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Positive Energy Representations of Gauge Groups I

Localization

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Abstract

This is the first in a series of papers on projective positive energy representations of gauge groups. Let $\Xi \to M$ be a principal fiber bundle, and let $\Gamma_c(M, \mathrm{Ad}(\Xi))$ be the group of compactly supported (local) gauge transformations. If P is a group of "space–time symmetries" acting on $\Xi \to M$, then a projective unitary representation of $\Gamma_c(M, \mathrm{Ad}(\Xi)) \rtimes P$ is of *positive energy* if every "timelike generator" $p_0 \in \mathfrak{p}$ gives rise to a Hamiltonian $H(p_0)$ whose spectrum is bounded from below. Our main result shows that in the absence of fixed points for the cone of timelike generators, the projective positive energy representations of the connected component $\Gamma_c(M, \mathrm{Ad}(\Xi))_0$ come from 1-dimensional P-orbits. For compact M this yields a complete classification of the projective positive energy representations in terms of lowest weight representations of affine Kac–Moody algebras. For noncompact M, it yields a classification under further restrictions on the space of ground states.

In the second part of this series we consider larger groups of gauge transformations, which contain also global transformations. The present results are used to localize the positive energy representations at (conformal) infinity.

Keywords. Infinite-dimensional Lie group, unitary representation, loop group, gauge group, positive energy representation, projective representation

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Chapter 1

Introduction

This is the first in a series of papers where we analyze the projective positive energy representations of gauge groups.

Our main motivation is the Wigner–Mackey classification [60, 113] of projective unitary representations of the Poincaré group. Every irreducible such representation is labeled by an $\mathrm{SO}^{\uparrow}(1, d-1)$-orbit in momentum space $\mathbb{R}^d$, together with an irreducible unitary representation of the corresponding little group. It is called a *positive energy* representation if for every 1-parameter group of timelike translations, the corresponding Hamilton operator is bounded from below. This excludes the tachyonic orbits, leaving the positive mass hyperboloids $p_\mu p^\mu = m^2$, the positive light cone $p_\mu p^\mu = 0$, $p_0 > 0$, and the origin $p = 0$. The corresponding little groups yield an intrinsic description of spin (for the positive mass hyperboloids) and helicity (for the positive light cone).

In this series of papers we aim to extend this picture with an infinite-dimensional group $\mathcal{G}$ of gauge transformations, placing internal symmetries and space-time symmetries on the same footing.

1.1 Outline of Part I and II of this series

For a gauge theory with structure group K, the fields over the space-time manifold M are associated to a principal K-bundle $\Xi \to M$. We consider the equivariant setting, where the group P of space-time symmetries acts by automorphisms on $\Xi \to M$, and the Lie algebra $\mathfrak{p}$ of P contains a distinguished cone $\mathcal{C} \subseteq \mathfrak{p}$ of "timelike generators". For Minkowski space, this is of course the Poincaré group P with the cone $\mathcal{C}$ of timelike translations.

The relevant group $\mathcal{G}$ of gauge transformations depends on the context. It always contains the group

$$\mathcal{G}_c := \Gamma_c(M, \mathrm{Ad}(\Xi))$$

of compactly supported vertical automorphisms of $\Xi \to M$, and it is this group that we will focus on in Part I of this series. In Part II we consider also global gauge transformations. The group $\mathcal{G}$ is then larger than $\mathcal{G}_c$, but it may be smaller than the group $\Gamma(M, \mathrm{Ad}(\Xi))$ of all vertical automorphisms due to boundary conditions at infinity.

A projective unitary representation of $\mathcal{G} \rtimes P$ assigns to every timelike generator $p_0 \in \mathcal{C}$ a one-parameter group of projective unitary transformations, and hence a selfadjoint Hamiltonian $H(p_0)$ that is well defined up to a constant.

Our main objective is to study the projective unitary representations of $\mathcal{G} \rtimes P$ that are of positive energy, in the sense that the Hamiltonians $H(p_0)$ are bounded from below.

Perhaps surprisingly, this places rather stringent restrictions on the representation theory of $\mathcal{G}$, leading to a complete classification in favorable cases.

1.1.1 Outline of Part I

In the first part of this series, we focus on the group $\mathcal{G}_c = \Gamma_c(M, \mathrm{Ad}(\Xi))$ of *compactly supported* gauge transformations. Our main result concerns the case where M has no fixed points for the cone $\mathcal{C}$ of timelike generators, and K is a 1-connected, semisimple Lie group.

Localization theorem. *For every projective positive energy representation $(\bar{\rho}, \mathcal{H})$ of the identity component $\Gamma_c(M, \mathrm{Ad}(\Xi))_0$, there exists a 1-dimensional, P-equivariantly embedded submanifold $S \subseteq M$ and a positive energy representation $\bar{\rho}_S$ of $\Gamma_c(S, \mathrm{Ad}(\Xi))$ such that the following diagram commutes,*

$$\begin{tikzcd}
\Gamma_c(M, \mathrm{Ad}(\Xi))_0 \arrow[r, "\bar{\rho}"] \arrow[d, "r_S"'] & \mathrm{PU}(\mathcal{H}) \\
\Gamma_c(S, \mathrm{Ad}(\Xi)), \arrow[ur, "\bar{\rho}_S"'] &
\end{tikzcd}$$

where the vertical arrow denotes restriction to S.

This effectively reduces the classification of projective positive energy representations to the 1-dimensional case. If M is compact, then

$$S = \bigcup_{j=1}^{k} S_j$$

is a finite union of circles. If K is noncompact and simple, then we show that all positive energy representations are trivial. If K is compact, then the group

$$\Gamma(S, \mathrm{Ad}(\Xi)) \cong \prod_{j=1}^{k} \Gamma(S_j, \mathrm{Ad}(\Xi))$$

is a finite product of twisted loop groups, yielding a complete classification in terms of tensor products of highest weight representations for the corresponding affine Kac–Moody algebras [32, 54, 94, 104].

To some extent these results generalize to the case of noncompact manifolds M, where S can then have infinitely many connected components. We are able to classify the projective positive energy representations under the additional assumption that they admit a cyclic ground state vector which is unique up to scalar. These *vacuum representations* are classified in terms of infinite tensor products of vacuum representations of affine Kac–Moody algebras. In particular, every such representation is of type I. Without the vacuum condition, the classification is considerably more involved. We study in detail the case where all connected components of S are circles. Under a geometric "spectral gap" condition, we reduce the classification of projective positive energy representations to the representation theory of UHF C^*-algebras, yielding a rich source of representations of type II and III.

1.1.2 Outline of Part II

In the second part of this series, we consider the case where $\mathcal{G}$ contains *global* as well as compactly supported gauge transformations. To study the projective positive energy representations, we use the exact sequence

$$1 \to \mathcal{G}_c \to \mathcal{G} \to \mathcal{G}/\mathcal{G}_c \to 1.$$

By the results from Part I on the positive energy representations of $\mathcal{G}_c$, the problem essentially reduces to the group $\mathcal{G}/\mathcal{G}_c$ of gauge transformations "at infinity".

Needless to say, the resulting representation theory is very sensitive to the boundary conditions at infinity. We focus on the situation where M is an *asymptotically simple* space-time in the sense of Penrose [24, 37, 91, 92], and $\mathcal{G}$ consists of gauge transformations that extend smoothly to the conformal boundary. For the motivating example of the Poincaré group acting on d-dimensional Minkowski space, we obtain the following detailed account of the projective positive energy representation theory.

Minkowski space in dimension $d > 2$. In this setting we show that the projective positive energy representations of $\mathcal{G}$ depend only on the 1-*jets* of the gauge transformation at spacelike infinity ι_0 and at past and future timelike infinity $\iota_{\pm}$. This reduces the problem to the classification of projective positive energy representations of the (finite-dimensional!) semidirect product

$$\big(\mathrm{SO}^{\uparrow}(1, d-1) \times K^3\big) \ltimes \big(\mathbb{R}^d \oplus (\mathfrak{k}^3 \otimes \mathbb{R}^{d*})\big),$$

where $\mathrm{SO}(1, d-1)$ acts on $\mathbb{R}^d$ in the usual fashion, and K acts on its Lie algebra $\mathfrak{k}$ by the adjoint representation. The three copies of K encode the values of the gauge transformation at ι_0 and $\iota_{\pm}$, whereas the three copies of the additive group $\mathfrak{k} \otimes \mathbb{R}^{d*}$ encode the derivatives.

In the special case where the derivatives act trivially, we recover a projective positive energy representations of the Poincaré group $\mathbb{R}^d \rtimes \mathrm{SO}^{\uparrow}(1, d-1)$, together with 3 projective unitary representations of the structure group K. More generally, by

Mackey's theorem of imprimitivity, the irreducible projective positive energy representations are labeled by an orbit of $\mathrm{SO}^{\uparrow}(1, d-1) \times K^3$ in $\mathbb{R}^d \oplus (\mathfrak{k}^3 \otimes \mathbb{R}^{d*})$ whose energy is bounded from below, together with a projective unitary representation of the corresponding little group. In general these little groups will not contain the three copies of K, giving rise to phenomena that are reminiscent of spontaneous symmetry breaking.

Minkowski space in dimension $d = 2$. In contrast to the higher dimensional case, the projective positive energy representations in $d = 2$ do *not* in general factor through a finite-dimensional Lie group. For simplicity, we consider the group $\mathcal{G}$ of gauge transformations that extend smoothly to the *conformal compactification* of 2-dimensional Minkowski space. Here the three points ι_0 and $\iota_\pm$ at space- and timelike infinity are collapsed to a single point I, and past and future null infinity $\mathcal{I}^-$ and $\mathcal{I}^+$ are identified along lightlike geodesics (cf. [92]). The boundary of this space is a union of two circles $\mathbb{S}^1_L$ and $\mathbb{S}^1_R$ (corresponding to left and right moving modes) that intersect transversally in a single point I. We prove that the positive energy representations of $\mathcal{G}$ depend only on the *values* of the gauge transformations at null infinity

$$\mathcal{I} = (\mathbb{S}^1_L \cup \mathbb{S}^1_R) \setminus \{I\},$$

and on the *2-jets* at the single point I.

At the Lie algebra level, the problem therefore reduces to classifying the projective positive energy representations of the abelian extension

$$0 \to |\mathfrak{k}| \to \mathfrak{g} \to \mathfrak{g}_{\mathrm{eq}} \to 0.$$

Here, the equalizer Lie algebra

$$\mathfrak{g}_{\mathrm{eq}} = \big\{(\xi, \eta) \in \Gamma(\mathbb{S}^1_L, \mathrm{ad}(\Xi)) \times \Gamma(\mathbb{S}^1_R, \mathrm{ad}(\Xi)); \xi(I) = \eta(I)\big\}$$

represents the values of the infinitesimal gauge transformations on the conformal boundary $\mathbb{S}^1_L \cup \mathbb{S}^1_R$, the abelian Lie algebra $|\mathfrak{k}|$ with underlying vector space $\mathfrak{k}$ represents the mixed second derivatives at I, and $\mathfrak{g}_{\mathrm{eq}}$ acts on $|\mathfrak{k}|$ by evaluating at I and composing with the adjoint representation.

Even in the untwisted case, where Ξ is the trivial K-bundle, the classification of the projective positive energy representations is by no means trivial. This is because the positive energy condition is *not* with respect to rigid rotations of $\mathbb{S}^1_{L/R}$, but with respect to the translations of the real projective line

$$\mathbb{S}^1_{L/R} = \mathbb{R} \cup \{I\}$$

fixing the point I at infinity.

Under the restrictive additional condition that the projective unitary representations are of positive energy with respect to rotations as well as translations, we obtain a classification in terms of highest weight representations of the two untwisted affine Kac–Moody algebras corresponding to $\mathbb{S}^1_L$ and $\mathbb{S}^1_R$, together with a projective positive energy representation of the finite-dimensional Lie group of 2-jets of gauge transformations at I.

Although the Kac–Moody representations are familiar from the construction of loop group nets in conformal field theory, the positive energy representations involving 2-jets (which are Poincaré invariant but not conformally invariant) appear to be a novel feature.

1.2 Structure of the present memoir

For a closed quantum system that is described by a Hilbert space $\mathcal{H}$, any two states that differ by a global phase are physically indistinguishable. The state space of the system is therefore described by the projective Hilbert space $\mathbb{P}(\mathcal{H})$. By Wigner's theorem, a connected Lie group G acts on the projective Hilbert space $\mathbb{P}(\mathcal{H})$ by projective unitary transformations, resulting in a projective unitary representation $\bar{\rho}\colon G \to \mathrm{PU}(\mathcal{H})$.

1.2.1 Positive energy representations

Since we are interested in the group of compactly supported gauge transformations, we need to work with *infinite-dimensional* Lie groups modeled on locally convex spaces, or *locally convex Lie groups* for short. In Chapter 2 we recall and extend some recent results from [52, 76] that allow us to go back and forth between smooth projective unitary representations of a locally convex Lie group G, smooth unitary representations of a central Lie group extension $G^\sharp$, and the derived representations of its Lie algebra $\mathfrak{g}^\sharp$.

In Chapter 3 we introduce projective *positive energy representations* in the context of a Lie group P that acts smoothly on G by automorphisms. For a distinguished *positive energy cone* $\mathcal{C} \subseteq \mathfrak{p}$, we require that the spectrum of the corresponding selfadjoint operators in the derived representation is bounded from below. Since a representation is of positive energy for the cone $\mathcal{C}$ if and only if it is of positive energy for the 1-parameter subgroups generated by $\mathcal{C} \subseteq \mathfrak{p}$, we can always reduce to the case $P = \mathbb{R}$, where the non-negative spectrum condition pertains to a single Hamilton operator H. Using the Borchers–Arveson theorem, we further reduce the classification to the *minimal* representations, where $H \geq 0$ is the smallest possible Hamilton operator with non-negative spectrum.

In Chapter 4 we then turn to our subject proper, namely the locally convex Lie group $\mathcal{G}_c$ of compactly supported gauge transformations. We consider the setting where M is a manifold, P is a Lie group acting smoothly on M, and $\mathcal{K} \to M$ is a bundle of 1-connected semisimple Lie groups that is equipped with a lift of this action. The group $\mathcal{G}_c = \Gamma_c(M, \mathcal{K})$ of compactly supported sections then carries a smooth action of P by automorphisms, and we consider the smooth projective unitary representations of the semidirect product $\Gamma_c(M, \mathcal{K}) \rtimes P$.

The motivating example is of course the case where $\mathcal{K} = \mathrm{Ad}(\Xi)$ is the adjoint bundle of a principal fiber bundle $\Xi \to M$, and $\Gamma_c(M, \mathrm{Ad}(\Xi))$ is the group of vertical automorphisms of Ξ that are trivial outside a compact subset of M. The reason for the minor generalization to bundles of Lie groups is purely technical; the reduction to simple structure groups in Section 4.2 is somewhat easier in that setting.

1.2.2 The localization theorem

The main result in the present memoir is the following localization result (a minor generalization of the one in Section 1.1.1), which essentially reduces the classification of projective positive energy representations to the 1-dimensional case.

Localization theorem (Theorem 7.19). *Let $(\bar{\rho}, \mathcal{H})$ be a projective positive energy representation of $\Gamma_c(M, \mathcal{K}) \rtimes P$. If the cone $\mathcal{C}$ has no fixed points in M, then there exists a 1-dimensional, P-equivariantly embedded submanifold $S \subseteq M$ such that on the connected component $\Gamma_c(M, \mathcal{K})_0$, the projective representation $\bar{\rho}$ factors through the restriction homomorphism $r_S \colon \Gamma_c(M, \mathcal{K})_0 \to \Gamma_c(S, \mathcal{K})$.*

We sketch the proof in the special case that the structure group K of $\mathcal{K}$ is a compact simple Lie group. The result for (not necessarily compact) semisimple Lie groups is reduced to the simple case in Section 4.2, and to the compact simple case in Section 6.1. We require K to be 1-connected, but this is by no means essential; results beyond 1-connected groups are discussed in Sections 7.1 and 8.3.

Further, we will assume without loss of generality that P is the additive group $\mathbb{R}$ of real numbers. The corresponding flow is then given by a non-vanishing vector field $\mathbf{v}_M$ on M, which lifts to a vector field $\mathbf{v}$ on $\mathcal{K}$. We denote the corresponding derivation of the gauge algebra by $D\xi := L_{\mathbf{v}}\xi$. The reduction from P to $\mathbb{R}$ is carried out in Section 7.5 by considering the 1-parameter subgroups of P that are generated by elements of the positive energy cone $\mathcal{C} \subseteq \mathfrak{p}$.

Step 1. Let $\mathfrak{K} \to M$ be the bundle of Lie algebras derived from $\mathcal{K} \to M$. Then, every smooth projective unitary representation of $\Gamma_c(M, \mathcal{K}) \rtimes \mathbb{R}$ gives rise to an $\mathbb{R}$-invariant 2-cocycle ω on the compactly supported gauge algebra $\Gamma_c(M, \mathfrak{K})$. In Section 4.3 we show that every such cocycle is cohomologous to one of the form

$$\omega(\xi, \eta) = \lambda(\kappa(\xi, d_\nabla \eta)) \quad \text{for } \xi, \eta \in \Gamma_c(M, \mathfrak{K}), \tag{1.1}$$

where ∇ is a Lie connection on $\mathfrak{K} \to M$, κ is a positive definite invariant bilinear form on the Lie algebra $\mathfrak{k}$ of K, and $\lambda \colon \Omega^1_c(M) \to \mathbb{R}$ is a closed current that is invariant under the flow.

Step 2. The positive energy condition for $(\rho, \mathcal{H})$ gives rise to a Cauchy–Schwarz inequality for the derived Lie algebra representation $\mathtt{d}\rho$. In Section 3.4 we show that if $[\xi, D\xi] = 0$ and $\omega(\xi, D) = 0$, then

$$\langle \psi, i\,\mathtt{d}\rho(D\xi)\psi\rangle^2 \leq 2\omega(\xi, D\xi)\langle \psi, H\psi\rangle \tag{1.2}$$

for every smooth unit vector ψ. Moreover, $\omega(\xi, D\xi)$ is non-negative. In Chapter 5 this is used to show that the closed current λ from (1.1) takes the form

$$\lambda(\alpha) = \int_M (i_{\mathbf{v}_M}\alpha)(x)d\mu(x)$$

for a flow-invariant regular Borel measure μ on M. In terms of this measure, the Cauchy–Schwarz inequality (1.2) becomes

$$\langle \psi, i\,\mathtt{d}\rho(L_{\mathbf{v}}\xi)\psi\rangle^2 \leq 2\langle \psi, H\psi\rangle \|L_{\mathbf{v}}\xi\|_{\mu}^2. \tag{1.3}$$

In other words, if ξ is in the image of the derivation

$$D = L_{\mathbf{v}},$$

then the expectation value of the unbounded operator $i\,\mathtt{d}\rho(\xi)$ is controlled in terms of the energy $\langle \psi, H\psi\rangle$, and the L^2-norm of ξ with respect to the measure μ. In fact, a small but important refinement allows one to control the expectation of $i\,\mathtt{d}\rho(\xi)$ in terms of similar data if ξ is not in the image of the derivation.

Step 3. In Chapter 6 we use the Cauchy–Schwarz estimate (1.3) and its refinement to show that

$$\pm\, i\,\mathtt{d}\rho(\xi) \leq \|\xi\|_{\nu}\mathbf{1} + \|\xi\|_{\mu}H \tag{1.4}$$

as unbounded operators. The measure ν is absolutely continuous with respect to μ, with a density that is upper semi-continuous and invariant under the flow. From a technical point of view, this is the heart of the proof. It allows us to extend $\mathtt{d}\rho$ to a positive energy representation of the Banach–Lie algebra $H^2_\partial(M, \mathcal{K})$ of sections that are twice differentiable in the direction of the flow, but only ν-*measurable* in the direction perpendicular to the flow.

Step 4. The final steps of the proof are carried out in Chapter 7. Every point in M admits a flow box $U_0 \times I \simeq U \subseteq M$, where the flow fixes all points in U_0 and acts by translation on the interval $I \subseteq \mathbb{R}$ for small times. Accordingly, the flow-invariant measure on U decomposes as

$$\mu = \mu_0 \otimes dt.$$

Since the sections in $H^2_\partial(U, \mathcal{K}) \subseteq H^2_\partial(M, \mathcal{K})$ need only be measurable in the direction perpendicular to the flow, we can continuously embed $C^\infty_c(I, \mathfrak{k})$ as a Lie subalgebra of $H^2_\partial(U, \mathcal{K})$ by multiplying with an indicator function χ_E for a Borel subset $E \subseteq U_0$ of finite measure. This yields a projective unitary representation of $C^\infty_c(I, \mathfrak{k})$ with central charge $2\pi\mu_0(E)$.

Step 5. Since the dense space of analytic vectors for H is analytic for the extension of $\mathtt{d}\rho$ to $H^2_\partial(M, \mathcal{K})$, the projective unitary representation of $C^\infty_c(I, \mathfrak{k})$ extends to the 1-connected Lie group G that integrates $C^\infty_c(I, \mathfrak{k})$. This gives rise to a smooth central $\mathbb{T}$-extension $G^\sharp \to G$. For every smooth map $\sigma\colon \mathbb{S}^2 \to G$, the pullback $\sigma^* G^\sharp \to \mathbb{S}^2$ is a principal circle bundle, and integrality of the corresponding Chern class implies that $2\pi\mu_0(E) \in \mathbb{N}_0$. Since this holds for every Borel set, we conclude that μ_0 is a locally finite sum of point measures, and hence that $\mu = \mu_0 \otimes dt$ is concentrated on a closed embedded submanifold $S_U \subseteq U$ of dimension 1. Since the argument is local, the measure μ is concentrated on a closed, embedded, 1-dimensional submanifold $S \subseteq M$. Using (1.4), one shows that $\mathtt{d}\rho$ vanishes on the ideal of sections that vanish μ-almost everywhere. This proves the theorem at the Lie algebra level. The result at the group level follows because $\Gamma_c(S, \mathcal{K})$ is 1-connected.

1.2.3 Classification of positive energy representations

For manifolds with a fixed point free $\mathbb{R}$-action, the Localization theorem effectively reduces the projective positive energy representation theory to the 1-dimensional setting.

Compact manifolds. For *compact* manifolds M, we show in Chapter 8 that the localization theorem leads to a full classification. Indeed, since

$$S = \bigcup_{j=1}^{k} S_j$$

is a finite union of periodic orbits S_j, the group $\Gamma(S, \mathcal{K})$ is a finite product of *twisted loop groups* $\Gamma(S_j, \mathcal{K})$. The projective positive energy representations of twisted loop groups are classified in Section 8.1, using the rich structure and representation theory of affine Kac–Moody Lie algebras [54], combined with the method of holomorphic induction for Fréchet–Lie groups developed in [77, 79].

This leads to a full classification of projective positive energy representations of $\Gamma(M, \mathcal{K})$, which is detailed in Section 8.2. Up to unitary equivalence, every irreducible projective positive energy representation is determined by the following data.

- Finitely many periodic $\mathbb{R}$-orbits $S_j \subseteq M$, each equipped with a central charge $c_j \in \mathbb{N}$.
- For every pair (S_j, c_j), an anti-dominant integral weight λ_j of the corresponding affine Kac–Moody algebra with central charge $c_j \in \mathbb{N}$.

Moreover, every projective positive energy representation is a direct sum of irreducible ones.

Noncompact manifolds. For *noncompact* manifolds M, the situation is somewhat more intricate. Here S is a union of countably many $\mathbb{R}$-orbits S_j, each of which is diffeomorphic to either $\mathbb{R}$ or $\mathbb{S}^1$. These two cases are considered separately in Chapter 9.

In Section 9.1 we consider the case where S consists of countably many lines. Since the bundle $\mathcal{K}$ trivializes over every line, the gauge group $\Gamma_c(S, \mathcal{K})$ is a weak direct product of countably many copies of $C_c^\infty(\mathbb{R}, K)$. In order to arrive at a (partial) classification, we impose the additional condition that the projective positive energy representation admit a cyclic ground state vector $\Omega \in \mathcal{H}$ that is unique up to a scalar. In Theorem 9.11 we show that these *vacuum representations* are classified up to unitary equivalence by a non-zero central charge $c_j \in \mathbb{N}$ for every connected component $S_j \simeq \mathbb{R}$. The proof proceeds by reducing to the (important) special case $M = \mathbb{R}$, where the classification is essentially due to Tanimoto [102].

In Section 9.2 we consider the case where S consists of infinitely many circles. Here we impose the much less restrictive condition that $\mathcal{H}$ is a *ground state representation*. This means that $\mathcal{H}$ is generated under $\Gamma_c(S, \mathcal{K})$ by the space of ground states, but we do not require these ground states to be unique. We show that under an (essentially geometric) *spectral gap* condition, every positive energy representation is automatically a ground state representation. Since $\Gamma_c(S, \mathcal{K})$ admits projective positive energy representations of Type II and III, it is necessary to consider factor representations instead of irreducible ones. If all orbits in M are periodic, we show that the minimal, factorial ground state representations of $\Gamma_c(M, \mathcal{K})$ are classified up to unitary equivalence by 3 pieces of data. The first two are the same as in the case of compact manifolds.

- Countably many periodic orbits $S_j \subseteq M$, equipped with a central charge $c_j \in \mathbb{N}$.
- For every pair (S_j, c_j) an anti-dominant integral weight λ_j of the corresponding affine Kac–Moody algebra with central charge c_j.

The integral weight λ_j gives rise to a unitary lowest weight representation $\mathcal{H}_{\lambda_j}$ of the corresponding affine Kac–Moody algebra. Using the ground state projections P_j, we consider the collection of finite tensor products of the compact operators $K(\mathcal{H}_{\lambda_j})$ as a directed system of C^*-algebras. Its direct limit

$$\mathcal{B} = \bigotimes_j K(\mathcal{H}_{\lambda_j})$$

has a distinguished ground state projection

$$P_\infty = \bigotimes_j P_j .$$

The third datum needed to characterize a minimal factorial ground state representation is the following.

- A factorial representation of $\mathcal{B}$ that is generated by fixed points of the projection P_∞.

Since $P_\infty \mathcal{B} P_\infty$ is a UHF C^*-algebra, this provides a rich supply of representations of type II and III, in marked contrast with the compact case.

1.3 Connection to the existing literature

Abelian structure groups. If the structure Lie algebra $\mathfrak{k}$ is merely assumed to be reductive, then it decomposes as a direct sum $\mathfrak{k} = \mathfrak{z} \oplus \mathfrak{k}'$, where $\mathfrak{z}$ is abelian and the commutator algebra $\mathfrak{k}$ is semisimple. Since this decomposition is invariant under all automorphism, we obtain a corresponding decomposition on the level of Lie algebra bundles $\mathfrak{K} \cong \mathfrak{Z} \oplus \mathfrak{K}'$. Accordingly, the Lie algebra $\mathfrak{g} = \Gamma_c(M, \mathfrak{K})$ decomposes as a direct sum $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}'$ and this decomposition is orthogonal with respect to any 2-cocycle because $\mathfrak{g}'$ is perfect. Therefore, the classification of the positive energy representations basically reduces to the cases where $\mathfrak{k}$ is semisimple and where $\mathfrak{k}$ is abelian. We refer to Solecki's paper [100] for some interesting results concerning groups of maps with values in the circle group, and to [98] for related results pertaining to defects in conformal field theory. G. Segal's paper [97] contains a number of interesting results on projective positive energy representations of loop groups with values in a torus.

Integrating representations of infinite-dimensional Lie groups. The technique to integrate representations of infinite-dimensional Lie algebras to groups by first verifying suitable estimates has already been used by R. Goodman and N. Wallach in [32] to construct the irreducible unitary positive energy representations of loop groups and diffeomorphism groups. Their technique has later been refined by V. Toledano–Laredo [104] to larger classes of infinite-dimensional Lie algebras. Related results on integrating Lie algebra representations can be found in [52].

Non-commutative distributions. In [3] an irreducible unitary representation of $\mathcal{G}_c = \Gamma_c(M, \mathrm{Ad}(\Xi))$ is called a *non-commutative distribution*. In view of the Borchers–Arveson theorem [13], an irreducible projective positive energy representation of $\mathcal{G}_c \rtimes_\alpha \mathbb{R}$ remains irreducible when restricted to $\mathcal{G}_c$. In this sense we contribute to the program outlined in [3] by classifying those non-commutative distributions for M compact and K compact semisimple for which extensions to positive energy representations exist.

Tensor product representations. For any, not necessary compact, Lie group K, the group $C(X, K)$ has unitary representations obtained as finite tensor products of evaluation representations. However, for some noncompact groups, such as $K = \widetilde{\mathrm{SU}}_{1,n}(\mathbb{C})$, one even has "continuous" tensor product representations which are irreducible and extend to groups of measurable maps (cf. [101] for finite-dimensional target groups, [48], [9], [15], [16], [21], [23, 108], [14, 107] for semisimple target groups, [35] for a general discussion and classification results for locally compact target groups, [6] for classification results for compact and nilpotent target groups, and [88] for an example where the target group $\mathrm{U}(\infty)$ is infinite-dimensional). In the algebraic context, these representations also appear in [47] which contains a classification of various types of unitary representations generalizing highest weight representations. All these representations are most naturally defined on groups of measurable maps, so that they neither require a topology nor a smooth structure on X.

Derivative and energy representations. One of the first references concerning unitary representations of groups of smooth maps such as $C^\infty(\mathbb{R}, \mathrm{SU}(2,\mathbb{C}))$ is [22], where the authors introduce the concept of a *derivative representation* which depends only on the derivatives up to some order N in some point $t_0 \in \mathbb{R}$. These ideas can be combined with continuous tensor product representations to obtain factorizable representations that do not extend to groups of measurable maps [89], [90]. Further, there exist factorial representations of mapping groups defined most naturally on groups of Sobolev H^1-maps, the so-called energy representations (cf. [45, 46], [2], [109], [3], [4], [99], [5], [1]).

Central extensions. The problem of classifying all smooth projective irreducible unitary representations of gauge groups is still wide open. Our treatment in Chapter 5 of the present memoir, as well as our earlier work on bounded representations [51], suggests that a classification of the central extensions of gauge algebras can be a key step towards this goal. The second Lie algebra homology of $\mathfrak{sl}_n(\mathcal{A})$ for a unital ring $\mathcal{A}$ is due to Bloch [10] and Kassel–Loday [56], and the full homology ring of $\mathfrak{gl}(\mathcal{A})$ was characterized in terms of the cyclic homology of $\mathcal{A}$ by Tsygan and Loday–Quillen [57, 58, 106]. Some of these arguments were adapted to $C^\infty(M, \mathfrak{k})$ with semisimple $\mathfrak{k}$ by Pressley–Segal [94, Section 4.2], and to $\mathcal{A} \otimes \mathfrak{k}$ for general Lie algebras $\mathfrak{k}$ by [36, 82, 115] in the setting where $\mathcal{A}$ is commutative. For non-trivial Lie algebra bundles, the universal central extension of the gauge algebra was obtained in [53] from the compactly supported trivial case [61] using a localization trick.

The case where M is a torus. In [105] (see also [3, Section 5.4]) Torresani studies projective unitary "highest weight representations" of $C^\infty(\mathbb{T}^d, \mathfrak{k})$, where $\mathfrak{k}$ is compact simple. Besides the finite tensor products of so-called evaluation representations (*elementary representations*) he finds finite tensor products of evaluation representations of $C^\infty(\mathbb{T}^d, \mathfrak{k}) \cong C^\infty(\mathbb{T}^{d-1}, C^\infty(\mathbb{T}, \mathfrak{k}))$, where the representations of the target algebra $C^\infty(\mathbb{T}, \mathfrak{k})$ are projective highest weight representations (*semi-elementary representations*). Our results in Chapter 8 reduce to this picture in the special case of a circle action on a torus.

Norm continuous representations. In a previous paper [50], we considered the related problem of classifying norm continuous unitary representations of the connected groups $\Gamma_c(M, \mathcal{K})_0$. In this case the problem also reduces to the case where $\mathfrak{k}$ is compact semisimple and the representations are linear rather than projective. For every irreducible representation ρ, there exists an embedded 0-dimensional submanifold S, i.e., a locally finite subset, $S \subseteq M$ such that ρ factors through the restriction map $\Gamma_c(M, \mathfrak{K}) \to \Gamma_c(S, \mathfrak{K}) \cong \mathfrak{k}^{(S)}$. If M is compact, it follows that ρ is a finite tensor product of irreducible representations obtained by composing an irreducible representation of $\mathfrak{k}$ with the evaluation in a point $s \in S$. In particular, it is finite-dimensional. If M is noncompact, then the bounded representation theory of the LF-Lie algebra $\Gamma_c(M, \mathfrak{K})$ is "wild" in the sense that there exist bounded factor representations of type II and III. The main result in [50] is a complete reduction of the classification of bounded irreducible representations to the classification of irreducible representations of UHF C^*-algebras.

Type III representations from noncompact orbits. For noncompact M, a different source of representations comes from the group $C_c^\infty(\mathbb{R}, K)$ corresponding to a single noncompact connected component of S. Here representations of Type III_1 were constructed in [19, 112]. Other results in this context have recently been obtained in [17], where solitonic representations of conformal nets on the circle are constructed from non-smooth diffeomorphisms. These in turn provide positive energy representations of $C_c^\infty(\mathbb{R}, K) \cong C_c^\infty(\mathbb{T} \setminus \{-1\}, K)$ which do not extend to loop group representations [17, Theorem 3.4, Section 4.2]. In particular, irreducible representations of this type are obtained.

Chapter 2

Projective representations of Lie groups

In this chapter, we introduce Lie groups modeled on locally convex vector spaces, or *locally convex Lie groups* for short. This is a generalization of the concept of a finite-dimensional Lie group that captures a wide range of interesting examples (cf. [71] for an overview), including gauge groups, our main object of study. We then summarize the central results from [52], which allow us to go back and forth between smooth projective unitary representations of a locally convex Lie group G and smooth unitary representations of a central Lie group extension $G^\sharp$ of G. On the identity component G_0, these are characterized by representations of the corresponding Lie algebra $\mathfrak{g}^\sharp$.

2.1 Locally convex Lie groups

Let E and F be locally convex spaces, $U \subseteq E$ open and $f : U \to F$ a map. Then, the *derivative of f at x in the direction h* is defined as

$$\partial_h f(x) := \lim_{t \to 0} \frac{1}{t}(f(x+th) - f(x))$$

whenever it exists. We set $Df(x)(h) := \partial_h f(x)$. The function f is called *differentiable at x* if $Df(x)(h)$ exists for all $h \in E$. It is called *continuously differentiable* if it is differentiable at all points of U and

$$Df : U \times E \to F, \quad (x, h) \mapsto Df(x)(h)$$

is a continuous map. Note that this implies that the maps $Df(x)$ are linear (cf. [30, Lemma 1.2.11]). The map f is called a *C^k-map*, $k \in \mathbb{N} \cup \{\infty\}$, if it is continuous, the iterated directional derivatives

$$D^j f(x)(h_1, \ldots, h_j) := (\partial_{h_j} \cdots \partial_{h_1} f)(x)$$

exist for all integers $1 \leq j \leq k$, $x \in U$ and $h_1, \ldots, h_j \in E$, and all maps

$$D^j f : U \times E^j \to F$$

are continuous. As usual, C^∞-maps are called *smooth*.

Once the concept of a smooth function between open subsets of locally convex spaces is established, it is clear how to define a locally convex smooth manifold (cf. [71], [30]).

Definition 2.1. A *locally convex Lie group* G is a group equipped with a smooth manifold structure modeled on a locally convex space for which the group multiplication and the inversion are smooth maps. Morphisms of locally convex Lie groups are smooth group homomorphisms.

We write $\mathbf{1} \in G$ for the identity element. The Lie algebra $\mathfrak{g}$ of G is identified with the tangent space $T_{\mathbf{1}}(G)$, and the Lie bracket is obtained by identification with the Lie algebra of left invariant vector fields. It is a *locally convex Lie algebra* in the following sense.

Definition 2.2. A *locally convex Lie algebra* is a locally convex vector space $\mathfrak{g}$ with a continuous Lie bracket $[\cdot,\cdot]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. Morphisms of locally convex Lie algebras are continuous Lie algebra homomorphisms.

Definition 2.3. A smooth map $\exp\colon \mathfrak{g} \to G$ is called an *exponential function* if each curve $\gamma_x(t) := \exp(tx)$ is a one-parameter group with $\gamma_x'(0) = x$. A Lie group G is said to be *locally exponential* if it has an exponential function for which there is an open 0-neighborhood U in $\mathfrak{g}$ mapped diffeomorphically by exp onto an open subset of G.

2.2 Smooth representations

Let G be a locally convex Lie group with Lie algebra $\mathfrak{g}$ and exponential function $\exp\colon \mathfrak{g} \to G$. In the context of Lie theory, it is natural to study *smooth* (projective) unitary representations on a complex Hilbert space $\mathcal{H}$. We take the scalar product on $\mathcal{H}$ to be linear in the *second* argument, and denote the group of unitary operators by $\mathrm{U}(\mathcal{H})$.

2.2.1 Unitary representations

A *unitary representation* $(\rho, \mathcal{H})$ of G is a Hilbert space $\mathcal{H}$ with a group homomorphism $\rho\colon G \to \mathrm{U}(\mathcal{H})$. A *unitary equivalence* between $(\rho, \mathcal{H})$ and $(\rho', \mathcal{H}')$ is a unitary transformation $U\colon \mathcal{H} \to \mathcal{H}'$ such that

$$U \circ \rho(g) = \rho(g)' \circ U \quad \text{for all } g \in G.$$

Definition 2.4 (Continuous unitary representations). A unitary representation $(\rho, \mathcal{H})$ is called *continuous* if the orbit map $G \to \mathcal{H}\colon g \mapsto \rho(g)\psi$ is continuous for all $\psi \in \mathcal{H}$ (see [78] for more details).

Definition 2.5 (Smooth unitary representations). We call $\psi \in \mathcal{H}$ a *smooth vector* if the orbit map $g \mapsto \rho(g)\psi$ is smooth, and write $\mathcal{H}^\infty \subseteq \mathcal{H}$ for the subspace of smooth vectors. We say that ρ is *smooth* if $\mathcal{H}^\infty$ is dense in $\mathcal{H}$ (see [52] for more details).

Every smooth representation is continuous. A representation ρ is called *bounded* if $\rho\colon G \to \mathrm{U}(\mathcal{H})$ is continuous with respect to the norm topology on $\mathrm{U}(\mathcal{H})$. Boundedness implies continuity, but many interesting continuous representations, including the (smooth!) positive energy representations that are the main focus of this memoir, are unbounded. For some recent results on the automatic smoothness of unbounded unitary representations satisfying certain spectral conditions such as semiboundedness (Definition 6.31), we refer to [114].

Definition 2.6. (Derived representation) For a smooth unitary representation $(\rho, \mathcal{H})$, the *derived representation* $\mathtt{d}\rho\colon \mathfrak{g} \to \mathrm{End}(\mathcal{H}^\infty)$ of the Lie algebra $\mathfrak{g}$ is defined by

$$\mathtt{d}\rho(\xi)\psi := \frac{d}{dt}\bigg|_{t=0} \rho(\exp t\xi)\psi.$$

Remark 2.7 (Selfadjoint generators). The closure of any operator $\mathtt{d}\rho(\xi)$ coincides with the infinitesimal generator of the unitary one-parameter group $\rho(\exp t\xi)$. In particular, the operators $i \cdot \mathtt{d}\rho(\xi)$ are essentially selfadjoint by Stone's theorem (cf. [95, Section VIII.4]).

2.2.2 Projective unitary representations

Let $\mathcal{H}$ be a Hilbert space. The projective Hilbert space is denoted by $\mathbb{P}(\mathcal{H})$, and its elements are denoted $[\psi] = \mathbb{C}\psi$ for non-zero $\psi \in \mathcal{H}$. We denote the projective unitary group by

$$\mathrm{PU}(\mathcal{H}) := \mathrm{U}(\mathcal{H})/\mathbb{T}\mathbf{1}$$

and write $\overline{U}$ for the image of $U \in \mathrm{U}(\mathcal{H})$ in $\mathrm{PU}(\mathcal{H})$.

A *projective unitary representation* $(\bar{\rho}, \mathcal{H})$ of a locally convex Lie group G is a complex Hilbert space $\mathcal{H}$ with a group homomorphism $\bar{\rho}\colon G \to \mathrm{PU}(\mathcal{H})$. A *unitary equivalence* between $(\bar{\rho}, \mathcal{H})$ and $(\bar{\rho}', \mathcal{H}')$ is a unitary transformation $U\colon \mathcal{H} \to \mathcal{H}'$ such that $\overline{U} \circ \bar{\rho}(g) = \bar{\rho}(g)' \circ \overline{U}$ for all $g \in G$.

A projective unitary representation yields an action of G on $\mathbb{P}(\mathcal{H})$. Since $\mathbb{P}(\mathcal{H})$ is a Hilbert manifold, we can use this to define continuous and smooth projective representations.

Definition 2.8 (Continuous projective unitary representations). A projective unitary representation $(\bar{\rho}, \mathcal{H})$ is called *continuous* if the orbit map $G \to \mathbb{P}(\mathcal{H})\colon g \mapsto \bar{\rho}(g)[\psi]$ is continuous for all $[\psi] \in \mathbb{P}(\mathcal{H})$.

Definition 2.9 (Smooth projective unitary representations). A ray $[\psi] \in \mathbb{P}(\mathcal{H})$ is called *smooth* if its orbit map $g \mapsto \bar{\rho}(g)[\psi]$ is smooth, and we denote the set of smooth rays by $\mathbb{P}(\mathcal{H})^\infty$. A projective unitary representation $(\bar{\rho}, \mathcal{H})$ is called *smooth* if $\mathbb{P}(\mathcal{H})^\infty$ is dense in $\mathcal{H}$ (cf. [52, 78]).

2.3 Central extensions

In this memoir, we are primarily interested in smooth projective unitary representations $\bar{\rho}\colon G \to \mathrm{PU}(\mathcal{H})$ of a locally convex Lie group G. We call $\bar{\rho}$ *linear* if it comes from a smooth unitary representation $\rho\colon G \to \mathrm{U}(\mathcal{H})$.

Although not every smooth projective unitary representation of G is linear, it can always be viewed as a smooth linear representation of a *central extension* of G by the circle group

$$\mathbb{T} \cong \mathbb{R}/2\pi\mathbb{Z}.$$

Definition 2.10 (Central group extensions). A *central extension* of G by $\mathbb{T}$ is an exact sequence

$$1 \to \mathbb{T} \to G^\sharp \to G \to 1$$

of locally convex Lie groups (the arrows are smooth group homomorphisms) such that the image of $\mathbb{T}$ is central in $G^\sharp$ and $G^\sharp \to G$ is a locally trivial principal $\mathbb{T}$-bundle. An *isomorphism* $\Phi\colon G^\sharp \to G^{\sharp\prime}$ of central $\mathbb{T}$-extensions is an isomorphism of locally convex Lie groups that induces the identity maps on G and $\mathbb{T}$.

For a smooth projective unitary representation $(\bar{\rho}, \mathcal{H})$ of G, the group

$$G^\sharp := \left\{(g, U) \in G \times \mathrm{U}(\mathcal{H}) : \bar{\rho}(g) = \overline{U}\right\} \tag{2.1}$$

is a central Lie group extension of G by $\mathbb{T}$ [52, Theorem 4.3]. Its smooth unitary representation

$$\rho\colon G^\sharp \to \mathrm{U}(\mathcal{H}), \quad (g, U) \mapsto U$$

reduces to $z \mapsto z\mathbf{1}$ on $\mathbb{T}$ and induces $\bar{\rho}$ on G. Since the restriction of ρ to the identity component $G_0^\sharp$ is determined by the derived Lie algebra representation [52, Proposition 3.4], it is worthwhile to take a closer look at central extensions of locally convex Lie algebras.

Definition 2.11 (Central Lie algebra extensions). A *central extension* of a locally convex Lie algebra $\mathfrak{g}$ by $\mathbb{R}$ is an exact sequence

$$0 \to \mathbb{R} \to \mathfrak{g}^\sharp \to \mathfrak{g} \to 0$$

of locally convex Lie algebras (the arrows are continuous Lie algebra homomorphisms) such that the image of $\mathbb{R}$ is central in $\mathfrak{g}^\sharp$. An *isomorphism* $\varphi\colon \mathfrak{g}^\sharp \to \mathfrak{g}^{\sharp\prime}$ of central extensions is an isomorphism of locally convex Lie algebras that induces the identity maps on $\mathfrak{g}$ and $\mathbb{R}$.

The group extensions of Definition 2.10 give rise to Lie algebra extensions in the sense of Definition 2.11. In order to classify the latter, we introduce continuous Lie algebra cohomology.

Definition 2.12. The *continuous Lie algebra cohomology* space $H^n(\mathfrak{g},\mathbb{R})$ of a locally convex Lie algebra $\mathfrak{g}$ is the cohomology of the complex $C^\bullet(\mathfrak{g},\mathbb{R})$, where $C^n(\mathfrak{g},\mathbb{R})$ consists of the continuous alternating linear maps $\mathfrak{g}^n \to \mathbb{R}$ with differential

$$\delta\colon C^n(\mathfrak{g},\mathbb{R}) \to C^{n+1}(\mathfrak{g},\mathbb{R})$$

defined by

$$\delta\omega(\xi_0,\dots,\xi_n) := \sum_{0\le i<j\le n} (-1)^{i+j}\omega\big([\xi_i,\xi_j],\xi_1,\dots,\widehat{\xi}_i,\dots,\widehat{\xi}_j,\dots,\xi_n\big).$$

The second Lie algebra cohomology $H^2(\mathfrak{g},\mathbb{R})$ classifies central extensions up to isomorphism. The 2-cocycle $\omega\colon \mathfrak{g}^2 \to \mathbb{R}$ gives rise to the Lie algebra

$$\mathfrak{g}^\sharp_\omega := \mathbb{R} \oplus_\omega \mathfrak{g}$$

with the Lie bracket

$$[(z,\xi),(z',\xi')] := \big(\omega(\xi,\xi'),[\xi,\xi']\big).$$

Equipped with the obvious maps $\mathbb{R} \to \mathfrak{g}^\sharp_\omega \to \mathfrak{g}$, this defines a central extension of $\mathfrak{g}$. Every central extension is isomorphic to one of this form, and two central extensions are isomorphic if and only if the corresponding cohomology classes $[\omega] \in H^2(\mathfrak{g},\mathbb{R})$ coincide [52, Proposition 6.3].

The following theorem collects some of the main results of our previous paper [52, Corollary 4.5, Theorem 7.3]. It allows us to go back and forth between smooth projective unitary representations of G, smooth unitary representations of a central extension $G^\sharp$ of G, and the corresponding representations of its Lie algebra $\mathfrak{g}^\sharp$.

Theorem 2.13 (Projective G-representations and linear $\mathfrak{g}^\sharp$-representations).

(a) *Every smooth projective unitary representation $(\bar{\rho},\mathcal{H})$ of G gives rise to a central extension $\mathbb{T} \to G^\sharp \to G$ of locally convex Lie groups, and a smooth unitary representation $(\rho,\mathcal{H})$ of $G^\sharp$. In turn, this gives rise to the central extension $\mathbb{R} \to \mathfrak{g}^\sharp \to \mathfrak{g}$ of locally convex Lie algebras and the derived representation $\mathtt{d}\rho\colon \mathfrak{g}^\sharp \to \mathrm{End}(\mathcal{H}^\infty)$ of $\mathfrak{g}^\sharp$ by essentially skew-adjoint operators.*

(b) *If G is connected, then $(\bar{\rho},\mathcal{H})$ and $(\bar{\rho}',\mathcal{H}')$ are unitarily equivalent if and only if the derived Lie algebra representations $(\mathtt{d}\rho,\mathcal{H}^\infty)$ and $(\mathtt{d}\rho',\mathcal{H}'^\infty)$ are unitarily equivalent. This means that there exists an isomorphism $\varphi\colon \mathfrak{g}^\sharp \to \mathfrak{g}^{\sharp\prime}$ of central extensions and a unitary isomorphism $U\colon \mathcal{H} \to \mathcal{H}'$ such that*

$$U\mathcal{H}^\infty \subseteq \mathcal{H}^{\infty\prime}$$

and

$$\mathtt{d}\rho'(\varphi(\xi)) \circ U = U \circ \mathtt{d}\rho(\xi) \quad \textit{for all } \xi \in \mathfrak{g}^\sharp.$$

2.4 Integration of projective representations

In this section we discuss the integrability of (projective) unitary representations of Banach–Lie algebras, based on the existence of analytic vectors. Here our main result is the integrability theorem for projective representations of Banach–Lie groups (Theorem 2.18) that we derive with the methods from [76].

Definition 2.14. Let $(\rho, \mathcal{D})$ be a representation of the topological Lie algebra $\mathfrak{g}$ on the pre-Hilbert space $\mathcal{D}$. We say that

(i) ρ is a $*$-*representation* if all operators $\rho(x)$, $x \in \mathfrak{g}$, are skew-symmetric,

(ii) ρ is *strongly continuous* if all the maps $\mathfrak{g} \to \mathcal{D}, x \mapsto \rho(x)\xi$ are continuous,

(iii) $\xi \in \mathcal{D}$ is an *analytic vector* if there exists a 0-neighborhood $U \subseteq \mathfrak{g}$ such that $\sum_{n=0}^{\infty} \frac{\|\rho(x)^n\xi\|}{n!} < \infty$ for every $x \in U$. The analytic vectors form a linear subspace $\mathcal{D}^\omega \subseteq \mathcal{D}$.

Remark 2.15. If $\mathfrak{g}$ is a Banach–Lie algebra, then [76, Proposition 4.10] implies that $\xi \in \mathcal{D}$ is an analytic vector if and only if it is an analytic vector for all operators $\rho(x)$, $x \in \mathfrak{g}$ in the sense that there exists an $s > 0$ such that

$$\sum_{n=0}^{\infty} \frac{s^n \|\rho(x)^n \xi\|}{n!} < \infty.$$

We shall need the following lemma that is not spelled out explicitly in [76].

Lemma 2.16. *Let $(\rho, \mathcal{D})$ be a strongly continuous $*$-representation of the Banach–Lie algebra $\mathfrak{g}$. Then, $\mathcal{D}^\omega$ is a $\rho(\mathfrak{g})$-invariant subspace.*

Proof. Following [76, Definition 3.2], we call a linear functional $\beta\colon U(\mathfrak{g}) \to \mathbb{C}$ on the enveloping algebra of $\mathfrak{g}$ an *analytic functional* if all n-linear maps

$$\mathfrak{g}^n \to \mathbb{C}, \quad (x_1, \ldots, x_n) \mapsto \beta(x_1 \cdots x_n)$$

are continuous and the series $\sum_{n=0}^{\infty} \frac{\beta(x^n)}{n!}$ converges for every x in a 0-neighborhood of $\mathfrak{g}$. According to [76, Proposition 6.3], a vector $\xi \in \mathcal{D}$ is analytic if and only if the functional $\beta_\xi(D) := \langle \xi, \rho(D)\xi \rangle$ is analytic, where $\rho\colon U(\mathfrak{g}) \to \mathrm{End}(\mathcal{D})$ denotes the extension of ρ to the enveloping algebra. For $\xi \in \mathcal{D}^\omega$ and $x \in \mathfrak{g}$, the functional

$$\beta_{\rho(x)\xi}(D) := \langle \rho(x)\xi, \rho(D)\rho(x)\xi \rangle = \beta_\xi((-x)Dx)$$

is also analytic by [76, Theorem 3.6], so that $\rho(x)\xi \in \mathcal{D}^\omega$ by [76, Proposition 6.3]. This shows that $\rho(\mathfrak{g})\mathcal{D}^\omega \subseteq \mathcal{D}^\omega$. ∎

To formulate the integrability theorem for projective representations, we first give a precise definition of a projective $*$-representation of a topological Lie algebra $\mathfrak{g}$.

Definition 2.17. Suppose that $\omega\colon \mathfrak{g}^2 \to \mathbb{R}$ is a continuous 2-cocycle and that

$$\mathfrak{g}^\sharp_\omega = \mathbb{R} \oplus_\omega \mathfrak{g}$$

is the corresponding central extension. Then, any $*$-representation $(\rho^\sharp, \mathcal{D})$ with

$$\rho^\sharp(1, 0) = i\mathbf{1}$$

leads to a linear map

$$\rho\colon \mathfrak{g} \to \operatorname{End}(\mathcal{D}), \quad \rho(x) := \rho^\sharp(0, x)$$

satisfying

$$[\rho(x), \rho(y)] = \rho([x, y]) + \omega(x, y)i\mathbf{1}.$$

We then call $(\rho, \mathcal{D})$ a *projective $*$-representation with cocycle ω.*

Theorem 2.18 (Integrability theorem for projective representations). *Let G be a 1-connected Banach–Lie group with Lie algebra $\mathfrak{g}$, and let $(\rho, \mathcal{D})$ be a projective strongly continuous $*$-representation of $\mathfrak{g}$ on the dense subspace $\mathcal{D}$ of the Hilbert space $\mathcal{H}$. If $\mathcal{D}$ contains a dense subspace of analytic vectors, then there exists a smooth projective unitary representation $\overline{\pi}\colon G \to \mathrm{PU}(\mathcal{H})$ on $\mathcal{H}$ with the property that $\overline{\pi}(\exp x) = q(e^{\overline{\rho(x)}})$ for $x \in \mathfrak{g}$, where $q\colon \mathrm{U}(\mathcal{H}) \to \mathrm{PU}(\mathcal{H})$ denotes the quotient map.*

Proof. We proceed as in the proof of [76, Theorem 6.8]. Using Lemma 2.16, we see that we may assume, without loss of generality, that $\mathcal{D} = \mathcal{D}^\omega$, so that $\mathcal{D}$ consists of analytic vectors. According to Nelson's theorem [86], the operators $\rho(x)$, $x \in \mathfrak{g}$, are essentially skew-adjoint, so that their closures generate unitary one-parameter groups. The same holds for the operators $\hat{\rho}(t, x)$, $(t, x) \in \mathfrak{g}^\sharp$. This leads to a map

$$\tilde{\pi}\colon \mathfrak{g}^\sharp \to \mathrm{U}(\mathcal{H}), \quad x \mapsto e^{\overline{\hat{\rho}(t,x)}} = e^{it} e^{\overline{\rho(x)}}.$$

From the proof of [76, Theorem 6.8], we immediately derive that

$$\tilde{\pi}((t, x) * (s, y)) = \tilde{\pi}(t, x)\tilde{\pi}(s, y)$$

holds for $(t, x), (s, y)$ in some open 0-neighborhood $U^\sharp \subseteq \mathfrak{g}^\sharp$. This implies that

$$q(e^{\overline{\rho(x*y)}}) = q(e^{\overline{\rho(x)}})q(e^{\overline{\rho(y)}})$$

for x, y in some open 0-neighborhood $U \subseteq \mathfrak{g}$. Now [12, Chapter 3, Section 6, Lemma 1.1] implies the existence of a unique homomorphism $\overline{\pi}\colon G \to \mathrm{PU}(\mathcal{H})$ such that $\overline{\pi}(\exp x) = q(e^{\overline{\rho(x)}})$ holds for all elements x in some 0-neighborhood of $\mathfrak{g}$.

That $\overline{\pi}$ is a smooth projective representation (Definition 2.5) follows from the analyticity of the orbit maps $G \to \mathbb{P}(\mathcal{H})$, $g \mapsto \overline{\pi}(g)[v]$ for $v \in \mathcal{D}^\omega$, which in turn follows from

$$\overline{\pi}(\exp x)[v] = [e^{\overline{\rho(x)}}v].$$

∎

2.5 Double extensions

Suppose that G is a locally convex Lie group and $\alpha\colon \mathbb{R} \to \mathrm{Aut}(G)$ a homomorphism defining a smooth $\mathbb{R}$-action on G. Then, the semidirect product $G \rtimes_\alpha \mathbb{R}$ is a Lie group with Lie algebra $\mathfrak{g} \rtimes_D \mathbb{R}$, where $D \in \mathrm{der}(\mathfrak{g})$ is the infinitesimal generator of the $\mathbb{R}$-action on $\mathfrak{g}$ induced by α.

If $\bar{\rho}\colon G \rtimes_\alpha \mathbb{R} \to \mathrm{PU}(\mathcal{H})$ is a smooth projective unitary representation, then Theorem 2.13 yields a central extension

$$\mathbb{T} \to \widehat{G} := (G \rtimes_\alpha \mathbb{R})^\sharp \to G \rtimes_\alpha \mathbb{R}$$

with a smooth unitary representation ρ of $\widehat{G}$ on $\mathcal{H}$ that induces $\bar{\rho}$. From Theorem 2.13, we see that the restriction of $\bar{\rho}$ to $(G \rtimes_\alpha \mathbb{R})_0$ is determined up to unitary equivalence by the derived representation $\mathtt{d}\rho$ of the central extension $\widehat{\mathfrak{g}} = (\mathfrak{g} \rtimes_D \mathbb{R})^\sharp$. We identify $\mathfrak{g}$ with the linear subspace $\{0\} \times \mathfrak{g} \times \{0\}$ of $\widehat{\mathfrak{g}}$. We write

$$C := (1, 0, 0) \quad \text{and} \quad D := (0, 0, 1)$$

for the central element and derivation in $\mathbb{R} \oplus_\omega (\mathfrak{g} \rtimes_D \mathbb{R})$ respectively, so that

$$\widehat{\mathfrak{g}} = \mathbb{R}C \oplus_\omega (\mathfrak{g} \rtimes \mathbb{R}D). \tag{2.2}$$

We trust that using the same symbol for the derivation $D \in \mathrm{der}(\mathfrak{g})$ and the Lie algebra element $D \in \widehat{\mathfrak{g}}$ that implements it will not lead to confusion. Note that in the representation $\mathtt{d}\rho$ of $\widehat{\mathfrak{g}}$, the central element C acts by $i\mathbf{1}$. Writing $(z, x, t) = zC + x + tD$, the bracket in $\widehat{\mathfrak{g}}$ takes the form

$$\begin{aligned}&[zC + x + tD, z'C + x' + t'D] \\ &\qquad = \big(\omega(x, x') + t\omega(D, x') - t'\omega(D, x)\big)C + ([x, x'] + tDx' - t'Dx).\end{aligned}$$

Chapter 3

Positive energy representations

In this chapter we introduce positive energy representations and some tools to handle them. In Section 3.1 we give the precise definition on both the linear and the projective level, and in Section 3.2 we define equivariant positive energy representations. In Section 3.3, we use the Borchers–Arveson theorem to reduce the classification of positive energy representations to the so-called *minimal* ones. Finally, in Section 3.4 we describe the key tool of this memoir in a first general form: the Cauchy–Schwarz estimates for projective positive energy representations. Here we will discuss them for general groups, but they will be refined in the context of gauge algebras in Section 5.3 below.

