z_L-\psi^z)\cdot\delta^z\sigma^y_L\nu\right|&\lesssim\big\langle(y-z)_L\big\rangle^{-d}\big(d_L(y)+d_L(z)\big)^{-d}\\\left|\int_{\partial B(y)}\psi^z\cdot(\delta^z\sigma^y_L-\delta^z\sigma^y)\nu\right|&\lesssim\langle y-z\rangle^{-2d}\big(d_L(y)+d_L(z)\big)^{-d}+L^{-d}\big\langle(y-z)_L\big\rangle^{-d}.\end{aligned}$$

where we have set for abbreviation $d_L(z):=\langle\operatorname{dist}(z,\partial Q_L)\rangle$. Inserting these estimates into the above, we get

$$\begin{aligned}A^2_L&\lesssim\int_{\mathbb{R}^d}\big(1-\gamma^2_{L,\rho}(z)\big)\langle z\rangle^{-2d}\,f_2(0,z)\,dz\\&\quad+L^{-d}\iint_{Q_L\times Q_L}\big\langle(y-z)_L\big\rangle^{-d}d_L(y)^{-d}\,f_2(y,z)\,dy\,dz.\end{aligned}$$

In terms of the two-point intensity, appealing to Lemma 1.1 (iii), recalling (4.29), and using that $\int_{Q_L} \langle y\rangle^{-d}\,dy \lesssim \log L$ and $\int_{Q_L} d_L(y)^{-d}\,dy \lesssim L^{d-1}$, we deduce

$$\begin{aligned} A_L^2 &\lesssim \lambda_2(\mathcal{P}) \int_{\mathbb{R}^d} \left(\frac{|z|}{L} \wedge 1\right) \langle z\rangle^{-2d}\,dz \\ &\quad + \lambda_2(\mathcal{P}) L^{-d} \iint_{Q_L\times Q_L} \big\langle (y-z)_L\big\rangle^{-d} d_L(y)^{-d}\,dy dz \\ &\lesssim \lambda_2(\mathcal{P}) \frac{\log L}{L}. \end{aligned} \tag{4.31}$$

It remains to estimate the last term A_L^3 in (4.27). Using again the trace estimates of Lemma 3.5, the decay estimates of Lemma 4.7, and the energy estimate (3.44), we find

$$\left| \int_{\partial B(y)} \psi_L^z \cdot \sigma_L^y \nu \right| \lesssim \big\langle (y-z)_L\big\rangle^{-d},$$

and thus

$$\begin{aligned} A_L^3 &\lesssim \lambda(\mathcal{P})^2 L^{-d} \iint_{Q_{L,\rho}\times (Q_L\setminus Q_{L,\rho})} \big\langle (y-z)_L\big\rangle^{-d}\,dy dz \\ &\lesssim \lambda(\mathcal{P})^2 \frac{\log L}{L}. \end{aligned}$$

Combining this with (4.27), (4.30), and (4.31), the conclusion (ii) follows.

Step 4. Uniform remainder estimate: proof of (iii). The starting point is the refined estimate (3.26) on remainders, which reads in this case

$$\begin{aligned} |R_L^2| &\le \mathbb{E}\bigg[L^{-d} \sum_n \int_{B(x_{n,L})} \Big| \sum_{m:m\neq n} \mathrm{D}(\psi_L^{\{m\}}) \Big|^2 \bigg] \\ &\quad + \bigg| \mathbb{E}\bigg[L^{-d} \sum_n \int_{B(x_{n,L})} \Big(\sum_{m:m\neq n} \mathrm{D}(\psi_L^{\{m\}}) \Big) : \mathrm{D}(\widehat{\psi}_{n,L}^{\{n\}}) \bigg] \bigg|, \end{aligned}$$

where we recall that $\widehat{\psi}_{n,L}^{\{n\}}$ is defined by (3.27). Expanding the square and separating the different intersection patterns, this can be rewritten as follows, in terms of multi-point densities,

$$\begin{aligned} |R_L^2| &\le L^{-d} \iint_{(Q_{L,\rho})^2} \bigg(\int_{B(x)} |\mathrm{D}(\psi_L^y)|^2 \bigg) f_2(x,y)\,dxdy \\ &\quad + L^{-d} \bigg| \iiint_{(Q_{L,\rho})^3} \bigg(\int_{B(x)} \mathrm{D}(\psi_L^y) : \mathrm{D}(\psi_L^z) \bigg) f_3(x,y,z)\,dxdydz \bigg| \\ &\quad + L^{-d} \bigg| \iint_{(Q_{L,\rho})^2} \bigg(\int_{B(x)} \mathrm{D}(\psi_L^y) : \mathrm{D}(\widehat{\psi}_{x,L}^x) \bigg) f_2(x,y)\,dxdy \bigg|, \end{aligned}$$

where we use the obvious notation for $\widehat{\psi}^{x}_{x,L}$ such that $\widehat{\psi}^{x_n}_{x_n,L} := \widehat{\psi}^{\{n\}}_{n,L}$. Replacing f_2, f_3 by their expansions (4.7) in terms of correlation functions, and noting that several contributions can be turned into boundary terms using identity $\int_{Q_L} \mathrm{D}(\psi^{y}_{L})\,dy = 0$, we obtain

$$
\begin{aligned}
|R^2_L| &\lesssim L^{-d} \iint_{(Q_{L,\rho})^2} \Big(\int_{B(x)} \big|\mathrm{D}(\psi^{y}_{L})\big|^2 \Big) f_2(x,y)\,dxdy \\
&+ L^{-d} \Big| \iiint_{(Q_{L,\rho})^3} \Big(\int_{B(x)} \mathrm{D}(\psi^{y}_{L}) : \mathrm{D}(\psi^{z}_{L}) \Big) \big(\lambda(\mathcal{P}) h_2(y,z) + h_3(x,y,z)\big)\,dxdydz \Big| \\
&+ L^{-d} \Big| \iint_{(Q_{L,\rho})^2} \Big(\int_{B(x)} \mathrm{D}(\psi^{y}_{L}) : \mathrm{D}(\widehat{\psi}^{x}_{x,L}) \Big) h_2(x,y)\,dxdy \Big| \\
&+ \lambda(\mathcal{P}) L^{-d} \Big| \iiint_{(Q_{L,\rho})^2 \times (Q_L \setminus Q_{L,\rho})} \Big(\int_{B(x)} \mathrm{D}(\psi^{y}_{L}) : \mathrm{D}(\psi^{z}_{L}) \Big) h_2(x,y)\,dxdydz \Big| \\
&+ \lambda(\mathcal{P})^3 L^{-d} \Big| \iiint_{Q_{L,\rho} \times (Q_L \setminus Q_{L,\rho})^2} \Big(\int_{B(x)} \mathrm{D}(\psi^{y}_{L}) : \mathrm{D}(\psi^{z}_{L}) \Big)\,dxdydz \Big| \\
&+ \lambda(\mathcal{P})^2 L^{-d} \Big| \iint_{Q_{L,\rho} \times (Q_L \setminus Q_{L,\rho})} \Big(\int_{B(x)} \mathrm{D}(\psi^{y}_{L}) : \mathrm{D}(\widehat{\psi}^{x}_{x,L}) \Big)\,dxdy \Big|.
\end{aligned}
$$

Using (3.28) to estimate $\widehat{\psi}^{x}_{x,L}$ in terms of ψ^{x}_{L},

$$
\int_{B(x)} \big|\mathrm{D}(\widehat{\psi}^{x}_{x,L})\big|^2 \lesssim \int_{B_{1+\rho}(x)} \big|\mathrm{D}(\psi^{x}_{L}) + E\big|^2 \lesssim 1,
$$

and appealing to the decay estimates of Lemma 4.7, we deduce

$$
\begin{aligned}
|R^2_L| &\lesssim L^{-d} \iint_{(Q_{L,\rho})^2} \big\langle (x-y)_L \big\rangle^{-2d} f_2(x,y)\,dxdy \\
&\quad + L^{-d} \iiint_{(Q_{L,\rho})^3} \big\langle (x-y)_L \big\rangle^{-d} \big\langle (x-z)_L \big\rangle^{-d} \\
&\qquad\qquad \times \big(\lambda(\mathcal{P}) \big|h_2(y,z)\big| + \big|h_3(x,y,z)\big|\big)\,dxdydz \\
&\quad + L^{-d} \iint_{(Q_{L,\rho})^2} \big\langle (x-y)_L \big\rangle^{-d} \big|h_2(x,y)\big|\,dxdy \\
&\quad + \lambda(\mathcal{P}) L^{-d} \iiint_{(Q_{L,\rho})^2 \times (Q_L \setminus Q_{L,\rho})} \big\langle (x-y)_L \big\rangle^{-d} \big\langle (x-z)_L \big\rangle^{-d} \\
&\qquad\qquad \times \big|h_2(x,y)\big|\,dxdydz \\
&\quad + \lambda(\mathcal{P})^3 L^{-d} \iiint_{Q_{L,\rho} \times (Q_L \setminus Q_{L,\rho})^2} \big\langle (x-y)_L \big\rangle^{-d} \big\langle (x-z)_L \big\rangle^{-d}\,dxdydz \\
&\quad + \lambda(\mathcal{P})^2 L^{-d} \iint_{Q_{L,\rho} \times (Q_L \setminus Q_{L,\rho})} \big\langle (x-y)_L \big\rangle^{-d}\,dxdy.
\end{aligned}
$$

In terms of multi-point intensities, appealing to Lemma 1.1 (ii)–(iii), and using (4.9) and the decay assumption (Mix^n_ω) to estimate correlation functions as in (4.26), the conclusion (iii) follows after straightforward computations. ∎

4.4.2 Explicit renormalization of $\bar{\mathbf{B}}^3$

The explicit renormalization of $\bar{\mathbf{B}}^2$ above is solely based on the simple and neat cancellation property (4.24). Higher-order cluster formulas require more subtle cancellations, which can only be captured after suitably decomposing corrector differences in terms of elementary single-particle contributions as in (4.20). Before turning to the general case and proving Theorem 4.3, we start with a detailed account of the third-order cluster coefficient $\bar{\mathbf{B}}^3$, which contains all the necessary new ingredients.

Proposition 4.10 (Renormalization of $\bar{\mathbf{B}}^3$). *Let* (H_ρ) *and* ($\mathrm{H}^{\mathrm{unif}}_\rho$) *hold, and assume for simplicity that particles are spherical with unit radius, $I_n = B(x_n)$. Let also the mixing assumption* (Mix^n_ω) *hold to order $n = 3$ with some non-increasing rate function $\omega \in C^\infty_b(\mathbb{R}^+)$ satisfying the Dini-type condition $\int_1^\infty \frac{\log t}{t}\omega(t)\,dt < \infty$, as well as the doubling condition $\omega(2t) \simeq \omega(t)$ for all $t \geq 0$. Then, the infinite-volume third-order cluster coefficient $\bar{\mathbf{B}}^3$ defined in* (3.13) *can be expressed as follows,*

$$
\begin{aligned}
&E : \bar{\mathbf{B}}^3 E \\
&= 3 \iint_{\mathbb{R}^d\times\mathbb{R}^d} \Big(\int_{\partial B} \big(\mathcal{J}^y \mathcal{J}^z_y(\psi^z + Ex)\big)\cdot \sigma^0 \nu \Big) \big(\lambda(\mathcal{P}) h_2(0,z) + h_3(0,y,z)\big)\, dydz \\
&\quad + 3 \iint_{\mathbb{R}^d\times\mathbb{R}^d} \Big(\int_{\partial B} (\mathcal{J}^y \mathcal{J}^z_y \delta^y \psi^z)\cdot \sigma^0 \nu \Big) \big(f_3(0,y,z) - \lambda(\mathcal{P}) f_2(y,z)\big)\, dydz \\
&\quad + 3 \iint_{\mathbb{R}^d\times\mathbb{R}^d} \Big(\int_{\partial B} \big(\mathcal{J}^y \mathcal{J}^z_y(\psi^z + Ex)\big)\cdot \delta^y \sigma^0 \nu \Big) \\
&\qquad\qquad \times \big(f_3(0,y,z) - \lambda(\mathcal{P}) f_2(0,y)\big)\, dydz \\
&\quad + 3 \iint_{\mathbb{R}^d\times\mathbb{R}^d} \Big(\int_{\partial B} (\mathcal{J}^z \delta^y \psi^z)\cdot \delta^y \sigma^0 \nu \Big) f_3(0,y,z)\, dydz \\
&\quad + 3 \iint_{\mathbb{R}^d\times\mathbb{R}^d} \Big(\int_{\partial B} (\mathcal{J}^y \mathcal{J}^z_y \delta^y \psi^z)\cdot \delta^y \sigma^0 \nu \Big) f_3(0,y,z)\, dydz \\
&\quad + \frac{3}{2} \iint_{\mathbb{R}^d\times\mathbb{R}^d} \Big(\int_{\partial B} \delta^{y,z}\psi^{\varnothing}\cdot \delta^{y,z}\sigma^0 \nu \Big) f_3(0,y,z)\, dydz,
\end{aligned}
\tag{4.32}
$$

where all the integrals are absolutely convergent and where we use the notation (4.15)*. In addition, the following estimates hold:*

(i) Uniform cluster estimate:

$$
|\bar{\mathbf{B}}^3_L| \lesssim \lambda_3(\mathcal{P}) + \int_1^\infty \frac{\log t}{t}\big(\omega(t) \wedge \lambda_3(\mathcal{P})\big)dt,
$$

hence, in case of an algebraic weight $\omega(t) \le Ct^{-\beta}$ *for some* $C, \beta > 0$,

$$|\bar{\mathbf{B}}_L^3| \lesssim \lambda_3(\mathcal{P})\big|\log \lambda(\mathcal{P})\big|^2.$$

(ii) Periodization error estimate:

$$|\bar{\mathbf{B}}_L^3 - \bar{\mathbf{B}}^3| \lesssim \frac{\log L}{L} + \omega(L)(\log L)^2 + \int_1^\infty \frac{\log t}{t+L}\,\omega(t)\,dt.$$

(iii) Uniform remainder estimate: *If* (Mix$_\omega^n$) *further holds with* $n = 5$, *then*

$$|R_L^3| \lesssim \lambda_3(\mathcal{P}) + \sum_{j=3}^{5} \int_1^\infty \frac{(\log t)^{j-2}}{t}\big(\omega(t) \wedge \lambda_j(\mathcal{P})\big)\,dt,$$

hence, in case of an algebraic weight $\omega(t) \le Ct^{-\beta}$ *for some* $C, \beta > 0$,

$$|R_L^3| \lesssim \lambda_3(\mathcal{P})\big|\log \lambda(\mathcal{P})\big|^2 + \lambda_4(\mathcal{P})\big|\log \lambda(\mathcal{P})\big|^3 + \lambda_5(\mathcal{P})\big|\log \lambda(\mathcal{P})\big|^4.$$

Proof. We split the proof into four steps. Given $E \in \mathbb{M}_0^{\mathrm{sym}}$ with $|E| = 1$, for notational convenience, we write $\bar{\mathbf{B}}_L^3$, $\bar{\mathbf{B}}^3$, and R_L^3 for $E : \bar{\mathbf{B}}_L^3 E$, $E : \bar{\mathbf{B}}^3 E$, and $E : R_L^3 E$.

Step 1. Reformulation of $\bar{\mathbf{B}}_L^3$:

$$\begin{aligned}
\bar{\mathbf{B}}_L^3 = E_L^3
&+ 3L^{-d} \iiint_{(Q_{L,\rho})^3} \bigg(\int_{\partial B(x)} (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \bar{\psi}_L^z) \cdot \sigma_L^x \nu \bigg) \\
&\qquad \times \big(\lambda(\mathcal{P}) h_2(x,z) + h_3(x,y,z)\big)\,dxdydz \\
&+ 3L^{-d} \iiint_{(Q_{L,\rho})^3} \bigg(\int_{\partial B(x)} (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \mathcal{J}_{L;z}^y \bar{\psi}_L^{y,z}) \cdot \sigma_L^x \nu \bigg) \\
&\qquad \times \big(f_3(x,y,z) - \lambda(\mathcal{P}) f_2(y,z)\big)\,dxdydz \\
&+ 3L^{-d} \iiint_{(Q_{L,\rho})^3} \bigg(\int_{\partial B(x)} (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \bar{\psi}_L^z) \cdot \delta^y \sigma_L^x \nu \bigg) \\
&\qquad \times \big(f_3(x,y,z) - \lambda(\mathcal{P}) f_2(x,y)\big)\,dxdydz \\
&+ 3L^{-d} \iiint_{(Q_{L,\rho})^3} \bigg(\int_{\partial B(x)} (\mathcal{J}_L^z \mathcal{J}_{L;z}^y \bar{\psi}_L^{y,z}) \cdot \delta^y \sigma_L^x \nu \bigg) f_3(x,y,z)\,dxdydz \\
&+ 3L^{-d} \iiint_{(Q_{L,\rho})^3} \bigg(\int_{\partial B(x)} (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \mathcal{J}_{L;z}^y \bar{\psi}_L^{y,z}) \cdot \delta^y \sigma_L^x \nu \bigg) f_3(x,y,z)\,dxdydz \\
&+ \frac{3}{2} L^{-d} \iiint_{(Q_{L,\rho})^3} \bigg(\int_{\partial B(x)} \delta^{y,z} \psi_L^{\varnothing} \cdot \delta^{y,z} \sigma_L^x \nu \bigg) f_3(x,y,z)\,dxdydz, \quad (4.33)
\end{aligned}$$

where we henceforth use the shorthand notation

$$\bar{\psi}_L^Y := \psi_L^Y + Ex,$$

and where E_L^3 stands for boundary terms,

$$\begin{aligned}
&E_L^3 \\
&:= 3\lambda(\mathcal{P})^3 L^{-d} \iiint_{Q_{L,\rho}\times(Q_L\setminus Q_{L,\rho})^2} \left(\int_{\partial B(x)} (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \bar{\psi}_L^z)\cdot \sigma_L^x \nu\right) dxdydz \\
&- 3\lambda(\mathcal{P}) L^{-d} \iiint_{(Q_{L,\rho})^2\times(Q_L\setminus Q_{L,\rho})} \left(\int_{\partial B(x)} (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \bar{\psi}_L^z)\cdot \sigma_L^x \nu\right) h_2(x,y)\, dxdydz \\
&- 3\lambda(\mathcal{P}) L^{-d} \iiint_{(Q_L\setminus Q_{L,\rho})\times(Q_{L,\rho})^2} \left(\int_{\partial B(x)} (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \bar{\psi}_L^z)\cdot \sigma_L^x \nu\right) h_2(y,z)\, dxdydz \\
&- 3\lambda(\mathcal{P}) L^{-d} \iiint_{(Q_L\setminus Q_{L,\rho})\times(Q_{L,\rho})^2} \left(\int_{\partial B(x)} (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \mathcal{J}_{L;z}^y \bar{\psi}_L^{y,z})\cdot \sigma_L^x \nu\right) \\
&\qquad\qquad \times f_2(y,z)\, dxdydz \\
&- 3\lambda(\mathcal{P}) L^{-d} \iiint_{(Q_{L,\rho})^2\times(Q_L\setminus Q_{L,\rho})} \left(\int_{\partial B(x)} (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \bar{\psi}_L^z)\cdot \delta^y \sigma_L^x \nu\right) f_2(x,y)\, dxdydz.
\end{aligned}$$

By definition, cf. (3.8), the finite-volume approximation $\bar{\mathbf{B}}_L^3$ is given by

$$\bar{\mathbf{B}}_L^3 = \frac{3}{2} L^{-d} \sum_{n,m,p}^{\neq} \mathbb{E}\left[\int_{\partial B(x_{n,L})} \delta^{\{m,p\}} \psi_L^{\varnothing} \cdot \sigma_L^{\{n,m,p\}} \nu\right].$$

Decomposing

$$\sigma_L^{\{n,m,p\}} = \sigma_L^{\{n\}} + \delta^{\{m\}}\sigma_L^{\{n\}} + \delta^{\{p\}}\sigma_L^{\{n\}} + \delta^{\{m,p\}}\sigma_L^{\{n\}},$$

this becomes by symmetry,

$$\begin{aligned}
\bar{\mathbf{B}}_L^3 = {} & \frac{3}{2} L^{-d} \sum_{n,m,p}^{\neq} \mathbb{E}\left[\int_{\partial B(x_{n,L})} \delta^{\{m,p\}} \psi_L^{\varnothing} \cdot \sigma_L^{\{n\}} \nu\right] \\
& + 3L^{-d} \sum_{n,m,p}^{\neq} \mathbb{E}\left[\int_{\partial B(x_{n,L})} \delta^{\{m,p\}} \psi_L^{\varnothing} \cdot \delta^{\{m\}}\sigma_L^{\{n\}} \nu\right] \\
& + \frac{3}{2} L^{-d} \sum_{n,m,p}^{\neq} \mathbb{E}\left[\int_{\partial B(x_{n,L})} \delta^{\{m,p\}} \psi_L^{\varnothing} \cdot \delta^{\{m,p\}}\sigma_L^{\{n\}} \nu\right].
\end{aligned}$$

In terms of multi-point densities, cf. (1.15), recalling the choice of the finite-volume approximation with $\mathcal{P}_L = \{x_n : x_n \in Q_{L,\rho}\}$, cf. (3.1), we can rewrite

$$\begin{aligned}\bar{\mathbf{B}}_L^3 &= \frac{3}{2}L^{-d}\iiint_{(Q_{L,\rho})^3}\Big(\int_{\partial B(x)}\delta^{y,z}\psi_L^{\varnothing}\cdot\sigma_L^x\nu\Big)f_3(x,y,z)\,dxdydz\\ &\quad+3L^{-d}\iiint_{(Q_{L,\rho})^3}\Big(\int_{\partial B(x)}\delta^{y,z}\psi_L^{\varnothing}\cdot\delta^y\sigma_L^x\nu\Big)f_3(x,y,z)\,dxdydz\\ &\quad+\frac{3}{2}L^{-d}\iiint_{(Q_{L,\rho})^3}\Big(\int_{\partial B(x)}\delta^{y,z}\psi_L^{\varnothing}\cdot\delta^{y,z}\sigma_L^x\nu\Big)f_3(x,y,z)\,dxdydz. \quad (4.34)\end{aligned}$$

It remains to further analyze the first two right-hand side terms and we split the proof into two further substeps. To capture cancellations, we shall expand $\delta^{y,z}\psi_L^{\varnothing}$ and $\delta^y\psi_L^x$ in terms of single-particle contributions as in (4.20).

Substep 1.1. Proof that

$$\begin{aligned}&L^{-d}\iiint_{(Q_{L,\rho})^3}\Big(\int_{\partial B(x)}\delta^{y,z}\psi_L^{\varnothing}\cdot\sigma_L^x\nu\Big)f_3(x,y,z)\,dxdydz\\ &\quad=E_L^{3,1}+2L^{-d}\iiint_{(Q_{L,\rho})^3}\Big(\int_{\partial B(x)}(\mathcal{J}_L^y\mathcal{J}_{L;y}^z\bar{\psi}_L^z)\cdot\sigma_L^x\nu\Big)\\ &\qquad\qquad\times\big(\lambda(\mathcal{P})h_2(x,z)+h_3(x,y,z)\big)\,dxdydz\\ &\qquad+2L^{-d}\iiint_{(Q_{L,\rho})^3}\Big(\int_{\partial B(x)}(\mathcal{J}_L^y\mathcal{J}_{L;y}^z\mathcal{J}_{L;z}^y\bar{\psi}_L^{y,z})\cdot\sigma_L^x\nu\Big)\\ &\qquad\qquad\times\big(f_3(x,y,z)-\lambda(\mathcal{P})f_2(y,z)\big)\,dxdydz, \quad (4.35)\end{aligned}$$

where we recall the shorthand notation $\bar{\psi}_L^Y = \psi_L^Y + Ex$, and where $E_L^{3,1}$ stands for boundary terms,

$$\begin{aligned}E_L^{3,1} &:= 2\lambda(\mathcal{P})^3L^{-d}\iiint_{Q_{L,\rho}\times(Q_L\setminus Q_{L,\rho})^2}\Big(\int_{\partial B(x)}(\mathcal{J}_L^y\mathcal{J}_{L;y}^z\bar{\psi}_L^z)\cdot\sigma_L^x\nu\Big)\,dxdydz\\ &-2\lambda(\mathcal{P})L^{-d}\iiint_{(Q_{L,\rho})^2\times(Q_L\setminus Q_{L,\rho})}\Big(\int_{\partial B(x)}(\mathcal{J}_L^y\mathcal{J}_{L;y}^z\bar{\psi}_L^z)\cdot\sigma_L^x\nu\Big)h_2(x,y)\,dxdydz\\ &-2\lambda(\mathcal{P})L^{-d}\iiint_{(Q_L\setminus Q_{L,\rho})\times(Q_{L,\rho})^2}\Big(\int_{\partial B(x)}(\mathcal{J}_L^y\mathcal{J}_{L;y}^z\bar{\psi}_L^z)\cdot\sigma_L^x\nu\Big)h_2(y,z)\,dxdydz\\ &-2\lambda(\mathcal{P})L^{-d}\\ &\quad\times\iiint_{(Q_L\setminus Q_{L,\rho})\times(Q_{L,\rho})^2}\Big(\int_{\partial B(x)}(\mathcal{J}_L^y\mathcal{J}_{L;y}^z\mathcal{J}_{L;z}^y\bar{\psi}_L^{y,z})\cdot\sigma_L^x\nu\Big)f_2(y,z)\,dxdydz.\end{aligned}$$

In view of (4.20), corrector differences can be decomposed as

$$\begin{aligned}\delta^{y,z}\psi_L^{\varnothing} &= \mathcal{J}_L^y\delta^z\psi_L^y+\mathcal{J}_L^z\delta^y\psi_L^z\\ &= \mathcal{J}_L^y\mathcal{J}_{L;y}^z\bar{\psi}_L^{y,z}+\mathcal{J}_L^z\mathcal{J}_{L;z}^y\bar{\psi}_L^{y,z}, \quad (4.36)\end{aligned}$$

and thus, further writing

$$\begin{aligned}\psi_L^{y,z} &= \psi_L^z + \delta^y \psi_L^z = \psi_L^z + \mathcal{J}_{L;z}^y \bar{\psi}_L^{y,z} \\ &= \psi_L^y + \delta^z \psi_L^y = \psi_L^y + \mathcal{J}_{L;y}^z \bar{\psi}_L^{y,z},\end{aligned}$$

we deduce

$$\begin{aligned}&\delta^{y,z} \psi_L^{\varnothing} \\ &= \mathcal{J}_L^y \mathcal{J}_{L;y}^z \bar{\psi}_L^z + \mathcal{J}_L^z \mathcal{J}_{L;z}^y \bar{\psi}_L^y + \mathcal{J}_L^y \mathcal{J}_{L;y}^z \mathcal{J}_{L;z}^y \bar{\psi}_L^{y,z} + \mathcal{J}_L^z \mathcal{J}_{L;z}^y \mathcal{J}_{L;y}^z \bar{\psi}_L^{y,z}. \end{aligned} \tag{4.37}$$

From this decomposition, we get by symmetry

$$\begin{aligned}&L^{-d} \iiint_{(Q_{L,\rho})^3} \left(\int_{\partial B(x)} \delta^{y,z} \psi_L^{\varnothing} \cdot \sigma_L^x \nu \right) f_3(x,y,z)\, dxdydz \\ &= 2L^{-d} \iiint_{(Q_{L,\rho})^3} \left(\int_{\partial B(x)} (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \bar{\psi}_L^z) \cdot \sigma_L^x \nu \right) f_3(x,y,z)\, dxdydz \\ &\quad + 2L^{-d} \iiint_{(Q_{L,\rho})^3} \left(\int_{\partial B(x)} (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \mathcal{J}_{L;z}^y \bar{\psi}_L^{y,z}) \cdot \sigma_L^x \nu \right) f_3(x,y,z)\, dxdydz. \end{aligned} \tag{4.38}$$

We are now in position to exploit cancellations properties. First note that Lemma 4.6 yields, recalling $\psi_L^z = \psi_L^0(\cdot - z)$,

$$\int_{Q_L} (\mathcal{J}_{L;y}^z \bar{\psi}_L^z)\, dz = 0, \tag{4.39}$$

and thus, for all $x, y \in \mathbb{R}^d$,

$$\int_{Q_L} \left(\int_{\partial B(x)} (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \bar{\psi}_L^z) \cdot \sigma_L^x \nu \right) dz = 0.$$

In addition, similarly to what was done in (4.24) for the renormalization of $\bar{\mathbf{B}}_L^2$, writing

$$\int_{\partial B(x)} (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \bar{\psi}_L^z) \cdot \sigma_L^x \nu = \int_{\partial B} (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \bar{\psi}_L^z)(\cdot + x) \cdot \sigma_L^0 \nu,$$

and using the condition $\int_{\partial B} \sigma_L^0 \nu = 0$, we find

$$\int_{Q_L} \left(\int_{\partial B(x)} (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \bar{\psi}_L^z) \cdot \sigma_L^x \nu \right) dx = 0, \tag{4.40}$$

and likewise,

$$\int_{Q_L} \left(\int_{\partial B(x)} (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \mathcal{J}_{L;z}^y \bar{\psi}_L^{y,z}) \cdot \sigma_L^x \nu \right) dx = 0.$$

Turning back to (4.38), replacing f_3 by its expansion (4.7) in terms of correlation functions, and using these three cancellation properties, the claim (4.35) follows.

Substep 1.2. Proof that

$$
\begin{aligned}
&L^{-d} \iiint_{(Q_{L,\rho})^3} \Big(\int_{\partial B(x)} \delta^{y,z} \psi_L^{\varnothing} \cdot \delta^y \sigma_L^x \nu \Big) f_3(x,y,z)\, dxdydz \\
&\quad = E_L^{3,2} + L^{-d} \iiint_{(Q_{L,\rho})^3} \Big(\int_{\partial B(x)} (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \bar{\psi}_L^z) \cdot \delta^y \sigma_L^x \nu \Big) \\
&\qquad\qquad \times \big(f_3(x,y,z) - \lambda(\mathcal{P}) f_2(x,y) \big) dxdydz \\
&\qquad + L^{-d} \iiint_{(Q_{L,\rho})^3} \Big(\int_{\partial B(x)} (\mathcal{J}_L^z \mathcal{J}_{L;z}^y \bar{\psi}_L^{y,z}) \cdot \delta^y \sigma_L^x \nu \Big) f_3(x,y,z)\, dxdydz \\
&\qquad + L^{-d} \iiint_{(Q_{L,\rho})^3} \Big(\int_{\partial B(x)} (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \mathcal{J}_{L;z}^y \bar{\psi}_L^{y,z}) \cdot \delta^y \sigma_L^x \nu \Big) f_3(x,y,z)\, dxdydz,
\end{aligned}
\tag{4.41}
$$

where $E_L^{3,2}$ stands for a boundary term,

$$
\begin{aligned}
&E_L^{3,2} := \\
&-\lambda(\mathcal{P}) L^{-d} \iiint_{(Q_{L,\rho})^2 \times (Q_L \setminus Q_{L,\rho})} \Big(\int_{\partial B(x)} (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \bar{\psi}_L^z) \cdot \delta^y \sigma_L^x \nu \Big) f_2(x,y)\, dxdydz.
\end{aligned}
$$

Inserting this into (4.34), together with (4.35), the claim (4.33) follows.

We turn to the proof of (4.41). As this term benefits from some additional decay due to the factor $\delta^y \sigma_L^x$, we only need the following (asymmetric) simpler version of (4.37),

$$
\delta^{y,z} \psi_L^{\varnothing} = \mathcal{J}_L^y \mathcal{J}_{L;y}^z \bar{\psi}_L^z + \mathcal{J}_L^z \mathcal{J}_{L;z}^y \bar{\psi}_L^{y,z} + \mathcal{J}_L^y \mathcal{J}_{L;y}^z \mathcal{J}_{L;z}^y \bar{\psi}_L^{y,z},
$$

which leads us to

$$
\begin{aligned}
&L^{-d} \iiint_{(Q_{L,\rho})^3} \Big(\int_{\partial B(x)} \delta^{y,z} \psi_L^{\varnothing} \cdot \delta^y \sigma_L^x \nu \Big) f_3(x,y,z)\, dxdydz \\
&\quad = L^{-d} \iiint_{(Q_{L,\rho})^3} \Big(\int_{\partial B(x)} (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \bar{\psi}_L^z) \cdot \delta^y \sigma_L^x \nu \Big) f_3(x,y,z)\, dxdydz \\
&\qquad + L^{-d} \iiint_{(Q_{L,\rho})^3} \Big(\int_{\partial B(x)} (\mathcal{J}_L^z \mathcal{J}_{L;z}^y \bar{\psi}_L^{y,z}) \cdot \delta^y \sigma_L^x \nu \Big) f_3(x,y,z)\, dxdydz \\
&\qquad + L^{-d} \iiint_{(Q_{L,\rho})^3} \Big(\int_{\partial B(x)} (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \mathcal{J}_{L;z}^y \bar{\psi}_L^{y,z}) \cdot \delta^y \sigma_L^x \nu \Big) f_3(x,y,z)\, dxdydz,
\end{aligned}
$$

and it remains to analyze the first right-hand side term. For that purpose, we use again the elementary cancellation property (4.39), now in form of

$$
\int_{Q_L} \Big(\int_{\partial B(x)} (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \bar{\psi}_L^z) \cdot \delta^y \sigma_L^x \nu \Big) dz = 0,
$$

which entails

$$
\begin{aligned}
&L^{-d} \iiint_{(Q_{L,\rho})^3} \Big(\int_{\partial B(x)} (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \bar{\psi}_L^z) \cdot \delta^y \sigma_L^x \nu \Big) f_3(x,y,z)\, dxdydz \\
&\quad = L^{-d} \iiint_{(Q_{L,\rho})^3} \Big(\int_{\partial B(x)} (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \bar{\psi}_L^z) \cdot \delta^y \sigma_L^x \nu \Big) \\
&\qquad\qquad \times \big(f_3(x,y,z) - \lambda(\mathcal{P}) f_2(x,y) \big)\, dxdydz \\
&\qquad - \lambda(\mathcal{P}) L^{-d} \iiint_{(Q_{L,\rho})^2 \times (Q_L \setminus Q_{L,\rho})} \Big(\int_{\partial B(x)} (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \bar{\psi}_L^z) \cdot \delta^y \sigma_L^x \nu \Big) \\
&\qquad\qquad \times f_2(x,y)\, dxdydz,
\end{aligned}
$$

and the claim (4.41) follows.

Step 2. Uniform estimates: proof of (i). As in the proof of Proposition 4.9 (i), appealing to the trace estimates of Lemma 2.5, the decay estimates of Lemma 4.7, and the energy estimate (3.44), formula (4.33) for $\bar{\mathbf{B}}_L^3$ can be estimated as follows,

$$
\begin{aligned}
&|\bar{\mathbf{B}}_L^3| \lesssim |E_L^3| \\
&+ L^{-d} \iiint_{(Q_L)^3} \langle (x-y)_L \rangle^{-d} \langle (y-z)_L \rangle^{-d} \big(\lambda(\mathcal{P}) |h_2(x,z)| + |h_3(x,y,z)| \big)\, dxdydz \\
&+ L^{-d} \iiint_{(Q_L)^3} \langle (x-y)_L \rangle^{-d} \langle (y-z)_L \rangle^{-2d} \big| f_3(x,y,z) - \lambda(\mathcal{P}) f_2(y,z) \big|\, dxdydz \\
&+ L^{-d} \iiint_{(Q_L)^3} \langle (x-z)_L \rangle^{-d} \langle (z-y)_L \rangle^{-d} \langle (y-x)_L \rangle^{-d} f_3(x,y,z)\, dxdydz \\
&+ L^{-d} \iiint_{(Q_L)^3} \langle (x-y)_L \rangle^{-2d} \langle (y-z)_L \rangle^{-2d} f_3(x,y,z)\, dxdydz,
\end{aligned}
$$

and, for boundary terms,

$$
\begin{aligned}
&|E_L^3| \lesssim \lambda(\mathcal{P})^3 L^{-d} \iiint_{Q_L \times (Q_L \setminus Q_{L,\rho})^2} \langle (x-y)_L \rangle^{-d} \langle (y-z)_L \rangle^{-d}\, dxdydz \\
&+ \lambda(\mathcal{P}) L^{-d} \iiint_{(Q_L \setminus Q_{L,\rho}) \times (Q_L)^2} \langle (x-y)_L \rangle^{-d} \langle (y-z)_L \rangle^{-d} |h_2(y,z)|\, dxdydz \\
&+ \lambda(\mathcal{P}) L^{-d} \iiint_{(Q_L \setminus Q_{L,\rho}) \times (Q_L)^2} \langle (x-y)_L \rangle^{-d} \langle (y-z)_L \rangle^{-2d} f_2(y,z)\, dxdydz.
\end{aligned}
$$

In terms of multi-point intensities, appealing to Lemma 1.1, using both (4.9) and the decay assumption (Mix_ω^n) to estimate correlation functions similarly to (4.26), we

deduce after straightforward computations

$$\begin{aligned}|\overline{\mathbf{B}}^3_L| &\lesssim |E^3_L| + \lambda_3(\mathcal{P}) + \int_1^\infty \frac{\log t}{t}\big(\omega(t)\wedge\lambda_3(\mathcal{P})\big)dt,\\ |E^3_L| &\lesssim \frac{\log L}{L}\bigg(\lambda_3(\mathcal{P}) + \int_1^L \frac{1}{t}\big(\omega(t)\wedge\lambda_3(\mathcal{P})\big)dt\bigg),\end{aligned} \tag{4.42}$$

and the conclusion (i) follows.

