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Integrating the Wigner Distribution on Subsets of the Phase Space, a Survey

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Abstract

We review several properties of integrals of the Wigner distribution on subsets of the phase space. Along our way, we provide a theoretical proof of the invalidity of Flandrin's conjecture, a fact already proven via numerical arguments in our joint paper [J. Fourier Anal. Appl. **26** (2020), no. 1, article no. 6] with B. Delourme and T. Duyckaerts. We use also the J. G. Wood and A. J. Bracken paper [J. Math. Phys. **46** (2005), no. 4, article no. 042103], for which we offer a mathematical perspective. We review thoroughly the case of subsets of the plane whose boundary is a conic curve and show that Mehler's formula can be helpful in the analysis of these cases, including for the higher-dimensional case investigated in the paper [J. Math. Phys. **51** (2010), no. 10, article no. 102101] by E. Lieb and Y. Ostrover. Using the Feichtinger algebra, we show that, generically in the Baire sense, the Wigner distribution of a pulse in $L^2(\mathbb{R}^n)$ does not belong to $L^1(\mathbb{R}^{2n})$, providing as a byproduct a large class of examples of subsets of the phase space $\mathbb{R}^{2n}$ on which the integral of the Wigner distribution is infinite. We study as well the case of convex polygons of the plane, with a rather weak estimate depending on the number of vertices, but independent of the area of the polygon.

Keywords. Wigner distribution, signal theory, pseudo-differential operators

Mathematics Subject Classification (2020). Primary 81S30; Secondary 35P05, 42B20, 47G10, 47G30, 47N70, 94A12

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Contents

Foreword

As indicated by the title of this memoir, this work is a survey of properties of integrals of the Wigner distribution on subsets of the phase space. Since it is quite lengthy, we wish in this foreword to describe the content of this article, browsing through the table of contents, expecting that the reader will find some organization with the way this memoir is written. In particular, we shall point here what is original in our survey (to the best of our knowledge) and what was well-known beforehand. There is no doubt that the fifty-five articles quoted in the references list are a small part of the literature on the topic and could be probably extended tenfold: we expect nevertheless that our choice of references will be enough to cover the most important contributions.

Chapter 1 is *Preliminaries and definitions* and is very classical. We have used J. Leray's book [31] and other lecture notes of this author at the *Collège de France* such as [30], L. Hörmander's four-volume treatise, *The analysis of linear partial differential operators* and, in particular, volume III, as well as K. Gröchenig's [16], *Foundations of time-frequency analysis*, along with G. B. Folland's [15], A. Unterberger's [50] and N. Lerner's [33]. Some details are given, in particular, on positive quantizations, but that chapter is far from being self-contained, which is probably unavoidable: the link of properties of the Wigner distribution and of the Weyl quantization of classical Hamiltonians is easy to obtain but turns out to be an important piece of information for our purpose.

Chapter 2 is stressing the link *Quantization of radial functions – Mehler's formula* and is also very classical: here also the link aforementioned is easy to get but gives some simplifications in the formulas providing the quantization of radial Hamiltonians: in one dimension for the configuration space (phase space $\mathbb{R}^2$), we are reduced to check simple integrals related to the Laguerre polynomials, following P. Flandrin's method in his 1988 article [13].

Chapter 3 is dealing with *Conics with eccentricity* < 1. The result for the disc in $\mathbb{R}^2$ is due to P. Flandrin and the result for the Euclidean ball in $\mathbb{R}^{2n}$ to E. Lieb and Y. Ostrover in [39]. Using Mehler's formula simplifies a little bit the presentation, but leaves open the case of anisotropic ellipsoids for which we formulate a conjecture.

Chapter 4 is dealing with *Epigraphs of Parabolas*. The results obtained in that chapter follow easily from Chapter 3 but nevertheless the precise diagonalisation proven there seems to be new. We formulate also a conjecture on anisotropic paraboloids which is closely related to the conjecture in Chapter 3.

Chapter 5 is concerned with *Conics with eccentricity* > 1. Many of the results in that chapter are contained in the paper [55] by J. G. Wood and A. J. Bracken; however since the latter article contains some formal calculations, using for instance test functions which do not belong to $L^2(\mathbb{R})$, we have made a mathematically sound presentation. As certainly the most important contribution of this work, we provide a "theoretical" disproof of Flandrin's conjecture on integrals of the Wigner distribution on convex subsets of the phase space: we find, in particular, some $a > 0$ and some function $u \in L^2(\mathbb{R})$ with norm 1 such that

$$\iint_{[0,a]^2} \mathcal{W}(u,u)(x,\xi)dxd\xi > 1,$$

where $\mathcal{W}(u,u)$ is the Wigner distribution of u. This fact was already proven in our joint paper [6] with B. Delourme and T. Duyckaerts, using a rigorous numerical argument.

Chapter 6 is entitled *Unboundedness is Baire generic* and most of its content is included in Chapter 12 of K. Gröchenig's book [16]. Using the Feichtinger algebra, we show that, generically in the Baire sense, the Wigner distribution of a pulse in $L^2(\mathbb{R}^n)$ does not belong to $L^1(\mathbb{R}^{2n})$, providing as a byproduct a large class of examples of subsets of the phase space $\mathbb{R}^{2n}$ on which the integral of the Wigner distribution is infinite. We raise a couple of questions, in particular, whether we can find a pulse $u \in L^2(\mathbb{R}^n)$ such that

$$E_+(u) = \big\{(x,\xi) \in \mathbb{R}^{2n}, \mathcal{W}(u,u)(x,\xi) > 0\big\} \text{ is connected.}$$

Chapter 7 is *Convex polygons in the plane*: we study there the sets defined by the intersection of N half-spaces in the plane $\mathbb{R}^2$ and the integrals of the Wigner distribution on these sets. We start with convex cones ($N = 2$) for which a complete result is known and we go on with triangles ($N = 3$) for which we find an upper bound: the integral of $\mathcal{W}(u,u)$ on a triangle of $\mathbb{R}^2$ for a normalized pulse in $L^2(\mathbb{R})$ is bounded above by a universal constant. We show also that the integral of $\mathcal{W}(u,u)$ on a convex polygon with N sides of $\mathbb{R}^2$ for a normalized pulse in $L^2(\mathbb{R})$ is bounded above by a universal constant $\times\sqrt{N}$. We raise a couple of questions: in particular it seems possible that the behaviour of convex subsets of the plane is such that there exists a constant $\alpha > 1$ such that

$$\text{for all } C \text{ convex subset of the plane } \mathbb{R}^2, \text{ for all } u \in L^2(\mathbb{R}) \text{ with } \|u\|_{L^2(\mathbb{R})} = 1,$$
$$\text{we have } \iint_C \mathcal{W}(u,u)(x,\xi)dxd\xi \le \alpha.$$

That would be a weak version of Flandrin's conjecture: the original Flandrin's conjecture was the above statement with $\alpha = 1$, which is untrue, but that does not rule out the existence of a number $\alpha > 1$ such that the above estimate holds true.

Chapter 8 is entitled *Open questions and Conjectures*: we review in that chapter the various conjectures that we meet along the text of the memoir, estimating the importance and difficulty of the various questions. Chapter A is an appendix containing only classical material, hopefully helping the reader by improving the self-containedness of this memoir.

Chapter 1

Preliminaries and definitions

1.1 The Wigner distribution

Let u, v be given functions in $L^2(\mathbb{R}^n)$. The function Ω, defined on $\mathbb{R}^n \times \mathbb{R}^n$ by

$$\mathbb{R}^n \times \mathbb{R}^n \ni (z, x) \mapsto u\Big(x + \frac{z}{2}\Big)\bar{v}\Big(x - \frac{z}{2}\Big) = \Omega(u, v)(x, z), \tag{1.1.1}$$

belongs to $L^2(\mathbb{R}^{2n})$ from the identity

$$\int_{\mathbb{R}^{2n}} |\Omega(u, v)(x, z)|^2 dx dz = \|u\|^2_{L^2(\mathbb{R}^n)} \|v\|^2_{L^2(\mathbb{R}^n)}. \tag{1.1.2}$$

We have also

$$\sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} |\Omega(x, z)| dz \le 2^n \|u\|_{L^2(\mathbb{R}^n)} \|v\|_{L^2(\mathbb{R}^n)}. \tag{1.1.3}$$

We may then give the following definition (the reader will find some reminders on the Fourier transformation in Section A.1 of our appendix).

Definition 1.1.1. Let u, v be given functions in $L^2(\mathbb{R}^n)$. We define the joint Wigner distribution $\mathcal{W}(u, v)$ as the partial Fourier transform with respect to z of the function Ω defined in (1.1.1). We have for $(x, \xi) \in \mathbb{R}^n_x \times \mathbb{R}^n_\xi$, using (1.1.3),

$$\mathcal{W}(u, v)(x, \xi) = \int_{\mathbb{R}^n} e^{-2i\pi z \cdot \xi} u\Big(x + \frac{z}{2}\Big)\bar{v}\Big(x - \frac{z}{2}\Big) dz. \tag{1.1.4}$$

The Wigner distribution of u is defined as $\mathcal{W}(u, u)$.

N.B. By inverse Fourier transformation we get, in a weak sense,

$$u(x_1) \otimes \bar{v}(x_2) = \int \mathcal{W}(u, v)\Big(\frac{x_1 + x_2}{2}, \xi\Big) e^{2i\pi(x_1 - x_2)\cdot\xi} d\xi. \tag{1.1.5}$$

Lemma 1.1.2. *Let u, v be given functions in $L^2(\mathbb{R}^n)$. The function $\mathcal{W}(u, v)$ belongs to $L^2(\mathbb{R}^{2n})$ and we have*

$$\|\mathcal{W}(u, v)\|_{L^2(\mathbb{R}^{2n})} = \|u\|_{L^2(\mathbb{R}^n)} \|v\|_{L^2(\mathbb{R}^n)}. \tag{1.1.6}$$

We have also

$$\overline{\mathcal{W}(u, v)(x, \xi)} = \mathcal{W}(v, u)(x, \xi), \tag{1.1.7}$$

so that $\mathcal{W}(u, u)$ is real-valued.

Proof. Note that the function $\mathcal{W}(u, v)$ is in $L^2(\mathbb{R}^{2n})$ and satisfies (1.1.6) from (1.1.2) and the definition of $\mathcal{W}$ as the partial Fourier transform of Ω. Property (1.1.7) is immediate and entails that $\mathcal{W}(u, u)$ is real-valued. ∎

Remark 1.1.3. We note also that the real-valued function $\mathcal{W}(u, u)$ can take negative values, choosing, for instance,

$$u_1(x) = xe^{-\pi x^2}$$

on the real line, we get

$$\mathcal{W}(u_1, u_1)(x, \xi) = 2^{1/2} e^{-2\pi(x^2+\xi^2)}\Big(x^2 + \xi^2 - \frac{1}{4\pi}\Big).$$

In fact, the real-valued function $\mathcal{W}(u, u)$ will take negative values unless u is a Gaussian function, thanks to a Theorem due to E. Lieb (see [37] and books [16] and [41]). As a matter of fact, this range of $\mathcal{W}(u, u)$ intersecting $\mathbb{R}_-$ for most "pulses" u in $L^2(\mathbb{R}^n)$ makes rather weird the qualification of $\mathcal{W}(u, u)$ as a "quasi-probability" (anyhow the emphasis must be on *quasi*, not on *probability*).

Remark 1.1.4. We have also by Fourier inversion formula, say for $u \in \mathscr{S}(\mathbb{R}^n)$,

$$u\Big(x + \frac{z}{2}\Big)\bar{u}\Big(x - \frac{z}{2}\Big) = \Omega(x, z) = \int \mathcal{W}(u, u)(x, \xi) e^{2i\pi z\cdot\xi} d\xi, \tag{1.1.8}$$

so that, with $z = 2x = y$, we get the reconstruction formula,

$$u(y)\bar{u}(0) = \int \mathcal{W}(u, u)\Big(\frac{y}{2}, \xi\Big) e^{2i\pi y\cdot\xi} d\xi,$$

as well as

$$|u(x)|^2 = \int \mathcal{W}(u, u)(x, \xi) d\xi, \quad |\hat{u}(\xi)|^2 = \int \mathcal{W}(u, u)(x, \xi) dx, \tag{1.1.9}$$

the former formula following from (1.1.8) and the latter from

$$\begin{aligned}\int \mathcal{W}(u, u)(x, \xi) dx &= \iint e^{-2i\pi z\xi} u\Big(x + \frac{z}{2}\Big)\bar{u}\Big(x - \frac{z}{2}\Big) dz dx \\ &= \iint e^{-2i\pi\xi(x_1 - x_2)} u(x_1)\bar{u}(x_2) dx_1 dx_2 = |\hat{u}(\xi)|^2.\end{aligned}$$

Lemma 1.1.5. *Let u be a function in $L^2(\mathbb{R}^n)$ which is even or odd. Then, $\mathcal{W}(u, u)$ is an even function.*

Proof. Using the notation

$$\check{u}(x) = u(-x), \tag{1.1.10}$$

we check

$$\begin{aligned}\mathcal{W}(u,v)(-x,-\xi) &= \int_{\mathbb{R}^n} e^{2i\pi z\cdot\xi} u\Big(-x+\frac{z}{2}\Big)\bar{v}\Big(-x-\frac{z}{2}\Big)dz \\ &= \int_{\mathbb{R}^n} e^{2i\pi z\cdot\xi} \check{u}\Big(x-\frac{z}{2}\Big)\bar{\check{v}}\Big(x+\frac{z}{2}\Big)dz \\ &= \int_{\mathbb{R}^n} e^{-2i\pi z\cdot\xi} \check{u}\Big(x+\frac{z}{2}\Big)\bar{\check{v}}\Big(x-\frac{z}{2}\Big)dz \\ &= \mathcal{W}(\check{u},\check{v})(x,\xi),\end{aligned}$$

so that if $\check{u} = \pm u$, we get $\mathcal{W}(u,u)(-x,-\xi) = \mathcal{W}(u,u)(x,\xi)$. ■

N.B. This lemma is a very particular case of the symplectic covariance property displayed below in (1.2.49).

N.B. In part 1 of volume IV in the collected works [54] of Eugene P. Wigner, we find the first occurrence of what will be called later on the *Wigner distribution* along with a physicist point of view.

It turns out that most of the properties of the Wigner distribution (in particular, Lemma 1.1.5) are inherited from its links with the Weyl quantization introduced by H. Weyl in 1926 in the first edition of [53] and our next remarks are devised to stress that link.

1.2 Weyl quantization, composition formulas, positive quantizations

1.2.1 Weyl quantization

The main goal of Hermann Weyl in his seminal paper [53] was to give a simple formula, also providing symplectic covariance, ensuring that real-valued Hamiltonians $a(x,\xi)$ get quantized by formally self-adjoint operators. The standard way of dealing with differential operators does not achieve that goal since for instance the standard quantization of the Hamiltonian $x\xi$ (indeed real-valued) is the operator xD_x, which is not symmetric (D_x is defined in (A.1.4)); H. Weyl's choice in that case was

$$x\xi \text{ should be quantized by the operator } \quad \frac{1}{2}(xD_x + D_x x), \quad \text{(indeed symmetric)},$$

and more generally, say for $a \in \mathscr{S}(\mathbb{R}^{2n})$, $u \in \mathscr{S}(\mathbb{R}^n)$, the quantization of the Hamiltonian $a(x,\xi)$, denoted by $\mathrm{Op_w}(a)$, should be given by the formula

$$(\mathrm{Op_w}(a)u)(x) = \iint e^{2i\pi(x-y)\cdot\xi} a\Big(\frac{x+y}{2},\xi\Big)u(y)dyd\xi.$$

For $v \in \mathscr{S}(\mathbb{R}^n)$, we may consider

$$\begin{aligned}\langle \mathrm{Op_w}(a)u, v\rangle_{L^2(\mathbb{R}^n)} &= \iiint a(x,\xi)e^{-2i\pi z\cdot\xi}u\Big(x+\frac{z}{2}\Big)\bar{v}\Big(x-\frac{z}{2}\Big)dzdxd\xi \\ &= \iint_{\mathbb{R}^n\times\mathbb{R}^n} a(x,\xi)\mathcal{W}(u,v)(x,\xi)dxd\xi,\end{aligned}$$

and the latter formula allows us to give the following definition.

Definition 1.2.1. Let $a \in \mathscr{S}'(\mathbb{R}^{2n})$. We define the Weyl quantization $\mathrm{Op_w}(a)$ of the Hamiltonian a, by the formula

$$(\mathrm{Op_w}(a)u)(x) = \iint e^{2i\pi(x-y)\cdot\xi}a\Big(\frac{x+y}{2},\xi\Big)u(y)dyd\xi,$$

to be understood weakly as

$$\langle \mathrm{Op_w}(a)u, \bar{v}\rangle_{\mathscr{S}'(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} = \langle a, \mathcal{W}(u,v)\rangle_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})}. \tag{1.2.1}$$

We note that the sesquilinear mapping

$$\mathscr{S}(\mathbb{R}^n)\times\mathscr{S}(\mathbb{R}^n) \ni (u,v) \mapsto \mathcal{W}(u,v) \in \mathscr{S}(\mathbb{R}^{2n}),$$

is continuous so that the above bracket of duality $\langle a, \mathcal{W}(u,v)\rangle_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})}$ makes sense. We note as well that a temperate distribution $a \in \mathscr{S}'(\mathbb{R}^{2n})$ gets quantized by a continuous operator $\mathrm{Op_w}(a)$ from $\mathscr{S}(\mathbb{R}^n)$ into $\mathscr{S}'(\mathbb{R}^n)$. This very general framework is not really useful since we want to compose our operators $\mathrm{Op_w}(a)\mathrm{Op_w}(b)$. A first step in this direction is to look for sufficient conditions ensuring that the operator $\mathrm{Op_w}(a)$ is bounded on $L^2(\mathbb{R}^n)$. Moreover, for $a \in \mathscr{S}'(\mathbb{R}^{2n})$ and b a polynomial in $\mathbb{C}[x,\xi]$, we have the composition formula,

$$\mathrm{Op_w}(a)\mathrm{Op_w}(b) = \mathrm{Op_w}(a\sharp b), \tag{1.2.2}$$

$$(a\sharp b)(x,\xi) = \sum_{k\ge 0}\frac{1}{(4i\pi)^k}\sum_{|\alpha|+|\beta|=k}\frac{(-1)^{|\beta|}}{\alpha!\beta!}(\partial_\xi^\alpha\partial_x^\beta a)(x,\xi)(\partial_x^\alpha\partial_\xi^\beta b)(x,\xi), \tag{1.2.3}$$

which involves here a finite sum. This follows from [33, formula (2.1.26)] where several generalizations can be found (see in particular in that reference the integral formula (2.1.18) which can be given a meaning for quite general classes of symbols). As a consequence of (1.2.3), we get that

$$\begin{aligned}&(a\sharp b) = \sum_{k\ge 0}\omega_k(a,b), \quad \omega_0(a,b) = ab, \quad \omega_1(a,b) = \frac{1}{4i\pi}\{a,b\}, \\ &\{a,b\} = \partial_\xi a\cdot\partial_x b - \partial_x a\partial_\xi b,\end{aligned} \tag{1.2.4}$$

where $\{a,b\}$ is called the *Poisson bracket* of a and b.

Proposition 1.2.2. *Let a be a tempered distribution on $\mathbb{R}^{2n}$. Then, we have*

$$\|\mathrm{Op_w}(a)\|_{\mathcal{B}(L^2(\mathbb{R}^n))} \le \min\bigl(2^n\|a\|_{L^1(\mathbb{R}^{2n})}, \|\hat{a}\|_{L^1(\mathbb{R}^{2n})}\bigr). \tag{1.2.5}$$

Proof. In fact, we have from (1.2.1), $u, v \in \mathscr{S}(\mathbb{R}^n)$,

$$\langle \mathrm{Op_w}(a)u, v\rangle_{L^2(\mathbb{R}^n)} = \iiint a(x,\xi)u(2x-y)\bar{v}(y)e^{-4i\pi(x-y)\cdot\xi}2^n dydxd\xi,$$

and we define for $(x,\xi) \in \mathbb{R}^{2n}$ the operator $\sigma_{x,\xi}$ by

$$(\sigma_{x,\xi}u)(y) = u(2x-y)e^{-4i\pi(x-y)\cdot\xi}. \tag{1.2.6}$$

Claim 1.2.3. The operator $\sigma_{x,\xi}$ (called *phase symmetry*, also known as the *Grossman–Royer operator*) is unitary and self-adjoint.

Proof of Claim 1.2.3. Indeed, we have

$$\begin{aligned}(\sigma_{x,\xi}^2 u)(y) &= (\sigma_{x,\xi}u)(2x-y)e^{-4i\pi(x-y)\cdot\xi}\\ &= u(2x-(2x-y))e^{-4i\pi(x-(2x-y))\cdot\xi}e^{-4i\pi(x-y)\cdot\xi}\\ &= u(y), \text{ so that } \sigma_{x,\xi}^2 = \mathrm{Id}.\end{aligned}$$

We have also

$$\begin{aligned}\langle \sigma_{x,\xi}^* u, v\rangle_{L^2(\mathbb{R}^n)} &= \langle u, \sigma_{x,\xi}v\rangle_{L^2(\mathbb{R}^n)}\\ &= \overline{\mathcal{W}(v,u)(x,\xi)} = \mathcal{W}(u,v)(x,\xi)\\ &= \langle \sigma_{x,\xi}u, v\rangle_{L^2(\mathbb{R}^n)},\end{aligned}$$

proving that $\sigma_{x,\xi}^* = \sigma_{x,\xi}$. ∎

We have thus

$$\mathrm{Op_w}(a) = 2^n \iint a(x,\xi)\sigma_{x,\xi}dxd\xi, \tag{1.2.7}$$

and the previous claim is proving the first estimate of the proposition. As a consequence of (1.2.7), we obtain that

$$(\mathrm{Op_w}(a))^* = \mathrm{Op_w}(\bar{a}), \text{ so that for } a \text{ real-valued, } (\mathrm{Op_w}(a))^* = \mathrm{Op_w}(a).$$

To prove the second estimate, we introduce the so-called ambiguity function $\mathcal{A}(u,v)$ as the inverse Fourier transform of the Wigner function $\mathcal{W}(u,v)$, so that for u, v in the Schwartz class, we have

$$(\mathcal{A}(u,v))(\eta,y) = \iint \mathcal{W}(u,v)(x,\xi)e^{2i\pi(x\cdot\eta+\xi\cdot y)}dxd\xi,$$

i.e.,

$$(\mathcal{A}(u,v))(\eta,y) = \int u\Big(x+\frac{y}{2}\Big)\bar{v}\Big(x-\frac{y}{2}\Big)e^{2i\pi x\cdot\eta}dx, \tag{1.2.8}$$

which reads as well as

$$(\mathcal{A}(u,v))(\eta,y) = \int u\Big(\frac{y}{2}+\frac{z}{2}\Big)\bar{\check{v}}\Big(\frac{y}{2}-\frac{z}{2}\Big)e^{2i\pi z\cdot\frac{\eta}{2}}dz2^{-n} = \mathcal{W}(u,\check{v})\Big(\frac{y}{2},-\frac{\eta}{2}\Big)2^{-n}. \tag{1.2.9}$$

N.B. The ambiguity function is called the *Fourier–Wigner transform* in G. B. Folland's book [15].

Remark 1.2.4. With $\Omega(u,v)$ defined by (1.1.1), we have

$$\mathcal{W}(u,v) = \mathcal{F}_2(\Omega(u,v)), \tag{1.2.10}$$

where $\mathcal{F}_2$ stands for the Fourier transformation with respect to the second variable. Taking the Fourier transform with respect to the second variable in the previous formula gives, with $\mathcal{F}_j$ (resp., $\mathcal{F}$) standing for the Fourier transform with respect to the jth variable (resp., all variables),

$$\mathcal{F}_2\mathcal{W} = \mathcal{C}_2\Omega, \quad \mathcal{F}\mathcal{W} = \mathcal{F}_1\mathcal{C}_2\Omega, \quad \mathcal{A} = \mathcal{C}\mathcal{F}\mathcal{W} = \mathcal{F}_1\mathcal{C}_1\Omega,$$

where $\mathcal{C}$ (resp., $\mathcal{C}_1$ or $\mathcal{C}_2$) stands for the "check" operator $\mathcal{C}$ in $\mathbb{R}^n\times\mathbb{R}^n$ given by (1.1.10) (resp., with respect to the first or second variable), the latter formula being (1.2.8).

Applying Plancherel formula on (1.2.1), we get

$$\langle \operatorname{Op_w}(a)u, v\rangle_{L^2(\mathbb{R}^n)} = \langle \hat{a}, \mathcal{A}(u,v)\rangle_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})}.$$

We note that a consequence of (1.2.3) is that for a linear form $L(x,\xi)$, we have

$$L\sharp L = L^2 \text{ and more generally } L^{\sharp N} = L^N.$$

As a result, considering for $(y,\eta)\in\mathbb{R}^{2n}$, the linear form $L_{\eta,y}$ defined by

$$L_{\eta,y}(x,\xi) = x\cdot\eta + \xi\cdot y,$$

we see that

$$\mathcal{A}(u,v)(\eta,y) = \langle \operatorname{Op_w}(e^{2i\pi(x\cdot\eta+\xi\cdot y)})u, v\rangle_{L^2(\mathbb{R}^n)},$$

and thus we get Hermann Weyl's original formula

$$\operatorname{Op_w}(a) = \iint \hat{a}(\eta,y)e^{i\operatorname{Op_w}(L_{\eta,y})}dyd\eta,$$

which implies the second estimate in the proposition. ∎

Proposition 1.2.5. *Let $a \in \mathscr{S}'(\mathbb{R}^{2n})$. The distribution kernel $k_a(x, y)$ of the operator* $\mathrm{Op_w}(a)$ *is*

$$k_a(x, y) = \hat{a}^{[2]}\Big(\frac{x+y}{2}, y - x\Big), \tag{1.2.11}$$

where $a^{[2]}$ stands for the Fourier transform of a with respect to the second variable. Let $k \in \mathscr{S}'(\mathbb{R}^{2n})$ be the distribution kernel of a continuous operator A from $\mathscr{S}(\mathbb{R}^n)$ into $\mathscr{S}'(\mathbb{R}^n)$. Then, the Weyl symbol a of A is

$$a(x, \xi) = \int e^{-2\pi i t \cdot \xi} k\Big(x + \frac{t}{2}, x - \frac{t}{2}\Big) dt,$$

where the integral sign means that we take the Fourier transform with respect to t of the distribution $k(x + \frac{t}{2}, x - \frac{t}{2})$ on $\mathbb{R}^{2n}$ (see Section A.1.1 for the definition of the Fourier transformation on tempered distributions).

Proof. With $u, v \in \mathscr{S}(\mathbb{R}^n)$, we have defined $\mathrm{Op_w}(a)$ via formula (1.2.1) and using Remark 1.2.4, we get

$$\begin{aligned}
\langle \mathrm{Op_w}(a)u, \bar{v}\rangle_{\mathscr{S}'(\mathbb{R}^n), \mathscr{S}(\mathbb{R}^n)} &= \big\langle a(x, \xi), \widehat{\Omega}^{[2]}(x, \xi)\big\rangle_{\mathscr{S}'(\mathbb{R}^{2n}), \mathscr{S}(\mathbb{R}^{2n})} \\
&= \Big\langle \hat{a}^{[2]}(t, z), u\Big(t + \frac{z}{2}\Big)\bar{v}\Big(t - \frac{z}{2}\Big)\Big\rangle_{\mathscr{S}'(\mathbb{R}^{2n}), \mathscr{S}(\mathbb{R}^{2n})} \\
&= \Big\langle \hat{a}^{[2]}\Big(\frac{x+y}{2}, y - x\Big), u(y)\bar{v}(x)\Big\rangle_{\mathscr{S}'(\mathbb{R}^{2n}), \mathscr{S}(\mathbb{R}^{2n})},
\end{aligned}$$

proving (1.2.11). As a consequence, we find that

$$k_a\Big(x + \frac{t}{2}, x - \frac{t}{2}\Big) = \hat{a}^{[2]}(x, -t),$$

and by Fourier inversion, this entails

$$\begin{aligned}
a(x, \xi) &= \mathrm{Fourier}_t\Big(k_a\Big(x + \frac{t}{2}, x - \frac{t}{2}\Big)\Big)(\xi) \\
&= \int e^{-2\pi i t \cdot \xi} k_a\Big(x + \frac{t}{2}, x - \frac{t}{2}\Big) dt,
\end{aligned} \tag{1.2.12}$$

where the integral sign means that we perform a Fourier transformation with respect to the variable t. ∎

A particular case of Segal's formula (see, e.g., [33, Theorem 2.1.2]) is with $\mathcal{F}$ standing for the Fourier transformation on $\mathbb{R}^n$,

$$\mathcal{F}^* \mathrm{Op_w}(a) \mathcal{F} = \mathrm{Op_w}\,(a(\xi, -x)).$$

1.2.2 The symplectic group

We define the canonical symplectic form σ on $\mathbb{R}^n \times \mathbb{R}^n$ by

$$\langle \sigma X, Y\rangle = [X, Y] = \xi \cdot y - \eta \cdot x \quad \text{with } X = (x, \xi), Y = (y, \eta). \tag{1.2.13}$$

The symplectic group[1] $\mathrm{Sp}(n, \mathbb{R})$ is the subgroup of $S \in \mathrm{Gl}(2n, \mathbb{R})$ such that

$$\forall X, Y \in \mathbb{R}^{2n}, \quad [SX, SY] = [X, Y], \text{ i.e., } S^*\sigma S = \sigma, \tag{1.2.14}$$

where S^* is the transpose and

$$\sigma = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. \tag{1.2.15}$$

It is easy to prove directly from (1.2.14) that $\mathrm{Sp}(1, \mathbb{R}) = \mathrm{Sl}(2, \mathbb{R})$.

Theorem 1.2.6. *Let n be an integer* ≥ 1. *The group* $\mathrm{Sp}(n, \mathbb{R})$ *is included in* $\mathrm{Sl}(2n, \mathbb{R})$ *and generated by the following mappings*

$$\begin{pmatrix} I_n & 0 \\ A & I_n \end{pmatrix}, \qquad \text{where } A \text{ is an } n \times n \text{ symmetric matrix,} \tag{1.2.16}$$

$$\begin{pmatrix} B^{-1} & 0 \\ 0 & B^* \end{pmatrix}, \qquad B \in \mathrm{Gl}(n, \mathbb{R}), \tag{1.2.17}$$

$$\begin{pmatrix} I_n & -C \\ 0 & I_n \end{pmatrix}, \qquad \text{where } C \text{ is an } n \times n \text{ symmetric matrix.} \tag{1.2.18}$$

For A, B, C *as above, the mapping*

$$\Xi_{A,B,C} = \begin{pmatrix} B^{-1} & -B^{-1}C \\ AB^{-1} & B^* - AB^{-1}C \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ A & I_n \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ 0 & B^* \end{pmatrix} \begin{pmatrix} I_n & -C \\ 0 & I_n \end{pmatrix}, \tag{1.2.19}$$

belongs to $\mathrm{Sp}(n, \mathbb{R})$. *Moreover, we define on* $\mathbb{R}^n \times \mathbb{R}^n$ *the generating function* S *of the symplectic mapping* $\Xi_{A,B,C}$ *by the identity*

$$S(x, \eta) = \frac{1}{2}(\langle Ax, x\rangle + 2\langle Bx, \eta\rangle + \langle C\eta, \eta\rangle) \text{ so that } \Xi\Big(\frac{\partial S}{\partial \eta} \oplus \eta\Big) = x \oplus \frac{\partial S}{\partial x}. \tag{1.2.20}$$

For a symplectic mapping Ξ, *to be of the form* (1.2.19) *is equivalent to the assumption that the mapping* $x \mapsto \pi_{\mathbb{R}^n \times \{0\}} \Xi(x \oplus 0)$ *is invertible from* $\mathbb{R}^n$ *to* $\mathbb{R}^n$; *moreover, if this mapping is not invertible, the symplectic mapping* Ξ *is the product of two mappings of the type* $\Xi_{A,B,C}$.

[1] This is obviously a group since for $S_1, S_2 \in Sp(n, \mathbb{R})$, the last equation in (1.2.14) implies that $|\det S| = 1$ and $[S_1 S_2^{-1} X, S_1 S_2^{-1} Y] = [S_2^{-1} X, S_2^{-1} Y] = [X, Y]$, since $[S_2^{-1} X, S_2^{-1} Y] = [S_2 S_2^{-1} X, S_2 S_2^{-1} Y] = [X, Y]$. We shall prove below that the determinant of a symplectic mapping is actually 1.

Proof. The expression of Ξ above as well as (1.2.20) follow from a simple direct computation left to the reader. The inclusion of the symplectic group in the special linear group follows from the statement on the generators. We consider now Ξ in $\mathrm{Sp}(n,\mathbb{R})$: we have

$$\Xi = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}, \quad \text{where } P, Q, R, S, \text{ are } n \times n \text{ matrices.} \tag{1.2.21}$$

The equation

$$\Xi^* \sigma \Xi = \sigma,$$

is satisfied with $\sigma = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$, which means

$$P^*R = (P^*R)^*, \quad Q^*S = (Q^*S)^*, \quad P^*S - R^*Q = I_n. \tag{1.2.22}$$

We can note also that the mapping $\Xi \mapsto \Xi^*$ is an isomorphism of $\mathrm{Sp}(n,\mathbb{R})$ since $\Xi \in \mathrm{Sp}(n,\mathbb{R})$ means

$$\Xi^*\sigma\Xi = \sigma \Longrightarrow \Xi^{-1}\sigma^{-1}(\Xi^*)^{-1} = \sigma^{-1} \Longrightarrow \Xi^{-1}(-\sigma^{-1})(\Xi^*)^{-1} = (-\sigma^{-1}),$$

and since $(-\sigma^{-1}) = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$, we get that $\Xi^* \in \mathrm{Sp}(n,\mathbb{R})$. As a result,

$$\Xi = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \in \mathrm{Sp}(n,\mathbb{R}), \tag{1.2.23}$$

is also equivalent to

$$PQ^* = (PQ^*)^*, \quad RS^* = (RS^*)^*, \quad PS^* - QR^* = I_n. \tag{1.2.24}$$

Let us assume that the mapping P is invertible, which is the assumption in the last statement of the theorem. We define then the mappings A, B, C by

$$A = RP^{-1}, \quad B = P^{-1}, \quad C = -P^{-1}Q,$$

so that we have

$$A^* = P^{*-1}R^*PP^{-1} = P^{*-1}P^*RP^{-1} = RP^{-1} = A,$$

as well as

$$C^* = -Q^*P^{*-1} = -P^{-1}PQ^*P^{*-1} = -P^{-1}QP^*P^{*-1} = -P^{-1}Q = C,$$

and

$$P = B^{-1}, \quad R = AB^{-1}, \quad Q = -B^{-1}C,$$
$$S = P^{*-1}(I_n + R^*Q) = B^*(I_n - B^{*-1}A^*B^{-1}C) = B^* - AB^{-1}C.$$

We have thus proven that any symplectic matrix Ξ as above such that P is invertible is indeed given by the product appearing in Theorem 1.2.6.

Let us now consider the case where a symplectic mapping Ξ (given by (1.2.23)) is such that $\det P = 0$; writing $\mathbb{R}^n = \ker P \oplus N$ we have that P is an isomorphism from N onto $\operatorname{ran} P$. Let $B_1 \in \mathrm{Gl}(n, \mathbb{R})$ such that $B_1 P$ is the identity on N (see footnote[2]). We have

$$\begin{pmatrix} B_1 & 0 \\ 0 & B_1^{*-1} \end{pmatrix} \begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} B_1 P & B_1 Q \\ B_1^{*-1} R & B_1^{*-1} S \end{pmatrix}. \tag{1.2.25}$$

If $p = \dim(\ker P)$, we have for the $n \times n$ matrix $B_1 P$ the following block decomposition

$$B_1 P = \begin{pmatrix} 0_{p,p} & 0_{p,n-p} \\ 0_{n-p,p} & I_{n-p} \end{pmatrix},$$

where $0_{r,s}$ stands for an $r \times s$ matrix with only 0 as an entry. On the other hand, we know from (1.2.22) that the mapping

$$(B_1 P)^* B_1^{*-1} R = P^* R$$

is symmetric. Writing $B_1^{*-1} R = \begin{pmatrix} \tilde{R}_{p,p} & \tilde{R}_{p,n-p} \\ \tilde{R}_{n-p,p} & \tilde{R}_{n-p,n-p} \end{pmatrix}$, where $\tilde{R}_{r,s}$ stands for an $r \times s$ matrix, this gives the symmetry of

$$\begin{pmatrix} 0_{p,p} & 0_{p,n-p} \\ 0_{n-p,p} & I_{n-p} \end{pmatrix} \begin{pmatrix} \tilde{R}_{p,p} & \tilde{R}_{p,n-p} \\ \tilde{R}_{n-p,p} & \tilde{R}_{n-p,n-p} \end{pmatrix} = \begin{pmatrix} 0_{p,p} & 0_{p,n-p} \\ \tilde{R}_{n-p,p} & \tilde{R}_{n-p,n-p} \end{pmatrix},$$

implying that $\tilde{R}_{n-p,p} = 0$. The symplectic matrix (1.2.25) is thus equal to

$$\begin{pmatrix} \begin{pmatrix} 0_{p,p} & 0_{p,n-p} \\ 0_{n-p,p} & I_{n-p} \end{pmatrix} & B_1 Q \\ \begin{pmatrix} \tilde{R}_{p,p} & \tilde{R}_{p,n-p} \\ 0_{n-p,p} & \tilde{R}_{n-p,n-p} \end{pmatrix} & B_1^{*-1} S \end{pmatrix}, \quad \text{where } B_1 Q \text{ and } B_1^{*-1} S \text{ are } n \times n \text{ blocks.}$$

The invertibility of (1.2.25) implies that $\tilde{R}_{p,p}$ is invertible. We consider now the $n \times n$ symmetric matrix

$$C = \begin{pmatrix} I_{p,p} & 0_{p,n-p} \\ 0_{n-p,p} & 0_{n-p,n-p} \end{pmatrix},$$

[2] This is indeed possible: choosing a supplement space M for $P(N)$, we have

$$\mathbb{R}^n = \underbrace{\ker P}_{\dim p} \oplus \underbrace{N}_{\dim n-p} = \underbrace{P(N)}_{\dim n-p} \oplus \underbrace{M}_{\dim p},$$

and we can define B_1 on $P(N)$ by $B_1(Px) = x$ (without ambiguity since for $x_1, x_2 \in N$ with $Px_1 = Px_2$ we get $x_1 - x_2 \in \ker P \cap N = \{0\}$) and $B_{1|M} : M \to \ker P$ can be chosen as an isomorphism, so that $B_1(P(N)) + B_1(M) = N + \ker P$, which implies $\operatorname{rank} B_1 = n$.

and the symplectic mapping

$$\begin{pmatrix} I_n & C \\ 0 & I_n \end{pmatrix} \begin{pmatrix} B_1 & 0 \\ 0 & B_1^{*-1} \end{pmatrix} \begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} I_n & C \\ 0 & I_n \end{pmatrix} \begin{pmatrix} B_1 P & B_1 Q \\ B_1^{*-1} R & B_1^{*-1} S \end{pmatrix}, \quad (1.2.26)$$

which is a symplectic mapping $\left(\begin{smallmatrix} P' & Q' \\ R' & S' \end{smallmatrix}\right)$ with

$$\begin{aligned} P' &= B_1 P + C B_1^{*-1} R \\ &= \begin{pmatrix} 0_{p,p} & 0_{p,n-p} \\ 0_{n-p,p} & I_{n-p} \end{pmatrix} + \begin{pmatrix} I_{p,p} & 0_{p,n-p} \\ 0_{n-p,p} & 0_{n-p,n-p} \end{pmatrix} \begin{pmatrix} \tilde{R}_{p,p} & \tilde{R}_{p,n-p} \\ 0_{n-p,p} & \tilde{R}_{n-p,n-p} \end{pmatrix} \\ &= \begin{pmatrix} \tilde{R}_{p,p} & \tilde{R}_{p,n-p} \\ 0_{n-p,p} & \tilde{I}_{n-p} \end{pmatrix}, \end{aligned}$$

which is an invertible mapping. From equation (1.2.26) and the first part of our discussion, we get that

$$\begin{pmatrix} P' & Q' \\ R' & S' \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ A' & I_n \end{pmatrix} \begin{pmatrix} B'^{-1} & 0 \\ 0 & B'^{*} \end{pmatrix} \begin{pmatrix} I_n & -C' \\ 0 & I_n \end{pmatrix},$$

with A', C' symmetric and B' invertible and

$$\Xi = \begin{pmatrix} B_1{}^{-1} & 0 \\ 0 & B_1^{*} \end{pmatrix} \begin{pmatrix} I_n & -C \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_n & 0 \\ A' & I_n \end{pmatrix} \begin{pmatrix} B'^{-1} & 0 \\ 0 & B'^{*} \end{pmatrix} \begin{pmatrix} I_n & -C' \\ 0 & I_n \end{pmatrix},$$

proving that the $\Xi_{A,B,C}$ generate the symplectic group and more precisely that every Ξ in the symplectic group is the product of at most two mappings of type $\Xi_{A,B,C}$. This completes the proof of Theorem 1.2.6. ■

Corollary 1.2.7. *We have* $\mathrm{Sp}(n, \mathbb{R}) \subset \mathrm{Sl}(2n, \mathbb{R})$.

Proof. Indeed, the symplectic mappings (1.2.16), (1.2.17), and (1.2.18) do have determinants equal to 1 and since Theorem 1.2.6 implies that they generate the symplectic group, this proves the sought result. ■

Remark 1.2.8. Of course for $n \geq 2$, $\mathrm{Sp}(n, \mathbb{R})$ is a proper subgroup of $\mathrm{Sl}(2n, \mathbb{R})$. Indeed, the following matrix:

$$\mathtt{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

has determinant 1 but fails to be symplectic: using notation (1.2.21), we see that the first and the third equation are satisfied, which is not the case for the second one.

N.B. Since the matrix $-I_{2n}$ belongs to $\mathrm{Sp}(n, \mathbb{R})$ ((1.2.14) holds trivially), we find that $S \in \mathrm{Sp}(n, \mathbb{R})$ is equivalent to $-S \in \mathrm{Sp}(n, \mathbb{R})$.

Claim 1.2.9. The symplectic group is also generated by the mappings

$$\begin{aligned}
&(x, \xi) \mapsto (B^{-1}x, B^*\xi), \quad && B \in \mathrm{Gl}(n, \mathbb{R}),\\
&(x, \xi) \mapsto (\xi, -x), && \\
&(x, \xi) \mapsto (x, \xi + Ax), && A \in \mathrm{Sym}(n, \mathbb{R}).
\end{aligned}$$

Another set of generators of the symplectic group is given by the mappings

$$\begin{aligned}
&(x, \xi) \mapsto (B^{-1}x, B^*\xi), \quad && B \in \mathrm{Gl}(n, \mathbb{R}),\\
&(x, \xi) \mapsto (\xi, -x), && \\
&(x, \xi) \mapsto (x - C\xi, \xi), && C \in \mathrm{Sym}(n, \mathbb{R}).
\end{aligned}$$

Proof. Indeed, we have for $C^* = C$ a real symmetric $n \times n$ matrix

$$\underbrace{\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}}_{\sigma^{-1}} \begin{pmatrix} I_n & -C \\ 0 & I_n \end{pmatrix} \underbrace{\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}}_{\sigma} = \begin{pmatrix} I_n & 0 \\ C & I_n \end{pmatrix},$$

proving the claim. ∎

Remark 1.2.10. The symplectic matrix

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} = 2^{-1/2} \begin{pmatrix} I_n & I_n \\ -I_n & I_n \end{pmatrix} 2^{-1/2} \begin{pmatrix} I_n & I_n \\ -I_n & I_n \end{pmatrix} = \Xi^2_{-I_n, 2^{1/2}I_n, -I_n}, \tag{1.2.27}$$

is not of the form $\Xi_{A,B,C}$ but is the square of such a matrix. It is also the case of all the mappings $(x_k, \xi_k) \mapsto (\xi_k, -x_k)$ with the other coordinates fixed. Similarly, the symplectic matrix

$$\begin{pmatrix} 0 & -I_n \\ I_n & I_n \end{pmatrix} = \begin{pmatrix} I_n & -I_n \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_n & 0 \\ I_n & I_n \end{pmatrix},$$

is not of the form $\Xi_{A,B,C}$ but is the product $\Xi_{0,I,I}\,\Xi_{I,I,0}$.

1.2.3 The metaplectic group

Proposition 1.2.11. *Let A, B, C be as in Theorem 1.2.6, and let S be the generating function of $\Xi_{A,B,C}$* (cf. (1.2.20)). *We define the operator $M_{A,B,C}$ on $\mathscr{S}(\mathbb{R}^n)$ by*

$$(M_{A,B,C}v)(x) = \int_{\mathbb{R}^n} e^{2i\pi S(x,\eta)} \hat{v}(\eta) d\eta (\det B)^{1/2}, \tag{1.2.28}$$

where $(\det B)^{1/2}$ *is a square-root of* $\det B$. *This operator is an automorphism of* $\mathscr{S}'(\mathbb{R}^n)$ *and of* $\mathscr{S}(\mathbb{R}^n)$ *which is unitary on* $L^2(\mathbb{R}^n)$, *and such that, for all* $a \in \mathscr{S}'(\mathbb{R}^{2n})$,

$$M^*_{A,B,C}\operatorname{Op}_{\mathrm{w}}(a)M_{A,B,C} = \operatorname{Op}_{\mathrm{w}}(a \circ \Xi_{A,B,C}), \tag{1.2.29}$$

where $\Xi_{A,B,C}$ *is defined in Theorem* 1.2.6.

N.B. We have for A, B, C as above,

$$(M_{A,I,0}v)(x) = e^{i\pi\langle Ax,x\rangle}v(x), \tag{1.2.30}$$

$$(M_{0,B,0}v)(x) = (\det B)^{1/2}v(Bx), \tag{1.2.31}$$

$$(M_{0,I,C}v)(x) = \big(e^{i\pi\langle CD_x,D_x\rangle}v\big)(x), \tag{1.2.32}$$

three operators which are obviously automorphisms of $\mathscr{S}(\mathbb{R}^n)$ and of $\mathscr{S}'(\mathbb{R}^n)$ as well as unitary operators in $L^2(\mathbb{R}^n)$.

Proof. Formula (1.2.29) is easily checked for each operator (1.2.30), (1.2.31), and (1.2.32). Since we have

$$\Xi_{A,B,C} = \Xi_{A,I,0}\ \Xi_{0,B,0}\ \Xi_{0,I,C}$$

and

$$M_{A,B,C} = M_{A,I,0}\ M_{0,B,0}\ M_{0,I,C}, \tag{1.2.33}$$

we get (1.2.29) and the proposition. ∎

Remark 1.2.12. We define

$$m(B) = \frac{\arg(\det B)}{\pi} = \begin{cases} \frac{k2\pi}{\pi} = 2k \in \{0,2\} \mod 4 & \text{for } \det B > 0, \\ \frac{k2\pi+\pi}{\pi} = 2k+1 \in \{1,3\} \mod 4 & \text{for } \det B < 0, \end{cases} \tag{1.2.34}$$

so that

$$\det B = |\det B| e^{i\pi m(B)}, \quad (\det B)^{1/2} \in |\det B|^{1/2}\big\{e^{i\frac{\pi}{2}m(B)}, e^{i\frac{\pi}{2}(m(B)+2)}\big\}.^{3}$$

We will consider $m(B)$ as an element of $\mathbb{Z}/4\mathbb{Z}$, so that the function $m(B) \mapsto e^{i\frac{\pi}{2}m(B)}$ is well-defined. For A, B, C as in Proposition 1.2.11, we may define

$$\big(M^{\{m(B)\}}_{A,B,C}v\big)(x) = e^{\frac{i\pi m(B)}{2}}|\det B|^{1/2}\int_{\mathbb{R}^n} e^{i\pi(Ax^2+2Bx\cdot\eta+C\eta^2)}\hat{v}(\eta)d\eta,^{4} \tag{1.2.35}$$

[3]This is a synthetic way to write

$$(\det B)^{1/2} \in \{(\pm 1)|\det B|^{1/2}\} \text{ if } \det B > 0, \quad (\det B)^{1/2} \in \{(\pm i)|\det B|^{1/2}\} \text{ if } \det B < 0.$$

[4]We can of course define $M^{\{m\}}_{A,B,C}$ for any m, but to stay in the metaplectic group (cf. Definition 1.2.13), we have to make sure that $m \in \{m(B), m(B)+2\}$ modulo 4.

but we shall omit the super-script $m(B)$ when we do not want to distinguish between the two roots of $\det B$. We note in particular that we have

$$M^{\{0\}}_{0,I_n,0} = \mathrm{Id}_{L^2(\mathbb{R}^n)}, \quad M^{\{2\}}_{0,I_n,0} = -\mathrm{Id}_{L^2(\mathbb{R}^n)},$$

and also with the notation (1.2.6),

$$M^{\{n\}}_{0,-I_n,0} = e^{\frac{i\pi n}{2}}\sigma_0, \quad M^{\{n+2\}}_{0,-I_n,0} = -e^{\frac{i\pi n}{2}}\sigma_0.$$

More generally, we have

$$\text{for } \det B > 0,\ M^{\{0\}}_{A,B,C} = -M^{\{2\}}_{A,B,C}, \quad \text{for } \det B < 0,\ M^{\{1\}}_{A,B,C} = -M^{\{3\}}_{A,B,C}. \tag{1.2.36}$$

We note also that for $B \in \mathrm{Gl}(n,\mathbb{R})$, we have

$$m(B^*) = m(B) = m(B^{-1}),$$

since $\det B = \det B^*$ and $\det(B^{-1}) = (\det B)^{-1}$ so that

$$\arg(\det B) = \arg(\det B^{-1}).$$

Moreover, we have for $B \in \mathrm{Gl}(n,\mathbb{R})$,

$$\det(-B) = (-1)^n \det B, \quad \arg(\det(-B)) = \begin{cases} \arg(\det B) & \text{if } n \text{ is even}, \\ \arg(\det B) + \pi & \text{if } n \text{ is odd}, \end{cases}$$

so that

$$m(-B) = n + m(B). \tag{1.2.37}$$

Examples. Let us start with a one-dimensional example: in Remark 1.2.10, we have seen, in particular, that

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \left\{2^{-1/2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\right\}^2, \quad 2^{-1/2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \Xi_{-1,2^{1/2},-1},$$

where we have used (1.2.19) to get the second equation. We have also with the notations of Theorem 1.2.6,

$$(M_{-1,2^{1/2},-1}v)(x) = \int_{\mathbb{R}} e^{2i\pi\frac{1}{2}(-x^2+2^{3/2}x\eta-\eta^2)}\hat{v}(\eta)d\eta 2^{1/4},$$

so that the kernel $k_1(x,y)$ of the operator $M_{-1,2^{1/2},-1}$ is

$$\begin{aligned} &k_1(x,y) \\ &= 2^{1/4}\int e^{i\pi(-x^2+2^{3/2}x\eta-\eta^2)}e^{-2i\pi y\eta}d\eta \underbrace{=}_{\text{use (A.1.7)}} 2^{1/4}e^{-i\pi/4}e^{i\pi(x^2+y^2)}e^{-2^{3/2}i\pi xy}, \end{aligned}$$

so that the kernel k_2 of the operator $(M_{-1,2^{1/2},-1})^2$ is (using again (A.1.7)),

$$k_2(x,y) = \int k_1(x,z)k_1(z,y)dz$$
$$= 2^{1/2}e^{-i\pi/2}e^{i\pi(x^2+y^2)}\int e^{2i\pi z^2}e^{-2i\pi z 2^{1/2}(x+y)}dz = e^{-i\pi/4}e^{-2i\pi xy},$$

so that

$$(M_{-1,2^{1/2},-1})^2 = e^{-i\pi/4}\mathcal{F}_1,$$

with $\mathcal{F}_1$ standing for the 1d Fourier transformation. We get similarly that in n dimensions,

$$(M_{-I_n,2^{1/2}I_n,-I_n})^2 = e^{-i\pi n/4}\mathcal{F}, \tag{1.2.38}$$

with $\mathcal{F}$ standing for the Fourier transformation. Similar expressions can be obtained for $\mathcal{F}_k$, the Fourier transformation with respect to the variable x_k in n dimensions, $k \in [\![1,n]\!]$ with

$$(M_{A_k,B_k,C_k})^2 = e^{-i\pi/4}\mathcal{F}_k,$$

where B_k is the $n \times n$ diagonal matrix with diagonal entries equal to 1 except for the kth equal to $2^{1/2}$, the $n \times n$ diagonal matrices $A_k = C_k$ with diagonal entries equal to 0, except for the kth equal to -1.

Definition 1.2.13. The metaplectic group $\mathrm{Mp}(n)$ is defined as the subgroup of the group of unitary operators on $L^2(\mathbb{R}^n)$ generated by

$M_{A,I,0}$, where A is an $n \times n$ symmetric matrix, cf. (1.2.30), (1.2.39)

$M_{0,B,0}$, with $B \in \mathrm{Gl}(n,\mathbb{R})$, with $(\det B)^{\frac12} = |\det B|^{\frac12} e^{\frac{i\pi m(B)}{2}}$, cf. (1.2.34), (1.2.31), (1.2.40)

$M_{0,I,C}$, where C is an $n \times n$ symmetric matrix, cf. (1.2.32). (1.2.41)

Claim 1.2.14. If M belongs to $\mathrm{Mp}(n)$, then $-M$ belongs to $\mathrm{Mp}(n)$.

Proof. According to (1.2.36), we have

$$M^{\{2\}}_{0,I_n,0} = -M^{\{0\}}_{0,I_n,0} = -\mathrm{Id}_{L^2(\mathbb{R}^n)}$$

so that $-\mathrm{Id}_{L^2(\mathbb{R}^n)}$ belongs to $\mathrm{Mp}(n)$, proving the claim. ∎

Proposition 1.2.15. *The metaplectic group* $\mathrm{Mp}(n)$ *is generated by*

$M_{A,I,0}$, *where A is an $n \times n$ symmetric matrix, cf.* (1.2.30), (1.2.42)

$M_{0,B,0}$, *with* $B \in \mathrm{Gl}(n,\mathbb{R})$, *with* $(\det B)^{\frac12} = |\det B|^{\frac12} e^{\frac{i\pi m(B)}{2}}$, *cf.* (1.2.34), (1.2.31), (1.2.43)

$e^{-\frac{i\pi n}{4}}\mathcal{F}$, *where $\mathcal{F}$ is the Fourier transformation.* (1.2.44)

Proof. We check for C symmetric $n \times n$ matrix,

$$\big(M^{\{0\}}_{C,I,0}(e^{-i\pi n/4}\mathcal{F} v)\big)(\eta) = e^{-i\pi n/4} e^{i\pi C\eta^2}\hat{v}(\eta),$$

so that

$$e^{i\pi n/4}\big(\mathcal{F}^{-1}(e^{-i\pi n/4}e^{i\pi C\eta^2}\hat{v}(\eta))\big)(x) = \int e^{2i\pi x\eta}e^{i\pi C\eta^2}\hat{v}(\eta)d\eta = (M^{\{0\}}_{0,I,C}v)(x),$$

yielding

$$e^{i\pi n/4}\mathcal{F}^{-1}M^{\{0\}}_{0,I,C}e^{-i\pi n/4}\mathcal{F} = M^{\{0\}}_{0,I,C},$$

so that the group generated by (1.2.42), (1.2.43), (1.2.44) contains (1.2.39), (1.2.40), and (1.2.41) and thus contains $\mathrm{Mp}(n)$. Moreover, (1.2.38) shows that (1.2.44) is included in $\mathrm{Mp}(n)$ so that the group generated by (1.2.42), (1.2.43), and (1.2.44) is included in $\mathrm{Mp}(n)$, proving the proposition. ∎

Remark 1.2.16. According to (A.1.6) in our appendix and to (1.2.36), we find

$$(e^{-i\pi n/4}\mathcal{F})^* = e^{i\pi n/4}\mathcal{F}\sigma_0 = e^{-i\pi n/4}\mathcal{F}e^{i\pi n/2}\sigma_0 = e^{-i\pi n/4}\mathcal{F}M^{\{n\}}_{0,-I_n,0}.$$

As a consequence, $e^{-i\pi n/4}\mathcal{F}, e^{-i\pi n/2}\sigma_0, e^{i\pi n/2}\sigma_0$ belong to the metaplectic group.

Lemma 1.2.17. *For $Y \in \mathbb{R}^{2n}$, we define the linear form L_Y on $\mathbb{R}^{2n}$ by*

$$L_Y(X) = \langle \sigma Y, X\rangle = [Y, X].$$

For any $M \in \mathrm{Mp}(n)$ there exists a unique $\chi \in \mathrm{Sp}(n,\mathbb{R})$ such that

$$\forall Y \in \mathbb{R}^{2n}, \quad M^*\mathrm{Op_w}(L_Y)M = \mathrm{Op_w}(L_{\chi^{-1}Y}). \tag{1.2.45}$$

Proof. Indeed, thanks to (1.2.29) and Definition 1.2.13, we can find $\chi \in \mathrm{Sp}(n,\mathbb{R})$ such that

$$M^*\mathrm{Op_w}(L_Y)M = \mathrm{Op_w}(L_Y\circ\chi) = \mathrm{Op_w}(L_{\chi^{-1}Y}),$$

since

$$(L_Y\circ\chi)(X) = \langle\sigma Y, \chi X\rangle = \langle\chi^*\sigma\chi\chi^{-1}Y, X\rangle = \langle\sigma\chi^{-1}Y, X\rangle = L_{\chi^{-1}Y}(X).$$

Moreover, if $\chi_1, \chi_2 \in \mathrm{Sp}(n,\mathbb{R})$ are such that for all $Y \in \mathbb{R}^{2n}$,

$$0 = \mathrm{Op_w}(L_{\chi_2^{-1}Y} - L_{\chi_1^{-1}Y}) = \mathrm{Op_w}(L_{(\chi_2^{-1}-\chi_1^{-1})Y}),$$

we get

$$L_{(\chi_2^{-1}-\chi_1^{-1})Y} = 0,$$

implying $\forall Y \in \mathbb{R}^{2n}, (\chi_2^{-1} - \chi_1^{-1})Y = 0$, i.e., $\chi_1 = \chi_2$. ∎

We can thus define a mapping

$$\Psi : \mathrm{Mp}(n) \to \mathrm{Sp}(n,\mathbb{R}) \quad \text{with } \Psi(M) = \chi \text{ satisfying (1.2.45).} \tag{1.2.46}$$

In particular, we have from (1.2.29) in Proposition 1.2.11 and (1.2.38) that

$$\Psi(M_{A,B,C}) = \Xi_{A,B,C}, \quad \Psi\big(e^{-\frac{i\pi n}{4}}\mathcal{F}\big) = \sigma = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. \tag{1.2.47}$$

Theorem 1.2.18. *The mapping* Ψ *defined in* (1.2.46) *is a surjective homomorphism of groups with kernel* $\{\pm \mathrm{Id}_{L^2(\mathbb{R}^n)}\}$.

Proof. This mapping is a homomorphism of groups: if M_1, M_2 belong to $\mathrm{Mp}(n)$, we have with $\chi_j = \Psi(M_j)$,

$$\begin{aligned}(M_1M_2)^*\mathrm{Op_w}(L_Y)M_1M_2 &= M_2^*\mathrm{Op_w}(L_{\chi_1^{-1}Y})M_2 \\ &= \mathrm{Op_w}(L_{\chi_2^{-1}\chi_1^{-1}Y}) = \mathrm{Op_w}(L_{(\chi_1\circ\chi_2)^{-1}Y}),\end{aligned}$$

proving that $\Psi(M_1M_2) = \Psi(M_1)\Psi(M_2)$. Moreover, the homomorphism Ψ is onto, thanks to (1.2.29) and Theorem 1.2.6. The kernel of Ψ is made with $M \in \mathrm{Mp}(n)$ such that for all $Y \in \mathbb{R}^{2n}$,

$$M^*\mathrm{Op_w}(L_Y)M = \mathrm{Op_w}(L_Y),$$

i.e.,

$$[\mathrm{Op_w}(L_Y), M] = 0,$$

so that, thanks to (1.2.3), (1.2.4), if $\mu(x,\xi)$ is the Weyl symbol of M (M is an endomorphism of $\mathscr{S}'(\mathbb{R}^n)$ and thus has a distribution kernel as well as a Weyl symbol via formula (1.2.12)), we get for all $(y,\eta) \in \mathbb{R}^{2n}$,

$$0 = \{\eta\cdot x - y\cdot\xi, \mu(x,\xi)\} \ \text{ so that } d\mu = 0,$$

and μ is a constant so that $M = c\,\mathrm{Id}_{L^2(\mathbb{R}^n)}$, necessarily with $|c| = 1$ (since M is unitary). Applying Theorem A.2.11 gives $c \in \{\pm 1\}$, concluding the proof. ∎

N.B. The proof of Theorem A.2.11 is relegated in our appendix, and requires some effort.

Corollary 1.2.19. *For* $\chi \in \mathrm{Sp}(n,\mathbb{R})$, *the fiber* $\Psi^{-1}\{\chi\}$ *contains exactly two metaplectic transformations and more precisely*

$$\Psi^{-1}\{\chi\} = \{M, -M\},$$

where M *is a metaplectic transformation.*

Proof. This corollary is an immediate consequence of Theorem 1.2.18. ∎

Theorem 1.2.20 (Symplectic covariance of the Weyl calculus). *Let a be in $\mathscr{S}'(\mathbb{R}^{2n})$ and let χ be in* $\mathrm{Sp}(n,\mathbb{R})$. *Then, for a metaplectic operator M such that $\Psi(M)=\chi$, we have*

$$M^*\mathrm{Op_w}(a)M = \mathrm{Op_w}(a\circ\chi). \tag{1.2.48}$$

For $u,v\in\mathscr{S}(\mathbb{R}^n)$, we have

$$\mathcal{W}(Mu,Mv)=\mathcal{W}(u,v)\circ\chi^{-1}, \tag{1.2.49}$$

where $\mathcal{W}$ is the Wigner distribution given in (1.1.4).

Proof. The first property follows from (1.2.29) and Definition 1.2.13 whereas (1.2.49) is a consequence of (1.2.1) and (1.2.48). ∎

We note also that for $Y=(y,\eta)\in\mathbb{R}^{2n}$, the symmetry S_Y is defined by

$$S_Y(X)=2Y-X$$

and is quantized by the phase symmetry σ_Y as defined by (1.2.6) with the formula

$$\mathrm{Op_w}(a\circ S_Y)=\sigma_Y^*\mathrm{Op_w}(a)\sigma_Y=\sigma_Y\mathrm{Op_w}(a)\sigma_Y. \tag{1.2.50}$$

Similarly, the translation T_Y is defined on the phase space by

$$T_Y(X)=X+Y$$

and is quantized by the *phase translation* τ_Y,

$$(\tau_{(y,\eta)}u)(x)=u(x-y)e^{2i\pi(x-\frac{y}{2})\cdot\eta}, \tag{1.2.51}$$

and we have

$$\mathrm{Op_w}(a\circ T_Y)=\tau_Y^*\mathrm{Op_w}(a)\tau_Y=\tau_{-Y}\mathrm{Op_w}(a)\tau_Y.$$

Remark 1.2.21. Property (1.2.49) can be extended to the affine symplectic group and we have with the phase translations defined in (1.2.51),

$$\forall(X,Y)\in\mathbb{R}^{2n}\times\mathbb{R}^{2n},\quad \mathcal{W}(\tau_Y u,\tau_Y v)(X)=\mathcal{W}(u,v)(X-Y).$$

We will define the *affine group* $\mathrm{Mp}_a(n)$ as the group of unitary transformations of $L^2(\mathbb{R}^n)$ generated by transformations (1.2.30), (1.2.31), and (1.2.32) and phase translations given by (1.2.51).

N.B. More information on the metaplectic group is given in J. Leray's book [31], the same author's articles [30,32], as well as A. Weil's paper [52] (see also V. S. Buslaev's article [5], K. Gröchenig's book [16, Chapter 9], H. Reiter's lecture notes [43]).

Theorem 1 in E. Lieb's classical article [37] gives a more precise version of (1.2.53), (1.2.54), and (1.2.55) below.

Theorem 1.2.22. *Let u, v be in $L^2(\mathbb{R}^n)$. Then, $\mathcal{W}(u,v)$ is a uniformly continuous function belonging to $L^2(\mathbb{R}^{2n}) \cap L^\infty(\mathbb{R}^{2n})$ and using the definitions* (1.2.51), (1.2.6) *for the phase translations and phase symmetry, we have*

$$\begin{aligned}\mathcal{W}(u,v)(X) &= 2^n \langle \sigma_X u, v\rangle_{L^2(\mathbb{R}^n)} = 2^n \langle \tau_X^* u, \tau_X \check{v}\rangle_{L^2(\mathbb{R}^n)} \\ &= 2^n \langle \sigma_0 \tau_{-2X} u, v\rangle_{L^2(\mathbb{R}^n)}, \end{aligned} \tag{1.2.52}$$

$$\|\mathcal{W}(u,v)\|_{L^2(\mathbb{R}^{2n})} = \|u\|_{L^2(\mathbb{R}^n)} \|v\|_{L^2(\mathbb{R}^n)}, \tag{1.2.53}$$

$$\forall p \in [1,+\infty], \quad \|\mathcal{W}(u,v)\|_{L^\infty(\mathbb{R}^{2n})} \le 2^n \|u\|_{L^p(\mathbb{R}^n)} \|v\|_{L^{p'}(\mathbb{R}^n)}. \tag{1.2.54}$$

More generally, for $q \ge 2$ and $r \in [q', q]$, we have[5]

$$\|\mathcal{W}(u,v)\|_{L^q(\mathbb{R}^{2n})} \le 2^{\frac{n(q-2)}{q}} \|u\|_{L^r(\mathbb{R}^n)} \|v\|_{L^{r'}(\mathbb{R}^n)}. \tag{1.2.55}$$

Moreover, we have

$$\lim_{\mathbb{R}^{2n} \ni X, |X| \to +\infty} \big[\mathcal{W}(u,v)(X)\big] = 0.$$

Proof. We have with $\check{v}(x) = v(-x) = (\sigma_0 v)(x)$,

$$\begin{aligned}\mathcal{W}(u,v)(x,\xi) &= 2^n \int u(x+z)\bar{v}(x-z) e^{-4i\pi z\xi} dz \\ &= 2^n \int u(z-(-x)) e^{2i\pi(z-\frac{-x}{2})(-\xi)} \bar{\check{v}}(z-x) e^{-2i\pi(z-\frac{x}{2})\xi} \\ &\quad \times e^{-4i\pi z\xi + 2i\pi(z-\frac{-x}{2})\xi + 2i\pi(z-\frac{x}{2})\xi} dz \\ &= 2^n \int (\tau_{(-x,-\xi)} u)(z) \overline{(\tau_{(x,\xi)} \check{v})(z)} dz = 2^n \langle \tau_{(x,\xi)}^* u, \tau_{(x,\xi)} \check{v}\rangle_{L^2(\mathbb{R}^n)},\end{aligned}$$

or for short

$$\mathcal{W}(u,v)(X) = 2^n \langle \tau_X^* u, \tau_X \check{v}\rangle_{L^2(\mathbb{R}^n)}.$$

As a consequence, we find from (1.2.7) that

$$\langle \mathrm{Op_w}(a) u, v\rangle = \int a(X) 2^n \langle \sigma_0 \tau_{2X}^* u, v\rangle dX,$$

and since $(\sigma_{x,\xi} u)(y) = u(2x-y) e^{-4i\pi(x-y)\cdot\xi}$, we can verify directly that

$$\sigma_0 \tau_{-2X} = \sigma_X. \tag{1.2.56}$$

Indeed, composing the translation of vector $-2X$ in $\mathbb{R}^{2n}$ with the symmetry with respect to 0, we have

$$Y \mapsto Y - 2X \mapsto 2X - Y = Y', \quad \frac{1}{2}(Y + Y') = X,$$

[5] We use the standard notation: for $p \in [1,+\infty]$ we define p' by the equality $\frac{1}{p} + \frac{1}{p'} = 1$.

that is the symmetry with respect to X. Quantifying this equality, we use

$$(\tau_{(-2x,-2\xi)}u)(z) = u(z+2x)e^{2i\pi(z-\frac{-2x}{2})(-2\xi)} = u(z+2x)e^{-4i\pi(z+x)\xi},$$

so that we obtain

$$\sigma_0(\tau_{(-2x,-2\xi)}u)(z) = u(-z+2x)e^{-4i\pi(-z+x)\xi} = (\sigma_{x,\xi}u)(z),$$

which proves (1.2.56) and thus (1.2.52). Formula (1.2.53) is already proven in (1.1.6) and (1.2.54) follows from (1.2.52), Hölder's inequality and the fact that τ_X is an endomorphism of $L^p(\mathbb{R}^n)$ with norm 1 (cf. the expression (1.2.51)). To prove (1.2.55) we note that from the expression (1.2.10), the Hausdorff–Young's inequality implies

$$\|\mathcal{W}(u,v)\|_{L^q\otimes L^q} \leq \|\Omega(u,v)\|_{L^q\otimes L^{q'}} \leq \||u|^{q'} * |v|^{q'}\|^{1/q'}_{L^{q/q'}} 2^{n\frac{q-2}{q}}, \tag{1.2.57}$$

and since Young's inequality[6] gives

$$\||u|^{q'} * |v|^{q'}\|_{L^{q/q'}} \leq \||u|^{q'}\|_{L^{a/q'}} \||v|^{q'}\|_{L^{b/q'}},$$

$a, b \geq q'$ with

$$1 - \frac{q'}{q} = 1 - \frac{q'}{a} + 1 - \frac{q'}{b},$$

i.e.,

$$q'\left(\frac{1}{a} + \frac{1}{b}\right) = 1 + \frac{q'}{q},$$

that is

$$\frac{1}{a} + \frac{1}{b} = 1,$$

so that

$$\||u|^{q'} * |v|^{q'}\|_{L^{q/q'}} \leq \|u\|^{q'}_{L^a} \|v\|^{q'}_{L^b},$$

in such a way that (1.2.57) yields

$$\|\mathcal{W}(u,v)\|_{L^q\otimes L^q} \leq 2^{n\frac{q-2}{q}} \|u\|_{L^a} \|v\|_{L^b}, \quad a, b \geq q', \quad \frac{1}{a} + \frac{1}{b} = 1,$$

which is (1.2.55). We are left with the proof of uniform continuity of $\mathcal{W}(u,v)$. We have for $X, Y \in \mathbb{R}^{2n}$,

$$\mathcal{W}(u,v)(Y) - \mathcal{W}(u,v)(X) = 2^n \langle (\sigma_Y - \sigma_X)u, v\rangle_{L^2(\mathbb{R}^n)},$$

and since $\sigma_Y^2 = \mathrm{Id}$ (see Claim 1.2.3), we find

$$\begin{aligned}\mathcal{W}(u,v)(Y) - \mathcal{W}(u,v)(X) &= 2^n \langle (\sigma_Y\sigma_X - \mathrm{Id})\sigma_X u, v\rangle_{L^2(\mathbb{R}^n)} \\ &= 2^n \langle \sigma_X u, (\sigma_X\sigma_Y - \mathrm{Id})v\rangle_{L^2(\mathbb{R}^n)}.\end{aligned}$$

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[6] For $p, q, r \in [1, +\infty]$ with $\frac{1}{p'} + \frac{1}{q'} = \frac{1}{r'}$, we have, $\|f * g\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q}$.

According to [33, formula (2.1.16)], we have

$$\sigma_X \sigma_Y = \tau_{2X-2Y} e^{4i\pi[Y,X]},$$

and this implies

$$|\mathcal{W}(u,v)(Y) - \mathcal{W}(u,v)(X)| \le 2^n \|u\|_{L^2(\mathbb{R}^n)} \|\tau_{2(X-Y)} v\|_{L^2(\mathbb{R}^n)}. \tag{1.2.58}$$

We have from (1.2.50),

$$\begin{aligned}\tau_{z,\zeta} v(x) - v(x) &= v(x-z) e^{2i\pi(x-\frac{z}{2})\zeta} - v(x)\\ &= \big(v(x-z) - v(x)\big) e^{2i\pi(x-\frac{z}{2})\zeta} + v(x)\big(e^{2i\pi(x-\frac{z}{2})\zeta} - 1\big),\end{aligned}$$

and thus

$$\begin{aligned}&\|\tau_Z v - v\|_{L^2(\mathbb{R}^n)}\\ &\quad\le \Big(\int |v(x-z) - v(x)|^2 dx\Big)^{1/2} + \Big(\int |v(x)|^2 |e^{2i\pi(x-\frac{z}{2})\zeta} - 1|^2 dx\Big)^{1/2}.\end{aligned}$$

We have the classical result, due to the density in L^2 of continuous compactly supported functions,

$$\lim_{\mathbb{R}^n \ni z \to 0} \int |v(x-z) - v(x)|^2 dx = 0,$$

and moreover the Lebesgue dominated convergence theorem implies

$$\lim_{(z,\zeta)\to(0,0)} \int \underbrace{|v(x)|^2}_{\in L^1(\mathbb{R}^n)} \underbrace{|e^{2i\pi(x-\frac{z}{2})\zeta} - 1|^2}_{\le 4} dx = 0,$$

so that

$$\lim_{\mathbb{R}^{2n} \ni Z \to 0} \|\tau_Z v - v\|_{L^2(\mathbb{R}^n)} = 0.$$

As a consequence, (1.2.58) implies the uniform continuity of $\mathcal{W}(u,v)$. Moreover, we have, for $\phi, \psi \in \mathscr{S}(\mathbb{R}^n)$,

$$\mathcal{W}(u,v) = \mathcal{W}(u-\phi, v) + \mathcal{W}(\phi, v-\psi) + \mathcal{W}(\phi,\psi),$$

so that

$$\begin{aligned}|\mathcal{W}(u,v)(x,\xi)| &\le \int \Big|(u-\phi)\Big(x+\frac{z}{2}\Big)\Big|\Big|v\Big(x-\frac{z}{2}\Big)\Big| dz\\ &\quad+ \iint \Big|(v-\psi)\Big(x-\frac{z}{2}\Big)\Big|\Big|\phi\Big(x+\frac{z}{2}\Big)\Big| dz + |\mathcal{W}(\phi,\psi)(x,\xi)|\\ &\le 2^n \|u-\phi\|_{L^2(\mathbb{R}^n)} \|v\|_{L^2(\mathbb{R}^n)} + 2^n \|v-\psi\|_{L^2(\mathbb{R}^n)} \|\phi\|_{L^2(\mathbb{R}^n)}\\ &\quad+ |\mathcal{W}(\phi,\psi)(x,\xi)|.\end{aligned}$$

We choose now sequences (ϕ_k), (ψ_k) of $\mathscr{S}(\mathbb{R}^n)$ converging respectively in $L^2(\mathbb{R}^n)$ towards u, v. We obtain for all $k \in \mathbb{N}$,

$$\begin{aligned}|\mathcal{W}(u,v)(x,\xi)| \le 2^n \|u-\phi_k\|_{L^2(\mathbb{R}^n)}\|v\|_{L^2(\mathbb{R}^n)} + 2^n\|v-\psi_k\|_{L^2(\mathbb{R}^n)}\|\phi_k\|_{L^2(\mathbb{R}^n)}\\ + |\mathcal{W}(\phi_k,\psi_k)(x,\xi)|,\end{aligned}$$

so that using that $\mathcal{W}(\phi_k,\psi_k)$ belongs to $\mathscr{S}(\mathbb{R}^{2n})$, we get

$$\begin{aligned}&\limsup_{\mathbb{R}^{2n}\ni X, |X|\to+\infty} \big[|\mathcal{W}(u,v)(X)|\big]\\ &\qquad\le 2^n\|u-\phi_k\|_{L^2(\mathbb{R}^n)}\|v\|_{L^2(\mathbb{R}^n)} + 2^n\|v-\psi_k\|_{L^2(\mathbb{R}^n)}\|\phi_k\|_{L^2(\mathbb{R}^n)},\end{aligned}$$

and thus, taking the limit when $k \to +\infty$, we obtain

$$\lim_{\mathbb{R}^{2n}\ni X, |X|\to+\infty} \big[|\mathcal{W}(u,v)(X)|\big] = 0,$$

completing the proof of Theorem 1.2.22. ∎

Remark 1.2.23. Let u be in $L^2(\mathbb{R}^n)$ be an even function. We then have

$$\mathcal{W}(u,u)(0,0) = 2^n\|u\|^2_{L^2(\mathbb{R}^n)} = \|\mathcal{W}(u,u)\|_{L^\infty(\mathbb{R}^{2n})}.$$

On the other hand, if u is odd, we have

$$\mathcal{W}(u,u)(0,0) = -2^n\|u\|^2_{L^2(\mathbb{R}^n)} = -\|\mathcal{W}(u,u)\|_{L^\infty(\mathbb{R}^{2n})},$$

showing that for odd functions the minimum of the Wigner distribution is negative (we assume $u \ne 0$ in $L^2(\mathbb{R}^n)$) and attained at 0. Let us check for instance the (odd) function u_1 of Remark 1.1.3. We have

$$\begin{aligned}2\|u_1\|^2_{L^2(\mathbb{R})} &= 2\int x^2 e^{-2\pi x^2}dx = 4\int_0^{+\infty}\frac{t}{2\pi}e^{-t}(2\pi)^{-1/2}\frac{1}{2}t^{-1/2}dt\\ &= \frac{2\Gamma(3/2)}{(2\pi)^{3/2}} = \frac{\Gamma(1/2)}{(2\pi)^{3/2}} = \frac{1}{2^{3/2}\pi} = -\mathcal{W}(u_1,u_1)(0,0).\end{aligned}$$

1.2.4 On weak versions of the Wigner distribution

Let u, v be in the space $\mathscr{S}'(\mathbb{R}^n)$ of tempered distributions. Then, we can define as above the tempered distribution $\Omega(u,v)$ in $\mathbb{R}^{2n}$: we set

$$\begin{aligned}&\langle \Omega(u,v)(x,z), \Phi(x,z)\rangle_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})}\\ &\qquad= \Big\langle u(x_1)\otimes \bar{v}(x_2), \Phi\Big(\frac{x_1+x_2}{2}, x_1-x_2\Big)\Big\rangle_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})},\end{aligned}$$

and then we define the Wigner distribution $\mathcal{W}(u,v)$ as the Fourier transform with respect to z of $\Omega(u,v)$, meaning that

$$\langle \mathcal{W}(u,v), \Psi\rangle_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})} = \langle \Omega(u,v), \mathcal{F}_2\Psi\rangle_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})},$$

where

$$(\mathcal{F}_2\Psi)(x,\xi) = \int_{\mathbb{R}^n} e^{-2i\pi z\cdot\xi}\Psi(x,z)dz.$$

Of course, $\mathcal{W}(u,v)$ is only a tempered distribution on $\mathbb{R}^{2n}$ and we have the inversion formula, using the notations of Remark 1.2.4,

$$\Omega(u,v) = \mathcal{F}_2\mathcal{C}_2\mathcal{W}(u,v).$$

The above remarks show that there is no difficulty to extend the definition of the joint Wigner distribution $\mathcal{W}(u,v)$ to the case where u,v are both tempered distributions on $\mathbb{R}^n$. Some properties are surviving from the L^2 theory, in particular the inversion formula, but one should be reasonably cautious at avoiding writing brackets of duality as integrals. Theorem 2 in [37] gives a more complete version of the following result.

Theorem 1.2.24. *Let $u \in L^2(\mathbb{R}^n)$ such that $\mathcal{W}(u,u) \in L^1(\mathbb{R}^{2n})$. Then, u belongs to $L^p(\mathbb{R}^n)$ for all $p \in [1,+\infty]$ and we have*

$$\|u\|_{L^1(\mathbb{R}^n)}\|u\|_{L^\infty(\mathbb{R}^n)} \le 2^n\|\mathcal{W}(u,u)\|_{L^1(\mathbb{R}^{2n})}.$$

N.B. We refer the reader to our Section 6.3 and, in particular, Theorem 6.3.3 showing that the set of u in $L^2(\mathbb{R}^n)$ such that $\mathcal{W}(u,u)$ belongs to $L^1(\mathbb{R}^{2n})$ is meager.

Proof. Thanks to Theorem 1.2.22, we have $\mathcal{W}(u,u) \in L^p(\mathbb{R}^{2n})$ for all $p \in [1,+\infty]$ and we have in a weak sense,

$$u\Big(x+\frac{z}{2}\Big)\bar{u}\Big(x-\frac{z}{2}\Big) = \int e^{2i\pi z\cdot\xi}\mathcal{W}(u,u)(x,\xi)d\xi,$$

so that

$$u(x_1)\bar{u}(x_2) = \int e^{2i\pi(x_1-x_2)\cdot\xi}\mathcal{W}(u,u)\Big(\frac{x_1+x_2}{2},\xi\Big)d\xi,$$

and thus we get

$$\int |u(x_1)||u(x_2)|dx_1 \le \iint \Big|\mathcal{W}(u,u)\Big(\frac{x_1+x_2}{2},\xi\Big)\Big|d\xi dx_1 = 2^n\|\mathcal{W}(u,u)\|_{L^1(\mathbb{R}^{2n})},$$

i.e.,

$$\|u\|_{L^1(\mathbb{R}^n)}\|u\|_{L^\infty(\mathbb{R}^n)} \le 2^n\|\mathcal{W}(u,u)\|_{L^1(\mathbb{R}^{2n})},$$

proving the lemma. ∎

1.2.5 Composition formulas

The following lemma is classical (see, e.g., [19], [46]); however we shall provide a proof for the convenience of the reader.

Lemma 1.2.25. *Let u, v, f, g be in $L^2(\mathbb{R}^n)$. Then*

$$\langle u, g\rangle_{L^2(\mathbb{R}^n)}\langle f, v\rangle_{L^2(\mathbb{R}^n)} = \iint \mathcal{W}(u,v)(x,\xi)\mathcal{W}(f,g)(x,\xi)dxd\xi. \tag{1.2.59}$$

In other words, the Weyl symbol of the rank-one operator $u \mapsto \langle u, g\rangle_{L^2(\mathbb{R}^n)} f$ is $\mathcal{W}(f,g)$. In particular, if $f = g$ is a unit vector in $L^2(\mathbb{R}^n)$ we find that $\mathcal{W}(f,f)$ is the Weyl symbol of the orthogonal projection onto $\mathbb{C} f$.