3.1 Positive energy representations

Let G be a locally convex Lie group with Lie algebra $\mathfrak{g}$ and let $\alpha\colon \mathbb{R} \to \mathrm{Aut}(G)$ be a homomorphism defining a smooth $\mathbb{R}$-action on G. Then, it also induces a smooth action $\alpha^{\mathfrak{g}}$ on $\mathfrak{g}$ and we write $D \in \mathrm{der}(\mathfrak{g})$ for its infinitesimal generator

$$Dx := \frac{d}{dt}\bigg|_{t=0} \alpha_t^{\mathfrak{g}}(x) \quad \text{for } x \in \mathfrak{g}.$$

In this section, we investigate smooth projective unitary representations of G that extend to projective *positive energy* representations of $G \rtimes_\alpha \mathbb{R}$.

Definition 3.1 (Projective positive energy representations). A smooth, projective, unitary representation $\bar{\rho}\colon G \rtimes_\alpha \mathbb{R} \to \mathrm{PU}(\mathcal{H})$ is called a *positive energy representation* if one (hence any) strongly continuous homomorphic lift $U\colon \mathbb{R} \to \mathrm{U}(\mathcal{H})$ of $\overline{U}\colon \mathbb{R} \to \mathrm{PU}(\mathcal{H}), t \mapsto \bar{\rho}(\mathbf{1}, t)$ has a generator

$$H := i\frac{d}{dt}\bigg|_{t=0} U_t$$

whose spectrum is bounded below. We then call H a *Hamiltonian* and note that $U_t = e^{-itH}$ holds in the sense of functional calculus.

Remark 3.2. By adding a constant, we can always choose a Hamiltonian H that satisfies $\mathrm{Spec}(H) \subseteq [0, \infty)$.

We have seen in Section 2.5 that every smooth projective unitary representation $\bar{\rho}$ of $G \rtimes_\alpha \mathbb{R}$ gives rise to a smooth linear representation $(\rho, \mathcal{H})$ of a locally convex

Lie group $\widehat{G} = (G \rtimes_\alpha \mathbb{R})^\sharp$, a central $\mathbb{T}$-extension of $G \rtimes_\alpha \mathbb{R}$ with Lie algebra

$$\widehat{\mathfrak{g}} = \mathbb{R} \oplus_\omega (\mathfrak{g} \rtimes_D \mathbb{R}) = \mathbb{R}C \oplus_\omega (\mathfrak{g} \rtimes \mathbb{R}D)$$

as in (2.2).

Definition 3.3 (Linear positive energy representations). Let $\rho\colon \widehat{G} \to \mathrm{U}(\mathcal{H})$ be a smooth unitary representation of $\widehat{G}$. Then, ρ gives rise to a derived representation $\mathtt{d}\rho$ of $\widehat{\mathfrak{g}}$ on the space $\mathcal{H}^\infty$ of smooth vectors. We call

$$H := i\,\mathtt{d}\rho(D)$$

the *Hamiltonian* and we say that ρ is a *positive energy representation* if

$$\mathtt{d}\rho(C) = i\mathbf{1} \quad \text{and if } \mathrm{Spec}(H) \subseteq [0,\infty).$$

Remark 3.4. (a) If $\mathtt{d}\rho(C) = i\mathbf{1}$ and $\mathrm{Spec}(H) \subseteq [E_0,\infty)$ is bounded below, we can always replace D by $D + E_0C$ to obtain a positive Hamiltonian. Note that this does not change the cocycle ω on $\mathfrak{g} \rtimes_D \mathbb{R}$, only the isomorphism between $\widehat{\mathfrak{g}}$ and $(\mathfrak{g} \rtimes_D \mathbb{R})^\sharp$.

(b) For a cocycle ω on $\mathfrak{g} \rtimes_D \mathbb{R}$, the relation

$$\omega(D,[\xi,\eta]) = \omega(D\xi,\eta) + \omega(\xi,D\eta)$$

shows that the linear functional $i_D\omega$ measures the non-invariance of the restriction of ω to $\mathfrak{g} \times \mathfrak{g}$ under the derivation D. It also shows that if the Lie algebra $\mathfrak{g}$ is perfect, then the linear functional $i_D\omega\colon \mathfrak{g} \to \mathbb{R}$ is completely determined by the restriction of ω to $\mathfrak{g} \times \mathfrak{g}$.

3.2 Equivariant positive energy representations

We will also need an equivariant version of positive energy representations. Let P be a Lie group with Lie algebra $\mathfrak{p}$ and let $\alpha\colon P \to \mathrm{Aut}(G)$ be a homomorphism defining a smooth P-action on G.

Definition 3.5 (Equivariant projective positive energy representations). A smooth, projective, unitary representation $\bar\rho\colon G \rtimes_\alpha P \to \mathrm{PU}(\mathcal{H})$ is called a *positive energy representation with respect to $p \in \mathfrak{p}$* if the projective representation

$$\bar\rho_p\colon G \rtimes_{\alpha\circ\exp_p} \mathbb{R} \to \mathrm{PU}(\mathcal{H})$$

defined by

$$\bar\rho_p(g,t) := \bar\rho(g,\exp(pt))$$

is of positive energy in the sense of Definition 3.1. The *positive energy cone* $\mathcal{C} \subseteq \mathfrak{p}$ is the set of all elements $p \in \mathfrak{p}$ for which $\bar\rho$ is a positive energy representation.

Note that $\mathcal{C}$ is an Ad_P-invariant cone. In particular, the representation $\bar{\rho}$ is of positive energy with respect to $p \in \mathfrak{p}$ if and only if it is of positive energy for all elements in the cone generated by the adjoint orbit $\mathrm{Ad}_P(p) \subseteq \mathfrak{p}$ of p.

The homomorphism $\alpha: P \to \mathrm{Aut}(G)$ can be twisted by an inner automorphism Ad_{g_0}, $g_0 \in G$, yielding

$$\alpha' = \mathrm{Ad}_{g_0} \alpha \mathrm{Ad}_{g_0}^{-1}.$$

Essentially, these inner twists do not affect the class of equivariant projective positive energy representations.

Proposition 3.6. *Let $(\bar{\rho}, \mathcal{H})$ be an equivariant projective positive energy representation of $G \rtimes_\alpha P$, and let*

$$\bar{U}_0 := \bar{\rho}(g_0).$$

Then

$$\bar{\rho}'(g, p) := \bar{U}_0 \bar{\rho}(\mathrm{Ad}_{g_0}^{-1}(g), p) \bar{U}_0^{-1}$$

is an equivariant projective positive energy representation of $G \rtimes_{\alpha'} P$ with the same restriction to G, and with the same positive energy cone $\mathcal{C} \subseteq \mathfrak{p}$.

Proof. To see that $\bar{\rho}'$ is a projective representation of $G \rtimes_{\alpha'} P$, one checks that the following is a commutative diagram of group homomorphisms:

$$\begin{tikzcd}
G \rtimes_\alpha P \arrow[r, "\bar{\rho}"] \arrow[d, "(\mathrm{Ad}_{g_0}, \mathrm{Id}_P)"'] & \mathrm{PU}(\mathcal{H}) \arrow[d, "\mathrm{Ad}_{U_0}"] \\
G \rtimes_{\alpha'} P \arrow[r, "\bar{\rho}'"] & \mathrm{PU}(\mathcal{H}).
\end{tikzcd}$$

For the positive energy condition, note that any lift $t \mapsto V_t$ of $t \mapsto \bar{\rho}(\exp(tp))$ yields a lift $t \mapsto U_0 V_t U_0^{-1}$ of $t \mapsto \bar{\rho}'(\exp(tp))$ whose generator has the same spectrum. ■

3.3 Minimal representations

The following refinement of the Borchers–Arveson theorem [13] will be used in the proof of Corollary 3.9 below.

Theorem 3.7. *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra. Further, let $(U_t)_{t \in \mathbb{R}}$ be a strongly continuous unitary one-parameter group on $\mathcal{H}$ for which $\mathcal{M}$ is invariant under conjugation with the operators U_t, so that we obtain a one-parameter group $\alpha: \mathbb{R} \to \mathrm{Aut}(\mathcal{M})$ by*

$$\alpha_t(M) := \mathrm{Ad}(U_t)M := U_t M U_t^* \quad \text{for } M \in \mathcal{M}.$$

If $U_t = e^{-itH}$ with $H \geq 0$, then the following assertions hold:

(i) *there exists a strongly continuous unitary one-parameter group $(V_t)_{t\in\mathbb{R}}$ in $\mathcal{M}$ with*
$$\mathrm{Ad}(V_t) = \alpha_t \quad \textit{and} \quad V_t = e^{-itH_0}$$
with $H_0 \geq 0$. It is uniquely determined by the requirement that it is minimal in the sense that, for any other one-parameter group $(V_t')_{t\in\mathbb{R}}$ with these properties, the central one-parameter group $V_t'V_{-t} = e^{-itZ}$ in $\mathcal{M}$ satisfies $Z \geq 0$,

(ii) *if $V_T = \mathbf{1}$ for some $T > 0$ and $\mathcal{F} \subseteq \mathcal{H}$ is an $\mathcal{M}$-invariant subspace, then the subspace*
$$\mathcal{F}_0 := \{\xi \in \mathcal{F} : H_0\xi = 0\}$$
is $\mathcal{M}$-generating in $\mathcal{F}$,

(iii) *if $\alpha_T = \mathrm{id}_{\mathcal{M}}$ for some $T > 0$, then $V_T = \mathbf{1}$.*

Proof. (i) This is the Borchers–Arveson theorem (see [11, Theorem II.4.6]; also [13, Theorem 3.2.46] and [8] for a detailed discussion).

(ii) If $V_T = \mathbf{1}$, then $\mathrm{Spec}(H_0) \subseteq \frac{2\pi}{T}\mathbb{Z}$. In particular, H_0 is diagonalizable. If $\mathcal{F}_0$ is not $\mathcal{M}$-generating in $\mathcal{F}$, then
$$\mathcal{E} := (\mathcal{M}\mathcal{F}_0)^\perp \cap \mathcal{F}$$
is a non-zero $\mathcal{M}$-invariant subspace of $\mathcal{F}$ with
$$\inf \mathrm{Spec}(H_0|_{\mathcal{E}}) \geq \frac{2\pi}{T}.$$
As
$$\mathcal{H}_0 := \ker H_0 \subseteq \mathcal{E}^\perp,$$
we also have $\mathcal{M}\mathcal{H}_0 \subseteq \mathcal{E}^\perp$. Since $\mathcal{M}\mathcal{H}_0$ is invariant under $\mathcal{M}$ and $\mathcal{M}'$, the orthogonal projection Z onto
$$\mathcal{H}_1 := (\mathcal{M}\mathcal{H}_0)^\perp$$
is central in $\mathcal{M}$. On this subspace we have $\inf \mathrm{Spec}(H_0|_{\mathcal{H}_1}) \geq \frac{2\pi}{T}$, so that
$$H := H_0 - Z\frac{2\pi}{T} \geq 0,$$
contradicting minimality.

(iii) If $\alpha_T = \mathrm{id}_{\mathcal{M}}$, then V_T is contained in the center $\mathcal{Z}(\mathcal{M}) = \mathcal{M} \cap \mathcal{M}'$ of $\mathcal{M}$. As $\mathcal{Z}(\mathcal{M})$ is a direct sum of L^∞-algebras, there exists a non-negative $Z \geq 0$ in $\mathcal{Z}(\mathcal{M})$ with $\mathrm{Spec}(Z) \subseteq [0, \frac{2\pi}{T}]$ and $V_T = e^{iTZ}$. Now
$$V_t' := e^{-it(H_0+Z)} = V_t e^{-itZ}$$

also has a non-negative generator $H_1 := H_0 + Z$ and satisfies $V'_T = \mathbf{1}$. In particular,

$$\mathrm{Spec}(H_1) \subseteq \frac{2\pi}{T}\mathbb{Z}.$$

We claim that the minimality of V implies that, for every $\varepsilon > 0$, the central support of the spectral projection $P := P^{H_0}[0,\varepsilon]$ of H_0 in $\mathcal{M}$ equals $\mathbf{1}$. To see this, note that the central support Q of P is the orthogonal projection onto the closed subspace generated by $\mathcal{M}P\mathcal{H}$. If this subspace is proper, then the restriction H_1 of H_0 to $\mathcal{H}_1 := (\mathbf{1} - Q)\mathcal{H}$ satisfies $H_1 \geq \varepsilon\mathbf{1}$, so that

$$H' := H_0 - \varepsilon(\mathbf{1} - Q) \geq 0.$$

The minimality of H_0 now yields $\mathbf{1} = Q$.

We now show that $\mathrm{Spec}(Z) \subseteq \{0, \frac{2\pi}{T}\}$, which implies that

$$V_T = V'_T e^{-iTZ} = V'_T = \mathbf{1}.$$

Assume that this is not the case. Then, there exists a non-zero spectral value $0 < a < \frac{2\pi}{T}$ of Z. Let $\varepsilon > 0$ be such that $0 < a - 2\varepsilon < a + 2\varepsilon < \frac{2\pi}{T}$ and consider the spectral projection $Q := P^Z([a-\varepsilon, a+\varepsilon])$ for Z, which is contained in $\mathcal{Z}(\mathcal{M})$. Since the central support of $P^{H_0}[0,\varepsilon]$ is $\mathbf{1}$, we have $QP^{H_0}([0,\varepsilon]) \neq 0$, so that

$$\mathrm{Spec}(QH_0) \cap [0,\varepsilon] \neq \emptyset.$$

Since $\mathrm{Spec}(QZ) \subseteq [a-\varepsilon, a+\varepsilon]$, this leads to

$$\mathrm{Spec}((H_0 + Z)Q) \cap [a-\varepsilon, a+2\varepsilon] \neq \emptyset.$$

This contradicts

$$\mathrm{Spec}((H_0 + Z)Q) = \mathrm{Spec}(H_1 Q) \subseteq \mathrm{Spec}(H_1) \subseteq \frac{2\pi}{T}\mathbb{Z}. \qquad \blacksquare$$

Using the Borchers–Arveson theorem, every smooth positive energy representation $(\rho, \mathcal{H})$ can be brought in the following standard form.

Definition 3.8 (Minimal representations). A positive energy representation $(\rho, \mathcal{H})$ of $\widehat{G}$ is called *minimal* if the 1-parameter group $U_t = \rho(\exp(tD))$ is minimal with respect to the von Neumann algebra $\rho(\widehat{G})''$.

Corollary 3.9. *Let $(\rho, \mathcal{H})$ be a positive energy representation of $\widehat{G}$ and let $G^\sharp \subseteq \widehat{G}$ be the inverse image of the subgroup G of $G \rtimes_\alpha \mathbb{R}$, so that $\widehat{G} \cong G^\sharp \rtimes \mathbb{R}$. Then, there exists a unitary 1-parameter group $(W_t)_{t\in\mathbb{R}}$ in the commutant $\rho(\widehat{G})'$ such that the representation $\rho_0(g,t) := \rho(g,t)W_t^{-1}$ has the following properties:*

(i) $\rho_0(\widehat{G})'' = \rho(G^\sharp)''$,

(ii) *if ρ is irreducible, then so is $\rho|_{G^\sharp}$,*

(iii) *if $\alpha_T = \mathrm{id}_G$, then $\rho_0(\mathbf{1}, T) = \mathbf{1}$ and, for every closed $\rho(\widehat{G})$-invariant subspace $\mathcal{F} \subseteq \mathcal{H}$, the subspace $\mathcal{F}_0 := \{\xi \in \mathcal{F} : H_0\xi = 0\}$ is $\widehat{G}$-generating in $\mathcal{F}$,*

(iv) *ρ_0 is a smooth positive energy representation.*

Proof. (i) Theorem 3.7 implies that $U_t := \rho(\exp tD)$ can be written as $U_t = V_t W_t$, where $(V_t)_{t\in\mathbb{R}}$ is a continuous unitary one-parameter group in the von Neumann algebra $\mathcal{M} := \rho(G^\sharp)''$ and $W_t \in \rho(G^\sharp)'$.

(ii) If ρ is irreducible, then Schur's Lemma implies that $W_t \in \mathbb{T}\mathbf{1}$, hence that the restriction $\rho|_{G^\sharp}$ remains irreducible.

(iii) follows from Theorem 3.7 (iii) and (ii).

(iv) As $V_t = \rho_0(\mathbf{1}, t)$ has a positive generator, ρ_0 also is a positive energy representation. It remains to see that ρ_0 is smooth. Since $(W_t)_{t\in\mathbb{R}}$ lies in the commutant $\rho(\widehat{G})'$, all its spectral subspaces are invariant under $\widehat{G}$. Therefore, ρ is a direct sum of subrepresentations for which W is norm continuous. We may therefore assume, without loss of generality, that W is norm continuous. Then, we can consider W as a smooth representation of $\widehat{G}$ and therefore $\rho_0(g,t) = \rho(g,t)W_{-t}$ is a smooth representation of $\widehat{G}$. ∎

In view of the factorization $\rho(g,t) = \rho_0(g,t)W_t$, we can adopt the point of view that we know all positive energy representations if we know the minimal ones. On the level of the irreducible representations, the only difference is a phase factor corresponding to the minimal energy level. In general, the ambiguity consists in unitary one-parameter groups of the commutant, and these can be classified in terms of spectral measures.

3.4 Cauchy–Schwarz estimates (general case)

We show that the requirement that a representation be of positive energy severely restricts the class of cocycles that may occur.

Let ρ be a positive energy representation of $\widehat{G}$. For a smooth unit vector $\psi \in \mathcal{H}^\infty$ the expectation values

$$\langle H\rangle_\psi := \langle \psi, H\psi\rangle \quad \text{and} \quad \langle i\,\mathrm{d}\rho(\xi)\rangle_\psi := \langle \psi, i\,\mathrm{d}\rho(\xi)\psi\rangle$$

of H and $\xi \in \mathfrak{g}$ are defined. The following is a non-commutative adaptation of [85, Theorem 2.8].

Lemma 3.10 (Cauchy–Schwarz estimate). *Let ρ be a positive energy representation of $\widehat{G}$, and let $\xi \in \mathfrak{g}$ be such that $[\xi, D\xi] = 0$. Then, for every unit vector $\psi \in \mathcal{H}^\infty$,* *we have*

$$\big(\langle i\, \mathrm{d}\rho(D\xi)\rangle_{\psi} + \omega(\xi, D)\big)^2 \leq 2\omega(\xi, D\xi)\langle H\rangle_{\psi},$$

and further $\omega(\xi, D\xi) \geq 0$.

Proof. Since $H = i\,\mathrm{d}\rho(D)$ has non-negative spectrum, the expectation value of the energy in the state defined by $\exp(t\,\mathrm{d}\rho(\xi))\psi$ is non-negative for all $t \in \mathbb{R}$;

$$0 \leq \langle H\rangle_{\exp(t\,\mathrm{d}\rho(\xi))\psi} = \langle e^{-t\,\mathrm{ad}_{\mathrm{d}\rho(\xi)}} H\rangle_{\psi}. \tag{3.1}$$

Since $[\xi, D\xi] = 0$, the exponential series terminates at order 2,

$$\begin{aligned}\exp(-t\,\mathrm{ad}_{\mathrm{d}\rho(\xi)})(H) &= i\,\mathrm{d}\rho(e^{-t\,\mathrm{ad}_\xi} D)\\ &= i\,\mathrm{d}\rho\Big(D + tD\xi - t\omega(\xi, D)C - \frac{t^2}{2}\omega(\xi, D\xi)C\Big)\\ &= H + t\big(i\,\mathrm{d}\rho(D\xi) + \omega(\xi, D)\big) + \frac{t^2}{2}\omega(\xi, D\xi),\end{aligned} \tag{3.2}$$

so that substitution in (3.1) yields the inequality

$$0 \leq \langle H\rangle_{\psi} + t\big(\langle i\,\mathrm{d}\rho(D\xi)\rangle_{\psi} + \omega(\xi, D)\big) + \frac{t^2}{2}\omega(\xi, D\xi) \quad \text{for } t \in \mathbb{R}.$$

The proposition now follows from the simple observation that $at^2 + bt + c \geq 0$ for all $t \in \mathbb{R}$ is equivalent to $0 \leq a, c$ and $b^2 \leq 4ac$. ∎

The Cauchy–Schwarz estimate will play an important role in the rest of the memoir. We will use it mainly in situations where $\omega(D, \mathfrak{g}) = \{0\}$, so that the bilinear form $(\xi, \eta) \mapsto \omega(\xi, D\eta)$ is symmetric. This is the case for gauge algebras (cf. Remark 5.8), but also more generally for locally convex Lie algebras with an admissible derivation in the sense of [52, Definition 9.1, Proposition 9.10].

In Chapter 5 we use Lemma 3.10 to show that $(\xi, \eta) \mapsto \omega(\xi, D\eta)$ is a positive semidefinite form on the gauge algebra $\mathfrak{g}$, and that every cocycle coming from a positive energy representation can be represented by a *measure* (Theorem 5.7). In Chapter 6, we make extensive use of the bound on the expectation value $\langle i\,\mathrm{d}\rho(D\xi)\rangle_{\psi}$ in terms of the average energy $\langle H\rangle_{\psi}$ afforded by Lemma 3.10. In fact, we shall need such bounds also for Lie algebra elements which are not in the image of D. The following refinement of the Cauchy–Schwarz estimate was designed for this purpose.

We start out with a proposition on Lie algebras which are *Mackey complete*, in the sense that every smooth curve $\zeta\colon [0, 1] \to \mathfrak{g}$ has a weak integral $\int_0^1 \zeta(t)dt$ in $\mathfrak{g}$. For a Mackey complete Lie algebra $\mathfrak{g}$, the operator

$$\int_0^1 e^{s\,\mathrm{ad}_y} ds$$

on $\mathfrak{g}$ is denoted $\frac{e^{\mathrm{ad}_y} - \mathbf{1}}{\mathrm{ad}_y}$.

Proposition 3.11. *Let $\widehat{\mathfrak{g}} = \mathbb{R} \oplus_\omega \mathfrak{g}$ be a central extension of a Mackey complete Lie algebra $\mathfrak{g}$ of the Lie group G with exponential function* exp. *Then, the adjoint action $\mathrm{Ad}^{\widehat{\mathfrak{g}}}$ of G on $\widehat{\mathfrak{g}}$ satisfies*

$$\mathrm{Ad}^{\widehat{\mathfrak{g}}}_{\exp y}(z, x) = \left(z + \omega\left(y, \frac{e^{\mathrm{ad}_y} - \mathbf{1}}{\mathrm{ad}_y}(x)\right), e^{\mathrm{ad}_y} x\right).$$

Proof. This is verified by solving the ODE

$$\gamma'(t) = [(0, y), \gamma(t)] \quad \text{with } \gamma(0) = (z, x).$$

Writing $\gamma(t) = (\alpha(t), e^{t\,\mathrm{ad}_y}(x))$, it leads to $\alpha'(t) = \omega(y, e^{t\,\mathrm{ad}_y} x)$. ∎

Lemma 3.12 (Refined Cauchy–Schwarz estimate). *Let $\mathfrak{g}$ be a Mackey complete Lie algebra, and let ρ be a positive energy representation of $\widehat{G}$. Let $\xi, \eta \in \mathfrak{g}$ be such that $[\xi, D\xi] = 0$ and $[\eta, D\eta] = 0$. Then, for all $s \in \mathbb{R}$, we have*

$$\begin{aligned}&\left(\left\langle i\,\mathtt{d}\rho(e^{-s\,\mathrm{ad}_\eta} D\xi)\right\rangle_\psi + \omega(\xi, D) + \omega\left(\frac{e^{-s\,\mathrm{ad}_\eta} - \mathbf{1}}{\mathrm{ad}_\eta}(D\xi), \eta\right)\right)^2 \\ &\qquad \leq 2\omega(\xi, D\xi)\left(\langle H\rangle_\psi + s(\langle i\,\mathtt{d}\rho(D\eta)\rangle_\psi + \omega(\eta, D)) + \frac{s^2}{2}\omega(\eta, D\eta)\right).\end{aligned}$$

In particular, if $\omega(\xi, D) = 0$, $\omega(\eta, D) = 0$ and $\omega(\mathrm{ad}^n_{\mathtt{d}\rho(\eta)}(D\xi), \eta) = 0$ for all $n \geq 0$, then

$$\left\langle i\,\mathtt{d}\rho\left(e^{-s\,\mathrm{ad}_\eta} D\xi\right)\right\rangle_\psi^2 \leq 2\omega(\xi, D\xi)\left(\langle H\rangle_\psi + s\langle i\,\mathtt{d}\rho(D\eta)\rangle_\psi + \frac{s^2}{2}\omega(\eta, D\eta)\right).$$

Proof. We write $W_{s,t} := \exp(t\,\mathtt{d}\rho(\xi))\exp(s\,\mathtt{d}\rho(\eta))$, and exploit the fact that the operator $H_{s,t} := W^*_{s,t} H W_{s,t}$ has non-negative spectrum. Repeated use of (3.2) on

$$H_{s,t} = \exp(-s\,\mathrm{ad}_{\mathtt{d}\rho(\eta)})\big(\exp(-t\,\mathrm{ad}_{\mathtt{d}\rho(\xi)})H\big)$$

yields

$$H_{s,t} = A_0(s) + A_1(s)t + A_2 t^2$$

with

$$\begin{aligned} A_0(s) &= H + s(i\,\mathtt{d}\rho(D\eta) + \omega(\eta, D)\mathbf{1}) + \frac{s^2}{2}\omega(\eta, D\eta)\mathbf{1}, \\ A_1(s) &= \omega(\xi, D)\mathbf{1} + \exp(-s\,\mathrm{ad}_{\mathtt{d}\rho(\eta)})(i\,\mathtt{d}\rho(D\xi)), \\ A_2 &= \frac{1}{2}\omega(\xi, D\xi)\mathbf{1}. \end{aligned}$$

With the preceding proposition, we obtain for $\exp(-s\,\mathrm{ad}_{\mathtt{d}\rho(\eta)})(i\,\mathtt{d}\rho(D\xi))$ the expression

$$i\,\mathtt{d}\rho(e^{-s\,\mathrm{ad}_\eta} D\xi) + \omega\left(\frac{e^{-s\,\mathrm{ad}_\eta} - \mathbf{1}}{\mathrm{ad}_\eta}(D\xi), \eta\right)\mathbf{1},$$

and thus

$$A_1(s) = \omega(\xi, D)\mathbf{1} + i\,\mathrm{d}\rho\big(e^{-s\,\mathrm{ad}_\eta}(D\xi)\big) + \omega\bigg(\frac{e^{-s\,\mathrm{ad}_\eta} - \mathbf{1}}{\mathrm{ad}_\eta}(D\xi), \eta\bigg)\mathbf{1}.$$

Consider the expectation value $\langle H_{s,t}\rangle_\psi \geq 0$. Setting

$$\alpha_0(s) := \langle A_0(s)\rangle_\psi, \quad \alpha_1(s) := \langle A_1(s)\rangle_\psi \quad \text{and} \quad \alpha_2 := \langle A_2\rangle_\psi,$$

we observe that

$$\langle H_{s,t}\rangle_\psi = \alpha_0(s) + \alpha_1(s)t + \alpha_2 t^2$$

is a non-negative polynomial in t of degree at most 2. From this, we obtain the inequality $\alpha_1(s)^2 \leq 4\alpha_2\alpha_0(s)$. This is the first inequality mentioned above, the second one is a direct consequence. ■

Chapter 4

Covariant extensions of gauge algebras

The results in the preceding chapter concerned the general level of Lie groups of the form $G \rtimes_\alpha \mathbb{R}$. Now we turn to the specifics of gauge groups. After introducing gauge groups and their Lie algebras in Section 4.1, we describe in Section 4.2 a procedure that provides a reduction from semisimple to simple structure Lie algebras, at the expense of replacing M by a finite covering manifold $\widehat{M}$. In Section 4.3, we recall the classification [51] of 2-cocycles for the extended gauge algebra $\mathfrak{g} \rtimes_D \mathbb{R}$.

4.1 Gauge groups and gauge algebras

Let $\mathcal{K} \to M$ be a smooth bundle of Lie groups, and let $\mathfrak{K} \to M$ be the corresponding Lie algebra bundle with fibers

$$\mathfrak{K}_x = \mathrm{Lie}(\mathcal{K}_x).$$

If M is connected, then the fibers $\mathcal{K}_x$ of $\mathcal{K} \to M$ are all isomorphic to a fixed structure group K, and the fibers $\mathfrak{K}_x$ of $\mathfrak{K}$ are isomorphic to its Lie algebra

$$\mathfrak{k} = \mathrm{Lie}(K).$$

Definition 4.1 (Gauge group). The *gauge group* is the group $\Gamma(M, \mathcal{K})$ of smooth sections of $\mathcal{K} \to M$, and the *compactly supported gauge group* is the group $\Gamma_c(M, \mathcal{K})$ of smooth compactly supported sections.

Definition 4.2 (Gauge algebra). We define the *gauge algebra* as the Fréchet–Lie algebra $\Gamma(M, \mathfrak{K})$ of smooth sections of $\mathfrak{K} \to M$, equipped with the pointwise Lie bracket. The *compactly supported gauge algebra* $\Gamma_c(M, \mathfrak{K})$ is the LF-Lie algebra of smooth compactly supported sections.

The compactly supported gauge group $\Gamma_c(M, \mathcal{K})$ is a locally convex Lie group, whose Lie algebra is the compactly supported gauge algebra $\Gamma_c(M, \mathfrak{K})$. It is locally exponential, with $\exp\colon \Gamma_c(M, \mathfrak{K}) \to \Gamma_c(M, \mathcal{K})$ given by pointwise exponentiation [51, Proposition 2.3].

Definition 4.3. In the following we write $\widetilde{\Gamma}_c(M, \mathcal{K})_0$ for the simply connected covering group of the identity component $\Gamma_c(M, \mathcal{K})_0$ and

$$q_\Gamma \colon \widetilde{\Gamma}_c(M, \mathcal{K})_0 \to \Gamma_c(M, \mathcal{K})_0$$

for the covering map. Then, $\widetilde{\Gamma}_c(M, \mathcal{K})_0$ has the same Lie algebra $\Gamma_c(M, \mathfrak{K})$ as the gauge group $\Gamma_c(M, \mathcal{K})$, and its exponential function Exp satisfies $q_\Gamma \circ \mathrm{Exp} = \exp$.

4.1.1 Gauge groups from principal fiber bundles

The motivating example of a gauge group is of course the group $\mathrm{Gau}(\Xi)$ of vertical automorphisms of a principal K-bundle $\pi\colon \Xi \to M$.

Definition 4.4. A *vertical automorphism* of a principal fiber bundle $\pi\colon \Xi \to M$ is a K-equivariant diffeomorphism $\alpha\colon \Xi \to \Xi$ such that $\pi \circ \alpha = \pi$. The group $\mathrm{Gau}(\Xi)$ of vertical automorphisms is called the *gauge group* of Ξ.

In order to interpret $\mathrm{Gau}(\Xi)$ as a gauge group in the sense of Definition 4.1, define the bundle of groups $\mathrm{Ad}(\Xi) \to M$ with typical fiber K by

$$\mathrm{Ad}(\Xi) := \Xi \times K/\sim,$$

where the relation $\sim$ is given by $(pk, h) \sim (p, khk^{-1})$ for $p \in \Xi$ and $k, h \in K$. We obtain an isomorphism

$$\mathrm{Gau}(\Xi) \simeq \Gamma(M, \mathrm{Ad}(\Xi))$$

by mapping the section $\sigma \in \Gamma(M, \mathrm{Ad}(\Xi))$ to the corresponding vertical automorphism $\alpha_\sigma \in \mathrm{Gau}(\Xi)$, defined by

$$\alpha_\sigma(p) = p \cdot k$$

if $\sigma(\pi(p))$ is the class of (p, k) in $\mathrm{Ad}(\Xi) = \Xi \times K/\sim$.

The bundle of Lie algebras associated to Ξ is the *adjoint bundle* $\mathrm{ad}(\Xi) \to M$, defined as the quotient

$$\mathrm{ad}(\Xi) := \Xi \times_{\mathrm{Ad}} \mathfrak{k}$$

of $\Xi \times \mathfrak{k}$ modulo the relation $(pk, X) \sim (p, \mathrm{Ad}_k(X))$ for $p \in \Xi$, $X \in \mathfrak{k}$ and $k \in K$. Here $\mathrm{Ad}_k \in \mathrm{Aut}(\mathfrak{k})$ is the Lie algebra automorphism induced by the group automorphism $h \mapsto khk^{-1}$.

The *compactly supported gauge group* $\mathrm{Gau}_c(\Xi) \subseteq \mathrm{Gau}(\Xi)$ is the group of vertical bundle automorphisms of Ξ that are trivial outside the preimage of some compact subset of M. Since it is isomorphic to $\Gamma_c(M, \mathrm{Ad}(\Xi))$, it is a locally convex Lie group with Lie algebra $\mathfrak{gau}_c(\Xi) = \Gamma_c(M, \mathrm{ad}(\Xi))$.

Remark 4.5. In applications to gauge theory on noncompact manifolds M, the relevant group $\mathcal{G}$ of gauge transformations may be smaller than $\mathrm{Gau}(\Xi)$ due to boundary conditions at infinity. One expects $\mathcal{G}$ to contain at least $\mathrm{Gau}_c(\Xi)$, or perhaps even some larger Lie group of gauge transformations specified by a decay condition at infinity (cf. [31, 110]). In Part II of this series of papers, we will focus on the case where $M = \mathbb{R}^d$ is Minkowski space, and $\mathcal{G} \subset \Gamma(\mathbb{R}^d, \mathrm{Ad}(\Xi))$ is the group of gauge transformations that extend continuously to the conformal completion of Minkowski space. If the extension of Ξ to the conformal completion is trivial, then $\mathcal{G}$ contains global as well as compactly supported gauge transformations.

4.1.2 Gauge groups and space-time symmetries

An *automorphism* of $\pi: \mathcal{K} \to M$ is a pair $(\gamma, \gamma_M) \in \mathrm{Diff}(\mathcal{K}) \times \mathrm{Diff}(M)$ with $\pi \circ \gamma = \gamma_M \circ \pi$, such that for each fiber $\mathcal{K}_x$, the map $\gamma|_{\mathcal{K}_x}: \mathcal{K}_x \to \mathcal{K}_{\gamma_M(x)}$ is a group homomorphism. Since γ_M is determined by γ, we will omit it from the notation. We denote the group of automorphisms of $\mathcal{K}$ by $\mathrm{Aut}(\mathcal{K})$.

Definition 4.6 (Geometric $\mathbb{R}$-actions). In the context of gauge groups, we will be interested in $\mathbb{R}$-actions $\alpha: \mathbb{R} \to \mathrm{Aut}(\Gamma(M, \mathcal{K}))$ which are of *geometric* type, i.e., derived from a 1-parameter group $\gamma: \mathbb{R} \to \mathrm{Aut}(\mathcal{K})$ by

$$\alpha_t(\sigma) := \gamma_{-t} \circ \sigma \circ \gamma_{M,t}.$$

The $\mathbb{R}$-action on $\Gamma(M, \mathcal{K})$ preserves the subgroup $\Gamma_c(M, \mathcal{K})_0$ on which it defines a smooth action. Moreover, it lifts to a smooth action on the simply connected covering group $\widetilde{\Gamma}_c(M, \mathcal{K})_0$ (cf. [62, Theorem VI.3]).

Remark 4.7. If $\mathcal{K}$ is of the form $\mathrm{Ad}(\Xi)$ for a principal fiber bundle $\Xi \to M$, then a 1-parameter group of automorphisms of Ξ induces a 1-parameter group of automorphisms of $\mathcal{K}$.

The 1-parameter group $\alpha: \mathbb{R} \to \mathrm{Aut}(\Gamma(M, \mathcal{K}))$ of group automorphisms differentiates to a 1-parameter group $\alpha^{\mathfrak{g}}: \mathbb{R} \to \mathrm{Aut}(\Gamma(M, \mathfrak{K}))$ of Lie algebra automorphisms given by

$$\alpha_t^{\mathfrak{g}}(\xi) = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \gamma_{-t} \circ e^{\varepsilon\xi} \circ \gamma_{M,t}.$$

The corresponding derivation $D := \frac{d}{dt}\big|_{t=0} \alpha_t^{\mathfrak{g}}$ of $\Gamma(M, \mathfrak{K})$ can be described in terms of the infinitesimal generator of γ,

$$\mathbf{v} := \frac{d}{dt}\bigg|_{t=0} \gamma_{-t} \in \mathcal{V}(\mathcal{K}).$$

We identify the element $\xi \in \Gamma(M, \mathfrak{K})$ with the vertical, fiberwise left invariant vector field $\Xi_\xi \in \mathcal{V}(\mathcal{K})$ defined by $\Xi_\xi(k_x) = \frac{d}{d\varepsilon}\big|_{\varepsilon=0} k_x e^{\varepsilon\xi(x)}$. Using the equality $[\mathbf{v}, \Xi_\xi] = \Xi_{D(\xi)}$, we write

$$D(\xi) = L_{\mathbf{v}}\xi.$$

For $\mathfrak{g} = \Gamma_c(M, \mathfrak{K})$, the Lie algebra $\mathfrak{g} \rtimes_D \mathbb{R}$ then has the bracket

$$[\xi \oplus t, \xi' \oplus t'] = \big([\xi, \xi'] + (tL_{\mathbf{v}}\xi' - t'L_{\mathbf{v}}\xi)\big) \oplus 0. \tag{4.1}$$

Remark 4.8. Alternatively, we can consider $\gamma: \mathbb{R} \to \mathrm{Aut}(\mathcal{K})$ as a smooth 1-parameter group of bisections of the gauge groupoid $\mathcal{G}(\mathcal{K}) \rightrightarrows M$, the Lie groupoid whose objects are points $x, y \in M$, and whose morphisms are Lie group isomorphisms $\mathcal{K}_x \to \mathcal{K}_y$. It gives rise to a smooth 1-parameter family $\dot{\gamma}$ of bisections of the Lie groupoid $\mathcal{G}(\mathfrak{K}) \rightrightarrows M$, whose morphisms from x to y are Lie algebra isomorphisms $\mathfrak{K}_x \to \mathfrak{K}_y$. Its generator $\mathbf{v} = -\frac{d}{dt}|_{t=0}\dot{\gamma}$ is thus a section of its Lie algebroid $\mathfrak{a}(\mathfrak{K}) \to M$, called the *Atiyah algebroid*. A section $\xi \in \Gamma(M, \mathfrak{K})$ can be considered as an element of $\Gamma(M, \mathfrak{der}(\mathfrak{K})) \subseteq \Gamma(M, \mathfrak{a}(\mathfrak{K}))$, and we interpret $L_{\mathbf{v}}\xi$ as the commutator $[\mathbf{v}, \xi]$ in $\Gamma(M, \mathfrak{a}(\mathfrak{K}))$. We will need this picture in Section 4.2, where the bundle of Lie groups is not available.

4.2 Reduction to simple structure algebras

In this memoir, we consider gauge algebras with a *semisimple* structure algebra $\mathfrak{k}$. The following theorem shows that, without further loss of generality, we may restrict attention to the case where $\mathfrak{k}$ is *simple*.

Theorem 4.9 (Reduction from semisimple to simple structure algebras). *If $\mathfrak{K} \to M$ is a smooth locally trivial bundle of Lie algebras with semisimple fibers, then there exists a finite cover $\widehat{M} \to M$ and a smooth locally trivial bundle of Lie algebras $\widehat{\mathfrak{K}} \to \widehat{M}$ with simple fibers such that there exist isomorphisms $\Gamma(M, \mathfrak{K}) \simeq \Gamma(\widehat{M}, \widehat{\mathfrak{K}})$ and $\Gamma_c(M, \mathfrak{K}) \simeq \Gamma_c(\widehat{M}, \widehat{\mathfrak{K}})$ of locally convex Lie algebras.*

This is proven in [51, Theorem 3.1]. In brief, one uses local trivializations of $\mathfrak{K} \to M$ to give a manifold structure to

$$\widehat{M} := \bigcup_{x \in M} \mathrm{Spec}(\mathfrak{K}_x),$$

where $\mathrm{Spec}(\mathfrak{K}_x)$ is the finite set of maximal ideals $I_x \subset \mathfrak{K}_x$. The bundle of Lie algebras is then defined by

$$\widehat{\mathfrak{K}} := \bigcup_{I_x \in \widehat{M}} \mathfrak{K}_x / I_x,$$

and one shows that the natural projection $\pi : \widehat{\mathfrak{K}} \to \widehat{M}$ is a locally trivial vector bundle. Note that the finite cover $\widehat{M} \to M$ is not necessarily connected, and that the isomorphism classes of the fibers of $\widehat{\mathfrak{K}} \to \widehat{M}$ are not necessarily the same over different connected components of $\widehat{M}$.

Remark 4.10. Since a smooth 1-parameter family of automorphisms of $\mathfrak{K} \to M$ acts naturally on the maximal ideals, we obtain a smooth action on the Lie algebra bundle $\widehat{\mathfrak{K}} \to \widehat{M}$. We denote the corresponding section of the Atiyah algebroid $\mathfrak{a}(\widehat{\mathfrak{K}}) \to \widehat{M}$ by $\widehat{\mathbf{v}} \in \Gamma(\widehat{M}, \mathfrak{a}(\widehat{\mathfrak{K}}))$, and we denote the corresponding vector field on $\widehat{M}$ by

$$\mathbf{v}_{\widehat{M}} := \pi_* \widehat{\mathbf{v}}.$$

Since $\widehat{\mathfrak{K}}$ has simple fibers, the Atiyah algebroid $\mathfrak{a}(\widehat{\mathfrak{K}})$ fits in the exact sequence

$$\widehat{\mathfrak{K}} \to \mathfrak{a}(\widehat{\mathfrak{K}}) \to T\widehat{M},$$

where the first map is given by the pointwise adjoint action, and the second by the anchor. Note that the action on $\widehat{M}$ is locally free or periodic if and only if the action on M is. In that case, the period on $\widehat{M}$ is a multiple of the period on M.

In many situations, the connected components of $\widehat{M}$ are diffeomorphic to M. However, non-trivial covers $\widehat{M} \to M$ do occur naturally, for example in connection to non-orientable 4-manifolds.

Example 4.11. If the fibers of $\mathfrak{K} \to M$ are simple, then $\widehat{M} = M$.

Example 4.12. If $\mathfrak{K} = M \times \mathfrak{k}$ is trivial, then $\widehat{M} = M \times \mathrm{Spec}(\mathfrak{k})$ and all connected components of $\widehat{M}$ are diffeomorphic to M.

Example 4.13. Suppose that M is connected, and that the typical fiber $\mathfrak{k}$ of $\mathfrak{K} \to M$ is a semisimple Lie algebra with r simple ideals that are mutually non-isomorphic. Then,

$$\widehat{M} = \bigsqcup_{i=1}^{r} M$$

is a disjoint union of copies of M.

Example 4.14 (Frame bundles of 4-manifolds). Let M be a 4-dimensional Riemannian manifold. Let $\Xi := \mathrm{OF}(M)$ be the principal $\mathrm{O}(4,\mathbb{R})$-bundle of orthogonal frames, and let $\mathfrak{K} = \mathrm{ad}(\Xi)$. Then, $K = \mathrm{O}(4,\mathbb{R})$ and $\mathfrak{k} = \mathfrak{so}(4,\mathbb{R})$ is isomorphic to

$$\mathfrak{su}_L(2,\mathbb{C}) \oplus \mathfrak{su}_R(2,\mathbb{C}).$$

The group $\pi_0(K)$ is of order 2, the non-trivial element acting by conjugation with $T = \mathrm{diag}(-1,1,1,1)$. Since this permutes the two simple ideals, the manifold $\widehat{M}$ is the orientable double cover of M. This is the disjoint union $\widehat{M} = M_L \sqcup M_R$ of two copies of M if M is orientable, and a connected twofold cover $\widehat{M} \to M$ if it is not.

4.3 Central extensions of gauge algebras

Let $\mathfrak{g}$ be the compactly supported gauge algebra $\Gamma_c(M,\mathfrak{K})$, where $\mathfrak{K} \to M$ is a Lie algebra bundle with simple fibers. In this section, we classify all possible central extensions of $\mathfrak{g} \rtimes_D \mathbb{R}$. This amounts to calculating the continuous second Lie algebra cohomology $H^2(\mathfrak{g} \rtimes_D \mathbb{R}, \mathbb{R})$ with trivial coefficients. In Chapter 5, we will characterize those cocycles coming from a positive energy representation.

4.3.1 Universal invariant symmetric bilinear forms

Let $\mathfrak{k}$ be a finite-dimensional, simple real Lie algebra. Then, its automorphism group $\mathrm{Aut}(\mathfrak{k})$ is a closed subgroup of $\mathrm{GL}(\mathfrak{k})$, hence a Lie group with Lie algebra $\mathrm{der}(\mathfrak{k}) \simeq \mathfrak{k}$.

Since $\mathfrak{k}$ acts trivially on the space

$$V(\mathfrak{k}) := S^2(\mathfrak{k})\big/(\mathfrak{k} \cdot S^2(\mathfrak{k}))$$

of $\mathfrak{k}$-coinvariants of the twofold symmetric tensor power $S^2(\mathfrak{k})$, the $\mathrm{Aut}(\mathfrak{k})$-representation on $V(\mathfrak{k})$ factors through $\pi_0(\mathrm{Aut}(\mathfrak{k}))$. The *universal $\mathfrak{k}$-invariant symmetric bilinear form* is defined by

$$\kappa\colon \mathfrak{k} \times \mathfrak{k} \to V(\mathfrak{k}), \quad \kappa(x, y) := [x \otimes_s y] = \frac{1}{2}[x \otimes y + y \otimes x].$$

We associate to $\lambda \in V(\mathfrak{k})^*$ the $\mathbb{R}$-valued, $\mathrm{der}(\mathfrak{k})$-invariant, symmetric, bilinear form

$$\kappa_\lambda := \lambda \circ \kappa.$$

This correspondence is a bijection between $V(\mathfrak{k})^*$ and the space of $\mathrm{der}(\mathfrak{k})$-invariant symmetric bilinear forms on $\mathfrak{k}$.

Since $\mathfrak{k}$ is simple, we have $V(\mathfrak{k}) \simeq \mathbb{C}$ if $\mathfrak{k}$ admits a complex structure, and $V(\mathfrak{k}) \simeq \mathbb{R}$ if it does not (cf. [84, Appendix B]). In the latter case, $\mathfrak{k}$ is called *absolutely simple*. The universal invariant symmetric bilinear form can be identified with the Killing form of the real Lie algebra $\mathfrak{k}$ if $V(\mathfrak{k}) \simeq \mathbb{R}$ and with the Killing form of the underlying complex Lie algebra if $V(\mathfrak{k}) \simeq \mathbb{C}$. In particular, in the important special case that $\mathfrak{k}$ is a compact simple Lie algebra, a universal invariant bilinear form $\kappa\colon \mathfrak{k} \times \mathfrak{k} \to V(\mathfrak{k})$ is the negative definite Killing form given by $\mathrm{tr}(\mathrm{ad}\, x\, \mathrm{ad}\, y)$. However, in the following, we shall always use the normalized invariant positive definite symmetric bilinear form κ that satisfies

$$\kappa(i\alpha^\vee, i\alpha^\vee) = 2 \tag{4.2}$$

for the coroots $\alpha^\vee$ corresponding to long roots in the root decomposition of $\mathfrak{k}_\mathbb{C}$ (cf. [68, 94] and Appendix A).

4.3.2 The flat bundle $\mathbb{V} = V(\mathfrak{K})$

If $\mathfrak{K} \to M$ is a bundle of Lie algebras with simple fibers, then we denote by $\mathbb{V} \to M$ the vector bundle with fibers $\mathbb{V}_x = V(\mathfrak{K}_x)$. It carries a canonical flat connection $\mathtt{d}$, defined by

$$\mathtt{d}\kappa(\xi, \eta) := \kappa(d_\nabla \xi, \eta) + \kappa(\xi, d_\nabla \eta) \quad \text{for } \xi, \eta \in \Gamma(M, \mathfrak{K}),$$

where ∇ is a *Lie connection* on $\mathfrak{K}$, meaning that

$$d_\nabla[\xi, \eta] = [d_\nabla \xi, \eta] + [\xi, d_\nabla \eta] \quad \text{for all } \xi, \eta \in \Gamma(M, \mathfrak{K}).$$

Since the fibers are assumed to be simple, any two Lie connections differ by a $\mathfrak{K}$-valued 1-form, so that the preceding definition is independent of the choice of ∇ (cf. [53]).

Let $\mathfrak{k}_i$ be the fiber of $\mathfrak{K}$ over a connected component M_i of M. If $\mathfrak{k}_i$ is absolutely simple (hence, in particular, when $\mathfrak{k}$ is compact), we have $V(\mathfrak{k}_i) \simeq \mathbb{R}$, and $\pi_0(\mathrm{Aut}(\mathfrak{k}))$ acts trivially on $V(\mathfrak{k}_i)$. In this case, $\mathbb{V} \to M_i$ is the trivial line bundle $M_i \times \mathbb{R} \to M_i$.

If $\mathfrak{k}_i$ possesses a complex structure, then $V(\mathfrak{k}_i) \simeq \mathbb{C}$, and $\alpha \in \mathrm{Aut}(\mathfrak{k}_i)$ flips the complex structure on $\mathbb{C}$ if and only if it flips the complex structure on $\mathfrak{k}_i$. In this case, $\mathbb{V} \to M_i$ is a vector bundle of real rank 2.

Remark 4.15. In the context of positive energy representations, we will see in Theorem 6.2 below that $\mathfrak{k}$ must be compact, so that $\mathbb{V} \to M$ is the trivial real line bundle. Although we need to consider the *a priori* possibility of non-trivial bundles, then, it will become clear in the course of our analysis that they will not give rise to positive energy representations.

4.3.3 Classification of central extensions

We define 2-cocycles $\omega_{\lambda,\nabla}$ on $\mathfrak{g} \rtimes_D \mathbb{R}$ whose classes span the cohomology group $H^2(\mathfrak{g} \rtimes_D \mathbb{R}, \mathbb{R})$. They depend on a $\mathbb{V}$-*valued* 1-*current* $\lambda \in \Omega^1_c(M, \mathbb{V})'$, and on a *Lie connection* ∇ on $\mathfrak{K}$. A 1-current $\lambda \in \Omega^1_c(M, \mathbb{V})'$ is said to be

(L1) *closed* if $\lambda(\mathrm{d}C^\infty_c(M, \mathbb{V})) = 0$,

(L2) $\mathbf{v}_M$*-invariant* if $\lambda(L_{\mathbf{v}_M}\Omega^1_c(M, \mathbb{V})) = \{0\}$.

Given a closed, $\mathbf{v}_M$-invariant current $\lambda \in \Omega^1_c(M, \mathbb{V})'$, we define the 2-cocycle $\omega_{\lambda,\nabla}$ on $\mathfrak{g} \rtimes_D \mathbb{R}$ by skew-symmetry and the equations

$$\omega_{\lambda,\nabla}(\xi, \eta) = \lambda(\kappa(\xi, d_\nabla \eta)), \tag{4.3}$$

$$\omega_{\lambda,\nabla}(D, \xi) = \lambda(\kappa(L_{\mathbf{v}}\nabla, \xi)), \tag{4.4}$$

where we write ξ for $(\xi, 0) \in \mathfrak{g} \rtimes_D \mathbb{R}$ and D for $(0, 1) \in \mathfrak{g} \rtimes_D \mathbb{R}$ as in (2.2). We define the $\mathrm{der}(\mathfrak{K})$-valued 1-form $L_{\mathbf{v}}\nabla \in \Omega^1(M, \mathrm{der}(\mathfrak{K}))$ by

$$(L_{\mathbf{v}}\nabla)_w(\xi) = L_{\mathbf{v}}(d_\nabla \xi)_w - \nabla_w L_{\mathbf{v}}\xi = L_{\mathbf{v}}(\nabla_w \xi) - \nabla_w L_{\mathbf{v}}\xi - \nabla_{[\mathbf{v}_M, w]}\xi \tag{4.5}$$

for all $w \in \mathcal{V}(M)$, $\xi \in \Gamma(M, \mathfrak{K})$. Since the fibers of $\mathfrak{K} \to M$ are simple, all derivations are inner, so we can identify $L_{\mathbf{v}}\nabla$ with an element of $\Omega^1(M, \mathfrak{K})$. Using the formulae

$$\mathrm{d}\kappa(\xi, \eta) = \kappa(d_\nabla \xi, \eta) + \kappa(\xi, d_\nabla \eta), \tag{4.6}$$

$$L_{\mathbf{v}_M}\kappa(\xi, \eta) = \kappa(L_{\mathbf{v}}\xi, \eta) + \kappa(\xi, L_{\mathbf{v}}\eta), \tag{4.7}$$

$$L_{\mathbf{v}}(d_\nabla \xi) - d_\nabla L_{\mathbf{v}}\xi = [L_{\mathbf{v}}\nabla, \xi], \tag{4.8}$$

it is not difficult to check that $\omega_{\lambda,\nabla}$ is a cocycle. Skew-symmetry follows from (4.6) and (L1). The vanishing of $\delta\omega_{\lambda,\nabla}$ on $\mathfrak{g}$ follows from (4.6), the derivation property of ∇ and invariance of κ. Finally, $i_D\delta\omega_{\lambda,\nabla} = 0$ follows from skew-symmetry, (4.8), (4.7), (L2) and the invariance of κ.

Note that the class $[\omega_{\lambda,\nabla}]$ in $H^2(\mathfrak{g} \rtimes_D \mathbb{R}, \mathbb{R})$ depends only on λ, not on ∇. Indeed, two connection 1-forms ∇ and ∇' differ by $A \in \Omega^1(M, \mathrm{der}(\mathfrak{K}))$. Using $\mathrm{der}(\mathfrak{K}) \simeq \mathfrak{K}$, we find

$$\omega_{\lambda,\nabla'} - \omega_{\lambda,\nabla} = \delta\chi_A \quad \text{with } \chi_A(\xi \oplus t) := \lambda(\kappa(A, \xi)).$$

According to the following theorem, every continuous Lie algebra 2-cocycle on $\mathfrak{g} \rtimes_D \mathbb{R}$ is cohomologous to one of the type $\omega_{\lambda,\nabla}$ as defined in (4.3) and (4.4).

Theorem 4.16 (Central extensions of extended gauge algebras). *Let $\mathcal{K} \to M$ be a bundle of Lie groups with simple fibers, equipped with a 1-parameter group of automorphisms with generator $\mathbf{v} \in \mathcal{V}(\mathcal{K})$. Let $\mathfrak{g} = \Gamma_c(M, \mathfrak{K})$ be the compactly supported gauge algebra, and let $\mathfrak{g} \rtimes_D \mathbb{R}$ be the Lie algebra (4.1). Then, the map $\lambda \mapsto [\omega_{\lambda,\nabla}]$ induces an isomorphism*

$$\big(\Omega^1_c(M, \mathbb{V})\big/(d\Omega^0_c(M, \mathbb{V}) + L_{\mathbf{v}_M}\Omega^1_c(M, \mathbb{V}))\big)' \xrightarrow{\sim} H^2(\mathfrak{g} \rtimes_D \mathbb{R}, \mathbb{R})$$

between the space of closed, $\mathbf{v}_M$-invariant $\mathbb{V}$-valued currents and $H^2(\mathfrak{g} \rtimes_D \mathbb{R}, \mathbb{R})$.

This is proven in [51, Theorem 5.3]. The proof relies heavily on the description of $H^2(\mathfrak{g}, \mathbb{R})$ provided in [53, Proposition 1.1].

Remark 4.17 (Temporal gauge). If the Lie connection ∇ on $\mathfrak{K}$ can be chosen so as to make $\mathbf{v} \in \mathcal{V}(\mathcal{K})$ horizontal, $\nabla_{\mathbf{v}_M}\xi = L_{\mathbf{v}}\xi$ for all $\xi \in \Gamma(M, \mathfrak{K})$, then equation (4.5) shows that $L_{\mathbf{v}}\nabla = i_{\mathbf{v}_M}R$, where R is the curvature of ∇. For such connections, (4.4) is equivalent to

$$\omega_{\lambda,\nabla}(D, \xi) = \lambda(\kappa(i_{\mathbf{v}_M}R, \xi)).$$

Chapter 5

Cocycles for positive energy representations

Having classified all the possible 2-cocycles on $\Gamma_c(M,\mathfrak{K}) \rtimes \mathbb{R}$, we now address the restrictions that are imposed on these cocycles by the Cauchy–Schwarz estimates from Section 3.4.

In Section 5.1 we derive a local normal form of the cocycle ω in a flow box around a point $m \in M$, where $\mathbf{v}_M \in \mathcal{V}(M)$ does not vanish. In Section 5.2, we use this to derive a global normal form for ω, provided that $\mathbf{v}_M$ is nowhere vanishing. It turns out that ω is characterized by a *measure* μ on the covering space $\widehat{M}$. In Section 5.3, we plug this information back into the Cauchy–Schwarz estimate. This yields the basic estimates needed for the continuity results in Chapter 6.

The setting of this chapter is as follows. As before, $\pi\colon \mathcal{K} \to M$ is a bundle of Lie groups with semisimple fibers, and $\mathfrak{K} \to M$ is the corresponding bundle of Lie algebras. We consider positive energy representations of $\widehat{G}$, where $G = \Gamma_c(M,\mathcal{K})$ is the compactly supported gauge group with Lie algebra $\mathfrak{g} = \Gamma_c(M,\mathfrak{K})$. In fact, we will work mainly at the Lie algebra level, so our results continue to hold for the slightly more general case that $G = \widetilde{\Gamma}_c(M,\mathcal{K})_0$ is the simply connected cover of the identity component. Using Section 4.2, we identify $\mathfrak{g} = \Gamma_c(M,\mathfrak{K})$ with $\mathfrak{g} = \Gamma_c(\widehat{M},\widehat{\mathfrak{K}})$, where $\widehat{\mathfrak{K}} \to \widehat{M}$ is a Lie algebra bundle with simple fibers over a covering space $\widehat{M}$ of M. We assume that the 1-parameter group of automorphisms is of geometric type in the sense of Definition 4.6. The analogs of the generators $\mathbf{v}_M \in \mathcal{V}(M)$ and $\mathbf{v} \in \Gamma(M,\mathfrak{a}(\mathfrak{K}))$ for $\widehat{\mathfrak{K}}$ are denoted by $\pi_*\widehat{\mathbf{v}} \in \mathcal{V}(\widehat{M})$ and $\widehat{\mathbf{v}} \in \Gamma(\widehat{M},\mathfrak{a}(\widehat{\mathfrak{K}}))$.

5.1 Local gauge algebras

The following simple lemma will be used extensively throughout the rest of the memoir. It gives a normal form for the pair $(\Gamma_c(M,\mathfrak{K}),\mathbf{v})$ in the neighborhood of a point $m \in M$ where the vector field $\mathbf{v}_M$ does not vanish.

Definition 5.1 (Good flowbox). A *good flowbox* is a $\mathbf{v}$-equivariant, local trivialization $(I \times U_0) \times K \to \mathcal{K}$ of $\mathcal{K}$ over an open neighborhood $U \subseteq M$ that is equivariantly diffeomorphic to $I \times U_0$. Here $I \subseteq \mathbb{R}$ is a bounded open interval, and $U_0 \subseteq \mathbb{R}^{n-1}$ is open. Note that for $n = 1$, we may take $U_0 = \{0\}$.