Step 3. Convergence result: proof of (ii). In terms of $\gamma^3_{L,\rho}(y,z) := L^{-d}|Q_{L,\rho}\cap(y+Q_{L,\rho})\cap(z+Q_{L,\rho})|$, using stationarity and recalling that $\delta^y\psi^z = \mathcal{J}^y_z\bar\psi^{y,z}$, the formula (4.32) for the infinite-volume cluster coefficient $\overline{\mathbf{B}}^3$ takes the equivalent form

$$\begin{aligned}\overline{\mathbf{B}}^3 &= E^4_L\\ &+ 3L^{-d}\iiint_{(Q_{L,\rho})^3}\bigg(\int_{\partial B(x)}(\mathcal{J}^y\mathcal{J}^z_y\bar\psi^z)\cdot\sigma^x\nu\bigg)\\ &\qquad\times\big(\lambda(\mathcal{P})h_2(x,z)+h_3(x,y,z)\big)\,dxdydz\\ &+ 3L^{-d}\iiint_{(Q_{L,\rho})^3}\bigg(\int_{\partial B(x)}(\mathcal{J}^y\mathcal{J}^z_y\mathcal{J}^y_z\bar\psi^{y,z})\cdot\sigma^x\nu\bigg)\\ &\qquad\times\big(f_3(x,y,z)-\lambda(\mathcal{P})f_2(y,z)\big)\,dxdydz\\ &+ 3L^{-d}\iiint_{(Q_{L,\rho})^3}\bigg(\int_{\partial B(x)}(\mathcal{J}^y\mathcal{J}^z_y\bar\psi^z)\cdot\delta^y\sigma^x\nu\bigg)\\ &\qquad\times\big(f_3(x,y,z)-\lambda(\mathcal{P})f_2(x,y)\big)\,dxdydz\\ &+ 3L^{-d}\iiint_{(Q_{L,\rho})^3}\bigg(\int_{\partial B(x)}(\mathcal{J}^z\mathcal{J}^y_z\bar\psi^{y,z})\cdot\delta^y\sigma^x\nu\bigg)f_3(x,y,z)\,dxdydz\\ &+ 3L^{-d}\iiint_{(Q_{L,\rho})^3}\bigg(\int_{\partial B(x)}(\mathcal{J}^y\mathcal{J}^z_{L;y}\mathcal{J}^y_z\bar\psi^{y,z})\cdot\delta^y\sigma^x\nu\bigg)f_3(x,y,z)\,dxdydz\\ &+ \frac{3}{2}L^{-d}\iiint_{(Q_{L,\rho})^3}\bigg(\int_{\partial B(x)}\delta^{y,z}\psi^\varnothing\cdot\delta^{y,z}\sigma^x\nu\bigg)f_3(x,y,z)\,dxdydz,\end{aligned}$$

where

$$\begin{aligned}E^4_L &:= 3\iint_{\mathbb{R}^d\times\mathbb{R}^d}(1-\gamma^3_{L,\rho}(y,z))\bigg(\int_{\partial B}(\mathcal{J}^y\mathcal{J}^z_y\bar\psi^z)\cdot\sigma^0\nu\bigg)\\ &\qquad\times\big(\lambda(\mathcal{P})h_2(0,z)+h_3(0,y,z)\big)\,dydz\\ &+ 3\iint_{\mathbb{R}^d\times\mathbb{R}^d}(1-\gamma^3_{L,\rho}(y,z))\bigg(\int_{\partial B}(\mathcal{J}^y\mathcal{J}^z_y\delta^y\psi^z)\cdot\sigma^0\nu\bigg)\\ &\qquad\times\big(f_3(0,y,z)-\lambda(\mathcal{P})f_2(y,z)\big)\,dydz\\ &+ 3\iint_{\mathbb{R}^d\times\mathbb{R}^d}(1-\gamma^3_{L,\rho}(y,z))\bigg(\int_{\partial B}(\mathcal{J}^y\mathcal{J}^z_y\bar\psi^z)\cdot\delta^y\sigma^0\nu\bigg)\\ &\qquad\times\big(f_3(0,y,z)-\lambda(\mathcal{P})f_2(0,y)\big)\,dydz\end{aligned}$$

$$
\begin{aligned}
&+3\iint_{\mathbb{R}^d\times\mathbb{R}^d}(1-\gamma^3_{L,\rho}(y,z))\Big(\int_{\partial B}(\mathcal{J}^z\delta^y\psi^z)\cdot\delta^y\sigma^0\nu\Big)f_3(0,y,z)\,dydz\\
&+3\iint_{\mathbb{R}^d\times\mathbb{R}^d}(1-\gamma^3_{L,\rho}(y,z))\Big(\int_{\partial B}(\mathcal{J}^y\mathcal{J}^z_y\delta^y\psi^z)\cdot\delta^y\sigma^0\nu\Big)f_3(0,y,z)\,dydz\\
&+\frac{3}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d}(1-\gamma^3_{L,\rho}(y,z))\Big(\int_{\partial B}\delta^{y,z}\psi^{\varnothing}\cdot\delta^{y,z}\sigma^0\nu\Big)f_3(0,y,z)\,dydz,
\end{aligned}
$$

so that the identity (4.33) for $\bar{\mathbf{B}}^3_L$ entails

$$
\begin{aligned}
&\bar{\mathbf{B}}^3_L-\bar{\mathbf{B}}^3=E^3_L-E^4_L\\
&+3L^{-d}\iiint_{(Q_{L,\rho})^3}\Big(\int_{\partial B(x)}(\mathcal{J}^y_L\mathcal{J}^z_{L;y}\bar\psi^z_L)\cdot\sigma^x_L\nu-(\mathcal{J}^y\mathcal{J}^z_y\bar\psi^z)\cdot\sigma^x\nu\Big)\\
&\qquad\qquad\times\big(\lambda(\mathcal{P})h_2(x,z)+h_3(x,y,z)\big)\,dxdydz\\
&+3L^{-d}\iiint_{(Q_{L,\rho})^3}\Big(\int_{\partial B(x)}(\mathcal{J}^y_L\mathcal{J}^z_{L;y}\mathcal{J}^y_{L;z}\bar\psi^{y,z}_L)\cdot\sigma^x_L\nu-(\mathcal{J}^y\mathcal{J}^z_y\mathcal{J}^y_z\bar\psi^{y,z})\cdot\sigma^x\nu\Big)\\
&\qquad\qquad\times\big(f_3(x,y,z)-\lambda(\mathcal{P})f_2(y,z)\big)\,dxdydz\\
&+3L^{-d}\iiint_{(Q_{L,\rho})^3}\Big(\int_{\partial B(x)}(\mathcal{J}^y_L\mathcal{J}^z_{L;y}\bar\psi^z_L)\cdot\delta^y\sigma^x_L\nu-(\mathcal{J}^y\mathcal{J}^z_y\bar\psi^z)\cdot\delta^y\sigma^x\nu\Big)\\
&\qquad\qquad\times\big(f_3(x,y,z)-\lambda(\mathcal{P})f_2(x,y)\big)\,dxdydz\\
&+3L^{-d}\iiint_{(Q_{L,\rho})^3}\Big(\int_{\partial B(x)}(\mathcal{J}^z_L\mathcal{J}^y_{L;z}\bar\psi^{y,z}_L)\cdot\delta^y\sigma^x_L\nu-(\mathcal{J}^z\mathcal{J}^y_z\bar\psi^{y,z})\cdot\delta^y\sigma^x\nu\Big)\\
&\qquad\qquad\times f_3(x,y,z)\,dxdydz\\
&+3L^{-d}\iiint_{(Q_{L,\rho})^3}\Big(\int_{\partial B(x)}(\mathcal{J}^y_L\mathcal{J}^z_{L;y}\mathcal{J}^y_{L;z}\bar\psi^{y,z}_L)\cdot\delta^y\sigma^x_L\nu-(\mathcal{J}^y\mathcal{J}^z_y\mathcal{J}^y_z\bar\psi^{y,z})\cdot\delta^y\sigma^x\nu\Big)\\
&\qquad\qquad\times f_3(x,y,z)\,dxdydz\\
&+\frac{3}{2}L^{-d}\iiint_{(Q_{L,\rho})^3}\Big(\int_{\partial B(x)}\delta^{y,z}\psi^{\varnothing}_L\cdot\delta^{y,z}\sigma^x_L\nu-\delta^{y,z}\psi^{\varnothing}\cdot\delta^{y,z}\sigma^x\nu\Big)\\
&\qquad\qquad\times f_3(x,y,z)\,dxdydz.
\end{aligned}
\tag{4.43}
$$

The first boundary contribution E^3_L is already estimated in (4.42). Noting that

$$
1-\gamma^3_{L,\rho}(y,z)\lesssim\frac{\langle y\rangle}{L}\wedge 1+\frac{\langle z\rangle}{L}\wedge 1,
$$

using the trace estimates of Lemma 3.5, the decay estimates of Lemma 4.7, the energy estimate (3.44), and Lemma 1.1 (iii), and further using (4.9) and the decay assumption (Mix^n_ω) to estimate correlation functions similarly to (4.26), we obtain for the second boundary contribution in (4.43),

$$
E^4_L\lesssim\frac{1}{L}\int_1^L(\log t)\,\omega(t)\,dt+\int_L^\infty\frac{\log t}{t}\,\omega(t)\,dt.
$$

It remains to estimate the remaining six right-hand side terms in (4.43). We focus on the first term, which is the most involved, and we skip the detail for the last five ones. We split the proof into two further substeps.

Substep 3.1. First periodization error term in (4.43): proof that

$$\begin{aligned}\Big| L^{-d} \iiint_{(Q_{L,\rho})^3} \Big(\int_{\partial B(x)} (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \bar\psi_L^z) \cdot \sigma_L^x \nu - (\mathcal{J}^y \mathcal{J}_y^z \bar\psi^z) \cdot \sigma^x \nu \Big) \\ \times \big(\lambda(\mathcal{P}) h_2(x,z) + h_3(x,y,z)\big)\, dx dy dz \Big| \\ \lesssim \omega(L)(\log L)^2 + \frac{1}{L} \int_1^L (\log t)\omega(t)\, dt. \end{aligned} \tag{4.44}$$

Decomposing

$$\begin{aligned}&(\mathcal{J}_L^y \mathcal{J}_{L;y}^z \bar\psi_L^z) \cdot \sigma_L^x \nu - (\mathcal{J}^y \mathcal{J}_y^z \bar\psi^z) \cdot \sigma^x \nu \\ &\quad = (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \bar\psi_L^z) \cdot (\sigma_L^x - \sigma^x)\nu + (\mathcal{J}_L^y - \mathcal{J}^y)\mathcal{J}_{L;y}^z \bar\psi_L^z \cdot \sigma^x \nu \\ &\qquad + \mathcal{J}^y (\mathcal{J}_{L;y}^z - \mathcal{J}_y^z)\bar\psi_L^z \cdot \sigma^x \nu + \mathcal{J}^y \mathcal{J}_y^z (\psi_L^z - \psi^z) \cdot \sigma^x \nu,\end{aligned}$$

and appealing to the trace estimates of Lemma 3.5, we find

$$\begin{aligned}&\Big| \int_{\partial B(x)} (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \bar\psi_L^z) \cdot \sigma_L^x \nu - (\mathcal{J}^y \mathcal{J}_y^z \bar\psi^z) \cdot \sigma^x \nu \Big| \\ &\lesssim \Big(\int_{B(x)} \big|\mathrm{D}(\mathcal{J}_L^y \mathcal{J}_{L;y}^z \bar\psi_L^z)\big|^2 \Big)^{\frac12} \Big(\int_{B_{1+\rho}(x)} \big|\mathrm{D}(\psi_L^x - \psi^x)\big|^2 \Big)^{\frac12} \\ &\quad + \Big(\int_{B(x)} \big|\mathrm{D}\big((\mathcal{J}_L^y - \mathcal{J}^y)\mathcal{J}_{L;y}^z \bar\psi_L^z\big)\big|^2 \Big)^{\frac12} \Big(\int_{B_{1+\rho}(x)} \big|\mathrm{D}(\psi^x)\big|^2 \Big)^{\frac12} \\ &\quad + \Big(\int_{B(x)} \big|\mathrm{D}\big(\mathcal{J}^y (\mathcal{J}_{L;y}^z - \mathcal{J}_y^z)\bar\psi_L^z\big)\big|^2 \Big)^{\frac12} \Big(\int_{B_{1+\rho}(x)} \big|\mathrm{D}(\psi^x)\big|^2 \Big)^{\frac12} \\ &\quad + \Big(\int_{B(x)} \big|\mathrm{D}\big(\mathcal{J}^y \mathcal{J}_y^z (\psi_L^z - \psi^z)\big)\big|^2 \Big)^{\frac12} \Big(\int_{B_{1+\rho}(x)} \big|\mathrm{D}(\psi^x)\big|^2 \Big)^{\frac12}.\end{aligned}$$

We then appeal to the decay estimates of Lemma 4.7, to the periodization error estimates of Lemma 4.8, and to the energy estimate (3.44), in form of

$$\begin{aligned}&\Big(\int_{B_{1+\rho}(x)} \big|\mathrm{D}(\psi_L^x - \psi^x)\big|^2 \Big)^{\frac12} \lesssim L^{-d}, \\ &\Big(\int_{B_{1+\rho}(x)} \big|\mathrm{D}(\mathcal{J}_L^y \mathcal{J}_{L;y}^z \bar\psi_L^z)\big|^2 \Big)^{\frac12} \lesssim \big\langle (x-y)_L \big\rangle^{-d} \big\langle (y-z)_L \big\rangle^{-d},\end{aligned}$$

$$\Big(\int_{B_{1+\rho}(x)} \big|\mathrm{D}\big((\mathcal{J}_L^y - \mathcal{J}^y)\mathcal{J}_{L;y}^z \bar\psi_L^z\big)\big|^2\Big)^{\frac12}$$
$$\lesssim \big\langle (y-z)_L\big\rangle^{-d}\big(\mathbb{1}_{|x-y|>\frac{L}{4}}\big\langle (x-y)_L\big\rangle^{-d} + \mathbb{1}_{|x-y|\le\frac{L}{4}} L^{-d}\big),$$
$$\Big(\int_{B_{1+\rho}(x)} \big|\mathrm{D}\big(\mathcal{J}^y(\mathcal{J}_{L;y}^z - \mathcal{J}_y^z)\bar\psi_L^z\big)\big|^2\Big)^{\frac12}$$
$$\lesssim \langle x-y\rangle^{-d}\big(\mathbb{1}_{|y-z|>\frac{L}{4}}\big\langle (y-z)_L\big\rangle^{-d} + \mathbb{1}_{|y-z|\le\frac{L}{4}} L^{-d}\big),$$
$$\Big(\int_{B_{1+\rho}(x)} \big|\mathrm{D}\big(\mathcal{J}^y\mathcal{J}_y^z(\psi_L^z - \psi^z)\big)\big|^2\Big)^{\frac12} \lesssim L^{-d}\langle x-y\rangle^{-d}\big\langle (y-z)_L\big\rangle^{-d},$$

so that the above becomes

$$\Big|\int_{\partial B(x)} (\mathcal{J}_L^y\mathcal{J}_{L;y}^z\bar\psi_L^z)\cdot\sigma_L^x v - (\mathcal{J}^y\mathcal{J}_y^z\bar\psi^z)\cdot\sigma^x v\Big|$$
$$\lesssim \mathbb{1}_{|x-y|>\frac{L}{4}}\big\langle (x-y)_L\big\rangle^{-d}\big\langle (y-z)_L\big\rangle^{-d} + \mathbb{1}_{|x-y|\le\frac{L}{4}} L^{-d}\big\langle (y-z)_L\big\rangle^{-d}$$
$$+ \mathbb{1}_{|y-z|>\frac{L}{4}}\langle x-y\rangle^{-d}\big\langle (y-z)_L\big\rangle^{-d} + \mathbb{1}_{|y-z|\le\frac{L}{4}} L^{-d}\langle x-y\rangle^{-d}.$$

Further, using (4.9) and the decay assumption (Mix$_\omega^n$) to estimate correlation functions similarly to (4.26), we get by symmetry

$$\Big|L^{-d}\iiint_{(Q_{L,\rho})^3}\Big(\int_{\partial B(x)} (\mathcal{J}_L^y\mathcal{J}_{L;y}^z\bar\psi_L^z)\cdot\sigma_L^x v - (\mathcal{J}^y\mathcal{J}_y^z\bar\psi^z)\cdot\sigma^x v\Big)$$
$$\times\big(\lambda(\mathcal{P})h_2(x,z) + h_3(x,y,z)\big)\,dxdydz\Big|$$
$$\lesssim L^{-d}\iiint_{(Q_L)^3}\mathbb{1}_{|x-y|>\frac{L}{4}}\big\langle (x-y)_L\big\rangle^{-d}\big\langle (y-z)_L\big\rangle^{-d}\omega\big(|x-z|\big)\,dxdydz$$
$$+ L^{-2d}\iiint_{(Q_L)^3}\mathbb{1}_{|x-y|\le\frac{L}{4}}\big\langle (y-z)_L\big\rangle^{-d}\omega\big(|x-z|\big)\,dxdydz. \tag{4.45}$$

We start with the first right-hand side term, which is the most delicate one to estimate. By properties of ω, we may decompose

$$\omega\big(|x-z|\big) \lesssim \mathbb{1}_{|x-z|>\frac{L}{4}}\omega(L) + \mathbb{1}_{|x-z|\le\frac{L}{4}}\omega\big(|x-z|\big).$$

The first contribution with $|x-z| > \frac{L}{4}$ is easily estimated,

$$\omega(L)L^{-d}\iiint_{(Q_L)^3}\mathbb{1}_{|x-y|>\frac{L}{4}}\mathbb{1}_{|x-z|>\frac{L}{4}}\big\langle (x-y)_L\big\rangle^{-d}\big\langle (y-z)_L\big\rangle^{-d}\,dxdydz$$
$$\lesssim \omega(L)(\log L)^2,$$

and we turn to the contribution with $|x-z| \le \frac{L}{4}$. For that purpose, we interpolate between two bounds for the integral with respect to y,

$$\begin{aligned}
&\int_{Q_L} \mathbb{1}_{|x-y|>\frac{L}{4}} \mathbb{1}_{|x-z|\le\frac{L}{4}} \langle (x-y)_L \rangle^{-d} \langle (y-z)_L \rangle^{-d} \, dy \\
&\quad \lesssim \mathbb{1}_{|x-z|\le\frac{L}{4}} \big(\langle (x-z)_L \rangle^{-d} \log\big(2 + |(x-z)_L|\big) \big) \wedge \big\langle \mathrm{dist}\big(\{x,z\}, \partial Q_L\big) \big\rangle^{-d} \\
&\quad \lesssim \big(\langle x-z \rangle + \mathrm{dist}\big(\{x,z\}, \partial Q_L\big) \big)^{-d} \log\big(2 + |x-z|\big),
\end{aligned}$$

where we further used that $(x-z)_L = x-z$ if $|x-z| < \frac{L}{4}$. By symmetry, we then get

$$\begin{aligned}
&L^{-d} \iiint_{(Q_L)^3} \mathbb{1}_{|x-y|>\frac{L}{4}} \mathbb{1}_{|x-z|\le\frac{L}{4}} \langle (x-y)_L \rangle^{-d} \langle (y-z)_L \rangle^{-d} \omega\big(|x-z|\big) \, dx dy dz \\
&\lesssim L^{-d} \iint_{(Q_L)^2} \big(\langle x-z \rangle + \mathrm{dist}\big(\{x\}, \partial Q_L\big) \big)^{-d} \log\big(2 + |x-z|\big) \omega\big(|x-z|\big) \, dx dz \\
&\lesssim \frac{1}{L} \int_1^L \log(t) \omega(t) \, dt, \qquad (4.46)
\end{aligned}$$

where the last bound follows from a straightforward computation, carefully distinguishing between the cases $\langle x-z \rangle \ge \mathrm{dist}(\{x\}, \partial Q_L)$ and $\langle x-z \rangle \le \mathrm{dist}(\{x\}, \partial Q_L)$. Indeed, on the one hand, the part with $\langle x-z \rangle \ge \langle \mathrm{dist}(\{x\}, \partial Q_L) \rangle$ can be estimated by

$$\begin{aligned}
&L^{-d} \iint_{(Q_L)^2} \mathbb{1}_{\mathrm{dist}(\{x\},\partial Q_L) \le \langle x-z \rangle} \langle x-z \rangle^{-d} \log\big(2 + |x-z|\big) \omega\big(|x-z|\big) \, dx dz \\
&\quad \lesssim L^{-d} \int_{Q_{2L}} \big(L^{d-1} \langle y \rangle \big) \langle y \rangle^{-d} \log\big(2 + |y|\big) \omega\big(|y|\big) \, dy \\
&\quad \lesssim \frac{1}{L} \int_1^L (\log r) \omega(r) \, dr,
\end{aligned}$$

and on the other hand the part with $\langle x-z \rangle \le \mathrm{dist}(\{x\}, \partial Q_L)$ is estimated by

$$\begin{aligned}
&L^{-d} \int_{Q_L} \big\langle \mathrm{dist}\big(\{x\}, \partial Q_L\big) \big\rangle^{-d} \\
&\qquad\qquad \times \bigg(\int_{\{z \in Q_L : \langle x-z \rangle \le \mathrm{dist}(\{x\}, \partial Q_L)\}} \log\big(2 + |x-z|\big) \omega\big(|x-z|\big) \, dz \bigg) dx \\
&\quad \lesssim L^{-d} \int_1^L r^{d-1} \langle L-r \rangle^{-d} \bigg(\int_1^{L-r} s^{d-1} (\log s) \omega(s) \, ds \bigg) dr \\
&\quad \lesssim \frac{1}{L} \int_{L/2}^L \langle L-r \rangle^{-d} \bigg(\int_1^{L-r} s^{d-1} (\log s) \omega(s) \, ds \bigg) dr \\
&\qquad + L^{-d} \int_1^L s^{d-1} (\log s) \omega(s) \, ds
\end{aligned}$$

$$\begin{aligned}
&\lesssim \frac{1}{L}\int_1^{L/2} s^{d-1}(\log s)\omega(s)\Big(\int_{L/2}^{L-s}\langle L-r\rangle^{-d}\,dr\Big)\,ds\\
&\quad + L^{-d}\int_1^L s^{d-1}(\log s)\omega(s)\,ds\\
&\lesssim \frac{1}{L}\int_1^L (\log s)\omega(s)\,ds,
\end{aligned}$$

which yields the bound (4.46). It remains to estimate the second right-hand side term in (4.45), for which we directly find

$$\begin{aligned}
&L^{-2d}\iiint_{(Q_L)^3}\mathbb{1}_{|x-y|\le\frac{L}{4}}\big\langle (y-z)_L\big\rangle^{-d}\omega\big(|x-z|\big)\,dxdydz\\
&\qquad\lesssim (\log L)L^{-d}\int_1^L t^{d-1}\omega(t)\,dt \lesssim \frac{1}{L}\int_1^L(\log t)\omega(t)\,dt.
\end{aligned}$$

Inserting these different estimates into (4.45), the claim (4.44) follows using the doubling property of ω.

Substep 3.2. Conclusion. The next four terms of (4.43) are estimated similarly to the first periodization error term, and we skip most details for brevity. We solely briefly comment on the last term in (4.43), which is slightly different as it involves second-order corrector differences. We claim that

$$\begin{aligned}
&\Big|L^{-d}\iiint_{(Q_{L,\rho})^3}\Big(\int_{\partial B(x)}\delta^{y,z}\psi_L^{\varnothing}\cdot\delta^{y,z}\sigma_L^x v-\delta^{y,z}\psi^{\varnothing}\cdot\delta^{y,z}\sigma^x v\Big)f_3(x,y,z)\,dxdydz\Big|\\
&\qquad\lesssim \lambda_3(\mathcal{P})\frac{1}{L}.
\end{aligned}\tag{4.47}$$

By (4.36), and arguing similarly to the case of $\delta^{y,z}\psi_L^x$, we can decompose

$$\begin{aligned}
\delta^{y,z}\psi_L^{\varnothing} &= \mathcal{J}_L^y\mathcal{J}_{L;y}^z\bar{\psi}_L^{y,z}+\mathcal{J}_L^z\mathcal{J}_{L;z}^y\bar{\psi}_L^{y,z},\\
\delta^{y,z}\psi_L^{x} &= \mathcal{J}_{L;x}^y\mathcal{J}_{L;x,y}^z\bar{\psi}_L^{x,y,z}+\mathcal{J}_{L;x}^z\mathcal{J}_{L;x;z}^y\bar{\psi}_L^{x,y,z},
\end{aligned}$$

so that, proceeding as in Substep 3.1 above, we can write $\delta^{y,z}\psi_L^{\varnothing}\cdot\delta^{y,z}\sigma_L^x-\delta^{y,z}\psi^{\varnothing}\cdot\delta^{y,z}\sigma^x$ as the difference of four terms, which can each be written as a telescopic sum of six terms involving "elementary" periodization errors. The most delicate of those terms is the following one,

$$\begin{aligned}
&L^{-d}\iiint_{(Q_L)^3}\Big(\int_{B(x)}\big|\mathrm{D}(\mathcal{J}^y\mathcal{J}_y^z\bar{\psi}^{y,z})\big|^2\Big)^{\frac12}\\
&\qquad\times\Big(\int_{B_{1+\rho}(x)}\big|\mathrm{D}\big(\mathcal{J}_x^z\mathcal{J}_{x,z}^y(\psi_L^{x,y,z}-\psi^{x,y,z})\big)\big|^2\Big)^{\frac12}f_3(x,y,z)\,dxdydz.
\end{aligned}$$

By the decay estimates of Lemma 4.7, the periodization error estimates of Lemma 4.8, and the energy estimate (3.44), and further appealing to Lemma 1.1 (iii) to control f_3 in terms of the multi-point intensity, this term is bounded by

$$L^{-d}\lambda_3(\mathcal{P})\iiint_{(Q_L)^3}\langle x-y\rangle^{-d}\langle x-z\rangle^{-d}\langle y-z\rangle^{-2d}\big|\operatorname{dist}(y,\partial Q_L)\big|^{-d}\,dxdydz$$
$$\lesssim \lambda_3(\mathcal{P})\frac{1}{L}.$$

All the other terms can be bounded similarly, and the claim (4.47) follows. This concludes the proof of (ii).

Step 4. Analysis of remainders. Starting from (3.26), expanding the products, and separating the different intersection patterns, we are led to the following, in terms of multi-point intensities,

$$\begin{aligned}
|R_L^3| \lesssim\ & L^{-d}\bigg|\int_{(Q_{L,\rho})^5}\bigg(\int_{B(x)}\mathrm{D}(\delta^{y,z}\psi_L^{\varnothing}):\mathrm{D}(\delta^{y',z'}\psi_L^{\varnothing})\bigg)\\
&\qquad\times f_5(x,y,z,y',z')\,dxdydzdy'dz'\bigg|\\
&+L^{-d}\bigg|\int_{(Q_{L,\rho})^4}\bigg(\int_{B(x)}\mathrm{D}(\delta^{y,z}\psi_L^{\varnothing}):\mathrm{D}(\delta^{y,z'}\psi_L^{\varnothing})\bigg)f_4(x,y,z,z')\,dxdydzdz'\bigg|\\
&+L^{-d}\bigg|\int_{(Q_{L,\rho})^3}\bigg(\int_{B(x)}\mathrm{D}(\delta^{y,z}\psi_L^{\varnothing}):\mathrm{D}(\delta^{z'}\widehat{\psi}_{x,L}^{x})\bigg)f_4(x,y,z,z')\,dxdydzdz'\bigg|\\
&+L^{-d}\int_{(Q_{L,\rho})^3}\bigg(\int_{B(x)}|\mathrm{D}(\delta^{y,z}\psi_L^{\varnothing})|^2\bigg)f_3(x,y,z)\,dxdydz\\
&+L^{-d}\bigg|\int_{(Q_{L,\rho})^3}\bigg(\int_{B(x)}\mathrm{D}(\delta^{y,z}\psi_L^{\varnothing}):\mathrm{D}(\delta^{y}\widehat{\psi}_{x,L}^{x})\bigg)f_3(x,y,z)\,dxdydz\bigg|\\
&+L^{-d}\bigg|\int_{(Q_{L,\rho})^3}\bigg(\int_{B(x)}\mathrm{D}(\delta^{y,z}\psi_L^{\varnothing}):\mathrm{D}(\widehat{\psi}_{x,L}^{x})\bigg)f_3(x,y,z)\,dxdydz\bigg|.
\end{aligned}$$

As in the analysis of $\bar{\mathbf{B}}_L^3$ in Step 1, cancellations are unravelled by decomposing $\delta^{y,z}\psi_L^{\varnothing}$ in terms of single-particle contributions. We leave the details to the reader. ∎

4.4.3 Higher-order explicit renormalization

Finally, we turn to the general higher-order case. The obtained renormalized formulas are not displayed in the statement as they take the form of intricate diagrammatic expansions and require notation that will be introduced in the proof.

Proposition 4.11 (Higher-order renormalizations). *Let* (H_ρ) *and* ($\mathrm{H}_\rho^{\mathrm{unif}}$) *hold, and assume for simplicity that particles are spherical with unit radius,* $I_n = B(x_n)$. *Let*

also the mixing assumption (Mix$^n_\omega$) *hold to order* $n = k+1 \geq 2$ *with rate* $\omega \in C^\infty_b(\mathbb{R}^+)$ *satisfying the Dini-type condition* $\int_1^\infty \frac{1}{t}(\log t)^{k-1}\omega(t)\,dt < \infty$. *Then, the infinite-volume* $(k+1)$*th-order cluster coefficient* $\bar{\mathbf{B}}^{k+1}$, *cf.* (3.13), *can be expressed by means of absolutely convergent integrals. In addition, in case of an algebraic rate* $\omega(t) \leq Ct^{-\beta}$ *for some* $C, \beta > 0$, *the following hold.*

(i) Uniform estimate:

$$|\bar{\mathbf{B}}^{k+1}_L| \lesssim \lambda_{k+1}(\mathcal{P})\big|\log \lambda(\mathcal{P})\big|^k.$$

(ii) Convergence result:

$$|\bar{\mathbf{B}}^{k+1}_L - \bar{\mathbf{B}}^{k+1}| \lesssim \frac{(\log L)^k}{L^{\beta\wedge 1}}.$$

(iii) Uniform remainder estimate: *If* (Mix$^n_\omega$) *further holds with* $n = 2k+1$ *with algebraic weight* $\omega(t) \leq Ct^{-\beta}$, *then*

$$|R^{k+1}_L| \lesssim \sum_{j=k}^{2k} \lambda_{j+1}(\mathcal{P})\big|\log \lambda(\mathcal{P})\big|^j.$$

Proof. Let $k \geq 1$ be fixed. By definition, cf. (3.8), the periodic approximation $\bar{\mathbf{B}}^{k+1}_L$ is given by

$$\bar{\mathbf{B}}^{k+1}_L = \frac{1}{2}(k+1)!\,L^{-d} \sum_{\sharp F = k+1} \sum_{n\in F} \mathbb{E}\bigg[\int_{\partial B(x_{n,L})} \delta^{F\setminus\{n\}}\psi^{\varnothing}_L \cdot \sigma^F_L \nu\bigg],$$

and thus, decomposing $\sigma^F_L = \sum_{G\subset F\setminus\{n\}} \delta^G \sigma^{\{n\}}_L$ for $n \in F$, we get by symmetry,

$$\bar{\mathbf{B}}^{k+1}_L = \frac{k+1}{2}\sum_{l=0}^{k}\binom{k}{l} L^{-d} \sum_{n_0,n_1,\ldots,n_k}^{\neq} \mathbb{E}\bigg[\int_{\partial B(x_{n_0,L})} \delta^{\{n_1,\ldots,n_k\}}\psi^{\varnothing}_L \cdot \delta^{\{n_{l+1},\ldots,n_k\}}\sigma^{\{n_0\}}_L \nu\bigg].$$

In terms of multi-point densities, cf. (1.15), recalling the choice (3.1) of the finite-volume approximation, and using the notation (4.15), this becomes

$$\bar{\mathbf{B}}^{k+1}_L = \frac{k+1}{2}\sum_{l=0}^{k}\binom{k}{l} L^{-d} \int_{(Q_{L,\rho})^{k+1}} \bigg(\int_{\partial B(x_0)} \delta^{x_1,\ldots,x_k}\psi^{\varnothing}_L \cdot \delta^{x_{l+1},\ldots,x_k}\sigma^{x_0}_L \nu\bigg) \times f_{k+1}(x_0,\ldots,x_k)\,dx_0\cdots dx_k. \quad (4.48)$$

We now need to capture enough cancellations to make these integrals absolutely summable uniformly in the large-volume limit. For that purpose, similarly to what we did for $\bar{\mathbf{B}}^3_L$ in the proof of Proposition 4.10, we shall proceed to a suitable expansion of $\delta^{x_1,\ldots,x_k}\psi^{\varnothing}_L$ in terms single-particle contributions. For general order k, it is conveniently expressed in terms of diagrams. We split the proof into six steps.

Step 1. Diagrammatic decomposition of $\delta^{x_1,\dots,x_k}\psi_L^{\varnothing}$. We start with some terminology and notation:

— We use the standard notation $[k] := \{1,\dots,k\}$ and for any subset $S \subset [k]$ we define $x_S := (x_i)_{i\in S}$.

— Given a sequence $I = (i_1,\dots,i_l)$ of indices, the first index i_1 is called the *root* of I, the last index i_l is its *endpoint*, and l is its *length*. The associated index set is denoted by $\langle I\rangle := \{i_1,\dots,i_l\}$ and we define the *cardinality* of I as $\sharp I := \sharp\langle I\rangle$. An index i is then said to belong to I (for short, $i \in I$) if it belongs to the index set $\langle I\rangle$, and we define $x_I := x_{\langle I\rangle} = (x_i)_{i\in\langle I\rangle}$. Two sequences I and J are said to be *disjoint* if there is no index belonging to both, that is, $\langle I\rangle \cap \langle J\rangle = \varnothing$.

— The *concatenation* of two sequences $I = (i_1,\dots,i_l)$ and $J = (j_1,\dots,j_m)$ is denoted by $I \uplus J := (i_1,\dots,i_l,j_1,\dots,j_m)$.

— A *string* of indices is defined as any sequence of distinct indices with length ≥ 1.

— Given a string $I = (i_1,\dots,i_l)$ and an index set S, we define the *elementary contribution of* I *given* S as the following composition of operators,

$$\mathcal{R}^I_{L;S}(x_{[k]}) := \mathcal{J}^{x_{i_1}}_{L;x_S}\mathcal{J}^{x_{i_2}}_{L;x_{i_1}}\mathcal{J}^{x_{i_3}}_{L;x_{i_1},x_{i_2}}\cdots\mathcal{J}^{x_{i_l}}_{L;x_{i_1},\dots,x_{i_{l-1}}}, \tag{4.49}$$

where we recall that the $\mathcal{J}^z_{L,Y}$'s are defined in (4.18).

— A *block* is defined as any sequence B of indices that takes the form

$$B = (b) \uplus I_1 \uplus \cdots \uplus I_r,$$

where $r \geq 0$ (for $r = 0$ we simply have $B = (b)$) and where $I_1,\dots,I_r$ are strings of length ≥ 2 with the following property: for all $1 \leq j \leq r$, the endpoint of I_j belongs to $(b) \uplus I_1 \uplus \cdots \uplus I_{j-1}$ but other elements of I_j do not.