Proof. Both functions $\mathcal{W}(u,v)$, $\mathcal{W}(f,g)$ belong to $L^2(\mathbb{R}^{2n})$, so that the integral on the right-hand side of (1.2.59) actually makes sense. Also, $\mathcal{W}(u,v)$ is the partial Fourier transform with respect to the variable z of $(x,z) \mapsto u(x+z/2)\bar{v}(x-z/2)$, thus applying Plancherel formula[7], we obtain that

$$\begin{aligned}
&\iint \mathcal{W}(u,v)(x,\xi)\mathcal{W}(f,g)(x,\xi)dxd\xi \\
&\quad = \iint u(x+z/2)\bar{v}(x-z/2)f(x-z/2)\bar{g}(x+z/2)dxdz \\
&\quad = \langle u, g\rangle_{L^2(\mathbb{R}^n)}\langle f, v\rangle_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

The last property follows from (1.2.1). ∎

Using [33, Section 2.1.5], we obtain that for $a, b \in \mathscr{S}(\mathbb{R}^{2n})$

$$\mathrm{Op_w}(a)\mathrm{Op_w}(b) = \iint_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} a(Y)b(Z)2^{2n}\sigma_Y\sigma_Z dYdZ.$$

We get

$$\mathrm{Op_w}(a)\mathrm{Op_w}(b) = \mathrm{Op_w}(a\sharp b), \tag{1.2.60}$$

[7] We refer of course to the formula

$$\langle \hat{u}, \hat{v}\rangle_{L^2(\mathbb{R}^n)} = \langle u, v\rangle_{L^2(\mathbb{R}^n)},$$

when using the *complex* Hilbert space $L^2(\mathbb{R}^n)$. Note however that formula (A.1.3) is using the *real* duality between $\mathscr{S}(\mathbb{R}^n)$ and $\mathscr{S}'(\mathbb{R}^n)$ so that to check, with $\mathscr{S}^*(\mathbb{R}^N)$ standing for the anti-dual of $\mathscr{S}(\mathbb{R}^N)$ (i.e., continuous anti-linear forms on $\mathscr{S}(\mathbb{R}^N)$), we have also

$$\begin{aligned}
\langle \widehat{T}, \widehat{\phi}\rangle_{\mathscr{S}^*(\mathbb{R}^N),\mathscr{S}(\mathbb{R}^N)} &= \langle \widehat{T}, \overline{\widehat{\phi}}\rangle_{\mathscr{S}'(\mathbb{R}^N),\mathscr{S}(\mathbb{R}^N)} = \langle T, \widehat{\overline{\widehat{\phi}}}\rangle_{\mathscr{S}'(\mathbb{R}^N),\mathscr{S}(\mathbb{R}^N)} \\
&= \langle T, \bar{\phi}\rangle_{\mathscr{S}'(\mathbb{R}^N),\mathscr{S}(\mathbb{R}^N)} = \langle T, \phi\rangle_{\mathscr{S}^*(\mathbb{R}^N),\mathscr{S}(\mathbb{R}^N)}.
\end{aligned}$$

with

$$(a\sharp b)(X) = 2^{2n} \iint_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} e^{-4i\pi[X-Y,X-Z]} a(Y)b(Z)dYdZ \tag{1.2.61}$$

$$= \iint_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} e^{-2i\pi\langle\Xi,Z\rangle} a\Big(X + \frac{\sigma^{-1}\Xi}{2}\Big) b(Z+X)d\Xi dZ \tag{1.2.62}$$

$$= \int_{\mathbb{R}^{2n}} e^{2i\pi\langle X,\Xi\rangle} a\Big(X + \frac{\sigma^{-1}\Xi}{2}\Big)\hat{b}(\Xi)d\Xi, \tag{1.2.63}$$

where $[\cdot,\cdot]$ is the symplectic form (1.2.13) and σ is (1.2.15). Formula (1.2.62) is interesting since very close to the group J^t defined in [33, formula (4.1.14)].

Lemma 1.2.26. *Let u_0, u_1, u_2, u_3 be in $L^2(\mathbb{R}^n)$. Then, we have for all $X \in \mathbb{R}^{2n}$,*

$$|\langle u_1, u_2\rangle_{L^2}||\mathcal{W}(u_0,u_3)(X)| \le 2^n \big(|\mathcal{W}(u_0,u_2)| * |\mathcal{W}(\check{u}_1,u_3)|\big)(X).$$

Proof. According to Lemma 1.2.25, we have for $v \in L^2(\mathbb{R}^n)$,

$$\begin{aligned} \mathrm{Op_w}(\mathcal{W}(u_0,u_2))\mathrm{Op_w}(\mathcal{W}(u_1,u_3))v &= \mathrm{Op_w}(\mathcal{W}(u_0,u_2))\big(\langle v,u_3\rangle_{L^2(\mathbb{R}^n)}u_1\big) \\ &= \langle v,u_3\rangle_{L^2(\mathbb{R}^n)}\langle u_1,u_2\rangle_{L^2(\mathbb{R}^n)}u_0 \\ &= \langle u_1,u_2\rangle_{L^2(\mathbb{R}^n)}\mathrm{Op_w}(\mathcal{W}(u_0,u_3))v, \end{aligned}$$

so that with the notation (1.2.60), we get

$$\mathcal{W}(u_0,u_2)\sharp\mathcal{W}(u_1,u_3) = \langle u_1,u_2\rangle_{L^2(\mathbb{R}^n)}\mathcal{W}(u_0,u_3), \tag{1.2.64}$$

and using (1.2.63), we get

$$\begin{aligned} &\big(\mathcal{W}(u_0,u_2)\sharp\mathcal{W}(u_1,u_3)\big)(x,\xi) \\ &= \iint e^{2i\pi(x\cdot\eta+\xi\cdot y)}\mathcal{W}(u_0,u_2)\Big(x-\frac{y}{2},\xi+\frac{\eta}{2}\Big)\overbrace{\mathcal{F}\big(\mathcal{W}(u_1,u_3)\big)(\eta,y)}^{\mathcal{A}(u_1,u_3)(-\eta,-y)}\,dyd\eta, \end{aligned}$$

where $\mathcal{F}$ stands for the Fourier transformation and $\mathcal{A}$ for the ambiguity function (cf. (1.2.8)). With formula (1.2.9), we obtain

$$\begin{aligned} &\big(\mathcal{W}(u_0,u_2)\sharp\mathcal{W}(u_1,u_3)\big)(x,\xi) \\ &\quad= \iint e^{4i\pi(-x\cdot\eta+\xi\cdot y)}\mathcal{W}(u_0,u_2)(x-y,\xi-\eta)\mathcal{W}(\check{u}_1,u_3)(y,\eta)dyd\eta 2^n, \end{aligned}$$

yielding from (1.2.64) for any $X \in \mathbb{R}^{2n}$,

$$\langle u_1,u_2\rangle_{L^2}\mathcal{W}(u_0,u_3)(X) = \int_{\mathbb{R}^{2n}} e^{4i\pi[X,Y]}\mathcal{W}(u_0,u_2)(X-Y)\mathcal{W}(\check{u}_1,u_3)(Y)dY2^n,$$

which implies the lemma. ∎

1.2.6 L^2-boundedness

Theorem 1.2.27. *Let a be a semi-classical symbol on $\mathbb{R}^{2n}$, i.e., a smooth function of (x,ξ) depending on $h \in (0,1]$ such that*

$$\forall l \in \mathbb{N}, \quad p_l(a) = \sup_{\substack{(x,\xi)\in\mathbb{R}^{2n}, h\in(0,1] \\ |\alpha|+|\beta|\le l}} |(\partial_x^\alpha \partial_\xi^\beta a)(x,\xi,h)| h^{-\frac{|\alpha|+|\beta|}{2}} < +\infty. \quad (1.2.65)$$

Then, the operator $\mathrm{Op_w}(a(x,\xi,h))$ is bounded on $L^2(\mathbb{R}^n)$ and such that

$$\|\mathrm{Op_w}(a(x,\xi,h))\|_{\mathcal{B}(L^2(\mathbb{R}^n))} \le c_n p_{\ell_n}(a),$$

where c_n and ℓ_n depend only on n.

Proof. Theorem 1.2 in A. Boulkhemair's article [3] is providing that result (and more) with

$$\ell_n = [n/2] + 1.$$

Note also that [33, Theorem 1.1.4] is providing an elementary proof of the above result for the ordinary quantization of a given by

$$\begin{aligned}(\mathrm{Op}_0(a)u)(x) &= \int e^{2i\pi x\cdot\xi} a(x,\xi,h)\hat{u}(\xi)d\xi \\ &= \iint e^{2i\pi(x-y)\cdot\xi} a(x,\xi,h)u(y)dyd\xi.\end{aligned}$$ ∎

N.B. Formula (1.2.63) appears as

$$(a \sharp b)(X) = \left[\mathrm{Op}_0\Big(a\big(X - \frac{\sigma\Xi}{2}\big)\Big)b\right](X),$$

where $\mathrm{Op}_0(\cdot)$ stands for the ordinary quantization in $2n$ dimensions.

The following classical result is a consequence of Theorem 1.2.27.

Theorem 1.2.28. *Let $C_b^\infty(\mathbb{R}^{2n})$ be the set of bounded smooth complex-valued functions on $\mathbb{R}^{2n}$ such that all derivatives are bounded and let a be in $C_b^\infty(\mathbb{R}^{2n})$. Then, the operator $\mathrm{Op_w}(a)$ is bounded on $L^2(\mathbb{R}^n)$ and the $\mathcal{B}(L^2(\mathbb{R}^n))$ norm of $\mathrm{Op_w}(a)$ is bounded above by a fixed semi-norm of a in the Fréchet space $C_b^\infty(\mathbb{R}^{2n})$.*

1.2.7 On the Heisenberg Uncertainty Relations

Let $u \in \mathscr{S}(\mathbb{R})$. We have, using the notations (A.1.4),

$$2\,\mathrm{Re}\langle D_x u, ixu\rangle_{L^2(\mathbb{R})} = \langle [D_x, ix]u, u\rangle_{L^2(\mathbb{R})} = \frac{1}{2\pi}\|u\|^2_{L^2(\mathbb{R})}, \quad (1.2.66)$$

implying, in particular,

$$\|D_x u\|_{L^2(\mathbb{R})}\|xu\|_{L^2(\mathbb{R})} \ge \frac{1}{4\pi}\|u\|^2_{L^2(\mathbb{R})},$$

which is an equality for $u(x) = e^{-\pi x^2}$; moreover we infer also from (1.2.66) that

$$\langle \pi(D_x^2 + x^2)u, u\rangle \ge \frac{1}{2}\|u\|^2_{L^2(\mathbb{R})},$$

and for

$$q_\mu(x,\xi) = \sum_{1\le j\le n} \mu_j(x_j^2 + \xi_j^2), \quad 0 \le \mu_1 \le \cdots \le \mu_n,$$

the inequality

$$\langle \operatorname{Op_w}\big(\pi q_\mu(x,\xi)\big)u, u\rangle_{L^2(\mathbb{R}^n)} \ge \|u\|^2_{L^2(\mathbb{R}^n)} \frac{1}{2} \underbrace{\sum_{1\le j\le n} \mu_j}_{\substack{\text{defined as}\\ \text{trace}_+(q_\mu)}}, \tag{1.2.67}$$

which is an equality for $u(x) = e^{-\pi|x|^2}$. Note that the above (optimal) inequality can be reformulated as

$$\iint_{\mathbb{R}^{2n}} \pi q_\mu(x,\xi)\mathcal{W}(u,u)(x,\xi)dxd\xi \ge \|u\|^2_{L^2(\mathbb{R}^n)}\frac{1}{2}\operatorname{trace}_+(q_\mu).$$

Note also that with the symplectic matrix σ defined by (1.2.15), the so-called fundamental matrix of q_μ is defined by

$$F_{q_\mu} = \sigma^{-1}Q_\mu = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}\begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix} = \begin{pmatrix} 0 & -M \\ M & 0 \end{pmatrix} \quad \text{with } M = \operatorname{diag}(\mu_1,\dots,\mu_n)$$

so that

$$\text{Spectrum } F_{q_\mu} = \{\pm i\mu_j\}_{1\le j\le n}, \quad \operatorname{trace}_+(q_\mu) = \sum_{\substack{\lambda \text{ eigenvalue of } F_{q_\mu}\\ \text{with } \operatorname{Im}\lambda>0}} \lambda/i.$$

With the notations

$$\begin{cases} C_j = D_{x_j} + ix_j, & \text{creation operators,} \\ C_j^* = D_{x_j} - ix_j, & \text{annihilation operators,} \end{cases}$$

we see that

$$\pi[C_j^*, C_j] = \pi[D_{x_j} - ix_j, D_{x_j} + ix_j] = I,$$

and

$$\operatorname{Op_w}(q_\mu) = \pi\sum_{1\le j\le n} \mu_j C_j C_j^* + \frac{1}{2}\operatorname{trace}_+(q_\mu),$$

which provides another proof of (1.2.67).

Lemma 1.2.29 (Quantum Mechanics must deal with unbounded operators[8]). *Let* $\mathbb{H}$ *be a Hilbert space and let* $J, K \in \mathcal{B}(\mathbb{H})$*; then the commutator* $[J, K] \neq \mathrm{Id}$.

Proof. Let J, K be bounded operators with $[J, K] = \mathrm{Id}$. Then, for all $N \in \mathbb{N}^*$, we have

$$[J, K^N] = N K^{N-1}. \tag{1.2.68}$$

Indeed, this is true for $N = 1$ and if it holds for some $N \geq 1$, we find that

$$\begin{aligned}[J, K^{N+1}] &= JK^N K - K^{N+1} J = [J, K^N]K + K^N JK - K^{N+1} J \\ &= [J, K^N]K + K^N(JK - KJ) = [J, K^N]K + K^N = (N+1)K^N.\end{aligned}$$

Note that (1.2.68) implies that for all $N \in \mathbb{N}^*$, we have $K^N \neq 0$: of course $K \neq 0$ since $[J, K] = \mathrm{Id}$ and if we had $K^N = 0$ for some $N \geq 2$, (1.2.68) would imply $K^{N-1} = 0$ and eventually $K = 0$. As a consequence, we get from (1.2.68) that for all $N \geq 2$,

$$N \|K^{N-1}\|_{\mathcal{B}(\mathbb{H})} \leq 2\|J\|_{\mathcal{B}(\mathbb{H})} \|K^N\|_{\mathcal{B}(\mathbb{H})} \leq 2\|J\|_{\mathcal{B}(\mathbb{H})} \|K\|_{\mathcal{B}(\mathbb{H})} \|K^{N-1}\|_{\mathcal{B}(\mathbb{H})},$$

implying since $\|K^{N-1}\|_{\mathcal{B}(\mathbb{H})} > 0$, that

$$\forall N \geq 2, \quad N \leq 2\|J\|\|K\|,$$

which is impossible and proves the lemma. ∎

Lemma 1.2.30 (Hardy's inequality: the study of non-self-adjoint operators may be useful to determine lowerbounds of self-adjoint operators). *Let* $n \in \mathbb{N}, n \geq 3$*; let* u *in* $L^2(\mathbb{R}^n)$ *such that* $\nabla u \in L^2(\mathbb{R}^n)$, $|x|^{-1}u \in L^2(\mathbb{R}^n)$*. Then, we have*

$$\|\nabla u\|^2_{L^2(\mathbb{R}^n)} \geq \left(\frac{n-2}{2}\right)^2 \||x|^{-1}u\|^2_{L^2(\mathbb{R}^n)}.$$

Proof. We write first

$$\begin{aligned}&\sum_{1\leq j\leq n} \|(D_{x_j} - i\phi_j)u\|^2_{L^2(\mathbb{R}^n)} \\ &\qquad = \langle |D|^2 u, u\rangle_{L^2(\mathbb{R}^n)} + \langle |\phi|^2 u, u\rangle_{L^2(\mathbb{R}^n)} - \frac{1}{2\pi}\langle (\operatorname{div}\phi)u, u\rangle_{L^2(\mathbb{R}^n)},\end{aligned}$$

so that with $\phi(x) = \frac{\nu x}{2\pi|x|^2}$, we get the operator inequality

$$|D|^2 + \frac{\nu^2}{4\pi^2|x|^2} \geq \frac{\nu(n-2)}{4\pi^2|x|^2}, \quad \text{so that } -\Delta \geq |x|^{-2} \underbrace{\nu(n-2-\nu)}_{\text{largest at } \nu=(n-2)/2},$$

proving the lemma. ∎

[8] Thus, QM must involve infinite-dimensional Hilbert spaces and unbounded operators on them.

N.B. A modern approach to the Heisenberg uncertainty principle should certainly begin with reading C. Fefferman's article [8] as well as E. Lieb's book [38].

1.2.8 Non-negative quantizations formulas

Lemma 1.2.31. *Let χ be an even function in $\mathscr{S}(\mathbb{R}^{2n})$ with $L^2(\mathbb{R}^{2n})$ norm equal to 1. We define*

$$\Gamma_\chi = \bar{\chi}\sharp\chi. \tag{1.2.69}$$

Then, the function Γ_χ belongs to $\mathscr{S}(\mathbb{R}^{2n})$, is real-valued even and is such that

$$\int_{\mathbb{R}^{2n}} \Gamma_\chi(X)dX = 1.$$

Let u be in $L^2(\mathbb{R}^n)$. Then, the convolution $\mathcal{W}(u,u) * \Gamma_\chi$ is non-negative. As a result, the operator with Weyl symbol $a * \Gamma_\chi$ is a non-negative operator whenever a is a non-negative function.

Proof. Following the book [33], the composition formula (1.2.61) is bilinear continuous from $\mathscr{S}(\mathbb{R}^{2n})^2$ into $\mathscr{S}(\mathbb{R}^{2n})$ and we have also

$$\overline{a\sharp b} = \bar{b}\sharp\bar{a}.$$

So that Γ_χ is indeed real-valued. Moreover, we have

$$\begin{aligned}\int_{\mathbb{R}^{2n}} \Gamma_\chi(X)dX &= 2^{2n}\iiint_{(\mathbb{R}^{2n})^3} e^{-4i\pi[X-Y,Y-Z]}\bar{\chi}(Y)\chi(Z)dYdZdX\\ &= \int |\chi(Y)|^2dY = 1,\end{aligned}$$

and

$$\begin{aligned}\Gamma_\chi(-X) &= 2^{2n}\iint_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} e^{-4i\pi[-X-Y,-X-Z]}\bar{\chi}(Y)\chi(Z)dYdZ\\ &= 2^{2n}\iint_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} e^{-4i\pi[-X+Y,-X+Z]}\bar{\chi}(Y)\chi(Z)dYdZ = \Gamma_\chi(X).\end{aligned}$$

We have also

$$\begin{aligned}&\big(\mathcal{W}(u,u) * \Gamma_\chi\big)(Y)\\ &= \int_{\mathbb{R}^{2n}} \mathcal{W}(u,u)(Y-X)\Gamma_\chi(X)dX = \int_{\mathbb{R}^{2n}} \mathcal{W}(u,u)(Y+X)\Gamma_\chi(X)dX\\ &= \int_{\mathbb{R}^{2n}} \mathcal{W}(u,u)(T_Y(X))\Gamma_\chi(X)dX = \int_{\mathbb{R}^{2n}} \mathcal{W}(\tau_{-Y}u,\tau_{-Y}u)(X)\Gamma_\chi(X)dX\\ &= \int_{\mathbb{R}^{2n}} \mathcal{W}(\tau_{-Y}u,\tau_{-Y}u)(X)(\bar{\chi}\sharp\chi)(X)dX\\ &= \langle \mathrm{Op_w}(\bar{\chi}\sharp\chi)\tau_{-Y}u,\tau_{-Y}u\rangle_{L^2(\mathbb{R}^n)} = \|\mathrm{Op_w}(\chi)\tau_{-Y}u\|^2_{L^2(\mathbb{R}^n)} \ge 0,\end{aligned}$$

proving the first statement of non-negativity. Let a be a non-negative function, say in $L^1(\mathbb{R}^{2n})$; we have

$$
\begin{aligned}
&\mathrm{Op_w}(a * \Gamma_\chi) \\
&= 2^n \iint a(Y)\Gamma_\chi(X-Y)\sigma_X \, dY dX = \int a(Y) \int (\bar{\chi}\sharp\chi)(X-Y) 2^n \sigma_X \, dX dY \\
&= \int a(Y) \int (\bar{\chi}\sharp\chi)(T_{-Y}(X)) 2^n \sigma_X \, dX dY = \int a(Y) \tau_Y \mathrm{Op_w}(\bar{\chi}\sharp\chi)\tau_{-Y} \, dY \\
&= \int a(Y) \tau_Y \mathrm{Op_w}(\bar{\chi}) \mathrm{Op_w}(\chi)\tau_{-Y} \, dY \\
&= \int a(Y) \underbrace{[\mathrm{Op_w}(\chi)\tau_{-Y}]^* [\mathrm{Op_w}(\chi)\tau_{-Y}]}_{\text{non-negative operator}} \, dY \geq 0,
\end{aligned}
$$

if $a(Y) \geq 0$ for all $Y \in \mathbb{R}^{2n}$ and this concludes the proof. ■

We can write as well

$$
\begin{aligned}
\mathrm{Op_w}(a * \Gamma_\chi) &= \int_{\mathbb{R}^{2n}} a(Y) \big[\tau_Y \mathrm{Op_w}(\chi)\tau_{-Y}\big]^* \big[\tau_Y \mathrm{Op_w}(\chi)\tau_{-Y}\big] dY \\
&= \int_{\mathbb{R}^{2n}} a(Y) \Sigma_\chi(Y) dY, \qquad (1.2.70)
\end{aligned}
$$

with

$$
\Sigma_\chi(Y) = [\tau_Y \mathrm{Op_w}(\chi)\tau_{-Y}]^* [\tau_Y \mathrm{Op_w}(\chi)\tau_{-Y}] = \big(\mathrm{Op_w}(\chi(\cdot - Y))\big)^* \mathrm{Op_w}(\chi(\cdot - Y)). \qquad (1.2.71)
$$

Remark 1.2.32. The Gaussian case in the previous lemma gives rise to the standard non-negativity properties of coherent states. In fact, choosing $\chi(X) = 2^n e^{-2\pi|X|^2}$, we see that χ is even, belongs to the Schwartz space and

$$
\|\chi\|^2_{L^2(\mathbb{R}^{2n})} = 2^{2n} \int_{\mathbb{R}^{2n}} e^{-4\pi|X|^2} dX = 2^{2n} 4^{-2n/2} = 1.
$$

We have also[9]

$$
\begin{aligned}
\Gamma_\chi(X) &= 2^{4n} \iint_{(\mathbb{R}^{2n})^2} e^{-4i\pi[X-Y, X-Z]} e^{-2\pi(|Y|^2+|Z|^2)} dY dZ \\
&= 2^{3n} \int_{\mathbb{R}^{2n}} e^{4i\pi[Y,X]} e^{-2\pi(|X+Y|^2+|Y|^2)} dY \\
&= 2^{3n} \int_{\mathbb{R}^{2n}} e^{4i\pi[Y,X]} e^{-2\pi(|Y+\frac{X}{2}|^2+|Y-\frac{X}{2}|^2)} dY \\
&= 2^{3n} e^{-\pi|X|^2} \int_{\mathbb{R}^{2n}} e^{4i\pi[Y,X]} e^{-4\pi|Y|^2} dY = 2^{3n} e^{-\pi|X|^2} 4^{-n} e^{-\pi|X|^2} = \chi(X).
\end{aligned}
$$

[9] [33, Proposition 4.1.1] is useful to compute the Fourier transform of Gaussian functions and is a notable asset of the Fourier normalization given in Section A.1.1.

In that case we find that $\mathrm{Op_w}(\chi)$ is a rank-one orthogonal projection on the fundamental state Ψ_0 of the harmonic oscillator $\pi(|D_x|^2 + |x|^2)$. According to (A.1.16) the one-dimensional kth Hermite function is

$$\psi_k(x) = \frac{(-1)^k}{2^k\sqrt{k!}} 2^{1/4} e^{\pi x^2} \left(\frac{d}{\sqrt{\pi}dx}\right)^k (e^{-2\pi x^2}), \tag{1.2.72}$$

so that $\Psi_0(x) = 2^{n/4} e^{-\pi|x|^2}$. We calculate

$$\begin{aligned}\Gamma(x,\xi) = \mathcal{W}(\Psi_0,\Psi_0)(x,\xi) &= 2^{n/2}\int_{\mathbb{R}^n} e^{-\pi(|x+z/2|^2+|x-z/2|^2)} e^{-2i\pi z\xi} dz \\ &= 2^{n/2} e^{-2\pi|x|^2}\int_{\mathbb{R}^n} e^{-\pi z^2/2} e^{-2i\pi z\xi} dz = 2^n e^{-2\pi|x|^2} e^{-2\pi|\xi|^2} = \chi(x,\xi).\end{aligned}$$

The anti-Wick quantization of a symbol a is defined as (see, e.g., M. Shubin's book [47])

$$\mathrm{Op_{aw}}(\mathrm{a}) = \int_{\mathbb{R}^{2n}} \mathrm{a}(Y)\Sigma_Y dY, \tag{1.2.73}$$

where Σ_Y is the rank-one orthogonal projection given by

$$\Sigma_{y,\eta} u = \langle u, \tau_{y,\eta}\Psi_0\rangle \tau_{y,\eta}\Psi_0.$$

Remark 1.2.33. It is interesting to notice that to produce non-negativity of the operator with Weyl symbol $\mathrm{a} * \Gamma_\chi$ when a is a non-negative function, we do not use the non-negativity of Γ_χ as a function, which by the way does not always hold (except in the Gaussian cases), but we use the fact that the quantization of Γ_χ is non-negative, as it is defined as $\mathrm{Op_w}(\bar{\chi}\sharp\chi) = (\mathrm{Op_w}(\chi))^* \,\mathrm{Op_w}(\chi)$.

Remark 1.2.34. Another important remark is concerned with the Taylor expansion of $\mathrm{a} * \Gamma_\chi$, we have

$$\begin{aligned}(\mathrm{a} * \Gamma_\chi)(X) &= \int \mathrm{a}(X-Y)\Gamma_\chi(Y)dY = \int \mathrm{a}(X+Y)\Gamma_\chi(Y)dY \\ &= \int \Big(\mathrm{a}(X) + \mathrm{a}'(X)Y + \int_0^1 (1-\theta)\mathrm{a}''(X+\theta Y)Y^2\Big)\Gamma_\chi(Y)dY \\ &= \mathrm{a}(X) + \iint_0^1 (1-\theta)\mathrm{a}''(X+\theta Y)Y^2\Gamma_\chi(Y)dY.\end{aligned}$$

As a result the difference $(\mathrm{a} * \Gamma_\chi) - \mathrm{a}$ depends only on the second derivative of a. If for instance a is a semi-classical symbol, i.e., a smooth function of (x,ξ) depending on $h \in (0,1]$ such that

$$\forall(\alpha,\beta) \in \mathbb{N}^n \times \mathbb{N}^n, \qquad \sup_{(x,\xi)\in\mathbb{R}^{2n}, h\in(0,1]} |(\partial_x^\alpha \partial_\xi^\beta \mathrm{a})(x,\xi,h)| h^{-\frac{|\alpha|+|\beta|}{2}} < +\infty, \tag{1.2.74}$$

then the difference $\mathrm{Op_{aw}}(\mathrm{a}) - \mathrm{Op_w}(\mathrm{a})$ is bounded on $L^2(\mathbb{R}^n)$ with an $O(h)$ operator-norm, so that if a happens also to be non-negative, we find

$$\mathrm{Op_w}(\mathrm{a}) = \underbrace{\mathrm{Op_w}(\mathrm{a}) - \mathrm{Op_w}(\mathrm{a} * \Gamma_\chi)}_{\substack{O(h)\\ \text{as an operator,}\\ \text{cf. Theorem 1.2.27}}} + \underbrace{\mathrm{Op_w}(\mathrm{a} * \Gamma_\chi)}_{\substack{\geq 0\\ \text{as an operator}}},$$

and we obtain a version of the so-called Sharp Gårding inequality,

$$\mathrm{Op_w}(\mathrm{a}) + Ch \geq 0 \quad \text{(as an operator).}$$

Theorem 1.2.35. *Let χ be an even function in the Schwartz space $\mathscr{S}(\mathbb{R}^{2n})$ with $L^2(\mathbb{R}^{2n})$ norm equal to 1 and let Γ_χ be given by (1.2.69). For $a \in L^\infty(\mathbb{R}^{2n})$, we define*

$$\mathrm{Op}(\chi, a) = \mathrm{Op_w}(a * \Gamma_\chi).$$

Then, $\mathrm{Op}(\chi, a)$ *is a bounded operator in $L^2(\mathbb{R}^n)$ and we have*

$$\|\mathrm{Op}(\chi, a)\|_{\mathcal{B}(L^2(\mathbb{R}^n))} \leq \|a\|_{L^\infty(\mathbb{R}^{2n})}. \tag{1.2.75}$$

Moreover, if a is valued in some interval J of the real line, we have the operator inequalities

$$\inf J \leq \mathrm{Op}(\chi, a) \leq \sup J. \tag{1.2.76}$$

In particular, if $a(x, \xi) \geq 0$ for all $(x, \xi) \in \mathbb{R}^{2n}$, we have the operator-inequality $\mathrm{Op}(\chi, a) \geq 0$.

N.B. The non-negativity of the anti-Wick quantization (1.2.73) and its avatars Husimi [25], Coherent States, Gabor wavelets (see, e.g., [11]), are particular cases of the above theorem. More information on this topic is available in Section 2.4 of the book [33]. Another remark is that this result can easily be extended to matrix-valued symbols as in Remark 2 page 79 of L. Hörmander's [24] and even to symbols valued in $\mathcal{B}(\mathbb{H})$, where $\mathbb{H}$ is a Hilbert space.

Proof. We start with Formulas (1.2.70), (1.2.71), entailing

$$\mathrm{Op}(\chi, a) = \int_{\mathbb{R}^{2n}} a(Y)\Sigma_\chi(Y)dY,$$

with $\Sigma_\chi(Y) = [\mathrm{Op_w}(\chi(\cdot - Y))]^* \mathrm{Op_w}(\chi(\cdot - Y)) = \tau_Y \mathrm{Op_w}(\bar{\chi} \sharp \chi)\tau_{-Y}$. We note that

$$\mathrm{Op}(\chi, 1) = \int_{\mathbb{R}^{2n}} \tau_Y \mathrm{Op_w}(\bar{\chi} \sharp \chi)\tau_{-Y} dY,$$

so has Weyl symbol $X \mapsto \int_{\mathbb{R}^{2n}} \Gamma_\chi(X - Y)dY = 1$ from Lemma 1.2.31 and thus $\mathrm{Op}(\chi, 1) = \mathrm{Id}$. We infer that for $u, v \in \mathscr{S}(\mathbb{R}^n)$,

$$\langle \mathrm{Op}(\chi, a)u, v\rangle_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^{2n}} a(Y)\langle \mathrm{Op_w}(\chi(\cdot - Y))u, \mathrm{Op_w}(\chi(\cdot - Y))v\rangle dY,$$

so that with any $\nu > 0$,

$$
\begin{aligned}
&|\langle \mathrm{Op}(\chi, a)u, v\rangle_{L^2(\mathbb{R}^n)}| \\
&\le \|a\|_{L^\infty(\mathbb{R}^{2n})} \int_{\mathbb{R}^{2n}} \frac{1}{2}\big(\nu \|\mathrm{Op_w}(\chi(\cdot - Y))u\|^2_{L^2(\mathbb{R}^n)} + \nu^{-1}\|\mathrm{Op_w}(\chi(\cdot - Y))v\|^2_{L^2(\mathbb{R}^n)}\big) dY \\
&= \|a\|_{L^\infty(\mathbb{R}^{2n})} \frac{1}{2}\big(\nu \langle \mathrm{Op}(\chi, 1)u, u\rangle_{L^2(\mathbb{R}^n)} + \nu^{-1}\langle \mathrm{Op}(\chi, 1)v, v\rangle_{L^2(\mathbb{R}^n)}\big) \\
&= \|a\|_{L^\infty(\mathbb{R}^{2n})} \frac{1}{2}\big(\nu \|u\|^2_{L^2(\mathbb{R}^n)} + \nu^{-1}\|v\|^2_{L^2(\mathbb{R}^n)}\big),
\end{aligned}
$$

and taking the infimum of the right-hand side with respect to ν, we obtain

$$
|\langle \mathrm{Op}(\chi, a)u, v\rangle_{L^2(\mathbb{R}^n)}| \le \|a\|_{L^\infty(\mathbb{R}^{2n})} \|u\|_{L^2(\mathbb{R}^n)} \|v\|_{L^2(\mathbb{R}^n)},
$$

proving (1.2.75). To prove (1.2.76), it is enough to prove the last statement in the theorem which follows immediately from (1.2.70), (1.2.71) since each operator Σ_Y is non-negative. The proof of the theorem is complete. ∎

It is nice to have examples of non-negative quantizations, but somehow more importantly, it is crucial to relate these quantizations to the mainstream quantization, that is to the Weyl quantization. This is what we do in the next theorem, dealing with semi-classical symbols.

Theorem 1.2.36 (Sharp Gårding inequality). *Let a be a function defined on $\mathbb{R}^n \times \mathbb{R}^n \times (0, 1]$ such that $a(x, \xi, h)$ is smooth for all $h \in (0, 1]$ and such that*

$$
\forall(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^n, \qquad \sup_{(x,\xi,h)\in\mathbb{R}^n\times\mathbb{R}^n\times(0,1]} |(\partial_x^\alpha \partial_\xi^\beta a)(x, \xi, h)| h^{-|\beta|} < +\infty. \tag{1.2.77}
$$

Let us assume that the function a is valued in $\mathbb{R}_+$. Then, there exists a constant C such that

$$
\mathrm{Op_w}(a) + Ch \ge 0.
$$

Proof. We have given a proof of this result in Remark 1.2.34 but with a different definition for a semi-classical symbol (see (1.2.74)). Starting with our definition above in (1.2.77), we define

$$
b(x, \xi, h) = a(h^{1/2}x, h^{-1/2}\xi, h),
$$

and we see that b satisfies the estimates (1.2.74) and is a non-negative function so that, applying Remark 1.2.34, we can find a constant C such that

$$
\mathrm{Op_w}(b) + Ch \ge 0.
$$

We note now that Segal's formula (1.2.48) applied to the symplectic mapping

$$
(x, \xi) \mapsto (h^{1/2}x, h^{-1/2}\xi),
$$

shows that $\mathrm{Op_w}(b)$ is unitarily equivalent to $\mathrm{Op_w}(a)$, providing the sought result. ∎

N.B. Several versions of the above theorem can be found in the literature, in particular, [24, Theorem 18.1.14]. The first proof of this result was given in 1966 by L. Hörmander in [21] for scalar-valued symbols and a proof for systems was given by P. Lax and L. Nirenberg in [28] on the same year. Far-reaching refinements of that inequality were given by C. Fefferman and D. H. Phong, who proved in [9] in 1978 that, under the same assumption as in Theorem 1.2.36 for scalar-valued symbols, they obtain the much stronger

$$\operatorname{Op}_{\mathrm{w}}(a) + Ch^2 \geq 0. \tag{1.2.78}$$

A thorough discussion of these questions is given in [24, Section 18.6] and in [33, Section 2.5] (see also [1]).

1.3 Examples

1.3.1 Hermite functions

We can easily calculate the Wigner distribution of Hermite functions and since the Wigner distributions respect tensor products as partial Fourier transforms, it is enough to do in one dimension. With ψ_k given in (1.2.72), the Wigner distribution $\mathcal{W}(\psi_k, \psi_k)$ appears as the Weyl symbol of $\mathbb{P}_{k;1} = \mathbb{P}_k$ as defined in (A.1.17). We find that the Weyl symbol of $\mathbb{P}_{0;n}$, following (A.3.2), is

$$2^n e^{-2\pi(|x|^2+|\xi|^2)}.$$

More generally, the paper [27] provides in one dimension

$$\mathcal{W}(\psi_k, \psi_k)(x, \xi) = (-1)^k 2e^{-2\pi(x^2+\xi^2)} L_k(4\pi(x^2 + \xi^2)), \tag{1.3.1}$$

where L_k is the standard Laguerre polynomial with degree k (see (A.4.1)). As a result, the Weyl symbol of $\mathbb{P}_{k;n}$ is equal to $\pi_{k,n}(x, \xi)$ with

$$\pi_{k,n}(x, \xi) = (-1)^k 2^n e^{-2\pi(|x|^2+|\xi|^2)} \sum_{\alpha\in\mathbb{N}^n, |\alpha|=k} \prod_{1\leq j\leq n} L_{\alpha_j}(4\pi(x_j^2 + \xi_j^2)).$$

Note that the leading term in the polynomial $(-1)^k L_k(t)$ is $t^k/k!$ and this implies that the set

$$\{(x, \xi) \in \mathbb{R}^2, \mathcal{W}(\psi_k, \psi_k)(x, \xi) < 0\},$$

where $\mathcal{W}(\psi_k, \psi_k)$ is given by (1.3.1) is a relatively compact open subset of $\mathbb{R}^2$. Indeed, we have

$$W(\psi_k, \psi_k)(X) = 2e^{-2\pi|X|^2} \left\{\frac{(4\pi|X|^2)^k}{k!}\right\} \underbrace{\left(1 + \sum_{0\leq l\leq k-1} a_l (4\pi|X|^2)^{-(k-l)}\right)}_{\geq 1/2 \text{ for } |X|\geq R_0}$$

which implies that

$$\{X \in \mathbb{R}^2, |X| \geq \max(R_0, 1)\} \subset \{X \in \mathbb{R}^2, W(\psi_k, \psi_k)(X) > 0\},$$

and thus

$$\{W(\psi_k, \psi_k)(X) \leq 0\} \subset \{|X| < \max(R_0, 1)\}.$$

1.3.2 One-sided exponentials

Let us define for $a > 0$, $f_a(t) = H(t)a^{1/2}e^{-at/2}$. We have

$$\begin{aligned}
\mathcal{W}(f_a, f_a)(x, \xi) &= aH(x) \int_{|z| \leq 2x} e^{-2i\pi z\xi} e^{-\frac{a}{2}(x+z/2)} e^{-\frac{a}{2}(x-z/2)} dz \\
&= aH(x)e^{-xa} \int_{|z| \leq 2x} e^{-2i\pi z\xi} dz \\
&= 2aH(x)e^{-xa} \int_0^{2x} \cos(z2\pi\xi) dz \\
&= aH(x)e^{-xa} \frac{\sin(4\pi x\xi)}{\pi\xi}.
\end{aligned}$$

We can check

$$\iint \mathcal{W}(f_a, f_a)(x, \xi) dx d\xi = \frac{a}{\pi} \int_{x=0}^{+\infty} e^{-ax} \int \frac{\sin(4\pi x\xi)}{\xi} d\xi dx = 1 = \|f_a\|^2_{L^2(\mathbb{R})},$$

and since

$$\int_{\mathbb{R}} \frac{\sin^2 t}{t^2} dt = \pi,$$

we verify (see Lemma 1.2.25 and (1.1.4)),

$$\iint \mathcal{W}(f_a, f_a)(x, \xi)^2 dx d\xi = \frac{a^2}{\pi^2} \int_{x=0}^{+\infty} e^{-2ax} \int \frac{\sin^2(4\pi x\xi)}{\xi^2} d\xi dx = 1 = \|f_a\|^4_{L^2(\mathbb{R})}.$$

On the other hand, the ambiguity function $\mathcal{A}(f_a, f_a)$ is the inverse Fourier transform of $\mathcal{W}$ and we have

$$\begin{aligned}
\mathcal{A}(f_a, f_a)(\eta, y) &= \frac{a}{\pi} \iint H(x)e^{-x(a-2i\pi\eta)} \frac{\sin\xi}{\xi} e^{2i\pi \frac{y}{4\pi x}\xi} dx d\xi \\
&= a \int_{|y|/2}^{+\infty} e^{-x(a-2i\pi\eta)} dx = \frac{ae^{-\frac{1}{2}|y|(a-2i\pi\eta)}}{a - 2i\pi\eta},
\end{aligned}$$

which corresponds to [17, formula (9)] noting that with our notations, we have

$$\mathcal{A}(f, f)(\eta, y) = \widetilde{\mathcal{A}}(f, f)(y, -\eta),$$

where $\widetilde{\mathcal{A}}(f, f)$ is the normalization chosen in [17]. Going back to the Wigner distribution, that simple example is interesting since we have

$$\begin{aligned}&\big\{(x,\xi), \mathcal{W}(f_a, f_a)(x,\xi) < 0\big\}\\ &= \bigcup_{k\in\mathbb{N}}\Big\{(x,\xi) \in (0,+\infty)\times\mathbb{R}^*, \frac{k}{2}+\frac{1}{4} < x|\xi| < \frac{k}{2}+\frac{1}{2}\Big\},\end{aligned}$$

and we see that the Lebesgue measure of

$$E_k = \Big\{(x,\xi) \in (0,+\infty)\times\mathbb{R}^*, \frac{k}{2}+\frac{1}{4} < x|\xi| < \frac{k}{2}+\frac{1}{2}\Big\},$$

is infinite since

$$|E_k| = 2\int_0^{+\infty}\frac{dx}{4x} = +\infty.$$

Moreover, the function $\mathcal{W}(f_a, f_a)(x,\xi)$ does not belong to $L^1(\mathbb{R}^2)$ since

$$\iint H(x)e^{-xa}\left|\frac{\sin(4\pi x\xi)}{\pi\xi}\right|dxd\xi \ge \iint_{(0,+\infty)^2} e^{-xa}\left|\frac{\sin\eta}{\pi\eta}\right|dxd\eta = +\infty.$$

As a consequence, we have, using the notation for $\alpha \in \mathbb{R}$,

$$\alpha_\pm = \max(\pm\alpha, 0),$$

$$\iint \big(\mathcal{W}(f_a, f_a)(x,\xi)\big)_+ dxd\xi = \iint \big(\mathcal{W}(f_a, f_a)(x,\xi)\big)_- dxd\xi = +\infty,$$

since the real-valued function $\mathcal{W}(f_a, f_a)$ does not belong to $L^1(\mathbb{R}^2)$ and is such that

$$\iint \mathcal{W}(f_a, f_a)(x,\xi)dxd\xi = \|f_a\|^2_{L^2(\mathbb{R})} = 1.$$

We will see in Section 6.4 several important consequences of that phenomenon for the quantization of the indicatrix of some subsets of $\mathbb{R}^2$, such as

$$E_\pm = \big\{(x,\xi), \pm\mathcal{W}(f_a, f_a)(x,\xi) > 0\big\}.$$

1.3.3 Box functions

We start with $\beta_0(t) = \mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]}(t)$, for which a straightforward calculation gives

$$\mathcal{W}(\beta_0, \beta_0)(x,\xi) = \mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]}(x)\frac{\sin(2\pi(1-2|x|)\xi)}{\pi\xi}.$$

More generally, for real parameters $a \le b$, defining

$$\beta(t) = (b-a)^{-1/2}\mathbf{1}_{[a,b]}(t)e^{2i\pi\omega t},$$

we find

$$\mathcal{W}(\beta,\beta)(x,\xi) = [(b-a)\pi(\xi-\omega)]^{-1} \times \Big(\mathbf{1}_{[a,\frac{a+b}{2}]}(x)\sin[4\pi(\xi-\omega)(x-a)] + \mathbf{1}_{[\frac{a+b}{2},b]}(x)\sin[4\pi(\xi-\omega)(b-x)]\Big).$$

Checking now $\beta_1(t) = \mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]}(t)\operatorname{sign} t$, we find after a simple (but this time a bit tedious) calculation,

$$\mathcal{W}(\beta_1,\beta_1)(x,\xi) = \mathbf{1}\left(|x| \le \frac{1}{4}\right)\frac{2\sin(4\pi|x|\xi) - \sin\big(2\pi(1-2|x|)\xi\big)}{\pi\xi} + \mathbf{1}\left(\frac{1}{4} \le |x| \le \frac{1}{2}\right)\frac{\sin(2\pi(1-2|x|)\xi)}{\pi\xi}.$$

1.4 Integrals of the Wigner distribution on subsets of the phase space

Lemma 1.4.1. *Let E be a measurable subset with finite Lebesgue measure of the phase space $\mathbb{R}^n \times \mathbb{R}^n$ and let $\mathbf{1}_E$ be the indicator function of the set E. Then, the operator with Weyl symbol $\mathbf{1}_E$ is bounded self-adjoint on $L^2(\mathbb{R}^n)$ and for any $u \in L^2(\mathbb{R}^n)$, we have*

$$\langle \operatorname{Op_w}(\mathbf{1}_E)u,u\rangle_{L^2(\mathbb{R}^n)} = \iint_E \mathcal{W}(u,u)(x,\xi)\,dx\,d\xi. \tag{1.4.1}$$

Proof. It follows immediately from (1.2.1) and (1.2.5). ∎

Remark 1.4.2. A consequence of the above formula is that a spectral analysis of the operator $\operatorname{Op_w}(\mathbf{1}_E)$ would display interesting extremalization properties for the right-hand side of (1.4.1); for instance, if

$$\lambda_- = \inf\big(\operatorname{spectrum}(\operatorname{Op_w}(\mathbf{1}_E))\big), \quad \lambda_+ = \sup\big(\operatorname{spectrum}(\operatorname{Op_w}(\mathbf{1}_E))\big),$$

we obtain that for u normalized in $L^2(\mathbb{R}^n)$, we have

$$\lambda_- \le \iint_E \mathcal{W}(u,u)(x,\xi)\,dx\,d\xi \le \lambda_+.$$

In particular, if λ_- is an eigenvalue related to a normalized eigenfunction u_-, (resp., if λ_+ is an eigenvalue related to a normalized eigenfunction u_+), we get for all u normalized in $L^2(\mathbb{R}^n)$,

$$\iint_E \mathcal{W}(u_-,u_-)(x,\xi)\,dx\,d\xi \le \iint_E \mathcal{W}(u,u)(x,\xi)\,dx\,d\xi \le \iint_E \mathcal{W}(u_+,u_+)(x,\xi)\,dx\,d\xi.$$

We shall see below several examples where the operator $\operatorname{Op_w}(\mathbf{1}_E)$ is bounded on $L^2(\mathbb{R}^n)$ with an E having infinite Lebesgue measure. We may note in particular that

$$\operatorname{Op_w}(\mathbf{1}_{\mathbb{R}^{2n}}) = \mathrm{Id},$$

and for a given non-zero linear form $L(x,\xi)$ on $\mathbb{R}^{2n}$ and

$$E = \{(x,\xi) \in \mathbb{R}^{2n}, L(x,\xi) \in J\}, \quad \text{where } J \text{ is a subset of } \mathbb{R}, \tag{1.4.2}$$

we may find affine symplectic coordinates (y,η) on $\mathbb{R}^{2n}$ such that $L(x,\xi) = y_1$, implying with (1.2.48) that $\operatorname{Op_w}(\mathbf{1}_E)$ is unitarily equivalent to the orthogonal projection $u \mapsto u(y)\mathbf{1}_J(y_1)$. Although in that case, the quantization of the indicatrix of E given by (1.4.2) is trivial, we shall see below that in many cases, including some rather explicit ones, the Weyl quantization of the rough Hamiltonian $\mathbf{1}_E(x,\xi)$ could be far from a projection and may have a rather complicated spectrum with a supremum which could be strictly larger than 1 and an infimum which could be negative.

In some sense, although we have the trivial identity $\mathbf{1}_E(x,\xi)^2 = \mathbf{1}_E(x,\xi)$, we shall see that the quantization process by the Weyl formula is destroying that property; to understand integrals of the Wigner distribution on subsets of the phase space, formula (1.4.1) forces us to consider the Weyl quantization of the function $\mathbf{1}_E(x,\xi)$ and the Heisenberg Uncertainty Principle shows that non-commutation properties are governing operators and these properties are of course distorting the classical identities satisfied by classical Hamiltonians.

We must point out as well that we do not have here at our disposal a semi-classical version of our quantization which could ensure some bridge between classical properties and operator-theoretic results as it is the case for the quantization of nice smooth semi-classical symbols depending on a small parameter h such as a C^∞ function $a(x,\xi,h)$ satisfying (1.2.77). In particular, for a symbol a satisfying (1.2.77), we have the following result: if for all $(x,\xi,h) \in \mathbb{R}^n \times \mathbb{R}^n \times (0,1]$ we have $a(x,\xi,h) \le 1$, then there exists a semi-norm C of the symbol a such that

$$\mathrm{Id} - \operatorname{Op_w}(a) + Ch^2 \ge 0,$$

i.e.,

$$\operatorname{Op_w}(a) \le \mathrm{Id} + Ch^2,$$

an inequality following from the Fefferman–Phong inequality (cf. (1.2.78)) which implies as well the following lemma.

Lemma 1.4.3. *Let a be a semi-classical symbol of order 0, i.e., a smooth function satisfying (1.2.77) such that for all $(x,\xi,h) \in \mathbb{R}^n \times \mathbb{R}^n \times (0,1]$ we have $0 \le a(x,\xi,h) \le 1$. Then, there exists a semi-norm C of the symbol a such that*

$$-Ch^2 \le \operatorname{Op_w}(a) \le \mathrm{Id} + Ch^2.$$

Chapter 2

Quantization of radial functions and Mehler's formula

This section and the following are essentially based upon the author's paper [36].

2.1 Basic formulas in one dimension

In this section, we work in one dimension and consider a function F in the Schwartz class of $\mathbb{R}$. We want to calculate somewhat explicitly the Weyl quantization of $F(x^2+\xi^2)$ and also extend that computation to the case where F is merely $L^\infty(\mathbb{R})$. We have, say for F in the Wiener algebra $\mathscr{W}(\mathbb{R}) = \text{Fourier}(L^1(\mathbb{R}))$,

$$\operatorname{Op_w}(F(x^2+\xi^2)) = \int_{\mathbb{R}} \widehat{F}(\tau)\operatorname{Op_w}(e^{2i\pi\tau(x^2+\xi^2)})d\tau,$$

as an absolutely converging integral of a function defined on $\mathbb{R}$ (equipped with the Lebesgue measure) valued in $\mathscr{B}(L^2(\mathbb{R}))$ (bounded endomorphisms of $L^2(\mathbb{R})$). In fact, applying Mehler's formula (A.3.1), we find

$$\underbrace{\operatorname{Op_w}(e^{2i\pi\tau(x^2+\xi^2)})}_{\substack{\text{operator with Weyl symbol}\\ e^{2i\pi\tau(x^2+\xi^2)}}} = \cos(\arctan\tau)\, \underbrace{e^{2i\pi(\arctan\tau)\operatorname{Op_w}(x^2+\xi^2)}}_{\substack{\text{exponential } e^{iM},\\ \text{with } M \text{ self-adjoint operator}\\ =2\pi(\arctan\tau)\operatorname{Op_w}(x^2+\xi^2)}},$$

so that, using the spectral decomposition (A.1.17) of the harmonic oscillator

$$\operatorname{Op_w}(\pi(x^2+\xi^2)),$$

we get,

$$\begin{aligned}\operatorname{Op_w}(F(x^2+\xi^2)) &= \int_{\mathbb{R}} \widehat{F}(\tau)\sum_{k\ge 0} e^{2i(\arctan\tau)(k+\frac12)}\mathbb{P}_k\frac{d\tau}{\sqrt{1+\tau^2}}\\ &= \sum_{k\ge 0}\int_{\mathbb{R}} \widehat{F}(\tau)e^{2i(k+\frac12)\arctan\tau}\frac{d\tau}{\sqrt{1+\tau^2}}\mathbb{P}_k,\end{aligned}$$

where the use of Fubini theorem is justified by

$$\int_{\mathbb{R}} |\widehat{F}(\tau)|\frac{d\tau}{\sqrt{1+\tau^2}} < +\infty, \quad \mathbb{P}_k \ge 0, \quad \sum_{k\ge 0}\mathbb{P}_k = \mathrm{Id}\,.$$

We have

$$\begin{aligned}&\int_{\mathbb{R}} \widehat{F}(\tau)e^{2i(k+\frac12)\arctan\tau}\frac{d\tau}{\sqrt{1+\tau^2}}\\ &\qquad = \int_{\mathbb{R}} \widehat{F}(\tau)(\cos(\arctan\tau)+i\,\sin(\arctan\tau))^{2k+1}\frac{d\tau}{\sqrt{1+\tau^2}},\end{aligned}$$

and, using Section A.8.1, we get

$$\int_{\mathbb{R}} \widehat{F}(\tau) e^{2i(k+\frac{1}{2})\arctan\tau} \frac{d\tau}{\sqrt{1+\tau^2}} = \int_{\mathbb{R}} \widehat{F}(\tau)(1+i\tau)^{2k+1} \frac{d\tau}{(1+\tau^2)^{k+1}}.$$

We have proven the following lemma.

Lemma 2.1.1. *Let F be a tempered distribution on $\mathbb{R}$ such that $\widehat{F}$ is locally integrable and such that*

$$\int_{\mathbb{R}} |\widehat{F}(\tau)| \frac{d\tau}{\sqrt{1+\tau^2}} < +\infty. \tag{2.1.1}$$

Then, the operator $\mathrm{Op_w}\left(F(x^2+\xi^2)\right)$ *has the spectral decomposition*

$$\begin{aligned}\mathrm{Op_w}\left(F(x^2+\xi^2)\right) &= \sum_{k\geq 0} \int_{\mathbb{R}} \frac{\widehat{F}(\tau)(1+i\tau)^{2k+1}}{(1+\tau^2)^{k+1}} d\tau \mathbb{P}_k \\ &= \sum_{k\geq 0} \int_{\mathbb{R}} \frac{\widehat{F}(\tau)(1+i\tau)^{k}}{(1-i\tau)^{k+1}} d\tau \mathbb{P}_k,\end{aligned}$$

where the orthogonal projections $\mathbb{P}_k$ are defined in (A.1.17).

2.2 Higher-dimensional questions

We work now in n dimensions and consider a function F in the Schwartz class of $\mathbb{R}$. We want to calculate somewhat explicitly the Weyl quantization of $F(\sum_{1\leq j\leq n} \mu_j (x_j^2+\xi_j^2))$, where the μ_j are positive parameters, denoted by

$$\mathrm{Op_w}\left(F\Big(\sum_{1\leq j\leq n} \mu_j(x_j^2+\xi_j^2)\Big)\right), \quad q_\mu(x,\xi) = \sum_{1\leq j\leq n} \mu_j(x_j^2+\xi_j^2),$$

and also extend that computation to the case where F is merely $L^\infty(\mathbb{R})$. We have, say for F in the Wiener algebra $\mathscr{W}(\mathbb{R}) = \mathrm{Fourier}(L^1(\mathbb{R}))$,

$$\mathrm{Op_w}\left(F(q_\mu(x,\xi))\right) = \int_{\mathbb{R}} \widehat{F}(\tau)\mathrm{Op_w}\left(e^{2i\pi\tau\sum_{1\leq j\leq n}\mu_j(x_j^2+\xi_j^2)}\right)d\tau,$$

as an absolutely converging integral of a function defined on $\mathbb{R}$ (equipped with the Lebesgue measure) valued in $\mathscr{B}(L^2(\mathbb{R}^n))$ (bounded endomorphisms of $L^2(\mathbb{R}^n)$). In fact, applying Mehler's formula (A.3.1), we find by tensorisation,

$$\underbrace{\mathrm{Op_w}\left(e^{2i\pi\tau\sum_{1\leq j\leq n}\mu_j(x_j^2+\xi_j^2)}\right)}_{\substack{\text{operator with Weyl symbol}\\ e^{2i\pi\tau q_\mu(x,\xi)}}} = \prod_{1\leq j\leq n} \cos(\arctan(\tau\mu_j))\underbrace{e^{2i\pi(\arctan(\tau\mu_j))\mathrm{Op_w}(x_j^2+\xi_j^2)}}_{\substack{\text{exponential } e^{iM_j},\\ \text{with } M_j \text{ self-adjoint operator}\\ =2\pi(\arctan(\tau\mu_j))\mathrm{Op_w}(x_j^2+\xi_j^2)}}, \tag{2.2.1}$$

so that, using the spectral decomposition (A.1.19) of the harmonic oscillator, we get

$$\begin{aligned}
&\mathrm{Op_w}\left(F(q_\mu(x,\xi))\right)\\
&\quad=\int_{\mathbb{R}}\widehat{F}(\tau)\sum_{\alpha\in\mathbb{N}^n}\prod_{1\le j\le n}e^{2i(\arctan(\tau\mu_j))(\alpha_j+\frac12)}\mathbb{P}_{\alpha_j}\frac{1}{\sqrt{1+(\tau\mu_j)^2}}d\tau\\
&\quad=\sum_{\alpha\in\mathbb{N}^n}\int_{\mathbb{R}}\widehat{F}(\tau)\prod_{1\le j\le n}e^{2i(\alpha_j+\frac12)\arctan(\tau\mu_j)}\frac{1}{\sqrt{1+(\tau\mu_j)^2}}d\tau\mathbb{P}_\alpha,
\end{aligned}$$

where the use of Fubini theorem is justified by

$$\int_{\mathbb{R}}|\widehat{F}(\tau)|\frac{d\tau}{\sqrt{1+\tau^2}}<+\infty,\quad \mathbb{P}_\alpha\ge 0,\quad \sum_\alpha \mathbb{P}_\alpha=\mathrm{Id}\,.$$

We have

$$\begin{aligned}
&\int_{\mathbb{R}}\widehat{F}(\tau)\prod_{1\le j\le n}e^{2i(\alpha_j+\frac12)\arctan(\tau\mu_j)}\frac{1}{\sqrt{1+(\tau\mu_j)^2}}d\tau\\
&=\int_{\mathbb{R}}\widehat{F}(\tau)\prod_{1\le j\le n}\left(\cos(\arctan(\mu_j\tau))+i\sin(\arctan(\mu_j\tau))\right)^{2\alpha_j+1}\frac{1}{\sqrt{1+(\tau\mu_j)^2}}d\tau,
\end{aligned}$$

and, using Section A.8.1, we get

$$\begin{aligned}
&\int_{\mathbb{R}}\widehat{F}(\tau)\prod_{1\le j\le n}e^{2i(\alpha_j+\frac12)\arctan(\tau\mu_j)}\frac{1}{\sqrt{1+(\tau\mu_j)^2}}d\tau\\
&\quad=\int_{\mathbb{R}}\widehat{F}(\tau)\prod_{1\le j\le n}\frac{(1+i\tau\mu_j)^{2\alpha_j+1}}{(1+(\tau\mu_j)^2)^{\alpha_j+\frac12}}\frac{1}{\sqrt{1+(\tau\mu_j)^2}}d\tau.
\end{aligned}$$

We have proven the following lemma.

Lemma 2.2.1. *Let F be a tempered distribution on $\mathbb{R}$ such that $\widehat{F}$ is locally integrable and such that*

$$\int_{\mathbb{R}}|\widehat{F}(\tau)|\frac{d\tau}{\sqrt{1+\tau^2}}<+\infty.$$

Then, the operator $\mathrm{Op_w}\big(F(\sum_{1\le j\le n}\mu_j(x_j^2+\xi_j^2))\big)$ has the spectral decomposition

$$\begin{aligned}
\mathrm{Op_w}\Big(F\Big(\sum_{1\le j\le n}\mu_j(x_j^2+\xi_j^2)\Big)\Big)&=\sum_{\alpha\in\mathbb{N}^n}\int_{\mathbb{R}}\widehat{F}(\tau)\prod_{1\le j\le n}\frac{(1+i\tau\mu_j)^{2\alpha_j+1}}{(1+\tau^2\mu_j^2)^{\alpha_j+1}}d\tau\mathbb{P}_\alpha\\
&=\sum_{\alpha\in\mathbb{N}^n}\int_{\mathbb{R}}\widehat{F}(\tau)\prod_{1\le j\le n}\frac{(1+i\tau\mu_j)^{\alpha_j}}{(1-i\tau\mu_j)^{\alpha_j+1}}d\tau\mathbb{P}_\alpha,
\end{aligned}$$

where $\mathbb{P}_\alpha$ is the rank-one orthogonal projection onto Ψ_α given by (A.1.18).

Lemma 2.2.2. *Let F be as in Lemma 2.2.2 and let us assume that all the μ_j are equal to μ (positive). Then*

$$\operatorname{Op_w}\Big(F\big(\mu\sum_{1\le j\le n}(x_j^2+\xi_j^2)\big)\Big)=\sum_{k\ge 0}\int_{\mathbb{R}}\widehat{F}(\tau)\frac{(1+i\tau\mu)^k}{(1-i\tau\mu)^{k+n}}d\tau\mathbb{P}_{k;n},$$

with

$$\mathbb{P}_{k;n}=\sum_{\substack{\alpha\in\mathbb{N}^n\\|\alpha|=k}}\mathbb{P}_\alpha,$$

where $\mathbb{P}_\alpha$ is the rank-one orthogonal projection onto Ψ_α given by (A.1.18).

Proof. With all the μ_j equal to $\mu>0$, we find

$$\prod_{1\le j\le n}\frac{(1+i\tau\mu_j)^{\alpha_j}}{(1-i\tau\mu_j)^{\alpha_j+1}}=\prod_{1\le j\le n}\frac{(1+i\tau\mu)^{\alpha_j}}{(1-i\tau\mu)^{\alpha_j+1}}=\frac{(1+i\tau\mu)^{|\alpha|}}{(1-i\tau\mu)^{|\alpha|+n}},$$

which depends only on $|\alpha|$, so that applying the previous lemma gives

$$\Big(F\big(\mu\sum_{1\le j\le n}(x_j^2+\xi_j^2)\big)\Big)^w=\sum_{k\ge 0}\int_{\mathbb{R}}\widehat{F}(\tau)\frac{(1+i\tau\mu)^k}{(1-i\tau\mu)^{k+n}}d\tau\mathbb{P}_{k;n},$$

giving the sought result. ∎

Chapter 3

Conics with eccentricity smaller than 1

3.1 Indicatrix of a disc

Let us assume now that with some $a \ge 0$,

$$F = \mathbf{1}_{[-\frac{a}{2\pi}, \frac{a}{2\pi}]},$$

so that

$$F(x^2 + \xi^2) = \mathbf{1}_{\{2\pi(x^2+\xi^2) \le a\}}.$$

According to Section A.8.1, we have

$$\widehat{F}(\tau) = \frac{\sin a\tau}{\pi\tau},$$

so that (2.1.1) holds true. We find in this case,

$$\mathrm{Op_w}(F(x^2+\xi^2)) = \sum_{k \ge 0} F_k(a)\mathbb{P}_k, \quad F_k(a) = \int_{\mathbb{R}} \frac{\sin a\tau}{\pi\tau} \frac{(1+i\tau)^k}{(1-i\tau)^{k+1}} d\tau, \quad (3.1.1)$$

so that (note that $F_k(a)$ is real-valued since F is real-valued and thus the operator $\mathrm{Op_w}(F(x^2+\xi^2))$ is self-adjoint), and for $a > 0$, using the result (A.8.2) in Section A.8.2, we obtain

$$\begin{aligned}
F_k'(a) &= \frac{1}{\pi} \int_{\mathbb{R}} \cos a\tau \frac{(1+i\tau)^k}{(1-i\tau)^{k+1}} d\tau \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} e^{ia\tau} \left\{ \frac{(1+i\tau)^k}{(1-i\tau)^{k+1}} + \frac{(1-i\tau)^k}{(1+i\tau)^{k+1}} \right\} d\tau \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} e^{ia\tau} \left\{ \frac{i^k(\tau-i)^k}{(-i)^{k+1}(\tau+i)^{k+1}} + \frac{(-i)^k(\tau+i)^k}{i^{k+1}(\tau-i)^{k+1}} \right\} d\tau \\
&= \frac{(-1)^k}{2i\pi} \int_{\mathbb{R}} e^{ia\tau} \left\{ -\frac{(\tau-i)^k}{(\tau+i)^{k+1}} + \frac{(\tau+i)^k}{(\tau-i)^{k+1}} \right\} d\tau.
\end{aligned}$$

We shall now calculate explicitly both integrals above: let $1 < R$ be given and let us consider the closed path (see Figure 3.1)

$$\gamma_R = [-R, R] \cup \underbrace{\{Re^{i\theta}\}_{0 \le \theta \le \pi}}_{\gamma_{2;R}}.$$

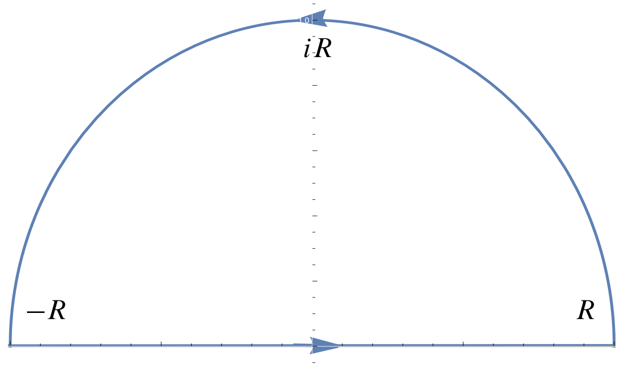

Figure 3.1. $\gamma_R = [-R, R] \cup \{Re^{i\theta}\}_{0\le\theta\le\pi}$.

We have

$$\frac{1}{2i\pi}\int_{\gamma_R} e^{ia\tau}\Big\{-\frac{(\tau-i)^k}{(\tau+i)^{k+1}}+\frac{(\tau+i)^k}{(\tau-i)^{k+1}}\Big\}d\tau = \mathtt{Res}\Big(e^{ia\tau}\frac{(\tau+i)^k}{(\tau-i)^{k+1}};i\Big)$$

$$= \frac{1}{k!}(\frac{d}{d\tau})^k\{e^{ia\tau}(\tau+i)^k\}_{|\tau=i},$$

and we note that, for $a>0$,

$$\lim_{R\to+\infty}\int_{\gamma_{2;R}} e^{ia\tau}\Big\{-\frac{(\tau-i)^k}{(\tau+i)^{k+1}}+\frac{(\tau+i)^k}{(\tau-i)^{k+1}}\Big\}d\tau = 0,$$

since for $R\ge 2$,

$$\int_0^\pi |e^{iaRe^{i\theta}}|\Big|-\frac{(Re^{i\theta}-i)^k}{(Re^{i\theta}+i)^{k+1}}+\frac{(Re^{i\theta}+i)^k}{(Re^{i\theta}-i)^{k+1}}\Big||iRe^{i\theta}|d\theta$$

$$\le \int_0^\pi e^{-aR\sin\theta}\Big|-\frac{(e^{i\theta}-iR^{-1})^k}{(e^{i\theta}+iR^{-1})^{k+1}}+\frac{(e^{i\theta}+iR^{-1})^k}{(e^{i\theta}-iR^{-1})^{k+1}}\Big|d\theta$$

$$\le \int_0^\pi e^{-aR\sin\theta}d\theta \sup_{0\le\rho\le 1/2}\Big\{\frac{(1+\rho)^k}{(1-\rho)^{k+1}}+\frac{(1+\rho)^k}{(1-\rho)^{k+1}}\Big\}.$$

For $a>0$, we obtain

$$\lim_{R\to+\infty}\int_0^\pi e^{-aR\sin\theta}d\theta = 0$$

by dominated convergence. As a result, we get

$$F_k'(a) = (-1)^k \frac{1}{k!}\left(\frac{d}{d\tau}\right)^k \{e^{ia\tau}(\tau+i)^k\}_{|\tau=i}$$
$$= (-1)^k \frac{1}{k!}\left(\frac{d}{\frac{i}{a}d\varepsilon}\right)^k \{e^{-a-\varepsilon}(i+i\frac{\varepsilon}{a}+i)^k\}_{|\varepsilon=0},$$

that is

$$F_k'(a) = \frac{(-1)^k}{k!}e^{-a}\left(\frac{d}{d\varepsilon}\right)^k \{e^{-\varepsilon}(2a+\varepsilon)^k\}_{|\varepsilon=0}.$$

We note that F_k' belongs to $L^1(\mathbb{R}_+)$ as the product of e^{-a} by a polynomial. We have also that

$$\lim_{a\to+\infty} F_k(a) = 1 \quad \text{(see Section A.8.3)},$$

and this yields

$$F_k(a) = 1 + \int_{+\infty}^{a} F_k'(b)db = 1 - \int_a^{+\infty} \frac{(-1)^k}{k!}e^{-b}\left(\frac{d}{d\varepsilon}\right)^k \{e^{-\varepsilon}(2b+\varepsilon)^k\}_{|\varepsilon=0}db,$$

so that

$$F_k(a) = 1 - e^{-a}P_k(a), \tag{3.1.2}$$

with

$$P_k(a) = \frac{(-1)^k}{k!}\int_0^{+\infty} e^{-t}\left(\frac{d}{d\varepsilon}\right)^k \{e^{-2\varepsilon}(a+t+\varepsilon)^k\}_{|\varepsilon=0}dt$$
$$= \frac{(-1)^k}{k!}\int_0^{+\infty} e^{t}\left(\frac{d}{d\varepsilon}\right)^k \{e^{-2\varepsilon-2t}(a+t+\varepsilon)^k\}_{|\varepsilon=0}dt$$
$$= \frac{(-1)^k}{k!}\int_0^{+\infty} e^{t}\left(\frac{d}{dt}\right)^k \{e^{-2t}(a+t)^k\}dt. \tag{3.1.3}$$

We see that P_k is a polynomial with leading monomial $\frac{2^k a^k}{k!}$ (by a direct computation) and $P_k(0) = 1$ (since $0 = F_k(0) = 1 - P_k(0)$) and moreover, using Laguerre polynomials (see, e.g., (A.4.1) in our Section A.4), we obtain

$$P_k(a) = \frac{(-1)^k}{k!}\int_0^{+\infty} e^{-t}e^{2t+2a}\left(\frac{d}{2dt}\right)^k \{e^{-2t-2a}(2a+2t)^k\}dt$$
$$= (-1)^k \int_0^{+\infty} e^{-t}L_k(2t+2a)dt, \tag{3.1.4}$$

and this gives in particular

$$\begin{aligned}
P_k'(a) &= (-1)^k \int_0^{+\infty} e^{-t} 2L_k'(2t+2a)dt \\
&= (-1)^k \left\{ [e^{-t} L_k(2t+2a)]_{t=0}^{t=+\infty} + \int_0^{+\infty} e^{-t} L_k(2t+2a)dt \right\} \\
&= (-1)^{k+1} L_k(2a) + P_k(a).
\end{aligned}$$

Moreover, we have from (3.1.3), for $k \geq 1$,

$$\begin{aligned}
P_k'(a) &= \frac{(-1)^k}{k!} \int_0^{+\infty} e^t \left(\frac{d}{dt}\right)^k \{e^{-2t} k(a+t)^{k-1}\} dt \\
&= \frac{(-1)^k}{k!} \int_0^{+\infty} e^t \frac{d}{dt} \left(\frac{d}{dt}\right)^{k-1} \{e^{-2t} k(a+t)^{k-1}\} dt \\
&= \frac{(-1)^k}{k!} \left\{ \left[e^t \left(\frac{d}{dt}\right)^{k-1} \{e^{-2t} k(a+t)^{k-1}\} \right]_{t=0}^{t=+\infty} \right. \\
&\qquad \left. - \int_0^{+\infty} e^t \left(\frac{d}{dt}\right)^{k-1} \{e^{-2t} k(a+t)^{k-1}\} dt \right\} \\
&= \frac{(-1)^{k-1}}{(k-1)!} \left(\frac{d}{dt}\right)^{k-1} \{e^{-2t}(a+t)^{k-1}\}_{|t=0} \\
&\quad + \frac{(-1)^{k-1}}{(k-1)!} \int_0^{+\infty} e^t \left(\frac{d}{dt}\right)^{k-1} \{e^{-2t}(a+t)^{k-1}\} dt \\
&= \frac{(-1)^{k-1}}{(k-1)!} e^{2t+2a} \left(\frac{d}{2dt}\right)^{k-1} \{e^{-2t-2a}(2a+2t)^{k-1}\}_{|t=0} \\
&\quad + \frac{(-1)^{k-1}}{(k-1)!} \int_0^{+\infty} e^t \left(\frac{d}{dt}\right)^{k-1} \{e^{-2t}(a+t)^{k-1}\} dt \\
&= (-1)^{k-1} L_{k-1}(2a) + P_{k-1}(a),
\end{aligned}$$

so that

$$\forall k \geq 1, \quad P_k'(a) = (-1)^{k-1} L_{k-1}(2a) + P_{k-1}(a) = (-1)^{k+1} L_k(2a) + P_k(a). \tag{3.1.5}$$

This implies for $N \geq 1$,

$$\sum_{1 \leq k \leq N} P_k(a) - \sum_{1 \leq k \leq N} (-1)^k L_k(2a) = \sum_{0 \leq k \leq N-1} P_k(a) + \sum_{0 \leq k \leq N-1} (-1)^k L_k(2a),$$

yielding

$$P_N(a) - \underbrace{P_0(a)}_{=1=L_0(a)} = \sum_{1\le k\le N} (-1)^k L_k(2a) + \sum_{0\le k\le N-1} (-1)^k L_k(2a),$$

and

$$P_N(a) = \sum_{0\le k\le N} (-1)^k L_k(2a) + \sum_{0\le k\le N-1} (-1)^k L_k(2a). \tag{3.1.6}$$

Note that the previous formula holds as well for $N = 0$, since $P_0 = 1 = L_0$.

Although the function $\mathbb{R}_+ \ni a \mapsto F_k(a)$ has no monotonicity properties, we prove below that $\mathbb{R}_+ \ni a \mapsto P_k(a)$ is indeed increasing. For that purpose, let us use (3.1.5), which implies

$$\begin{aligned} P_k'(a) &= (-1)^{k-1} L_{k-1}(2a) + P_{k-1}(a), \quad k \ge 1,\\ P_{k-1}(a) &= P_{k-2}(a) + (-1)^{k-2} L_{k-2}(2a) + (-1)^{k-1} L_{k-1}(2a), \quad k \ge 2,\\ P_k'(a) &= 2(-1)^{k-1} L_{k-1}(2a) + (-1)^{k-2} L_{k-2}(2a) + P_{k-2}(a), \quad k \ge 2. \end{aligned}$$

We claim that for $k \ge 1$,

$$P_k'(a) = 2 \sum_{0\le l\le k-1} (-1)^l L_l(2a). \tag{3.1.7}$$

That property holds for $k = 1$ since $P_1(a) = 1 + 2a$: we check $P_1'(a) = 2$. Moreover, we have

$$\begin{aligned} P_{k+1}'(a) &= (-1)^k L_k(2a) + P_k(a) \quad \text{(from the first equation in (3.1.5))}\\ \text{(using (3.1.6))} \quad &= (-1)^k L_k(2a) + \sum_{0\le l\le k} (-1)^l L_l(2a) + \sum_{0\le l\le k-1} (-1)^l L_l(2a)\\ &= 2 \sum_{0\le l\le k} (-1)^l L_l(2a), \end{aligned}$$

which is the sought formula. As a byproduct we find from (A.4.2)

$$\forall a \ge 0, \quad P_k'(a) \ge 0,$$

which implies that for $a \ge 0$, $P_k(a) \ge P_k(0) = 1$. We have proven the following lemma.

Lemma 3.1.1. *The polynomial*

$$P_k(a) = e^a(1 - F_k(a))$$

is increasing on $\mathbb{R}_+$,

$$P_k(0) = 1.$$

Let us take a look at the first P_k: we have

$$
\begin{aligned}
P_0(a) &= 1,\\
P_1(a) &= 1 + 2a,\\
P_2(a) &= 1 + 2a^2,\\
P_3(a) &= 1 + 2a - 2a^2 + \frac{4a^3}{3},\\
P_4(a) &= 1 + 4a^2 - \frac{8a^3}{3} + \frac{2a^4}{3},\\
P_5(a) &= 1 + 2a - 4a^2 + \frac{16a^3}{3} - 2a^4 + \frac{4a^5}{15},\\
P_6(a) &= 1 + 6a^2 - 8a^3 + \frac{14a^4}{3} - \frac{16a^5}{15} + \frac{4a^6}{45},\\
P_7(a) &= 1 + 2a - 6a^2 + 12a^3 - \frac{26a^4}{3} + \frac{44a^5}{15} - \frac{4a^6}{9} + \frac{8a^7}{315},\\
P_8(a) &= 1 + 8a^2 - 16a^3 + \frac{44a^4}{3} - \frac{32a^5}{5} + \frac{64a^6}{45} - \frac{16a^7}{105} + \frac{2a^8}{315},\\
P_9(a) &= 1 + 2a - 8a^2 + \frac{64a^3}{3} - \frac{68a^4}{3} + \frac{184a^5}{15} - \frac{32a^6}{9} + \frac{176a^7}{315}\\
&\quad - \frac{2a^8}{45} + \frac{4a^9}{2835},\\
P_{10}(a) &= 1 + 10a^2 - \frac{80a^3}{3} + \frac{100a^4}{3} - \frac{64a^5}{3} + \frac{344a^6}{45} - \frac{496a^7}{315} + \frac{58a^8}{315}\\
&\quad - \frac{32a^9}{2835} + \frac{4a^{10}}{14175},\\
P_{11}(a) &= 1 + 2a - 10a^2 + \frac{100a^3}{3} - \frac{140a^4}{3} + \frac{104a^5}{3} - \frac{664a^6}{45} + \frac{1184a^7}{315}\\
&\quad - \frac{26a^8}{45} + \frac{148a^9}{2835} - \frac{4a^{10}}{1575} + \frac{8a^{11}}{155925},\\
P_{12}(a) &= 1 + 12a^2 - 40a^3 + \frac{190a^4}{3} - \frac{160a^5}{3} + \frac{1184a^6}{45} - \frac{2512a^7}{315} + \frac{478a^8}{315}\\
&\quad - \frac{512a^9}{2835} + \frac{184a^{10}}{14175} - \frac{16a^{11}}{31185} + \frac{4a^{12}}{467775}.
\end{aligned}
$$

We note as well that

$$
P_k(x) = \sum_{0 \le m \le k} \frac{x^m}{m!} \sum_{m \le l \le k} 2^l (-1)^{k-l} \binom{k}{l},
$$

since from (3.1.3),

$$\begin{aligned}
P_k(a) &= \frac{(-1)^k}{k!}\int_0^{+\infty} e^t\left(\frac{d}{dt}\right)^k\{e^{-2t}(a+t)^k\}dt\\
&= (-1)^k\sum_{0\le m\le k}\int_0^{+\infty} e^{-t}\frac{(-2)^{k-m}}{(k-m)!}\frac{k!}{(k-m)!m!}(a+t)^{k-m}dt\\
&= (-1)^k\sum_{0\le m\le k}\int_0^{+\infty} e^{-t}\frac{(-2)^{k-m}}{(k-m)!}\frac{k!}{(k-m)!m!}\sum_{0\le l\le k-m} a^l t^{k-l-m}\binom{k-m}{l}dt\\
&= (-1)^k\sum_{\substack{0\le m\le k\\0\le l\le k-m}}\frac{(-2)^{k-m}}{(k-m)!}\frac{k!}{(k-m)!m!}a^l(k-l-m)!\binom{k-m}{l}\\
&= \sum_{0\le l+m\le k}\frac{(-1)^m 2^{k-m}}{(k-m)!}\frac{k!}{m!}a^l\frac{1}{l!} = \sum_{0\le l\le k}\frac{a^l}{l!}\sum_{l\le m'\le k}(-1)^{k-m'}2^{m'}\binom{k}{m'},
\end{aligned}$$

which is the sought formula.

Lemma 3.1.2. *With the polynomial P_k defined by (3.1.4), we have*

$$\begin{cases} P_k(a) = 2\sum_{0\le l\le k-1}(-1)^l L_l(2a) + (-1)^k L_k(2a),\\ P_k'(a) = 2\sum_{0\le l\le k-1}(-1)^l L_l(2a).\end{cases}$$

Proof. We may use the already proven (3.1.6), (3.1.7), but we may also prove this directly by induction on k. ∎

Proposition 3.1.3. *Let F_k be given by (3.1.2) with P_k defined by (3.1.3). We have*

$$\begin{aligned}
&F_k(a) = 1 - e^{-a}P_k(a) \le 1 - e^{-a} = F_0(a) \quad \textit{for } a \ge 0,\\
&F_k'(a) = e^{-a}\big(P_k(a) - P_k'(a)\big) = e^{-a}(-1)^k L_k(2a),\\
&F_k'(0) = (-1)^k, \quad \lim_{a\to+\infty} F_k'(a) = 0_+, \quad F_k(0) = 0, \lim_{a\to+\infty} F_k(a) = 1_-. \qquad (3.1.8)
\end{aligned}$$

Proof. We use (3.1.2), (3.1.7), and (3.1.6) for the three first equalities, Lemma 3.1.1 for the first inequality. The fourth equality follows from $L_k(0) = 1$, while the fifth is due to the fact that the leading monomial of $(-1)^k L_k(2a)$ is $2^k a^k/k!$. The two last equalities are a consequence of the first line. ∎

Remark 3.1.4. The zeroes of F_k' on the positive half-line are the positive zeroes of the Laguerre polynomial L_k divided by 2. When k is even (resp., odd) the function F_k is positive increasing (resp., negative decreasing) near 0, then oscillates with changes of monotonicity at each a such that $L_k(2a) = 0$ and when $2a$ is larger than the largest zero of L_k, the function F_k is increasing, smaller than 1, with limit 1 at infinity. Typically, we have $F_{2l}(0) = 0$, $F'_{2l}(0) = +1$,

$$0 < a_{1,2l} < a_2 < \cdots < a_{2l-1,2l} < a_{2l,2l}, \quad \text{the zeroes of } L_{2l}(2a), \tag{3.1.9}$$

F_{2l} vanishes simply at $b_0 = 0$ and at $b_j \in (a_j, a_{j+1})$ for $1 \le j \le 2l-1$, also at $b_{2l} > a_{2l}$: $2l+1$ zeroes with a positive (resp., negative) derivative at $b_0, b_2, \ldots, b_{2l}$ (resp., at $b_1, b_3, \ldots, b_{2l-1}$). Moreover, we have $F_{2l+1}(0) = 0$, $F'_{2l+1}(0) = -1$,

$$0 < a_{1,2l+1} < a_{2,2l+1} < \cdots < a_{2l,2l+1} < a_{2l+1,2l+1}, \quad \text{the zeroes of } L_{2l+1}(2a), \tag{3.1.10}$$

F_{2l+1} vanishes simply at $b_0 = 0$ and at $b_j \in (a_j, a_{j+1})$ for $1 \le j \le 2l$, also at $b_{2l+1} > a_{2l+1}$: $2l+2$ zeroes with a positive (resp., negative) derivative at $b_1, b_3, \ldots, b_{2l+1}$ (resp., at $b_0, b_2, \ldots, b_{2l}$).

We note as well that a consequence of the previous remark is that

$$\min_{a \ge 0} F_{2l}(a) = \min_{1 \le j \le l} \{F_{2l}(a_{2j,2l})\},$$
$$\min_{a \ge 0} F_{2l+1}(a) = \min_{0 \le j \le l} \{F_{2l+1}(a_{2j+1,2l+1})\},$$

where $(a_{p,k})_{1 \le p \le k}$ are defined in (3.1.9), (3.1.10).

Theorem 3.1.5. *Let $a \ge 0$ be given and let*

$$D_a = \left\{(x, \xi) \in \mathbb{R}^2, x^2 + \xi^2 \le \frac{a}{2\pi}\right\}. \tag{3.1.11}$$

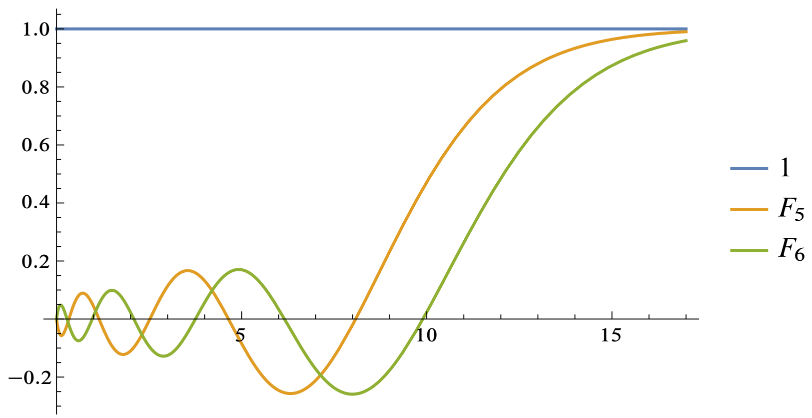

Figure 3.2. Functions F_5, F_6.

Then, we have

$$\mathrm{Op_w}(\mathbf{1}_{D_a}) = \sum_{k\ge 0} F_k(a)\mathbb{P}_k \le 1 - e^{-a}.$$

Proof. An immediate consequence of (3.1.1), (3.1.8). Note that the inequality in the above theorem is due to P. Flandrin in [13] (see also the related references [20], [14]). ∎

Curves. Let us display some curves of $\mathbb{R}_+ \ni a \mapsto F_k(a) = 1 - e^{-a} P_k(a)$.

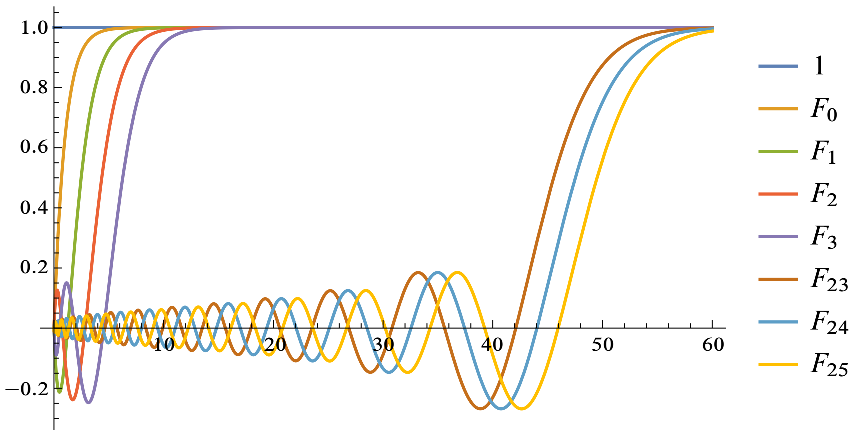

Figure 3.3. Functions F_k.