In particular, we have coordinates $t := x_0$ for I and $\vec{x} := (x_1,\ldots,x_{n-1})$ for U_0 such that $\mathbf{v}_M \in \mathcal{V}(U)$ corresponds to $\partial_t \in \mathcal{V}(I \times U_0)$.

Lemma 5.2. *For any point $m \in M$ with $\mathbf{v}_M(m) \neq 0$, there exists a good flowbox $U \simeq I \times U_0$ containing m. Under the trivialization $U \times \mathfrak{k} \to \mathfrak{K}|_U$, the induced* *isomorphism* $C^\infty_c(U,\mathfrak{k}) \simeq \Gamma_c(U,\mathfrak{K})$ *yields an inclusion*

$$I_U\colon C^\infty_c(U,\mathfrak{k}) \rtimes_{\partial_t} \mathbb{R} \hookrightarrow \Gamma_c(M,\mathfrak{K}) \rtimes_D \mathbb{R}.$$

Proof. Since $\mathbf{v}_M(m) \neq 0$, we can find a neighborhood $U \subseteq M$ of m and local coordinates $t, x_1, \ldots, x_{n-1}$ such that the vector field $\mathbf{v}_M$ on U is of the form ∂_t. We may assume that $U \simeq I \times U_0$ where $U_0 \subseteq \mathbb{R}^{n-1}$ corresponds to $t = 0$ and $I \subseteq \mathbb{R}$ corresponds to $\vec{x} = 0$. We choose U_0 sufficiently small for there to exist a trivialization $\Phi\colon U_0 \times K \to \mathcal{K}|_{U_0}$, which we then extend to a trivialization $U \times K \simeq \mathcal{K}|_U$ over U by $(t,x,k) \mapsto \gamma(-t)\Phi(x,k)$. As $\frac{d}{dt}|_{t=0}\gamma(-t) = \mathbf{v}$, the vector field $\mathbf{v} \in \mathcal{V}(\mathcal{K})$ is horizontal in this trivialization. ■

We consider $\mathfrak{g}_U := C^\infty_c(U,\mathfrak{k})$ as a subalgebra of $\mathfrak{g} = \Gamma_c(M,\mathfrak{K})$ and wish to study the restriction $\mathtt{d}\rho_U$ of the representation $\mathtt{d}\rho$ to the subalgebra

$$\widehat{\mathfrak{g}}_U := \mathbb{R} \oplus_\omega (\mathfrak{g}_U \rtimes_{\partial_t} \mathbb{R}).$$

Note that the subalgebra $\widehat{\mathfrak{g}}_U \hookrightarrow \widehat{\mathfrak{g}}$ does not correspond to a Lie subgroup of $\widehat{G}$ unless U is γ-invariant, so we cannot work at the level of Lie groups.

If $A \in \Omega^1(U,\mathfrak{k})$ is the local connection 1-form corresponding to the Lie connection ∇, then up to coboundaries, by (4.3) and (4.4) the restriction ω_U of ω to $\mathfrak{g}_U \rtimes_{\partial_t} \mathbb{R}$ takes the form

$$\omega_U(fX, gY) \simeq \lambda_U(\kappa(fX, \mathtt{d}g \cdot Y + g[A,Y])) \tag{5.1}$$

$$\omega_U(\partial_t, fX) \simeq \lambda_U(\kappa(\partial_t A, fX)), \tag{5.2}$$

for some $\lambda_U \in \Omega^1_c(U, V(\mathfrak{k}))'$, where $f, g \in C^\infty_c(U,\mathbb{R})$ and $X, Y \in \mathfrak{k}$.

Proposition 5.3. *Let $m \in M$ be a point with $\mathbf{v}_M(m) \neq 0$ and let $U \simeq I \times U_0$ be a good flowbox (cf. Definition 5.1). Let $\iota\colon U_0 \hookrightarrow M$ be the corresponding inclusion. Then, the map $\Omega^1_c(U, V(\mathfrak{k})) \to \Gamma_c(U_0, \iota^*T^*M \otimes V(\mathfrak{k}))$, $\beta \mapsto \overline{\beta}$, defined by the integration*

$$\overline{\beta}(x_1,\ldots,x_{n-1}) := \int_{-\infty}^{\infty} \beta(t, x_1, \ldots, x_{n-1})dt,$$

yields a split exact sequence

$$0 \to L_{\partial_t}\Omega^1_c(U, V(\mathfrak{k})) \hookrightarrow \Omega^1_c(U, V(\mathfrak{k})) \to \Gamma_c(U_0, \iota^*T^*M \otimes V(\mathfrak{k})) \to 0$$

of locally convex spaces. In particular, $\lambda_U\colon \Omega^1_c(U, V(\mathfrak{k})) \to \mathbb{R}$ factors through a continuous linear map $\overline{\lambda}_{U_0}\colon \Gamma_c(U_0, \iota^*T^*M \otimes V(\mathfrak{k})) \to \mathbb{R}$.

Proof. The second statement follows from the first because

$$\lambda_U(L_{\partial_t}\Omega^1_c(U, V(\mathfrak{k}))) = \{0\}$$

by Theorem 4.16. The kernel of $\beta \mapsto \bar{\beta}$ is precisely $L_{\partial_t}\Omega^1_c(U, V(\mathfrak{k}))$ by the fundamental theorem of calculus. A bump function $\varphi \in C^\infty_c(I,\mathbb{R})$ of integral 1 yields the required continuous right inverse $\Gamma_c(U_0, \iota^*T^*M \otimes V(\mathfrak{k})) \to \Omega^1_c(U, V(\mathfrak{k}))$ for the integration map by sending $\vec{x} \mapsto \beta(\vec{x})$ to $(t,\vec{x}) \mapsto \varphi(t)\beta(x_1,\dots,x_{n-1})$. ■

For $X = Y$ we obtain with (5.1) the relation

$$\omega(\partial_t fX, fX) = \lambda_U\big(\partial_t f \cdot \mathrm{d}f \cdot \kappa(X,X)\big).$$

Unlike (5.1), which holds only modulo coboundaries, this equation is exact because $(\partial_t f)X$ and fX commute. Lemma 3.10 (the Cauchy–Schwarz estimate) then yields

$$-\lambda_U\big(\partial_t f \cdot \mathrm{d}f \cdot \kappa(X,X)\big) \geq 0. \tag{5.3}$$

This allows us to characterize λ_U as follows.

Proposition 5.4. *Let $m \in M$ be a point with $\mathbf{v}_M(m) \neq 0$. Then, there exists an open neighborhood $U \subseteq M$ of m such that, for each $X \in \mathfrak{k}$, there exists a unique $\mathbf{v}_M$-invariant positive locally finite regular Borel measure $\mu_{U,X}$ on U such that the functional $\lambda_{U,X} \in \Omega^1_c(U,\mathbb{R})'$ defined by $\lambda_{U,X}(\beta) := -\lambda_U(\beta\cdot\kappa(X,X))$ takes the form*

$$\lambda_{U,X}(\beta) = \int_U (i_{\mathbf{v}_M}\beta)d\mu_{U,X}(m).$$

Proof. Introduce coordinates $x_0 := t$ and $\vec{x} := (x_1,\dots,x_{n-1})$ on $U \simeq I \times U_0$ as in Definition 5.1. Define $\lambda_{U,i} \in C^\infty_c(U,\mathbb{R})'$, $i = 0,\dots,n-1$, by $\lambda_{U,i}(f) := \lambda_{U,X}(f\,\mathrm{d}x_i)$ and let $\lambda_i \in C^\infty_c(U_0,\mathbb{R})'$ be the corresponding distribution on U_0 (cf. Proposition 5.3), so

$$\lambda_{U,i}(f) = \lambda_i(\bar{f})$$

with

$$\bar{f}(\vec{x}) := \int_I f(t,\vec{x})dt.$$

Then,

$$\lambda_{U,X}(f\,\mathrm{d}g) = \sum_{i=0}^{n-1}\lambda_i(\overline{f\partial_i g}) \quad \text{for all } f,g \in C^\infty_c(U,\mathbb{R}).$$

Equation (5.3) then yields

$$\lambda_0\big(\overline{(\partial_t f)^2}\big) + \sum_{i=1}^{n-1}\lambda_i\big(\overline{\partial_t f\partial_i f}\big) \geq 0. \tag{5.4}$$

First, we show that $\lambda_0(h^2) \geq 0$ for any h in $C^\infty_c(U_0,\mathbb{R})$. Note that every element B of $C^\infty_c(I,\mathbb{R})$ satisfies

$$\int_I B\partial_t B dt = 0.$$

We choose $B \neq 0$, normalize it by

$$\int_I (\partial_t B)^2 dt = 1$$

and define

$$f(t, \vec{x}) := B(t)h(\vec{x}).$$

We then have

$$\overline{(\partial_t f)^2} = h^2 \quad \text{and} \quad \overline{\partial_t f \partial_i f} = h\partial_i h \int_I B\partial_t B dt = 0 \quad \text{for } i \geq 1.$$

Therefore, (5.4) yields $\lambda_0(h^2) \geq 0$ as required.

Since λ_0 extends[1] to a positive linear functional on $C_c(U_0, \mathbb{R})$, Riesz' representation theorem [96, Theorems 2.14 and 2.18] yields a unique locally finite regular Borel measure μ_0 on U_0 such that $\lambda_0(f) = \int_{U_0} f d\mu_0(x)$. This implies

$$\lambda_{U,0}(f) = \int_U f(u) d\mu_{U,X}(u),$$

with $\mu_{U,X}$ the product of μ_0 with the Lebesgue measure on I.

To finish the proof, we now prove that $\lambda_i = 0$ for $i > 0$. It suffices to show that $\lambda_i(h^2) = 0$ for all $h \in C_c^\infty(U_0, \mathbb{R})$. Choose $B_C, B_S \in C_c^\infty(I, \mathbb{R})$ so that

$$\int_I B_S(t) B_C'(t) dt = 1,$$

choose $C, S \in C_c^\infty(U_0, \mathbb{R})$ so that

$$C(x) = \cos\left(\sum_{i=1}^n k_i x^i\right) \quad \text{and} \quad S(x) = \sin\left(\sum_{i=1}^n k_i x^i\right)$$

for $x \in \operatorname{supp}(h)$, $k_i \in \mathbb{Z}$, and set

$$f(t, \vec{x}) := h(\vec{x})\big(B_C(t)C(\vec{x}) + B_S(t)S(\vec{x})\big).$$

Then, with

$$E := \int_I \big(|B_C'(t)| + |B_S'(t)|\big)^2 dt,$$

we have

$$0 \leq \overline{(\partial_t f)^2} = h^2(\vec{x}) \int_I \big(B_C'(t)C(\vec{x}) + B_S'(t)S(\vec{x})\big)^2 dt \leq E h^2(x).$$

---

[1]For every compact $S \subseteq U_0$, there exists a $\varphi \in C_c^\infty(U_0, \mathbb{R})$ with $\varphi|_S > 1$. With $L_S = \lambda_0(\varphi^2)$, it then follows from the inequality $\lambda_0(\|f\|_\infty \varphi^2 \pm f) \geq 0$ that $|\lambda_0(f)| \leq L_S \|f\|_\infty$ for all f with support in S.

Making repeated use of

$$\int_I F(t,\vec{x})\partial_t F(t,\vec{x})dt = 0 \quad \text{and} \quad \int_I B'_C B_S + B'_S B_C\, dt = 0,$$

we find, for $i = 1, \dots, n-1$,

$$\overline{\partial_t f \partial_i f} = k_i h^2.$$

Equation (5.4) then yields

$$\lambda_0\big(\overline{(\partial_t f)^2}\big) + \sum_{i=1}^{n-1} k_i \lambda_i(h^2) \geq 0 \quad \text{for all } k_i \in \mathbb{Z}, \tag{5.5}$$

where the function f depends on the k_i. As $\lambda_0(\overline{(\partial_t f)^2}) \leq E\lambda_0(h^2)$, the non-negative term $\lambda_0(\overline{(\partial_t f)^2})$ is bounded by a number that does not depend on k_i. It therefore follows from inequality (5.5) that $\lambda_i(h^2) = 0$ for all $i > 0$, as was to be proven. ∎

5.2 Infinitesimally free $\mathbb{R}$-actions

In Section 4.2, we saw that $\Gamma_c(M, \mathfrak{K})$ is isomorphic to the gauge algebra $\Gamma_c(\widehat{M}, \widehat{\mathfrak{K}})$, where $\widehat{\mathfrak{K}} \to \widehat{M}$ is a Lie algebra bundle with *simple* fibers over a cover $\widehat{M} \to M$. The decomposition $\widehat{M} = \bigsqcup_{i=1}^r \widehat{M_i}$ in connected components therefore gives rise to a direct sum decomposition

$$\Gamma_c(M, \mathfrak{K}) = \bigoplus_{i=1}^r \Gamma_c(\widehat{M}_i, \widehat{\mathfrak{K}}), \tag{5.6}$$

where $\widehat{\mathfrak{K}} \to \widehat{M}_i$ is a Lie algebra bundle with simple fibers isomorphic to $\mathfrak{k}_i$.

5.2.1 Reduction to compact simple structure algebras

If $\mathbf{v}_M$ is non-vanishing, then we can restrict attention to the terms in (5.6) where $\mathfrak{k}_i$ is a compact simple Lie algebra.

Corollary 5.5. *Suppose that $\mathfrak{k}_i$ is not compact, and let $m \in \widehat{M}_i$ be a point such that $\pi_* \widehat{\mathbf{v}}_m \neq 0$. Let $U \subseteq \widehat{M}_i$ be as in Proposition 5.4 and let $\lambda_U \in \Omega^1_c(U, V(\mathfrak{k}_i))'$ be as in (5.1) and (5.2). Then, $\lambda_U \colon \Omega^1_c(U, V(\mathfrak{k})) \to \mathbb{R}$ is zero. Consequently, ω_U is cohomologous to zero on $\Gamma_c(U, \widehat{\mathfrak{K}})$.*

Proof. It suffices to show that $\mu_{U,X} = 0$ for all $X \in \mathfrak{k}_i$. If $X, Y \in \mathfrak{k}_i$ with $\kappa(X,X) = -\kappa(Y,Y)$, then $\mu_{U,X} = -\mu_{U,Y}$ implies $\mu_{U,X} = \mu_{U,Y} = 0$. If $\mathfrak{k}_i$ is a complex Lie algebra, i.e., $V(\mathfrak{k}_i) \simeq \mathbb{C}$ (cf. Section 4.3.1), then the previous argument with $Y = iX$ yields $\mu_{U,X} = 0$ for all $X \in \mathfrak{k}_i$. If $V(\mathfrak{k}_i) \simeq \mathbb{R}$, then $\mathfrak{k}_i$ is noncompact if and only if $\{\kappa(X, X); X \in \mathfrak{k}_i\} = \mathbb{R}$. Therefore, the same reasoning applies. ■

Corollary 5.6. *If ρ is a positive energy representation of $\widehat{G}$ and $\mathbf{v}_M$ has no zeros, then ω is cohomologous to a cocycle that vanishes on the subalgebras $\Gamma_c(\widehat{M}_i, \widehat{\mathfrak{K}})$, where $\mathfrak{k}_i$ is noncompact.*

Proof. By Theorem 4.16 applied to $\Gamma_c(\widehat{M}, \widehat{\mathfrak{K}})$, the class $[\omega] \in H^2(\mathfrak{g} \rtimes_D \mathbb{R}, \mathbb{R})$ is uniquely determined by a $\mathbb{V}$-valued current $\lambda\colon \Omega_c(\widehat{M}, \mathbb{V}) \to \mathbb{R}$. Since $\mathbf{v}_M$ is everywhere non-zero, the same holds for $\pi_*\hat{\mathbf{v}}$. If $\mathfrak{k}_i$ is noncompact, by Corollary 5.5, $\widehat{M}_i$ can be covered with open sets U_{ij} such that λ vanishes on $\Omega_c(U_{ij}, \mathbb{V})$. As every element of $\Omega_c(\widehat{M}_i, \mathbb{V})$ can be written as a finite sum of elements of $\Omega_c(U_{ij}, \mathbb{V})$, the current λ vanishes on $\Omega_c(\widehat{M}_i, \mathbb{V})$. ■

5.2.2 Reduction of currents to measures

Let $\rho\colon \widehat{G} \to \mathrm{U}(\mathcal{H})$ be a positive energy representation, where $G = \widetilde{\Gamma}_c(M, \mathcal{K})_0$ is the simply connected Lie group with Lie algebra $\mathfrak{g} = \Gamma_c(M, \mathfrak{K})$, which covers the identity component of the compactly supported gauge group. This gives rise to a Lie algebra cocycle ω on $\mathfrak{g} \rtimes_D \mathbb{R}$. Using the results of Section 4.2, we identify the gauge Lie algebra $\mathfrak{g} = \Gamma_c(M, \mathfrak{K})$ with $\mathfrak{g} = \Gamma_c(\widehat{M}, \widehat{\mathfrak{K}})$, where $\widehat{\mathfrak{K}} \to \widehat{M}$ is a Lie algebra bundle with simple fibers. The cocycle ω can then be represented by a *measure* on $\widehat{M}$.

Theorem 5.7. *Suppose that $\mathbf{v}_M$ has no zeros, and that ω is a 2-cocycle on $\mathfrak{g} \rtimes_D \mathbb{R}$ induced by a positive energy representation $\rho\colon \widehat{G} \to \mathrm{U}(\mathcal{H})$. Then, there exists a positive, regular, locally finite Borel measure μ on $\widehat{M}$ invariant under the flow $\gamma_{\widehat{M}}$ on $\widehat{M}$ induced by $\gamma_{\mathcal{K}}$, such that ω is cohomologous to the 2-cocycle $\omega_{\mu,\nabla}$, given by*

$$\omega_{\mu,\nabla}(\xi, \eta) = -\int_{\widehat{M}} \kappa(\xi, \nabla_{\widehat{\mathbf{v}_M}}\eta)\,d\mu(m), \tag{5.7}$$

$$\omega_{\mu,\nabla}(D, \xi) = -\int_{\widehat{M}} \kappa(i_{\widehat{\mathbf{v}_M}}(L_{\hat{\mathbf{v}}}\nabla), \xi)\,d\mu(m) \quad \textit{for } \xi, \eta \in \Gamma_c(\widehat{M}, \widehat{\mathfrak{K}}). \tag{5.8}$$

The support of μ is contained in the union of the connected components $\widehat{M}_i$ where the fibers of $\widehat{\mathfrak{K}}$ are compact simple Lie algebras. In (5.7) and (5.8), we identify κ with the positive definite invariant bilinear form normalized as in (4.2).

Proof. As $\mathbf{v}_M$ is nowhere zero, we can cover $\widehat{M}$ by good flowboxes $U \subseteq \widehat{M}$ in the sense of Definition 5.1. In the corresponding local trivialization $\Gamma_c(U, \widehat{\mathfrak{K}}) \simeq C_c^\infty(U, \mathfrak{k})$ (cf. Lemma 5.2), we may assume that $\mathfrak{k}$ is compact by Corollary 5.5. We normalize κ as in (4.2) and define μ_U as $\mu_{U,X}$ for any $X \in \mathfrak{k}$ with $\kappa(X, X) = 1$. If U and U' are two such open sets, then the measures μ_U and $\mu_{U'}$ from Proposition 5.4

coincide on the intersection $U \cap U'$, as both measures are uniquely determined by the cocycle ω. The measures μ_U thus splice together to form a positive regular locally finite Borel measure on $\widehat{M}$. Equations (5.7) and (5.8) then follow immediately from (4.3), (4.4) in Section 4.3.3, and (5.1), (5.2). ∎

Remark 5.8. As the cohomology class $[\omega_{\lambda,\nabla}]$ is independent of the choice of the Lie connection, we are free to choose ∇ so that $\hat{\mathbf{v}}$ is horizontal. In that case, we have

$$i_{\widehat{\mathbf{v}_M}}(L_{\hat{\mathbf{v}}}\nabla) = 0 \quad \text{and} \quad L_{\hat{\mathbf{v}}}\xi = \nabla_{\pi_*\hat{\mathbf{v}}}\xi$$

(cf. Remark 4.17). Equation (5.8) then becomes

$$\omega_{\mu,\nabla}(D,\xi) = 0.$$

From Examples 4.11-4.14 in Section 4.2, we obtain the following.

Example 5.9. If $\mathfrak{K} \to M$ has simple fibers, then $\widehat{M} = M$. The class $[\omega_{\mu,\nabla}]$ then corresponds to a measure μ on M. It vanishes on the connected components of M where the fibers of $\mathfrak{K} \to M$ are noncompact.

Example 5.10. Suppose that M is connected, and that the typical fiber $\mathfrak{k} = \bigoplus_{i=1}^r \mathfrak{k}_i$ is the direct sum of r mutually non-isomorphic simple ideals $\mathfrak{k}_i$. Then, $\widehat{M}$ is the disjoint union of r copies of M. The class $[\omega_{\mu,\nabla}]$ is then given by r measures μ_i on M, one for each simple ideal. The same holds if $\mathfrak{K} = M \times \mathfrak{k}$ is trivial, and the $\mathfrak{k}_i$ are not necessarily non-isomorphic.

Example 5.11 (Frame bundles of 4-manifolds). (cf. Examples 4.14). Suppose that M is a Riemannian 4-manifold, and $\mathfrak{K} = \mathrm{ad}(\mathrm{OF}(M))$ is the adjoint bundle of its orthogonal frame bundle. If M is orientable, then $\omega_\mu = \omega_{\mu_L} + \omega_{\mu_R}$ is the sum of two cocycles with measures μ_L and μ_R on M corresponding to the simple factors $\mathfrak{su}_L(2,\mathbb{C})$ and $\mathfrak{su}_R(2,\mathbb{C})$ of $\mathfrak{so}(4,\mathbb{R})$. If M is not orientable, then ω_μ is described by a single measure μ on the orientable cover $\widehat{M} \to M$.

5.3 Cauchy–Schwarz estimates revisited

Using the explicit form of the cocycles determined in Theorem 5.7, we revisit the Cauchy–Schwarz estimates of Section 3.4. In this section, we assume that $\mathfrak{K} \to M$ has semisimple fibers, and that the vector field $\mathbf{v}_M$ on M is nowhere vanishing. As before, we identify $\Gamma_c(M,\mathfrak{K})$ with $\Gamma_c(\widehat{M},\widehat{\mathfrak{K}})$, where $\widehat{\mathfrak{K}} \to \widehat{M}$ has simple fibers.

Define the positive semidefinite symmetric bilinear form on $\mathfrak{g} = \Gamma_c(\widehat{M},\widehat{\mathfrak{K}})$ by

$$\langle \xi,\eta\rangle_\mu := \int_{\widehat{M}} \kappa(\xi,\eta)d\mu(m). \tag{5.9}$$

Using Theorem 5.7 and Remark 5.8, we may replace ω by $\omega_{\mu,\nabla}$ for a Lie connection ∇ on $\widehat{\mathfrak{K}}$ that makes $\widehat{\mathbf{v}}$ horizontal. In that case, we have $i_{\widehat{\mathbf{v}_M}}(L_{\widehat{\mathbf{v}}}\nabla) = 0$ and $L_{\widehat{\mathbf{v}}}\xi = \nabla_{\pi_*\widehat{\mathbf{v}}}\xi$ (cf. Remark 4.17). We may thus assume, without loss of generality, that the cocycle associated to a positive energy representation takes the form

$$\omega(\xi,\eta) = -\langle \xi, L_{\widehat{\mathbf{v}}}\eta\rangle_\mu = \langle L_{\widehat{\mathbf{v}}}\xi, \eta\rangle_\mu, \quad \omega(D,\xi) = 0. \tag{5.10}$$

The Cauchy–Schwarz estimate (Lemma 3.10) can now be reformulated as follows.

Lemma 5.12 (Cauchy–Schwarz Estimate). *Let ρ be a positive energy representation of $\widehat{G}$, where $G = \widetilde{\Gamma}_c(M,\mathcal{K})_0$ is the simply connected gauge group. If the vector field $\mathbf{v}_M$ on M has no zeros, then, after replacing the linear lift $\mathtt{d}\rho\colon \mathfrak{g} \to \mathrm{End}(\mathcal{H}^\infty)$ of the projective representation $\overline{\mathtt{d}\rho}$ of $\mathfrak{g}$ by $\mathtt{d}\rho + i\chi\mathbf{1}$ for some continuous linear functional $\chi\colon \mathfrak{g} \to \mathbb{R}$, we have*

$$\langle i\,\mathtt{d}\rho(L_{\widehat{\mathbf{v}}}\xi)\rangle_\psi^2 \leq 2\langle H\rangle_\psi \|L_{\widehat{\mathbf{v}}}\xi\|_\mu^2 \quad \text{for all } \xi \in \mathfrak{g} \text{ with } [L_{\widehat{\mathbf{v}}}\xi, \xi] = 0 \tag{5.11}$$

and every unit vector $\psi \in \mathcal{H}^\infty$.

Proof. First we observe that the passage from ω to an equivalent cocycle corresponds to replacing the subspace $\mathfrak{g} \subseteq \widehat{\mathfrak{g}}$ by the subspace $\chi(\xi)C + \xi$, $\xi \in \mathfrak{g}$, where $\chi\colon \mathfrak{g} \to \mathbb{R}$ is a continuous linear functional. For the representation $\mathtt{d}\rho$ this changes the value of $\mathtt{d}\rho(\xi)$ by adding $i\chi(\xi)$, so that we can achieve a cocycle of the form (5.10) by Theorem 5.7. Now we apply Lemma 3.10 with $i_D\omega_{\mu,\nabla} = 0$ and $\omega_{\mu,\nabla}(\xi, D\xi) = \|L_{\widehat{\mathbf{v}}}\xi\|_\mu^2$. ■

In the same vein, the refined Cauchy–Schwarz estimate, Lemma 3.12, can be reformulated as follows.

Lemma 5.13. *Under the assumptions of Lemma 5.12, we have*

$$\begin{aligned}\Big(\big\langle i\,\mathtt{d}\rho(e^{-s\,\mathrm{ad}_\eta}(L_{\widehat{\mathbf{v}}}\xi))\big\rangle_\psi - \Big\langle \frac{e^{-s\,\mathrm{ad}_\eta} - \mathbf{1}}{\mathrm{ad}_\eta}(L_{\widehat{\mathbf{v}}}\xi), L_{\widehat{\mathbf{v}}}\eta\Big\rangle_\mu\Big)^2 \\ \leq 2\|L_{\widehat{\mathbf{v}}}\xi\|_\mu^2\Big(\langle H\rangle_\psi + s\langle i\,\mathtt{d}\rho(L_{\widehat{\mathbf{v}}}\eta)\rangle_\psi + \frac{s^2}{2}\|L_{\widehat{\mathbf{v}}}\eta\|_\mu^2\Big)\end{aligned} \tag{5.12}$$

for all $s \in \mathbb{R}$, and for all $\xi, \eta \in \Gamma_c(\widehat{M}, \widehat{\mathfrak{K}})$ such that $[\xi, L_{\widehat{\mathbf{v}}}\xi] = 0$ and $[\eta, L_{\widehat{\mathbf{v}}}\eta] = 0$.

Chapter 6

Continuity properties

Having determined which cocycles are compatible with the Cauchy–Schwarz estimates, we now turn to the classification of positive energy representations for the central extensions that correspond to these cocycles. This will be achieved in Chapter 7, using continuity and extension results developed in the present chapter.

In this chapter, we assume that the flow $\mathbf{v}_M$ is nowhere vanishing. Further, we assume that the fibers of $\mathfrak{K} \to M$ are *simple* Lie algebras. This entails no loss of generality compared to semisimple fibers, as one can switch to the Lie algebra bundle $\widehat{\mathfrak{K}} \to \widehat{M}$ in that case by the results in Section 4.2.

In Section 6.1, we use the Cauchy–Schwarz estimate 5.12 to further reduce the problem to the case where $\mathfrak{K}$ has *compact* simple fibers. In Section 6.2, we use the refined Cauchy–Schwarz estimate of Lemma 5.13 to bound $i\,\mathtt{d}\rho(\xi)$ in terms of the Hamilton operator H, the L^2-norm $\|\xi\|_\mu$ with respect to the measure μ of Theorem 5.7, and the L^2-norm $\|\xi\|_{B\mu}$ with respect to the product of μ with a suitable upper semi-continuous function $B\colon M \to \mathbb{R}^+$. In Section 6.3, we interpret these estimates as a continuity property, and use this to define an extension of $\mathtt{d}\rho$ to a space $H^1_{B\mu}(M,\mathfrak{K})$ of sections that are differentiable in the direction of the orbits, but merely measurable in the transversal direction. In Section 6.4, we construct a subspace $H^1_\partial(M,\mathfrak{K})$ of bounded sections that is closed under the pointwise Lie bracket. Finally, in Section 6.5, we show that by extending to $H^1_{B\mu}(M,\mathfrak{K})$ and restricting to $H^1_\partial(M,\mathfrak{K})$, one obtains a representation of the latter Lie algebra that is compatible with the Hamiltonian H. On a subalgebra $H^2_\partial(M,\mathfrak{K})$ of sections that are twice differentiable in the orbit direction, we then show that there is a dense set of uniformly analytic vectors. In Chapter 7, this will be needed in order to integrate the Lie algebra representation to the group level.

6.1 Reduction to compact simple structure algebras

As a direct consequence of Lemma 5.12, we see that $\mathtt{d}\rho(L_{\mathbf{v}}\xi)$ vanishes for all $\xi \in \mathfrak{g}$ with $[\xi, L_{\mathbf{v}}\xi] = 0$ and $\|L_{\mathbf{v}}\xi\|_\mu = 0$. We use this to show that every positive energy representation factors through a gauge algebra with *compact* structure algebra.

Proposition 6.1. *For $\mathfrak{g} = \Gamma_c(M,\mathfrak{K})$ with $\mathbf{v}_M$ without zeros, we have*

$$\mathfrak{g} = D\mathfrak{g} + [D\mathfrak{g}, D\mathfrak{g}].$$

Considered as subsets of $\widehat{\mathfrak{g}} = \mathbb{R} \oplus_\omega (\mathfrak{g} \rtimes_D \mathbb{R})$, with ω as in Theorem 5.7, we have

$$\mathbb{R} \oplus_\omega \mathfrak{g} = D\mathfrak{g} + [D\mathfrak{g}, D\mathfrak{g}].$$

Proof. By a partition of unity argument, it suffices to prove this for $\mathfrak{g} = C_c^\infty(U, \mathfrak{k})$, where $U = I \times U_0$ is a good flowbox (cf. Definition 5.1) and

$$D\xi = \frac{d}{dt}\xi$$

(cf. Lemma 5.2). If $f \in C_c^\infty(U, \mathfrak{k})$ and $X \in \mathfrak{k}$, then fX lies in $D\mathfrak{g} \subseteq \mathfrak{g}$ if and only if

$$f_0(x) := \int_{-\infty}^{\infty} f(t, x) dt$$

is zero in $C_c^\infty(U_0, \mathbb{R})$. Fix $\zeta \in C_c^\infty(I, \mathbb{R})$ with $\int_{-\infty}^{\infty} \zeta(t) dt = 0$ and $\int_{-\infty}^{\infty} \zeta^2(t) dt = 1$. Then,

$$fX = (f - \zeta^2 f_0)X + \zeta^2 f_0 X \quad \text{with } (f - \zeta^2 f_0)X \in D\mathfrak{g}.$$

To show that $\zeta^2 f_0 X \in [D\mathfrak{g}, D\mathfrak{g}]$, choose $\chi \in C_c^\infty(U_0, \mathfrak{k})$ such that $\chi|_{\mathrm{supp}(f_0)} = 1$, and choose $Y_i, Z_i \in \mathfrak{k}$ such that $X = \sum_{i=1}^r [Y_i, Z_i]$. Since

$$\sum_{i=1}^{r} [\zeta f_0 Y_i, \zeta \chi Z_i] = \zeta^2 f_0 X$$

with $\zeta f_0 Y_i, \zeta \chi Z_i \in D\mathfrak{g}$, we have

$$fX = (f - \zeta^2 f_0)X + \sum_{i=1}^{r} [\zeta f_0 Y_i, \zeta \chi Z_i] \in D\mathfrak{g} + [D\mathfrak{g}, D\mathfrak{g}]. \tag{6.1}$$

This holds for the Lie bracket in $\mathfrak{g}$ as well as for the Lie bracket in $\widehat{\mathfrak{g}}$. The relation

$$\int_{-\infty}^{\infty} \zeta \frac{d}{dt} \zeta dt = 0$$

implies $\omega(\zeta f_0 Y_i, \zeta \chi Z_i) = 0$. This shows that $\mathfrak{g} = D\mathfrak{g} + [D\mathfrak{g}, D\mathfrak{g}]$ in $\mathfrak{g}$ and also $\mathfrak{g} \subseteq D\mathfrak{g} + [D\mathfrak{g}, D\mathfrak{g}]$ in $\widehat{\mathfrak{g}}$. Since ω is not identically zero on $D\mathfrak{g} \times D\mathfrak{g}$, the subspace $D\mathfrak{g} + [D\mathfrak{g}, D\mathfrak{g}]$ of $\widehat{\mathfrak{g}}$ cannot be proper and thus $\mathbb{R}C \subseteq D\mathfrak{g} + [D\mathfrak{g}, D\mathfrak{g}]$. This shows that

$$\widehat{\mathfrak{g}} = \mathbb{R}C + \mathfrak{g} = D\mathfrak{g} + [D\mathfrak{g}, D\mathfrak{g}].$$ ∎

Theorem 6.2 (Reduction to compact structure algebra). *Let $M_i \subseteq M$ be a connected component such that the (simple) fibers of $\mathfrak{K}|_{M_i}$ are not compact. Suppose that $\mathbf{v}_M$ is non-vanishing on M_i. Then, after twisting by a functional $\chi \in \Gamma_c(M_i, \mathfrak{K})'$ if necessary, every positive energy representation $d\rho$ of $\Gamma_c(M, \mathfrak{K})$ vanishes on the ideal $\Gamma_c(M_i, \mathfrak{K})$.*

Proof. By a partition of unity argument, it suffices to consider the restriction of $d\rho$ to $C_c^\infty(U, \mathfrak{k})$ for a good flowbox $U \subseteq M_i$ (cf. Definition 5.1). Every $\xi \in DC_c^\infty(U, \mathfrak{k})$ is a finite sum of elements of the form $f'X$, with $f \in C_c^\infty(U, \mathbb{R})$ and $X \in \mathfrak{k}$. Since $\mathfrak{k}$ is noncompact, μ vanishes on M_i by Theorem 5.7. Since

$$\|f'X\|_\mu = 0 \quad \text{and} \quad [fX_i, f'X_i] = 0,$$

it follows from Lemma 5.12 that, after twisting by χ so as to change ω to $\omega_{\mu,\nabla}$, we have $\mathtt{d}\rho(f'X_i) = 0$. Since $\mathtt{d}\rho(DC_c^\infty(U, \mathfrak{k})) = \{0\}$, Proposition 6.1 yields

$$\mathtt{d}\rho(C_c^\infty(U, \mathfrak{k})) = \{0\}.$$

Thus, $\mathtt{d}\rho(\Gamma_c(M_i, \mathfrak{K})) = \{0\}$, as required. ∎

This shows that we can restrict attention to bundles $\mathfrak{K} \to M$ with *compact* simple fibers. (Note that the result requires a non-zero vector field on M, so this is compatible with the unitary highest weight representations of $C^\infty(\mathbb{S}^1, \mathfrak{su}_{1,n-1}(\mathbb{C}))$ studied in [47].) In conjunction with Proposition 6.1, Lemma 5.12 can also be used to prove the following.

Corollary 6.3. *If* $\mathfrak{g} = \Gamma_c(M, \mathfrak{K})$*, where* $\mathfrak{K} \to M$ *has compact simple fibers, then, after twisting by* $\chi \in \Gamma_c(M, \mathfrak{K})'$ *if necessary, every positive energy representation* $\mathtt{d}\rho$ *of* $\widehat{\mathfrak{g}}$ *vanishes on the ideal*

$$I_\mu := \big\{\xi \in \mathfrak{g};\, \mu(\{x \in M;\, \xi(x) \neq 0\}) = 0\big\}$$

of sections that vanish μ*-almost everywhere.*

Proof. By a partition of unity argument, we may assume that $\mathfrak{g} = C_c^\infty(U, \mathfrak{k})$, with $U \subseteq M$ a good flowbox (Definition 5.1). Let $\xi \in I_\mu$ and consider the open subset $\mathcal{O}_\xi := \{x \in M;\, \xi(x) \neq 0\}$, which is the "open support" of ξ. Since ξ is a linear combination of terms fX with smaller or equal open support, we may assume that $\xi = fX$ for $f \in C_c^\infty(U, \mathbb{R})$ and $X \in \mathfrak{k}$. If $fX \in D\mathfrak{g}$, then $fX \in I_\mu$ implies $\|fX\|_\mu = 0$ and hence $\mathtt{d}\rho(fX) = 0$ by Lemma 5.12. Decompose fX as in equation (6.1),

$$fX = (f - \zeta^2 f_0)X + \sum_{i=1}^{r} [\zeta f_0 Y_i, \zeta\chi Z_i].$$

As $\mathcal{O}_\xi$ is open and $\mu = dt \otimes \mu_0$, we have $\mu(\mathcal{O}_\xi) = 0$ if and only if $\mu_0(p(\mathcal{O}_\xi))$ vanishes, where $p\colon U \to U_0$ is the projection on the orbit space. Now $(f - \zeta^2 f_0)X$ and $\zeta f_0 Y_i$ are in $D\mathfrak{g}$ and vanish outside $p^{-1}p(\mathcal{O}_\xi)$, so that their images under $\mathtt{d}\rho$ vanish. Indeed, as these are both of the form $L_{\mathrm{v}}\eta$ with $\|L_{\mathrm{v}}\eta\|_\mu = 0$ and $[L_{\mathrm{v}}\eta, \eta] = 0$, this follows from Lemma 5.12. We conclude that $\mathtt{d}\rho(fX) = 0$, as required. ∎

6.2 Extending the estimates from $D\mathfrak{g}$ to $\mathfrak{g}$

To see that $\mathtt{d}\rho$ factors through a linear map on $\mathfrak{g}/I_\mu$, we used the Cauchy–Schwarz estimate of Lemma 5.12. Using the *refined* Cauchy–Schwarz estimate of Lemma 5.13, we then extend $\mathrm{d}\rho$ to a linear map on $\overline{\mathfrak{g}/I_\mu}$, the L^2-completion of $\mathfrak{g}/I_\mu$ with respect to the measure μ.

Note that an extension to the subspace $\overline{D\mathfrak{g}/I_\mu} \subseteq \overline{\mathfrak{g}/I_\mu}$ can already be achieved using the "ordinary" Cauchy–Schwarz estimate of Lemma 5.12. Indeed, for $\xi \in D\mathfrak{g}$, one can use (5.11) to show that $\mathrm{d}\rho(\xi)$ satisfies an operator inequality of the form

$$\pm\, i\,\mathrm{d}\rho(\xi) \leq \|\xi\|_\mu(\alpha\mathbf{1} + \beta H) \tag{6.2}$$

for certain constants $\alpha, \beta > 0$. With this, one can prove that $\mathrm{d}\rho \colon D\mathfrak{g}/I_\mu \to \mathrm{End}(\mathscr{H}^\infty)$ is weakly continuous with respect to the L^2-topology on $D\mathfrak{g}/I_\mu$, and that it extends to the L^2-completion $\overline{D\mathfrak{g}/I_\mu}$.

In order to extend $\mathrm{d}\rho$ to the full space $\overline{\mathfrak{g}/I_\mu}$, however, we will need an analog of (6.2) that holds not just for $\xi \in D\mathfrak{g}$, but for all $\xi \in \mathfrak{g}$. This is Proposition 6.16, which we prove using the refined Cauchy–Schwarz estimate of Lemma 5.13.

6.2.1 The local gauge algebra with fibers $\mathfrak{k} = \mathfrak{su}(2, \mathbb{C})$

First, we restrict our attention to the compact structure algebra $\mathfrak{k} = \mathfrak{su}(2, \mathbb{C})$. We will later derive the general case from this example. Let $\kappa(a, b) = -\mathrm{tr}(ab)$ be the invariant bilinear form on $\mathfrak{k}$, normalized so that elements x with

$$\mathrm{Spec}(\mathrm{ad}\, x) = \{0, \pm 2i\} \;\text{ satisfy }\; \kappa(x, x) = 2.$$

Further, let $U' \subset U$ be a *good pair of flowboxes* in the sense of the following definition. We write $U \Subset V$ if the closure of U is contained in an open subset of V.

Definition 6.4 (Good pair of flowboxes). Let $U' \simeq I' \times U_0'$ and $U \simeq I \times U_0$ be good flowboxes in the sense of Definition 5.1, and let $U' \subset U$. We call $U' \subset U$ a *good pair of flowboxes* if $I' \Subset I$ and $U_0' \Subset U_0$.

Remark 6.5. Note that $U_0' = U_0 = \{0\}$ is allowed! Unless specified otherwise, we assume that $I' = (-T'/2, T'/2)$ and $I = (-T/2, T/2)$ with $0 < T' < T < \infty$.

Remark 6.6. Recall that M is equipped with a flow-invariant measure μ, which takes the form $dt \otimes \mu_0$ on $I \times U_0$. To a good pair of flowboxes, we can therefore assign the number

$$\frac{T}{T - T'}\frac{\mu_0(U_0)}{T'} = \frac{\mu(U)}{T'(T - T')},$$

which will play a significant role throughout this chapter. If this is not too large, we think of the flowboxes as "sufficiently quadratic".

In view of Lemma 5.2, we restrict attention to the case where the Lie algebra is $\mathfrak{g} = C_c^\infty(I \times U_0, \mathfrak{k})$, and $\mathbf{v} = \partial_t$. For $z \in C_c^\infty(U', \mathbb{C})$, we define $\xi(z) \in \mathfrak{g}$ by

$$\xi(z)(t, u) := \begin{pmatrix} 0 & z(t, u) \\ -\bar{z}(t, u) & 0 \end{pmatrix} \tag{6.3}$$

and note that $[\xi, \frac{\partial}{\partial t}\xi] = 0$. We also consider the element $\eta \in \mathfrak{g}$ defined by

$$\eta(t,u) := \chi(u) \begin{pmatrix} i\tau(t) & 0 \\ 0 & -i\tau(t) \end{pmatrix}, \tag{6.4}$$

where $\tau \in C_c^\infty(I, \mathbb{R})$ and $\chi \in C_c^\infty(U_0, \mathbb{R})$ are such that $\tau(t) = t$ for $t \in I'$ and $\chi(u) = 1$ for $u \in U_0'$. It also satisfies

$$\left[\tau, \frac{\partial}{\partial t}\tau\right] = 0.$$

Thus, $\chi(u)\tau(t) = t$ on U', hence, in particular, on the support of every $z \in C_c^\infty(U', \mathbb{C})$.

On $C_c^\infty(\mathbb{R}, \mathbb{C})$, we define the usual scalar product

$$\langle f, g\rangle_{dt} := \int_{-\infty}^{\infty} \overline{f(t)} g(t) dt$$

and the Fourier transform

$$\hat{f}(k) := \int_{-\infty}^{\infty} f(t) e^{-ikt} dt, \quad k \in \mathbb{R}.$$

For $z \in C_c^\infty(U, \mathbb{C})$, we will denote by $\hat{z}(k, u)$ the "parallel" Fourier transform, i.e., the Fourier transform of $t \mapsto z(t, u)$ evaluated at k.

We can choose τ such that $\|\tau'\|_{dt}^2$ is arbitrarily close to $\frac{TT'}{T-T'}$, and we can choose $0 \le \chi \le 1$ so that $\|\chi\|_{\mu_0}^2 \le \mu_0(U_0)$. Thus, $\|\chi\tau'\|_\mu^2$ can be chosen arbitrarily close to $\frac{TT'}{T-T'}\mu_0(U_0)$. Therefore, for every $\varepsilon > 0$, there exists an $\eta \in C_c^\infty(U, \mathbb{C})$ as in (6.4) satisfying

$$\|L_{\mathrm{v}}\eta\|_\mu^2 = 2\frac{TT'}{T-T'}\mu_0(U_0) + \varepsilon. \tag{6.5}$$

For η as in (6.4), Lemma 5.13 yields the following estimate.

Proposition 6.7. *Let $z \in C_c^\infty(U', \mathbb{C})$, and let $k \in \mathbb{R}$ be such that $\hat{z}(k, u) = 0$ for all $u \in U_0'$. Then, we have*

$$\langle i\, d\rho(\xi(z))\rangle_\psi^2 \le 4\|z\|_\mu^2 \left(\langle H\rangle_\psi - \frac{1}{2}k\langle i\, d\rho(L_{\mathrm{v}}\eta)\rangle_\psi + \frac{1}{8}k^2\|L_{\mathrm{v}}\eta\|_\mu^2 \right).$$

Proof. Since $[\xi(z), \xi(z)'] = 0$ and $[\eta, \eta'] = 0$, we may apply Lemma 5.13. First, we evaluate the left-hand side of inequality (5.12). Since $\tau(t)\chi(u) = t$ on $\mathrm{supp}(\xi(z))$, we have $\mathrm{ad}_\eta(\xi(z)) = \xi(2tiz)$. Since $L_{\mathrm{v}}\xi(z) = \xi(z')$, we have

$$e^{-s\,\mathrm{ad}_\eta}(L_{\mathrm{v}}\xi(z)) = \xi(z' e^{-2its}).$$

As $\kappa(\xi(z), L_{\mathrm{v}}\eta) = 0$ for all $z \in C_c^\infty(U', \mathbb{C})$, we have

$$\left\langle \frac{e^{-s\,\mathrm{ad}_\eta} - \mathbf{1}}{\mathrm{ad}_\eta}(L_{\mathrm{v}}\xi(z)), L_{\mathrm{v}}\eta \right\rangle_\mu = \left\langle \xi\left(\frac{e^{-2tis} - 1}{2it} z'\right), L_{\mathrm{v}}\eta \right\rangle_\mu = 0.$$

On the right-hand side of inequality (5.12), we have $\|L_{\mathbf{v}}\xi(z)\|_\mu^2 = 2\|z'\|_\mu^2$. We thus obtain

$$\left\langle i\,\mathrm{d}\rho(\xi(z'e^{-2ist}))\right\rangle_\psi^2 \leq 4\|z'\|_\mu^2\left(\langle H\rangle_\psi + s\langle i\,\mathrm{d}\rho(L_{\mathbf{v}}\eta)\rangle_\psi + \frac{s^2}{2}\|L_{\mathbf{v}}\eta\|_\mu^2\right)$$

for all $s \in \mathbb{R}$ and $z \in C_c^\infty(U', \mathbb{C})$. Note that $w \in C_c^\infty(U', \mathbb{C})$ is of the form $w = z'e^{-2ist}$ for some $z \in C_c^\infty(U, \mathbb{C})$ if and only if the parallel Fourier transform $\widehat{w}(k, u)$ vanishes for $k = -2s$. Since in that case $\|w\|_\mu^2 = \|z'\|_\mu^2$, the proposition follows. ∎

We thus obtain a 1-parameter family of inequalities indexed by $k \in \mathbb{R}$, the case $k = 0$ reducing to the Cauchy–Schwarz estimate because $\widehat{z}(0, u) = 0$ is equivalent to $\xi(z) \in D\mathfrak{g}$. The idea of the following proposition is to lift the requirement that the Fourier transform vanish by showing that every $z \in C_c^\infty(U', \mathbb{C})$ can be written, in a controlled way, as the sum of two functions whose parallel Fourier transform vanishes for some $k \in \mathbb{R}$.

Proposition 6.8. *There exist $a, b \in \mathbb{R}$ such that, for all $z \in C_c^\infty(U', \mathbb{C})$ for which $U' = I' \times U_0'$ contained in $U = I \times U_0$, we have*

$$\left\langle i\,d\rho(\xi(z))\right\rangle_\psi^2 \leq (a + b\langle H\rangle_\psi)\|\xi(z)\|_\mu^2 \tag{6.6}$$

for constants a and b that depend on the interval lengths $T = |I|$ and $T' = |I'|$ and on $\mu_0(U_0)$, but not on z or ψ.

Proof. Let k be an arbitrary real number not equal to zero, and choose a function $\zeta \in C_c^\infty(I', \mathbb{C})$ with $\widehat{\zeta}(0) \neq 0$ and $\widehat{\zeta}(k) = 0$. (Such functions certainly exist. For instance, one can choose $\zeta(t) = \alpha'(t)e^{ikt}$ for some $\alpha \in C_c^\infty(I', \mathbb{R})$ with $\widehat{\zeta}(0) = \widehat{\alpha'}(-k) = -ik\widehat{\alpha}(-k) \neq 0$.) If we split z into $z = z_0 + z_k$ with

$$z_k(t, u) := \widehat{z}(0, u)\widehat{\zeta}(0)^{-1}\zeta(t) \quad \text{and} \quad z_0 := z - z_k,$$

then $\widehat{z}_0(0, u) = 0$ and $\widehat{z}_k(k, u) = 0$. We apply Proposition 6.7 separately to both terms on the right-hand side of

$$|\langle i\,\mathrm{d}\rho(\xi(z))\rangle_\psi| \leq |\langle i\,\mathrm{d}\rho(\xi(z_0))\rangle_\psi| + |\langle i\,\mathrm{d}\rho(\xi(z_k))\rangle_\psi|$$

to obtain

$$|\langle i\,\mathrm{d}\rho(\xi(z))\rangle_\psi| \leq 2\|z_0\|_\mu\sqrt{\langle H\rangle_\psi} + 2\|z_k\|_\mu\sqrt{\langle H\rangle_\psi + \frac{k^2}{4}\frac{TT'}{T - T'}\mu_0(U_0)}. \tag{6.7}$$

Indeed, the term $k\langle i\,\mathrm{d}\rho(L_{\mathbf{v}}\eta)\rangle_\psi$ can be assumed non-positive by switching k with $-k$ and ζ with $\bar{\zeta}$ if necessary. The term $\|L_{\mathbf{v}}\eta\|_\mu^2$ is then estimated by (6.5), and we take the limit $\varepsilon \downarrow 0$.

Since $|\widehat{z}(0,u)|^2 \leq T'\|z(\cdot,u)\|^2_{dt}$, we have $\|\widehat{z}(0,\cdot)\|^2_{\mu_0} \leq T'\|z\|^2_\mu$. It follows that $\|z_k\|_\mu$ can be estimated in terms of $\|z\|_\mu$ as

$$\|z_k\|_\mu = \|\widehat{z}(\cdot,0)\|_{\mu_0}\|\widehat{\zeta}(0)^{-1}\zeta\|_{dt} \leq \sqrt{T'}\|\widehat{\zeta}(0)^{-1}\zeta\|_{dt}\|z\|_\mu.$$

Similarly, $\|z_0\|_\mu$ can be estimated in terms of $\|z\|_\mu$ by means of

$$\|z_0\|_\mu \leq \|z\|_\mu + \|z_k\|_\mu$$

and the above estimate on $\|z_k\|_\mu$. Substituting this into (6.7), we derive the estimate

$$\langle i\,\mathrm{d}\rho(\xi(z))\rangle^2_\psi \leq 4\|z\|^2_\mu\big(1+2\sqrt{T'}\|\widehat{\zeta}(0)^{-1}\zeta\|_{dt}\big)^2\bigg(\langle H\rangle_\psi + \frac{k^2}{4}\frac{TT'}{T-T'}\mu_0(U_0)\bigg). \tag{6.8}$$

Since $\|\xi(z)\|^2_\mu = 2\|z\|^2_\mu$, equation (6.8) is equivalent to (6.6) with the constants

$$a := 2\bigg(\frac{k^2}{4}\frac{TT'}{T-T'}\mu_0(U_0)\bigg)\big(1+2\sqrt{T'}\|\widehat{\zeta}(0)^{-1}\zeta\|_{dt}\big)^2, \tag{6.9}$$

$$b := 2\big(1+2\sqrt{T'}\|\widehat{\zeta}(0)^{-1}\zeta\|_{dt}\big)^2. \tag{6.10}$$

This completes the proof. ■

For $\xi(z)$ of the form (6.3) in a gauge algebra $\mathfrak{g} = C^\infty_c(U',\mathfrak{k})$ with $\mathfrak{k} = \mathfrak{su}(2,\mathbb{C})$, we can now prove an operator inequality of the form (6.2).

Proposition 6.9. *There exist constants $a, b \in \mathbb{R}$, depending on T, T' and $\mu_0(U_0)$, such that for all α, β with $\alpha^2 \geq a$ and $2\alpha\beta \geq b$, we have*

$$\pm\, i\,\mathrm{d}\rho(\xi(z)) \leq \|\xi(z)\|_\mu(\alpha\mathbf{1} + \beta H) \quad \textit{for } z \in C^\infty_c(U',\mathbb{C}) \tag{6.11}$$

as an inequality of unbounded operators on $\mathcal{H}$ with domain containing $\mathcal{H}^\infty$.

Proof. Note that the inequality (6.11) is equivalent to

$$\langle\psi, i\,\mathrm{d}\rho(\xi(z))\psi\rangle^2 \leq \|\xi(z)\|^2_\mu\langle\psi,(\alpha\mathbf{1}+\beta H)\psi\rangle^2 \quad \text{for all } \psi \in \mathcal{H}^\infty.$$

As $\beta^2\langle\psi, H\psi\rangle^2 \geq 0$, this follows from Proposition 6.8 under the above conditions on α and β. ■

Remark 6.10. The estimate (6.11) is rather crude for large energies, in the sense that one expects $\mathrm{d}\rho(\xi) \sim \sqrt{H}$, not $\mathrm{d}\rho(\xi) \sim H$.

It will be convenient to gain more control over the constants a and b in Proposition 6.8, and the constants α, β in Proposition 6.9. For this, we need to remove the dependence on ζ in (6.9) and (6.10).

Proposition 6.11. *The constants a and b in Proposition 6.8 can be chosen as*

$$a = \frac{T}{T-T'}\left(\frac{\mu_0(U_0)}{T'}\right)\nu^2 b, \tag{6.12}$$

with

$$b = 2\left(1 + \frac{2}{\sqrt{1-(\sin(\nu)/\nu)^2}}\right)^2. \tag{6.13}$$

Here, $\nu > 0$ can be chosen at will.

Remark 6.12. It will be convenient to choose $\nu = \pi$. Then, b attains its minimal value $b = 18$, and $a = 18\pi^2 \frac{T}{T-T'}\frac{\mu_0(U_0)}{T'}$.

Proof. In (6.9) and (6.10), we need to minimize the expression $\sqrt{T'}\|\widehat{\zeta}(0)^{-1}\zeta\|_{dt}$ over all $\zeta \in C_c^\infty(I', \mathbb{C})$ with $\widehat{\zeta}(k) = 0$ and $\widehat{\zeta}(0) \neq 0$, where $k \in \mathbb{R}^\times$ is arbitrary. Since $\widehat{\zeta}(k) = \langle e^{ikt}, \zeta\rangle_{dt}$ and $\widehat{\zeta}(0) = \langle 1, \zeta\rangle_{dt}$, this amounts to maximizing

$$F(\zeta) := \left(\sqrt{T'}\|\widehat{\zeta}(0)^{-1}\zeta\|_{dt}\right)^{-1} = \frac{|\langle 1, \zeta\rangle_{dt}|}{\|1\|_{dt}\|\zeta\|_{dt}}.$$

Since F is continuous on $L^2(I') \setminus \{0\}$, and $C_c^\infty(I', \mathbb{C})$ is dense in $L^2(I')$, $F(\zeta_{\max})$ is maximal on the projection $\zeta_{\max}$ of 1 on the orthogonal complement of the function $e^{ikt} \in L^2(I')$. This is essentially a two-dimensional problem in the space spanned by

$$e_0 := \frac{1}{\sqrt{T'}}1 \quad \text{and} \quad e_k := \frac{1}{\sqrt{T'}}e^{ikt},$$

with

$$\langle e_0, e_0\rangle = \langle e_k, e_k\rangle = 1 \quad \text{and} \quad \langle e_0, e_k\rangle = \frac{\sin(kT'/2)}{kT'/2}. \tag{6.14}$$

It follows that $\zeta_{\max} = e_0 - \langle e_k, e_0\rangle e_k$, and $F(\zeta_{\max}) = \sqrt{1 - |\langle e_0, e_k\rangle|^2}$. Using (6.14), we find

$$F(\zeta_{\max}) = \sqrt{1 - \left(\frac{\sin(kT'/2)}{kT'/2}\right)^2}. \tag{6.15}$$

Equations (6.12) and (6.13) are now obtained from (6.9) and (6.10) with $k = 2\nu/T'$ by substituting the maximal value (6.15) for $F(\zeta) = (\sqrt{T'}\|\widehat{\zeta}(0)^{-1}\zeta\|_{dt})^{-1}$. ∎

6.2.2 The local gauge algebra with compact simple fibers

We now extend Proposition 6.8 to the case where $\mathfrak{k}$ is an arbitrary compact simple Lie algebra. With $I' \times U_0'$ and $I \times U_0$ a good pair of flowboxes (cf. Definition 6.4), we consider $\mathfrak{g}_{U'} := C_c^\infty(I' \times U_0', \mathfrak{k})$ and $\mathfrak{g}_U := C_c^\infty(I \times U_0, \mathfrak{k})$ as subalgebras of the Lie algebra $\mathfrak{g} = \Gamma_c(M, \mathfrak{K})$.

Lemma 6.13. *Let $\mathrm{d}\rho$ be a positive energy representation of $\widehat{\mathfrak{g}}$, and let $\eta > 0$. Then, we have*

$$\pm\, i\, \mathrm{d}\rho(\xi) \leq \|\xi\|_{\mu}\big(K(\eta)\mathbf{1} + \eta H\big) \quad \textit{for all } \xi \in \mathfrak{g}_{U'}, \tag{6.16}$$

where $K(\eta)$ is a constant independent of ξ. More precisely,

$$K(\eta) = \max\left(9d_{\mathfrak{k}}/\eta, 3\pi\sqrt{2d_{\mathfrak{k}}\frac{T}{T-T'}\frac{\mu_0(U_0)}{T'}}\right), \tag{6.17}$$

where $d_{\mathfrak{k}}$ is the dimension of $\mathfrak{k}$.