— Given a block $B = (b) \uplus I_1 \uplus \cdots \uplus I_r$ and an index set S, we define the *elementary contribution of* B *given* S as the following composition of operators,

$$\begin{aligned}&\mathcal{C}^B_{L;S}(x_{[k]})\\ &:= \mathcal{J}^{x_b}_{L;x_S}\mathcal{R}^{I_1}_{L;\{b\}}(x_{[k]})\mathcal{R}^{I_2}_{L;\langle(b)\uplus I_1\rangle}(x_{[k]})\cdots\mathcal{R}^{I_n}_{L;\langle(b)\uplus I_1\uplus\cdots\uplus I_{n-1}\rangle}(x_{[k]}).\end{aligned} \tag{4.50}$$

In these terms, we claim that $\delta^{x_1,\dots,x_j}\psi_L^{\varnothing}$ can be decomposed as follows,

$$\begin{aligned}&\delta^{x_1,\dots,x_k}\psi_L^{\varnothing}\\ &\quad= \sum_{r=1}^{k}\sum_{B_1,\dots,B_r}\mathcal{C}^{B_1}_{L;\varnothing}(x_{[k]})\mathcal{C}^{B_2}_{L;\langle B_1\rangle}(x_{[k]})\cdots\mathcal{C}^{B_r}_{L;\langle B_{r-1}\rangle}(x_{[k]})\,\bar{\psi}_L^{x_{B_r}},\end{aligned} \tag{4.51}$$

where we recall the shorthand notation $\bar{\psi}_L^{x_{B_r}} = \psi_L^{x_{B_r}} + Ex$, and where the sum $\sum_{B_1,\dots,B_r}$ runs over all r-tuples of disjoint blocks $B_1,\dots,B_r$ such that

$$\langle B_1 \uplus \cdots \uplus B_r\rangle = [k].$$

Note that this sum (4.51) is obviously finite, uniformly in L. Any sequence of indices of $[k]$ can be viewed as a walk on the vertex set $[k]$, thus inducing a (traversable) graph on $[k]$ where edges are defined by successive elements of the sequence (with possible multiplicities). In this view, each term in (4.51) can be conveniently represented by a corresponding diagram, cf. Figure 4.1; as we shall see, these graphical representations will prove crucial in estimating the different terms. This decomposition of corrector differences can be understood as a variant of the method of reflections [37]: while the latter allows us to decompose multi-particle solutions as an infinite expansion involving iterations of single-particle operators, the present decomposition is always finite, it still involves multi-particle solutions, but it has a simple enough structure to unravel explicit cancellations.

We turn to the proof of (4.51). More precisely, we shall prove the following seemingly simpler statement: for all disjoint index sets $S, T \subset [k]$ with $S \neq \varnothing$, we have

$$\delta^{x_S} \psi_L^{x_T} = \sum_{b \in S} \mathcal{J}_{L;x_T}^{x_b} \delta^{x_{S\setminus\{b\}}} \bar{\psi}_L^{x_b} + \sum_{\substack{I \text{ string} \\ \langle I \rangle \subset S}} \sum_{c \in T} \mathcal{R}_{L;T}^{I \uplus (c)} \underbrace{\delta^{x_{S\setminus\langle I\rangle}} \bar{\psi}_L^{x_I, x_T}}_{\clubsuit}. \tag{4.52}$$

We quickly argue that this indeed implies (4.51). First, we iteratively replace the corrector difference $\clubsuit$ in (4.52), using (4.52) itself, to the effect that

$$\begin{aligned}
&\delta^{x_S} \psi_L^{x_T} \\
&= \sum_{l \geq 0} \sum_{\substack{I_1, \dots, I_l \text{ disjoint strings} \\ \langle I_1 \rangle, \dots, \langle I_l \rangle \subset S}} \sum_{b \in S \setminus \langle I_1 \uplus \cdots \uplus I_l \rangle} \\
&\quad \times \sum_{\substack{c_1, \dots, c_l \\ \forall j : c_j \in T \cup \langle I_1 \uplus \cdots \uplus I_{j-1} \rangle}} \mathcal{R}_{L;T}^{I_1 \uplus (c_1)}(x_{[k]}) \mathcal{R}_{L;T \cup \langle I_1 \rangle}^{I_2 \uplus (c_2)}(x_{[k]}) \\
&\qquad \cdots \mathcal{R}_{L;T \cup \langle I_1 \uplus \cdots \uplus I_{l-1} \rangle}^{I_l \uplus (c_l)}(x_{[k]}) \mathcal{J}_{L;x_{T \cup \langle I_1 \uplus \cdots \uplus I_l \rangle}}^{x_b} \delta^{x_{S \setminus \langle I_1 \uplus \cdots \uplus I_l \uplus (b) \rangle}} \bar{\psi}_L^{x_b} \\
&\quad + \sum_{l \geq 1} \sum_{\substack{I_1, \dots, I_l \text{ disjoint strings} \\ \langle I_1 \rangle \cup \cdots \cup \langle I_l \rangle = S}} \sum_{\substack{c_1, \dots, c_l \\ \forall j : c_j \in T \cup \langle I_1 \uplus \cdots \uplus I_{j-1} \rangle}} \mathcal{R}_{L;T}^{I_1 \uplus (c_1)}(x_{[k]}) \mathcal{R}_{L;T \cup \langle I_1 \rangle}^{I_2 \uplus (c_2)}(x_{[k]}) \\
&\qquad \cdots \mathcal{R}_{L;T \cup \langle I_1 \uplus \cdots \uplus I_{l-1} \rangle}^{I_l \uplus (c_l)}(x_{[k]}) \bar{\psi}_L^{x_{S \cup T}}.
\end{aligned}$$

In particular, recalling the notation (4.49) for elementary contributions $\mathcal{R}_{L;S}^{I}$, and recognizing the definition (4.50) of block contributions, we deduce for all disjoint index sets $S, T \subset [k]$ with $S \neq \varnothing$,

$$\begin{aligned}
&\sum_{b \in S} \mathcal{J}_{L;x_T}^{x_b} \delta^{x_{S\setminus\{b\}}} \bar{\psi}_L^{x_b} \\
&= \sum_{\substack{B \text{ block} \\ \langle B \rangle \subset S}} \mathcal{C}_{L;T}^{B}(x_{[k]}) \Big(\sum_{b \in S \setminus \langle B \rangle} \mathcal{J}_{L;x_{\langle B \rangle}}^{x_b} \delta^{x_{S\setminus(\langle B\rangle \cup \{b\})}} \bar{\psi}_L^{x_b} \Big) + \sum_{\substack{B \text{ block} \\ \langle B \rangle = S}} \mathcal{C}_{L;T}^{B}(x_{[k]}) \bar{\psi}_L^{x_S}.
\end{aligned}$$

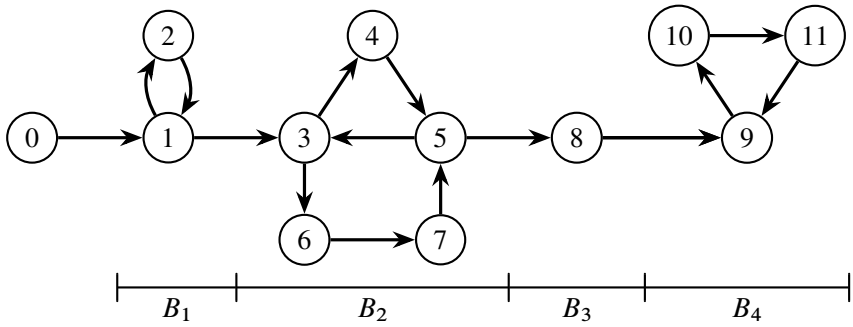


Figure 4.1. Each term in (4.51) can be represented by means of a directed graph on the index set $\{0\} \cup [k]$, where edges are given by pairs of consecutive elements in $B_1 \uplus \cdots \uplus B_r$ with possible multiplicities and where we include the edge $(0, b)$ with b the root of B_1. In this way, an edge to i corresponds to an operator $\mathcal{J}^{x_i}_{L;x_T}$ in (4.51). For instance, the above diagram is associated with blocks $B_1 = (1) \uplus (2,1)$, $B_2 = (3) \uplus (4,5,3) \uplus (6,7,5)$, $B_3 = (8)$, and $B_4 = (9) \uplus (10,11,9)$.

Iterating this identity, starting from (4.20) in form of

$$\delta^{x_S} \psi^{\varnothing}_L = \sum_{b \in S} \mathcal{J}^{x_b}_L \delta^{x_{S\setminus\{b\}}} \bar{\psi}^{x_b}_L,$$

the claim (4.51) follows.

We are left with the proof of (4.52). Given disjoint index sets $S, T \subset [k]$ with $S \neq \varnothing$, in view of (4.20), corrector differences can be decomposed as

$$\delta^{x_S} \psi^{x_T}_L = \sum_{b \in S} \mathcal{J}^{x_b}_{L;x_T} \delta^{x_{S\setminus\{b\}}} \bar{\psi}^{x_b,x_T}_L.$$

Decomposing $\bar{\psi}^{x_b,x_T}_L = \bar{\psi}^{x_b}_L + (\psi^{x_b,x_T}_L - \psi^{x_b}_L)$, this becomes

$$\delta^{x_S} \psi^{x_T}_L = \sum_{b \in S} \mathcal{J}^{x_b}_{L;x_T} \delta^{x_{S\setminus\{b\}}} \bar{\psi}^{x_b}_L + \sum_{b \in S} \mathcal{J}^{x_b}_{L;x_T} \delta^{x_{S\setminus\{b\}}} (\psi^{x_b,x_T}_L - \psi^{x_b}_L), \tag{4.53}$$

and it remains to further decompose the last right-hand side term. For that purpose, in view of Lemma 3.4, for all $D \subset S$ with $D \neq \varnothing$, we note that $\delta^{x_{S\setminus D}} (\psi^{x_D,x_T}_L - \psi^{x_D}_L)$ satisfies

$$\begin{aligned}
&- \triangle \delta^{x_{S\setminus D}} (\psi^{x_D,x_T}_L - \psi^{x_D}_L) \\
&+ \nabla \delta^{x_{S\setminus D}} \big(\Sigma^{x_D,x_T}_L \mathbb{1}_{Q_L \setminus \bigcup_{i \in D\cup T} B(x_i)} - \Sigma^{x_D}_L \mathbb{1}_{Q_L \setminus \bigcup_{b \in D} B(x_b)} \big) \\
&\quad = - \sum_{b \in D} \delta_{\partial B(x_b)} \delta^{x_{S\setminus D}} (\sigma^{x_D,x_T}_L - \sigma^{x_D}_L) \\
&\qquad - \sum_{i \in S\setminus D} \delta_{\partial B(x_i)} \delta^{x_{S\setminus D\cup\{i\}}} (\sigma^{x_D,x_i,x_T}_L - \sigma^{x_D,x_i}_L) - \sum_{i \in T} \delta_{\partial B(x_i)} \delta^{x_{S\setminus D}} \sigma^{x_D,x_T}_L,
\end{aligned}$$

which then allows us to write

$$\begin{aligned}\delta^{x_{S\setminus D}}(\psi_L^{x_D,x_T}-\psi_L^{x_D})\\ =\sum_{i\in S\setminus D}\mathcal{J}_{L;x_D}^{x_i}\delta^{x_{S\setminus D\cup\{i\}}}(\psi_L^{x_D,x_i,x_T}-\psi_L^{x_D,x_i})+\sum_{i\in T}\mathcal{J}_{L;x_D}^{x_i}\delta^{x_{S\setminus D}}\bar{\psi}_L^{x_D,x_T}.\end{aligned}$$

Using iteratively this identity for D exhausting S, we obtain upon recognizing the definition (4.49) of elementary contributions,

$$\sum_{b\in S}\mathcal{J}_{L;x_T}^{x_b}\delta^{x_{S\setminus\{b\}}}(\psi_L^{x_b,x_T}-\psi_L^{x_b})=\sum_{\substack{I\text{ string}\\ \langle I\rangle\subset S}}\sum_{c\in T}\mathcal{R}_{L;T}^{I\uplus(c)}(x_{[k]})\delta^{x_{S\setminus\langle I\rangle}}\psi_L^{x_I,x_T}.$$

Inserting this into (4.53), the claim (4.52) follows.

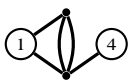
Step 2. Estimation of block contributions and graphical notation. Let B be a block of indices with root b and endpoint f. By definition of elementary block contributions, cf. (4.50), for any index set S that is disjoint from $\langle B\rangle$, Lemma 4.7 yields

$$\Big(\int_{B(z)}\big|\nabla\mathcal{C}_{L;S}^{B}(x_{[k]})\zeta\big|^2\Big)^{\frac12}\lesssim\big\langle(z-x_b)_L\big\rangle^{-d}D_B(x_B)\Big(\int_{B(x_f)}|\nabla\zeta|^2\Big)^{\frac12},\tag{4.54}$$

where for any sequence $C=(c_1,\dots,c_m)$ we define

$$D_C(x_C):=\big\langle(x_{c_1}-x_{c_2})_L\big\rangle^{-d}\cdots\big\langle(x_{c_{m-1}}-x_{c_m})_L\big\rangle^{-d}.\tag{4.55}$$

As such contributions will be combined in intricate ways in the sequel, we introduce a convenient graphical notation. Integration variables are represented by small black circles and frozen variables by small white circles. The index of a frozen variable is occasionally indicated inside the corresponding white circle. A solid line between two vertices i and j represents a factor $\langle(x_i-x_j)_L\rangle^{-d}$. In particular, multiple edges correspond to powers of this factor. For instance, we have



$$\begin{aligned}=\int_{(Q_L)^2}\big\langle(x_1-x_2)_L\big\rangle^{-d}\big\langle(x_2-x_3)_L\big\rangle^{-2d}\big\langle(x_3-x_1)_L\big\rangle^{-d}\\ \times\big\langle(x_3-x_4)_L\big\rangle^{-d}\,dx_2dx_3.\end{aligned}$$

When evaluating integrals with borderline factors $\langle(x_i-x_j)_L\rangle^{-d}$, we naturally obtain logarithmic factors, for which we shall use the shorthand notation

$$\mathcal{L}_L\big((z_i)_{i\in J}\big):=\log\big(2+\max_{i,j\in J}\big|(z_i-z_j)_L\big|\big).$$

This is combined into our graphical notation as follows: a symbolic prefactor $\mathcal{L}$ in front of a diagram indicates that a factor $\mathcal{L}_L(x_S)$ is to be included into the corresponding integral, where S stands for the set of all implemented indices. For instance, for

any power $\mu \geq 0$, we have

$$\mathscr{L}^{\mu}\ \textcircled{1}\!\!\textcircled{4} = \int_{(Q_L)^2} \mathscr{L}_L(x_1,x_2,x_3,x_4)^{\mu} \big\langle (x_1-x_2)_L \big\rangle^{-d} \big\langle (x_2-x_3)_L \big\rangle^{-2d} \times \big\langle (x_3-x_1)_L \big\rangle^{-d} \big\langle (x_3-x_4)_L \big\rangle^{-d}\, dx_2 dx_3.$$

Noting that for any $\gamma > 0$ and $\mu \geq 0$ a direct evaluation of integrals yields

$$\int_{Q_L} \mathscr{L}_L(x,y,z)^{\mu} \big\langle (x-y)_L \big\rangle^{-d} \big\langle (y-z)_L \big\rangle^{-\gamma}\, dy \lesssim \begin{cases} \big\langle (x-z)_L \big\rangle^{-d} \mathscr{L}_L(x,z)^{\mu} & \text{if } \gamma > d, \\ \big\langle (x-z)_L \big\rangle^{-d} \mathscr{L}_L(x,z)^{\mu+1} & \text{if } \gamma = d, \\ \big\langle (x-z)_L \big\rangle^{-\gamma} \mathscr{L}_L(x,z)^{\mu+1} & \text{if } \gamma < d, \end{cases}$$

we deduce with our graphical notation

$$\begin{aligned} \mathscr{L}^{\mu}\ &\circ\!\!-\!\!\bullet\!\!-\!\!\circ \lesssim \mathscr{L}^{\mu+1}\ \circ\!\!-\!\!\circ \\ \mathscr{L}^{\mu}\ &\circ\!\!\frown\!\!\bullet\!\!-\!\!\circ \lesssim \mathscr{L}^{\mu}\ \circ\!\!-\!\!\circ \end{aligned} \tag{4.56}$$

which allows for instance to estimate graphically

$$\textcircled{1}\!\!\textcircled{4} \lesssim \textcircled{1}\!\!\frown\!\!\textcircled{4} \lesssim \textcircled{1}\!\!-\!\!\textcircled{4} = \big\langle (x_1-x_4)_L \big\rangle^{-d}.$$

The counting of logarithmic factors in the sequel will be quite trivial as we shall notice that at most one logarithmic factor appears each time a vertex disappears in the graphical representation. This rough bound can often be improved, but it suffices for our purposes.

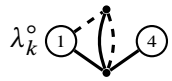
We need to add one more ingredient to our graphical notation. Indeed, in the sequel, we replace the density function f_{k+1} in (4.48) by its expansion (4.7) in terms of correlation functions, and we estimate the latter by appealing both to the decay assumption (Mix^n_{ω}) and to the uniform bound (4.9). This leads us to combine products of the form $D_C(x_C)$ with products of factors of the form

$$\omega\big((x_i - x_j)_L\big) \wedge \lambda_{p_i}(\mathcal{P})$$

for some $p_i \geq 0$. In our graphical notation, such a factor is represented by a dashed line between vertices i and j. In principle, the value p_i should be included in the notation to precise the value of the edge. For convenience, we rather use a simplified notation: for a diagram with s dashed lines, a symbolic prefactor λ_k° indicates that the dashed lines correspond to factors $(\omega((\cdot)_L) \wedge \lambda_{p_i}(\mathcal{P}))_{1 \leq i \leq s}$ with any $p_1, \ldots, p_s \geq 1$ satisfying

$$p_1 + \cdots + p_s = k,$$

and we take the sum over the different possible choices of such p_i's. For instance,

λ_k° 

$$= \sum_{\substack{p_1,p_2\geq 1\\ p_1+p_2=k}} \int_{(Q_L)^2} \big(\omega\big((x_1-x_2)_L\big)\wedge\lambda_{p_1}(\mathcal{P})\big)\big(\omega\big((x_2-x_3)_L\big)\wedge\lambda_{p_2}(\mathcal{P})\big) \times \big\langle (x_2-x_3)_L\big\rangle^{-d}\big\langle (x_3-x_1)_L\big\rangle^{-d}\big\langle (x_3-x_4)_L\big\rangle^{-d}\, dx_2dx_3.$$

In addition, a symbolic prefactor λ_k' in front of a diagram with s dashed lines indicates that the whole expression is multiplied by a factor $\lambda_{p_0}(\mathcal{P})$ and that the dashed lines correspond to factors $(\omega((\cdot)_L)\wedge\lambda_{p_i}(\mathcal{P}))_{1\leq i\leq s}$ with any $p_0,\dots,p_s\geq 1$ satisfying $p_0+\cdots+p_s=k$, where we again take the sum over all possible choices. In other words,

$$\big(\lambda_k'\big) = \sum_{p=1}^{k} \lambda_p(\mathcal{P}) \times \big(\lambda_{k-p}^\circ\big).$$

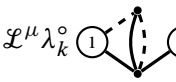
As obviously $\omega((\cdot)_L)\wedge\lambda_{p_i}(\mathcal{P})\leq\lambda_{p_i}(\mathcal{P})$, we get with our notation

λ_k° $\leq \lambda_k'$ $= \lambda_k(\mathcal{P})$ (4.57)

Next, we combine this with the notation $\mathcal{L}$ for logarithmic factors: in front of a diagram with a prefactor λ_k°, a symbolic prefactor $\mathcal{L}$ indicates that either a factor $\mathcal{L}(x_S)$ is to be included into the corresponding integral, where S stands for the set of implemented indices, or that one of the factors $\omega((x_i-x_j)_L)\wedge\lambda_{p_i}(\mathcal{P})$ is to be replaced by

$$\big(\omega\big((x_i-x_j)_L\big)\mathcal{L}_L(x_i,x_j)\big)\wedge\big(\lambda_{p_i}(\mathcal{P})\big|\log\lambda(\mathcal{P})\big|\big),$$

and we take the sum over the two choices. Powers of $\mathcal{L}$ are defined accordingly: for instance, for any $\mu\geq 0$,

$\mathcal{L}^\mu\lambda_k^\circ$ 

$$= \sum_{\substack{\mu_0,\mu_1,\mu_2\geq 0\\ \mu_0+\mu_1+\mu_2=\mu}} \sum_{\substack{p_0,p_1,p_2\geq 1\\ p_0+p_1+p_2=k}} \int_{(Q_L)^2} \mathcal{L}_L(x_1,x_2,x_3,x_4)^{\mu_0} \times \Big(\big(\omega\big((x_1-x_2)_L\big)\mathcal{L}_L(x_1,x_2)^{\mu_1}\big)\wedge\big(\lambda_{p_1}(\mathcal{P})\big|\log\lambda(\mathcal{P})\big|^{\mu_1}\big)\Big) \times \Big(\big(\omega\big((x_2-x_3)_L\big)\mathcal{L}(x_2,x_3)^{\mu_2}\big)\wedge\big(\lambda_{p_2}(\mathcal{P})\big|\log\lambda(\mathcal{P})\big|^{\mu_2}\big)\Big) \times \big\langle (x_2-x_3)_L\big\rangle^{-d}\big\langle (x_3-x_1)_L\big\rangle^{-d}\big\langle (x_3-x_4)_L\big\rangle^{-d}\, dx_2dx_3.$$

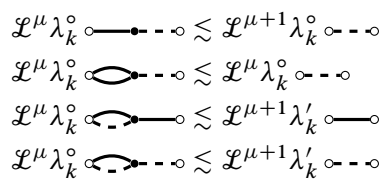
When $\mathcal{L}$ is in front of a diagram with a prefactor λ_k', we add the possibility of multiplying the whole expression by a factor $|\log\lambda(\mathcal{P})|$: for all $\mu\geq 0$, this means

$$\big(\mathcal{L}^\mu\lambda_k'\big) = \sum_{\nu=0}^{\mu}\sum_{p=1}^{k} \lambda_p(\mathcal{P})\big|\log\lambda(\mathcal{P})\big|^\nu \times \big(\mathcal{L}^{\mu-\nu}\lambda_{k-p}^\circ\big).$$

In case of an algebraic rate $\omega(t) = Ct^{-\beta}$ for some $\beta \in (0, d)$, a direct evaluation of integrals yields, for all $\lambda > 0$ and $\mu, \nu \geq 0$,

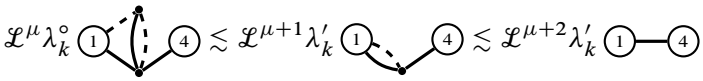
$$\int_{Q_L} \mathcal{L}_L(x,y,z)^{\mu} \big\langle (x-y)_L \big\rangle^{-d} \Big(\big(\omega\big((y-z)_L\big) \mathcal{L}_L(y,z)^{\nu} \big) \wedge \big(\lambda \big| \log \lambda \big|^{\nu} \big) \Big)\, dy$$
$$\lesssim \big(\omega\big((x-z)_L\big) \mathcal{L}_L(x-z)^{\mu+\nu+1} \big) \wedge \big(\lambda \big| \log \lambda \big|^{\mu+\nu+1} \big).$$

With our graphical notation, recalling Lemma 1.1 (ii), this estimate and similar computations yield for all $\mu \geq 0$,



(4.58)

which allows for instance to estimate graphically



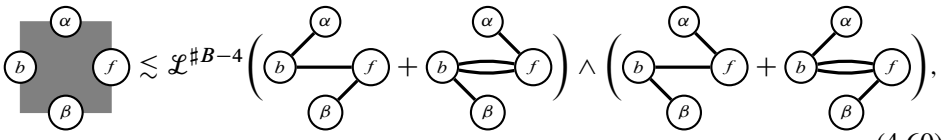
$$\leq \lambda_k(\mathcal{P}) \big\langle (x_1 - x_4)_L \big\rangle^{-d} \big(\big| \log \lambda(\mathcal{P}) \big| + \mathcal{L}_L(x_1, x_4) \big)^{\mu+2}.$$

This notation will be used abundantly in the sequel.

Step 3. Partial integration on blocks. Let B be a block of indices with root b and endpoint f. We shall establish the following key estimate for partial integrals on B: for all $\alpha, \beta \in \langle B \rangle$,

$$\int_{(Q_L)^{\sharp\langle B\rangle \setminus \{b,f,\alpha,\beta\}}} D_B(x_B)\, dx_{\langle B\rangle \setminus \{b,f,\alpha,\beta\}}$$
$$\lesssim_B \mathcal{L}_L(x_b, x_f, x_\alpha, x_\beta)^{\sharp\langle B\rangle \setminus \{b,f,\alpha,\beta\}}$$
$$\times \big(D_{(\alpha,b,f,\beta)}(x_{[k]}) + D_{(\alpha,f,b,\beta)}(x_{[k]}) \big\langle (x_b - x_f)_L \big\rangle^{-d} \big)$$
$$\wedge \big(D_{(\alpha,f,b,\beta)}(x_{[k]}) + D_{(\alpha,b,f,\beta)}(x_{[k]}) \big\langle (x_b - x_f)_L \big\rangle^{-d} \big). \tag{4.59}$$

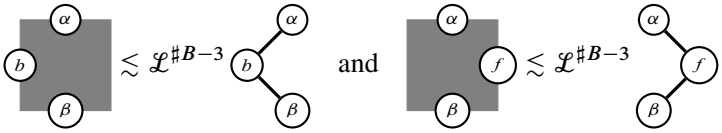
With the above graphical notation, if b, f, α, β are all distinct, this can be written as



(4.60)

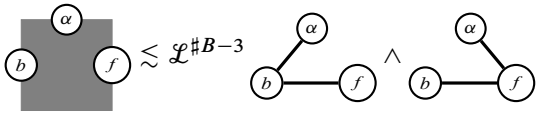
where henceforth gray squares stand for integration on a generic block. As these estimates will be abundantly used in the sequel, we also display the important special

case when one further integrates over b or f, which is deduced by applying (4.56),

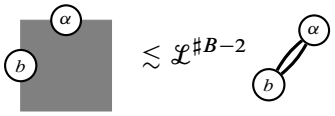


(4.61)

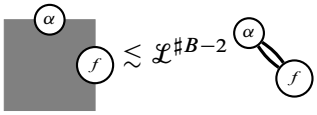
and we further display the special cases when $\alpha = \beta$,



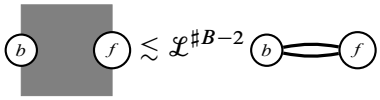
(4.62)



(4.63)



(4.64)



(4.65)

We shall in fact prove a much more precise estimates, see (4.67) below, but the above convenient estimates will be enough for our purposes. Powers of the logarithmic factor in each of these estimates is equal to the difference between the numbers of vertices in the left-hand side and in the right-hand side: indeed, in view of (4.56), each vertex that is integrated yields at most one logarithmic factor. This could in fact be improved in (4.62)–(4.65) based on (4.67), but we shall not need such refinements.

Before turning to the proof, we make a notational comment. A special role is of course played in the above estimates by the root b and by the endpoint f of the block. In the sequel, even when vertices are not labeled explicitly, as e.g. in (4.67), we take the convention that the root and the endpoint are always drawn respectively on the left and on the right sides of the square (or at one of these two locations in case they coincide), while all other distinguished vertices are drawn indistinctly on the upper and lower sides.

We turn to the proof of (4.59). For that purpose, we shall study geometric properties of the graph $\mathfrak{G}$ associated with the block B. Letting

$$B = (b_1, b_2, \dots, b_l)$$

with $b_1 = b$ and $b_l = f$, we recall that we define vertices of $\mathfrak{G}$ as the elements of the index set $\langle B \rangle = \{b_i\}_{1 \le i \le l}$, and edges of $\mathfrak{G}$ as pairs of consecutive indices (b_i, b_{i+1}) with $1 \le i < l$. Note that $\mathfrak{G}$ is connected and may have multiple edges

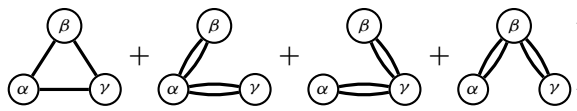
but no self-loop. We shall repeatedly use the following observation: as edges of $\mathfrak{G}$ are defined from the block B, we note that b and f have odd degree ≥ 3 and that other vertices have even degree ≥ 2 (where the degree of a vertex is the number of unoriented edges containing that vertex; see e.g. Figure 4.2). We split the proof into three further substeps.

Substep 3.1. Cyclic estimate. In the spirit of (4.55), for a graph $\mathfrak{H}$ on the index set $[k]$, we define

$$D_{\mathfrak{H}}(x_{[k]}) := \prod_{(i,j)\in\mathfrak{H}} \big\langle (x_i - x_j)_L \big\rangle^{-d},$$

where the notation $(i,j) \in \mathfrak{H}$ means that (i,j) is an edge of $\mathfrak{H}$. Provided that $\mathfrak{H}$ is Eulerian (that is, provided that $\mathfrak{H}$ is connected and that each vertex has even degree), we claim that for all vertices $\alpha, \beta, \gamma \in \langle \mathfrak{H} \rangle$,

$$\int_{(Q_L)^{\sharp\mathfrak{H}\setminus\{\alpha,\beta,\gamma\}}} D_{\mathfrak{H}}(x_{[k]})\, dx_{\langle\mathfrak{H}\rangle\setminus\{\alpha,\beta,\gamma\}} \lesssim_{\mathfrak{H}} \mathcal{L}^{\sharp\mathfrak{H}\setminus\{\alpha,\beta,\gamma\}} \Big(\quad + \quad + \quad + \quad \Big). \tag{4.66}$$



We will use standard terminology from graph theory, which we recall here for clarity: a walk is a sequence of edges joining a sequence of vertices; a trail is a walk in which all edges are distinct (taking edge multiplicity into account); a path is a trail in which all vertices are also distinct; a circuit is a trail in which the first and last vertices coincide; a cycle is a circuit in which only the first and last vertices coincide.

We turn to the proof of (4.66) and we argue by induction on the size of $\mathfrak{H}$. Assume that α, β, γ are distinct (the cases $\alpha = \beta \neq \gamma$ and $\alpha = \beta = \gamma$ can be treated similarly and are skipped for brevity). The result is straightforward if $\sharp\mathfrak{H} = 3$ as no integral is performed in that case. We turn to the case $\sharp\mathfrak{H} > 3$. As $\mathfrak{H}$ is Eulerian, there is a circuit that covers $\mathfrak{H}$ (that is, a circuit that visits every edge of $\mathfrak{H}$ exactly once). Removing some subcircuits, we deduce that one of the following two possibilities must hold up to a permutation of α, β, γ:

(a) either there is a cycle $\mathfrak{C}$ visiting α, β, γ;

(b) or there is a cycle $\mathfrak{C}_1$ visiting α, β and a cycle $\mathfrak{C}_2$ visiting α, γ such that vertices of $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are all distinct except α.

Both cases can be treated similarly and we focus on the first one for brevity. Let $\mathfrak{C}$ be a cycle visiting α, β, γ. Denote by $\mathfrak{H}'$ the (possibly empty) subgraph of $\mathfrak{H}$ induced by the complement of the edge set of the cycle $\mathfrak{C}$. As $\mathfrak{H}$ is Eulerian and as $\mathfrak{C}$ is a cycle, we notice that $\mathfrak{H}'$ is the union of Eulerian subgraphs $\mathfrak{H}'_1, \dots, \mathfrak{H}'_s$ that are edge-disjoint. We may then decompose

$$D_{\mathfrak{H}}(x_{[k]}) = D_{\mathfrak{C}}(x_{[k]}) D_{\mathfrak{H}'_1}(x_{[k]}) \cdots D_{\mathfrak{H}'_s}(x_{[k]}).$$

For all $1 \le i \le s$, there is a vertex j_i of $\mathfrak{H}_i'$ that also belongs to the cycle $\mathfrak{C}$. Summing separately over repeated variables, we may then estimate

$$\int_{(Q_L)^{\sharp\mathfrak{H}\setminus\{\alpha,\beta,\gamma\}}} D_{\mathfrak{H}}(x_{[k]})\, dx_{\langle\mathfrak{H}\rangle\setminus\{\alpha,\beta,\gamma\}}$$
$$\lesssim \int_{(Q_L)^{\sharp\mathfrak{C}\setminus\{\alpha,\beta,\gamma\}}} D_{\mathfrak{C}}(x_{[k]}) \left(\prod_{i=1}^{s} \int_{(Q_L)^{\sharp\mathfrak{H}_i'-1}} D_{\mathfrak{H}_i'}(x_{[k]})\, dx_{\langle\mathfrak{H}_i'\rangle\setminus\{j_i\}} \right) dx_{\langle\mathfrak{C}\rangle\setminus\{\alpha,\beta,\gamma\}}.$$

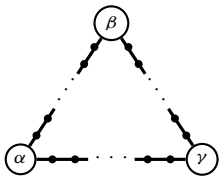
As the $\mathfrak{H}_i'$'s are strict Eulerian subgraphs of $\mathfrak{H}$, an induction argument allows to assume that the claim (4.66) is already known to hold for $\mathfrak{H}$ replaced by any of the $\mathfrak{H}_i'$'s. In particular, upon integration, this entails

$$\int_{(Q_L)^{\sharp\mathfrak{H}_i'-1}} D_{\mathfrak{H}_i'}(x_{[k]})\, dx_{\langle\mathfrak{H}_i'\rangle\setminus\{j_i\}} \lesssim \mathfrak{L}^{\sharp\mathfrak{H}_i'-1}.$$

The above then reduces to

$$\int_{(Q_L)^{\sharp\mathfrak{H}\setminus\{\alpha,\beta,\gamma\}}} D_{\mathfrak{H}}(x_{[k]})\, dx_{\langle\mathfrak{H}\rangle\setminus\{\alpha,\beta,\gamma\}}$$
$$\lesssim \mathfrak{L}^{\sum_i(\sharp\mathfrak{H}_i'-1)} \int_{(Q_L)^{\sharp\mathfrak{C}\setminus\{\alpha,\beta,\gamma\}}} D_{\mathfrak{C}}(x_{[k]})\, dx_{\langle\mathfrak{C}\rangle\setminus\{\alpha,\beta,\gamma\}},$$

where the right-hand side is now simply an integral of the form



Using (4.56) to evaluate the integrals, noting that the number of appearing logarithmic factors is bounded by the length of $\mathfrak{C}$ minus 3 and that the length of $\mathfrak{C}$ is bounded by the total number of vertices minus the number of vertices not in the cycle (that is, $\sum_i(\sharp\mathfrak{H}_i'-1)$), the claim (4.66) follows. More precisely, we obtain in this way the first right-hand side term in (4.66), while other terms correspond to case (b) above.

Substep 3.2. Path decomposition of the graph $\mathfrak{G}$ associated with a block B. We show that, if $b \neq f$, there exist three edge-disjoint trails $\mathfrak{L}^1, \mathfrak{L}^2, \mathfrak{L}^3$ that cover $\mathfrak{G}$ (that is, the union of their vertex sets is the vertex set of $\mathfrak{G}$ and the disjoint union of their edge sets is the edge set of $\mathfrak{G}$). We refer to Figure 4.2 for an illustrative example.

As b and f have odd degree ≥ 3 and as all other vertices of $\mathfrak{G}$ have even degree, we can find a trail $\mathfrak{L}^1$ from b to f. Without loss of generality, we can assume that b and f are visited only once by $\mathfrak{L}^1$. Then consider the subgraph $\mathfrak{G}'$ of $\mathfrak{G}$ induced by the complement of the edge set of $\mathfrak{L}^1$. By construction, all vertices of $\mathfrak{G}'$ now have

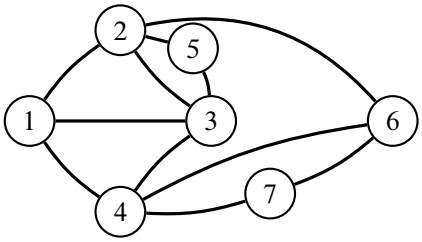


Figure 4.2. This graph represents the block $B = (1) \uplus (2,3,1) \uplus (4,3) \uplus (5,2) \uplus (6,4) \uplus (7,6)$. The path decomposition of Substep 2.3 can be chosen in this case as $\mathfrak{L}^1 = (1,2,6)$, $\mathfrak{L}^2 = (1,4,7,6)$, and $\mathfrak{L}^3 = (1,3,2,5,3,4,6)$.

even degree, and the definition of the block B ensures that $\mathfrak{G}'$ must be connected. This allows to find two other disjoint trails $\mathfrak{L}^2$, $\mathfrak{L}^3$ from b to f in $\mathfrak{G}'$.