3.2 Indicatrix of a Euclidean ball

The following result displays an explicit spectral decomposition on the Hermite basis for the Weyl quantization of the characteristic function of Euclidean balls.

Theorem 3.2.1. *Let $a \ge 0$ be given and let*

$$\mathcal{Q}_{a,n} = \mathrm{Op_w}(\mathbf{1}\{2\pi(|x|^2 + |\xi|^2) \le a\}),$$

be the Weyl quantization of the characteristic function of the Euclidean ball of $\mathbb{R}^{2n}$ with center 0 and radius $\sqrt{a/(2\pi)}$. Then, we have

$$\mathcal{Q}_{a,n} = \sum_{k\ge 0} F_{k;n}(a)\mathbb{P}_{k;n},$$

with $\mathbb{P}_{k;n} = \sum_{\alpha\in\mathbb{N}^n, |\alpha|=k} \mathbb{P}_\alpha$, where $\mathbb{P}_\alpha$ is the orthogonal projection onto Ψ_α (defined in (A.1.18)), with $|\alpha| = \sum_{1\le j\le n} \alpha_j = k$ and

$$F_{k;n}(a) = \int_{\mathbb{R}} \frac{\sin a\tau}{\pi\tau} \frac{(1+i\tau)^k}{(1-i\tau)^{k+n}} d\tau.$$

The spectral decomposition of the previous theorem allows a simple recovery of the result of the article [39] by E. Lieb and Y. Ostrover.

Theorem 3.2.2. *Let $a \ge 0, \mathcal{Q}_{a,n}, F_{k;n}$ be defined above. Then, we have*

$$F_{k;n}(a) \le 1 - \frac{1}{\Gamma(n)} \int_a^{+\infty} e^{-t} t^{n-1} dt = 1 - \frac{\Gamma(n,a)}{\Gamma(n)}, \tag{3.2.1}$$

and thus we have

$$\mathcal{Q}_{a,n} \le 1 - \frac{\Gamma(n,a)}{\Gamma(n)}, \tag{3.2.2}$$

where the incomplete Gamma function $\Gamma(\cdot,\cdot)$ is defined in (A.8.3).

Proof of Theorems 3.2.1 and 3.2.2. We use the results of (the previous) Section 3.1: Let us assume now that with some $a \ge 0$,

$$F = \mathbf{1}_{[-\frac{a}{2\pi}, \frac{a}{2\pi}]},$$

so that

$$F(|x|^2 + |\xi|^2) = \mathbf{1}\{2\pi(|x|^2 + |\xi|^2) \le a\}.$$

According to Section A.8.1, we have $\widehat{F}(\tau) = \frac{\sin a\tau}{\pi\tau}$, so that (2.1.1) holds true. We find in this case, following the results of Lemma 2.2.2,

$$\begin{gathered}\operatorname{Op_w}\left(F(|x|^2 + |\xi|^2)\right) = \sum_{k\ge 0} F_{k;n}(a)\mathbb{P}_{k;n}, \quad \mathbb{P}_{k;n} = \sum_{\alpha\in\mathbb{N}^n, |\alpha|=k} \mathbb{P}_\alpha, \\ F_{k;n}(a) = \int_{\mathbb{R}} \frac{\sin a\tau}{\pi\tau} \frac{(1+i\tau)^k}{(1-i\tau)^{k+n}} d\tau, \end{gathered}\tag{3.2.3}$$

where $\mathbb{P}_\alpha$ is the orthogonal projection onto Ψ_α (defined in (A.1.18)), with

$$|\alpha| = \sum_{1\le j\le n} \alpha_j = k.$$

This completes the proof of Theorem 3.2.1.

We postpone the proof of Theorem 3.2.2 until after settling a couple of lemmas.

Lemma 3.2.3. *Let $(k,n) \in \mathbb{N} \times \mathbb{N}^*$. With $F_{k;n}(a)$ given by (3.2.3), we have*

$$F_{k;n}(a) = 1 - e^{-a} P_{k,n}(a), \quad \textit{where } P_{k;n} \textit{ is the polynomial}$$

$$P_{k;n}(a) = \frac{(-1)^{k+n-1}}{(k+n-1)!} \int_0^{+\infty} e^{-t}(t+a)^{n-1} \left\{ e^s \left(\frac{d}{ds}\right)^{n+k-1} \left[s^k e^{-s}\right]\right\}_{|s=2t+2a} dt, \tag{3.2.4}$$

$$P_{k;n}(a) = \frac{(-1)^{k+n-1}}{(k+n-1)! 2^{n-1}} \int_0^{+\infty} (t+a)^{n-1} e^t \left(\frac{d}{dt}\right)^{n+k-1} \{(t+a)^k e^{-2t}\} dt.$$

Proof of Lemma 3.2.3. The lemma holds true for $n = 1$ from Proposition 3.1.3. We have for $a > 0$, $n \geq 2$,

$$\begin{aligned}
F'_{k;n}(a) &= \frac{1}{\pi} \int_{\mathbb{R}} \cos a\tau \frac{(1+i\tau)^k}{(1-i\tau)^{k+n}} d\tau \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} e^{ia\tau} \frac{(1+i\tau)^k}{(1-i\tau)^{k+n}} d\tau + \frac{1}{2\pi} \int_{\mathbb{R}} e^{ia\tau} \frac{(1-i\tau)^k}{(1+i\tau)^{k+n}} d\tau \\
&= \frac{i}{2i\pi} \int_{\mathbb{R}} e^{ia\tau} \frac{i^k(\tau-i)^k}{(-i)^{k+n}(\tau+i)^{k+n}} d\tau + \frac{i}{2i\pi} \int_{\mathbb{R}} e^{ia\tau} \frac{(-i)^k(\tau+i)^k}{i^{k+n}(\tau-i)^{k+n}} d\tau,
\end{aligned}$$

so that

$$\begin{aligned}
F'_{k;n}(a) &= i^{1-n}(-1)^k \mathtt{Res}\left(e^{ia\tau} \frac{(\tau+i)^k}{(\tau-i)^{k+n}}; i\right) \\
&= \frac{i^{1-n}(-1)^k}{(k+n-1)!} \left(\frac{d}{d\tau}\right)^{k+n-1} \left\{e^{ia\tau}(\tau+i)^k\right\}_{|\tau=i},
\end{aligned}$$

and thus

$$\begin{aligned}
F'_{k;n}(a) &= \frac{i^{1-n}(-1)^k}{(k+n-1)!} \left(\frac{d}{\frac{i}{a}d\varepsilon}\right)^{k+n-1} \left\{e^{-a-\varepsilon}\left(i + i\frac{\varepsilon}{a} + i\right)^k\right\}_{|\varepsilon=0} \\
&= \frac{i^{1-n}(-1)^k a^{n-1}}{i^{n-1}(k+n-1)!} \left(\frac{d}{d\varepsilon}\right)^{k+n-1} \left\{e^{-a-\varepsilon}(2a+\varepsilon)^k\right\}_{|\varepsilon=0} \\
&= e^a \frac{(-1)^{k+n-1} a^{n-1}}{(k+n-1)!} \left(\frac{d}{2d\varepsilon}\right)^{k+n-1} \left\{e^{-2a-2\varepsilon}(2a+2\varepsilon)^k\right\}_{|\varepsilon=0},
\end{aligned}$$

that is

$$\begin{aligned}
F'_{k;n}(t) &= \frac{(-1)^{k+n-1}}{(k+n-1)!} e^t t^{n-1} \left(\frac{d}{ds}\right)^{k+n-1} \left\{e^{-s}s^k\right\}_{|s=2t} \\
&= \frac{(-1)^{k+n-1}}{(k+n-1)!2^{n-1}} e^t t^{n-1} \left(\frac{d}{dt}\right)^{k+n-1} \left\{e^{-2t}t^k\right\}.
\end{aligned}$$

We have also that $\lim_{a\to+\infty} F_{k;n}(a) = 1$ (following the arguments of Section 3.1) and this yields

$$\begin{aligned}
F_{k;n}(a) &= 1 - \frac{(-1)^{k+n-1}}{(k+n-1)!2^{n-1}} \int_a^{+\infty} e^t t^{n-1} \left(\frac{d}{dt}\right)^{k+n-1} \left\{e^{-2t}t^k\right\} dt \\
&= 1 - e^{-a} \frac{(-1)^{k+n-1}}{(k+n-1)!2^{n-1}} \\
&\quad \times \int_0^{+\infty} (t+a)^{n-1} e^t \left(\frac{d}{dt}\right)^{k+n-1} \left\{e^{-2t}(t+a)^k\right\} dt,
\end{aligned}$$

concluding the proof of the lemma. ∎

Let us go back to formula (3.2.4), written as

$$\frac{(-1)^{k+n-1}}{2^{n-1}}\int_0^{+\infty} e^{-t}\left\{\frac{(2t+2a)^{n-1}}{(k+n-1)!}(\frac{d}{d\varepsilon}-1)^{n+k-1}\left[(\varepsilon+2t+2a)^k\right]\right\}_{|\varepsilon=0} dt$$

$$= P_{k;n}(a) = \frac{(-1)^{k+n-1}}{2^{n-1}}\int_0^{+\infty} e^{-t}L^{1-n}_{k+n-1}(2t+2a)dt, \qquad (3.2.5)$$

where the generalized Laguerre polynomial L^{1-n}_{k+n-1} is defined by (A.4.5) (note that $1-n+k+n-1=k$ which is not negative).

Lemma 3.2.4. *Let $n\in\mathbb{N}^*$, $k\in\mathbb{N}$ and let $P_{k;n}$ be the polynomial defined in Lemma 3.2.3 (and thus in (3.2.5)). Then, we have*

$$P_{k;n}(X)-P'_{k;n}(X)=\frac{(-1)^{k+n-1}}{2^{n-1}}L^{1-n}_{k+n-1}(2X), \quad P_{k;n}(0)=1, \qquad (3.2.6)$$

$$\textit{for } n\geq 2, \quad P'_{k;n}=P_{k;n-1}. \qquad (3.2.7)$$

Proof. From (3.2.5), we find

$$\begin{aligned}
P'_{k;n}(a) &= \frac{(-1)^{k+n-1}}{2^{n-1}}\int_0^{+\infty} e^{-t}2(L^{1-n}_{k+n-1})'(2t+2a)dt\\
&= \frac{(-1)^{k+n-1}}{2^{n-1}}\left\{\left[e^{-t}(L^{1-n}_{k+n-1})(2t+2a)\right]_{t=0}^{t=+\infty}+\int_0^{+\infty} e^{-t}L^{1-n}_{k+n-1}(2t+2a)dt\right\}\\
&= \frac{(-1)^{k+n}}{2^{n-1}}L^{1-n}_{k+n-1}(2a)+P_{k;n}(a),
\end{aligned}$$

and since $0=F_{k;n}(0)=1-P_{k;n}(0)$, this proves (3.2.6). Using now (3.2.5) and (A.4.7), we find that

$$\begin{aligned}
P_{k;n}(a) &= \frac{(-1)^{k+n}}{2^{n-1}}\int_0^{+\infty}\frac{d}{dt}\{e^{-t}\}L^{1-n}_{k+n-1}(2t+2a)dt\\
&= \frac{(-1)^{k+n}}{2^{n-1}}\left\{\left[e^{-t}L^{1-n}_{k+n-1}(2t+2a)\right]_{t=0}^{t=+\infty}\right.\\
&\qquad\left.-\int_0^{+\infty} e^{-t}2(L^{1-n}_{k+n-1})'(2t+2a)dt\right\}\\
&= \frac{(-1)^{k+n}}{2^{n-1}}\left\{-L^{1-n}_{k+n-1}(2a)+\int_0^{+\infty} e^{-t}2(L^{2-n}_{k+n-2})(2t+2a)dt\right\}\\
&= \underbrace{\frac{(-1)^{k+n-1}}{2^{n-1}}L^{1-n}_{k+n-1}(2a)}_{\substack{P_{k;n}(a)-P'_{k;n}(a)\\ \text{from (3.2.6)}}}+\underbrace{\frac{(-1)^{k+n-2}}{2^{n-2}}\int_0^{+\infty} e^{-t}L^{2-n}_{k+n-2}(2t+2a)dt}_{\substack{P_{k;n-1}(a)\\ \text{from (3.2.5)}}},
\end{aligned}$$

so that for $n\geq 2$, $k\in\mathbb{N}$, we obtain (3.2.7), completing the proof of the lemma. ∎

Lemma 3.2.5. *Let $k, n, P_{k;n}$ be as in Lemma 3.2.4. Then, we have*

$$\forall j \in [\![0, n-1]\!], \quad \left(\frac{d}{dX}\right)^j P_{k;n} = P_{k;n-j}. \tag{3.2.8}$$

Moreover, for all $a \geq 0$ and all $k \in \mathbb{N}$,

$$P_{k;n}(a) \geq P_{0;n}(a) = \frac{1}{(n-1)!}\int_0^{+\infty} e^{-t}(t+a)^{n-1}dt = e^a \frac{\Gamma(n,a)}{\Gamma(n)}. \tag{3.2.9}$$

Proof. Formula (3.2.8) follows immediately by induction from (3.2.7) since the latter is proving (3.2.8) for $j = 1, n \geq 2, k \in \mathbb{N}$. Assuming that (3.2.8) holds true for some $1 \leq j < n$, all $k \in \mathbb{N}$, we have $P_{k;n}^{(j)} = P_{k,n-j}$ and if $j + 1 < n$, we obtain from (3.2.7) that

$$P_{k,n-j-1} = P'_{k,n-j} = P_{k;n}^{(j+1)},$$

proving (3.2.8). The property (3.2.9) holds true for $n=1$. From (3.2.7), $P_{k;n+1}(0)=1$, we find that $P_{k;n+1}(a) = 1 + \int_0^a P_{k;n}(s)ds$ and assuming that (3.2.9) holds true for n, we obtain for $a \geq 0$,

$$\begin{aligned}
P_{k;n+1}(a) &\geq 1 + \int_0^a \frac{1}{(n-1)!}\int_0^{+\infty} e^{-t}(t+s)^{n-1}dtds \\
&= 1 + \int_0^{+\infty} e^{-t}\left[\frac{(t+s)^n}{n!}\right]_{s=0}^{s=a} dt \\
&= 1 + \frac{1}{n!}\int_0^{+\infty} e^{-t}((t+a)^n - t^n)dt = \frac{1}{n!}\int_0^{+\infty} e^{-t}(t+a)^n dt,
\end{aligned}$$

completing the proof of the lemma. ∎

We can now prove Theorem 3.2.2, since

$$F_{k;n}(a) = 1 - e^{-a}P_{k;n}(a)$$

the estimate (3.2.8) implies indeed

$$F_{k;n}(a) \leq \frac{\Gamma(n,a)}{\Gamma(n)},$$

concluding the proof. ∎

Remark 3.2.6. Our methods of proof in one and more dimensions are quite similar.

- Using Mehler's formula, we diagonalise in the Hermite basis the quantization of the indicatrix of the Euclidean ball

$$D_{a;n} = \left\{(x,\xi) \in \mathbb{R}^{2n}, 2\pi\left(|x|^2 + |\xi|^2\right) \leq a\right\}.$$

- Once we get the diagonalisation

$$\operatorname{Op}_{\mathrm{w}}(\mathbf{1}_{D_{a;n}}) = \sum_{k \in \mathbb{N}} F_{k;n}(a) \mathbb{P}_{k;n},$$

  we study explicitly the functions $F_{k;n}$ and prove that

$$F_{k;n}(a) = 1 - e^{-a} P_{k;n}(a),$$

  where $P_{k;n}$ is a polynomial given in terms of the generalized Laguerre polynomials

$$P_{k;n}(a) = \frac{(-1)^{k+n-1}}{2^{n-1}} \int_0^{+\infty} e^{-t} L_{k+n-1}^{1-n}(2t + 2a) dt.$$

- Following the Flandrin paper [13], we use Feldheim inequality in [12] to tackle the case $n = 1$, and next we use an induction on n, made possible by the relationship between the standard and the generalized Laguerre polynomials. It is interesting to note that the functions $F_{k;n}$ have no monotonicity properties: with value 0 at 0, they have an oscillatory behavior for $a \le a_{k,n}$ and for a large enough, increase monotonically to 1 (see for instance Figures 3.2 and 3.3 in the 1D case); the inequality

$$F_{k;n}(a) \le 1 - e^{-a}$$

  holds true for all $a \ge 0$ in all dimensions. On the other hand, the polynomials $P_{k;n}$ are increasing and larger than 1 on the positive half-line.

The key ingredients are thus Mehler's formula and Feldheim inequality, but it should be pointed out that the arguments proving Feldheim inequality (formula (6.8) and Theorem 12) in the R. Askey and G. Gasper's article [2] are also based upon a version of Mehler's formula which appears thus as the basic result for our investigation. The paper [39] by E. Lieb and Y. Ostrover has a slightly different line of arguments and takes advantage of symmetry properties of the sphere. We shall go back to this in a situation where the symmetry is absent, such as for some general ellipsoids.

3.3 Ellipsoids in the phase space

3.3.1 Preliminaries

We provide below a couple of remarks on ellipsoids in higher dimensions. Let us first recall a particular case of in [24, Theorem 21.5.3].

Theorem 3.3.1 (Symplectic reduction of quadratic forms). *Let q be a positive-definite quadratic form on $\mathbb{R}^n \times \mathbb{R}^n$ equipped with the canonical symplectic form* (1.2.13).

Then, there exists S in the symplectic group $\mathrm{Sp}(n,\mathbb{R})$ *of* $\mathbb{R}^{2n}$ *and* $\mu_1,\dots,\mu_n$ *positive such that for all* $X=(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n$,

$$q(SX)=\sum_{1\le j\le n}\mu_j(x_j^2+\xi_j^2). \tag{3.3.1}$$

Note that an interesting consequence of this theorem is that, considering a general ellipsoid in $\mathbb{R}^{2n}$ (with center of gravity at 0),

$$\mathbb{E}=\{X\in\mathbb{R}^{2n},q(X)\le 1\},$$

where q is a positive definite quadratic form, we are able to find symplectic coordinates such that q is given by (3.3.1). Note however that no further simplification is possible and that the μ_j are symplectic invariants of $\mathbb{E}$. Note that the volume of $\mathbb{E}$ is given by

$$|\mathbb{E}|_{2n}=\frac{\pi^n}{n!\mu_1\cdots\mu_n}.$$

3.3.2 Spectral decomposition for the quantization of the characteristic function of the ellipsoid

Let $a_1,\dots,a_n$ be positive numbers. We consider the ellipsoid $E(a_1,\dots,a_n)$ given by

$$E(a)=E(a_1,\dots,a_n)=\left\{(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n,2\pi\sum_{1\le j\le n}\frac{x_j^2+\xi_j^2}{a_j}\le 1\right\}. \tag{3.3.2}$$

We define on $\mathbb{R}^n$ the function

$$F(X_1,\dots,X_n)=\mathbf{1}_{[-1,1]}\left(\frac{2\pi}{a_1}X_1+\cdots+\frac{2\pi}{a_n}X_n\right).$$

Theorem 3.3.2. *Let* $a=(a_j)_{1\le j\le n}$ *be positive numbers and let* $E(a)$ *be defined by* (3.3.2)*. Then, we have*

$$\mathrm{Op_w}(\mathbf{1}_{E(a)})=\sum_{\alpha\in\mathbb{N}^n}F_\alpha(a)\mathbb{P}_\alpha,$$

where $\mathbb{P}_\alpha$ *is defined in* (A.1.19) *and* $F_\alpha(a)=1-K_\alpha(a)$*, with*

$$K_\alpha(a)=\int_{\substack{\sum t_j/a_j\ge 1\\ t_j\ge 0}}e^{-(t_1+\cdots+t_n)}\prod_{1\le j\le n}(-1)^{\alpha_j}L_{\alpha_j}(2t_j)dt, \tag{3.3.3}$$

Remark 3.3.3. For all $\alpha\in\mathbb{N}^n$, the functions F_α,K_α are holomorphic on

$$\mathcal{U}=\{a\in\mathbb{C}^n,\forall j\in[\![1,n]\!],\mathrm{Re}\,a_j>0\}. \tag{3.3.4}$$

Indeed, let K be a compact subset of $\mathcal{U}$; there exists $\rho > 0$ such that

$$\forall (a_1, \dots, a_n) \in K, \quad \min_{1 \le j \le n} \operatorname{Re} a_j \ge \rho,$$

and as a result for $a \in K$, we have for $s \in \mathbb{R}^n_+$

$$\left| e^{-(a_1 s_1 + \dots + a_n s_n)} \prod_{1 \le j \le n} (-1)^{\alpha_j} L_{\alpha_j}(2a_j s_j) \right| \le e^{-\rho(s_1 + \dots + s_n)} C_{K,\alpha} (1 + |s|)^{|\alpha|},$$

so that

$$\begin{aligned} &\int_{\substack{\sum s_j \ge 1 \\ s_j \ge 0}} \sup_{a \in K} \left| e^{-(a_1 s_1 + \dots + a_n s_n)} \prod_{1 \le j \le n} (-1)^{\alpha_j} L_{\alpha_j}(2a_j s_j) \right| ds \\ &\quad \le \int_{\substack{\sum s_j \ge 1 \\ s_j \ge 0}} e^{-\rho(s_1 + \dots + s_n)} C_{K,\alpha} (1 + |s|)^{|\alpha|} ds \\ &\quad \le C_{K,\alpha} \int_{\mathbb{R}^n} e^{-\rho \sigma_n |s|} (1 + |s|)^{|\alpha|} ds < +\infty. \end{aligned}$$

Since we have

$$K_\alpha(a) = \int_{\substack{\sum s_j \ge 1 \\ s_j \ge 0}} e^{-(a_1 s_1 + \dots + a_n s_n)} \prod_{1 \le j \le n} (-1)^{\alpha_j} L_{\alpha_j}(2a_j s_j) ds \, a_1 \cdots a_n,$$

this proves the sought holomorphy.

Proof of Theorem 3.3.2. We have

$$\begin{aligned} \operatorname{Op}_{\mathrm{w}}(\mathbf{1}_{E(a)}) &= \operatorname{Op}_{\mathrm{w}} \left(F(x_1^2 + \xi_1^2, \dots, x_n^2 + \xi_n^2) \right) \\ &= \int_{\mathbb{R}^n} \widehat{F}(\tau) \operatorname{Op}_{\mathrm{w}} \left(e^{2i\pi \sum_j \tau_j (x_j^2 + \xi_j^2)} \right) d\tau \\ &= \sum_{\alpha \in \mathbb{N}^n} \int_{\mathbb{R}^n} \widehat{F}(\tau) \prod_{1 \le j \le n} \frac{(1 + i\tau_j)^{2\alpha_j + 1}}{(1 + \tau_j^2)^{\alpha_j + 1}} d\tau \mathbb{P}_\alpha \\ &= \sum_{\alpha \in \mathbb{N}^n} \int_{\mathbb{R}^n} \widehat{F}(\tau) \prod_{1 \le j \le n} \frac{(1 + i\tau_j)^{\alpha_j}}{(1 - i\tau_j)^{\alpha_j + 1}} d\tau \mathbb{P}_\alpha, \end{aligned}$$

where $\mathbb{P}_\alpha$ is defined in (A.1.19). On the other hand, we have

$$\begin{aligned} \widehat{F}(\tau) &= \int e^{-2i\pi\tau\cdot x} \mathbf{1}_{[-1,1]} \Big(\frac{2\pi}{a_1} x_1 + \dots + \frac{2\pi}{a_n} x_n \Big) dx_1 \cdots dx_n \\ &= a_1 \cdots a_n (2\pi)^{-n} \int e^{-i \sum_j \tau_j a_j y_j} \mathbf{1}_{[-1,1]} \Big(\sum y_j \Big) dy, \end{aligned}$$

so that, with M_k defined in (A.4.3), using (A.4.4), we get

$$
\begin{aligned}
&\mathrm{Op_w}(\mathbf{1}_{E(a)})\\
&= a_1 \cdots a_n \sum_{\alpha\in\mathbb{N}^n} \iint_{\mathbb{R}^n\times\mathbb{R}^n} e^{-i2\pi \sum_j \tau_j a_j y_j} \mathbf{1}_{[-1,1]}\big(\sum y_j\big) dy\\
&\quad\times \prod_{1\le j\le n} \frac{(1+i2\pi\tau_j)^{\alpha_j}}{(1-i2\pi\tau_j)^{\alpha_j+1}} d\tau \mathbb{P}_\alpha\\
&= a_1 \cdots a_n \sum_{\alpha\in\mathbb{N}^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i2\pi \sum_j \tau_j a_j y_j} \mathbf{1}_{[-1,1]}\big(\sum y_j\big) dy \prod_{1\le j\le n} \overline{\widehat{G}_{\alpha_j}(\tau_j)} d\tau \mathbb{P}_\alpha\\
&= a_1 \cdots a_n \sum_{\alpha\in\mathbb{N}^n} \int_{\mathbb{R}^n} \mathbf{1}_{[-1,1]}\big(\sum y_j\big) \prod_{1\le j\le n} G_{\alpha_j}(a_j y_j) dy \mathbb{P}_\alpha\\
&= \sum_{\alpha\in\mathbb{N}^n} \int_{\mathbb{R}^n} \mathbf{1}_{[-1,1]}\big(\sum t_j/a_j\big) \prod_{1\le j\le n} (-1)^{\alpha_j} H(t_j) e^{-t_j} L_{\alpha_j}(2t_j) dt \mathbb{P}_\alpha,
\end{aligned}
$$

with

$$
\begin{aligned}
F_\alpha(a) &= \int_{\mathbb{R}^n} \Big(1 - \mathbf{1}_{[1,+\infty]}\big(\sum t_j/a_j\big)\Big) \prod_{1\le j\le n} (-1)^{\alpha_j} H(t_j) e^{-t_j} L_{\alpha_j}(2t_j) dt\\
&= 1 - \int_{\mathbb{R}^n} \mathbf{1}_{[1,+\infty]}\big(\sum t_j/a_j\big) \prod_{1\le j\le n} (-1)^{\alpha_j} H(t_j) e^{-t_j} L_{\alpha_j}(2t_j) dt, \quad (3.3.5)
\end{aligned}
$$

where we have used that

$$
P_{k;1}(0) = 1 \quad \text{(cf. Lemma 3.1.1)},
$$

so that setting

$$
K_\alpha(a) = \int_{\substack{\sum t_j/a_j \ge 1\\ t_j\ge 0}} e^{-(t_1+\cdots+t_n)} \prod_{1\le j\le n} (-1)^{\alpha_j} L_{\alpha_j}(2t_j) dt,
$$

we have $F_\alpha(a) = 1 - K_\alpha(a)$, concluding the proof of the theorem. ∎

Remark 3.3.4. We have from (3.3.5)

$$
F_\alpha(a_1,\dots,a_n) = \int_{\mathbb{R}^n} \mathbf{1}_{[0,1]}\Big(\sum_{1\le j\le n} s_j\Big) \prod_{1\le j\le n} (-1)^{\alpha_j} H(s_j) e^{-a_j s_j} L_{\alpha_j}(2a_j s_j) a_j ds,
$$

and since the set

$$
\Big\{ s\in\mathbb{R}^n_+, \sum_{1\le j\le n} s_j \le 1 \Big\}
$$

is compact, we obtain that F_α is an entire function, as well as K_α which is indeed given by (3.3.3) on the open subset $\mathcal{U}$ defined in (3.3.4).

Lemma 3.3.5. *With the notations of Theorem 3.3.2, we have with $\mu_j = 1/a_j$,*

$$F_\alpha(a) = \Big(\prod_{1\le j\le n} a_j\Big)\int_{\mathbb{R}} \frac{\sin\tau}{\pi\tau}\Big(\prod_{1\le j\le n}\frac{(a_j+i\tau)^{\alpha_j}}{(a_j-i\tau)^{\alpha_j+1}}\Big)d\tau$$
$$= \int_{\mathbb{R}} \frac{\sin\tau}{\pi\tau}\Big(\prod_{1\le j\le n}\frac{(1+i\tau\mu_j)^{\alpha_j}}{(1-i\tau\mu_j)^{\alpha_j+1}}\Big)d\tau. \tag{3.3.6}$$

Proof. Mehler's formula implies in one dimension that

$$\mathrm{Op_w}(e^{2\pi i\tau(x^2+\xi^2)}) = (1+\tau^2)^{-1/2}\exp\big[2\pi i(\arctan\tau)(x^2+D_x^2)\big],$$

and a simple tensorisation gives

$$\mathrm{Op_w}(e^{2\pi i\tau\sum_j \mu_j(x_j^2+\xi_j^2)})$$
$$= \prod_j (1+(\tau\mu_j)^2)^{-1/2}\exp\Big[2\pi i\sum_j(\arctan(\tau\mu_j))(x_j^2+D_{x_j}^2)\Big],$$

so that we have

$$\mathrm{Op_w}\Big(F\big(\sum_j \mu_j(x_j^2+\xi_j^2)\big)\Big)$$
$$= \int_{\mathbb{R}} \widehat{F}(\tau)\mathrm{Op_w}\Big(e^{2\pi i\tau\sum_j\mu_j(x_j^2+\xi_j^2)}\Big)d\tau$$
$$= \int_{\mathbb{R}} \widehat{F}(\tau)\prod_j(1+(\tau\mu_j)^2)^{-1/2}\exp\Big[2\pi i\sum_j(\arctan(\tau\mu_j))(x_j^2+D_{x_j}^2)\Big]d\tau$$
$$= \sum_{\alpha\in\mathbb{N}^n}\int_{\mathbb{R}} \widehat{F}(\tau)\Big(\prod_j(1+(\tau\mu_j)^2)^{-1/2}\exp\Big[2i(\arctan(\tau\mu_j))\Big(\alpha_j+\frac{1}{2}\Big)\Big]\Big)d\tau\mathbb{P}_\alpha$$
$$= \sum_{\alpha\in\mathbb{N}^n}\int_{\mathbb{R}} \widehat{F}(\tau)\Big(\prod_j(1+(\tau\mu_j)^2)^{-1/2}\frac{(1+i\tau\mu_j)^{2\alpha_j+1}}{(1+(\tau\mu_j)^2)^{\alpha_j+\frac{1}{2}}}\Big)d\tau\mathbb{P}_\alpha$$
$$= \sum_{\alpha\in\mathbb{N}^n}\int_{\mathbb{R}} \widehat{F}(\tau)\Big(\prod_{1\le j\le n}\frac{(1+i\tau\mu_j)^{\alpha_j}}{(1-i\tau\mu_j)^{\alpha_j+1}}\Big)d\tau\mathbb{P}_\alpha,$$

and for $F(t) = \mathbf{1}_{[-1,1]}(2\pi t)$, we find $\widehat{F}(\tau) = \frac{\sin\tau}{\pi\tau}$ and the sought result. ∎

Remark 3.3.6. It is also possible to provide a direct checking for the above lemma, since with the notations (A.4.3), (A.4.4), we have

$$\frac{(1+i\tau\mu_j)^{\alpha_j}}{(1-i\tau\mu_j)^{\alpha_j+1}} = \widecheck{\widehat{G_{\alpha_j}}}(\tau\mu_j/(2\pi)),$$

and thus

$$\begin{aligned}F_\alpha(a) &= \int_{\mathbb{R}} \widehat{F}(\tau) \prod_j \widecheck{G_{\alpha_j}}(\tau\mu_j/(2\pi))d\tau \\ &= \int_{\mathbb{R}} \widehat{F}(\tau) \int_{\mathbb{R}^n} \prod_j (-1)^{\alpha_j} L_{\alpha_j}(2t_j)H(t_j)e^{-t_j} e^{2\pi i \tau \mu_j t_j/(2\pi)} dt d\tau \\ &= \int_{\mathbb{R}^n} \prod_j (-1)^{\alpha_j} L_{\alpha_j}(2t_j)H(t_j)e^{-t_j} F\Big(\sum_j \mu_j t_j/2\pi\Big)dt.\end{aligned}$$

Now, since we have

$$F\Big(\sum_j \mu_j t_j/2\pi\Big) = \mathbf{1}_{[-1,1]}\Big(\sum_j \mu_j t_j\Big),$$

this fits with the expression of F_α in Theorem 3.3.2.

Remark 3.3.7. Another interesting remark is that the expression (3.3.6) depends obviously only on $|\alpha|$ and $a = a_1 = \cdots = a_n$ in the case where all the a_j are equal: indeed, in that case, we have with $\mu = 1/a$,

$$\prod_{1\le j\le n} \frac{(1+i\tau\mu_j)^{\alpha_j}}{(1-i\tau\mu_j)^{\alpha_j+1}} = \frac{(1+i\tau\mu)^{|\alpha|}}{(1-i\tau\mu)^{|\alpha|+n}},$$

and this gives another (*a posteriori*) justification of our calculations in the isotropic case of Section 3.2. On the other hand, we get also the identity

$$F_{0_{\mathbb{N}^n}}(a_1,\dots,a_n) = \int_{\mathbb{R}} \frac{\sin\tau}{\pi\tau} \operatorname{Re}\Big(\prod_{1\le j\le n} (1-i\tau\mu_j)^{-1}\Big)d\tau,$$

where the explicit expression (3.3.7) is given for the left-hand side.

Lemma 3.3.8. *With the notations of Theorem 3.3.2, the function $K_{\alpha_1,\dots,\alpha_n}(a_1,\dots,a_n)$ is symmetric in the variables $(\alpha_1,a_1,\dots,\alpha_n,a_n)$, i.e., for a permutation π of $\{1,\dots,n\}$, we have*

$$K_{\alpha_{\pi(1)},\dots,\alpha_{\pi(n)}}(a_{\pi(1)},\dots,a_{\pi(n)}) = K_{\alpha_1,\dots,\alpha_n}(a_1,\dots,a_n).$$

Proof. Formula (3.3.3) yields

$$K_\alpha(a) = \int_{\substack{\sum s_j\ge 1\\ s_j\ge 0}} \prod_{1\le j\le n} \big(e^{-a_j s_j} a_j (-1)^{\alpha_j} L_{\alpha_j}(2a_j s_j)\big)ds,$$

and the domain of integration is invariant by permutation of the variables, entailing the sought result. ∎

Lemma 3.3.9. *With the notations of Theorem 3.3.2, we have*

$$
\begin{aligned}
&K_{\alpha_1,\ldots,\alpha_n}(a_1,\ldots,a_n) = e^{-a_n} P_{\alpha_n}(a_n)\\
&\quad + \int_0^{a_n} (-1)^{\alpha_n} L_{\alpha_n}(2t_n) e^{-t_n} K_{\alpha_1,\ldots,\alpha_{n-1}}\big(a_1(1-t_n/a_n),\ldots,a_{n-1}(1-t_n/a_n)\big)dt_n\\
&= e^{-a_n} P_{\alpha_n}(a_n)\\
&\quad + \int_0^{1} (-1)^{\alpha_n} L_{\alpha_n}(2a_n\theta) e^{-\theta a_n} K_{\alpha_1,\ldots,\alpha_{n-1}}\big(a_1(1-\theta),\ldots,a_{n-1}(1-\theta)\big)d\theta a_n.
\end{aligned}
$$

Proof. The domain of integration is the disjoint union

$$
\left\{\frac{t_1}{a_1}+\cdots+\frac{t_{n-1}}{a_{n-1}} \geq 1-\frac{t_n}{a_n}, t_j \geq 0, 0 \leq \frac{t_n}{a_n} \leq 1\right\} \sqcup \left\{\frac{t_n}{a_n} > 1, t_j \geq 0, 1 \leq j \leq n-1\right\},
$$

so that

$$
\begin{aligned}
&K_{\alpha_1,\ldots,\alpha_n}(a_1,\ldots,a_n) = e^{-a_n} P_{\alpha_n}(a_n)\\
&\quad + \int_0^{a_n} (-1)^{\alpha_n} L_{\alpha_n}(2t_n) e^{-t_n} K_{\alpha_1,\ldots,\alpha_{n-1}}\big(a_1(1-t_n/a_n),\ldots,a_{n-1}(1-t_n/a_n)\big)dt_n\\
&= e^{-a_n} P_{\alpha_n}(a_n)\\
&\quad + \int_0^{1} (-1)^{\alpha_n} L_{\alpha_n}(2a_n\theta) e^{-\theta a_n} K_{\alpha_1,\ldots,\alpha_{n-1}}\big(a_1(1-\theta),\ldots,a_{n-1}(1-\theta)\big)d\theta a_n,
\end{aligned}
$$

which is the sought result. ∎

Lemma 3.3.10. *With the notations of Theorem 3.3.2, we have, assuming that the $(a_j)_{1\leq j\leq n}$ are positive distinct numbers,*

$$
K_{0,\ldots,0}(a_1,\ldots,a_n) = \sum_{1\leq j\leq n} e^{-a_j} \frac{\prod_{k\neq j} a_k}{\prod_{k\neq j}(a_k-a_j)}. \tag{3.3.7}
$$

Proof. The latter formula is true for $n=1$ since we have

$$
K_0(a_1) = e^{-a_1}.
$$

We have also

$$
\begin{aligned}
K_{0\in\mathbb{N}^n}(a_1,\ldots,a_n) &= e^{-a_n} + a_n \int_0^1 e^{-\theta a_n} K_{0\in\mathbb{N}^{n-1}}\big(a_1(1-\theta),\ldots,a_{n-1}(1-\theta)\big)d\theta\\
&= e^{-a_n} + a_n \int_0^1 e^{-\theta a_n} \sum_{1\leq j\leq n-1} e^{-a_j(1-\theta)} \frac{\prod_{k\neq j} a_k}{\prod_{k\neq j}(a_k-a_j)} d\theta\\
&= e^{-a_n} + a_n \sum_{1\leq j\leq n-1} \frac{\prod_{k\neq j} a_k}{\prod_{k\neq j}(a_k-a_j)} \int_0^1 e^{-\theta a_n} e^{-a_j(1-\theta)} d\theta
\end{aligned}
$$

$$
\begin{aligned}
&= e^{-a_n} + \sum_{1\le j\le n-1} \frac{a_n \prod_{k\neq j} a_k}{\prod_{k\neq j}(a_k - a_j)} e^{-a_j} \int_0^1 e^{\theta(a_j - a_n)} d\theta \\
&= e^{-a_n} + \sum_{1\le j\le n-1} \frac{a_n \prod_{k\neq j} a_k}{\prod_{k\neq j}(a_k - a_j)} e^{-a_j} \frac{e^{a_j - a_n} - 1}{a_j - a_n} \\
&= e^{-a_n} + \sum_{1\le j\le n-1} \frac{a_n \prod_{k\neq j} a_k}{\prod_{k\neq j}(a_k - a_j)} \frac{e^{-a_n} - e^{-a_j}}{(a_j - a_n)} \\
&= e^{-a_n}\left(1 + \sum_{1\le j\le n-1} \frac{a_n \prod_{k\neq j} a_k}{\prod_{k\neq j}(a_k - a_j)} \frac{1}{(a_j - a_n)}\right) \\
&\quad + \sum_{1\le j\le n-1} \frac{a_n \prod_{k\neq j} a_k}{\prod_{k\neq j}(a_k - a_j)} \frac{e^{-a_j}}{(a_n - a_j)}.
\end{aligned}
$$

We need to prove that

$$
\left(1 + \sum_{1\le j\le n-1} \frac{a_n \prod_{k\neq j, 1\le k\le n-1} a_k}{\prod_{k\neq j, 1\le k\le n-1}(a_k - a_j)} \frac{1}{(a_j - a_n)}\right) = \frac{\prod_{1\le l\le n-1} a_l}{\prod_{1\le l\le n-1}(a_l - a_n)}.
$$

That is

$$
\prod_{1\le l\le n-1} a_l = \prod_{1\le l\le n-1}(a_l - a_n)\left(1 + \sum_{1\le j\le n-1} \frac{a_n \prod_{k\neq j, 1\le k\le n-1} a_k}{\prod_{k\neq j, 1\le k\le n-1}(a_k - a_j)} \frac{1}{(a_j - a_n)}\right),
$$

which is

$$
\begin{aligned}
\prod_{1\le l\le n-1} a_l &= \prod_{1\le l\le n-1}(a_l - a_n) \\
&\quad + \sum_{1\le j\le n-1} \frac{a_n \prod_{k\neq j, 1\le k\le n-1} a_k}{\prod_{k\neq j, 1\le k\le n-1}(a_k - a_j)} \frac{\prod_{1\le l\le n-1}(a_l - a_n)}{(a_j - a_n)},
\end{aligned}
$$

i.e.,

$$
\prod_{1\le l\le n-1} a_l = \prod_{1\le l\le n-1}(a_l - a_n) + \sum_{1\le j\le n-1} \frac{a_n \prod_{k\neq j, 1\le k\le n-1} a_k(a_k - a_n)}{\prod_{k\neq j, 1\le k\le n-1}(a_k - a_j)}. \tag{3.3.8}
$$

Let us reformulate (3.3.8) as an equality between polynomials (to be proven) with

$$
\prod_{1\le l\le n-1}(a_l - X) + \sum_{1\le j\le n-1} \frac{X \prod_{k\neq j, 1\le k\le n-1} a_k(a_k - X)}{\prod_{k\neq j, 1\le k\le n-1}(a_k - a_j)} - \prod_{1\le l\le n-1} a_l = 0, \tag{3.3.9}
$$

and let us assume that the $(a_j)_{1\le j\le n-1}$ are distinct and different from 0. The polynomial $\mathcal{Q}$ on the left-hand side has degree less than $n-1$ and we have

$$\mathcal{Q}(0) = 0 \quad \forall j \in [\![1, n-1]\!],$$
$$\mathcal{Q}(a_j) = \frac{a_j \prod_{k\ne j, 1\le k\le n-1} a_k(a_k - a_j)}{\prod_{k\ne j,1\le k\le n-1}(a_k - a_j)} - \prod_{1\le l\le n-1} a_l = 0,$$

so that $\mathcal{Q}$ has degree less than $n-1$ with n distinct roots and this proves the identity (3.3.9) when the $(a_j)_{1\le j\le n-1}$ are distinct and all different from 0, proving (3.3.7) in that case; of course we may assume that all a_j are positive and noting from (3.3.3) that K_α is continuous on $(\mathbb{R}_+^*)^n$, we get formula (3.3.7) in all cases where all the a_j are positive, concluding the proof of the lemma. ∎

Lemma 3.3.11. *With the notations of Theorem 3.3.2, we have, assuming $0 < a_1 \le \cdots \le a_n$, the inequality*

$$K_{0\in\mathbb{N}^n}(a_1,\dots,a_n) \ge \sum_{1\le j\le n} e^{-a_j}\frac{\prod_{1\le l<j} a_l}{(j-1)!} \ge e^{-\min_{1\le j\le n} a_j} = \max_{1\le j\le n} e^{-a_j}.$$

Remark 3.3.12. The above estimate is sharp in the sense that when all the a_j are equal to the same $a > 0$, we have proven in (3.2.1) that

$$\begin{aligned}
K_0(a) &= \frac{e^{-a}}{(n-1)!}\int_0^{+\infty} e^{-s}(s+a)^{n-1}ds\\
&= e^{-a}\sum_{0\le l\le n-1}\frac{a^l}{(n-1-l)!l!}\Gamma(n-l)\\
&= e^{-a}\sum_{0\le l\le n-1}\frac{a^l}{l!} = e^{-a}\sum_{1\le j\le n}\frac{a^{j-1}}{(j-1)!}\\
&= \sum_{1\le j\le n} e^{-a_j}\frac{\prod_{1\le l<j} a_l}{(j-1)!}\Big|_{a_1=\cdots=a_n=a}.
\end{aligned}$$

Proof. The property is true for $n=1$ since $K_0(a_1) = e^{-a_1}$. We check the case $n=2$ with $a_1 < a_2$, and we find

$$\begin{aligned}
K_{(0,0)}(a_1,a_2) &= e^{-a_1} + \int_0^{a_1} e^{-t_1}e^{-a_2(1-t_1/a_1)}dt_1\\
&= e^{-a_1} + e^{-a_2}\frac{e^{a_2-a_1}-1}{\frac{a_2}{a_1}-1} = e^{-a_1} + e^{-a_2}a_1\frac{e^{a_2-a_1}-1}{a_2-a_1}\\
&\ge e^{-a_1} + e^{-a_2}a_1.
\end{aligned}$$

Let us consider for some $n \geq 3$, $0 < a_1 < \cdots < a_n$ and inductively,

$$
\begin{aligned}
&K_{0\in\mathbb{N}^n}(a_1,\ldots,a_n)\\
&\quad= e^{-a_1}P_0(a_1) + \int_0^{a_1} e^{-t_1} K_{0\in\mathbb{N}^{n-1}}\big(a_2(1-t_1/a_1),\ldots,a_n(1-t_1/a_1)\big)dt_1\\
&\quad= e^{-a_1}P_0(a_1) + a_1\int_0^{1} e^{-a_1\theta} K_{0\in\mathbb{N}^{n-1}}\big(a_2(1-\theta),\ldots,a_n(1-\theta)\big)d\theta\\
&\quad\geq e^{-a_1} + a_1\int_0^1 e^{-a_1\theta}\sum_{2\leq j\leq n} e^{-a_j(1-\theta)}\frac{\prod_{2\leq l<j} a_l}{(j-2)!}(1-\theta)^{j-2}d\theta\\
&\quad= e^{-a_1} + \sum_{2\leq j\leq n} e^{-a_j}\underbrace{\Big(a_1\prod_{2\leq l<j} a_l\Big)}_{\prod_{1\leq k<j} a_k}\int_0^1 e^{(a_j-a_1)\theta}\frac{1}{(j-2)!}(1-\theta)^{j-2}d\theta\\
&\quad\geq e^{-a_1} + \sum_{2\leq j\leq n} e^{-a_j}\Big(\prod_{1\leq k<j} a_k\Big)\int_0^1 \frac{1}{(j-2)!}(1-\theta)^{j-2}d\theta\\
&\quad= e^{-a_1} + \sum_{2\leq j\leq n} e^{-a_j}\Big(\prod_{1\leq k<j} a_k\Big)\frac{1}{(j-1)!},
\end{aligned}
$$

concluding the proof of the lemma. ■

Remark 3.3.13. The reader may have noticed that it is not obvious on formula (3.3.7)

$$K_{0,\ldots,0}(a_1,\ldots,a_n) = \sum_{1\leq j\leq n} e^{-a_j}\frac{\prod_{k\neq j} a_k}{\prod_{k\neq j}(a_k-a_j)},$$

that K_0 is an entire function. Let us start with taking a look at

$$
\begin{aligned}
&K_{0,0}(a_1,a_2)\\
&= \frac{e^{-a_1}a_2}{a_2-a_1} + \frac{e^{-a_2}a_1}{a_1-a_2} = \frac{a_2e^{-a_1}-a_1e^{-a_2}}{a_2-a_1}\\
&= e^{-\frac{(a_1+a_2)}{2}}\frac{a_2e^{-\frac{a_1}{2}+\frac{a_2}{2}}-a_1e^{-\frac{a_2}{2}+\frac{a_1}{2}}}{a_2-a_1}\\
&= e^{-\frac{(a_1+a_2)}{2}}\frac{a_2(\cosh\frac{a_2-a_1}{2}+\sinh\frac{a_2-a_1}{2})-a_1(\cosh\frac{a_1-a_2}{2}+\sinh\frac{a_1-a_2}{2})}{a_2-a_1}\\
&= e^{-\frac{(a_1+a_2)}{2}}\Big[\cosh\Big(\frac{a_2-a_1}{2}\Big)+\frac{(a_2+a_1)\sinh(\frac{a_2-a_1}{2})}{a_2-a_1}\Big]\\
&= e^{-\frac{(a_1+a_2)}{2}}\Big[\cosh\Big(\frac{a_2-a_1}{2}\Big)+\frac{\frac{1}{2}(a_2+a_1)\sinh(\frac{a_2-a_1}{2})}{\frac{a_2-a_1}{2}}\Big]\\
&= e^{-\frac{(a_1+a_2)}{2}}\Big[\cosh\Big(\frac{a_2-a_1}{2}\Big)+\frac{1}{2}(a_2+a_1)\,\mathrm{shc}\Big(\frac{a_2-a_1}{2}\Big)\Big], \qquad (3.3.10)
\end{aligned}
$$

where shc stands for the even entire function defined by

$$\operatorname{shc} t = \frac{\sinh t}{t}.$$

We have also from Lemma 3.3.5

$$F_\alpha(a) = \int_{\mathbb{R}} \frac{\sin\tau}{\pi\tau}\Big(\prod_{1\le j\le n} \frac{(1+i\tau\mu_j)^{\alpha_j}}{(1-i\tau\mu_j)^{\alpha_j+1}}\Big)d\tau,$$

and defining the function $F_\alpha(a,\lambda)$ as the absolutely converging integral,

$$F_\alpha(a,\lambda) = \int_{\mathbb{R}} \frac{\sin(\lambda\tau)}{\pi\tau}\Big(\prod_{1\le j\le n} \frac{(1+i\tau\mu_j)^{\alpha_j}}{(1-i\tau\mu_j)^{\alpha_j+1}}\Big)d\tau, \quad F_\alpha(a) = F_\alpha(a,1),$$

we get

$$\begin{aligned}
\frac{\partial F_\alpha}{\partial\lambda}(a,\lambda) &= \frac{1}{\pi}\int_{\mathbb{R}} \cos(\lambda\tau)\Big(\prod_{1\le j\le n} \frac{(1+i\tau\mu_j)^{\alpha_j}}{(1-i\tau\mu_j)^{\alpha_j+1}}\Big)d\tau \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda\tau}\Big(\prod_{1\le j\le n} \frac{(1+i\tau\mu_j)^{\alpha_j}}{(1-i\tau\mu_j)^{\alpha_j+1}}\Big)d\tau \\
&\quad + \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda\tau}\Big(\prod_{1\le j\le n} \frac{(1-i\tau\mu_j)^{\alpha_j}}{(1+i\tau\mu_j)^{\alpha_j+1}}\Big)d\tau \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda\tau}\Big(\prod_{1\le j\le n} \frac{(1+i\tau\mu_j)^{\alpha_j}}{(1-i\tau\mu_j)^{\alpha_j+1}} + \prod_{1\le j\le n} \frac{(1-i\tau\mu_j)^{\alpha_j}}{(1+i\tau\mu_j)^{\alpha_j+1}}\Big)d\tau \\
&= i\sum_{1\le j\le n} \operatorname{Res}\Big(e^{i\lambda\tau}\prod_{1\le j\le n} \frac{(1-i\tau\mu_j)^{\alpha_j}}{(1+i\tau\mu_j)^{\alpha_j+1}}; \tau = i/\mu_j = ia_j\Big) \\
&= i\sum_{1\le j\le n} \operatorname{Res}\Big(e^{i\lambda\tau}\prod_{1\le j\le n} \frac{(-i\mu_j)^{\alpha_j}(ia_j+\tau)^{\alpha_j}}{(i\mu_j)^{\alpha_j+1}(-ia_j+\tau)^{\alpha_j+1}}; \tau = ia_j\Big) \\
&= \frac{1}{i^{n-1}}\sum_{1\le j\le n} \operatorname{Res}\Big(e^{i\lambda\tau}\prod_{1\le j\le n} (-1)^{\alpha_j}\frac{a_j(ia_j+\tau)^{\alpha_j}}{(\tau-ia_j)^{\alpha_j+1}}; \tau = ia_j\Big),
\end{aligned}$$

so that assuming that the a_j are positive and distinct, we get

$$\begin{aligned}
\frac{\partial F_\alpha}{\partial\lambda}(a,\lambda) &= \frac{1}{i^{n-1}}\Big(\prod_{1\le k\le n} a_k\Big)\sum_{1\le j\le n}\frac{1}{\alpha_j!} \\
&\times\Big(\frac{d}{d\tau}\Big)^{\alpha_j}\Big(e^{i\lambda\tau}(-1)^{\alpha_j}(ia_j+\tau)^{\alpha_j}\prod_{1\le k\le n, k\ne j}(-1)^{\alpha_k}\frac{(ia_k+\tau)^{\alpha_k}}{(\tau-ia_k)^{\alpha_k+1}}\Big)_{|\tau=ia_j}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{i^{n-1}}\Big(\prod_{1\le k\le n} a_k\Big) \sum_{1\le j\le n} \frac{1}{\alpha_j!}\\
&\times \Big(\frac{d}{i d\sigma}\Big)^{\alpha_j}\Big(e^{-\lambda\sigma}(-1)^{\alpha_j}(i a_j + i\sigma)^{\alpha_j} \prod_{1\le k\le n, k\ne j} (-1)^{\alpha_k}\frac{(i a_k + i\sigma)^{\alpha_k}}{(i\sigma - i a_k)^{\alpha_k+1}}\Big)_{|\sigma=a_j}\\
&= (-1)^{n-1+|\alpha|}\Big(\prod_{1\le k\le n} a_k\Big) \sum_{1\le j\le n} \frac{1}{\alpha_j!}\\
&\times \Big(\frac{d}{d\sigma}\Big)^{\alpha_j}\Big(e^{-\lambda\sigma}(a_j + \sigma)^{\alpha_j} \prod_{1\le k\le n, k\ne j} \frac{(a_k + \sigma)^{\alpha_k}}{(\sigma - a_k)^{\alpha_k+1}}\Big)_{|\sigma=a_j}\\
&= \Big(\prod_{1\le k\le n} a_k\Big) \sum_{1\le j\le n} \frac{(-1)^{\alpha_j}}{\alpha_j!}\\
&\times \Big(\frac{d}{d\sigma}\Big)^{\alpha_j}\Big(e^{-\lambda\sigma}(a_j + \sigma)^{\alpha_j} \prod_{1\le k\le n, k\ne j} \frac{(a_k + \sigma)^{\alpha_k}}{(a_k - \sigma)^{\alpha_k+1}}\Big)_{|\sigma=a_j}.
\end{aligned}
$$

Since $F_\alpha(a, +\infty) = 1$, thanks to Lemma A.1.7, we find eventually that

$$
\begin{aligned}
&F_\alpha(a) = F_\alpha(a, 1) = \int_{+\infty}^{1} \frac{\partial F_\alpha}{\partial \lambda}(a, \lambda) d\lambda + 1 = 1 - K_\alpha(a),\\
&K_\alpha(a) = \Big(\prod_{1\le k\le n} a_k\Big) \sum_{1\le j\le n} \frac{(-1)^{\alpha_j}}{\alpha_j!}\\
&\times \int_1^{+\infty} \Big(\frac{d}{d\sigma}\Big)^{\alpha_j}\Big(e^{-\lambda\sigma}(a_j + \sigma)^{\alpha_j} \prod_{1\le k\le n, k\ne j} \frac{(a_k + \sigma)^{\alpha_k}}{(a_k - \sigma)^{\alpha_k+1}}\Big)_{|\sigma=a_j} d\lambda\\
&= \sum_{1\le j\le n} \frac{(-1)^{\alpha_j}}{\alpha_j!}\\
&\times \int_1^{+\infty} e^{-\lambda a_j}\Big(\frac{d}{d\sigma} - \lambda\Big)^{\alpha_j}\Big((a_j + \sigma)^{\alpha_j} a_j \prod_{1\le k\le n, k\ne j} \frac{(a_k + \sigma)^{\alpha_k} a_k}{(a_k - \sigma)^{\alpha_k+1}}\Big)_{|\sigma=a_j} d\lambda\\
&= \sum_{1\le j\le n} \frac{(-1)^{\alpha_j}}{\alpha_j!}\\
&\times \int_1^{+\infty} e^{-\lambda a_j}\Big(\frac{d}{d\sigma} - \lambda\Big)^{\alpha_j}\Big((a_j + \sigma)^{\alpha_j} \prod_{1\le k\le n, k\ne j} \frac{(a_k + \sigma)^{\alpha_k}}{(a_k - \sigma)^{\alpha_k+1}}\Big)_{|\sigma=a_j} d\lambda\\
&= \sum_{1\le j\le n} \frac{(-1)^{\alpha_j}}{\alpha_j!}\\
&\times \int_{a_j}^{+\infty} e^{-t_j}\Big(\frac{d}{d a_j s} - \frac{t_j}{a_j}\Big)^{\alpha_j}\Big((a_j + a_j s)^{\alpha_j} \prod_{1\le k\le n, k\ne j} \frac{a_k(a_k + a_j s)^{\alpha_k}}{(a_k - a_j s)^{\alpha_k+1}}\Big)_{|s=1} dt_j
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{1\le j\le n} \frac{(-1)^{\alpha_j}}{\alpha_j!} \\
&\times \int_{a_j}^{+\infty} e^{-t}\Big(\frac{d}{ds}-t\Big)^{\alpha_j}\Big((1+s)^{\alpha_j} \prod_{1\le k\le n, k\neq j} \frac{a_k(a_k+a_js)^{\alpha_k}}{(a_k-a_js)^{\alpha_k+1}}\Big)_{|s=1} dt \\
&= \sum_{1\le j\le n} \frac{(-1)^{\alpha_j}}{\alpha_j!} \\
&\times \int_{a_j}^{+\infty} e^{-t}\Big(\frac{d}{ds}-1\Big)^{\alpha_j}\Big((t+s)^{\alpha_j} \prod_{1\le k\le n, k\neq j} \frac{a_k(a_k+a_js/t)^{\alpha_k}}{(a_k-a_js/t)^{\alpha_k+1}}\Big)_{|s=t} dt \\
&= \sum_{1\le j\le n} \frac{(-1)^{\alpha_j}}{\alpha_j!} \int_{a_j}^{+\infty} e^{-t} \\
&\times \Big(\frac{d}{d(s+t)}-1\Big)^{\alpha_j}\Big((t+s)^{\alpha_j} \prod_{1\le k\le n, k\neq j} \frac{ta_k(t(a_k-a_j)+a_j(s+t))^{\alpha_k}}{(t(a_k+a_j)-a_j(s+t))^{\alpha_k+1}}\Big)_{|s+t=2t} dt \\
&= \sum_{1\le j\le n} (-1)^{\alpha_j} \int_{a_j}^{+\infty} e^{-t} \\
&\times \Big(\frac{d}{ds}-1\Big)^{\alpha_j}\Big(\frac{s^{\alpha_j}}{\alpha_j!} \prod_{1\le k\le n, k\neq j} \frac{ta_k(t(a_k-a_j)+a_js)^{\alpha_k}}{(t(a_k+a_j)-a_js)^{\alpha_k+1}}\Big)_{|s=2t} dt \\
&= \sum_{1\le j\le n} (-1)^{\alpha_j} e^{-a_j} \int_{0}^{+\infty} e^{-t} \\
&\times \Big(\frac{d}{ds}-1\Big)^{\alpha_j}\Big(\frac{s^{\alpha_j}}{\alpha_j!} \prod_{1\le k\le n, k\neq j} \frac{(t+a_j)a_k\big((t+a_j)(a_k-a_j)+a_js\big)^{\alpha_k}}{\big((t+a_j)(a_k+a_j)-a_js\big)^{\alpha_k+1}}\Big)_{|s=2t+2a_j} dt.
\end{aligned}
$$

We have also to deal with

$$
\prod_{1\le k\le n, k\neq j} \frac{(t+a_j)a_k\big((t+a_j)(a_k-a_j)+a_js\big)^{\alpha_k}}{\big((t+a_j)(a_k+a_j)-a_js\big)^{\alpha_k+1}}
$$

and

$$
\begin{gathered}
\big((t+a_j)(a_k+a_j)-a_j(2t+2a_j)\big) = a_j(a_k+a_j)-2a_j^2+t(a_k-a_j) = (t+a_j)(a_k-a_j) \\
(t+a_j)(a_k+a_j)-a_js = (t+a_j)(a_k-a_j)+a_j(2t+2a_j-s)
\end{gathered}
$$

so that

$$
\begin{aligned}
K_\alpha(a) &= \sum_{1\le j\le n} (-1)^{\alpha_j} e^{-a_j} \int_0^{+\infty} e^{-t} \\
&\times \Big(\frac{d}{ds}-1\Big)^{\alpha_j}\Big(\frac{s^{\alpha_j}}{\alpha_j!} \prod_{1\le k\le n, k\neq j} \frac{(t+a_j)a_k((t+a_j)(a_k+a_j)+a_j(s-2t-2a_j))^{\alpha_k}}{((t+a_j)(a_k-a_j)-a_j(s-2t-2a_j))^{\alpha_k+1}}\Big)_{|s=2t+2a_j} dt.
\end{aligned}
\tag{3.3.11}
$$

3.4 A conjecture on integrals of products of Laguerre polynomials

We formulate in this section a conjecture on the behaviour of the functions $K_\alpha(a)$; as displayed in the previous sections, we know several useful elements for the analysis of these functions, including some quite explicit expression. However, in the non-isotropic case, we were not able to prove the estimate $F_\alpha(a) \le 1$, equivalent to $K_\alpha(a) \ge 0$, except for the case $\alpha = 0$. We are thus reduced to conjectural statements.

Conjecture 3.4.1. Let $n \ge 1$ be an integer and let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$. For $a = (a_1, \dots, a_n) \in (0, +\infty)^n$, we define

$$K_\alpha(a) = \int_{\substack{t=(t_1,\dots,t_n)\in\mathbb{R}_+^n \\ \sum_{1\le j\le n} t_j/a_j \ge 1}} e^{-(t_1+\dots+t_n)} \prod_{1\le j\le n} (-1)^{\alpha_j} L_{\alpha_j}(2t_j) dt,$$

where L_k stands for the classical Laguerre polynomial

$$L_k(X) = \left(\frac{d}{dX} - 1\right)^k \frac{X^k}{k!}.$$

Then, we conjecture that, assuming $0 < a_1 \le \dots \le a_n$, we have

$$K_\alpha(a) \ge \sum_{1\le j\le n} e^{-a_j} \frac{\prod_{1\le l<j} a_l}{(j-1)!}. \tag{3.4.1}$$

Remark 3.4.2. A slightly stronger and more symmetrical version of the above conjecture is that for n, α, a, K_α as above, we have

$$K_\alpha(a) \ge K_0(a). \tag{3.4.2}$$

It is indeed stronger since we have proven in Lemma 3.3.11 that $K_0(a)$ is greater than the right-hand side of (3.4.1).

Theorem 3.4.3. *The previous conjecture is a proven theorem in the following cases.*

(1) *When* $n = 1$.
(2) *For all* $n \ge 1$, *when all the* a_j *are equal.*
(3) *For all* $n \ge 1$, *when* $\alpha = 0_{\mathbb{N}^n}$.
(4) *When* $n = 2$ *and* $\min(\alpha_1, \alpha_2) = 0$.

Proof. (1) When $n = 1$, we have proven above (in Proposition 3.1.3) that for $\alpha \in \mathbb{N}$, $a > 0$,

$$K_\alpha(a) = e^{-a} P_\alpha(a) \ge e^{-a},$$

which is indeed (3.4.2) in that case. With the notations of Theorem 3.1.5 (and in particular where D_a is defined in (3.1.11)) this implies

$$\operatorname{Op_w}(\mathbf{1}_{D_a}) \le 1 - e^{-a},$$

an inequality due to P. Flandrin in 1988 paper [13].

(2) Assuming that all the a_j are equal to $a > 0$, we have proven in Theorem 3.2.2 that for $\alpha \in \mathbb{N}^n$, $|\alpha| = \sum_{1\le j\le n} \alpha_j$,

$$K_\alpha(a,\dots,a) \ge \frac{\Gamma(n,a)}{\Gamma(n)} = e^{-a} \sum_{1\le j\le n} \frac{a^{j-1}}{(j-1)!} = K_0(a,\dots,a),$$

since from (3.3.3), we have

$$\begin{aligned} K_0(a,\dots,a) &= \int_{\substack{\sum t_j \ge a\\ t_j\ge 0}} e^{-(t_1+\cdots+t_n)} dt \\ &= \int_{\substack{t_n\ge a\\ t_j\ge 0}} e^{-(t_1+\cdots+t_n)} dt + \int_0^a e^{-t_n} \int_{\sum t_j \ge a-t_n} e^{-(t_1+\cdots+t_{n-1})} dt \\ \text{(inductively)} \quad &= e^{-a} + \int_0^a e^{-t_n} e^{-(a-t_n)} \sum_{1\le j\le n-1} \frac{(a-t_n)^{j-1}}{(j-1)!} dt_n \\ &= e^{-a}\left(1 + \sum_{1\le j\le n-1} \frac{a^j}{j!}\right) = e^{-a} \sum_{1\le j\le n} \frac{a^{j-1}}{(j-1)!}, \end{aligned}$$

proving (3.4.2) in that case. With

$$D_{(a)} = \left\{(x,\xi) \in \mathbb{R}^{2n}, 2\pi \frac{|x|^2+|\xi|^2}{a} \le 1\right\},$$

this implies that

$$\mathrm{Op_w}(\mathbf{1}_{D_{(a)}}) \le 1 - e^{-a} \sum_{1\le j\le n} \frac{a^{j-1}}{(j-1)!},$$

an inequality proven in the 2010 article [39] by E. Lieb and Y. Ostrover.

(3) When $\alpha = 0_{\mathbb{N}^n}$, we have proven (3.4.1) in Lemma 3.3.11.

(4) When $n = 2$, from the case $n = 1$ we have $K_{\alpha_2}(a_2) = e^{-a_2} P_{\alpha_2}(a_2)$, so that from Lemma 3.3.9, we obtain

$$\begin{aligned} &K_{\alpha_1,\alpha_2}(a_1,a_2) \\ &\quad = e^{-a_1} P_{\alpha_1}(a_1) + a_1 \int_0^1 e^{-\theta a_1 - (1-\theta) a_2} (-1)^{\alpha_1} L_{\alpha_1}(2\theta a_1) P_{\alpha_2}(a_2(1-\theta)) d\theta, \end{aligned}$$

and if α_1=0, it means that

$$\begin{aligned} K_{0,\alpha_2}(a_1,a_2) &= e^{-a_1} + a_1 \int_0^1 e^{-\theta a_1 - (1-\theta) a_2} P_{\alpha_2}(a_2(1-\theta)) d\theta \\ &\ge e^{-a_1} + a_1 \int_0^1 e^{-\theta a_1 - (1-\theta) a_2} d\theta = K_{0,0}(a_1,a_2), \end{aligned}$$

and the reasoning is identical for $\alpha_2 = 0$, concluding the proof of the theorem. ∎

We are interested in the Weyl quantization of the indicatrix of

$$D_{a_1,\dots,a_n} = \Big\{(x,\xi) \in \mathbb{R}^{2n}, 2\pi \sum_{1\le j\le n} \frac{x_j^2+\xi_j^2}{a_j} \le 1\Big\}, \quad a_j > 0,$$

and we have a weaker conjecture.

Conjecture 3.4.4 (A weak form of Conjecture 3.4.1). With n, α, a, K_α as in Conjecture 3.4.1, we conjecture that

$$K_\alpha(a) \ge 0. \tag{3.4.3}$$

Note that inequality (3.4.3) is equivalent to

$$\mathrm{Op_w}(\mathbf{1}_{D_{a_1,\dots,a_n}}) \le 1.$$

Remark 3.4.5. In the first place, although the second conjecture is much weaker than the first, there is no reason to believe that the weak conjecture should be easier to prove than the first: in particular, in the known cases, it is indeed the proof of the precise statement (3.4.1) which leads to (3.4.3) and we are not aware of a direct proof of (3.4.3), even in one dimension.

A summary of our knowledge on the functions K_α. As proven in Remarks 3.3.3 and 3.3.4, the functions K_α are entire functions given on the open subset (3.3.4) by formula (3.3.3) (see also formula (3.3.10)). Moreover, the function $F_\alpha(a) = 1 - K_\alpha(a)$ can be expressed as a simple integral for $a_j > 0$,

$$F_\alpha(a_1,\dots,a_n) = \int_{\mathbb{R}} \frac{\sin\tau}{\pi\tau}\Bigg(\prod_{1\le j\le n} \frac{(1+i\tau\mu_j)^{\alpha_j}}{(1-i\tau\mu_j)^{\alpha_j+1}}\Bigg) d\tau, \quad \mu_j = \frac{1}{a_j},$$

and we have an explicit expression of the function K_α as a sum of simple integrals in (3.3.11). However, having an explicit expression does not mean much and for instance, we do have several explicit expressions for the Laguerre polynomials but inequality (A.4.2) remains very hard work, requiring a deep understanding of these polynomials. We have also an induction formula in Lemma 3.3.9. As a further remark, we have the following

Lemma 3.4.6. *Let n, α, a, K_α as in Conjecture 3.4.1. Then, we have*

$$\lim_{a_n\to+\infty} K_{\alpha_1,\dots,\alpha_{n-1},\alpha_n}(a_1,\dots,a_{n-1},a_n) = K_{\alpha_1,\dots,\alpha_{n-1}}(a_1,\dots,a_{n-1}), \tag{3.4.4}$$

$$\lim_{a_1\to 0_+} K_{\alpha_1,\alpha_2,\dots,\alpha_n}(a_1,a_2,\dots,a_n) = 1. \tag{3.4.5}$$

Proof. Formula (3.3.3) and the Lebesgue dominated convergence theorem imply the first equality (3.4.4). Lemma 3.3.9, in which we may swap the variables a_1 and a_n

gives for $a_1 > 0$

$$K_{\alpha_1,\alpha_2,\dots,\alpha_n}(a_1, a_2, \dots, a_n) = e^{-a_1} P_{\alpha_1}(a_1) \\ + a_1 \int_0^1 e^{-\theta a_1} (-1)^{\alpha_1} L_{\alpha_1}(2a_1\theta) K_{\alpha_2,\dots,\alpha_n}\big(a_2(1-\theta), \dots, a_n(1-\theta)\big) d\theta,$$

and since P_{α_1} is a polynomial such that $P_{\alpha_1}(0) = 1$, we get (3.4.5). ■

Reasons to believe in the conjecture. This is true in one dimension, also in n dimensions for spheres and it is a quadratic problem in the sense that ellipsoids are convex subsets of $\mathbb{R}^{2n}$ characterized by an inequality

$$\big\{X \in \mathbb{R}^{2n}, p(X) \le 0\big\},$$

where p is a polynomial of degree 2 with a positive-definite quadratic part. We shall see below in this memoir that convexity of a set A does not guarantee that the quantization $\operatorname{Op_w}(\mathbf{1}_A)$ is smaller than 1 as an operator and that Flandrin's conjecture is not true, but it is hard to believe that such a phenomenon could occur for ellipsoids. We must point out a specific feature of anisotropy related to Mehler's formula (2.2.1): if all the μ_j are equal to the same $\mu > 0$ (this is the isotropic case), then, with $q_\mu(x,\xi) = \mu(|x|^2 + |\xi|^2)$, we have

$$\operatorname{Op_w}(e^{2i\pi\tau q_\mu(x,\xi)}) = \phi(\tau\mu) e^{2i\arctan(\tau\mu)\sum_{1\le j\le n}\pi(x_j^2 + D_j^2)},$$

where $\phi(\tau\mu)$ is a scalar quantity. As a consequence, if we quantize $F(q_\mu(x,\xi))$, we get

$$\operatorname{Op_w}\big(F\big(q_\mu(x,\xi)\big)\big) = \int_{\mathbb{R}} \widehat{F}(\tau)\phi(\tau\mu) e^{2i\frac{\arctan(\tau\mu)}{\mu}\pi \operatorname{Op_w}(q_\mu)} d\tau,$$

and thus

$$\operatorname{Op_w}\big(F(q_\mu(x,\xi))\big) = \widetilde{F}(\operatorname{Op_w}(q_\mu)), \quad \widetilde{F}(\lambda) = \int_{\mathbb{R}} \widehat{F}(\tau)\phi(\tau\mu) e^{2i\pi\frac{\arctan(\tau\mu)}{\mu}\lambda} d\tau,$$

and $\operatorname{Op_w}\big(F(q_\mu(x,\xi))\big)$ appears as a function of the self-adjoint operator $\operatorname{Op_w}(q_\mu)$. Following the same route in the anisotropic case, we get, with

$$q_\mu(x,\xi) = \sum_{1\le j\le n} \mu_j(x_j^2 + \xi_j^2),$$

$$\operatorname{Op_w}\big(F(q_\mu(x,\xi))\big) = \int_{\mathbb{R}} \widehat{F}(\tau)\phi(\tau\mu) e^{2i\pi\sum_{1\le j\le n}\left(\frac{\arctan(\tau\mu_j)}{\mu_j}\right)\mu_j(x_j^2 + D_j^2)} d\tau,$$

and since $\frac{1}{\mu_j}\arctan(\tau\mu_j)$ does depend on μ_j (and not only on τ), the operator $\operatorname{Op_w}$ $(F(q_\mu(x,\xi)))$ is not a function of the self-adjoint operator $\operatorname{Op_w}(q_\mu)$.

As a final comment on the strongest form of the Conjecture (3.4.2), we would say that it could be seen as a property of the Laguerre polynomials, known in the case $n = 1$, where it stands as follows: we define for $k \in \mathbb{N}$, the polynomial P_k by

$$P_k(x) = \int_0^{+\infty} e^{-t}(-1)^k L_k(2x + 2t)dt,$$

and we have $P_k(0) = 1$ from (A.4.4). Moreover, we have the inequality (equivalent to (3.4.2) for $n = 1$)

$$\forall x \geq 0, \quad P_k(x) \geq P_k(0). \tag{3.4.6}$$

We note that $e^{-x}P_k(x) = \int_x^{+\infty} e^{-s}(-1)^k L_k(2s)ds$, so that the unique solution P_k of the Initial Value Problem for the ODE

$$P_k(x) - P_k'(x) = (-1)^k L_k(2x), \quad P_k(0) = 1,$$

does satisfy (3.4.6). We note that from Lemma 3.1.2, we have

$$P_k'(X) = 2 \sum_{0 \leq l < k} (-1)^l L_l(2X),$$

so that (3.4.6) is a consequence of Feldheim inequality (A.4.2). Let us reformulate (3.4.2), using the polynomials P_k: for $a_j \geq 0$,

$$\begin{aligned} K_\alpha(a) &= \int_{\substack{t=(t_1,\dots,t_n)\in\mathbb{R}_+^n \\ \sum_{1\leq j\leq n} t_j/a_j \geq 1}} \prod_{1\leq j\leq n} \frac{\partial}{\partial t_j}\{-e^{-t_j}P_{\alpha_j}(t_j)\}dt \\ &\geq K_0(a) = \int_{\substack{t=(t_1,\dots,t_n)\in\mathbb{R}_+^n \\ \sum_{1\leq j\leq n} t_j/a_j \geq 1}} \prod_{1\leq j\leq n} \frac{\partial}{\partial t_j}\{-e^{-t_j}\}dt, \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\int H\Big(1 - \sum_{1\leq j\leq n} s_j\Big) \prod_{1\leq j\leq n} H(s_j)\frac{\partial}{\partial s_j}\{-e^{-a_j s_j}P_{\alpha_j}(a_j s_j)\}ds \\ &\leq \int H\Big(1 - \sum_{1\leq j\leq n} s_j\Big) \prod_{1\leq j\leq n} a_j H(s_j)e^{-a_j s_j}ds, \end{aligned}$$

where $H = \mathbf{1}_{\mathbb{R}_+}$ (Heaviside function). This is equivalent to

$$\begin{aligned} &\int H\Big(1 - \sum_{1\leq j\leq n} s_j\Big) \prod_{1\leq j\leq n} H(s_j)e^{-a_j s_j}\Big(a_j - \frac{\partial}{\partial s_j}\Big)\{P_{\alpha_j}(a_j s_j)\}ds \\ &\leq \int H\big(1 - \sum_{1\leq j\leq n} s_j\big) \prod_{1\leq j\leq n} a_j H(s_j)e^{-a_j s_j}ds, \end{aligned}$$

i.e., to

$$\int H\Big(1-\sum_{1\le j\le n} s_j\Big)\prod_{1\le j\le n} H(s_j)e^{-a_j s_j}$$
$$\times\Big(\prod_{1\le j\le n} a_j - \prod_{1\le j\le n}\Big(a_j-\frac{\partial}{\partial s_j}\Big)\big\{P_{\alpha_j}(a_j s_j)\big\}\Big)ds \ge 0.$$

Note that for $n = 1$, it means for $a \ge 0$,

$$\begin{aligned}
&\int_0^1 e^{-as}(a - aP_k(as) + aP_k'(as))ds \\
&\quad = 1 - e^{-a} + \int_0^1 \frac{d}{ds}\{e^{-as}P_k(as)\} \\
&\quad = 1 - e^{-a} + e^{-a}P_k(a) - P_k(0) = e^{-a}(P_k(a) - 1) \ge 0,
\end{aligned}$$

which holds true from (3.4.6).

Remark 3.4.7. There are several classical results on products of Laguerre polynomials, in particular, the article [7], *On some expansions in Laguerre polynomials* by A. Erdélyi and also the paper [40], *Linearization of the products of the generalized Lauricella polynomials and the multivariate Laguerre polynomials via their integral representations* by Shuoh-Jung Liu, Shy-Der Lin, Han-Chun Lu and H. M. Srivastava. However, it seems that the non-negativity of the polynomials $P_{\alpha;1}$, $P'_{\alpha;1}$, do not suffice to tackle the conjecture in two dimensions and more.

Chapter 4

Parabolas

4.1 Preliminary remarks

We start with a picture, demonstrating that the epigraph of a parabola is an increasing union of ellipses (see Figure 4.1). It is easy to see that the epigraph of a parabola, i.e., the set $\{(x, \xi) \in \mathbb{R}^2, \xi > x^2\}$ is a countable increasing union of ellipses in the sense that

$$\mathscr{P} = \{(x, \xi) \in \mathbb{R}^2, \xi > x^2\} = \bigcup_{k \geq 1} \underbrace{\{(x, \xi) \in \mathbb{R}^2, \xi > x^2 + k^{-2}\xi^2\}}_{\mathcal{E}_k}. \tag{4.1.1}$$

Note that for $k \geq 1$ we have $\mathcal{E}_k \subset \mathcal{E}_{k+1} \subset \mathscr{P}$ since $x^2 + k^{-2}\xi^2 \geq x^2 + (k+1)^{-2}\xi^2 > x^2$, from the fact that $\xi > 0$ on $\mathcal{E}_k$. Moreover, if $\xi > x^2$ and $k > \xi/\sqrt{\xi - x^2}$, we get $(x, \xi) \in \mathcal{E}_k$.

Remark 4.1.1. The ellipse $\mathcal{E}_k$ is symplectically equivalent to a circle with area $\frac{\pi k^3}{4}$ since

$$\begin{aligned} x^2 + k^{-2}\xi^2 - \xi &= x^2 + k^{-2}\left(\xi - \frac{k^2}{2}\right)^2 - \frac{k^2}{4} = (\lambda^{-1}y)^2 + k^{-2}\left(\lambda\eta - \frac{k^2}{2}\right)^2 - \frac{k^2}{4} \\ &= \lambda^{-2}y^2 + \lambda^2 k^{-2}\left(\eta - \frac{k^2}{2\lambda}\right)^2 - \frac{k^2}{4}, \end{aligned}$$

so that choosing λ such that $\lambda^{-2} = \lambda^2 k^{-2}$, e.g., $\lambda = \sqrt{k}$, we get

$$x^2 + k^{-2}\xi^2 - \xi = k^{-1}\left(y^2 + \left(\eta - \frac{k^2}{2\lambda}\right)^2\right) - \frac{k^2}{4},$$

and $\mathcal{E}_k = \{(y, \zeta) \in \mathbb{R}^2, y^2 + \zeta^2 < \frac{k^3}{4}\}$, where (y, ζ) are the affine symplectic coordinates

$$y = xk^{1/2}, \quad \zeta = \xi k^{-1/2} - \frac{k^{3/2}}{2}.$$

Lemma 4.1.2. *Let $u \in \mathscr{S}(\mathbb{R})$. Then, $\mathcal{W}(u, u)$ belongs to $\mathscr{S}(\mathbb{R}^2)$ and with $\mathcal{E}, \mathcal{E}_k$ defined by (4.1.1), we have*

$$\iint_{\xi > x^2} \mathcal{W}(u, u)(x, \xi) dx d\xi = \lim_{k \to +\infty} \iint_{\mathcal{E}_k} \mathcal{W}(u, u)(x, \xi) dx d\xi \leq \|u\|^2_{L^2(\mathbb{R})}.$$

Proof. Since $\mathcal{W}(u, u)$ belongs to $\mathscr{S}(\mathbb{R}^{2n}) \subset L^1(\mathbb{R}^{2n})$, we may apply the Lebesgue dominated convergence theorem and (4.1.1) to obtain the equality in the lemma. On

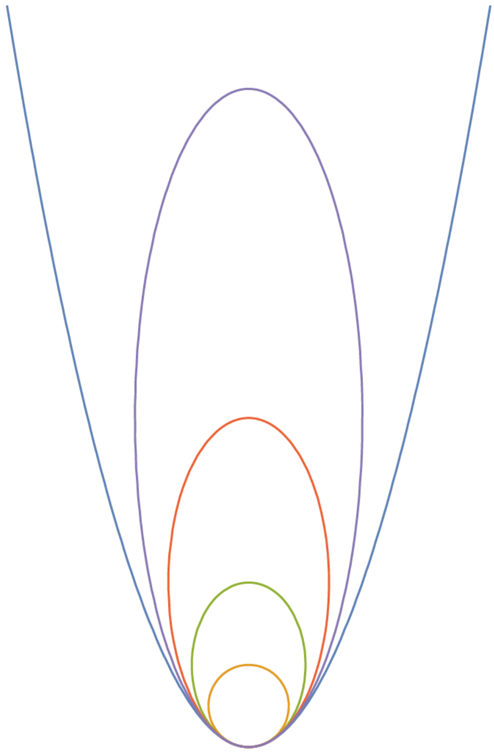

Figure 4.1. The epigraph of a parabola is an increasing union of ellipses.

the other hand, Theorem 3.1.5 and Remark 4.1.1 imply

$$\iint_{\mathcal{E}_k} \mathcal{W}(u,u)(x,\xi)dxd\xi = \langle \mathrm{Op_w}(\mathbf{1}_{\mathcal{E}_k})u,u\rangle \le \big(1-e^{-\frac{\pi k^3}{2}}\big)\|u\|^2_{L^2(\mathbb{R})} \le \|u\|^2_{L^2(\mathbb{R})},$$

and the sought result. ∎

Remark 4.1.3. Moreover, Theorem 3.1.5 and the expression of $F_0(a) = 1-e^{-a}$ imply that with ψ_0 defined in (A.1.16), we have

$$\iint_{\mathcal{E}_k} \mathcal{W}(\psi_0,\psi_0)(x,\xi)dxd\xi = \langle \mathrm{Op_w}(\mathbf{1}_{\mathcal{E}_k})\psi_0,\psi_0\rangle = \|\psi_0\|^2_{L^2(\mathbb{R})}(1-e^{-\pi k^3/3}),$$

so that from Lemma 4.1.2, we have $\iint_{\mathcal{P}} \mathcal{W}(\psi_0,\psi_0)(x,\xi)dxd\xi = \|\psi_0\|^2_{L^2(\mathbb{R})}$, entailing

$$\sup_{\phi\in\mathscr{S}(\mathbb{R}),\|\phi\|_{L^2(\mathbb{R})}=1} \iint_{\mathcal{P}} \mathcal{W}(\phi,\phi)(x,\xi)dxd\xi = 1.$$

Remark 4.1.4. We want to study the operator with Weyl symbol $H(\xi - x^2)$ ($H = \mathbf{1}_{\mathbb{R}_+}$ is the Heaviside function) and since $\xi - x^2$ is a polynomial with degree less than 2, see from (1.2.3) that $\operatorname{Op_w}\big(H(\xi - x^2)\big)$ commutes with

$$D_x - x^2 = e^{2\pi i x^3/3} D_x e^{-2\pi i x^3/3},$$

and the latter has (continuous) spectrum $\mathbb{R}$: we expect thus that $\operatorname{Op_w}\big(H(\xi - x^2)\big)$ should have continuous spectrum and be conjugated to a Fourier multiplier.