Proof. Using the root decomposition of $\mathfrak{k}_{\mathbb{C}}$ with respect to the complexification $\mathfrak{t}_{\mathbb{C}}$ of a maximal abelian subalgebra $\mathfrak{t} \subseteq \mathfrak{k}$, one obtains a basis $(X_1, \ldots, X_{d_{\mathfrak{k}}})$ of $\mathfrak{k}$ with $\kappa(X_i, X_j) = 2\delta_{ij}$, where $-\kappa$ is the Killing form of $\mathfrak{k}$ and such that every X_j is contained in some $\mathfrak{su}(2,\mathbb{C})$-triple in $\mathfrak{k}$ [42, Proposition 6.45]. Every $\xi \in \mathfrak{g}_{U'}$ can then be written as $\xi = \sum_i f_i X_i$ for $f_i \in C_c^{\infty}(U_0' \times I', \mathbb{R})$. Since every X_i is contained in an $\mathfrak{su}(2,\mathbb{C})$-triple, we can apply Proposition 6.9 to $f_i X_i \in \mathfrak{g}_{U'}$ with $z = f_i$. We obtain

$$\pm i\,\mathrm{d}\rho(fX_i) \leq \|fX_i\|_{\mu}(\alpha\mathbf{1} + \beta H),$$

and thus

$$\pm i\,\mathrm{d}\rho(\xi) \leq \left(\sum_{i=1}^{d_{\mathfrak{k}}} \|fX_i\|_{\mu}\right)(\alpha\mathbf{1} + \beta H).$$

As the different terms $f_i X_i$ are orthogonal, we have $\sum_{i=1}^{d_{\mathfrak{k}}} \|f_i X_i\|_{\mu} \leq \sqrt{d_{\mathfrak{k}}}\|\xi\|_{\mu}$, and we obtain

$$\pm\, i\,\mathrm{d}\rho(\xi) \leq \|\xi\|_{\mu}\big(\sqrt{d_{\mathfrak{k}}}\alpha\mathbf{1} + \sqrt{d_{\mathfrak{k}}}\beta H\big). \tag{6.18}$$

By Proposition 6.9, we are allowed to choose any α and β with $\alpha^2 \geq a$ and $2\alpha\beta \geq b$. Following Remark 6.12, we take $a = 18\pi^2\lambda(\mu_0(U_0)/T')$ and $b = 18$. The inequality (6.18) therefore holds for any value of $\beta > 0$ if we set

$$\alpha = \max\left(9/\beta, 3\pi\sqrt{2\frac{T}{T-T'}\frac{\mu_0(U_0)}{T'}}\right). \tag{6.19}$$

Inequality (6.16) now follows from (6.18) with $\beta = \eta/\sqrt{d_{\mathfrak{k}}}$ and $K(\eta) = \sqrt{d_{\mathfrak{k}}}\alpha$. ∎

Proposition 6.14. *For all $\xi \in \mathfrak{g}_{U'}$ and $t > 0$, the spectrum of $tH \pm i\,\mathrm{d}\rho(\xi)$ is bounded below. More precisely,*

$$-\max\left(9d_{\mathfrak{k}}\|\xi\|_{\mu}^2/t, 3\pi\|\xi\|_{\mu}\sqrt{2d_{\mathfrak{k}}\frac{T}{T-T'}\frac{\mu_0(U_0)}{T'}}\right) \leq \inf\big(\mathrm{Spec}(tH \pm i\,\mathrm{d}\rho(\xi))\big). \tag{6.20}$$

Proof. If $\|\xi\|_{\mu} = 0$, then $\mathrm{d}\rho(\xi) = 0$ by Corollary 6.3. In that case, (6.20) simply follows from the fact that H has non-negative spectrum. If $\|\xi\|_{\mu} \neq 0$, we apply Lemma 6.13 with $\eta = t/\|\xi\|_{\mu}$. ∎

6.2.3 Global estimates and the bounding function

We need to derive suitable estimates of the type (6.16) globally, on the full Lie algebra $\Gamma_c(M,\mathfrak{K})$ rather than merely on $C_c^\infty(U,\mathfrak{k})$. In this section, we show how to do this for compact as well as noncompact manifolds M, under the assumption that $\mathbf{v}_M$ is nowhere vanishing.

For compact manifolds, we will derive an estimate of the form (6.16), albeit with a larger constant $K(\eta)$. In the noncompact case, however, the expression $\|\xi\|_\mu K(\eta)$ in (6.16) will have to be replaced by $\|\xi\|_{B_\varepsilon\mu}$, where $B_\varepsilon\colon M \to \mathbb{R}^+$ is a suitable upper semi-continuous function on M that is invariant under the flow, and $\|\xi\|_{B_\varepsilon\mu}$ is the L^2-norm of $\xi \in \Gamma_c(M,\mathfrak{K})$ with respect to the measure $B_\varepsilon\mu$,

$$\|\xi\|^2_{B_\varepsilon\mu} = \langle\xi,\xi\rangle_{B_\varepsilon\mu}, \quad \langle\xi,\eta\rangle_{B_\varepsilon\mu} = \int_M \kappa(\xi,\eta)B_\varepsilon(m)d\mu(m). \tag{6.21}$$

In this setting, we will prove that

$$\pm i\,\mathrm{d}\rho(\xi) \leq \|\xi\|_{B_\varepsilon\mu}\mathbf{1} + \varepsilon\|\xi\|_\mu H \quad \text{for all } \xi \in \Gamma_c(M,\mathfrak{K}).$$

Note that, since $\mathbf{v}_M$ is nowhere vanishing on M, every $m \in M$ is contained in a good pair of flow boxes in the sense of Definition 6.4.

Definition 6.15. For $m \in M$, define $b(m)$ as the infimum of the set of numbers $\frac{T}{T-T'}\frac{\mu_0(U_0)}{T'}$, ranging over all good pairs of flowboxes $U' \Subset U$ containing m.

Proposition 6.16. *The function $b\colon M \to \mathbb{R}^+$ is invariant under the flow $(\gamma_{M,t})_{t\in\mathbb{R}}$. Further, it is upper semi-continuous, hence, in particular, measurable.*

Proof. The invariance under the flow follows from the fact that $U' \subset U$ is a good pair of flow boxes around m if and only if $\gamma_{M,t}(U') \subset \gamma_{M,t}(U)$ is a good pair of flow boxes around $\gamma_{M,t}(m)$. For the upper semi-continuity, note that for every $\varepsilon > 0$, there is a good pair of flowboxes $U' \subset U$ around m such that

$$\frac{T}{T-T'}\frac{\mu_0(U_0)}{T'} < b(m) + \varepsilon.$$

For every m' in the open neighborhood U' of m, we thus have $b(m') \leq b(m) + \varepsilon$. ∎

Theorem 6.17. *Let $\mathrm{d}\rho$ be a positive energy representation of $\widehat{\mathfrak{g}}$, and let $\varepsilon > 0$. Then, we have*

$$\pm\, i\,\mathrm{d}\rho(\xi) \leq \|\xi\|_{B_\varepsilon\mu}\mathbf{1} + \varepsilon\|\xi\|_\mu H \quad \textit{for every } \xi \in \Gamma_c(M,\mathfrak{K}). \tag{6.22}$$

Here, $B_\varepsilon\colon M \to \mathbb{R}^+$ is the upper semi-continuous function defined by

$$B_\varepsilon(m) := \max\left(81 d_{\mathfrak{k}}^2(d_M+1)^4/\varepsilon^2,\, 18\pi^2 d_{\mathfrak{k}}(d_M+1)^2 b(m)\right), \tag{6.23}$$

with $b(m)$ as in Definition 6.15. It is invariant under the flow $(\gamma_{M,t})_{t\in\mathbb{R}}$.

Proof. Let d be a Riemannian metric on M for which M is complete, so that closed bounded subsets of M are compact by the Hopf–Rinow theorem. Let $V \subseteq M$ be the compact support of $\xi \in \Gamma_c(M, \mathfrak{K})$. Since b is upper semi-continuous, the functions $\beta_n : M \to \mathbb{R}^+$ defined by

$$\beta_n(m) := \sup\{b(m'); d(m, m') \leq 1/n\}$$

constitute a decreasing sequence converging pointwise to b as $n \to \infty$. We now show that the functions β_n are upper semi-continuous. To see this, note that, for every $m_0 \in M$ and every $\varepsilon > 0$, there exists an $n \in \mathbb{N}$ with $b(m) < b(m_0) + \varepsilon/2$ for m in the closed ball $\overline{W}_{2/n}(m_0)$ with radius $2/n$ around m_0. Since this ball is compact, it contains finitely many m_i such that it is covered by open neighborhoods $\mathcal{O}_i$ of m_i such that $b(m) \leq b(m_i) + \varepsilon/2$ for all $m \in \mathcal{O}_i$. If $d(m, m_0) < \frac{1}{n}$, then $W_{1/n}(m) \subseteq \bigcup_i \mathcal{O}_i$, so that $\beta_n(m) < \beta_n(m_0) + \varepsilon$. In particular, β_n is measurable, and bounded on the compact set V.

For every $n \in \mathbb{N}$, choose a cover of V by finitely many open balls $W_{r_i}(m_i)$ of radius $r_i \leq 1/n$ around m_i, with the property that $W_{r_i}(m_i) \subseteq U' \Subset U$ for a good pair $U' \Subset U$ of flow boxes with $\frac{T}{T-T'} \frac{\mu_0(U_0)}{T} \leq b(m_i) + 1/n$. Since $b(m_i) \leq \beta_n(m)$ for all $m \in W_{r_i}(m_i)$, it follows that

$$\frac{T}{T-T'} \frac{\mu_0(U_0)}{T} \leq \beta_n(m) + 1/n \quad \text{for all } m \in W_{r_i}(m_i). \tag{6.24}$$

By the Brouwer–Lebesgue paving principle [44, Theorem V1], there exists a finite subcover $(W_j)_{j \in J}$ with the property that every point $m \in V$ is contained in at most $d_M + 1$ sets.

Let φ_j be a partition of unity with respect to $(W_j)_{j \in J}$. By Lemma 6.13, applied to $\eta := \varepsilon/(d_M + 1)$, we obtain $\pm i\, \mathrm{d}\rho(\varphi_j \xi) \leq \|\varphi_j \xi\|_\mu (K_j(\eta)\mathbf{1} + \eta H)$, where $K_j(\eta)$ is given by (6.17) for a good pair of flowboxes $U' \Subset U$ containing W_j. From (6.17) and (6.24), we find that

$$B_{n,\eta}(m) := \max\left((9d_{\mathfrak{k}}/\eta)^2, 18\pi^2 d_{\mathfrak{k}}(\beta_n(m) + 1/n)\right) \geq K_j(\eta)^2 \quad \text{for all } m \in W_j.$$

As $\|\varphi_j \xi\|_\mu K(\eta) \leq \|\varphi_j \xi\|_{B_{n,\eta}\mu}$, we have $\pm i\, \mathrm{d}\rho(\varphi_j \xi) \leq \|\varphi_j \xi\|_{B_{n,\eta}\mu} \mathbf{1} + \eta \|\varphi_j \xi\|_\mu H$ for all $j \in J$, and thus

$$\pm i\, \mathrm{d}\rho(\xi) \leq \Big(\sum_{j \in J} \|\varphi_j \xi\|_{B_{n,\eta}\mu}\Big)\mathbf{1} + \eta\Big(\sum_{j \in J} \|\varphi_j \xi\|_\mu\Big) H.$$

Since $\|(\varphi_j \xi)(m)\|_\kappa \leq \|\xi(m)\|_\kappa$, and since at most $d_M + 1$ of the values $\varphi_j(m)$ are non-zero, it follows that

$$\sum_{j \in J} \|\varphi_j \xi\|_\mu \leq (d_M + 1)\|\xi\|_\mu \quad \text{and} \quad \sum_{j \in J} \|\varphi_j \xi\|_{B_{n,\eta}\mu} \leq (d_M + 1)\|\xi\|_{B_{n,\eta}\mu},$$

so that

$$\pm i\,\mathtt{d}\rho(\xi) \leq (d_M + 1)(\|\xi\|_{B_{n,\eta}\mu}\mathbf{1} + \eta\|\xi\|_\mu H). \tag{6.25}$$

To obtain (6.22) from (6.25), recall that $\eta := \varepsilon/(d_M + 1)$. The second term on the right is thus $(d_M+1)\eta\|\xi\|_\mu = \varepsilon\|\xi\|_\mu$, as required. For the first term, note that $\beta_n + 1/n$ is a bounded, decreasing sequence converging pointwise to b on V. The bounded, decreasing sequence $(d_M + 1)^2 B_{n,\eta}(m)$ thus converges to $B_\varepsilon(m)$ in (6.22), where $\varepsilon = (d_M + 1)\eta$. By the dominated convergence theorem, we find that, for $n \to \infty$, the squared norm $((d_M + 1)\|\xi\|_{B_{n,\eta}})^2$ approaches

$$\int_V \|\xi\|_\kappa^2 (d_M + 1)^2 B_{n,\eta} d\mu(m) \to \int_V \|\xi\|_\kappa^2 B_\varepsilon d\mu(m) = \|\xi\|_{B_\varepsilon\mu}^2.$$

Since (6.25) holds for every $n \in \mathbb{N}$, the proposition follows. ■

Note that if the function $b\colon M \to \mathbb{R}^+$ of Definition 6.15 is bounded, then so is B_ε. If we define $K(\varepsilon)^2 := \|B_\varepsilon\|_\infty$, then we recover the inequality

$$\pm i\,\mathtt{d}\rho(\xi) \leq \|\xi\|_\mu(K(\varepsilon)\mathbf{1} + \varepsilon H), \tag{6.26}$$

since $\|\xi\|_{B_\varepsilon\mu} \leq K(\varepsilon)\|\xi\|_\mu$. This happens, in particular, if M is compact because the upper semi-continuous function B_ε is then automatically bounded.

Corollary 6.18. *Suppose that M is compact and $\mathbf{v}_M$ is nowhere vanishing on M. Then, for every $\varepsilon > 0$, there exists a constant $K(\varepsilon) > 0$ such that (6.26) holds for all $\xi \in \Gamma(M, \mathfrak{K})$.*

Another important situation in which B_ε is bounded is for product manifolds of the form $M = \mathbb{R} \times \Sigma$.

Corollary 6.19. *Suppose that $M \simeq \mathbb{R} \times \Sigma$ with $\mathbf{v}_M = \frac{\partial}{\partial t}$. Then, the inequality (6.26) holds for the compactly supported gauge algebra $\mathfrak{g} = \Gamma_c(M, \mathfrak{K})$, with constant $K(\varepsilon) = 9d_{\mathfrak{k}}(d_M + 1)^2/\varepsilon$ depending on M and $\mathfrak{K}$ only through the dimension.*

Proof. For $(t, x) \in \mathbb{R} \times \Sigma$, choose $U_0' \Subset U_0 \subseteq \Sigma$ with $U_0 \subseteq \Sigma$ relatively compact, and $x \in U_0'$. For T' sufficiently large, (t, x) is contained in the good pair of flowboxes $U' = U_0' \times (-T'/2, T'/2)$, and $U = U_0 \times (-T/2, T/2)$ for $T = 2T'$. Since

$$\frac{T}{T - T'} \frac{\mu_0(U_0)}{T'} = 2\mu_0(U_0)/T'$$

approaches 0 for $T' \to \infty$, it follows that $b(t, x) = 0$. In particular,

$$B_\varepsilon(m) = 81 d_{\mathfrak{k}}^2 (d_M + 1)^4/\varepsilon^2$$

is constant, and the result follows. ■

6.3 Extending representations to Sobolev spaces

In this section, we extend the map $\mathrm{d}\rho$ to the Hilbert completion $L^2_{B\mu}(M,\mathfrak{K})$ of $\mathfrak{g}/I_\mu$ with respect to the inner product (6.21) corresponding to $B_\varepsilon\mu$.

Note that since $\|\xi\|_\mu$ is dominated by a multiple of $\|\xi\|_{B_\varepsilon\mu}$, the inner product $\langle\xi,\eta\rangle_\mu$ is continuous on $L^2_{B\mu}(M,\mathfrak{K})$. As the difference between $\|\xi\|_{B_\varepsilon\mu}$ and $\|\xi\|_{B_{\widetilde{\varepsilon}}\mu}$ for $\varepsilon,\widetilde{\varepsilon}>0$ is a multiple of $\|\xi\|_\mu$, the space $L^2_{B\mu}(M,\mathfrak{K})$ and its topology are independent of ε. (This is why we omit ε from the notation in $L^2_{B\mu}(M,\mathcal{K})$.)

6.3.1 The completion $L^2_{B\mu}(M,\mathfrak{K})$ in L^2-norm

We use Theorem 6.17 to extend $\mathrm{d}\rho$ from $\mathfrak{g}$ to $L^2_{B\mu}(M,\mathfrak{g})$. Define

$$\Gamma_\xi = \|\xi\|_{B_\varepsilon\mu}\mathbf{1} + \varepsilon\|\xi\|_\mu H,$$

and note that its domain $\mathcal{D}(\Gamma_\xi)$ is contained in the domain $\mathcal{D}(H)$ of H. With this notation, (6.22) becomes

$$0 \le \Gamma_\xi \pm i\,\mathrm{d}\rho(\xi), \tag{6.27}$$

as an inequality of unbounded operators on $\mathcal{H}^\infty$. Further, define

$$A := \mathbf{1} + H \quad \text{with } \mathcal{D}(A) = \mathcal{D}(H). \tag{6.28}$$

Proposition 6.20. *Let $0<\varepsilon\le 1$. There exists a map r from $L^2_{B\mu}(M,\mathfrak{K})$ to the unbounded, skew-symmetric operators on $\mathcal{H}$ such that $\mathcal{D}(r(\xi))$ contains $\mathcal{D}(H)$ for all $\xi\in L^2_{B\mu}(M,\mathfrak{K})$, $r(\xi)|_{\mathcal{H}^\infty} = \mathrm{d}\rho(\xi)$ for all $\xi\in\mathfrak{g}$, and, for all $\psi\in\mathcal{D}(H)$, the functional*

$$L^2_{B\mu}(M,\mathfrak{K}) \to \mathbb{C} \ \text{ defined by } \ \xi \mapsto \langle r(\xi)\rangle_\psi$$

is continuous. Furthermore, there exists a continuous map

$$\lambda\colon L^2_{B\mu}(M,\mathfrak{K}) \to B(\mathcal{H})$$

into the bounded operators such that $\|\lambda(\xi)\|\le\|\xi\|_{B_\varepsilon\mu}$, $\lambda(\xi)$ is skew-hermitian, $\lambda(\xi)$ leaves the domain of $A^{1/2}$ invariant, and

$$r(\xi) = A^{1/2}\lambda(\xi)A^{1/2},$$

as an equality of unbounded operators on $\mathcal{D}(H)$.

Proof. Let ξ_n be a sequence in $\mathfrak{g}/I_\mu$ for which $\|\xi-\xi_n\|_{B_\varepsilon\mu}\to 0$, and hence also $\|\xi-\xi_n\|_\mu\to 0$. Without loss of generality, we assume that $\|\xi-\xi_n\|_{B_\varepsilon\mu}\le\frac{1}{2}$ and $\varepsilon\|\xi-\xi_n\|_\mu\le\frac{1}{2}$ for all n, so that

$$\Gamma_\xi - \Gamma_{\xi_n} + A \ge \frac{1}{2}A. \tag{6.29}$$

Define the sesquilinear forms

$$B_n^{\pm}\colon \mathcal{H}^\infty \times \mathcal{H}^\infty \to \mathbb{C}, \quad B_n^{\pm}(\psi, \chi) := \big\langle \psi, ((\Gamma_\xi + A) \pm i\,\mathrm{d}\rho(\xi_n))\chi \big\rangle.$$

The forms $B_n^{\pm}$ are positive definite; combining (6.29) with inequality (6.27) applied to ξ_n, we find

$$B_n^{\pm}(\psi, \psi) \geq \langle \psi, (\Gamma_\xi - \Gamma_{\xi_n} + A)\psi \rangle \geq \frac{1}{2}\langle \psi, A\psi \rangle \geq \frac{1}{2}\|\psi\|^2. \tag{6.30}$$

By (6.27) and the convergence of ξ_n, we find that $B_n^+(\psi, \psi)$ is a Cauchy sequence for every $\psi \in \mathcal{H}^\infty$,

$$|B_n^+(\psi, \psi) - B_m^+(\psi, \psi)| = |\langle \psi, i\,\mathrm{d}\rho(\xi_n - \xi_m)\psi \rangle| \leq \langle \psi, \Gamma_{\xi_n - \xi_m}\psi \rangle \to 0.$$

It follows that $B^+(\psi, \chi) := \lim_{n\to\infty} B_n^+(\psi, \chi)$ defines a positive definite, sesquilinear form $\mathcal{H}^\infty \times \mathcal{H}^\infty \to \mathbb{C}$. Here we use that the estimate (6.30) is independent of n. The same argument applies to $B^-(\psi, \chi) := \lim_{n\to\infty} B_n^-(\psi, \chi)$. Note that

$$\frac{1}{2}\langle \psi, A\psi \rangle \leq B^{\pm}(\psi, \psi) \leq \langle \psi, (2\Gamma_\xi + A)\psi \rangle \leq c_\xi \langle \psi, A\psi \rangle \tag{6.31}$$

for some $c_\xi > 0$. The forms $B^{\pm}$ therefore extend uniquely to closed, sesquilinear forms $\overline{B}^{\pm}\colon \mathcal{D}(A^{1/2}) \times \mathcal{D}(A^{1/2}) \to \mathbb{C}$. In turn, the forms $\overline{B}^{\pm}$ define a Friedrichs extension; a closed, possibly unbounded positive operator $b^{\pm}(\xi)\colon \mathcal{D}(H) \to \mathcal{H}$, such that $\overline{B}^{\pm}(\psi, \chi) = \langle \psi, b^{\pm}(\xi)\chi \rangle$ for all $\psi, \chi \in \mathcal{D}(H)$ (cf. [25, Appendix I.A.2]). Set

$$r(\xi) := \frac{1}{2i}(b^+(\xi) - b^-(\xi)).$$

Since $b^+(\xi)$ and $b^-(\xi)$ are selfadjoint, $r(\xi)$ is skew-symmetric. If $\xi \in \mathfrak{g}$, then

$$\langle \psi, r(\xi)\chi \rangle = \langle \psi, \mathrm{d}\rho(\xi)\chi \rangle \quad \text{for all } \psi, \chi \in \mathcal{H}^\infty,$$

so $r(\xi)$ is an extension of $\mathrm{d}\rho(\xi)$.

Define

$$\lambda(\xi)\colon D(A^{1/2}) \to D(A^{1/2}), \quad \lambda(\xi) := A^{-1/2} r(\xi) A^{-1/2}.$$

Then, for $\psi, \chi \in D(A^{1/2})$, we have $A^{-1/2}\psi, A^{-1/2}\chi \in \mathcal{D}(H)$. Therefore,

$$\langle \psi, \lambda(\xi)\chi \rangle = -\langle \lambda(\xi)\psi, \chi \rangle = \frac{1}{2i}(\overline{B}^+ - \overline{B}^-)(A^{-1/2}\psi, A^{-1/2}\chi). \tag{6.32}$$

By (6.31) and Cauchy–Schwarz, we have

$$|\overline{B}^{\pm}(\psi, \chi)| \leq c_\xi \|A^{1/2}\psi\| \|A^{1/2}\chi\|,$$

so $|\langle\psi, \lambda(\xi)\chi\rangle| \leq c_\xi \|\psi\|\|\chi\|$ by (6.32). Therefore, $\lambda(\xi)$ extends to a hermitian operator on $\mathcal{H}$. As such, the operator norm $\|\lambda(\xi)\|$ is the supremum of $|\langle\psi, \lambda(\xi)\psi\rangle|$ over all ψ in the unit sphere of $\mathcal{H}$. For $\psi \in A^{1/2}\mathcal{H}^\infty$, (6.27) yields

$$|\langle\psi, \lambda(\xi)\psi\rangle| = \lim_{n\to\infty} |\langle A^{-1/2}\psi, \mathrm{d}\rho(\xi_n)A^{-1/2}\psi\rangle| \leq \langle\psi, A^{-1/2}\Gamma_\xi A^{-1/2}\psi\rangle.$$

We claim that

$$A^{-1/2}\Gamma_\xi A^{-1/2} \leq \|\xi\|_{B_\varepsilon\mu} \quad \text{for } 0 < \varepsilon < 1. \tag{6.33}$$

In fact, since Γ_ξ and A commute, this is equivalent to $\Gamma_\xi \leq A\|\xi\|_{B\mu}$, which in turn is equivalent to

$$\|\xi\|_{B_\varepsilon\mu}\mathbf{1} + \varepsilon\|\xi\|_\mu H \leq \|\xi\|_{B_\varepsilon\mu}(\mathbf{1} + H)$$

and this to $\varepsilon\|\xi\|_\mu \leq \|\xi\|_{B_\varepsilon\mu}$, which, for $\varepsilon < 1$, follows from the estimate

$$B_\varepsilon \geq 81d_{\mathfrak{k}}^2(d_M + 1)^4/\varepsilon^2 > 1.$$

With (6.33), we find

$$|\langle\psi, \lambda(\xi)\psi\rangle| \leq \|\xi\|_{B_\varepsilon\mu}\|\psi\|^2 \quad \text{for } \psi \in A^{1/2}\mathcal{H}^\infty.$$

To prove that $\|\lambda(\xi)\| \leq \|\xi\|_{B_\varepsilon\mu}$, it therefore suffices to show that $A^{1/2}\mathcal{H}^\infty$ is dense in $\mathcal{H}$. First, we show that $A\mathcal{H}^\infty$ is dense in $\mathcal{H}$. Since $\exp(itA) = e^{it}\exp(itH)$ leaves the space $\mathcal{H}^\infty$ of smooth vectors invariant, the restriction A_0 of A to $\mathcal{H}^\infty$ is essentially selfadjoint [95, Section VIII.4]. Suppose that $\psi \perp A_0\mathcal{H}^\infty$. Then, $\psi \in \mathcal{D}(A_0^*) = \mathcal{D}(A)$, and $A_0^*\psi = A\psi = 0$. Since A is injective, $\psi = 0$ and $A\mathcal{H}^\infty$ is dense in $\mathcal{H}$. Applying the contraction $A^{-1/2}$, we find that $A^{1/2}\mathcal{H}^\infty$ is dense in $A^{-1/2}\mathcal{H}$. Since $A^{-1/2}\mathcal{H} = \mathcal{D}(A^{1/2})$ is dense in $\mathcal{H}$, we conclude that $A^{1/2}\mathcal{H}^\infty$ is dense in $\mathcal{H}$, as required. ∎

For $s \in \mathbb{R}$, denote by $\mathcal{H}_s$ the Hilbert space completion of $\mathcal{D}(A^s)$ with respect to the inner product

$$\langle\psi, \chi\rangle_s = \langle A^s\psi, A^s\chi\rangle.$$

Denote the corresponding norm by $\|\psi\|_s = \|A^s\psi\|$, and denote the norm of a continuous operator $A\colon \mathcal{H}_s \to \mathcal{H}_t$ by $\|A\|_{s,t}$. As

$$r(\xi) = A^{1/2}\lambda(\xi)A^{1/2}$$

with $\|\lambda(\xi)\| \leq \|\xi\|_{B_\varepsilon\mu}$, the operator $r(\xi)\colon \mathcal{D}(A) \to \mathcal{H}$ extends to a bounded operator $r(\xi)\colon \mathcal{H}_{1/2} \to \mathcal{H}_{-1/2}$, with

$$\|r(\xi)\psi\|_{-1/2} \leq \|\xi\|_{B_\varepsilon\mu}\|\psi\|_{1/2}. \tag{6.34}$$

We thus have

$$\|r(\xi)\|_{1/2,-1/2} \leq \|\xi\|_{B_\varepsilon\mu}.$$

6.3.2 The completion in Sobolev norm

Note that convergence of ξ_n to ξ in $L^2_{B\mu}(M,\mathfrak{K})$ only implies *weak* operator convergence of $r(\xi_n)$ to $r(\xi)$, as operators on the pre-Hilbert space $\mathcal{D}(H)$. In this section, we define a subspace $H^1_{B\mu}(M,\mathfrak{K})$ of $L^2_{B\mu}(M,\mathfrak{K})$ where convergence to ξ implies *strong* convergence to $r(\xi)$.

Definition 6.21 (Parallel Sobolov spaces). For $k \geq 0$, the *parallel Sobolev norm* q_k is defined by

$$q_k(\xi) := \sum_{n=0}^{k} \|\xi\|_n, \quad \text{where } \|\xi\|_n := \|D^n\xi\|_{B_\varepsilon\mu}.$$

The *parallel Sobolev space* $H^k_{B\mu}(M,\mathfrak{K}) \subseteq L^2_{B\mu}(M,\mathfrak{K})$ is the Banach completion of $\mathfrak{g}/I_\mu$ with respect to the norm q_k.

Proposition 6.22. *Let r be as in Proposition 6.20. If $\xi \in H^k_{B\mu}(M,\mathfrak{K})$, then $r(\xi)$ maps $\mathcal{D}(H^{k+1})$ into $\mathcal{D}(H^k)$. For $k = 1$, we have*

$$[H, r(\xi)] = ir(D\xi) \tag{6.35}$$

as an equality of unbounded operators on $\mathcal{D}(H^2)$. Furthermore, if $\xi \in H^k_{B\mu}(M,\mathfrak{K})$, then $r(\xi)$ extends to a continuous operator $\mathcal{H}_{k+1/2} \to \mathcal{H}_{k-1/2}$ with

$$\|r(\xi)\psi\|_{k-1/2} \leq \sum_{j=0}^{k} \binom{k}{j} \|\xi\|_j \|\psi\|_{k-j+1/2}. \tag{6.36}$$

Finally, for all $\xi \in H^1_{B\mu}(M,\mathfrak{K})$, the skew-symmetric operator $r(\xi)$ is essentially skew-adjoint.

Proof. We prove that for $\xi \in H^k_{B\mu}(M,\mathfrak{K})$, $r(\xi)$ maps $\mathcal{D}(H^{k+1})$ into $\mathcal{D}(H^k)$. We proceed by induction on k. Since $H^0_{B\mu}(M,\mathfrak{K}) = L^2_{B\mu}(M,\mathfrak{K})$, the case $k = 0$ follows from Proposition 6.20. Suppose that the statement holds for all $\xi \in H^k_{B\mu}(M,\mathfrak{K})$. For $\xi \in H^{k+1}_{B\mu}(M,\mathfrak{K})$ and $\psi \in \mathcal{D}(H^{k+2})$, we show that $r(\xi)\psi \in \mathcal{D}(H^{k+1})$. Since H^{k+1} is selfadjoint, it suffices to show that $\chi \mapsto \langle r(\xi)\psi, H^{k+1}\chi\rangle$ is a continuous, linear functional on $\mathcal{H}^\infty$ with respect to the subspace topology induced by the inclusion in $\mathcal{H}$.

Let $\xi_n \in \mathfrak{g}/I_\mu$ be a sequence such that $\xi_n \to \xi$ and $D\xi_n \to D\xi$ in $L^2_{B\mu}(M,\mathfrak{K})$. Since $Hr(\xi_n) = r(\xi_n)H + ir(D\xi_n)$ on $\mathcal{H}^\infty$, we have

$$\begin{aligned}\langle r(\xi)\psi, H^{k+1}\chi\rangle &= -\lim_{n\to\infty}\langle\psi, r(\xi_n)H^{k+1}\chi\rangle \\ &= -\lim_{n\to\infty}\langle H\psi, r(\xi_n)H^k\chi\rangle + \lim_{n\to\infty}\langle\psi, ir(D\xi_n)H^k\chi\rangle \\ &= \langle r(\xi)H\psi + ir(D\xi)\psi, H^k\chi\rangle. \end{aligned} \tag{6.37}$$

As $\psi \in \mathcal{D}(H^{k+2})$, both $H\psi$ and ψ are in $\mathcal{D}(H^{k+1})$. Since $\xi \in H^{k+1}_{B\mu}(M, \mathfrak{K})$, we have $\xi, D\xi \in H^k_{B\mu}(M, \mathfrak{K})$, so that $r(\xi)H\psi + ir(D\xi)\psi \in \mathcal{D}(H^k)$ by the induction hypothesis. From (6.37), we thus find that

$$\langle r(\xi)\psi, H^{k+1}\chi\rangle = \big\langle H^k(r(\xi)H\psi + ir(D\xi)\psi), \chi\big\rangle,$$

which is manifestly continuous in the variable χ. We conclude that $r(\xi)$ maps the domain $\mathcal{D}(H^{k+2})$ to $\mathcal{D}(H^{k+1})$. Moreover, for $k = 0$, we find that

$$Hr(\xi) - r(\xi)H - ir(D\xi)$$

vanishes on $\mathcal{D}(H^2)$.

The inequality (6.36) is proven in a similar fashion. Assume by induction that (6.36) holds for all $\xi \in H^k_{B\mu}(M, \mathfrak{K})$ and $\psi \in \mathcal{H}_{k+1/2}$, the case $k = 0$ being (6.34). We recall that $\|\psi\|_s = \|A^s\psi\|$ with $A = \mathbf{1} + H$ (see (6.28)). For $\xi \in H^{k+1}_{B\mu}(M, \mathfrak{K})$ and $\psi \in \mathcal{H}_{k+3/2}$, we use $Ar(\xi)\psi = r(\xi)A\psi + ir(D\xi)\psi$ to see that

$$\|r(\xi)\psi\|_{k+1/2} = \|Ar(\xi)\psi\|_{k-1/2} \leq \|r(\xi)A\psi\|_{k-1/2} + \|r(D\xi)\|_{k-1/2}.$$

By the induction hypothesis with $\|A\psi\|_{k-j+1/2} = \|\psi\|_{(k+1)-j+1/2}$ (for the first term) and $\|D\xi\|_j = \|\xi\|_{j+1}$ (for the second), we find that $\|r(\xi)\psi\|_{k+1/2}$ is bounded by

$$\sum_{j=0}^{k} \binom{k}{j} \|\xi\|_j \|\psi\|_{(k+1)-j+1/2} + \sum_{j=0}^{k} \binom{k}{j} \|\xi\|_{j+1} \|\psi\|_{k-j+1/2}.$$

Since $\binom{k}{j} + \binom{k}{j-1} = \binom{k+1}{j}$, we have

$$\|r(\xi)\psi\|_{k+1/2} \leq \sum_{j=0}^{k+1} \binom{k+1}{j} \|\xi\|_j \|\psi\|_{k+1-j+1/2},$$

as required.

Finally, if $\xi \in H^1_{B\mu}(M, \mathfrak{K})$, then $\xi, D\xi \in L^2_{B\mu}(M, \mathfrak{K})$. By (6.34), the operators $r(\xi)$ and $[A, r(\xi)] = ir(D\xi)$ from $\mathcal{D}(H)$ to $\mathcal{H}$ extend continuously to bounded operators $\mathcal{H}_{1/2} \to \mathcal{H}_{-1/2}$. It then follows from a result of Nelson [87, Proposition 2] that $r(\xi)$ is essentially skew-adjoint. ■

If we estimate $\|\xi\|_j \leq q_k(\xi)$ and $\|\psi\|_{k-j+1/2} \leq \|\psi\|_{k+1/2}$ in (6.36), we find that $r(\xi)\colon \mathcal{H}_{k+1/2} \to \mathcal{H}_{k-1/2}$ satisfies

$$\|r(\xi)\psi\|_{k-1/2} \leq 2^k q_k(\xi)\|\psi\|_{k+1/2},$$

so the linear map $H^k_{B\mu}(M, \mathfrak{K}) \times \mathcal{H}_{k+1/2} \to \mathcal{H}_{k-1/2}$ defined by $(\xi, \psi) \mapsto r(\xi)\psi$ is jointly continuous. For $k = 1$, we find from (6.36) the slightly stronger estimate

$$\|r(\xi)\psi\| \leq \|r(\xi)\psi\|_{1/2} \leq q_1(\xi)\|A^{3/2}\psi\|. \tag{6.38}$$

In particular, convergence of ξ_n to ξ in $H^1_{B\mu}(M,\mathfrak{K})$ implies *strong* convergence of $r(\xi_n)$ to $r(\xi)$ on $\mathcal{D}(A^{3/2})$.

6.4 Sobolev–Lie algebras

Having established that the positive energy representation $\mathrm{d}\rho$ extends to a continuous map r on $H^k_{B\mu}(M,\mathfrak{K})$, we would like to determine whether r gives rise to a Lie algebra representation. Since the spaces $H^k_{B\mu}(M,\mathfrak{K})$ do not inherit the Lie algebra structure from $\mathfrak{g}/I_\mu$, we introduce two spaces of *bounded* Sobolev sections of $\mathfrak{K} \to M$, both equipped with the pointwise Lie bracket.

For an open subset $N \subseteq M$, we define the Lie algebra $H^k_b(N,\mathfrak{K})$ of bounded parallel Sobolev sections, and a certain subalgebra $H^k_\partial(N,\mathfrak{K})$ of sections that vanish to order k at the boundary of the 1-point compactification of N. As before, the underlying measure is the restriction to N of the flow-invariant measure $B_\varepsilon\mu$ on M. For convenience of notation, we will denote this measure by $\nu = B_\varepsilon\mu$.

6.4.1 The Lie algebra $L^2_b(N,\mathfrak{K})$ of bounded L^2-sections

Let N be an open subset of M, and let ξ be a measurable section of $\mathfrak{K} \to N$. Then,

$$\|\xi\|_\kappa = \sqrt{\kappa(\xi,\xi)}$$

is a measurable function on N. We define $\|\xi\|_\infty$ to be the essential supremum of $\|\xi\|_\kappa$ with respect to ν, and we define $L^\infty(N,\mathfrak{K})$ to be the Lie algebra of equivalence classes of essentially bounded, measurable sections of $\mathfrak{K} \to N$. This is a Banach–Lie algebra with respect to the norm $\|\xi\|_\infty$, and the Lie bracket coming from the pointwise bracket of sections. We define $L^2_b(N,\mathfrak{K})$ to be the space of equivalence classes of sections which are both essentially bounded and square integrable with respect to ν. Since both $L^2(N,\mathfrak{K})$ and $L^\infty(N,\mathfrak{K})$ are complete, it follows that $L^2_b(N,\mathfrak{K})$ is a Banach space with respect to the norm $\|\xi\|_\nu + \|\xi\|_\infty$.

Let $c_\mathfrak{k}$ be a constant such that

$$\|[X,Y]\|_\kappa \leq c_\mathfrak{k}\|X\|_\kappa\|Y\|_\kappa \tag{6.39}$$

for all $X, Y \in \mathfrak{k}$. Then, we find

$$\|[\xi,\eta]\|_\nu \leq c_\mathfrak{k}\|\xi\|_\infty\|\eta\|_\nu, \tag{6.40}$$

$$\|[\xi,\eta]\|_\infty \leq c_\mathfrak{k}\|\xi\|_\infty\|\eta\|_\infty. \tag{6.41}$$

It follows that the Lie bracket $[\cdot,\cdot]\colon L^2_b(N,\mathfrak{K}) \times L^2_b(N,\mathfrak{K}) \to L^\infty(N,\mathfrak{K})$ takes values in $L^2_b(N,\mathfrak{K})$ and is continuous with respect to the norm $p_0(\xi) := \|\xi\|_\nu + \|\xi\|_\infty$. In particular, $L^2_b(N,\mathfrak{K})$ is a Banach–Lie algebra, and the inclusion $L^2_b(N,\mathfrak{K}) \hookrightarrow L^\infty(N,\mathfrak{K})$ is a continuous homomorphism of Banach–Lie algebras. If $N \subseteq N'$, then $L^2_b(N,\mathfrak{K})$ is a subalgebra of $L^2_b(N',\mathfrak{K})$ in the natural fashion.

6.4.2 The "parallel" Sobolev–Lie algebras $H_b^k(N, \mathfrak{K})$

Recall from Definition 4.6 that a one-parameter group $(\gamma_t)_{t\in\mathbb{R}}$ of automorphisms of $\mathfrak{K} \to M$ gives rise to a one-parameter group $(\alpha_t)_{t\in\mathbb{R}}$ of automorphisms of

$$\mathfrak{g} = \Gamma_c(M, \mathfrak{K}).$$

In the same way, we obtain a one-parameter group of automorphisms of $L_b^2(M, \mathfrak{K})$.

Indeed, since the Killing form is invariant under automorphisms, $\|\alpha_t(\xi)\|_\kappa = \|\xi\|_\kappa \circ \gamma_{M,t}$, so that, in particular, $\|\alpha_t(\xi)\|_\infty = \|\xi\|_\infty$. Further, since the measure $\nu = B_\varepsilon\mu$ is invariant under the flow $\gamma_{M,t}$ (Theorem 5.7), we find $\|\alpha_t(\xi)\|_\nu = \|\xi\|_\nu$.

Since α_t is a one-parameter group of unitary transformations of the Hilbert space $L_\nu^2(M, \mathfrak{K})$, it is generated by a skew-adjoint operator D. We define $H_\nu^1(N, \mathfrak{K})$ to be the intersection of its domain with $L_\nu^2(N, \mathfrak{K})$, and we define $H_b^1(N, \mathfrak{K})$ to be the space of all $\xi \in H_\nu^1(N, \mathfrak{K})$, where both $\|\xi\|_\infty$ and $\|D\xi\|_\infty$ are finite. In other words, $H_b^1(N, \mathfrak{K})$ is the space of equivalence classes of essentially bounded, square integrable sections ξ of $\mathfrak{K} \to N$ such that the L^2-limit

$$D(\xi) := L^2\text{-}\lim_{t\to 0} \frac{1}{t}(\alpha_t(\xi) - \xi)$$

exists, and $\|D(\xi)\|_\infty$ is finite.

Proposition 6.23. *The space $H_b^1(N, \mathfrak{K})$ is a Lie subalgebra of $L_b^2(N, \mathfrak{K})$, and the generator $D\colon H_b^1(N, \mathfrak{K}) \to L_b^2(N, \mathfrak{K})$ satisfies*

$$D([\xi, \eta]) = [D(\xi), \eta] + [\xi, D(\eta)] \quad \textit{for all } \xi, \eta \in H_b^1(N, \mathfrak{K}). \tag{6.42}$$

Proof. Let $\xi, \eta \in H_b^1(N, \mathfrak{K})$, and denote by L^2-lim the limit with respect to the norm $\|\xi\|_\nu$. First, we show that $[\xi, \eta]$ is in the domain of D:

$$\begin{aligned} D([\xi, \eta]) &= L^2\text{-}\lim_{t\to 0} \frac{1}{t}(\alpha_t([\xi, \eta]) - [\xi, \eta]) \\ &= L^2\text{-}\lim_{t\to 0}[D\xi, \alpha_t(\eta)] + L^2\text{-}\lim_{t\to 0}\left[\frac{1}{t}(\alpha_t(\xi) - \xi) - D(\xi), \alpha_t(\eta)\right] \\ &\quad + L^2\text{-}\lim_{t\to 0}[\xi, \frac{1}{t}(\alpha_t(\eta) - \eta)] = [D\xi, \eta] + [\xi, D\eta]. \end{aligned}$$

In the last step, we used the inequality (6.40) three times. Since $\|D\xi\|_\infty$ is bounded and $L^2\text{-}\lim_{t\to 0}\alpha_t(\eta) = \eta$, it follows from (6.40) that the first term is given by

$$L^2\text{-}\lim_{t\to 0}[D\xi, \alpha_t(\eta)] = [D\xi, \eta].$$

Similarly, since $\|\xi\|_\infty$ is bounded and $L^2\text{-}\lim_{t\to 0}\frac{1}{t}(\alpha_t(\eta) - \eta) = D(\eta)$, the third term equals $[\xi, D(\eta)]$. To see that the second term is zero, note that $\|\alpha_t(\eta)\|_\infty = \|\eta\|_\infty$. It then follows from (6.40) and the fact that $L^2\text{-}\lim_{t\to 0}\frac{1}{t}(\alpha_t(\xi) - \xi) - D(\xi) = 0$.

This shows not only that $[\xi,\eta]$ is in the domain of D, but also that (6.42) holds. By (6.41), it follows that $\|D([\xi,\eta])\|_\infty \le c_{\mathfrak{k}}(\|D\xi\|_\infty\|\eta\|_\infty + \|\xi\|_\infty\|D\eta\|_\infty)$ is finite, so that $[\xi,\eta] \in H^1_b(N,\mathfrak{K})$. ∎

This allows us to define parallel Sobolev–Lie algebras of order $k \in \mathbb{N}$. We set

$$H^0_b(N,\mathfrak{K}) := L^2_b(N,\mathfrak{K}),$$

and define $H^1_b(N,\mathfrak{K})$ as above. For $k \ge 2$, we define $H^k_b(N,\mathfrak{K})$ as

$$H^{k-1}_b(N,\mathfrak{K}) \cap D^{-1}(H^{k-1}_b(N,\mathfrak{K})).$$

In other words, ξ is in $H^k_b(N,\mathfrak{K})$ if both ξ and $D\xi$ are in $H^{k-1}_b(N,\mathfrak{K})$.

Proposition 6.24. *The space $H^k_b(N,\mathfrak{K})$ is a Lie subalgebra of $H^{k-1}_b(N,\mathfrak{K})$.*

Proof. The proof is by induction on k, where $k = 1$ is Proposition 6.23. If $\xi,\eta \in H^k_b(N,\mathfrak{K})$, then $\xi, D(\xi), \eta, D(\eta) \in H^{k-1}_b(N,\mathfrak{K})$. Since $H^{k-1}_b(N,\mathfrak{K})$ is a Lie algebra, it follows that $D([\xi,\eta]) = [D(\xi),\eta] + [\xi, D(\eta)]$ is in $H^{k-1}_b(N,\mathfrak{K})$. Thus, $[\xi,\eta] \in H^k_b(N,\mathfrak{K})$, as required. ∎

On $H^k_b(N,\mathfrak{K})$, we define for every $n \in \{0,\dots,k\}$ the derived norms

$$\|\xi\|_{n,\infty} := \|D^n\xi\|_\infty \quad \text{and} \quad \|\xi\|_n := \|D^n\xi\|_\nu.$$

The parallel C^k-norm q_{C^k} and the parallel Sobolev norm q_k are defined by

$$q_{C^k}(\xi) := \sum_{n=0}^{k} \|\xi\|_{n,\infty} \quad \text{and} \quad q_k(\xi) := \sum_{n=0}^{k} \|\xi\|_n, \tag{6.43}$$

respectively. We equip $H^k_b(N,\mathfrak{K})$ with the topology derived from the combined norm

$$p_k(\xi) := \sum_{n=0}^{k} \|\xi\|_{n,\infty} + \|\xi\|_n. \tag{6.44}$$

Note that for $\xi \in H^k_b(N,\xi)$, we have $p_{k-1}(\xi) \le p_k(\xi)$, but also $p_{k-1}(D(\xi)) \le p_k(\xi)$. It follows that both the inclusion $\iota\colon H^{k+1}_b(N,\mathfrak{K}) \hookrightarrow H^k_b(N,\mathfrak{K})$ and the derivative $D\colon H^{k+1}_b(N,\mathfrak{K}) \to H^k_b(N,\mathfrak{K})$ are continuous.

Proposition 6.25. *For every $k \ge 0$, $H^k_b(N,\mathfrak{K})$ is a Banach–Lie algebra with respect to the norm p_k. The Lie bracket is separately continuous with respect to the Sobolev norm q_k.*

Proof. By the derivation property and (6.39), we have

$$\|D^n([\xi,\eta])\|_\kappa \le c_{\mathfrak{k}} \sum_{j=0}^{n} \binom{n}{j} \|D^j\xi\|_\kappa \|D^{n-j}\eta\|_\kappa.$$

Since $\|[\xi,\eta]\|_n = \|D^n([\xi,\eta])\|_\nu$ and $\|[\xi,\eta]\|_{n,\infty} = \|D^n([\xi,\eta])\|_\infty$, it follows that

$$\|[\xi,\eta]\|_n \le c_{\mathfrak{k}} \sum_{j=0}^{n} \binom{n}{j} \|\xi\|_{j,\infty}\|\eta\|_{n-j},$$

$$\|[\xi,\eta]\|_{n,\infty} \le c_{\mathfrak{k}} \sum_{j=0}^{n} \binom{n}{j} \|\xi\|_{j,\infty}\|\eta\|_{n-j,\infty}.$$

Taking $n = k$ and estimating the binomial coefficients by 2^k, it follows that

$$q_k([\xi,\eta]) \le 2^k c_{\mathfrak{k}} q_{C^k}(\xi) q_k(\eta), \tag{6.45}$$

$$q_{C^k}([\xi,\eta]) \le 2^k c_{\mathfrak{k}} q_{C^k}(\xi) q_{C^k}(\eta). \tag{6.46}$$

This shows that the Lie bracket is continuous for the norm p_k, and separately continuous for the Sobolev norm q_k.

To show that $H^k_b(N,\mathfrak{K})$ is complete, we note that $H^0_b(N,\mathfrak{K}) = L^2_b(N,\mathfrak{K})$ is a Banach space, and proceed by induction on k. Let $\xi_n \in H^k_b(N,\mathfrak{K})$ be a sequence with $p_k(\xi_n - \xi_m) \to 0$. Then, $p_{k-1}(\xi_n - \xi_m) \to 0$ and $p_{k-1}(D(\xi_n) - D(\xi_m)) \to 0$, so by induction, there exist $\xi, \Xi \in H^{k-1}_b(N,\mathfrak{K})$ with

$$p_{k-1}(\xi - \xi_n) \to 0 \quad \text{and} \quad p_{k-1}(\Xi - D(\xi_n)) \to 0.$$

Since $D\colon H^1_\nu(M,\mathfrak{K}) \to L^2(M,\mathfrak{K})$ is the generator of a strongly continuous 1-parameter group of unitary operators, Stone's theorem implies that it is selfadjoint, and hence, in particular, closed. It follows that ξ lies in the domain of D, and $D(\xi) = \Xi$ lies in $H^{k-1}_b(N,\mathfrak{K})$. Thus, $\xi \in H^k_b(N,\mathfrak{K})$, and

$$p_k(\xi - \xi_n) \le p_{k-1}(\xi - \xi_n) + p_{k-1}(D(\xi) - D(\xi_n)) \to 0. \qquad \blacksquare$$

We denote by $H^\infty_b(N,\mathfrak{K})$ the Fréchet–Lie algebra arising from the inverse limit of the Banach–Lie algebras $H^k_b(N,\mathfrak{K})$ with respect to the natural inclusions

$$\iota\colon H^{k+1}_b(N,\mathfrak{K}) \hookrightarrow H^k_b(N,\mathfrak{K}).$$

The derivative $D\colon H^\infty_b(N,\mathfrak{K}) \to H^\infty_b(N,\mathfrak{K})$ is a continuous derivation, giving rise to the Fréchet–Lie algebra $H^\infty_b(N,\mathfrak{K}) \rtimes \mathbb{R}D$.

6.4.3 Boundary conditions and the Lie algebras $H^k_\partial(N,\mathfrak{K})$

Let $H^1_\partial(N,\mathfrak{K})$ be the closure of $\Gamma_c(N,\mathfrak{K})$ in $H^1_b(N,\mathfrak{K})$ with respect to the parallel Sobolev norm $q_1(\xi) = \|\xi\|_\nu + \|\xi\|_{1,\nu}$.

Proposition 6.26. *The space $H^1_\partial(N,\mathfrak{K})$ is a closed Lie subalgebra of $H^1_b(N,\mathfrak{K})$. In particular, it is a Banach–Lie algebra with respect to the subspace topology, induced by the norm $p_1(\xi)$ of* (6.44).

Proof. Since $H^1_\partial(N,\mathfrak{K})$ is by definition closed with respect to the Sobolev norm $q_1(\xi)$, it is a fortiori closed with respect to the larger norm $p_1(\xi)$ that defines the Banach space topology on $H^1_b(N,\mathfrak{K})$. As $H^1_\partial(N,\mathfrak{K})$ is a closed subspace of a Banach space, it is a Banach space itself.

It remains to show that $H^1_\partial(N,\mathfrak{K})$ is closed under the Lie bracket. For every $\xi \in H^1_b(N,\mathfrak{K})$, the linear operator $\mathrm{ad}_\xi\colon H^1_b(N,\mathfrak{K}) \to H^1_b(N,\mathfrak{K})$ is continuous with respect to the norm $q_1(\xi)$, as

$$q_1(\mathrm{ad}_\xi(\eta)) \leq 2c_{\mathfrak{k}} q_{C^1}(\xi) q_1(\eta)$$

by (6.45). If $\xi \in \Gamma_c(N,\mathfrak{K})$, then $\mathrm{ad}(\xi)$ maps $\Gamma_c(N,\mathfrak{K})$ to $\Gamma_c(N,\mathfrak{K})$. As ad_ξ is continuous for the norm q_1, it also maps $H^1_\partial(N,\mathfrak{K})$ to $H^1_\partial(N,\mathfrak{K})$. It follows that, for all $\eta \in H^1_\partial(N,\mathfrak{K})$, ad_η maps $\Gamma_c(N,\mathfrak{K})$ to $H^1_\partial(N,\mathfrak{K})$. By continuity with respect to q_1, it therefore maps $H^1_\partial(N,\mathfrak{K})$ to $H^1_\partial(N,\mathfrak{K})$, and we conclude that $H^1_\partial(N,\mathfrak{K})$ is closed under the Lie bracket. ∎

For $k \geq 2$, we define $H^k_\partial(N,\mathfrak{K})$ as the space of all $\xi \in H^k_b(N,\mathfrak{K})$ such that both ξ and $D(\xi)$ lie in $H^{k-1}_\partial(N,\mathfrak{K})$.

Proposition 6.27. *The space $H^k_\partial(N,\mathfrak{K})$ is a closed Lie subalgebra of $H^k_b(N,\mathfrak{K})$. In particular, it is a Banach–Lie algebra with respect to the subspace topology, induced by the norm $p_k(\xi)$ of* (6.44)*.*

Proof. We proceed by induction on k, the case $k = 1$ being Proposition 6.26. Recall that both the inclusion $\iota\colon H^k_b(N,\mathfrak{K}) \hookrightarrow H^{k-1}_b(N,\mathfrak{K})$ and the derivative

$$D\colon H^k_b(N,\mathfrak{K}) \to H^{k-1}_b(N,\mathfrak{K})$$

are continuous. Since

$$H^k_\partial(N,\mathfrak{K}) = \iota^{-1}(H^{k-1}_\partial(N,\mathfrak{K})) \cap D^{-1}(H^{k-1}_\partial(N,\mathfrak{K}))$$

is the intersection of two closed subspaces, it is a closed subspace of $H^k_b(N,\mathfrak{K})$ itself. To show that it is closed under the Lie bracket, suppose that $\xi,\eta \in H^k_\partial(N,\mathfrak{K})$, so that $\xi,\eta,D\xi,D\eta \in H^{k-1}_\partial(N,\mathfrak{K})$. As $H^{k-1}_\partial(N,\mathfrak{K})$ is a Lie algebra, it follows that $[\xi,\eta]$ and $D([\xi,\eta]) = [D(\xi),\eta] + [\xi,D(\eta)]$ are both in $H^{k-1}_\partial(N,\mathfrak{K})$. From this, one sees that also $[\xi,\eta] \in H^k_\partial(N,\mathfrak{K})$. ∎

Note that the 2-cocycle $\omega(\xi,\eta) = \langle D\xi,\eta\rangle_\mu$ on $\mathfrak{g}$ is continuous for the Sobolev norm $q_1(\xi)$ and hence extends uniquely to $H^k_\partial(N,\mathfrak{K})$. This defines a continuous central extension of $H^k_\partial(N,\mathfrak{K})$,

$$\mathbb{R}C \oplus_\omega H^k_\partial(N,\mathfrak{K}).$$

Define the Fréchet–Lie algebra $H^{\infty}_{\partial}(N,\mathfrak{K})$ as the inverse limit of the Banach–Lie algebras $H^{k}_{\partial}(N,\mathfrak{K})$ under the natural inclusions $H^{k}_{\partial}(N,\mathfrak{K}) \to H^{k-1}_{\partial}(N,\mathfrak{K})$. Since $D\colon H^{\infty}_{\partial}(N,\mathfrak{K}) \to H^{\infty}_{\partial}(N,\mathfrak{K})$ is a continuous derivation, we obtain the double extension of Fréchet–Lie algebras

$$\big(\mathbb{R}C \oplus_{\omega} H^{\infty}_{\partial}(N,\mathfrak{K})\big) \rtimes \mathbb{R}D.$$

6.4.4 Intervals and blocks

Suppose that $N \simeq \Sigma \times I$, where $I \subseteq \mathbb{R}$ is an open, not necessarily finite interval with the Lebesgue measure dt, and Σ is a $(d_M - 1)$-dimensional manifold with locally finite measure ν_0. The bundle $\mathfrak{K}|_N \simeq N \times \mathfrak{k}$ is trivial, and the translation by t' sends (x,t) to $(x, t-t')$ wherever it is defined. In this cartesian product situation, it will be useful to separate the variables in Σ from those in I.

Define a bilinear map

$$T\colon L^2_b(\Sigma,\mathbb{R}) \times L^2_b(I,\mathfrak{k}) \to L^2_b(N,\mathfrak{k}), \quad T(f,\xi)(x,t) = f(x)\xi(t).$$

It is continuous since $\|f\xi\|_{\nu} = \|f\|_{\nu_0}\|\xi\|_{dt}$ and $\|f\xi\|_{\infty} = \|f\|_{\infty}\|\xi\|_{\infty}$.

Proposition 6.28. *The product $T(f,\xi) = f\xi$ defines a continuous bilinear map*

$$T\colon L^2_b(\Sigma,\mathbb{R}) \times H^k_{\partial}(I,\mathfrak{k}) \to H^k_{\partial}(N,\mathfrak{k}).$$

Proof. Since $\|f\xi\|_{\nu} = \|f\|_{\nu_0}\|\xi\|_{dt}$, and since time translation acts only on ξ, it follows that $f\xi \in \mathcal{D}(D)$ if and only if $\xi \in \mathcal{D}(D)$, and $D(f\xi) = fD(\xi)$. From this, one derives that T maps $L^2_b(\Sigma,\mathbb{R}) \times H^k_b(I,\mathfrak{k})$ to $H^k_b(N,\mathfrak{k})$, with $\|f\xi\|_n = \|f\|_{\nu_0}\|\xi\|_n$ and $\|f\xi\|_{n,\infty} = \|f\|_{\infty}\|\xi\|_{n,\infty}$.

Suppose that $\xi \in H^1_{\partial}(I,\mathfrak{k})$, so that there exists a sequence $\xi_n \in C^{\infty}_c(I,\mathfrak{k})$ with $\|\xi - \xi_n\|_{dt} \to 0$ and $\|D(\xi) - D(\xi_n)\|_{dt} \to 0$. For every $f \in L^2_b(\Sigma,\mathbb{R})$, it is possible to find a sequence $f_n \in C^{\infty}_c(\Sigma,\mathbb{R})$ with $\|f - f_n\|_{\nu_0} \to 0$. Then

$$\|f\xi - f_n\xi_n\|_{\nu} \leq \|f - f_n\|_{\nu_0}\|\xi\|_{dt} + \|f_n\|_{\nu_0}\|\xi - \xi_n\|_{dt} \to 0.$$

Similarly, one finds that $\|D(f\xi) - D(f_n\xi_n)\|_{\nu} = \|fD(\xi) - f_nD(\xi_n)\| \to 0$. It follows that T maps $L^2_b(\Sigma,\mathbb{R}) \times H^1_{\partial}(I,\mathfrak{k})$ to $H^1_{\partial}(N,\mathfrak{k})$. From $D(f\xi) = fD(\xi)$, one then finds that it maps $L^2_b(\Sigma,\mathbb{R}) \times H^k_{\partial}(I,\mathfrak{k})$ to $H^k_{\partial}(N,\mathfrak{k})$. ∎

In Lemma 7.10, we will need the above result in the following form.