Next, assume that a vertex $\alpha \in \langle \mathfrak{G} \rangle$ is not visited by any of the three constructed trails $\mathfrak{L}^1$, $\mathfrak{L}^2$, $\mathfrak{L}^3$. Recalling that $\mathfrak{G}$ is connected, a degree argument as above ensures that there exists a circuit $\mathfrak{K}$ from α to itself that is disjoint from the trails $\mathfrak{L}^1$, $\mathfrak{L}^2$, $\mathfrak{L}^3$ and that crosses at least one of them. A detour via $\mathfrak{K}$ is then easily added to those trails in such a way that they remain disjoint and that at least one of them now visits α. Repeating this construction, we are led to edge-disjoint trails $\mathfrak{L}^1$, $\mathfrak{L}^2$, $\mathfrak{L}^3$ that visit all vertices of $\mathfrak{G}$.

Finally, consider the subgraph $\mathfrak{G}''$ of $\mathfrak{G}$ induced by the complement of the union of the edge sets of $\mathfrak{L}^1$, $\mathfrak{L}^2$, $\mathfrak{L}^3$. By construction, all vertices of $\mathfrak{G}''$ have even degree, which allows us to write $\mathfrak{G}''$ as a union of edge-disjoint circuits. Adding detours via these circuits, we can assume that the trails $\mathfrak{L}^1$, $\mathfrak{L}^2$, $\mathfrak{L}^3$ cover the whole graph $\mathfrak{G}$, and the claim follows.

Substep 3.3. Proof of (4.59). Let $\mathfrak{L}^1, \mathfrak{L}^2, \mathfrak{L}^3$ be three covering edge-disjoint trails from b to f as constructed above. Given a vertex α, distinguishing between the number of paths from b to f to which α belongs, and removing cycles, we get the following four possibilities:

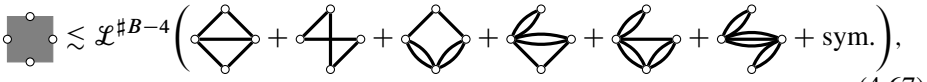
(a) either there exists a cycle $\mathfrak{C}$ from b or from f that visits α and there exist three paths $\mathfrak{K}^1$, $\mathfrak{K}^2$, $\mathfrak{K}^3$ from b to f, such that $\mathfrak{C}$, $\mathfrak{K}^1$, $\mathfrak{K}^2$, $\mathfrak{K}^3$ are edge-disjoint and cross each other only at b or f;

(b) or there exists a path $\mathfrak{K}^1$ from b to f that visits α and there exist two other paths $\mathfrak{K}^2$, $\mathfrak{K}^3$ from b to f that do not, such that $\mathfrak{K}^1$, $\mathfrak{K}^2$, $\mathfrak{K}^3$ are edge-disjoint and cross each other only at b or f;

(c) or there exist two paths $\mathfrak{K}^1$, $\mathfrak{K}^2$ from b to f that visit α and there exists another path $\mathfrak{K}^3$ from b to f that does not, such that $\mathfrak{K}^1$, $\mathfrak{K}^2$, $\mathfrak{K}^3$ are edge-disjoint and cross each other only at α, b or f;

(d) or there exist three paths $\mathfrak{K}^1$, $\mathfrak{K}^2$, $\mathfrak{K}^3$ from b to f that visit α and that are edge-disjoint and cross each other only at α, b and f.

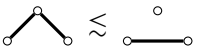
Given another vertex β, and distinguishing between corresponding cases, we obtain three distinguished paths $\mathfrak{K}^1$, $\mathfrak{K}^2$, $\mathfrak{K}^3$ from b to f that may visit or not α and β, in different possible orders, and we obtain up to two cycles $\mathfrak{C}^1$, $\mathfrak{C}^2$ from b or f visiting α or β. The subgraph of $\mathfrak{G}$ induced by the complement of the union of the edge sets of those three paths and possible cycles is necessarily a disjoint union of Eulerian graphs and can be removed by duplicating variables as in Substep 3.1. It remains to consider the union of those three paths and possible cycles. Considering different patterns and using (4.56) to estimate consecutive edges along each path between frozen vertices b, f, α, β, we are led to



(4.67)

where for brevity "sym." stands for the sum of all other graphs obtained by reflecting the six pictured graphs with respect to the vertical axis, the horizontal axis, or both (which corresponds to permuting α and β, b and f, or both). In fact, the analysis of all possible patterns produces a larger number of terms, but we claim that all others are bounded by the above. For instance, another possible pattern corresponds to the case of three paths $\mathfrak{K}^1$, $\mathfrak{K}^2$, $\mathfrak{K}^3$ from b to f visiting both α and β, where $\mathfrak{K}^1$, $\mathfrak{K}^2$ visit α before β while $\mathfrak{K}^3$ visit them in reverse order: we claim that the corresponding contribution can be bounded as follows,

which is indeed bounded by the right-hand side of (4.67). This bound follows from



which is itself nothing but the triangle inequality

$$\big\langle (x_1 - x_3)_L \big\rangle \le \big\langle (x_1 - x_2)_L \big\rangle + \big\langle (x_2 - x_3)_L \big\rangle,$$

post-processed into

$$\big\langle (x_1 - x_3)_L \big\rangle \le 2 \big\langle (x_1 - x_2)_L \big\rangle \big\langle (x_2 - x_3)_L \big\rangle$$

and raised to the power $-d$. A straightforward similar inspection of all other possible patterns shows that the bound (4.67) indeed holds; we skip the detail for brevity.

Finally, removing a few edges in (4.67), we are led in particular to the claim (4.60). The claims (4.61)–(4.65) further follow as straightforward corollaries after integrations using (4.56).

Step 4. Approximate cancellation of translation-invariant averages on given blocks. Let B be a block of indices with root b and endpoint f, and let S, T be disjoint

index sets with $(S \cup T) \cap \langle B \rangle = \varnothing$. Let $m := \sharp B$ and $s := \sharp S$. For all x_B, x_S, let $\zeta_{L;x_B}^{x_S} \in H^1_{\mathrm{per}}(Q_L)^d$ satisfy (4.17) at $z = x_f$, and assume that ζ_L is equivariant under translations in the sense that

$$\zeta_{L;x_B+[z]_B}^{x_S+[z]_S}(\cdot + z) = \zeta_{L;x_B}^{x_S}, \quad \text{for all } z \in \mathbb{R}^d,$$

where $[z]_B$ (resp. $[z]_S$) stands for the element of $(\mathbb{R}^d)^m$ (resp. $(\mathbb{R}^d)^s$) with all coordinates equal to z. Then, for any function h on $(Q_L)^{m+s}$ that is translation-invariant in the sense that $h(x_B + [z]_B, x_S + [z]_S) = h(x_B, x_S)$ for all $z \in \mathbb{R}^d$, we have for any linear functional $F : H^1_{\mathrm{per}}(Q_L)^d \to \mathbb{R}$,

$$\begin{aligned}
&\left| \int_{(Q_L)^{m+s}} F\big[\mathcal{C}_{L;T}^{B}(x_{[k]})\zeta_{L;x_B}^{x_S}\big] h(x_B, x_S)\, dx_B dx_S \right| \\
&\qquad \le \int_{Q_L} \Bigg(\int_{(Q_L+x_b)^{m+s-1}\setminus(Q_L)^{m+s-1}} \big|F\big[\mathcal{C}_{L;T}^{B}(x_{[k]})\zeta_{L;x_B}^{x_S}\big]\big| \\
&\qquad\qquad \times \big(\big|h_L(x_B, x_S)\big| + \big|h(x_B, x_S)\big|\big)\, dx_{\langle B\rangle\setminus\{b\}} dx_S \Bigg) dx_b, \qquad (4.68)
\end{aligned}$$

where we have defined the periodization $h_L(z) := h(z_L)$ where $z_L \in (Q_L)^{m+s}$ stands for the reduction of $z \in (\mathbb{R}^d)^{m+s}$ modulo $(L\mathbb{Z}^d)^{m+s}$. Note that we do not obtain an exact cancellation in general for such a symmetric average on a block, but this bound reduces it to a boundary term.

We turn to the proof of (4.68). Set for abbreviation $C := \langle B \rangle \setminus \{b\}$. By definition of elementary block contributions, cf. (4.50), we can write

$$\mathcal{C}_{L;T}^{B}(x_{[k]})\zeta_{L;x_B}^{x_S} = \mathcal{J}_{L;x_T}^{x_b} \xi_{L;x_b}^{x_C;x_S}, \qquad (4.69)$$

for some function $\xi_{L;x_b}^{x_C;x_S}$ that satisfies (4.17) at $z = x_b$ and is such that ξ_L is equivariant under translations. The left-hand side of (4.68) then becomes

$$\begin{aligned}
&\int_{(Q_L)^{m+s}} F\big[\mathcal{C}_{L;T}^{B}(x_{[k]})\zeta_{L;x_B}^{x_S}\big]\, h(x_B, x_S)\, dx_B dx_S \\
&\quad = \int_{(Q_L)^{m+s}} F\big[\mathcal{J}_{L;x_T}^{x_b} \xi_{L;x_b}^{x_C;x_S}\big] h(x_b, x_C, x_S)\, dx_b dx_C dx_S,
\end{aligned}$$

and thus, using the equivariance of ξ under translations,

$$\begin{aligned}
&\int_{(Q_L)^{m+s}} F\big[\mathcal{C}_{L;T}^{B}(x_{[k]})\zeta_{L;x_B}^{x_S}\big] h(x_B, x_S)\, dx_B dx_S \\
&\quad = \int_{(Q_L)^{m+s}} F\big[\mathcal{J}_{L;x_T}^{x_b} \big(\xi_{L;0}^{x_C-[x_b]_C;x_S-[x_b]_S}(\cdot - x_b)\big)\big] h(x_b, x_C, x_S)\, dx_b dx_C dx_S.
\end{aligned}$$

Replacing h by its periodization h_L (which we can on $(Q_L)^{m+s}$), changing variables and using periodicity, the above becomes in these terms,

$$\begin{aligned}&\int_{(Q_L)^{m+s}} F\big[\mathcal{C}^B_{L;T}(x_{[k]})\zeta^{x_S}_{L;x_B}\big]h(x_B,x_S)\,dx_B dx_S\\&\quad=\int_{(Q_L)^{m+s}} F\big[\mathcal{J}^{x_b}_{L;x_T}\big(\xi^{x_C;x_S}_{L;0}(\cdot-x_b)\big)\big]\\&\qquad\qquad\times h_L\big(x_b,x_C+[x_b]_C,x_S+[x_b]_S\big)\,dx_b dx_C dx_S.\end{aligned}$$

If $h_L(x_b,x_C+[x_b]_C,x_S+[x_b]_S)$ were replaced by $h(0,x_C,x_S)$ in the integrand, the cancellation property of Lemma 4.6 would precisely entail that the integral vanishes (this would have been the case if we had considered a periodization in law of $\mathcal{I}$ rather than (3.1)). Adding and subtracting $h(0,x_C,x_S)$, we deduce

$$\begin{aligned}&\int_{(Q_L)^{m+s}} F\big[\mathcal{C}^B_{L;T}(x_{[k]})\zeta^{x_S}_{L;x_B}\big]\,h(x_B,x_S)\,dx_B dx_S\\&\quad=\int_{(Q_L)^{m+s}} F\big[\mathcal{J}^{x_b}_{L;x_T}\big(\xi^{x_C;x_S}_{L;0}(\cdot-x_b)\big)\big]\\&\qquad\times\big(h_L\big(x_b,x_C+[x_b]_C,x_S+[x_b]_S\big)-h(0,x_C,x_S)\big)\,dx_b dx_C dx_S.\end{aligned}$$

If x_b, x_C, x_S are such that

$$(x_b,x_C+[x_b]_C,x_S+[x_b]_S)\in(Q_L)^{m+s},$$

then the definition of the periodization h_L and the translation invariance of h imply that the integrand vanishes. This leads us to the bound

$$\begin{aligned}&\Big|\int_{(Q_L)^{m+s}} F\big[\mathcal{C}^B_{L;T}(x_{[k]})\zeta^{x_S}_{L;x_B}\big]\,h(x_B,x_S)\,dx_B dx_S\Big|\\&\quad\le\int_{Q_L}\Big(\int_{(Q_L)^{m+s-1}\setminus(Q_L-x_b)^{m+s-1}}\big|F\big[\mathcal{J}^{x_b}_{L;x_T}\big(\xi^{x_C;x_S}_{L;0}(\cdot-x_b)\big)\big]\big|\\&\qquad\times\big(\big|h_L\big(x_b,x_C+[x_b]_C,x_S+[x_b]_S\big)\big|+\big|h(0,x_C,x_S)\big|\big)\,dx_{\{B\}\setminus\{b\}}dx_S\Big)\,dx_b.\end{aligned}$$

Using again (4.69) and the equivariance of ξ, the claim (4.68) follows.

Step 5. Uniform estimates: proof of (i). The starting point is the decomposition (4.48) of $\bar{\mathbf{B}}^{k+1}_L$. For brevity, we shall focus on the term corresponding to $l=k$ in (4.48), that is,

$$\begin{aligned}&\bar{\mathbf{C}}^{k+1}_L:=\\&\frac{k+1}{2}L^{-d}\int_{(Q_{L,\rho})^{k+1}}\Big(\int_{\partial B(x_0)}\delta^{x_1,\dots,x_k}\psi^{\varnothing}_L\cdot\sigma^{x_0}_L\nu\Big)f_{k+1}(x_0,\dots,x_k)\,dx_0\cdots dx_k,\end{aligned}$$

while the other terms are simpler to estimate due to the additional decay given by the factor $\delta^{x_{l+1},\dots,x_k}\sigma_L^{x_0}$. Inserting the diagrammatic decomposition (4.51), we get

$$\bar{\mathbf{C}}_L^{k+1} = \frac{k+1}{2}\sum_{r=1}^{k}\sum_{B_1,\dots,B_r}\bar{\mathbf{C}}_L^{k+1;r}(B_1,\dots,B_r),$$

where we recall that the sum runs over all r-tuples of disjoint blocks $B_1,\dots,B_r$ such that $\langle B_1 \uplus \cdots \uplus B_r\rangle = [k]$, and where we have set for abbreviation

$$\begin{aligned}\bar{\mathbf{C}}_L^{k+1;r}(B_1,\dots,B_r) :=\\ L^{-d}\int_{(Q_{L,\rho})^{k+1}}\Big(\int_{\partial B(x_0)}\big(\mathcal{C}_{L;\varnothing}^{B_1}(x_{[k]})\mathcal{C}_{L;\langle B_1\rangle}^{B_2}(x_{[k]})\cdots\mathcal{C}_{L;\langle B_{r-1}\rangle}^{B_r}(x_{[k]})\bar{\psi}_L^{x_{B_r}}\big)\cdot\sigma_L^{x_0}\nu\Big)\\ \times f_{k+1}(x_0,x_{[k]})\,dx_0dx_{[k]}.\end{aligned}$$

Let such $B_1,\dots,B_r$ be fixed. Replacing f_{k+1} by its expansion (4.7) in terms of correlation functions, we find

$$\begin{aligned}&\bar{\mathbf{C}}_L^{k+1;r}(B_1,\dots,B_r)\\ &= \sum_{\pi} L^{-d}\int_{(Q_{L,\rho})^{k+1}}\Big(\prod_{H\in\pi}h_{\sharp H}(x_H)\Big)\\ &\times\Big(\int_{\partial B(x_0)}\big(\mathcal{C}_{L;\varnothing}^{B_1}(x_{[k]})\mathcal{C}_{L;\langle B_1\rangle}^{B_2}(x_{[k]})\cdots\mathcal{C}_{L;\langle B_{r-1}\rangle}^{B_r}(x_{[k]})\bar{\psi}_L^{x_{B_r}}\big)\cdot\sigma_L^{x_0}\nu\Big)\,dx_0dx_{[k]},\end{aligned}\tag{4.70}$$

where π runs over all partitions of the index set $\{0\}\cup[k]$ and where H runs over all cells of the partition π.

We shall say that a partition π of $\{0\}\cup[k]$ is *covering* for $B_1,\dots,B_r$ if there is no "separating" index $1\le\alpha\le r$ such that each cell $H\in\pi$ is included either in $\{0\}\cup\bigcup_{i=1}^{\alpha-1}\langle B_i\rangle$ or in $\bigcup_{i=\alpha}^{r}\langle B_i\rangle$. We denote by $\mathcal{K}(B_1,\dots,B_r)$ the set of such partitions. Using the approximate cancellation property (4.68), and further noting as in (4.40) that

$$\int_{Q_L}\Big(\int_{\partial B(x_0)}\big(\mathcal{C}_{L;\varnothing}^{B_1}(x_{[k]})\mathcal{C}_{L;\langle B_1\rangle}^{B_2}(x_{[k]})\cdots\mathcal{C}_{L;\langle B_{r-1}\rangle}^{B_r}(x_{[k]})\psi_L^{x_{B_r}}\big)\cdot\sigma_L^{x_0}\nu\Big)dx_0 = 0,$$

we note that only covering partitions produce nontrivial terms in (4.70): contributions from non-covering partitions either vanish or are reduced to boundary terms. Therefore, we naturally decompose

$$\begin{aligned}&\big|\bar{\mathbf{C}}_L^{k+1;r}(B_1,\dots,B_r)\big|\\ &\le\sum_{\pi\in\mathcal{K}(B_1,\dots,B_r)}\big|\bar{\mathbf{C}}_L^{k+1;r}(B_1,\dots,B_r;\pi)\big| + \sum_{\pi\notin\mathcal{K}(B_1,\dots,B_r)}\big|\bar{\mathbf{C}}_L^{k+1;r}(B_1,\dots,B_r;\pi)\big|,\end{aligned}$$

where we have set for abbreviation

$$\begin{aligned}&\bar{\mathbf{C}}_L^{k+1;r}(B_1,\dots,B_r;\pi)\\&:= L^{-d}\int_{(Q_{L,\rho})^{k+1}}\Big(\prod_{H\in\pi}h_{\sharp H}(x_H)\Big)\\&\times\Big(\int_{\partial B(x_0)}\big(\bar{\mathcal{C}}_{L;\varnothing}^{B_1}(x_{[k]})\bar{\mathcal{C}}_{L;\langle B_1\rangle}^{B_2}(x_{[k]})\cdots\bar{\mathcal{C}}_{L;\langle B_{r-1}\rangle}^{B_r}(x_{[k]})\bar{\psi}_L^{x_{B_r}}\big)\cdot\sigma_L^{x_0}\nu\Big)\,dx_0dx_{[k]}.\end{aligned}$$

We split the proof into two further substeps, separately considering the two types of contributions.

Substep 5.1. Main contributions: in case of an algebraic rate $\omega(t)\le Ct^{-\beta}$ for some $C,\beta>0$, we have for all $\pi\in\mathcal{K}(B_1,\dots,B_r)$,

$$\big|\bar{\mathbf{C}}_L^{k+1;r}(B_1,\dots,B_r;\pi)\big|\lesssim\lambda_{k+1}(\mathcal{P})\big|\log\lambda(\mathcal{P})\big|^k. \tag{4.71}$$

Without loss of generality, we may assume $\beta\in(0,d)$ (so we can appeal to (4.58)). Using the boundary conditions for $\psi_L^{x_0}$ and the incompressibility constraints to smuggle in arbitrary constants in the different factors, and then appealing to the trace estimates of Lemma 2.5, we find

$$\begin{aligned}&\big|\bar{\mathbf{C}}_L^{k+1;r}(B_1,\dots,B_r;\pi)\big|\\&\lesssim L^{-d}\int_{(Q_{L,\rho})^{k+1}}\Big(\prod_{H\in\pi}\big|h_{\sharp H}(x_H)\big|\Big)\\&\quad\times\Big(\int_{B(x_0)}\big|\nabla\bar{\mathcal{C}}_{L;\varnothing}^{B_1}(x_{[k]})\bar{\mathcal{C}}_{L;\langle B_1\rangle}^{B_2}(x_{[k]})\cdots\bar{\mathcal{C}}_{L;\langle B_{r-1}\rangle}^{B_r}(x_{[k]})\bar{\psi}_L^{x_{B_r}}\big|^2\Big)^{\frac12}dx_0dx_{[k]}.\end{aligned}$$

For all $1\le l\le r$, denote by b_l the root of B_l and by f_l its endpoint, and set for notational convenience $f_0:=0$. Iterating the bound (4.54), we then get

$$\begin{aligned}&\big|\bar{\mathbf{C}}_L^{k+1;r}(B_1,\dots,B_r;\pi)\big|\\&\quad\lesssim L^{-d}\int_{(Q_{L,\rho})^{k+1}}\Big(\prod_{H\in\pi}\big|h_{\sharp H}(x_H)\big|\Big)\\&\qquad\times\Big(\prod_{l=1}^r\big\langle(x_{f_{l-1}}-x_{b_l})_L\big\rangle^{-d}D_{B_l}(x_{B_l})\Big)\,dx_0dx_{[k]}.\end{aligned}\tag{4.72}$$

Next, we examine the structure of the product of correlation functions. Given a covering partition $\pi\in\mathcal{K}(B_1,\dots,B_r)$, we can construct a sequence of intertwined pairings $(m_i,m_i')_{1\le i\le s}$ (for some integer $s\ge1$) such that

— $(m_j)_{1\le j\le s}$ and $(m_j')_{1\le j\le s}$ are increasing, $m_1=0$, and $m_s'=r$;

— $m_{i-1}'<m_{i+1}$ for all $1<i<s$, and $m_i\le m_{i-1}'$ for all $1<i\le s$;

— for all i there is a cell $H \in \pi$ such that $H \cap \langle B_{m_i} \rangle \neq \varnothing$ and $H \cap \langle B_{m'_i} \rangle \neq \varnothing$ (with the understanding that $B_0 = \{0\}$).

The construction is as follows: Starting from $m_1 = 0$, we define m'_1 as the maximum index m such that there is $H \in \pi$ with $\{0\} \cap H \neq \varnothing$ and $\langle B_m \rangle \cap H \neq \varnothing$, which is well defined by the covering assumption for π with index $\alpha = 1$. Once m_i and m'_i are defined for some $i \geq 1$, if $m'_i < r$, we define m'_{i+1} as the maximum index m such that there is $H \in \pi$ with $(\{0\} \cup \bigcup_{l \leq m'_i} \langle B_l \rangle) \cap H \neq \varnothing$ and $\langle B_m \rangle \cap H \neq \varnothing$, which is well defined by the covering assumption for π with index $\alpha = m'_i + 1 \leq r$ and satisfies $m'_{i+1} > m'_i$ by construction. Next, we define m_{i+1} as the minimum index m such that there is $H \in \pi$ with $\langle B_m \rangle \cap H \neq \varnothing$ and $\langle B_{m'_{i+1}} \rangle \cap H \neq \varnothing$. We continue the construction until $m'_s = r$ is reached. We claim that by construction we have $m'_{i-1} < m_{i+1} \leq m'_i$ for all i (which, since $(m'_j)_j$ is increasing, implies that $(m_j)_j$ is increasing as well). On the one hand, we indeed have $m_{i+1} \leq m'_i$ by definition of m'_{i+1}. On the other hand, we must have $m_{i+1} > m'_{i-1}$ since the inequality $m_{i+1} \leq m'_{i-1}$ would imply $m'_i = m'_{i+1}$ and contradict the strict monotonicity of the sequence $(m'_j)_j$.

With this construction of intertwined pairings $(m_i, m'_i)_{1 \leq i \leq s}$, we can choose a sequence of distinct blocks $(H_i)_{1 \leq i \leq s}$ of π such that $\langle B_{m_i} \rangle \cap H_i \neq \varnothing$ and $\langle B_{m'_i} \rangle \cap H_i \neq \varnothing$ for all i (recall that $B_0 = \{0\}$). We may then pick indices $j_i, j'_i \in \{0\} \cup [k]$ such that $j_i \in \langle B_{m_i} \rangle \cap H_i$ and $j'_i \in \langle B_{m'_i} \rangle \cap H_i$ for all i. In these terms, appealing both to (4.9) and to the decay assumption (Mix$^n_\omega$) with $n = k + 1$, the product of correlation functions in (4.72) can be bounded for instance as follows (up to integration, as in (4.26)),

$$\prod_{H \in \pi} \big| h_{\sharp H}(x_H) \big| \lesssim \lambda_{p_0}(\mathcal{P}) \prod_{i=1}^{s} \big(\omega(x_{j_i} - x_{j'_i}) \wedge \lambda_{p_i}(\mathcal{P}) \big), \tag{4.73}$$

for some $p_0, \ldots, p_s \geq 1$ with $\sum_{i=0}^{s} p_i = k + 1$. We then define the following concatenations of blocks between paired indices: for $1 \leq i < s$,

$$\begin{aligned} A_i &:= (f_{m'_{i-1}}) \uplus B_{m'_{i-1}+1} \uplus \cdots \uplus B_{m_{i+1}-1} \uplus (b_{m_{i+1}}), \\ A'_i &:= B_{m_{i+1}} \uplus \cdots \uplus B_{m'_i}, \end{aligned}$$

with the convention $m'_0 := 0$, and

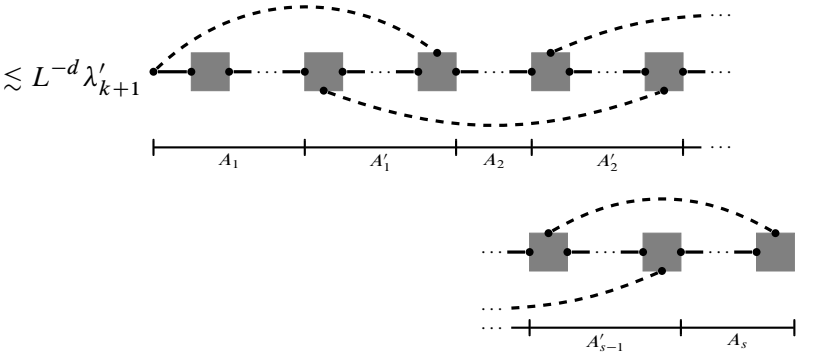
$$A_s := (f_{m'_{s-1}}) \uplus B_{m'_{s-1}+1} \uplus \cdots \uplus B_{m_s}, \quad A'_s := \varnothing.$$

In these terms, inserting (4.73) into (4.72), the integral can be reorganized as

$$\begin{aligned} &\big| \bar{\mathbf{C}}_L^{k+1;r}(B_1, \ldots, B_r; \pi) \big| \lesssim \lambda_{p_0}(\mathcal{P}) \\ &\times L^{-d} \int_{(Q_{L,\rho})^{k+1}} \left(\prod_{i=1}^{s} D_{A_i}(x_{A_i}) D_{A'_i}(x_{A'_i}) \big(\omega(x_{j_i} - x_{j'_i}) \wedge \lambda_{p_i}(\mathcal{P}) \big) \right) dx_0 dx_{[k]}. \end{aligned}$$

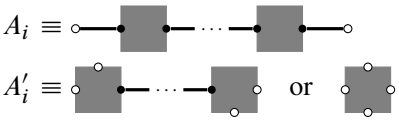
We emphasize that the coupled indices j_i's and j_i''s belong to A_i''s and can thus intersect A_i's only at their endpoints. In terms of the graphical representation introduced in Step 3, the latter integral can be represented generically in the following way,

$$\left|\bar{\mathbf{C}}_L^{k+1;r}(B_1,\dots,B_r;\pi)\right|$$



(4.74)

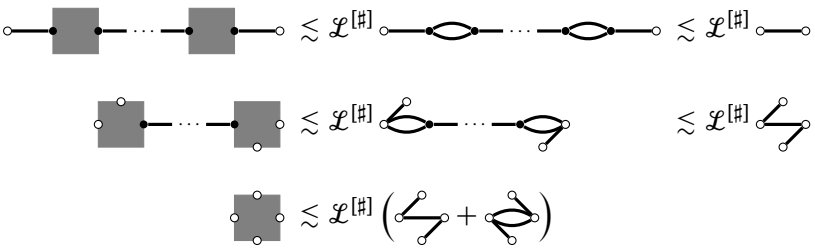
where we further delineate the concatenations of blocks A_i's and A_i''s. In particular, note these take the generic forms



where the second possibility for A_i' corresponds to the case when

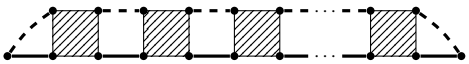
$$m_{i+1} = m_i'.$$

In order to estimate (4.74), we first perform integration on A_i's and A_i''s: using (4.60)–(4.65) to estimate the integral on each block, and using (4.56) to estimate consecutive edges, we find

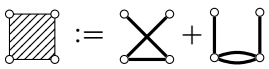


where henceforth we use the shorthand notation $\mathfrak{L}^{[\sharp]}$ for a power of the logarithmic factor that can change from an occurrence to another and stands for the difference between the numbers of vertices in the left-hand side and in the right-hand side.

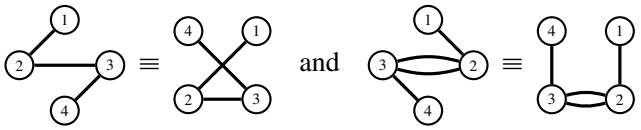
Inserting this into (4.74), we are led to

$$\big|\bar{\mathbf{C}}_L^{k+1;r}(B_1,\dots,B_r;\pi)\big| \lesssim L^{-d}\,\mathscr{L}^{[\sharp]}\lambda'_{k+1}$$  (4.75)

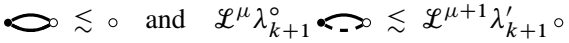
where for abbreviation hatched boxes are given by



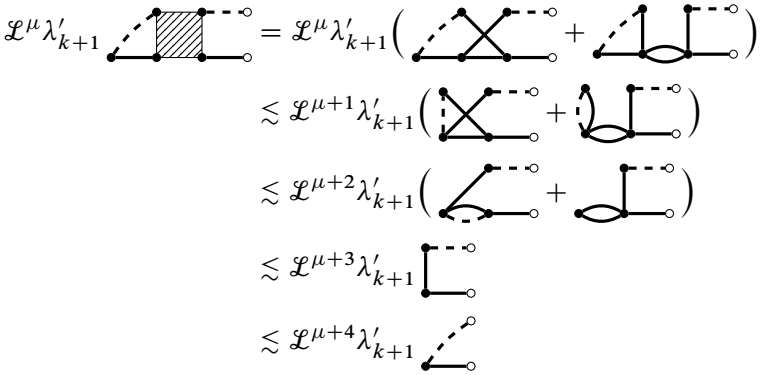
which we obtain by reorganizing the graphs as follows,



It remains to evaluate the right-hand side in (4.75). Using the graphical rules (4.56), (4.57), and (4.58), and noting that direct integrations yield



we can estimate



Iterating this estimate, the right-hand side of (4.75) can now be estimated as follows,

$$\big|\bar{\mathbf{C}}_L^{k+1;r}(B_1,\dots,B_r;\pi)\big| \lesssim L^{-d}\,\mathscr{L}^{[\sharp]}\lambda'_{k+1}\bullet$$

As the number of vertices in the left-hand side is equal to $k+1$ while only one vertex remains in the right-hand side, recalling our notation for $\mathscr{L}$ and λ'_k, and noting that free integration yields $\bullet = L^d$, we get

$$\big|\bar{\mathbf{C}}_L^{k+1;r}(B_1,\dots,B_r;\pi)\big| \lesssim L^{-d}\,\mathscr{L}^{k}\lambda'_{k+1}\bullet = \lambda_{k+1}(\mathcal{P})\big|\log\lambda(\mathcal{P})\big|^k,$$

that is, (4.71).

Substep 5.2. Boundary terms: in case of an algebraic rate $\omega(t) \le Ct^{-\beta}$ for $C, \beta > 0$, we have for all $\pi \notin \mathcal{K}(B_1, \dots, B_r)$,

$$\begin{aligned} &\big|\bar{\mathbf{C}}_L^{k+1;r}(B_1, \dots, B_r; \pi)\big| \\ &\quad \lesssim \begin{cases} \big(\lambda_{k+1}(\mathcal{P})\big|\log \lambda(\mathcal{P})\big|^{k-1}\big) \wedge \frac{(\log L)^{k-1}}{L^{\beta \wedge 1}} & \text{if } \beta \neq 1, \\ \big(\lambda_{k+1}(\mathcal{P})\big|\log \lambda(\mathcal{P})\big|^{k}\big) \wedge \frac{(\log L)^{k}}{L} & \text{if } \beta = 1. \end{cases} \end{aligned} \tag{4.76}$$

By definition, given $\pi \notin \mathcal{K}(B_1, \dots, B_r)$, we can consider the largest separating index $\alpha \le r$ such that each cell $H \in \pi$ is included either in $\{0\} \cup \bigcup_{i=1}^{\alpha-1} \langle B_i \rangle$ or in $\bigcup_{i=\alpha}^{r} \langle B_i \rangle$. Setting $\widehat{B} := B_\alpha \uplus \cdots \uplus B_r$, the choice of α ensures that π can be restricted to a partition $\widehat{\pi}$ of the index subset $\langle \widehat{B} \rangle \subset [k]$ such that $\widehat{\pi}$ is covering for $B_\alpha, \dots, B_r$. Arguing as in (4.72) and estimating the integrals over the first blocks $\{0\}, B_1, \dots, B_{\alpha-1}$ brutally as in Section 3.8 without taking any advantage of the decay of correlation functions, we get

$$\begin{aligned} &\big|\bar{\mathbf{C}}_L^{k+1;r}(B_1, \dots, B_r; \pi)\big| \\ &\lesssim (\log L)^\alpha \Big(\prod_{H \in \pi \setminus \widehat{\pi}} \lambda_{\sharp H}(\mathcal{P}) \Big) \\ &\times L^{-d} \int_{Q_L} \Big(\int_{(Q_L + x_b)^{\sharp \widehat{B}-1} \setminus (Q_L)^{\sharp \widehat{B}-1}} \Big(\prod_{H \in \widehat{\pi}} \big|h_{\sharp H}(x_H)\big| \Big) D_{\widehat{B}}(x_{\widehat{B}})\, dx_{\langle \widehat{B} \rangle \setminus \{b\}} \Big)\, dx_b \end{aligned}$$

It remains to show that the remaining integral is a boundary term that is algebraically small as $L \uparrow \infty$, so that in particular the logarithmic prefactor $(\log L)^\alpha$ plays no role. For that purpose, we first note that

$$\mathbb{1}_{(Q_L + x_b)^{\sharp \widehat{B}-1} \setminus (Q_L)^{\sharp \widehat{B}-1}}(x_{\langle \widehat{B} \rangle \setminus \{b\}}) \le \sum_{j \in \langle \widehat{B} \rangle \setminus \{b\}} \mathbb{1}_{(Q_L + x_b) \setminus Q_L}(x_j),$$

so the above can be bounded by

$$\begin{aligned} &\big|\bar{\mathbf{C}}_L^{k+1;r}(B_1, \dots, B_r; \pi)\big| \\ &\lesssim (\log L)^\alpha \Big(\prod_{H \in \pi \setminus \widehat{\pi}} \lambda_{\sharp H} \Big) \\ &\quad \times \sum_{j \in \langle \widehat{B} \rangle \setminus \{b\}} L^{-d} \int_{Q_L} \int_{(Q_L + x_b) \setminus Q_L} \Big(\int_{(Q_L + x_b)^{\sharp \widehat{B}-2}} \Big(\prod_{H \in \widehat{\pi}} \big|h_{\sharp H}(x_H)\big| \Big) \\ &\qquad\qquad \times D_{\widehat{B}}(x_{\widehat{B}})\, dx_{\langle \widehat{B} \rangle \setminus \{b,j\}} \Big) dx_j\, dx_b. \end{aligned}$$

As $\widehat{\pi}$ is covering for $B_\alpha, \dots, B_r$, that is, $\widehat{\pi} \in \mathcal{K}(B_\alpha, \dots, B_r)$, similar arguments based on the graphical representation as in Substep 5.1 allow to estimate the integral over

$\langle \widehat{B} \rangle \setminus \{b, j\}$, to the effect of

$$\begin{aligned} &\big|\bar{\mathbf{C}}_L^{k+1;r}(B_1, \ldots, B_r; \pi)\big| \\ &\lesssim (\log L)^{\alpha} \sum_{j \in \langle \widehat{B} \rangle \setminus \{b\}} L^{-d} \int_{Q_L} \int_{(Q_L + x_b) \setminus Q_L} \Big(\mathscr{L}^{\sharp \widehat{B} - 2} \lambda'_{k+1} \; b \; j \Big) \, dx_j \, dx_b. \end{aligned} \tag{4.77}$$

In order to estimate this integral, we note that

$$\begin{aligned} &L^{-d} \int_{Q_L} \int_{(Q_L + x_b) \setminus Q_L} \Big(\lambda^{\circ}_{k+1} \; b \; j \Big) \, dx_j \, dx_b \\ &= L^{-d} \int_{Q_L} \int_{(Q_L + x_b) \setminus Q_L} \big\langle (x_b - x_j)_L \big\rangle^{-d} \big(\omega\big((x_b - x_j)_L\big) \wedge \lambda_{k+1}(\mathcal{P}) \big) \, dx_j \, dx_b \\ &= L^{-d} \int_{Q_L} \int_{Q_L \setminus (Q_L - x_b)} \langle y \rangle^{-d} \big(\omega(y) \wedge \lambda_{k+1}(\mathcal{P}) \big) \, dy \, dx_b \\ &= L^{-d} \int_{Q_L} \langle y \rangle^{-d} \big(\omega(y) \wedge \lambda_{k+1}(\mathcal{P}) \big) \big| Q_L \setminus (Q_L - y) \big| \, dy, \end{aligned}$$

and thus, using (4.29) in form of $L^{-d} |Q_L \setminus (Q_L - y)| \lesssim \frac{|y|}{L} \wedge 1$, in case of an algebraic rate $\omega(t) \le C t^{-\beta}$ for some $C, \beta > 0$,

$$\begin{aligned} &L^{-d} \int_{Q_L} \int_{(Q_L + x_b) \setminus Q_L} \Big(\lambda^{\circ}_{p} \; b \; j \Big) \, dx_j \, dx_b \\ &\lesssim L^{-1} \int_{Q_L} \langle y \rangle^{1-d} \big(\omega(y) \wedge \lambda_p(\mathcal{P}) \big) \, dy \\ &\lesssim \begin{cases} \lambda_p(\mathcal{P}) \wedge L^{-\beta \wedge 1} & \text{if } \beta \neq 1, \\ \big(\lambda_p(\mathcal{P}) \big| \log \lambda(\mathcal{P}) \big| \big) \wedge \frac{\log L}{L} & \text{if } \beta = 1. \end{cases} \end{aligned}$$

Now turning back to the right-hand side in (4.77), repeating the above computation after including logarithmic factors, and noting that $\alpha + \sharp \widehat{B} \le \sharp B = k + 1$, the claim (4.76) follows.