4.2 Calculation of the kernel

The Weyl symbol of the operator $\operatorname{Op_w}(\mathbf{1}_{\mathscr{P}})$ is

$$H(\xi - x^2),$$

($\mathscr{P}$ is defined in (4.1.1), H is the Heaviside function $H = \mathbf{1}_{\mathbb{R}_+}$), corresponding to the distribution kernel $k_{\mathscr{P}}(x, y)$ obtained from Proposition 1.2.5 by (we use freely integrals meaning only Fourier transform in the distributional sense),

$$\begin{aligned}
k_{\mathscr{P}}(x, y) &= \int e^{2i\pi(x-y)\xi} H\left(\xi - \left(\frac{x+y}{2}\right)^2\right) d\xi = \int e^{2i\pi(x-y)(\xi + (\frac{x+y}{2})^2)} H(\xi) d\xi \\
&= e^{2i\pi(x-y)(\frac{x+y}{2})^2} \frac{1}{2}\left(\delta_0(y-x) + \frac{1}{i\pi(y-x)}\right) \\
&= \frac{\delta_0(y-x)}{2} + \frac{e^{2i\pi(x-y)(\frac{x+y}{2})^2}}{2i\pi(y-x)}.
\end{aligned}$$

We have

$$\begin{aligned}
4(x-y)\left(\frac{x+y}{2}\right)^2 &= (x^2 - y^2)(x+y) = x^3 - y^3 + x^2 y - y^2 x \\
&= \frac{4}{3}(x^3 - y^3) + \frac{1}{3}(y-x)^3,
\end{aligned}$$

so that

$$k_{\mathscr{P}}(x, y) = e^{i\frac{2\pi}{3}x^3}\left(\frac{\delta_0(y-x)}{2} + \frac{e^{i\frac{\pi}{2}\frac{1}{3}(y-x)^3}}{2i\pi(y-x)}\right) e^{-i\frac{2\pi}{3}y^3},$$

and the operator $\operatorname{Op_w}(\mathbf{1}_{\mathscr{P}})$ is unitarily equivalent to the operator with kernel

$$\tilde{k}(x, y) = \frac{\delta_0(y-x)}{2} + \frac{e^{i\frac{\pi}{6}(y-x)^3}}{2i\pi(y-x)}.$$

We have proven the following result.

Lemma 4.2.1. *The operator with Weyl symbol* $\mathbb{R}^2 \ni (x, \xi) \mapsto \mathbf{1}_{\mathbb{R}_+}(\xi - x^2)$ *has the distribution kernel*

$$k_{\mathcal{P}}(x, y) = e^{i\frac{2\pi}{3}x^3}\left(\frac{\delta_0(y-x)}{2} + \frac{e^{i\frac{\pi}{6}(y-x)^3}}{2i\pi(y-x)}\right)e^{-i\frac{2\pi}{3}y^3},$$

and is thus unitarily equivalent to

$$\frac{\mathrm{Id}}{2} + \textit{convolution with} \quad \frac{ie^{-i\pi t^3/6}}{2\pi}\mathrm{pv}\frac{1}{t}. \tag{4.2.1}$$

Lemma 4.2.2. *The distribution* $\frac{ie^{-i\pi t^3/6}}{2\pi}\mathrm{pv}\frac{1}{t}$ *has the Fourier transform*

$$\frac{1}{2\pi}\int \frac{\sin(2\pi a s\tau + \frac{s^3}{3})}{s}ds, \quad a = (2/\pi)^{1/3}.$$

The operator (4.2.1) *is the Fourier multiplier* $\omega(D_t)$ *with*

$$\omega(\tau) = \frac{1}{2}\left(1 + \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\sin(s\eta + \frac{s^3}{3})}{s}ds\right), \quad \eta = 2^{4/3}\pi^{2/3}\tau.$$

Proof. We calculate in the distribution sense ($t = as, a = (2/\pi)^{1/3}$),

$$\begin{aligned}\int e^{-2i\pi t\tau} i\frac{e^{-i\pi t^3/6}}{2\pi t}dt &= \frac{i}{2\pi}\int e^{-2i\pi a s\tau}\frac{e^{-i\pi a^3 s^3/6}}{s}ds\\ &= \frac{i}{2\pi}\int \frac{(-i)\sin(\frac{s^3}{3} + 2\pi a s\tau)}{s}ds\\ &= \frac{1}{2\pi}\int \frac{\sin(2\pi a s\tau + \frac{s^3}{3})}{s}ds,\end{aligned}$$

so that with $\eta = 2\pi a\tau$, we get

$$\omega(\tau) = \frac{1}{2}\left(1 + \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\sin(s\eta + \frac{s^3}{3})}{s}ds\right) = \frac{1}{2}\big(1 - F(\eta)\big) = G(\eta),$$

proving the lemma. ∎

Lemma 4.2.3. *We have, with* $\eta = 2^{4/3}\pi^{2/3}\tau$,

$$\omega(\tau) = \frac{1}{2}\left(1 + \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\sin(s\eta + \frac{s^3}{3})}{s}ds\right) = G(\eta), \quad \omega(0) = \frac{2}{3} = G(0), \tag{4.2.2}$$

$$G'(\eta) = \frac{1}{2\pi}\int_{\mathbb{R}}\cos\left(s\eta + \frac{s^3}{3}\right)ds = \mathrm{Re}\,\frac{1}{2\pi}\int_{\mathbb{R}}\exp i\left(s\eta + \frac{s^3}{3}\right)ds = \mathtt{Ai}(\eta), \tag{4.2.3}$$

$$G(\eta) = \frac{2}{3} + \int_0^\eta \mathtt{Ai}(\xi)d\xi, \tag{4.2.4}$$

where Ai *is Airy function defined as the inverse Fourier transform of* $t \mapsto e^{i(2\pi t)^3/3}$.

Proof. We have

$$\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\sin(\frac{s^3}{3})}{s}ds=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\sin(\sigma)}{3^{1/3}\sigma^{1/3}}3^{1/3}\frac{1}{3}\sigma^{-2/3}d\sigma=\frac{1}{3\pi}\int_{-\infty}^{+\infty}\frac{\sin\sigma}{\sigma}d\sigma=\frac{1}{3}, \tag{4.2.5}$$

proving (4.2.2). We have also

$$G(\eta)=\frac{1}{2}+\operatorname{Im}\Big[\text{Inverse Fourier Transform}\Big\{y\mapsto e^{i(2\pi y)^3/3}\operatorname{pv}\Big(\frac{1}{2\pi y}\Big)\Big\}\Big],$$

and thus

$$\begin{aligned}G'(\eta)&=\operatorname{Im}\Big[\text{Inverse Fourier Transform}\big\{y\mapsto e^{i(2\pi y)^3/3}i\big\}\Big]\\&=\operatorname{Im}\Big(\int e^{2i\pi y\eta}e^{i(2\pi y)^3/3}i\,dy\Big)=\operatorname{Im}\Big(\frac{1}{2\pi}\int e^{it\eta}e^{it^3/3}i\,dt\Big)=\mathtt{Ai}(\eta),\end{aligned}$$

which is (4.2.3), implying (4.2.4). ∎

Lemma 4.2.4. *With G defined in Lemma 4.2.3, we get that G is an entire function, real-valued on the real line such that*

$$\lim_{\eta\to+\infty}G(\eta)=1,\qquad\lim_{\eta\to-\infty}G(\eta)=0, \tag{4.2.6}$$

and moreover with η_0 the largest zero of the Airy function ($\eta_0\approx-2.33811$), the function G has an absolute minimum at η_0 with $G(\eta_0)\approx-0.274352$,

$$\forall\eta\in\mathbb{R},\quad G(\eta_0)\le G(\eta)<1. \tag{4.2.7}$$

Proof. The first statements follow from Lemma 4.2.3 and (4.2.6) is implied by (4.2.4) and (A.7.18), (A.7.22). The strict inequality in (4.2.7) follows for $\eta\ge0$ from (4.2.3) since Ai is positive on $[0,+\infty)$ so that G is strictly increasing there from $G(0)=2/3$ to $G(+\infty)=1$. The other statements are proven in Section A.7 of the appendix. ∎

4.3 The main result

Collecting the results of Lemmas 4.2.1, 4.2.2, 4.2.3, 4.2.4, and of Section A.7 in the appendix, we have proven the following theorem.

Theorem 4.3.1. *Let $H(\xi-x^2)=\mathbf{1}\{(x,\xi)\in\mathbb{R}^2,\xi\ge x^2\}$ be the indicatrix of the epigraph of the parabola with equation $\xi=x^2$. Then, the operator with Weyl symbol $H(\xi-x^2)$ is unitary equivalent to the Fourier multiplier $G(2^{4/3}\pi^{2/3}\tau)$, where*

$$G(\eta)=\frac{2}{3}+\int_0^\eta\mathtt{Ai}(\xi)d\xi=\int_{-\infty}^\eta\mathtt{Ai}(\xi)d\xi,\quad(\mathtt{Ai}\text{ is the Airy function}).$$

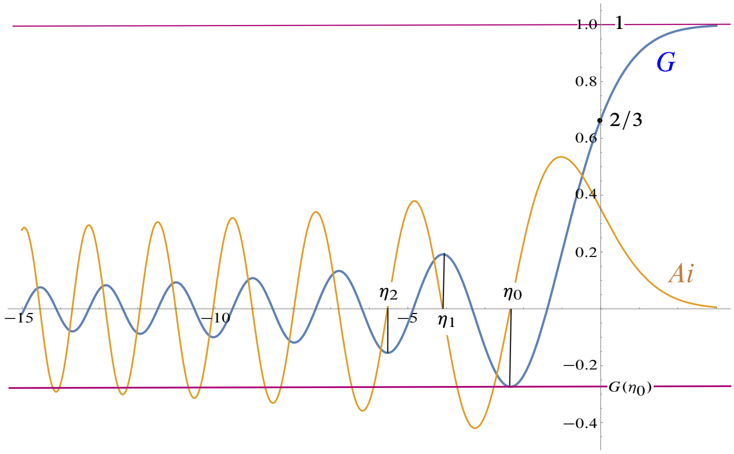

Figure 4.2. The function G and its derivative `Ai`. More details on G are given in Appendix A.7, Figure A.1.

The function G is entire on $\mathbb{C}$, real-valued on the real line (see Figure 4.2) and such that

$$G(\mathbb{R}) = [G(\eta_0), 1),$$

where η_0 is the largest zero of the Airy function. We have

$$\eta_0 \approx -2.338107410,$$
$$G(\eta_0) \approx -0.2743520591.$$

The operator with Weyl symbol $H(\xi - x^2)$ is self-adjoint bounded on $L^2(\mathbb{R})$ with norm 1, with spectrum equal to $[G(\eta_0), 1]$ (continuous spectrum) and

$$\forall u \in L^2(\mathbb{R}), \quad G(\eta_0)\|u\|^2_{L^2(\mathbb{R})} \le \iint_{\xi \ge x^2} \mathcal{W}(u,u)(x,\xi)\,dx\,d\xi \le \|u\|^2_{L^2(\mathbb{R})}.$$

4.4 Paraboloids, a conjecture

We are interested now in multi-dimensional versions of the previous results, namely, we would like to find a bound for integrals of the Wigner distribution on paraboloids of $\mathbb{R}^{2n}$ for $n \ge 2$. Let us start with recalling in [24, Theorem 21.5.3], a version of which was given in our Theorem 3.3.1 in the positive-definite case.

4.4.1 On non-negative quadratic forms

Theorem 4.4.1 (Symplectic reduction of quadratic forms, [24, Theorem 21.5.3]). *Let q be a non-negative quadratic form on $\mathbb{R}^n \times \mathbb{R}^n$ equipped with the canonical* *symplectic form* (1.2.13). *Then, there exists* S *in symplectic group* $\mathrm{Sp}(n,\mathbb{R})$ *of* $\mathbb{R}^{2n}$,

$$r \in \{0,\dots,n\},\ \mu_1,\dots,\mu_r \text{ positive, and } s \in \mathbb{N} \text{ such that } r+s \le n,$$

so that for all $X = (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$,

$$q(SX) = \sum_{1\le j\le r} \mu_j(x_j^2+\xi_j^2) + \sum_{r+1\le j\le r+s} x_j^2.$$

Definition 4.4.2. Let $n \in \mathbb{N}^*$ and let $\mathbb{R}^{2n}$ be equipped with the canonical symplectic form (1.2.13). Let q be a non-negative quadratic form on $\mathbb{R}^{2n}$ with rank $2n-1$ and T be a non-zero vector in $\mathbb{R}^{2n}$ such that $q(\sigma T) = 0$. A paraboloid $\mathscr{P}$ of $\mathbb{R}^{2n}$ with vertex 0 and shape (q,T) is defined by

$$\mathscr{P} = \left\{X \in \mathbb{R}^{2n}, q(X) \le [X,T]\right\}.$$

A paraboloid $\mathscr{Q}$ with vertex $m \in \mathbb{R}^{2n}$ and shape (q,T) is defined as

$$\mathscr{Q} = \mathscr{P} + m,$$

where $\mathscr{P}$ is a paraboloid with vertex 0 and shape (q,T).

Remark 4.4.3. We can find some symplectic coordinates such that

$$q(X) - [X,T] = \sum_{1\le j\le r} \mu_j(x_j^2+\xi_j^2) + \sum_{r+1\le j\le r+s} x_j^2 + \sum_{1\le j\le n}(x_j\tau_j - \xi_j t_j),$$

with $2r+s = 2n-1$. We can get rid of the linear terms $x_j\tau_j - \xi_j t_j$ when $1 \le j \le r$ by writing

$$\mu_j(x_j^2+\xi_j^2) + x_j\tau_j - \xi_j t_j = \mu_j\left(x_j + \frac{\tau_j}{2\mu_j}\right)^2 + \mu_j\left(\xi_j - \frac{t_j}{2\mu_j}\right)^2 - \frac{1}{4\mu_j}(t_j^2+\tau_j^2),$$

and also of $x_j\tau_j$ for $r+1 \le j \le r+s$, since

$$x_j^2 + x_j\tau_j = \left(x_j + \frac{\tau_j}{2}\right)^2 - \frac{\tau_j^2}{4}.$$

We are left with using affine symplectic coordinates (y,η) so that

$$\begin{aligned} q(X) - [X,T] = &\sum_{1\le j\le r} \mu_j(y_j^2+\eta_j^2) + \sum_{r+1\le j\le r+s} y_j^2 - \sum_{r+1\le j\le r+s} \eta_j t_j \\ &+ \sum_{r+s+1\le j\le n} (y_j\tau_j - \eta_j t_j) - a. \end{aligned}$$

Since we have $2r+s = 2n-1$, we get $r+s+1 = 2n-r$: we cannot have $r+s+1 \le n$ since it would imply that $2n-r \le n$ and thus $r \ge n$, which is incompatible with $2r+s=2n-1, r,s\ge 0$. We get then that $s=2l+1, r=n-1-l$ and since $r+s\le n, 1\le s$, we have $l=0, s=1, r=n-1$, and

$$q(X)-[X,T]=\sum_{1\le j\le n-1}\mu_j(y_j^2+\eta_j^2)+y_n^2-\eta_n t_n-a,$$

and $t_n\in\mathbb{R}^*$. With $y_n=t^{1/3}\tilde y_n, \eta_n=t^{-1/3}\tilde\eta_n$, we get

$$q(X)-[X,T]=\sum_{1\le j\le n-1}\mu_j(y_j^2+\eta_j^2)+t^{2/3}(\tilde y_n^2-\tilde\eta_n-at^{-2/3}),$$

and the inequality $q(X)-[X,T]\le 0$ is equivalent to

$$\sum_{1\le j\le n-1}t^{-2/3}\mu_j(y_j^2+\eta_j^2)+\tilde y_n^2\le\tilde\eta_n+at^{-2/3}.$$

We can thus assume *ab initio* that our paraboloid is given by the inequality

$$\sum_{1\le j\le n-1}\nu_j(x_j^2+\xi_j^2)+x_n^2\le\xi_n.$$

4.4.2 On the kernel for the paraboloid

We shall consider the paraboloid

$$\mathscr{P}_n=\Big\{(x,\xi)\in\mathbb{R}^{2n}, x_n^2+\sum_{1\le j\le n-1}(x_j^2+\xi_j^2)\le\xi_n\Big\}.$$

We have with $X'=(x';\xi')=(x_1,\dots,x_{n-1};\xi_1,\dots,\xi_{n-1})$,

$$\begin{aligned}
P&=\operatorname{Op_w}\big(H(\xi_n-x_n^2-|X'|^2)\big)=\int_{\mathbb{R}}\widehat H(\tau)\operatorname{Op_w}(e^{2i\pi\tau(\xi_n-x_n^2)})\operatorname{Op_w}(e^{-2i\pi\tau|X'|^2})d\tau\\
&=\sum_{k\ge0}\int_{\mathbb{R}}\widehat H(\tau)\mathbb{P}_{k;n-1}\otimes\operatorname{Op_w}(e^{2i\pi\tau(\xi_n-x_n^2)})e^{-i(\arctan\tau)(2k+n-1)}(1+\tau^2)^{-\frac{(n-1)}{2}}d\tau\\
&=\frac12\operatorname{Id}+\frac{1}{2i\pi}\sum_{k\ge0}\mathbb{P}_{k;n-1}\\
&\quad\otimes\int_{\mathbb{R}}\operatorname{Op_w}(e^{2i\pi\tau(\xi_n-x_n^2)})\frac1\tau\left(\frac{1-i\tau}{(1+\tau^2)^{1/2}}\right)^{2k+n-1}(1+\tau^2)^{-\frac{(n-1)}{2}}d\tau\\
&=\frac12\operatorname{Id}+\frac12\sum_{k\ge0}\mathbb{P}_{k;n-1}\otimes\int_{\mathbb{R}}\operatorname{Op_w}(e^{2i\pi\tau(\xi_n-x_n^2)})\frac{(1-i\tau)^k}{i\pi\tau(1+i\tau)^{k+n-1}}d\tau.
\end{aligned}$$

Let $k(x_n,y_n)$ be the kernel of the operator in the integral, we have

$$k(x_n,y_n)=e^{\frac{2i\pi}{3}(x_n^3-y_n^3)}e^{-\frac{i\pi}{6}(x_n-y_n)^3}\frac{i}{\pi(x_n-y_n)}\frac{(1+i(x_n-y_n))^k}{(1-i(x_n-y_n))^{k+n-1}}.$$

As a result, we find that P is unitarily equivalent to $\tilde{P}$, with

$$2\tilde{P} = \sum_{k\ge 0} \mathbb{P}_{k;n-1} \otimes \left(I_n + \text{convolution with } \frac{ie^{-\frac{i\pi}{6}x_n^3}}{\pi x_n} \frac{(1+ix_n)^k}{(1-ix_n)^{k+n-1}} \right).$$

We define

$$\begin{aligned}\omega_{k,n-1}(\tau) &= \frac{1}{2} + \int \frac{ie^{-\frac{i\pi}{6}t^3}}{2\pi t} \frac{(1+it)^k}{(1-it)^{k+n-1}} e^{-2i\pi t\tau} dt\\ &= \frac{1}{2} + \int \frac{e^{\frac{i\pi}{6}t^3}}{2i\pi t} \frac{(1-it)^k}{(1+it)^{k+n-1}} e^{2i\pi t\tau} dt,\end{aligned}$$

and we get that

$$\tilde{P} = \sum_{k\ge 0} \mathbb{P}_{k;n-1} \otimes \omega_{k,n-1}(D_{x_n}).$$

We note that for $n = 1$, the sum is reduced to $k = 0$ with $\mathbb{P}_{0;0} = I$, so that we recover formula (4.2.2) with $\omega_{0,0} = \omega$. We find also that

$$\omega'_{k,n-1}(\tau) = \int e^{\frac{i\pi}{6}t^3} \frac{(1-it)^k}{(1+it)^{k+n-1}} e^{2i\pi t\tau} dt, \tag{4.4.1}$$

in the sense that the inverse Fourier transform of $t \mapsto e^{\frac{i\pi}{6}t^3} \frac{(1-it)^k}{(1+it)^{k+n-1}}$ is the distribution derivative of $\omega_{k,n-1}$. Going back to the normalization of Lemma 4.2.3, we have, with $\eta = 2^{4/3}\pi^{2/3}\tau$,

$$\begin{aligned}&G_{k,n-1}(\eta) = \omega_{k,n-1}(\tau),\\ &G'_{k,n-1}(\eta) = 2^{-4/3}\pi^{-2/3} \int e^{\frac{i\pi}{6}t^3} \frac{(1-it)^k}{(1+it)^{k+n-1}} e^{2^{-\frac{1}{3}} i\pi^{\frac{1}{3}} t\eta} dt,\\ &\underbrace{=}_{t=\pi^{-\frac{1}{3}}2^{\frac{1}{3}}s} \frac{1}{2\pi} \int e^{\frac{is^3}{3}} \frac{(1-i\pi^{-1/3}2^{1/3}s)^k}{(1+i\pi^{-1/3}2^{1/3}s)^{k+n-1}} e^{is\eta} ds := A_{k,n-1}(\eta).\end{aligned}$$

We have $A_{0,0} = \mathtt{Ai}$ and $A_{k,n-1}$ is an entire function, real-valued on the real line; we have

$$G_{k,n-1}(\eta) = \int_{-\infty}^{\eta} A_{k,n-1}(\xi)d\xi, \quad G_{k,n-1}(+\infty) = 1.$$

Remark 4.4.4. We claim that the asymptotic properties of the functions $A_{k,n-1}$ are analogous to the properties of the standard Airy function and we have indeed from (4.4.1),

$$\omega'_{k,n-1}(\tau) = (1-iD)^k(1+iD)^{-k-n+1}\mathcal{F}^{-1}\big(e^{\frac{i\pi}{6}t^3}\big).$$

We claim as well that

$$-\frac{1}{2} < \inf_{k\geq 0, \eta\in\mathbb{R}} G(\eta) < 0, \qquad \sup_{k\geq 0, \eta\in\mathbb{R}} G(\eta) = 1,$$

so that $\tilde{P}$ is bounded on $L^2(\mathbb{R}^n)$ and

$$\int_{\xi_n \geq x_n^2 + \sum_{1\leq j\leq n-1}(x_j^2+\xi_j^2)} \mathcal{W}(u,u)(x,\xi)\,dx\,d\xi \leq \|u\|^2_{L^2(\mathbb{R}^n)}.$$

Chapter 5

Conics with eccentricity greater than 1

We want to consider now integrals of the Wigner distribution on "hyperbolic" convex subsets of the plane such as

$$\mathscr{C}_\sigma = \{(x, \xi) \in \mathbb{R}^2, x\xi \geq \sigma, x \geq 0\}, \tag{5.0.1}$$

where σ is a non-negative parameter. It is convenient to start with the limit-case where $\sigma = 0$ and $\mathscr{C}_0 = \{(x, \xi) \in \mathbb{R}^2, x \geq 0, \xi \geq 0\}$ (we will label $\mathscr{C}_0$ as the *quarter-plane*). The indicator function of $\mathscr{C}_0$ is $H(x)H(\xi)$ where $H = \mathbf{1}_{\mathbb{R}_+}$ is the Heaviside function.

N.B. The reader will see a great similarity between our calculations below in this section and the J. G. Wood and A. J. Bracken paper [55] (see also [4]). This article is very important for the problem at stake – Integrating the Wigner distribution on subsets of the phase space – and was a wealthy source of information for us, although as a mathematician, the author has a quite rigid relationship with calculations, and feels the need to justify formal manipulations; for instance, we may point out that the test functions used in [55] are homogeneous distributions of type

$$x_\pm^{-\frac{1}{2}+i\omega}, \quad \omega \in \mathbb{R},$$

which are not in $L^2(\mathbb{R})$ (not even in L^2_{loc}), a situation which raises some difficulties, first when you try to normalize in L^2 these test functions and also when trying to give a non-formal meaning to their images under the operator with Weyl symbol $H(x)H(\xi)$, images which are not clearly defined. In our joint paper [6] with B. Delourme and T. Duyckaerts, proving that Flandrin's conjecture is not true, we followed numerical arguments which were quite apart from the arguments of [55]. However, in this memoir, we do follow many of the arguments of [55], along with avoiding formal calculations.

5.1 The quarter-plane, a counterexample to Flandrin's conjecture

5.1.1 Preliminaries

We study in this section the operator

$$A_0 = \mathrm{Op}_{\mathrm{w}}(H(x)H(\xi)), \tag{5.1.1}$$

where $H = \mathbf{1}_{\mathbb{R}_+}$, that is the Weyl quantization of the characteristic function of the first quarter of the plane.

Lemma 5.1.1. *The operator A_0 given by* (5.1.1) *is bounded self-adjoint on* $L^2(\mathbb{R})$.

Proof. Since the Weyl symbol of A_0 is real-valued, A_0 is formally self-adjoint and it is enough to prove that A_0 is bounded on $L^2(\mathbb{R})$. Let us start with recalling the classical formulas

$$\begin{aligned}\widehat{H}(t) &= \frac{\delta_0(t)}{2} + \frac{1}{2i\pi}\operatorname{pv}\Big(\frac{1}{t}\Big),\\ \widehat{\operatorname{sign}} &= \frac{1}{i\pi}\operatorname{pv}\Big(\frac{1}{t}\Big),\end{aligned}$$

useful below. The kernel[1] of A_0 is

$$k_0(x,y) = H(x+y)\widehat{H}(y-x) = H(x+y)\frac{1}{2}\Big(\delta_0(y-x) + \frac{1}{i\pi}\operatorname{pv}\frac{1}{y-x}\Big). \tag{5.1.2}$$

For $\lambda > 0$, we define $A_{0,\lambda} = \big(H(x)\mathbf{1}_{[0,\lambda]}(\xi)\big)^w$, whose distribution-kernel is the $L^\infty(\mathbb{R}^{2n})$ function

$$k_{0,\lambda}(x,y) = H(x+y)e^{i\pi(x-y)\lambda}\frac{\sin(\pi(x-y)\lambda)}{\pi(x-y)}.$$

We can thus notice that

$$\begin{aligned}k_{0,\lambda}(x,y) = {}& \overbrace{H(x)H(y)e^{i\pi(x-y)\lambda}\frac{\sin(\pi(x-y)\lambda)}{\pi(x-y)}}^{k^\flat_{0,\lambda}(x,y)}\\ &+ \underbrace{H(x+y)\big(H(-x)H(y) + H(x)H(-y)\big)\frac{\sin(\pi(x-y)\lambda)}{\pi(x-y)}e^{i\pi(x-y)\lambda}}_{k^\sharp_{0,\lambda}(x,y)},\end{aligned}$$

and the operator with distribution-kernel $k^\flat_{0,\lambda}$ is

$$H\operatorname{Op_w}(\mathbf{1}_{[0,\lambda]}(\xi))H, \text{ that is } H\mathbf{1}_{[0,\lambda]}(D)H,$$

---

[1] There is no difficulty at defining the product $S\big((x+y)/2\big)T(x-y)$ for S,T tempered distributions on the real line since we may use the tensor product with

$$\begin{aligned}&\Big\langle S\Big(\frac{x+y}{2}\Big)T(x-y), \Phi(x,y)\Big\rangle_{\mathscr{S}'(\mathbb{R}^2),\mathscr{S}(\mathbb{R}^2)}\\ &\quad= \Big\langle S(x_1)\otimes T(x_2), \Phi\Big(x_1+\frac{x_2}{2}, x_1-\frac{x_2}{2}\Big)\Big\rangle_{\mathscr{S}'(\mathbb{R}^2),\mathscr{S}(\mathbb{R}^2)}.\end{aligned}$$

However, we shall not use directly formula (5.1.2), since want to avoid formal manipulation involving for instance meaningless products such as $H(x)H(y)k_0(x,y)$. We refer the reader to Remark 5.1.2 for more details on this matter.

where H stands for the operator of multiplication by the Heaviside function H. On the other hand, the operator with distribution kernel $k^\sharp_{0,\lambda}$ is such that

$$\begin{aligned}|k^\sharp_{0,\lambda}(x,y)| &\le H(x+y)\frac{H(-x)H(y)+H(x)H(-y)}{\pi|x-y|}\\ &= H(x+y)\frac{H(-x)H(y)}{\pi(y-x)} + H(x+y)\frac{H(x)H(-y)}{\pi(x-y)}.\end{aligned}$$

According to Proposition A.5.1 in Appendix A.7, the Hardy operator and the modified Hardy operators are bounded on $L^2(\mathbb{R})$ and we obtain that, for $\phi, \psi \in \mathscr{S}(\mathbb{R}^n)$, with $H = H(x), \check{H} = H(-x)$,

$$\begin{aligned}&\left|\iint H(x)\mathbf{1}_{[0,\lambda]}(\xi)W(\phi,\psi)(x,\xi)dxd\xi\right|\\ &\le \|H\phi\|_{L^2(\mathbb{R})}\|H\psi\|_{L^2(\mathbb{R})} + \frac{1}{2}\|H\phi\|_{L^2(\mathbb{R})}\|\check{H}\psi\|_{L^2(\mathbb{R})} + \frac{1}{2}\|\check{H}\phi\|_{L^2(\mathbb{R})}\|H\psi\|_{L^2(\mathbb{R})}\end{aligned}$$

so that

$$\begin{aligned}&|\langle A_0\phi,\psi\rangle_{\mathscr{S}^*(\mathbb{R}),\mathscr{S}(\mathbb{R})}|\\ &= \left|\iint H(x)H(\xi)\overbrace{W(\phi,\psi)(x,\xi)}^{\in\mathscr{S}(\mathbb{R}^2)}dxd\xi\right|\\ &= \lim_{\lambda\to+\infty}\left|\iint H(x)\mathbf{1}_{[0,\lambda]}(\xi)W(\phi,\psi)(x,\xi)dxd\xi\right|\\ &\le \|H\phi\|_{L^2(\mathbb{R})}\|H\psi\|_{L^2(\mathbb{R})} + \frac{1}{2}\|H\phi\|_{L^2(\mathbb{R})}\|\check{H}\psi\|_{L^2(\mathbb{R})}\\ &\quad + \frac{1}{2}\|\check{H}\phi\|_{L^2(\mathbb{R})}\|H\psi\|_{L^2(\mathbb{R})},\end{aligned}\tag{5.1.3}$$

yielding the L^2-boundedness of the operator A_0, and this concludes the proof of the lemma. ∎

Remark 5.1.2. That cumbersome detour with the operator $A_{0,\lambda}$ is useful to ensure that the operator A is indeed bounded on $L^2(\mathbb{R})$. The kernel k_0 of A_0 is a distribution of order 1 and the product $H(x)H(y)k_0(x,y)$ is not a priori meaningful, even when k is a Radon measure.

Even a wave-front-set approach, which would allow the product $H(x)\mathrm{pv}(1/(y-x))$, does not offer a meaning for the product $H(x)H(y)\mathrm{pv}(1/(y-x))$ since the wave-front-set of $\mathrm{pv}(1/(y-x))$ is located on the conormal of the first diagonal (i.e., $\{(x,x;\xi,-\xi)\}_{x\in\mathbb{R},\xi\in\mathbb{R}^*}$), whereas the wave-front set at $(0,0)$ of $H(x)H(y)$ contains all directions and in particular is antipodal to the conormal of the diagonal at $(0,0)$.

However, with the proven L^2-boundedness of A_0, then the products of operators HA_0H, $\check{H}A_0H$, $HA_0\check{H}$, $\check{H}A_0\check{H}$ make sense and for instance we may approximate in the strong-operator-topology the operator HA_0H by the operator $\chi(\cdot/\varepsilon)A\chi(\cdot/\varepsilon)$, where χ is a smooth function supported in $[1,+\infty)$ and equal to 1 on $[2,+\infty)$. We have indeed

$$HAH = (H-\chi(\cdot/\varepsilon))AH + \chi(\cdot/\varepsilon)A(H-\chi(\cdot/\varepsilon)) + \chi(\cdot/\varepsilon)A\chi(\cdot/\varepsilon),$$

so that for $u \in L^2(\mathbb{R})$, $HAHu = \lim_{\varepsilon\to 0_+} \chi(\cdot/\varepsilon)A\chi(\cdot/\varepsilon)u$. The operator with kernel

$$H(x+y)\chi(x/\varepsilon)\chi(y/\varepsilon)\mathrm{pv}\frac{1}{i\pi(y-x)} = \chi(x/\varepsilon)\chi(y/\varepsilon)\mathrm{pv}\frac{1}{i\pi(y-x)},$$

converges strongly towards the operator $H(\operatorname{sign} D)H$.

Proposition 5.1.3. *Let $A_0 = \operatorname{Op_w}(H(x)H(\xi))$ be the operator with Weyl symbol $H(x)H(\xi)$, a priori sending $\mathscr{S}(\mathbb{R})$ into $\mathscr{S}'(\mathbb{R})$. Then, A_0 can be uniquely extended to a self-adjoint bounded operator on $L^2(\mathbb{R})$ with*

$$\|A_0\|_{\mathcal{B}(L^2(\mathbb{R}))} \le \frac{1+\sqrt{2}}{2} \approx 1.207 \tag{5.1.4}$$

N.B. The bound above can be significantly improved (see Proposition 5.4.4 for optimal bounds) and moreover we will show below that the spectrum of A_0 actually intersects $(1,+\infty)$. In fact, it is easier to start with the information that A_0 is indeed bounded on $L^2(\mathbb{R})$.

Proof. The $L^2(\mathbb{R})$-boundedness of A_0 is given by Lemma 5.1.1. We are left with proving the bound (5.1.4): we note that (5.1.3) implies

$$|\langle A_0u, u\rangle_{L^2(\mathbb{R})}| \le \|Hu\|^2_{L^2(\mathbb{R})} + \|Hu\|_{L^2(\mathbb{R})}\|\check{H}u\|_{L^2(\mathbb{R})},$$

proving the proposition, since the eigenvalues of the quadratic form $\mathbb{R}^2 \ni (x_1,x_2) \mapsto x_1^2 + x_1x_2$ are $(1\pm\sqrt{2})/2$. ∎

We can do much better and actually diagonalise the operator A_0, using as in Proposition A.5.1 logarithmic coordinates on each half-line. We state a lemma on “diagonal” terms whose proof is already given above.

Lemma 5.1.4 (Diagonal terms). *Let A_0 be the operator with Weyl symbol $H(x)H(\xi)$. With H standing as well for the operator of multiplication by $H(x)$, we have*

$$HA_0H = HH(D)H = H\frac{(\mathrm{Id}+\operatorname{sign} D)}{2}H.$$

Lemma 5.1.5 (Off-diagonal terms). *Let $B_0 = 2\operatorname{Re}\check{H}A_0H = \check{H}A_0H + HA_0\check{H}$. Then, we have for all $u \in L^2(\mathbb{R})$,*

$$|\langle B_0u, u\rangle_{L^2(\mathbb{R})}| \le \frac{1}{2}\|Hu\|_{L^2(\mathbb{R})}\|\check{H}u\|_{L^2(\mathbb{R})}. \tag{5.1.5}$$

Proof of Lemma 5.1.5. For $u \in \mathscr{S}(\mathbb{R})$ such that $0 \notin \operatorname{supp} u$, we define for $t \in \mathbb{R}$,

$$\phi_1(t) = u(e^t)e^{t/2}, \quad \phi_2(t) = u(-e^t)e^{t/2}, \tag{5.1.6}$$

so that

$$\begin{aligned} \|Hu\|^2_{L^2(\mathbb{R})} &= \|\phi_1\|^2_{L^2(\mathbb{R})}, \\ \|\check{H}u\|^2_{L^2(\mathbb{R})} &= \|\phi_2\|^2_{L^2(\mathbb{R})}. \end{aligned}$$

We have

$$\begin{aligned} \langle B_0 u, u\rangle_{L^2(\mathbb{R})} &= \iint \frac{H(x+y)(\check{H}(x)H(y) + H(x)\check{H}(y))}{2i\pi(y-x)} u(y)\bar{u}(x)dydx \\ &= \iint \frac{H(-e^s + e^t)e^{\frac{s+t}{2}}}{2i\pi(e^t + e^s)} \phi_1(t)\bar{\phi}_2(s)dsdt \\ &\quad - \iint \frac{H(e^s - e^t)e^{\frac{s+t}{2}}}{2i\pi(e^t + e^s)} \phi_2(t)\bar{\phi}_1(s)dsdt \\ &= \iint \frac{H(t-s)}{4i\pi\cosh(\frac{s-t}{2})} \phi_1(t)\bar{\phi}_2(s)dsdt \\ &\quad - \iint \frac{H(s-t)}{4i\pi\cosh(\frac{s-t}{2})} \phi_2(t)\bar{\phi}_1(s)dsdt, \end{aligned}$$

so that

$$\langle B_0 u, u\rangle_{L^2(\mathbb{R})} = \langle \tilde{S}_0 * \phi_1, \phi_2\rangle_{L^2(\mathbb{R})} + \langle S_0 * \phi_2, \phi_1\rangle_{L^2(\mathbb{R})}$$

and

$$\tilde{S}_0(t) = \frac{\check{H}(t)}{4i\pi\cosh(t/2)}, \quad S_0(t) = \frac{iH(t)}{4\pi\cosh(t/2)}. \tag{5.1.7}$$

We calculate

$$\int_0^{+\infty} \frac{dt}{4\pi\cosh(t/2)} = \frac{1}{2\pi}[\arctan(\sinh(t/2))]_0^{+\infty} = \frac{1}{4} = \int_{-\infty}^0 \frac{dt}{4\pi\cosh(t/2)},$$

so that

$$|\langle B_0 u, u\rangle_{L^2(\mathbb{R})}| \le \frac{1}{2}\|\phi_1\|_{L^2(\mathbb{R})}\|\phi_2\|_{L^2(\mathbb{R})} = \frac{1}{2}\|Hu\|_{L^2(\mathbb{R})}\|\check{H}u\|_{L^2(\mathbb{R})},$$

proving the estimate of the lemma for $u \in \mathscr{S}(\mathbb{R})$ such that $0 \notin \operatorname{supp} u$. We use now that we already know that B_0 is a bounded self-adjoint operator on $L^2(\mathbb{R})$: let u be a function in $L^2(\mathbb{R})$ and let $(\phi_k)_{k\ge 1}$ be a sequence[2] in $\mathscr{S}(\mathbb{R})$ such that each ϕ_k vanishes in a neighborhood of 0 so that $\lim_k \phi_k = u$ in $L^2(\mathbb{R})$. We find that

$$\begin{aligned}
&|\langle B_0 u, u\rangle_{L^2(\mathbb{R})}| \\
&\le |\langle B_0(u-\phi_k), u\rangle_{L^2(\mathbb{R})}| + |\langle B_0\phi_k, u-\phi_k\rangle_{L^2(\mathbb{R})}| + |\langle B_0\phi_k, \phi_k\rangle_{L^2(\mathbb{R})}| \\
&\le \|B_0\|_{\mathcal{B}(L^2(\mathbb{R}))}\big(\|u-\phi_k\|_{L^2(\mathbb{R})}\|u\|_{L^2(\mathbb{R})} + \|u-\phi_k\|_{L^2(\mathbb{R})}\|\phi_k\|_{L^2(\mathbb{R})}\big) \\
&\quad + \frac{1}{2}\|H\phi_k\|_{L^2(\mathbb{R})}\|\check{H}\phi_k\|_{L^2(\mathbb{R})},
\end{aligned}$$

providing readily the result of the lemma since the multiplication by H and $\check{H}$ are bounded operators on $L^2(\mathbb{R})$. ∎

Remark 5.1.6. The estimate (5.1.5) and Lemma 5.1.4 are already improving (5.1.4), since the eigenvalues of the quadratic form $\mathbb{R}^2 \ni (x_1, x_2) \mapsto x_1^2 + \frac{1}{2}x_1x_2$ are $(2 \pm \sqrt{5})/4$, so that the right-hand side of (5.1.4) can be replaced by $(2+\sqrt{5})/4 \approx 1.059$. Anyhow, we shall provide below a diagonalisation of A_0 and optimal bounds.

N.B. We shall be a little faster in the sequel on the "cumbersome" detours to avoid formal multiplication of kernels by Heaviside functions but the reader should keep in mind that it is an important point to secure $L^2(\mathbb{R})$-boundedness *before* any further manipulation of the kernels.

5.1.2 An isometric isomorphism

Remark 5.1.7. The mapping Ψ defined by

$$\begin{aligned}
\Psi\colon L^2(\mathbb{R}) &\to L^2(\mathbb{R};\mathbb{C}^2) \\
u &\mapsto \big((Hu)(e^t)e^{t/2}, (\check{H}u)(-e^t)e^{t/2}\big)
\end{aligned} \tag{5.1.8}$$

is an isometric isomorphism of Hilbert spaces: indeed, we have

$$\|u\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} |u(e^t)|^2 e^t\,dt + \int_{\mathbb{R}} |u(-e^t)|^2 e^t\,dt.$$

Moreover, if $(\phi_1, \phi_2) \in L^2(\mathbb{R};\mathbb{C}^2)$, we may define for $x \in \mathbb{R}^*$

$$u(x) = H(x)\phi_1(\ln x)x^{-1/2} + \check{H}(x)\phi_2(\ln|x|)|x|^{-1/2},$$

and we have

$$\Psi(u)(t) = \big(\phi_1(t), \phi_2(t)\big).$$

[2] Such a sequence is easy to find: a first step is to find a sequence $(\tilde{\phi}_k)_{k\ge 1}$ in the Schwartz space converging in $L^2(\mathbb{R})$ towards u, then consider with a given $\omega \in C^\infty(\mathbb{R};[0,1])$ such that $\omega(t) = 0$ for $|t| \le 1$ and $\omega(t) = 1$ for $|t| \ge 2$, $\phi_k(x) = \omega(kx)\tilde{\phi}_k(x)$.

Remark 5.1.8. Using Lemma 5.1.4 and notations (5.1.6) we see that

$$\begin{aligned}
\langle HA_0Hu,u\rangle_{L^2(\mathbb{R})} &= \frac{1}{2}\|\phi_1\|^2_{L^2(\mathbb{R})} + \iint \frac{1}{2i\pi}\mathrm{pv}\frac{e^{(s+t)/2}}{e^t-e^s}\phi_1(t)\bar{\phi}_1(s)dsdt\\
&= \frac{1}{2}\|\phi_1\|^2_{L^2(\mathbb{R})} + \iint \frac{1}{4i\pi}\mathrm{pv}\frac{1}{\sinh(\frac{t-s}{2})}\phi_1(t)\bar{\phi}_1(s)dsdt\\
&= \int_{\mathbb{R}} |\hat{\phi}_1(\tau)|^2\Big(\frac{1}{2}+\widehat{T}_0(\tau)\Big)d\tau,
\end{aligned}$$

with

$$T_0(t) = \frac{t}{4\sinh(t/2)}\mathrm{pv}\frac{i}{\pi t}. \tag{5.1.9}$$

We have

$$\widehat{T}_0 = \operatorname{sign} * \rho_0 \quad \text{with } \rho_0(\tau) = \int \frac{t}{4\sinh(t/2)}e^{-2i\pi t\tau}dt, \tag{5.1.10}$$

and we note that the function ρ_0 belongs to $\mathscr{S}(\mathbb{R})$, as the Fourier transform of a function in $\mathscr{S}(\mathbb{R})$. Also, we have

$$\int \rho_0(\tau)d\tau = \hat{\rho}_0(0) = \frac{1}{2},$$

and this yields with $\frac{d}{d\tau}\{\frac{1}{2}+\widehat{T}_0\} = 2\rho_0$ (which follows from (5.1.10)) and

$$\frac{1}{2}+\widehat{T}_0(\tau) = 1 - \int_\tau^{+\infty} 2\rho_0(\tau')d\tau', \tag{5.1.11}$$

since

$$\frac{d}{d\tau}\left\{\frac{1}{2}+\widehat{T}_0+\int_\tau^{+\infty} 2\rho_0(\tau')d\tau'\right\} = 0 \quad \text{and} \quad \lim_{\tau\to+\infty}(\operatorname{sign} * \rho_0)(\tau) = \frac{1}{2}.$$

Theorem 5.1.9. *Let A_0 be the operator with Weyl symbol $H(x)H(\xi)$. The operator A_0 is bounded self-adjoint on $L^2(\mathbb{R})$ so that we may define, with Ψ defined in (5.1.8),*

$$\widetilde{A}_0 = \Psi A_0 \Psi^{-1}.$$

The operator $\widetilde{A}_0$ is the Fourier multiplier on $L^2(\mathbb{R};\mathbb{C}^2)$ given by the matrix

$$\mathcal{M}_0(\tau) = \begin{pmatrix} \frac{1}{2}+\widehat{T}_0(\tau) & \widehat{S}_0(\tau)\\ \overline{\widehat{S}_0(\tau)} & 0\end{pmatrix}, \tag{5.1.12}$$

where T_0, S_0 are defined respectively in (5.1.9), (5.1.7). In particular, we have with $\Phi = (\phi_1,\phi_2) \in L^2(\mathbb{R};\mathbb{C}^2)$,

$$\langle \widetilde{A}_0\Phi,\Phi\rangle_{L^2(\mathbb{R};\mathbb{C}^2)} = \int_{\mathbb{R}} e^{2i\pi t\tau}\langle \mathcal{M}_0(\tau)\hat{\Phi}(\tau),\hat{\Phi}(\tau)\rangle_{\mathbb{C}^2}d\tau.$$

Remark 5.1.10. As a consequence of Theorem 5.1.9, we find that the spectrum of the self-adjoint bounded operator A_0 is the closure of the set of eigenvalues of the matrices $\mathcal{M}_0(\tau)$ when τ runs on the real line.

Proof. The proof follows readily from Remarks 5.1.7, 5.1.8 and Lemmas 5.1.4, 5.1.5. ■

Lemma 5.1.11. *Let $\mathcal{N}$ be a 2×2 Hermitian matrix*

$$\mathcal{N} = \begin{pmatrix} a_{11} & a_{12} \\ \overline{a_{12}} & 0 \end{pmatrix}.$$

Then, the eigenvalues $\lambda_- \leq \lambda_+$ of $\mathcal{N}$ are such that

$$\lambda_- < 0 < 1 < \lambda_+, \tag{5.1.13}$$

if and only if

$$a_{12} \neq 0 \quad \text{and} \quad |a_{12}|^2 > 1 - a_{11}. \tag{5.1.14}$$

Proof. The characteristic polynomial of $\mathcal{N}$ is $p(\lambda) = \lambda^2 - a_{11}\lambda - |a_{12}|^2$ and since a_{11} is real-valued, has two real roots $\lambda_- \leq \lambda_+$. If (5.1.14) holds true, the roots are distinct and

$$p(0) = -|a_{12}|^2 < 0, \quad p(1) = 1 - a_{11} - |a_{12}|^2 < 0,$$

implying (5.1.13). Conversely, if (5.1.13) is satisfied, then $p(0)$, $p(1)$ are both negative, implying (5.1.14), completing the proof of the lemma. ■

Lemma 5.1.12. *Let us define for $\omega \in \mathbb{R}$,*

$$I(\omega) = \frac{1}{4\pi} \int_0^{+\infty} \frac{\sin(t\omega)}{\cosh(t/2)} dt.$$

Then, we have

$$I(\omega) = \frac{1}{4\pi\omega} + O(\omega^{-3}), \quad |\omega| \to +\infty.$$

Proof. Indeed, we have for $\omega \in \mathbb{R}^*$,

$$\begin{aligned} I(\omega) &= -\frac{1}{4\pi\omega} \int_0^{+\infty} \frac{\frac{d}{dt}\cos(t\omega)}{\cosh(t/2)} dt \\ &= \frac{1}{4\pi\omega}\left(1 - \int_0^{+\infty} \frac{\cos(t\omega)}{(\cosh(t/2))^2} \frac{1}{2} \sinh(t/2) dt\right) \\ &= \frac{1}{4\pi\omega}(1 + g(\omega)), \end{aligned}$$

with

$$
\begin{aligned}
g(\omega) &= -\int_0^{+\infty} \frac{d}{\omega dt}\{\sin(t\omega)\}\operatorname{sech}(t/2)\frac{1}{2}\tanh(t/2)dt \\
&= \frac{1}{2\omega}\int_0^{+\infty} \sin(t\omega)\frac{d}{dt}\big\{\operatorname{sech}(t/2)\tanh(t/2)\big\}dt \\
&= -\frac{1}{2\omega^2}\int_0^{+\infty} \frac{d}{dt}\{\cos(t\omega)\}\frac{d}{dt}\big\{\operatorname{sech}(t/2)\tanh(t/2)\big\}dt \\
&= \frac{1}{2\omega^2}\left\{\int_0^{+\infty} \cos(t\omega)\frac{d^2}{dt^2}\big\{\operatorname{sech}(t/2)\tanh(t/2)\big\}dt + \frac{1}{2}\right\} = O(\omega^{-2}),
\end{aligned}
$$

proving the lemma. ■

Proposition 5.1.13. *The matrix $\mathcal{M}_0(\tau)$ defined in (5.1.12) is equal to*

$$
\mathcal{M}_0(\tau) = \begin{pmatrix} a_{11}(\tau) & a_{12}(\tau) \\ \overline{a_{12}(\tau)} & 0 \end{pmatrix}, \tag{5.1.15}
$$

with

$$
1 - a_{11}(\tau) = \int_\tau^{+\infty} 2\rho_0(\tau')d\tau', \quad a_{12}(\tau) = \frac{i}{4\pi}\int_0^{+\infty} \frac{e^{-2i\pi\tau t}}{\cosh(t/2)}dt. \tag{5.1.16}
$$

We have

$$
1 - a_{11}(\tau) = O(\tau^{-N}) \quad \textit{for any } N \textit{ when } \tau \to +\infty, \tag{5.1.17}
$$

$$
\operatorname{Re}(a_{12}(\tau)) = \frac{1}{8\pi^2\tau} + O(\tau^{-3}) \quad \textit{when } \tau \to +\infty. \tag{5.1.18}
$$

Proof. Formulas (5.1.15), (5.1.16) follow from Theorem 5.1.9, (5.1.11), and (5.1.7). The estimates (5.1.17) follow from the fact that ρ_0 belongs to the Schwartz class and (5.1.18) is a reformulation of Lemma 5.1.12. ■

Theorem 5.1.14. *Let A_0 be the operator with Weyl symbol $H(x)H(\xi)$, where H is the Heaviside function. Then, A_0 is a bounded self-adjoint operator on $L^2(\mathbb{R})$ such that*

$$
\inf\big(\operatorname{spectrum}(A_0)\big) < 0 < 1 < \sup\big(\operatorname{spectrum}(A_0)\big). \tag{5.1.19}
$$

Proof. Using Remark 5.1.10 and Proposition 5.1.13 we find that for τ large enough, Conditions (5.1.14) are satisfied, proving readily (5.1.19). ■

Corollary 5.1.15 (A counterexample to Flandrin's conjecture). *There exists a function $\phi_0 \in \mathscr{S}(\mathbb{R})$, with $L^2(\mathbb{R})$ norm equal to 1 such that*

$$
\iint_{x\geq 0, \xi\geq 0} \mathcal{W}(\phi_0, \phi_0)(x,\xi)dx d\xi > 1.
$$

There exists $a > 0$ such that $\iint_{0\leq x\leq a, 0\leq\xi\leq a} \mathcal{W}(\phi_0, \phi_0)(x,\xi)dx d\xi > 1$.

Remark 5.1.16. In [13, page 2178], we find the sentence *"it is conjectured that*

$$\forall u \in L^2(\mathbb{R}), \qquad \iint_{\mathcal{C}} \mathcal{W}(u,u)(x,\xi)dxd\xi \le \|u\|^2_{L^2(\mathbb{R})}, \tag{5.1.20}$$

is true for any convex domain $\mathcal{C}$", a quite mild commitment for the validity of (5.1.20), although that statement was referred to later on as *Flandrin's conjecture* in the literature. The second part of the above corollary is providing a disproof of that conjecture based upon an "abstract" argument used in the proof of Theorem 5.1.14; the result of that corollary was already known via a numerical analysis argument after our joint work [6] with B. Delourme and T. Duyckaerts.

Proof. From Theorem 5.1.14, we find $u_0 \in L^2(\mathbb{R})$ such that

$$\|u_0\|^2_{L^2(\mathbb{R})} < \langle A_0 u_0, u_0 \rangle.$$

Let $\psi \in \mathscr{S}(\mathbb{R})$: we have

$$\begin{aligned}|\langle A_0 u_0, u_0\rangle - \langle A_0 \psi, \psi\rangle| &= |\langle A_0(u_0 - \psi), u_0\rangle + \langle A_0\psi, u_0 - \psi\rangle| \\ &\le \|A_0\|_{\mathcal{B}(L^2(\mathbb{R}))}\|u_0 - \psi\|_{L^2(\mathbb{R})}\big(\|u_0\|_{L^2(\mathbb{R})} + \|\psi\|_{L^2(\mathbb{R})}\big),\end{aligned}$$

and thus if $(\psi_k)_{k\ge 1}$ is a sequence of $\mathscr{S}(\mathbb{R})$ converging towards u_0 in $L^2(\mathbb{R})$, we get

$$\begin{aligned}&\|u_0\|^2_{L^2(\mathbb{R})} < \langle A_0 u_0, u_0\rangle \\ &\quad \le \langle A_0\psi_k, \psi_k\rangle + \underbrace{\|A_0\|_{\mathcal{B}(L^2(\mathbb{R}))}\|u_0 - \psi_k\|_{L^2(\mathbb{R})}\big(\|u_0\|_{L^2(\mathbb{R})} + \|\psi_k\|_{L^2(\mathbb{R})}\big)}_{=\sigma_k, \text{ goes to 0 when } k\to+\infty.}.\end{aligned}$$

There exists $k_0 \ge 1$ such that for $k \ge k_0$, we have

$$0 \le \sigma_k \le \frac{1}{2}\big(\langle A_0 u_0, u_0\rangle - \|u_0\|^2_{L^2(\mathbb{R})}\big) = \frac{\varepsilon_0}{2}, \qquad \varepsilon_0 > 0.$$

We obtain that for $k \ge k_0$,

$$\|u_0\|^2_{L^2(\mathbb{R})} < \langle A_0 u_0, u_0\rangle \le \langle A_0\psi_k, \psi_k\rangle + \frac{\varepsilon_0}{2},$$

and thus

$$\begin{aligned}\|\psi_k\|^2_{L^2(\mathbb{R})} &= \underbrace{\|\psi_k\|^2_{L^2(\mathbb{R})} - \|u_0\|^2_{L^2(\mathbb{R})}}_{=\theta_k, \text{ goes to 0 when } k\to+\infty} + \|u_0\|^2_{L^2(\mathbb{R})} \\ &= \theta_k + \langle A_0 u_0, u_0\rangle - \varepsilon_0 \le \theta_k + \langle A_0\psi_k, \psi_k\rangle + \frac{\varepsilon_0}{2} - \varepsilon_0 \\ &= \langle A_0\psi_k, \psi_k\rangle + \theta_k - \frac{\varepsilon_0}{2}.\end{aligned}$$

Choosing now $k \geq k_0$ and k large enough to have $\theta_k < \varepsilon_0/4$, we get

$$\|\psi_k\|^2_{L^2(\mathbb{R})} \leq \langle A_0\psi_k, \psi_k\rangle - \frac{\varepsilon_0}{4} < \langle A_0\psi_k, \psi_k\rangle,$$

and since for $\tilde{\phi} = \psi_k$, the Wigner distribution $\mathcal{W}(\tilde{\phi}, \tilde{\phi})$ belongs to $\mathscr{S}(\mathbb{R}^2)$, we have

$$\|\tilde{\phi}\|^2_{L^2(\mathbb{R})} < \langle A_0\tilde{\phi}, \tilde{\phi}\rangle = \iint H(x)H(\xi)\mathcal{W}(\tilde{\phi}, \tilde{\phi})(x, \xi)dxd\xi,$$

and noting that this strict inequality above implies that $\tilde{\phi} \neq 0$, we may set $\phi_0 = \tilde{\phi}/\|\tilde{\phi}\|$ and get the first statement in the corollary.

N.B. The proof above is complicated by the fact that the identity

$$\langle a^w u, u\rangle_{L^2(\mathbb{R}^n)} = \iint_{\mathbb{R}^{2n}} a(x, \xi)\mathcal{W}(u, u)(x, \xi)dxd\xi,$$

is valid a priori for $u \in \mathscr{S}(\mathbb{R}^n)$ (and in that case $\mathcal{W}(u, u)$ belongs to $\mathscr{S}(\mathbb{R}^{2n})$), but could be meaningless as a Lebesgue integral even for $\mathrm{Op_w}(a)$ bounded on $L^2(\mathbb{R}^n)$ and $u \in L^2(\mathbb{R}^n)$, since we shall have $\mathcal{W}(u, u) \in L^2(\mathbb{R}^{2n})$ but not in $L^1(\mathbb{R}^{2n})$ (we shall see in Chapter 6 that generically the Wigner distribution of a pulse u in $L^2(\mathbb{R}^n)$ does *not* belong to $L^1(\mathbb{R}^{2n})$).

Since $\mathcal{W}(\phi, \phi)$ belongs to the Schwartz space of $\mathbb{R}^2$, the Lebesgue dominated convergence theorem provides the last statement in the corollary. ■

N.B. The reader will notice that the results of the incoming Section 5.2 in the special case $\sigma = 0$ imply the results of Section 5.1, which could be then erased, say at the second reading. However, as far as the first – and maybe only – reading is concerned, we checked that most of the computational arguments in the next section are much more involved and it seemed worthwhile to the author to avoid unnecessary complications for the disproof of Flandrin's conjecture via the quarter-plane example and set apart the more involved examples of the hyperbolic regions tackled in Section 5.2.

5.2 Hyperbolic regions

We consider in this section the (5.0.1) set $\mathcal{C}_\sigma$ with a non-negative σ.

5.2.1 A preliminary observation

We want to consider the operator A_σ with Weyl symbol $H(x)H(x\xi - \sigma)$ and as in Section 5.1.1, we would like to secure the fact that A_σ is bounded on $L^2(\mathbb{R})$.

Claim 5.2.1. For all $\sigma \ge 0$ the operator A_σ is bounded self-adjoint on $L^2(\mathbb{R})$.

Proof of the claim. Let us choose

$$\chi_0 \in C^\infty(\mathbb{R};[0,1]) \quad \text{with} \begin{cases} \chi_0(t) = 0 & \text{for } t \le 1, \\ \chi_0(t) = 1 & \text{for } t \ge 2. \end{cases} \tag{5.2.1}$$

For $\phi, \psi \in \mathscr{S}(\mathbb{R})$, we have

$$\begin{aligned} &\langle (A_0 - A_\sigma)\phi, \psi\rangle_{\mathscr{S}^*(\mathbb{R}),\mathscr{S}(\mathbb{R})} \\ &= \iint H(x)H(\xi)H(\sigma - x\xi) \underbrace{\mathcal{W}(\phi,\psi)(x,\xi)}_{\in \mathscr{S}(\mathbb{R}^2)} dx d\xi \\ &= \lim_{\varepsilon\to 0_+} \iint \chi_0(x/\varepsilon)H(\xi)H(\sigma - x\xi)\mathcal{W}(\phi,\psi)(x,\xi) dx d\xi. \end{aligned} \tag{5.2.2}$$

The kernel $k_{\sigma,\varepsilon}$ of the operator with Weyl symbol $\chi_0(x/\varepsilon)H(\xi)H(\sigma - x\xi)$ is

$$\ell_{\sigma,\varepsilon}(x,y) = \chi_0\left(\frac{x+y}{2\varepsilon}\right) e^{2i\pi\sigma\frac{x-y}{x+y}} \frac{\sin(\frac{2\pi\sigma(x-y)}{x+y})}{\pi(x-y)},$$

and we have

$$\begin{aligned} &\iint \ell_{\sigma,\varepsilon}(x,y)\phi(y)\bar{\psi}(x) dy dx \\ &= \iint \chi_0\left(\frac{x+y}{2\varepsilon}\right) e^{2i\pi\sigma\frac{x-y}{x+y}} \frac{\sin(\frac{2\pi\sigma(x-y)}{x+y})}{\pi(x-y)} \phi(y)\bar{\psi}(x) dx dy \\ &= \iint \chi_0\left(\frac{x+y}{2}\right) e^{2i\pi\sigma\frac{x-y}{x+y}} \frac{\sin(\frac{2\pi\sigma(x-y)}{x+y})}{\pi\varepsilon(x-y)} \phi(\varepsilon y)\bar{\psi}(\varepsilon x)\varepsilon^2 dx dy \\ &= \iint \underbrace{\chi_0\left(\frac{x+y}{2}\right) e^{2i\pi\sigma\frac{x-y}{x+y}} \frac{\sin(\frac{2\pi\sigma(x-y)}{x+y})}{\pi(x-y)}}_{m_\sigma(x,y)} \underbrace{\phi(\varepsilon y)\varepsilon^{1/2}}_{\phi_\varepsilon(y)} \underbrace{\bar{\psi}(\varepsilon x)\varepsilon^{1/2}}_{\bar{\psi}_\varepsilon(x)} dy dx. \end{aligned} \tag{5.2.3}$$

We note that, assuming as we may that $\sigma > 0$,

$$\begin{aligned} &|m_\sigma(x,y)H(x)H(y)| \\ &= \chi_0\left(\frac{x+y}{2}\right) \left| \frac{\sin(\frac{2\pi\sigma(x-y)}{x+y})}{\frac{2\pi\sigma(x-y)}{x+y}} \right| \frac{2\sigma H(x)H(y)}{x+y} \le \frac{2\sigma H(x)H(y)}{x+y}, \end{aligned} \tag{5.2.4}$$

and

$$|m_\sigma(x,y)\check{H}(x)H(y)| = \chi_0\left(\frac{x+y}{2}\right) \left| \frac{\sin(\frac{2\pi\sigma(x-y)}{x+y})}{\pi(x-y)} \right| \check{H}(x)H(y) \le \frac{\check{H}(x)H(y)}{\pi(y-x)}, \tag{5.2.5}$$

as well as

$$|m_\sigma(x,y)\check{H}(y)H(x)| = \chi_0\Bigl(\frac{x+y}{2}\Bigr)\left|\frac{\sin(\frac{2\pi\sigma(x-y)}{x+y})}{\pi(x-y)}\right|\check{H}(y)H(x) \le \frac{\check{H}(y)H(x)}{\pi(x-y)}. \tag{5.2.6}$$

As a consequence, since we have also $m_\sigma(x,y)\check{H}(x)\check{H}(y) \equiv 0$, the inequalities (5.2.4), (5.2.5), (5.2.6), the identities (5.2.3), (5.2.2) and Proposition A.5.1 imply that

$$\begin{aligned}|\langle (A_0 - A_\sigma)\phi, \psi\rangle_{\mathscr{S}^*(\mathbb{R}),\mathscr{S}(\mathbb{R})}| &\le 2\pi\sigma \underbrace{\|H\phi_\varepsilon\|_{L^2(\mathbb{R})}}_{\|H\phi\|_{L^2(\mathbb{R})}}\|H\psi_\varepsilon\|_{L^2(\mathbb{R})} \\ &+ \underbrace{\|\check{H}\phi_\varepsilon\|_{L^2(\mathbb{R})}}_{\|\check{H}\phi\|_{L^2(\mathbb{R})}}\|H\psi_\varepsilon\|_{L^2(\mathbb{R})} + \|H\phi_\varepsilon\|_{L^2(\mathbb{R})}\|\check{H}\psi_\varepsilon\|_{L^2(\mathbb{R})},\end{aligned}$$

proving that $A_0 - A_\sigma$ is bounded on $L^2(\mathbb{R})$; with Proposition 5.1.3, this implies that A_σ is also bounded on $L^2(\mathbb{R})$, proving the claim. ∎

N.B. With that important piece of information in Claim 5.2.1, we shall be less strict in manipulations of kernels and accept below some abuse of language in these matters.

The Weyl quantization of $\mathbf{1}_{\mathscr{C}_\sigma}$ has the kernel

$$k_\sigma(x,y) = H(x+y)e^{4i\pi\sigma(\frac{x-y}{x+y})}\frac{1}{2}\Bigl(\delta_0(y-x) + \frac{1}{i\pi}\mathrm{pv}\frac{1}{y-x}\Bigr), \tag{5.2.7}$$

a formula to be compared to (5.1.2). Using the Schwartz function ϕ_0 of Corollary 5.1.15, we get from Lebesgue dominated convergence theorem that for σ small enough

$$\langle \mathrm{Op_w}(\mathbf{1}_{\mathscr{C}_\sigma})\phi_0, \phi_0\rangle_{L^2(\mathbb{R})} = \iint_{x\xi\ge\sigma, x>0} \mathcal{W}(\phi_0,\phi_0)(x,\xi)dxd\xi > 1.$$

However, this argument does not work for large positive σ and we must go back to a direct calculation.

5.2.2 Diagonal terms

Denoting by A_σ the operator with kernel (5.2.7) (and Weyl symbol $H(x\xi-\sigma)H(x)$), we find that for $u \in \mathscr{S}(\mathbb{R})$, $u_+ = Hu$, we have

$$\begin{aligned}&\langle A_\sigma Hu, Hu\rangle_{L^2(\mathbb{R})} \\ &= \iint e^{4i\pi\sigma(\frac{x-y}{x+y})}\frac{1}{2}\Bigl(\delta_0(y-x) + \frac{1}{i\pi}\mathrm{pv}\frac{1}{y-x}\Bigr)u_+(y)\bar{u}_+(x)dydx \\ &= \frac{1}{2}\|u_+\|^2_{L^2(\mathbb{R}_+)} + \iint_{\mathbb{R}^2} e^{4i\pi\sigma(\frac{e^s-e^t}{e^s+e^t})}\frac{1}{2i\pi}\mathrm{pv}\frac{1}{e^t-e^s}u_+(e^t)\bar{u}_+(e^s)e^{s+t}dsdt \\ &= \frac{1}{2}\|u_+\|^2_{L^2(\mathbb{R}_+)} + \iint_{\mathbb{R}^2} e^{4i\pi\sigma\tanh(\frac{s-t}{2})}\frac{1}{2i\pi}\mathrm{pv}\frac{e^{(s+t)/2}}{e^t-e^s}\phi_1(t)\bar{\phi}_1(s)dsdt,\end{aligned}$$

with

$$\phi_1(t) = u_+(e^t)e^{t/2},$$

so that

$$\|\phi_1\|_{L^2(\mathbb{R})} = \|u_+\|_{L^2(\mathbb{R}_+)}.$$

We get

$$\langle A_\sigma Hu, Hu\rangle_{L^2(\mathbb{R})} = \frac{1}{2}\|u\|^2_{L^2(\mathbb{R}_+)} + \frac{1}{4i\pi}\iint_{\mathbb{R}^2} \frac{e^{4i\pi\sigma\tanh(\frac{s-t}{2})}}{\sinh(\frac{t-s}{2})}\phi(t)\bar{\phi}(s)dsdt,$$

and noting that $\sinh x = xC(x)$, with C even such that $1/C \in \mathscr{S}(\mathbb{R})$, we find

$$\begin{aligned}\langle A_\sigma Hu, Hu\rangle_{L^2(\mathbb{R})} &= \frac{1}{2}\|\phi_1\|^2_{L^2(\mathbb{R})} - \frac{1}{2i\pi}\iint_{\mathbb{R}^2} \frac{e^{4i\pi\sigma\tanh(\frac{s-t}{2})}}{(s-t)C(\frac{s-t}{2})}\phi(t)\bar{\phi}(s)dsdt \\ &= \frac{1}{2}\|\phi_1\|^2_{L^2(\mathbb{R})} + \langle T_\sigma * \phi_1, \phi_1\rangle_{L^2(\mathbb{R})} \\ &= \int_{\mathbb{R}} |\hat{\phi}_1(\tau)|^2\Big(\frac{1}{2} + \widehat{T}_\sigma(\tau)\Big)d\tau, \end{aligned} \tag{5.2.8}$$

with

$$T_\sigma(t) = \frac{1}{4}\frac{te^{4i\pi\sigma\tanh(\frac{t}{2})}}{\sinh(t/2)}\mathrm{pv}\frac{i}{\pi t}. \tag{5.2.9}$$

We note that

$$\widehat{T}_\sigma(\tau) = \operatorname{sign} * \rho_\sigma,$$

with

$$\rho_\sigma(\tau) = \frac{1}{4}\int \frac{te^{4i\pi\sigma\tanh(\frac{t}{2})}}{\sinh(t/2)}e^{-2i\pi t\tau}dt, \quad \rho_\sigma \in \mathscr{S}(\mathbb{R}), \tag{5.2.10}$$

since the function

$$\mathbb{R} \ni t \mapsto \frac{te^{4i\pi\sigma\tanh(\frac{t}{2})}}{\sinh(t/2)}$$

belongs to the Schwartz space[3]. Note also that the function ρ_σ is real-valued on the real line. This entails that

$$\frac{d}{d\tau}\Big\{\frac{1}{2} + \widehat{T}_\sigma\Big\} = 2\rho_\sigma, \tag{5.2.11}$$

and since

$$\rho_\sigma(\tau) = \frac{1}{4}\mathcal{F}\Big\{t \mapsto \frac{te^{4i\pi\sigma\tanh(t/2)}}{\sinh(t/2)}\Big\},$$

[3]Indeed, the iterated derivatives of tanh are polynomials of tanh (check this by induction on the order of derivatives) and thus bounded on the real line; since the function $t \mapsto t/\sinh(t/2)$ belongs to the Schwartz space, this proves that the above product is in $\mathscr{S}(\mathbb{R})$.

implying

$$\int_{\mathbb{R}} \rho_\sigma(\tau) d\tau = \frac{1}{2},$$

we get that

$$\lim_{\tau \to \pm\infty} \widehat{T}_\sigma(\tau) = \pm\frac{1}{2}. \tag{5.2.12}$$

This yields that

$$\frac{1}{2} + \widehat{T}_\sigma(\tau) - 1 = \int_{+\infty}^{\tau} 2\rho_\sigma(\tau')d\tau' = -1 + \int_{-\infty}^{\tau} 2\rho_\sigma(\tau')d\tau',$$

where the last equality follows from (5.2.12): indeed, we have for $\tau > 0$, from (5.2.11),

$$\frac{1}{2} + \widehat{T}_\sigma(\tau) - 1 = \int_{+\infty}^{\tau} 2\rho_\sigma(\tau')d\tau' = -1 + \int_{-\infty}^{\tau} 2\rho_\sigma(\tau')d\tau', \tag{5.2.13}$$

and for $\tau < 0$,

$$\frac{1}{2} + \widehat{T}_\sigma(\tau) = \int_{-\infty}^{\tau} 2\rho_\sigma(\tau')d\tau' = 1 + \int_{+\infty}^{\tau} 2\rho_\sigma(\tau')d\tau'.$$

We note that

$$\forall N \in \mathbb{N}, \quad \sup_{\tau \in \mathbb{R}} |\tau|^N \left| \frac{1}{2} + \widehat{T}_\sigma(\tau) - H(\tau) \right| < +\infty. \tag{5.2.14}$$

Indeed, for $\tau > 0$, we have, using $\rho_\sigma \in \mathscr{S}(\mathbb{R})$,

$$\left| \tau^N \int_{+\infty}^{\tau} \rho_\sigma(\tau')d\tau' \right| \le \int_{\tau}^{+\infty} |\rho_\sigma(\tau')| \tau'^N d\tau' \le \int_{0}^{+\infty} |\rho_\sigma(\tau')| \tau'^N d\tau < +\infty.$$

Also, for $\tau < 0$, we have

$$\left| \tau^N \int_{-\infty}^{\tau} \rho_\sigma(\tau')d\tau' \right| \le \int_{-\infty}^{\tau} |\rho_\sigma(\tau')| |\tau'|^N d\tau' \le \int_{-\infty}^{0} |\rho_\sigma(\tau')| |\tau'|^N d\tau < +\infty.$$

This means that the Fourier multiplier $\frac{1}{2} + \widehat{T}_\sigma(\tau)$ is somehow "exponentially close" to $H(\tau)$ for large values of $|\tau|$ and in particular close to 1 for large positive values of τ. We have also

$$\begin{aligned} \widehat{T}_\sigma(\tau) &= \frac{i}{4\pi} \int_{\mathbb{R}} e^{-2i\pi\tau t} \frac{e^{4i\pi\sigma \tanh(\frac{t}{2})}}{\sinh(t/2)} dt \\ &= \frac{1}{2\pi} \int_0^{+\infty} \frac{\sin(2\pi t\tau - 4\pi\sigma \tanh(t/2))}{\sinh(t/2)} dt. \end{aligned} \tag{5.2.15}$$

The next lemma provides more precise estimates than (5.2.14).

Lemma 5.2.2. *Let $\tau > 0, \sigma \ge 0$. Defining $a_{11}(\tau,\sigma) = \frac{1}{2} + \widehat{T}_\sigma(\tau)$ as given by (5.2.9), we have*

$$|1 - a_{11}(\tau,\sigma)| \le 2e^{-\pi^2\tau}e^{4\pi\sigma}. \tag{5.2.16}$$

Proof. Using (5.2.13) and Lemma A.6.3, we find that for $\tau > 0$,

$$\begin{aligned}
|1 - a_{11}(\tau,\sigma)| &\le 2\int_\tau^{+\infty} |\rho_\sigma(\tau')| d\tau' \\
&\le 2\int_\tau^{+\infty} |\rho_\sigma(\tau')| d\tau' \\
&\le 12e^{4\pi\sigma}\int_\tau^{+\infty} e^{-\pi^2\tau'} d\tau' \\
&= e^{4\pi\sigma}\frac{12}{\pi^2}e^{-\pi^2\tau},
\end{aligned}$$

entailing the sought result. ∎

5.2.3 Off-diagonal terms

We want now to check the off-diagonal terms: we have with $u \in \mathscr{S}(\mathbb{R})$,

$$\begin{gathered}
u_+ = Hu, \quad u_- = \check{H}u, \\
\phi_1(t) = u_+(e^t)e^{t/2}, \quad \phi_2(t) = u_-(-e^t)e^{t/2},
\end{gathered}$$

and

$$\begin{aligned}
&\langle A_\sigma \check{H}u, Hu\rangle_{L^2(\mathbb{R})} \\
&\quad = \iint e^{4i\pi\sigma(\frac{x-y}{x+y})}\frac{H(x+y)\check{H}(y)H(x)}{2i\pi}\mathrm{pv}\frac{1}{y-x}u_-(y)\bar{u}_+(x)dydx \\
&\quad = \iint e^{4i\pi\sigma(\frac{e^s+e^t}{e^s-e^t})}\frac{H(e^s-e^t)}{2i\pi}\mathrm{pv}\frac{1}{-e^t-e^s}\phi_2(t)\bar{\phi}_1(s)e^{\frac{t+s}{2}}dtds \\
&\quad = \iint e^{4i\pi\sigma\coth(\frac{s-t}{2})}\frac{iH(s-t)}{4\pi}\frac{1}{\cosh(\frac{t-s}{2})}\phi_2(t)\bar{\phi}_1(s)dtds \\
&\quad = \frac{i}{4\pi}\iint e^{4i\pi\sigma\coth(\frac{s-t}{2})}H(s-t)\frac{1}{\cosh(\frac{s-t}{2})}\phi_2(t)\bar{\phi}_1(s)dtds \\
&\quad = \langle S_\sigma * \phi_2, \phi_1\rangle_{L^2(\mathbb{R})},
\end{aligned} \tag{5.2.17}$$

with

$$S_\sigma(t) = \frac{i}{4\pi}H(t)\frac{e^{4i\pi\sigma\coth(\frac{t}{2})}}{\cosh(\frac{t}{2})}. \tag{5.2.18}$$

We have also that

$$
\begin{aligned}
\widehat{S}_\sigma(\tau) &= \frac{i}{4\pi}\int H(t)\frac{e^{4i\pi\sigma\coth(\frac{t}{2})}}{\cosh(\frac{t}{2})}e^{-2i\pi t\tau}dt\\
&= \frac{i}{4\pi}\int_0^{+\infty}\frac{\cos(4\pi\sigma\coth(t/2)-2\pi t\tau)}{\cosh(\frac{t}{2})}dt\\
&\quad -\frac{1}{4\pi}\int_0^{+\infty}\frac{\sin(4\pi\sigma\coth(t/2)-2\pi t\tau)}{\cosh(\frac{t}{2})}dt\\
&= \frac{i}{4\pi}\int_0^{+\infty}\frac{\cos(2\pi t\tau-4\pi\sigma\coth(t/2))}{\cosh(\frac{t}{2})}dt\\
&\quad +\frac{1}{4\pi}\int_0^{+\infty}\frac{\sin(2\pi t\tau-4\pi\sigma\coth(t/2))}{\cosh(t/2)}dt. \qquad (5.2.19)
\end{aligned}
$$

Note that from (5.2.9), (5.2.10), we have

$$
\widehat{T}_\sigma(\tau) = \frac{i}{4\pi}\int\frac{e^{4i\pi\sigma\tanh(\frac{t}{2})}}{\sinh(t/2)}e^{-2i\pi t\tau}dt = \frac{1}{2\pi}\int_0^{+\infty}\frac{\sin(2\pi t\tau-4\pi\sigma\tanh(t/2))}{\sinh(t/2)}dt.
$$

5.2.4 An isometric isomorphism

Theorem 5.2.3. *Let $\sigma\ge 0$ be given, let $\mathcal{C}_\sigma$ be the set defined by (5.0.1) and let A_σ be the operator with Weyl symbol $\mathbf{1}_{\mathcal{C}_\sigma}$, (whose kernel is given by (5.2.7)). The operator A_σ is bounded self-adjoint on $L^2(\mathbb{R})$ so that we may define, with Ψ defined in (5.1.8),*

$$
\widetilde{A}_\sigma = \Psi A_\sigma\Psi^{-1}.
$$

The operator $\widetilde{A}_\sigma$ is the Fourier multiplier on $L^2(\mathbb{R};\mathbb{C}^2)$ given by the matrix

$$
\mathcal{M}_\sigma(\tau) = \begin{pmatrix}\frac{1}{2}+\widehat{T}_\sigma(\tau) & \widehat{S}_\sigma(\tau)\\ \overline{\widehat{S}_\sigma(\tau)} & 0\end{pmatrix}, \qquad (5.2.20)
$$

where T_σ, S_σ are defined respectively in (5.2.9), (5.2.15), (5.2.18). In particular, we have with $\Phi=(\phi_1,\phi_2)\in L^2(\mathbb{R};\mathbb{C}^2)$,

$$
\langle\widetilde{A}_\sigma\Phi,\Phi\rangle_{L^2(\mathbb{R};\mathbb{C}^2)} = \int_{\mathbb{R}} e^{2i\pi t\tau}\langle\mathcal{M}_\sigma(\tau)\hat{\Phi}(\tau),\hat{\Phi}(\tau)\rangle_{\mathbb{C}^2}d\tau. \qquad (5.2.21)
$$

Proof. We have

$$
\begin{aligned}
&\operatorname{kernel}(HA_\sigma H) = e^{4i\pi\sigma\frac{x-y}{x+y}}H(x)H(y)\widehat{H}(y-x),\\
&\operatorname{kernel}(\check{H}A_\sigma H + HA_\sigma\check{H})\\
&\qquad = e^{4i\pi\sigma\frac{x-y}{x+y}}H(x+y)\big(\check{H}(x)H(y)+H(x)\check{H}(y)\big)\frac{1}{2i\pi(y-x)},\\
&\check{H}A_\sigma\check{H} = 0.
\end{aligned}
$$

Proposition A.5.1 in Appendix A.7 is readily giving the L^2-boundedness (and self-adjointness) of

$$\check{H} A_\sigma H + H A_\sigma \check{H}.$$

We find also that $H A_\sigma H - \frac{H}{2}$ has kernel

$$e^{4i\pi\sigma\frac{x-y}{x+y}} H(x) H(y) \frac{1}{2i\pi(y-x)},$$

and thus it is enough to study the operator with kernel

$$e^{4i\pi\sigma\frac{e^s-e^t}{e^s+e^t}} \frac{e^{\frac{s+t}{2}}}{2i\pi(e^t-e^s)} = e^{4i\pi\sigma\tanh(\frac{s-t}{2})} \frac{1}{4i\pi\sinh(\frac{t-s}{2})},$$

which is a convolution operator by

$$T_\sigma(t) = e^{4i\pi\sigma\tanh(\frac{t}{2})} \frac{t}{4\sinh(\frac{t}{2})} \mathrm{pv}\frac{i}{\pi t},$$

given by (5.2.9). Formula (5.2.10) implies in particular that $\widehat{T_\sigma}$ is bounded (and real-valued) on the real line, entailing eventually the boundedness and self-adjointness of A_σ. Formulas (5.2.8), (5.2.17), and (5.2.18) are providing (5.2.21), completing the proof of the theorem. ∎

5.2.5 The main result on hyperbolic regions

Theorem 5.2.4. *Let $\sigma \geq 0$ be given and let A_σ be the operator defined in Theorem 5.2.3. Then, A_σ is a bounded self-adjoint operator on $L^2(\mathbb{R})$ such that*

$$\inf\big(\mathrm{spectrum}(A_\sigma)\big) < 0 < 1 < \sup\big(\mathrm{spectrum}(A_\sigma)\big).$$

The spectrum of A_σ is the closure of the set of eigenvalues of $\mathcal{M}_\sigma(\tau)$ for τ running on the real line.

Remark 5.2.5. It is enough to prove that, with a given $\sigma \geq 0$, there exists $\tau \in \mathbb{R}$ such that $\mathcal{M}_\sigma(\tau)$ satisfies (5.1.14).

Proof. We have from (5.2.20), (5.2.15), and (5.2.19),

$$\mathcal{M}_\sigma(\tau) = \begin{pmatrix} \frac{1}{2} + \frac{1}{2\pi}\int_0^{+\infty} \frac{\sin(2\pi t\tau - 4\pi\sigma\tanh(t/2))}{\sinh(t/2)} dt & \cdot & \frac{i}{4\pi}\int_0^{+\infty} \frac{e^{-2i\pi(t\tau - \frac{2\sigma}{\tanh(t/2)})}}{\cosh(t/2)} dt \\ \frac{1}{4i\pi}\int_0^{+\infty} \frac{e^{2i\pi(t\tau - \frac{2\sigma}{\tanh(t/2)})}}{\cosh(t/2)} dt & \cdot & 0 \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}(\tau,\sigma) & a_{12}(\tau,\sigma) \\ a_{21}(\tau,\sigma) & a_{22}(\tau,\sigma) \end{pmatrix}. \tag{5.2.22}$$

On the other hand, we have

$$\overline{a_{12}} = a_{21} = \frac{1}{4i\pi}\int_0^{+\infty} \frac{e^{2i\pi(t\tau - \frac{2\sigma}{\tanh(t/2)})}}{\cosh(t/2)}dt, \tag{5.2.23}$$

so that

$$\operatorname{Re} a_{12}(\tau,\sigma) = \frac{1}{4\pi}\int_0^{+\infty} \frac{\sin[2\pi(t\tau - \frac{2\sigma}{\tanh(\frac{t}{2})})]}{\cosh(\frac{t}{2})}dt. \tag{5.2.24}$$

We note that the function

$$t \mapsto \frac{e^{2i\pi(t\tau - \frac{2\sigma}{\tanh(t/2)})}}{\cosh(t/2)},$$

is holomorphic on $\mathbb{C}\backslash i\pi\mathbb{Z}$, with simple poles at $(2\mathbb{Z}+1)i\pi$ (zeroes of $\cosh(t/2)$) and essential singularities at $2\mathbb{Z}i\pi$ (zeroes of $\sinh(t/2)$). We shall need a more explicit quantitative expression for a_{21} to obtain a precise asymptotic result which could be compared to the estimate (5.2.16). The next lemma is proven in [55]; we provide a proof here for the convenience of the reader.