Corollary 6.29. *Let $E \subseteq \Sigma$ be a subset of finite measure, and let χ_E be the corresponding indicator function. Then, the map $\iota_E\colon H^k_{\partial}(I,\mathfrak{k}) \to H^k_{\partial}(N,\mathfrak{K})$ defined by $\iota_E(\xi)(x,t) = \chi_E(x)\xi(t)$ is a continuous Lie algebra homomorphism.*

6.5 The continuous extension theorem

It follows from Proposition 6.20 that the Lie algebra representation $\mathrm{d}\rho$ extends from $\mathfrak{g} = \Gamma_c(M, \mathfrak{K})$ to $L^2_{B\mu}(M, \mathfrak{K})$. In the following theorem, we show that this extension yields a Lie algebra representation of $\mathbb{R}C \oplus_\omega H^1_\partial(M, \mathfrak{K})$, which is compatible with the derivation $D\colon H^1_\partial(M, \mathfrak{K}) \to L^2_b(M, \mathfrak{K})$.

Theorem 6.30 (Continuous extension). *Let ρ be a positive energy representation of $\widehat{G}$ with derived representation $\mathrm{d}\rho$, and let $N \subseteq M$ be an open subset.*

(a) *There exists a linear map r from $L^2_b(N, \mathfrak{K})$ to the unbounded, skew-symmetric operators on $\mathcal{H}$ with domain $\mathcal{D}(H)$ such that $r(\xi)\psi$ coincides with $\mathrm{d}\rho(\xi)\psi$ for all $\xi \in \Gamma_c(N, \mathfrak{K})$ and $\psi \in \mathcal{H}^\infty$.*

(b) *This defines a representation of the Banach–Lie algebra $\mathbb{R}C \oplus_\omega H^1_\partial(N, \mathfrak{K})$ by essentially skew-adjoint operators. For $\xi, \eta \in H^1_\partial(N, \mathfrak{K})$, the operators $r(\xi)$ and $r(\eta)$ map $\mathcal{D}(H^2)$ to $\mathcal{D}(H)$. On $\mathcal{D}(H^2)$, we have the commutator relation*
$$[r(\xi), r(\eta)] = r([\xi, \eta]) + i\omega(\xi, \eta)\mathbf{1}, \tag{6.47}$$
where $\omega(\xi, \eta) = \langle D\xi, \eta\rangle_\mu$.

(c) *If $\xi \in H^1_\partial(N, \mathfrak{K})$, then $D\xi \in L^2_b(N, \mathfrak{K})$ and*
$$[\mathrm{d}\rho(D), r(\xi)] = r(D\xi).$$

In particular, we obtain a positive energy representation of the Fréchet–Lie algebra $(\mathbb{R}C \oplus_\omega H^\infty_\partial(N, \mathfrak{K})) \rtimes \mathbb{R}D$.

Proof. The derived representation $\mathrm{d}\rho$ is defined on the Lie algebra
$$\widehat{\mathfrak{g}} = \big(\mathbb{R}C \oplus_\omega \mathfrak{g}\big) \rtimes \mathbb{R}D.$$

By Proposition 6.20, we obtain an extension r of $\mathrm{d}\rho$ to $L^2_{B\mu}(M, \mathfrak{K})$, hence, in particular, to $L^2_b(N, \mathfrak{K})$. From Proposition 6.22, we see that $r(\xi)$ is essentially skew-adjoint for ξ in the smaller space $H^1_{B\mu}(M, \mathfrak{K}) \subseteq L^2_{B\mu}(M, \mathfrak{K})$, and that $[\mathrm{d}\rho(D), r(\xi)] = r(\xi')$ for all $\xi \in H^1_{B\mu}(M, \mathfrak{K})$, hence, in particular, for $\xi \in H^1_\partial(N, \mathfrak{K}) \subseteq H^1_{B\mu}(M, \mathfrak{K})$.

By Cauchy–Schwarz and the inequality (6.38), we have
$$|\langle r(\xi)\psi, r(\eta)\chi\rangle| \leq \|A^{3/2}\psi\|\|A^{3/2}\chi\| q_1(\xi) q_1(\chi) \tag{6.48}$$
for all $\psi, \chi \in \mathcal{D}(H^2)$ and $\xi, \eta \in H^1_{B\mu}(M, \mathfrak{g})$, where $A := \mathbf{1} + H$ and q_1 is the parallel Sobolev norm of (6.43). Further, by Proposition 6.22, the products $r(\xi)r(\eta)$ and $r(\eta)r(\xi)$ are well defined on $\mathcal{D}(H^2)$. Since
$$\langle \psi, [r(\xi), r(\eta)]\chi\rangle = -\langle r(\xi)\psi, r(\eta)\chi\rangle + \langle r(\eta)\psi, r(\xi)\chi\rangle,$$

it follows that the bilinear form

$$H^1_{B\mu}(M,\mathfrak{K}) \times H^1_{B\mu}(M,\mathfrak{K}) \to \mathbb{C}, \quad (\xi,\eta) \mapsto \langle \psi, [r(\xi), r(\eta)]\chi \rangle$$

is continuous with respect to the parallel Sobolev norm q_1. In particular, its restriction to

$$H^1_\partial(N,\mathfrak{K}) \subseteq H^1_{B\mu}(M,\mathfrak{K})$$

is continuous with respect to q_1.

Similarly, using Cauchy–Schwarz and (6.38), we find for $\xi, \eta \in H^1_\partial(N,\mathfrak{K})$ that

$$|\langle \chi, r([\xi,\eta])\psi \rangle| \le \|\chi\| \|A^{3/2}\psi\| q_1([\xi,\eta]).$$

Since the Lie bracket on $H^1_\partial(N,\mathfrak{K})$ is *separately* continuous for the norm q_1 by Proposition 6.25, it follows that the bilinear form $H^1_\partial(N,\mathfrak{K}) \times H^1_\partial(N,\mathfrak{K}) \to \mathbb{C}$ defined by

$$(\xi,\eta) \mapsto \langle \chi, r([\xi,\eta])\psi \rangle$$

is *separately* continuous with respect to q_1.

As the cocycle $\omega(\xi,\eta) = \langle D\xi, \eta \rangle_\mu$ extends to a bilinear map on $H^1_\partial(N,\mathfrak{K})$ that is continuous with respect to q_1, the bilinear form

$$(\xi,\eta) \mapsto \big\langle \chi, \big([r(\xi), r(\eta)] - r([\xi,\eta]) - i\omega(\xi,\eta)\big)\psi \big\rangle$$

is *separately* continuous for the q_1-topology. Since it vanishes on the dense subset $\Gamma_c(N,\mathfrak{K}) \subseteq H^1_\partial(N,\mathfrak{K})$, it is identically zero. It follows that

$$[r(\xi), r(\eta)]\psi = r([\xi,\eta])\psi + i\omega(\xi,\eta)\psi$$

for all $\psi \in \mathcal{D}(H^2)$. The operator $r([\xi,\eta]) + i\omega(\xi,\eta)\mathbf{1}$ with domain containing $\mathcal{D}(H)$ is thus an essentially skew-adjoint extension of the operator $[r(\xi), r(\eta)]$ with domain $\mathcal{D}(H^2)$. ■

6.5.1 Semibounded representations

The concept of a semibounded representation, introduced in [73,75], is much stronger than that of a positive energy condition. As results in [81] show, it provides enough regularity to lead to a sufficient supply of C^*-algebraic tools to decompose representations as direct integrals.

Definition 6.31 (Semibounded representations). A smooth representation $(\rho, \mathcal{H})$ of a locally convex Lie group G is called *semibounded* if the function

$$s_\rho\colon \mathfrak{g} \to \mathbb{R} \cup \{\infty\}, \quad s_\rho(x) := \sup(\mathrm{Spec}(i\,\mathrm{d}\rho(x))) \tag{6.49}$$

is bounded on a neighborhood of some point $x_0 \in \mathfrak{g}$. Then, the set W_ρ of all such points x_0 is an open $\mathrm{Ad}(G)$-invariant convex cone in $\mathfrak{g}$.

For Lie groups G which are locally exponential or whose Lie algebra $\mathfrak{g}$ is barrelled[1], a semibounded representation is bounded if and only if $W_\rho = \mathfrak{g}$ [73, Theorem 3.1 and Proposition 3.5]. The positive energy representation r of $H^1_\partial(N, \mathfrak{K})$ fulfills the following semiboundedness condition.

Proposition 6.32. *Let $\xi \in L^2_{B\mu}(M, \mathfrak{K})$, and let $t > 0$. Then*

$$-\|\xi\|_{B_1\mu} - 9\|\xi\|^2_\mu d_\mathfrak{k}(d_M + 1)^2/t \leq \inf(\mathrm{Spec}(ir(tD \oplus \xi))).$$

In particular, the spectrum of $tH \pm ir(\xi)$ is bounded below for every $t > 0$, and this bound is uniform on an open neighborhood of D in $L^2_{B\mu}(M, \mathfrak{K}) \rtimes \mathbb{R}D$.

Proof. Using Proposition 6.20, one finds that the map $L^2_{B\mu}(M, \mathfrak{K}) \to \mathbb{C}$ defined by

$$\xi \mapsto \langle \Gamma_\xi \pm ir(\xi)\rangle_\psi$$

is continuous for every $\psi \in \mathcal{D}(H)$, and every $\varepsilon > 0$. It is non-negative on the dense subspace $\Gamma_c(M, \mathfrak{K})$ by Theorem 6.17, and hence on all of $L^2_{B\mu}(M, \mathfrak{K})$ by continuity. If $\|\xi\|_\mu = 0$, then $r(\xi) = 0$, and the proposition holds trivially. If $\|\xi\|_\mu \neq 0$ and $\varepsilon := t/\|\xi\|_\mu$, then

$$\Gamma_\xi = tH + \|\xi\|_{B_\varepsilon\mu}\mathbf{1}$$

satisfies $0 \leq \langle \Gamma_\xi \pm ir(\xi)\rangle_\psi$, and thus

$$-\|\xi\|_{B_\varepsilon\mu}\|\psi\|^2 \leq \langle \psi, tH \pm ir(\xi), \psi\rangle.$$

Since

$$\|\xi\|_{B_\varepsilon\mu} \leq \|\xi\|_{B_1\mu} + 9\|\xi\|_\mu d_\mathfrak{k}(d_M + 1)^2/\varepsilon,$$

the result follows by substituting $\varepsilon = t/\|\xi\|_\mu$. ∎

Corollary 6.33. *The positive energy representation $\mathrm{d}\rho$ of the Lie algebra*

$$\big(\mathbb{R}C \oplus_\omega \Gamma_c(M, \mathfrak{K})\big) \rtimes \mathbb{R}D$$

is semibounded and the cone W_ρ contains the open half space

$$\big(\mathbb{R}C \oplus_\omega \Gamma_c(M, \mathfrak{K})\big) - \mathbb{R}_+ D.$$

Proof. This follows from Proposition 6.32 because $\mathrm{d}\rho$ comes from a group representation, the central element C is represented by a constant, and the inclusion $\Gamma_c(M, \mathfrak{K}) \hookrightarrow L^2_{B\mu}(M, \mathfrak{K})$ is continuous. ∎

[1] These are the locally convex spaces for which the Uniform Boundedness Principle holds. All Fréchet spaces and locally convex direct limits of Fréchet spaces are barrelled, which includes, in particular, LF spaces of test functions on noncompact manifolds.

6.5.2 Analytic vectors

A vector ψ in a Banach space $\mathcal{X}$ is called *analytic* for an unbounded operator A on $\mathcal{X}$ if $\psi \in \bigcap_{n\in\mathbb{N}} \mathcal{D}(A^n)$, and the series $\sum_{n=0}^{\infty} \frac{s^n}{n!}\|A^n\psi\|$ has positive radius of convergence $R_A > 0$.

Lemma 6.34. *Let $\xi \in H^2_\partial(N,\mathfrak{K})$, and consider H and $r(\xi)$ as unbounded operators on $\mathcal{H}$. If $\psi \in \mathcal{H}$ is an analytic vector for H, then it is also analytic for $r(\xi)$. If ψ has radius of convergence R_H for H, then the exponential series*

$$\exp(r(\xi))\psi = \sum_{n=0}^{\infty} \frac{1}{n!} r(\xi)^n \psi$$

is absolutely convergent on the ball defined by

$$p_2(\xi) < -\frac{1}{2c_{\mathfrak{k}}} \log\left(1 - \frac{(2c_{\mathfrak{k}})^2}{(c_{\mathfrak{k}}+1)^2}\left(1 - \exp\left(-\frac{(c_{\mathfrak{k}}+1)^2}{2c_{\mathfrak{k}}} R_H\right)\right)\right). \tag{6.50}$$

Proof. We apply [86, Theorem 1] to $r(\xi)$ and $A = \mathbf{1} + H$, considered as unbounded operators on the Banach space $\mathcal{H}_{1/2}$. For $\xi \in H^1_{B\mu}(M,\mathfrak{K})$ and $\psi \in \mathcal{D}(H^2) \subseteq \mathcal{H}_{1/2}$, the inequality (6.38) yields

$$\|r(\xi)\psi\|_{1/2} \leq q_1(\xi)\|A\psi\|_{1/2}. \tag{6.51}$$

By (6.35), we have $\mathrm{ad}_{r(\xi)} A = -i r(D\xi)$. If $\xi \in H^2_\partial(N,\mathfrak{K})$, then by definition, both ξ and $D\xi$ are in $H^1_\partial(N,\mathfrak{K})$. It follows that also $\mathrm{ad}^{n-1}_\xi(D\xi) \in H^1_\partial(N,\mathfrak{K})$ for $n \geq 1$. By (6.47) and induction, we find

$$\mathrm{ad}^n_{r(\xi)}(A) = -i\,\mathrm{ad}^{n-1}_{r(\xi)}(r(D\xi)) = -i\, r(\mathrm{ad}^{n-1}_\xi(D\xi)) + \omega(\xi, \mathrm{ad}^{n-2}_\xi(D\xi))\mathbf{1} \tag{6.52}$$

as an equality of unbounded operators from $\mathcal{D}(H^2)$ to $\mathcal{H}_{1/2}$. From (6.40) and (6.45), we infer that

$$\|\mathrm{ad}^n_\xi(D\xi)\|_{B_\varepsilon\mu} \leq (c_{\mathfrak{k}}\|\xi\|_\infty)^n \|D\xi\|_{B_\varepsilon\mu}, \tag{6.53}$$

$$q_1\big(\mathrm{ad}^n_\xi(D\xi)\big) \leq (2c_{\mathfrak{k}} q_{C^1}(\xi))^n q_1(D\xi). \tag{6.54}$$

Next we estimate $\|\mathrm{ad}^n_{r(\xi)}(A)\psi\|_{1/2}$. Applying (6.52) and noting that

$$|\omega(\xi,\eta)| = |\langle D\xi,\eta\rangle_\mu| \leq \|D\xi\|_\mu\|\eta\|_\mu \quad \text{and} \quad \|D\xi\|_\mu \leq \|D\xi\|_{B_\varepsilon\mu},$$

the second term on the right-hand side of (6.52) satisfies

$$\|\omega(\xi, \mathrm{ad}^{n-2}_\xi(D\xi))\psi\|_{1/2} \leq (c_{\mathfrak{k}}\|\xi\|_\infty)^{n-2}\|D\xi\|^2_{B_\varepsilon\mu}\|\psi\|_{1/2}. \tag{6.55}$$

Applying (6.51) and (6.54) to the first term on the right-hand side of (6.52), we find

$$\|r(\mathrm{ad}^{n-1}_\xi(D\xi))\psi\|_{1/2} \leq (2c_{\mathfrak{k}} q_{C^1}(\xi))^{n-1} q_1(D\xi)\|A\psi\|_{1/2}. \tag{6.56}$$

Combining (6.55) and (6.56) with (6.52), and using that $\|\psi\|_{1/2} \le \|A\psi\|_{1/2}$, we find

$$\|\mathrm{ad}^n_{r(\xi)}(A)\psi\|_{1/2} \le c_n \|A\psi\|_{1/2},$$

with

$$c_n = q_1(D\xi)\big(\|D\xi\|_{B_\varepsilon\mu} + (2c_{\mathfrak{k}} q_{C^1}(\xi))\big)(2c_{\mathfrak{k}} q_{C^1}(\xi))^{n-2}. \tag{6.57}$$

Since the series

$$\nu(s) := \sum_{n=1}^{\infty} \frac{c_n}{n!} s^n$$

has positive radius of convergence, we may now fix some $t_0 > 0$ with $\nu(t_0) < 1$ and assume that $0 \le s, t \le t_0$. Applying [86, Theorem 1] to $\mathcal{H}_{1/2}$ guarantees that for

$$\varpi(s) := \int_0^s (1 - \nu(t))^{-1} dt,$$

we have

$$\sum_{n=0}^{\infty} \frac{s^n}{n!} \|r(\xi)^n \psi\|_{1/2} \le \sum_{n=0}^{\infty} \frac{(c \cdot \varpi(s))^n}{n!} \|A^n \psi\|_{1/2}, \quad \text{with } c := q_1(\xi)$$

as in (6.51). Since $\|r(\xi)^n\psi\| \le \|r(\xi)^n\psi\|_{1/2}$ and $\|A^n\psi\|_{1/2} \le \|A^{n+1}\psi\|$, this yields

$$\sum_{n=0}^{\infty} \frac{s^n}{n!} \|r(\xi)^n \psi\| \le \sum_{n=0}^{\infty} \frac{(c \cdot \varpi(s))^n}{n!} \|A^{n+1} \psi\|. \tag{6.58}$$

To get an explicit estimate on the radius of convergence, note that all norms of (derivatives of) ξ occurring in (6.57) are dominated by $p_2(\xi)$ (cf. (6.44)). The estimate $c_n \le ab^n$ with

$$a := (1 + 2c_{\mathfrak{k}})/(2c_{\mathfrak{k}})^2 \quad \text{and} \quad b := 2c_{\mathfrak{k}} p_2(\xi)$$

yields $\nu(s) \le a(e^{bs} - 1)$. Accordingly, $\nu(t_0) < 1$ is ensured if

$$bt_0 < \log\left(1 + \frac{1}{a}\right) = -\log\left(1 - \frac{1}{1+a}\right).$$

In particular, $s = 1$ is allowed if $p_2(\xi) < \frac{1}{2c_{\mathfrak{k}}} \log(1 + \frac{1}{a})$. Substituting this in

$$\varpi(s) = \int_0^s (1 - \nu(t))^{-1} dt$$

and integrating, we obtain

$$\varpi(s) \le -\frac{1}{(1+a)b} \log((1+a)e^{-bs} - a). \tag{6.59}$$

If ψ is an analytic vector for H, it is analytic for $A = \mathbf{1} + H$ with the same radius of convergence R_H. The right-hand side of (6.58) therefore converges absolutely if $c \cdot \varpi(s) < R_H$, where $c = q_1(\xi)$. Since $q_1(\xi) \leq p_2(\xi)$, we find $\frac{c}{b} \leq \frac{1}{2c_{\mathfrak{k}}}$, and hence $\frac{c}{(1+a)b} \leq 2c_{\mathfrak{k}}/(c_{\mathfrak{k}} + 1)^2$. Substituting this in (6.59), we find that $c \cdot \varpi(s) \leq R_H$ if

$$bs \leq -\log\left(1 - \frac{1}{1+a}\left(1 - \exp\left(-\frac{(c_{\mathfrak{k}} + 1)^2}{2c_{\mathfrak{k}}} R_H\right)\right)\right) < -\log\left(1 - \frac{1}{1+a}\right).$$

Putting $s = 1$, and substituting a and b in the above equation, we find that (6.58) converges if $p_2(\xi)$ satisfies (6.50). ■

Chapter 7

The localization theorem

In this section, we use the continuity and analyticity results from Chapter 6 to prove a *localization theorem*. Our main result reduces the classification of positive energy representations of the identity component $\Gamma_c(M, \mathcal{K})_0$ to the case where the base manifold M is one-dimensional. We start in the setting of a fixed point free $\mathbb{R}$-action on the manifold M, and extend this to more general Lie group actions in Section 7.5.

7.1 Statement and discussion of the theorem

Theorem 7.1 (Localization theorem). *Let $\pi\colon \mathcal{K} \to M$ be a Lie group bundle whose fibers are 1-connected semisimple. Let $\gamma_{\mathcal{K}}\colon \mathbb{R} \to \mathrm{Aut}(\mathcal{K})$ be a homomorphism that defines a smooth action on $\mathcal{K}$, and induces a fixed-point free flow γ_M on M. Then, for every projective positive energy representation*

$$\bar{\rho}\colon \Gamma_c(M, \mathcal{K})_0 \to \mathrm{PU}(\mathcal{H})$$

of the connected gauge group $\Gamma_c(M, \mathcal{K})_0$, there exists a one-dimensional, closed, embedded, flow-invariant submanifold $S \subseteq M$ such that $\bar{\rho}$ factors through a projective positive energy representation $\bar{\rho}_S$ of the connected Lie group $\Gamma_c(S, \mathcal{K})$. The diagram

$$\begin{tikzcd} \Gamma_c(M, \mathcal{K})_0 \arrow[r, "\bar{\rho}"] \arrow[d, "r_S"] & \mathrm{PU}(\mathcal{H}) \\ \Gamma_c(S, \mathcal{K}) \arrow[ur, "\bar{\rho}_S"'] & \end{tikzcd}$$

commutes, where $r_S\colon \Gamma_c(M, \mathcal{K})_0 \to \Gamma_c(S, \mathcal{K})$ is the restriction homomorphism.

Remark 7.2. It is convenient to define $\Gamma_c(\emptyset, \mathcal{K}) := \{\mathbf{1}\}$, so that the above theorem holds for the trivial representation with $S = \emptyset$.

Remark 7.3 (Localization for the simply connected cover). In fact, we will prove a slightly stronger result: every projective positive energy representation

$$\bar{\rho}\colon \widetilde{\Gamma}_c(M, \mathcal{K})_0 \to \mathrm{PU}(\mathcal{H})$$

of the simply connected cover of $\Gamma_c(M, \mathcal{K})_0$ factors through $\widetilde{r}_S := r_S \circ q_\Gamma$, where $q_\Gamma\colon \widetilde{\Gamma}_c(M, \mathcal{K})_0 \to \Gamma_c(M, \mathcal{K})$ is the covering map and

$$r_S\colon \Gamma_c(M, \mathcal{K})_0 \to \Gamma_c(S, \mathcal{K})$$

the restriction. This strengthening of Theorem 7.1 is needed in Part II, where we handle localization for gauge groups on manifolds with boundary.

Note that M is not required to be compact or connected, and that the fibers of $\mathcal{K} \to M$ are not required to be compact. The result for noncompact M is a major feature, which we will use extensively later on (see Chapter 9 and Part II). Allowing noncompact fibers, however, is not a big step. Indeed, noncompact simple fibers result in trivial representations by Theorem 6.2, so we already know that the theorem holds with $S = \emptyset$ in that case. Before proceeding with the proof in Section 7.2, we show that the assumption of 1-connectedness of the fibers is not essential.

Remark 7.4 (Non-simply connected fibers). Suppose that the typical fibers of the bundle $\mathcal{K} \to M$ are connected, but not necessarily simply connected. Let K_i be the typical fiber over the connected component M_i of M, and let $\widetilde{K}_i$ be its 1-connected universal cover. The kernel $\pi_1(K_i)$ of the covering map $\widetilde{K}_i \twoheadrightarrow K_i$ is a finite, central subgroup, yielding a central extension

$$\pi_1(K_i) \hookrightarrow \widetilde{K}_i \twoheadrightarrow K_i. \tag{7.1}$$

The natural inclusion $\mathrm{Aut}(K_i) \hookrightarrow \mathrm{Aut}(\widetilde{K}_i)$, obtained by the canonical lift of automorphisms, yields a Lie group bundle $\widetilde{\mathcal{K}}_i \to M_i$ with fiber $\widetilde{K}_i$ over each M_i, and hence a Lie group bundle $\widetilde{\mathcal{K}} \to M$ over M. It comes with a natural bundle map $\widetilde{\mathcal{K}} \to \mathcal{K}$ over the identity of M, which restricts to the universal covering map on every fiber. The kernel $\mathcal{Z} \subseteq \widetilde{\mathcal{K}}$ of this map is a bundle of discrete, abelian groups, whose fibers over M_i are isomorphic to $\pi_1(K_i)$. Analogous to (7.1), we thus obtain an exact sequence of Lie group bundles

$$\mathcal{Z} \hookrightarrow \widetilde{\mathcal{K}} \twoheadrightarrow \mathcal{K}.$$

The 1-parameter group $\gamma_{\mathcal{K}} : \mathbb{R} \to \mathrm{Aut}(\mathcal{K})$ lifts to $\gamma_{\widetilde{\mathcal{K}}} : \mathbb{R} \to \mathrm{Aut}(\widetilde{\mathcal{K}})$ with the same infinitesimal generator $\mathbf{v} \in \Gamma(M, \mathfrak{a}(\mathfrak{K}))$ (cf. Remark 4.8). As every smooth section $\xi \in \Gamma_c(M, \mathcal{K})_0$ lifts to a section of $\widetilde{\mathcal{K}}$ because the natural map $\Gamma_c(M, \widetilde{\mathcal{K}}) \to \Gamma_c(M, \mathcal{K})$ is a covering morphism of Lie groups, the projection $\widetilde{\mathcal{K}} \to \mathcal{K}$ yields a surjective Lie group homomorphism, and hence an exact sequence

$$\Gamma_c(M, \mathcal{Z}) \hookrightarrow \Gamma_c(M, \widetilde{\mathcal{K}}) \to \Gamma_c(M, \mathcal{K}). \tag{7.2}$$

Since the fibers of $\mathcal{Z}$ are discrete, the group $\Gamma_c(M_i, \mathcal{Z}_i)$ of compactly supported sections of $\mathcal{Z}_i \to M_i$ is trivial if M_i is noncompact. If M_i is compact, $\Gamma_c(M_i, \mathcal{Z}_i)$ can be identified with $\pi_1(K_i)^{\pi_1(M_i)}$, the fixed point subgroup of $\pi_1(K_i)$ under the monodromy action $\pi_1(M_i) \to \mathrm{Aut}(\pi_1(K_i))$. We thus obtain an isomorphism

$$\Gamma_c(M, \mathcal{Z}) \simeq \prod\nolimits'_{i \in I} \pi_1(K_i)^{\pi_1(M_i)} \tag{7.3}$$

of discrete groups where $\prod'_{i \in I} \pi_1(K_i)^{\pi_1(M_i)}$ denotes the weak direct product of the finite abelian groups $\pi_1(K_i)^{\pi_1(M_i)}$ (all tuples with finitely many non-zero entries), running over all i for which the connected component M_i is compact. In particular, it follows from (7.2) and (7.3) that projective positive energy representations of $\Gamma_c(M, \mathcal{K})_0$ correspond to projective positive energy representations of $\Gamma_c(M, \widetilde{\mathcal{K}})_0$ that are trivial on

$$Z_{[M]} := \Gamma_c(M, \mathcal{Z}) \cap \Gamma_c(M, \widetilde{\mathcal{K}})_0.$$

Note that the embedding $S \hookrightarrow M$ yields a "diagonal" morphism $Z_{[M]} \to Z_{[S]}$. The term "diagonal" is justified by the special case where $\mathcal{K}$ is a trivial bundle over a compact, connected manifold M. Then, the embedded 1-dimensional submanifold $\emptyset \neq S \subseteq M$ is the disjoint union of N circles, and $Z_{[M]} \simeq \pi_1(K)$ can literally be identified with the diagonal subgroup of $Z_{[S]} \simeq \pi_1(K)^N$.

Combining Theorem 7.1 with Remark 7.4, we obtain a localization theorem for bundles whose fibers are not necessarily simply connected.

Corollary 7.5 (Localization theorem for non-simply connected fibers). *Suppose that the fibers of $\mathcal{K} \to M$ are connected, but not necessarily simply connected. Then, $\bar{\rho}$ arises by factorization from a projective positive energy representation of* $\Gamma_c(S, \widetilde{\mathcal{K}})$ *that is trivial on the image of $Z_{[M]}$ in $Z_{[S]}$.*

Remark 7.6 (Abelian groups). In the localization Theorem 7.1 we have assumed that the fiber Lie group K is semisimple. We now explain why this is crucial and that there is no localization for abelian target groups, so that the localization theorem does not extend to bundles with general compact fiber Lie algebras. To this end, let $K = (\mathfrak{k}, +)$ be a finite-dimensional real vector space and fix a positive definite scalar product κ on $\mathfrak{k}$. Further, let M be a smooth manifold and consider the Lie group $G := \mathfrak{g} := C_c^\infty(M, \mathfrak{k})$, which can be identified with the group of compactly supported sections of the trivial bundle $\mathcal{K} = M \times K$. We also fix a smooth flow $\gamma_M : \mathbb{R} \to \mathrm{Diff}(M)$, its generator $\mathbf{v}_M \in \mathcal{V}(M)$, and a γ_M-invariant positive Radon measure μ on M. Then

$$\kappa_{\mathfrak{g}}(\xi, \eta) := \int_M \kappa(\xi, \eta) d\mu$$

defines a positive semidefinite scalar product on $\mathfrak{g}$, invariant under the $\mathbb{R}$-action on $\mathfrak{g}$ given by

$$\alpha_t \xi := \xi \circ \gamma_M(t),$$

whose infinitesimal generator is $D\xi = \mathcal{L}_{\mathbf{v}_M} \xi$. Then

$$\omega(\xi, \eta) := \kappa_{\mathfrak{g}}(D\xi, \eta) = \int_M \kappa(\mathcal{L}_{\mathbf{v}_M} \xi, \eta) d\mu$$

is an $\mathbb{R}$-invariant skew-symmetric form on the abelian Lie algebra $\mathfrak{g}$, hence a Lie algebra 2-cocycle. Combining in [85, Theorems 3.2 and 5.9], it now follows that all these cocycles are obtained from projective positive energy representations of the groups $G \rtimes_\alpha \mathbb{R}$. This shows that, for abelian fibers, no restrictions on the measure μ exist.

Example 7.7. We consider the Lie algebra $\mathfrak{g} = C^\infty(\mathbb{T}^d, \mathfrak{k})$, $\mathfrak{k}$ compact simple and $\alpha_t(\xi) = \xi \circ \gamma_t$, where

$$\gamma_t(z_1, \ldots, z_d) = \left(e^{2\pi i t\theta_1} z_1, \ldots, e^{2\pi i t\theta_{d-1}} z_{d-1}, e^{2\pi i t} z_d\right).$$

This means that $\mathbf{v}_M$ is the invariant vector field on the Lie group $\mathbb{T}^d \cong \mathbb{R}^d/\mathbb{Z}^d$ with exponential function

$$\exp(x_1, \ldots, x_d) = \left(e^{2\pi i x_1}, \ldots, e^{2\pi i x_d}\right)$$

whose value in $\mathbf{1}$ is given by $x = (\theta_1, \ldots, \theta_{d-1}, 1)$. This action has a closed orbit if and only if the one-parameter group $A := \exp(\mathbb{R}x)$ is closed, which is equivalent to $\theta_j \in \mathbb{Q}$ for all j.

If this condition is satisfied, then $A \cong \mathbb{T}$ and the α-orbits are the A-cosets in the group $\mathbb{T}^d$. This situation is also studied by Torresani in [105]. If this condition is not satisfied, then the localization theorem implies that there are no non-trivial projective positive energy representations.

Remark 7.8. The localization theorem also yields partial information for flows with fixed points, and for manifolds with boundary.

(a) If the vector field $\mathbf{v}_M$ has zeros, then

$$M^\times := \{x \in M : \mathbf{v}_M(x) \neq 0\}$$

is an open flow-invariant submanifold of M and the localization theorem applies to the bundle $\mathfrak{K}|_{M^\times}$. In this context, this theorem does not provide a complete reduction to the one-dimensional case for two reasons. One is that the representations of $\Gamma_c(M^\times, \mathfrak{K})$ do not uniquely determine those of $\Gamma_c(M, \mathfrak{K})$ and the other reason is that the 1-dimensional submanifold S of $M^\times$ need not be closed in M, so that the extendability of the representation of $\Gamma_c(M^\times, \mathfrak{K})$ to the Lie algebra $\Gamma_c(M, \mathfrak{K})$ provides "boundary conditions at infinity" for the corresponding representations of $\Gamma_c(S, \mathfrak{K})$. We will further explore these boundary conditions in future work.

(b) Similarly, if $\bar{M}$ is a manifold with boundary, then both its interior $M = \bar{M} \setminus \partial M$ and its boundary ∂M are invariant under the flow. In Part II of this series of papers, we apply the localization theorem to M and ∂M separately, and combine the information to obtain classification results for positive energy representations of the gauge group $\Gamma(\bar{M}, \mathcal{K})$. The main challenge here is that although every projective unitary representation of $\Gamma(\bar{M}, \mathcal{K})$ automatically restricts to $\Gamma_c(M, \mathcal{K})$, we heavily rely on the positive energy condition to obtain a representation of $\Gamma_c(\partial M, \mathcal{K})$.

Example 7.9. A typical example of a flow with fixed points is the 2-sphere $M = \mathbb{S}^2$, where

$$\gamma_{M,t}(x, y, z) = \begin{pmatrix} \cos(t) & \sin(t) & 0 \\ -\sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{7.4}$$

is the rotation around the z-axis with unit angular velocity, and $P = \mathbb{S}^2 \times \mathfrak{k}$ is the trivial bundle. The lift of the infinitesimal action is then given by

$$\mathbf{v}(x, y, z) = (y\partial_x - x\partial_y) + A(x, y, z), \tag{7.5}$$

where the first part is the horizontal lift of the infinitesimal action corresponding to (7.4), and the second part is the vertical vector field corresponding to a smooth function $A: \mathbb{S}^2 \to \mathfrak{k}$.

Then, $M^\times = \mathbb{S}^2 \setminus \{(0, 0, \pm 1)\}$, and the integral curves on $\mathbb{S}^2$ are precisely the circles of latitude. Therefore, S is either compact and a finite union of circles, or it is noncompact and an infinite union of circles. More precisely,

$$S = \big\{(x, y, z) \in \mathbb{S}^2 : z \in J\big\},$$

where $J \subset (-1, 1)$ is a discrete set that has at most two accumulation points ± 1, corresponding to the two fixed points of the circle action. We return to this example in Section 9.3.

7.2 Localization at the Lie algebra level

The remainder of this chapter is devoted to the proof of Theorem 7.1. We start by proving the statement at the level of Lie algebras. This proceeds through several lemmas. In the first one, relying heavily on Theorem 6.30 and Lemma 6.34, we derive integrality results for the flow-invariant measure μ of Section 5.2.2.

Lemma 7.10. *Suppose that the fibers of $\mathfrak{K} \to M$ are simple Lie algebras. Consider a good flow box $U \simeq U_0 \times I \subseteq M$ around $x \in M$ in the sense of Definition 5.1, so that the restriction of the invariant measure μ to $U \simeq U_0 \times I$ takes the form*

$$\mu|_U = \mu_0 \otimes dt.$$

Then, for every measurable subset $E \subseteq U_0$,

$$\mu_0(E) \in \frac{1}{2\pi}\mathbb{N}_0.$$

Proof. We may assume without loss of generality that the fibers of $\mathfrak{K}$ over U are compact, as $\mu_0(E)$ would otherwise be zero by Corollary 5.5. Let $\chi_E : U_0 \to \{0, 1\}$ be the indicator function of E. Consider the Lie algebra homomorphism

$$\iota_E : \mathbb{R}C \oplus_\omega H^2_\partial(I, \mathfrak{k}) \to \mathbb{R}C \oplus_\omega H^2_\partial(U, \mathfrak{k}), \quad zC \oplus \xi \mapsto zC \oplus \chi_E \xi$$

whose continuity follows from Corollary 6.29. If we pull back the representation r of $\mathbb{R}C \oplus_\omega H^2_\partial(U, \mathfrak{k})$ of Theorem 6.30 along ι_E, we obtain a projective $*$-representation of the Banach–Lie algebra $\mathfrak{h} := H^2_\partial(I, \mathfrak{k})$. By Lemma 6.34, its space of analytic vectors is dense in $\mathcal{H}$.

Since $\mathfrak{h}$ consists of functions $I \to \mathfrak{k}$ and it contains $C_c^\infty(I, \mathfrak{k})$, the fact that $\mathfrak{z}(\mathfrak{k}) = \{0\}$ implies that the center of $\mathfrak{h}$ is trivial. As $\mathfrak{h}$ is a Banach–Lie algebra, it is in particular locally exponential, so there exists a 1-connected Lie group H with Lie algebra $\mathfrak{h}$ by [71, Theorem IV.3.8] (see [30] for a complete proof).

Now Theorem 2.18 provides a smooth, projective, unitary representation

$$\pi \colon H \to \mathrm{PU}(\mathcal{H}).$$

By Theorem 5.7, the corresponding Lie algebra cocycle is given by

$$\begin{aligned} \omega(\xi, \eta) &= -\int_U \kappa(\chi_E \xi, \nabla_{\mathbf{v}_M}(\chi_E \eta)) d\mu = -\int_{U_0 \times I} \kappa(\chi_E \xi, \chi_E \eta') d\mu_0 dt \\ &= -\mu_0(E) \int_I \kappa(\xi, \eta') dt = \mu_0(E) \int_I \kappa(\xi', \eta) dt. \end{aligned}$$

Theorem 2.13 now implies the existence of a central Lie group extension $H^\sharp$ of H by $\mathbb{T} \cong \mathbb{R}/2\pi\mathbb{Z}$ with Lie algebra $\mathfrak{h}^\sharp_\omega = \mathbb{R}C \oplus_\omega \mathfrak{h}$.

This in turn implies integrality conditions on the values of $\mu_0(E)$. To see how these can be obtained, we associate to ω the corresponding left invariant 2-form Ω on H with $\Omega_{\mathbf{1}} = \omega$. This form defines a *period homomorphism*

$$\mathrm{per}_\omega \colon \pi_2(H) \to \mathbb{R}, \quad [\sigma] \mapsto \int_\sigma \Omega$$

(cf. [69, Definition 5.8]) and [69, Lemma 5.11] implies that

$$\mathrm{im}(\mathrm{per}_\omega) \subseteq 2\pi\mathbb{Z}.$$

Since the rescaling map

$$\gamma \colon C_c^\infty(I, \mathfrak{k}) \to C_c^\infty((-\pi, \pi), \mathfrak{k}), \quad \gamma(\xi)(\theta) = \xi\left(\frac{T}{2\pi}\theta\right)$$

from the interval $I = (-T/2, T/2)$ to the interval $(-\pi, \pi)$ is an isomorphism of Lie algebras, the cocycle $\int_I \kappa(\xi', \eta) dt$ on $C_c^\infty(I, \mathfrak{k})$ has the same period group as the cocycle $\int_{-\pi}^{\pi} \kappa(\xi', \eta) d\theta$ on $C_c^\infty((-\pi, \pi), \mathfrak{k})$. In [70, Lemma V.11], it was shown that this, in turn, has the same period group as the cocycle $\int_{-\pi}^{\pi} \kappa(\xi', \eta) d\theta$ on $C_c^\infty(\mathbb{S}^1, \mathfrak{k})$. By [68, Theorem II.5], the period group of $\frac{1}{2\pi} \int_{-\pi}^{\pi} \kappa(\xi', \eta) d\theta$ is $2\pi\mathbb{Z}$, provided that κ is normalized as in (4.2). Combining all this, we conclude that $\mu_0(E) \in \frac{1}{2\pi}\mathbb{Z}$. ∎

As the measure $2\pi\mu_0$ takes integral values, the following proposition shows that it is automatically discrete.

Proposition 7.11. *Let ζ be a locally finite, regular Borel measure on a locally compact space Σ. If ζ takes values in $\mathbb{N}_0 \cup \{\infty\}$, then there exists a locally finite subset $\Lambda \subseteq \Sigma$ and natural numbers $c_x = \zeta(\{x\})$ such that*

$$\zeta = \sum_{x \in \Lambda} c_x \delta_x.$$

Proof. By regularity, ζ is determined by its values on compact subsets, so it suffices to assume that Σ is compact and to show that, in this case, ζ is a finite sum of Dirac measures.

Let $\mathcal{F}$ be the family of compact subsets of full measure. For $F_1, F_2 \in \mathcal{F}$, we have

$$\mu(F_1 \setminus F_2) = \mu(F_2 \setminus F_1) = 0,$$

so that $F_1 \cap F_2$ also has full measure. This shows that $\mathcal{F}$ is closed under finite intersections. We show that

$$C := \bigcap_{F \in \mathcal{F}} F$$

has full measure. Let V be an open set containing C. Since the open complements F^c cover the compact set V^c, there exist finitely many F_i such that $F_1^c \cup \cdots \cup F_k^c \supseteq V^c$, and hence $F_1 \cap \cdots \cap F_k \subseteq V$. Since $\mathcal{F}$ is closed under finite intersections, every open set V containing C has full measure. By regularity, we conclude that C has full measure itself.

Pick $x \in C$. For any open neighborhood U of x in C, the minimality of C implies that $\zeta(C \setminus U) < \zeta(C)$, so that $\zeta(U) > 0$. Let U be an open neighborhood of x in C for which $\zeta(U)$ is minimal; here we use that the values of ζ are contained in $\mathbb{N}_0$. For any smaller open neighborhood $V \subseteq U$ of x in C we then have $\zeta(V) = \zeta(U)$ and therefore $\zeta(U \setminus V) = 0$. This implies that $\zeta(K) = 0$ for any compact subset $K \subseteq U \setminus \{x\}$ and hence that $\zeta(U \setminus \{x\}) = 0$ by the regularity of ζ. Now the minimality of C entails that $C = \{x\} \cup (C \setminus U)$. Since $x \in C$ was arbitrary, it follows that C is discrete, hence finite: $C = \{x_1, \ldots, x_k\}$. Accordingly, the restriction of ζ to a compact subset is the finite sum

$$\zeta = \sum_{j=1}^{k} \zeta(\{x_j\}) \delta_{x_j}$$

of Dirac measures. ∎

Recall from Theorem 4.9 that the bundle $\mathfrak{K} \to M$ of semisimple Lie algebras gives rise to a bundle $\widehat{\mathfrak{K}} \to \widehat{M}$ of simple Lie algebras with $\Gamma_c(M, \mathfrak{K}) \simeq \Gamma_c(\widehat{M}, \widehat{\mathfrak{K}})$. By Remark 4.10, it inherits the 1-parameter group of automorphisms.

Lemma 7.12. *If the flow on M has no fixed points, then the support $\widehat{S}$ of μ is a one-dimensional, flow-invariant, closed embedded submanifold of $\widehat{M}_{\mathrm{cpt}}$, the part of $\widehat{M}$ over which the fibers of $\widehat{\mathfrak{K}}$ are compact.*

Proof. Since the flow on M has no fixed points, the vector field $\mathbf{v}_M$ on M has no zeros. As the same holds for its lift to $\widehat{M}$, every point $x \in \widehat{M}$ is contained in a good flow box $U \cong U_0 \times I$ in the sense of Definition 5.1. In any such flow box, the measure μ is of the form $\mu_0 \otimes dt$, where μ_0 is a regular measure on U_0. From Lemma 7.10 and Proposition 7.11, we conclude that μ_0 has finite support in U_0, so that $\widehat{S} \cap U \cong F \times I$, where $F \subseteq U_0$ is a finite subset. This implies that $\widehat{S}$ is a one-dimensional, closed embedded submanifold invariant under the flow on $\widehat{M}$. The final statement follows from Theorem 6.2. ∎

Combined with Corollary 6.3, this shows that Theorem 7.1 holds at the level of Lie algebras.

Lemma 7.13. *There exists a 1-dimensional, closed, embedded, flow-invariant submanifold $S \subseteq M$ such that the projective positive energy representation $d\rho$ of the Lie algebra $\Gamma_c(M, \mathfrak{K})$ factors through the restriction map $r_S^{\mathfrak{k}}\colon \Gamma_c(M, \mathfrak{K}) \to \Gamma_c(S, \mathfrak{K})$.*

Proof. Combining Lemma 7.12 with Corollary 6.3 and Theorem 6.2, we conclude that the projective Lie algebra representation $d\rho$ of $\Gamma_c(\widehat{M}, \widehat{\mathfrak{K}})$ vanishes on the ideal

$$J_{\widehat{S}} := \{\xi \in \Gamma_c(\widehat{M}, \widehat{\mathfrak{K}}) : \xi|_{\widehat{S}} = 0\}.$$

It follows that the projective positive energy representation of $\Gamma_c(M, \mathfrak{K})$ vanishes on $J_S := \{\xi \in \Gamma_c(M, \mathfrak{K}) : \xi|_S = 0\}$, where $S \subseteq M$ is the image of $\widehat{S}$ under the finite, $\mathbb{R}$-equivariant covering map $\widehat{M} \to M$. Since $\widehat{S} \subseteq \widehat{M}$ is a 1-dimensional, closed, embedded, flow-invariant submanifold, the same holds for $S \subseteq M$. This implies that the projective representation factors through the restriction map

$$r_S^{\mathfrak{k}}\colon \Gamma_c(M, \mathfrak{K}) \to \Gamma_c(S, \mathfrak{K}),$$

which is a quotient map of locally convex spaces. ∎

7.3 Twisted loop groups

Let S be a one-dimensional, embedded, flow-invariant submanifold of M. Then, it is the disjoint union $S = \bigsqcup_{j \in J} S_j$ of its connected components S_j, which are either diffeomorphic to $\mathbb{R}$ (for a non-periodic orbit), or to $\mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$ (for a periodic orbit).

Fix $j \in J$ and let $K = K_j$ denote the fiber of $\mathcal{K}|_{S_j}$. If $S_j \cong \mathbb{R}$, then the bundle $\mathcal{K}|_{S_j}$ is trivial, i.e., equivalent to

$$S_j \times K \cong \mathbb{R} \times K.$$

This trivialization can be achieved $\mathbb{R}$-equivariantly, using an integral curve in the corresponding frame bundle $\mathrm{Aut}(\mathcal{K}) \to \mathbb{R}$, a principal bundle with fiber $\mathrm{Aut}(K)$.

The action of $\mathbb{R}$ on $\mathcal{K}$ is then simply given by

$$\gamma_t(x,k) = (x+t,k) \quad \text{for } t,x \in \mathbb{R} \text{ and } k \in K. \tag{7.6}$$

If $S_j \cong \mathbb{S}^1$ is a periodic orbit, then the universal covering map $q_j \colon \widetilde{S}_j \to S_j$ can be identified with the quotient map $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$. If the period of the orbit S_j is T, then we scale the $\mathbb{R}$-action on $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ by $1/T$, yielding

$$\gamma_{\mathbb{S}^1,t}([x]) = [x + t/T].$$

We have seen above that the pullback $q_j^*(\mathcal{K}|_{S_j})$ is equivariantly equivalent to the trivial bundle $\mathbb{R} \times K$ on which $\mathbb{R}$ acts by translation in the first factor. The action of the fundamental group $\pi_1(S_j) \cong \mathbb{Z}$ on $\mathbb{R} \times K$ is given by bundle automorphisms that commute with the $\mathbb{R}$-action; there exists an automorphism $\Phi \in \mathrm{Aut}(K)$ such that

$$n \cdot (x,k) = (x+n, \Phi^{-n}(k)) \quad \text{for all } n \in \mathbb{Z}.$$

Accordingly, we have an equivariant isomorphism

$$\mathcal{K}|_{S_j} \cong (\mathbb{R} \times K)/\sim,$$

where

$$(x,k) \sim (x+n, \Phi^{-n}(k))$$

for all $x \in \mathbb{R}, k \in K$ and $n \in \mathbb{Z}$. We write the equivalence classes as $[x,k]$, and we denote the K-bundle over $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ obtained in this way by

$$\mathcal{K}_\Phi := (\mathbb{R} \times K)/\sim, \quad \text{with } \mathcal{K}_\Phi \to \mathbb{R}/\mathbb{Z}$$

given by

$$[x,k] \mapsto [x] = x + \mathbb{Z}.$$

The $\mathbb{R}$-action is given in these terms by

$$\gamma_t([x,k]) = [x + t/T, k].$$

Note that

$$\gamma_T([x,k]) = [x+1,k] = [x, \Phi(k)],$$

so that Φ can be interpreted as a *holonomy*.

Recall that, for two automorphisms $\Phi, \Psi \in \mathrm{Aut}(K)$, the corresponding K-bundles $\mathcal{K}_\Phi$ and $\mathcal{K}_\Psi$ are equivalent if and only if the classes $[\Phi]$ and $[\Psi]$ are conjugate in the component group $\pi_0(\mathrm{Aut}(K))$, and they are $\mathbb{R}$-equivariantly isomorphic if and only if Φ and Ψ are conjugate in $\mathrm{Aut}(K)$. Indeed, any isomorphism $\Gamma_{\Psi,\Phi} \colon \mathcal{K}_\Phi \to \mathcal{K}_\Psi$ inducing the identity on the base is of the form

$$\Gamma_{\Psi,\Phi}([x,k]) = [x, \zeta_x(k)],$$

where $\zeta\colon \mathbb{R} \to \mathrm{Aut}(K)$ is smooth and satisfies

$$\zeta_{x+1} = \Psi^{-1} \circ \zeta_x \circ \Phi \quad \text{for all } x \in \mathbb{R}. \tag{7.7}$$

Such a smooth curve ζ exists if and only if $[\Phi]$ and $[\Psi]$ are conjugate in the finite group $\pi_0(\mathrm{Aut}(K))$. In particular, the set of equivalence classes of group bundles with fiber K over $\mathbb{S}^1$ corresponds to the set of conjugacy classes in the group $\pi_0(\mathrm{Aut}(K))$, which is finite for a semisimple compact Lie group K. This follows from the compactness of the group $\mathrm{Aut}(K) \subseteq \mathrm{Aut}(\widetilde{K}) \cong \mathrm{Aut}(\mathfrak{k})$ as a subgroup of $\mathrm{GL}(\mathfrak{k})$ preserving the scalar product κ.

The bundle isomorphism $\Gamma_{\Psi,\Phi}$ is $\mathbb{R}$-equivariant if and only if the function ζ is constant. Accordingly, the two bundles $\mathcal{K}_\Phi$ and $\mathcal{K}_\Psi$ are $\mathbb{R}$-equivariantly isomorphic if and only if Φ and Ψ are conjugate in $\mathrm{Aut}(K)$, so that equivariant isomorphism classes of principal K-bundles over $\mathbb{S}^1$ correspond to conjugacy classes in the group $\mathrm{Aut}(K)$ (cf. [94, Section 4.4] and [20, Section 9]).

The group $\Gamma_c(\mathbb{R}/\mathbb{Z}, \mathcal{K}_\Phi)$ is isomorphic to the twisted loop group

$$\mathcal{L}_\Phi(K) := \left\{\xi \in C^\infty(\mathbb{R}, K) : (\forall x \in \mathbb{R})\xi(x+1) = \Phi^{-1}(\xi(x))\right\} \tag{7.8}$$

with Lie algebra

$$\mathcal{L}_\varphi(\mathfrak{k}) := \left\{\xi \in C^\infty(\mathbb{R}, \mathfrak{k}) : (\forall x \in \mathbb{R})\,\xi(x+1) = \varphi^{-1}(\xi(x))\right\}, \tag{7.9}$$

where $\varphi \in \mathrm{Aut}(\mathfrak{k})$ is the automorphism of $\mathfrak{k}$ induced by Φ. The $\mathbb{R}$-action on $\mathcal{L}_\varphi(\mathfrak{k})$ is given by

$$\alpha_t(\xi)(x) = \xi(x + t/T) \quad \text{and} \quad D\xi = \frac{1}{T}\xi'.$$

In some situations it is convenient to use a slightly different normalization for which Φ is of finite order, but then the $\mathbb{R}$-action becomes more complicated. If K is compact, then $\mathrm{Aut}(K)$ is compact as well. In this case, there exists a finite subgroup $F \subseteq \mathrm{Aut}(K)$ with $\mathrm{Aut}(K) = F\,\mathrm{Aut}(K)_0$ (see [42, Theorem 6.36]) and we may choose a representative Φ_0 of $[\Phi] \in \pi_0(\mathrm{Aut}(K))$ in such a way that $\Phi_0 \in F$.

If $\Gamma_{\Phi,\Phi_0}\colon \mathcal{K}_{\Phi_0} \to \mathcal{K}_\Phi$ is a group bundle isomorphism specified by the smooth curve $\zeta\colon \mathbb{R} \to \mathrm{Aut}(K)$ satisfying

$$\zeta_{x+1} = \Phi^{-1}\zeta_x\Phi_0 \quad \text{for } x \in \mathbb{R}$$

(see (7.7)), then the $\mathbb{R}$-action on $\Gamma_c(\mathbb{R}/\mathbb{Z}, \mathcal{K}_{\Phi_0}) \cong \mathcal{L}_{\Phi_0}(K)$ takes the form

$$\widetilde{\alpha}_t(\xi)(x) = \zeta_x^{-1}\zeta_{x+t/T}\xi(x + t/T) \quad \text{for } \xi \in \mathcal{L}_{\Phi_0}(K).$$

On the Lie algebra level we obtain the corresponding derivation given by

$$\widetilde{D}\xi = \frac{1}{T}(\xi' + \delta^l(\zeta)\xi),$$

where

$$\delta^l(\zeta)\colon \mathbb{R} \to \mathbf{L}(\mathrm{Aut}(K)) = \mathrm{der}(\mathfrak{k}), \quad \delta^l(\zeta)_x = \frac{d}{dt}\Big|_{t=0} \zeta_x^{-1}\zeta_{t+x}$$

is the left logarithmic derivative of ζ. Identifying $\mathfrak{k}$ via the adjoint representation with $\mathrm{der}(\mathfrak{k})$, we obtain a smooth curve $A\colon \mathbb{R} \to \mathfrak{k}$ with $\mathrm{ad} \circ A = \delta^l(\zeta)$ for which

$$\widetilde{D}\xi = \frac{1}{T}(\xi' + [A, \xi]). \tag{7.10}$$

Note that A belongs to the twisted loop algebra $\mathcal{L}_{\varphi_0}(\mathfrak{k})$; since $\zeta_{x+1} = \Phi^{-1}\zeta_x\Phi_0$, we have

$$\zeta_{x+1}^{-1}\zeta_{x+1+t} = \Phi_0^{-1}(\zeta_x^{-1}\zeta_{x+t})\Phi_0,$$

and hence

$$\delta^l(\zeta)_{x+1} = \varphi_0^{-1}\delta^l(\zeta)_x\varphi_0.$$

It follows that the curve A satisfies

$$A_{x+1} = \varphi_0^{-1}A_x,$$

so that $A \in \mathcal{L}_{\varphi_0}(\mathfrak{k})$.

Remark 7.14. We denote by $\mathcal{L}_\Phi^\sharp(K)_c$ the central $\mathbb{T}$-extension of $\mathcal{L}_\Phi(K)$ corresponding to the Lie algebra cocycle

$$\omega(\xi, \eta) = \frac{c}{2\pi}\int_0^1 \kappa(\xi', \eta)dt, \quad c \in \mathbb{Z}$$

with period group $2\pi c\mathbb{Z}$ (see the discussion in Section 7.2). If the central charge c is 1, we omit the subscript and simply write $\mathcal{L}_\Phi^\sharp(K)$. Since the Lie algebra $\mathcal{L}_\varphi(\mathfrak{k})$ of $\mathcal{L}_\Phi(K)$ is perfect [62, Theorem VI.3] implies that the $\mathbb{R}$-action α on $\mathcal{L}_\Phi(K)$ lifts to a smooth $\mathbb{R}$-action $\alpha^\sharp$ on $\mathcal{L}_\Phi^\sharp(K)_c$, and we obtain a double extension of the form

$$\widehat{\mathcal{L}}_\Phi(K)_c \cong \mathcal{L}_\Phi^\sharp(K)_c \rtimes_{\alpha^\sharp} \mathbb{R}.$$

The c-fold cover $\mathbb{T} \twoheadrightarrow \mathbb{T}\colon z \mapsto z^c$ extends to a c-fold cover $\mathcal{L}_\Phi^\sharp(K) \twoheadrightarrow \mathcal{L}_\Phi^\sharp(K)_c$, for which the following diagram commutes:

$$\begin{tikzcd}
\mathbb{T} \arrow[r] \arrow[d, "z^c"'] & \mathcal{L}_\Phi^\sharp(K) \arrow[r] \arrow[d] & \mathcal{L}_\Phi(K) \arrow[d, "\mathrm{id}"] \\
\mathbb{T} \arrow[r] & \mathcal{L}_\Phi^\sharp(K)_c \arrow[r] & \mathcal{L}_\Phi(K).
\end{tikzcd}$$

Using this covering map, we can identify the representations of $\mathcal{L}_\Phi(K)_c$ with those representations of $\mathcal{L}_\Phi(K)$ for which the roots $\{z \in \mathbb{T}; z^c = 1\} \subseteq \mathbb{T}$ of order c act trivially.

7.4 Localization at the group level

To obtain the localization result at the group level, we need the following factorization lemma.

Lemma 7.15. *Let $r\colon G \to H$ be an open, surjective morphism of locally exponential Lie groups, and let $R\colon G \to U$ be a continuous homomorphism of topological groups such that*

$$\mathbf{L}(\ker R) := \{x \in \mathfrak{g} : \exp(\mathbb{R}x) \subseteq \ker R\} \supseteq \ker \mathbf{L}(r) = \mathbf{L}(\ker r).$$

Then, R factors through a continuous homomorphism $\overline{R}\colon G/(\ker r)_0 \to U$ and r induces a covering morphism $G/(\ker r)_0 \to H$ of Lie groups.

Proof. In view of [71, Proposition IV.3.4] (see [30] for a complete proof), $N := \ker r$ is a closed, locally exponential Lie subgroup of G. In particular, its identity component N_0 is open in N, so that the isomorphism $G/N \to H$ of locally exponential Lie groups leads to a covering morphism $G/N_0 \to H$ ([71, Theorem IV.3.5]). For every $x \in \mathbf{L}(N)$, we have $\exp(\mathbb{R}x) \subseteq \ker R$, so that $N_0 = \langle \exp \mathbf{L}(N) \rangle \subseteq \ker R$. Therefore, R factors through G/N_0. ■

Lemma 7.16. *Let $S \subseteq M$ be a closed, 1-dimensional submanifold and suppose that the fibers of $\mathcal{K}|_S \to S$ are 1-connected, semisimple Lie groups. Then, $\Gamma_c(S, \mathcal{K})$ is 1-connected.*

For $S \cong \mathbb{R}/T\mathbb{Z} \cong \mathbb{S}^1$, it follows in particular that, for a 1-connected Lie group K and an automorphism $\Phi \in \mathrm{Aut}(K)$, the twisted loop group.

$$\mathcal{L}_\Phi^T(K) := \{\xi \in C^\infty(\mathbb{R}, K) : (\forall t \in \mathbb{R})\, \xi(t+T) = \Phi^{-1}(\xi(t))\} \tag{7.11}$$

is 1-connected.