Step 6. Strategy for (ii) and (iii). Both for (ii) and (iii), the arguments are similar to what we already did so far, and require no new insight. We omit lengthy details for brevity.

We start with (ii). In view of the estimation (4.76) for boundary terms, it remains to estimate the convergence of terms corresponding to covering partitions in (4.70) in the large-volume limit. For that purpose, we appeal to the periodization error estimates of Lemma 4.8, as in the proof of Proposition 4.9 (ii).

We turn to (iii). The starting point is the refined estimate (3.26) on R_L^{k+1}. In the spirit of the proof of Proposition 4.9 (iii), a decomposition of the right-hand side in (3.26) can be performed in the same way as what we did above for $\bar{\mathbf{B}}_L^{k+1}$. ∎

4.5 Optimality of cluster estimates

This section is devoted to the proof of Theorem 4.4, which shows that logarithmic factors are optimal in general in our estimation of cluster coefficients, e.g. Proposition 4.11 (i). As will be clear in the proof below, logarithmic factors are related to the lack of continuity of the Helmholtz projection in $\mathrm{L}^\infty(\mathbb{R}^d)$.

Proof of Theorem 4.4. Let Assumptions (H_ρ), ($\mathrm{H}_\rho^{\mathrm{unif}}$), and (Indep) hold, and assume that the correlation function satisfies the Dini condition (4.5). We split the proof into two steps.

Step 1. Proof of (i). Appealing to Proposition 4.9 in form of the explicit formula (4.21) for $\overline{\mathbf{B}}^2$, and estimating the second right-hand side term as in the proof of Proposition 4.9 (i), we find

$$\left| E : \overline{\mathbf{B}}^2 E - \int_{\mathbb{R}^d} \mathbb{E}\bigg[\int_{\partial I^\circ} \psi^z \cdot \sigma^0 \nu \bigg] h_2(0,z)\, dz \right| \lesssim \lambda_2(\mathcal{P}), \tag{4.78}$$

where σ^0 and ψ^z are associated respectively with single particles at I° and at $z + I^{\circ\prime}$, where I° and $I^{\circ\prime}$ are iid copies of the same random shape. Replacing ψ^z by its Taylor expansion, using the boundary conditions for σ^0, and using standard decay properties of ψ^z, we find

$$\left| \int_{\partial I^\circ} \psi^z \cdot \sigma^0 \nu - \mathrm{D}(\psi^z)(0) : \int_{\partial I^\circ} \sigma^0 \nu \otimes x \right| \lesssim \langle z \rangle^{-d-1}.$$

Inserting this into (4.78) together with (4.9), and recalling the shorthand notation

$$\begin{aligned} E : \widehat{\mathbf{B}}^1 E &= \frac{1}{2} \mathbb{E}\bigg[\int_{\partial I^\circ} Ex \cdot \sigma^\circ \nu \bigg] \\ &= \mathbb{E}\bigg[\int_{\mathbb{R}^d} |\mathrm{D}(\psi^\circ)|^2 \bigg], \end{aligned}$$

cf. (2.6), we get

$$\left| E : \overline{\mathbf{B}}^2 E - (2\widehat{\mathbf{B}}^1 E) : \left(\int_{\mathbb{R}^d} \mathbb{E}\big[\mathrm{D}(\psi^z)(0)\big] h_2(0,z)\, dz \right) \right| \lesssim \lambda_2(\mathcal{P}). \tag{4.79}$$

Next, we further analyze $\mathrm{D}(\psi^z)(0)$. In view of Lemma 3.3, we note that ψ^z satisfies in $\mathbb{R}^d$,

$$-\triangle \psi^z + \nabla(\Sigma^z \mathbb{1}_{\mathbb{R}^d \setminus (z+I^{\circ\prime})}) = -\delta_{\partial(z+I^{\circ\prime})} \sigma^z \nu.$$

In terms of the Stokeslet G for the free Stokes equation, Green's representation formula then yields

$$\nabla_i \psi^z(0) = -\int_{\partial(z+I^{\circ\prime})} \nabla_i G(-\cdot)\, \sigma^z \nu.$$

Replacing $\nabla_i G$ by its Taylor expansion, using the boundary conditions for σ^z, and using standard decay properties of G, we find

$$\left|\nabla_i \psi^z(0) - \nabla^2_{ij} G(-z) \int_{\partial(z+I^{\circ\prime})} (\cdot - z)_j \, \sigma^z \, \nu \right| \lesssim \langle z \rangle^{-d-1},$$

and therefore, taking the expectation, noting that $\sigma^z = \sigma^0(\cdot - z)$, and recognizing $\widehat{\mathbf{B}}^1 E$ again,

$$\left|\mathbb{E}\big[\nabla_i \psi^z_k(0)\big] - (2\widehat{\mathbf{B}}^1 E)_{lj} \nabla^2_{ij} G_{kl}(-z)\right| \lesssim \langle z \rangle^{-d-1}.$$

Inserting this into (4.79) together with (4.9) again, we get

$$\left| E : \overline{\mathbf{B}}^2 E - (2\widehat{\mathbf{B}}^1 E)_{lj} (2\widehat{\mathbf{B}}^1 E)_{ki} \left(\text{p. v.} \int_{\mathbb{R}^d} \nabla^2_{ij} G_{kl}(z) h_2(0,z)\, dz \right) \right| \lesssim \lambda_2(\mathcal{P}), \tag{4.80}$$

where the notation p. v. stands for the principal value. It remains to analyze the integral term in the left-hand side. As h_2 satisfies the Dini condition (4.5), this integral is absolutely summable. Further, assuming that the point process $\mathcal{P}$ is statistically isotropic, the correlation function $h_2(0,\cdot)$ is radial. By symmetry, this entails p. v. $\int_{\mathbb{R}^d} \nabla^2_{ij} G_{kl}(z)\, h_2(0,z)\, dz = 0$, and the conclusion (i) follows.

Step 2. Proof of (ii). In view of (4.80), as $2\widehat{\mathbf{B}}^1 E$ does not vanish, it suffices to construct a point process $\mathcal{P}$ that satisfies Assumptions (H_ρ) and ($\mathrm{H}^{\mathrm{unif}}_\rho$), has decay of correlations (4.5) with algebraic rate $\omega(t) \le C t^{-\beta}$ for some $C, \beta > 0$, and satisfies the local independence condition $\lambda_2(\mathcal{P}) \simeq \lambda(\mathcal{P})^2 \ll 1$, such that the integral term in the left-hand side of (4.80) satisfies

$$\left| \text{p. v.} \int_{\mathbb{R}^d} \nabla^2 G(z)\, h_2(0,z)\, dz \right| \gtrsim \lambda(\mathcal{P})^2 \big|\log \lambda(\mathcal{P})\big|, \tag{4.81}$$

with the logarithmic factor. We shall consider spherical particles, $I^\circ = B$, and we start with the construction of the correlation function h_2. For that purpose, first note that we can find a smooth bounded function

$$g : \mathbb{S}^{d-1} \to [0,1]$$

such that

$$\left| \int_{\partial B} \nabla^2 G(e) g(e)\, d\theta(e) \right| \gtrsim 1, \tag{4.82}$$

where $d\theta$ stands for the Lebesgue measure on ∂B, and we then define

$$h(z) := \frac{g\big(\frac{z}{|z|}\big)}{1 + \lambda^2 |z|^{2d+1}}.$$

Using (4.82), a computation in spherical coordinates yields

$$\left| \int_{|z|>2(1+\rho)} \nabla^2 G(z) h(z)\, dz \right| = \left(\int_{2(1+\rho)}^{\infty} r^{-1} (1+\lambda^2 r^{2d+1})^{-1} dr \right) \left| \int_{\partial B} \nabla^2 G(e) g(e)\, d\theta(e) \right| \gtrsim |\log \lambda|,$$

which proves (4.81) if the point process is chosen with intensity $\lambda(\mathcal{P}) = \lambda$ and with two-point correlation function h_2 given by

$$h_2(x,y) + \lambda^2 := \lambda^2 (h(x-y)+1) \mathbb{1}_{|x-y|>2(1+\rho)}.$$

In particular, this choice also yields

$$h_2(0,z)\mathbb{1}_{|z|>2(1+\rho)} \geq 0, \quad \sup_z \int_{Q(z)} \left|h_2(0,\cdot)\right| \lesssim \lambda^2, \quad \left|h_2(0,z)\right| \lesssim \langle z \rangle^{-2d-1}.$$

It remains to prove that this choice of h_2 can be realized as the correlation function of a point process with intensity $\lambda(\mathcal{P}) = \lambda$ and satisfying (H_ρ) and ($\mathrm{H}_\rho^{\mathrm{unif}}$): this is precisely the subject of Proposition 4.12 below. ∎

The construction of a point process with given intensity and given two-point density function is easily done under suitable positivity conditions, e.g. following [43]. In the present setting, more care is needed to further ensure stationarity and ergodicity of the constructed point process. Note that we use here a sufficient positivity condition that is much stronger than the one in [43], but is easier to handle and suffices for our purposes.

Proposition 4.12 (Realizability of point processes). *Let $\lambda, \rho > 0$ and let $h \in \mathrm{L}^\infty(\mathbb{R}^d)$ be nonnegative with $h(x) \to 0$ uniformly as $|x| \uparrow \infty$. Then, there exists a strongly mixing stationary point process $\mathcal{P} = \{x_n\}_n$ on $\mathbb{R}^d$ with intensity λ and two-point density*

$$f_2(x,y) := \lambda^2 \big(h(x-y)+1\big) \mathbb{1}_{|x-y|>2(1+\rho)}, \tag{4.83}$$

such that $|x_n - x_m| \geq 2(1+\rho)$ almost surely for all $n \neq m$.

Proof. Let $\mathcal{M}_\rho$ denote the set of locally finite point sets $\{z_n\}_n$ with $|z_n - z_m| \geq 2(1+\rho)$ for all $n \neq m$. It is easily checked that $\mathcal{M}_\rho$ is compact for the topology of convergence of point sets restricted to compact domains (this coincides with the vague topology when viewing point sets $\{z_n\}_n$ as measures $\sum_n \delta_{z_n}$). Consider the space $V := C(\mathcal{M}_\rho)$, and denote by V_0 the dense vector subset of polynomials with continuous coefficients on $\mathcal{M}_\rho$, that is, the subset of functions $P^N : \mathcal{M}_\rho \to \mathbb{R}$ of the form

$$P^N\big(\{z_n\}_n\big) = P_0^N + \sum_{k=1}^{N} \sum_{n_1,\ldots,n_k}^{\neq} P_k^N(z_{n_1},\ldots,z_{n_k}), \tag{4.84}$$

with $P_0^N \in \mathbb{R}$ and $P_k^N \in C_c(S_\rho^k)$ for $1 \le k \le N$, where we use the shorthand notation

$$S_\rho^k := \big\{(z_1, \dots, z_k) \in (\mathbb{R}^d)^k : |z_n - z_m| \ge 2(1+\rho) \text{ for all } n \ne m\big\}.$$

In order to construct a point process with the two-point density f_2 given by (4.83), we shall further prescribe all its multi-point density functions. For convenience, these are chosen in form of Mayer cluster expansions with vanishing higher-order correlations: for all $k \ge 1$,

$$\begin{aligned} &f_k(z_1, \dots, z_k) \\ &:= \lambda^k \, \mathbb{1}_{S_\rho^k}(z_1, \dots, z_k) \\ &\quad \times \left(1 + \sum_{j=1}^{k/2} \frac{1}{2^j \, j!} \sum_{1 \le n_1, \dots, n_{2j} \le k}^{\neq} h(z_{n_1} - z_{n_2}) \cdots h(z_{n_{2\ell-1}} - z_{n_{2j}})\right). \end{aligned} \tag{4.85}$$

Next, we define in these terms a linear map $L : V_0 \to \mathbb{R}$ as follows: for any polynomial P^N of the form (4.84), we set

$$L(P^N) := P_0^N + \sum_{k=1}^N \int_{S_\rho^k} P_k^N \, f_k. \tag{4.86}$$

We argue that L is a positive linear functional on V_0, hence it is also continuous on V_0 with respect to the topology of V. Indeed, for any polynomial P^N of the form (4.84) with $P^N \ge 0$ pointwise, if we evaluate it at the points of a Poisson point process with intensity λ, and if we compute the expectation, we find

$$P_0^N + \sum_{k=1}^N \lambda^k \int_{S_\rho^k} P_k^N \ge 0,$$

hence, noting that the positivity of h entails $f_k \ge \lambda^k \mathbb{1}_{S_\rho^k}$ for all $k \ge 1$, we get

$$L(P^N) \ge P_0^N + \sum_{k=1}^N \lambda^k \int_{S_\rho^k} P_k^N \ge 0,$$

thus proving the claimed positivity.

As V_0 is dense in V, we can extend L uniquely into a positive linear functional

$$L : V \to \mathbb{R}.$$

Next, by the Riesz–Markov–Kakutani representation theorem, there exists a random element in $\mathcal{M}_\rho$, that is, a random point process $\mathcal{P} = \{x_n\}_n$, such that

$$\mathbb{E}\big[P(\mathcal{P})\big] = L(P) \quad \text{for all } P \in V.$$

Testing this relation with polynomials, and using (4.86), we deduce that for all $k \geq 1$ the k-point density function of the point process $\mathcal{P}$ coincides with f_k. In particular, it has intensity $f_1 = \lambda$ and two-point density

$$f_2(x, y) = \lambda^2 \big(h(x - y) + 1\big) \mathbb{1}_{|x-y|>2(1+\rho)}$$

as desired. In addition, L is translation-invariant by definition, hence $\mathcal{P}$ is stationary.

It remains to check that $\mathcal{P}$ is strongly mixing. For that purpose, we compute the covariance of $\sigma(\mathcal{P})$-measurable random variables. Choose a polynomial P^N of the form (4.84), and let $R > 0$ be such that P_k^N is supported in $(B_R)^k$ for all $1 \leq k \leq N$. For $|x| > 2R$, as we have $B_R \cap (x + B_R) = \varnothing$, we can compute

$$\begin{aligned}
&\mathrm{Cov}\big[P^N(\mathcal{P} + x); P^N(\mathcal{P})\big] \\
&\quad = (P_0^N)^2 + \sum_{k\geq 1} \sum_{j=0}^{k} \int_{S_\rho^k} \big(P_j^N(\cdot + x, \dots, \cdot + x) \otimes P_{k-j}^N\big)\big(f_k - f_j \otimes f_{k-j}\big).
\end{aligned}$$

The definition (4.85) of f_k easily leads to

$$\begin{aligned}
&\left| \int_{S_\rho^k} \big(P_j^N(\cdot + x, \dots, \cdot + x) \otimes P_{k-j}^N\big)\big(f_k - f_j \otimes f_{k-j}\big) \right| \lesssim \\
&\Big(\sup_{z \in B_{2R}} h(x + z)\Big)(2\lambda)^k \big(1 + \|h\|_{\mathrm{L}^\infty(\mathbb{R}^d)}\big)^{\frac{k}{2}-1} \|P_j^N\|_{\mathrm{L}^1((\mathbb{R}^d)^j)} \|P_{k-j}^N\|_{\mathrm{L}^1((\mathbb{R}^d)^{k-j})}.
\end{aligned}$$

As by assumption $h(x) \to 0$ as $|x| \uparrow \infty$, we get $\mathrm{Cov}[P^N(\mathcal{P} + x); P^N(\mathcal{P})] \to 0$. By a density argument, the same holds if P^N is replaced by any element of V, which proves that $\mathcal{P}$ is strongly mixing. ∎

Chapter 5

Conclusion

In this last chapter, we recall, reformulate, and comment our main findings on the validity of Einstein's formula and of higher-order cluster expansions for the effective viscosity, as obtained in Chapters 2, 3, and 4.

5.1 Cluster expansion of the effective viscosity

We start with the validity of Einstein's formula and the associated error estimates as proved in Chapter 2, cf. Theorem 2.1. The three important features of this result are the generality in terms of probabilistic assumptions (mere qualitative ergodicity under Assumption $(\mathrm{H}_\rho^{\mathrm{unif}})$), the sharpness of the error estimate (5.1), and the possibility for particles to touch (under Assumption $(\mathrm{H}_{\rho,\kappa}^{\mathrm{mom}})$ or $(\mathrm{H}_{\rho,\kappa}^{\mathrm{perc}})$).

Theorem 5.1 (Einstein's formula). *Under Assumption (H_ρ), provided that Assumption $(\mathrm{H}_\rho^{\mathrm{unif}})$, $(\mathrm{H}_{\rho,\kappa}^{\mathrm{mom}})$, or $(\mathrm{H}_{\rho,\kappa}^{\mathrm{perc}})$ holds for some $\rho > 0$ and $\kappa > 1$, we have*

$$\begin{aligned}\big|\overline{\mathbf{B}} - (\mathrm{Id} + \overline{\mathbf{B}}^1)\big| &\lesssim_\rho \lambda_2(\mathcal{P}) \log\left(2 + \frac{\lambda(\mathcal{P})}{\lambda_2(\mathcal{P})\big(\ell(\mathcal{P})+1\big)^d}\right) \\ &\quad + \begin{cases} 0 & \text{in case of } (\mathrm{H}_\rho^{\mathrm{unif}}), \\ \mathcal{K}_\kappa\, \lambda_2(\mathcal{P})^{1-\frac{1}{\kappa}} \lambda(\mathcal{P})^{\frac{1}{\kappa}} & \text{in case of } (\mathrm{H}_{\rho,\kappa}^{\mathrm{mom}}) \text{ or } (\mathrm{H}_{\rho,\kappa}^{\mathrm{perc}}), \end{cases} \end{aligned} \tag{5.1}$$

where $\overline{\mathbf{B}}^1$ satisfies

$$|\overline{\mathbf{B}}^1| \simeq \lambda(\mathcal{P}),$$

and is defined for all $E \in \mathbb{M}_0^{\mathrm{sym}}$ by

$$E : \overline{\mathbf{B}}^1 E := \sum_n \mathbb{E}\bigg[\frac{\mathbb{1}_{0\in I_n}}{|I_n|} \int_{\mathbb{R}^d} \big|\mathrm{D}(\psi_E^{\{n\}})\big|^2\bigg],$$

where $\psi_E^{\{n\}}$ is the unique decaying solution of the single-particle problem (1.5). In particular, the estimate $|\overline{\mathbf{B}} - (\mathrm{Id} + \overline{\mathbf{B}}^1)| = o(\lambda(\mathcal{P}))$ holds provided the point process $\mathcal{P}$ satisfies $\lambda_2(\mathcal{P}) = o(\lambda(\mathcal{P}))$.

In order to address the optimality of this estimate, one needs to identify the next term in the expansion. In Chapters 3 and 4, we have further investigated higher-order expansions of the effective viscosity in form of cluster expansions. The upcoming result, which summarizes Theorems 4.1 and 4.3 in Chapter 4, gives the optimal

order of magnitude of the cluster coefficients and of the remainder. The two important features of this result are the generality of the point processes (to be compared with results in Section 5.2 below) and the sharpness of the estimates. The main achievement is the explicit understanding of the needed renormalizations to all orders, solving a problem that was still open in the physics community.

Theorem 5.2 (Cluster expansion in general dilute setting). *On top of* (H_ρ) *and* ($\mathrm{H}_\rho^{\mathrm{unif}}$), *assume that the inclusion process is* α*-mixing in the sense of* (Mix) *with algebraic rate* ω. *Then, for all* $k \geq 1$, *the following holds for the effective viscosity,*

$$\left| \bar{\mathbf{B}} - \mathrm{Id} - \sum_{j=1}^{k} \frac{1}{j!} \bar{\mathbf{B}}^j \right| \lesssim_k \sum_{l=k+1}^{2k+1} \lambda_l(\mathcal{P}) \left| \log \lambda(\mathcal{P}) \right|^{l-1},$$

$$|\bar{\mathbf{B}}^j| \lesssim_j \lambda_j(\mathcal{P}) \left| \log \lambda(\mathcal{P}) \right|^{j-1}, \quad \text{for all } 1 \leq j \leq k,$$

where the cluster coefficients $\{\bar{\mathbf{B}}^j\}_j$ *are defined in* (3.13) *by means of finite-volume approximations. If in addition the independence assumption* (Indep) *holds for particle shapes, renormalized formulas can be given for cluster coefficients in form of absolutely convergent multiple integrals, cf. Section* 4.4, *and the following quantitative convergence result holds for finite-volume approximations* $\{\bar{\mathbf{B}}_L^j\}_j$*: in case of an algebraic* α*-mixing rate* $\omega(t) \leq C t^{-\beta}$ *for some* $C, \beta > 0$,

$$|\bar{\mathbf{B}}_L^j - \bar{\mathbf{B}}^j| \lesssim_j \frac{(\log L)^{j-1}}{L^{\beta \wedge 1}}.$$

Note that the bound on $\bar{\mathbf{B}}_2$ in Theorem 5.2 essentially coincides with the estimate on the remainder in Theorem 5.1, which contrasts with the results of Lemma 1.2 in the short-range setting by a logarithmic correction. Optimality of the latter is addressed in Theorem 4.4, which we presently recall.

Theorem 5.3. *About the optimality of estimates on* $\bar{\mathbf{B}}_2$, *the following statements hold.*

(i) Isotropic setting: *On top of Assumptions* (H_ρ), ($\mathrm{H}_\rho^{\mathrm{unif}}$), *and* (Indep), *assume that the* 2*-point correlation function* $h_2(x, y) := f_2(x, y) - \lambda(\mathcal{P})^2$ *satisfies the following decay assumption,*

$$\iint_{B(x)\times B(y)} |h_2| \leq \omega\big(|x - y|\big),$$

with some rate ω *satisfying the Dini condition* $\int_1^\infty t^{-1}\omega(t)\,dt < \infty$. *If in addition the point process* $\mathcal{P}$ *is statistically isotropic, which entails that the correlation function is radial, then the following improved estimate holds,*

$$|\bar{\mathbf{B}}^2| \lesssim \lambda_2(\mathcal{P}).$$

(ii) Optimality in the general setting: *There exists an inclusion process* $\mathcal{I}$ *that satisfies Assumptions* (H_ρ), ($\mathrm{H}_\rho^{\mathrm{unif}}$), (Indep), *and* (4.5), *as well as the local independence*

condition $\lambda_2(\mathcal{P}) \simeq \lambda(\mathcal{P})^2 \ll 1$*, such that we have*

$$|\overline{\mathbf{B}}^2| \simeq \lambda_2(\mathcal{P})\big|\log \lambda_2(\mathcal{P})\big|.$$

Based on the explicit renormalization of higher-order cluster coefficients, it appears that Theorem 5.3 (ii) readily extends to higher orders, demonstrating the optimality of cluster estimates in Theorem 5.2.

To conclude this section, let us apply and confront Theorems 5.1, 5.2, and 5.3 to some specific families of inclusion processes displaying multi-point intensities with different scaling laws. We start with the construction.

• *Construction of inclusion processes* $\{\mathcal{I}_{\beta,\lambda}\}_{\beta,\lambda}$*:* We define a family of point processes $\{\mathcal{P}_{\beta,\lambda}\}_{\beta,\lambda}$ with parameters $0 \le \beta \le 1$ and $0 < \lambda \ll 1$ as follows. Consider a hardcore Poisson process $\mathcal{P}' = \{x_n'\}_n$ with radius 6 and with intensity $\lambda(\mathcal{P}') = \lambda$, see e.g. [17, Section 3.4] using Penrose's graphical construction [57]. Next, independently choose a sequence $\{y_n\}_n$ of iid random points that are uniformly distributed in $B_4 \setminus B_3$, and, given $\beta \in [0,1]$, also independently choose a sequence $\{b_{n,\beta}\}_n$ of iid Bernoulli variables with parameter $\lambda^\beta = \mathbb{P}[b_{n,\beta} = 1]$. The desired point processes and spherical inclusion processes are then defined by

$$\mathcal{P}_{\beta,\lambda} := \mathcal{P}' \cup \big\{x_n' + y_n : b_{n,\beta} = 1\big\}, \quad \mathcal{I}_{\beta,\lambda} := \bigcup_{x\in\mathcal{P}_{\beta,\lambda}} B(x).$$

• *Properties of the processes:* $\mathcal{I}_{\beta,\lambda}$ satisfies (H_ρ) and ($\mathrm{H}_\rho^{\mathrm{unif}}$) (with $\rho = 1$) as well as (Indep). In addition, the point process $\mathcal{P}_{\beta,\lambda}$ is statistically isotropic and α-mixing with exponential rate uniformly with respect to β, λ (e.g. [16, Proposition 1.4 (iii)] and [17, Proposition 3.5]), so that Theorem 5.2 applies. A direct computation shows that the multi-point intensities scale as follows,

$$\lambda(\mathcal{P}_{\beta,\lambda}) \simeq \lambda, \quad \lambda_2(\mathcal{P}_{\beta,\lambda}) \simeq \lambda^{1+\beta}, \quad \lambda_3(\mathcal{P}_{\beta,\lambda}) \simeq \lambda^{2+\beta},$$

and more generally $\lambda_{2k}(\mathcal{P}_{\beta,\lambda}) \simeq_k \lambda^{k(1+\beta)}$ and $\lambda_{2k+1}(\mathcal{P}_{\beta,\lambda}) \simeq_k \lambda^{1+k(1+\beta)}$. In particular the minimal local independence condition $\lambda_3(\mathcal{P}_{\beta,\lambda}) \ll \lambda_2(\mathcal{P}_{\beta,\lambda}) \ll \lambda(\mathcal{P}_{\beta,\lambda})$ holds for $\beta > 0$.

• *Second-order cluster expansion:* We denote by $\overline{\mathbf{B}}_{\beta,\lambda}$ the effective viscosity associated with $\mathcal{I}_{\beta,\lambda}$. Theorem 5.2 implies that

$$\Big|\overline{\mathbf{B}}_{\beta,\lambda} - \Big(\mathrm{Id} + \overline{\mathbf{B}}^1_{\beta,\lambda} + \frac{1}{2}\overline{\mathbf{B}}^2_{\beta,\lambda}\Big)\Big| \lesssim \lambda^{2+\beta}|\log\lambda|^2,$$

where $|\overline{\mathbf{B}}^1_{\beta,\lambda}| \simeq \lambda$ and $|\overline{\mathbf{B}}^2_{\beta,\lambda}| \simeq \lambda^{1+\beta}$ (cf. (2.6) and Theorem 5.3 (i)). In particular, discarding $\overline{\mathbf{B}}^2_{\beta,\lambda}$ in the above yields the following (completely new) sharp error estimate for Einstein's formula in this setting: for all $0 \le \beta \le 1$ and $\lambda \ll 1$,

$$\big|\overline{\mathbf{B}}_{\beta,\lambda} - (\mathrm{Id} + \overline{\mathbf{B}}^1_{\beta,\lambda})\big| \simeq \lambda^{1+\beta} \simeq |\overline{\mathbf{B}}^1_{\beta,\lambda}|^{1+\beta}.$$

In this example, Einstein's formula is thus accurate whenever $\beta > 0$, which illustrates the full range of the condition $\lambda_2(\mathcal{P}_{\beta,\lambda}) = o(\lambda(\mathcal{P}_{\beta,\lambda}))$ in Theorem 5.1.

5.2 Summability of the cluster expansion

Finally, we consider the following two specific one-parameter dilution procedures, for which our results can be substantially strengthened using the uniform $\ell^1 - \ell^2$ energy estimates of Theorem 3.9: more precisely, logarithmic corrections in cluster estimates can be removed in that case and the full cluster expansion is absolutely converging.

(Dilat) *Dilution by geometric dilation:* Given a point process $\mathcal{P} = \{x_n\}_n$ and random inclusions $I_n = x_n + I_n^\circ$ satisfying (H_ρ), we consider the dilated process $\mathcal{P}_s = \{sx_n\}_n$ and the corresponding inclusions $I_{n,s} = sx_n + I_n^\circ$. The latter has minimal distance $\ell(\mathcal{P}_s) = s\ell(\mathcal{P}) \simeq s$, still satisfies ($H_\rho$), and further satisfies (H_ρ^{unif}) with minimal interparticle distance

$$\inf_{n\neq m} \operatorname{dist}(I_{n,s}, I_{m,s}) \geq \inf_{n\neq m} |sx_n - sx_m| - 2 \geq s\ell(\mathcal{P}) - 2 \gtrsim s,$$

provided $s \gg 1$. Its multi-point intensities take the form

$$\lambda_j(\mathcal{P}_s) = s^{-jd}\lambda_j(\mathcal{P}) \quad \text{for all } j \geq 1.$$

(Delet) *Dilution by random deletion:* Given a point process $\mathcal{P} = \{x_n\}_n$ and random inclusions $I_n = x_n + I_n^\circ$ satisfying (H_ρ) and (H_ρ^{unif}), the Bernoulli deletion scheme consists in keeping each inclusion only with given probability $p \in [0, 1]$. More precisely, we attach to the inclusions iid Bernoulli variables $\{b_n^{(p)}\}_n$, independent of $\mathcal{P}$, $\mathcal{I}$, with parameter

$$p = \mathbb{P}\big[b_n^{(p)} = 1\big],$$

and we define the corresponding decimated process

$$\mathcal{P}^{(p)} := \{x_n\}_{n\in N^{(p)}}, \quad \mathcal{I}^{(p)} := \bigcup_{n\in N^{(p)}} I_n, \quad N^{(p)} := \{n : b_n^{(p)} = 1\}.$$

This decimated process still satisfies (H_ρ) and (H_ρ^{unif}), and its multi-point intensities are given by

$$\lambda_j(\mathcal{P}^{(p)}) = p^j\lambda_j(\mathcal{P}) \quad \text{for all } j \geq 1.$$

In these one-parameter settings, dilute expansions of the effective viscosity amount to expansions with respect to the dilution parameters s^{-1} or p. Given a random set of particles $\mathcal{I} = \bigcup_n I_n$ centered at the points of $\mathcal{P} = \{x_n\}_n$, we shall consider both

dilution procedures at once, defining the dilated decimated process

$$\mathcal{P}_s^{(p)} := \{x_{n,s}\}_{n\in N^{(p)}}, \quad \mathcal{I}_s^{(p)} := \bigcup_{n\in N^{(p)}} I_{n,s}, \quad x_{n,s} := sx_n, \quad I_{n,s} := x_{n,s} + I_n^\circ.$$

As a consequence of Theorem 3.2, together with (3.78) and (3.79) in Section 3.7, we obtain the following summability result and estimates for the cluster expansion of the effective viscosity $\bar{\mathbf{B}}_s^{(p)}$ associated with $\mathcal{I}_s^{(p)}$. In particular, it shows that the scaling of cluster coefficients coincides in this case with that of Lemma 1.2 for the short-range setting: indeed, we have

$$|\bar{\mathbf{B}}_s^{(p),j}| = p^j |\bar{\mathbf{B}}_s^j| \lesssim_j (ps^{-d})^j \simeq \lambda_j(\mathcal{P}_s^{(p)}).$$

We emphasize that no mixing assumption is required here.

Theorem 5.4 (Cluster expansion for one-parameter dilution procedures). *Under* (H_ρ) *and* ($\mathrm{H}_\rho^{\mathrm{unif}}$)*, for the specific dilution models* (Dilat) *and* (Delet) *above, with dilation parameter* s *and Bernoulli parameter* p*, the cluster expansion of the effective viscosity is uniformly summable in the following sense: there exists a constant* C *(only depending on* d, ρ*) such that for all* $0 \le ps^{-d} < \frac{1}{C}$ *the effective viscosity satisfies*

$$\bar{\mathbf{B}}_s^{(p)} = \mathrm{Id} + \sum_{j=1}^{\infty} \frac{p^j}{j!} \bar{\mathbf{B}}_s^j, \quad |\bar{\mathbf{B}}_s^j| \le j!\,(Cs^{-d})^j \quad \text{for all } j \ge 1, \tag{5.2}$$

where the cluster coefficients $\{\bar{\mathbf{B}}_s^j\}_j$ *are defined in* (3.13) *by means of finite-volume approximations.*

Remarks 5.5. We comment on the analyticity of the effective viscosity with respect to dilution parameters.

(a) In case of the random deletion model (Delet), the expansion (5.2) yields the local analyticity of $p \mapsto \bar{\mathbf{B}}^{(p)}$ at $p = 0$. Local analyticity can, in fact, be established on the whole interval $0 \le p \le 1$; the reader is referred to [15] for a similar result in the scalar setting.

(b) In case of the dilation model (Dilat), the expansion (5.2) does not yield the analyticity of the map $s^{-d} \mapsto \bar{\mathbf{B}}_s$ since the rescaled coefficients $\{s^{dj}\bar{\mathbf{B}}_s^j\}_j$ also depend on s. By means of multipole expansions, the maps

$$s^{-1} \mapsto s^{dj}\bar{\mathbf{B}}_s^j$$

can be checked to be analytic themselves, as well as $s^{-1} \mapsto \bar{\mathbf{B}}_s$. For a more direct approach to expansions in s^{-1}, we refer to the recent work [59] in the scalar setting; see also [9].

To illustrate Remark 5.5 (b), we display the first term of the monopole expansion for the second-order coefficient $\bar{\mathbf{B}}_s^2$. In particular, as is natural, we note that $\bar{\mathbf{B}}_s^2$ can be expressed to leading order in terms of the single-particle problem only, and

it coincides with the formula obtained in [27, Proposition 5.6] in case of spherical inclusions.