Lemma 5.2.6. *Let $\tau > 0, \sigma \geq 0$ be given and let $a_{21}(\tau,\sigma)$ be given by (5.2.23). We have*

$$\begin{aligned}\operatorname{Re} a_{21}(\tau,\sigma) = \frac{e^{-2\pi^2\tau}}{4\pi}\Bigg\{&\int_0^{\pi}\Big(\frac{e^{2\pi(t\tau-2\sigma\tan(t/2))}-1}{\sin(t/2)} + \frac{\sinh(t/2)-\sin(t/2)}{\sinh(t/2)\sin(t/2)}\Big)dt \\ &+ \int_0^{\pi}\frac{1-\cos 2\pi(t\tau - 2\sigma\tanh(t/2))}{\sinh(t/2)}dt \\ &- \int_{\pi}^{+\infty}\frac{\cos 2\pi(t\tau-2\sigma\tanh(t/2))}{\sinh(t/2)}dt\Bigg\}.\end{aligned} \tag{5.2.25}$$

Proof of Lemma 5.2.6. Let $0 < \varepsilon < \pi/2 < \pi < R$ be given. We consider the closed path $\gamma_{\varepsilon,R}$ of $\mathbb{C}\backslash i\pi\mathbb{Z}$ with $\operatorname{index}_{\gamma_{\varepsilon,R}}(i\pi\mathbb{Z}) \equiv 0$,

$$\begin{aligned}\gamma_{\varepsilon,R} = {}& [\varepsilon, R] \cup [R, R+i\pi] \cup [R+i\pi, \varepsilon+i\pi] \\ &\cup \{i\pi + \varepsilon e^{i\theta}\}_{0\geq\theta\geq-\pi/2} \cup i[\pi-\varepsilon,\varepsilon] \cup \{\varepsilon e^{i\theta}\}_{\pi/2\geq\theta\geq 0},\end{aligned} \tag{5.2.26}$$

and we have

$$\oint_{\gamma_{\varepsilon,R}} \frac{e^{2i\pi(z\tau-\frac{2\sigma}{\tanh(z/2)})}}{\cosh(z/2)}dz = 0. \tag{5.2.27}$$

We note as well that

$$\begin{aligned}I_2 &= \oint_{[R,R+i\pi]} \frac{e^{2i\pi(z\tau-\frac{2\sigma}{\tanh(z/2)})}}{\cosh(z/2)}dz = i\int_0^{\pi}\frac{e^{2i\pi((R+it)\tau - \frac{2\sigma}{\tanh(\frac{R+it}{2})})}}{\cosh(\frac{R+it}{2})}dt \\ &= ie^{2i\pi R\tau}\int_0^{\pi} e^{-2\pi t\tau}e^{-4i\pi\sigma\frac{1+e^{-R-it}}{1-e^{-R-it}}}\frac{2dt}{e^{\frac{R+it}{2}}(1+e^{-R-it})},\end{aligned} \tag{5.2.28}$$

so that

$$|I_2| \le 2e^{-R/2}\int_0^{\pi} e^{4\pi\sigma\,\mathrm{Im}\left(\frac{1+e^{-R-it}}{1-e^{-R-it}}\right)}\frac{dt}{|1-e^{-R}|},$$

and since

$$\mathrm{Im}\left(\frac{1+e^{-R-it}}{1-e^{-R-it}}\right) = \mathrm{Im}\,\frac{(1+e^{-R-it})(1-e^{-R+it})}{|1-e^{-R-it}|^2} = \frac{-2e^{-R}\sin t}{|1-e^{-R-it}|^2} \le 0,$$

we get

$$|I_2| \le e^{-R/2}\frac{2\pi}{1-e^{-R}}, \quad \text{where } I_2 \text{ is defined in (5.2.28).} \tag{5.2.29}$$

We note for future reference the standard formulas,

$$\cosh\left(\frac{i\pi}{2}+z\right) = i\sinh z, \quad \sinh\left(\frac{i\pi}{2}+z\right) = i\cosh z, \quad \tanh\left(\frac{i\pi}{2}+z\right) = \coth z, \tag{5.2.30}$$

and we check now

$$\begin{aligned} I_4 &= -\int_{-\pi/2}^{0} \frac{e^{2i\pi((i\pi+\varepsilon e^{i\theta})\tau - 2\sigma\coth(\frac{i\pi+\varepsilon e^{i\theta}}{2}))}}{\cosh\frac{i\pi+\varepsilon e^{i\theta}}{2}} i\varepsilon e^{i\theta}\,d\theta \\ &= -e^{-2\pi^2\tau}\int_{-\pi/2}^{0} \frac{e^{2i\pi(\varepsilon e^{i\theta}\tau - 2\sigma\tanh(\frac{\varepsilon e^{i\theta}}{2}))}}{i\sinh\frac{\varepsilon e^{i\theta}}{2}} i\varepsilon e^{i\theta}\,d\theta, \end{aligned} \tag{5.2.31}$$

and since

$$\left|\frac{e^{2i\pi(\varepsilon e^{i\theta}\tau - 2\sigma\tanh(\frac{\varepsilon e^{i\theta}}{2}))}}{i\sinh\frac{\varepsilon e^{i\theta}}{2}} i\varepsilon e^{i\theta}\right| \le 2\max_{|z|\le\pi/2}\left|\frac{z}{\sinh z}\right| e^{\pi^2\tau}e^{4\pi\sigma\sup_{|z|\le\pi/4}\left|\frac{\sinh z}{\cosh z}\right|},$$

the Lebesgue dominated convergence theorem gives

$$\lim_{\varepsilon\to 0_+} I_4 = -\pi e^{-2\pi^2\tau}. \tag{5.2.32}$$

Defining now

$$I_6 = -\int_0^{\pi/2} \frac{e^{2i\pi(\varepsilon e^{i\theta}\tau - 2\sigma\coth(\frac{\varepsilon e^{i\theta}}{2}))}}{\cosh\frac{\varepsilon e^{i\theta}}{2}} i\varepsilon e^{i\theta}\,d\theta, \tag{5.2.33}$$

and noting that

$$\begin{aligned} 4\pi\sigma\,\mathrm{Im}\coth\left(\frac{\varepsilon e^{i\theta}}{2}\right) &= 4\pi\sigma\,\mathrm{Im}\,\frac{1+e^{-\varepsilon e^{i\theta}}}{1-e^{-\varepsilon e^{i\theta}}} = 4\pi\sigma\,\mathrm{Im}\,\frac{(1+e^{-\varepsilon e^{i\theta}})(1-e^{-\varepsilon e^{-i\theta}})}{|1-e^{-\varepsilon e^{i\theta}}|^2} \\ &= 4\pi\sigma\,\mathrm{Im}\,\frac{e^{-\varepsilon e^{i\theta}} - e^{-\varepsilon e^{-i\theta}}}{|1-e^{-\varepsilon e^{i\theta}}|^2} = 4\pi\sigma\,\mathrm{Im}\,\frac{e^{-\varepsilon\cos\theta}(e^{-i\varepsilon\sin\theta} - e^{i\varepsilon\sin\theta})}{|1-e^{-\varepsilon e^{i\theta}}|^2} \\ &= 4\pi\sigma e^{-\varepsilon\cos\theta}\,\mathrm{Im}\,\frac{(-2i)\sin(\varepsilon\sin\theta)}{|1-e^{-\varepsilon e^{i\theta}}|^2} = -4\pi\sigma e^{-\varepsilon\cos\theta}\frac{2\sin(\varepsilon\sin\theta)}{|1-e^{-\varepsilon e^{i\theta}}|^2} \le 0, \end{aligned}$$

we get that

$$|I_6| \le \int_0^{\pi/2} \frac{e^{-2\pi\varepsilon\tau\sin\theta}}{\min_{|z|\le\pi/4}|\cosh z|} d\theta\varepsilon \le \varepsilon \frac{\pi/2}{\min_{|z|\le\pi/4}|\cosh z|},$$

entailing

$$\lim_{\varepsilon\to 0_+} I_6 = 0. \tag{5.2.34}$$

With

$$I_1 = \oint_{[\varepsilon,R]} \frac{e^{2i\pi(z\tau - \frac{2\sigma}{\tanh(z/2)})}}{\cosh(z/2)} dz, \tag{5.2.35}$$

we have from (5.2.23)

$$\lim_{\substack{\varepsilon\to 0_+\\ R\to+\infty}} I_1 = 4i\pi a_{21}. \tag{5.2.36}$$

We define now

$$\begin{aligned}
I_5 &= -\oint_{[i\varepsilon, i(\pi-\varepsilon)]} \frac{e^{2i\pi(z\tau - \frac{2\sigma}{\tanh(z/2)})}}{\cosh(z/2)} dz = -\int_\varepsilon^{\pi-\varepsilon} \frac{e^{2i\pi(it\tau - \frac{2\sigma}{\tanh(it/2)})}}{\cosh(it/2)} i\,dt\\
&= -\int_\varepsilon^{\pi-\varepsilon} e^{-2\pi t\tau} \frac{e^{\frac{-4i\pi\sigma}{i\tan(t/2)}}}{\cos(t/2)} i\,dt = -i\int_\varepsilon^{\pi-\varepsilon} e^{-2\pi t\tau} \frac{e^{\frac{-4\pi\sigma}{\tan(t/2)}}}{\cos(t/2)} dt\\
&= -i\int_\varepsilon^{\pi-\varepsilon} e^{-2\pi(\pi-s)\tau} \frac{e^{-\frac{4\pi\sigma}{\tan((\pi-s)/2)}}}{\cos((\pi-s)/2)} ds\\
&= -ie^{-2\pi^2\tau}\int_\varepsilon^{\pi-\varepsilon} e^{2\pi s\tau} \frac{e^{-\frac{4\pi\sigma\sin(s/2)}{\cos(s/2)}}}{\sin(s/2)} ds,
\end{aligned}$$

so that

$$I_5 = -ie^{-2\pi^2\tau}\int_\varepsilon^{\pi-\varepsilon} e^{2\pi s\tau} \frac{e^{-4\pi\sigma\tan(s/2)}}{\sin(s/2)} ds. \tag{5.2.37}$$

We have also

$$\begin{aligned}
I_3 &= \oint_{[R+i\pi, \varepsilon+i\pi]} \frac{e^{2i\pi(z\tau - \frac{2\sigma}{\tanh(z/2)})}}{\cosh(z/2)} dz\\
&= -\int_\varepsilon^R \frac{e^{2i\pi((t+i\pi)\tau - \frac{2\sigma}{\tanh((t+i\pi)/2)})}}{\cosh((t+i\pi)/2)} dt,
\end{aligned} \tag{5.2.38}$$

so that using Formulas (5.2.30), we get

$$I_3 = -e^{-2\pi^2\tau}\int_\varepsilon^R \frac{e^{2i\pi(t\tau - 2\sigma\tanh(t/2))}}{i\,\sinh(t/2)} dt,$$

and

$$I_3 + I_5 = ie^{-2\pi^2\tau}\left(\int_\varepsilon^R \frac{e^{2i\pi(t\tau-2\sigma\tanh(t/2))}}{\sinh(t/2)}dt - \int_\varepsilon^{\pi-\varepsilon} e^{2\pi t\tau}\frac{e^{-4\pi\sigma\tan(t/2)}}{\sin(t/2)}dt\right)$$

$$= ie^{-2\pi^2\tau}\left\{\int_\varepsilon^{\pi-\varepsilon}\left(\frac{e^{2i\pi(t\tau-2\sigma\tanh(t/2))}}{\sinh(t/2)} - \frac{e^{2\pi(t\tau-2\sigma\tan(t/2))}}{\sin(t/2)}\right)dt\right.$$

$$\left. + \int_{\pi-\varepsilon}^R \frac{e^{2i\pi(t\tau-2\sigma\tanh(t/2))}}{\sinh(t/2)}dt\right\}. \tag{5.2.39}$$

From (5.2.27), (5.2.26), (5.2.28), (5.2.31), (5.2.33), (5.2.35), and (5.2.37), (5.2.38), we find that

$$I_1 = -I_2 - (I_3 + I_5) - I_4 - I_6,$$

so that taking the limit of both sides[4] when $\varepsilon \to 0_+$, $R \to +\infty$ we get, thanks to (5.2.36), (5.2.29), (5.2.39), (5.2.32), and (5.2.34),

$$4i\pi a_{21} = -ie^{-2\pi^2\tau}\left\{\int_0^\pi\left(\frac{e^{2i\pi(t\tau-2\sigma\tanh(t/2))}}{\sinh(t/2)} - \frac{e^{2\pi(t\tau-2\sigma\tan(t/2))}}{\sin(t/2)}\right)dt\right.$$

$$\left. + \int_\pi^{+\infty}\frac{e^{2i\pi(t\tau-2\sigma\tanh(t/2))}}{\sinh(t/2)}dt\right\} + \pi e^{-2\pi^2\tau},$$

implying that

$$a_{21} = \frac{e^{-2\pi^2\tau}}{4\pi}\left\{\int_0^\pi\left(-\frac{e^{2i\pi(t\tau-2\sigma\tanh(t/2))}}{\sinh(t/2)} + \frac{e^{2\pi(t\tau-2\sigma\tan(t/2))}}{\sin(t/2)}\right)dt\right.$$

$$\left. - \int_\pi^{+\infty}\frac{e^{2i\pi(t\tau-2\sigma\tanh(t/2))}}{\sinh(t/2)}dt\right\} - \frac{i}{4}e^{-2\pi^2\tau}$$

that is

$$a_{21} = \frac{e^{-2\pi^2\tau}}{4\pi}\int_0^\pi\left(\frac{e^{2\pi(t\tau-2\sigma\tan(t/2))}}{\sin(t/2)} - \frac{\cos 2\pi(t\tau-2\sigma\tanh(t/2))}{\sinh(t/2)}\right)dt$$

$$- \frac{e^{-2\pi^2\tau}}{4\pi}\int_\pi^{+\infty}\frac{\cos 2\pi(t\tau-2\sigma\tanh(t/2))}{\sinh(t/2)}dt$$

$$- i\frac{e^{-2\pi^2\tau}}{4\pi}\int_0^\pi\frac{\sin 2\pi(t\tau-2\sigma\tanh(t/2))}{\sinh(t/2)}dt - \frac{i}{4}e^{-2\pi^2\tau}$$

$$- i\frac{e^{-2\pi^2\tau}}{4\pi}\int_\pi^{+\infty}\frac{\sin 2\pi(t\tau-2\sigma\tanh(t/2))}{\sinh(t/2)}dt, \tag{5.2.40}$$

[4] $I_1, I_2, I_4, I_6, I_3 + I_5$ do have limits when $\varepsilon \to 0_+$, $R \to +\infty$.

yielding

$$\begin{aligned}\operatorname{Re} a_{21} &= \frac{e^{-2\pi^2\tau}}{4\pi}\int_0^{\pi}\left(\frac{e^{2\pi(t\tau-2\sigma\tan(t/2))}}{\sin(t/2)} - \frac{\cos 2\pi(t\tau-2\sigma\tanh(t/2))}{\sinh(t/2)}\right)dt \\ &\quad - \frac{e^{-2\pi^2\tau}}{4\pi}\int_{\pi}^{+\infty}\frac{\cos 2\pi(t\tau-2\sigma\tanh(t/2))}{\sinh(t/2)}dt,\end{aligned}$$

completing the proof of Lemma 5.2.6. ■

Remark 5.2.7. Formula (5.2.40) also yields

$$\begin{aligned}\operatorname{Im} a_{12} = -\operatorname{Im} a_{21} = \frac{e^{-2\pi^2\tau}}{4\pi}\Bigg\{&\int_0^{\pi}\frac{\sin 2\pi(t\tau-2\sigma\tanh(t/2))}{\sinh(t/2)}dt + \pi \\ &+\int_{\pi}^{+\infty}\frac{\sin 2\pi(t\tau-2\sigma\tanh(t/2))}{\sinh(t/2)}dt\Bigg\},\end{aligned}$$

and since from (5.2.22), we have

$$a_{11} = \frac{1}{2} + \frac{1}{2\pi}\int_0^{+\infty}\frac{\sin(2\pi t\tau - 4\pi\sigma\tanh(t/2))}{\sinh(t/2)}dt,$$

this gives

$$\operatorname{Im} a_{12} = \frac{e^{-2\pi^2\tau}}{4\pi}\left(2\pi\left(a_{11}-\frac{1}{2}\right)+\pi\right) = \frac{e^{-2\pi^2\tau}}{2}a_{11}. \tag{5.2.41}$$

To complete the proof of Theorem 5.2.4, it will be enough, according to Lemma 5.1.11, to prove that, for $\tau \to +\infty$, $|a_{12}|^2 \gg 1 - a_{11}$. To achieve that, we note from (5.2.41) that the imaginary part of a_{12} is useless and we shall prove simply that

$$(\operatorname{Re} a_{12})^2 \gg 1 - a_{11}.$$

To get this we are going to use (5.2.16) and a precise asymptotic behavior for $(\operatorname{Re} a_{12})^2$ displayed in the next lemma and issued from the explicit formula (5.2.25).

Lemma 5.2.8. *Let $\tau \geq 1, \sigma \geq 0$ be given and let $a_{21}(\tau,\sigma)$ be given by* (5.2.23). *We have then*

$$\operatorname{Re} a_{21}(\tau,\sigma) \geq \frac{e^{-8\pi\sqrt{\tau}\sqrt{\sigma}}}{8\pi^3\tau} - \frac{1}{2\pi}e^{-2\pi^2\tau}. \tag{5.2.42}$$

Proof of Lemma 5.2.8. Since for $t \geq 0$ we have $\sinh(t/2) - \sin(t/2) \geq 0$, we get from (5.2.25),

$$\begin{aligned}\operatorname{Re} a_{21}(\tau,\sigma) &\geq \frac{e^{-2\pi^2\tau}}{4\pi}\left\{\int_0^{\pi}\frac{e^{2\pi(t\tau-2\sigma\tan(t/2))}-1}{\sin(t/2)}dt - \int_{\pi}^{+\infty}\frac{1}{\sinh(t/2)}dt\right\} \\ &= \frac{e^{-2\pi^2\tau}}{4\pi}\int_0^{\pi}\frac{e^{2\pi(t\tau-2\sigma\tan(t/2))}-1}{\sin(t/2)}dt - \frac{e^{-2\pi^2\tau}}{2\pi}\ln\left(\coth\frac{\pi}{4}\right).\end{aligned}$$

Let us define

$$\omega = 2\pi\tau, \quad \kappa = 2\pi\sigma, \quad \nu = \kappa^{1/2}\omega^{-1/2}, \quad \phi_\nu(s) = s - \nu^2 \tan s. \tag{5.2.43}$$

We have

$$2\pi\big(t\tau - 2\sigma\tan(t/2)\big) = 2\pi\tau(t - 2\nu^2\tan(t/2)) = 4\pi\tau\left(\frac{t}{2} - \nu^2\tan\frac{t}{2}\right) = 2\omega\phi_\nu(t/2).$$

We have thus

$$\operatorname{Re} a_{21}(\tau,\sigma) \geq \frac{e^{-\pi\omega}}{2\pi}\int_0^{\pi/2}\frac{e^{2\omega\phi_\nu(s)}-1}{\sin s}ds - \frac{e^{-\pi\omega}}{2\pi}\underbrace{\ln\left(\coth\frac{\pi}{4}\right)}_{\approx 0.421908}. \tag{5.2.44}$$

Defining

$$\psi_\nu(\omega) = \frac{e^{-\pi\omega}}{2\pi}\int_0^{\pi/2}\frac{e^{2\omega\phi_\nu(s)}-1}{\sin s}ds, \tag{5.2.45}$$

we can use (5.2.43), (5.2.44), and (A.6.13) to get whenever $\tau > 0$,

$$2\pi \operatorname{Re} a_{21}(\tau,\sigma) \geq \frac{e^{-8\pi\sqrt{\tau}\sqrt{\sigma}}}{\pi^2\tau}\left(\frac{1}{2} - \frac{1}{4\tau}\right) - e^{-2\pi^2\tau},$$

so that for $\tau \geq 1$ we find

$$2\pi \operatorname{Re} a_{21}(\tau,\sigma) \geq \frac{e^{-8\pi\sqrt{\tau}\sqrt{\sigma}}}{4\pi^2\tau} - e^{-2\pi^2\tau},$$

yielding the lemma. ∎

We eventually go back to the proof of Theorem 5.2.4: let $\sigma > 0$ be given. From Lemma 5.2.8 and (5.2.16), we have for $\tau \geq 1$,

$$|1 - a_{11}(\tau,\sigma)| \leq 2e^{-\pi^2\tau}e^{4\pi\sigma},$$

$$\operatorname{Re} a_{21}(\tau,\sigma) \geq \frac{e^{-8\pi\sqrt{\tau}\sqrt{\sigma}}}{8\pi^3\tau} - \frac{1}{2\pi}e^{-2\pi^2\tau} = \frac{e^{-8\pi\sqrt{\tau}\sqrt{\sigma}}}{8\pi^3\tau}\left(1 - \frac{4\pi^2\tau e^{8\pi\sqrt{\tau}\sqrt{\sigma}}}{e^{2\pi^2\tau}}\right).$$

This entails that for $\tau \geq \tau_0(\sigma)$, we have

$$\operatorname{Re} a_{21}(\tau,\sigma) \geq \frac{e^{-8\pi\sqrt{\tau}\sqrt{\sigma}}}{16\pi^3\tau}, \tag{5.2.46}$$

and thus $a_{21} \neq 0$ and

$$|a_{21}(\sigma,\tau)|^2 \geq \frac{e^{-16\pi\sqrt{\tau}\sqrt{\sigma}}}{2^8\pi^6\tau^2} > |1 - a_{11}(\tau,\sigma)|, \tag{5.2.47}$$

where the last inequality above holds true (thanks to (5.2.16)) whenever

$$2e^{-\pi^2\tau}e^{4\pi\sigma} < \frac{e^{-16\pi\sqrt{\tau}\sqrt{\sigma}}}{2^8\pi^6\tau^2},$$

which is indeed true for $\tau \ge \tau_1(\sigma)$. As a result for $\tau \ge \max(4\sigma, 4, \tau_0(\sigma), \tau_1(\sigma))$, we obtain that (5.2.47) is satisfied so that Remark 5.2.5 implies the result of Theorem 5.2.4, completing our proof. ∎

Remark 5.2.9. The functions $\tau_0(\sigma), \tau_1(\sigma)$ can be determined rather easily, the first one by the condition

$$\tau \ge \tau_0(\sigma) \Longrightarrow \frac{4\pi^2\tau e^{8\pi\sqrt{\tau}\sqrt{\sigma}}}{e^{2\pi^2\tau}} \le \frac{1}{2},$$

whereas the second one should satisfy

$$\tau \ge \tau_1(\sigma) \Longrightarrow e^{4\pi\sigma}2^9\pi^6\tau^2e^{16\pi\sqrt{\tau}\sqrt{\sigma}} < e^{\pi^2\tau}.$$

5.3 Comments and further results

5.3.1 Qualitative explanations on the various computations

We would like to go back to our proofs that

$$|a_{12}(\tau,\sigma)|^2 \gg |1 - a_{11}(\tau,\sigma)|, \quad \tau \to +\infty, \tag{5.3.1}$$

which is our key argument via Lemma 5.1.11 and give a couple of qualitative explanations which may enlighten the calculations. It is of course much simpler to begin with the case $\sigma = 0$: in that case, according to Proposition 5.1.13 and (5.1.10), we have

$$1 - a_{11}(\tau, 0) = \int_\tau^{+\infty} 2\rho_0(\tau')d\tau', \quad 2\rho_0(\tau) = \int \underbrace{\left(\frac{t/2}{\sinh(t/2)}\right)}_{\substack{=f_0(t),\ f_0 \in \mathscr{S}(\mathbb{R}) \\ \text{holomorphic} \\ \text{on } |\operatorname{Im} t| < 2\pi.}} e^{-2i\pi t\tau}ds,$$

so that $2\rho_0(\tau) = \hat{f}_0(\tau)$. We get thus readily that ρ_0 belongs to the Schwartz space, as the Fourier transform of a function in the Schwartz space and this implies in particular that $1 - a_{11}(\tau, 0)$ has fast decay towards 0 when $\tau \to +\infty$, as proven in Proposition 5.1.13. We note also that (5.2.41) gives $\operatorname{Im} a_{12}(\tau, 0)^2 = e^{-4\pi^2\tau}a_{11}(\tau, 0)^2/4$, and since the limit of a_{11} is 1, we do not expect any help from the imaginary part of a_{12} to proving (5.3.1). Turning our attention to $\operatorname{Re} a_{12}$ in (5.1.18), we have

$$4\pi \operatorname{Re} a_{21}(\tau, 0) = \int_0^{+\infty} \frac{\sin(2\pi t\tau)}{\cosh(t/2)}dt, \tag{5.3.2}$$

which is the sine-Fourier transform of the function $t \mapsto H(t)\operatorname{sech}(t/2) = g_0(t)$, which has a singularity at $t = 0$: as a consequence, thanks to Lemma A.1.1, the Fourier transform $\widehat{g_0}$ cannot be rapidly decreasing, cannot even belong to $L^1(\mathbb{R})$ (that would imply that g_0 is continuous). Moreover, the sine-Fourier transform above is the Fourier transform of the odd part of g_0, $g_{\text{odd}}(t) = \operatorname{sech}(t/2)\operatorname{sign} t$, which is also singular at 0, thus $\widehat{g_{\text{odd}}}$ cannot be rapidly decreasing and is an odd function, which is enough to prove, without more calculations, that (5.3.1) holds true. In Section 5.1, we used a more explicit argument, with providing an equivalent of (5.3.2) equal to $1/(2\pi\tau)$ near $+\infty$. Summing-up, (5.3.1) in the case $\sigma = 0$ follows from the existence of a singularity of the function g_0 above, which is discontinuous at 0.

Let us now take a look at the case $\sigma > 0$, which turns out to be more computationally involved. We have from (5.2.23)

$$\begin{aligned}4\pi i a_{21}(\tau, \sigma) &= \int_{\mathbb{R}} H(t)\operatorname{sech}(t/2) e^{-i4\pi\sigma\coth(t/2)} e^{2i\pi t\tau} dt = \check{\widehat{g_\sigma}}(\tau),\\ g_\sigma(t) &= H(t)\operatorname{sech}(t/2) e^{-i4\pi\sigma\coth(t/2)}.\end{aligned}$$

The single discontinuity at $t = 0$ of g_σ when $\sigma > 0$ is much wilder than for $\sigma = 0$: in the latter case, we had only a jump discontinuity with different limits on both sides, whereas when $\sigma > 0$, we have an essential discontinuity with an oscillatory behaviour in $(-1, +1)$ when $t \to 0_+$ for the real and imaginary parts of a_{12}. However, g_σ belongs to all $L^p(\mathbb{R})$, $p \in [1, +\infty]$, so that its Fourier transform belongs to $L^p(\mathbb{R})$, $p \in [2, +\infty]$: we expect then that both sides of (5.3.1) have limit 0 for $\tau \to +\infty$ and we must prove that $1 - a_{11}$ decays much faster than a_{12}. Looking at a slightly simplified model and using the notations (5.2.43), we define for ω, ν positive, a function α presumably close to $4\pi i a_{21}$, given by

$$\alpha(\omega, \nu) = \int_0^{+\infty} e^{i2\omega\mu_\nu(s)}\operatorname{sech}(s) ds, \quad \mu_\nu(s) = s - \frac{\nu^2}{s}, \quad \mu_\nu'(s) = 1 + \frac{\nu^2}{s^2}.$$

Trying our hand with the stationary phase method, we look at

$$\begin{aligned}\alpha(\omega, \nu) &= \frac{1}{2i\omega}\int_0^{+\infty} \frac{d}{ds}\{e^{i2\omega\mu_\nu(s)}\}\frac{\operatorname{sech}(s)}{\mu_\nu'(s)} ds\\ &= \frac{1}{2i\omega}\int_0^{+\infty} \frac{d}{ds}\{e^{i2\omega\mu_\nu(s)}\}\frac{s^2\operatorname{sech}(s)}{s^2 + \nu^2} ds\\ &= \frac{i}{2\omega}\int_0^{+\infty} e^{i2\omega\mu_\nu(s)}\frac{d}{ds}\left\{\frac{s^2\operatorname{sech}(s)}{s^2 + \nu^2}\right\},\end{aligned}$$

since the boundary term vanishes. Iterating that computation shows that $\alpha(\omega, \nu) = O_\sigma(\omega^{-N})$ for all N when $\omega \to +\infty$, meaning that the information of fast decay for $1 - a_{11}$ will not suffice to get (5.3.1). Also, it is worth noticing that no fast decay of the function α occurs when $\omega \to -\infty$, otherwise Lemma A.1.1 would give smoothness for the function $s \mapsto e^{-2i\kappa/s} H(s) \operatorname{sech} s$: in fact, we see also that for $\sigma > 0$, $\tau = -\lambda$, $\lambda > 0$, we have

$$2\pi i a_{21}(-\lambda, \sigma) = \int_0^{+\infty} \operatorname{sech}(s) e^{-i4\pi\sigma \coth(s)} e^{-4i\pi s\lambda} ds,$$

and the phase function is $\tilde{\mu}(s) = -4i\pi(s\lambda + \sigma \coth(s))$ and we have

$$\frac{d}{ds}\{s\lambda + \sigma \coth(s)\} = \lambda - \frac{\sigma(1 - \tanh^2 s)}{\tanh^2 s} = \frac{(\lambda + \sigma)\tanh^2 s - \sigma}{\tanh^2 s},$$

which does vanish at $\tanh s = \sigma/(\lambda + \sigma)$. As a result we could say that, for $\sigma > 0$, the C^∞ wave-front-set (see, e.g., [23, Section 8.1]) of the function g_σ is reduced to $\{0\} \times (-\infty, 0)$. It turns out that we can show that the Gevrey-2 wave-front-set of g_σ is $\{0\} \times \mathbb{R}^*$, and it is expressed via the lowerbound estimate (5.2.42); the route that we took for proving this was an explicit calculation of $\operatorname{Re} a_{12}$, following the paper [55]. Finally, the upper bound (5.2.16) can be improved as

$$|1 - a_{11}(\tau, \sigma)| \le C_{\sigma,\varepsilon} e^{-(\pi-\varepsilon)2\pi\tau}, \quad \varepsilon > 0,$$

and is expressing the fact that function

$$t \mapsto \frac{t e^{4i\pi\sigma \tanh(\frac{t}{2})}}{\sinh(\frac{t}{2})}$$

is analytic on the real line, with a radius of convergence on the real line bounded below by π (cf. Proposition A.1.2).

5.3.2 More results and examples: ℓ^p balls, corners

For a, ϕ_0 like in Corollary 5.1.15, defining

$$\Omega_p = \left\{(x, \xi) \in \mathbb{R}^2, \left|x - \frac{a}{2}\right|^p + \left|\xi - \frac{a}{2}\right|^p < \left(\frac{a}{2}\right)^p\right\},$$

since $\mathcal{W}(\phi_0, \phi_0) \in \mathscr{S}(\mathbb{R}^2)$, we get

$$\lim_{p \to +\infty} \iint_{\Omega_p} \mathcal{W}(\phi_0, \phi_0)(x, \xi) dx d\xi = \iint_{[0,a]^2} \mathcal{W}(\phi_0, \phi_0)(x, \xi) dx d\xi > \|\phi_0\|^2_{L^2(\mathbb{R})},$$

proving that the spectrum of $\operatorname{Op_w}(\mathbf{1}_{\Omega_p})$ intersects $(1, +\infty)$ for p large enough, showing that a counterexample to Flandrin's conjecture can be a convex analytic open bounded set. Moreover, defining

$$Q_a = \{(x, \xi) \in \mathbb{R}^2, |x| + |\xi| \le a/\sqrt{2}\},$$

we note that Q_a is obtained by rotation and translation of $[0,a]^2$ so that we can find ϕ_1 in the Schwartz space such that

$$\iint_{Q_a} \mathcal{W}(\phi_1,\phi_1)(x,\xi)\,dx\,d\xi > \|\phi_1\|^2_{L^2(\mathbb{R})}.$$

Since we have

$$\begin{aligned}\lim_{p\to 1} \iint_{|x|^p+|\xi|^p\le(a/\sqrt{2})^p} &\mathcal{W}(\phi_1,\phi_1)(x,\xi)\,dx\,d\xi \\ &= \iint_{Q_a} \mathcal{W}(\phi_1,\phi_1)(x,\xi)\,dx\,d\xi > \|\phi_1\|^2_{L^2(\mathbb{R})},\end{aligned}$$

we get that for $p-1$ small enough we have

$$\iint_{|x|^p+|\xi|^p\le(a/\sqrt{2})^p} \mathcal{W}(\phi_1,\phi_1)(x,\xi)\,dx\,d\xi > \|\phi_1\|^2_{L^2(\mathbb{R})},$$

proving that ℓ^p balls are counterexamples to Flandrin's conjecture for $p-1$ or $1/p$ small enough.

Convex affine cones with aperture strictly less than π of $\mathbb{R}^2$ are translations and rotations of

$$\Sigma_{\theta_0} = \left\{(x,\xi)\in\mathbb{R}^2\backslash(\mathbb{R}_-\times\{0\}), \arg(x+i\xi)\in(0,\theta_0)\right\} \quad \text{for some } \theta_0\in(0,\pi). \tag{5.3.3}$$

The vertex of Σ_{θ_0} and its rotations is defined as 0 and the vertex of the translation of vector T_0 of Σ_{θ_0} is defined as T_0. We note that all convex affine cones with aperture strictly less than π are symplectically equivalent in $\mathbb{R}^2$, since Σ_{θ_0} is symplectically equivalent to (the interior of) the quarter plane $\Sigma_{\pi/2}$: indeed, let θ_0 be in $(0,\pi)$; the symplectic matrix M_{θ_0} defined by

$$M_{\theta_0} = \begin{pmatrix} 1 & -\operatorname{cotan}\theta_0 \\ 0 & 1\end{pmatrix},$$

is such that $M_{\theta_0}\binom{1}{0} = \binom{1}{0}$, $M_{\theta_0}\begin{pmatrix}\cos\theta_0\\ \sin\theta_0\end{pmatrix} = \binom{0}{\sin\theta_0}$, proving that

$$M_{\theta_0}\Sigma_{\theta_0} = \Sigma_{\pi/2}.$$

The next result follows from [6, Theorem 1.3] and shows that many counterexamples to Flandrin's conjecture can be obtained.

Theorem 5.3.1. *Let K be a subset of the closure of a convex affine cone with aperture strictly less than π and vertex X_0 such that K contains a neighborhood of the vertex* *in the cone*[5]*. Then, there exists* $\lambda > 0$ *such that, with*

$$K_\lambda = X_0 + \lambda(K - X_0),$$

there exists $\phi \in \mathscr{S}(\mathbb{R})$ *such that*

$$\iint_{K_\lambda} \mathcal{W}(\phi, \phi)(x, \xi) dx d\xi > \|\phi\|^2_{L^2(\mathbb{R})}. \tag{5.3.4}$$

N.B. Note that (5.3.4) implies that ϕ is not the zero function. Also, taking K convex produces another counterexample to Flandrin's conjecture since K_λ will be then convex, but we do not need that assumption to proving the result.

Proof. There is no loss of generality at assuming $X_0 = 0$ and

$$[0, \rho_0]^2 \subset K \subset \overline{\Sigma}_{\pi/2}, \quad \rho_0 > 0.$$

Using Corollary 5.1.15, we find $\phi_0 \in \mathscr{S}(\mathbb{R})$ (so that $\mathcal{W}(\phi_0, \phi_0) \in \mathscr{S}(\mathbb{R}^2)$) such that

$$\lim_{\lambda \to +\infty} \iint_{K_\lambda} \mathcal{W}(\phi_0, \phi_0)(x, \xi) dx d\xi = \iint_{\Sigma_{\pi/2}} \mathcal{W}(\phi_0, \phi_0)(x, \xi) dx d\xi > \|\phi_0\|^2_{L^2(\mathbb{R})},$$

implying for λ large enough that $\iint_{K_\lambda} \mathcal{W}(\phi_0, \phi_0)(x, \xi) dx d\xi > \|\phi_0\|^2_{L^2(\mathbb{R})}$, which is the sought result. ∎

5.4 Numerics

Definition 5.4.1. Let $\sigma \geq 0$ be given. With the 2×2 Hermitian matrix $\mathcal{M}_\sigma$ given by (5.2.22), we define for $\tau \in \mathbb{R}$,

$$\lambda_+(\tau, \sigma) = \frac{1}{2}\Big(a_{11}(\tau, \sigma) + \sqrt{a_{11}^2(\tau, \sigma) + 4|a_{12}(\tau, \sigma)|^2}\Big),$$
$$\lambda_-(\tau, \sigma) = \frac{1}{2}\Big(a_{11}(\tau, \sigma) - \sqrt{a_{11}^2(\tau, \sigma) + 4|a_{12}(\tau, \sigma)|^2}\Big).$$

Remark 5.4.2. According to (5.2.41), we have

$$\lambda_+(\tau, \sigma) = \frac{1}{2}\Big(a_{11}(\tau, \sigma) + \sqrt{a_{11}^2(\tau, \sigma)(1 + e^{-4\pi^2\tau}) + 4(\operatorname{Re} a_{12}(\tau, \sigma))^2}\Big), \tag{5.4.1}$$
$$\lambda_-(\tau, \sigma) = \frac{1}{2}\Big(a_{11}(\tau, \sigma) - \sqrt{a_{11}^2(\tau, \sigma)\big(1 + e^{-4\pi^2\tau}\big) + 4(\operatorname{Re} a_{12}(\tau, \sigma))^2}\Big), \tag{5.4.2}$$

so that the knowledge of a_{11} and $\operatorname{Re} a_{12}$ suffices for expressing $\lambda_\pm$.

[5] We shall say that the set K has a corner.

An immediate consequence of Theorem 5.2.4 is the following theorem.

Theorem 5.4.3. *Let $\sigma \ge 0$ be given and let A_σ be the self-adjoint operator bounded in $L^2(\mathbb{R})$ defined in Theorem 5.2.4. With the notations of Definition 5.4.1, we have*

$$M_\sigma := \sup\{\text{spectrum}(A_\sigma)\} = \sup_{\tau\in\mathbb{R}} \lambda_+(\tau,\sigma), \tag{5.4.3}$$

$$m_\sigma := \inf\{\text{spectrum}(A_\sigma)\} = \inf_{\tau\in\mathbb{R}} \lambda_-(\tau,\sigma). \tag{5.4.4}$$

Moreover, for all $\sigma \ge 0$ we have

$$m_\sigma < 0 < 1 < M_\sigma.$$

5.4.1 The quarter-plane: $\sigma = 0$

Of course, as shown by the respective calculations of Sections 5.1 and 5.2, the case $\sigma = 0$, dealing with the quarter-plane is much simpler than the cases where $\sigma > 0$. Nonetheless, we know explicitly a spectral decomposition of the operator with Weyl symbol $H(x)H(\xi)$ from Theorem 5.2.3, but we can calculate without difficulty numerical expressions of M_0, m_0 as defined in (5.4.3), (5.4.4).

Proposition 5.4.4. *We have from (A.6.22), (5.2.24),*

$$a_{11}(\tau,0) = \frac{1}{1+e^{-4\pi^2\tau}}, \quad \operatorname{Re} a_{12}(\tau,0) = \frac{1}{4\pi}\int_0^{+\infty} \sin(2\pi t\tau)\operatorname{sech}(t/2)dt,$$

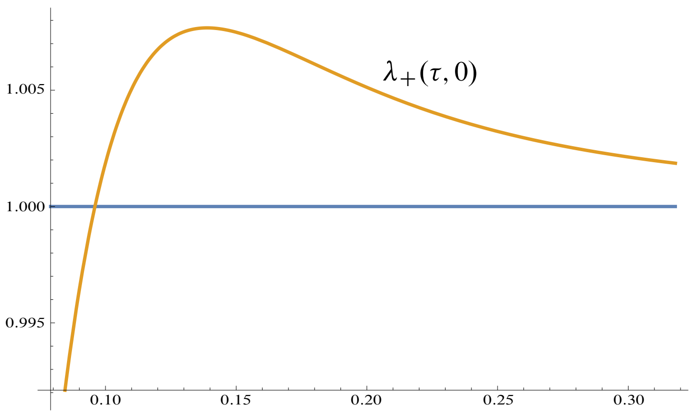

Figure 5.1. The function $\tau \mapsto \lambda_+(\tau,0)$ near its maximum, well above 1.

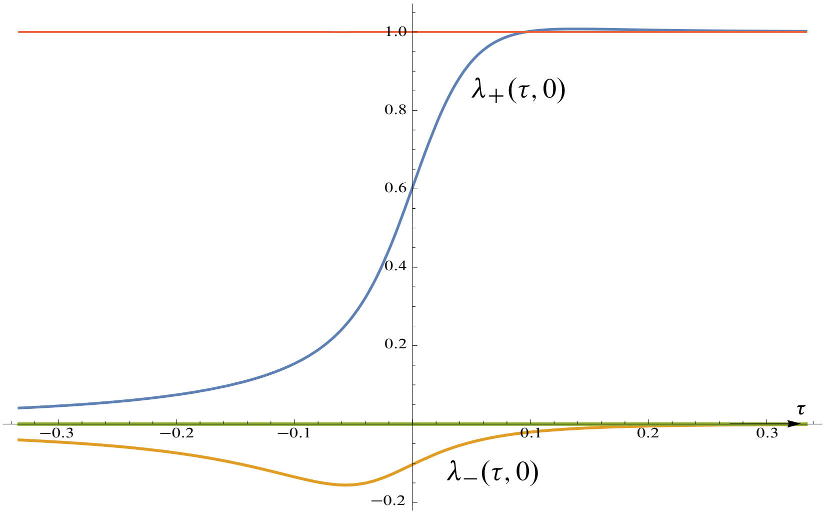

Figure 5.2. The functions $\tau \mapsto \lambda_+(\tau, 0), \lambda_-(\tau, 0)$.

and we can use these formulas and (5.4.1)*,* (5.4.2)*,* (5.4.3)*, and* (5.4.4) *to calculate numerically*

$$M_0 \approx \mathbf{1.00767997007003}, \quad (\lambda_+(\tau, 0) \textit{ at } \tau \approx 0.138815397930141),$$
$$m_0 \approx -0.155939843191243, \quad (\lambda_-(\tau, 0) \textit{ at } \tau \approx -0.0566304954736227).$$

5.4.2 On hyperbolic regions

We want now to tackle the case $\sigma > 0$. In order to use the expressions (A.6.22), (5.2.25) respectively for a_{11} and a_{12}, we need first to evaluate the residue term in (A.6.22). The mapping $z \mapsto \tanh z$ is a biholomorphism of neighborhoods of 0 in the complex plane, so that we have for z near the origin,

$$\zeta = \tanh z, \quad d\zeta = (1-\zeta^2)dz, \quad z = \operatorname{arcth} \zeta = \frac{1}{2} \ln\left(\frac{1+\zeta}{1-\zeta}\right),$$
$$\frac{e^{2i\omega z - 2i\kappa \coth z}}{\cosh z} dz = \left(\frac{1+\zeta}{1-\zeta}\right)^{i\omega} e^{-2i\frac{\kappa}{\zeta}} \frac{2}{\left(\frac{1+\zeta}{1-\zeta}\right)^{1/2} + \left(\frac{1-\zeta}{1+\zeta}\right)^{1/2}} \frac{d\zeta}{(1-\zeta^2)}$$
$$= (1+\zeta)^{-\frac{1}{2}+i\omega}(1-\zeta)^{-\frac{1}{2}-i\omega} e^{-2i\frac{\kappa}{\zeta}} d\zeta,$$

so that

$$\mathtt{Res}\left(\frac{e^{2i\omega z - 2i\kappa \coth z}}{\cosh z}, 0\right) = \mathtt{Res}((1+\zeta)^{-\frac{1}{2}+i\omega}(1-\zeta)^{-\frac{1}{2}-i\omega} e^{-2i\frac{\kappa}{\zeta}}, 0). \quad (5.4.5)$$

Proposition 5.4.5. *Let $\sigma \ge 0$ be given. Then, for any $\tau \in \mathbb{R}$, using the notations, $\omega = 2\pi\tau$, $\kappa = 2\pi\sigma$, we have, for any $\rho \in (0, 1)$,*

$$a_{11}(\tau, \sigma) = \frac{1}{1 + e^{-2\pi\omega}} + \frac{e^{-\pi\omega}}{1 + e^{-2\pi\omega}} \frac{\rho}{2\pi} \times \operatorname{Im}\left\{ \int_{-\pi}^{\pi} \exp\left(i\omega \operatorname{Log}\left(\frac{1 + \rho e^{i\theta}}{1 - \rho e^{i\theta}}\right)\right) \frac{e^{-\frac{2i\kappa e^{-i\theta}}{\rho}} e^{i\theta}}{\sqrt{1 - \rho^2 e^{2i\theta}}} d\theta \right\}. \tag{5.4.6}$$

$$\operatorname{Re} a_{21}(\tau, \sigma) = \frac{e^{-\pi\omega}}{2\pi} \left\{ 2 \int_0^{\pi/2} \frac{e^{(s\omega - \kappa \tan s)} \sinh(s\omega - \kappa \tan s)}{\sin s} ds + \ln\left(\coth \frac{\pi}{4}\right) + 2 \int_0^{\pi/2} \frac{\sin^2(s\omega - \kappa \tanh s)}{\sinh s} ds - \int_{\pi/2}^{+\infty} \frac{\cos 2(s\omega - \kappa \tanh s)}{\sinh s} ds \right\}, \tag{5.4.7}$$

$$\operatorname{Im} a_{12}(\tau, \sigma) = \frac{e^{-\pi\omega}}{2} a_{11}(\tau, \sigma). \tag{5.4.8}$$

Proof. Formula (5.4.6) follows from (5.4.5) and (A.6.22) whereas (5.4.7) is (5.2.25) after a change of variable $t = 2s$, where the second integral term inside the brackets is evaluated (cf. Lemma A.6.1); formula (5.4.8) is a reminder of (5.2.41). ∎

N.B. Our choice for ρ in the numerical calculations of (5.4.6) is $\rho = 3/4$, which is a good compromise between using a value of ρ clearly away from 1 (to avoid singularities coming from small denominators in the Log term) and minimize the oscillations and size coming from the term $\exp(-2i\kappa\rho^{-1}e^{-i\theta})$; note that the modulus of the latter is

$$\exp(-2\kappa\rho^{-1} \sin\theta),$$

which is a smooth function of ρ (flat at 0) when $\theta \in [0, \pi]$, but is unbounded for $\rho \to 0_+$ when $\theta \in (-\pi, 0)$. There is no surprise here since although the residue does not depend on the choice of $\rho \in (0, 1)$, we cannot get the value of that residue by letting ρ go to 0 because of the part of the path in the lower half-plane. The argument of $\exp(-2i\kappa\rho^{-1}e^{-i\theta})$ is $-2\kappa\rho^{-1}\cos\theta$ and taking ρ too small would be devastating for the calculations because of the strong oscillations triggered by the term $\exp(-2i\kappa\rho^{-1}\cos\theta)$ all over the circle. Of course for the evaluation of $\operatorname{Log}(\frac{1+\rho e^{i\theta}}{1-\rho e^{i\theta}})$ is easier for ρ small, but we have to take into account the constraints in that direction mentioned above.

Remark 5.4.6. It seems easier numerically for the evaluation of a_{11} to use (5.4.6) rather than any other expression (see, e.g., Lemma 5.2.2, (5.2.22), (A.6.14)). However, the following formula could be interesting, theoretically and numerically: recall-

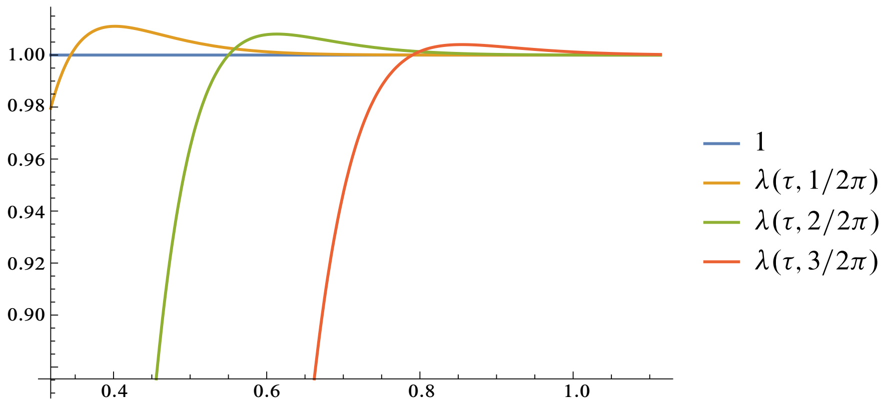

Figure 5.3. Functions $\lambda_+(\tau, \kappa/2\pi)$ with $\kappa = 1, 2, 3$: their maxima are strictly greater than 1.

ing that $\operatorname{sinc} x = \frac{\sin x}{x}$, we have from (5.2.22)

$$\begin{aligned} a_{11}(\tau,\sigma) = \frac{1}{2} + \frac{2\omega}{\pi} \int_0^{+\infty} \operatorname{sinc}(2\omega s) \frac{s}{\sinh s} \cos(2\kappa \tanh s) ds \\ - \frac{2\kappa}{\pi} \int_0^{+\infty} \operatorname{sinc}(2\kappa s) \frac{1}{\cosh s} \cos(2\omega s) ds, \end{aligned} \tag{5.4.9}$$

but it turns out that numerical calculations involving (5.4.9) seem to be less reliable than the methods using (5.4.6).

We can also take a look at the following curves.

Remark 5.4.7. In the above figure, in order to put the three curves on the same picture, we have used three different logarithmic scales on the vertical axis, namely, we have drawn

$$\tau \mapsto 1 + \alpha_j \operatorname{Log}(\lambda_+(\tau, \sigma_j)), \quad 1 \le j \le 3, \sigma_j = j/2\pi, \alpha_1 = 20, \alpha_2 = 100, \alpha_3 = 500.$$

Of course, we have

$$1 + \alpha_j \operatorname{Log}(\lambda_+(\tau, \sigma_j)) > 1 \Longleftrightarrow \operatorname{Log}(\lambda_+(\tau, \sigma_j)) > 0 \Longleftrightarrow \lambda_+(\tau, \sigma_j) > 1,$$

so that the piece of curves in Figure 5.3 which are above 1 are indeed corresponding to curves of $\tau \mapsto \lambda_+(\tau, \sigma_j)$ which go strictly above the threshold 1. We have also

$$\begin{aligned} &\max_\tau \lambda_+(\tau, \sigma_1) \approx 1 + 55 \times 10^{-5} && \text{at } \tau \approx 0.402030, \\ &\max_\tau \lambda_+(\tau, \sigma_2) \approx 1 + 8 \times 10^{-5} && \text{at } \tau \approx 0.613262, \\ &\max_\tau \lambda_+(\tau, \sigma_3) \approx 1 + 10^{-5} && \text{at } \tau \approx 0.854746. \end{aligned}$$

We are glad to have a theoretical proof of Theorem 5.2.4 since the numerical analysis of cases where σ is large, say larger than 10, seems to be very difficult to achieve, at least through a standard use of `Mathematica`. The reason for that is quite clear since using our Lemma 5.1.11, we did study the function β defined by

$$\beta(\tau, \sigma) = |a_{12}(\tau, \sigma)|^2 + a_{11}(\tau, \sigma) - 1, \tag{5.4.10}$$

and proved that for each $\sigma \geq 0$ there exists $T_0(\sigma)$ such that for all $\tau \geq T_0(\sigma)$ we have $\beta(\tau, \sigma) > 0$ and $a_{12}(\tau, \sigma) \neq 0$. Thanks to Lemma 5.2.2 and (5.2.46) we knew that for $\tau \geq T_0(\sigma)$, we had

$$|1 - a_{11}| \leq 2e^{-\pi^2\tau} e^{4\pi\sigma} \ll \frac{e^{-16\pi\sqrt{\tau}\sqrt{\sigma}}}{2^8\pi^6\tau^2} \leq (\operatorname{Re} a_{21})^2 \leq |a_{12}|^2,$$

where the second inequality $\ll$ is in fact comparing for σ fixed two exponential decays. The numerical analysis of that inequality is certainly quite difficult when σ and τ are large since both sides are converging to zero quite fast for σ fixed and $\tau \to +\infty$; of course taking the logarithm of both sides looks quite reasonable, but in practice does not seem really easy numerically. When $\sigma = 0$, the situation is much better, since we had to compare (cf. Section 5.3.1) an exponential decay $|1 - a_{11}| \leq 2e^{-\pi^2\tau}$ to a polynomial decay

$$|\operatorname{Re} a_{12}|^2 \sim \frac{1}{2^6\pi^4\tau^2}, \quad \tau \to +\infty,$$

and this could be an *a posteriori* explanation for which our numerical argument in [6] worked smoothly to disprove Flandrin's conjecture. So to pick up the quarter-plane ((5.0.1) with $\sigma = 0$) to produce a counterexample to that conjecture was indeed a very wise choice: if you choose instead $\mathscr{C}_\sigma$ for σ large, our Theorem 5.2.4 shows that it is also a counterexample to Flandrin's conjecture[6], but we have a theoretical proof for that Theorem and if we depended on a numerical analysis, it is quite likely that checking numerically the positivity of the function β defined in (5.4.10) could be rather difficult, even say for $\sigma = 10$.

[6]As a convex subset of the plane on which the integral of the Wigner distribution of some normalized pulse is strictly larger than 1.

Chapter 6

Unboundedness is Baire generic

In this section, we show that for plenty of subsets E of the phase space $\mathbb{R}^{2n}$, the operator $\operatorname{Op_w}(\mathbf{1}_E)$ is not bounded on $L^2(\mathbb{R}^n)$.

6.1 Preliminaries

6.1.1 Prolegomena

Lemma 6.1.1. *Let $u, v \in L^2(\mathbb{R}^n)$ and let $\mathcal{W}(u,u)$, $\mathcal{W}(v,v)$, be their Wigner distributions. Then, we have*

$$\|\mathcal{W}(u,u) - \mathcal{W}(v,v)\|_{L^2(\mathbb{R}^{2n})} \le \|u-v\|_{L^2(\mathbb{R}^n)}\big(\|u\|_{L^2(\mathbb{R}^n)} + \|v\|_{L^2(\mathbb{R}^n)}\big).$$

As a consequence if a sequence (u_k) is converging in $L^2(\mathbb{R}^n)$, then the sequence $(\mathcal{W}(u_k,u_k))$ converges in $L^2(\mathbb{R}^{2n})$ towards $\mathcal{W}(u,u)$.

Proof. We have by sesquilinearity $\mathcal{W}(u,u) - \mathcal{W}(v,v) = \mathcal{W}(u-v,u) + \mathcal{W}(v,u-v)$, so that

$$\begin{aligned}\|\mathcal{W}(u,u) - \mathcal{W}(v,v)\|_{L^2(\mathbb{R}^{2n})} &\le \|\mathcal{W}(u-v,u)\|_{L^2(\mathbb{R}^{2n})} + \|\mathcal{W}(v,u-v)\|_{L^2(\mathbb{R}^{2n})} \\ &\underbrace{=}_{(1.1.6)} \|u-v\|_{L^2(\mathbb{R}^n)}\big(\|u\|_{L^2(\mathbb{R}^n)} + \|v\|_{L^2(\mathbb{R}^n)}\big),\end{aligned}$$

proving the lemma. ∎

Lemma 6.1.2. *Let (u_k) be a converging sequence in $L^2(\mathbb{R}^n)$ with limit u. Let us assume that there exists $C_0 \ge 0$ such that*

$$\forall k \in \mathbb{N}, \quad \iint |\mathcal{W}(u_k,u_k)(x,\xi)| dx d\xi \le C_0.$$

Then, we have $\iint |\mathcal{W}(u,u)(x,\xi)| dx d\xi \le C_0$.

Proof. Let $R > 0$ be given. We check

$$\begin{aligned}&\iint_{|x|^2+|\xi|^2\le R^2} |\mathcal{W}(u,u)(x,\xi) - \mathcal{W}(u_k,u_k)(x,\xi)| dx d\xi \\ &\quad\le \iint_{|x|^2+|\xi|^2\le R^2} |\mathcal{W}(u-u_k,u)(x,\xi)| dx d\xi \\ &\qquad + \iint_{|x|^2+|\xi|^2\le R^2} |\mathcal{W}(u_k,u-u_k)(x,\xi)| dx d\xi \\ &\quad\le \sqrt{|\mathbb{B}^{2n}| R^{2n}}\big(\|\mathcal{W}(u-u_k,u)\|_{L^2(\mathbb{R}^{2n})} + \|\mathcal{W}(u_k,u-u_k)\|_{L^2(\mathbb{R}^{2n})}\big) \\ &\quad= \sqrt{|\mathbb{B}^{2n}| R^{2n}}\|u-u_k\|_{L^2(\mathbb{R}^n)}\big(\|u\|_{L^2(\mathbb{R}^n)} + \|u_k\|_{L^2(\mathbb{R}^n)}\big),\end{aligned}$$

and thus

$$\begin{aligned}
&\iint_{|x|^2+|\xi|^2\le R^2} |\mathcal{W}(u,u)(x,\xi)|dxd\xi\\
&\quad\le \iint_{|x|^2+|\xi|^2\le R^2} |\mathcal{W}(u_k,u_k)(x,\xi)|dxd\xi\\
&\qquad+\sqrt{|\mathbb{B}^{2n}|R^{2n}}\|u-u_k\|_{L^2(\mathbb{R}^n)}\big(\|u\|_{L^2(\mathbb{R}^n)}+\|u_k\|_{L^2(\mathbb{R}^n)}\big)\\
&\quad\le C_0+\sqrt{|\mathbb{B}^{2n}|R^{2n}}\|u-u_k\|_{L^2(\mathbb{R}^n)}\big(\|u\|_{L^2(\mathbb{R}^n)}+\|u_k\|_{L^2(\mathbb{R}^n)}\big),
\end{aligned}$$

implying for all $R>0$,

$$\iint_{|x|^2+|\xi|^2\le R^2} |\mathcal{W}(u,u)(x,\xi)|dxd\xi \le C_0,$$

and thus the sought result. ■

6.1.2 An explicit construction

We just calculate in this section $\mathcal{W}(v_0,v_0)$ for $v_0=\mathbf{1}_{[-1/2,1/2]}$.

Remark 6.1.3. When u is supported in a closed convex set J, we have in the integral (1.1.4) defining $\mathcal{W}$, $x\pm\frac{z}{2}\in J\Rightarrow x\in J$, so that supp $\mathcal{W}(u,u)\subset J\times\mathbb{R}^n$.

We have

$$\mathcal{W}(v_0,v_0)(x,\xi)=\int_{\substack{-1/2\le x+z/2\le 1/2\\ -1/2\le x-z/2\le 1/2}} e^{2i\pi z\xi}dz,$$

and the integration domain is

$$-\min(1-2x,1+2x)=\max(-1-2x,2x-1)\le z\le\min(1-2x,1+2x),$$

which is empty unless $1-2x, 1+2x\ge 0$, i.e., $x\in[-1/2,+1/2]$, and moreover we have the equivalence

$$1-2x\le 1+2x \Longleftrightarrow x\ge 0,$$

so that

$$\begin{aligned}
&\mathcal{W}(v_0,v_0)(x,\xi)\\
&\quad= H(x)\int_{-(1-2x)}^{1-2x} e^{2i\pi z\xi}dz + H(-x)\int_{-(1+2x)}^{1+2x} e^{2i\pi z\xi}dz\\
&\quad= H(x)\frac{e^{2i\pi\xi(1-2x)}-e^{-2i\pi\xi(1-2x)}}{2i\pi\xi}+H(-x)\frac{e^{2i\pi\xi(1+2x)}-e^{-2i\pi\xi(1+2x)}}{2i\pi\xi}\\
&\quad= \mathbf{1}_{[0,1/2]}(x)\frac{\sin(2\pi\xi(1-2x))}{\pi\xi}+\mathbf{1}_{[-1/2,0]}\frac{\sin(2\pi\xi(1+2x))}{\pi\xi}. \qquad (6.1.1)
\end{aligned}$$

More generally for a, b, ω real numbers with $a < b$ and

$$u_{a,b,\omega}(x) = (b-a)^{-1/2}\mathbf{1}_{[a,b]}(x)e^{2i\pi\omega x}, \tag{6.1.2}$$

we have

$$\begin{aligned}
&\mathcal{W}(u_{a,b,\omega}, u_{a,b,\omega})(x,\xi) \\
&\quad = \frac{\left(\mathbf{1}_{[a,\frac{a+b}{2}]}(x)\sin[4\pi(\xi-\omega)(x-a)] + \mathbf{1}_{[\frac{a+b}{2},b]}(x)\sin[4\pi(\xi-\omega)(b-x)]\right)}{(b-a)\pi(\xi-\omega)}.
\end{aligned}$$

We check now, using (6.1.1), for $N > 0$,

$$\begin{aligned}
\iint |\mathcal{W}(v_0, v_0)(x,\xi)|dxd\xi &\ge \int_{0\le x\le 1/4}\int_0^N \left|\frac{\sin(2\pi\xi(1-2x))}{\pi\xi}\right| d\xi dx \\
&= \int_{0\le x\le 1/4}\int_0^{N2\pi(1-2x)} \left|\frac{\sin\eta}{\pi\eta}\right| d\eta dx \\
&\ge \int_{0\le x\le 1/4}\int_0^{N\pi} \left|\frac{\sin\eta}{\pi\eta}\right| d\eta dx = \frac{1}{4}\int_0^{N\pi}\left|\frac{\sin\eta}{\pi\eta}\right| d\eta,
\end{aligned}$$

so that

$$\iint |\mathcal{W}(v_0, v_0)(x,\xi)|dxd\xi = +\infty. \tag{6.1.3}$$

Proposition 6.1.4. *Let a, b, ω be real numbers with $a < b$ and let us define $u_{a,b,\omega}$ by* (6.1.2). *Then, we have*

$$\iint \left|\mathcal{W}(u_{a,b,\omega}, u_{a,b,\omega})(x,\xi)\right| dxd\xi = +\infty. \tag{6.1.4}$$

N.B. Since $u_{a,b,\omega}$ is a normalized $L^2(\mathbb{R})$ function, we also have from (1.1.6), (1.1.9) that the real-valued $\mathcal{W}(u_{a,b,\omega}, u_{a,b,\omega})$ does satisfy

$$\begin{aligned}
\int\left|\int \mathcal{W}(u_{a,b,\omega}, u_{a,b,\omega})(x,\xi)dx\right| d\xi &= \int\left|\int \mathcal{W}(u_{a,b,\omega}, u_{a,b,\omega})(x,\xi)d\xi\right| dx \\
&= \|u_{a,b,\omega}\|^2_{L^2(\mathbb{R})} = 1, \\
\iint \mathcal{W}(u_{a,b,\omega}, u_{a,b,\omega})(x,\xi)^2 dxd\xi &= \|u_{a,b,\omega}\|^4_{L^2(\mathbb{R})} = 1.
\end{aligned}$$

We shall see in the next sections that most of the time in the Baire Category sense, we have for $u \in L^2(\mathbb{R}^n)$, $\iint |\mathcal{W}(u,u)(x,\xi)|dxd\xi = +\infty$.

Proof. The proof is already given above for $v_0 = u_{-1/2,1/2,0}$. Moreover, we have with

$$\alpha = \frac{1}{b-a}, \quad \beta = \frac{b+a}{2(a-b)},$$

the formula

$$v_0(y) = e^{-2i\pi\omega(y-\beta)\alpha^{-1}} u_{a,b,\omega}((y-\beta)\alpha^{-1})\alpha^{-1/2},$$

so that $u_{a,b,\omega} = \mathcal{M} v_0$, where $\mathcal{M}$ belongs to the group $\mathrm{Mp}_a(n)$. (cf. Section 1.2.1) and the covariance property (1.2.49) shows that the already proven (6.1.3) implies (6.1.4). ■

6.2 Modulation spaces

In this section, we use the Feichtinger algebra M^1, introduced in [10] (the terminology *Feichtinger algebra* goes back to the book [44]). The survey article [26] by M. S. Jakobsen is a good source for recent developments of the theory as well as Chapter 12 in the K. Gröchenig's book [16]. We refer the reader to the paper [18] by K. Gröchenig and M. Leinert as well as to J. Sjöstrand's article [48] for the use of modulation spaces to proving a non-commutative Wiener lemma.

6.2.1 Preliminary lemmas

The following lemmas in this subsection are well-known (see, e.g., [16, Theorem 11.2.5]). However, we provide a proof for the self-containedness of our survey.

Lemma 6.2.1. *Let ϕ_0 be a non-zero function in $\mathscr{S}(\mathbb{R}^n)$. For $u \in \mathscr{S}'(\mathbb{R}^n)$ the following properties are equivalent.*

(i) $u \in \mathscr{S}(\mathbb{R}^n)$.

(ii) $\mathcal{W}(u, \phi_0) \in \mathscr{S}(\mathbb{R}^{2n})$.

(iii) $\forall N \in \mathbb{N}, \sup_{X \in \mathbb{R}^{2n}} |\mathcal{W}(u, \phi_0)(X)|(1 + |X|)^N < +\infty$.

Proof. Let us assume (i) holds true; with $\Omega(u, \phi_0)$ defined in (1.1.1), we find that $\Omega(u, \phi_0)$ belongs to $\mathscr{S}(\mathbb{R}^{2n})$, thus as well as its partial Fourier transform $\mathcal{W}(u, \phi_0)$, proving (ii). We have obviously that (ii) implies (iii). Let us now assume that (iii) holds true. Using (1.1.5), we find

$$u(x_1)\bar{\phi}_0(x_2) = \int \mathcal{W}(u, \phi_0)\Big(\frac{x_1 + x_2}{2}, \xi\Big) e^{2i\pi(x_1 - x_2)\xi} d\xi,$$

and thus

$$\begin{aligned} u(x_1)\|\phi_0\|^2_{L^2(\mathbb{R}^n)} &= \iint \mathcal{W}(u, \phi_0)\Big(\frac{x_1 + x_2}{2}, \xi\Big) e^{2i\pi(x_1 - x_2)\xi} \phi_0(x_2) d\xi dx_2 \\ &= \iint \mathcal{W}(u, \phi_0)(y, \xi) e^{4i\pi(x_1 - y)\xi} \phi_0(2y - x_1) d\xi dy 2^n, \end{aligned}$$

so that the latter equality, the fact that ϕ_0 belongs to $\mathscr{S}(\mathbb{R}^n)$ imply (i) by differentiation under the integral sign, concluding the proof of the lemma. ■

Lemma 6.2.2. *Let ϕ_0, ϕ_1 be non-zero functions in $L^2(\mathbb{R}^n)$. Let $u \in L^2(\mathbb{R}^n)$ such that $\mathcal{W}(u, \phi_0)$ belongs to $L^1(\mathbb{R}^{2n})$. Then, $\mathcal{W}(u, \phi_1)$ belongs as well to $L^1(\mathbb{R}^{2n})$.*

Proof. According to Lemma 1.2.26 applied to $u_0 = u, u_1 = u_2 = \phi_0, u_3 = \phi_1$, we have

$$\|\phi_0\|_{L^2}^2 \mathcal{W}(u, \phi_1) \in L^1(\mathbb{R}^{2n}),$$

since $\mathcal{W}(u, \phi_0)$ belongs to $L^1(\mathbb{R}^{2n})$ as well as $\mathcal{W}(\check{\phi}_0, \phi_1)$. ■

Lemma 6.2.3. *Let $u \in L^2(\mathbb{R}^n)$. The following properties are equivalent.*

(i) *For all $\phi \in \mathscr{S}(\mathbb{R}^n)$, we have $\mathcal{W}(u, \phi) \in L^1(\mathbb{R}^{2n})$.*

(ii) *For a non-zero $\phi \in \mathscr{S}(\mathbb{R}^n)$, we have $\mathcal{W}(u, \phi) \in L^1(\mathbb{R}^{2n})$.*

(iii) *$\mathcal{W}(u, u)$ belongs to $L^1(\mathbb{R}^{2n})$.*

Proof. We have obviously (i)⇒(ii) and, conversely, Lemma 6.2.2 yields (ii)⇒(i). Assuming (i) and using Lemma 1.2.26 with $u_0 = u_3 = u, u_1 = u_2 = \phi \in \mathscr{S}(\mathbb{R}^n)$, we get

$$\|\phi\|_{L^2}^2 |\mathcal{W}(u, u)(X)| \le 2^n \big(|\mathcal{W}(u, \phi)| * |\mathcal{W}(\check{\phi}, u)|\big)(X),$$

so that choosing a non-zero ϕ in the Schwartz space, we obtain (iii). Conversely, assuming (iii) and using again Lemma 1.2.26 with $u_0 = u_2 = u, u_3 = \phi \in \mathscr{S}(\mathbb{R}^n)$, $u_1 = \psi \in \mathscr{S}(\mathbb{R}^n)$, we find

$$|\langle \psi, u\rangle_{L^2}| |\mathcal{W}(u, \phi)(X)| \le 2^n \big(\underbrace{|\mathcal{W}(u, u)|}_{\in L^1(\mathbb{R}^{2n})} * |\underbrace{\mathcal{W}(\check{\psi}, \phi)}_{\in \mathscr{S}(\mathbb{R}^{2n})}|\big)(X). \tag{6.2.1}$$

Assuming as we may $u \ne 0$, we can choose $\psi \in \mathscr{S}(\mathbb{R}^n)$ such that

$$\langle \psi, u\rangle_{L^2} \ne 0,$$

so that (6.2.1) implies (i). ■

Lemma 6.2.4. *Let $u_1, u_2, u_3 \in L^2(\mathbb{R}^n)$. Then, we have the inversion formula,*

$$\mathrm{Op_w}\,(\mathcal{W}(u_1, u_2))u_3 = \langle u_3, u_2\rangle_{L^2(\mathbb{R}^n)} u_1.$$

Proof. It is an immediate consequence of Lemma 1.2.25. ■

6.2.2 The space $M^1(\mathbb{R}^n)$

Definition 6.2.5. The space $M^1(\mathbb{R}^n)$ is defined as the set of $u \in L^2(\mathbb{R}^n)$ such that, for all $\phi \in \mathscr{S}(\mathbb{R}^n)$, $\mathcal{W}(u, \phi)$ belongs to $L^1(\mathbb{R}^{2n})$. According to Lemma 6.2.3, $M^1(\mathbb{R}^n)$ is also the set of $u \in L^2(\mathbb{R}^n)$ such that $\mathcal{W}(u, u) \in L^1(\mathbb{R}^{2n})$ as well as the set of $u \in L^2(\mathbb{R}^n)$ such that, for a non-zero $\phi \in \mathscr{S}(\mathbb{R}^n)$, $\mathcal{W}(u, \phi)$ belongs to $L^1(\mathbb{R}^{2n})$.

Proposition 6.2.6. *Let ψ_0 be the standard fundamental state of the harmonic oscillator $\pi(D_x^2 + x^2)$ given by*

$$\psi_0(x) = 2^{n/4} e^{-\pi|x|^2}. \tag{6.2.2}$$

Then, $M^1(\mathbb{R}^n) \ni u \mapsto \|\mathcal{W}(u,\psi_0)\|_{L^1(\mathbb{R}^{2n})}$ is a norm on $M^1(\mathbb{R}^n)$. Let ψ be a non-zero function in $\mathscr{S}(\mathbb{R}^n)$: then $M^1(\mathbb{R}^n) \ni u \mapsto \|\mathcal{W}(u,\psi)\|_{L^1(\mathbb{R}^{2n})}$ is a norm on $M^1(\mathbb{R}^n)$, equivalent to the previous norm.

Proof. The homogeneity and triangle inequality are immediate, let us check the separation: let $u \in L^2(\mathbb{R}^n)$ such that $\mathcal{W}(u,\psi) = 0$. Then, we have

$$0 = \langle \operatorname{Op_w}(\mathcal{W}(u,\psi))\psi, u\rangle_{L^2(\mathbb{R}^n)} = \|u\|^2_{L^2(\mathbb{R}^n)} \|\psi\|^2_{L^2(\mathbb{R}^n)},$$

proving the sought result. Let ψ be a non-zero function in $\mathscr{S}(\mathbb{R}^n)$; according to Lemma 1.2.26 applied to $u_0 = u, u_1 = u_2 = \psi_0, u_3 = \psi$, we find

$$|\mathcal{W}(u,\psi)(X)| \le 2^n \big(|\mathcal{W}(u,\psi_0)| * |\mathcal{W}(\psi_0,\psi)|\big)(X),$$

so that we have

$$\begin{aligned} \|\mathcal{W}(u,\psi)\|_{L^1(\mathbb{R}^{2n})} &\le 2^n \|\mathcal{W}(\psi_0,\psi)\|_{L^1(\mathbb{R}^{2n})} \|\mathcal{W}(u,\psi_0)\|_{L^1(\mathbb{R}^{2n})}, \\ \|\mathcal{W}(u,\psi_0)\|_{L^1(\mathbb{R}^{2n})} &\le 2^n \|\mathcal{W}(\psi,\psi_0)\|_{L^1(\mathbb{R}^{2n})} \|\mathcal{W}(u,\psi)\|_{L^1(\mathbb{R}^{2n})}, \end{aligned} \tag{6.2.3}$$

proving the equivalence of norms. ■

Proposition 6.2.7. *The space $M^1(\mathbb{R}^n)$, equipped with the equivalent norms of Proposition 6.2.6, is a Banach space. The space $\mathscr{S}(\mathbb{R}^n)$ is dense in $M^1(\mathbb{R}^n)$.*

Proof. Let $(u_k)_{k\ge1}$ be a Cauchy sequence in $M^1(\mathbb{R}^n)$: it means that $(\mathcal{W}(u_k,\psi_0))_{k\ge1}$ is a Cauchy sequence in $L^1(\mathbb{R}^{2n})$, thus such that

$$\lim_k \mathcal{W}(u_k,\psi_0) = U \quad \text{in } L^1(\mathbb{R}^{2n}). \tag{6.2.4}$$

On the other hand, from Lemma 1.2.25, we have $u_k - u_l = \operatorname{Op_w}(\mathcal{W}(u_k - u_l,\psi_0))\psi_0$, so that

$$\begin{aligned} &\|u_k - u_l\|_{L^2(\mathbb{R}^n)} \\ &\quad \le \|\operatorname{Op_w}(\mathcal{W}(u_k - u_l,\psi_0))\|_{\mathcal{B}(L^2(\mathbb{R}^n))} \underbrace{\le}_{\text{cf. (1.2.5)}} 2^n \|\mathcal{W}(u_k - u_l,\psi_0)\|_{L^1(\mathbb{R}^{2n})}, \end{aligned}$$

implying that $(u_k)_{k\ge1}$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$, thus converging towards a function u in $L^2(\mathbb{R}^n)$. Since from (1.1.6), we have $\|\mathcal{W}(u_k - u,\psi_0)\|_{L^2(\mathbb{R}^{2n})} = \|u_k - u\|_{L^2(\mathbb{R}^n)}$, we obtain as well that

$$\lim_k \mathcal{W}(u_k,\psi_0) = \mathcal{W}(u,\psi_0) \quad \text{in } L^2(\mathbb{R}^{2n}),$$

and this implies along with (6.2.4) that $U = \mathcal{W}(u, \psi_0)$ in $\mathscr{S}'(\mathbb{R}^{2n})$. As a result, we have $\mathcal{W}(u, \psi_0) \in L^1(\mathbb{R}^{2n})$, so that $u \in M^1(\mathbb{R}^n)$ and

$$\lim_k \mathcal{W}(u_k, \psi_0) = \mathcal{W}(u, \psi_0) \quad \text{in } L^1(\mathbb{R}^{2n}),$$

entailing convergence towards u for the sequence $(u_k)_{k\ge 1}$ in $M^1(\mathbb{R}^n)$ and the sought completeness. We are left with the density question and we start with a calculation.

Claim 6.2.8. With the phase symmetry $\sigma_{y,\eta}$ given by (1.2.6) and ψ_0 by (6.2.2) we have for $X, Y \in \mathbb{R}^{2n}$,

$$\mathcal{W}(\sigma_Y \psi_0, \psi_0)(X) = 2^n e^{-2\pi|X-Y|^2} e^{-4i\pi[X,Y]}, \tag{6.2.5}$$

where the symplectic form is given in (1.2.13).

Proof of the Claim. We have indeed

$$\begin{aligned}
\mathcal{W}(\sigma_{y,\eta}\psi_0, \psi_0)(x,\xi) &= \int (\sigma_{y,\eta}\psi_0)\Big(x + \frac{z}{2}\Big)\psi_0\Big(x - \frac{z}{2}\Big) e^{-2i\pi z\cdot\xi} dz \\
&= \int \psi_0\Big(2y - x - \frac{z}{2}\Big) e^{4i\pi\eta\cdot(x+\frac{z}{2}-y)} \psi_0\Big(x - \frac{z}{2}\Big) e^{-2i\pi z\cdot\xi} dz \\
&= 2^{n/2} \int e^{-\pi(|2y-x-\frac{z}{2}|^2 + |x-\frac{z}{2}|^2)} e^{2i\pi z\cdot(\eta-\xi)} dz e^{4i\pi\eta\cdot(x-y)} \\
&= 2^{n/2} e^{4i\pi\eta\cdot(x-y)} \int e^{-\frac{\pi}{2}(|2y-z|^2 + |2(y-x)|^2)} e^{2i\pi z\cdot(\eta-\xi)} dz \\
&= 2^{n/2} e^{4i\pi\eta\cdot(x-y)} e^{-2\pi|y-x|^2} e^{4i\pi y\cdot(\eta-\xi)} 2^{n/2} e^{-2\pi|\eta-\xi|^2},
\end{aligned}$$

which is the sought formula. ∎

Let u be a function in $M^1(\mathbb{R}^n)$. For $\varepsilon > 0$ we define

$$u_\varepsilon(x) = \int_{\mathbb{R}^{2n}} \mathcal{W}(u, \psi_0)(Y) e^{-\varepsilon|Y|^2} 2^n (\sigma_Y \psi_0)(x) dY,$$

and we have

$$\mathcal{W}(u_\varepsilon, \psi_0)(X) = \int_{\mathbb{R}^{2n}} \mathcal{W}(u, \psi_0)(Y) e^{-\varepsilon|Y|^2} 2^n \mathcal{W}(\sigma_Y \psi_0, \psi_0)(X) dY,$$

so that Lemma 6.2.1 and (6.2.5) imply readily that u_ε belongs to the Schwartz space. Moreover, we have

$$u = \operatorname{Op}_{\mathrm{w}}(\mathcal{W}(u, \psi_0))\psi_0,$$

from Lemma 6.2.4 and thus

$$\mathcal{W}(u, \psi_0)(X) = \int_{\mathbb{R}^{2n}} \mathcal{W}(u, \psi_0)(Y) 2^n \mathcal{W}(\sigma_Y \psi_0, \psi_0)(X) dY,$$

so that

$$\int_{\mathbb{R}^{2n}} |\mathcal{W}(u_\varepsilon, \psi_0)(X) - \mathcal{W}(u, \psi_0)(X)| dX$$
$$\leq 2^n \iint_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} \underbrace{|\mathcal{W}(u, \psi_0)(Y)||\mathcal{W}(\sigma_Y \psi_0, \psi_0)(X)|}_{\in L^1(\mathbb{R}^{4n}) \text{ from (6.2.5) and } u \in M^1(\mathbb{R}^n)} \underbrace{\left(1 - e^{-\varepsilon|Y|^2}\right)}_{\in[0,1]} dY dX.$$

The Lebesgue dominated convergence theorem shows that the integral above tends to 0 when $\varepsilon \to 0_+$, proving the convergence in $M^1(\mathbb{R}^n)$ of the sequence (u_ε), which completes the proof of the density. ∎

Theorem 6.2.9. *Let $\mathcal{M}$ be an element of the metaplectic group $Mp(n)$ (Definition 1.2.13). Then, $\mathcal{M}$ is an isomorphism of $M^1(\mathbb{R}^n)$ and we have for $u \in M^1(\mathbb{R}^n)$, $\phi \in \mathscr{S}(\mathbb{R}^n)$,*

$$\mathcal{W}(\mathcal{M}u, \mathcal{M}\phi) = \mathcal{W}(u, \phi) \circ S^{-1}, \tag{6.2.6}$$

where $\mathcal{M}$ is in the fiber of the symplectic transformation S. In particular, the space $M^1(\mathbb{R}^n)$ is invariant by the Fourier transformation and partial Fourier transformations, by the rescaling (1.2.31), by the transformations (1.2.30), (1.2.32) and also by the phase translations (1.2.51) and phase symmetries (1.2.6).

Proof. Formula (6.2.6) follows readily from (1.2.49) and if u belongs to $M^1(\mathbb{R}^n)$, we find that

$$\mathcal{W}(\mathcal{M}u, \underbrace{\mathcal{M}\psi_0}_{\in\mathscr{S}(\mathbb{R}^n)}) = \underbrace{\mathcal{W}(u, \psi_0)}_{\in L^1(\mathbb{R}^{2n})} \circ S^{-1},$$

and since $\det S = 1$, we have

$$\|\mathcal{W}(\mathcal{M}u, \mathcal{M}\psi_0)\|_{L^1(\mathbb{R}^{2n})} = \|\mathcal{W}(u, \psi_0)\|_{L^1(\mathbb{R}^{2n})},$$

implying that $\mathcal{W}(\mathcal{M}u, \mathcal{M}\psi_0)$ belongs to $L^1(\mathbb{R}^{2n})$ so that, thanks to Definition 6.2.5, we get that $\mathcal{M}u$ belongs to $M^1(\mathbb{R}^n)$. The same properties are true for $\mathcal{M}^{-1}$. ∎

Remark 6.2.10. From Definition 6.2.5, we see that, for $u \in M^1(\mathbb{R}^n)$, we have

$$\mathcal{W}(u, u) \in L^1(\mathbb{R}^{2n}),$$

and this implies, thanks to Theorem 1.2.24, that $M^1(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$. Moreover, we have

$$\mathcal{F}\left(M^1(\mathbb{R}^n)\right) \subset M^1(\mathbb{R}^n),$$

since for $u \in M^1(\mathbb{R}^n)$, we have

$$\mathcal{W}(\hat{u}, \psi_0) = \mathcal{W}(\hat{u}, \hat{\psi}_0)$$

and thanks to (6.2.6),

$$\|\mathcal{W}(\hat{u}, \hat{\psi}_0)\|_{L^1(\mathbb{R}^{2n})} = \|\mathcal{W}(u, \psi_0)\|_{L^1(\mathbb{R}^{2n})}.$$

As a consequence we find

$$\mathcal{F}\big(M^1(\mathbb{R}^n)\big) \subset M^1(\mathbb{R}^n) = \mathcal{F}^2\mathcal{C}(M^1(\mathbb{R}^n)) = \mathcal{F}^2(M^1(\mathbb{R}^n)) \subset \mathcal{F}\big(M^1(\mathbb{R}^n)\big),$$

and consequently

$$M^1(\mathbb{R}^n) = \mathcal{F}(M^1(\mathbb{R}^n)) \subset \mathcal{F}(L^1(\mathbb{R}^n)) \subset C_{(0)}(\mathbb{R}^n),$$

where the latter inclusion is due to the Riemann–Lebesgue lemma with $C_{(0)}(\mathbb{R}^n)$ standing for the space of continuous functions with limit 0 at infinity. Moreover, for $u \in M^1(\mathbb{R}^n)$ and ψ_0 given by (6.2.2), we get from (1.1.5),

$$u(x_1)\bar{\psi}_0(x_2) = \int \mathcal{W}(u, \psi_0)\Big(\frac{x_1 + x_2}{2}, \xi\Big)e^{2i\pi(x_1-x_2)\cdot\xi}d\xi,$$

so that

$$u(x_1) = \iint \mathcal{W}(u, \psi_0)(y, \eta)e^{4i\pi(x_1-y)\cdot\eta}\bar{\psi}_0(2y - x_1)dyd\eta 2^n,$$

implying

$$\|u\|_{L^1(\mathbb{R}^n)} \le \|\mathcal{W}(u, \psi_0)\|_{L^1(\mathbb{R}^{2n})}2^{\frac{5n}{4}}, \tag{6.2.7}$$

and similarly for $p \in [1, +\infty]$,

$$\|u\|_{L^p(\mathbb{R}^n)} \le \|\mathcal{W}(u, \psi_0)\|_{L^1(\mathbb{R}^{2n})}2^{\frac{5n}{4}} p^{-\frac{n}{2p}},$$

yielding the continuous injection of $M^1(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$.