Proof. If S has connected components $(S_j)_{j\in J}$ with typical fiber K_j of $\mathcal{K}|_{S_j}$, then

$$\Gamma_c(S, \mathcal{K}) \cong {\prod}'_{j\in J} \Gamma_c(S_j, \mathcal{K}). \tag{7.12}$$

(We refer to [26, Proposition 7.3] for a discussion of weak direct products of Lie groups.)

If $S_j \simeq \mathbb{S}^1$, then $\Gamma_c(S_j, \mathcal{K})$ is isomorphic to the twisted loop group $\mathcal{L}_{\Phi_j}^T(K_j)$, where Φ_j is an automorphism of K_j. Since $\pi_0(K_j)$, $\pi_1(K_j)$ vanish, $\pi_2(K_j)$ vanishes as well[1]. The long exact sequence of homotopy groups corresponding to the

[1] Since K_j is homotopy equivalent to a maximal compact subgroup, this follows from Cartan's theorem [64, Theorem 3.7].

Serre fibration $\mathrm{ev}_0\colon \mathcal{L}^T_{\Phi_j}(K_j) \to K_j$ thus yields an isomorphism between the homotopy groups π_0 and π_1 of $\mathcal{L}^T_{\Phi_j}(K_j)$ and $\mathcal{L}^T_{\Phi_j}(K_j)_* := \ker(\mathrm{ev}_0)$. Since the inclusion $\mathcal{L}^T_{\Phi_j}(K_j)_* \hookrightarrow \mathcal{L}^T_{\Phi_j}(K_j)_{*,\mathrm{ct}}$ into the group of continuous, based, twisted loops is a homotopy equivalence by [84, Corollary 3.4], and since $\pi_m(\mathcal{L}^T_{\Phi_j}(K_j)_{*,\mathrm{ct}}) \simeq \pi_m(\Omega K_j) \simeq \pi_{m+1}(K_j)$ for $m \in \mathbb{N}_0$ (cf. [84, page 391]), we conclude that $\mathcal{L}^T_{\Phi_j}(K_j)$ is 1-connected.

If $S_j \simeq \mathbb{R}$, then $\Gamma_c(S_j, \mathcal{K}) \simeq C^\infty_c(\mathbb{R}, K_j)$ is 1-connected by [70, Theorem A.10]. From [28, Proposition 3.3], we then conclude that the locally exponential Lie group (7.12) is 1-connected. ∎

With these topological considerations out of the way, we now complete the proof of the localization theorem.

Proof of Theorem 7.1. In Lemma 7.13, we showed that the projective positive energy representation $\mathrm{d}\rho$ of $\Gamma_c(M, \mathfrak{K})$ factors through the restriction map

$$r^{\mathfrak{k}}_S\colon \Gamma_c(M, \mathfrak{K}) \to \Gamma_c(S, \mathfrak{K}),$$

so it remains to prove the corresponding factorization on the group level. For this, apply Lemma 7.15 to the locally exponential Lie groups $G = \widetilde{\Gamma}_c(M, \mathcal{K})_0$ and $H = \Gamma_c(S, \mathcal{K})$ (which are both 1-connected by Lemma 7.16), and the topological group

$$U = \mathrm{PU}(\mathcal{H}).$$

The homomorphism r is the homomorphism $\tilde{r}_S\colon \widetilde{\Gamma}_c(M, \mathcal{K})_0 \to \Gamma_c(S, \mathcal{K})$, induced by the restriction $r_S\colon \Gamma_c(M, \mathcal{K})_0 \to \Gamma_c(S, \mathcal{K})$, and R is the projective representation $\bar{\rho}\colon \widetilde{\Gamma}_c(M, \mathcal{K})_0 \to \mathrm{PU}(\mathcal{H})$. We conclude that $\bar{\rho}$ factors through a projective positive energy representation of the 1-connected Lie group $\Gamma_c(S, \mathcal{K})$.

Since every representation of $\Gamma_c(M, \mathcal{K})_0$ defines by pullback a representation of its simply connected covering, the assertion also follows for representations of this group. This concludes the proof of the theorem. ∎

7.5 Localization for equivariant representations

In this section we extend the localization Theorem 7.1 to the *equivariant* setting, where the action of $\mathbb{R}$ on M is replaced by a smooth action of a Lie group P on M. The positive energy condition (cf. Section 3.2) then refers not to an $\mathbb{R}$-action, but to the *positive energy cone* $\mathcal{C} \subseteq \mathfrak{p}$ inside the Lie algebra $\mathfrak{p}$ of P.

Let M be a manifold, let P be a Lie group acting smoothly on M, and let $\mathcal{K} \to M$ be a bundle of 1-connected, semisimple Lie groups that is equipped with a lift of this action. We denote the P-action on M by $\gamma_M\colon P \to \mathrm{Diff}(M)$, its lift to $\mathcal{K}$ by

$\gamma\colon P \to \mathrm{Aut}(\mathcal{K})$, and the corresponding action on the compactly supported gauge group by $\alpha\colon P \to \mathrm{Aut}(\Gamma_c(M, \mathcal{K}))$. On the infinitesimal level, the action of P on M gives rise to the action $\mathbf{v}_M\colon \mathfrak{p} \to \mathcal{V}(M)$, $p \mapsto \mathbf{v}_M^p$ of the Lie algebra $\mathfrak{p} := \mathbf{L}(P)$.

Let $(\bar{\rho}, \mathcal{H})$ be a smooth, projective, positive energy representation of the semidirect product $\Gamma_c(M, \mathcal{K}) \rtimes_\alpha P$ (cf. Definition 3.5), with positive energy cone $\mathcal{C} \subseteq \mathfrak{p}$.

Definition 7.17. The *fixed point set* $\Sigma \subseteq M$ of the positive energy cone $\mathcal{C} \subseteq \mathfrak{p}$ (a closed convex invariant cone in $\mathfrak{p}$) is defined as

$$\Sigma := \big\{m \in M : (\forall p \in \mathcal{C})\mathbf{v}_M^p(m) = 0\big\}.$$

Since the positive energy cone $\mathcal{C}$ is Ad_P-invariant, its fixed point set Σ is a closed, P-invariant subset of M. In the following we first consider the fixed-point-free scenario $\Sigma = \emptyset$, and return to the general case in [49, Part II].

Definition 7.18. Let $\bar{\rho}$ be a smooth, projective, unitary representation of $\Gamma_c(M, \mathcal{K})$. The *support* of $\bar{\rho}$, denoted $\mathrm{supp}(\bar{\rho})$, is defined as the complement of the union of all open subsets $U \subseteq M$ for which the kernel of $\bar{\rho}$ contains the normal subgroup $\Gamma_c(U, \mathcal{K})$. Similarly, the *support* of $\mathtt{d}\rho$ is the complement of the union of all open sets $U \subseteq M$ such that the kernel of $\mathtt{d}\rho$ contains $\Gamma_c(U, \mathfrak{K})$.

Note that the support is a closed subset of M. If the representation $\bar{\rho}$ extends to the semidirect product $\Gamma_c(M, \mathcal{K}) \rtimes_\alpha P$, then the support of $\bar{\rho}$ is invariant under the action of P on M. This leads to severe restrictions for positive energy representations.

Theorem 7.19 (Equivariant localization theorem). *Let $(\bar{\rho}, \mathcal{H})$ be a smooth, projective, positive energy representation of $\Gamma_c(M, \mathcal{K})_0 \rtimes_\alpha P$, and suppose that $\mathcal{C}$ has no fixed points. Then, there exists a 1-dimensional, P-equivariantly embedded submanifold $S \subseteq M$ such that $\bar{\rho}$ factors through the restriction homomorphism*

$$r_S\colon \Gamma_c(M, \mathcal{K})_0 \to \Gamma_c(S, \mathcal{K}).$$

Remark 7.20 (Equivariant localization for the simply connected cover). Since the P-action on $\Gamma_c(M, \mathcal{K})$ preserves the identity component $\Gamma_c(M, \mathcal{K})_0$, it lifts to the simply connected cover $\widetilde{\Gamma}_c(M, \mathcal{K})_0$. In this context the same result remains valid: every smooth, projective, positive energy representation $\bar{\rho}$ of $\widetilde{\Gamma}_c(M, \mathcal{K}) \rtimes_\alpha P$ factors through the homomorphism $\widetilde{r}_S\colon \widetilde{\Gamma}_c(M, \mathcal{K})_0 \to \Gamma_c(S, \mathcal{K})$ obtained by composing the restriction r_S with the covering map.

Proof. For every $p \in \mathcal{C}$, let $U_p \subseteq M$ be the open set of points in M where $\mathbf{v}_M^p$ is non-vanishing. Applying Lemma 7.13 to the manifold U_p, with the gauge group $\Gamma_c(U_p, \mathcal{K})$ and the $\mathbb{R}$-action $\alpha_p(t) := \alpha(\exp(tp))$, one finds an embedded, 1-dimensional submanifold $S_p \subseteq U_p$ such that the projective Lie algebra representation $\mathtt{d}\rho$ factors through the restriction map $r_{S_p}^{\mathfrak{k}}\colon \Gamma_c(U_p, \mathfrak{K}) \to \Gamma_c(S_p, \mathfrak{K})$. The support of

$\mathtt{d}\rho|_{\Gamma_c(U_p,\mathfrak{K})}$ is thus contained in S_p. It actually equals S_p because the cocycle on $\Gamma_c(U_p, \mathfrak{K})$ is given by a measure with support S_p. The sets S_p and $S_{p'}$ therefore coincide on $U_p \cap U_{p'}$, so the union $S = \bigcup_{p\in\mathcal{C}} S_p$ is a 1-dimensional, closed embedded submanifold of M. Here we use that the U_p cover M because $\mathcal{C}$ has no common fixed point. Since $gS_p = S_{\mathrm{Ad}_g(p)}$ for every $g \in P$, the union S is P-invariant.

Let $I_S := \{\xi \in \Gamma_c(M, \mathfrak{K}); \xi|_S = 0\}$ be the vanishing ideal of S in $\Gamma_c(M, \mathfrak{K})$. Since any $\xi \in I_S$ can be written as a finite sum of $\xi_p \in I_{S_p} \subseteq \Gamma_c(U_p, \mathfrak{K})$, and since the restriction of $\mathtt{d}\rho$ to $\Gamma_c(U_p, \mathfrak{K})$ vanishes on I_{S_p}, we conclude that $\mathtt{d}\rho$ vanishes on I_S. From Lemma 7.15 and Lemma 7.16, we then find (as in the proof of Theorem 7.1) that $\bar{\rho}$ factors through the restriction $\Gamma_c(M, \mathcal{K})_0 \to \Gamma_c(S, \mathcal{K})$ and that the corresponding assertion holds for representations of the covering group $\widetilde{\Gamma}_c(M, \mathcal{K})_0$. ■

The building blocks for the positive energy representations therefore come from actions of P on 1-dimensional manifolds on which $\mathcal{C}$ has no fixed point. According to the classification of hyperplane subalgebras of finite-dimensional Lie algebras [40, 41], an effective action of a connected finite-dimensional Lie group P on a simply connected one-dimensional manifold is of one of the following 3 types:

- the action of $P = \mathbb{R}$ on the line $\mathbb{R}$,
- the action of the affine group $P = \mathrm{Aff}(\mathbb{R})_0$ on the real line $\mathbb{R}$,
- the action of $P = \widetilde{\mathrm{SL}}(2, \mathbb{R})$ on the real line $\mathbb{R}$, considered as the simply connected cover of $\mathbb{P}_1(\mathbb{R}) \cong \mathbb{S}^1$.

In the infinite-dimensional context, the action of the simply connected covering group $P = \widetilde{\mathrm{Diff}}_+(\mathbb{S}^1)$ on $\mathbb{R} \cong \widetilde{\mathbb{S}}^1$ is a natural example.

Chapter 8

The classification for M compact

If the flow γ_M on M has no fixed points, the localization Theorem 7.1 reduces the classification of projective positive energy representations of the identity component $\Gamma_c(M, \mathcal{K})_0$ of the compactly supported gauge group to the situation where the base manifold is a closed, embedded, flow-invariant submanifold $S \subseteq M$ of dimension one.

The connected components of S are either diffeomorphic to $\mathbb{R}$ (for a non-periodic orbit), or to $\mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$ (for a periodic orbit). Since a gauge group on $\mathbb{R}$ is equivariantly isomorphic to $C_c^\infty(\mathbb{R}, K)$ (with $\mathbb{R}$ acting by translation), and a gauge group on $\mathbb{S}^1$ is equivariantly isomorphic to a twisted loop group (with $\mathbb{R}$ acting by rotation), the gauge group on S is a product of twisted loop groups and groups of the form $C_c^\infty(\mathbb{R}, K)$.

In this chapter, we describe the complete classification of positive energy representations for twisted loop groups. This leads to a classification of the positive energy representations of $\Gamma_c(M, \mathcal{K})_0$ for which the one-dimensional submanifold S is compact. Since this is automatically the case if M is compact, we arrive at a complete classification in this setting.

8.1 Positive energy representation of twisted loop groups

We now describe the complete classification of projective positive energy representations for twisted loop groups.

In this section K denotes a 1-connected compact (hence semisimple) Lie group, $\Phi \in \mathrm{Aut}(K)$ is an automorphism of *finite order* $\Phi^N = \mathrm{id}_K$, and $\varphi = \mathbf{L}(\Phi) \in \mathrm{Aut}(\mathfrak{k})$ is the corresponding automorphism of $\mathfrak{k}$. We further assume that the invariant form κ on $\mathfrak{k}$ is normalized in such a way that

$$\kappa(i\alpha^\vee, i\alpha^\vee) = 2$$

for all long roots α. We denote the (twisted) loop groups and algebras by $\mathcal{L}_\Phi(K)$ and $\mathcal{L}_\varphi(\mathfrak{k})$ respectively, as in (7.8) and (7.9). The (double) extensions with $c = 1$ are denoted by $\mathcal{L}_\Phi^\sharp(K)$ and $\widehat{\mathcal{L}}_\Phi(K)$, cf. Remark 7.14.

Definition 8.1. We call a positive energy representation $(\rho, \mathcal{H})$ of $\widehat{\mathcal{L}}_\Phi(K)$

(i) *basic* if $U_t := \rho(\exp tD) \subseteq \rho(\mathcal{L}_\Phi^\sharp(K))''$ for every $t \in \mathbb{R}$,

(ii) *periodic* if $U_T = \mathbf{1}$ for some $T > 0$.

Note that if ρ is minimal (Definition 3.8), then it is in particular basic.

Remark 8.2. If $(\rho, \mathcal{H})$ is periodic with $U_T = \mathbf{1}$, then [79, Lemma 5.1] implies that the space $\mathcal{H}^\infty$ of smooth vectors is invariant under the operators

$$p_n(v) := \frac{1}{T}\int_0^T e^{-2\pi i n t/T} U_t v \, dt.$$

These are orthogonal projections onto the eigenvectors of $H = i\,\mathtt{d}\rho(D)$ for the eigenvalues $-2\pi n/T$, $n \in \mathbb{Z}$.

Recall from Section 7.3 that with the identification $\mathbb{S}^1 \simeq \mathbb{R}/\mathbb{Z}$ and with $\Phi^N = \mathrm{id}_K$, we have

$$D(\xi) = \frac{1}{T}(\xi' + [A, \xi]).$$

It will be convenient to introduce the derivative

$$\mathbf{d}(\xi) = \frac{d}{dx}\xi, \quad \text{so that } D = \frac{1}{T}(\mathbf{d} + \mathrm{ad}_A). \tag{8.1}$$

Remark 8.3 (Independence of positive energy condition from lift of $\mathbb{R}$-action). From Proposition 6.32, applied to $M = \mathbb{R}/\mathbb{Z}$, it follows that a smooth representation of $\mathcal{L}^\sharp_\Phi(K)$ is of positive energy with respect to the derivation D if and only if it is of positive energy with respect to the derivation $\mathbf{d}$. Then, the representation is semibounded in the sense of Definition 6.31. As this holds for $D = \frac{1}{T}(\mathbf{d} + \mathrm{ad}_L)$ with any $T > 0$ and $L \in \mathcal{L}_\varphi(\mathfrak{k})$, the positive energy condition does not depend on the choice of the vector field $\mathbf{v}$ on $\mathcal{K}_\Phi = \mathbb{R} \times_\Phi K$ lifting the vector field $\mathbf{v}_M = \frac{1}{T}\frac{d}{dt}$ on $\mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$.

From $\Phi^N = \mathrm{id}_K$, we immediately derive that $\varphi^N = \mathrm{id}_{\mathfrak{k}}$. For $\widehat{\mathfrak{g}} = \widehat{\mathcal{L}}_\varphi(\mathfrak{k})$, we define the canonical triangular decomposition by

$$\widehat{\mathfrak{g}}_{\mathbb{C}} = \widehat{\mathfrak{g}}_{\mathbb{C}}^+ \oplus \widehat{\mathfrak{g}}_{\mathbb{C}}^0 \oplus \widehat{\mathfrak{g}}_{\mathbb{C}}^-$$

with

$$\widehat{\mathfrak{g}}_{\mathbb{C}}^\pm := \overline{\sum_{\pm n > 0} \widehat{\mathfrak{g}}_{\mathbb{C}}^n},$$

where

$$\widehat{\mathfrak{g}}_{\mathbb{C}}^n := \ker\left(\mathbf{d} + \frac{2\pi i n}{N}\mathbf{1}\right) \quad \text{for } n \in \mathbb{Z}$$

(see (A.1) in the appendix). For $\mathfrak{g} = \mathcal{L}_\varphi(\mathfrak{k})$, we have the analogous decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}^+ \oplus \mathfrak{g}_{\mathbb{C}}^0 \oplus \mathfrak{g}_{\mathbb{C}}^-$$

with

$$\mathfrak{g}_{\mathbb{C}}^+ = \widehat{\mathfrak{g}}_{\mathbb{C}}^+ \quad \text{and} \quad \mathfrak{g}_{\mathbb{C}}^- = \widehat{\mathfrak{g}}_{\mathbb{C}}^-.$$

For a smooth unitary representation of $\widehat{\mathcal{L}}_\Phi(K)$, we define its *minimal energy subspace* with respect to $i\,\mathtt{d}\rho(\mathbf{d})$ by

$$\mathcal{E} := \overline{(\mathcal{H}^\infty)^{\mathfrak{g}_{\mathbb{C}}^-}} \quad \text{for } (\mathcal{H}^\infty)^{\mathfrak{g}_{\mathbb{C}}^-} := \left\{\psi \in \mathcal{H}^\infty : (\forall x \in \mathfrak{g}_{\mathbb{C}}^-)\,\mathtt{d}\rho(x)\psi = 0\right\}. \tag{8.2}$$

Lemma 8.4. *For every smooth positive energy representation* $(\rho, \mathcal{H})$ *of* $\widehat{\mathcal{L}}_{\Phi}(K)$*, the subspace* $\mathcal{E}$ *is generating for* $\mathcal{L}_{\Phi}^{\sharp}(K)$.

Proof. Note that $\mathcal{E}$ is defined in terms of $\rho|_{\mathcal{L}_{\Phi}^{\sharp}(K)}$. In view of Corollary 3.9 and the fact that $\alpha_N = \mathrm{id}_{\mathcal{L}_{\Phi}(K)}$, we may therefore assume, without loss of generality, that ρ is periodic.

Let $\mathcal{H}' \subseteq \mathcal{H}$ denote the smallest closed $\mathcal{L}_{\Phi}^{\sharp}(K)$-invariant subspace containing $\mathcal{E}$. Then, $\mathcal{H}'$ is U-invariant, and the representation of $\widehat{\mathcal{L}}_{\Phi}(K)$ on $(\mathcal{H}')^{\perp}$ is also a positive energy representation. If $(\mathcal{H}')^{\perp} \neq \{0\}$, then its minimal energy subspace $\mathcal{F}$ is non-zero by Remark 8.2, and since it contains smooth vectors, we obtain a contradiction to $\mathcal{F} \perp \mathcal{E}$. Therefore, $(\mathcal{H}')^{\perp} = \{0\}$ and the subspace $\mathcal{E}$ is $\mathcal{L}_{\Phi}^{\sharp}(K)$-generating. ■

We now abbreviate

$$G := \mathcal{L}_{\Phi}(K), \quad \widehat{G} := \widehat{\mathcal{L}}_{\Phi}(K) \quad \text{and} \quad G^{\sharp} := \mathcal{L}_{\Phi}^{\sharp}(K) \tag{8.3}$$

and denote the corresponding groups of fixed points by

$$L = K^{\Phi}, \quad \widehat{L} := \mathrm{Fix}_{\alpha}(\widehat{G}) \cong \mathbb{T} \times K^{\Phi} \times \mathbb{R}, \quad L^{\sharp} := \widehat{L} \cap G^{\sharp} \cong \mathbb{T} \times L.$$

From the discussion in [79, Section 5.2 and Appendix C], it follows that the homogeneous space $G/L \cong \widehat{G}/\widehat{L} \cong G^{\sharp}/L^{\sharp}$ carries the structure of a complex Fréchet manifold on which $\widehat{G}$ acts analytically, and the tangent space in the base point is isomorphic to the quotient space $\widehat{\mathfrak{g}}_{\mathbb{C}}/(\widehat{\mathfrak{g}}_{\mathbb{C}}^{0} + \mathfrak{g}_{\mathbb{C}}^{+})$. For any bounded unitary representation (ρ^{L}, E) of $\widehat{L}$, we then obtain a holomorphic vector bundle $\mathbb{E} := \widehat{G} \times_{\widehat{L}} E$ over $\widehat{G}/\widehat{L}$. We write $\Gamma_{\mathrm{hol}}(G/L, \mathbb{E})$ for the space of holomorphic sections of $\mathbb{E}$.

Definition 8.5 (Holomorphically induced representations). A unitary representation $(\rho, \mathcal{H})$ of $\widehat{G}$ is said to be *holomorphically induced* from (ρ^{L}, E) if there exists a G-equivariant linear injection $\Psi\colon \mathcal{H} \to \Gamma_{\mathrm{hol}}(G/L, \mathbb{E})$ such that the adjoint of the evaluation map

$$\mathrm{ev}_{\mathbf{1}\widehat{L}}\colon \mathcal{H} \to E = \mathbb{E}_{\mathbf{1}\widehat{L}}$$

defines an isometry $\mathrm{ev}_{\mathbf{1}\widehat{L}}^{*}\colon E \hookrightarrow \mathcal{H}$. If there exists a unitary representation $(\rho, \mathcal{H})$ holomorphically induced from (ρ^{L}, E), then it is uniquely determined [77, Definition 3.10]. We then call the representation (ρ^{L}, E) of $\widehat{L}$ *(holomorphically) inducible*. The same statements apply to $G^{\sharp}$ and $L^{\sharp}$.

Let $\mathfrak{t}^{\circ} \subseteq \mathfrak{k}^{\varphi}$ be maximal abelian, so that

$$\mathfrak{t} = \mathbb{R}C \oplus \mathfrak{t}^{\circ} \oplus \mathbb{R}\mathbf{d}$$

is maximal abelian in $\widehat{\mathcal{L}}_{\varphi}(\mathfrak{k})$. We write $T^{\sharp} = \mathbb{T} \times T^{\circ}$ for the torus group with Lie algebra $\mathfrak{t}^{\sharp} = \mathbb{R}C \oplus \mathfrak{t}^{\circ}$. Let Δ^{+} be a positive system for the affine Kac–Moody Lie algebra $\widehat{\mathcal{L}}_{\varphi}(\mathfrak{k}_{\mathbb{C}})$ with respect to the Cartan subalgebra $\mathfrak{t}_{\mathbb{C}}$ such that, for all $\alpha \in \Delta$, the relation $\alpha(i\mathbf{d}) > 0$ implies $\alpha \in \Delta^{+}$ (cf. Appendix A and [38, Chapter X]).

Proposition 8.6. *A bounded representation* (ρ^L, E) *of*

$$L^\sharp = \exp(\mathbb{R}C) \times L \cong \mathbb{T} \times L$$

is holomorphically inducible if and only if

$$\mathtt{d}\rho^L([z^*, z]) \geq 0 \quad \text{for all } z \in \mathfrak{g}_{\mathbb{C}}^n,\ n > 0. \tag{8.4}$$

In particular, the irreducible, holomorphically inducible representations of $L^\sharp$ *are parametrized by the anti-dominant, integral weights* λ *of the form*

$$\lambda = (\lambda(C), \lambda_0, 0) \in i\mathfrak{t}^* \tag{8.5}$$

of the affine Kac–Moody Lie algebra $\widehat{\mathcal{L}}_\varphi(\mathfrak{k}_{\mathbb{C}})$ *with respect to the Cartan subalgebra* $\mathfrak{t}_{\mathbb{C}}$ *and the positive system* Δ^+*. Here the central charge* $c := -i\lambda(C)$ *is contained in* $\mathbb{N}_0$*, and for every central charge* c *there are only finitely many such representations with* $\lambda(C) = ic$

Proof. Since the representation ρ^L of the compact group $L^\sharp$ is a direct sum of irreducible representations, we may assume that it is a representation with lowest weight λ with respect to the positive system of roots Δ_0^+ of $(\mathfrak{k}_{\mathbb{C}}^\varphi, \mathfrak{t}_{\mathbb{C}}^\circ)$.

The necessity of (8.4) follows from [79, Proposition 5.6]. To show that λ is anti-dominant for $(\widehat{\mathcal{L}}_\varphi(\mathfrak{k}_{\mathbb{C}}), \mathfrak{t}_{\mathbb{C}}, \Delta^+)$, we need that $\lambda((\alpha, n)^\vee) \leq 0$ for $(\alpha, n) \in \Delta_+$. We distinguish the cases $n > 0$ and $n = 0$. If $n > 0$, we use (A.2) in Appendix A, to see that (8.4) implies $\lambda((\alpha, n)^\vee) \leq 0$ for $0 \neq \alpha \in \Delta_0$, the root system of $(\mathfrak{k}_{\mathbb{C}}^\varphi, \mathfrak{t}_{\mathbb{C}}^\circ)$. For $n = 0$, the assertion follows from $\lambda(\beta^\vee) \leq 0$ for $\beta \in \Delta_0^+$.

Next, we prove the integrality of λ. For $\alpha \neq 0$, the relation

$$\exp(2\pi i (\alpha, n)^\vee) = \mathbf{1} \tag{8.6}$$

in $T^\sharp$ follows from the fact that

$$\mathfrak{k}(\alpha, n) := \operatorname{span}_{\mathbb{R}}\{x \otimes e_n - x^* \otimes e_{-n}, i(x \otimes e_n + x^* \otimes e_{-n}), i(\alpha, n)^\vee\} \cong \mathfrak{su}(2, \mathbb{C}).$$

Since λ corresponds to a character of $T^\sharp$, the relation (8.6) implies that

$$\exp(2\pi i \lambda((\alpha, n)^\vee)) = 1,$$

so that $\lambda((\alpha, n)^\vee) \in \mathbb{Z}$. We conclude that λ is anti-dominant integral.

We now argue that every integral, anti-dominant weight λ as in (8.5) specifies a holomorphically inducible representation (ρ^L, E_λ) of $L^\sharp$. In fact, the unitarity of the corresponding lowest weight module $L(\lambda, -\Delta^+)$ of the affine Kac–Moody algebra $\widehat{\mathcal{L}}_\varphi(\mathfrak{k}_{\mathbb{C}})$ ([54, Theorem 11.7]) can be used as in the proof of [79, Theorem 5.10] to see with [79, Theorem C.6] that (ρ^L, E_λ) is holomorphically inducible. ∎

The following theorem is well-known for untwisted loop groups $\mathcal{L}(K)$, but we did not find an appropriate statement in the literature for the twisted case. It requires some refined methods based on holomorphic induction which we draw from [79].

Theorem 8.7. *If K is 1-connected and $(\rho, \mathcal{H})$ is a positive energy representation of $\widehat{\mathcal{L}}_\Phi(K)$, then its restriction to $\mathcal{L}_\Phi^\sharp(K)$ is a finite direct sum of factor representations of type I, hence, in particular, a direct sum of irreducible representations.*

Proof. Since the assertion only refers to the restriction $\rho|_{\mathcal{L}_\Phi^\sharp(K)}$, we may assume, without loss of generality, that $\rho = \rho_0$ is minimal (Definition 3.8 and Theorem 3.7). Then, $\alpha_N = \mathrm{id}_{\mathcal{L}_\Phi(K)}$ implies that ρ is periodic and that every subrepresentation is generated by the fixed points of

$$U_t = \rho(\mathbf{1}, t) = e^{-itH}.$$

In view of Remark 8.2, the space $\mathcal{H}^\infty$ of smooth vectors for $\widehat{G} = \widehat{\mathcal{L}}_\Phi(K)$ (see (8.3)) is invariant under the projections $p_n : \mathcal{H} \to \mathcal{H}_n$ onto the eigenspaces of $H = i\,\mathtt{d}\rho(D)$. Since $\rho = \rho_0$ is minimal, we have $\mathcal{H}_n = \{0\}$ for $n < 0$ and $\mathcal{H}_0$ is generating. Now $\mathcal{H}_0 \cap \mathcal{H}^\infty$ is contained in $\mathcal{E}$, the closure of $(\mathcal{H}^\infty)^{\mathfrak{g}_{\mathbb{C}}^-}$ from (8.2). As the intersection $\mathcal{H}_0 \cap \mathcal{H}^\infty$ is dense in $\mathcal{H}_0$, we have $\mathcal{H}_0 \subseteq \mathcal{E}$.

Recall that

$$\widehat{L} = \mathrm{Fix}_\alpha(\widehat{G}) \cong L \times \mathbb{R} \cong \mathbb{T} \times K^\Phi \times \mathbb{R}.$$

As K^Φ is compact and $U_N = \mathbf{1}$ follows from $\varphi^N = \mathrm{id}_{\mathfrak{k}}$, $\rho(\widehat{L})$ is a compact subgroup of $\mathrm{U}(\mathcal{H})$. Hence, the $\widehat{L}$-invariant subspace $\mathcal{H}_0 \subseteq \mathcal{E}$ is a direct sum of finite-dimensional subrepresentations. In particular, it decomposes into isotypic components $\mathcal{E}_j := E_j \otimes \mathcal{M}_j$, $j \in J$, where $\mathcal{M}_j \cong B(E_j, \mathcal{H}_0)^{\widehat{L}}$ is the multiplicity space of the (finite-dimensional) irreducible representation (ρ_j^L, E_j). It also follows that the representation of $\widehat{L}$ on each $\mathcal{E}_j$ is semisimple in the algebraic sense and that the irreducible subrepresentations are of the form $E_j \otimes \psi$, $\psi \in \mathcal{M}_j$. As a consequence, every $\widehat{L}$-invariant subspace of $\mathcal{E}_j$ is of the form $E_j \otimes \mathcal{M}_j'$ for a linear subspace $\mathcal{M}_j' \subseteq \mathcal{M}_j$.

The dense subspace $(\mathcal{H}^\infty)^{\mathfrak{g}_{\mathbb{C}}^-}$ of $\mathcal{E}$ is invariant under the projections onto the isotypic components because they are given by integration over a compact group[1]. This implies that $\mathcal{E}_j \cap (\mathcal{H}^\infty)^{\mathfrak{g}_{\mathbb{C}}^-}$ is dense in $\mathcal{E}_j$. In view of the preceding discussion, we thus obtain

$$\mathcal{E}_j \cap (\mathcal{H}^\infty)^{\mathfrak{g}_{\mathbb{C}}^-} \cong E_j \otimes \mathcal{M}_j^\infty$$

for a dense linear subspace $\mathcal{M}_j^\infty \subseteq \mathcal{M}_j$. In view of Lemma 8.4, we now have to show that, for every $\psi \in \mathcal{M}_j^\infty$, the subspace $E_j \otimes \psi \subseteq \mathcal{E}$ generates an irreducible subrepresentation of

$$G^\sharp = \mathcal{L}_\Phi^\sharp(K).$$

[1] This follows by differentiation under the integral sign, see [30, Proposition 1.3.23].

For the untwisted case, i.e., $\Phi = \mathrm{id}_K$, this follows from [68, Proposition VII.1]. For the twisted case we have to invoke the machinery of holomorphic induction described in Definition 8.5. For the following argument, observe that $\mathcal{L}^\sharp_\Phi(K)$ is connected by Lemma 7.16. On the finite-dimensional subspace $E := E_j \otimes \psi \subseteq (\mathcal{H}^\infty)^{\mathfrak{g}_{\mathbb{C}}^-}$, the representation of $\widehat{L}$ is bounded. Hence, [79, Theorem C.3] implies that the $\widehat{G}$-subrepresentation $(\rho', \mathcal{H}')$ of $(\rho, \mathcal{H})$ generated by E is holomorphically induced from the $\widehat{L}$-representation (ρ^L, E). In view of [79, Theorem C.2], the irreducibility of (ρ^L, E) implies the irreducibility of $(\rho', \mathcal{H}')$.

We have seen in the proof of Proposition 8.6 that the holomorphically inducible irreducible representations ρ^L of $\widehat{L}$ are parametrized by a set of anti-dominant integral weights of an affine Kac–Moody algebra $\widehat{\mathcal{L}}_\psi(\mathfrak{k}_{\mathbb{C}})$ with a fixed central charge. This implies the finiteness of the possible types. ∎

The following corollary can be used to deal with gauge groups if the structure group K is not 1-connected. It covers in particular the case $K = \mathrm{Aut}(\mathfrak{k})$ that arises from structure groups of Lie algebra bundles $\mathfrak{K} \to \mathbb{S}^1$.

Corollary 8.8 (Non-connected fibers). *If K is a compact Lie group with simple Lie algebra and $(\rho, \mathcal{H})$ a positive energy representation of $\widehat{\mathcal{L}}_\Phi(K)$, then its restriction to $\mathcal{L}^\sharp_\Phi(K)$ is a finite direct sum of factor representations of type I, hence, in particular, a direct sum of irreducible representations.*

Proof. Since K is compact with simple Lie algebra, the groups $\pi_0(K)$ and $\pi_1(K)$ are finite. Therefore, the exact sequence

$$\mathbf{1} \to \pi_1(K)/\operatorname{im}(\pi_1(\Phi) - \mathrm{id}) \hookrightarrow \pi_0(\mathcal{L}_\Phi(K)) \twoheadrightarrow \pi_0(K)^\Phi \to \mathbf{1}$$

from [84, Remark 2.6 (a)] implies that $\pi_0(\mathcal{L}_\Phi(K))$ is finite. The identity component $\mathcal{L}_\Phi(K)_0$ is isomorphic to $\mathcal{L}_\Phi(\widetilde{K}_0)$, where $\widetilde{K}_0$ is the simply connected covering of the identity component K_0 of K. Now the assertion follows by combining Theorem 8.7 with Theorem C.1. ∎

Remark 8.9 (Explicit aspects of the Borchers–Arveson theorem).

(a) Let $(\rho, \mathcal{H})$ be a positive energy representation of $\widehat{\mathcal{L}}_\Phi(K)$ for which the restriction $\rho^\sharp$ to $\mathcal{L}^\sharp_\Phi(K)$ is isotypic. Then, the proof of Theorem 8.7 shows that $\rho^\sharp$ is holomorphically induced from $(\rho^L, \mathcal{E})$, where $\mathcal{E} \cong E \otimes \mathcal{M}$ and (ρ^L, E) is an irreducible representation of $L^\sharp$, and hence of K^Φ.

That the representation is basic, $U_{\mathbb{R}} \subseteq \rho(G^\sharp)''$, is equivalent to $U_{\mathbb{R}}$ commuting with the commutant $\rho(G^\sharp)'$. Since the restriction to $\mathcal{E}$ yields an isomorphism $\rho(G^\sharp)' \to \rho^L(L^\sharp)' = \rho^L(L)'$ ([79, Theorem C.2]) and $\mathcal{E}$ is invariant under $U_{\mathbb{R}}$, the inclusion $U_{\mathbb{R}} \subseteq (\rho(G^\sharp)')'$ is equivalent to

$$U_{\mathbb{R}}|_{\mathcal{E}} \subseteq (\rho^L(L)')' = B(E) \otimes \mathbf{1}.$$

Since $\widehat{L} = \mathbb{T} \times K^\Phi \times \mathbb{R}$, where K^Φ is considered as a subgroup of constant sections, we have $U_\mathbb{R}|_\mathcal{E} \subseteq \rho(\widehat{L})'$. The representation is therefore basic if and only if $U_\mathbb{R}|_\mathcal{E}$ is contained in

$$\rho(\widehat{L})' \cap \rho^L(L)'' = \mathbb{C}\mathbf{1},$$

that is, if and only if U acts on $\mathcal{E}$ by a character.

(b) We construct an example which is not basic, but which is factorial on $G^\sharp$. Let $(\rho, \mathcal{H})$ be an irreducible positive energy representation of $\widehat{G} = \widehat{\mathcal{L}}_\Phi(K)$. For any non-trivial character $\chi\colon \mathbb{R} \to \mathbb{T}$, the representation $\rho \oplus (\widehat{\chi} \otimes \rho)$ with $\widehat{\chi}(g,t) := \chi(t)$ is factorial on $G^\sharp$, but not on $\widehat{G}$.

8.2 The classification theorem for compact base manifolds

Let M be a manifold on which the flow γ_M has no fixed points, and let K be a compact, connected, simple Lie group. We now obtain a full classification of the projective positive energy representations of $\Gamma_c(M, \mathcal{K})_0$ in the case where M is compact, by combining the localization Theorem 7.1 with the results on twisted loop groups from Section 8.1.

8.2.1 One-dimensional manifolds with compact components

By Theorem 7.1 and Corollary 7.5, every projective positive energy representation of $\Gamma_c(M, \mathcal{K})_0$ factors through the gauge group $\Gamma_c(S, \widetilde{\mathcal{K}})$ of a 1-dimensional, $\mathbb{R}$-equivariantly closed embedded submanifold $S \subseteq M$. If S is compact, then it is the disjoint union of finitely many circles S_j on which $\mathbb{R}$ acts with period T_j.

In this section we assume that S is a (not necessarily finite) union of circles. The restricted gauge group $G := \Gamma_c(S, \widetilde{\mathcal{K}})$ is then a restricted direct product of twisted loop groups $\mathcal{L}_{\Phi_j}(\widetilde{K}_j)$, where $\widetilde{K}_j$ is the 1-connected cover of the structure group K_j of $\mathcal{K}|_{S_j}$. On the Lie algebra level, we have a direct sum of Lie algebras

$$\mathfrak{g} \cong \bigoplus_{j \in J} \mathcal{L}_{\varphi_j}(\mathfrak{k}_j).$$

As in (8.1), the infinitesimal generator D of the $\mathbb{R}$-action acts on $\xi \in \mathfrak{g}$ by

$$D(\xi) = \bigoplus_{j \in J} \frac{1}{T_j}(\mathbf{d}_j \xi_j + [A_j, \xi_j]),$$

where $A_j \in \mathcal{L}_{\varphi_j}(\mathfrak{k})$ is determined by the $\mathbb{R}$-action according to (7.10).

Let $(\mathrm{d}\rho, \mathcal{H})$ be a positive energy representation of $\mathfrak{g}^\sharp = \mathbb{R}C \oplus_\omega \mathfrak{g}$ with cocycle

$$\omega(\xi, \eta) = \sum_{j \in J} \frac{c_j}{2\pi} \int_0^1 \kappa(\xi_j', \eta_j)dt \quad \text{with } c_j \in \mathbb{N}_0.$$

For each j, $\mathrm{d}\rho$ restricts to a positive energy representation of the centrally extended twisted loop algebra $\mathcal{L}^\sharp_{\varphi_j}(\mathfrak{k}_j)$ with central charge c_j. By Proposition 8.6 and Remark 8.3 (cf. [94, Chapter 9] for the untwisted case), the irreducible positive energy representations of $\mathcal{L}^\sharp_{\varphi_j}(\mathfrak{k}_j)$ with central charge c_j are precisely the irreducible unitary lowest weight representations $(\mathrm{d}\rho_\lambda, \mathcal{H}_\lambda)$ with integral anti-dominant weight λ satisfying $\lambda(C) = ic_j$. Since there are finitely many of these, the representation can be written as a finite sum

$$\mathcal{H} = \bigoplus_\lambda \mathcal{H}^j_\lambda \otimes \mathcal{M}^j_\lambda \tag{8.7}$$

where the sum runs over the integral anti-dominant weights of $\mathcal{L}^\sharp_{\varphi_j}(\mathfrak{k}_j)$ with central charge c_j (cf. (8.5)) and $\mathcal{L}^\sharp_{\varphi_j}(\mathfrak{k}_j)$ acts trivially on the multiplicity space $\mathcal{M}^j_\lambda$ (Theorem 8.7).

Now suppose that $(\rho, \mathcal{H})$ is a positive energy factor representation of $G^\sharp$. Then, the restriction to a normal subgroup

$$G_j := \mathcal{L}_{\Phi_j}(\widetilde{K})$$

decomposes discretely with finitely many isotypes (Theorem 8.7). For a subset $F \subseteq J$, we denote the corresponding normal subgroup of G by

$$G_F := \bigoplus_{j\in F} \mathcal{L}_{\Phi_j}(\widetilde{K}_j).$$

Since G_j commutes with $G_{J\setminus\{j\}}$, the factoriality of ρ on $G^\sharp$ implies that the restriction of ρ to $G^\sharp_j$ is factorial as well. Hence, there is only one summand in (8.7), and we have

$$\mathcal{H} = \mathcal{H}^j_\lambda \otimes \mathcal{H}'$$

for some multiplicity space $\mathcal{H}'$. Although a priori we only have a single operator H for all components S_j, we now obtain an operator $\mathrm{d}\rho(\mathbf{d}_j)$ satisfying

$$[\mathrm{d}\rho(\mathbf{d}_j), \mathrm{d}\rho(\xi_i)] = \delta_{ij}\mathrm{d}\rho(\xi'_j)$$

from the minimal implementation[2] in Corollary 3.9.

Since

$$H' := H - \frac{i}{T_j}\mathrm{d}\rho(\mathbf{d}_j + A_j)$$

commutes with $\widehat{\mathcal{L}}_{\varphi_j}(\mathfrak{k})$, we obtain a positive energy representation on $\mathcal{H}'$ with Hamiltonian H', but now for the group $G^\sharp_{J\setminus\{j\}}$. Continuing this way, we obtain for each

[2]One could also use the Segal–Sugawara construction ([55, Section 3] and [32]), but this leads to a non-zero minimal eigenvalue; see Section 9.1.1 for more details.

$j \in J$ an integral anti-dominant weight λ_j of central charge c_j, and for each finite subset $F \subseteq J$ a tensor product decomposition

$$\rho = \rho_F \otimes \rho'_F, \quad \mathcal{H} = \mathcal{H}_F \otimes \mathcal{H}'_F \quad \text{with } \mathcal{H}_F := \bigotimes_{j \in F} \mathcal{H}_{\lambda_j} \tag{8.8}$$

into positive energy representations for the gauge groups $G^\sharp_F$ and $G^\sharp_{J \setminus F}$.

8.2.2 Compact base manifolds

For gauge groups over a compact base manifold M, we thus obtain the following classification result. It contains in particular Torresani's classification for linear flows on a torus; see [105] and [3, Section 5.4].

Theorem 8.10. *Let M be a compact manifold with a fixed point free $\mathbb{R}$-action γ_M, and let $\mathcal{K} \to M$ be a bundle of Lie groups with compact, simple, connected fibers. Let $\bar{\rho}\colon \Gamma(M, \mathcal{K})_0 \rtimes \mathbb{R} \to \mathrm{PU}(\mathcal{H})$ be a minimal projective positive energy representation with respect to a lift γ of the $\mathbb{R}$-action to $\mathcal{K}$. Then, there exist finitely many $\mathbb{R}$-orbits $S_j \subseteq M$, $j \in J$, with central charge $c_j \in \mathbb{N}_0$ such that $\bar{\rho}$ arises by factorization from an isotypic positive energy representation ρ_S of*

$$\widehat{G} = G^\sharp \rtimes \mathbb{R},$$

where

$$G := \Gamma(S, \widetilde{\mathcal{K}}) \simeq \prod_{j \in J} \mathcal{L}_{\Phi_j}(\widetilde{K}_j).$$

If ρ_S is irreducible, then

$$\mathcal{H} = \bigotimes_{j \in J} \mathcal{H}_{\lambda_j}$$

is a tensor product of lowest weight representations $(\rho_{\lambda_j}, \mathcal{H}_{\lambda_j})$ of the corresponding affine Kac–Moody group $\widehat{\mathcal{L}}_{\Phi_j}(\widetilde{K}_j)$, where λ_j is an integral anti-dominant weight of central charge c_j. On the level of the Lie algebra

$$\widehat{\mathfrak{g}} = \mathbb{R}C \times_\omega \Big(\bigoplus_{j \in J} \mathcal{L}_{\varphi_j}(\mathfrak{k}) \rtimes \mathbb{R}D\Big),$$

the central element acts by $d\rho(C) = i\mathbf{1}$, and the generator D acts by

$$d\rho(D) = \sum_{j \in J} \frac{1}{T_j} d\rho_{\lambda_j}(\mathbf{d}_j + A_j),$$

where $A_j \in \mathcal{L}_{\varphi_j}(\mathfrak{k})$ is specified by the $\mathbb{R}$-action on G.

Proof. Since M is compact, the $\mathbb{R}$-invariant, embedded, one-dimensional submanifold S is a union of finitely many periodic orbits. By Corollary 7.5, the projective positive energy representation $\bar{\rho}$ of $\Gamma(S, \mathcal{K})_0$ thus arises by factorization from a projective positive energy representation of $G = \Gamma(S, \widetilde{\mathcal{K}})$, which is trivial on the image Z of the diagonal group $Z_{[M]}$ in $Z_{[S]}$ (cf. Remark 7.4). It then follows from (8.8) and the discussion in Section 8.2.1 that every factorial positive energy representation is a multiple of a product of lowest weight representations as described above. The only thing left to check is that this representation restricts to a character on the image Z of $Z_{[M]}$ in G. Since Z is a subgroup of the central group $Z_{[S]} = \prod_{j\in J} \pi_1(K_j)^{\Phi_j}$, it is in particular contained in the group $\prod_{j\in J} (\widetilde{K}_j)^{\Phi_j}$ of constant sections, which is connected by [39, Theorem 12.4.26]. Its Lie algebra $\bigoplus_{j\in J} \mathfrak{k}^{\varphi_j}$ is contained in the radical of the cocycle ω. Since $\mathcal{L}(\widetilde{K}_j)^{\Phi_j}$ is 1-connected (Lemma 7.16), this implies that Z is not only central in G, but also in $\widehat{G}$. In particular, every factor representation restricts to a character on Z. ■

Remark 8.11 (Semisimple groups). In Theorem 8.10, the restriction to simple fibers is by no means essential. For Lie group bundles $\mathcal{K} \to M$ with compact semisimple, 1-connected fibers, the representation still localizes to an embedded 1-dimensional submanifold $S \subseteq M$ by Theorem 7.1. As M is compact, S consists of finitely many circles S_j. Since the fibers of $\mathcal{K} \to M$ are 1-connected, the passage from M to the finite cover $\widehat{M}$ (Theorem 4.9) yields not only a Lie algebra bundle $\widehat{\mathfrak{K}} \to \widehat{M}$, but also a Lie group bundle $\widehat{\mathcal{K}} \to \widehat{M}$ with simple, compact fibers. By the same argument as in Remark 4.10, the $\mathbb{R}$-action on $\mathcal{K} \to M$ lifts to $\widehat{\mathcal{K}} \to \widehat{M}$. Applying Theorem 8.10 to $\widehat{\mathcal{K}} \to \widehat{M}$, we find that the minimal factorial positive energy representations are again multiples of the irreducible ones. The latter are now parametrized by embedded circles $\widehat{S}_{j,r} \subseteq \widehat{M}$, together with an integral anti-dominant weight $\lambda_{j,r}$ with central charge $\lambda_{j,r}(C) = ic_{j,r}$. Here, the circle $\widehat{S}_{j,r} \subseteq \widehat{M}$ is a finite cover of the circle $S_j \subseteq M$. The weight $\lambda_{j,r}$ is associated to the Kac–Moody algebra $\widehat{\mathcal{L}}_{\Phi_{j,r}}(\mathfrak{k}_{j,r})$, where $\mathfrak{k}_{j,r}$ is a simple ideal in the semisimple Lie algebra $\mathfrak{k}_j$, and $\Phi_{j,r}$ is the smallest power of the holonomy around S_j that maps $\mathfrak{k}_{j,r}$ to itself.

8.3 Extensions to non-connected groups

In this section we discuss several phenomena related to non-connected variants of the group G. Dealing with non-connected groups is typically more complicated because they may not have a simply connected covering group, nor do central extensions or representations of the identity component always extend to the whole group.

This suggests the following classification scheme to deal with projective positive energy representations of $G \rtimes_\alpha \mathbb{R}$ if G is not connected.

- Determine which central extensions of G_0 extend to the non-connected groups G.

- Determine which of these do this in an $\mathbb{R}$-equivariant fashion. This leads us to central extensions of the non-connected group $G \rtimes_\alpha \mathbb{R}$.
- Determine the irreducible positive energy representations of the non-connected groups $\widehat{G}$ in terms of the representations of $\widehat{G}_0$ (this may be carried out with Mackey's method of unitary induction, as in [104]).

The following factorization theorem reduces the classification of the irreducible representations to the corresponding problem for the identity component G_0 and the group $\pi_0(G)$ of connected components. It shows in particular that no additional difficulties arise if K is a 1-connected simple group. We shall use the notation

$$G \to \pi_0(G), \quad g \mapsto [g]$$

for the quotient map.

Theorem 8.12 (Factorization theorem for non-connected gauge groups). *Suppose that K is a 1-connected simple compact Lie group, that M is compact and that $\mathbf{v}_M$ has no zeros. Then, every positive energy representation $(\rho, \mathcal{H})$ of $G^\sharp = \Gamma(M, \mathcal{K})^\sharp$ can be written as $\rho(g) = \rho'(g)\zeta([g])$, where ρ' factors through a 1-dimensional, closed, $\mathbb{R}$-equivariantly embedded submanifold $S \subseteq M$, and $\zeta\colon \pi_0(G) \to \mathrm{U}(\mathcal{H})$ is a representation that commutes with $\rho'(G_0^\sharp) = \rho(G_0^\sharp)$. In particular, every irreducible positive energy representation of $G^\sharp$ is of the form $\rho' \otimes \zeta$ where both ρ' and ζ are irreducible, and, conversely, any such tensor product is irreducible.*

Proof. Let $(\rho, \mathcal{H})$ be a positive energy representation of $G^\sharp$. From Theorem 8.10 we know that the restriction of ρ to $G_0^\sharp$ factors through an evaluation homomorphism

$$\mathrm{ev}\colon G \to G_S := \Gamma(S, \mathcal{K}) \cong \prod_{j \in J} \mathcal{L}_{\Phi_j}(K),$$

that is, there exists a positive energy representation ρ_1 of $G_S^\sharp$ such that

$$\rho|_{G_0^\sharp} = \rho_1 \circ \mathrm{ev}\,|_{G_0^\sharp}.$$

Since K is 1-connected, the groups $\mathcal{L}_{\Phi_j}(K)$ are connected and therefore G_S is connected. Then, $\rho' := \rho_1 \circ \mathrm{ev}$ is a positive energy representation of $G^\sharp$ that coincides with ρ on $G_0^\sharp$.

This construction shows in particular that $\pi_0(G)$ acts trivially on the set of equivalence classes of irreducible positive energy representation of $G_0^\sharp$. Indeed, for every irreducible representation ρ_1 of $G_S^\sharp$, the representation ρ' extends the representation $\rho_1 \circ \mathrm{ev}\,|_{G_0^\sharp}$ to a representation of $G^\sharp$ on the same space.

As every positive energy representation of $G^\sharp$ decomposes on $G_0^\sharp$ into irreducible ones (Theorem 8.10), it follows that $\rho(G^\sharp)$ preserves all the $G_0^\sharp$ isotypic subspaces

$\mathcal{H}_j \cong \mathcal{F}_j \otimes \mathcal{M}_j$, $j \in J$, and on these the representation of $G_0^\sharp$ has the form $\rho_j \otimes \mathbf{1}$. Extending ρ_j to a representation $\widetilde{\rho}_j$ of $G^\sharp$, the restriction of ρ from $\mathcal{H}$ to $\mathcal{H}_j$ takes the form $\widetilde{\rho}_j \otimes \zeta_j$, where $\zeta_j : \pi_0(G^\sharp) \cong \pi_0(G) \to \mathrm{U}(\mathcal{M}_j)$ is a unitary representation on the multiplicity space. Putting everything together, we obtain a factorization $\rho = \rho' \otimes \zeta$, where ζ is a representation of $\pi_0(G)$ that commutes with $\rho(G_0^\sharp)$.

In view of Schur's Lemma, our construction shows in particular that the representation ρ is irreducible if and only if it is isotypical on $G_0^\sharp$, that is, $\mathcal{H} = \mathcal{F} \otimes \mathcal{M}$, and the representation ζ of $\pi_0(G)$ on $\mathcal{M}$ is irreducible. ■

Remark 8.13. (a) If K is connected but not simply connected and $\mathfrak{k}$ is a compact simple Lie algebra, then the classification in [104] shows that not all central extensions of $\mathcal{L}(K)_0$ extend to the whole group $\mathcal{L}(K)$, so that the situation becomes more complicated. Likewise, irreducible projective positive energy representations of $\mathcal{L}(K)_0$ do not in general extend to the whole group $\mathcal{L}(K)$. In [104] one finds a classification of the irreducible projective positive energy representations of the groups $\mathcal{L}(K)$ for connected simple groups K. Here the new difficulty is that the group $\pi_0(\mathcal{L}(K)) \cong \pi_1(K)$ acts non-trivially on the alcove whose intersection with the weight lattice classifies the irreducible projective positive energy representations of the connected group $\mathcal{L}(K)_0 \cong \mathcal{L}(\widetilde{K})$ for a fixed central charge.

(b) If we start with a projective representation of the non-connected gauge group $\Gamma_c(M, \mathcal{K})$, we get a representation of the image of $\Gamma_c(M, \mathcal{K})$ in $\Gamma_c(S, \mathcal{K})$, which is a restricted direct product of twisted loop groups. It maps $\Gamma_c(M, \mathcal{K})_0$ onto the identity component, but additional information is contained in the images of the other connected components. We then get a projective representation of a Lie group whose Lie algebra is $\Gamma_c(S, \mathfrak{K})$ and whose group of connected components is an image of $\pi_0(\Gamma_c(M, \mathcal{K}))$. Its action on the Lie algebra does not permute the ideals of the type $\mathcal{L}_\varphi(\mathfrak{k})$, so it acts on each twisted loop algebra separately by the adjoint action of some element of $\mathcal{L}_\Phi(K)$. This suggests that one needs a description of those Lie algebra cocycles ω on $\Gamma_c(M, \mathfrak{K})$ that actually correspond to central Lie group extensions of the full group $\Gamma_c(M, \mathcal{K})$. Here the obstructions lie in $H^3(\pi_0(\Gamma_c(M, \mathcal{K})), \mathbb{T})$. We refer to [72] for further details on such obstructions and for methods of their computation.

(c) For a bundle of Lie groups $\mathcal{K} \to M$, passing to the simply connected covering of the structure group K may not always be possible. For this, an obstruction class in $H^3(M, \pi_1(K))$ has to vanish (see [83]). Since $\pi_1(K)$ is finite for semisimple compact groups K, this is a torsion class. So for a discrete central subgroup $D \subseteq K$, every bundle with structure group K factorizes to a bundle with structure group K/D, but in general, not all bundles with structure group K/D are of this form.

Chapter 9

The classification for M noncompact

Even in the noncompact case, the techniques developed so far open up a number of new perspectives. The localization Theorem 7.1 allows us to restrict attention to a 1-dimensional invariant submanifold $S \subseteq M$. If M is noncompact, then S can have infinitely many connected components S_j, each of which is diffeomorphic to either $\mathbb{R}$ or $\mathbb{S}^1$. We consider these two cases separately.

In Section 9.1 we consider the case where S consists of infinitely many lines. In order to arrive at a (partial) classification, we impose the additional condition that the positive energy representation $(\bar{\rho}, \mathcal{H})$ admits a cyclic ground state vector $\Omega \in \mathcal{H}$ that is unique up to scalar. In Theorem 9.11 we show that these *vacuum representations* are classified up to unitary equivalence by a central charge $c_j \in \mathbb{N}_0$ for every connected component $S_j \simeq \mathbb{R}$. The proof proceeds by reducing to the (important) special case $M = \mathbb{R}$, where the classification is essentially due to Tanimoto [102].

In Section 9.2 we consider the case where S consists of infinitely many circles. Here we impose the much less restrictive condition that $\mathcal{H}$ is a *ground state representation*. This means that $\mathcal{H}$ is generated by the space of ground states, but we do not require these ground states to be unique. We show that this condition is automatically satisfied if the periods (9.6) of the $\mathbb{R}$-action are uniformly bounded. In Theorem 9.16 we classify this type of representations in terms of C^*-algebraic data, using techniques similar to those used in [50] for norm-continuous representations. The possibility of an infinite-dimensional space of ground states gives rise to interesting phenomena, such as factor representations of type II and III.

Finally, in Section 9.3, we briefly explore a simple situation where the $\mathbb{R}$-action has a fixed point. The main thing we wish to point out is that the lift of the $\mathbb{R}$-action *at the fixed point* has a qualitative influence on the type of representation theory that one encounters. In Part II of this series we develop the necessary tools to resolve the positive energy representation theory in more detail.

9.1 Infinitely many lines

In contrast to the case of (twisted) loop groups, the classification of projective positive energy representations of $C_c^\infty(\mathbb{R}, K)$, for K a compact 1-connected simple Lie group, is an open problem—closely related to the classification problem for representations of loop group nets (cf. [103, 111] and Remark 9.4).

A large class of examples can be obtained by restricting projective positive energy representations of the loop group $G := \mathcal{L}(K)$ to $G_{\mathrm{cs}} := C_c^\infty(\mathbb{R}, K)$, where the latter is considered as a subgroup by identifying the circle with the real projective line $\mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$. The restriction of an irreducible projective positive energy representation of $\mathcal{L}(K)$ remains irreducible, essentially by [103, Corollary IV.1.3.3]. In Section 9.1.1 we show that the restriction remains of positive energy as well. This is not a priori clear, since the positive energy is defined in terms of rotations of the circle for G and in terms of translations on the real line for G_{cs}.

It is not true that *all* projective unitary positive energy representations of G_{cs} arise by restriction in this way, and the classification remains an open problem. We can, however, classify the projective positive energy representations under the additional assumption that they admit a cyclic ground state vector which is unique up to scalar. These *vacuum representations* were classified by Tanimoto for the Lie algebra of $\mathfrak{k}$-valued Schwartz functions [102], and in Section 9.1.2 we use Theorem 6.30 to push these results to the compactly supported setting.