Proposition 5.6 (Leading order of monopole expansion). *On top of* (H_ρ) *and* ($\mathrm{H}_\rho^{\mathrm{unif}}$), *assume that particles have independent shapes, cf.* (Indep), *and that the two-point correlation function* $h_2 = f_2 - \lambda(\mathcal{P})^2$ *satisfies the decay assumption*

$$\iint_{B(x)\times B(y)} |h_2| \le \omega\big(|x-y|\big),$$

with some rate ω *satisfying the Dini condition* $\int_1^\infty t^{-1}\omega(t)\,dt < \infty$. *Consider the dilated process* $\mathcal{P}_s$, *cf.* (Dilat), *and the associated second-order cluster coefficient* $\overline{\mathbf{B}}_s^2$ *defined in Theorem* 5.4. *Then, we have*

$$\big|\overline{\mathbf{B}}_s^2 - s^{-2d}\overline{\mathbf{B}}^{2,1}\big| \lesssim s^{-2d-1},$$

and the leading-order contribution $\overline{\mathbf{B}}^{2,1}$ *is given by the following reduced formula,*

$$E : \overline{\mathbf{B}}^{2,1} E = (2\widehat{\mathbf{B}}^1 E) : \bigg(\text{p. v.} \int_{\mathbb{R}^d} \mathcal{G}(z)\, h_2(0,z)\, dz\bigg)(2\widehat{\mathbf{B}}^1 E),$$

where $\widehat{\mathbf{B}}^1 E$ *is defined in* (2.6), *where the notation* p. v. *stands for the principal value, and where the* 4*-tensor field* $\mathcal{G}$ *is given by* $M : \mathcal{G}(z)M = M_{jk}M_{lm}\nabla^2_{km}G_{jl}(z)$ *in terms of the standard Stokeslet*

$$G(z) = \frac{|z|^{2-d}}{2(d-2)|\partial B|}\Big(\mathrm{Id} + (d-2)\frac{z\otimes z}{|z|^2}\Big).$$

In case of spherical particles, $I_n = B(x_n)$, *we thus have*

$$E : \overline{\mathbf{B}}^{2,1} E = (d+2)^2|B|\ \text{p. v.} \int_{\mathbb{R}^d} \bigg(\frac{d+2}{2}\frac{(z\cdot Ez)^2}{|z|^{d+4}} - \frac{|Ez|^2}{|z|^{d+2}}\bigg) h_2(0,z)\, dz.$$

Proof. Starting from the renormalized formula (4.21) in Proposition 4.9, repeating the proof of (4.80) to decompose the first contribution, and using (4.25) to estimate the second one, we are led to

$$\begin{aligned}\bigg|E : \overline{\mathbf{B}}_s^2 E - (2\widehat{\mathbf{B}}^1 E)_{lj}(2\widehat{\mathbf{B}}^1 E)_{ki}\bigg(\text{p. v.} \int_{\mathbb{R}^d} \nabla^2_{ij}G_{kl}(z) h_2(0,z)\, dz\bigg)\bigg| \\ \lesssim \int_{\mathbb{R}^d} \langle z\rangle^{-d-1}\big|h_{2,s}(0,z)\big|\, dz + \int_{\mathbb{R}^d}\langle z\rangle^{-2d} f_{2,s}(0,z)\, dz.\end{aligned}$$

Using that the two-point density and the correlation for the dilated process $\mathcal{P}_s$ take the form

$$f_{2,s}(0,z) = s^{-2d} f_2(0, s^{-1}z), \quad h_{2,s}(0,z) = s^{-2d} h_2(0, s^{-1}z),$$

and changing variables, the conclusion follows by scaling. In case of spherical particles, we appeal to the proof of Proposition 2.2 for the explicit computation of $\widehat{\mathbf{B}}^1 E$. ∎

Finally, we revisit a recent result by Gérard-Varet [26] that displays to second order similar estimates as for the random deletion procedure, cf. (5.2), but only assuming some specific structure of the multi-point densities up to order 5, thus contrasting with Theorem 5.4. As a corollary of Proposition 4.11, we establish the following result, which constitutes an extension of [26] to higher orders with new, optimal error bounds. Note indeed that for $k = 2$ the result (5.3) below yields an error bound $O(p^3)$, which improves on the bound $O(p^{\frac{5}{2}})$ obtained in [26].

Corollary 5.7. *Let $\mathcal{P}$ satisfy Assumptions (H_ρ) and ($\mathrm{H}_\rho^{\mathrm{unif}}$), and let $\mathcal{I}$ satisfy the independence assumption (Indep). Given $k \geq 2$, assume that there exists $0 < p \leq 1$ such that the multi-point density functions of $\mathcal{P}$ can be written as*

$$f_j = p^j f_j^\circ \quad \text{for all } 1 \leq j \leq 2k+1,$$

for some functions $(f_j^\circ)_{1\leq j\leq 2k+1}$. Further, define functions $(h_j^\circ)_{1\leq j\leq 2k+1}$ through the correlation/density relation (4.8) starting from $(f_j^\circ)_{1\leq j\leq 2k+1}$ and assume that they satisfy (Mix_ω^n) to order $n = 2k+1$ with algebraic rate ω. Then, we have

$$\left| \overline{\mathbf{B}} - \mathrm{Id} - \sum_{j=1}^{k} \frac{1}{j!} \overline{\mathbf{B}}^j \right| \lesssim_k p^{k+1}, \quad |\overline{\mathbf{B}}^j| \lesssim_j p^j \quad \text{for all } 1 \leq j \leq k. \tag{5.3}$$

where the multiplicative constants are independent of p.

Proof. The assumption $f_j = p^j f_j^\circ$ entails $h_j = p^j h_j^\circ$, where h_j° is assumed to satisfy (Mix_ω^n). Further, writing $(f_j^\circ)_j$ in terms of $(h_j^\circ)_j$ by means of (4.7), the assumption (Mix_ω^n) for the latter yields

$$\lambda_j^\circ := \sup_{z_1,\dots,z_j} \fint_{Q(z_1)\times\cdots\times Q(z_j)} f_j^\circ \lesssim_j 1,$$

where the bound only depends on j, ω, and on the constant function f_1°. In this setting, the bounds of Proposition 4.11 (i)–(iii) now take the form

$$|\overline{\mathbf{B}}^j| \lesssim p^j \overline{\lambda}_j^\circ |\log \lambda^\circ|^{j-1},$$
$$|R^{k+1}| \lesssim \sum_{j=k}^{2k+1} p^{j+1} \overline{\lambda}_{j+1}^\circ |\log \lambda^\circ|^j \lesssim p^{k+1},$$

and the conclusion readily follows. ∎

Appendix A

Stokeslet estimates with rigid inclusions

This appendix is dedicated to the proof of several estimates on the behavior of the fluid velocity generated by a localized force dipole in the presence of a finite number of rigid inclusions. In other words, it concerns the Stokeslet for the Stokes problem with rigid inclusions, and we shall prove in particular Lemmas 3.10, 4.7, and 4.8.

A.1 Main results

For convenience, we start by recalling the notation of Section 4.3.2. Given a set $Y \subset Q_L$ of "background" positions with

$$\operatorname{dist}\big(B(y), B(y')\big) > 2\rho, \ \operatorname{dist}\big(B(y), \partial Q_L\big) > \rho, \ \text{for all } y, y' \in Y, \ y \neq y', \quad \text{(A.1)}$$

we denote by $\psi_L^Y \in H^1_{\mathrm{per}}(Q_L)^d$ the solution of the following periodic corrector problem, using the shorthand notation $\sigma_L^Y := \sigma(\psi_L^Y + Ex, \Sigma_L^Y)$,

$$\begin{cases} -\triangle \psi_L^Y + \nabla \Sigma_L^Y = 0, & \text{in } Q_L \setminus \bigcup_{y\in Y} B(y), \\ \operatorname{div}(\psi_L^Y) = 0, & \text{in } Q_L \setminus \bigcup_{y\in Y} B(y), \\ \mathrm{D}(\psi_L^Y + Ex) = 0, & \text{in } \bigcup_{y\in Y} B(y), \\ \int_{\partial B(y)} \sigma_L^Y \nu = 0, & \forall y \in Y, \\ \int_{\partial B(y)} \Theta(x-y)\cdot \sigma_L^Y \nu = 0, & \forall \Theta \in \mathbb{M}^{\mathrm{skew}}, \ \forall y \in Y. \end{cases}$$

Next, we turn to elementary single-particle contributions $\{\mathcal{J}^z_{L;Y}\}_{z,Y}$: Given a "tagged" position $z \in Q_L$, given $(\zeta, P) \in H^1(B_{1+\rho}(z))^d \times \mathrm{L}^2(B_{1+\rho}(z) \setminus B(z))$ satisfying the following Stokes equations in a neighborhood of $B(z)$,

$$\begin{cases} -\triangle \zeta + \nabla P = 0, & \text{in } B_{1+\rho}(z) \setminus B(z), \\ \operatorname{div}(\zeta) = 0, & \text{in } B_{1+\rho}(z) \setminus B(z), \\ \mathrm{D}(\zeta) = 0, & \text{in } B(z), \\ \int_{\partial B(z)} \sigma(\zeta, P)\nu = 0, & \\ \int_{\partial B(z)} \Theta(x-z)\cdot \sigma(\zeta, P)\nu = 0, & \forall \Theta \in \mathbb{M}^{\mathrm{skew}}, \end{cases} \quad \text{(A.2)}$$

and given a finite subset $Y \subset Q_L$ of "background" positions satisfying (A.1), we define $\mathcal{J}^z_{L;Y}\zeta \in H^1_{\mathrm{per}}(Q_L)^d$ as the solution of the following Stokes problem with force

dipole localized around z and rigid inclusions around points of Y,

$$\begin{cases} -\triangle \mathcal{J}^z_{L;Y}\zeta + \nabla \mathcal{Q}^z_{L;Y}\zeta = -\delta_{\partial B^L(z)}\sigma(\zeta,P)\nu, & \text{in } Q_L \setminus \bigcup_{y\in Y} B(y), \\ \operatorname{div}(\mathcal{J}^z_{L;Y}\zeta) = 0, & \text{in } Q_L \setminus \bigcup_{y\in Y} B(y), \\ \mathrm{D}(\mathcal{J}^z_{L;Y}\zeta) = 0, & \text{in } \bigcup_{y\in Y\setminus Y^z} B(y), \\ \int_{\partial B(y)} \sigma(\mathcal{J}^z_{L;Y}\zeta, \mathcal{Q}^z_{L;Y}\zeta)\nu = 0, & \forall y \in Y \setminus Y_z, \\ \int_{\partial B(y)} \Theta(x-y)\cdot\sigma(\mathcal{J}^z_{L;Y}\zeta, \mathcal{Q}^z_{L;Y}\zeta)\nu = 0, & \forall \Theta \in \mathbb{M}^{\mathrm{skew}},\ \forall y \in Y \setminus Y_z, \\ \mathcal{J}^z_{L;Y}\zeta = V_z + \Theta_z(x-z), & \text{in } \bigcup_{y\in Y_z} B(y), \\ & \text{for some } V_z \in \mathbb{R}^d, \Theta_z \in \mathbb{M}^{\mathrm{skew}}, \\ \sum_{y\in Y_z} \int_{\partial B(y)} \sigma(\mathcal{J}^z_{L;Y}\zeta, \mathcal{Q}^z_{L;Y}\zeta)\nu & \\ = \sum_{y\in Y_z} \int_{B(y)\cap \partial B^L(z)} \sigma(\zeta,P)\nu, & \\ \sum_{y\in Y_z} \int_{\partial B(y)} \Theta(x-z)\cdot\sigma(\mathcal{J}^z_{L;Y}\zeta, \mathcal{Q}^z_{L;Y}\zeta)\nu & \\ = \sum_{y\in Y_z} \int_{B(y)\cap \partial B^L(z)} \Theta(x-z)\cdot\sigma(\zeta,P)\nu, & \forall \Theta \in \mathbb{M}^{\mathrm{skew}}, \end{cases} \tag{A.3}$$

where we recall that $B^L(z) = (B(z) + L\mathbb{Z}^d) \cap Q_L$ stands for the periodization of the ball $B(z)$ in Q_L, where we have set $Y_z := \{y \in Y : B(y) \cap B^L(z) \neq \varnothing\}$, and where we have implicitly extended (ζ, P) periodically to $B_{1+\rho}(z) + L\mathbb{Z}^d$. The solution $\mathcal{J}^z_{L;Y}\zeta$ is only defined up to a rigid motion in Q_L, which we fix by further choosing

$$\int_{Q_L} \mathcal{J}^z_{L;Y}\zeta = 0, \quad \int_{Q_L} \nabla \mathcal{J}^z_{L;Y}\zeta \in \mathbb{M}^{\mathrm{sym}}_0.$$

Note that $\mathcal{J}^z_{L;Y}\zeta$ depends of course on the pair (ζ, P), not only on ζ, but we leave the pressure field implicit in the notation for convenience. We refer to Section 4.3.2 for motivation of the above equations (A.3), and we recall that it reduces to the following simpler equations when $\{z\} \cup Y$ satisfies (A.1) (meaning that z neither gets close to background positions Y nor to the cell boundary ∂Q_L),

$$\begin{cases} -\triangle \mathcal{J}^z_{L;Y}\zeta + \nabla \mathcal{Q}^z_{L;Y}\zeta = -\delta_{\partial B(z)}\sigma(\zeta,P)\nu, & \text{in } Q_L \setminus \bigcup_{y\in Y} B(y), \\ \operatorname{div}(\mathcal{J}^z_{L;Y}\zeta) = 0, & \text{in } Q_L \setminus \bigcup_{y\in Y} B(y), \\ \mathrm{D}(\mathcal{J}^z_{L;Y}\zeta) = 0, & \text{in } \bigcup_{y\in Y} B(y), \\ \int_{\partial B(y)} \sigma(\mathcal{J}^z_{L;Y}\zeta, \mathcal{Q}^z_{L;Y}\zeta)\nu = 0, & \forall y \in Y, \\ \int_{\partial B(y)} \Theta(x-y)\cdot\sigma(\mathcal{J}^z_{L;Y}\zeta, \mathcal{Q}^z_{L;Y}\zeta)\nu = 0, & \forall \Theta \in \mathbb{M}^{\mathrm{skew}},\ \forall y \in Y. \end{cases} \tag{A.4}$$

We further define

$$\mathcal{J}^z_L\zeta := \mathcal{J}^z_{L;\varnothing}\zeta,$$

for which the Stokes problem (A.3) reduces to

$$-\triangle \mathcal{J}_L^z \zeta + \nabla \mathcal{Q}_L^z \zeta = -\delta_{\partial B^L(z)} \sigma(\zeta, P)\nu, \quad \operatorname{div}(\mathcal{J}_L^z \zeta) = 0, \quad \text{in } Q_L, \tag{A.5}$$

and we define $\mathcal{J}_Y^z \zeta$, $\mathcal{J}^z \zeta$ as the corresponding operators on whole space, that is, with $B^L(z)$ and Q_L replaced by $B(z)$ and $\mathbb{R}^d$, respectively, in (A.3) and (A.5).

With the above notation, we start by recalling the statement of Lemma 4.7 regarding the optimal decay properties of the Stokeslets $\{\mathcal{J}_{L;Y}^z\}_{z,Y}$. Note that Lemma 3.10 is a particular case of this result, using notation (4.15), when $\{z\} \cup Y$ satisfies (A.1). The proof is displayed in Section A.2.

Lemma A.1 (Decay of Stokeslets with rigid inclusions). *Let $z \in \mathbb{R}^d$, let (ζ, P) satisfy (A.2) at z, and let $Y \subset Q_L$ satisfy (A.1). Then, we have for all $x \in Q_L$,*

$$\begin{aligned} \left(\int_{B^L(x)} \left| \mathrm{D}(\mathcal{J}_{L;Y}^z \zeta) \right|^2 \right)^{\frac{1}{2}} &\lesssim_{\sharp Y} \langle (x-z)_L \rangle^{-d} \left(\int_{B_{1+\rho}(z)} \left| \mathrm{D}(\zeta) \right|^2 \right)^{\frac{1}{2}}, \\ \left(\int_{B(x)} \left| \mathrm{D}(\mathcal{J}_Y^z \zeta) \right|^2 \right)^{\frac{1}{2}} &\lesssim_{\sharp Y} \langle x-z \rangle^{-d} \left(\int_{B_{1+\rho}(z)} \left| \mathrm{D}(\zeta) \right|^2 \right)^{\frac{1}{2}}. \end{aligned} \tag{A.6}$$

A similar argument leads us to the following version of the mean-value property for Stokes equations in the presence of a finite number of rigid inclusions. The proof is displayed in Section A.3.

Lemma A.2 (Mean-value property with rigid inclusions). *Let $Y \subset Q_L$ satisfy (A.1) and let $w \in H^1(Q_L)^d$ satisfy the following free steady Stokes equations in Q_L,*

$$\begin{cases} -\triangle w + \nabla P = 0, & \text{in } Q_L \setminus \bigcup_{y \in Y} B(y), \\ \operatorname{div}(w) = 0, & \text{in } Q_L \setminus \bigcup_{y \in Y} B(y), \\ \mathrm{D}(w) = 0, & \text{in } \bigcup_{y \in Y} B(y), \\ \int_{\partial B(y)} \sigma(w, P)\nu = 0, & \forall y \in Y, \\ \int_{\partial B(y)} \Theta(x-y) \cdot \sigma(w, P)\nu = 0, & \forall y \in Y, \ \forall \Theta \in \mathbb{M}^{\mathrm{skew}}. \end{cases} \tag{A.7}$$

Then, we have for all $B(x) \subset Q_L$,

$$\int_{B^L(x)} \left| \mathrm{D}(w) \right|^2 \lesssim_{\sharp Y} \langle \operatorname{dist}(x, \partial Q_L) \rangle^{-d} \int_{Q_L} \left| \mathrm{D}(w) \right|^2. \tag{A.8}$$

Finally, we recall the statement of Lemma 4.8 regarding the error $\mathcal{J}_{L;Y}^z - \mathcal{J}_Y^z$ between periodized and whole-space Stokeslets. The proof makes heavy use of the above mean-value property and is displayed in Section A.4. The stated bounds are not optimal: finer estimates are given in the proof, but this simplified statement is good enough for our purposes.

Lemma A.3 (Periodization error). *Let $z \in Q_L$, let (ζ, P) satisfy (A.2) at z, and let $Y \subset Q_L$ be a finite subset such that $\{z\} \cup Y$ satisfies (A.1). Then, we have for all $x \in Q_L$*

$$\Big(\int_{B^L_{1+\rho}(x)} \big|\mathrm{D}(\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_Y\zeta)\big|^2\Big)^{\frac12} \lesssim_{\sharp Y} \Big(\int_{B_{1+\rho}(z)} \big|\mathrm{D}(\zeta)\big|^2\Big)^{\frac12} \times \Big(\mathbb{1}_{|x-z|>\frac{L}{4}}\big|(x-z)_L\big|^{-d} + \mathbb{1}_{|x-z|\le\frac{L}{4}}\big|\mathrm{dist}(Y\setminus\{x,z\},\partial Q_L)\big|^{-d}\Big), \quad \text{(A.9)}$$

where we recall the notation $\mathrm{dist}(\varnothing,\partial Q_L) = L$ *and* $B^L_r(z) = (B_r(z) + L\mathbb{Z}^d) \cap Q_L$. *In addition,*

$$\Big(\int_{B^L_{1+\rho}(x)} \big|\mathrm{D}(\psi^Y_L - \psi^Y)\big|^2\Big)^{\frac12} \lesssim_{\sharp Y} \Big(\big|\mathrm{dist}(x,\partial Q_L)\big| + \big|\mathrm{dist}\big(Y\setminus\{x\},\partial Q_L\big)\big|\Big)^{-d}. \quad \text{(A.10)}$$

In the above three lemmas, the multiplicative constants in the estimates crucially depend on the finite number of rigid particles: in Lemma A.1, for instance, a quick inspection of the proof shows that the multiplicative constant can be bounded by $C^{\sharp Y}(\sharp Y)!^{3/2}$. Although these deterministic results fail in general for an unbounded number of rigid inclusions, we refer the reader to [19] where corresponding results are proved to hold in a suitable annealed sense in case of a stationary and ergodic random ensemble of rigid inclusions.

A.2 Decay of Stokeslets with rigid inclusions

This section is devoted to the proof of Lemma A.1 (hence of Lemmas 3.10 and 4.7). We argue by comparing $\mathcal{J}^z_{L;Y}\zeta$ to $\mathcal{J}^z_{L;Y_z}\zeta$ (recall $Y_z = \{y \in Y : B(y) \cap B^L(z) \neq \varnothing\}$), which is a variant of the solution $\mathcal{J}^z_L\zeta$ of the corresponding problem without rigid inclusions. Equation (A.3) for $\mathcal{J}^z_{L;Y_z}\zeta$ reads

$$\begin{cases} -\triangle\mathcal{J}^z_{L;Y_z}\zeta + \nabla\mathcal{Q}^z_{L;Y_z}\zeta = -\delta_{\partial B^L(z)}\sigma(\zeta,P)\nu, & \text{in } Q_L\setminus\bigcup_{y\in Y_z}B(y),\\ \mathrm{div}(\mathcal{J}^z_{L;Y_z}\zeta) = 0, & \text{in } Q_L\setminus\bigcup_{y\in Y_z}B(y),\\ \mathcal{J}^z_{L;Y_z}\zeta = V_z + \Theta_z(x-z), & \text{in } \bigcup_{y\in Y_z}B(y)\\ & \text{for some } V_z\in\mathbb{R}^d, \Theta_z\in\mathbb{M}^{\mathrm{skew}},\\ \sum_{y\in Y_z}\int_{\partial B(y)}\sigma(\mathcal{J}^z_{L;Y_z}\zeta,\mathcal{Q}^z_{L;Y_z}\zeta)\nu & \\ = \sum_{y\in Y_z}\int_{B(y)\cap\partial B^L(z)}\sigma(\zeta,P)\nu, & \\ \sum_{y\in Y_z}\int_{\partial B(y)}\Theta(x-z)\cdot\sigma(\mathcal{J}^z_{L;Y_z}\zeta,\mathcal{Q}^z_{L;Y_z}\zeta)\nu & \\ = \sum_{y\in Y_z}\int_{B(y)\cap\partial B^L(z)}\Theta(x-z)\cdot\sigma(\zeta,P)\nu, & \forall\Theta\in\mathbb{M}^{\mathrm{skew}}. \end{cases} \quad \text{(A.11)}$$

We split the proof into three steps: we first apply elliptic regularity to unravel the decay properties of $\mathcal{J}^z_{L;Y_z}\zeta$, and then estimate the difference $\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_{L;Y_z}\zeta$ in the last two steps. Let $z \in Q_L$, let ζ satisfy (A.2) at z, and let $Y \subset Q_L$ satisfy (A.1).

Step 1. Proof that for all $x \in Q_L$,

$$\left(\int_{B^L(x)} \left|\mathrm{D}(\mathcal{J}^z_{L;Y_z}\zeta)\right|^2\right)^{\frac{1}{2}} \lesssim \langle (x-z)_L\rangle^{-d}\left(\int_{B_{1+\rho}(z)} \left|\mathrm{D}(\zeta)\right|^2\right)^{\frac{1}{2}}. \tag{A.12}$$

The argument is based on elliptic regularity via a duality argument, in a form that is similar to the proof of Theorem 3.9 in Section 3.5.2. By an energy estimate for $\mathcal{J}^z_{L;Y_z}\zeta$, the claim (A.12) is trivial if

$$\left|(x-z)_L\right| \lesssim 1,$$

and we shall focus on the case when

$$r := \frac{1}{2}\left|(x-z)_L\right| > 2(1+\rho). \tag{A.13}$$

By definition (A.11), we then note that $\mathcal{J}^z_{L;Y_z}\zeta$ satisfies the free steady Stokes equation in $B^L_r(x) = (B_r(x) + L\mathbb{Z}^d) \cap Q_L$, which is the periodization of the ball $B_r(x)$ in Q_L. Elliptic regularity in form of Lemma 2.6 then yields

$$\int_{B^L(x)} \left|\mathrm{D}(\mathcal{J}^z_{L;Y_z}\zeta)\right|^2 \lesssim r^{-d} \int_{B^L_r(x)} \left|\mathrm{D}(\mathcal{J}^z_{L;Y_z}\zeta)\right|^2. \tag{A.14}$$

Next, by duality, the right-hand side can be written as

$$\int_{B^L_r(x)} \left|\mathrm{D}(\mathcal{J}^n_{L;Y_z}\zeta)\right|^2 = \sup\left\{\left(\int_{Q_L} h : \mathrm{D}(\mathcal{J}^z_{L;Y_z}\zeta)\right)^2 : h \in \mathrm{L}^2(Q_L)^{d\times d}_{\mathrm{sym}}, \right.$$
$$\left. \|h\|_{\mathrm{L}^2(Q_L)} = 1,\ \operatorname{supp} h \subset B^L_r(x)\right\}. \tag{A.15}$$

Given a test function $h \in \mathrm{L}^2(Q_L)^{d\times d}_{\mathrm{sym}}$ with $\operatorname{supp} h \subset B^L_r(x)$, let $w_{L;h} \in H^1_{\mathrm{per}}(Q_L)^d$ be the solution of the auxiliary Stokes problem

$$\begin{cases} -\triangle w_{L;h} + \nabla Q_{L;h} = \operatorname{div}(h), & \text{in } Q_L \setminus \bigcup_{y\in Y_z} B(y), \\ \operatorname{div}(w_{L;h}) = 0, & \text{in } Q_L \setminus \bigcup_{y\in Y_z} B(y), \\ w_{L;h} = V_z + \Theta_z(x-z), & \text{in } \bigcup_{y\in Y_z} B(y), \\ & \text{for some } V_z \in \mathbb{R}^d, \Theta_z \in \mathbb{M}^{\mathrm{skew}}, \\ \sum_{y\in Y_z} \int_{\partial B(y)} \sigma(w_{L;h}, Q_{L;h})\nu = 0, & \\ \sum_{y\in Y_z} \int_{\partial B(y)} \Theta(x-z)\cdot\sigma(w_{L;h}, Q_{L;h})\nu = 0, & \forall \Theta \in \mathbb{M}^{\mathrm{skew}}. \end{cases} \tag{A.16}$$

These equations are indeed well posed since by (A.13) the support $B_r^L(x)$ of the force term h does not intersect the rigid inclusions $\bigcup_{y\in Y_z} B(y)$. By Lemma 3.3, $w_{L;h}$ satisfies the following relation in Q_L,

$$-\triangle w_{L;h}+\nabla\big(\mathbb{1}_{Q_L\setminus\bigcup_{y\in Y_z} B(y)}\, Q_{L;h}\big)=\operatorname{div}(h)-\sum_{y\in Y_z}\delta_{\partial B(y)}\sigma(w_{L;h},Q_{L;h})\nu. \quad (A.17)$$

Similarly, the defining equation (A.11) for $\mathcal{J}^z_{L;Y_z}\zeta$ yields in Q_L,

$$\begin{aligned}&-\triangle\mathcal{J}^z_{L;Y_z}\zeta+\nabla\big(\mathbb{1}_{Q_L\setminus\bigcup_{y\in Y_z} B(y)}\mathcal{Q}^z_{L;Y_z}\big)\\&=-\mathbb{1}_{Q_L\setminus\bigcup_{y\in Y_z} B(y)}\delta_{\partial B^L(z)}\sigma(\zeta,P)\nu-\sum_{y\in Y_z}\delta_{\partial B(y)}\sigma(\mathcal{J}^z_{L;Y_z}\zeta,\mathcal{Q}^z_{L;Y_z}\zeta)\nu. \quad (A.18)\end{aligned}$$

Testing (A.17) with $\mathcal{J}^z_{L;Y_z}\zeta$ and (A.18) with $w_{L;h}$, we are led to

$$\begin{aligned}\int_{Q_L} h:\mathrm{D}(\mathcal{J}^z_{L;Y_z}\zeta)=&\int_{\partial B^L(z)\setminus\bigcup_{y\in Y_z} B(y)} w_{L;h}\cdot\sigma(\zeta,P)\nu\\&+\sum_{y\in Y_z}\int_{\partial B(y)} w_{L;h}\cdot\sigma(\mathcal{J}^z_{L;Y_z}\zeta,\mathcal{Q}^z_{L;Y_z}\zeta)\nu\\&+\sum_{y\in Y_z}\int_{\partial B(y)}\mathcal{J}^z_{L;Y_z}\zeta\cdot\sigma(w_{L;h},Q_{L;h})\nu,\end{aligned}$$

and thus, using the boundary conditions in (A.11) and (A.16),

$$\int_{Q_L} h:\mathrm{D}(\mathcal{J}^z_{L;Y_z}\zeta)=\int_{\partial B^L(z)} w_{L;h}\cdot\sigma(\zeta,P)\nu.$$

Recalling that (ζ,P) satisfies (A.2) and is implicitly extended by Q_L-periodicity, using the boundary conditions and the incompressibility constraints to smuggle in arbitrary constants in the different factors, as in the proof of (3.30), and appealing to the trace estimates of Lemma 2.5, we find

$$\bigg(\int_{Q_L} h:\mathrm{D}(\mathcal{J}^z_{L;Y_z}\zeta)\bigg)^2\lesssim\bigg(\int_{B^L(z)}\big|\mathrm{D}(w_{L;h})\big|^2\bigg)\bigg(\int_{B_{1+\rho}(z)}\big|\mathrm{D}(\zeta)\big|^2\bigg). \quad (A.19)$$

As equation (A.16) entails that $w_{L;h}$ satisfies the free steady Stokes equation in $B_r^L(z)$, elliptic regularity in form of Lemma 2.6 yields

$$\int_{B^L(z)}\big|\mathrm{D}(w_{L;h})\big|^2\lesssim r^{-d}\int_{Q_L}\big|\mathrm{D}(w_{L;h})\big|^2,$$

and thus, combining this with an energy estimate for (A.16),

$$\int_{B^L(z)}\big|\mathrm{D}(w_{L;h})\big|^2\lesssim r^{-d}\int_{Q_L}|h|^2.$$

Combining this with (A.14), (A.15), and (A.19), the claim (A.12) follows.

Step 2. Proof that for all $x \in Q_L$,

$$\int_{B^L(x)} \left|\mathrm{D}(\mathcal{J}^z_{L;Y}\zeta)\right|^2 \lesssim \Big(\sum_{y\in\{x\}\cup(Y\setminus Y_z)} \langle (y-z)_L\rangle^{-2d}\Big) \int_{B_{1+\rho}(z)} \left|\mathrm{D}(\zeta)\right|^2. \tag{A.20}$$

In view of Lemma 3.3, the defining equation (A.3) for $\mathcal{J}^z_{L;Y}\zeta$ yields in Q_L,

$$\begin{aligned} &-\triangle \mathcal{J}^z_{L;Y}\zeta + \nabla\big(\mathbb{1}_{Q_L\setminus\bigcup_{y\in Y} B(y)}\mathcal{Q}^z_{L;Y}\big) \\ &\quad = -\mathbb{1}_{Q_L\setminus\bigcup_{y\in Y_z} B(y)}\delta_{\partial B^L(z)}\sigma(\zeta,P)\nu - \sum_{y\in Y}\delta_{\partial B(y)}\sigma(\mathcal{J}^z_{L;Y}\zeta,\mathcal{Q}^z_{L;Y}\zeta)\nu. \end{aligned}$$

Subtracting (A.18) entails in Q_L

$$\begin{aligned} &-\triangle(\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_{L;Y_z}\zeta) + \nabla\big(\mathbb{1}_{Q_L\setminus\bigcup_{y\in Y} B(y)}\mathcal{Q}^z_{L;Y} - \mathbb{1}_{Q_L\setminus\bigcup_{y\in Y_z} B(y)}\mathcal{Q}^z_{L;Y_z}\big) \\ &\quad = -\sum_{y\in Y\setminus Y_z}\delta_{\partial B(y)}\sigma(\mathcal{J}^z_{L;Y}\zeta,\mathcal{Q}^z_{L;Y}\zeta)\nu \\ &\qquad -\sum_{y\in Y_z}\delta_{\partial B(y)}\big(\sigma(\mathcal{J}^z_{L;Y}\zeta,\mathcal{Q}^z_{L;Y}\zeta)\nu - \sigma(\mathcal{J}^z_{L;Y_z}\zeta,\mathcal{Q}^z_{L;Y_z}\zeta)\nu\big). \end{aligned} \tag{A.21}$$

Testing this equation with $\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_{L;Y_z}\zeta$ itself, and using the boundary conditions in (A.3) and (A.11), we obtain the energy identity

$$\begin{aligned} &2\int_{Q_L} \left|\mathrm{D}(\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_{L;Y_z}\zeta)\right|^2 \\ &\qquad = \sum_{y\in Y\setminus Y_z}\int_{\partial B(y)} \mathcal{J}^z_{L;Y_z}\zeta\cdot\sigma(\mathcal{J}^z_{L;Y}\zeta,\mathcal{Q}^z_{L;Y}\zeta)\nu. \end{aligned}$$

Further, using the boundary conditions and the incompressibility constraints to smuggle in arbitrary constants in the different factors, as in the proof of (3.30), and appealing to the trace estimates of Lemma 2.5, we deduce

$$\begin{aligned} &\int_{Q_L} \left|\mathrm{D}(\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_{L;Y_z}\zeta)\right|^2 \\ &\quad \lesssim \sum_{y\in Y\setminus Y_z}\Big(\int_{B(y)} \left|\mathrm{D}(\mathcal{J}^z_{L;Y_z}\zeta)\right|^2\Big)^{\frac12}\Big(\int_{B_{1+\rho}(y)} \left|\mathrm{D}(\mathcal{J}^z_{L;Y}\zeta)\right|^2\Big)^{\frac12}. \end{aligned}$$

Decomposing $\mathcal{J}^z_{L;Y}\zeta = (\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_{L;Y_z}\zeta) + \mathcal{J}^z_{L;Y_z}\zeta$ in the last factor, using the triangle inequality and Young's inequality, we are led to

$$\int_{Q_L} \left|\mathrm{D}(\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_{L;Y_z}\zeta)\right|^2 \lesssim \sum_{y\in Y\setminus Y_z}\int_{B_{1+\rho}(y)} \left|\mathrm{D}(\mathcal{J}^z_{L;Y_z}\zeta)\right|^2.$$

The triangle inequality then yields for all $x \in Q_L$,

$$\int_{B^L(x)} \left|\mathrm{D}(\mathcal{J}^z_{L;Y}\zeta)\right|^2 \lesssim \int_{B^L(x)} \left|\mathrm{D}(\mathcal{J}^z_{L;Y_z}\zeta)\right|^2 + \int_{Q_L} \left|\mathrm{D}(\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_{L;Y_z}\zeta)\right|^2$$
$$\lesssim \int_{B^L(x)} \left|\mathrm{D}(\mathcal{J}^z_{L;Y_z}\zeta)\right|^2 + \sum_{y\in Y\setminus Y_z} \int_{B_{1+\rho}(y)} \left|\mathrm{D}(\mathcal{J}^z_{L;Y_z}\zeta)\right|^2,$$

which yields the claim (A.20) in combination with (A.12).