Theorem 6.2.11. *The space $M^1(\mathbb{R}^n)$ is a Banach algebra for convolution and for pointwise multiplication.*

Proof. Let $u, v \in M^1(\mathbb{R}^n)$; then the convolution $u * v$ makes sense and belongs to all $L^p(\mathbb{R}^n)$ for $p \in [1, +\infty]$, since we have $u \in L^1(\mathbb{R}^n)$. We calculate

$$\mathcal{W}(u * v, \psi_0)(x, \xi) = \int_{\mathbb{R}^n} u(y)\mathcal{W}(\tau_y v, \psi_0)(x, \xi)dy, \quad (\tau_y v)(x) = v(x - y),$$

so that

$$\|\mathcal{W}(u * v, \psi_0)\|_{L^1(\mathbb{R}^{2n})} \le \int_{\mathbb{R}^n} |u(y)|\|\mathcal{W}(\tau_y v, \psi_0)\|_{L^1(\mathbb{R}^{2n})}dy,$$

and since we have

$$\mathcal{W}(\tau_y v, \psi_0)(x, \xi) = \mathcal{W}(v, \tau_y\psi_0)(x, \xi)e^{-4i\pi y\cdot\xi},$$

we get

$$\|\mathcal{W}(u * v, \psi_0)\|_{L^1(\mathbb{R}^{2n})} \le \int_{\mathbb{R}^n} |u(y)| \|\mathcal{W}(v, \tau_y \psi_0)\|_{L^1(\mathbb{R}^{2n})} dy,$$

so that using (6.2.3), we obtain

$$\begin{aligned} &\|\mathcal{W}(u * v, \psi_0)\|_{L^1(\mathbb{R}^{2n})} \\ &\quad \le \int_{\mathbb{R}^n} |u(y)| 2^n \|\mathcal{W}(\psi_0, \tau_y \psi_0)\|_{L^1(\mathbb{R}^{2n})} dy \|\mathcal{W}(v, \psi_0)\|_{L^1(\mathbb{R}^{2n})}. \end{aligned}$$

We can check now that

$$\mathcal{W}(\psi_0, \tau_y \psi_0)(x, \xi) = 2^n e^{-2\pi(\xi^2 + (x - \frac{y}{2})^2)} e^{2i\pi \xi y},$$

so that

$$\begin{aligned} \|\mathcal{W}(u * v, \psi_0)\|_{L^1(\mathbb{R}^{2n})} &\le 2^n \|u\|_{L^1(\mathbb{R}^n)} \|\mathcal{W}(v, \psi_0)\|_{L^1(\mathbb{R}^{2n})} \\ &\underbrace{\le}_{(6.2.7)} 2^{\frac{9n}{4}} \|\mathcal{W}(u, \psi_0)\|_{L^1(\mathbb{R}^{2n})} \|\mathcal{W}(v, \psi_0)\|_{L^1(\mathbb{R}^{2n})}, \end{aligned} \tag{6.2.8}$$

proving that $M^1(\mathbb{R}^n)$ is a Banach algebra for convolution when equipped with the norm

$$N(u) = 2^{\frac{9n}{4}} \|\mathcal{W}(u, \psi_0)\|_{L^1(\mathbb{R}^{2n})}. \tag{6.2.9}$$

On the other hand, for $u, v \in M^1(\mathbb{R}^n)$, the pointwise product $u \cdot v$ makes sense and belongs to $L^1(\mathbb{R}^n)$ (since both functions are in $L^2(\mathbb{R}^n)$) and we have

$$u \cdot v = \mathcal{C}\mathcal{F}(\hat{u} * \hat{v}),$$

so that

$$\mathcal{W}(u \cdot v, \psi_0)(x, \xi) = \mathcal{W}(\mathcal{C}\mathcal{F}(\hat{u} * \hat{v}), \psi_0)(x, \xi) = \mathcal{W}(\mathcal{F}(\hat{u} * \hat{v}), \check{\psi}_0)(-x, -\xi),$$

and since $\psi_0 = \hat{\psi}_0$ is also even, we get

$$\begin{aligned} \|\mathcal{W}(u \cdot v, \psi_0)\|_{L^1(\mathbb{R}^{2n})} &= \|\mathcal{W}(\mathcal{F}(\hat{u} * \hat{v}), \mathcal{F}\psi_0)\|_{L^1(\mathbb{R}^{2n})} \\ &\underbrace{=}_{\text{cf. (1.2.49)}} \|\mathcal{W}\big(\hat{u} * \hat{v}, \psi_0\big)\|_{L^1(\mathbb{R}^{2n})} \\ &\underbrace{\le}_{(6.2.8)} 2^{\frac{9n}{4}} \|\mathcal{W}(\hat{u}, \hat{\psi}_0)\|_{L^1(\mathbb{R}^{2n})} \|\mathcal{W}(\hat{v}, \hat{\psi}_0)\|_{L^1(\mathbb{R}^{2n})} \\ &= 2^{\frac{9n}{4}} \|\mathcal{W}(u, \psi_0)\|_{L^1(\mathbb{R}^{2n})} \|\mathcal{W}(v, \psi_0)\|_{L^1(\mathbb{R}^{2n})}, \end{aligned}$$

proving as well that $M^1(\mathbb{R}^n)$ is a Banach algebra for pointwise multiplication with the norm (6.2.9). ∎

6.3 Most pulses give rise to a non-integrable Wigner distribution

In the sequel, n is an integer ≥ 1.

Lemma 6.3.1. *We have with ψ_0 given by (6.2.2),*

$$M^1(\mathbb{R}^n) = \Big\{u \in L^2(\mathbb{R}^n), \iint_{\mathbb{R}^{2n}} |\mathcal{W}(u,\psi_0)(x,\xi)| dx d\xi < +\infty\Big\}.$$

Then, $M^1(\mathbb{R}^n)$ is an F_σ of $L^2(\mathbb{R}^n)$ with empty interior.

Proof. We have $M^1(\mathbb{R}^n) = \bigcup_{N\in\mathbb{N}} \Phi_N$ with

$$\Phi_N = \Big\{u \in L^2(\mathbb{R}^n), \iint_{\mathbb{R}^{2n}} |\mathcal{W}(u,\psi_0)(x,\xi)| dx d\xi \le N\Big\}.$$

The set Φ_N is a closed subset of $L^2(\mathbb{R}^n)$ since if $(u_k)_{k\ge 1}$ is a sequence in Φ_N which converges in $L^2(\mathbb{R}^n)$ with limit u, we get for $R \ge 0$,

$$\begin{aligned}
&\iint_{|(x,\xi)|\le R} |\mathcal{W}(u,\psi_0)(x,\xi)| dx d\xi \\
&\le \iint_{|(x,\xi)|\le R} |\mathcal{W}(u-u_k,\psi_0)(x,\xi)| dx d\xi + \iint_{|(x,\xi)|\le R} |\mathcal{W}(u_k,\psi_0)(x,\xi)| dx d\xi \\
&\le \|u-u_k\|_{L^2(\mathbb{R}^n)} (|\mathbb{B}^{2n}| R^{2n})^{1/2} + N,
\end{aligned}$$

implying $\iint_{|(x,\xi)|\le R} |\mathcal{W}(u,\psi_0)(x,\xi)| dx d\xi \le N$, and this for any R, so that we obtain $u \in \Phi_N$. The interior of Φ_N is empty, since if it were not the case, as Φ_N is also convex and symmetric, 0 would be an interior point of Φ_N in $L^2(\mathbb{R}^n)$ and we would find $\rho_0 > 0$ such that

$$\|u\|_{L^2(\mathbb{R}^n)} \le \rho_0 \Longrightarrow \iint_{\mathbb{R}^{2n}} |\mathcal{W}(u,\psi_0)(x,\xi)| dx d\xi \le N,$$

and thus for any non-zero $u \in L^2(\mathbb{R}^n)$, we would have

$$\iint_{\mathbb{R}^{2n}} |\mathcal{W}(u,\psi_0)(x,\xi)| dx d\xi \|u\|^{-1}_{L^2(\mathbb{R}^n)} \rho_0 \le N$$

and thus

$$\|u\|_{M^1(\mathbb{R}^n)} \le N\rho_0^{-1} \|u\|_{L^2(\mathbb{R}^n)},$$

implying as well $L^2(\mathbb{R}^n) = M^1(\mathbb{R}^n)$ which is untrue, thanks to the examples of Section 6.1.2, e.g., (6.1.3), and this proves that the interior of Φ_N is actually empty. Now the Baire Category Theorem implies that the F_σ set $M^1(\mathbb{R}^n)$ is a subset of $L^2(\mathbb{R}^n)$ with empty interior. ∎

Let us give another decomposition of the space $M^1(\mathbb{R}^n)$.

Lemma 6.3.2. *According to Lemma 6.2.3, we have*

$$M^1(\mathbb{R}^n) = \left\{u \in L^2(\mathbb{R}^n), \iint_{\mathbb{R}^n\times\mathbb{R}^n} |\mathcal{W}(u,u)(x,\xi)|dxd\xi < +\infty\right\}.$$

Then, defining

$$\mathcal{F}_N = \left\{u \in L^2(\mathbb{R}^n), \iint_{\mathbb{R}^n\times\mathbb{R}^n} |\mathcal{W}(u,u)(x,\xi)|dxd\xi \le N\right\}, \tag{6.3.1}$$

each $\mathcal{F}_N$ is a closed subset of $L^2(\mathbb{R}^n)$ with empty interior.

Proof. We have $\mathscr{F} = M^1(\mathbb{R}^n) = \bigcup_{N\in\mathbb{N}} \mathcal{F}_N$. The set $\mathcal{F}_N$ is a closed subset of $L^2(\mathbb{R}^n)$ since if $(u_k)_{k\ge1}$ is a sequence in $\mathcal{F}_N$ which converges in $L^2(\mathbb{R}^n)$ with limit u, we have

$$\forall k \ge 1, \quad \iint_{\mathbb{R}^n\times\mathbb{R}^n} |\mathcal{W}(u_k,u_k)(x,\xi)|dxd\xi \le N,$$

so that we may apply Lemma 6.1.2 with $C_0 = N$, and readily get that u belongs to $\mathcal{F}_N$. We have also that $\text{interior}_{L^2(\mathbb{R}^n)}(\mathcal{F}_N) \subset \text{interior}_{L^2(\mathbb{R}^n)}(M^1(\mathbb{R}^n)) = \emptyset$. ∎

Theorem 6.3.3. *Defining*

$$\mathcal{G} = \left\{u \in L^2(\mathbb{R}^n), \iint_{\mathbb{R}^n\times\mathbb{R}^n} |\mathcal{W}(u,u)(x,\xi)|dxd\xi = +\infty\right\} = L^2(\mathbb{R}^n)\backslash M^1(\mathbb{R}^n), \tag{6.3.2}$$

we obtain that the set $\mathcal{G}$ is a dense G_δ subset of $L^2(\mathbb{R}^n)$.

Proof. It follows immediately from Lemma 6.3.2 and formula $\{\mathring{A}\}^c = \overline{A^c}$, yielding for $\mathcal{F}_N$ defined in (6.3.1), $L^2(\mathbb{R}^n) = \{\text{interior}(\bigcup_{\mathbb{N}} \mathcal{F}_N)\}^c = \overline{\bigcap_{\mathbb{N}} \mathcal{F}_N^c}$. ∎

Remark 6.3.4. It is interesting to note that the space $M^1(\mathbb{R}^n)$ is not reflexive, as it can be identified to ℓ^1 via Wilson bases, but it is a dual space. It turns out that both properties are linked to the fact that $M^1(\mathbb{R}^n)$ is an F_σ of $L^2(\mathbb{R}^n)$ as proven by Lemmas 6.3.1 and 6.3.2: if $\mathbb{X}$ is a reflexive Banach space continuously included in a Hilbert space $\mathbb{H}$, it is always an F_σ of $\mathbb{H}$, since we may write

$$\mathbb{X} = \bigcup_{N\in\mathbb{N}} N\mathbf{B}_{\mathbb{X}},$$

where $\mathbf{B}_{\mathbb{X}}$ is the closed unit ball of $\mathbb{X}$ and $N\mathbf{B}_{\mathbb{X}}$ is $\mathbb{H}$-closed since it is weakly compact (for the topology $\sigma(\mathbb{H},\mathbb{H})$); we cannot use that abstract argument in the case of the non-reflexive $M^1(\mathbb{R}^n)$, so we produced a direct elementary proof above. Also, it can be proven that if $\mathbb{X}$ is a Banach space continuously included in a Hilbert space $\mathbb{H}$, so that $\mathbb{X}$ is an F_σ of $\mathbb{H}$, then $\mathbb{X}$ must have a predual. As a result, the fact that $M^1(\mathbb{R}^n)$ has a predual appears as a consequence of the fact that $M^1(\mathbb{R}^n)$ is an F_σ of $L^2(\mathbb{R}^n)$.

6.4 Consequences on integrals of the Wigner distribution

Lemma 6.4.1. *Let $\mathscr{G}$ be defined in (6.3.2) and let $u \in \mathscr{G}$. Then, the positive and negative part of the real-valued $\mathcal{W}(u,u)$ are such that*

$$\iint \mathcal{W}(u,u)_+(x,\xi)dxd\xi = \iint \mathcal{W}(u,u)_-(x,\xi)dxd\xi = +\infty.$$

Proof. For $h \in (0,1]$, we define the symbol

$$a(x,\xi,h) = e^{-h(x^2+\xi^2)},$$

and we see that it is a semi-classical symbol in the sense (1.2.65). Let us start a *reductio ad absurdum* and assume $\iint \mathcal{W}(u,u)_-(x,\xi)dxd\xi < +\infty$, (which implies since $u \in \mathscr{G}$, $\iint \mathcal{W}(u,u)_+(x,\xi)dxd\xi = +\infty$). We note that

$$\langle \mathrm{Op_w}(a(x,\xi,h))u,u\rangle_{L^2(\mathbb{R}^n)} = \iint \underbrace{a(x,\xi,h)}_{\in L^2(\mathbb{R}^{2n})} \underbrace{\mathcal{W}(u,u)(x,\xi)}_{\in L^2(\mathbb{R}^{2n})} dxd\xi,$$

and thanks to Theorem 1.2.27, we have also

$$\sup_{h\in(0,1]} |\langle \mathrm{Op_w}(a(x,\xi,h))u,u\rangle_{L^2(\mathbb{R}^n)}| \le \sigma_n \|u\|^2_{L^2(\mathbb{R}^n)},$$

so that

$$\begin{aligned}\iint e^{-h(x^2+\xi^2)}\mathcal{W}(u,u)(x,\xi)dxd\xi + \iint e^{-h(x^2+\xi^2)}\mathcal{W}(u,u)_-(x,\xi)dxd\xi \\ = \iint e^{-h(x^2+\xi^2)}\mathcal{W}(u,u)_+(x,\xi)dxd\xi,\end{aligned}$$

and thus with $\theta_h \in [-1,1]$, we have

$$\begin{aligned}\theta_h\sigma_n\|u\|^2_{L^2(\mathbb{R}^n)} + \iint e^{-h(x^2+\xi^2)}\mathcal{W}(u,u)_-(x,\xi)dxd\xi \\ = \iint e^{-h(x^2+\xi^2)}\mathcal{W}(u,u)_+(x,\xi)dxd\xi.\end{aligned} \tag{6.4.1}$$

Choosing $h = 1/m, m \in \mathbb{N}^*$, we note that

$$e^{-\frac{1}{m}(x^2+\xi^2)}\mathcal{W}(u,u)_+(x,\xi) \le e^{-\frac{1}{m+1}(x^2+\xi^2)}\mathcal{W}(u,u)_+(x,\xi).$$

From the Beppo–Levi Theorem (see, e.g., [34, Theorem 1.6.1]), we get that

$$\lim_{m\to+\infty}\iint e^{-\frac{1}{m}(x^2+\xi^2)}\mathcal{W}(u,u)_+(x,\xi)dxd\xi = \iint \mathcal{W}(u,u)_+(x,\xi)dxd\xi = +\infty.$$

However, the left-hand side of (6.4.1) is bounded above by

$$\sigma_n \|u\|^2_{L^2(\mathbb{R}^n)} + \iint \mathcal{W}(u,u)_-(x,\xi)dxd\xi, \quad \text{which is finite,}$$

triggering a contradiction. We may now study the case where

$$\iint \mathcal{W}(u,u)_+(x,\xi)dxd\xi < +\infty, \quad \iint \mathcal{W}(u,u)_-(x,\xi)dxd\xi = +\infty.$$

The identity (6.4.1) still holds true with a left-hand side going to $+\infty$ when h goes to 0 whereas the right-hand side is bounded. This concludes the proof of the lemma. ∎

N.B. A shorter *heuristic* argument would be that the identity

$$\iint \mathcal{W}(u,u)(x,\xi)dxd\xi = \|u\|^2_{L^2(\mathbb{R}^n)} \quad \text{and} \quad \iint |\mathcal{W}(u,u)(x,\xi)|dxd\xi = +\infty$$

should imply the lemma, but the former integral is not absolutely converging, so that argument fails to be completely convincing since we need to give a meaning to the first integral.

Theorem 6.4.2. *Defining $\mathcal{G} = L^2(\mathbb{R}^n)\backslash M^1(\mathbb{R}^n)$ (cf. (6.3.2)) we find that the set $\mathcal{G}$ is a dense G_δ set in $L^2(\mathbb{R}^n)$ and for all $u \in \mathcal{G}$, we have*[1]

$$\iint \mathcal{W}(u,u)_+(x,\xi)dxd\xi = \iint \mathcal{W}(u,u)_-(x,\xi)dxd\xi = +\infty, \tag{6.4.2}$$

Defining[2]

$$E_\pm(u) = \{(x,\xi) \in \mathbb{R}^{2n}, \pm\mathcal{W}(u,u)(x,\xi) > 0\}, \tag{6.4.3}$$

we have for all $u \in \mathcal{G}$,

$$\iint_{E_\pm(u)} \mathcal{W}(u,u)(x,\xi)dxd\xi = \pm\infty, \tag{6.4.4}$$

and both sets $E_\pm(u)$ are open subsets of $\mathbb{R}^{2n}$ with infinite Lebesgue measure.

Proof. The first statements follow from Theorem 6.3.3 and Lemma 6.4.1. As far as (6.4.4) is concerned, we note that $\mathcal{W}(u,u) > 0$ (resp., < 0) on $E_+(u)$ (resp., $E_-(u)$), so that Theorem 6.3.3 implies (6.4.4). Moreover, $E_\pm(u)$ are open subsets of $\mathbb{R}^{2n}$ since, thanks to Theorem 1.2.22, the function $\mathcal{W}(u,u)$ is continuous; also, both subsets have infinite Lebesgue measure from (6.4.2) since $\mathcal{W}(u,u)$ belongs to $L^2(\mathbb{R}^{2n})$. ∎

[1] Note that $\mathcal{W}(u,u)$ is real-valued.

[2] Thanks to Theorem 1.2.22, the function $\mathcal{W}(u,u)$ is a continuous function, so it makes sense to consider its pointwise values.

Remark 6.4.3. There are many other interesting properties and generalizations of the space M^1 and in particular a close link between the Bargmann transform, the Fock spaces and modulation spaces: we refer the reader to Remark 5 on page 243 in Section 11.4 of [16], to our Section 1.2.8 in this memoir and to Section 2.4 of [33].

Remark 6.4.4. As a consequence of the previous theorem, we could say that for any *generic* u in $L^2(\mathbb{R}^n)$ (i.e., any $u \in \mathscr{G} = L^2(\mathbb{R}^n)\backslash M^1(\mathbb{R}^n)$), we can find open sets E_+, E_- such that the real-valued $\pm\mathcal{W}(u,u)$ is positive on $E_\pm$ and

$$\iint_{E_\pm} \mathcal{W}(u,u)(x,\xi)dxd\xi = \pm\infty.$$

We shall see in the next section some results on polygons in the plane and for instance, we shall be able to prove that there exists a "universal number" $\mu_3^+ > 1$ such that for any triangle[3] $\mathcal{T}$ in the plane, we have

$$\forall u \in L^2(\mathbb{R}), \qquad \iint_{\mathcal{T}} \mathcal{W}(u,u)(x,\xi)dxd\xi \le \mu_3^+ \|u\|^2_{L^2(\mathbb{R})}. \tag{6.4.5}$$

Note in particular that we will show that (6.4.5) holds true regardless of the area of the triangle (which could be infinite according to our definition of a triangle). Although that type of result may look pretty weak, it gets enhanced by Theorem 6.4.2 which proves that no triangle in the plane could be a set $E_+(u)$ (cf. (6.4.3)) for a generic u in $L^2(\mathbb{R})$.

[3] We define a triangle as the intersection of three half-planes, which includes of course the convex envelope of three points, but also the set with infinite area $\{(x,\xi) \in \mathbb{R}^2, x \ge 0, \xi \ge 0, x+\xi \ge \lambda\}$ for some $\lambda > 0$.

Chapter 7

Convex polygons of the plane

7.1 Convex cones

We have seen in Proposition 5.4.4 and Theorem 5.2.4 that the self-adjoint bounded operator with Weyl symbol $H(x)H(\xi)$ does satisfy

$$\begin{aligned} \mu_2^- = m_0 = \lambda_{\min}\big(\operatorname{Op}_{\mathrm{w}}(H(x)H(\xi))\big) &\le \operatorname{Op}_{\mathrm{w}}(H(x)H(\xi)) \\ &\le \lambda_{\max}\big(\operatorname{Op}_{\mathrm{w}}(H(x)H(\xi))\big) = M_0 = \mu_2^+, \\ [\mu_2^-, \mu_2^+] = \operatorname{spectrum}&\big(\operatorname{Op}_{\mathrm{w}}(H(x)H(\xi))\big), \end{aligned} \tag{7.1.1}$$

with

$$\mu_2^- \approx -0.155939843191243, \quad \mu_2^+ \approx 1.00767997007003. \tag{7.1.2}$$

This result is true as well for the characteristic function of any convex cone (which is not a half-plane nor the full plane) in the plane since we can map it to the quarter plane by a transformation in $\mathrm{Sl}(2,\mathbb{R}) = \mathrm{Sp}(1,\mathbb{R})$. On the other hand, a concave cone is the complement of a convex cone and the diagonalisation offered by Theorem 5.2.3 proves that the spectrum of the Weyl quantization of the indicatrix of a concave cone is

$$1 - \operatorname{Spectrum}\big(\operatorname{Op}_{\mathrm{w}}(H(x)H(\xi))\big).$$

We may sum-up the situation by the following theorem.

Theorem 7.1.1. *Let Σ_θ be a convex cone in $\mathbb{R}^2$ with aperture $\theta \in [0, 2\pi]$ (cf. (5.3.3)) and let $\mathcal{A}_\theta$ be the self-adjoint bounded operator with the indicator function of Σ_θ as a Weyl symbol.*

(1) *If $\theta = 0$, we have $\mathcal{A}_\theta = 0$.*

(2) *If $\theta \in (0, \pi)$, the operator $\mathcal{A}_\theta$ is unitarily equivalent to $\operatorname{Op}_{\mathrm{w}}(H(x)H(\xi))$, thus with spectrum $[\mu_2^-, \mu_2^+]$ with $\mu_2^- < 0 < 1 < \mu_2^+$, as given in Theorem 5.2.4.*

(3) *If $\theta = \pi$, Σ_π is a half-space and $\mathcal{A}_\pi$ is a proper orthogonal projection, thus with spectrum $\{0, 1\}$.*

(4) *If $\theta \in (\pi, 2\pi)$, Σ_θ is a concave cone and the operator $\mathcal{A}_\theta$ is unitarily equivalent to*

$$\mathrm{Id} - \operatorname{Op}_{\mathrm{w}}(H(x)H(\xi)),$$

thus with spectrum $[1 - \mu_2^+, 1 - \mu_2^-]$ (see footnote[1]).

(5) *If $\theta = 2\pi$, we have $\mathcal{A}_{2\pi} = \mathrm{Id}$.*

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[1] So that we have in particular, from (2), the inequalities $1 - \mu_2^+ < 0 < 1 < 1 - \mu_2^-$.

Remark 7.1.2. It is only in the trivial cases $\theta \in \{0, \pi, 2\pi\}$ that A_θ is an orthogonal projection. These cases are also characterized (among cones) by the fact that the spectrum of $\mathcal{A}_\theta$ is included in $[0, 1]$.

Remark 7.1.3. It is interesting to remark that all operators $\mathcal{A}_\theta$ for $\theta \in (0, \pi)$ are unitarily equivalent and thus with constant spectrum $[\mu_2^-, \mu_2^+]$ as given in Theorem 5.2.4. Nevertheless, the sequence $(\mathcal{A}_\theta)_{0<\theta<\pi}$ is weakly converging to the orthogonal projection $\mathcal{A}_\pi$ whose spectrum is $\{0, 1\}$: indeed for $\phi \in \mathscr{S}(\mathbb{R})$, $\psi \in \mathscr{S}(\mathbb{R})$, we have

$$\langle \mathcal{A}_\theta \phi, \psi \rangle_{L^2(\mathbb{R})} = \iint_{\Sigma_\theta} \underbrace{\mathcal{W}(\phi, \psi)}_{\in \mathscr{S}(\mathbb{R}^2)}(x, \xi) dx d\xi,$$

and thus the Lebesgue dominated convergence theorem implies that

$$\lim_{\theta \to \pi_-} \langle \mathcal{A}_\theta \phi, \psi \rangle_{L^2(\mathbb{R})} = \langle \mathcal{A}_\pi \phi, \psi \rangle_{L^2(\mathbb{R})}. \tag{7.1.3}$$

On the other hand, for $u, v \in L^2(\mathbb{R})$ and sequences $(\phi_k)_{k\geq 1}, (\psi_k)_{k\geq 1}$ in $\mathscr{S}(\mathbb{R})$ with respective limits u, v in $L^2(\mathbb{R})$, we have

$$\langle \mathcal{A}_\theta u, v \rangle_{L^2(\mathbb{R})} = \langle \mathcal{A}_\theta (u - \phi_k), v \rangle_{L^2(\mathbb{R})} + \langle \mathcal{A}_\theta \phi_k, v - \psi_k \rangle_{L^2(\mathbb{R})} + \langle \mathcal{A}_\theta \phi_k, \psi_k \rangle_{L^2(\mathbb{R})},$$

so that

$$\begin{aligned}
&\langle \mathcal{A}_\theta u, v \rangle_{L^2(\mathbb{R})} - \langle \mathcal{A}_\pi u, v \rangle_{L^2(\mathbb{R})} \\
&\quad = \langle \mathcal{A}_\theta (u - \phi_k), v \rangle_{L^2(\mathbb{R})} + \langle \mathcal{A}_\theta \phi_k, v - \psi_k \rangle_{L^2(\mathbb{R})} + \langle \mathcal{A}_\theta \phi_k, \psi_k \rangle_{L^2(\mathbb{R})}, \\
&\qquad - \langle \mathcal{A}_\pi (u - \phi_k), v \rangle_{L^2(\mathbb{R})} - \langle \mathcal{A}_\pi \phi_k, v - \psi_k \rangle_{L^2(\mathbb{R})} - \langle \mathcal{A}_\pi \phi_k, \psi_k \rangle_{L^2(\mathbb{R})},
\end{aligned}$$

implying

$$\begin{aligned}
&|\langle \mathcal{A}_\theta u, v \rangle_{L^2(\mathbb{R})} - \langle \mathcal{A}_\pi u, v \rangle_{L^2(\mathbb{R})}| \\
&\quad \leq (\mu_2^+ + 1)\big(\|u - \phi_k\|_{L^2(\mathbb{R})} \|v\|_{L^2(\mathbb{R})} + \|v - \psi_k\|_{L^2(\mathbb{R})} \|\phi_k\|_{L^2(\mathbb{R})}\big) \\
&\qquad + |\langle \mathcal{A}_\theta \phi_k, \psi_k \rangle_{L^2(\mathbb{R})} - \langle \mathcal{A}_\pi \phi_k, \psi_k \rangle_{L^2(\mathbb{R})}|,
\end{aligned}$$

and thus, using (7.1.3), we get

$$\begin{aligned}
&\limsup_{\theta \to 0_+} |\langle \mathcal{A}_\theta u, v \rangle_{L^2(\mathbb{R})} - \langle \mathcal{A}_\pi u, v \rangle_{L^2(\mathbb{R})}| \\
&\qquad \leq (\mu_2^+ + 1)\big(\|u - \phi_k\|_{L^2(\mathbb{R})} \|v\|_{L^2(\mathbb{R})} + \|v - \psi_k\|_{L^2(\mathbb{R})} \|\phi_k\|_{L^2(\mathbb{R})}\big).
\end{aligned}$$

Taking now the infimum with respect to k of the right-hand side in the above inequality, we obtain indeed the weak convergence

$$\lim_{\theta \to 0_+} \langle \mathcal{A}_\theta u, v \rangle_{L^2(\mathbb{R})} = \langle \mathcal{A}_\pi u, v \rangle_{L^2(\mathbb{R})}.$$

Of course, we cannot have strong convergence of the bounded self-adjoint $\mathcal{A}_\theta$ towards (the bounded self-adjoint) A_π because of their respective spectra and the same lines can be written on the weak limit 0 when $\theta \to 0_+$ of $\mathcal{A}_\theta$.

7.2 Triangles

We may consider general "triangles" in the plane that we define as

$$\mathcal{T}_{L_1,L_2,L_3}^{c_1,c_2,c_3} = \big\{(x,\xi) \in \mathbb{R}^2, L_j(x,\xi) \ge c_j,\ j \in \{1,2,3\}\big\},$$

c_j are real numbers and L_j are linear forms. To avoid degenerate situations, we shall assume that

$$\text{for } j \ne k, \quad dL_j \wedge dL_k \ne 0, \quad |\mathcal{T}_{L_1,L_2,L_3}^{c_1,c_2,c_3}| > 0 \quad \text{and} \quad \mathcal{T}_{L_1,L_2,L_3}^{c_1,c_2,c_3} \text{ is not a cone.} \tag{7.2.1}$$

Note that this includes standard triangles (convex envelope of three non-colinear points) but also sets with infinite area such as

$$\big\{(x,\xi) \in \mathbb{R}^2, x \ge 0, \xi \ge 0, x + \xi \ge \lambda\big\}, \quad \text{where } \lambda \text{ is a positive parameter.} \tag{7.2.2}$$

Without loss of generality, we may assume that $L_1(x,\xi) - c_1 = x$, $L_2(x,\xi) - c_2 = \xi$, so that

$$\mathcal{T}_{L_1,L_2,L_3}^{c_1,c_2,c_3} = \big\{(x,\xi) \in \mathbb{R}^2, x \ge 0, \xi \ge 0, ax + b\xi \ge \nu\big\},$$

where a, b, λ are real parameters with $a \ne 0, b \ne 0$ from the assumption (7.2.1); using the symplectic mapping $(x,\xi) \mapsto (\mu x, \xi/\mu)$ with $\mu = \sqrt{|b/a|}$, we see that the condition $ax + b\xi \ge \nu$ becomes

$$x \operatorname{sign} a + \xi \operatorname{sign} b \ge \lambda = \nu/\sqrt{|ab|}, \quad \text{i.e.} \quad \begin{cases} x + \xi & \ge \tilde{\nu}, \\ x - \xi & \ge \tilde{\nu}, \\ -x + \xi & \ge \tilde{\nu}, \\ -x - \xi & \ge \tilde{\nu}. \end{cases}$$

The first case requires $\tilde{\nu} > 0$ and the other cases $\tilde{\nu} < 0$. The only case with finite area is the fourth case

$$\mathcal{T}_{4,\lambda} = \big\{(x,\xi) \in \mathbb{R}^2, x \ge 0, \xi \ge 0, x + \xi \le \lambda\big\} \text{ triangle with area } \lambda^2/2,\ \lambda > 0. \tag{7.2.3}$$

The second case is

$$\mathcal{T}_{2,\lambda} = \big\{(x,\xi) \in \mathbb{R}^2, x \ge 0, \xi \ge 0, x - \xi \ge -\lambda\big\}, \quad \lambda > 0, \tag{7.2.4}$$

the third case is

$$\mathcal{T}_{3,\lambda} = \big\{(x,\xi) \in \mathbb{R}^2, x \ge 0, \xi \ge 0, \xi - x \ge -\lambda\big\}, \quad \lambda > 0, \tag{7.2.5}$$

and the first case is

$$\mathcal{T}_{1,\lambda} = \big\{(x,\xi) \in \mathbb{R}^2, x \ge 0, \xi \ge 0, \xi + x \ge \lambda\big\}, \quad \lambda > 0. \tag{7.2.6}$$

Proposition 7.2.1. *Let $\mathcal{T}_{4,\lambda}$ be a triangle with finite non-zero area in the plane given by* (7.2.3)*, where λ is a positive parameter. Then, the operator $\operatorname{Op_w}(\mathbf{1}_{\mathcal{T}_{4,\lambda}})$ is unitarily equivalent to the operator with kernel*

$$\tilde{k}_{4,\lambda}(x,y) = \mathbf{1}_{[0,\lambda]}\Big(\frac{x+y}{2}\Big)\frac{\sin(\pi(x-y)(\lambda - \frac{x+y}{2}))}{\pi(x-y)}. \tag{7.2.7}$$

The operator $\operatorname{Op_w}(\mathbf{1}_{\mathcal{T}_{4,\lambda}})$ is self-adjoint and bounded on $L^2(\mathbb{R})$ so that

$$\|\operatorname{Op_w}(\mathbf{1}_{\mathcal{T}_{4,\lambda}})\|_{\mathcal{B}(L^2(\mathbb{R}))} \le \frac{1}{2}\Big(\mu_2^+ + \sqrt{1+(\mu_2^+)^2}\Big) := \tilde{\mu}_3, \tag{7.2.8}$$

where μ_2^+ is given in (7.1.1).

Proof. The kernel $k_{4,\lambda}$ of $\operatorname{Op_w}(\mathbf{1}_{\mathcal{T}_{4,\lambda}})$ is such that

$$\begin{aligned}
k_{4,\lambda}(x,y) &= \mathbf{1}_{[0,\lambda]}\Big(\frac{x+y}{2}\Big)\int_0^{\lambda-\frac{x+y}{2}} e^{2i\pi(x-y)\xi}\,d\xi \\
&= \mathbf{1}_{[0,\lambda]}\Big(\frac{x+y}{2}\Big)\frac{(e^{2i\pi(x-y)(\lambda-\frac{x+y}{2})}-1)}{2i\pi(x-y)} \\
&= e^{i\pi(\lambda x-\frac{x^2}{2})}\mathbf{1}_{[0,\lambda]}\Big(\frac{x+y}{2}\Big)\frac{\sin(\pi(x-y)(\lambda-\frac{x+y}{2}))}{\pi(x-y)}e^{-i\pi(\lambda y-\frac{y^2}{2})},
\end{aligned}$$

proving (7.2.7). We note now that the kernel of the operator with Weyl symbol $H(\xi)$ $H(\lambda-\xi-x)$ is

$$\ell_\lambda(x,y) = e^{i\pi(\lambda x-\frac{x^2}{2})}H\Big(\lambda - \frac{x+y}{2}\Big)\frac{\sin(\pi(x-y)(\lambda-\frac{x+y}{2}))}{\pi(x-y)}e^{-i\pi(\lambda y-\frac{y^2}{2})},$$

and that

$$\operatorname{Op_w}\big(H(\xi)H(\lambda-\xi-x)\big)$$

is unitarily equivalent to the operator $\operatorname{Op_w}\big(H(x)H(\xi)\big)$ as given by Theorem 7.1.1. We get then

$$\begin{aligned}
k_{4,\lambda}(x,y) = H(x+y)\ell_\lambda(x,y) &= H(x)\ell_\lambda(x,y)H(y) \\
&\quad + H(x+y)\big(H(x)\check{H}(y)+\check{H}(x)H(y)\big)H\Big(\lambda-\frac{x+y}{2}\Big) \\
&\quad \times \frac{\sin(\pi(x-y)(\lambda-\frac{x+y}{2}))}{\pi(x-y)} \times e^{i\pi(\lambda x-\frac{x^2}{2})}e^{-i\pi(\lambda y-\frac{y^2}{2})},
\end{aligned}$$

and we have thus

$$\operatorname{Op_w}(\mathbf{1}_{\mathcal{T}_{4,\lambda}}) = H\operatorname{Op_w}\left(H(\xi)H(\lambda-\xi-x)\right)H + \Omega_\lambda,$$

where the kernel $\omega_\lambda(x,y)$ of the operator Ω_λ verifies

$$\begin{aligned}|\omega_\lambda(x,y)| &\le \frac{H(x+y)\big(H(x)\check{H}(y)+\check{H}(x)H(y)\big)}{\pi|x-y|}\\ &= \frac{H(x+y)\big(H(x)\check{H}(y)+\check{H}(x)H(y)\big)}{\pi(|x|+|y|)}.\end{aligned}$$

We obtain, thanks to Proposition A.5.1 (2), that

$$\iint |\omega_\lambda(x,y)||u(y)||u(x)|dydx \le \|\check{H}u\|_{L^2(\mathbb{R})}\|Hu\|_{L^2(\mathbb{R})}.$$

As a result, we find that

$$|\langle \operatorname{Op_w}(\mathbf{1}_{\mathcal{T}_{4,\lambda}})u,u\rangle_{L^2(\mathbb{R})}| \le \mu_2^+\|Hu\|^2_{L^2(\mathbb{R})} + \|\check{H}u\|_{L^2(\mathbb{R})}\|Hu\|_{L^2(\mathbb{R})},$$

proving (7.2.8). ∎

Proposition 7.2.2. *Let $\mathcal{T}_{1,\lambda}$ be a triangle with infinite area in the plane given by (7.2.6), where λ is a positive parameter. Then, the operator $\operatorname{Op_w}(\mathbf{1}_{\mathcal{T}_{1,\lambda}})$ is unitarily equivalent to the operator with kernel*

$$\tilde{k}_{1,\lambda}(x,y) = \mathbf{1}_{[0,\lambda]}\left(\frac{x+y}{2}\right)\frac{\sin\!\big(\pi(x-y)(\lambda-\frac{x+y}{2})\big)}{\pi(x-y)}.$$

The operator $\operatorname{Op_w}(\mathbf{1}_{\mathcal{T}_{1,\lambda}})$ is self-adjoint and bounded on $L^2(\mathbb{R})$ so that

$$\|\operatorname{Op_w}(\mathbf{1}_{\mathcal{T}_{1,\lambda}})\|_{\mathcal{B}(L^2(\mathbb{R}))} \le \frac{1}{2}\left(\mu_2^+ + \sqrt{\frac{1}{4}+(\mu_2^+)^2}\right) \approx 1.066294188078, \tag{7.2.9}$$

where μ_2^+ is given in (7.1.1).

Proof. We note that the kernel of the operator $\operatorname{Op_w}(H(x+\xi-\lambda)H(\xi))$ is

$$\ell_1(x,y) = e^{2i\pi(x-y)\max(0,\lambda-\frac{x+y}{2})}\frac{1}{2}\left(\delta_0(y-x)+\frac{1}{i\pi(y-x)}\right),$$

so that

$$\operatorname{Op_w}(\mathbf{1}_{\mathcal{T}_{1,\lambda}}) = H\underbrace{\operatorname{Op_w}(H(x+\xi-\lambda)H(\xi))}_{\substack{\text{unitarily equivalent to}\\ \operatorname{Op_w}(H(x)H(\xi))}}H + \Omega_{1,\lambda}, \tag{7.2.10}$$

where the kernel $\omega_{1,\lambda}$ of the operator $\Omega_{1,\lambda}$ is equal to

$$H(x+y)\big(H(x)\check{H}(y)+\check{H}(x)H(y)\big)\ell_1(x,y),$$

and such that

$$|\omega_{1,\lambda}(x,y)| \le H(x+y)\frac{(H(x)\check{H}(y)+\check{H}(x)H(y))}{2\pi(|x|+|y|)},$$

and, thanks to Proposition A.5.1 (2), we get from (7.2.10) that

$$|\langle \mathrm{Op_w}(\mathbf{1}_{\mathcal{T}_{1,\lambda}})u,u\rangle_{L^2(\mathbb{R})}| \le \mu_2^+\|Hu\|^2_{L^2(\mathbb{R})} + \frac{1}{2}\|\check{H}u\|_{L^2(\mathbb{R})}\|Hu\|_{L^2(\mathbb{R})},$$

which gives (7.2.9). ∎

We leave for the reader to check the two other cases (7.2.4), (7.2.5), which are very similar as well as the degenerate cases excluded by (7.2.1), which are in fact easier to tackle.

Theorem 7.2.3. *Let*

$$\mathscr{T} = \left\{\mathcal{T}^{c_1,c_2,c_3}_{L_1,L_2,L_3}\right\}_{\substack{c_j\in\mathbb{R},\ L_j\\ \text{linear form on } \mathbb{R}^2}}$$

be the set of triangles of $\mathbb{R}^2$. For all $\mathcal{T}\in\mathscr{T}$, the operator $\mathrm{Op_w}(\mathbf{1}_{\mathcal{T}})$ is bounded on $L^2(\mathbb{R})$, self-adjoint and we have

$$\begin{aligned}1.007680 \approx \mu_2^+ &= \sup_{\mathcal{C}\ \text{cone}} \|\mathrm{Op_w}(\mathbf{1}_{\mathcal{C}})\|_{\mathcal{B}(L^2(\mathbb{R}))}\\ \le \mu_3^+ &= \sup_{\mathcal{T}\ \text{triangle}} \|\mathrm{Op_w}(\mathbf{1}_{\mathcal{T}})\|_{\mathcal{B}(L^2(\mathbb{R}))} \le \tilde{\mu}_3 \approx 1.213668.\end{aligned}$$

N.B. The L^2 boundedness is easy to prove since it is obvious for triangles with finite areas and in the case of triangles with infinite area, we may note that in the case (7.2.6) (resp., (7.2.4), (7.2.5)) they are the union of two cones (resp., one cone) with a strip $[0,1]\times\mathbb{R}_+$. What matters most in the above statement is the effective explicit bound. Our result does not give an explicit value for μ_3^+ and it is quite likely that the bound given by $\tilde{\mu}_3$ is way too large.

Proof. The second inequality is proven in Propositions 7.2.1 and 7.2.2, whereas the first inequality is a consequence of Theorem 5.3.1. ∎

Remark 7.2.4. This implies that for any $u\in L^2(\mathbb{R})$ and any $\mathcal{T}\in\mathscr{T}$, we have

$$\left|\iint_{\mathcal{T}} \mathcal{W}(u,u)(x,\xi)\,dx\,d\xi\right| \le \tilde{\mu}_3\|u\|^2_{L^2(\mathbb{R})}, \quad \text{with } \tilde{\mu}_3 \approx 1.213668.$$

7.3 Convex polygons

We want to tackle now the general case of a convex polygon in the plane. We consider

$$L_1, \dots, L_N,$$

to be N linear forms of x, ξ ($L_j(x,\xi) = a_j\xi - \alpha_j x = [(x,\xi);(a_j,\alpha_j)]$) and $c_1, \dots, c_N$ some real constants. We consider the convex polygon

$$\mathcal{P} = \{(x,\xi) \in \mathbb{R}^2, \forall j \in \{1,\dots,N\}, L_j(x,\xi) - c_j \geq 0\}, \tag{7.3.1}$$

so that

$$\mathbf{1}_{\mathcal{P}}(x,\xi) = \prod_{1 \leq j \leq N} H\big(L_j(x,\xi) - c_j\big).$$

Definition 7.3.1. Let $N \in \mathbb{N}^*$, let $L_1, \dots, L_N$ be linear forms on $\mathbb{R}^2$ and let $c_1, \dots, c_N$ be real numbers. The polygon with N sides $\mathcal{P}^{c_1,\dots,c_N}_{L_1,\dots,L_N}$ is defined by (7.3.1). We shall denote by $\mathscr{P}_N$ the set of all polygons with N sides.

N.B. Since we may take some $L_j = 0$ in (7.3.1), we see that $\mathscr{P}_N \subset \mathscr{P}_{N+1}$.

Note as above that it includes some convex subsets of the plane with infinite area such as (7.2.2).

Theorem 7.3.2. *Let $\mathscr{P}_N$ be the set of convex polygons with N sides of the plane $\mathbb{R}^2$. We define*

$$\mu_N^+ = \sup_{\mathcal{P} \in \mathscr{P}_N} \|\mathrm{Op_w}(\mathbf{1}_{\mathcal{P}})\|_{\mathcal{B}(L^2(\mathbb{R}))}.$$

Then, μ_2^+ is given by Theorem 5.2.4 and

$$\forall N \geq 3, \quad \mu_N^+ \leq \sqrt{N/2}.$$

Proof. Using an affine symplectic transformation, we may assume that $L_N(x,\xi) - c_N = x$, so that

$$\mathbf{1}_{\mathcal{P}}(x,\xi) = H(x) \prod_{1 \leq j \leq N-1} H\big(a_j\xi - \alpha_j x - c_j\big),$$

and the kernel of the operator $\mathrm{Op_w}(\mathbf{1}_{\mathcal{P}})$ is

$$k_N(x,y) = H(x+y) \int e^{2i\pi(x-y)\xi} \prod_{1 \leq j \leq N-1} H\left(a_j\xi - \alpha_j\Big(\frac{x+y}{2}\Big) - c_j\right) d\xi.$$

As a result, we have

$$k_N(x,y) = H(x+y) k_{N-1}(x,y),$$

where k_{N-1} is the kernel of $\operatorname{Op}_w(\mathbf{1}_{\mathscr{P}_{N-1}})$, where

$$\mathscr{P}_{N-1} = \big\{(x,\xi) \in \mathbb{R}^2, \forall j \in \{1,\dots,N-1\}, L_j(x,\xi) - c_j \ge 0\big\}.$$

We may assume inductively that for any convex polygon $\mathscr{P}_k$ with $k \le N-1$ sides, there exist μ_k^+ such that

$$\operatorname{Op}_w(\mathbf{1}_{\mathscr{P}_k}) \le \mu_k^+,$$

where μ_k^+ depends only on k and not on the area of the polygon, a fact already proven for $k = 1, 2, 3$. We note that with $A_N = \operatorname{Op}_w(\mathbf{1}_{\mathscr{P}_N})$, we have with H standing for the operator of multiplication by $H(x)$,

$$HA_N H = HA_{N-1}H, \quad A_{N-1} = \operatorname{Op}_w(\mathbf{1}_{\mathscr{P}_{N-1}}),$$

since the kernel of $HA_N H$ is

$$H(x)H(y)k_N(x,y) = H(x+y)H(x)H(y)k_{N-1}(x,y) = H(x)H(y)k_{N-1}(x,y).$$

Also, we have, with $\check{H}(x) = H(-x)$, that $\check{H}A_N\check{H} = 0$, since the kernel of that operator is

$$\check{H}(x)\check{H}(y)H(x+y)k_{N-1}(x,y) = 0.$$

We have thus

$$A_N = HA_{N-1}H + 2\operatorname{Re}\check{H}A_N H, \tag{7.3.2}$$

and the kernel of $2\operatorname{Re}\check{H}A_N H$ is

$$\omega_N(x,y) = H(x+y)\big(\check{H}(x)H(y) + \check{H}(y)H(x)\big)k_{N-1}(x,y).$$

We calculate now

$$k_{N-1}(x,y) = \int e^{2i\pi(x-y)\xi} \prod_{1\le j\le N-1} H\Big(a_j\xi - \alpha_j\big(\frac{x+y}{2}\big) - c_j\Big)d\xi.$$

We check first the j such that $a_j = 0$ (and thus $\alpha_j \ne 0$)[2]. Without loss of generality, we may assume that this happens for $1 \le j < N_0$ so that with some interval J of the real line, $\tilde{\alpha}_j = \alpha_j/a_j, \tilde{c}_j = c_j/a_j$,

$$k_{N-1}(x,y) = \mathbf{1}_J\big(\frac{x+y}{2}\big)\int e^{2i\pi(x-y)\xi} \prod_{\substack{N_0\le j\le N-1\\ a_j>0}} H\Big(\xi - \tilde{\alpha}_j\big(\frac{x+y}{2}\big) - \tilde{c}_j\Big)$$
$$\times \prod_{\substack{N_0\le j\le N-1\\ a_j<0}} \check{H}\Big(\xi - \tilde{\alpha}_j\big(\frac{x+y}{2}\big) - \tilde{c}_j\Big)d\xi.$$

[2]In this induction proof, we may assume that all the linear forms L_j, $1 \le j \le N$ are different from 0, otherwise we may use the induction hypothesis.

We note that the integration domain is

$$\psi\Big(\frac{x+y}{2}\Big) = \max_{\substack{N_0 \le j \le N-1 \\ a_j > 0}} \Big(\tilde{\alpha}_j\Big(\frac{x+y}{2}\Big) + \tilde{c}_j\Big)$$
$$\le \xi \le \min_{\substack{N_0 \le j \le N-1 \\ a_j < 0}} \tilde{\alpha}_j\Big(\frac{x+y}{2}\Big) + \tilde{c}_j = -\phi\Big(\frac{x+y}{2}\Big),$$

with ϕ, ψ convex piecewise affine functions; since $\phi + \psi$ is also a convex function, we get the – convex – constraint $(\phi + \psi)((x+y)/2) \le 0$, so that $(x+y)/2$ must belong to a subinterval $\tilde{J}$ of the interval J. As a result we get that

$$\begin{aligned}
&k_{N-1}(x,y) \\
&= \mathbf{1}_{\tilde{J}}\Big(\frac{x+y}{2}\Big)\frac{e^{-2i\pi(x-y)\phi(\frac{x+y}{2})} - e^{2i\pi(x-y)\psi(\frac{x+y}{2})}}{2i\pi(x-y)} \\
&= \mathbf{1}_{\tilde{J}}\Big(\frac{x+y}{2}\Big)e^{-i\pi(x-y)(\phi-\psi)(\frac{x+y}{2})}\frac{e^{-i\pi(x-y)(\phi+\psi)(\frac{x+y}{2})} - e^{i\pi(x-y)(\phi+\psi)(\frac{x+y}{2})}}{2i\pi(x-y)} \\
&= \mathbf{1}_{\tilde{J}}\Big(\frac{x+y}{2}\Big)e^{-i\pi(x-y)(\phi-\psi)(\frac{x+y}{2})}\frac{\sin(\pi(x-y)(\phi+\psi)(\frac{x+y}{2}))}{\pi(y-x)},
\end{aligned}$$

and thus the kernel of $2\,\mathrm{Re}\,\check{H} A_N H$ is

$$\omega_N(x,y) = H(x+y)\big(\check{H}(x)H(y) + \check{H}(y)H(x)\big)\mathbf{1}_{\tilde{J}}\Big(\frac{x+y}{2}\Big)$$
$$\times\, e^{-i\pi(x-y)(\phi-\psi)(\frac{x+y}{2})}\frac{\sin(\pi(x-y)(\phi+\psi)(\frac{x+y}{2}))}{\pi(y-x)},$$

so that, thanks to Proposition A.5.1 (2),

$$2\,\mathrm{Re}\langle \check{H} A_N H u, u\rangle \le \|Hu\|\|\check{H}u\|,$$

and with (7.3.2) we obtain, $\langle A_N u, u\rangle \le \mu_{N-1}^+\|Hu\|^2 + \|Hu\|\|\check{H}u\|$, and we get

$$\mu_N^+ \le \frac{\mu_{N-1}^+ + \sqrt{(\mu_{N-1}^+)^2 + 1}}{2}.$$

This implies that

$$\forall N \ge 3, \quad \mu_N^+ \le \sqrt{N/2},$$

since it is true for $N = 3$ and[3] if we assume that it is true for some $N \ge 3$, we get

$$\mu_{N+1}^+ \le \frac{\mu_N^+ + \sqrt{(\mu_N^+)^2 + 1}}{2} \le \frac{1}{2}\Big(\sqrt{\frac{N}{2}} + \sqrt{\frac{N+2}{2}}\Big) \le \sqrt{\frac{N+1}{2}},$$

[3]Indeed, we have $\mu_3^+ \le \tilde{\mu}_3 < 1.2137 < 1.2247 \approx \sqrt{3/2}$.

where the latter inequality follows from the concavity of the square-root function since we have for a concave function F,

$$\frac{1}{2}\frac{N}{2}+\frac{1}{2}\frac{N+2}{2}=\frac{N+1}{2}$$

and thus

$$\frac{1}{2}F\Big(\frac{N}{2}\Big)+\frac{1}{2}F\Big(\frac{N+2}{2}\Big)\le F\Big(\frac{N+1}{2}\Big).$$

The proof of Theorem 7.3.2 is complete. ∎

Remark 7.3.3. The above result is weak by its dependence on the number of sides, but it should be pointed out that it is independent of the area of the polygon (which could be infinite). Another general comment is concerned with convexity: although Flandrin's conjecture is not true, there is still something special about convex subsets of the phase space and it is in particular interesting that an essentially explicit calculation of the kernel of the operator $\mathrm{Op_w}(\mathbf{1}_{\mathscr{P}})$ is tractable when $\mathscr{P}$ is a polygon with N sides of $\mathbb{R}^2$. Something analogous could probably be done with convex polytopes of $\mathbb{R}^{2n}$.

7.4 Symbols supported in a half-space

Theorem 7.4.1. (1) *Let A be a bounded self-adjoint operator on $L^2(\mathbb{R}^n)$ such that its Weyl symbol $a(x,\xi)$ is supported in $\mathbb{R}_+\times\mathbb{R}^{2n-1}$. Then, with $\check H$ standing for the orthogonal projection onto*

$$\big\{u\in L^2(\mathbb{R}^n),\operatorname{supp}u\subset\mathbb{R}_-\times\mathbb{R}^{n-1}\big\},$$

we have $\check H A\check H=0$.

(2) *Let A be as above; if A is a non-negative operator, then with $H=I-\check H$, we have*

$$\check H A=A\check H=0,\quad A=HAH,$$

N.B. We have seen explicit examples of bounded self-adjoint operators such that the Weyl symbol is supported in $x\ge 0$ but for which $\check H AH\ne 0$: the quarter-plane operator (see Section 5.1) has the Weyl symbol $H(x)H(\xi)$, the kernel of

$$\check H\mathrm{Op_w}(H(x)H(\xi))H\ \text{ is }\ \check H(x)H(y)H(x+y)\frac{1}{2i\pi}\mathrm{pv}\frac{1}{y-x},$$

which is not the zero distribution and, according to the above result, this alone implies that

$$\mathrm{Op_w}\left(H(x)H(\xi)\right)$$

cannot be non-negative.

Proof. Let us prove first that $\check{H} A \check{H} = 0$; let $\phi, \psi \in C_c^\infty(\mathbb{R}^n)$ such that

$$\operatorname{supp}\phi \cup \operatorname{supp}\psi \subset (-\infty, 0) \times \mathbb{R}^{n-1}.$$

Since the Wigner distribution $\mathcal{W}(\phi, \psi)$ belongs to $\mathscr{S}(\mathbb{R}^{2n})$ and is given by the integral

$$\mathcal{W}(\phi, \psi)(x, \xi) = \int_{\mathbb{R}^n} \phi\Big(x + \frac{z}{2}\Big)\bar{\psi}\Big(x - \frac{z}{2}\Big)e^{-2i\pi z\cdot\xi}dz,$$

we infer right away[4] that $\operatorname{supp}\mathcal{W}(\phi, \psi) \subset (-\infty, 0) \times \mathbb{R}^{2n-1}$. We know also that

$$\langle A\phi, \psi\rangle_{L^2(\mathbb{R}^n)} = \langle A\phi, \psi\rangle_{\mathscr{S}'(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} = \langle a, \mathcal{W}(\phi, \psi)\rangle_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})} = 0.$$

As a result, the $L^2(\mathbb{R}^n)$ bounded operator $\check{H} A \check{H}$ is such that, for $u, v \in L^2(\mathbb{R}^n)$, ϕ, ψ as above,

$$\begin{aligned}\langle \check{H} A \check{H} u, v\rangle_{L^2(\mathbb{R}^n)} &= \langle \check{H} A \check{H} \check{H} u, \check{H} v\rangle_{L^2(\mathbb{R}^n)} \\ &= \langle \check{H} A \check{H}(\check{H} u - \phi), \check{H} v\rangle_{L^2(\mathbb{R}^n)} + \langle \check{H} A \check{H}\phi, \check{H} v - \psi\rangle_{L^2(\mathbb{R}^n)} \\ &\quad + \underbrace{\langle \check{H} A \check{H}\phi, \psi\rangle_{L^2(\mathbb{R}^n)}}_{\langle A\phi,\psi\rangle_{L^2(\mathbb{R}^n)}=0},\end{aligned}$$

so that

$$\begin{aligned}&|\langle \check{H} A \check{H} u, v\rangle_{L^2(\mathbb{R}^n)}| \\ &\le \|A\|_{\mathcal{B}(L^2(\mathbb{R}^n))}\big(\|\check{H} u - \phi\|_{L^2(\mathbb{R}^n)}\|v\|_{L^2(\mathbb{R}^n)} + \|\check{H} v - \psi\|_{L^2(\mathbb{R}^n)}\|\phi\|_{L^2(\mathbb{R}^n)}\big).\end{aligned}$$

Using now that the set $\{\phi \in C_c^\infty(\mathbb{R}^n), \operatorname{supp}\phi \subset (-\infty, 0) \times \mathbb{R}^{n-1}\}$ is dense[5] in

$$\big\{w \in L^2(\mathbb{R}^n), \operatorname{supp} w \subset (-\infty, 0] \times \mathbb{R}^{n-1}\big\}, \tag{7.4.1}$$

---

[4]In the integrand, we must have, $x_1 + \frac{z_1}{2} \le -\varepsilon_0 < 0$, $x_1 - \frac{z_1}{2} \le -\varepsilon_1 < 0$ and thus $x_1 \le -(\varepsilon_0 + \varepsilon_1)/2$

[5]Let χ_0 be a function satisfying (5.2.1) and let w be in the set (7.4.1). Let $(\phi_k)_{k\ge1}$ be a sequence in $C_c^\infty(\mathbb{R}^n)$ converging in $L^2(\mathbb{R}^n)$ towards w; the function defined by

$$\tilde{\phi}_k(x) = \chi_0(-kx_1)\phi_k(x),$$

belongs to $C_c^\infty(\mathbb{R}^n)$, is supported in $(-\infty, -1/k] \times \mathbb{R}^{n-1}$, and that sequence converges in $L^2(\mathbb{R}^n)$ towards w since

$$\|\tilde{\phi}_k - w\|_{L^2(\mathbb{R}^n)} \le \underbrace{\|\chi_0(-kx_1)\big(\phi_k(x) - w(x)\big)\|_{L^2(\mathbb{R}^n)}}_{\le\|\phi_k - w\|_{L^2(\mathbb{R}^n)}\to 0 \text{ when } k\to+\infty.} + \|(\chi_0(-kx_1) - 1)w(x)\|_{L^2(\mathbb{R}^n)}$$

and $\|(\chi_0(-kx_1) - 1)w(x)\|^2_{L^2(\mathbb{R}^n)} \le \int \mathbf{1}\{-\frac{2}{k} \le x_1 \le 0\}|w(x)|^2 dx$ which has also limit 0 when k goes to $+\infty$ by the Lebesgue dominated convergence theorem.

we obtain that $\langle \check{H} A \check{H} u, v \rangle_{L^2(\mathbb{R}^n)} = 0$ and the first result. Let us assume that the operator A is non-negative. We have

$$A = B^2, \quad B = B^* \text{ bounded self-adjoint.}$$

It implies with $L^2(\mathbb{R}^n)$ norms and dot-products,

$$\begin{aligned}
\langle Au, u \rangle &= \langle HAHu, u \rangle + 2\,\mathrm{Re}\langle \check{H} AHu, \check{H} u \rangle \\
&= \langle HBBHu, u \rangle + 2\,\mathrm{Re}\langle \check{H} BBHu, \check{H} u \rangle \\
&= \|BHu\|^2 + 2\,\mathrm{Re}\langle BHu, B\check{H} u \rangle \\
&= \|BHu + B\check{H} u\|^2 - \|B\check{H} u\|^2 \\
&= \|Bu\|^2 - \|B\check{H} u\|^2 = \langle Au, u \rangle - \|B\check{H} u\|^2,
\end{aligned}$$

and thus $B\check{H} = 0$, so that $\check{H} B = 0$ and thus $\check{H} B^2 = \check{H} A = 0 = A\check{H}$, so that $\check{H} AH = 0 = HA\check{H}$, and $A = HAH$, concluding the proof of (2). ■

Corollary 7.4.2. *Let A be a bounded self-adjoint operator on $L^2(\mathbb{R}^n)$ such that its Weyl symbol is supported in $\mathbb{R}_+ \times \mathbb{R}^{2n-1}$ and such that* $\mathrm{Re}(\check{H} AH) \neq 0$, *then the spectrum of A intersects* $(-\infty, 0)$.

Proof. We have from (1) in the previous theorem,

$$A = (H + \check{H})A(H + \check{H}) = HAH + 2\,\mathrm{Re}\, HA\check{H},$$

and from (2), if A were non-negative, we would have $A\check{H} = 0$ and $\mathrm{Re}\, HA\check{H} = 0$, contradicting the assumption. ■

Remark 7.4.3. If $\mathscr{C}$ is a compact convex body of $\mathbb{R}^{2n}$, we may use the fact (see, e.g., [45]) that

$$\mathscr{C} = \bigcap_{\substack{\mathfrak{H}_j \text{ closed half-spaces} \\ \text{containing } K}} \mathfrak{H}_j .$$

Then, of course $\mathrm{Op_w}(\mathbf{1}_\mathscr{C})$ is a bounded self-adjoint operator on $L^2(\mathbb{R}^n)$, and if $\mathfrak{H}_j$ is defined by

$$\mathfrak{H}_j = \big\{(x, \xi) \in \mathbb{R}^2, L_j(x, \xi) \geq c_j\big\},$$

where L_j is a linear form on $\mathbb{R}^2$ and c_j a real constant, we obtain with the symplectic covariance of the Weyl calculus, setting

$$\mathscr{H}_j(x, \xi) = H\big(L_j(x, \xi) - c_j\big),$$

that for all $\mathfrak{H}_j$ closed half-spaces containing $\mathscr{C}$, we have

$$\mathrm{Op_w}(\mathbf{1}_\mathscr{C}) = \mathrm{Op_w}(H_j)\mathrm{Op_w}(\mathbf{1}_\mathscr{C})\mathrm{Op_w}(H_j) + 2\,\mathrm{Re}\,\mathrm{Op_w}(\check{H}_j)\mathrm{Op_w}(\mathbf{1}_\mathscr{C})\mathrm{Op_w}(H_j),$$

where $\check{H}(x, \xi) = H(-L_j(x, \xi) + c_j)$.

Chapter 8

Open questions and conjectures

In this section, we review the rather long list of conjectures formulated in the text and we try to classify their statements by rating their respective interest, relevance and difficulty. We should keep in mind that the study of $\mathrm{Op_w}(\mathbf{1}_E)$ for a subset E of the phase space is highly correlated to some particular set of special functions related to E: Hermite functions and Laguerre polynomials for ellipses, Airy functions for parabolas, homogeneous distributions for hyperbolas and so on. It is quite likely that the "shape" of E will determine the type of special functions to be studied to getting a diagonalisation of the operator $\mathrm{Op_w}(\mathbf{1}_E)$.

8.1 Anisotropic ellipsoids and paraboloids

Conjecture 8.1.1. Let E be an ellipsoid in $\mathbb{R}^{2n}$ equipped with its canonical symplectic structure. Then, the operator $\mathrm{Op_w}(\mathbf{1}_E)$ is bounded on $L^2(\mathbb{R}^n)$ (which is obvious from (1.2.5)) and we have

$$\mathrm{Op_w}(\mathbf{1}_E) \le \mathrm{Id}\,. \tag{8.1.1}$$

A sharp version of this result was proven for $n = 1$ in the 1988 P. Flandrin's article [13], and was improved to an isotropic higher-dimensional setting in paper [39] by E. Lieb and Y. Ostrover. Without isotropy, it remains a conjecture. As described in more details in Section 3.4, it can be reformulated as a problem on Laguerre polynomials. That conjecture is a very natural one and it would be quite surprising that a counterexample to (8.1.1) could occur from an anisotropic ellipsoid[1]. We introduced in Section 4.4 a conjecture on anisotropic paraboloids directly related to Conjecture 8.1.1.

Conjecture 8.1.2. Let E be an anisotropic paraboloid in $\mathbb{R}^{2n}$ equipped with its canonical symplectic structure. Then, the operator $\mathrm{Op_w}(\mathbf{1}_E)$ is bounded on $L^2(\mathbb{R}^n)$ and we have

$$\mathrm{Op_w}(\mathbf{1}_E) \le \mathrm{Id}\,. \tag{8.1.2}$$

In terms of special functions, it is related to a property of Airy-type functions. As a contrast with ellipses, we do not expect (8.1.2) to leave any room for improvement whereas (8.1.1) can certainly be improved with its right-hand side replaced by a smaller operator as in (3.2.2).

[1] We mean by anisotropic ellipsoid a set of type (3.3.2) where $0 < a_1 < a_2 < \cdots < a_n$.

8.2 Balls for the ℓ^p norm

We have seen in Section 5.3.2 that the quantization of the indicatrix of an ℓ^p ball could have a spectrum intersecting $(1, +\infty)$ when $p \neq 2$. More generally one could raise the following question.

Question 8.2.1. Let $p \in [1, +\infty]$, $p \neq 2$ and let $\mathbb{B}_p^{2n}$ be the unit ℓ^p ball in $\mathbb{R}^{2n}$. For $\lambda > 0$, we define the operator

$$P_{n,p,\lambda} = \mathrm{Op_w}(\mathbf{1}_{\lambda \mathbb{B}_p^{2n}}).$$

Is it possible to say something on the spectrum of the operator $P_{n,p,\lambda}$, even in a two-dimensional phase space ($n = 1$)? Is there an asymptotic behaviour for the upper bound of the spectrum of $P_{n,p,\lambda}$ when λ goes to $+\infty$?

8.3 On generic pulses in $L^2(\mathbb{R}^n)$

We have seen that the set $\mathscr{G}$ defined in (6.3.2) is generic in the Baire category sense, but our explicit examples were quite simplistic.

Question 8.3.1. Let $\mathscr{G}$ be defined in (6.3.2). Does there exist $u \in \mathscr{G}$ such that the set $E_+(u)$ (defined in (6.4.3)) is connected?

8.4 On convex bodies

Conjecture 8.4.1. For $N \geq 2$, we define

$$\mu_N^+ = \sup_{\substack{\mathscr{P} \text{ convex bounded} \\ \text{polygon with } N \text{ sides}}} \mathrm{Spectrum}(\mathrm{Op_w}(\mathbf{1}_{\mathscr{P}})).$$

Then, the sequence $(\mu_N^+)_{N \geq 2}$ is increasing[2] and there exists $\alpha > 0$ such that

$$\forall N \geq 2, \quad \mu_N^+ \leq \alpha \ln N.$$

N.B. Theorem 7.3.2 is a small step in this direction.

A stronger version of Conjecture 8.4.1 would be the following conjecture.

Conjecture 8.4.2. We define

$$\mu^+ = \sup_{\substack{\mathscr{C} \text{ convex} \\ \text{bounded}}} \mathrm{Spectrum}(\mathrm{Op_w}(\mathbf{1}_{\mathscr{C}})).$$

Then, we have $\mu^+ < +\infty$.

---

[2]According to our Definition 7.3.1 of the set $\mathscr{P}_N$ of polygons with N sides is increasing with respect to N.

The invalid Flandrin's conjecture was $\mu^+ = 1$ and we know now that $\mu_+ \geq \mu_2^+ > 1$ as given by (7.1.2).

Question 8.4.3. There is a diagonalisation of quantization of the indicator function of ellipsoids, paraboloids, and hyperbolic regions. Is there a non-quadratic example of diagonalisation?

Note that the quarter-plane, studied in Section 5.1, is somehow a degenerate hyperbolic region, but could be seen as a first answer to the above question. In the phase space $\mathbb{R}^{2n}$, an argument of homogeneity, similar to the one used for the quarter-plane, can probably be useful for handling integrals of the Wigner distribution on cones.

Question 8.4.4. The value of μ_2^+ is known explicitly, but for μ_3^+, we have only the upperbound $\tilde{\mu}_3$ as given by Theorem 7.2.3. Is it possible to determine explicitly the value of μ_3^+, either by answering Question 8.4.3, or via another argument?

Conjecture 8.4.5. Let $\mathscr{C}$ be a proper closed convex subset of $\mathbb{R}^2$ with positive Lebesgue measure such that $\operatorname{Op_w}(\mathbf{1}_{\mathscr{C}})$ is bounded self-adjoint on $L^2(\mathbb{R})$ (that assumption is useless if Conjecture 8.4.2 is proven) with a spectrum included in $[0, 1]$. Then, $\mathscr{C}$ is the strip $[0, 1] \times \mathbb{R}$, up to an affine symplectic map.

All the explicitly available examples are compatible with that conjecture (see also Remark 7.1.2) and the second part of Theorem 7.4.1 is also an indication in that direction. It would be nice in that instance to reach a spectral characterization of a subset modulo the affine symplectic group.

Appendix A

A.1 Fourier transform, Weyl quantization, harmonic oscillator

A.1.1 Fourier transform

For $f \in \mathscr{S}(\mathbb{R}^N)$, we define its Fourier transform by

$$\hat{f}(\xi) = \int_{\mathbb{R}^N} e^{-2i\pi x\cdot\xi} f(x)dx, \tag{A.1.1}$$

and we obtain the inversion formula

$$f(x) = \int_{\mathbb{R}^N} e^{2i\pi x\cdot\xi} \hat{f}(\xi)d\xi. \tag{A.1.2}$$

Both formulas can be extended to tempered distributions: for $T \in \mathscr{S}'(\mathbb{R}^N)$, we define the tempered distribution $\widehat{T}$ by

$$\langle \widehat{T}, \phi\rangle_{\mathscr{S}'(\mathbb{R}^N),\mathscr{S}(\mathbb{R}^N)} = \langle T, \hat{\phi}\rangle_{\mathscr{S}'(\mathbb{R}^N),\mathscr{S}(\mathbb{R}^N)}. \tag{A.1.3}$$

Note also that with this normalization, it is natural to introduce the operators D_x^α defined for $\alpha \in \mathbb{N}^N$ by

$$D_x^\alpha u = D_{x_1}^{\alpha_1} \cdots D_{x_N}^{\alpha_n} u, \quad D_{x_j} u = \frac{\partial u}{2i\pi \partial x_j}, \tag{A.1.4}$$

so that

$$\widehat{D_x^\alpha u} = \xi^\alpha \hat{u}(\xi),$$

with

$$\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_N^{\alpha_N}.$$

It follows readily from (A.1.1), (A.1.2), and (A.1.3) that for $u \in \mathscr{S}'(\mathbb{R}^n)$, the inversion formula

$$\check{\hat{\hat{u}}} = u, \tag{A.1.5}$$

holds true, where the distribution $\check{u}$ (extending (1.1.10)) is defined by

$$\langle \check{u}, \phi\rangle_{\mathscr{S}'(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} = \langle u, \check{\phi}\rangle_{\mathscr{S}'(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)}.$$

Using (1.2.6) and denoting the Fourier transformation by $\mathscr{F}$, (A.1.5) read

$$\sigma_0\mathscr{F}^2 = \mathrm{Id}, \quad [\mathscr{F}, \sigma_0] = 0, \text{ so that } \mathscr{F}^* = \mathscr{F}^{-1} = \sigma_0\mathscr{F} = \mathscr{F}\sigma_0. \tag{A.1.6}$$

This normalization yields simple formulas for the Fourier transform of Gaussian functions: for A a real-valued symmetric positive definite $n \times n$ matrix, we define the function v_A in the Schwartz space by

$$v_A(x) = e^{-\pi \langle Ax, x\rangle},$$

and we have

$$\widehat{v_A}(\xi) = (\det A)^{-1/2} e^{-\pi \langle A^{-1}\xi, \xi\rangle}.$$

Similarly, when B is a real-valued symmetric non-singular $n \times n$ matrix, the function w_B defined by

$$w_B(x) = e^{i\pi \langle Bx, x\rangle}$$

is in $L^\infty(\mathbb{R}^n)$ and thus a tempered distribution and we have

$$\widehat{w_B}(\xi) = |\det B|^{-1/2} e^{\frac{i\pi}{4} \operatorname{sign} B} e^{-i\pi \langle B^{-1}\xi, \xi\rangle}, \tag{A.1.7}$$

where sign B stands for the *s*ignature of B that is, with $\mathtt{E}$ the set of eigenvalues of B (which are real and non-zero),

$$\operatorname{sign} B = \underbrace{\operatorname{Card}(\mathtt{E} \cap \mathbb{R}_+)}_{\nu_+(B)} - \underbrace{\operatorname{Card}(\mathtt{E} \cap \mathbb{R}_-)}_{\nu_-(B) = \operatorname{index}(B)}.$$

The integer $\nu_-(B)$ is called the *index* of B, noted index (B); formula (A.1.7) can be written as

$$e^{-i\pi n/4} \mathcal{F}\left(e^{i\pi\langle Bx, x\rangle}\right) = i^{-\operatorname{index} B} |\det B|^{-1/2} e^{-i\pi\langle B^{-1}\xi, \xi\rangle}, \tag{A.1.8}$$

since $\nu_+ + \nu_- = n$ (as B is non-singular),

$$e^{\frac{i\pi n}{4}} e^{-\frac{i\pi\nu_-}{2}} = e^{\frac{i\pi}{4}(\nu_+ + \nu_- - 2\nu_-)} = e^{\frac{i\pi}{4}\operatorname{sign}(B)}.$$

We note also that

$$\operatorname{sign}(\det B) = (-1)^{\operatorname{index} B},$$

so that

$$\left(i^{-\operatorname{index} B} |\det B|^{-1/2}\right)^2 = (-1)^{\nu_-} |\det B|^{-1} = \operatorname{sign}(\det B)|\det B|^{-1} = (\det B)^{-1},$$

and thus the prefactor $i^{-\operatorname{index} B}|\det B|^{-1/2}$ in the right-hand side of (A.1.8) is a square-root of $1/\det B$.

With H standing for the characteristic function of $\mathbb{R}_+$, we have

$$1 = H + \check{H}, \quad \delta_0 = \widehat{H} + \hat{\check{H}},$$
$$D\operatorname{sign} = \frac{\delta_0}{i\pi}, \quad \widehat{D\operatorname{sign}} = \frac{1}{i\pi}, \quad \xi\widehat{\operatorname{sign}} = \frac{1}{i\pi}, \quad \widehat{\operatorname{sign}} = \frac{1}{i\pi}\operatorname{pv}\frac{1}{\xi}, \quad \text{(principal value)}$$

the latter formula following from the fact that

$$\xi\left(\widehat{\text{sign}} - \text{pv}\frac{1}{i\pi\xi}\right) = 0,$$

which implies

$$\widehat{\text{sign}} - \text{pv}\frac{1}{i\pi\xi} = c\delta_0 = 0,$$

since $\widehat{\text{sign}} - \frac{1}{i\pi\xi}$ is odd. We infer from that

$$\widehat{H} - \widehat{\check{H}} = \widehat{\text{sign}} = \text{pv}\frac{1}{i\pi\xi},$$

and

$$\widehat{H} = \frac{\delta_0}{2} + \text{pv}\frac{1}{2i\pi\xi}.$$

Lemma A.1.1. *Let T be a compactly supported distribution on $\mathbb{R}^n$ such that*

$$\forall N \in \mathbb{N}, \quad \langle\xi\rangle^N \widehat{T}(\xi) \quad \textit{is bounded,} \quad \textit{with } \langle\xi\rangle = \sqrt{1+|\xi|^2}. \tag{A.1.9}$$

Then, T is a C^∞ function.

Proof. Note that $\widehat{T}$ is an entire function, as the Fourier transform of a compactly supported distribution. Moreover, from (A.1.9) with $N = n+1$, we get that $\widehat{T}$ belongs to $L^1(\mathbb{R}^n)$ and thus T is a continuous function. Moreover, we have for any $\alpha \in \mathbb{N}^n$,

$$(D_x^\alpha T)(x) = \int e^{2i\pi x\cdot\xi} \underbrace{\xi^\alpha \widehat{T}(\xi)}_{\in L^1(\mathbb{R}^n)} d\xi,$$

so that T is a C^∞ function. ∎

Proposition A.1.2. *Let $\rho > 0$ and let f be a holomorphic function on a neighborhood of $\{z \in \mathbb{C}, |\operatorname{Im} z| \le \rho\}$ such that*

$$\forall y \in [-\rho, \rho], \quad \int |f(x+iy)|dx < +\infty, \tag{A.1.10}$$

$$\lim_{R\to+\infty} \int_{|y|\le\rho} |f(\pm R + iy)|dy = 0. \tag{A.1.11}$$

Then, we have

$$\forall \xi \in \mathbb{R}, \quad |\hat{f}(\xi)| \le Ce^{-2\pi\rho|\xi|},$$

with

$$C = \max(C_+, C_-), \quad C_\pm = \int_{\mathbb{R}} |f(x \pm i\rho)|dx.$$

Conversely, if f is a bounded measurable function such that $\hat{f}(\xi)$ is $O(e^{-2\pi r|\xi|})$ for some $r > 0$, then f is holomorphic on $\{z \in \mathbb{C}, |\operatorname{Im} z| < r\}$.

Proof. If f is holomorphic near $\{z \in \mathbb{C}, |\operatorname{Im} z| \le \rho\}$, satisfies (A.1.10) and (A.1.11), then Cauchy's formula shows that for $|y| \le \rho$,

$$\begin{aligned}
\int_{\mathbb{R}} e^{-2i\pi(x+iy)\xi} f(x+iy)dx &= e^{2\pi y\xi} \lim_{R\to+\infty} \int_{-R}^{R} e^{-2i\pi x\xi} f(x+iy)dx \\
&= \lim_{R\to+\infty} \int_{[-R+iy,R+iy]} e^{-2i\pi z\xi} f(z)dz \\
&= \lim_{R\to+\infty} \int_{[-R+iy,-R]\cup[-R,R]\cup[R,R+iy]} e^{-2i\pi z\xi} f(z)dz \\
&= \hat{f}(\xi) + \lim_{R\to+\infty} \Big(\int_0^y e^{-2i\pi(R+it)\xi} f(R+it)idt \\
&\qquad - \int_0^y e^{-2i\pi(-R+it)\xi} f(-R+it)idt \Big).
\end{aligned}$$

We have for $|y| \le \rho$,

$$\left| \int_0^y e^{-2i\pi(\pm R+it)\xi} f(\pm R+it)idt \right| \le \int_{|t|\le\rho} |f(\pm R+it)|dt e^{2\pi\rho|\xi|},$$

which goes to 0 when R goes to $+\infty$, thanks to (A.1.11), so that for all $y \in [-\rho, \rho]$, we have

$$\int_{\mathbb{R}} e^{-2i\pi(x+iy)\xi} f(x+iy)dx = \hat{f}(\xi),$$

which implies for $y = -\rho \operatorname{sign} \xi$ (taken as 0, if $\xi = 0$)

$$|\hat{f}(\xi)| \le \int_{\mathbb{R}} |f(x \mp i\rho)|dx\, e^{-2\pi\rho|\xi|} \underbrace{\le}_{\text{from (A.1.10)}} Ce^{-2\pi\rho|\xi|},$$

proving the first part of the proposition. Let us consider now a function f in $L^\infty(\mathbb{R})$ such that $\hat{f}(\xi)$ is $O(e^{-2\pi r|\xi|})$ for some $r > 0$, and let $\rho \in (0, r)$. We have $f(x) = \int e^{2i\pi x\xi} \hat{f}(\xi)d\xi$ and for $|y| \le \rho$, we have $\int_{\mathbb{R}} e^{2\pi|y||\xi|}|\hat{f}(\xi)|d\xi < +\infty$, so that f is holomorphic on $\{z \in \mathbb{C}, |\operatorname{Im} z| < r\}$ with

$$f(x+iy) = \int_{\mathbb{R}} e^{2i\pi(x+iy)\xi} \hat{f}(\xi)d\xi,$$

concluding the proof. ∎

A.1.2 Weyl quantization

Let $a \in \mathscr{S}'(\mathbb{R}^{2n})$. We have defined the operator $\operatorname{Op_w}(a)$, continuous from $\mathscr{S}(\mathbb{R}^n)$ into $\mathscr{S}'(\mathbb{R}^n)$, in Section 1.2.1 with the formula

$$\langle \operatorname{Op_w}(a)u, \bar{v}\rangle_{\mathscr{S}'(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} = \langle a, \mathcal{W}(u,v)\rangle_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})},$$

where the Wigner function $\mathcal{W}(u, v)$ is defined in Definition 1.1.1. We note that the sesquilinear mapping $\mathscr{S}(\mathbb{R}^n) \times \mathscr{S}(\mathbb{R}^n) \ni (u, v) \mapsto \mathcal{W}(u, v) \in \mathscr{S}(\mathbb{R}^{2n})$ is continuous so that the above bracket of duality $\langle a, \mathcal{W}(u, v)\rangle_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})}$, makes sense. We note as well that a temperate distribution $a \in \mathscr{S}'(\mathbb{R}^{2n})$ gets quantized by a continuous operator $\operatorname{Op_w}(a)$ from $\mathscr{S}(\mathbb{R}^n)$ into $\mathscr{S}'(\mathbb{R}^n)$.

Lemma A.1.3. *Let a be a tempered distribution on $\mathbb{R}^{2n}$ and let b be a polynomial of degree d on $\mathbb{R}^{2n}$. Then, we have*

$$a \sharp b = \sum_{0\le k\le d} \omega_k(a, b), \quad \textit{with}$$

$$\omega_k(a, b) = \frac{1}{(4i\pi)^k} \sum_{|\alpha|+|\beta|=k} \frac{(-1)^{|\beta|}}{\alpha!\beta!} (\partial_\xi^\alpha \partial_x^\beta a)(x, \xi)(\partial_x^\alpha \partial_\xi^\beta b)(x, \xi), \tag{A.1.12}$$

$$\omega_k(b, a) = (-1)^k \omega_k(a, b). \tag{A.1.13}$$

The Weyl symbol of the commutator $[\operatorname{Op_w}(a), \operatorname{Op_w}(b)]$ *is*

$$c(a, b) = 2 \sum_{\substack{0\le k\le d\\ k \text{ odd}}} \omega_k(a, b).$$

If the degree of b is smaller than 2, we have

$$c(a, b) = 2\omega_1(a, b) = \frac{1}{2\pi i}\{a, b\},$$

and if a is a function of b, the commutator $[\operatorname{Op_w}(a), \operatorname{Op_w}(b)] = 0$.

Remark A.1.4. In particular, if $q(x, \xi)$ is a quadratic polynomial and $a(x, \xi) = H(1 - q(x, \xi))$, is the characteristic function of the set $\{(x, \xi), q(x, \xi) \le 1\}$, then we have $[\operatorname{Op_w}(a), \operatorname{Op_w}(q)] = 0$.