Finally, in Section 9.1.3, we classify the vacuum state representations for a noncompact manifold M with a free $\mathbb{R}$-action. The proof proceeds by identifying the restricted gauge group $\Gamma_c(S, \mathcal{K})$ with the weak product

$$\prod\nolimits_j' C_c^\infty(S_j, K),$$

where j labels the connected components $S_j \simeq \mathbb{R}$. We then use the results from Appendix D, where we show that the classification of vacuum representations for a weak product of Lie groups reduces to the same problem for each of its factors.

9.1.1 Restriction from $\mathcal{L}(K)$ to $C_c^\infty(\mathbb{R}, K)$

By identifying the circle $\mathbb{S}^1$ with the real projective line $\mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$, we can consider $G_{\mathrm{cs}} := C_c^\infty(\mathbb{R}, K)$ as a subgroup of the loop group $G := \mathcal{L}(K)$.

Note that the natural $\mathbb{R}$-action by translations on G_{cs} does not agree with the $\mathbb{R}$-action by rigid rotations on G. In terms of the real projective line, the rotation action of $\mathbb{R}/\mathbb{Z}$ is given by the fractional linear maps

$$R_t(x) = \frac{\cos \pi t \cdot x + \sin \pi t}{-\sin \pi t \cdot x + \cos \pi t}, \quad x \in \mathbb{R} \cup \{\infty\},\ [t] \in \mathbb{R}/\mathbb{Z},$$

whereas the translation action of is given by $T_t(x) = x + t$.

Proposition 9.1 (Restriction of positive energy representations). *Let $(\rho, \mathcal{H})$ be an irreducible positive energy representation of $\mathcal{L}^\sharp(K)$ with respect to the $\mathbb{R}$-action by rotations. Then, the restriction of ρ to $C_c^\infty(\mathbb{R}, K)^\sharp$ is an irreducible positive energy with respect to the $\mathbb{R}$-action by translations.*

We first prove that the restriction remains irreducible, and then continue with the positive energy condition.

Proof of irreducibility. Let $G_* := \{\xi \in G : \xi(\infty) = \mathbf{1}\}$ be the subgroup of based loops. Since $\rho(G_*^\sharp)'' = \mathbb{C}\mathbf{1}$ by[1] [103, Corollary IV.1.3.3], it suffices to show that $\rho(G_{\mathrm{cs}}^\sharp)$ is dense in $\rho(G_*^\sharp)$ for the strong operator topology. By [79, Appendix A], the representation of $G^\sharp$ extends to a smooth representation of the Banach–Lie group $H^1(\mathbb{S}^1, K)$ of H^1-loops, whose Lie algebra is the space $H^1(\mathbb{S}^1, \mathfrak{k})$ of H^1-functions $\xi\colon \mathbb{S}^1 \to \mathfrak{k}$. Since these are the absolutely continuous functions whose derivatives are L^2, the derivative $\xi \mapsto \xi'$ maps the subspace $H^1_*(\mathbb{S}^1, \mathfrak{k})$ of H^1-functions that vanish in the base point homeomorphically to

$$L^2_*(\mathbb{S}^1, \mathfrak{k}) = \left\{\xi \in L^2(\mathbb{S}^1, \mathfrak{k}) : \int_{\mathbb{S}^1} \xi(t)dt = 0\right\}.$$

In this space the subspace $\{\eta' : \eta \in C_c^\infty(\mathbb{R}, \mathfrak{k})\}$ is easily seen to be dense. Since $G_*^\sharp$ is connected, this implies that $\rho(G_{\mathrm{cs}}^\sharp)$ is dense in $\rho(G_*^\sharp)$. ∎

To prove the positive energy condition for the restriction, we need to compare the generator $\mathbf{d}_0$ of rigid rotations with the generator $\mathbf{d}_1$ of translations. In $\mathfrak{sl}(2, \mathbb{R})$, these are given by

$$\mathbf{d}_0 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{d}_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{9.1}$$

The fact that $\mathbf{d}_0$ and $\mathbf{d}_1$ generate the same Ad-invariant closed convex cone in the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ leads to the following characterization (cf. [59, Section 1.3]).

Lemma 9.2. *For a unitary representation $(\rho, \mathcal{H})$ of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$, the generator $i\, d\rho(\mathbf{d}_0)$ is bounded from below if and only if $i\, d\rho(\mathbf{d}_1)$ is bounded from below. Moreover, if this is the case, then $i\, d\rho(\mathbf{d}_0) \geq 0$ and $i\, d\rho(\mathbf{d}_1) \geq 0$.*

In particular, an $\widetilde{\mathrm{SL}}(2, \mathbb{R})$-representation is of positive energy for $\mathbf{d}_0$ if and only if it is of positive energy for $\mathbf{d}_1$. To prove that the restriction from $\mathcal{L}(K)^\sharp$ to $C_c^\infty(\mathbb{R}, K)$ is of positive energy with respect to $\mathbf{d}_1$, it therefore suffices to extend the action by rigid rotations to an action of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$. This is done using the Segal–Sugawara construction.

Proof of positive energy. Recall from [32, Section 7] and [33] that every irreducible projective positive energy representation $(\bar{\rho}_\lambda, \mathcal{H}_\lambda)$ of $\mathcal{L}(K)$ with lowest weight λ extends to a projective representation of the semidirect product

$$\mathcal{L}(K) \rtimes \mathrm{Diff}_+(\mathbb{S}^1),$$

where $\mathrm{Diff}_+(\mathbb{S}^1)$ acts on $\mathcal{L}(K)$ by $\alpha_\varphi(\xi) := \xi \circ \varphi^{-1}$. The cocycle

$$\omega(\xi, \eta) = \frac{c}{2\pi} \int_{\mathbb{S}^1} \kappa(\xi'(t), \eta(t))dt \tag{9.2}$$

[1] Alternatively, one can use [102, Theorem 6.4], which uses [7, Corollary 1.2.3].

is easily seen to be invariant under the action of $\mathrm{Diff}_+(\mathbb{S}^1)$, but it is much harder to verify the covariance of the representations $\bar{\rho}_\lambda$. In [32, Section 7.2], the representation of the Virasoro algebra obtained from the Segal–Sugawara construction is integrated to a group representation. Since this respects the semidirect product structure of $\mathcal{L}(K) \rtimes \mathrm{Diff}_+(\mathbb{S}^1)$, it follows in particular that

$$\bar{\rho}_\lambda \circ \varphi \cong \bar{\rho}_\lambda \quad \text{for every } \varphi \in \mathrm{Diff}_+(\mathbb{S}_1). \tag{9.3}$$

By Schur's Lemma and the irreducibility of $\bar{\rho}_\lambda$, the projective representation $\bar{\rho}^P$ of $\mathrm{Diff}_+(\mathbb{S}^1)$ on $\mathscr{H}_\lambda$ is uniquely determined by the intertwining property

$$\bar{\rho}^P(\varphi)\bar{\rho}_\lambda(\xi)\bar{\rho}^P(\varphi)^{-1} = \bar{\rho}(\alpha_\varphi \xi) \quad \text{for } \xi \in \mathcal{L}(K), \varphi \in \mathrm{Diff}_+(\mathbb{S}^1).$$

Since $\mathrm{Diff}_+(\mathbb{S}^1)$ contains the group of rigid rotations with respect to which $\bar{\rho}_\lambda$ is a positive energy representation, the Hamiltonian $H = i\,\mathrm{d}\rho^P(\mathbf{d}_0)$ associated to $\mathbf{d}_0$ is bounded below. Since ρ^P is a positive energy representation of the Virasoro group, it restricts to a positive energy representation of its subgroup $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, the simply connected cover of the group $\mathrm{PSL}(2,\mathbb{R}) \subseteq \mathrm{Diff}_+(\mathbb{S}^1)$ of fractional linear transformations of $\mathbb{S}^1 \cong \mathbb{P}_1(\mathbb{R})$. By Lemma 9.2, the generator $i\,\mathrm{d}\rho^P(\mathbf{d}_1)$ then has non-negative spectrum. ■

Remark 9.3. Since the cocycle (9.2) is invariant under the action of $\mathrm{Diff}_+(\mathbb{S}^1)$, twisting $\bar{\rho}_\lambda$with $\varphi \in \mathrm{Diff}_+(\mathbb{S}^1)$ leads to an irreducible projective unitary representation $\bar{\rho}_\lambda \circ \varphi$ with the same central charge c. By Proposition 8.6, there are only finitely many types of such representations satisfying the positive energy condition. If we knew a priori that this twist preserves the positive energy condition (which is presently not the case), then we could bypass the integration procedure in [32], and construct the projective representation of $\mathrm{Diff}_+(\mathbb{S}^1)$ as follows.

By the Epstein–Hermann–Thurston theorem, $\mathrm{Diff}(M)_0$ is a simple group for every compact connected smooth manifold M (see [18]). In particular, $\mathrm{Diff}_+(\mathbb{S}^1)$ is a simple group. Since it has no normal subgroup of finite index, it acts trivially on any finite set. This implies that $\bar{\rho}_\lambda \circ \varphi \cong \bar{\rho}_\lambda$ for every $\varphi \in \mathrm{Diff}_+(\mathbb{S}^1)$. The unitaries that implement this equivalence constitute a projective unitary representation of $\mathrm{Diff}_+(\mathbb{S}^1)$.

Remark 9.4. The class of positive energy representations is by no means exhausted by the representations of Proposition 9.1. We briefly sketch the construction of a class of type III_1 factor representations, following [19, 112].

Recall from [32, Section 7.2] that an irreducible positive energy vacuum representation ρ of $G^\sharp = \mathcal{L}^\sharp(K)$ gives rise to a vacuum representation ρ^P of $\mathrm{Diff}_+(\mathbb{S}^1)^\sharp$. If we lift the $\mathbb{R}$-action by translations along the 2-fold covering $q\colon \mathbb{S}^1 \to \mathbb{R} \cup \{\infty\} \cong \mathbb{S}^1$, we obtain a flow on $\mathbb{S}^1$ with exactly two fixed points. Its generator $\mathbf{v}$ is obtained from the vector field $\mathbf{d}_0$ generating rigid rotations by multiplication with a non-negative function. This implies that the operator $i\,\mathrm{d}\rho^P(\mathbf{v})$ is bounded from below.

Let $I \subseteq \mathbb{S}^1$ be one of the two connected components of $q^{-1}(\mathbb{R})$ and identify

$$G_{\mathrm{cs}} = C_c^\infty(\mathbb{R}, K)$$

with $C_c^\infty(I, K)$. Then, the restriction of ρ to $G_{\mathrm{cs}}^\sharp$ is a factor representation of type III_1. Combining this with the one-parameter group generated by the vector field $\mathbf{v}$, we obtain a projective positive energy representation of $G_{\mathrm{cs}}^\sharp \rtimes \mathbb{R}$ with respect to the translation action on $\mathbb{R}$ ([112, Proposition 3.2]). We refer to [17, 112] for further details (see also the Remark after [103, Theorem IV.2.2.1]).

More generally, we may consider smooth vector fields $\mathbf{v} \in \mathcal{V}(\mathbb{S}^1)$ which are non-negative multiples $f\mathbf{d}_0$, $f \geq 0$, of the generator $\mathbf{d}_0$ of rigid rotations. For vacuum representations of $\mathcal{L}(K) \rtimes \mathrm{Diff}_+(\mathbb{S}^1)$, the corresponding selfadjoint operator $i\,\mathrm{d}\rho^P(\mathbf{v})$ is bounded from below (cf. [19]). If $I \subseteq \mathbb{S}^1$ is an open interval on which $\mathbf{v}$ has no zeros but for which $\mathbf{v}$ vanishes in the boundary ∂I, then we obtain an embedding

$$C_c^\infty(\mathbb{R}, \mathfrak{k}) \rtimes \mathbb{R} \cong C_c^\infty(I, \mathfrak{k}) \rtimes \mathbb{R} \hookrightarrow \mathcal{L}(\mathfrak{k}) \rtimes \mathbb{R}\mathbf{v}$$

that integrates to the group level, where we obtain a projective positive energy representation of $C_c^\infty(\mathbb{R}, K)$.

9.1.2 Vacuum representations of $C_c^\infty(\mathbb{R}, K)$

Although the classification of projective positive energy representations $(\bar{\rho}, \mathcal{H})$ of $C_c^\infty(\mathbb{R}, K)$ is an open problem in general, it can be resolved under the additional assumption that $\mathcal{H}$ admits a unique, cyclic ground state.

Definition 9.5. Let $(\rho, \mathcal{H})$ be a positive energy representation of $\widehat{G}$.

(a) A *ground state vector* is a vector $\Omega \in \mathcal{D}(H) \subseteq \mathcal{H}$ such that $H\Omega = E_0\Omega$ for $E_0 := \inf(\mathrm{spec}(H))$. We denote the space of ground state vectors by $\mathcal{E}$.

(b) A *ground state representation* is a positive energy representation $(\rho, \mathcal{H})$ that is generated by its space of ground states, in the sense that the linear span of $\rho(\widehat{G})\mathcal{E}$ is dense in $\mathcal{H}$.

(c) A *vacuum representation* is a ground state representation where the ground state is unique up to scalar, $\mathcal{E} = \mathbb{C}\Omega$.

At the Lie algebra level, we obtain analogous definitions if we replace the requirement that $\rho(\widehat{G})\mathcal{E}$ is dense in $\mathcal{H}$ by the requirement that

$$\mathcal{U}(\mathfrak{g})\Omega = \mathcal{U}(\mathfrak{g}^\sharp)\Omega$$

is dense in $\mathcal{H}$. Although the translation between these two concepts requires some caution, the two notions turn out to be compatible for positive energy representations.

Proposition 9.6. *Let $(\rho, \mathcal{H})$ be a positive energy representation of $\widehat{G}$ with ground state vector Ω. Then, $\mathcal{U}(\mathfrak{g}^\sharp)\Omega$ is dense in $\mathcal{H}$ if and only if Ω is cyclic under $\rho(G^\sharp)$.*

Proof. For a closed interval $I \subseteq \mathbb{R}$, let

$$G_I := \{\xi \in G = C_c^\infty(\mathbb{R}, K) : \xi(\mathbb{R} \setminus I) = \{e\}\}$$

denote the Fréchet–Lie subgroup of maps supported by I. We claim that the Lie group $G_I^\sharp$ is BCH, i.e., it is locally exponential and its Lie algebra $\mathfrak{g}_I^\sharp$ is BCH, which means that the Baker–Campbell–Hausdorff series defines an analytic local multiplication on a 0-neighborhood of $\mathfrak{g}$ ([30, Theorem 15.7.1]). For G_I this follows from [30, Example 7.1.4 (c)] because the BCH property is inherited from the target group K. Further [30, Theorem 15.4.19] implies that the centrally extended Lie algebra $\mathfrak{g}_I^\sharp$ is also locally exponential and the proof of this theorem shows that the analyticity of the local multiplication is inherited by the central extension.

Lemma 6.34 implies that Ω is an analytic vector for each element in $\mathfrak{g}_I^\sharp$, so that [76, Proposition 4.10] further entails that Ω is an analytic vector for $G_I^\sharp$. Hence, the closure of $\mathcal{U}(\mathfrak{g}_I^\sharp)\Omega$ is $G_I^\sharp$-invariant. As the interval I was arbitrary, the closure of $\mathcal{U}(\mathfrak{g}^\sharp)\Omega$ is invariant under $G^\sharp$, hence also under[2] $\widehat{G}$, because Ω is an H-eigenvector. This shows that $\mathcal{U}(\mathfrak{g}^\sharp)\Omega$ is dense in $\mathcal{H}$ if and only if Ω is cyclic under $\rho(G^\sharp)$. ∎

The vacuum representations for the Lie algebra $\mathfrak{g}_{\mathcal{S}} = \mathcal{S}(\mathbb{R}, \mathfrak{k})$ of $\mathfrak{k}$-valued Schwartz functions have been classified by Yoh Tanimoto.

Theorem 9.7 (Tanimoto's classification theorem; [102, Corollary 5.8]). *Let $(\pi, \mathcal{H}^\infty)$ be a vacuum representation of $\widehat{\mathfrak{g}}_{\mathcal{S}}$ with respect to the $\mathbb{R}$-action by translations. Suppose that for all $\psi, \chi \in \mathcal{H}^\infty$, the functional $\xi \mapsto \langle \psi, \pi(\xi)\chi\rangle$ is a tempered distribution. Then, $(\pi, \mathcal{H}^\infty)$ is characterized up to unitary equivalence by its central charge $c \in \mathbb{N}_0$.*

Using the continuity results from Chapter 6, we show that Tanimoto's classification theorem remains true for the smaller Lie algebra $\mathfrak{g}_{\mathrm{cs}} := C_c^\infty(\mathbb{R}, \mathfrak{k})$ of compactly supported smooth $\mathfrak{k}$-valued functions. This is an important improvement because the relevant Lie algebra for the classification of ground states of loop group nets is not $\mathcal{S}(\mathbb{R}, \mathfrak{k})$, but $C_c^\infty(\mathbb{R}, \mathfrak{k})$ (cf. [102, Section 6]).

As usual, we denote

$$\widehat{\mathfrak{g}}_{\mathrm{cs}} := (\mathbb{R}C \oplus_\omega \mathfrak{g}_{\mathrm{cs}}) \rtimes \mathbb{R}D \quad \text{and} \quad \widehat{\mathfrak{g}}_{\mathcal{S}} := (\mathbb{R}C \oplus_\omega \mathfrak{g}_{\mathcal{S}}) \rtimes \mathbb{R}D,$$

[2]For the concept of an analytic map to make sense, we need the group to be analytic. Since the $\mathbb{R}$-action on G need not be analytic, the semidirect product $G \rtimes_\alpha \mathbb{R}$ is in general not an analytic Lie group. In particular, $\widehat{G}$ need not be an analytic Lie group.

where D acts by infinitesimal translations. Since the inclusion of $\mathcal{S}(\mathbb{R}, \mathfrak{k})$ in $H^1_\partial(\mathbb{R}, \mathfrak{k})$ is continuous, the following is an immediate consequence of Theorem 6.30.

Proposition 9.8. *Let $(\rho, \mathcal{H})$ be a positive energy representation of the group*

$$\widehat{G}_{\mathrm{cs}} := (C^\infty_c(\mathbb{R}, K) \rtimes \mathbb{R})^\sharp.$$

Then, the derived representation $\mathtt{d}\rho$ of $\widehat{\mathfrak{g}}_{\mathrm{cs}}$ extends uniquely to a positive energy representation r of $\widehat{\mathfrak{g}}_{\mathcal{S}}$ such that, for all $\psi, \chi \in \mathcal{H}^\infty$, the functional $\xi \mapsto \langle \psi, r(\xi)\chi\rangle$ is a tempered distribution.

Combined with Theorem 9.7, this immediately yields the classification of vacuum representations in the compactly supported setting.

Theorem 9.9 (Vacuum representations of $C^\infty_c(\mathbb{R}, K)$). *Let K be a 1-connected, compact, simple Lie group and $G_{\mathrm{cs}} = C^\infty_c(\mathbb{R}, K)$. Then, a vacuum representation $(\rho, \mathcal{H})$ of $\widehat{G}_{\mathrm{cs}}$ is characterized up to unitary equivalence by its central charge $c \in \mathbb{N}_0$.*

Proof. By Proposition 9.6, the derived representation $\mathtt{d}\rho$ of $\widehat{\mathfrak{g}}_{\mathrm{cs}}$ is a vacuum representation which by Proposition 9.8 extends to a continuous representation of the Lie algebra $\widehat{\mathfrak{g}}_{\mathcal{S}}$. By [102, Corollary 5.8], the latter is determined up to isomorphism by its central charge $c \in \mathbb{N}_0$. Since G_{cs} is connected (Lemma 7.16), the representation ρ of $\widehat{G}_{\mathrm{cs}}$ is uniquely determined by its derived Lie algebra representation (Theorem 2.13), and the result follows. ■

In Section 9.1.1, we saw that the restriction of an irreducible positive energy representation of $\mathcal{L}^\sharp(K)$ (with respect to rotations) yields an irreducible positive energy representation of $C^\infty_c(\mathbb{R}, K)^\sharp$ (with respect to translations). We now show that the unique vacuum representation of $C^\infty_c(\mathbb{R}, K)^\sharp$ with central charge c arises by restriction of the irreducible positive energy representation of $\mathcal{L}^\sharp(K)$ with lowest weight

$$\lambda = (ic, 0, 0).$$

Proposition 9.10. *The irreducible lowest weight representation of $\mathcal{L}^\sharp(K)$ with lowest weight λ restricts to a vacuum representation of $C^\infty_c(\mathbb{R}, K)^\sharp$ if and only if the restriction λ_0 of λ to $i\mathfrak{t}$ is zero.*

Proof. Recall from Section 9.1.1 that every irreducible projective positive energy representation of $\mathcal{L}(K) \rtimes \mathbb{R}\mathbf{d}_0$ with lowest weight λ extends to $\mathcal{L}(K) \rtimes \mathrm{Diff}_+(\mathbb{S}^1)$. By Lemma 9.2, this induces a unitary representation of

$$\widetilde{\mathrm{SL}}(2, \mathbb{R}) \subseteq \mathrm{Diff}_+(\mathbb{S}^1)^\sharp,$$

which is of positive energy not only with respect to $\mathbf{d}_0 \in \mathfrak{sl}(2, \mathbb{R})$, but also with respect to $\mathbf{d}_1 \in \mathfrak{sl}(2, \mathbb{R})$ (cf. (9.1)).

Recall from Remark 8.9 that the space $\mathcal{E}_0$ of ground states for $i\,\mathrm{d}\rho(\mathbf{d}_0)$ is an irreducible unitary K-representation. Its lowest weight λ_0 is the restriction of λ to $i\mathfrak{t}$. By the formula in [32, Theorem 3.5 (iii)], the minimal eigenvalue of $H_0 = i\,\mathrm{d}\rho(\mathbf{d}_0)$ is a positive multiple of the Casimir eigenvalue for K on $\mathcal{E}_0$. In particular, it vanishes if and only if $\lambda_0 = 0$, which is the case if and only if $\dim \mathcal{E}_0 = 1$,

$$\inf \mathrm{Spec}(H_0) = 0 \iff \lambda_0 = 0 \iff \dim \mathcal{E}_0 = 1. \tag{9.4}$$

By a result of Mautner and Moore [63, 66],

$$\ker(\mathrm{d}\rho(\mathbf{d}_0)) = \ker(\mathrm{d}\rho(\mathbf{d}_1)) \tag{9.5}$$

coincides with the subspace of vectors that are fixed under $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ (See Appendix E for a simplified direct proof.). If $\lambda_0 = 0$, the ground state for $H_1 := i\,\mathrm{d}\rho(\mathbf{d}_1)$ is therefore unique up to a scalar.

Conversely, suppose that the space $\mathcal{E}_1$ of ground states for H_1 is non-trivial. Since the adjoint orbit through $\mathbf{d}_1$ contains $\mathbb{R}^+\mathbf{d}_1$, the spectrum of $H_1 = i\,\mathrm{d}\rho(\mathbf{d}_1)$ is scale invariant. Any ground state $H_1\Omega = E\Omega$ then has $E = 0$, and will satisfy $H_0\Omega = 0$ by (9.5). Since H_0 is non-negative, it has minimal eigenvalue zero, the space $\mathcal{E}_0$ of ground states for $i\,\mathrm{d}\rho(\mathbf{d}_0)$ is one-dimensional. We conclude that $\lambda_0 = 0$, and that $\mathcal{E}_1 \subseteq \mathcal{E}_0$ is one-dimensional as well. ∎

9.1.3 Vacuum representations for noncompact manifolds

Let $\mathcal{K} \to M$ be a bundle of 1-connected simple compact Lie groups over a 2nd countable manifold M, equipped with a smooth $\mathbb{R}$-action by automorphisms.

Theorem 9.11. *If the action of $\mathbb{R}$ on M is free, then up to unitary equivalence, there is a bijective correspondence between the following.*

(a) *Smooth projective unitary representations $\bar{\rho}\colon \Gamma_c(M,\mathcal{K})_0 \to \mathrm{PU}(\mathcal{H})$ extending to a vacuum representation of $\Gamma_c(M,\mathcal{K})_0^\sharp \rtimes_\alpha \mathbb{R}$ with smooth ground state vector Ω.*

(b) *Closed, embedded, 1-dimensional flow-invariant submanifolds S, together with a non-zero central charge $c_j \in \mathbb{N}$ for every connected component $S_j \simeq \mathbb{R}$ of S.*

Under this correspondence we have

$$(\mathcal{H},\Omega) = \bigotimes_{j\in J}(\mathcal{H}_j,\Omega_j) \quad \textit{and} \quad \rho(g) = \bigotimes_{j\in J}\rho_j(g|_{S_j}),$$

where $(\rho_j,\mathcal{H}_j,\Omega_j)$ is the restriction to $C_c^\infty(\mathbb{R},K) \simeq \Gamma_c(S_j,\mathcal{K})$ of the lowest weight representation of $\mathcal{L}^\sharp(K)$ with lowest weight $\lambda = (c_j,0,0)$ and J is the countable set of connected components of S.

Proof. By the localization Theorem 7.1, every projective positive energy representation $\bar{\rho}$ factors through $\Gamma_c(S, \mathcal{K})$ for a closed, embedded, 1-dimensional submanifold $S \subseteq M$. It follows that ρ factors through $\Gamma_c(S, \mathcal{K})^\sharp$. Since M is 2^{nd} countable, S has at most countably many connected components S_j, $j \in J$, and the freeness of the action implies that each of these is $\mathbb{R}$-equivariantly isomorphic to $\mathbb{R}$. By Lemma D.3, the Lie group $\Gamma_c(S, \mathcal{K})$ is isomorphic to the weak product

$$G := \prod\nolimits'_{j \in J} G_j, \quad \text{with } G_j = \Gamma_c(S_j, \mathcal{K}).$$

The cocycle $\psi : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ on $\mathfrak{g} := \mathbf{L}(G) = \bigoplus_{j \in J} \mathfrak{g}_j$ vanishes on $\mathfrak{g}_i \times \mathfrak{g}_j$ for $i \neq j$. Since every G_j is connected, this implies that

$$G^\sharp \cong \left(\prod\nolimits'_{j \in J} G_j^\sharp \right) \Big/ N,$$

where $N \subseteq \prod'_{j \in J} \mathbb{T}_j$ is the kernel of the smooth character

$$\chi : \prod\nolimits'_{j \in J} \mathbb{T}_j \to \mathbb{T}, \quad (z_j)_{j \in J} \mapsto \prod_{j \in J} z_j.$$

The vacuum representations of $G^\sharp$ therefore correspond to vacuum representations of the weak product $\prod'_{j \in J} G_j^\sharp$ such that the central subgroup $\prod'_{j \in J} \mathbb{T}_j$ acts by χ. We may assume, without loss of generality, that the ground state energy is zero, $H\Omega = 0$.

By Theorem D.6, every vacuum representation $(\rho, \mathcal{H}, \Omega)$ of the weak product $\prod'_{j \in J} G_j^\sharp$ is a product of vacuum representations $(\rho_j, \mathcal{H}_j, \Omega_j)$ of $G_j^\sharp$, and by Proposition D.7, ρ is smooth with smooth ground state vector Ω if and only if all the ρ_i are smooth with smooth ground state Ω_j.

Since ρ_j is irreducible by Proposition D.5, its restriction to the central subgroup $\mathbb{T}_j \subseteq G_j^\sharp$ is a character $\chi_j : \mathbb{T}_j \to \mathbb{T}$. The product $\rho = \bigotimes_{j \in J} \rho_j$ acts by χ on the center $\prod'_{j \in J} \mathbb{T}_j$ if and only if $\chi_j(z) = z\mathbf{1}$ for all $j \in J$.

Using the free $\mathbb{R}$-action to identify $\mathcal{K}|_{S_j}$ with $\mathbb{R} \times K$, we obtain an $\mathbb{R}$-equivariant isomorphism between $G_j = \Gamma_c(S_j, \mathcal{K})$ and $C_c^\infty(\mathbb{R}, K)$ (cf. Section 7.3). By Theorem 9.9, the vacuum representations of $G_j^\sharp$ are characterized up to unitary equivalence by their central charge $c_j \in \mathbb{N}_0$, and by Proposition 9.10, $(\rho_j, \mathcal{H}_j, \Omega_j)$ is unitarily equivalent to the restriction to $C_c^\infty(\mathbb{R}, K)$ of the lowest weight representation of $\mathcal{L}^\sharp(K)$ with $\lambda = (c_j, 0, 0)$. If $c_j = 0$, then the corresponding representation is trivial, so we can omit both S_j and c_j from the description. ∎

9.2 Infinitely many circles

We continue with the case where all connected components S_j of S are circles. In marked contrast with the case of infinitely many lines, the projective positive energy representations associated to a single connected component S_j are well understood, allowing us to classify the projective positive energy representations of $\Gamma_c(S, \mathcal{K})$ under the much weaker condition that the Hilbert space $\mathcal{H}$ is generated by the space $\mathcal{E}$ of ground states. These are the *ground state representations* of Definition 9.5.

As before, we assume that K is a 1-connected compact Lie group, which is not a serious restriction as long as K is connected (cf. Remark 7.4). In Section 9.2.1 we describe the *spectral gap condition*, an essentially geometric sufficient condition for all positive energy representations to be generated by the space of ground states. The main result of this section is Theorem 9.16 in Section 9.2.2, where we describe the ground state representations in terms of the representation theory of UHF C^*-algebras.

9.2.1 The spectral gap condition

Following the line of reasoning in Chapter 8, we associate to every compact connected component S_j a "local" Hamiltonian H_j. If these local Hamiltonians have a uniform spectral gap, we say that $(\rho, \mathcal{H})$ satisfies the *spectral gap condition*. We show that this (essentially geometric) condition guarantees that the positive energy representations are generated by their space of ground states.

We continue with the notation

$$G = \Gamma_c(S, \mathcal{K}) \cong \prod\nolimits'_{j\in J} \mathcal{L}_{\Phi_j}(K_j),$$

where $\prod'_{j\in J}$ denotes the weak direct product as in Section D.1. As in Section 7.3, we identify S_j with $\mathbb{R}/\mathbb{Z}$, where the time translation $\gamma_{S,t}$ acts on $[x_j] \in S_j$ by

$$\gamma_{S,t}([x_j]) = \left[x_j + \frac{t}{T_j}\right].$$

The derivation acts on $\xi_j \in \mathcal{L}_{\Phi_j}(\mathfrak{k}_j)$ by

$$D\xi_j = \frac{1}{T_j}(\mathbf{d}_j\xi_j + [A_j, \xi_j]).$$

By choosing a suitable parametrization of $\mathcal{K}|_{S_j}$, we may assume that A_j is constant (see [79, Proposition 2.14] or [65, Section 5.2]) and lies in the maximal abelian subalgebra $\mathfrak{t}^\circ$ of $\mathfrak{k}^{\varphi_j}$ (Theorem B.2). By acting with the φ_j-twisted Weyl group $\mathcal{W}$, i.e., the Weyl group of the underlying Kac–Moody Lie algebra, we may also assume that $\mathbf{d}_j + A_j$ lies in the positive Weyl chamber, i.e., $(\alpha, n)(i(\mathbf{d}_j + A_j)) \geq 0$ for all positive roots $(\alpha, n) \in \Delta^+$ ([65, Section 3] and Appendix A).

In the following $(\rho_{\lambda_j}, \mathcal{H}_{\lambda_j})$ denotes the irreducible positive energy representation of

$$G_j^\sharp \cong \mathcal{L}^\sharp_{\Phi_j}(K_j) \cong \Gamma(S_j, \mathcal{K})^\sharp$$

with lowest weight λ_j (cf. Section 8.2). Then, the minimal eigenspace V_j^0 of $\mathbf{d}_j$ in $\mathcal{H}_{\lambda_j}$ is an irreducible K^Φ-representation. Since A_j is anti-dominant, the minimal eigenspace W_j^0 of H_j (which is also finite-dimensional by Kac–Moody theory) contains all weight vectors v_μ in V_j^0 with $\mu(A_j) = 0$. Note that W_j^0 is 1-dimensional for generic A_j and increases in dimension as $\mathbf{d}_j + A_j$ is contained in a smaller face of the Weyl chamber (or, equivalently, as A_j is contained in a smaller face of the Weyl alcove), and that $W_j^0 = V_j^0 = V_{\lambda_j^0}$ if $A_j = 0$. We denote the orthogonal projection $\mathcal{H}_{\lambda_j} \to V_j^0$ by P_j, and for a finite subset $F \subseteq J$, we set

$$P_F := \prod_{j \in F} P_j.$$

Let $(\rho, \mathcal{H})$ be a factorial projective positive energy representation of $G^\sharp$. Recall from Section 8.2.1 that, for every finite subset $F \subseteq J$, we have a tensor product decomposition $\mathcal{H} = \mathcal{H}_F \otimes \mathcal{H}'_F$. Here

$$\mathcal{H}_F = \bigotimes_{j \in F} \mathcal{H}_{\lambda_j}$$

is a positive energy representation of $G_F^\sharp$ with Hamiltonian

$$H_F = \sum_{j \in F} H_{\lambda_j},$$

where

$$H_{\lambda_j} = i\,\mathrm{d}\rho_{\lambda_j}\left(\frac{1}{T_j}(\mathbf{d}_j + A_j)\right) - \frac{i}{T_j}\lambda_j(\mathbf{d}_j + A_j)\mathbf{1}$$

is the minimal non-negative Hamiltonian on $\mathcal{H}_{\lambda_j}$ from Section 3.3. The other factor $\mathcal{H}'_F$ is a minimal positive energy representation of $G^\sharp_{J\setminus F}$ with Hamiltonian H', and we have

$$H = H_F \otimes \mathbf{1} + \mathbf{1} \otimes H'.$$

The ground states for a $\widehat{G}$-representation (in the sense of Definition 9.5) can be characterized in terms of the "local" Hamiltonians as follows.

Lemma 9.12. *For a factorial minimal positive energy representation $(\rho, \mathcal{H})$ of $\widehat{G}$, a vector $\Omega \in \mathcal{D}(H) \subseteq \mathcal{H}$ is a ground state vector if and only if $H_{\lambda_j}\Omega = 0$ for every $j \in J$.*

Proof. "$\Rightarrow$": Suppose first that Ω is a ground state vector. Then, $0 \leq H_{\lambda_j} \leq H$ implies that $H_{\lambda_j}\Omega = 0$.

"$\Leftarrow$": Conversely, suppose that $H_{\lambda_j}\Omega = 0$ holds for all $j \in J$. By minimality, the cyclic subspace generated by Ω under $G^\sharp$ is $\widehat{G}$-invariant and the corresponding representation on this subspace is minimal. We may therefore assume that Ω is cyclic.

For every finite subset $F \subseteq J$, Ω is fixed by the operators $V_t^F := e^{-itH_F}$, $t \in \mathbb{R}$. These operators satisfy

$$V_t^F \rho(g) V_{-t}^F = \rho(\alpha_t(g)) \quad \text{for } g \in G_F^\sharp,\ t \in \mathbb{R}.$$

For any finite superset $F' \supseteq F$ we then have

$$V_t^{F'} \rho(g)\Omega = \rho(\alpha_t(g))\Omega = V_t^F \rho(g)\Omega \quad \text{for } g \in G_F^\sharp.$$

This means that V_t^F and $V_t^{F'}$ coincide on the closed subspace $\mathcal{H}^F$ generated by $\rho(G_F^\sharp)\Omega$. Since the union of these subspaces is dense in $\mathcal{H}$, we obtain a unitary one-parameter group $(V_t)_{t\in\mathbb{R}}$ on $\mathcal{H}$ whose restriction to $\mathcal{H}^F$ coincides with $(V_t^F)_{t\in\mathbb{R}}$. This implies that

$$V_t \rho(g) V_{-t} = \rho(\alpha_t(g)) \quad \text{for } g \in G^\sharp,\ t \in \mathbb{R}.$$

Write $V_t = e^{-it\widetilde{H}}$ for a positive selfadjoint operator $\widetilde{H}$. Then, our construction shows that $\widetilde{H}$ coincides with $H_F = \sum_{j\in F} H_{\lambda_j}$ on $\mathcal{H}^F$, and thus $\widetilde{H} \geq 0$. By minimality of H, we have $0 \leq H \leq \widetilde{H}$, so that $\widetilde{H}\Omega = 0$ leads to $H\Omega = 0$. ∎

Definition 9.13 (Spectral gap). We say that the family $(\lambda_j, A_j, T_j)_{j\in J}$ satisfies the *spectral gap condition* if there exists a positive real number ΔE such that, for every $j \in J$,

$$\mathrm{Spec}(H_{\lambda_j}) \subseteq \{0\} \cup [\Delta E, \infty).$$

The spectral gap condition is essentially geometric in nature. Recall that for $m \in M$, the $\mathbb{R}$-action γ_t yields a group automomorphism $\gamma_t(m)\colon \mathcal{K}_m \to \mathcal{K}_{\gamma_M(m)}$. The spectral gap condition is automatically satisfied if the period

$$T(m) := \inf\big\{t > 0; \gamma_{M,t}(m) = m, \gamma_t = \mathrm{Id} \in \mathrm{Aut}(\mathcal{K}_m)\big\} \tag{9.6}$$

is uniformly bounded on M. Indeed, the $\mathbb{R}$-action on $\Gamma(S_j, \mathcal{K})$ then has period $T_j \leq \sup_{m\in M} T(m)$, so the spectrum of $\mathbf{d}_j + A_j$ in every minimal unitary positive energy representation will be contained in $(2\pi i/T_j)\mathbb{Z}$.

Proposition 9.14 (Spectral gaps yield ground state vectors). *Let $(\rho, \mathcal{H})$ be a factorial minimal positive energy representation of $\widehat{G}$ such that the corresponding family $(\lambda_j, A_j, T_j)_{j\in J}$ satisfies the spectral gap condition with some $\Delta E > 0$. Then, $\mathcal{H}$ is generated under $G^\sharp$ by the subspace* $\ker H$ *of ground state vectors.*

Proof. The minimality implies that 0 is the infimum of the spectrum of H, so that the spectral projection

$$P := P([0, \Delta E/2])$$

is non-zero. First we show that $P\mathcal{H}$ is contained in the kernel of every H_{λ_j}. In fact, the operator $H - H_{\lambda_j}$ is non-negative. Since the minimal non-zero spectral value of

H_{λ_j} is $\geq \Delta E$, it follows that $P\mathcal{H} \subseteq \ker H_{\lambda_j}$. Lemma 9.12 now shows that $H\Omega = 0$. Therefore, $\mathcal{F} := P\mathcal{H}$ coincides with the subspace $\ker H$ of ground state vectors.

Next we show that $\mathcal{F}$ is generating under $G^\sharp$. Let $\mathcal{H}^1 \subseteq \mathcal{H}$ be the closed subspace generated by $\mathcal{F}$ under $G^\sharp$. Then, we obtain a $G^\sharp$-invariant decomposition $\mathcal{H} = \mathcal{H}^1 \oplus \mathcal{H}^2$. Minimality of ρ now implies that it is also $\widehat{G}$-invariant, so that the Hamiltonian H decomposes accordingly as $H = H^1 \oplus H^2$. Since $\mathcal{F} \cap \mathcal{H}^2 = \{0\}$, we obtain $\mathcal{H}^2 = \{0\}$ by minimality of H^2 and the first part of the proof. This shows that $\mathcal{H} = \mathcal{H}^1$ is generated by $\mathcal{F}$ under $G^\sharp$. ■

9.2.2 Classification in terms of UHF C^*-algebras

As in [50], where we dealt with norm continuous representations of gauge groups, we aim at a description of the factor representations of positive energy in terms of C^*-algebras. As semiboundedness is crucial to obtain corresponding C^*-algebras ([81]), we first observe that positive energy representations are semibounded (cf. Definition 6.31).

Applying Corollary 6.33 with $M = S$, we immediately obtain the following result.

Theorem 9.15. *If all connected components of S are compact, then every projective positive energy representation $(\rho, \mathcal{H})$ of*

$$\Gamma_c(S, \mathcal{K}) \cong \prod\nolimits'_{j\in J} \mathcal{L}_{\Phi_j}(K_j)$$

is semibounded with the affine hyperplane $\Gamma_c(S,\mathfrak{K})^\sharp - D$ contained in the open cone W_ρ, so that W_ρ is an open half space. In particular, it is a positive energy representation for all derivations

$$D_A := D - \operatorname{ad} A, \quad A \in \Gamma_c(S, \mathfrak{K}).$$

Let $(\rho, \mathcal{H})$ be a factorial minimal positive energy representation of $\widehat{G}$ and let $(\lambda_j, A_j, T_j)_{j\in J}$ be as above. Since the projection $P_j \colon \mathcal{H}_{\lambda_j} \to V_j^0$ onto the minimal energy space for H_{λ_j} in $\mathcal{H}_{\lambda_j}$ is finite-dimensional, P_j is a compact operator. We may therefore consider the direct limit

$$\mathcal{B} := \bigotimes_{j\in J} \big(K(\mathcal{H}_{\lambda_j}), P_j\big) \tag{9.7}$$

of the C^*-algebras

$$\mathcal{B}_F := \bigotimes_{j\in F} \big(K(\mathcal{H}_{\lambda_j}), P_j\big), \quad F \subseteq J \text{ finite},$$

where the tensor product of the non-unital algebras $K(\mathcal{H}_{\lambda_j})$ is constructed as in [34] with the inclusions

$$\mathcal{B}_{F_1} \hookrightarrow \mathcal{B}_{F_2},$$
$$A \mapsto A \otimes \bigotimes_{j \in F_2 \setminus F_1} P_j$$

for finite subsets $F_1 \subseteq F_2$ of J. We write $B \otimes \bigotimes_{j \in J \setminus F} P_j$ for the image of $B \in \mathcal{B}_F$ in $\mathcal{B}$ and

$$P_\infty := \bigotimes_{j \in J} P_j.$$

If J is finite, then $\mathcal{B} \cong \mathcal{B}_J$ and the above tensor product is finite. The C^*-algebra $\mathcal{B}$ carries a natural one-parameter group of automorphisms $(\alpha_t^{\mathcal{B}})_{t \in \mathbb{R}}$ specified by

$$\alpha_t^{\mathcal{B}}(B) = e^{-itH_F} B e^{itH_F} \quad \text{for } t \in \mathbb{R},\ B \in \mathcal{B}_F,$$

which fixes the projection P_∞.

Since every ground state representation can be written as a direct sum of cyclic ones, we may assume, without loss of generality, that $\mathcal{H}$ has a cyclic ground state $\Omega \in \mathcal{H}$. This defines a state of $\mathcal{B}$ by

$$\omega(B) := \langle \Omega, B\Omega \rangle \quad \text{for } B \in \mathcal{B}_F$$

because P_j projects onto the kernel of H_{λ_j} which contains Ω. Conversely, if $(\pi, \mathcal{H})$ is a representation of the C^*-algebra $\mathcal{B}$ that is generated by a vector Ω with

$$\pi(P_\infty)\Omega = \Omega,$$

then we obtain commuting representations of the multiplier algebras $B(\mathcal{H}_{\lambda_j})$ of $K(\mathcal{H}_{\lambda_j})$. In particular, we recover a unitary representation of the restricted product

$${\prod}'_{j \in J} \mathrm{U}(\mathcal{H}_{\lambda_j}),$$

and hence, a unitary representation of $G^\sharp$. This representation extends canonically to a minimal positive energy representation of $\widehat{G}$, where the Hamiltonian H is determined uniquely by

$$e^{-itH}\pi(B)\Omega = \pi(\alpha_t^{\mathcal{B}}(B))\Omega \quad \text{for } B \in \mathcal{B}.$$

The representations constructed above are now positive energy representations for the C^*-dynamical system $(\mathcal{B}, \mathbb{R}, \alpha^{\mathcal{B}})$ generated by ground states (cf. [13]).

From this correspondence, we derive the following noncompact analog of Theorem 8.10.

Theorem 9.16. *Let $\mathcal{B}$ be the C^*-algebra constructed for $(\lambda_j, A_j, T_j)_{j\in J}$ with a possibly infinite index set J as above. Then, the above construction yields a one-to-one correspondence between the following.*

(a) *Isomorphism classes of minimal factorial positive energy representations of $\widehat{G}$ corresponding to the family $(\lambda_j, A_j, T_j)_{j\in J}$.*

(b) *Isomorphism classes of factorial representations of $\mathcal{B}$ that are generated by fixed points of the projection P_∞.*

Proof. "(a)⇒(b)": Let $(\rho, \mathcal{H})$ be a factorial minimal positive energy representation of $\widehat{G}$ corresponding to the family $(\lambda_j, A_j, T_j)_{j\in J}$. As J is at most countably infinite, we may assume, without loss of generality, that $J = \mathbb{N}$ (the case of finite J is proved along the same lines) and put $\mathcal{B}_n := \mathcal{B}_{F_n}$ for $F_n = \{1, \dots, n\}$. Then, we inductively choose factorizations of $(\rho, \mathcal{H})$ as $(\rho_{F_n} \otimes \rho'_{F_n}, \mathcal{H}_{F_n} \otimes \mathcal{H}'_{F_n})$ with

$$\mathcal{H}_{F_n} = \mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_n}, \quad \rho_{F_n} \cong \rho_{\lambda_1} \otimes \cdots \otimes \rho_{\lambda_n}.$$

We then obtain a consistent sequence of representations of the C^*-algebras $\mathcal{B}_n$ on the subspaces $\mathcal{H}_n := \mathcal{H}_{F_n} \otimes \mathcal{E}'_n$, where $\mathcal{E}'_n \subseteq \mathcal{H}'_{F_n}$ is the minimal eigenspace of H'_F on $\mathcal{H}'_{F_n}$, by

$$\pi_n \colon \mathcal{B}_n \to B(\mathcal{H}_{F_n} \otimes \mathcal{E}'_n), \quad \pi_n(B) := B \otimes \mathbf{1} \quad \text{for } B \in \mathcal{B}_n.$$

As the union of the subspaces $(\mathcal{H}_n)_{n\in\mathbb{N}}$ is dense in $\mathcal{H}$, we thus obtain a non-degenerate representation $(\pi, \mathcal{H})$ of $\mathcal{B}$ satisfying

$$\rho_{F_n}(g)\pi_n(B) = \pi_n(\rho_{F_n}(g)B) \quad \text{for } g \in G^\sharp_{F_n}, B \in \mathcal{B}_n, \tag{9.8}$$

and $\pi(P_\infty)$ is the projection onto the minimal eigenspace of H. Note that (9.8) determines the representation ρ uniquely in terms of the representation $(\pi, \mathcal{H})$ of $\mathcal{B}$.

"(b)⇒(a)": Suppose, conversely, that $(\pi, \mathcal{H})$ is a factorial representation of $\mathcal{B}$ generated by the subspace $\mathcal{E} := P_\infty\mathcal{H}$. Then, the union of the closed subspaces

$$\mathcal{H}_n := \pi(\mathcal{B}_n)\pi(P_\infty)\mathcal{H}$$

is dense in $\mathcal{H}$. Since the representation of $\mathcal{B}_n \cong \mathcal{K}(\mathcal{H}_{F_n})$ on $\mathcal{H}_n$ is non-degenerate, we obtain consistent factorizations

$$\mathcal{H}_n \cong \mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_n} \otimes \mathcal{E}'_n \cong \mathcal{H}_{F_n} \otimes \mathcal{E}'_n \quad \text{with } \pi(B) = B \otimes \mathbf{1}_{\mathcal{E}'_n},\ B \in \mathcal{B}_n.$$

This implies the existence of smooth unitary representations ρ_n of the groups $G^\sharp_{F_n}$ on $\mathcal{H}_n$ which are uniquely determined by

$$\rho_n(g)\pi(B)\pi(P_\infty) = \pi(\rho_{F_n}(g)B)\pi(P_\infty) \quad \text{for } g \in G^\sharp_{F_n}, B \in \mathcal{B}_n.$$

The uniqueness implies that $\rho_{n+1}(g)|_{\mathcal{H}_n} = \rho_n(g)$ for $g \in G^\sharp_{F_n}$, so that we obtain a unitary representation of $G^\sharp = \bigcup_F G^\sharp_F$ on $\mathcal{H}$ which naturally extends to $\widehat{G} = G^\sharp \rtimes \mathbb{R}$. Its continuity follows from [27, Lemma 4.4].

For the smoothness we use [114, Theorem 2.9]: The Lie algebra $\widehat{\mathfrak{g}}$ is the union of the subalgebras $\widehat{\mathfrak{g}}_{F_n}$, and the representation is smooth on the corresponding subgroup $\widehat{G}_{F_n}$. Further, the element $D \in \bigcap_n \widehat{\mathfrak{g}}_{F_n}$ lies in the interior of the open cones

$$W_{\rho|_{\widehat{G}_{F_n}}} \supseteq \mathfrak{g}^\sharp_{F_n} + (0,\infty)D,$$

which are open half spaces (Theorem 9.15). To apply Zellner's theorem, we have to show that the groups $\widehat{G}_{F_n}$ have the Trotter property, i.e., for any two elements x, y in the Lie algebra, we have

$$\exp(t(x+y)) = \lim_{n\to\infty}\left(\exp\left(\frac{t}{n}x\right)\exp\left(\frac{t}{n}y\right)\right)^n$$

in the sense of uniform convergence on compact subsets of $\mathbb{R}$. We first use [80, Theorem 4.11] to see that $G_{F_n} \rtimes \mathbb{R}$ has the Trotter property; as these groups are C^0-regular ([29, Theorem J]) and [80, Theorem 4.15] implies that the central extension $\widehat{G}_{F_n}$ also has the Trotter property. As any two elements $x, y \in \widehat{\mathfrak{g}}$ are contained in some $\widehat{\mathfrak{g}}_{F_n}$, the group $\widehat{G}$ also has the Trotter property. Therefore, [114, Theorem 2.9(a)] implies that the dense subspace $\mathcal{D}^\infty(\mathtt{d}\rho(D))$ of smooth vectors of the Hamiltonian coincides with $\mathcal{D}^\infty_c(\widehat{\mathfrak{g}})$, the set of all vectors ξ in the common domain of all finite products of elements in $\widehat{\mathfrak{g}}$, for which all maps

$$\widehat{\mathfrak{g}}^n \to \mathcal{H}, \quad (x_1,\ldots,x_n) \mapsto \mathtt{d}\rho(x_1)\cdots\mathtt{d}\rho(x_n)\xi$$

are continuous and n-linear. As the subgroup $G^\sharp$ is locally exponential (see [74, Lemma 4.3]) now implies that ξ is a smooth vector for $G^\sharp$, and since it is also smooth for $H = i\overline{\mathtt{d}\rho(D)}$, [74, Theorem 7.2] further entails that it is smooth for $\widehat{G}$. This proves the smoothness of ρ.

Clearly, the two constructions are mutually inverse, up to unitary equivalence. ■

Remark 9.17. (a) By Lemma 9.12, the preceding theorem covers all minimal factorial representations for which $(\lambda_j, A_j, T_j)_{j\in J}$ satisfies the spectral gap condition.

(b) The projection $P_\infty \in \mathcal{B}$ defines the hereditary subalgebra $\mathcal{A} := P_\infty \mathcal{B} P_\infty$ onto which

$$\varepsilon\colon \mathcal{B} \to \mathcal{A}, \quad B \mapsto P_\infty B P_\infty$$

defines a conditional expectation, so, in particular, a completely positive map. From this perspective, the representations specified in Theorem 9.16 are precisely those obtained by Stinespring dilation from the completely positive maps that have the form $\omega = \pi \circ \varepsilon$, where $(\pi, \mathcal{F})$ is a non-degenerate representation of $\mathcal{A}$.

For $n_j := \operatorname{tr} P_j$, we have

$$\mathcal{A} \cong \bigotimes_{j \in J} M_{n_j}(\mathbb{C}),$$

showing that $\mathcal{A}$ is a UHF algebra [93]. The representation theory of these algebras also appears naturally in the context of norm continuous representations of gauge groups (cf. [50]). If infinitely many of the n_j are > 1, this leads to factor representations of type II and III. So the situation depends on the size of the minimal energy spaces in $\mathcal{H}_{\lambda_j}$. In particular, we obtain factorial representations as infinite tensor products corresponding to factorial product states on $\mathcal{A}$ because they correspond to product states on $\mathcal{B}$. We refer to [50] for details on the connection between norm-continuous representations of the restricted product $\prod'_{j \in J} K_j$ of the compact groups K_j and representations of infinite tensor products of matrix algebras.

9.3 A simple example with fixed points

In Part II of this series, we will focus on the type of phenomena one encounters when the $\mathbb{R}$-action on M is *not* fixed point free. To give a preview of the problems one encounters there, we briefly revisit the simple example of the circle action on $\mathbb{S}^2$, lifted to an $\mathbb{R}$-action on the trivial bundle $\mathcal{K} = \mathbb{S}^2 \times K$ (cf. Example 7.9). The fixed points are then the “north pole” $n = (0, 0, 1)$ and the “south pole” $(0, 0, -1)$.

Since every projective positive energy representation of $G = C^\infty(\mathbb{S}^2, K)$ restricts to a projective positive energy representation of the normal subgroup

$$G^\times = C_c^\infty(\mathbb{S}^2 \setminus \{n, s\}, K),$$

we can apply the techniques developed so far to $G^\times$. The two problems that remain are then to determine if a representation extends from $G^\times$ to G, and, if so, to classify the possible extensions. We will pursue these problems elsewhere, and for the moment content ourselves with describing the representation theory of $G^\times$. Although the Lie algebra bundle $\mathfrak{K} \to \mathbb{S}^2$ is trivial, the $\mathbb{R}$-action (7.5) on $\mathfrak{K}$ that covers the circle action on $\mathbb{S}^2$ will in general not be trivializable. It turns out that the lift of the circle action at the fixed points $n, s \in \mathbb{S}^2$ has a qualitative effect on the positive energy representation theory of $G^\times$.

By Theorem 7.1, every projective positive energy representation of $G^\times$ factors through a projective positive energy representation of $C_c^\infty(S, K)$, where

$$S = \big\{(x, y, z) \in \mathbb{S}^2 : z \in J\big\}$$

is a union of circles labeled by a discrete subset $J \subset (-1, 1)$ that has at most two accumulation points ± 1, corresponding to the fixed points n and s. Recall from

Example 7.9 that the fundamental vector field for the $\mathbb{R}$-action is of the form

$$\mathbf{v}(x,y,z) = (y\partial_x - x\partial_y) + A(x,y,z),$$

with $A(x,y,z) \in \mathfrak{k}$. For simplicity, consider first the case where $A \in \mathfrak{k}$ is *independent* of (x,y,z). If we identify the loop algebras of the various circles in the obvious manner, then the infinitesimal $\mathbb{R}$-action is represented by the *same* element $\mathbf{d}_j + A_j = \mathbf{d} + A$ for every circle S_j. It follows that $(\lambda_j, A_j, T_j) = (\lambda_j, A, 2\pi)$, so the C^*-algebra $\mathcal{A} = P_\infty \mathcal{B} P_\infty$ that governs the ground state representations is essentially determined by a sequence λ_j of anti-dominant integral weights for the affine Kac–Moody algebra $\widehat{\mathcal{L}}(\mathfrak{k})$.

The operators $H_j = i\pi_{\lambda_j}(\mathbf{d} + A)$ are readily seen to satisfy the spectral gap property 9.13. Indeed, the operators $i\pi_{\lambda_j}(\mathbf{d})$ on $\mathcal{H}_{\lambda_j}$ have a uniform spectral gap because the $\mathbb{R}$-action on $\mathbb{S}^2$ is periodic. Since $A \in \mathfrak{k}$ has a uniform spectral gap in all finite-dimensional lowest weight representations, it also has a uniform spectral gap in the minimal eigenspaces W_j^0 of the operators $i\pi_{\lambda_j}(\mathbf{d})$. The spectral gap for H_j then follows from the fact that $\mathbf{d}$ commutes with A in $\widehat{\mathcal{L}}(\mathfrak{k})$.

By Proposition 9.14, every factorial positive energy representation of $\widehat{G}^\times$ is a ground state representation, so the factorial projective positive energy representations are completely classified by Theorem 9.16.

(a) If A is an inner point of the Weyl chamber, then the minimal eigenspace of H_j in an irreducible $\mathfrak{k}$-representation is always 1-dimensional, $W_j^0 = \mathbb{C}\Omega_j$. In this case every projective irreducible positive energy representation is a *vacuum representation*, and it is of the form

$$(\mathcal{H}, \Omega) = \bigotimes_{j \in j} (\mathcal{H}_{\lambda_j}, \Omega_j) \tag{9.9}$$

by the results in Section D.2. Moreover, every factorial positive energy representation of $\widehat{G}^\times$ is of type I, i.e., a direct sum of irreducible representations. This follows from the fact that, if in the construction of Section 9.2 all projections P_j are of rank 1, then the projection $P_\infty \in \mathcal{B}$ has the property that the subalgebra $P_\infty \mathcal{B} P_\infty$ is one-dimensional. In particular, $P_\infty a P_\infty = \varphi(a) P_\infty$ defines a state of $\mathcal{B}$ and every representation $(\pi, \mathcal{H})$ of $\mathcal{B}$ generated by the range of $\pi(P_\infty)$ is a multiple of the GNS representation $(\pi_\varphi, \mathcal{H}_\varphi)$. For unit vectors $\Omega_j \in \mathrm{im}(P_j)$, (9.7) implies that

$$(\mathcal{H}_\varphi, \Omega_\varphi) \cong \bigotimes_{j \in J} (\mathcal{H}_{\lambda_j}, \Omega_j),$$

that the representation π_φ is faithful, and that $\pi_\varphi(\mathcal{B}) \cong K(\mathcal{H}_\varphi)$.

(b) If A lies in at least one face of the Weyl alcove, then the space W_j^0 of ground states need not be 1-dimensional. The projective positive energy factor representations of $\mathfrak{g}^\times$ are then classified by the lowest weights λ_j, together with a representation of the UHF C^*-algebra

$$\mathcal{A} = \bigotimes_{j \in J} B(W_j^0).$$

If W_j^0 is of dimension > 1 for infinitely many j, then this is an infinite tensor product of matrix algebras. By [93] it follows that $G^\times$ admits factor representations of type II and III.

If $A(x, y, z)$ is not constant and $A(n)$ and $A(s)$ are inner points of the Weyl alcove, then the situation remains qualitatively the same as in (a). Indeed, the holonomy with respect to A on S_j will approach $\exp(A(n))$ or $\exp(A(s))$ as $z_j \to \pm 1$, so the spectral gap condition holds for all but finitely many circles. In this case one finds a tensor product decomposition analogous to (9.9), where all but finitely terms are vacuum representations. In particular, the space of ground states is finite-dimensional.

However, if either $A(n)$ or $A(s)$ is not an inner point of the Weyl alcove, then the spectral gap condition need no longer be satisfied. The ground state representations can still be classified in the manner outlined above, but these can no longer be expected to exhaust the positive energy representations.