Step 3. Conclusion. We argue by induction on the cardinality of $Y \setminus Y_z$ for (A.6). If $\sharp(Y \setminus Y_z) = 0$, that is, if $Y = Y_z$, the conclusion (A.6) already follows from (A.12). Given $n \geq 1$, we assume that (A.6) holds whenever $\sharp(Y \setminus Y_z) < n$, and we shall show that it also holds when $\sharp(Y \setminus Y_z) = n$. Let $Y \subset Q_L$ be fixed with $\sharp(Y \setminus Y_z) = n$. For any $S \subset Y \setminus Y_z$, the same argument as for (A.21) yields in $Q_L \setminus \bigcup_{y\in S} B(y)$

$$-\triangle(\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_{L;Y_z\cup S}\zeta) + \nabla\big(\mathbb{1}_{Q_L\setminus\bigcup_{y\in Y} B(y)}\mathcal{Q}^z_{L;Y} - \mathbb{1}_{Q_L\setminus\bigcup_{y\in Y_z\cup S} B(y)}\mathcal{Q}^z_{L;Y_z\cup S}\big)$$
$$= -\sum_{y\in Y\setminus(Y_z\cup S)} \delta_{\partial B(y)}\sigma(\mathcal{J}^z_{L;Y}\zeta, \mathcal{Q}^z_{L;Y}\zeta)\nu$$
$$-\sum_{y\in Y_z} \delta_{\partial B(y)}\sigma\big(\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_{L;Y_z\cup S}\zeta, \mathcal{Q}^z_{L;Y}\zeta - \mathcal{Q}^z_{L;Y_z\cup S}\zeta\big)\nu.$$

As $\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_{L;Y_z\cup S}\zeta$ is further rigid in $\bigcup_{y\in S} B(y)$, this implies, by definition of $\{\mathcal{J}^y_{L;S}\}_y$,

$$\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_{L;Y_z\cup S}\zeta = \sum_{y\in Y\setminus(Y_z\cup S)} \mathcal{J}^y_{L;S}\mathcal{J}^z_{L;Y}\zeta + \sum_{y\in Y_z} \mathcal{J}^y_{L;S}(\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_{L;Y_z\cup S}\zeta),$$

which we may further decompose as

$$\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_{L;Y_z\cup S}\zeta = \sum_{y\in Y\setminus(Y_z\cup S)} \mathcal{J}^y_{L;S}(\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_{L;Y_z\cup S\cup\{y\}}\zeta)$$
$$+ \sum_{y\in Y\setminus(Y_z\cup S)} \mathcal{J}^y_{L;S}\mathcal{J}^z_{L;Y_z\cup S\cup\{y\}}\zeta + \sum_{y\in Y_z} \mathcal{J}^y_{L;S}(\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_{L;Y_z\cup S}\zeta).$$

Iterating this identity, we find

$$\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_{L;Y_z}\zeta$$
$$= \sum_{l=1}^{n} \sum_{y_1,\dots,y_l\in Y\setminus Y_z}^{\neq} \mathcal{J}^{y_1}_L \mathcal{J}^{y_2}_{L;\{y_1\}} \cdots \mathcal{J}^{y_l}_{L;\{y_1,\dots,y_{l-1}\}} \mathcal{J}^z_{L;Y_z\cup\{y_1,\dots,y_l\}}\zeta$$
$$+ \sum_{l=1}^{n} \sum_{y_1,\dots,y_{l-1}\in Y\setminus Y_z}^{\neq} \sum_{y\in Y_z} \mathcal{J}^{y_1}_L \mathcal{J}^{y_2}_{L;\{y_1\}} \cdots \mathcal{J}^{y_{l-1}}_{L;\{y_1,\dots,y_{l-2}\}} \mathcal{J}^{y}_{L;\{y_1,\dots,y_{l-1}\}}$$
$$\times \big(\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_{L;Y_z\cup\{y_1,\dots,y_{l-1}\}}\zeta\big).$$

We now appeal to the induction hypothesis in form of (A.6) for the terms $\mathcal{J}^y_{L;\{y_1,\dots,y_j\}}$ and $\mathcal{J}^z_{L;Y_z\cup\{y_1,\dots,y_j\}}$ for all $1 \le j < n$ and $y \in Y$, to the suboptimal decay estimate (A.20) for $\mathcal{J}^z_{L;Y_z\cup\{y_1,\dots,y_n\}}$ (which only appears in the first right-hand sum when $l = n$). Recalling that

$$\big|(y-z)_L\big| \le 2 \quad \text{for all } y \in Y_z,$$

this yields for all $x \in Q_L$, after straightforward simplifications,

$$\begin{aligned}\Big(\int_{B^L(x)} \big|\mathrm{D}(\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_{L;Y_z}\zeta)\big|^2\Big)^{\frac12}&\\ \lesssim_n \Big(\int_{B_{1+\rho}(z)} \big|\mathrm{D}(\zeta)\big|^2\Big)^{\frac12}&\\ \times \sum_{l=0}^{n} \sum_{y_1,\dots,y_l\in Y\setminus Y_z}^{\neq} \big\langle (x-y_1)_L\big\rangle^{-d}\big\langle (y_1-y_2)_L\big\rangle^{-d}\cdots\big\langle (y_l-z)_L\big\rangle^{-d}.&\end{aligned}$$

The conclusion (A.6) now follows from the bound

$$\big\langle (a-b)_L\big\rangle^{-d}\big\langle (b-c)_L\big\rangle^{-d} \lesssim \big\langle (a-c)_L\big\rangle^{-d}$$

for all $a, b, c \in Q_L$. ∎

A.3 Mean-value property with rigid inclusions

This section is devoted to the proof of Lemma A.2. We split the proof into two steps. Let $Y \subset Q_L$ satisfy (A.1) and let $(w, P) \in H^1(Q_L)^d \times \mathrm{L}^2(Q_L)$ satisfy (A.7) in Q_L.

Step 1. Proof that for all $x \in Q_L$,

$$\int_{B^L(x)} \big|\mathrm{D}(w)\big|^2 \lesssim_{\sharp Y} \Big(\sum_{y\in\{x\}\cup Y} \big\langle \operatorname{dist}(y,\partial Q_L)\big\rangle^{-d}\Big) \int_{Q_L} \big|\mathrm{D}(w)\big|^2. \tag{A.22}$$

For that purpose, we shall compare w to the solution

$$\tilde w \in w + H^1_{\mathrm{per}}(Q_L)^d$$

of the free steady Stokes equations without rigid particles in Q_L,

$$-\triangle \tilde w + \nabla \tilde P = 0, \quad \operatorname{div}(\tilde w) = 0, \quad \text{in } Q_L. \tag{A.23}$$

In view of Lemma 3.3, the equations (A.7) for w yield the following relation in Q_L,

$$-\triangle w + \nabla(\mathbb{1}_{Q_L\setminus\bigcup_{y\in Y} B(y)} P) = -\sum_{y\in Y} \delta_{\partial B(y)}\sigma(w,P)\nu.$$

Subtracting (A.23), we deduce that the difference $w-\tilde{w}\in H^1_{\rm per}(Q_L)$ satisfies

$$-\triangle(w-\tilde{w})+\nabla\big(\mathbb{1}_{Q_L\setminus\bigcup_{y\in Y}B(y)}P-\widetilde{P}\big)=-\sum_{y\in Y}\delta_{\partial B(y)}\sigma(w,P)\nu. \tag{A.24}$$

Testing this equation with $w-\tilde{w}$ and using the boundary conditions in (A.7), we obtain the energy identity

$$2\int_{Q_L}\big|\mathrm{D}(w-\tilde{w})\big|^2=\sum_{y\in Y}\int_{\partial B(y)}\tilde{w}\cdot\sigma(w,\mathcal{Q})\nu.$$

Further, using the boundary conditions and the incompressibility constraints to smuggle in arbitrary constants in the different factors, as in the proof of (3.30), and appealing to the trace estimates of Lemma 2.5, we get

$$\int_{Q_L}\big|\mathrm{D}(w-\tilde{w})\big|^2\lesssim\sum_{y\in Y}\bigg(\int_{B(y)}\big|\mathrm{D}(\tilde{w})\big|^2\bigg)^{\frac12}\bigg(\int_{B_{1+\rho}(y)}\big|\mathrm{D}(w)\big|^2\bigg)^{\frac12}. \tag{A.25}$$

Decomposing

$$w=(w-\tilde{w})+\tilde{w}$$

in the last factor, using the triangle inequality and Young's inequality, we are led to

$$\int_{Q_L}\big|\mathrm{D}(w-\tilde{w})\big|^2\lesssim\sum_{y\in Y}\int_{B_{1+\rho}(y)}\big|\mathrm{D}(\tilde{w})\big|^2.$$

and thus, by the triangle inequality, for all $x\in Q_L$,

$$\int_{B^L(x)}\big|\mathrm{D}(w)\big|^2\lesssim\int_{B^L(x)}\big|\mathrm{D}(\tilde{w})\big|^2+\sum_{y\in Y}\int_{B_{1+\rho}(y)}\big|\mathrm{D}(\tilde{w})\big|^2. \tag{A.26}$$

Rather decomposing $\tilde{w}=w-(w-\tilde{w})$, we note that (A.25) also yields the energy estimate

$$\int_{Q_L}\big|\mathrm{D}(\tilde{w})\big|^2\lesssim\int_{Q_L}\big|\mathrm{D}(w)\big|^2. \tag{A.27}$$

As $\tilde{w}$ satisfies the free steady Stokes equations in Q_L, cf. (A.23), the mean-value property of Lemma 2.6 yields for all $x\in Q_L$,

$$\int_{B^L(x)}\big|\mathrm{D}(\tilde{w})\big|^2\lesssim\big\langle\operatorname{dist}(x,\partial Q_L)\big\rangle^{-d}\int_{Q_L}\big|\mathrm{D}(\tilde{w})\big|^2,$$

and thus, combined with (A.27),

$$\int_{B^L(x)}\big|\mathrm{D}(\tilde{w})\big|^2\lesssim\big\langle\operatorname{dist}(x,\partial Q_L)\big\rangle^{-d}\int_{Q_L}\big|\mathrm{D}(w)\big|^2. \tag{A.28}$$

Inserting this into (A.26), the claim (A.22) follows.

Step 2. Conclusion. Given $S \subset Y$, we denote by

$$w_S \in w + H^1_{\mathrm{per}}(Q_L)^d$$

the solution of the free steady Stokes problem with rigid inclusions at points of S only,

$$\begin{cases} -\triangle w_S + \nabla P_S = 0, & \text{in } Q_L \setminus \bigcup_{y\in S} B(y), \\ \operatorname{div}(w_S) = 0, & \text{in } Q_L \setminus \bigcup_{y\in S} B(y), \\ \mathrm{D}(w_S) = 0, & \text{in } \bigcup_{y\in S} B(y), \\ \int_{\partial B(y)} \sigma(w_S, P_S)\nu = 0, & \forall y \in S, \\ \int_{\partial B(y)} \Theta(x-y)\cdot\sigma(w_S, P_S)\nu = 0, & \forall y \in S,\ \forall \Theta \in \mathbb{M}^{\mathrm{skew}}. \end{cases}$$

In particular, we recover $w_Y = w$ and $w_\varnothing = \tilde{w}$ as defined in (A.23). The result (A.22) of Step 1 yields in this case, for all $x \in Q_L$,

$$\int_{B^L(x)} \left|\mathrm{D}(w_S)\right|^2 \lesssim_{\sharp S} \Big(\sum_{y\in\{x\}\cup S} \langle \operatorname{dist}(y, \partial Q_L)\rangle^{-d} \Big) \int_{Q_L} \left|\mathrm{D}(w_S)\right|^2.$$

Noting that a similar argument to the case of (A.27) further yields the energy estimate

$$\int_{Q_L} \left|\mathrm{D}(w_S)\right|^2 \lesssim \int_{Q_L} \left|\mathrm{D}(w)\right|^2,$$

we deduce for all $x \in Q_L$,

$$\int_{B^L(x)} \left|\mathrm{D}(w_S)\right|^2 \lesssim_{\sharp S} \Big(\sum_{y\in\{x\}\cup S} \langle \operatorname{dist}(y, \partial Q_L)\rangle^{-d} \Big) \int_{Q_L} \left|\mathrm{D}(w)\right|^2. \tag{A.29}$$

We shall now decompose w in terms of this sequence $(w_S)_{S\subset Y}$. Arguing as for (A.24), we note that for any $S \subset Y$ the following relation holds in $Q_L \setminus \bigcup_{y\in S} B(y)$,

$$-\triangle(w - w_S) + \nabla(P - P_S) = -\sum_{y\in Y\setminus S} \delta_{\partial B(y)}\sigma(w, P)\nu.$$

As $w - w_S$ is rigid in $\bigcup_{y\in S} B(y)$, this allows us to decompose

$$w - w_S = \sum_{y\in Y\setminus S} \mathcal{J}^y_{L;S} w,$$

and thus, iterating this identity and starting with $w_\varnothing = \tilde{w}$,

$$w = \tilde{w} + \sum_{l=1}^{\sharp Y} \sum_{y_1,\dots,y_l\in Y}^{\neq} \mathcal{J}^{y_1}_L \mathcal{J}^{y_2}_{L;\{y_1\}} \cdots \mathcal{J}^{y_l}_{L;\{y_1,\dots,y_{l-1}\}} w_{\{y_1,\dots,y_l\}}.$$

Appealing to the decay estimates for $\{\mathcal{J}^y_{L;S}\}_{y,S}$ in Lemma A.1, and to (A.28) and (A.29), we get after straightforward simplifications, for all $x \in Q_L$,

$$\int_{B^L(x)} \big|\mathrm{D}(w)\big|^2 \lesssim_{\sharp Y} \int_{Q_L} \big|\mathrm{D}(w)\big|^2$$
$$\times \sum_{l=0}^{\sharp Y} \sum_{y_1,\dots,y_l \in Y}^{\neq} \big\langle (x-y_1)_L \big\rangle^{-2d} \big\langle (y_1-y_2)_L \big\rangle^{-2d} \cdots \big\langle (y_{l-1}-y_l)_L \big\rangle^{-2d} \big\langle \mathrm{dist}(y_l, \partial Q_L) \big\rangle^{-d}.$$

Using that $\langle (a-b)_L \rangle^{-d} \langle (b-c)_L \rangle^{-d} \lesssim \langle (a-c)_L \rangle^{-d}$ for all $a, b, c \in Q_L$, and noting that the infimum over $c \in \partial Q_L$ further yields

$$\big\langle (a-b)_L \big\rangle^{-d} \big\langle \mathrm{dist}(b, \partial Q_L) \big\rangle^{-d} \lesssim \big\langle \mathrm{dist}(a, \partial Q_L) \big\rangle^{-d},$$

the conclusion (A.8) follows. ∎

A.4 Periodization errors

This section is devoted to the proof of Lemma A.3. We split the proof into three steps. Let $z \in Q_L$, let ζ satisfy (A.2) at z, and let $Y \subset Q_L$ be such that $\{z\} \cup Y$ satisfies (A.1).

Step 1. Proof that for all $x \in Q_L$,

$$\int_{B^L_{1+\rho}(x)} \big|\mathrm{D}(\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_Y\zeta)\big|^2 \lesssim_{\sharp Y} \int_{B_{1+\rho}(z)} \big|\mathrm{D}(\zeta)\big|^2$$
$$\times \min\Big\{ \big\langle (x-z)_L \big\rangle^{-2d} \wedge \Big(\big\langle \mathrm{dist}\big(x, \partial Q_L(a)\big) \big\rangle^{-d} \big\langle \mathrm{dist}\big(z, \partial Q_L(a)\big) \big\rangle^{-d} \Big) : a \in \mathbb{R}^d,\ x, z \in Q_L(a),\ Y \subset Q_L(a) \Big\}. \quad \text{(A.30)}$$

It suffices to prove this estimate for $a = 0$, that is,

$$\int_{B^L_{1+\rho}(x)} \big|\mathrm{D}(\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_Y\zeta)\big|^2 \lesssim_{\sharp Y} \int_{B_{1+\rho}(z)} \big|\mathrm{D}(\zeta)\big|^2$$
$$\times \Big(\big\langle (x-z)_L \big\rangle^{-2d} \wedge \big(\big\langle \mathrm{dist}(x, \partial Q_L) \big\rangle^{-d} \big\langle \mathrm{dist}(z, \partial Q_L) \big\rangle^{-d} \big) \Big),$$

as the claim (A.30) then follows by translating the underlying cell Q_L, which does indeed not change the equations provided that the translated cell still contains the relevant points x, z, Y. Further, noting that Lemma A.1 together with the triangle inequality yields

$$\int_{B^L_{1+\rho}(x)} \big|\mathrm{D}(\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_Y\zeta)\big|^2 \lesssim_{\sharp Y} \big\langle (x-z)_L \big\rangle^{-2d} \int_{B_{1+\rho}(z)} \big|\mathrm{D}(\zeta)\big|^2,$$

it only remains to prove for all $x \in Q_L$,

$$\int_{B^L_{1+\rho}(x)} \big|\mathrm{D}(\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_Y\zeta)\big|^2 \lesssim_{\sharp Y} \big\langle \mathrm{dist}(x,\partial Q_L)\big\rangle^{-d} \big\langle \mathrm{dist}(z,\partial Q_L)\big\rangle^{-d} \int_{B_{1+\rho}(z)} \big|\mathrm{D}(\zeta)\big|^2. \tag{A.31}$$

As $\{z\} \cup Y$ satisfies (A.1), we recall that $\mathcal{J}^z_{L;Y}\zeta$ satisfies the simpler Stokes problem (A.4) (and likewise for $\mathcal{J}^z_Y\zeta$). The difference $\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_Y\zeta$ then satisfies the free steady Stokes equations (A.7). Applying the mean-value property of Lemma A.2 to this equation, we get for all $x \in Q_L$,

$$\int_{B^L_{1+\rho}(x)} \big|\mathrm{D}(\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_Y\zeta)\big|^2 \lesssim_{\sharp Y} \big\langle \mathrm{dist}(x,\partial Q_L)\big\rangle^{-d} \int_{Q_L} \big|\mathrm{D}(\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_Y\zeta)\big|^2. \tag{A.32}$$

In order to estimate the last integral, taking some inspiration from the proof of (2.33), we note that it is convenient to further compare $\mathcal{J}^z_{L;Y}\zeta$ and $\mathcal{J}^z_Y\zeta$ to the solution of the corresponding Neumann problem in Q_L: we define

$$\mathcal{J}^z_{N;Y}\zeta \in H^1(Q_L)^d$$

as the solution of

$$\begin{cases} -\triangle \mathcal{J}^z_{N;Y}\zeta + \nabla \mathcal{Q}^z_{N;Y}\zeta = -\delta_{\partial B^L(z)}\sigma(\zeta,P)\nu, & \text{in } Q_L \setminus \bigcup_{y\in Y} B(y), \\ \mathrm{div}(\mathcal{J}^z_{N;Y}\zeta) = 0, & \text{in } Q_L \setminus \bigcup_{y\in Y} B(y), \\ \sigma(\mathcal{J}^z_{N;Y}\zeta, \mathcal{Q}^z_{N;Y}\zeta)\nu = 0, & \text{on } \partial Q_L, \\ \mathrm{D}(\mathcal{J}^z_{N;Y}\zeta) = 0, & \text{in } \bigcup_{y\in Y} B(y), \\ \int_{\partial B(y)} \sigma(\mathcal{J}^z_{N;Y}\zeta, \mathcal{Q}^z_{N;Y}\zeta)\nu = 0, & \forall y \in Y, \\ \int_{\partial B(y)} \Theta(x-y)\cdot\sigma(\mathcal{J}^z_{N;Y}\zeta, \mathcal{Q}^z_{N;Y}\zeta)\nu = 0, & \forall y \in Y,\ \forall \Theta \in \mathbb{M}^{\mathrm{skew}}. \end{cases} \tag{A.33}$$

In these terms, we start by estimating

$$\int_{Q_L} \big|\mathrm{D}(\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_Y\zeta)\big|^2 \le 2\int_{Q_L} \big|\mathrm{D}(H_1)\big|^2 + 2\int_{Q_L} \big|\mathrm{D}(H_2)\big|^2, \tag{A.34}$$

where we have set for abbreviation

$$H_1 := \mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_{N;Y}\zeta,$$
$$H_2 := \mathcal{J}^z_Y\zeta - \mathcal{J}^z_{N;Y}\zeta.$$

We denote by P_1, P_2 the corresponding pressure differences. In view of (A.4) and (A.33), (H_1, P_1) satisfies

$$\begin{cases} -\triangle H_1 + \nabla P_1 = 0, & \text{in } Q_L \setminus \bigcup_{y\in Y} B(y), \\ \operatorname{div}(H_1) = 0, & \text{in } Q_L \setminus \bigcup_{y\in Y} B(y), \\ \sigma(H_1, P_1)\nu = \sigma(\mathcal{J}^z_{L;Y}\zeta, \mathcal{Q}^z_{L;Y}\zeta)\nu, & \text{on } \partial Q_L, \\ \mathrm{D}(H_1) = 0, & \text{in } \bigcup_{y\in Y} B(y), \\ \int_{\partial B(y)} \sigma(H_1, P_1)\nu = 0, & \forall y \in Y, \\ \int_{\partial B(y)} \Theta(x-y)\cdot\sigma(H_1, P_1)\nu = 0, & \forall y \in Y,\ \forall \Theta \in \mathbb{M}^{\mathrm{skew}}, \end{cases} \tag{A.35}$$

for which the energy identity takes the form

$$2\int_{Q_L} \left|\mathrm{D}(H_1)\right|^2 = \int_{\partial Q_L} H_1 \cdot \sigma(\mathcal{J}^z_{L;Y}\zeta, \mathcal{Q}^z_{L;Y}\zeta)\nu,$$

hence, recalling $H_1 = \mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_{N;Y}\zeta$ and the periodicity of $\mathcal{J}^z_{L;Y}\zeta$,

$$2\int_{Q_L} \left|\mathrm{D}(H_1)\right|^2 = -\int_{\partial Q_L} \mathcal{J}^z_{N;Y}\zeta \cdot \sigma(\mathcal{J}^z_{L;Y}\zeta, \mathcal{Q}^z_{L;Y}\zeta)\nu. \tag{A.36}$$

By Lemma 3.3 and (A.11), $\mathcal{J}^z_{L;Y}\zeta$ satisfies in Q_L

$$\begin{aligned} &-\triangle \mathcal{J}^z_{L;Y}\zeta + \nabla\big(\mathbb{1}_{\mathbb{R}^d\setminus\bigcup_{y\in Y} B(y)}\mathcal{Q}^z_{L;Y}\big) \\ &\quad = -\delta_{\partial B(z)}\,\sigma(\zeta, P)\nu - \sum_{y\in Y} \delta_{\partial B(y)}\sigma(\mathcal{J}^z_{L;Y}\zeta, \mathcal{Q}^z_{L;Y}\zeta)\nu, \end{aligned}$$

whereas, by (A.33), $\mathcal{J}^z_{N;Y}\zeta$ satisfies

$$\begin{aligned} &-\triangle \mathcal{J}^z_{N;Y}\zeta + \nabla\big(\mathbb{1}_{\mathbb{R}^d\setminus\bigcup_{y\in Y} B(y)}\mathcal{Q}^z_{N;Y}\big) \\ &\quad = -\delta_{\partial B(z)}\sigma(\zeta, P)\nu - \sum_{y\in Y} \delta_{\partial B(y)}\sigma(\mathcal{J}^z_{N;Y}\zeta, \mathcal{Q}^z_{N;Y}\zeta)\nu. \end{aligned}$$

Testing the first relation with $\mathcal{J}^z_{N;Y}\zeta$, testing the second one with $\mathcal{J}^z_{L;Y}\zeta$, and using boundary conditions, we find

$$\begin{aligned} &\int_{\partial Q_L} \mathcal{J}^z_{N;Y}\zeta \cdot \sigma(\mathcal{J}^z_{L;Y}\zeta, \mathcal{Q}^z_{L;Y}\zeta)\nu \\ &\quad = 2\int \mathrm{D}(\mathcal{J}^z_{N;Y}\zeta) : \mathrm{D}(\mathcal{J}^z_{L;Y}\zeta) + \int_{\partial B(z)} \mathcal{J}^z_{N;Y}\zeta \cdot \sigma(\zeta, P)\nu \\ &\quad = \int_{\partial B(z)} (\mathcal{J}^z_{N;Y}\zeta - \mathcal{J}^z_{L;Y}\zeta)\cdot\sigma(\zeta, P)\nu, \end{aligned}$$

so that identity (A.36) becomes

$$2\int_{Q_L}\left|\mathrm{D}(H_1)\right|^2=\int_{\partial B(z)}H_1\cdot\sigma(\zeta,P)\nu. \tag{A.37}$$

Using the boundary conditions and the incompressibility constraint to smuggle in arbitrary constants in the different factors, as in the proof of (3.30), and appealing to the trace estimates of Lemma 2.5, we find

$$\int_{Q_L}\left|\mathrm{D}(H_1)\right|^2\lesssim\left(\int_{B(z)}\left|\mathrm{D}(H_1)\right|^2\right)^{\frac12}\left(\int_{B_{1+\rho}(z)}\left|\mathrm{D}(\zeta)\right|^2\right)^{\frac12}.$$

Applying the mean-value property of Lemma A.2 to equation (A.35) for H_1, and using Young's inequality, we deduce

$$\int_{Q_L}\left|\mathrm{D}(H_1)\right|^2\lesssim\left\langle\mathrm{dist}(z,\partial Q_L)\right\rangle^{-d}\int_{B_{1+\rho}(z)}\left|\mathrm{D}(\zeta)\right|^2. \tag{A.38}$$

Likewise, repeating the argument in favor of (A.37), this time for H_2, we obtain

$$2\int_{Q_L}\left|\mathrm{D}(H_2)\right|^2=\int_{\partial B(z)}H_2\cdot\sigma(\zeta,P)\nu+\int_{\partial Q_L}\mathcal{J}_Y^z\zeta\cdot\sigma(\mathcal{J}_Y^z\zeta,\mathcal{Q}_Y^z\zeta)\nu,$$

or equivalently, using the free steady Stokes equations for $\mathcal{J}_Y^z\zeta$ in $\mathbb{R}^d\setminus Q_L$ and integrating by parts to reformulate the second right-hand side term,

$$\begin{aligned}2\int_{Q_L}\left|\mathrm{D}(H_2)\right|^2&=\int_{\partial B(z)}H_2\cdot\sigma(\zeta,P)\nu-2\int_{\mathbb{R}^d\setminus Q_L}\left|\mathrm{D}(\mathcal{J}_Y^z\zeta)\right|^2\\&\le\int_{\partial B(z)}H_2\cdot\sigma(\zeta,P)\nu.\end{aligned}$$

Arguing as for H_1, we may then deduce

$$\int_{Q_L}\left|\mathrm{D}(H_2)\right|^2\lesssim\left\langle\mathrm{dist}(z,\partial Q_L)\right\rangle^{-d}\int_{B_{1+\rho}(z)}\left|\mathrm{D}(\zeta)\right|^2.$$

Combined with (A.32), (A.34), and (A.38), this yields the claim (A.31).

Step 2. Proof of (A.9). We claim that the conclusion (A.9) is a simple post-processing of (A.30). As (A.9) trivially follows from (A.30) if $|x-z|>\frac{L}{4}$, it remains to consider the case when $|x-z|\le\frac{L}{4}$. In that case, we can choose $q\in\frac{L}{4}\mathbb{Z}^d$ with $|q|_\infty\le\frac{L}{4}$ such that $x,z\in Q_{\frac12 L}(q)$. We then construct a translation vector a componentwise: First, for all directions $1\le i\le d$ with $q_i=0$, we set $a_i:=0$. Second, for all i with $q_i=\frac{L}{4}$, we set $a_i:=\mathrm{dist}(Y\setminus\{x,z\},P_L^{i,-})$, where $P_L^{i,-}$ is the cubic facet $\{v\in\partial Q_L:v_i=-\frac{L}{2}\}$. Third, for all i with $q_i=-\frac{L}{4}$, we set $a_i:=-\mathrm{dist}(Y\setminus\{x,z\},P_L^{i,+})$, where $P_L^{i,+}$ is the facet $\{v\in\partial Q_L:v_i=\frac{L}{2}\}$. With this construction of a, we find

that $Y \setminus \{x, z\}$ is included in the translated cube $Q_L(a)$ (and actually intersects its boundary). Moreover, we find

$$\begin{aligned}\operatorname{dist}\big(x, \partial Q_L(a)\big) &\geq \operatorname{dist}(x, \partial Q_L) + \inf_i |a_i| \\ &\geq \operatorname{dist}(x, \partial Q_L) + \operatorname{dist}\big(Y \setminus \{x, z\}, \partial Q_L\big),\end{aligned}$$

and similarly

$$\operatorname{dist}\big(z, \partial Q_L(a)\big) \geq \operatorname{dist}(z, \partial Q_L) + \operatorname{dist}\big(Y \setminus \{x, z\}, \partial Q_L\big).$$

In particular, we get

$$\big\langle \operatorname{dist}\big(x, \partial Q_L(a)\big)\big\rangle^{-d} \big\langle \operatorname{dist}\big(z, \partial Q_L(a)\big)\big\rangle^{-d} \leq \big\langle \operatorname{dist}\big(Y \setminus \{x, z\}, \partial Q_L\big)\big\rangle^{-2d},$$

so that the conclusion (A.9) indeed follows from (A.30).

Step 3. Proof of (A.10). We shall prove the following refined version of (A.10): for all $x \in Q_L$,

$$\begin{aligned}&\int_{B^L_{1+\rho}(x)} \big|\mathrm{D}(\psi^Y_L - \psi^Y)\big|^2 \\ &\quad\lesssim \min\Big\{\big\langle \operatorname{dist}\big(x, \partial Q_L(a)\big)\big\rangle^{-d} \big\langle \operatorname{dist}\big(Y, \partial Q_L(a)\big)\big\rangle^{-d} : \\ &\qquad\qquad a \in \mathbb{R}^d,\ x \in Q_L(a),\ Y \subset Q_L(a)\Big\}. \end{aligned} \tag{A.39}$$

Arguing similarly to Step 2, it is easily seen that the translation a can be suitably chosen so that this estimate yields the conclusion (A.10). In order to prove (A.39), it suffices, in fact, to prove it for $a = 0$, that is,

$$\int_{B^L_{1+\rho}(x)} \big|\mathrm{D}(\psi^Y_L - \psi^Y)\big|^2 \lesssim \big\langle \operatorname{dist}(x, \partial Q_L)\big\rangle^{-d} \big\langle \operatorname{dist}(Y, \partial Q_L)\big\rangle^{-d}, \tag{A.40}$$

as the claim (A.39) then follows by translating the underlying cell Q_L, which does indeed not change the equations provided that the translated cell still contains x, Y.

We turn to the proof of (A.40). As the difference

$$\psi^Y_L - \psi^Y$$

satisfies a free steady Stokes problem of the form (A.7), we may apply the mean-value property of Lemma A.2 to the effect that for all $x \in Q_L$,

$$\int_{B^L_{1+\rho}(x)} \big|\mathrm{D}(\psi^Y_L - \psi^Y)\big|^2 \lesssim \big\langle \operatorname{dist}(x, \partial Q_L)\big\rangle^{-d} \int_{Q_L} \big|\mathrm{D}(\psi^Y_L - \psi^Y)\big|^2. \tag{A.41}$$

In order to estimate the last integral, we argue similarly to Step 1 by further comparing ψ^Y_L, ψ^Y to the solution of the corresponding Neumann problem in Q_L: we define

$\psi_N^Y \in H^1(Q_L)^d$ as the solution of

$$\begin{cases} -\triangle \psi_N^Y + \nabla \Sigma_N^Y = 0, & \text{in } Q_L \setminus \bigcup_{y\in Y} B(y), \\ \operatorname{div}(\psi_N^Y) = 0, & \text{in } Q_L \setminus \bigcup_{y\in Y} B(y), \\ \sigma(\psi_N^Y, \Sigma_N^Y)\nu = 0, & \text{on } \partial Q_L, \\ \mathrm{D}(\psi_N^Y + Ex) = 0, & \text{in } \bigcup_{y\in Y} B(y), \\ \int_{\partial B(y)} \sigma(\psi_N^Y, \Sigma_N^Y)\nu = 0, & \forall y \in Y, \\ \int_{\partial B(y)} \Theta(x-y)\cdot\sigma(\psi_N^Y, \Sigma_N^Y)\nu = 0, & \forall y \in Y,\ \forall \Theta \in \mathbb{M}^{\mathrm{skew}}. \end{cases}$$

In these terms, we start by estimating

$$\int_{Q_L} \left|\mathrm{D}(\psi_L^Y - \psi^Y)\right|^2 \le 2\int_{Q_L} \left|\mathrm{D}(G_1)\right|^2 + 2\int_{Q_L} \left|\mathrm{D}(G_2)\right|^2, \tag{A.42}$$

where we have set for abbreviation

$$G_1 := \psi_L^Y - \psi_N^Y, \quad G_2 := \psi^Y - \psi_N^Y.$$

We denote by R_1, R_2 the corresponding pressure differences. Similarly, as in Step 1, energy identities take the form

$$2\int_{Q_L} \left|\mathrm{D}(G_1)\right|^2 = \sum_{y\in Y} \int_{\partial B(y)} E(x-y)\cdot\sigma(G_1, R_1)\nu,$$
$$2\int_{Q_L} \left|\mathrm{D}(G_2)\right|^2 = \sum_{y\in Y} \int_{\partial B(y)} E(x-y)\cdot\sigma(G_2, R_2)\nu - 2\int_{\mathbb{R}^d\setminus Q_L} \left|\mathrm{D}(\psi^Y)\right|^2,$$

and we deduce by means of trace estimates, for both $i = 1, 2$,

$$\int_{Q_L} \left|\mathrm{D}(G_i)\right|^2 \lesssim \sum_{y\in Y} \left(\int_{B_{1+\rho}(y)} \left|\mathrm{D}(G_i)\right|^2 \right)^{\frac{1}{2}}.$$

Hence, applying the mean-value property of Lemma A.2 to G_1, G_2, together with Young's inequality,

$$\int_{Q_L} \left|\mathrm{D}(G_i)\right|^2 \lesssim_{\sharp Y} \sum_{y\in Y} \left\langle \operatorname{dist}(y, \partial Q_L)\right\rangle^{-d}.$$

Combined with (A.41) and (A.42), this yields the claim (A.40), and concludes the proof. ∎

Appendix B

Finite-volume approximation of the effective viscosity

This appendix is devoted to the proof of an algebraic convergence rate for the finite-volume approximation $\overline{\mathbf{B}}_L$ of the effective viscosity $\overline{\mathbf{B}}$ under an algebraic α-mixing condition, as announced in Remark 4.2.

Proposition B.1 (Convergence rate for $\overline{\mathbf{B}}_L$). *On top of* (H_ρ)*, assume that the algebraic mixing condition* (Mix) *holds. Then there exists* $\gamma \in (0, \beta)$ *(only depending on* d, ρ *and on the mixing exponent* β*) such that for all* L*,*

$$|\overline{\mathbf{B}}_L - \overline{\mathbf{B}}| \lesssim L^{-\gamma}.$$

The proof displayed below closely follows the monograph [3] by Armstrong, Kuusi, and Mourrat (albeit in the more general version [4] for α-mixing coefficients), based on the original argument [5] by Armstrong and Smart. We identify a suitable subadditive quantity J that satisfies all the requirements of [3,4] in the present Stokes context: the definition (B.4) and Lemma B.2 below constitute the only new insight w.r.t. [3], and the conclusion follows from elementary adaptations of the arguments in [3,4]. Although we could have used the same subadditive quantity as in [3], we have chosen to use a subadditive quantity J built on the approximations (2.9) and (2.10) that we used to prove Einstein's formula, that is, in the form of (B.3) below. This choice, which is specific to our problem, makes some of the upcoming arguments technically simpler than in [3], in particular avoiding the use of convex duality.

Let $E \in \mathbb{M}_0$ be fixed with $|E| = 1$. We say that a bounded domain $U \subset \mathbb{R}^d$ is *suitable* if

$$\operatorname{dist}(\mathcal{I} \cap U, \partial U) > \rho.$$

Consider the following weakly closed subsets of $H^1(U)^d$,

$$\mathcal{H}(U) := \big\{u \in H^1(U)^d : \operatorname{div}(\phi) = 0, \text{ and } \mathrm{D}(\phi + Ex) = 0 \text{ on } \mathcal{I} \cap U\big\},$$
$$\mathcal{H}_\circ(U) := H^1_0(U)^d \cap \mathcal{H}(U),$$

and the following minimization problems (note that $\psi_*(U)$ is only defined up to a rigid motion),

$$\psi_*(U) := \arg\min \left\{ \int_U \big|\mathrm{D}(\phi)\big|^2 : \phi \in \mathcal{H}(U) \right\}, \tag{B.1}$$

$$\psi_\circ(U) := \arg\min \left\{ \int_U \big|\mathrm{D}(\phi)\big|^2 : \phi \in \mathcal{H}_0(U) \right\}. \tag{B.2}$$

Recalling that the fattened inclusions $\{I_n + \rho B\}_n$ are disjoint, we define the modified cubes

$$U_L(x) := \Big(Q_L(x) \setminus \bigcup_{n:x_n \notin Q_L(x)} (I_n + \rho B)\Big) \cup \Big(\bigcup_{n:x_n \in Q_L(x)} (I_n + \rho B)\Big),$$

which satisfy by definition $Q_{L-2(1+\rho)} \subset U_L(x) \subset Q_{L+2(1+\rho)}$ and $\mathcal{I} \cap \partial U_L(x) = \varnothing$. The family $\{U_L(x)\}_{x \in L\mathbb{Z}^d}$ constitutes a partition of $\mathbb{R}^d$. Setting $U_L = U_L(0)$, we then consider the following alternative finite-volume approximations of the effective viscosity $\overline{\mathbf{B}}$,

$$\begin{aligned} E : \widetilde{\mathbf{B}}_{L,*} E &= 1 + \mathbb{E}\Big[\fint_{U_L} \big|\mathrm{D}\big(\psi_*(U_L)\big)\big|^2 \Big], \\ E : \widetilde{\mathbf{B}}_{L,\circ} E &= 1 + \mathbb{E}\Big[\fint_{U_L} \big|\mathrm{D}\big(\psi_\circ(U_L)\big)\big|^2 \Big]. \end{aligned} \tag{B.3}$$

Since $\mathcal{H}_\circ(U_L) \subset \mathcal{H}(U_L)$, we have $E : \widetilde{\mathbf{B}}_{L,*} E \le E : \widetilde{\mathbf{B}}_{L,\circ} E$. We then define a random set function J for suitable sets U via

$$J(U) := \fint_U \big|\mathrm{D}\big(\psi_\circ(U)\big)\big|^2 - \big|\mathrm{D}\big(\psi_*(U)\big)\big|^2. \tag{B.4}$$

The following lemma collects elementary properties of J. In particular, item (iii) states that $U \mapsto |U| J(U)$ is subadditive.