Proof. Applying (1.2.2), (1.2.3), we obtain that this lemma follows from (A.1.13), that we check now

$$\begin{aligned}
(4i\pi)^k \omega_k(a, b) &= \sum_{|\alpha|+|\beta|=k} \frac{(-1)^{|\beta|}}{\alpha!\beta!} (\partial_\xi^\alpha \partial_x^\beta a)(x, \xi)(\partial_x^\alpha \partial_\xi^\beta b)(x, \xi)\\
&= \sum_{|\alpha|+|\beta|=k} \frac{(-1)^{|\alpha|}}{\alpha!\beta!} (\partial_\xi^\beta \partial_x^\alpha a)(x, \xi)(\partial_x^\beta \partial_\xi^\alpha b)(x, \xi)\\
&= \sum_{|\alpha|+|\beta|=k} \frac{(-1)^{k-|\beta|}}{\alpha!\beta!} (\partial_\xi^\beta \partial_x^\alpha a)(x, \xi)(\partial_x^\beta \partial_\xi^\alpha b)(x, \xi)\\
&= (-1)^k (4i\pi)^k \omega_k(b, a),
\end{aligned}$$

which is the sought result. ∎

Remark A.1.5. We can note that formula (1.2.61) is non-local in the sense that for $a, b \in \mathscr{S}(\mathbb{R}^{2n})$ with disjoint supports, although all $\omega_k(a,b)$ (given by (A.1.12)) are identically 0, the function $a\sharp b$ (which belongs to $\mathscr{S}(\mathbb{R}^{2n})$) is different from 0; let us give an example. Let $\chi_0 \in C_c^\infty(\mathbb{R}; [0,1])$ with support $[-1+\varepsilon_0, 1-\varepsilon_0]$ with $\varepsilon_0 \in (0,1)$ and let us consider in $\mathbb{R}^2$,

$$a(x,\xi) = \chi_0(x)e^{-\pi\xi^2}, \quad b(x,\xi) = \chi_0(x-2)e^{-\pi\xi^2},$$

so that a, b both belong to $\mathscr{S}(\mathbb{R}^2)$ and

$$\operatorname{supp} a = [-1+\varepsilon_0, 1-\varepsilon_0] \times \mathbb{R}, \quad \operatorname{supp} b = [1+\varepsilon_0, 3-\varepsilon_0] \times \mathbb{R},$$

so that the supports are disjoint and all $\omega_k(a,b)$ are identically vanishing. We check now

$$\begin{aligned}
&(a\sharp b)(x,\xi)\\
&= 4\iiiint \chi_0(y)e^{-\pi\eta^2}\chi_0(z-2)e^{-\pi\zeta^2}e^{-4i\pi(\xi-\eta)(x-z)}e^{4i\pi(x-y)(\xi-\zeta)}dy\,d\eta\,dz\,d\zeta\\
&= 4\iint \chi_0(y)\chi_0(z-2)e^{-4\pi(x-z)^2}e^{-4\pi(x-y)^2}e^{4i\pi\xi(z-x+x-y)}dy\,dz\\
&= 4\Big(\int \chi_0(y)e^{-4i\pi\xi y}e^{-4\pi(x-y)^2}dy\Big)\Big(\int \chi_0(z)e^{4i\pi\xi z}e^{-4\pi(x-2-z)^2}dz\Big),
\end{aligned}$$

so that

$$(a\sharp b)(0,0) = 4\underbrace{\Big(\int \chi_0(y)e^{-4\pi y^2}dy\Big)}_{>0}\underbrace{\Big(\int \chi_0(z)e^{-4\pi(2+z)^2}dz\Big)}_{>0} > 0.$$

A.1.3 Some explicit computations

We may also calculate with

$$\begin{aligned}
u_a(x) &= (2a)^{1/4}e^{-\pi a x^2}, a > 0, \qquad \text{(A.1.14)}\\
\mathcal{W}(u_a,u_a)(x,\xi) &= (2a)^{1/2}\int e^{-2i\pi z\cdot\xi}e^{-\pi a|x-\frac{z}{2}|^2}e^{-\pi a|x+\frac{z}{2}|^2}dz\\
&= (2a)^{1/2}\int e^{-2i\pi z\cdot\xi}e^{-2\pi a x^2}e^{-\pi a z^2/2}dz\\
&= (2a)^{1/2}e^{-2\pi a x^2}2^{1/2}a^{-1/2}e^{-\pi\frac{2}{a}\xi^2}\\
&= 2e^{-2\pi(ax^2+a^{-1}\xi^2)},
\end{aligned}$$

which is also a Gaussian function on the phase space (and positive function). The calculation of

$$\mathcal{W}(u_a', u_a')(x,\xi)$$

is interesting since we have

$$
\begin{aligned}
4\pi^2 \langle D_x b^w D_x u_a, \bar{u}_a \rangle_{\mathscr{S}'(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} &= \langle b^w u'_a, \bar{u}'_a \rangle_{\mathscr{S}'(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} \\
&= \langle b, \mathcal{W}(u'_a, u'_a) \rangle_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})},
\end{aligned}
$$

and for $b(x,\xi)$ real-valued we have

$$
\xi \sharp b \sharp \xi = \left(\xi b + \frac{b'_x}{4i\pi}\right)\sharp \xi = \xi^2 b + \frac{b'_x \xi}{4i\pi} - \frac{\partial_x}{4i\pi}\left(\xi b + \frac{b'_x}{4i\pi}\right) = \xi^2 b + \frac{b''_{xx}}{16\pi^2},
$$

so that

$$
4\pi^2 \iint 2e^{-2\pi(ax^2+a^{-1}\xi^2)}\left(\xi^2 b + \frac{b''_{xx}}{16\pi^2}\right) dx d\xi = \langle b, \mathcal{W}(u'_a, u'_a)\rangle,
$$

proving that

$$
\begin{aligned}
\mathcal{W}(u'_a, u'_a)(x,\xi) &= 2e^{-2\pi(ax^2+a^{-1}\xi^2)} 4\pi^2\xi^2 + \frac{1}{4} 2\partial_x^2 \left(e^{-2\pi(ax^2+a^{-1}\xi^2)}\right) \\
&= 2e^{-2\pi(ax^2+a^{-1}\xi^2)}\left(4\pi^2\xi^2 + \frac{1}{4}((-4\pi a x)^2 - 4\pi a)\right) \\
&= 8\pi^2 e^{-2\pi(ax^2+a^{-1}\xi^2)} a \left(a^{-1}\xi^2 + ax^2 - \frac{1}{4\pi}\right).
\end{aligned}
$$

We obtain that the function $\mathcal{W}(u'_a, u'_a)$ is negative on

$$
a^{-1}\xi^2 + ax^2 < \frac{1}{4\pi},
$$

which has area $1/4$. We may note as well for consistency for u_a given by (A.1.14), we have

$$
u'_a = (2a)^{1/4}(-2\pi a x)e^{-\pi a x^2}, \quad \|u'_a\|^2_{L^2} = \pi a,
$$

and

$$
\begin{aligned}
\iint \mathcal{W}(u'_a, u'_a)(x,\xi) dx d\xi &= 8\pi^2 a \iint e^{-2\pi(y^2+\eta^2)}\left(y^2 + \eta^2 - \frac{1}{4\pi}\right) dy d\eta \\
&= \frac{8\pi^2 a}{8\pi} = \pi a = \|u'_a\|^2_{L^2}.
\end{aligned}
$$

For $\lambda > 0$ and $a \in \mathscr{S}'(\mathbb{R}^{2n})$, we define

$$
a_\lambda(x,\xi) = a(\lambda^{-1}x, \lambda\xi),
$$

and we find that

$$
\begin{aligned}
&(a_\lambda)^w = U_\lambda^* a^w U_\lambda, \qquad\qquad (\text{A.1.15}) \\
&\text{for} \quad f \in \mathscr{S}(\mathbb{R}^n), (U_\lambda f)(x) = f(\lambda x)\lambda^{n/2}, \quad U_\lambda^* = U_{\lambda^{-1}} = (U_\lambda)^{-1}.
\end{aligned}
$$

We note that the above formula is a particular case of *Segal's formula* (see, e.g., [33, Theorem 2.1.2]).

A.1.4 The harmonic oscillator

The harmonic oscillator $\mathscr{H}_n$ in n dimensions is defined as the operator with Weyl symbol $\pi(|x|^2 + |\xi|^2)$ and thus from (A.1.15), we find that

$$\mathscr{H} = U_{\sqrt{2\pi}} \frac{1}{2}\big(|x|^2 + 4\pi^2|\xi|^2\big)^w U^*_{\sqrt{2\pi}} = U_{\sqrt{2\pi}} \frac{1}{2}(-\Delta + |x|^2) U^*_{\sqrt{2\pi}}.$$

We shall define in one dimension the Hermite function of level $k \in \mathbb{N}$, by

$$\psi_k(x) = \frac{(-1)^k}{2^k\sqrt{k!}} 2^{1/4} e^{\pi x^2} \left(\frac{d}{\sqrt{\pi}\,dx}\right)^k (e^{-2\pi x^2}), \tag{A.1.16}$$

and we find that $(\psi_k)_{k\in\mathbb{N}}$ is a Hilbertian orthonormal basis on $L^2(\mathbb{R})$. The one-dimensional harmonic oscillator can be written as

$$\mathscr{H}_1 = \sum_{k\geq 0} \left(\frac{1}{2} + k\right)\mathbb{P}_k, \tag{A.1.17}$$

where $\mathbb{P}_k$ is the orthogonal projection onto ψ_k.

In n dimensions, we consider a multi-index $(\alpha_1, \dots, \alpha_n) = \alpha \in \mathbb{N}^n$ and we define on $\mathbb{R}^n$, using the one-dimensional (A.1.16),

$$\Psi_\alpha(x) = \prod_{1\leq j\leq n} \psi_{\alpha_j}(x_j), \quad \mathscr{E}_k = \mathrm{Vect}\{\Psi_\alpha\}_{\alpha\in\mathbb{N}^n, |\alpha|=k}, \quad |\alpha| = \sum_{1\leq j\leq n} \alpha_j. \tag{A.1.18}$$

We note that the dimension of $\mathscr{E}_{k,n}$ is

$$\binom{k+n-1}{n-1},$$

and that (A.1.17) holds with $\mathbb{P}_{k;n}$ standing for the orthogonal projection onto $\mathscr{E}_{k,n}$; the lowest eigenvalue of $\mathscr{H}_n$ is $n/2$ and the corresponding eigenspace is one-dimensional in all dimensions, although in two and more dimensions, the eigenspaces corresponding to the eigenvalue $\frac{n}{2} + k, k \geq 1$ are multi-dimensional with dimension $\binom{k+n-1}{n-1}$. The n-dimensional harmonic oscillator can be written as

$$\mathscr{H}_n = \sum_{k\geq 0} \left(\frac{n}{2} + k\right)\mathbb{P}_{k;n},$$

where $\mathbb{P}_{k;n}$ stands for the orthogonal projection onto $\mathscr{E}_{k,n}$ defined above. We have in particular

$$\mathbb{P}_{k;n} = \sum_{\alpha\in\mathbb{N}^n, |\alpha|=k} \mathbb{P}_\alpha, \quad \text{where } \mathbb{P}_\alpha \text{ is the orthogonal projection onto } \Psi_\alpha. \tag{A.1.19}$$

A.1.5 On the spectrum of the anisotropic harmonic oscillator

The standard n-dimensional harmonic oscillator is the operator

$$\mathcal{H}_n = \pi \sum_{1\le j\le n} (D_j^2 + x_j^2), \quad D_j = \frac{1}{2\pi i}\partial_{x_j},$$

and its spectral decomposition is

$$\mathcal{H} = \sum_{k\ge 0}\left(\frac{n}{2}+k\right)\mathbb{P}_{k;n}, \quad \mathbb{P}_{k;n} = \sum_{\alpha\in\mathbb{N}^n, \alpha_1+\dots+\alpha_n=k} \mathbb{P}_{\alpha_1}\otimes\dots\otimes\mathbb{P}_{\alpha_n},$$

where $\mathbb{P}_{\alpha_j}$ stands for the orthogonal projection onto the one-dimensional Hermite function with level α_j. Now let us consider for $\mu = (\mu_1, \dots, \mu_n)$ with $\mu_j > 0$, the operator

$$\mathcal{H}_{(\mu)} = \pi \sum_{1\le j\le n} \mu_j (D_j^2 + x_j^2) = \pi \mathrm{Op}_{\mathrm{w}}(q_\mu(x,\xi)),$$

with

$$q_\mu(x,\xi) = \sum_{1\le j\le n} \mu_j (x_j^2 + \xi_j^2).$$

With the notation $|\mu| = \sum_{1\le j\le n}\mu_j$ and $\mu\cdot\alpha = \sum_{1\le j\le n}\mu_j\alpha_j$, we have

$$\mathcal{H}_{(\mu)} = \sum_{\alpha\in\mathbb{N}^n}\left(\frac{|\mu|}{2}+\mu\cdot\alpha\right)\underbrace{(\mathbb{P}_{\alpha_1}\otimes\dots\otimes\mathbb{P}_{\alpha_n})}_{\mathbb{P}_\alpha},$$

so that the eigenspaces are the same as for $\mathcal{H}_n$ but the arithmetic properties of μ make possible that all eigenvalues $(\frac{|\mu|}{2}+\mu\cdot\alpha)$ are simple. For instance for

$$n = 2, 0 < \mu_1 < \mu_2, \quad \frac{\mu_2}{\mu_1} \notin \mathbb{Q},$$

if $\beta \in \mathbb{Z}^2$ is such that $\mu_1\beta_1 + \mu_2\beta_2 = 0$, this implies that $\beta = 0$ and thus that all the eigenvalues of $\mathcal{H}_{(\mu)}$ are simple.

Remark A.1.6. If $0 < \mu_1 \le \dots \le \mu_n$ and if for all $j \in [2, n]$ we have $\mu_j/\mu_1 \in \mathbb{N}$, we then have for $\alpha \in \mathbb{N}^n$,

$$\alpha\cdot\mu = \mu_1 \underbrace{\left(\alpha_1 + \sum_{2\le j\le n}\frac{\alpha_j\mu_j}{\mu_1}\right)}_{\beta_1} = \beta\cdot\mu, \quad \beta = (\beta_1, 0, \dots, 0) \in \mathbb{N}^n.$$

Sinus cardinal. It is a classical result of Distribution Theory that the weak limit when $\lambda \to +\infty$ of the sinus cardinal $\frac{\sin(\lambda x)}{x}$ is $\pi\delta_0$, where δ_0 is the Dirac mass at 0, but we wish to extend that result to more general test functions.

Lemma A.1.7. *Let f be a function in $L^1_{loc}(\mathbb{R})$ such that*

$$\int_{|\tau|\ge 1} \frac{|f(\tau)|}{|\tau|} d\tau < +\infty \quad \textit{and} \quad \exists a \in \mathbb{C} \textit{ so that } \int_{|\tau|\le 1} \frac{|f(\tau)-a|}{|\tau|} d\tau < +\infty.$$

Then, we have

$$\lim_{\lambda\to+\infty} \int_{\mathbb{R}} \frac{\sin(\lambda\tau)}{\pi\tau} f(\tau) d\tau = a. \tag{A.1.20}$$

N.B. In particular, if f is a Hölderian function such that $f(\tau)/\tau \in L^1(\{|\tau| \ge 1\})$ we get that the left-hand side of (A.1.20) equals $f(0)$.

Proof. Let χ_0 be a function in $C_c^\infty(\mathbb{R})$ equal to 1 near the origin and let us define $\chi_1 = 1 - \chi_0$. We have

$$\int_{\mathbb{R}} \frac{\sin(\lambda\tau)}{\pi\tau} f(\tau) d\tau = \int_{\mathbb{R}} \frac{\sin(\lambda\tau)}{\pi} \underbrace{\frac{\big(f(\tau)-a\big)}{\tau}\chi_0(\tau)}_{\in L^1(\mathbb{R})} d\tau + a \int_{\mathbb{R}} \frac{\sin(\lambda\tau)}{\pi\tau}\chi_0(\tau) d\tau$$
$$+ \int_{\mathbb{R}} \frac{\sin(\lambda\tau)}{\pi} \underbrace{f(\tau)\tau^{-1}\chi_1(\tau)}_{\in L^1(\mathbb{R})} d\tau,$$

so that the limit when $\lambda \to +\infty$ of the first and the third integral is zero, thanks to the Riemann–Lebesgue lemma. We note also that

$$\frac{\sin(\lambda\tau)}{\pi\tau} = \widehat{\mathbf{1}_{[-\frac{\lambda}{2\pi},\frac{\lambda}{2\pi}]}}(\tau),$$

and applying Plancherel's formula to the second integral yields

$$\int_{\mathbb{R}} \frac{\sin(\lambda\tau)}{\pi\tau}\chi_0(\tau) d\tau = \int_{|t|\le\lambda/(2\pi)} \widehat{\chi_0}(t) dt,$$

whose limit when $\lambda \to +\infty$ is $\int_{\mathbb{R}} \widehat{\chi_0}(t)dt = \chi_0(0) = 1$, thanks to the Lebesgue dominated convergence theorem, completing the proof of the lemma. ∎

A.2 Further properties of the metaplectic group

A.2.1 Another set of generators for the metaplectic group

Definition A.2.1. Let P, L, Q be $n \times n$ real matrices such that $P = P^*$, $Q = Q^*$ and $\det L \neq 0$. We define the operator $\mathcal{M}_{P,L,Q}$ by the formula

$$(\mathcal{M}_{P,L,Q}u)(x) = e^{-i\pi n/4} (\det L)^{1/2} \int_{\mathbb{R}^n} e^{i\pi\{\langle Px,x\rangle - 2\langle Lx,y\rangle + \langle Qy,y\rangle\}} u(y) dy.$$

N.B. In that definition, $(\det L)^{1/2}$ stands for a choice of a square-root of the real number $\det L$, that is $\pm\sqrt{\det L}$ if $\det L > 0$ and $\pm i\sqrt{-\det L}$ if $\det L < 0$.

With $m(L) \in \mathbb{Z}/4\mathbb{Z}$ defined by (1.2.34) we shall also define

$$\big(\mathcal{M}_{P,L,Q}^{\{m(L)\}}u\big)(x) = e^{-\frac{i\pi n}{4}} e^{\frac{i\pi m(L)}{2}} |\det L|^{1/2} \int_{\mathbb{R}^n} e^{i\pi\{\langle Px,x\rangle - 2\langle Lx,y\rangle + \langle Qy,y\rangle\}} u(y) dy.$$

Proposition A.2.2. *The operator $\mathcal{M}_{P,L,Q}$ given in Definition A.2.1 is an automorphism of $\mathscr{S}(\mathbb{R}^n)$ and of $\mathscr{S}'(\mathbb{R}^n)$ which is a unitary operator on $L^2(\mathbb{R}^n)$ belonging to the metaplectic group (cf. Definition 1.2.13). Moreover, the metaplectic group is generated by the set*

$$\{\mathcal{M}_{P,L,Q}\}_{\substack{P=P^*, Q=Q^* \\ \det L \neq 0}}.$$

Proof. Using the notation (1.2.28) and (1.2.37), we see that[1]

$$M_{A,B,C}^{\{m(B)\}} = \mathcal{M}_{A,-B,C}^{\{m(B)+n\}} \mathcal{F} e^{-i\pi n/4}, \quad \mathcal{M}_{P,L,Q}^{\{m(L)\}} = M_{P,-L,Q}^{\{m(L)-n\}} \big(\mathcal{F} e^{-i\pi n/4}\big)^{-1}, \tag{A.2.1}$$

and (1.2.44) imply that the set $\{\mathcal{M}_{P,L,Q}\}$ is included in $\mathrm{Mp}(n)$ (second formula in (A.2.1)) whereas the fact that

$$\mathcal{F} e^{-i\pi n/4} = \mathcal{M}_{0,I_n,0}^{\{0\}},$$

the first formula in (A.2.1) and Definition 1.2.13 imply that $\mathrm{Mp}(n)$ is generated by the set $\{\mathcal{M}_{P,L,Q}\}$, proving the proposition. ∎

Remark A.2.3. From (A.2.1), we deduce, noting $m(I_n) \in \{0, 2\}, m(-I_n) \in \{n, n + 2\}$,

$$-\mathrm{Id}_{L^2(\mathbb{R}^n)} = M_{0,I_n,0}^{\{2\}} = \mathcal{M}_{0,-I_n,0}^{\{n+2\}} \mathcal{M}_{0,I_n,0}^{\{0\}},$$

so that

$$\mathcal{M}_{P,L,Q}^{\{m(L)+2\}} = -\mathcal{M}_{P,L,Q}^{\{m(L)\}} = \mathcal{M}_{0,-I_n,0}^{\{n+2\}} \mathcal{M}_{0,I_n,0}^{\{0\}} \mathcal{M}_{P,L,Q}^{\{m(L)\}}.$$

---

[1]We note that $m(B) + n \in \{m(-B), m(-B) + 2\}$ modulo 4: indeed, we have modulo 4

$$\begin{cases} \text{for } n \text{ even,} & \underbrace{\{0,2\}}_{\det B>0} + n = \underbrace{\{0,2\}}_{\det(-B)>0}, \quad \underbrace{\{1,3\}}_{\det B<0} + n = \underbrace{\{1,3\}}_{\det(-B)<0}, \\ \text{for } n \text{ odd,} & \underbrace{\{0,2\}}_{\det B>0} + n = \underbrace{\{1,3\}}_{\det(-B)<0}, \quad \underbrace{\{1,3\}}_{\det B<0} + n = \underbrace{\{0,2\}}_{\det(-B)>0}. \end{cases}$$

We have also $m(L) - n \in \{m(-L), m(-L) + 2\}$ since we know already (from the above in that footnote) that $m(L) - n \in \{m(-L), m(-L) + 2\} - 2n$, which gives $m(L) - n \in \{m(-L), m(-L) + 2\}$ for n even; for $n = 2l + 1$ odd we get the same result since

$$m(L) - n \in \{m(-L), m(-L) + 2\} - 4l - 2 = \{m(-L) - 2, m(-L)\} = \{m(-L) + 2, m(-L)\}.$$

Lemma A.2.4. *With the homomorphism Ψ defined in (1.2.46) and defining*

$$\Lambda_{P,L,Q} = \Psi(\mathcal{M}_{P,L,Q}),$$

we find that

$$\Lambda_{P,L,Q} = \begin{pmatrix} L^{-1}Q & L^{-1} \\ PL^{-1}Q - L^* & PL^{-1} \end{pmatrix}.$$

Proof. Indeed, from the second formula in (A.2.1), (1.2.38), (1.2.27), and (1.2.47) we get that

$$\Lambda_{P,L,Q} = \Xi_{P,-L,Q}\,\Xi^{-2}_{-I_n,2^{1/2}I_n,-I_n} = \begin{pmatrix} -L^{-1} & L^{-1}Q \\ -PL^{-1} & -L^* + PL^{-1}Q \end{pmatrix}\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

providing the sought result. ∎

Lemma A.2.5. *Let P_j, L_j, Q_j, $j = 1, 2$ be as in Definition A.2.1 and let us assume that*

$$\mathcal{M}_{P_1,L_1,Q_1}\mathcal{M}_{P_2,L_2,Q_2} = e^{i\phi}\,\mathrm{Id}_{L^2(\mathbb{R}^n)}, \quad \phi \in \mathbb{R}. \tag{A.2.2}$$

Then, we have

$$P_1 + Q_2 = Q_1 + P_2 = 0, \quad L_2 = -L_1^*, \quad e^{i\phi} \in \{\pm 1\}. \tag{A.2.3}$$

Proof. The assumption (A.2.2) implies that both sides of the equality belong to $\mathrm{Mp}(n)$ and

$$\Lambda_{P_1,L_1,Q_1}\Lambda_{P_2,L_2,Q_2} = \Psi\big(e^{i\phi}\,\mathrm{Id}_{L^2(\mathbb{R}^n)}\big) = I_{2n},$$

where the last equality follows from the fact that $e^{i\phi}\,\mathrm{Id}_{L^2(\mathbb{R}^n)}$ commutes with every operator $\mathrm{Op_w}(L_Y)$ given in Lemma 1.2.17. We have thus

$$\begin{pmatrix} L_1^{-1}Q_1 & L_1^{-1} \\ P_1L_1^{-1}Q_1 - L_1^* & P_1L_1^{-1} \end{pmatrix}\begin{pmatrix} L_2^{-1}Q_2 & L_2^{-1} \\ P_2L_2^{-1}Q_2 - L_2^* & P_2L_2^{-1} \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix},$$

so that

$$\begin{aligned}
\text{first line} \times \text{second column:} \quad & L_1^{-1}Q_1L_2^{-1} + L_1^{-1}P_2L_2^{-1} = 0 \Longrightarrow Q_1 + P_2 = 0,\\
\text{second line} \times \text{first column:} \quad & (P_1L_1^{-1}Q_1 - L_1^*)L_2^{-1}Q_2 + P_1L_1^{-1}(P_2L_2^{-1}Q_2 - L_2^*) = 0,\\
\text{second line} \times \text{second column:} \quad & (P_1L_1^{-1}Q_1 - L_1^*)L_2^{-1} + P_1L_1^{-1}P_2L_2^{-1} = I_n,
\end{aligned}$$

which gives

$$(P_1L_1^{-1}Q_1 - L_1^*)L_2^{-1} + P_1L_1^{-1}\underbrace{P_2}_{-Q_1}L_2^{-1} = I_n \Rightarrow -L_1^*L_2^{-1} = I_n \Rightarrow L_2 = -L_1^*,$$

$$P_1L_1^{-1}Q_1L_2^{-1}Q_2\underbrace{-L_1^*L_2^{-1}}_{I_n}Q_2 + P_1L_1^{-1}\underbrace{P_2}_{-Q_1}L_2^{-1}Q_2 - P_1\underbrace{L_1^{-1}L_2^*}_{-I_n} = 0 \Rightarrow P_1 + Q_2 = 0,$$

providing the sought formulas in (A.2.3), except for the last one. Let κ_j be the kernel of $\mathcal{M}_{P_j,L_j,Q_j}$ and let $\kappa = \kappa_1 \circ \kappa_2$ be the kernel of the composition (in the left-hand side of (A.2.2)). We have consequently

$$\begin{aligned}
&\kappa(x,y)\\
&= (\det L_1)^{1/2}(\det L_2)^{1/2}e^{-i\pi n/2}\int e^{i\pi\{P_1x^2-2L_1x\cdot z+Q_1z^2+P_2z^2-2L_2z\cdot y+Q_2y^2\}}dz\\
&= (\det L_1)^{1/2}(\det(-L_1^*))^{1/2}e^{-i\pi n/2}e^{i\pi\{P_1x^2-P_1y^2\}}\int e^{-2i\pi\{z\cdot(L_1x+L_2^*y)\}}dz\\
&= (\det L_1)^{1/2}(\det(-L_1^*))^{1/2}e^{-i\pi n/2}e^{i\pi\{P_1x^2-P_1y^2\}}\delta_0(L_1x+L_2^*y)\\
&= (\det L_1)^{1/2}(\det(-L_1^*))^{1/2}e^{-i\pi n/2}e^{i\pi\{P_1x^2-P_1y^2\}}\delta_0(x-y)|\det L_1|^{-1},
\end{aligned}$$

entailing

$$\begin{aligned}
e^{i\phi}\delta_0(x-y)\underbrace{=}_{\text{(A.2.2)}}\kappa(x,y) &= e^{i\frac{\pi}{2}(m(L_1)+m(L_1^*)+n)}\delta_0(x-y)e^{-i\pi n/2}\\
&= e^{i\pi m(L_1)}\delta_0(x-y),
\end{aligned}$$

proving that $e^{i\phi} = e^{i\pi m(L_1)} \in \{\pm 1\}$. The proof of the lemma is complete. ■

Claim A.2.6. Let P, L, Q be as in Definition A.2.1. Then, we have

$$(\mathcal{M}_{P,L,Q}^{\{m(L)\}})^{-1} = \mathcal{M}_{-Q,-L^*,-P}^{\{n-m(L)\}}, \tag{A.2.4}$$

and moreover $n - m(L) \in \{m(-L^*), m(-L^*)+2\}$ modulo 4.

Proof of Claim A.2.6. Indeed, calculating the kernel κ of $\mathcal{M}_{P,L,Q}^{\{m(L)\}}\mathcal{M}_{-Q,-L^*,-P}^{\{n-m(L)\}}$, we get

$$\begin{aligned}
\kappa(x,y) &= e^{\frac{i\pi}{2}(m(L)+n-m(L)-n)}|\det L|\int e^{i\pi\{Px^2-2Lx\cdot z+Qz^2-Qz^2+2L^*z\cdot y-Py^2\}}dz\\
&= |\det L|e^{i\pi\{Px^2-Py^2\}}\delta_0(Lx-Ly) = \delta_0(x-y),
\end{aligned}$$

so that

$$\mathcal{M}_{P,L,Q}^{\{m(L)\}}\mathcal{M}_{-Q,-L^*,-P}^{\{n-m(L)\}} = \mathrm{Id}_{L^2(\mathbb{R}^n)}$$

and since $\mathcal{M}_{P,L,Q}$ is unitary, this proves (A.2.4). The last assertion is equivalent to $m(L) \in \{n - m(-L^*), n - m(-L^*) - 2\}$. Since the latter set is equal to $\{-m(L), -m(L)-2\}$ and the mapping

$$\mathbb{Z}/4\mathbb{Z} \ni x \mapsto -x \in \mathbb{Z}/4\mathbb{Z},$$

leaves invariant the sets $\{0,2\}, \{1,3\}$, we obtain the sought result, concluding the proof of the claim. ■

Proposition A.2.7. *Let P_j, L_j, Q_j, $j = 1, 2$ be as in Definition A.2.1 and let us assume that*

$$\det(Q_1 + P_2) \neq 0.$$

Then, there exist P, L, Q, as in Definition A.2.1 such that

$$\mathcal{M}_{P_1,L_1,Q_1}^{\{m(L_1)\}} \mathcal{M}_{P_2,L_2,Q_2}^{\{m(L_2)\}} = \mathcal{M}_{P,L,Q}^{\{m(L_1)+m(L_2)-\text{index}\,(Q_1+P_2)\}}.$$

More precisely, we have

$$\begin{aligned} &P = P_1 - L_1^*(Q_1 + P_2)^{-1} L_1, \quad Q = Q_2 - L_2(Q_1 + P_2)^{-1} L_2^*, \\ &L = L_2(Q_1 + P_2)^{-1} L_1. \end{aligned}$$

Moreover, we have

$$m(L_1) + m(L_2) - \text{index}\,(Q_1 + P_2) \in \{m(L), m(L) + 2\} \mod 4.$$

Proof. The kernel κ of $\mathcal{M}_{P_1,L_1,Q_1} \mathcal{M}_{P_2,L_2,Q_2}$ is

$$\begin{aligned} &\kappa(x, y) \\ &= (\det L_1)^{1/2} (\det L_2)^{1/2} e^{-i\pi n/2} \int e^{i\pi\{P_1 x^2 - 2L_1 x \cdot z + Q_1 z^2 + P_2 z^2 - 2L_2 z \cdot y + Q_2 y^2\}} dz \\ &= (\det L_1)^{1/2} (\det L_2)^{1/2} e^{-i\pi n/2} e^{i\pi\{P_1 x^2 + Q_2 y^2\}} \\ &\quad \times \int e^{-2i\pi(L_1 x + L_2^* y)\cdot z} e^{i\pi(Q_1+P_2)z^2} dz \\ &= (\det L_1)^{1/2} (\det L_2)^{1/2} e^{-i\pi n/2} e^{i\pi\{P_1 x^2 + Q_2 y^2\}} e^{-i\pi(Q_1+P_2)^{-1}(L_1 x + L_2^* y)^2} \\ &\quad \times |\det(Q_1 + P_2)|^{-1/2} e^{i\frac{\pi}{4}\,\text{sign}(Q_1+P_2)}, \end{aligned}$$

according to formula (A.1.7) (see also (A.1.8)), noting that the matrix $Q_1 + P_2$ is real symmetric and non-singular. As a result, we have

$$\begin{aligned} &\kappa(x, y) \\ &= e^{i\pi\{(P_1 - L_1^*(Q_1+P_2)^{-1}L_1)x^2 + (Q_2 - L_2(Q_1+P_2)^{-1}L_2^* y^2)\}} e^{-2i\pi\{L_2(Q_1+P_2)^{-1}L_1 x\cdot y\}} \\ &\quad \times (\det L_1)^{1/2} (\det L_2)^{1/2} e^{-i\pi n/2} |\det(Q_1 + P_2)|^{-1/2} e^{i\frac{\pi}{4}\,\text{sign}(Q_1+P_2)}. \end{aligned}$$

We note that, with E_{12} standing for the eigenvalues of $Q_1 + P_2$,

$$\nu_+ = \text{Card}(\mathrm{E}_{12} \cap \mathbb{R}_+), \quad \nu_- = \text{Card}(\mathrm{E}_{12} \cap \mathbb{R}_-) = \text{index}\,(Q_1 + P_2),$$

implying that the kernel κ is given by

$$\kappa(x, y) = e^{i\frac{\pi}{2}\left(m(L_1)+m(L_2)-n+\frac{1}{2}(\nu_+-\nu_-)\right)} |\det L|^{1/2} e^{i\pi\{Px^2 - 2Lx\cdot y + Qy^2\}}, \quad \text{(A.2.5)}$$

with

$$P = P_1 - L_1^*(Q_1 + P_2)^{-1}L_1, \quad Q = Q_2 - L_2(Q_1 + P_2)^{-1}L_2^*, \tag{A.2.6}$$
$$L = L_2(Q_1 + P_2)^{-1}L_1.$$

Checking the unit factor in front of the right-hand side of (A.2.5), we note that $\nu_+ + \nu_- = n$ since $Q_1 + P_2$ is non-singular and we get

$$\begin{aligned} e^{i\frac{\pi}{2}(m(L_1)+m(L_2)-n+\frac{1}{2}(\nu_+-\nu_-))} &= e^{-\frac{i\pi n}{4}} e^{i\frac{\pi}{2}(m(L_1)+m(L_2)-\frac{n}{2}+\frac{1}{2}(\nu_+-\nu_-))} \\ &= e^{-\frac{i\pi n}{4}} e^{i\frac{\pi}{2}(m(L_1)+m(L_2)-\nu_-)}. \end{aligned}$$

We have also, since index $(Q_1 + P_2) =$ index $(Q_1 + P_2)^{-1}$,

$$\begin{aligned} \left(e^{i\frac{\pi}{2}(m(L_1)+m(L_2)-\nu_-)}\right)^2 &= \operatorname{sign}(\det L_1)\operatorname{sign}(\det L_2)(-1)^{\nu_-} \\ &= \operatorname{sign}(\det L_1)\operatorname{sign}(\det L_2)\operatorname{sign}(\det(Q_1 + P_2)^{-1}) \\ &= \operatorname{sign}(\det L), \end{aligned}$$

entailing that

$$\kappa(x, y) = e^{-\frac{i\pi n}{4}}(\det L)^{1/2} e^{i\pi\{Px^2 - 2Lx\cdot y + Qy^2\}},$$

concluding the proof of the proposition. ∎

Lemma A.2.8. *Let P_j, L_j, Q_j, $j = 1, 2, 3$ be as in Definition A.2.1. Then, there exist (P', L', Q'), (P'', L'', Q'') as in Definition A.2.1 such that*

$$\mathcal{M}_{P_1,L_1,Q_1}\mathcal{M}_{P_2,L_2,Q_2}\mathcal{M}_{P_3,L_3,Q_3} = \mathcal{M}_{P',L',Q'}\mathcal{M}_{P'',L'',Q''}. \tag{A.2.7}$$

Proof. If $\det(Q_1 + P_2) \neq 0$, Lemma A.2.7 implies that $\mathcal{M}_{P_1,L_1,Q_1}\mathcal{M}_{P_2,L_2,Q_2} = \mathcal{M}_{P',L',Q'}$ so that (A.2.7) is satisfied with $(P'', L'', Q'') = (P_3, L_3, Q_3)$. We may thus assume in the sequel that $\det(Q_1 + P_2) = 0$. Then, the kernel of $Q_1 + P_2$ is of dimension $r \in [\![1, n]\!]$; let us define J_r as the orthogonal projection onto $\ker(Q_1 + P_2)$.

Claim A.2.9. The matrix $J_r + (Q_1 + P_2)^2$ is positive definite (thus invertible).

Proof. Indeed, if $J_r x + (Q_1 + P_2)^2 x = 0$, we obtain by taking the dot-product with x that

$$\|J_r x\|^2 + \|(Q_1 + P_2)x\|^2 = 0 \Longrightarrow x \in \ker(Q_1 + P_2),\ J_r x = 0 \Longrightarrow x = 0.$$

This matrix is also non-negative, proving the claim. ∎

Let us define the real $n \times n$ symmetric matrix

$$P = \mu L_2\left[J_r + (Q_1 + P_2)^2\right]^{-1} L_2^* - Q_2, \tag{A.2.8}$$

where μ is a positive parameter to be chosen later; we note that $P + Q_2$ is invertible. Also, we have

$$L_2^*\big(Q_2 + P\big)^{-1} L_2 - (Q_1 + P_2) = \mu^{-1}\big[J_r + (Q_1 + P_2)^2\big] - (Q_1 + P_2),$$

which is invertible if μ (is different from 0 and) does not meet the spectrum of $Q_1 + P_2$ (see footnote[2]). We have also

$$\begin{aligned} P - P_3 &= \mu L_2\big[J_r + (Q_1 + P_2)^2\big]^{-1} L_2^* - (Q_2 + P_3) \\ &= L_2\big\{\mu\big[J_r + (Q_1 + P_2)^2\big]^{-1} - L_2^{-1}(Q_2 + P_3)L_2^{*-1}\big\}L_2^*, \end{aligned}$$

which is invertible for μ large enough[3]. Eventually, defining

$$\lambda_0 = \max(\text{Spectrum}\,|Q_2 + P_1|),$$

the condition

$$\mu > \max\big\{\lambda_0, \|L_2^{-1}(Q_2 + P_3)L_2^{*-1}\|, \|L_2^{-1}(Q_2 + P_3)L_2^{*-1}\|\lambda_0^2\big\},$$

implies that, with P given by (A.2.8), we obtain that the matrices

$$P + Q_2, \quad Q_1 + P_2 - L_2^*\big(Q_2 + P\big)^{-1} L_2, \quad P - P_3 \text{ are invertible.} \tag{A.2.9}$$

Using now Lemma A.2.7 and the first property in (A.2.9), we get that we can find $\tilde{P}, \tilde{L}, \tilde{Q}$ as in Definition A.2.1 such that

$$\mathcal{M}_{P_2,L_2,Q_2}\mathcal{M}_{P,I_n,0} = \mathcal{M}_{\tilde{P},\tilde{L},\tilde{Q}},$$

with (thanks to (A.2.6)),

$$\tilde{P} = P_2 - L_2^*(Q_2 + P)^{-1}L_2.$$

We check now

$$\mathcal{M}_{P_1,L_1,Q_1}\mathcal{M}_{P_2,L_2,Q_2}\mathcal{M}_{P,I_n,0} = \mathcal{M}_{P_1,L_1,Q_1}\mathcal{M}_{\tilde{P},\tilde{L},\tilde{Q}},$$

---

[2]The symmetric matrices $Q_1 + P_2$ and J_r can be diagonalised simultaneously so that the invertibility of

$$\mu^{-1}\big[J_r + (Q_1 + P_2)^2\big] - (Q_1 + P_2)$$

is equivalent to $\mu \neq 0$, $\mu^{-1}\lambda_j^2 \neq \lambda_j$, i.e., $\mu \neq \lambda_j$, where the λ_j are the non-zero eigenvalues of $Q_1 + P_2$.

[3]Indeed, the eigenvalues of $[J_r + (Q_1 + P_2)^2]^{-1}$ are 1 and λ_j^{-2} where the λ_j are the non-zero eigenvalues of $Q_1 + P_2$. To secure the invertibility of $P - P_3$, it is thus enough to have

$$\min(\mu, \mu\lambda_j^{-2}) > \|L_2^{-1}(Q_2 + P_3)L_2^{*-1}\|,$$

where the λ_j are the non-zero eigenvalues of $Q_1 + P_2$.

and we note that

$$Q_1 + \tilde{P} = Q_1 + P_2 - L_2^*(Q_2 + P)^{-1} L_2 \quad \text{is invertible,}$$

thanks to the second property in (A.2.9) so that, from Lemma A.2.7, we can find P', L', Q' as in Definition A.2.1 such that

$$\mathcal{M}_{P_1,L_1,Q_1} \mathcal{M}_{\tilde{P},\tilde{L},\tilde{Q}} = \mathcal{M}_{P',L',Q'},$$

and this yields

$$\mathcal{M}_{P_1,L_1,Q_1} \mathcal{M}_{P_2,L_2,Q_2} \mathcal{M}_{P,I_n,0} = \mathcal{M}_{P',L',Q'}. \tag{A.2.10}$$

Finally, we check

$$\underbrace{\mathcal{M}^{-1}_{P,I_n,0}}_{\substack{=\mathcal{M}_{0,-I_n,-P} \\ \text{cf. Claim A.2.6}}} \mathcal{M}_{P_3,L_3,Q_3} = \mathcal{M}_{0,-I_n,-P} \mathcal{M}_{P_3,L_3,Q_3},$$

and since $-P + P_3$ is invertible (thanks to the third property in (A.2.9)), we obtain, using once again Lemma A.2.7, that we can find P'', L'', Q'' as in Definition A.2.1 such that

$$\mathcal{M}^{-1}_{P,I_n,0} \mathcal{M}_{P_3,L_3,Q_3} = \mathcal{M}_{P'',L'',Q''}. \tag{A.2.11}$$

Gathering the information above, we find that

$$\begin{aligned} &\mathcal{M}_{P_1,L_1,Q_1} \mathcal{M}_{P_2,L_2,Q_2} \mathcal{M}_{P_3,L_3,Q_3} \\ &\quad = \underbrace{\mathcal{M}_{P_1,L_1,Q_1} \mathcal{M}_{P_2,L_2,Q_2} \mathcal{M}_{P,I_n,0}}_{\mathcal{M}_{P',L',Q'},\ \text{(A.2.10)}} \underbrace{\mathcal{M}^{-1}_{P,I_n,0} \mathcal{M}_{P_3,L_3,Q_3}}_{\mathcal{M}_{P'',L'',Q''},\ \text{(A.2.11)}}, \end{aligned}$$

which ends the proof of the lemma. ■

Proposition A.2.10. *The metaplectic group* Mp(n) *is equal to the set*

$$\left\{ \mathcal{M}_{P_1,L_1,Q_1} \mathcal{M}_{P_2,L_2,Q_2} \right\}_{\substack{P_j = P_j^*, Q_j = Q_j^* \\ \det L_j \neq 0}}.$$

In other words, every metaplectic operator of Mp(n) *is the product of two operators of type* $\mathcal{M}_{P,L,Q}$ *as given by Definition* A.2.1.

Proof. From Proposition A.2.2, the metaplectic group is generated by the $\mathcal{M}_{P,L,Q}$ and since the inverse of $\mathcal{M}_{P,L,Q}$ is $\mathcal{M}_{-Q,-L^*,-P}$, thanks to Claim A.2.6, it is enough to check the products

$$\mathcal{M}_{P_1,L_1,Q_1} \cdots \mathcal{M}_{P_N,L_N,Q_N}$$

for $N \geq 3$. Lemma A.2.8 is tackling the case $N = 3$ and a trivial recurrence on N provides the result of the proposition. ■

Theorem A.2.11. *Let M be an element of* $\mathrm{Mp}(n)$ *such that* $M = e^{i\phi}\,\mathrm{Id}_{L^2(\mathbb{R}^n)}, \phi \in \mathbb{R}$. *Then, $e^{i\phi}$ belongs to the set $\{-1,1\}$. In other words, the intersection of the metaplectic group with the unit circle (identified to the unitary operators in $L^2(\mathbb{R}^n)$ defined by the mappings $v \mapsto zv$ where $z \in \mathbb{S}^1 \subset \mathbb{C}$) is reduced to the set $\{-1,1\}$.*

Proof. Using Proposition A.2.10, the result follows from Lemma A.2.5. ∎

We may go back to the description given by Proposition 1.2.11 and Definition 1.2.13.

Proposition A.2.12. *The metaplectic group* $\mathrm{Mp}(n)$ *is equal to the set*

$$\{M_{A_1,B_1,C_1} M_{A_2,B_2,C_2}\}_{\substack{A_j=A_j^*, C_j=C_j^*,\\ \det B_j \neq 0}},$$

where the operators $M_{A,B,C}$ are defined in Proposition 1.2.11.

Proof. Let M be in $\mathrm{Mp}(n)$. We have

$$\begin{aligned} M &= (M_{A_1,B_1,C_1})^{\pm 1} \cdots (M_{A_N,B_N,C_N})^{\pm 1} \\ &\underbrace{=}_{(A.2.1)} \left(\mathcal{M}_{A_1,-B_1,C_1} e^{-i\pi n/4}\mathcal{F}\right)^{\pm 1} \cdots \left(\mathcal{M}_{A_N,-B_N,C_N} e^{-i\pi n/4}\mathcal{F}\right)^{\pm 1} \\ &= \left(\mathcal{M}_{A_1,-B_1,C_1}\mathcal{M}_{0,I_n,0}\right)^{\pm 1} \cdots \left(\mathcal{M}_{A_N,-B_N,C_N}\mathcal{M}_{0,I_n,0}\right)^{\pm 1}, \end{aligned}$$

and since from Claim A.2.6, we have

$$\mathcal{M}_{A,B,C}^{-1} = \mathcal{M}_{-C,-B^*,-A},$$

we find that M is in fact a product of $2N$ terms of type $\mathcal{M}_{P,L,Q}$, and thanks to Proposition A.2.10, we get

$$\begin{aligned} M &= \mathcal{M}_{P_1,L_1,Q_1}\mathcal{M}_{P_2,L_2,Q_2} = \underbrace{\mathcal{M}_{P_1,L_1,Q_1} e^{-i\pi n/4}\mathcal{F}}_{M_{P_1,-L_1,Q_1}} \underbrace{\left(e^{-i\pi n/4}\mathcal{F}\right)^{-1}\mathcal{M}_{P_2,L_2,Q_2}}_{\left(\mathcal{M}_{-Q_2,-L_2^*,-P_2} e^{-i\pi n/4}\mathcal{F}\right)^{-1}} \\ &= M_{P_1,-L_1,Q_1}\left(M_{-Q_2,-L_2^*,-P_2}\right)^{-1} \\ &= M_{P_1,-L_1,0}M_{0,I_n,Q_1}\left(M_{-Q_2,-L_2^*,0}M_{0,I_n,-P_2}\right)^{-1} \\ &= M_{P_1,-L_1,0}M_{0,I_n,Q_1}M_{0,I_n,P_2}\left(M_{-Q_2,-L_2^*,0}\right)^{-1} \\ &= M_{P_1,-L_1,0}M_{0,I_n,Q_1+P_2}\left(M_{-Q_2,-L_2^*,0}\right)^{-1} \qquad \text{(cf. formula (1.2.33))} \\ &= M_{P_1,-L_1,Q_1+P_2}M_{A'',B'',0} \quad \text{(cf. Lemma A.2.14 below in the next subsection)}, \end{aligned}$$

proving the proposition. ∎

A.2.2 On some subgroups of the metaplectic group

We have seen in (1.2.24), (1.2.22) some equivalent conditions for a matrix

$$\Xi = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \quad \text{where } P, Q, R, S \text{ are } n \times n \text{ real matrices,} \tag{A.2.12}$$

to be symplectic. We note here that when $\Xi \in \mathrm{Sp}(n, \mathbb{R})$, we have

$$\Xi^{-1} = \begin{pmatrix} S^* & -Q^* \\ -R^* & P^* \end{pmatrix}, \tag{A.2.13}$$

as it is easily checked from (1.2.24), (1.2.22). When $\det P \neq 0$, we proved that $\Xi = \Xi_{A,B,C}$ as defined in (1.2.19). Also from (A.2.13), we get that if $\det S \neq 0$ we have

$$\Xi^{-1} = \Xi_{A,B,C},$$

so that

$$\Xi = \begin{pmatrix} I_n & C \\ 0 & I_n \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & B^{*-1} \end{pmatrix} \begin{pmatrix} I_n & 0 \\ -A & I_n \end{pmatrix}.$$

Some other properties of the same type are available when $\det Q$ or $\det R$ are different from 0. Indeed, we have for $\Xi \in \mathrm{Sp}(n, \mathbb{R})$ and σ given by (1.2.15),

$$\Xi\sigma = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \sigma = \begin{pmatrix} -Q & P \\ -S & R \end{pmatrix} \underbrace{=}_{\text{if } \det Q \neq 0} \Xi_{A,B,C}, \tag{A.2.14}$$

so that

$$\Xi = -\Xi_{A,B,C}\sigma = \begin{pmatrix} I_n & 0 \\ A & I_n \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ 0 & B^* \end{pmatrix} \begin{pmatrix} I_n & -C \\ 0 & I_n \end{pmatrix} \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

If we have $\det R \neq 0$, using the two first equalities in (A.2.14), we get that $(\Xi\sigma)^{-1} = \Xi_{A,B,C}$, which gives

$$\Xi = \begin{pmatrix} I_n & C \\ 0 & I_n \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & B^{*-1} \end{pmatrix} \begin{pmatrix} I_n & 0 \\ -A & I_n \end{pmatrix} \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

However, it is indeed possible when $n \geq 2$ to have a symplectic matrix in $\mathrm{Sp}(n, \mathbb{R})$ in the form (A.2.12) such that all blocks are singular, as shown in the following remark.

Remark A.2.13. The 4×4 matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$$

belongs to $\mathrm{Sp}(2, \mathbb{R})$ although all the block 2×2 matrices P, Q, R, S, are singular (with rank 1).

Lemma A.2.14. *With $M_{A,B,C}$ defined in Proposition 1.2.11, the sets*

$$\mathcal{L} = \{M_{A,B,0}\}_{\substack{A=A^*,\\ \det B\neq 0}}, \quad \mathcal{R} = \{M_{0,B,C}\}_{\substack{C=C^*,\\ \det B\neq 0}}, \tag{A.2.15}$$

are subgroups of the metaplectic group (cf. Definition 1.2.13).

Proof. Indeed, $\mathcal{L}$ contains the identity of $L^2(\mathbb{R}^n)$ and we have for $v \in L^2(\mathbb{R}^n)$,

$$\begin{aligned}
M_{A_1,B_1,0}M^{-1}_{A_2,B_2,0}v &= M_{A_1,B_1,0}\big\{M_{0,B_2^{-1},0}\{e^{-i\pi A_2x^2}v(x)\}\big\}\\
&= M_{A_1,B_1,0}\big\{e^{-i\pi B_2^{*-1}A_2B_2^{-1}x^2}v(B_2^{-1}x)\big\}(\det B_2)^{-1/2}\\
&= e^{i\pi A_1x^2}e^{-i\pi B_1^*B_2^{*-1}A_2B_2^{-1}B_1x^2}v(B_2^{-1}B_1x)(\det B_1)^{1/2}(\det B_2)^{-1/2}\\
&= e^{i\pi(A_1-B_1^*B_2^{*-1}A_2B_2^{-1}B_1)x^2}v(B_2^{-1}B_1x)(\det B_1)^{1/2}(\det B_2)^{-1/2}\\
&= (M_{A_1-B_1^*B_2^{*-1}A_2B_2^{-1}B_1,B_2^{-1}B_1,0}v)(x),
\end{aligned}$$

so that $M_{A_1,B_1,0}M^{-1}_{A_2,B_2,0}$ belongs to the set $\mathcal{L}$ in (A.2.15), proving that $\mathcal{L}$ is indeed a subgroup of the metaplectic group. We note also that the bijective mapping

$$\mathcal{L} \ni M \mapsto F^*MF \in \mathcal{R}, \tag{A.2.16}$$

(F stands for the Fourier transformation) sends $\mathcal{L}$ onto $\mathcal{R}$ since we have

$$\begin{aligned}
F^*M_{A,B,0}F &= F^*M_{A,I_n,0}FF^*M_{0,B,0}F = M_{0,I_n,A}M_{0,B^{*-1},0}\\
&= M_{0,B^{*-1},B^{*-1}AB^{-1}}.
\end{aligned} \tag{A.2.17}$$

Moreover, the mapping (A.2.16) is obviously one-to-one and is also onto since, given $B_1 \in \mathrm{Gl}(n,\mathbb{R})$ and C_1 a symmetric $n\times n$ matrix, we see from (A.2.17) that

$$F^*M_{B_1^{-1}C_1B_1^{*-1},B_1^{*-1},0}F = M_{0,B_1,C_1}.$$

The mapping (A.2.16) also extends to a group isomorphism of $\mathrm{Mp}(n)$, proving the lemma. ∎

Remark A.2.15. We may note that

$$\begin{aligned}
(M_{A_1,B_1,0}M_{A_2,B_2,0}v)(x) &= e^{i\pi A_1x^2}(M_{A_2,B_2,0}v)(B_1x)(\det B_1)^{1/2}\\
&= e^{i\pi(A_1+B_1^*A_2B_1)x^2}v(B_2B_1x)(\det B_1)^{1/2}(\det B_2)^{1/2}\\
&= (M_{A_1+B_1^*A_2B_1,B_2B_1,0}v)(x),
\end{aligned}$$

so that the internal binary operation $\star$ can be defined on the set $\{(A,B)\}_{\substack{A=A^*\\ \det B\neq 0}}$ as

$$(A_1,B_1)\star(A_2,B_2) = (A_1+B_1^*A_2B_1, B_2B_1),$$

for which the identity is $(0, I_n)$ and the inverse

$$(A,B)^{-1} = (-B^{*-1}AB^{-1}, B^{-1}).$$

Remark A.2.16. A consequence of Lemma A.2.14 is, with Ψ defined in (1.2.46), that

$$\{\Psi(M_{A,B,0})\}_{\substack{A=A^* \\ \det B\neq 0}} = \{\Xi_{A,B,0}\}_{\substack{A=A^* \\ \det B\neq 0}}, \quad \{\Psi(M_{0,B,C})\}_{\substack{C=C^* \\ \det B\neq 0}} = \{\Xi_{0,B,C}\}_{\substack{C=C^* \\ \det B\neq 0}},$$

are subgroups of the symplectic group $\mathrm{Sp}(n,\mathbb{R})$.

Proposition A.2.17. *The metaplectic group* $\mathrm{Mp}(n)$ *is equal to the set*

$$\left\{M_{A_1,B_1,C_1} M_{A_2,B_2,C_2}\right\}_{\substack{A_j=A_j^*, C_j=C_j^* \\ \det B_j\neq 0}}.$$

In other words, every metaplectic operator of $\mathrm{Mp}(n)$ *is the product of two operators of type* $M_{A,B,C}$ *as given by Proposition* 1.2.11.

Proof. Let $M \in \mathrm{Mp}(n)$; using Proposition A.2.10, we may assume that

$$\begin{aligned}
M &= \mathcal{M}_{P_1,L_1,Q_1}\mathcal{M}_{P_2,L_2,Q_2} \\
&= \mathcal{M}_{P_1,L_1,Q_1}\mathcal{F}e^{-i\pi n/4}\left(\mathcal{F}e^{-i\pi n/4}\right)^{-1}\mathcal{M}_{P_2,L_2,Q_2} \\
\text{(A.2.1)} &= M_{P_1,-L_1,Q_1}\left(\mathcal{M}^{-1}_{P_2,L_2,Q_2}\mathcal{F}e^{-i\pi n/4}\right)^{-1} \\
\text{(Claim A.2.6)} &= M_{P_1,-L_1,Q_1}\left(\mathcal{M}_{-Q_2,-L_2^*,-P_2}\mathcal{F}e^{-i\pi n/4}\right)^{-1} \\
\text{(A.2.1), (1.2.33)} &= M_{P_1,-L_1,Q_1}M^{-1}_{-Q_2,L_2^*,-P_2} \\
&= M_{P_1,-L_1,0}M_{0,I_n,Q_1}\left(M_{-Q_2,L_2^*,0}M_{0,I_n,-P_2}\right)^{-1} \\
&= M_{P_1,-L_1,0}M_{0,I_n,Q_1}M_{0,I_n,P_2}M^{-1}_{-Q_2,L_2^*,0} \\
&= M_{P_1,-L_1,0}M_{0,I_n,Q_1+P_2}M^{-1}_{-Q_2,L_2^*,0} \\
&= M_{P_1,-L_1,Q_1+P_2}M^{-1}_{-Q_2,L_2^*,0} \\
\text{(using Lemma A.2.14)} &= M_{P_1,-L_1,Q_1+P_2}M_{A',B',0},
\end{aligned}$$

proving the sought result. ∎

Remark A.2.18. We have used two different sets of generators of the metaplectic group. First the set $\mathscr{G}_1 = \{M^{\{m(B)\}}_{A,B,C}\}$ given by (1.2.35) which is somewhat natural, also allowing us to recover the operator $e^{-i\pi n/4}\mathcal{F}$ where the phase factor appears via formula (1.2.38). The Identity appears clearly as $M^{\{0\}}_{0,I_n,0}$, but the inverse of $M^{\{m(B)\}}_{A,B,C}$ cannot always be expressed within $\mathscr{G}_1$.

Also, we have the set $\mathscr{G}_2 = \left\{\mathcal{M}^{\{m(B)\}}_{A,B,C}\right\}$ given in Definition A.2.1, which incorporates a phase prefactor $e^{-i\pi n/4}$, looking a priori rather arbitrary but of course necessary for the sequel (this prefactor is also suggested by (1.2.38)); here to express the identity, we need to write it as $\mathcal{M}^{\{0\}}_{0,I_n,0}\mathcal{M}^{\{n\}}_{0,-I_n,0}$, but the inverse of $\mathcal{M}^{\{m(B)\}}_{A,B,C}$ is easily obtained by Claim A.2.6 within $\mathscr{G}_2$. Certainly the description given by $\mathscr{G}_2$ is much better, in particular because the calculations leading to Lemma A.2.5 and Proposition A.2.7 are rather easy as well as the proof of Lemma A.2.8; a statement analogous to Proposition A.2.10 for $\mathscr{G}_1$ is true (cf. Proposition A.2.12), but its proof is quite indirect and relies heavily on the results for $\mathscr{G}_2$.

A.3 Mehler's formula

We provide here a couple of statements related to the so-called Mehler's formula, appearing as particular cases of L. Hörmander's study in [22] (see also the more recent K. Pravda-Starov' article [42]). In the general framework, we consider a complex-valued quadratic form Q on the phase space $\mathbb{R}^{2n}$ such that $\operatorname{Re} Q \le 0$: we want to quantize the Gaussian function (here X stands for (x, ξ)) $\mathbf{a}(X) = e^{\langle QX,X\rangle}$, and to relate the operator with Weyl symbol $\mathbf{a}$ to the operator

$$\exp\left\{\operatorname{Op_w}(\langle QX, X\rangle)\right\}.$$

Lemma A.3.1. *For* $\operatorname{Re} t \ge 0$, $t \notin i\pi(2\mathbb{Z}+1)$, *we have in n dimensions,*

$$\big(\cosh(t/2)\big)^n \exp -t\pi \operatorname{Op_w}(|x|^2 + |\xi|^2) = \operatorname{Op_w}\left(e^{-2\tanh(\frac{t}{2})\pi(x^2+\xi^2)}\right).$$

In particular, for $t = -2is$, $s \in \mathbb{R}$, $s \notin \frac{\pi}{2}(1+2\mathbb{Z})$, we have in n dimensions

$$(\cos s)^n \exp\big(2i\pi s \operatorname{Op_w}(|x|^2 + |\xi|^2)\big) = \operatorname{Op_w}\left(e^{2i\pi \tan s(|x|^2+|\xi|^2)}\right). \tag{A.3.1}$$

Lemma A.3.2. *For any* $z \in \mathbb{C}$, $\operatorname{Re} z \ge 0$, *we have in n dimensions*

$$\operatorname{Op_w}\big(\exp -\big(2z\pi\left(|\xi|^2 + |x|^2\right)\big)\big) = \frac{1}{(1+z)^n} \sum_{k\ge 0} \left(\frac{1-z}{1+z}\right)^k \mathbb{P}_{k;n}, \tag{A.3.2}$$

where $\mathbb{P}_{k;n}$ is defined in Section A.1.4 and the equality holds between $L^2(\mathbb{R}^n)$-bounded operators.

We provide first a proof of a particular case of the results of [22].

Lemma A.3.3. *For* $\operatorname{Re} t \ge 0$, $t \notin i\pi(2\mathbb{Z}+1)$, *we have in n dimensions,*

$$(\cosh(t/2))^n \exp -t\pi \operatorname{Op_w}(|x|^2 + |\xi|^2) = \operatorname{Op_w}\left(e^{-2\tanh(\frac{t}{2})\pi(x^2+\xi^2)}\right). \tag{A.3.3}$$

Proof. By tensorisation, it is enough to prove that formula for $n = 1$, which we assume from now on. We define

$$L = \xi + ix, \quad \bar{L} = \xi - ix, \quad M(t) = \beta(t)\operatorname{Op_w}(e^{-\alpha(t)\pi L\bar{L}}),$$

where α, β are smooth functions of t to be chosen below. Assuming $\beta(0) = 1, \alpha(0) = 0$, we find that $M(0) = \mathrm{Id}$ and

$$\dot{M} + \pi \mathrm{Op_w}(|L|^2)M = \mathrm{Op_w}\big(\dot{\beta}e^{-\alpha\pi|L|^2} - \beta\dot{\alpha}\pi|L|^2e^{-\alpha\pi|L|^2} + \pi(|L|^2)\sharp\beta e^{-\alpha\pi|L|^2}\big).$$

We have from (1.2.3), since $\partial_x\partial_\xi|L|^2 = 0$,

$$\begin{aligned}
|L|^2\sharp e^{-\alpha\pi|L|^2} &= |L|^2e^{-\alpha\pi|L|^2} + \frac{1}{4i\pi}\overbrace{\{|L|^2, e^{-\alpha\pi|L|^2}\}}^{=0} \\
&\quad + \frac{1}{(4i\pi)^2}\frac{1}{2}\big(\partial_\xi^2(|L|^2)\partial_x^2 e^{-\alpha\pi|L|^2} + \partial_x^2(|L|^2)\partial_\xi^2 e^{-\alpha\pi|L|^2}\big) \\
&= |L|^2e^{-\alpha\pi|L|^2} \\
&\quad + \frac{1}{(4i\pi)^2}\frac{1}{2}e^{-\alpha\pi|L|^2}\big(2((-2\alpha\pi x)^2-2\alpha\pi)+2((-2\alpha\pi\xi)^2-2\alpha\pi)\big) \\
&= |L|^2e^{-\alpha\pi|L|^2}\left(1-\frac{4\alpha^2\pi^2}{16\pi^2}\right) + \frac{\alpha\pi}{4\pi^2}e^{-\alpha\pi|L|^2},
\end{aligned}$$

so that

$$\begin{aligned}
&\dot{M} + \pi\mathrm{Op_w}(|L|^2)M \\
&\quad = \mathrm{Op_w}\bigg(\dot{\beta}e^{-\alpha\pi|L|^2} - \beta\dot{\alpha}\pi|L|^2e^{-\alpha\pi|L|^2} \\
&\qquad\qquad + \pi\beta|L|^2e^{-\alpha\pi|L|^2}\left(1-\frac{4\alpha^2\pi^2}{16\pi^2}\right) + \frac{\alpha\pi\beta}{4\pi}e^{-\alpha\pi|L|^2}\bigg) \\
&\quad = \mathrm{Op_w}\left(e^{-\alpha\pi|L|^2}\left\{|L|^2\left(-\pi\dot{\alpha}\beta + \pi\beta\left(1-\frac{\alpha^2}{4}\right)\right) + \dot{\beta} + \frac{\alpha\beta}{4}\right\}\right).
\end{aligned}$$

We solve now

$$\dot{\alpha} = 1 - \frac{\alpha^2}{4}, \quad \alpha(0) = 0 \Longleftrightarrow \alpha(t) = 2\tanh(t/2),$$

and

$$4\dot{\beta} + \alpha\beta = 0, \quad \beta(0) = 1 \Longleftrightarrow \beta(t) = \frac{1}{\cosh(t/2)}.$$

We obtain that

$$\dot{M} + \pi\mathrm{Op_w}(|L|^2)M = 0,\ M(0) = \mathrm{Id},$$

and this implies

$$\beta(t)\mathrm{Op_w}(e^{-\alpha(t)\pi L\bar{L}}) = M(t) = \exp -t\pi(|L|^2)^w,$$

which proves (A.3.3). ∎

In particular, for $t = -2is, s \in \mathbb{R}, s \notin \frac{\pi}{2}(1+2\mathbb{Z})$, we have in n dimensions

$$(\cos s)^n \exp\big(2i\pi s \mathrm{Op_w}(|x|^2+|\xi|^2)\big) = \mathrm{Op_w}\left(e^{2i\pi \tan s(|x|^2+|\xi|^2)}\right). \tag{A.3.4}$$

Lemma A.3.4. *For any $z \in \mathbb{C}$, $\operatorname{Re} z \geq 0$, we have in n dimensions*

$$\mathrm{Op_w}\left(\exp -\big(2z\pi(|\xi|^2+|x|^2)\big)\right) = \frac{1}{(1+z)^n}\sum_{k\geq 0}\left(\frac{1-z}{1+z}\right)^k \mathbb{P}_{k;n},$$

where $\mathbb{P}_{k;n}$ is defined in Section A.1.4 and the equality holds between $L^2(\mathbb{R}^n)$-bounded operators.

Proof. Starting from (A.3.4), we get for $\tau \in \mathbb{R}$, in n dimensions,

$$(\cos(\arctan\tau))^n \exp\big(2i\pi \arctan\tau \mathrm{Op_w}(|x|^2+|\xi|^2)\big) = \mathrm{Op_w}\left(e^{2i\pi\tau(|x|^2+|\xi|^2)}\right),$$

so that using the spectral decomposition of the (n-dimensional) harmonic oscillator and (A.8.1), we get

$$(1+\tau^2)^{-n/2}\sum_{k\geq 0} e^{2i(\arctan\tau)(k+\frac{n}{2})}\mathbb{P}_{k;n} = \mathrm{Op_w}\left(e^{2i\pi\tau(|x|^2+|\xi|^2)}\right),$$

which implies

$$(1+\tau^2)^{-n/2}\sum_{k\geq 0}\frac{(1+i\tau)^{2k+n}}{(1+\tau^2)^{k+\frac{n}{2}}}\mathbb{P}_{k;n} = \mathrm{Op_w}\left(e^{2i\pi\tau(|x|^2+|\xi|^2)}\right),$$

entailing

$$\sum_{k\geq 0}\frac{(1+i\tau)^k}{(1-i\tau)^{k+n}}\mathbb{P}_{k;n} = \mathrm{Op_w}\left(e^{2i\pi\tau(|x|^2+|\xi|^2)}\right),$$

proving the lemma by analytic continuation (we may refer the reader as well to [50, pages 204–205] and note that for any $z \in \mathbb{C}, \operatorname{Re} z \geq 0$, we have $|\frac{1-z}{1+z}| \leq 1$). ∎

A.4 Laguerre polynomials

A.4.1 Classical Laguerre polynomials

The Laguerre polynomials $\{L_k\}_{k\in\mathbb{N}}$ are defined by

$$L_k(x) = \sum_{0\leq l\leq k}\frac{(-1)^l}{l!}\binom{k}{l}x^l = e^x\frac{1}{k!}\left(\frac{d}{dx}\right)^k\{x^k e^{-x}\} = \left(\frac{d}{dx}-1\right)^k\left\{\frac{x^k}{k!}\right\}, \tag{A.4.1}$$

and we have

$$
\begin{aligned}
L_0 &= 1,\\
L_1 &= -X + 1,\\
L_2 &= \frac{1}{2}(X^2 - 4X + 2),\\
L_3 &= \frac{1}{6}(-X^3 + 9X^2 - 18X + 6),\\
L_4 &= \frac{1}{24}(X^4 - 16X^3 + 72X^2 - 96X + 24),\\
L_5 &= \frac{1}{120}(-X^5 + 25X^4 - 200X^3 + 600X^2 - 600X + 120),\\
L_6 &= \frac{1}{720}(X^6 - 36X^5 + 450X^4 - 2400X^3 + 5400X^2 - 4320X + 720),\\
L_7 &= \frac{-X^7 + 49X^6 - 882X^5 + 7350X^4 - 29400X^3 + 52920X^2 - 35280X + 5040}{5040}.
\end{aligned}
$$

We get also easily from the above definition that

$$L'_{k+1} = L'_k - L_k,$$

since with $T = d/dX - 1$

$$L'_k - L_k = TL_k = T^{k+1}\Big(\frac{X^k}{k!}\Big) = T^{k+1}\Big(\frac{d}{dX}\frac{X^{k+1}}{(k+1)!}\Big) = \frac{d}{dX}L_{k+1}.$$

Formula (6.8) and Theorem 12 in the R. Askey and G. Gasper's article [2] provide the inequalities

$$\forall k \in \mathbb{N}, \forall x \geq 0, \qquad \sum_{0\leq l\leq k} (-1)^l L_l(x) \geq 0. \tag{A.4.2}$$

This result follows as well from formula (73) in the 1940 paper [12] by E. Feldheim. Let us calculate the Fourier transform of the Laguerre polynomials, we have

$$L_k(x) = \Big(\frac{d}{dx} - 1\Big)^k \Big\{\frac{x^k}{k!}\Big\},$$

so that

$$\widehat{L_k}(\xi) = (2i\pi\xi - 1)^k \Big(\frac{-1}{2i\pi}\Big)^k \frac{\delta_0^{(k)}}{k!} = \frac{(-1)^k}{k!}\Big(\xi - \frac{1}{2i\pi}\Big)^k \delta_0^{(k)}(\xi).$$

As a result, defining for $k \in \mathbb{N}, t \in \mathbb{R}$,

$$M_k(t) = (-1)^k H(t) e^{-t} L_k(2t), \quad H = \mathbf{1}_{\mathbb{R}_+}, \tag{A.4.3}$$

we find, using the homogeneity of degree $-k-1$ of $\delta_0^{(k)}$,

$$\begin{aligned}
\widehat{M_k}(\tau) &= \frac{1}{2}\frac{(-1)^k}{k!}\left(\frac{\tau}{2}-\frac{1}{2i\pi}\right)^k \delta_0^{(k)}\left(\frac{\tau}{2}\right) * \frac{(-1)^k}{1+2i\pi\tau} \\
&= (-1)^k (\frac{d}{d\sigma})^k \left\{\frac{(\sigma-\frac{1}{i\pi})^k/k!}{1+2i\pi(\tau-\sigma)}\right\}_{|\sigma=0} \\
\widehat{M_k}(\tau) &= \sum_l (-1)^k \binom{k}{l} \frac{(\sigma-\frac{1}{i\pi})^{k-l}}{(k-l)!}\frac{(k-l)!(2i\pi)^{k-l}}{\big(1+2i\pi(\tau-\sigma)\big)^{1+k-l}}{}_{|\sigma=0} \\
&= \sum_l (-1)^k \binom{k}{l} \frac{(-2)^{k-l}}{\big(1+2i\pi\tau\big)^{1+k-l}} \\
&= \frac{(-1)^k}{(1+2i\pi\tau)} \sum_l \binom{k}{l} \frac{(-2)^{k-l}}{\big(1+2i\pi\tau\big)^{k-l}} \\
&= \frac{(-1)^k}{(1+2i\pi\tau)}\left(1-\frac{2}{\big(1+2i\pi\tau\big)}\right)^k \\
&= \frac{(-1)^k}{(1+2i\pi\tau)}\left(\frac{-1+2i\pi\tau}{1+2i\pi\tau}\right)^k = \frac{1}{(1+2i\pi\tau)}\left(\frac{1-2i\pi\tau}{1+2i\pi\tau}\right)^k
\end{aligned}$$

so that

$$\widehat{M_k}(\tau) = \frac{(1-2i\pi\tau)^k}{(1+2i\pi\tau)^{k+1}} = \frac{(1-2i\pi\tau)^{2k+1}}{(1+4\pi^2\tau^2)^{k+1}}. \tag{A.4.4}$$

A.4.2 Generalized Laguerre polynomials

Let α be a complex number and let k be a non-negative integer such that $\alpha+k \notin (-\mathbb{N}^*)$. We define the generalized Laguerre polynomial L_k^α by

$$L_k^\alpha(x) = x^{-\alpha}e^x\left(\frac{d}{dx}\right)^k\left\{e^{-x}\frac{x^{k+\alpha}}{k!}\right\} = x^{-\alpha}\left(\frac{d}{dx}-1\right)^k\left\{\frac{x^{k+\alpha}}{k!}\right\}. \tag{A.4.5}$$

We note that L_k^α is indeed a polynomial with degree k with the formula

$$\begin{aligned}
L_k^\alpha(x) &= \sum_{k_1+k_2=k} \frac{1}{k!}\binom{k}{k_1}(-1)^{k_2}\Gamma(k+\alpha+1)\frac{x^{k-k_1}}{\Gamma(k+\alpha+1-k_1)} \\
&= \sum_{0\le k_1\le k} \frac{(-1)^{k_2}}{k_1!(k-k_1)!}\Gamma(k+\alpha+1)\frac{x^{k-k_1}}{\Gamma(k+\alpha+1-k_1)} \\
&= \sum_{0\le l\le k} \binom{k+\alpha}{k-l}\frac{(-1)^l x^l}{l!}.
\end{aligned} \tag{A.4.6}$$

N.B. We recall that the function $1/\Gamma$ is an entire function with simple zeroes at $-\mathbb{N}$. As a result to make sense for the binomial coefficient

$$\binom{k+\alpha}{k-l} = \frac{\Gamma(k+\alpha+1)}{(k-l)!\Gamma(l+\alpha+1)},$$

we need to make sure that $k+\alpha+1 \notin -\mathbb{N}$, i.e., $\alpha \notin -\mathbb{N}^* - k$.

Lemma A.4.1. *Let $\alpha \in \mathbb{C}\backslash(-\mathbb{N}^*)$ and let k be a non-negative integer. For $\alpha = 0$, we have $L_k^\alpha = L_k$, where L_k is the classical Laguerre polynomial defined in* (A.4.1)*. Moreover, we have for $l \le k$,*

$$\left(\frac{d}{dX}\right)^l L_k^\alpha = (-1)^l L_{k-l}^{\alpha+l}. \tag{A.4.7}$$

Proof. Indeed, we have from (A.4.6)

$$\begin{aligned}\left(\frac{d}{dX}\right)^l L_k^\alpha &= (-1)^l \sum_{l\le m\le k} \binom{k+\alpha}{k-m} \frac{(-1)^{m-l}X^{m-l}}{(m-l)!}\\ &= (-1)^l \sum_{0\le r\le k-l} \binom{k-l+\alpha+l}{k-r-l} \frac{(-1)^r X^r}{r!} = (-1)^l L_{k-l}^{\alpha+l},\end{aligned}$$

proving the sought formula. ∎

A.5 Singular integrals

Proposition A.5.1. (1) *The (Hardy) operator with distribution kernel*

$$\frac{H(x)H(y)}{\pi(x+y)}$$

is self-adjoint bounded on $L^2(\mathbb{R})$ with spectrum $[0,1]$ *and thus norm 1.*

(2) *The (modified Hardy) operators with respective distribution kernels*

$$H(x-y)\frac{H(x)H(y)}{\pi(x+y)}, \quad H(y-x)\frac{H(x)H(y)}{\pi(x+y)},$$

are bounded on $L^2(\mathbb{R})$ with norm $1/2$.

Proof. Let us prove (1): for $\phi \in L^2(\mathbb{R}_+)$, we define for $t \in \mathbb{R}$, $\tilde{\phi}(t) = \phi(e^t)e^{t/2}$, and we have to check the kernel

$$\frac{e^{t/2}e^{s/2}}{\pi(e^t+e^s)} = \frac{1}{\pi(e^{(t-s)/2}+e^{-(t-s)/2})} = \frac{1}{2\pi}\operatorname{sech}\left(\frac{t-s}{2}\right),$$

which is a convolution kernel. Using now the classical formula

$$\int e^{-2i\pi x\xi} \operatorname{sech} x dx = \pi \operatorname{sech}(\pi^2\xi),$$

we get that

$$\frac{1}{2\pi}\int \operatorname{sech}\left(\frac{t}{2}\right)e^{-2i\pi t\tau}dt = \operatorname{sech}(\pi^2 2\tau),$$

a smooth function whose range is $(0, 1]$, proving the first part of the proposition. To obtain (2), we observe with the notations $\phi(t) = u(e^t)e^{t/2}$, $\psi(s) = v(e^s)e^{s/2}$ that we have to check

$$\begin{aligned}&\iint H(s-t)\frac{e^{t/2}e^{s/2}}{\pi(e^t+e^s)}\phi(t)\bar{\psi}(s)dtds\\ &\quad= \iint \frac{H(s-t)}{\pi(e^{(t-s)/2}+e^{-(t-s)/2})}\phi(t)\bar{\psi}(s)dtds = \langle R*\phi, \psi\rangle_{L^2(\mathbb{R})},\end{aligned}$$

with

$$R(t) = \frac{H(t)}{2\pi\cosh(t/2)}, \quad \widehat{R}(\tau) = \frac{1}{2\pi}\int_0^{+\infty}\operatorname{sech}(t/2)e^{-2i\pi t\tau}dt,$$

so that[4]

$$|\widehat{R}(\tau)| \le \widehat{R}(0) = \frac{1}{2\pi}\int_0^{+\infty}\operatorname{sech}(t/2)dt = \frac{1}{2},$$

yielding the sought result. ∎

A.6 On some auxiliary functions

A.6.1 A preliminary quadrature

Lemma A.6.1. *We have*

$$\int_0^{\pi/2}(\csc s - \operatorname{csch} s)ds = \int_{\pi/2}^{+\infty}\operatorname{csch} s ds = \operatorname{Log}\left(\coth\frac{\pi}{4}\right),$$

with $\csc s = 1/\sin s, \operatorname{csch} s = 1/\sinh s$.

Proof. Note that the function $[0, \pi/2] \ni s \mapsto \frac{\sinh s - \sin s}{\sinh s \sin s}$, is continuous. Moreover, we have

$$\int\frac{ds}{\sin s} = \frac{1}{2}\operatorname{Log}\left(\frac{1-\cos s}{1+\cos s}\right) \quad\text{and}\quad \int\frac{ds}{\sinh s} = \frac{1}{2}\operatorname{Log}\left(\frac{\cosh s - 1}{\cosh s + 1}\right),$$

[4] We recall that $\frac{d}{ds}\arctan(\sinh s) = \operatorname{sech} s$.

so that

$$\int_{\varepsilon}^{\pi/2} (\csc s - \operatorname{csch} s) ds$$

$$= \frac{1}{2}\left[\operatorname{Log}\left(\frac{1-\cos s}{1+\cos s}\right)\right]_{\varepsilon}^{\pi/2} - \left[\frac{1}{2}\operatorname{Log}\left(\frac{\cosh s - 1}{\cosh s + 1}\right)\right]_{\varepsilon}^{\pi/2}$$

$$= \frac{1}{2}\operatorname{Log}\underbrace{\left(\left(\frac{1+\cos\varepsilon}{1-\cos\varepsilon}\right)\left(\frac{\cosh\varepsilon - 1}{\cosh\varepsilon + 1}\right)\right)}_{=\frac{(2+O(\varepsilon^2))(\frac{\varepsilon^2}{2}+O(\varepsilon^4))}{(\frac{\varepsilon^2}{2}+O(\varepsilon^4))(2+O(\varepsilon^2))}\to 1 \text{ for } \varepsilon\to 0} + \frac{1}{2}\operatorname{Log}\left(\frac{\cosh\frac{\pi}{2}+1}{\cosh\frac{\pi}{2}-1}\right),$$

so that we obtain

$$\int_0^{\pi/2} (\csc s - \operatorname{csch} s) ds = \frac{1}{2}\operatorname{Log}\left(\frac{e^{\pi/2}+e^{-\pi/2}+2}{e^{\pi/2}+e^{-\pi/2}-2}\right) = \operatorname{Log}\frac{\cosh(\pi/4)}{\sinh(\pi/4)},$$

which is the first result. Also, we have $\int_{\pi/2}^{+\infty} \operatorname{csch} s ds = \frac{1}{2}\operatorname{Log}(\frac{\cosh(\pi/2)+1}{\cosh(\pi/2)-1})$, yielding the second result. ∎

A.6.2 Study of the function ρ_σ

We study in this section the real-valued Schwartz function ρ_σ given in (5.2.10). Using the notations

$$\omega = 2\pi\tau, \quad \kappa = 2\pi\sigma, \quad \nu = \sqrt{\kappa/\omega}, \tag{A.6.1}$$

we have

$$\rho_\sigma(\tau) = \int_{\mathbb{R}} \frac{s}{\sinh s} e^{2i\omega(s-\nu^2\tanh s)} ds = \int_{\mathbb{R}} \frac{s}{\sinh s}\cos\big(2\omega(s-\nu^2\tanh s)\big) ds.$$

Defining the holomorphic function F by

$$F(z) = \frac{z}{\sinh z} e^{2i\omega(z-\nu^2\tanh z)}, \tag{A.6.2}$$

we see that F has simple poles at $i\pi\mathbb{Z}^*$ and essential singularities at $i\pi(\frac{1}{2}+\mathbb{Z})$. We already know that the function ρ_σ belongs to the Schwartz space, but we want to prove a more precise exponential decay. We start with the calculation of

$$\int_{\mathbb{R}+i\frac{\pi}{4}} F(z)dz = \int_{\mathbb{R}} \frac{t+i\frac{\pi}{4}}{\sinh(t+i\frac{\pi}{4})} e^{2i\omega(t+i\frac{\pi}{4}-\nu^2\tanh(t+i\frac{\pi}{4}))} dt$$

$$= e^{-\pi\omega/2} 2\sqrt{2}\int_{\mathbb{R}} \frac{t+i\frac{\pi}{4}}{(1+i)e^t-(1-i)e^{-t}} e^{2i\omega t} e^{-2i\omega\nu^2\frac{e^t(1+i)-e^{-t}(1-i)}{e^t(1+i)+e^{-t}(1-i)}} dt$$

$$= e^{-\pi\omega/2}\sqrt{2}\int_{\mathbb{R}} \frac{t+i\frac{\pi}{4}}{\sinh t + i\cosh t} e^{2i\omega t} e^{-2i\omega\nu^2\frac{e^t(1+i)-e^{-t}(1-i)}{e^t(1+i)+e^{-t}(1-i)}} dt.$$

We have

$$\operatorname{Im}\left(\frac{e^t(1+i)-e^{-t}(1-i)}{e^t(1+i)+e^{-t}(1-i)}\right)=\operatorname{Im}\left(\frac{\sinh t+i\cosh t}{\cosh t+i\sinh t}\right)=\frac{1}{\cosh^2 t+\sinh^2 t},$$

so that

$$\begin{aligned}\left|\int_{\mathbb{R}+i\frac{\pi}{4}}F(z)dz\right|&\leq e^{-\frac{\pi\omega}{2}}\sqrt{2}\int_{\mathbb{R}}\frac{\sqrt{t^2+(\frac{\pi}{4})^2}}{\sqrt{\sinh^2 t+\cosh^2 t}}e^{\frac{2\omega\nu^2}{\sinh^2 t+\cosh^2 t}}dt\\&=e^{-\frac{\pi\omega}{2}}\sqrt{2}e^{2\kappa}\int_{\mathbb{R}}\frac{\sqrt{t^2+(\frac{\pi}{4})^2}}{\sqrt{\sinh^2 t+\cosh^2 t}}dt\leq 6e^{-\frac{\pi\omega}{2}}e^{2\kappa}.\end{aligned}\tag{A.6.3}$$

Claim A.6.2. We have

$$\lim_{R\to+\infty}\oint_{[R,R+i\pi/4]}F(z)dz=\lim_{R\to+\infty}\oint_{[-R,-R+i\pi/4]}F(z)dz=0.$$

Proof of Claim A.6.2. We note first that

$$\oint_{[-R,-R+i\pi/4]}F(z)dz=-\overline{\oint_{[R,R+i\pi/4]}F(z)dz},$$

so that it is enough to prove one equality. Indeed, for $R>0$, we have

$$\oint_{[R,R+i\pi/4]}F(z)dz=\int_0^{\pi/4}\frac{R+it}{\sinh(R+it)}e^{2i\omega(R+it-\nu^2\tanh(R+it))}i\,dt,$$

so that

$$\begin{aligned}&\left|\oint_{[R,R+i\pi/4]}F(z)dz\right|\\&\quad\leq\int_0^{\pi/4}\frac{2\sqrt{R^2+t^2}}{|e^{R+it}||1-e^{-2R-2it}|}e^{-2\omega t}e^{2\kappa\operatorname{Im}(\tanh(R+it))}dt\\&\quad\leq e^{-R}\frac{\sqrt{4R^2+\pi^2/4}}{1-e^{-2R}}\int_0^{\pi/4}e^{2\kappa\left|\frac{1-e^{-2R-2it}}{1+e^{-2R-2it}}\right|}dt\\&\quad\leq e^{-R}\frac{\sqrt{4R^2+\pi^2/4}}{1-e^{-2R}}\frac{\pi}{4}e^{\frac{4\kappa}{(1-e^{-2R})}},\end{aligned}$$

proving the claim. ■

Lemma A.6.3. *We have for $\tau>0,\sigma\geq 0$, ρ_σ given in (5.2.10),*

$$|\rho_\sigma(\tau)|\leq 6e^{-\pi^2\tau}e^{4\pi\sigma}.\tag{A.6.4}$$

Proof. We have, with the notations (A.6.1), F given in (A.6.2) and $\gamma_R = [-R, -R + i\frac{\pi}{4}] \cup [-R + i\frac{\pi}{4}, R + i\frac{\pi}{4}] \cup [R + i\frac{\pi}{4}, R]$,

$$\rho_\sigma(\tau) = \lim_{R\to+\infty} \int_{[-R,R]} F(s)ds = \lim_{R\to+\infty} \left(\oint_{\gamma_R} F(z)dz \right) \underbrace{=}_{\text{Claim (A.6.2)}} \oint_{\mathbb{R}+\frac{i\pi}{4}} F(z)dz,$$

so that (A.6.3) implies the lemma. ∎

A.6.3 On the function ψ_ν

Let $\nu \in (0, 1)$ be given. We study first the function ϕ_ν defined on $[0, \pi/2)$ by

$$\phi_\nu(s) = s - \nu^2 \tan s, \quad \text{so that} \quad \phi_\nu'(s) = 1 - \nu^2(1 + \tan^2 s) = \frac{\cos^2 s - \nu^2}{\cos^2 s},$$

so that

$$\begin{array}{c|ccccccccc} s & 0 & & s_\nu & & t_\nu & & \frac{\pi}{2} \\ \hline \phi_\nu'(s) & 1-\nu^2 & + & 0 & - & & - & \\ \phi_\nu(s) & 0 & \nearrow & \phi_\nu(s_\nu) & \searrow & 0 & \searrow & -\infty \end{array} \tag{A.6.5}$$

We have

$$\begin{cases} s_\nu = \arccos \nu = \frac{\pi}{2} - \nu + O(\nu^3), \\ \phi_\nu(s_\nu) = \arccos \nu - \nu\sqrt{1-\nu^2} = \frac{\pi}{2} - 2\nu + O(\nu^3), \end{cases} \quad \text{for } \nu \to 0. \tag{A.6.6}$$

The function ϕ_ν is concave on $(0, \pi/2)$ since we have there

$$\phi_\nu''(s) = -\nu^2(-2)(\cos s)^{-3}(-\sin s) = -\nu^2 2(\cos s)^{-3} \sin s \le 0.$$

We have defined in (5.2.45)

$$\psi_\nu(\omega) = \frac{e^{-\pi\omega}}{2\pi} \int_0^{\pi/2} \frac{e^{2\omega\phi_\nu(s)} - 1}{\sin s} ds. \tag{A.6.7}$$

Let us start with an elementary lemma.