Appendix A

Twisted loop algebras and groups

Let K be a simple compact Lie group, $\Phi \in \mathrm{Aut}(K)$ and $\varphi = \mathbf{L}(\Phi) \in \mathrm{Aut}(\mathfrak{k})$. We assume that $\varphi^N = \mathrm{id}_K$ and let

$$\mathcal{L}_\Phi^T(K) := \left\{\xi \in C^\infty(\mathbb{R}, K) : (\forall t \in \mathbb{R})\xi(t+T) = \Phi^{-1}(\xi(t))\right\}$$

be the corresponding *twisted loop group*. The rotation action $(\alpha_t f)(s) := f(s+t)$ satisfies

$$\alpha_{NT} = \mathrm{id}_{\mathcal{L}_\Phi(K)}.$$

The Lie algebra of $\mathcal{L}_\varphi^T(K)$ is the twisted loop algebra

$$\mathcal{L}_\varphi^T(\mathfrak{k}) := \left\{\xi \in C^\infty(\mathbb{R}, \mathfrak{k}) : (\forall t \in \mathbb{R})\xi(t+T) = \varphi^{-1}(\xi(t))\right\}.$$

Accordingly, we obtain

$$\widehat{\mathcal{L}}_\varphi^T(\mathfrak{k}) := \left(\mathbb{R} \oplus_\omega \mathcal{L}_\varphi^T(\mathfrak{k})\right) \rtimes_D \mathbb{R}, \quad D\xi = \xi',$$

where

$$\omega(\xi, \eta) := \frac{c}{2\pi T}\int_0^T \kappa(\xi'(t), \eta(t))dt$$

for some $c \in \mathbb{Z}$ (the *central charge*). Here κ is the Killing form of $\mathfrak{k}$, normalized as in (4.2) by $\kappa(i\alpha^\vee, i\alpha^\vee) = 2$ for the coroots corresponding to long roots.

We write

$$\mathcal{L}_\varphi^{T,\sharp}(\mathfrak{k}) = \mathbb{R} \oplus_\omega \mathcal{L}_\varphi^T(\mathfrak{k}).$$

Let $\mathfrak{t}^\circ \subseteq \mathfrak{k}^\varphi$ be a maximal abelian subalgebra, so that $\mathfrak{z}_\mathfrak{k}(\mathfrak{t}^\circ)$ is maximal abelian in $\mathfrak{k}$ by [79, Lemma D.2] (see also [54]). Then, $\mathfrak{t} = \mathbb{R} \oplus \mathfrak{t}^\circ \oplus \mathbb{R}$ is maximal abelian in $\widehat{\mathcal{L}}_\varphi^T(\mathfrak{k})$ and the corresponding set of roots Δ can be identified with the set of pairs (α, n), where

$$(\alpha, n)(z, h, s) := (0, \alpha, n)(z, h, s) = \alpha(h) + is\frac{2\pi n}{NT}, \quad n \in \mathbb{Z}, \alpha \in \Delta_n. \tag{A.1}$$

Here, $\Delta_n \subseteq i(\mathfrak{t}^\circ)^*$ is the set of $\mathfrak{t}^\circ$-weights in

$$\mathfrak{k}_\mathbb{C}^n = \left\{x \in \mathfrak{k}_\mathbb{C} : \varphi^{-1}(x) = e^{2\pi in/N}x\right\}.$$

For $(\alpha, n) \neq (0, 0)$, the corresponding root space is

$$\mathcal{L}_\varphi^{T,\sharp}(\mathfrak{k}_\mathbb{C})^{(\alpha,n)} = \mathfrak{k}_\mathbb{C}^{(\alpha,n)} \otimes e_n = (\mathfrak{k}_\mathbb{C}^\alpha \cap \mathfrak{k}_\mathbb{C}^n) \otimes e_n, \quad \text{where } e_n(t) = e^{\frac{2\pi int}{NT}}.$$

The set

$$\Delta^\times = \{(\alpha, n) : 0 \neq \alpha \in \Delta_n, n \in \mathbb{Z}\}$$

has an N-fold layer structure

$$\Delta^\times = \bigcup_{n=0}^{N-1} \Delta_n^\times \times (n + N\mathbb{Z}), \quad \text{where } \Delta_n^\times := \Delta_n \setminus \{0\}.$$

For $n \in \mathbb{Z}$ and $x \in \mathfrak{k}_{\mathbb{C}}^{(\alpha,n)}$ with $[x, x^*] = \alpha^\vee$, the element $e_n \otimes x \in \mathcal{L}_\varphi^{T,\sharp}(\mathfrak{k}_{\mathbb{C}})^{(\alpha,n)}$ satisfies $(e_n \otimes x)^* = e_{-n} \otimes x^*$, which leads to the coroot

$$[e_n \otimes x, (e_n \otimes x)^*] = (\alpha, n)^\vee = \left(-i \frac{cn}{NT} \frac{\|\alpha^\vee\|^2}{2}, \alpha^\vee, 0 \right) = \alpha^\vee - \frac{icn}{NT} \frac{\|\alpha^\vee\|^2}{2} C, \tag{A.2}$$

where $C = (1, 0, 0)$. Here, we have used that

$$\omega(e_n \otimes x, e_{-n} \otimes x^*) = \frac{icn}{NT} \kappa(x, x^*)$$

and

$$\kappa(x, x^*) = \frac{1}{2}\kappa([\alpha^\vee, x], x^*) = \frac{1}{2}\kappa(\alpha^\vee, [x, x^*]) = \frac{1}{2}\kappa(\alpha^\vee, \alpha^\vee) = -\frac{1}{2}\|\alpha^\vee\|^2.$$

Since $\mathfrak{k}$ is simple, $\Delta^\times$ does not decompose into two mutually orthogonal proper subsets ([79, Lemma D.3]), so that

$$\widehat{\mathcal{L}}_\varphi^T(\mathfrak{k})_{\mathbb{C}}^{\mathrm{alg}} := \mathfrak{t}_{\mathbb{C}} + \sum_{(\alpha,n) \in \Delta} \mathcal{L}_\varphi^{T,\sharp}(\mathfrak{k}_{\mathbb{C}})^{(\alpha,n)}$$

is an affine Kac–Moody–Lie algebra (see [54, Theorem 8.5] and [38, Chapter X]). In this context the root (α, n) is real if and only if $\alpha \neq 0$. Choosing a positive system $\Delta^+ \subseteq \Delta$ such that the roots (α, n), $n > 0$, are positive, the lowest weights of unitary lowest weight representations of $\widehat{\mathcal{L}}_\varphi^T(\mathfrak{k})$ are the anti-dominant integral weights

$$\mathcal{P}(\mathfrak{t}, \Delta^+) := \{\lambda \in i\mathfrak{t}^* : (\forall(\alpha, n))\, 0 \neq \alpha, (\alpha, n) \in \Delta^+ \Rightarrow \lambda((\alpha, n)^\vee) \in \mathbb{N}_0\}.$$

Note that, for $n > 0$, we have

$$\lambda((\alpha, n)^\vee) = \lambda(\alpha^\vee) + \frac{cn}{NT} \frac{\|\alpha^\vee\|^2}{2},$$

so that we obtain $c > 0$ as a necessary condition for the existence of non-trivial unitary lowest weight modules.

Appendix B

Twisted conjugacy classes in compact groups

In this appendix we collect some more details concerning twisted conjugacy classes in compact groups.

A *Cartan subgroup* of a compact Lie group K is an abelian subgroup S topologically generated by a single element s ($s^{\mathbb{Z}}$ is dense in S) which has finite index in its normalizer $N_K(S) = \{k \in K : kSk^{-1} = S\}$.

Remark B.1. (a) For any Cartan subgroup S, the identity component S_0 is an abelian compact Lie group, hence a torus, and since tori are divisible, the short exact sequence $S_0 \hookrightarrow S \twoheadrightarrow \pi_0(S)$ splits, so that $S \cong S_0 \times \pi_0(S)$. By construction, $\pi_0(S)$ is a finite cyclic group. If $s_0 \in S_0$ is a topological generator, then, for every $N \in \mathbb{Z}$, the closure of $s_0^{N\mathbb{Z}}$ is a closed subgroup of finite index in S_0, hence equal to S_0. This implies that the topological generators of S are the elements of the form $s = (s_0, s_1) \in S_0 \times \pi_0(S)$, where s_0 is a topological generator of S_0 and s_1 is a generator of the cyclic group $\pi_0(S)$.

(b) By [14, Proposition IV.4.2], every element $k \in K$ is contained in a Cartan subgroup S such that the connected component kS_0 generates $\pi_0(S)$. The preceding discussion now shows that there exists an element $s_0 \in S_0$ such that $z := ks_0$ is a topological generator of S. Now [14, Proposition IV.4.3] implies that every element $g \in kK_0 = zK_0$ is conjugate to an element of kS_0.

Theorem B.2. *Let K be a compact connected Lie group and $\Phi \in \mathrm{Aut}(K)$ be an automorphism of finite order $N \in \mathbb{N}$. We consider the twisted conjugation action of K on itself given by*

$$g * k := gk\Phi(g)^{-1} \quad \textit{for } g, k \in K.$$

Then, the orbit of every element in K under this action intersects a maximal torus T^Φ of the subgroup K^Φ of Φ-fixed points.

Proof. We consider the compact Lie group $K_1 := K \rtimes \Phi^{\mathbb{Z}}$, where $\Phi^{\mathbb{Z}} \subseteq \mathrm{Aut}(K)$ is the finite subgroup generated by Φ. For $g, k \in K$, we then have

$$(g, \mathbf{1})(k, \Phi)(g, \mathbf{1})^{-1} = (gk\Phi(g)^{-1}, \Phi),$$

so that the conjugacy classes in the coset $K \times \{\Phi\} \subseteq K_1$ correspond to the Φ-twisted conjugacy classes in K.

According to Remark B.1 (b), the element $(\mathbf{1}, \Phi) \in K_1$ is contained in a Cartan subgroup S which is generated by an element of the form $z = (s_0, \Phi)$. As S_0 is abelian and commutes with $(\mathbf{1}, \Phi)$, it is contained in K^{Φ}. Let $T^{\Phi} \subseteq K^{\Phi}$ be a maximal torus containing S_0. Then, T^{Φ} commutes with S, so that the finiteness of $N_K(S)/S$ shows that $T^{\Phi} \subseteq S_0$. We conclude that

$$S = T^{\Phi} \times \Phi^{\mathbb{Z}}$$

is a Cartan subgroup of K_1. Therefore, Remark B.1 (b) implies that every Φ-twisted conjugacy class in K intersects $S_0 = T^{\Phi} \subseteq K^{\Phi}$. ■

We refer to [65] for more details on twisted conjugacy classes in compact groups, representatives, and stabilizer groups.

Remark B.3. If Φ is not of finite order, then the situation is more complicated. If, however, K is a compact Lie group with semisimple Lie algebra, then $\mathrm{Aut}(K)$ is a compact group with the same Lie algebra and one can apply the theory of Cartan subgroups of compact Lie groups to $\mathrm{Aut}(K)$.

Appendix C

Restricting representations to normal subgroups

Theorem C.1. *Let G be a group, and let $N \trianglelefteq G$ be a normal subgroup of finite index. Suppose that $(\pi, \mathcal{H})$ is a unitary representation of G whose restriction $\pi|_N$ decomposes discretely with finitely many isotypic components. Then, the same holds for π.*

Proof. We consider the two von Neumann algebras

$$\mathcal{N} := \pi(N)'' \subseteq \mathcal{M} := \pi(G)''.$$

Let

$$\mathcal{H} = \bigoplus_{j=1}^{m} \mathcal{H}_j, \quad \text{with } \mathcal{H}_j = \mathcal{F}_j \otimes \mathcal{C}_j$$

be the isotypic decomposition for N, where the representations $(\rho_j, \mathcal{F}_j)$ of N are irreducible and N acts on $\mathcal{H}_j$ by $\pi_j := \rho_j \otimes \mathbf{1}$. Then

$$\mathcal{N}' = \pi(N)' \cong \bigoplus_{j=1}^{m} B(\mathcal{C}_j).$$

The conjugation action of G on $\mathcal{N}'$ factors through an action of the finite group G/N. We have to show that $\mathcal{M}' = (\mathcal{N}')^{G/N}$ also is a finite direct sum of full operator algebras.

Let $F := \{[\rho_j] : j = 1, \ldots, m\} \subseteq \widehat{N}$ be the support of the restriction $\rho|_N$. This set decomposes under the natural action of G/N on the unitary dual $\widehat{N}$ into finitely many orbits $F_1, \ldots, F_k$. The group G permutes the isotypic subspaces $\mathcal{H}_j$ of N and, accordingly,

$$\pi_k \cong \pi_j \circ c_g^{-1}|_N \ \text{ if and only if } \ \pi(g)\mathcal{H}_j = \mathcal{H}_k.$$

This follows from the relation $\pi_k(n)\pi(g) = \pi(g)\pi_j(g^{-1}ng)$ for $g \in G, n \in N$. We conclude that

$$P_j := \big\{g \in G : \pi(g)\mathcal{H}_j = \mathcal{H}_j\big\} = \big\{g \in G : \rho_j \circ c_g \cong \rho_j\big\}.$$

For every $g \in P_j$, we thus obtain a unitary operator $U_g \colon \mathcal{F}_j \to \mathcal{F}_j$ such that

$$\rho \circ c_g = U_g \rho U_g^{-1}.$$

Since $\mathcal{F}_j$ is irreducible, U_g is well defined modulo $\mathbb{T}$. The projective unitary representation $\bar{\rho}_j(g) = [U_g]$ of P_j yields a central extension $q_j \colon P_j^\sharp \to P_j$, a homomorphic lift $N \hookrightarrow P_j^\sharp$ and an extension $\rho_j^\sharp\colon P_j^\sharp \to \mathrm{U}(\mathcal{F}_j)$ of the unitary representation ρ_j of N to $P_j^\sharp$. Accordingly, the representation of $P_j^\sharp$ on $\mathcal{H}_j$ takes the form

$$\pi_j(q_j(p)) = \rho_j^\sharp(p) \otimes \beta_j(p),$$

where $\beta_j\colon P_j^\sharp \to \mathrm{U}(\mathcal{C}_j)$ a unitary representation with $\ker\beta_j \supseteq N$.

Let $\mathcal{H} = \mathcal{H}^1 \oplus \cdots \oplus \mathcal{H}^k$ denote the decomposition of $\mathcal{H}$ under G, corresponding to the decomposition of F under G/N, so that each subspace $\mathcal{H}^j$ is a sum of certain subspaces $\mathcal{H}_\ell$. We may assume, without loss of generality, that G/N acts transitively on F, i.e., that

$$\mathcal{H} = \operatorname{span}(\pi(G)\mathcal{H}_1) = \bigoplus_{[g]\in G/P_1} \pi(g)\mathcal{H}_1.$$

This means that $(\pi, \mathcal{H})$ is induced from the representation $\rho_1^\sharp \otimes \beta_1$ of P_1 on $\mathcal{H}_1$.

The subspace $\mathcal{H}_1$ is generating for G, and hence separating for the commutant $\mathcal{M}'$. As $\mathcal{H}_1$ is isotypic for N, the commutant $\mathcal{M}' \subseteq \mathcal{N}'$ leaves $\mathcal{H}_1$ invariant; likewise all subspaces $\mathcal{H}_j$ are $\mathcal{M}'$-invariant. Since an operator $A \in B(\mathcal{H}_1)$ extends to an element of $\mathcal{M}'$ if and only if it commutes with P_1, we have

$$\mathcal{M}' \cong (\rho_1^\sharp \otimes \beta_1)(P_1)' = \mathbf{1} \otimes \beta(P_1^\sharp)'.$$

Since $\beta(P_1^\sharp)$ is a finite group, the assertion follows from the fact that every unitary representation of a finite group decomposes discretely with finitely many isotypes. ■

Appendix D

Vacuum representations

In this appendix, we show that vacuum representations of weak products of topological groups arise as products of vacuum representations.

D.1 Weak products and $\mathbb{R}$-actions

The *weak product* of a sequence $(G_n)_{n\in\mathbb{N}}$ of topological groups is defined as

$$G := \prod\nolimits'_{n\in\mathbb{N}} G_n = \bigcup_{N=1}^{\infty} G^N, \quad G^N = G_1 \times \cdots \times G_N,$$

where the group structure is inherited from the product group $\prod_{n\in\mathbb{N}} G_n$. However, we will need a topology that is finer than the product topology. We equip G with the *box topology*, for which a basis of e-neighborhoods consists of the sets $G \cap \prod_{n=1}^{\infty} U_n$, where $U_n \subseteq G_n$ is an e-neighborhood in G_n. By [27, Lemma 4.4], this turns G into a topological group, and G is the direct limit in the category of topological groups of the increasing sequence of subgroups G^N, endowed with the product topology.

To study vacuum representations of weak products, consider a sequence of topological groups $(G_n, \mathbb{R}, \alpha_n)_{n\in\mathbb{N}}$ with homomorphisms $\alpha_n : \mathbb{R} \to \mathrm{Aut}(G_n)$ that defines a continuous action of $\mathbb{R}$ on G_n. The homomorphisms $\alpha_n : \mathbb{R} \to \mathrm{Aut}(G_n)$ combine to a homomorphism $\alpha : \mathbb{R} \to \mathrm{Aut}(G)$ by

$$\alpha_t(g_1, \ldots, g_N, e, \ldots) := (\alpha_{1,t}(g_1), \ldots, \alpha_{N,t}(g_N), e, \ldots),$$

where $\mathrm{Aut}(G)$ denotes the group of topological automorphisms.

Proposition D.1. *The above map α is a continuous action of $\mathbb{R}$ on G.*

Proof. To see this, we first note that all orbit maps are continuous because the subgroups G^N carry the product topology. Since all automorphisms α_t are continuous by [27, Lemma 4.4], it suffices to verify continuity of the action in all pairs $(0, g) \in \mathbb{R} \times G^N$. So we have to find for every sequence $(U_n)_{n\in\mathbb{N}}$ of e-neighborhoods in G_n an $\varepsilon > 0$ and a sequence of e-neighborhoods $V_n \subseteq G_n$ such that

$$\alpha_{n,t}(g_n V_n) \subseteq g_n U_n \quad \text{for } |t| < \varepsilon, n \in \mathbb{N}.$$

As $[-1, 1] \subseteq \mathbb{R}$ is compact, we find for every $n \in \mathbb{N}$ an identity neighborhood $V_n \subseteq W_n \subseteq G_n$ such that $W_n W_n \subseteq U_n$ and $\alpha_{n,t}(V_n) \subseteq W_n$ for $|t| \leq 1$. For $n \leq N$ we now choose $\varepsilon > 0$ in such a way that $\alpha_{n,t}(g_n) \in g_n W_n$ holds for $|t| \leq \varepsilon$. Then

$$\alpha_{n,t}(g_n V_n) = \alpha_{n,t}(g_n)\alpha_{n,t}(V_n) \subseteq g_n W_n W_n \subseteq g_n U_n$$

holds for $|t| < \varepsilon$ and $n \leq N$. For $n > N$, we have $g_n = e$ and

$$\alpha_{n,t}(g_n V_n) = \alpha_{n,t}(V_n) \subseteq W_n \subseteq U_n \quad \text{for } |t| \leq \varepsilon.$$

Therefore, α defines a continuous action on the weak direct product G. ■

If, in addition, the groups G_n are Lie groups, then the box topology on G is compatible with a Lie group structure on G ([27, Remark 4.3]).

Lemma D.2. *If all groups G_n are locally exponential, then α defines a smooth action on G.*

Proof. By [27, Remark 4.3], the group G is locally exponential as well. Therefore, it suffices to show that the $\mathbb{R}$-action on the Lie algebra $\mathfrak{g} \cong \bigoplus_{n\in\mathbb{N}} \mathfrak{g}_n$ (the locally convex direct sum), is smooth. Let $D_n \in \mathrm{der}(\mathfrak{g}_n)$ denote the infinitesimal generator of the smooth actions α^n on $\mathfrak{g}_n$. Then

$$\alpha(t,x) = (e^{tD_n} x_n)_{n\in\mathbb{N}} = e^{tD} x \quad \text{for } D(x_n) = (D_n x_n)$$

and the tangent map of α is given by

$$\mathrm{d}\alpha(t,x)(s,y) = sD(\alpha(t,x)) + \alpha(t,y).$$

As $D\colon \mathfrak{g} \to \mathfrak{g}$ is a continuous linear operator, we inductively obtain from the continuity of α (Proposition D.1) that α is C^k for each $k \in \mathbb{N}$, and hence that α is smooth. ■

The weak products encountered in this memoir are mostly of the following form.

Lemma D.3. *Suppose that the smooth manifold S has countably many connected components and that $\mathcal{K} \to S$ is a Lie group bundle. Then, the Lie group $\Gamma_c(\mathcal{K})$ is isomorphic to the restricted Lie group product $\prod'_{n\in\mathbb{N}} \Gamma_c(\mathcal{K}|_{S_n})$.*

Proof. Since the groups $G = \Gamma_c(\mathcal{K})$ and $G_n = \Gamma_c(\mathcal{K}|_{S_n})$ are locally exponential, it suffices to verify that the Lie algebra $\mathfrak{g} = \Gamma_c(\mathfrak{K})$ is the locally convex direct sum of the ideals $\mathfrak{g}_n = \Gamma_c(\mathfrak{K}|_{S_n})$. That the summation map

$$\Phi\colon \quad \bigoplus_{n\in\mathbb{N}} \Gamma_c(\mathfrak{K}|_{S_n}) \to \mathfrak{g}$$

is continuous follows from the universal property of the locally convex direct sum. That its inverse Φ^{-1} is also continuous, follows from its continuity on the Fréchet subspaces $\Gamma_D(\mathcal{K})$, where $D \subseteq S$ is compact, because any compact subset intersects at most finitely many connected components. ■

D.2 Vacuum representations

Let G be a topological group, and let $\alpha: \mathbb{R} \to \mathrm{Aut}(G)$ be a homomorphism that defines a continuous action of $\mathbb{R}$ on G.

Definition D.4. A triple $(\rho, \mathcal{H}, \Omega)$ is called a *vacuum representation* of $(G, \mathbb{R}, \alpha)$, if $\rho: G \rtimes_\alpha \mathbb{R} \to \mathrm{U}(\mathcal{H})$ is a continuous unitary representation, $\Omega \in \mathcal{H}$ is a G-cyclic unit vector, and the selfadjoint operator H, defined by $U_t := \rho(e, t) = e^{-itH}$ for $t \in \mathbb{R}$, satisfies $\ker(H - E_0\mathbf{1}) = \mathbb{C}\Omega$ for $E_0 = \inf(\mathrm{spec}(H))$.

The following is an immediate consequence of [8, Proposition 5.4].

Proposition D.5. *For a vacuum representation* $(\rho, \mathcal{H}, \Omega)$ *of* $(G, \mathbb{R}, \alpha)$*, the following assertions hold:*

(a) $U_\mathbb{R} \subseteq \rho(G)''$,

(b) *the representation* $\rho|_G$ *of* G *on* $\mathcal{H}$ *is irreducible.*

Proof. (a) The one-parameter group $(U_t^0)_{t\in\mathbb{R}}$ defined by $U_t^0 := e^{itE_0} U_t$ is minimal for the von Neumann algebra $\rho(G)''$ (cf. Definition 3.8) by [8, Proposition 5.4], hence contained in $\rho(G)''$, and this implies (a).

(b) From (a) it follows that the closed subspace

$$\mathbb{C}\Omega = \ker(H - E_0\mathbf{1}) \subseteq \mathcal{H}$$

is invariant under the commutant $\mathcal{M}' := \rho(G)'$ of $\mathcal{M} := \rho(G)''$. As Ω is generating for $\mathcal{M}$, it is separating for $\mathcal{M}'$, so that $\dim \ker(H_0 - E_0\mathbf{1}) = 1$ leads to $\mathcal{M}' = \mathbb{C}\mathbf{1}$. Now the assertion follows from Schur's Lemma. ■

Let $(G_n, \mathbb{R}, \alpha_n)_{n\in\mathbb{N}}$ be a sequence of topological groups, with for each $n \in \mathbb{N}$ a homomorphism $\alpha_n: \mathbb{R} \to \mathrm{Aut}(G_n)$ that defines a continuous action of $\mathbb{R}$ on G_n. The following theorem identifies the vacuum representations of the weak product $(G, \mathbb{R}, \alpha)$ in terms of vacuum representations of the triples $(G_n, \mathbb{R}, \alpha_n)$.

Theorem D.6. *For any sequence* $(\rho_n, \mathcal{H}_n, \Omega_n)$ *of vacuum representations of* $(G_n, \mathbb{R}, \alpha_n)$ *with minimal energy* $E_0 = 0$*, the infinite tensor product*

$$(\mathcal{H}, \Omega) := \bigotimes_{n=1}^{\infty} (\mathcal{H}_n, \Omega_n) \tag{D.1}$$

carries a continuous vacuum representation of $(G, \mathbb{R}, \alpha)$*, defined by*

$$\rho(g_1, \ldots, g_n, e, \ldots) := \rho_1(g_1) \otimes \cdots \otimes \rho_n(g_n) \otimes \mathbf{1}_{n+1} \otimes \cdots. \tag{D.2}$$

Conversely, every vacuum representation of $(G, \mathbb{R}, \alpha)$ *with* $E_0 = 0$ *is equivalent to such a representation.*

Proof. First, we prove that if all $(\rho_n, \mathcal{H}_n, \Omega_n)$ are vacuum representations, then so is their infinite tensor product. Since the Ω_n are unit vectors, the infinite tensor product Hilbert space $\mathcal{H}$ is defined. It contains the subspaces

$$\mathcal{H}^N := \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N \otimes \Omega_{N+1} \otimes \cdots \cong \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N,$$

whose union is dense in $\mathcal{H}$. On $\mathcal{H}^N$, the representation ρ^N of $G^N \rtimes \mathbb{R}$, defined by

$$\rho^N((g_1, \ldots, g_N), t) := \rho_1(g_1, t) \otimes \cdots \otimes \rho_N(g_N, t),$$

is continuous with cyclic vector $\Omega = \otimes_{n=1}^{\infty} \Omega_n$. The representation $(\rho, \mathcal{H})$ of G now is a direct limit of the representations $(\rho^N, \mathcal{H}^N)$ of the subgroups G^N, hence a continuous unitary representation. Further, the invariance of Ω_n under the one-parameter group $U_t^n := \rho_n(e, t)$ implies that

$$U_t(v_1 \otimes \cdots \otimes v_N \otimes \Omega_{N+1} \otimes \cdots) := U_t^1 v_1 \otimes \cdots \otimes U_t^N v_N \otimes \Omega_{N+1} \otimes \cdots \qquad \text{(D.3)}$$

defines a continuous unitary one-parameter group on $\mathcal{H}$ satisfying

$$U_t \rho(g) U_t^* = \rho(\alpha_t(g)) \quad \text{for } g \in G, t \in \mathbb{R}.$$

By $\rho(g, t) := \rho(g) U_t$, we thus obtain a continuous unitary representation of G on $\mathcal{H}$ for which Ω is a G-cyclic unit vector fixed by the one-parameter group $(U_t)_{t \in \mathbb{R}}$. Writing

$$U_t = e^{-itH} \quad \text{and} \quad U_t^n = e^{-itH_n}$$

for selfadjoint operators $H_n \geq 0$, (D.3) implies that $H \geq 0$. To see that $\ker H = \mathbb{C}\Omega$, we decompose

$$\mathcal{H} = \mathcal{H}^N \otimes \mathcal{K}_N \quad \text{for } N \in \mathbb{N}.$$

Accordingly,

$$U_t = V_t \otimes W_t \quad \text{with } V_t = U_t^1 \otimes \cdots \otimes U_t^N,$$

and both one-parameter groups $(V_t)_{t \in \mathbb{R}}$ and $(W_t)_{t \in \mathbb{R}}$ have positive generators H_V and H_W. From [8, Lemma A.3] we thus infer that

$$H = (H_V \otimes \mathbf{1}_{\mathcal{K}^N}) + (\mathbf{1}_{\mathcal{H}^N} \otimes H_W)$$

in the sense of unbounded operators, hence, in particular, that

$$\mathcal{D}(H) = (\mathcal{D}(H_V) \otimes \mathcal{K}^N) \cap (\mathcal{H}^N \otimes \mathcal{D}(H_W)).$$

We conclude that, for every $N \in \mathbb{N}$,

$$\ker H \subseteq \ker H_V \otimes \mathcal{K}^N = \Omega_1 \otimes \cdots \otimes \Omega_N \otimes \mathcal{K}^N,$$

and this shows that

$$\ker H \subseteq \bigcap_N \Omega_1 \otimes \cdots \otimes \Omega_N \otimes \mathcal{K}^N = \mathbb{C}\Omega.$$

Therefore, $(\rho, \mathcal{H}, \Omega)$ is a vacuum representation of $(G, \mathbb{R}, \alpha)$.

Now we assume, conversely, that $(\rho, \mathcal{H}, \Omega)$ is a vacuum representation of the triple $(G, \mathbb{R}, \alpha)$. Then, the subspace

$$\mathcal{H}^N := \overline{\operatorname{span} \rho(G^N)\Omega}$$

carries a vacuum representation of $(G^N, \mathbb{R}, \alpha^N)$. In particular, this representation is irreducible by Proposition D.5. The group G is a topological product

$$G = G^N \times G^{>N}, \quad \text{where } G^{>N} := \prod\nolimits'_{n>N} G_n,$$

and the representation ρ is irreducible by Proposition D.5. Since its restriction to G^N carries an irreducible subrepresentation, the restriction to G^N is factorial of type I, hence of the form

$$\rho|_{G^N} = \rho^N \otimes \mathbf{1}$$

with respect to some factorization $\mathcal{H} = \mathcal{H}^N \otimes \mathcal{K}^N$. Starting with $N = 1$ and proceeding inductively, we see that

$$\rho^N \cong \rho_1 \otimes \cdots \otimes \rho_N$$

for vacuum representations $(\rho_n, \mathcal{H}_n, \Omega_n)$ of $(G_n, \mathbb{R}, \alpha_n)$. In particular, we obtain factorizations

$$\Omega = \Omega^N \otimes \tilde{\Omega}_N = \Omega_1 \otimes \cdots \otimes \Omega_N \otimes \tilde{\Omega}_N,$$

so that we may identify $\mathcal{H}^N$ with the subspace

$$\mathcal{H}^N \otimes \tilde{\Omega}_N \subseteq \mathcal{H}.$$

As Ω is G-cyclic, the union of these G^N-invariant subspaces is dense in $\mathcal{H}$. This implies that the vacuum representation $(\rho, \mathcal{H}, \Omega)$ is equivalent to the infinite tensor product $\otimes_{n\in\mathbb{N}}(\rho_n, \mathcal{H}_n, \Omega_n)$ of the ground state representations $(\rho_n, \mathcal{H}_n, \Omega_n)$. This completes the proof. ∎

The following allows us to reduce the classification of smooth vacuum representations to the local case, under the assumption that the ground state is smooth.

Proposition D.7. *Suppose that the G_n are Lie groups and that the $\mathbb{R}$-actions on G_n are smooth. Then, the vacuum representation $(\rho, \mathcal{H}, \Omega)$ is smooth with smooth vector Ω if and only if the vacuum representations $(\rho_n, \mathcal{H}_n, \Omega_n)$ are smooth with smooth vector Ω_n.*

Proof. If $(\rho, \mathcal{H}, \Omega)$ is a smooth representation with $\Omega \in \mathcal{H}^\infty$, then Ω will be a smooth vector for every $(\mathcal{H}_n, \rho_n, \Omega)$ as well. Since Ω is cyclic in $\mathcal{H}_n$, the latter will be a smooth representation.

Suppose, conversely, that the vacuum representations $(\rho_n, \mathcal{H}_n, \Omega_n)$ are smooth, and that $\Omega_n \in \mathcal{H}_n^\infty$ for all $n \in \mathbb{N}$. From Theorem D.6, we know that the tensor product representation $(\rho, \mathcal{H}, \Omega)$ is continuous and cyclic. To show that the vacuum representation $(\rho, \mathcal{H}, \Omega) = \bigotimes_{n=1}^\infty(\rho_n, \mathcal{H}_n, \Omega_n)$ is smooth with smooth vector $\Omega \in \mathcal{H}^\infty$, it suffices by [74, Theorem 7.2] to show that $\varphi(g) := \langle \Omega, \rho(g)\Omega\rangle$ is a smooth function from G to $\mathbb{C}$.

Note that φ is the infinite product $\prod_{n=1}^\infty \varphi_n(g_n)$ of the smooth, positive definite functions $\varphi_n \colon G_n \to \mathbb{C}$ defined by $\varphi_n(g) := \langle \Omega_n, \rho_n(g)\Omega_n\rangle$. To see that $\varphi \colon G \to \mathbb{C}$ is smooth, note that it can be decomposed into the smooth maps

$$G = \prod\nolimits'_{n\in\mathbb{N}} G_n \xrightarrow{\Phi_1} \mathbf{1} + \prod\nolimits'_{n\in\mathbb{N}} \mathbb{C} \xrightarrow{\Phi_2} \mathbf{1} + \ell^1(\mathbb{N}) \xrightarrow{\Phi_3} \mathbb{C},$$

where $\mathbf{1} = (1)_{n\in\mathbb{N}}$ and

$$\Phi_1((g_n)) = (\varphi_n(g_n)), \quad \Phi_2((z_n)) = (z_n), \quad \Phi_3((z_n)) = \prod_{n\in\mathbb{N}} z_n.$$

Here, the smoothness of Φ_1 follows from the compatibility with the box manifold structure, Φ_2 is continuous affine, and Φ_3 is holomorphic. It follows that

$$\varphi = \Phi_3 \circ \Phi_2 \circ \Phi_1$$

is smooth, and hence that $(\rho, \mathcal{H}, \Omega)$ is a smooth vacuum representation with smooth vector Ω. ∎

Appendix E

Ergodic property of 1-parameter subgroups of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$

We give a simplified proof for the following characterization of the ergodic property for 1-parameter subgroups of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ due to Mautner and Moore. Define the 1-parameter groups $x(t)$, $y(t)$ and $h(t)$ in $\mathrm{SL}(2,\mathbb{R})$ by

$$x(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad y(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \quad \text{and} \quad h(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix},$$

and let $\tilde{x}(t)$, $\tilde{y}(t)$ and $\tilde{h}(t)$ be their lift to $\widetilde{\mathrm{SL}}(2,\mathbb{R})$.

Lemma E.1. *Let $(\pi,\mathcal{H})$ be a continuous unitary representation of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, and let $\Omega \in \mathcal{H}$ be a unit vector. Then, the following are equivalent:*

(a) $\pi(\tilde{x}(t))\Omega = \Omega$ *for all* $t \in \mathbb{R}$,

(b) $\pi(\tilde{h}(t))\Omega = \Omega$ *for all* $t \in \mathbb{R}$,

(c) $\pi(g)\Omega = \Omega$ *for all* $g \in \widetilde{\mathrm{SL}}(2,\mathbb{R})$.

This well-known result plays an important role in ergodic theory. It is due to Calvin Moore [66,67], and in the proof below we almost literally follow his argument for the implication (a)⇒(b) from [67, page 7]. The implication (b)⇒(c) is due to Mautner [63], and this is implicitly used by Moore in [67], and by Howe and Moore in their seminal paper [43]. In his proof, Mautner uses the classification of irreducible unitary representations of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. We bypass this with a simple argument.

Proof. For (a)⇒(b), let $w(t) = x(t)y(-t^{-1})x(t)$, and note that we have

$$h(t) = w(e^t)w(1)^{-1} \quad \text{for all } t \in \mathbb{R}.$$

If we define $\tilde{w}(t) := \tilde{x}(t)\tilde{y}(-t^{-1})\tilde{x}(t)$, then the curve $t \mapsto \tilde{w}(e^t)\tilde{w}(1)^{-1}$ covers $h(t)$. Since it is the identity for $t=0$, we have $\tilde{w}(e^t)\tilde{w}(1)^{-1} = \tilde{h}(t)$. Since $\|\pi(\tilde{w}(t))\Omega\| = 1$ for all $t \neq 0$, it follows from

$$\lim_{|t|\to\infty} \langle \pi(\tilde{w}(t))\Omega, \Omega\rangle = \lim_{|t|\to\infty} \langle \pi(\tilde{y}(-t^{-1}))\Omega, \Omega\rangle = 1$$

that $\lim_{|t|\to\infty} \pi(\tilde{w}(t))\Omega = \Omega$. So for $\psi = \pi(\tilde{w}(1))\Omega$, we find $\lim_{t\to\infty} \pi(\tilde{h}(t))\psi = \Omega$. For every $s \in \mathbb{R}$ we thus have

$$\Omega = \lim_{t\to\infty} \pi(\tilde{h}(s+t))\psi = \pi(\tilde{h}(s)) \lim_{t\to\infty} \pi(\tilde{h}(t))\psi = \pi(\tilde{h}(s))\Omega,$$

so Ω is fixed by $\tilde{h}(s)$ for all $s \in \mathbb{R}$.

For (b)⇒(a), note that since

$$x(te^{-2s}) = h(-s)x(t)h(s) \quad \text{for all } s, t \in \mathbb{R},$$

the same equation $\widetilde{x}(te^{-2s}) = \widetilde{h}(-s)\widetilde{x}(t)\widetilde{h}(s)$ holds in $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ (both sides are the identity for $s = t = 0$). The invariance of Ω under the 1-parameter group $\widetilde{h}$ then implies

$$\langle \pi(\widetilde{x}(te^{-2s}))\Omega, \Omega\rangle = \langle \pi(\widetilde{x}(t))\Omega, \Omega\rangle.$$

Since $\lim_{s\to\infty} \widetilde{x}(te^{-2s})$ is the identity, we have $\langle \pi(\widetilde{x}(t))\Omega, \Omega\rangle = 1$, and it follows that $\pi(\widetilde{x}(t))\Omega = \Omega$ for all $t \in \mathbb{R}$.

Since $h(s)y(t)h(-s) = y(te^{-2s})$, a similar argument shows that if Ω is fixed by $\widetilde{h}$, then it is fixed by $\widetilde{y}$. It follows that if either (a) or (b) hold, then Ω is fixed by $\widetilde{x}(t)$, $\widetilde{y}(t)$ and $\widetilde{h}(t)$ alike, and hence by the group $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ that they generate. ∎

References

[1] S. Albeverio, B. K. Driver, M. Gordina, and A. M. Vershik, Equivalence of the Brownian and energy representations. *J. Math. Sci.* **219** (2016), no. 5, 612–630

[2] S. Albeverio and R. Høegh-Krohn, The energy representation of Sobolev–Lie groups. *Compositio Math.* **36** (1978), no. 1, 37–51

[3] S. Albeverio, R. J. Høegh-Krohn, J. A. Marion, D. H. Testard, and B. S. Torrésani, *Noncommutative distributions. Unitary representation of gauge groups and algebras*. Monogr. Textbooks Pure Appl. Math. 175, Marcel Dekker, New York, 1993

[4] S. Albeverio and B. Torrésani, Some remarks on representations of jet groups and gauge groups. *J. Math. Phys.* **35** (1994), no. 9, 4897–4908

[5] H. Ando, On the local structure of the representation of a local gauge group. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **13** (2010), no. 2, 223–242

[6] H. Araki, Factorizable representation of current algebra. Non commutative extension of the Lévy–Kinchin formula and cohomology of a solvable group with values in a Hilbert space. *Publ. Res. Inst. Math. Sci.* **5** (1969/70), 361–422

[7] H. Baumgärtel, *Operator algebraic methods in quantum field theory. A series of lectures*. Akademie, Berlin, 1995

[8] D. Beltiţă, H. Grundling, and K.-H. Neeb, Covariant representations for possibly singular actions on C^*-algebras. *Dissertationes Math.* **549** (2020), 94 pp.

[9] F. A. Berezin, Representations of the continuous direct product of universal coverings of the group of motions of the complex ball. *Trans. Mosc. Math. Soc.* **36** (1979), 281–289

[10] S. Bloch, The dilogarithm and extensions of Lie algebras. In *Algebraic K-theory, Evanston 1980 (Proc. Conf., Northwestern Univ., Evanston, Ill., 1980)*, pp. 1–23, Lecture Notes in Math. 854, Springer, Berlin, 1981

[11] H.-J. Borchers, *Translation group and particle representations in quantum field theory*. Lecture Notes in Phys. New Ser. m Monogr. 40, Springer, Berlin, 1996

[12] N. Bourbaki, *Lie groups and Lie algebras. Chapters 1–3*. Elem. Math. (Berlin), Springer, Berlin, 1989

[13] O. Bratteli and D. W. Robinson, *Operator algebras and quantum statistical mechanics. 1*. 2nd edn., Texts Monogr. Phys., Springer, New York, 1987

[14] T. Bröcker and T. tom Dieck, *Representations of compact Lie groups*. Grad. Texts in Math. 98, Springer, New York, 1985

[15] V. Chari and A. Pressley, New unitary representations of loop groups. *Math. Ann.* **275** (1986), no. 1, 87–104

[16] V. Chari and A. Pressley, Unitary representations of the maps $S^1 \to \mathrm{su}(N, 1)$. *Math. Proc. Cambridge Philos. Soc.* **102** (1987), no. 2, 259–272

[17] S. Del Vecchio, S. Iovieno, and Y. Tanimoto, Solitons and nonsmooth diffeomorphisms in conformal nets. *Comm. Math. Phys.* **375** (2020), no. 1, 391–427

[18] D. B. A. Epstein, The simplicity of certain groups of homeomorphisms. *Compositio Math.* **22** (1970), 165–173

[19] C. J. Fewster and S. Hollands, Quantum energy inequalities in two-dimensional conformal field theory. *Rev. Math. Phys.* **17** (2005), no. 5, 577–612

[20] D. S. Freed, M. J. Hopkins, and C. Teleman, Loop groups and twisted K-theory III. *Ann. of Math. (2)* **174** (2011), no. 2, 947–1007

[21] E. Galina and A. Kaplan, On unitary representations of nilpotent gauge groups. *Comm. Math. Phys.* **236** (2003), no. 2, 187–198

[22] I. M. Gelfand and M. I. Graev, Representations of quaternion groups over locally compact fields and function fields. *Funkcional. Anal. i Priložen.* **2** (1968), no. 1, 20–35

[23] I. M. Gelfand, M. I. Graev, and A. M. Veršik, Representations of the group of functions taking values in a compact Lie group. *Compositio Math.* **42** (1980), no. 2, 217–243

[24] R. Geroch, E. H. Kronheimer, and R. Penrose, Ideal points in space-time. *Proc. Roy. Soc. London Ser. A* **327** (1972), 545–567

[25] J. Glimm and A. Jaffe, *Quantum physics. A functional integral point of view*. 2nd edn., Springer, New York, 1987

[26] H. Glöckner, Lie groups of measurable mappings. *Canad. J. Math.* **55** (2003), no. 5, 969–999

[27] H. Glöckner, Direct limits of infinite-dimensional Lie groups compared to direct limits in related categories. *J. Funct. Anal.* **245** (2007), no. 1, 19–61

[28] H. Glöckner, Homotopy groups of ascending unions of infinite-dimensional manifolds. 2008, arXiv:0812.4713v2

[29] H. Glöckner, Measurable regularity properties of infinite-dimensional lie groups. 2015, arXiv:1601.02568

[30] H. Glöckner and K.-H. Neeb, Infinite-dimensional Lie groups. Book in preparation

[31] G. A. Goldin, Lectures on diffeomorphism groups in quantum physics. In *Contemporary problems in mathematical physics*, pp. 3–93, World Scientific Publishing, Teaneck, NJ, 2004

[32] R. Goodman and N. R. Wallach, Structure and unitary cocycle representations of loop groups and the group of diffeomorphisms of the circle. *J. Reine Angew. Math.* **347** (1984), 69–133

[33] R. Goodman and N. R. Wallach, Projective unitary positive-energy representations of Diff(S^1). *J. Funct. Anal.* **63** (1985), no. 3, 299–321

[34] H. Grundling and K.-H. Neeb, Full regularity for a C^*-algebra of the canonical commutation relations. *Rev. Math. Phys.* **21** (2009), no. 5, 587–613

[35] A. Guichardet, *Symmetric Hilbert spaces and related topics. Infinitely divisible positive definite functions. Continuous products and tensor products. Gaussian and Poissonian stochastic processes*. Lecture Notes in Math. 261, Springer, New York, 1972

[36] A. Haddi, Homologie des algèbres de Lie étendues à une algèbre commutative. *Comm. Algebra* **20** (1992), no. 4, 1145–1166

[37] S. W. Hawking and G. F. R. Ellis, *The large scale structure of space-time*. Cambridge Monogr. Math. Phys. 1, Cambridge University Press, Cambridge, 1973

[38] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*. Pure Appl. Math. 80, Academic Press, New York, 1978

[39] J. Hilgert and K.-H. Neeb, *Structure and geometry of Lie groups*. Springer Monogr. Math., Springer, New York, 2012

[40] K. H. Hofmann, Lie algebras with subalgebras of co-dimension one. *Illinois J. Math.* **9** (1965), 636–643

[41] K. H. Hofmann, Hyperplane subalgebras of real Lie algebras. *Geom. Dedicata* **36** (1990), no. 2-3, 207–224

[42] K. H. Hofmann and S. A. Morris, *The structure of compact groups. A primer for the student—a handbook for the expert*. De Gruyter Stud. Math. 25, De Gruyter, Berlin, 1998

[43] R. E. Howe and C. C. Moore, Asymptotic properties of unitary representations. *J. Functional Analysis* **32** (1979), no. 1, 72–96

[44] W. Hurewicz and H. Wallman, *Dimension theory*. Princeton Math. Ser. 4, Princeton University Press, Princeton, NJ, 1941

[45] R. S. Ismagilov, On unitary representations of the group $C_0^\infty(X, G)$, $G = \mathrm{SU}_2$. *Math. USSR-Sb.* **29** (1976), 105–117

[46] R. S. Ismagilov, *Representations of infinite-dimensional groups*. Transl. Math. Monogr. 152, American Mathematical Society, Providence, RI, 1996

[47] H. P. Jakobsen and V. Kac, A new class of unitarizable highest weight representations of infinite-dimensional Lie algebras. II. *J. Funct. Anal.* **82** (1989), no. 1, 69–90

[48] H. P. Jakobsen and V. G. Kac, A new class of unitarizable highest weight representations of infinite-dimensional Lie algebras. In *Nonlinear equations in classical and quantum field theory (Meudon/Paris, 1983/1984)*, pp. 1–20, Lecture Notes in Phys. 226, Springer, Berlin, 1985

[49] B. Janssens and K.-H. Neeb, Positive energy representations of gauge groups II: Conformal completion of space-time. In preparation

[50] B. Janssens and K.-H. Neeb, Norm continuous unitary representations of Lie algebras of smooth sections. *Int. Math. Res. Not. IMRN* **2015** (2015), no. 18, 9081–9137

[51] B. Janssens and K.-H. Neeb, Covariant central extensions of gauge Lie algebras. In *Stochastic geometric mechanics*, pp. 101–114, Springer Proc. Math. Stat. 202, Springer, Cham, 2017

[52] B. Janssens and K.-H. Neeb, Projective unitary representations of infinite-dimensional Lie groups. *Kyoto J. Math.* **59** (2019), no. 2, 293–341

[53] B. Janssens and C. Wockel, Universal central extensions of gauge algebras and groups. *J. Reine Angew. Math.* **682** (2013), 129–139

[54] V. G. Kac, *Infinite-dimensional Lie algebras*. 2nd edn., Cambridge University Press, Cambridge, 1985

[55] V. G. Kac and A. K. Raina, *Bombay lectures on highest weight representations of infinite-dimensional Lie algebras*. Adv. Ser. Math. Phys. 2, World Scientific, Singapore, 1987

[56] C. Kassel and J.-L. Loday, Extensions centrales d'algèbres de Lie. *Ann. Inst. Fourier (Grenoble)* **32** (1983), no. 4, 119–142

[57] J.-L. Loday and D. Quillen, Homologie cyclique et homologie de l'algèbre de Lie des matrices. *C. R. Acad. Sci. Paris Sér. I Math.* **296** (1983), no. 6, 295–297

[58] J.-L. Loday and D. Quillen, Cyclic homology and the Lie algebra homology of matrices. *Comment. Math. Helv.* **59** (1984), no. 4, 569–591

[59] R. Longo, Real Hilbert subspaces, modular theory, $\mathrm{SL}(2,\mathbb{R})$ and CFT. In *Von Neumann algebras in Sibiu*, pp. 33–91, Theta Ser. Adv. Math. 10, Theta, Bucharest, 2008

[60] G. W. Mackey, Imprimitivity for representations of locally compact groups. I. *Proc. Nat. Acad. Sci. U.S.A.* **35** (1949), 537–545

[61] P. Maier, Central extensions of topological current algebras. In *Geometry and analysis on finite- and infinite-dimensional Lie groups (Będlewo, 2000)*, pp. 61–76, Banach Center Publ. 55, Institute of Mathematics of the Polish Academy of Sciences, Warsaw, 2002

[62] P. Maier and K.-H. Neeb, Central extensions of current groups. *Math. Ann.* **326** (2003), no. 2, 367–415

[63] F. I. Mautner, Geodesic flows on symmetric Riemann spaces. *Ann. of Math. (2)* **65** (1957), 416–431

[64] M. Mimura, Homotopy theory of Lie groups. In *Handbook of algebraic topology*, pp. 951–991, North-Holland, Amsterdam, 1995

[65] S. Mohrdieck and R. Wendt, Integral conjugacy classes of compact Lie groups. *Manuscripta Math.* **113** (2004), no. 4, 531–547

[66] C. C. Moore, Ergodicity of flows on homogeneous spaces. *Amer. J. Math.* **88** (1966), 154–178

[67] C. C. Moore, Restrictions of unitary representations to subgroups and ergodic theory: Group extensions and group cohomology. In *Group Representations in Math. and Phys. (Battelle Seattle 1969 Rencontres)*, pp. 1–35, Lecture Notes in Phys. 6, Springer, Berlin, 1970

[68] K.-H. Neeb, Borel–Weil theory for loop groups. In *Infinite dimensional Kähler manifolds (Oberwolfach, 1995)*, pp. 179–229, DMV Sem. 31, Birkhäuser, Basel, 2001

[69] K.-H. Neeb, Central extensions of infinite-dimensional Lie groups. *Ann. Inst. Fourier (Grenoble)* **52** (2002), no. 5, 1365–1442

[70] K.-H. Neeb, Current groups for non-compact manifolds and their central extensions. In *Infinite dimensional groups and manifolds*, pp. 109–183, IRMA Lect. Math. Theor. Phys. 5, De Gruyter, Berlin, 2004

[71] K.-H. Neeb, Towards a Lie theory of locally convex groups. *Jpn. J. Math.* **1** (2006), no. 2, 291–468

[72] K.-H. Neeb, Non-abelian extensions of infinite-dimensional Lie groups. *Ann. Inst. Fourier (Grenoble)* **57** (2007), no. 1, 209–271

[73] K.-H. Neeb, Semi-bounded unitary representations of infinite-dimensional Lie groups. In *Infinite dimensional harmonic analysis IV*, pp. 209–222, World Scientific, Hackensack, NJ, 2009

[74] K.-H. Neeb, On differentiable vectors for representations of infinite dimensional Lie groups. *J. Funct. Anal.* **259** (2010), no. 11, 2814–2855

[75] K.-H. Neeb, Semibounded representations and invariant cones in infinite dimensional Lie algebras. *Confluentes Math.* **2** (2010), no. 1, 37–134

[76] K.-H. Neeb, On analytic vectors for unitary representations of infinite dimensional Lie groups. *Ann. Inst. Fourier (Grenoble)* **61** (2012), no. 5, 1839–1874

[77] K.-H. Neeb, Holomorphic realization of unitary representations of Banach–Lie groups. In *Lie groups: structure, actions, and representations*, pp. 185–223, Progr. Math. 306, Birkhäuser/Springer, New York, 2013

[78] K.-H. Neeb, Positive energy representations and continuity of projective representations for general topological groups. *Glasg. Math. J.* **56** (2014), no. 2, 295–316

[79] K. H. Neeb, Semibounded unitary representations of double extensions of Hilbert-loop groups. *Ann. Inst. Fourier (Grenoble)* **64** (2014), no. 5, 1823–1892

[80] K.-H. Neeb and H. Salmasian, Differentiable vectors and unitary representations of Fréchet–Lie supergroups. *Math. Z.* **275** (2013), no. 1-2, 419–451

[81] K.-H. Neeb, H. Salmasian, and C. Zellner, Smoothing operators and C^*-algebras for infinite dimensional Lie groups. *Internat. J. Math.* **28** (2017), no. 5, article no. 1750042, 32 pp.

[82] K.-H. Neeb and F. Wagemann, The second cohomology of current algebras of general Lie algebras. *Canad. J. Math.* **60** (2008), no. 4, 892–922

[83] K.-H. Neeb, F. Wagemann, and C. Wockel, Making lifting obstructions explicit. *Proc. Lond. Math. Soc. (3)* **106** (2013), no. 3, 589–620

[84] K.-H. Neeb and C. Wockel, Central extensions of groups of sections. *Ann. Global Anal. Geom.* **36** (2009), no. 4, 381–418

[85] K.-H. Neeb and C. Zellner, Oscillator algebras with semi-equicontinuous coadjoint orbits. *Differential Geom. Appl.* **31** (2013), no. 2, 268–283

[86] E. Nelson, Analytic vectors. *Ann. of Math. (2)* **70** (1959), 572–615

[87] E. Nelson, Time-ordered operator products of sharp-time quadratic forms. *J. Functional Analysis* **11** (1972), 211–219

[88] G. I. Olshanskiĭ, Spherical functions and characters on the group $U(\infty)^X$. *Russian Math. Surveys* **37** (1982), 233–234

[89] K. R. Parthasarathy and K. Schmidt, Factorisable representations of current groups and the Araki–Woods imbedding theorem. *Acta Math.* **128** (1972), no. 1-2, 53–71

[90] K. R. Parthasarathy and K. Schmidt, A new method for constructing factorisable representations for current groups and current algebras. *Comm. Math. Phys.* **50** (1976), no. 2, 167–175

[91] R. Penrose, Zero rest-mass fields including gravitation: Asymptotic behaviour. *Proc. Roy. Soc. London Ser. A* **284** (1965), 159–203

[92] R. Penrose and W. Rindler, *Spinors and space-time Vol. 2. Spinor and twistor methods in space-time geometry*. Cambridge Monogr. Math. Phys., Cambridge University Press, Cambridge, 1986

[93] R. T. Powers, Representations of uniformly hyperfinite algebras and their associated von Neumann rings. *Ann. Math. (2)* **86** (1967), 138–171

[94] A. Pressley and G. Segal, *Loop groups*. Oxford Math. Monogr., Oxford University Press, Oxford, 1986

[95] M. Reed and B. Simon, *Methods of modern mathematical physics. I. Functional analysis*. Academic Press, New York-London, 1972

[96] W. Rudin, *Real and complex analysis*. 3rd edn., McGraw-Hill, New York, 1987

[97] G. Segal, Unitary representations of some infinite-dimensional groups. *Comm. Math. Phys.* **80** (1981), no. 3, 301–342

[98] S. H. Shah, *Bicoloured torus loop groups*. Ph.D. thesis, Universiteit Utrecht, Nederland, 2017

[99] Y. Shimada, On irreducibility of the energy representation of the gauge group and the white noise distribution theory. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **8** (2005), no. 2, 153–177

[100] S. Solecki, Unitary representations of the groups of measurable and continuous functions with values in the circle. *J. Funct. Anal.* **267** (2014), no. 9, 3105–3124

[101] R. F. Streater, A continuum analogue of the lattice gas. *Comm. Math. Phys.* **12** (1969), 226–232

[102] Y. Tanimoto, Ground state representations of loop algebras. *Ann. Henri Poincaré* **12** (2011), no. 4, 805–827

[103] V. Toledano Laredo, *Fusion of positive energy representations of* $L\mathrm{Spin}(2n)$. Ph.D. thesis, University of Cambridge, Cambridge, 1997, arXiv:math/0409044

[104] V. Toledano Laredo, Positive energy representations of the loop groups of non-simply connected Lie groups. *Comm. Math. Phys.* **207** (1999), no. 2, 307–339

[105] B. Torrésani, Unitary positive energy representations of the gauge group. *Lett. Math. Phys.* **13** (1987), no. 1, 7–15

[106] B. L. Tsygan, The homology of matrix Lie algebras over rings and the Hochschild homology. *Russian Math. Surv.* **38** (1983), no. 2, 198–199

[107] A. M. Veršik, I. M. Gelfand, and M. I. Graev, Representations of the group $\mathrm{SL}(2, \mathbf{R})$, where $\mathbf{R}$ is a ring of functions. *Uspehi Mat. Nauk* **28** (1973), no. 5(173), 83–128

[108] A. M. Veršik, I. M. Gelfand, and M. I. Graev, Irreducible representations of the group G^X and cohomologies. *Funct. Anal. Appl.* **8** (1974), no. 2, 151–153

[109] N. R. Wallach, On the irreducibility and inequivalence of unitary representations of gauge groups. *Compositio Math.* **64** (1987), no. 1, 3–29

[110] B. Walter, Weighted diffeomorphism groups of Banach spaces and weighted mapping groups. *Dissertationes Math.* **484** (2012), 128

[111] A. Wassermann, Operator algebras and conformal field theory. III. Fusion of positive energy representations of $\mathrm{LSU}(N)$ using bounded operators. *Invent. Math.* **133** (1998), no. 3, 467–538

[112] M. Weiner, Conformal covariance and positivity of energy in charged sectors. *Comm. Math. Phys.* **265** (2006), no. 2, 493–506

[113] E. Wigner, On unitary representations of the inhomogeneous Lorentz group. *Ann. of Math. (2)* **40** (1939), no. 1, 149–204

[114] C. Zellner, On the existence of regular vectors. In *Representation theory—current trends and perspectives*, pp. 747–763, EMS Ser. Congr. Rep., EMS Press, Zürich, 2017

[115] P. Zusmanovich, The second homology group of current Lie algebras. *Astérisque* (1994), no. 226, 435–452. Errata on arXiv:0812.2625

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MEMOIRS OF THE EUROPEAN MATHEMATICAL SOCIETY

Bas Janssens, Karl-Hermann Neeb

Positive Energy Representations of Gauge Groups I

Localization

This is the first in a series of papers on projective positive energy representations of gauge groups. Let $\Xi \to M$ be a principal fiber bundle, and let $\Gamma_c(M, \mathrm{Ad}(\Xi))$ be the group of compactly supported (local) gauge transformations. If P is a group of "space–time symmetries" acting on $\Xi \to M$, then a projective unitary representation of $\Gamma_c(M, \mathrm{Ad}(\Xi)) \rtimes P$ is of *positive energy* if every "timelike generator" $p_0 \in \mathfrak{p}$ gives rise to a Hamiltonian $H(p_0)$ whose spectrum is bounded from below. Our main result shows that in the absence of fixed points for the cone of timelike generators, the projective positive energy representations of the connected component $\Gamma_c(M, \mathrm{Ad}(\Xi))_0$ come from 1-dimensional P-orbits. For compact M this yields a complete classification of the projective positive energy representations in terms of lowest weight representations of affine Kac–Moody algebras. For noncompact M, it yields a classification under further restrictions on the space of ground states.

In the second part of this series we consider larger groups of gauge transformations, which contain also global transformations. The present results are used to localize the positive energy representations at (conformal) infinity.

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