Lemma B.2 (Properties of J). (i) *Recalling the definition* (B.3) *of finite-volume approximations* $\widetilde{\mathbf{B}}_{L,*}, \widetilde{\mathbf{B}}_{L,\circ}$ *of the effective viscosity, there exists* $C > 0$ *such that*

$$E : \widetilde{\mathbf{B}}_{L,*} E - CL^{-1} \le E : \overline{\mathbf{B}} E \le E : \widetilde{\mathbf{B}}_{L,\circ} E + CL^{-1}, \tag{B.5}$$

$$E : \widetilde{\mathbf{B}}_{L,*} E - CL^{-1} \le E : \overline{\mathbf{B}}_{L+2(1+\rho)} E \le E : \widetilde{\mathbf{B}}_{L,\circ} E + CL^{-1}. \tag{B.6}$$

(ii) *For all suitable* U,

$$J(U) = \fint_U \big|\mathrm{D}\big(\psi_\circ(U) - \psi_*(U)\big)\big|^2. \tag{B.7}$$

(iii) *For all disjoint suitable sets* $U^1, \dots, U^k$, *setting* $U = \mathrm{int}(\bigcup_j \overline{U^j})$,

$$|U| J(U) \le \sum_{j=1}^k |U^j| J(U^j). \tag{B.8}$$

In addition, setting $\delta\psi(U) := \psi_\circ(U) - \psi_*(U)$,

$$\sum_{j=1}^k \big\|\mathrm{D}\big(\delta\psi(U) - \delta\psi(U^j)\big)\big\|^2_{\mathrm{L}^2(U^j)} = \sum_{j=1}^k |U^j| \big(J(U^j) - J(U)\big). \tag{B.9}$$

Proof. We split the proof into three steps.

Step 1. Proof of (i). We start with the proof of (B.6), that is, the comparison of $\widetilde{\mathbf{B}}_{L,*}$, $\widetilde{\mathbf{B}}_{L,\circ}$ with the periodic approximation $\overline{\mathbf{B}}_{L+2(1+\rho)}$. First, we extend $\psi_\circ(U_L)$ by zero on $Q_{L+2(1+\rho)} \setminus U_L^-$, which makes it a $Q_{L+2(1+\rho)}$-periodic function, and thus, testing variational problems,

$$\fint_{Q_{L+2(1+\rho)}} \big|\mathrm{D}(\psi_{L+2(1+\rho)})\big|^2 \le \fint_{Q_{L+2(1+\rho)}} \big|\mathrm{D}\big(\psi_\circ(U_L)\big)\big|^2 = \frac{|U_L|}{\big(L+2(1+\rho)\big)^d} \fint_{U_L} \big|\mathrm{D}\big(\psi_\circ(U_L)\big)\big|^2,$$

which yields, in view of $|L^{-d}|U_L| - 1| \lesssim L^{-1}$,

$$\fint_{Q_{L+2(1+\rho)}} \big|\mathrm{D}(\psi_{L+2(1+\rho)})\big|^2 \le (1 + CL^{-1}) \fint_{U_L} \big|\mathrm{D}\big(\psi_\circ(U_L)\big)\big|^2.$$

Second, as the restriction $\psi_{L+2(1+\rho)}|_{U_L}$ belongs to $\mathcal{H}(U_L)$ and is thus an admissible test function in (B.1), we similarly obtain

$$\begin{aligned}\fint_{U_L} \big|\mathrm{D}\big(\psi_*(U_L)\big)\big|^2 &\le \fint_{U_L} \big|\mathrm{D}(\psi_{L+2(1+\rho)})\big|^2 \\ &\le \frac{\big(L+2(1+\rho)\big)^d}{|U_L|} \fint_{Q_{L+2(1+\rho)}} \big|\mathrm{D}(\psi_{L+2(1+\rho)})\big|^2 \\ &\le (1+CL^{-1}) \fint_{Q_{L+2(1+\rho)}} \big|\mathrm{D}(\psi_{L+2(1+\rho)})\big|^2.\end{aligned}$$

The claim (B.6) follows from the combination of these two estimates with the following energy bounds, cf. (3.45),

$$\mathbb{E}\bigg[\fint_{Q_{L+2(1+\rho)}} \big|\mathrm{D}(\psi_{L+2(1+\rho)})\big|^2\bigg] + \mathbb{E}\bigg[\fint_{U_L} \big|\mathrm{D}\big(\psi_\circ(U_L)\big)\big|^2\bigg] + \mathbb{E}\big[\big|\mathrm{D}(\psi)\big|^2\big] \lesssim \lambda(\mathcal{P}). \tag{B.10}$$

We turn to the proof of (B.5). Since the restriction $\psi|_{U_L}$ belongs to $\mathcal{H}(U_L)$ and is thus an admissible test function in (B.1), we find by stationarity of $\mathrm{D}(\psi)$,

$$\begin{aligned}\mathbb{E}\bigg[\fint_{U_L} \big|\mathrm{D}\big(\psi_*(U_L)\big)\big|^2\bigg] &\le \mathbb{E}\bigg[\fint_{U_L} \big|\mathrm{D}(\psi)\big|^2\bigg] \le \mathbb{E}\bigg[\frac{L^d}{|U_L|} \fint_{Q_L} \big|\mathrm{D}(\psi)\big|^2\bigg] \\ &\le (1+CL^{-1})\mathbb{E}\big[\big|\mathrm{D}(\psi)\big|^2\big].\end{aligned}$$

For the converse inequality, we appeal to a cut-and-paste argument. The starting point is the following convergence, cf. [18],

$$\mathbb{E}\big[\big|\mathrm{D}(\psi)\big|^2\big] = \lim_{k\uparrow\infty} \mathbb{E}\bigg[\fint_{U_{kL}} \big|\mathrm{D}\big(\psi_\circ(U_{kL})\big)\big|^2\bigg].$$

Since $\widetilde{\psi}_\circ(U_{kL}) := \sum_j \psi_\circ(U_L(z_j))\mathbb{1}_{U_L(z_j)}$ belongs to $\mathcal{H}_0(U_{kL})$, where $\{U_L(z_j)\}_j$ is a partition of U_{kL}, we obtain for all k, by stationarity of $z \mapsto U_L(z)$,

$$\mathbb{E}\Big[\fint_{U_{kL}} \big|\mathrm{D}\big(\psi_\circ(U_{kL})\big)\big|^2 \Big] \le \sum_j \mathbb{E}\Big[\frac{|U_L(z_j)|}{|U_{kL}|} \fint_{U_L(z_j)} \big|\mathrm{D}\big(\psi_\circ\big(U_L(z_j)\big)\big)\big|^2 \Big]$$
$$\le (1 + CL^{-1})\mathbb{E}\Big[\fint_{U_L} \big|\mathrm{D}\big(\psi_\circ(U_L)\big)\big|^2 \Big].$$

The claim (B.5) follows from the combination of these three properties with the above energy bounds (B.10).

Step 2. Proof of (ii). By definition,

$$J(U) = \fint_U \mathrm{D}\big(\psi_\circ(U) - \psi_*(U)\big) : \mathrm{D}\big(\psi_\circ(U) + \psi_*(U)\big).$$

Since $\psi_\circ(U), \psi_*(U) \in \mathcal{H}(U)$, the difference $\psi_\circ(U) - \psi_*(U)$ is a suitable test function for the Euler–Lagrange equation of the minimization problem (B.1) defining $\psi_*(U)$, which yields

$$\int_U \mathrm{D}\big(\psi_\circ(U) - \psi_*(U)\big) : \mathrm{D}\big(\psi_*(U)\big) = 0,$$

and the claim (B.7) follows.

Step 3. Proof of (iii). We start with the proof of (B.8). Since the minimization problem (B.2) defines a subadditive set function due to the gluing property of $\mathcal{H}_0(U)$, and since the minimization problem (B.1) defines a superadditive function due to the restriction property of $\mathcal{H}(U)$, the function J is subadditive as the difference of a subadditive and of a superadditive function.

We turn to the proof of (B.9). The starting point is (B.7) for U^j, which yields

$$\begin{aligned} |U^j| J(U^j) - \int_{U^j} \big|\mathrm{D}\big(\delta\psi(U)\big)\big|^2 &= \int_{U^j} \mathrm{D}\big(\delta\psi(U^j) - \delta\psi(U)\big) : \mathrm{D}\big(\delta\psi(U^j) + \delta\psi(U)\big) \\ &= \int_{U^j} \big|\mathrm{D}\big(\delta\psi(U^j) - \delta\psi(U)\big)\big|^2 \\ &\quad + 2\int_{U^j} \mathrm{D}\big(\delta\psi(U^j) - \delta\psi(U)\big) : \mathrm{D}\big(\delta\psi(U)\big). \end{aligned} \tag{B.11}$$

We decompose the second right-hand side term into $2\sum_{k=1}^4 I_{k,j}$, in terms of

$$I_{1,j} = \int_{U^j} \mathrm{D}\big(\psi_\circ(U^j) - \psi_\circ(U)\big) : \mathrm{D}\big(\psi_\circ(U)\big),$$

$$I_{2,j} = -\int_{U^j} \mathrm{D}\big(\psi_\circ(U^j) - \psi_\circ(U)\big) : \mathrm{D}\big(\psi_*(U)\big),$$
$$I_{3,j} = \int_{U^j} \mathrm{D}\big(\psi_*(U)\big) : \mathrm{D}\big(\psi_\circ(U) - \psi_*(U)\big),$$
$$I_{4,j} = -\int_{U^j} \mathrm{D}\big(\psi_*(U^j)\big) : \mathrm{D}\big(\psi_\circ(U) - \psi_*(U)\big).$$

Since $\psi_\circ(U)|_{U^j}, \psi_*(U)|_{U^j} \in \mathcal{H}(U^j)$, the difference $(\psi_\circ(U) - \psi_*(U))|_{U^j}$ is a suitable test function for the Euler–Lagrange equation for $\psi_*(U^j)$, which yields $I_{4,j} = 0$. Likewise, since $\psi_\circ(U), \sum_j \psi_\circ(U^j)\mathbb{1}_{U^j} \in \mathcal{H}_0(U) \subset \mathcal{H}(U)$, we find both $\sum_j I_{1,j} = 0$ and $\sum_j I_{2,j} = 0$. In addition, since $\psi_\circ(U), \psi_*(U) \in \mathcal{H}(U)$, we find $\sum_j I_{3,j} = 0$. This entails

$$\sum_j \int_{U^j} \mathrm{D}\big(\delta\psi(U^j) - \delta\psi(U)\big) : \mathrm{D}\big(\delta\psi(U)\big) = 0.$$

Summing (B.11) over j, inserting the above, and recalling the identity (B.7), the claim (B.9) follows. ∎

For all $n \geq 0$, we set $U^n := U_{3^n}$ and define the discrepancy

$$\tau_n := \mathbb{E}\big[J(U^n)\big] - \mathbb{E}\big[J(U^{n+1})\big].$$

In contrast with [3], the set U^n is now random, so that subadditivity does not directly imply $\tau_n \geq 0$. This is however true up to an error $O(3^{-n})$, as we briefly argue. Choose a partition $\{U_j^n := U_{3^n}(z_j)\}_j$ of the set U^{n+1}. Taking the expectation of (B.9) applied to this decomposition of U^{n+1}, we find

$$\begin{aligned} 0 &\leq \mathbb{E}\Big[\sum_j \big\|\mathrm{D}\big(\delta\psi(U^{n+1}) - \delta\psi(U_j^n)\big)\big\|^2_{\mathrm{L}^2(U_j^n)}\Big] \\ &= \sum_j \mathbb{E}\big[|U_j^n|\big(J(U_j^n) - J(U^{n+1})\big)\big], \end{aligned} \tag{B.12}$$

whereas by the deterministic bounds $|3^d|U^n| - |U^{n+1}|| \lesssim 3^{n(d-1)}$ and $J(U_j^n) \lesssim 1$ we have for some constant $C \simeq 1$,

$$\begin{aligned} &\sum_j \mathbb{E}\big[|U_j^n|\big(J(U_j^n) - J(U^{n+1})\big)\big] \\ &\quad\lesssim 3^{nd}\big(\mathbb{E}\big[J(U^n)\big] - \mathbb{E}\big[J(U^{n+1})\big] + C3^{-n}\big). \end{aligned} \tag{B.13}$$

The combination of (B.12) and (B.13) yields the claim in form of

$$\overline{\tau}_n := \tau_n + C3^{-n} \geq 0. \tag{B.14}$$

The crux of the approach is the following control of the variance of averages of $\mathrm{D}(\delta\psi(U))$ in terms of τ_n. In view of Lemma B.2, the proof is identical to that of [3, Lemma 2.13] (albeit in the α-mixing version of [4], further arguing as in (B.14) and absorbing the additional error term).

Lemma B.3. *There exist $C, \varepsilon > 0$ (only depending on d, ρ, β) such that for all n,*

$$\mathrm{Var}\bigg[\fint_{U^n} \mathrm{D}\big(\delta\psi(U^n)\big) \bigg] \le C 3^{-\varepsilon n} + C \sum_{m=0}^{n} 3^{-\varepsilon(n-m)} \overline{\tau}_m.$$

Recall the following version of Korn's inequality: for any bounded domain $D \subset \mathbb{R}^d$, for all divergence-free fields $v \in \mathrm{L}^2(D)$, we have

$$\inf_{\substack{\kappa\in\mathbb{R}^d \\ \Theta\in\mathbb{M}^{\mathrm{skew}}}} \int_D \big|v(x) - \kappa - \Theta x\big|^2 \, dx \lesssim_D \big\|\mathrm{D}(v)\big\|^2_{H^{-1}(D)},$$

where the multiplicative constant only depends on the regularity of D. In contrast with Poincaré's inequality, the infimum over $\Theta \in \mathbb{M}^{\mathrm{skew}}$ allows to have the symmetrized gradient in the right-hand side instead of the full gradient. By the so-called multiscale Poincaré inequality in [3, Proposition 1.12], using the above Korn inequality instead of [3, Lemma 1.13], Lemma B.3 yields the following estimate as in [3, Lemma 2.15]. This is simpler than the statement in [3] since there is no convex duality involved.

Lemma B.4. *There exist $C, \varepsilon > 0$ (only depending on d, ρ, β) such that for all n,*

$$\mathbb{E}\bigg[\inf_{\substack{\kappa\in\mathbb{R}^d \\ \Theta\in\mathbb{M}^{\mathrm{skew}}}} \fint_{U^{n+1}} \big|\delta\psi(U^{n+1})(x) - \kappa - \Theta x\big|^2 dx \bigg] \le C 3^{2n} \Big(3^{-\varepsilon n} + \sum_{m=0}^{n} 3^{-\varepsilon(n-m)} \overline{\tau}_m \Big).$$

Next, we deduce the following estimate on J as in [3, Lemma 2.16] by means of the Caccioppoli inequality. As the latter inequality in the present Stokes context involves the pressure, the proof slightly differs from [3] and is included below.

Lemma B.5. *There exist $C, \varepsilon > 0$ (only depending on d, ρ, β) such that for all n,*

$$\mathbb{E}\big[J(U^n)\big] \le C 3^{-\varepsilon n} + C \sum_{m=0}^{n} 3^{-\varepsilon(n-m)} \overline{\tau}_m.$$

Proof. Caccioppoli's inequality in form of e.g. [19, Section 4.4, Step 1] yields for all $K \ge 1$, for any constants $c \in \mathbb{R}$, $\kappa \in \mathbb{R}^d$, and $\Theta \in \mathbb{M}^{\mathrm{skew}}$,

$$\begin{aligned} \fint_{U^n} \big|\mathrm{D}\big(\delta\psi(U^{n+1})\big)\big|^2 \lesssim K^2 3^{-2n} \fint_{U^{n+1}} \big|\delta\psi(U^{n+1})(x) - \kappa - \Theta x\big|^2 dx \\ + K^{-2} \fint_{U^{n+1}} \big|\delta\Sigma(U^{n+1}) - c\big|^2 \mathbb{1}_{\mathbb{R}^d\setminus I}, \end{aligned} \tag{B.15}$$

where $\delta\Sigma(U^{n+1})$ is the difference of the pressures associated with $\psi_\circ(U^{n+1})$ and $\psi_*(U^{n+1})$. Appealing to a local pressure estimate in form of e.g. [19, Lemma 3.3], and recalling Lemma B.2 (ii), we find

$$\inf_{c\in\mathbb{R}} \fint_{U^{n+1}} \big|\delta\Sigma(U^{n+1}) - c\big|^2 \mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}} \lesssim \fint_{U^{n+1}} \big|\mathrm{D}\big(\delta\psi(U^{n+1})\big)\big|^2 = J(U^{n+1}). \tag{B.16}$$

Taking the infimum over c, κ, Θ in (B.15), taking the expectation, inserting (B.16), and using Lemma B.4, we obtain

$$\mathbb{E}\bigg[\fint_{U^n} \big|\mathrm{D}\big(\delta\psi(U^{n+1})\big)\big|^2\bigg] \lesssim K^2\bigg(3^{-\varepsilon n} + \sum_{m=0}^{n} 3^{-\varepsilon(n-m)}\overline{\tau}_m\bigg) + K^{-2}\mathbb{E}\big[J(U^{n+1})\big],$$

and thus, in view of (B.14),

$$\begin{aligned}&\mathbb{E}\bigg[\fint_{U^n} \big|\mathrm{D}\big(\delta\psi(U^{n+1})\big)\big|^2\bigg]\\ &\quad\lesssim K^2\bigg(3^{-\varepsilon n} + \sum_{m=0}^{n} 3^{-\varepsilon(n-m)}\overline{\tau}_m\bigg) + K^{-2}\big(\mathbb{E}\big[J(U^n)\big] + 3^{-n}\big).\end{aligned} \tag{B.17}$$

Next, we argue that

$$\mathbb{E}\big[J(U^n)\big] \lesssim \mathbb{E}\bigg[\fint_{U^n} \big|\mathrm{D}\big(\delta\psi(U^{n+1})\big)\big|^2\bigg] + \overline{\tau}_n. \tag{B.18}$$

For that purpose, we first note that the definition of J yields

$$\begin{aligned}&\mathbb{E}\big[J(U^n)\big] - \mathbb{E}\bigg[\fint_{U^n} \big|\mathrm{D}\big(\delta\psi(U^{n+1})\big)\big|^2\bigg]\\ &\quad= \mathbb{E}\bigg[\fint_{U^n} \mathrm{D}\big(\delta\psi(U^n) - \delta\psi(U^{n+1})\big) : \mathrm{D}\big(\delta\psi(U^n) + \delta\psi(U^{n+1})\big)\bigg]\\ &\quad\lesssim \mathbb{E}\bigg[\fint_{U^n} \big|\mathrm{D}\big(\delta\psi(U^n) - \delta\psi(U^{n+1})\big)\big|^2\bigg]^{\frac12} \big(\mathbb{E}\big[J(U^n)\big] + \mathbb{E}\big[J(U^{n+1})\big]\big)^{\frac12}.\end{aligned}$$

In order to control the first factor, we appeal to (B.12) and (B.13) in form of

$$\mathbb{E}\bigg[\sum_j \big\|\mathrm{D}\big(\delta\psi(U_j^n) - \delta\psi(U^{n+1})\big)\big\|^2_{\mathrm{L}^2(U_j^n)}\bigg] \lesssim 3^{nd}\overline{\tau}_n.$$

Further, using the definition (B.14) of $\overline{\tau}_n$ to reformulate the second factor, we deduce

$$\mathbb{E}\big[J(U^n)\big] - \mathbb{E}\bigg[\fint_{U^n} \big|\mathrm{D}\big(\delta\psi(U^{n+1})\big)\big|^2\bigg] \lesssim (\overline{\tau}_n)^{\frac12}\big(\mathbb{E}\big[J(U^n)\big] + \overline{\tau}_n\big)^{\frac12},$$

and the claim (B.18) follows.

Choosing $K \simeq 1$ large enough, (B.17) and (B.18) combine to

$$\mathbb{E}\big[J(U^n)\big] \lesssim \mathbb{E}\bigg[\fint_{U^n} \big|\mathrm{D}\big(\delta\psi(U^{n+1})\big)\big|^2 \bigg] + \overline{\tau}_n \lesssim 3^{-(\varepsilon\wedge 1)n} + \sum_{m=0}^{n} 3^{-\beta(n-m)}\overline{\tau}_m,$$

and the conclusion follows. ∎

We may now proceed to the proof of Proposition B.1, which follows from Lemma B.5 by iteration.

Proof of Proposition B.1. Set $F_n := 3^{-\frac{1}{2}\varepsilon n}\sum_{m=0}^{n} 3^{\frac{1}{2}\varepsilon m}\mathbb{E}\big[J(U^m)\big]$. In terms of τ_n, recognizing a telescoping sum, we find

$$\begin{aligned} F_n - F_{n+1} &= 3^{-\frac{1}{2}\varepsilon n}\sum_{m=0}^{n} 3^{\frac{1}{2}\varepsilon m}\mathbb{E}\big[J(U^m)\big] - 3^{-\frac{1}{2}\varepsilon(n+1)}\sum_{m=0}^{n+1} 3^{\frac{1}{2}\varepsilon m}\mathbb{E}\big[J(U^m)\big] \\ &= 3^{-\frac{1}{2}\varepsilon n}\sum_{m=0}^{n} 3^{\frac{1}{2}\varepsilon m}\tau_m - 3^{-\frac{1}{2}\varepsilon(n+1)}\mathbb{E}\big[J(U^0)\big], \end{aligned}$$

and thus, using (B.14) and $\mathbb{E}\big[J(U^0)\big] \lesssim 1$,

$$F_n - F_{n+1} \geq 3^{-\frac{1}{2}\varepsilon n}\sum_{m=0}^{n} 3^{\frac{1}{2}\varepsilon m}\overline{\tau}_m - C3^{-\frac{1}{2}\varepsilon n}. \tag{B.19}$$

Similarly, we find

$$\begin{aligned} F_{n+1} &\leq 3^{-\frac{1}{2}\varepsilon(n+1)}\sum_{m=1}^{n+1} 3^{\frac{1}{2}\varepsilon m}\mathbb{E}\big[J(U^m)\big] + C3^{-\frac{1}{2}\varepsilon(n+1)} \\ &\leq 3^{-\frac{1}{2}\varepsilon(n+1)}\sum_{m=1}^{n+1} 3^{\frac{1}{2}\varepsilon m}\big(\mathbb{E}\big[J(U^{m-1})\big] + C3^{-(m-1)}\big) + C3^{-\frac{1}{2}\varepsilon(n+1)} \\ &\leq F_n + C3^{-\frac{1}{2}\varepsilon n}, \end{aligned}$$

which, by Lemma B.5, turns into

$$\begin{aligned} F_{n+1} &\leq C3^{-\frac{1}{2}\varepsilon n}\sum_{m=0}^{n} 3^{\frac{1}{2}\varepsilon m}\bigg(3^{-\varepsilon m} + \sum_{k=0}^{m} 3^{-\varepsilon(m-k)}\overline{\tau}_k\bigg) + C3^{-\frac{1}{2}\varepsilon n} \\ &\leq C3^{-\frac{1}{2}\varepsilon n} + C3^{-\frac{1}{2}\varepsilon n}\sum_{m=0}^{n} C3^{\frac{1}{2}\varepsilon m}\overline{\tau}_m. \end{aligned}$$

Combining this with (B.19), we obtain

$$F_{n+1} \leq C(F_n - F_{n+1}) + C3^{-\frac{1}{2}\varepsilon n},$$

and thus

$$F_{n+1} \leq \frac{C}{C+1}(F_n + 3^{-\frac{1}{2}\varepsilon n}).$$

By iteration, this yields $F_n \leq C\,3^{-\gamma n}$ for some $\gamma > 0$, and thus $\mathbb{E}[J(U^n)] \leq C\,3^{-\gamma n}$ and $\tau_n \leq C\,3^{-\gamma n}$. Since $\mathbb{E}[J(U^n)] = E : (\widetilde{\mathbf{B}}_{3^n,\circ} - \widetilde{\mathbf{B}}_{3^n,*})E$, this implies

$$0 \leq E : (\widetilde{\mathbf{B}}_{L,\circ} - \widetilde{\mathbf{B}}_{L,*})E \leq L^{-\gamma}.$$

Combined with Lemma B.2 (i), this yields the conclusion. ∎

References

[1] Y. Almog and H. Brenner, Global homogenization of a dilute suspension of sphere. 1998, unpublished manuscript

[2] H. Ammari, P. Garapon, H. Kang, and H. Lee, Effective viscosity properties of dilute suspensions of arbitrarily shaped particles. *Asymptot. Anal.* **80** (2012), no. 3-4, 189–211

[3] S. Armstrong, T. Kuusi, and J.-C. Mourrat, *Quantitative stochastic homogenization and large-scale regularity*. Grundlehren Math. Wiss. 352, Springer, Cham, 2019

[4] S. N. Armstrong and J.-C. Mourrat, Lipschitz regularity for elliptic equations with random coefficients. *Arch. Ration. Mech. Anal.* **219** (2016), no. 1, 255–348

[5] S. N. Armstrong and C. K. Smart, Quantitative stochastic homogenization of convex integral functionals. *Ann. Sci. Éc. Norm. Supér. (4)* **49** (2016), no. 2, 423–481

[6] G. K. Batchelor, Sedimentation in a dilute dispersion of spheres. *J. Fluid Mech.* **52** (1972), no. 2, 245–268

[7] G. K. Batchelor and J. T. Green, The determination of the bulk stress in suspension of spherical particles to order c^2. *J. Fluid Mech.* **56** (1972), 401–427

[8] G. K. Batchelor and J. T. Green, The hydrodynamic interaction of two small freely-moving spheres in a linear flow field. *J. Fluid Mech.* **56** (1972), no. 2, 375–400

[9] V. L. Berdichevskiĭ, *Variational principles in continuum mechanics* (in Russian). Nauka, Moscow, 1983

[10] R. Clausius, *Die mechanische Behandlung der Elektricität*. Vieweg, Braunshweig, 1879

[11] P. Doukhan, *Mixing*. Lect. Notes Stat. 85, Springer, New York, 1994

[12] M. Duerinckx, *Topics in the Mathematics of Disordered Media*. Ph.D. thesis, Université Libre de Bruxelles & Université Pierre et Marie Curie, 2017

[13] M. Duerinckx, Effective viscosity of random suspensions without uniform separation. *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **39** (2022), no. 5, 1009–1052

[14] M. Duerinckx and A. Gloria, The Clausius–Mossotti formula. *Asymptot. Anal.* (2023), DOI 10.3233/ASY-231840

[15] M. Duerinckx and A. Gloria, Analyticity of homogenized coefficients under Bernoulli perturbations and the Clausius–Mossotti formulas. *Arch. Ration. Mech. Anal.* **220** (2016), no. 1, 297–361

[16] M. Duerinckx and A. Gloria, Multiscale functional inequalities in probability: concentration properties. *ALEA Lat. Am. J. Probab. Math. Stat.* **17** (2020), no. 1, 133–157

[17] M. Duerinckx and A. Gloria, Multiscale functional inequalities in probability: constructive approach. *Ann. H. Lebesgue* **3** (2020), 825–872

[18] M. Duerinckx and A. Gloria, Corrector equations in fluid mechanics: effective viscosity of colloidal suspensions. *Arch. Ration. Mech. Anal.* **239** (2021), no. 2, 1025–1060

[19] M. Duerinckx and A. Gloria, Quantitative homogenization theory for random suspensions in steady Stokes flow. *J. Éc. polytech. Math.* **9** (2022), 1183–1244

[20] M. Duerinckx and A. Gloria, A short proof of Gevrey regularity for homogenized coefficients of the Poisson point process. *C. R. Math. Acad. Sci. Paris* **360** (2022), 909–918

[21] M. Duerinckx and A. Gloria, Continuum percolation in stochastic homogenization and the effective viscosity problem. *Arch. Ration. Mech. Anal.* **247** (2023), no. 2, article no. 26

[22] A. Einstein, Eine neue Bestimmung der Moleküldimensionen. *Ann. Phys.* **19** (1906), no. 2, 289–306

[23] A. Einstein, Berichtigung zu meiner Arbeit: "eine neue Bestimmung der Moleküldimensionen". *Ann. Phys.* **339** (1911), no. 3, 591–592

[24] N. A. Frankel and A. Acrivos, On the viscosity of a concentrated suspension of solid spheres. *Chem. Eng. Sci.* **22** (1967), no. 6, 847–853

[25] G. P. Galdi, *An introduction to the mathematical theory of the Navier–Stokes equations. Steady-state problems*. 2nd edn., Springer Monogr. Math., Springer, New York, 2011

[26] D. Gérard-Varet, Derivation of the Batchelor–Green formula for random suspensions. *J. Math. Pures Appl. (9)* **152** (2021), 211–250

[27] D. Gérard-Varet and M. Hillairet, Analysis of the viscosity of dilute suspensions beyond Einstein's formula. *Arch. Ration. Mech. Anal.* **238** (2020), no. 3, 1349–1411

[28] D. Gérard-Varet and R. M. Höfer, Mild assumptions for the derivation of Einstein's effective viscosity formula. *Comm. Partial Differential Equations* **46** (2021), no. 4, 611–629

[29] D. Gérard-Varet and A. Mecherbet, On the correction to Einstein's formula for the effective viscosity. *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **39** (2022), no. 1, 87–119

[30] A. Giunti, C. Gu, J.-C. Mourrat, and M. Nitzschner, Smoothness of the diffusion coefficients for particle systems in continuous space. *Commun. Contemp. Math.* **25** (2023), no. 3, article no. 2250027

[31] É. Guazzelli and J. F. Morris, *A physical introduction to suspension dynamics*. Cambridge Texts Appl. Math., Cambridge University Press, Cambridge, 2012

[32] B. M. Haines and A. L. Mazzucato, A proof of Einstein's effective viscosity for a dilute suspension of spheres. *SIAM J. Math. Anal.* **44** (2012), no. 3, 2120–2145

[33] M. Hillairet and D. Wu, Effective viscosity of a polydispersed suspension. *J. Math. Pures Appl. (9)* **138** (2020), 413–447

[34] E. J. Hinch, An averaged-equation approach to particle interactions in a fluid suspension. *J. Fluid Mech.* **83** (1977), no. 4, 695–720

[35] E. J. Hinch and L. G. Leal, The effect of brownian motion on the rheological properties of a suspension of non-spherical particles. *J. Fluid Mech.* **52** (1972), no. 4, 683–712

[36] R. M. Höfer, Sedimentation of inertialess particles in Stokes flows. *Comm. Math. Phys.* **360** (2018), no. 1, 55–101

[37] R. M. Höfer, Convergence of the method of reflections for particle suspensions in Stokes flows. *J. Differential Equations* **297** (2021), 81–109

[38] R. M. Höfer and J. J. L. Velázquez, The method of reflections, homogenization and screening for Poisson and Stokes equations in perforated domains. *Arch. Ration. Mech. Anal.* **227** (2018), no. 3, 1165–1221

[39] P.-E. Jabin and F. Otto, Identification of the dilute regime in particle sedimentation. *Comm. Math. Phys.* **250** (2004), no. 2, 415–432

[40] G. B. Jeffery, The motion of ellipsoidal particles immersed in a viscous fluid. *Proc. Roy. Soc. A* **102** (1922), 161–179

[41] D. J. Jeffrey and A. Acrivos, The rheological properties of suspensions of rigid particles. *AiChE J.* **22** (1976), no. 3, 417–432

[42] J. Keller, L. Rubenfeld, and J. Molyneux, Extremum principles for slow viscous flows with applications to suspensions. *J. Fluid Mech.* **30** (1967), 97–125

[43] T. Kuna, J. L. Lebowitz, and E. R. Speer, Necessary and sufficient conditions for realizability of point processes. *Ann. Appl. Probab.* **21** (2011), no. 4, 1253–1281

[44] L. G. Leal and E. J. Hinch, The effect of weak brownian rotations on particles in shear flow. *J. Fluid Mech.* **46** (1971), no. 4, 685–703

[45] T. Lévy and E. Sánchez-Palencia, Einstein-like approximation for homogenization with small concentration. II. Navier–Stokes equation. *Nonlinear Anal.* **9** (1985), no. 11, 1255–1268

[46] H. A. Lorentz, *The theory of electrons and its applications to the phenomena of light and radiant heat*. B. U. Teubner, Leipzig, 1909 (reprinted by Dover, New York, 1952)

[47] L. Lorenz, Ueber die Refraktionskonstante. *Ann. Phys. Chem.* **11** (1880), 70–103

[48] K. Z. Markov, Elementary micromechanics of heterogeneous media. In *Heterogeneous media*, pp. 1–162, Model. Simul. Sci. Eng. Technol., Birkhäuser, Boston, MA, 2000

[49] J. C. Maxwell, *A treatise on electricity and magnetism. Vol. 1*. Oxf. Class. Texts Phys. Sci., The Clarendon Press, New York, 1998

[50] G. W. Milton, *The theory of composites*. Cambridge Monogr. Appl. Comput. Math. 6, Cambridge University Press, Cambridge, 2002

[51] O. F. Mossotti, Sur les forces qui régissent la constitution intérieure des corps. Aperçu pour servir à la détermination de la cause et des lois de l'action moléculaire. *Ac. Sci. Torino* **22** (1836), 1–36

[52] O. F. Mossotti, Discussione analitica sul'influenza che l'azione di un mezzo dielettrico ha sulla distribuzione dell'elettricità alla superficie di più corpi elettrici disseminati in esso. *Mem. Mat. Fis. della Soc. Ital. di Sci. in Modena* **24** (1850), 49–74

[53] J.-C. Mourrat, First-order expansion of homogenized coefficients under Bernoulli perturbations. *J. Math. Pures Appl. (9)* **103** (2015), no. 1, 68–101

[54] B. Niethammer and R. Schubert, A local version of Einstein's formula for the effective viscosity of suspensions. *SIAM J. Math. Anal.* **52** (2020), no. 3, 2561–2591

[55] K. Nunan and J. Keller, Effective viscosity of a periodic suspension. *J. Fluid Mech.* **142** (1984), 269–287

[56] R. W. O'Brien, A method for the calculation of the effective transport properties of suspensions of interacting particles. *J. Fluid Mech.* **91** (1979), no. 1, 17–39

[57] M. D. Penrose, Random parking, sequential adsorption, and the jamming limit. *Comm. Math. Phys.* **218** (2001), no. 1, 153–176

[58] J. Perrin, Les preuves de la réalité moléculaire (étude spéciale des émulsions). Report read at the 1911 Solvay conference

[59] J. Pertinand, A fixed-point approach to Clausius–Mossotti formulas. In preparation

[60] E. Sánchez-Palencia, Einstein-like approximation for homogenization with small concentration. I. Elliptic problems. *Nonlinear Anal.* **9** (1985), no. 11, 1243–1254

[61] N. Straumann, On Einstein's doctoral thesis. 2005, arXiv:physics/0504201

[62] S. Torquato, *Random heterogeneous materials*. Interdiscip. Appl. Math. 16, Springer, New York, 2002

[63] M. Zuzovsky, P. Adler, and H. Brenner, Spatially periodic suspensions of convex particles in linear shear flows. III. Dilute arrays of spheres suspended in Newtonian fluids. *Phys. Fluids* **26** (1983), 1714–1723

MEMOIRS OF THE EUROPEAN MATHEMATICAL SOCIETY

Mitia Duerinckx, Antoine Gloria

On Einstein's Effective Viscosity Formula

In his PhD thesis, Einstein derived an explicit first-order expansion for the effective viscosity of a Stokes fluid with a suspension of small rigid particles at low density. His formal derivation relied on two implicit assumptions: (i) there is a scale separation between the size of the particles and the observation scale; and (ii) at first order, dilute particles do not interact with one another. In mathematical terms, the first assumption amounts to the validity of a homogenization result defining the effective viscosity tensor, which is now well understood. Next, the second assumption allowed Einstein to approximate this effective viscosity at low density by considering particles as being isolated. The rigorous justification is, in fact, quite subtle as the effective viscosity is a nonlinear nonlocal function of the ensemble of particles and as hydrodynamic interactions have borderline integrability.

In the present memoir, we establish Einstein's effective viscosity formula in the most general setting. In addition, we pursue the low-density expansion to arbitrary order in form of a cluster expansion, where the summation of hydrodynamic interactions crucially requires suitable renormalizations. In particular, we justify a celebrated result by Batchelor and Green on the second-order correction and we explicitly describe all higher-order renormalizations for the first time. In some specific settings, we further address the summability of the whole cluster expansion. Our approach relies on a combination of combinatorial arguments, variational analysis, elliptic regularity, probability theory, and diagrammatic integration methods.

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