Lemma A.6.4. *Let $\lambda > 0$ be given. Defining*

$$J(\lambda) = e^{-\lambda} \int_0^\lambda \frac{e^\sigma - 1}{\sigma} d\sigma,$$

we have

$$J(\lambda) = \lambda^{-1} + O(\lambda^{-2}), \quad \lambda \to +\infty, \tag{A.6.8}$$

$$\forall \lambda > 0, \quad J(\lambda) \ge \lambda^{-1} - \lambda^{-2}. \tag{A.6.9}$$

Proof. Indeed, we have for $\lambda > 0$,

$$\lambda J(\lambda) = \lambda e^{-\lambda} \sum_{k\geq 1} \int_0^{\lambda} \frac{\sigma^{k-1}}{k!} d\sigma = \lambda e^{-\lambda} \sum_{k\geq 1} \frac{\lambda^k}{k!k} = e^{-\lambda} \sum_{k\geq 1} \frac{\lambda^{k+1}}{(k+1)!} \frac{k+1}{k}$$
$$= e^{-\lambda} \sum_{k\geq 1} \frac{\lambda^{k+1}}{(k+1)!} + e^{-\lambda} \sum_{k\geq 1} \frac{\lambda^{k+1}}{(k+1)!} \frac{1}{k}$$
$$= e^{-\lambda}(e^{\lambda} - 1 - \lambda) + \lambda^{-1} \underbrace{\left(e^{-\lambda} \sum_{k\geq 1} \frac{\lambda^{k+2}}{(k+1)!} \frac{1}{k} \right)}_{R(\lambda)}, \tag{A.6.10}$$

with

$$0 \leq R(\lambda) \leq e^{-\lambda} \sum_{k\geq 1} \frac{\lambda^{k+2}}{(k+2)!} \frac{k+2}{k} \leq e^{-\lambda} \left(e^{\lambda} - 1 - \lambda - \frac{\lambda^2}{2} \right) \times 3 = O(1), \tag{A.6.11}$$

so that

$$\lambda J(\lambda) = e^{-\lambda}(e^{\lambda} - 1 - \lambda) + \lambda^{-1} O(1) = 1 + \lambda^{-1} O(1) - (1+\lambda)e^{-\lambda} = 1 + \lambda^{-1} O(1),$$

proving (A.6.8). Note also that (A.6.10), (A.6.11) imply, since $R(\lambda) \geq 0$,

$$\lambda J(\lambda) \geq 1 - e^{-\lambda}(1 + \lambda),$$

so that $J(\lambda) \geq \lambda^{-1} - e^{-\lambda}(1 + \lambda^{-1})$, and thus[5] the sought result (A.6.9). ∎

Remark A.6.5. Considering now the function φ_0 defined by

$$\varphi_0(\omega) = \frac{e^{-\pi\omega}}{2\pi} \int_0^{\pi/2} \frac{e^{2\omega s} - 1}{\sin s} ds,$$

we find that, for $\omega \geq 0$, using Lemma A.6.4,

$$\varphi_0(\omega) \geq \frac{e^{-\pi\omega}}{2\pi} \int_0^{\pi/2} \frac{e^{2\omega s} - 1}{s} ds = \frac{e^{-\pi\omega}}{2\pi} \int_0^{\pi\omega} \frac{e^{\sigma} - 1}{\sigma} d\sigma = \frac{1}{2\pi} J(\pi\omega),$$

so that

$$\varphi_0(\omega) \geq \frac{1}{2\pi^2\omega} - \frac{1}{2\pi^3\omega^2}.$$

It is our goal now to prove a minoration of the same flavour for the function (A.6.7) defined above.

[5] We leave for the reader to check that for $\lambda > 0$, $e^{-\lambda}(1 + \lambda^{-1}) \leq \lambda^{-2}$, which boils down to study $q(\lambda) = e^{-\lambda}(\lambda^2 + \lambda)$ reaching its maximum for $\lambda \in \mathbb{R}_+$, at $\lambda_0 = (1 + \sqrt{5})/2$ with $q(\lambda_0) \approx 0.84 < 1$.

Assuming $\nu \in (0, 1/2)$, we have $\frac{\pi}{3} < s_\nu < t_\nu < \frac{\pi}{2}$ (s_ν, t_ν are defined in (A.6.5), ψ_ν in (A.6.7)),

$$\begin{aligned}
2\pi e^{\pi\omega}\psi_\nu(\omega) &= \int_0^{t_\nu} \frac{e^{2\omega\phi_\nu(s)} - 1}{\sin s} ds + \int_{t_\nu}^{\pi/2} \frac{e^{2\omega\phi_\nu(s)} - 1}{\sin s} ds \\
&\geq \underbrace{\int_0^{t_\nu} \frac{e^{2\omega\phi_\nu(s)} - 1}{\sin s} ds}_{\text{on } (0,t_\nu),\ \phi_\nu(s) \geq 0} - \int_{t_\nu}^{\pi/2} \frac{ds}{\sin s} \\
&\geq \int_0^{s_\nu} \frac{e^{2\omega\phi_\nu(s)} - 1}{\sin s} ds - \int_{\pi/3}^{\pi/2} \frac{ds}{\sin s} \\
&= \underbrace{\int_0^{s_\nu} \frac{e^{2\omega\phi_\nu(s)} - 1}{\sin s} ds}_{\substack{\text{on } (0,s_\nu) \\ \phi_\nu(s)>0 \text{ and } \phi_\nu'(s)>0}} - \frac{\ln 3}{2}.
\end{aligned} \tag{A.6.12}$$

Claim A.6.6. For $s \in (0, \pi/2)$, we have $\phi_\nu(s) \geq \phi_\nu'(s)\sin s$. Moreover, for $s \in (0, s_\nu)$, we have $\frac{1}{\sin s} \geq \frac{\phi_\nu'(s)}{\phi_\nu(s)}$.

Proof of Claim A.6.6. Indeed, we have

$$\begin{aligned}
\phi_\nu(s) - \phi_\nu'(s)\sin s &= s - \nu^2 \tan s - \sin s + \nu^2(1 + \tan^2 s)\sin s \\
&= \nu^2(\sin s + \sin s \tan^2 s - \tan s) + s - \sin s \\
&= \nu^2\left(\frac{\sin s}{\cos^2 s} - \frac{\sin s}{\cos s}\right) + s - \sin s \\
&= \frac{\nu^2 \sin s}{\cos^2 s}(1 - \cos s) + s - \sin s \geq 0, \quad \text{for } s \in (0, \pi/2).
\end{aligned}$$

The last part of the claim follows from the first part and the fact that $\sin s$ and $\phi_\nu(s)$ are both positive on $(0, s_\nu)$. ∎

Going back now to (A.6.12), we obtain that for $\nu \in (0, 1/2)$ and $\omega > 0$, we have

$$\begin{aligned}
2\pi e^{\pi\omega}\psi_\nu(\omega) &\geq \int_0^{s_\nu} \frac{e^{2\omega\phi_\nu(s)} - 1}{\phi_\nu(s)} \phi_\nu'(s) ds - \frac{\ln 3}{2} \\
&= \int_0^{2\omega\phi_\nu(s_\nu)} \frac{e^\sigma - 1}{\sigma} d\sigma - \frac{\ln 3}{2} = e^{2\omega\phi_\nu(s_\nu)} J(2\omega\phi_\nu(s_\nu)) - \frac{\ln 3}{2},
\end{aligned}$$

so that, using (A.6.9), we get

$$\psi_\nu(\omega) \geq \frac{1}{2\pi} e^{-\pi\omega} e^{2\omega\phi_\nu(s_\nu)} \left(\frac{1}{2\omega\phi_\nu(s_\nu)} - \frac{1}{(2\omega\phi_\nu(s_\nu))^2}\right) - \frac{\ln 3}{2}\frac{1}{2\pi}e^{-\pi\omega},$$

and since $\phi_\nu(s_\nu) = \frac{\pi}{2} - \varepsilon_\nu$, with $\varepsilon_\nu \in (0, \pi/2)$, we find also that ε_ν is a concave function[6] of $\nu \in (0,1)$ and

$$\frac{\pi\nu}{2} \le \varepsilon_\nu \le 2\nu$$

so that

$$2\phi_\nu(s_\nu) = \pi - 2\varepsilon_\nu \in [\pi - 4\nu, \pi - \pi\nu],$$

so that for $\nu \in (0, 1/2]$, we have[7] (assuming $\omega > 0$),

$$\begin{aligned}\psi_\nu(\omega) &\ge \frac{1}{2\pi} e^{-\pi\omega} e^{\omega(\pi - 2\varepsilon_\nu)} \left(\frac{1}{\omega(\pi - 2\varepsilon_\nu)} - \frac{1}{(\omega(\pi - 2\varepsilon_\nu))^2} \right) - \frac{\ln 3}{2} \frac{1}{2\pi} e^{-\pi\omega}, \\ &\ge \frac{1}{2\pi} e^{-4\nu\omega} \left(\frac{1}{\omega\pi} - \frac{1}{\omega^2(\pi-2)^2} \right) - \frac{\ln 3}{2} \frac{1}{2\pi} e^{-\pi\omega},\end{aligned}$$

We recall the notations (A.6.1), so that $\nu = \sqrt{\kappa/\omega}$, i.e., $\nu\omega = \sqrt{\kappa\omega}$ and we get

$$\forall \omega > 0, \quad \psi_\nu(\omega) \ge \frac{1}{2\pi} e^{-4\sqrt{\kappa\omega}} \left(\frac{1}{\pi\omega} - \frac{1}{\omega^2} \right) - \frac{\ln 3}{2} \frac{1}{2\pi} e^{-\pi\omega}, \quad \nu = \sqrt{\kappa/\omega}. \tag{A.6.13}$$

A.6.4 An explicit expression for a_{11}

According to (5.2.22), we have

$$a_{11}(\tau, \sigma) = \frac{1}{2} + \frac{1}{2\pi} \int_0^{+\infty} \frac{\sin(2\pi t\tau - 4\pi\sigma \tanh(t/2))}{\sinh(t/2)} dt. \tag{A.6.14}$$

We have used in Section 5.2 the equivalent expression $a_{11}(\tau, \sigma) = \frac{1}{2} + \widehat{T_\sigma}(\tau)$, where T_σ is defined in (5.2.9) and we were able to prove the estimate in Lemma 5.2.2. It turns out that (A.6.4) is not optimal, and it is interesting to give an "explicit" expression for a_{11} as displayed in [55]. Using the notations (A.6.1), we can write (A.6.14) as

$$\begin{aligned}a_{11}(\tau, \sigma) &= \frac{1}{2} + \frac{1}{4\pi} \int_{\mathbb{R}} \operatorname{Im} \frac{\exp i(\omega t - 2\kappa \tanh(t/2))}{\sinh(t/2)} dt \\ &= \frac{1}{2} + \operatorname{Im} \lim_{R \to +\infty} \frac{1}{2\pi} \int_{[-R,R]} \frac{\exp 2i(\omega s - \kappa \tanh s)}{\sinh s} ds.\end{aligned} \tag{A.6.15}$$

[6]We have from (A.6.6),

$$\varepsilon_\nu = \frac{\pi}{2} - \arccos\nu + \nu\sqrt{1-\nu^2}, \quad \frac{d\varepsilon_\nu}{d\nu} = 2\sqrt{1-\nu^2}, \quad \frac{d^2\varepsilon_\nu}{d\nu^2} = -2\nu/\sqrt{1-\nu^2} < 0,$$

so that the concavity gives $\frac{\pi}{2}\nu \le \varepsilon_\nu \le 2\nu$.

[7]We know that $\omega(\pi - 2\varepsilon_\nu) \ge \omega(\pi - 4\nu) \ge \omega(\pi - 2)$ so that to ensure $\omega(\pi - 2\varepsilon_\nu) \ge 4$, it suffices to assume $\omega \ge 4/(\pi - 2)$.

Defining the holomorphic function G by

$$G(z) = \frac{\exp 2i(\omega z - \kappa \tanh z)}{2\pi \sinh z}, \tag{A.6.16}$$

we see that G has simple poles at $i\pi\mathbb{Z}$ and essential singularities at $i\pi(\frac{1}{2}+\mathbb{Z})$. For $R \in \mathbb{R}_+ \backslash \frac{\pi}{2}\mathbb{Z}$, $\varepsilon \in (0, \pi/2)$, we have

$$\begin{aligned}
&\oint_{[-R,-\varepsilon]\cup[\varepsilon,R]} G(z)dz + \oint_{\substack{\gamma_\varepsilon^- \\ \gamma_\varepsilon^-(\theta)=\varepsilon e^{i\theta} \\ -\pi\le t\le 0}} G(z)dz + \oint_{\substack{\gamma_R^+ \\ \gamma_R^+(\theta)=Re^{i\theta} \\ 0\le t\le \pi}} G(z)dz \\
&\quad = 2i\pi \sum_{\substack{k\in\mathbb{N} \\ k\pi<2R}} \mathtt{Res}(G, ik\pi/2).
\end{aligned} \tag{A.6.17}$$

Claim A.6.7. We have

$$\lim_{\varepsilon\to 0} \oint_{\gamma_\varepsilon^-} G(z)dz = \frac{i}{2}.$$

Proof. Indeed, we have

$$\begin{aligned}
&\int_{-\pi}^{0} \frac{\exp 2i(\omega\varepsilon e^{i\theta} - \kappa \tanh(\varepsilon e^{i\theta}))}{2\pi \sinh(\varepsilon e^{i\theta})} i\varepsilon e^{i\theta} d\theta \\
&\quad = \frac{i}{2\pi} \int_{-\pi}^{0} \frac{e^{2i\omega\varepsilon e^{i\theta}} \varepsilon e^{i\theta}}{\sinh(\varepsilon e^{i\theta})} \exp(-2i\kappa \tanh(\varepsilon e^{i\theta}))d\theta,
\end{aligned}$$

and since the function $z \mapsto \frac{ze^{2i\omega z}}{\sinh z} e^{-2i\kappa \tanh z}$ is holomorphic near 0 with value 1 at 0, we get the result of the claim. ∎

Lemma A.6.8. *We have*

$$\lim_{\mathbb{N}\ni m\to+\infty} \operatorname{Im}\left(\oint_{\gamma^+_{\frac{\pi}{4}+m\frac{\pi}{2}}} G(z)dz \right) = 0.$$

Proof. Indeed, we have with $R = \frac{\pi}{4} + m\frac{\pi}{2}$,

$$\begin{aligned}
&\operatorname{Im} \int_0^{\pi} \frac{\exp 2i(\omega Re^{i\theta} - \kappa \tanh(Re^{i\theta}))}{2\pi \sinh(Re^{i\theta})} iRe^{i\theta} d\theta \\
&= \frac{R}{\pi} \operatorname{Re} \int_0^{\pi} \frac{e^{2i\omega R\cos\theta} e^{-2R\omega \sin\theta} e^{i\theta}}{1 - e^{-2Re^{i\theta}}} e^{-Re^{i\theta}} \exp(-2i\kappa \tanh(Re^{i\theta}))d\theta \\
&= \frac{2R}{\pi} \int_0^{\pi/2} \operatorname{Re}\left\{ \frac{e^{2i\omega R\cos\theta} e^{-2R\omega \sin\theta} e^{i\theta}}{1 - e^{-2Re^{i\theta}}} e^{-Re^{i\theta}} \exp(-2i\kappa \tanh(Re^{i\theta})) \right\} d\theta,
\end{aligned}$$

so that

$$\operatorname{Im}\left(\oint_{\gamma^{+}_{\frac{\pi}{4}+m\frac{\pi}{2}}} G(z)dz\right)$$
$$= \frac{2R}{\pi}\int_0^{\pi/2} e^{-R\cos\theta}e^{-2R\omega\sin\theta}$$
$$\times \operatorname{Re}\left\{\frac{e^{2i\omega R\cos\theta}e^{i\theta}}{1-e^{-2Re^{i\theta}}}e^{-iR\sin\theta}\exp\left(-2i\kappa\tanh(Re^{i\theta})\right)\right\}d\theta. \quad \text{(A.6.18)}$$

We have also

$$\tanh(Re^{i\theta}) = \frac{1-e^{-2Re^{i\theta}}}{1+e^{-2Re^{i\theta}}}. \quad \text{(A.6.19)}$$

Claim A.6.9. Defining for $m \in \mathbb{N}, \theta \in [0,\pi]$, $g_m(\theta) = 1 - e^{-(\frac{\pi}{2}+m\pi)e^{i\theta}}$, we find that

$$\inf_{\substack{\theta\in[0,\pi]\\ m\in\mathbb{N}}} |g_m(\theta)| = \beta_0 > 0, \qquad \inf_{\substack{\theta\in[0,\pi]\\ m\in\mathbb{N}}} |2-g_m(\theta)| = \beta_1 > 0.$$

Proof of Claim A.6.9. If it were not the case, we could find sequences $\theta_l \in [0,\pi], m_l \in \mathbb{N}$ such that

$$\lim_{l\to+\infty} e^{-(\frac{\pi}{2}+m_l\pi)e^{i\theta_l}} = 1. \quad \text{(A.6.20)}$$

Taking the logarithm of the modulus of both sides, we would get

$$\lim_{l\to+\infty}\left(\frac{\pi}{2}+m_l\pi\right)\cos\theta_l = 0,$$

i.e.,

$$\cos\theta_l = \frac{\varepsilon_l}{\frac{\pi}{2}+m_l\pi}, \qquad \lim_{l\to+\infty}\varepsilon_l = 0.$$

Going back to (A.6.20), we find then

$$\lim_{l\to+\infty} e^{-i(\frac{\pi}{2}+m_l\pi)\sin\theta_l} = 1,$$

i.e., since $\sin\theta_l \geq 0$,

$$\lim_{l\to+\infty}\exp -i\left\{\left(\frac{\pi}{2}+m_l\pi\right)\left(1-\frac{\varepsilon_l^2}{(\frac{\pi}{2}+m_l\pi)^2}\right)^{1/2}\right\} = 1,$$

implying $\lim_{l\to+\infty} e^{-i(\frac{\pi}{2}+m_l\pi)} = 1$, which is not possible since

$$e^{-i(\frac{\pi}{2}+m_l\pi)} = -i(-1)^{m_l} \in \{\pm i\},$$

proving the first inequality of the claim. The second inequality follows from the same *reductio ad absurdum*, starting with

$$\lim_{l \to +\infty} e^{-(\frac{\pi}{2}+m_l\pi)e^{i\theta_l}} = -1,$$

ending-up with an impossibility since $-1 \notin \{\pm i\}$. ■

As a consequence of Claim A.6.9 and (A.6.19), we obtain for $R = \frac{\pi}{4} + m\frac{\pi}{2}$, $\theta \in (0, \pi)$,

$$|\tanh(Re^{i\theta})| \le \frac{2}{\beta_1}.$$

Formula (A.6.18) gives then

$$\left| \operatorname{Im}\left(\oint_{\gamma^+_{\frac{\pi}{4}+m\frac{\pi}{2}}} G(z)dz \right) \right| \le \frac{2R}{\pi} \int_0^{\pi/2} e^{-R\cos\theta} e^{-2R\omega\sin\theta} \frac{1}{\beta_0} \exp\left(4\kappa/\beta_1\right) d\theta,$$

where for $\omega > 0$, the right-hand side goes to zero when R goes to $+\infty$, completing the proof of Lemma A.6.8. ■

Lemma A.6.10. *With G defined in (A.6.16), we have*

$$2\pi \sum_{k\in\mathbb{N}} \operatorname{Res}(G, ik\pi/2) = \frac{1}{1+e^{-2\pi\omega}} + \frac{e^{-\pi\omega}}{i(1+e^{-2\pi\omega})} \operatorname{Res}\left(\frac{e^{2i\omega z - 2i\kappa\coth z}}{\cosh z}, 0 \right). \tag{A.6.21}$$

Proof. We have $\operatorname{Res}(G, ik\pi/2) = \operatorname{Res}(G_k, 0)$ and with $k = 2l$,

$$G_k(z) = \frac{\exp 2i(\omega(z + \frac{ik\pi}{2}) - \kappa\tanh(z + \frac{ik\pi}{2}))}{2\pi\sinh(z + \frac{ik\pi}{2})} = \frac{e^{-2l\pi\omega} e^{2i\omega z} e^{-2i\kappa\tanh z}}{2\pi(-1)^l \sinh z},$$

so that

$$\operatorname{Res}(G_{2l}, 0) = \frac{(-1)^l e^{-2l\pi\omega}}{2\pi},$$

whereas for $k = 2l + 1$, we have

$$\begin{aligned} G_{2l+1}(z) &= \frac{\exp 2i(\omega(z + il\pi + \frac{i\pi}{2}) - \kappa\tanh(z + il\pi + \frac{i\pi}{2}))}{2\pi\sinh(z + il\pi + \frac{i\pi}{2})} \\ &= \frac{e^{-(2l+1)\pi\omega} e^{2i\omega z} e^{-2i\kappa\coth z}}{2\pi(-1)^l i\cosh z}, \end{aligned}$$

so that

$$\operatorname{Res}(G_{2l+1}, 0) = \frac{(-1)^l e^{-(2l+1)\pi\omega}}{2\pi i} \operatorname{Res}\left(\frac{e^{2i\omega z - 2i\kappa\coth z}}{\cosh z}, 0 \right),$$

yielding

$$
\begin{aligned}
&2\pi \sum_{k\in\mathbb{N}} \mathrm{Res}(G, ik\pi/2) \\
&= \sum_{l\in\mathbb{N}} (-1)^l e^{-2l\pi\omega} + \sum_{l\in\mathbb{N}} \frac{(-1)^l e^{-(2l+1)\pi\omega}}{i} \mathrm{Res}\left(\frac{e^{2i\omega z - 2i\kappa \coth z}}{\cosh z}, 0\right), \\
&= \frac{1}{1+e^{-2\pi\omega}} + \frac{e^{-\pi\omega}}{i(1+e^{-2\pi\omega})} \mathrm{Res}\left(\frac{e^{2i\omega z - 2i\kappa \coth z}}{\cosh z}, 0\right),
\end{aligned}
$$

concluding the proof of the lemma. ∎

Proposition A.6.11. *Using the notations* (A.6.1)*, with a_{11} defined in* (A.6.14) *(see also* (A.6.15)*), we have for $\tau > 0, \sigma \geq 0$,*

$$
a_{11}(\tau,\sigma) = \frac{1}{1+e^{-2\pi\omega}} + \frac{e^{-\pi\omega}}{1+e^{-2\pi\omega}} \operatorname{Im}\left\{\mathrm{Res}\left(\frac{e^{2i(\omega z - \kappa \coth z)}}{\cosh z}, 0\right)\right\}. \tag{A.6.22}
$$

Proof. Taking the imaginary part of both sides in (A.6.17), and letting $R \to +\infty$, $\varepsilon \to 0_+$, we get, using (A.6.21), (A.6.15), Claim A.6.7,

$$
a_{11} - \frac{1}{2} + \operatorname{Im}\frac{i}{2} = \operatorname{Im} i\left(\frac{1}{1+e^{-2\pi\omega}} + \frac{e^{-\pi\omega}}{i(1+e^{-2\pi\omega})}\mathrm{Res}\left(\frac{e^{2i\omega z - 2i\kappa \coth z}}{\cosh z}, 0\right)\right),
$$

which is (A.6.22). ∎

Remark A.6.12. In particular, when $\sigma = 0$, we find for $\tau > 0$

$$
1 - a_{11}(\tau, 0) = \frac{e^{-4\pi^2\tau}}{1+e^{-4\pi^2\tau}},
$$

and since (5.2.24) implies that

$$
\begin{aligned}
2\pi \operatorname{Re} a_{12}(\tau,0) &= \int_0^{+\infty} \frac{\sin(4\pi t\tau)}{\cosh t} dt = \operatorname{Im}\langle e^{i4\pi\tau t} H(t), \operatorname{sech} t\rangle_{\mathscr{S}'(\mathbb{R}_t),\mathscr{S}(\mathbb{R}_t)} \\
&= \operatorname{Im} \frac{1}{4i\pi\tau}\left\langle \frac{d}{dt}\{e^{i4\pi\tau t}\} H(t), \operatorname{sech} t\right\rangle \\
&= \operatorname{Im} \frac{1}{4i\pi\tau}\left(\left\langle \frac{d}{dt}\{e^{i4\pi\tau t} H(t)\}, \operatorname{sech} t\right\rangle - \langle \delta_0, \operatorname{sech}\rangle\right) \\
&= \frac{1}{4\pi\tau} - \operatorname{Im}\frac{1}{4i\pi\tau}\langle e^{i4\pi\tau t} H(t), \operatorname{sech}'(t)\rangle \\
&= \frac{1}{4\pi\tau} + O(\tau^{-3}), \quad \tau \to +\infty,
\end{aligned}
$$

we readily find that

$$
\operatorname{Re} a_{12}(\tau, 0) \gg 1 - a_{11}(\tau, 0), \quad \tau \to +\infty,
$$

providing another proof of Theorem 5.2.4 in the case $\sigma = 0$.

Remark A.6.13. Equation (5.2.41) gives also $\operatorname{Im} a_{12}(\tau,\sigma) = \frac{e^{-2\pi^2\tau}}{2} a_{11}(\tau,\sigma)$, where (5.2.22) gives, using the notations (A.6.1),

$$\begin{aligned}\operatorname{Im} a_{12}(\tau,\sigma) &= \frac{1}{4\pi}\int_0^{+\infty} \frac{\cos(t\omega - 2\kappa\coth(t/2))}{\cosh(t/2)}dt \\ &= \frac{1}{2\pi}\int_0^{+\infty}\frac{\cos(2(t\omega-\kappa\coth t))}{\cosh t}dt = \frac{1}{4\pi}\int_{\mathbb{R}}\frac{\cos(2(t\omega-\kappa\coth t))}{\cosh t}dt.\end{aligned}$$

With G given by (A.6.16), we note that

$$\tilde{G}(z) = \frac{ie^{\pi\omega}}{2}G\left(z+\frac{i\pi}{2}\right) = \frac{\exp 2i(\omega z - \kappa\coth z)}{4\pi\cosh z},$$

a holomorphic function with simple poles at $i\pi(\frac{1}{2}+\mathbb{Z})$ and essential singularities at $i\pi\mathbb{Z}$. Following now for $\tilde{G}$ the track of G in Claim A.6.7, Lemmas A.6.8, A.6.10, and Proposition A.6.11, we get

$$\operatorname{Im} a_{12}(\tau,\sigma) = \lim_{\substack{m\to+\infty\\ \varepsilon\to 0_+}} \operatorname{Re}\oint_{[-R_m,-\varepsilon]\cup[\varepsilon,R_m]}\tilde{G}(z)dz,\quad R_m = \frac{\pi}{4}+m\frac{\pi}{2}, \tag{A.6.23}$$

and we have also

$$\begin{aligned}&\oint_{[-R_m,-\varepsilon]\cup[\varepsilon,R_m]}\tilde{G}(z)dz - \oint_{\substack{\gamma_\varepsilon^+\\ \gamma_\varepsilon^+(\theta)=\varepsilon e^{i\theta}\\ 0\le t\le\pi}}\tilde{G}(z)dz + \oint_{\substack{\gamma_{R_m}^+\\ \gamma_{R_m}^+(\theta)=R_m e^{i\theta}\\ 0\le t\le\pi}}\tilde{G}(z)dz \\ &= 2i\pi\sum_{1\le k\le m}\mathtt{Res}(\tilde{G}, ik\pi/2) = -\pi e^{\pi\omega}\sum_{1\le k\le m}\mathtt{Res}\left(G\left(\zeta+\frac{ik\pi}{2}+\frac{i\pi}{2}\right),0\right) \\ &= -\pi e^{\pi\omega}\sum_{2\le l\le m+1}\mathtt{Res}\left(G\left(\zeta+\frac{il\pi}{2}\right),0\right).\end{aligned} \tag{A.6.24}$$

Claim A.6.14. We have $\lim_{\varepsilon\to 0}\oint_{\gamma_\varepsilon^+}\tilde{G}(z)dz = 0$.

Proof. Indeed, we have $-2i\kappa\coth\varepsilon e^{i\theta} = -2i\kappa\frac{1+e^{-2\varepsilon e^{i\theta}}}{1-e^{-2\varepsilon e^{i\theta}}}$ and for $\theta\in(0,\pi)$,

$$\begin{aligned}\operatorname{Im}\left(\frac{1+e^{-2\varepsilon e^{i\theta}}}{1-e^{-2\varepsilon e^{i\theta}}}\right) &= \operatorname{Im}\frac{(1+e^{-2\varepsilon e^{i\theta}})(1-e^{-2\varepsilon e^{-i\theta}})}{|1-e^{-2\varepsilon e^{i\theta}}|^2} = \operatorname{Im}\frac{e^{-2\varepsilon e^{i\theta}}-e^{-2\varepsilon e^{-i\theta}}}{|1-e^{-2\varepsilon e^{i\theta}}|^2} \\ &= e^{-2\varepsilon\cos\theta}\operatorname{Im}\frac{e^{-2\varepsilon i\sin\theta}-e^{2\varepsilon i\sin\theta}}{|1-e^{-2\varepsilon e^{i\theta}}|^2} \\ &= e^{-2\varepsilon\cos\theta}\operatorname{Im}\frac{-2i\sin(2\varepsilon\sin\theta)}{|1-e^{-2\varepsilon e^{i\theta}}|^2} \\ &= -2e^{-2\varepsilon\cos\theta}\frac{\sin(2\varepsilon\sin\theta)}{|1-e^{-2\varepsilon e^{i\theta}}|^2}\le 0,\quad \text{if } \varepsilon\le\pi/4,\end{aligned}$$

so that $|e^{-2i\kappa \coth \varepsilon e^{i\theta}}| \leq 1$, implying

$$4\pi \left| \oint_{\gamma_\varepsilon^+} \tilde{G}(z)dz \right| \leq \int_0^\pi \frac{|e^{i\omega\varepsilon e^{i\theta}}|}{|\cosh \varepsilon e^{i\theta}|} \varepsilon |ie^{i\theta}| d\theta = \varepsilon \int_0^\pi \frac{e^{-\omega\varepsilon \sin\theta}}{|\cosh \varepsilon e^{i\theta}|} d\theta,$$

which goes to zero when $\varepsilon \to 0_+$, concluding the proof of Claim A.6.14. ∎

Claim A.6.15. We have $\lim_{\mathbb{N}\ni m\to+\infty} \oint_{\gamma^+_{\frac{\pi}{4}+m\frac{\pi}{2}}} \tilde{G}(z)dz = 0$.

Proof. Indeed, we have, using Claim A.6.9,

$$|\coth(R_m e^{i\theta})| = \left| \frac{1+e^{-2R_m e^{i\theta}}}{1-e^{-2R_m e^{i\theta}}} \right| \leq \begin{cases} \frac{1+e^{-2R_m \cos\theta}}{\beta_0} \leq \frac{2}{\beta_0}, & \text{for } \theta \in [0, \pi/2], \\ \left| \frac{1+e^{2R_m e^{i\theta}}}{1-e^{2R_m e^{i\theta}}} \right| \leq \frac{2}{\beta_0}, & \text{for } \theta \in [\frac{\pi}{2}, \pi], \end{cases}$$

so that

$$\begin{aligned} &|\tilde{G}(R_m e^{i\theta}) i R_m e^{i\theta}| \\ &\leq R_m e^{4\kappa/\beta_0} e^{-2\omega R_m \sin\theta} \begin{cases} \left| \frac{2e^{-R_m e^{i\theta}}}{1+e^{-2R_m e^{i\theta}}} \right| \leq \frac{2e^{-R_m \cos\theta}}{\beta_1} & \text{for } \theta \in [0, \frac{\pi}{2}], \\ \left| \frac{2e^{R_m e^{i\theta}}}{1+e^{2R_m e^{i\theta}}} \right| \leq \frac{2e^{R_m \cos\theta}}{\beta_1} & \text{for } \theta \in [\frac{\pi}{2}, \pi], \end{cases} \\ &\leq \frac{2R_m}{\beta_1} e^{4\kappa/\beta_0} e^{-2\omega R_m \sin\theta - R_m |\cos\theta|}, \end{aligned}$$

which goes to 0 when m goes to $+\infty$, proving the claim. ∎

Using (A.6.21), we calculate now

$$\begin{aligned} &2\pi \sum_{l\geq 2} \mathtt{Res}(G(\zeta + \frac{il\pi}{2}), 0) \\ &= \frac{1}{1+e^{-2\pi\omega}} + \frac{e^{-\pi\omega}}{i(1+e^{-2\pi\omega})} \mathtt{Res}\left(\frac{e^{2i\omega z - 2i\kappa \coth z}}{\cosh z}, 0 \right) \\ &\quad - 2\pi \big(\mathtt{Res}(G, i\pi/2) + \mathtt{Res}(G, 0) \big) \\ &= \frac{1}{1+e^{-2\pi\omega}} + \frac{e^{-\pi\omega}}{i(1+e^{-2\pi\omega})} \mathtt{Res}\left(\frac{e^{2i\omega z - 2i\kappa \coth z}}{\cosh z}, 0 \right) \\ &\quad + ie^{-\pi\omega} \mathtt{Res}\left(\frac{e^{2i\omega z - 2i\kappa \coth z}}{\cosh z}, 0 \right) - 1 \\ &= -\frac{e^{-2\pi\omega}}{1+e^{-2\pi\omega}} - i\left(\frac{e^{-\pi\omega}}{1+e^{-2\pi\omega}} - e^{-\pi\omega} \right) \mathtt{Res}\left(\frac{e^{2i\omega z - 2i\kappa \coth z}}{\cosh z}, 0 \right) \\ &= -\frac{e^{-2\pi\omega}}{1+e^{-2\pi\omega}} + ie^{-\pi\omega} \left(\frac{e^{-2\pi\omega}}{1+e^{-2\pi\omega}} \right) \mathtt{Res}\left(\frac{e^{2i\omega z - 2i\kappa \coth z}}{\cosh z}, 0 \right), \end{aligned}$$

so that from (A.6.23), (A.6.24), Claims A.6.14 and A.6.15, we obtain

$$\begin{aligned}
&\operatorname{Im} a_{12}(\tau,\sigma)\\
&= -\pi e^{\pi\omega}\frac{1}{2\pi}\Bigg(-\frac{e^{-2\pi\omega}}{1+e^{-2\pi\omega}} - e^{-\pi\omega}\left(\frac{e^{-2\pi\omega}}{1+e^{-2\pi\omega}}\right)\\
&\qquad\qquad \times \operatorname{Im}\left\{\mathtt{Res}\left(\frac{e^{2i\omega z - 2i\kappa \coth z}}{\cosh z}, 0\right)\right\}\Bigg)\\
&= e^{\pi\omega}\frac{1}{2}\left(\frac{e^{-2\pi\omega}}{1+e^{-2\pi\omega}} + e^{-\pi\omega}\left(\frac{e^{-2\pi\omega}}{1+e^{-2\pi\omega}}\right)\operatorname{Im}\left\{\mathtt{Res}\left(\frac{e^{2i\omega z - 2i\kappa \coth z}}{\cosh z}, 0\right)\right\}\right),
\end{aligned}$$

so that

$$\operatorname{Im} a_{12}(\tau,\sigma) = \frac{e^{-\pi\omega}}{2(1+e^{-2\pi\omega})} + \frac{e^{-2\pi\omega}}{2(1+e^{-2\pi\omega})}\operatorname{Im}\left\{\mathtt{Res}\left(\frac{e^{2i\omega z - 2i\kappa \coth z}}{\cosh z}, 0\right)\right\}, \tag{A.6.25}$$

recovering (A.6.22) from (5.2.41).

N.B. We note that

$$\mathtt{Res}\left(\frac{e^{2i\omega z - 2i\kappa \coth z}}{\cosh z}, 0\right) = \frac{1}{2}\mathtt{Res}\left(\frac{e^{i(\omega z - 2\kappa \coth(z/2))}}{\cosh(z/2)}, 0\right),$$

so that (A.6.25) corroborates formula (A14) in [55]; however, we were not able to understand formulas (A10), (A11), and (20) in [55].

A.7 Airy function

A.7.1 Standard results on the Airy function

We collect in this section a couple of classical results on the Airy function (see, e.g., Definition 7.6.8 in Section 7.6 of [23] or the references [51], [49], [29]). For all the statements of this section whose proofs are not included, we refer the reader to Chapter 9 of [35].

Definition A.7.1. The Airy function $\mathtt{Ai}$ is defined as the inverse Fourier transform of $\xi \mapsto e^{i(2\pi\xi)^3/3}$.

Proposition A.7.2. *For any $h > 0$ and all $x \in \mathbb{C}$, we have*

$$\mathtt{Ai}(x) = \frac{1}{2\pi}\int e^{\frac{i}{3}(\xi+ih)^3} e^{ix(\xi+ih)}d\xi = e^{-xh}e^{\frac{h^3}{3}}\frac{1}{2\pi}\int e^{-h\xi^2}e^{i(\frac{\xi^3}{3}-\xi h^2)}e^{ix\xi}d\xi.$$

We note that the function $\mathbb{R} \ni \xi \mapsto e^{\frac{i}{3}(\xi+ih)^3}$ belongs to the Schwartz space for any $h > 0$ since

$$\frac{i}{3}(\xi+ih)^3 = -h\xi^2 + \frac{h^3}{3} + i\left(\frac{\xi^3}{3} - \xi h^2\right),$$

so that $e^{\frac{i}{3}(\xi+ih)^3} = e^{-h\xi^2}e^{i(\frac{\xi^3}{3}-\xi h^2)}e^{h^3/3}$.

Theorem A.7.3. *The Airy function* $\mathtt{Ai}$ *is an entire function on* $\mathbb{C}$*, real-valued on the real line, which is the unique solution of the initial value problem for the Airy equation*

$$\mathtt{Ai}''(x) - x\,\mathtt{Ai}(x) = 0, \quad \mathtt{Ai}(0) = \frac{3^{-1/6}\Gamma(1/3)}{2\pi}, \quad \mathtt{Ai}'(0) = -\frac{3^{1/6}\Gamma(2/3)}{2\pi}. \tag{A.7.1}$$

We have also, for any $x \in \mathbb{C}$*,*

$$\mathtt{Ai}(x) = \frac{1}{\pi}\int_0^{+\infty} e^{-\xi^3/3} e^{-x\xi/2} \cos\left(\frac{x\xi\sqrt{3}}{2} + \frac{\pi}{6}\right) d\xi,$$

and the power series expansion of the Airy function is

$$\mathtt{Ai}(x) = \frac{1}{\pi 3^{2/3}} \sum_{k\geq 0} \frac{(3^{1/3}x)^k}{k!} \Gamma\left(\frac{k+1}{3}\right) \sin\left(2(k+1)\frac{\pi}{3}\right).$$

Lemma A.7.4. *For* $x \in \mathbb{C}\backslash\mathbb{R}_-$*, we have*

$$\mathtt{Ai}(x) = \frac{1}{2\pi} e^{-\frac{2}{3}x^{3/2}} \int_{\mathbb{R}} e^{-x^{1/2}\xi^2} e^{i\xi^3/3} d\xi. \tag{A.7.2}$$

Proof. Using Proposition A.7.2, we get (A.7.2) for $x > 0$ (choosing $h = x^{1/2}$), and then we may use an analytic continuation argument. ∎

Theorem A.7.5. *For all* $M \in \mathbb{N}$*, for all* $x \in \mathbb{C}\backslash\mathbb{R}_-$*, we have*

$$\mathtt{Ai}(x) = \frac{1}{2\pi} e^{-\frac{2x^{3/2}}{3}} x^{-1/4} \left\{ \sum_{0\leq l\leq M} \frac{(-1)^l}{3^{2l}(2l)!} \Gamma\left(3l + \frac{1}{2}\right) x^{-3l/2} + R_M(x) \right\},$$

with $$|R_M(x)| \leq \frac{\Gamma\left(3M+3+\frac{1}{2}\right)}{3^{2M+2}(2M+2)!} |x|^{-\left(\frac{3(M+1)}{2}\right)} \left(\cos\left(\frac{\arg x}{2}\right)\right)^{-3(M+1)-\frac{1}{2}}. \tag{A.7.3}$$

For $x < 0$*, we have*

$$\mathtt{Ai}(x) = \frac{1}{|x|^{1/4}\sqrt{\pi}} \left(\sin\left(\frac{\pi}{4} + \frac{2}{3}|x|^{3/2}\right) + O(|x|^{-3/2}) \right), \tag{A.7.4}$$

$$\mathtt{Ai}'(x) = -\frac{|x|^{1/4}}{\sqrt{\pi}} \left(\cos\left(\frac{\pi}{4} + \frac{2}{3}|x|^{3/2}\right) + O(|x|^{-3/2}) \right). \tag{A.7.5}$$

Lemma A.7.6. *With* $j = e^{2i\pi/3}$ *we have for all* $x \in \mathbb{C}$*,*

$$\mathtt{Ai}(x) + j\,\mathtt{Ai}(jx) + j^2\,\mathtt{Ai}(j^2x) = 0.$$

In particular, for $r \geq 0$*, we have*

$$\mathtt{Ai}(-r) = 2\,\mathrm{Re}\big(e^{\frac{i\pi}{3}}\,\mathtt{Ai}(re^{\frac{i\pi}{3}})\big). \tag{A.7.6}$$

Lemma A.7.7. *The zeroes of the Airy function are simple and located on* $(-\infty, 0)$. *We shall use the notation*

$$\mathtt{Ai}^{-1}(\{0\}) = \{\eta_k\}_{k\geq 0}, \quad \eta_{k+1} < \eta_k < 0, \quad \lim_{k\to+\infty} \eta_k = -\infty.$$

The largest zero of Ai *is* $\eta_0 \approx -2.338107410$ *and* $\mathtt{Ai}(\eta)$ *is positive for* $\eta > \eta_0$. *We have also for all* $k \geq 0$,

$$\mathtt{Ai}(\eta_{2k+1}) = 0,\ \mathtt{Ai}'(\eta_{2k+1}) < 0,\ \ \mathtt{Ai}(\eta_{2k}) = 0,\ \mathtt{Ai}'(\eta_{2k}) > 0,$$

$$\mathtt{Ai}(\eta) < 0 \ \textit{for}\ \eta \in (\eta_{2k+1}, \eta_{2k}),\ \mathtt{Ai}(\eta) > 0\, \textit{for}\ \eta \in (\eta_{2k+2}, \eta_{2k+1}), \quad \text{(A.7.7)}$$

$$\mathtt{Ai}''(\eta) > 0\, \textit{for}\ \eta \in (\eta_{2k+1}, \eta_{2k}),\ \mathtt{Ai}''(\eta) < 0\, \textit{for}\ \eta \in (\eta_{2k+2}, \eta_{2k+1}). \text{(A.7.8)}$$

N.B. The simplicity of the zeroes of the Airy function holds true for any non-zero solution of the Airy differential equation $y'' = xy$. The solutions of this ODE are analytic functions and if a is a double zero, we have $y(a) = y'(a) = 0$ and thus from the Airy equation, we get $y''(a) = 0$; we may then prove by induction on $k \geq 1$ that $y^{(l)}(a) = 0$ for $0 \leq l \leq k+1$: it is proven for $k = 1$, and if true for some $k \geq 1$, we get

$$y^{(k+2)}(x) = (xy(x))^{(k)} \Longrightarrow y^{(k+2)}(a) = 0,$$

proving the final step in the induction; as a consequence, the function has a zero of infinite order, which is impossible for a non-zero analytic function. Assertion (A.7.8) follows from the Airy differential equation (A.7.1), from (A.7.7) and $\eta_{2k} < 0$.

Remark A.7.8. For $M = 0$, $|\arg x| \leq \pi/3$, we have

$$|R_0(x)| \leq \frac{\Gamma\left(3+\frac{1}{2}\right)}{3^2(2)!}|x|^{-\frac{3}{2}}\left(\frac{\sqrt{3}}{2}\right)^{-\frac{7}{2}} = |x|^{-\frac{3}{2}}\sqrt{\pi}\frac{5}{3^{11/4}\sqrt{2}} \leq |x|^{-\frac{3}{2}} \times 0.305455,$$

so that

$$|R_0(x)| \leq 0.305455|x|^{-3/2} \quad \text{if } |\arg x| \leq \pi/3,$$
$$\text{and for } |x| \geq 12,\ |\arg x| \leq \pi/3 \ \text{ we have } \ |R_0(x)| \leq 0.007349.$$

We get then for $\lambda > 0$, using (A.7.6)

$$\begin{aligned}\mathtt{Ai}(-\lambda) &= \frac{1}{\pi}\operatorname{Re}\Big(e^{i\pi/3}\lambda^{-1/4}e^{-i\frac{2}{3}\lambda^{3/2}}\big(\sqrt{\pi}e^{-i\pi/12} + R_0(\lambda e^{i\pi/3})\big)\Big)\\ &= \frac{1}{\sqrt{\pi}}\lambda^{-1/4}\cos\left(\frac{\pi}{4} - \frac{2}{3}\lambda^{3/2}\right) + \frac{1}{\pi}\operatorname{Re}\{\lambda^{-1/4}R_0(re^{i\pi/3})e^{i\pi/4}e^{-i\frac{2}{3}\lambda^{3/2}}\}\\ &= \frac{1}{\sqrt{\pi}}\lambda^{-1/4}\left(\sin\left(\frac{\pi}{4}+\frac{2}{3}\lambda^{3/2}\right)+\frac{1}{\sqrt{\pi}}\operatorname{Re}\left\{R_0(\lambda e^{i\pi/3})e^{i\pi/4}e^{-i\frac{2}{3}\lambda^{3/2}}\right\}\right),\end{aligned}$$

so that

$$\text{for } \lambda > 0, \quad \mathtt{Ai}(-\lambda) = \frac{1}{\sqrt{\pi}}\lambda^{-1/4}\left(\sin\left(\frac{\pi}{4} + \frac{2}{3}\lambda^{3/2}\right) + \tilde{R}_0(\lambda)\right), \tag{A.7.9}$$

$$\text{with} \quad |\tilde{R}_0(\lambda)| \le \lambda^{-3/2} \times 0.172335,$$

$$\text{and for } \lambda \ge 12, \quad |\tilde{R}_0(\lambda)| \le 0.004146. \tag{A.7.10}$$

Remark A.7.9. For $M = 1$, $|\arg x| \le \pi/3$, we have

$$\begin{aligned} |R_1(x)| &\le \frac{\Gamma\left(6+\frac{1}{2}\right)}{3^4(4)!}|x|^{-3}\left(\frac{\sqrt{3}}{2}\right)^{-6-\frac{1}{2}} \\ &= |x|^{-3}\sqrt{\pi}\frac{11!}{2^{21/2}\times 3^{37/4}\times 5} \le |x|^{-3}\times 0.377203, \end{aligned}$$

and

$$\text{for } |x| \ge 12, \quad |R_1(x)| \le 0.000219,$$

so that

$$\begin{aligned} \mathtt{Ai}(-r) &= \frac{1}{\sqrt{\pi}}r^{-1/4}\left(\sin\left(\frac{\pi}{4}+\frac{2}{3}r^{3/2}\right) + \frac{\Gamma(7/2)}{18\sqrt{\pi}}\sin\left(\frac{2}{3}r^{3/2}-\frac{\pi}{4}\right)r^{-3/2}\right. \\ &\qquad \left. + \frac{1}{\sqrt{\pi}}\operatorname{Re}\left\{R_1(re^{i\pi/3})e^{i\pi/4}e^{-i\frac{2}{3}r^{3/2}}\right\}\right) \\ &= \frac{1}{\sqrt{\pi}}r^{-1/4}\left(\sin\left(\frac{\pi}{4}+\frac{2}{3}r^{3/2}\right) + \frac{\Gamma(7/2)}{18\sqrt{\pi}}\sin\left(\frac{2}{3}r^{3/2}-\frac{\pi}{4}\right)r^{-3/2}\right. \\ &\qquad \left. + \frac{1}{\sqrt{\pi}}\tilde{R}_1(r)\right), \end{aligned}$$

so that

$$\text{for } r > 0, \quad |\tilde{R}_1(r)| \le r^{-3}\times 0.377203, \tag{A.7.11}$$

$$\text{for } r \ge 12, \quad |\tilde{R}_1(r)| \le 0.000219.$$

We find for $\lambda > 0$,

$$\begin{aligned} G(-\lambda) = \int_{\lambda}^{+\infty} \frac{1}{r^{1/4}\sqrt{\pi}}&\left(\sin\left(\frac{\pi}{4}+\frac{2}{3}r^{3/2}\right) + \frac{\Gamma(7/2)}{18\sqrt{\pi}}r^{-3/2}\sin\left(\frac{2}{3}r^{3/2}-\frac{\pi}{4}\right)\right. \\ &\left. + \frac{1}{\sqrt{\pi}}\tilde{R}_1(r)\right)dr, \end{aligned} \tag{A.7.12}$$

and we have

$$\begin{aligned} &\int_{\lambda}^{+\infty} \frac{1}{r^{3/4}\sqrt{\pi}}r^{1/2}\sin\left(\frac{\pi}{4}+\frac{2}{3}r^{3/2}\right)dr \\ &\quad = \cos\left(\frac{\pi}{4}+\frac{2}{3}\lambda^{3/2}\right)\frac{1}{\lambda^{3/4}\sqrt{\pi}} - \frac{3}{4}\int_{\lambda}^{+\infty}\frac{1}{r^{7/4}\sqrt{\pi}}\cos\left(\frac{\pi}{4}+\frac{2}{3}r^{3/2}\right)dr, \end{aligned}$$

as well as

$$
\begin{aligned}
&-\frac{3}{4}\int_{\lambda}^{+\infty}\frac{1}{r^{7/4}\sqrt{\pi}}\cos\left(\frac{\pi}{4}+\frac{2}{3}r^{3/2}\right)dr\\
&\quad=-\frac{3}{4}\int_{\lambda}^{+\infty}\frac{1}{r^{9/4}\sqrt{\pi}}r^{1/2}\cos\left(\frac{\pi}{4}+\frac{2}{3}r^{3/2}\right)dr\\
&\quad=\frac{3}{4\sqrt{\pi}}\sin\left(\frac{\pi}{4}+\frac{2}{3}\lambda^{3/2}\right)\lambda^{-9/4}-\frac{3}{4\sqrt{\pi}}\frac{9}{4}\int_{\lambda}^{+\infty}r^{-13/4}\sin\left(\frac{\pi}{4}+\frac{2}{3}r^{3/2}\right)dr,
\end{aligned}
$$

so that

$$
\begin{aligned}
\int_{\lambda}^{+\infty}\frac{1}{r^{1/4}\sqrt{\pi}}\sin\left(\frac{\pi}{4}+\frac{2}{3}r^{3/2}\right)dr&=\cos\left(\frac{\pi}{4}+\frac{2}{3}\lambda^{3/2}\right)\frac{1}{\lambda^{3/4}\sqrt{\pi}}\\
+\frac{3}{4\sqrt{\pi}}\sin\left(\frac{\pi}{4}+\frac{2}{3}\lambda^{3/2}\right)\lambda^{-9/4}&-\frac{3}{4\sqrt{\pi}}\frac{9}{4}\int_{\lambda}^{+\infty}r^{-13/4}\sin\left(\frac{\pi}{4}+\frac{2}{3}r^{3/2}\right)dr.
\end{aligned}
\tag{A.7.13}
$$

We have also

$$
\begin{aligned}
&\int_{\lambda}^{+\infty}\frac{1}{r^{1/4}}\frac{\Gamma(7/2)}{18\pi}r^{-3/2}\sin\left(\frac{2}{3}r^{3/2}-\frac{\pi}{4}\right)dr\\
&\quad=\frac{\Gamma(7/2)}{18\pi}\int_{\lambda}^{+\infty}r^{-7/4}\sin\left(\frac{2}{3}r^{3/2}-\frac{\pi}{4}\right)dr\\
&\quad=-\frac{\Gamma(7/2)}{18\pi}\cos\left(\frac{2}{3}\lambda^{3/2}-\frac{\pi}{4}\right)\lambda^{-9/4}\\
&\qquad+\frac{\Gamma(7/2)}{18\pi}\frac{9}{4}\int_{\lambda}^{+\infty}\cos\left(\frac{2}{3}r^{3/2}-\frac{\pi}{4}\right)r^{-13/4}dr,
\end{aligned}
\tag{A.7.14}
$$

so that (A.7.13), (A.7.14), and (A.7.12) entail

$$
\begin{aligned}
G(-\lambda)=\cos\left(\frac{\pi}{4}+\frac{2}{3}\lambda^{3/2}\right)&\frac{1}{\lambda^{3/4}\sqrt{\pi}}+\frac{3}{4\sqrt{\pi}}\sin\left(\frac{\pi}{4}+\frac{2}{3}\lambda^{3/2}\right)\lambda^{-9/4}\\
&-\frac{3}{4\sqrt{\pi}}\frac{9}{4}\int_{\lambda}^{+\infty}r^{-13/4}\sin\left(\frac{\pi}{4}+\frac{2}{3}r^{3/2}\right)dr\\
&-\frac{\Gamma(7/2)}{18\pi}\cos\left(\frac{2}{3}\lambda^{3/2}-\frac{\pi}{4}\right)\lambda^{-9/4}\\
&+\frac{\Gamma(7/2)}{18\pi}\frac{9}{4}\int_{\lambda}^{+\infty}\cos\left(\frac{2}{3}r^{3/2}-\frac{\pi}{4}\right)r^{-13/4}dr\\
&+\frac{1}{\pi}\int_{\lambda}^{+\infty}r^{-1/4}\tilde{R}_1(r).
\end{aligned}
$$

We get then

$$
\begin{aligned}
G(-\lambda) = \frac{\lambda^{-3/4}}{\sqrt{\pi}}\Bigg(&\cos\left(\frac{\pi}{4}+\frac{2}{3}\lambda^{3/2}\right)+\frac{3}{4}\sin\left(\frac{\pi}{4}+\frac{2}{3}\lambda^{3/2}\right)\lambda^{-6/4} \\
&-\frac{3}{4}\times\frac{9}{4}\lambda^{3/4}\int_{\lambda}^{+\infty} r^{-13/4}\sin\left(\frac{\pi}{4}+\frac{2}{3}r^{3/2}\right)dr \\
&-\frac{\Gamma(7/2)}{18\sqrt{\pi}}\cos\left(\frac{2}{3}\lambda^{3/2}-\frac{\pi}{4}\right)\lambda^{-6/4} \\
&+\frac{\Gamma(7/2)}{18\sqrt{\pi}}\frac{9}{4}\lambda^{3/4}\int_{\lambda}^{+\infty}\cos\left(\frac{2}{3}r^{3/2}-\frac{\pi}{4}\right)r^{-13/4}dr \\
&+\frac{\lambda^{3/4}}{\sqrt{\pi}}\int_{\lambda}^{+\infty} r^{-1/4}\tilde{R}_1(r)\Bigg),
\end{aligned}
$$

so that

$$
G(-\lambda) = \frac{\lambda^{-3/4}}{\sqrt{\pi}}\left(\cos\left(\frac{\pi}{4}+\frac{2}{3}\lambda^{3/2}\right)+\lambda^{-3/2}S_1(\lambda)\right), \tag{A.7.15}
$$

with

$$
|S_1(\lambda)| \le \frac{3}{4}+\frac{3}{4}+\frac{\Gamma(7/2)}{18\sqrt{\pi}}+\frac{\Gamma(7/2)}{18\sqrt{\pi}}+\frac{4}{9\sqrt{\pi}}\times 0.377203 \le 1.80293,
$$

where we have used (A.7.11) for the bound of the last term above. As a consequence, if $\lambda \ge 12$, we get that

$$
|\lambda^{-3/2}S_1(\lambda)| \le 0.0433716. \tag{A.7.16}
$$

This is allowing us to extend the proof of Lemma A.7.15 to all values. Note that the first 10 values (and more) are accessible numerically.

Since we have

$$
\eta_9 = -12.82877675 < -12,
$$

formulas (A.7.9), (A.7.10), (A.7.15), and (A.7.16) imply the following result.

Lemma A.7.10. *With* `Ai` *and* G *defined above, we have for* $-\lambda \le \eta_9$

$$
\begin{aligned}
\mathtt{Ai}(-\lambda) &= \frac{1}{\sqrt{\pi}}\lambda^{-1/4}\left(\sin\left(\frac{\pi}{4}+\frac{2}{3}\lambda^{3/2}\right)+\tilde{R}_0(\lambda)\right), \\
|\tilde{R}_0(\lambda)| &\le \lambda^{-3/2}\times 0.172335 \le 0.004146, \qquad\qquad (A.7.17)\\
G(-\lambda) &= \frac{\lambda^{-3/4}}{\sqrt{\pi}}\left(\cos\left(\frac{\pi}{4}+\frac{2}{3}\lambda^{3/2}\right)+\tilde{S}_1(\lambda)\right), \\
|\tilde{S}_1(\lambda)| &\le \lambda^{-3/2}\times 1.80293 \le 0.0433716.
\end{aligned}
$$

A.7.2 More on the Airy function

Proposition A.7.11. *We have*

$$\int_0^{+\infty} \mathtt{Ai}(x)dx = \frac{1}{3}. \tag{A.7.18}$$

Proof. According to Theorem A.7.5, the Airy function Ai is rapidly decreasing on the positive half-line and thus belongs to $L^1(\mathbb{R}_+)$, so that the integral in (A.7.18) makes sense. Also, we have from Theorem A.7.5 and the Lebesgue dominated convergence theorem that,

$$\int_0^{+\infty} \mathtt{Ai}(x)dx = \lim_{h\to 0_+} \int_0^{+\infty} \mathtt{Ai}(x)e^{xh}dxe^{-h^3/3}, \tag{A.7.19}$$

and we shall now calculate the right-hand side of (A.7.19). We have for $h > 0$,

$$\begin{aligned}\int_0^{+\infty} \mathtt{Ai}(x)e^{xh}dxe^{-h^3/3} &= \int_0^{+\infty} \frac{1}{2\pi} \int e^{-h\xi^2} e^{i(\frac{\xi^3}{3}-\xi h^2)} e^{ix\xi} d\xi dx \\ &= \int_0^{+\infty} \widehat{\psi_h}(-x)dx,\end{aligned}$$

with

$$\psi_h(\xi) = e^{-h(2\pi\xi)^2} e^{i(\frac{(2\pi\xi)^3}{3}-(2\pi\xi)h^2)}, \tag{A.7.20}$$

so that

$$\begin{aligned}\int_0^{+\infty} \mathtt{Ai}(x)e^{xh}dxe^{-h^3/3} &= \left\langle \frac{\delta_0}{2} - \frac{1}{2\pi i}\mathrm{pv}\frac{1}{\xi}, \psi_h \right\rangle_{\mathscr{S}',\mathscr{S}} \\ &= \frac{1}{2} - \frac{1}{2\pi i}\left\langle \mathrm{pv}\frac{1}{\xi}, e^{-h(2\pi\xi)^2} e^{i(\frac{(2\pi\xi)^3}{3}-(2\pi\xi)h^2)} \right\rangle \\ &= \frac{1}{2} - \frac{1}{2\pi}\left\langle \mathrm{pv}\frac{1}{\xi}, e^{-h\xi^2} \sin\left(\frac{\xi^3}{3} - \xi h^2\right) \right\rangle.\end{aligned}$$

We note at this point that, according to (4.2.5), the right-hand side of the above equality is for $h = 0$ equal to

$$\frac{1}{2} - \frac{1}{2\pi}\frac{\pi}{3} = \frac{1}{3},$$

so that, with (A.7.19), we are left to proving that

$$\lim_{h\to 0_+} \left\langle \mathrm{pv}\frac{1}{\xi}, e^{-h\xi^2} \sin\left(\frac{\xi^3}{3} - \xi h^2\right) \right\rangle = \frac{\pi}{3}. \tag{A.7.21}$$

We have

$$\int \frac{\sin(\frac{\xi^3}{3} - \xi h^2)}{\xi} e^{-h\xi^2} d\xi = \frac{\pi}{3} + \int \frac{\sin(\frac{\xi^3}{3} - \xi h^2)e^{-h\xi^2} - \sin(\frac{\xi^3}{3})}{\xi} d\xi$$

$$= \frac{\pi}{3} + \underbrace{\int \frac{\sin(\frac{\xi^3}{3})}{\xi}\big(\cos(\xi h^2)e^{-h\xi^2} - 1\big)d\xi}_{I_1(h)} - \underbrace{\int \frac{\sin(\xi h^2)}{\xi} \cos\left(\frac{\xi^3}{3}\right)e^{-h\xi^2} d\xi}_{I_2(h)}.$$

We have

$$I_{1,1}(h) = \int_1^{+\infty} \frac{\xi^2 \sin(\frac{\xi^3}{3})}{\xi^3}\big(\cos(\xi h^2)e^{-h\xi^2} - 1\big)d\xi$$

$$= \int_1^{+\infty} \frac{\frac{d}{d\xi}(\cos(\frac{\xi^3}{3}))}{\xi^3}\big(\cos(\xi h^2)e^{-h\xi^2} - 1\big)d\xi,$$

and a simple integration by parts[8] shows that $\lim_{h\to 0} I_{1,1}(h) = 0$; we have also trivially that

$$0 = \lim_{h\to 0} \int_0^1 \frac{\xi^2 \sin(\frac{\xi^3}{3})}{\xi^3}\big(\cos(\xi h^2)e^{-h\xi^2} - 1\big)d\xi.$$

On the other hand, we have

$$|I_2(h)| \le \int h^2 e^{-h\xi^2} d\xi = O(h^{3/2}),$$

which completes the proof of (A.7.21) as well as the proof of Proposition A.7.11. ■

Lemma A.7.12. *We have*

$$\lim_{R\to+\infty} \int_{-R}^0 \mathtt{Ai}(x)dx = \frac{2}{3}. \tag{A.7.22}$$

Proof. Using (A.7.4), we find for $R \ge 1$,

$$\int_{-R}^0 \mathtt{Ai}(x)dx = \int_0^R \mathtt{Ai}(-r)dr = \int_0^1 \mathtt{Ai}(-r)dr$$
$$+ \int_1^R \left(\frac{1}{r^{1/4}\sqrt{\pi}} \sin\left(\frac{\pi}{4} + \frac{2}{3}r^{3/2}\right) + O(r^{-7/4})\right)dr,$$

proving that the limit in the left-hand side of (A.7.22) is existing.

---

[8]The boundary term is easy to handle and for the derivative falling on ξ^{-3}, we use that $|\cos(\xi h^2)e^{-h\xi^2} - 1| \le 2$; if the derivative falls on the other term we get

$$\int_1^{+\infty} \frac{\cos(\frac{\xi^3}{3})}{\xi^3}\big(2h\xi\cos(\xi h^2)e^{-h\xi^2} + e^{-h\xi^2}\sin(\xi h^2)h^2\big)d\xi,$$

which goes trivially to 0 with h.

Claim A.7.13. We have

$$\lim_{h\to 0_+}\int_{-\infty}^{0}\mathtt{Ai}(x)e^{xh}dx=\int_{-\infty}^{0}\mathtt{Ai}(x)dx.$$

Proof of Claim A.7.13. We have

$$\int_{-\infty}^{0}\mathtt{Ai}(x)e^{xh}dx=\int_{-\infty}^{-1}\mathtt{Ai}(x)e^{xh}dx+\underbrace{\int_{-1}^{0}\mathtt{Ai}(x)e^{xh}dx}_{\text{with limit }\int_{-1}^{0}\mathtt{Ai}(x)dx},$$

and using (A.7.4), we have only to check

$$\begin{aligned}
&\int_{-\infty}^{-1}|x|^{-1/4}e^{xh+i\frac{2}{3}|x|^{3/2}}dx\\
&\quad=\int_{1}^{+\infty}t^{-1/4}e^{-th+i\frac{2}{3}t^{3/2}}dt\\
&\quad=-\int_{1}^{+\infty}\frac{d}{dt}\{e^{-th+i\frac{2}{3}t^{3/2}}\}(h-it^{1/2})^{-1}t^{-1/4}dt\\
&\quad=e^{-h+i\frac{2}{3}}(h-i)^{-1}\\
&\qquad+\int_{1}^{+\infty}e^{-th+i\frac{2}{3}t^{3/2}}\left((h-it^{1/2})^{-2}\frac{i}{2}t^{-3/4}-(h-it^{1/2})^{-1}\frac{1}{4}t^{-5/4}\right)dt,
\end{aligned}$$

and since the absolute value of the integrand in the last integral is bounded above by $\frac{3}{4}t^{-7/4}$, we get the result of the claim. ■

With (A.7.19), (A.7.20), this gives

$$\begin{aligned}
\int_{-\infty}^{+\infty}\mathtt{Ai}(x)dx&=\lim_{h\to 0_+}\int_{-\infty}^{+\infty}\mathtt{Ai}(x)e^{xh}dxe^{-h^3/3}\\
&=\lim_{h\to 0_+}\left(\int_{\mathbb{R}}\widehat{\psi_h}(-\xi)d\xi=\psi_h(0)\right)=1,
\end{aligned}$$

and Proposition A.7.11 provides the result of the lemma. ■

A.7.3 Asymptotic expansion for the function G defined in (4.2.4)

Lemma A.7.14. *With G defined in (4.2.4), we have*

$$G(-\lambda)=\lambda^{-3/4}\pi^{-1/2}\sin\left(\frac{3\pi}{4}+\frac{2}{3}\lambda^{3/2}\right)+O(\lambda^{-9/4}),\quad \lambda\to+\infty.$$

Proof. Property (A.7.22) and (A.7.4) give for $\eta = -\lambda < 0$,

$$G(\eta) = \frac{2}{3} + \int_0^\eta \mathtt{Ai}(\xi)d\xi = \int_{-\infty}^\eta \mathtt{Ai}(\xi)d\xi = \int_\lambda^{+\infty} \mathtt{Ai}(-r)dr$$

$$= \int_\lambda^{+\infty} 2\,\mathrm{Re}\big(e^{\frac{i\pi}{3}}\,\mathtt{Ai}(e^{\frac{i\pi}{3}}r)\big)dr$$

(we have used (A.7.6); we use now (A.7.3) for $M = 1, x \in e^{i\pi/3}\mathbb{R}_+$)

$$= \int_\lambda^{+\infty} \left(\frac{1}{r^{1/4}\sqrt{\pi}} \sin\left(\frac{\pi}{4} + \frac{2}{3}r^{3/2}\right) + \frac{\Gamma(7/2)}{3^2 2\pi} r^{-7/4} \sin\left(\frac{2}{3}r^{3/2} - \frac{\pi}{4}\right)\right.$$
$$\left. + O(r^{-13/4})\right)dr$$

$$= (2/3)^{1/2}\pi^{-1/2} \int_{\frac{2}{3}\lambda^{3/2}}^{+\infty} s^{-1/2} \sin\left(\frac{\pi}{4} + s\right)ds$$
$$+ \frac{(2/3)^{3/2}\Gamma(7/2)}{3^2 2\pi} \int_{\frac{2}{3}\lambda^{3/2}}^{+\infty} s^{-3/2} \sin\left(s - \frac{\pi}{4}\right)ds + O(\lambda^{-9/4}).$$

We integrate by parts in the first integral with

$$\int_{\frac{2}{3}\lambda^{3/2}}^{+\infty} s^{-1/2} \sin\left(\frac{\pi}{4} + s\right)ds$$
$$= -\int_{\frac{2}{3}\lambda^{3/2}}^{+\infty} s^{-1/2} \frac{d}{ds}\left\{\cos\left(\frac{\pi}{4} + s\right)\right\}ds$$
$$= \left(\frac{2}{3}\lambda^{3/2}\right)^{-1/2} \cos\left(\frac{\pi}{4} + \frac{2}{3}\lambda^{3/2}\right)$$
$$+ \int_{\frac{2}{3}\lambda^{3/2}}^{+\infty} (-1/2)s^{-3/2}\cos(\pi/4 + s)ds.$$

We have to deal with two integrals of type

$$\int_{\lambda^{3/2}}^{+\infty} s^{-3/2} \frac{d}{i\,ds} e^{is}ds$$
$$= i(\lambda^{3/2})^{(-3/2)} e^{i\lambda^{3/2}} - \frac{1}{i}\int_{\lambda^{3/2}}^{+\infty} (-3/2)s^{-5/2}e^{is}ds = O(\lambda^{-9/4}).$$

Eventually we find

$$G(-\lambda) = \lambda^{-3/4}\pi^{-1/2}\cos\left(\frac{\pi}{4} + \frac{2}{3}\lambda^{3/2}\right) + O(\lambda^{-9/4}). \quad \blacksquare$$

With $(\eta_k)_{k\geq 0}$ standing for the decreasing sequence of the zeroes of the Airy function (cf. Lemma A.7.7), we have the following table of variation for the function G.

η	$-\infty$	$\cdots$	η_{2k+2}		η_{2k+1}		η_{2k}	$\cdots$	η_1		η_0		$+\infty$
$G''(\eta) = \mathtt{Ai}'(\eta)$	0	$\cdots$	$+$		$-$		$+$	$\cdots$	$-$		$+$		0
$G'(\eta) = \mathtt{Ai}(\eta)$	0	$\cdots$	0	$+$	0	$-$	0	$\cdots$	0	$-$	0	$+$	0
$G(\eta)$	0	$\cdots$	$G(\eta_{2k+2})$	$\nearrow$	$G(\eta_{2k+1})$	$\searrow$	$G(\eta_{2k})$	$\cdots$	$G(\eta_1)$	$\searrow$	$G(\eta_0)$	$\nearrow$	1

	$\boldsymbol{\eta}$	$\mathbf{G}(\boldsymbol{\eta})$
η_4	-7.944133589	-0.1187912133
η_3	-6.786708100	0.1333996865
η_2	-5.520559828	-0.1550343634
η_1	-4.087949444	0.1917571397
η_0	-2.338107410	-0.2743520591
η_9	-12.82877675	0.08315615192
η_8	-11.93601556	-0.08775971160
η_7	-11.00852430	0.09322050200
η_6	-10.04017434	-0.09984115980
η_5	-9.022650854	0.1080976882

Lemma A.7.15. *The zeroes of the function G on the real line are simple and make a decreasing sequence of negative numbers $(\xi_l)_{l\leq 0}$ such that*

$$\cdots \eta_{2k+2} < \xi_{2k+2} < \eta_{2k+1} < \xi_{2k+1} < \eta_{2k} < \xi_{2k} \cdots, \quad \xi_0 \approx -1.38418. \tag{A.7.23}$$

The largest ten zeroes of G are given by the following:

$$\begin{aligned} &\xi_0 = -1.38418, && \xi_1 = -3.33004, && \xi_2 = -4.86074, && \xi_3 = -6.18885, \\ &\xi_4 = -7.39024, && \xi_5 = -8.5022, && \xi_6 = -10.5366, && \xi_7 = -11.4826, \\ &\xi_8 = -12.3913, && \xi_9 = -13.2679. \end{aligned}$$

For all $k \in \mathbb{N}$, we have

$$G(\eta_{2k}) < 0 < G(\eta_{2k+1}), \tag{A.7.24}$$

and $G(\eta_{2k})$ (resp., $G(\eta_{2k+1})$) is a local minimum (resp., maximum) of G near η_{2k} (resp., η_{2k+1}). Moreover, $G(\eta_0)$ is an absolute minimum of the function G on the real line.

N.B. We claim also that

$$|G(\eta_{2k})| > G(\eta_{2k+1}) > |G(\eta_{2k+2})|, \tag{A.7.25}$$

but shall not provide a complete proof for that statement, which is anyway not needed is our Section 4.3.

Proof. In the first place, we know that $G(\eta_0) < 0$ and G strictly increases on $[\eta_0, +\infty)$ so that $\xi_0 \approx -1.38418$ is defined as the unique zero of G on $(\eta_0, 0)$ since $G(0) = 2/3$. We may note that we found in particular that $\forall \eta > \eta_0$, $1 > G(\eta) > G(\eta_0)$. Also, the first ten zeroes of G are simple and satisfy (A.7.23), (A.7.24), and (A.7.25). Moreover, using Lemma A.7.10, we obtain that for $\lambda \geq 12$,

$$\begin{aligned} G(-\lambda) = 0 &\Longrightarrow \left| \cos\left(\frac{3\pi}{4} + \frac{2}{3}\lambda^{3/2}\right)\right| \leq 0.0433716, \\ \mathtt{Ai}(-\lambda) = 0 &\Longrightarrow \left| \sin\left(\frac{\pi}{4} + \frac{2}{3}\lambda^{3/2}\right)\right| \leq 0.004146, \end{aligned}$$

As a result, if $-\lambda$ is a double zero of G we must have both inequalities above, which is impossible. As a result all zeroes of G are simple[9] and located on $(-\infty, 0)$. Let us consider the interval $[\eta_{2k+1}, \eta_{2k}]$, we have

$$\mathtt{Ai}(\eta_{2k+1}) = \mathtt{Ai}(\eta_{2k}) = 0, \quad \mathtt{Ai}'(\eta_{2k+1}) < 0 < \mathtt{Ai}'(\eta_{2k}), \quad \mathtt{Ai}'' > 0 \text{ on } (\eta_{2k+1}, \eta_{2k}).$$

As a result, we obtain that G has a local minimum at η_{2k} and a local maximum at η_{2k+1}. Moreover, we find from (A.7.17) in Lemma A.7.10 and $k \geq 5$ that

$$\max\left(\left|\sin\left(\frac{\pi}{4} + \frac{2}{3}|\eta_{2k}|^{3/2}\right)\right|, \left|\sin\left(\frac{\pi}{4} + \frac{2}{3}|\eta_{2k+1}|^{3/2}\right)\right|\right) \leq 0.004146$$

which implies that

$$\min\left(\left|\cos\left(\frac{\pi}{4} + \frac{2}{3}|\eta_{2k}|^{3/2}\right)\right|, \left|\cos\left(\frac{\pi}{4} + \frac{2}{3}|\eta_{2k+1}|^{3/2}\right)\right|\right) \geq 0.99999.$$

We know that $\mathtt{Ai}'(\eta_{2k}) > 0$, which implies, thanks[10] to (A.7.5)

$$\cos\left(\frac{\pi}{4} + \frac{2}{3}|\eta_{2k}|^{3/2}\right) \leq -0.99999, \quad \cos\left(\frac{\pi}{4} + \frac{2}{3}|\eta_{2k+1}|^{3/2}\right) \geq 0.99999,$$

[9]It is not hard to obtain an asymptotic version of this, namely the same result for λ large enough. However, asymptotic methods provide asymptotic results and to get a result at a finite distance, we had to use the numerical results of Lemma A.7.10, grounded on a numerical estimate of the constants appearing in Theorem A.7.5.

[10]Here this is proven if k is large enough from (A.7.5), and we leave to the reader the proof of a numerical estimate analogous to Lemma A.7.10 for the derivative of the Airy function. A

and Lemma A.7.10 implies that $G(\eta_{2k}) < 0 < G(\eta_{2k+1})$, which is (A.7.24). Since the function G is strictly monotone decreasing on the interval $[\eta_{2k+1}, \eta_{2k}]$, it has a unique simple zero ξ_{2k+1} on the interior of this interval. Analogously, we can prove that on the interval $[\eta_{2k+2}, \eta_{2k+1}]$, it has a unique simple zero ξ_{2k+2} on the interior of this interval, proving that the sequence of zeroes of the function G is decreasing strictly with

$$\eta_{2k+2} < \xi_{2k+2} < \eta_{2k+1} < \xi_{2k+1} < \eta_{2k} < \xi_{2k}, \quad k \geq 0.$$

We shall prove a weaker statement than (A.7.25): we know that $|G(\eta_l)| < |G(\eta_0)|$ for $1 \leq l \leq 9$ from the numerical values obtained above. Moreover, if $\lambda \geq 12$ we find

$$|G(-\lambda)| \leq \lambda^{-3/4}\pi^{-1/2}(1 + 0.0433716) \leq 0.0913016 < |G(\eta_0)| = 0.2743520591,$$

proving indeed that $G(\eta_0)$ is the absolute minimum of the function G on the real line, since the desired estimate is proven for $\eta > \eta_0$ and for $\eta < \eta_0$, either $G(\eta) \geq 0$, or $-0.0913016 \leq G(\eta) < 0$ if $\eta \leq -12$. As said above, the values less than 12 are treated directly by a numerical calculation. The proof of the lemma is complete. ■

A.8 Miscellaneous formulas

A.8.1 Some elementary formulas

We define for $\tau \in \mathbb{R}$,

$$\arctan \tau = \int_0^\tau \frac{dt}{1+t^2},$$

and we note that $\arctan \tau \in (-\pi/2, \pi/2)$,

$$\forall \tau \in \mathbb{R}, \quad \tan(\arctan \tau) = \tau, \quad \forall \theta \in (-\pi/2, \pi/2), \quad \arctan(\tan\theta) = \theta.$$

Moreover, we have for $\tau \in \mathbb{R}$,

$$e^{i \arctan \tau} = \frac{1}{\sqrt{1+\tau^2}}(1 + i\tau), \tag{A.8.1}$$

since for $\theta \in (-\pi/2, \pi/2)$, $\tau = \tan\theta$, we have

direct estimate is possible, using (A.7.2) and the identity (to be differentiated) for $\lambda > 0$,

$$\mathtt{Ai}(-\lambda) = \frac{\lambda^{-1/4}}{\sqrt{\pi}}\left\{\sin\left(\frac{\pi}{4} + \frac{2}{3}\lambda^{3/2}\right) + a_0(\lambda)\lambda^{-3/2}\right\},$$

$$a_0(\lambda) = \frac{\lambda^{3/2}}{\pi} e^{i(\frac{\pi}{3} - \frac{2}{3}\lambda^{3/2})} \int_{\mathbb{R}} e^{-\xi^2 \lambda^{1/2} e^{i\pi/6}} \big(\cos(\xi^3/3) - 1\big) d\xi.$$

$$1 + \tau^2 = \frac{1}{\cos^2 \theta}$$

and thus

$$\cos\theta > 0 \Longrightarrow \cos\theta = \frac{1}{\sqrt{1+\tau^2}} \Longrightarrow -\sin\theta = -\frac{1}{2}(1+\tau^2)^{-3/2} 2\tau(1+\tau^2),$$

so that

$$e^{i\theta} = \frac{1}{\sqrt{1+\tau^2}}(1+i\tau).$$

Let $a \in \mathbb{R}_+$ be given. The Fourier transform of $\mathbf{1}_{[-a,a]}$ is

$$\int_{-a}^{a} e^{-2i\pi x\xi} dx = 2\int_0^a \cos(2\pi x\xi) dx = \frac{2}{2\pi\xi}[\sin(2\pi x\xi)]_{x=0}^{x=a} = \frac{\sin(2\pi a\xi)}{\pi\xi}.$$

A.8.2 Taking the derivative of F_k on $\mathbb{R}_+$

We have, using a parity argument,

$$F_k(a) = \int_{\mathbb{R}} \frac{\sin a\tau}{\pi\tau} \frac{(1+i\tau)^{2k+1}}{(1+\tau^2)^{k+1}} d\tau = \sum_{0\le 2l\le 2k} \int_{\mathbb{R}} \frac{\sin a\tau}{\pi\tau} \frac{\binom{2k+1}{2l}(-1)^l \tau^{2l}}{(1+\tau^2)^{k+1}} d\tau.$$

We see also that $1 + 2k + 2 - 2l = 2k + 3 - 2l \ge 3$ so that we can take the derivative of F_k and get

$$F_k'(a) = \sum_{0\le 2l\le 2k} \int_{\mathbb{R}} \frac{\cos a\tau}{\pi} \frac{\binom{2k+1}{2l}(-1)^l \tau^{2l}}{(1+\tau^2)^{k+1}} d\tau = \frac{1}{\pi}\int_{\mathbb{R}} (\cos a\tau)\operatorname{Re}\left(\frac{(1+i\tau)^k}{(1-i\tau)^{k+1}}\right) d\tau,$$

with absolutely converging integrals. For $a > 0$, we have

$$F_k'(a) = \frac{1}{\pi}\int_{\mathbb{R}} (\cos a\tau) \frac{(1+i\tau)^k}{(1-i\tau)^{k+1}} d\tau, \tag{A.8.2}$$

since

$$\lim_{\lambda\to+\infty} \int_{-\lambda}^{\lambda} \frac{\tau^j \cos(a\tau)}{(1+\tau^2)^{k+1}} d\tau \quad \text{makes sense for } j \le 2k+1 \text{ (and vanishes for } j \text{ odd).}$$

A.8.3 A proof of the weak limit

We have for $u \in \mathscr{S}(\mathbb{R}^n)$, according to (1.2.1),

$$\langle(\mathbf{1}\{2\pi(x^2+\xi^2) \le a\})^w u, u\rangle = \iint_{2\pi(x^2+\xi^2)\le a} \mathcal{W}(u,u)(x,\xi) dx d\xi,$$

so that implies

$$\sum_{k\geq 0} F_k(a)\langle \mathbb{P}_k u, u\rangle_{L^2(\mathbb{R}^n)} = \iint_{2\pi(x^2+\xi^2)\leq a} \mathcal{W}(u,u)(x,\xi)dxd\xi.$$

Choosing now $u = u_k$ as a normalized eigenfunction of the harmonic oscillator with eigenvalue $k+\frac{1}{2}$, we obtain

$$F_k(a) = \iint_{2\pi(x^2+\xi^2)\leq a} \mathcal{W}(u_k,u_k)(x,\xi)dxd\xi.$$

Since the function $(x,\xi)\mapsto \mathcal{W}(u_k,u_k)(x,\xi)$ belongs to the Schwartz class of $\mathbb{R}^{2n}$, we find that

$$\lim_{a\to+\infty} F_k(a) = \iint_{\mathbb{R}^{2n}} \mathcal{W}(u_k,u_k)(x,\xi)dxd\xi = \|u_k\|^2_{L^2(\mathbb{R}^n)} = 1,$$

which is the sought formula.

A.8.4 A different normalization for the Wigner function

The paper [39] is using a different normalization for the Wigner distribution in n dimensions with

$$\widetilde{\mathcal{W}}(u,v)(x,\xi) = (2\pi)^{-n}\int_{\mathbb{R}^n} u\Big(x+\frac{z}{2}\Big)\bar{v}\Big(x-\frac{z}{2}\Big)e^{-iz\cdot\xi}dz.$$

The relationship with definition (1.1.4) is $\widetilde{\mathcal{W}}(u,v)(x,\xi) = \mathcal{W}(u,v)(x,\frac{\xi}{2\pi})(2\pi)^{-n}$. As a result, we find that

$$\mathcal{E}_{lo}(\mathbb{B}^{2n}(R)) = \sup_{\|u\|_{L^2(\mathbb{R}^n)}=1} \iint_{|x|^2+|\xi|^2\leq R^2} \widetilde{\mathcal{W}}(u,u)(x,\xi)dxd\xi,$$

is equal to

$$\begin{aligned}&\sup_{\|u\|_{L^2(\mathbb{R}^n)}=1} \iint_{|x|^2+4\pi^2|\xi|^2\leq R^2} \mathcal{W}(u,u)(x,\xi)dxd\xi\\ &\quad= \sup_{\|u\|_{L^2(\mathbb{R}^n)}=1} \iint_{2\pi(|x|^2+|\xi|^2)\leq R^2} \mathcal{W}(u,u)(x,\xi)dxd\xi,\end{aligned}$$

and we have proven here that for $u\in L^2(\mathbb{R}^n)$ with norm 1

$$\begin{aligned}&\iint_{|x|^2+|\xi|^2\leq \frac{a}{2\pi}=\frac{R^2}{2\pi}} \mathcal{W}(u,u)(x,\xi)dxd\xi\\ &\qquad\leq 1-\frac{1}{(n-1)!}\int_a^{+\infty} e^{-t}t^{n-1}dt = 1-\frac{\Gamma(n,R^2)}{\Gamma(n)},\end{aligned}$$

where the upper incomplete Gamma function $\Gamma(z, x)$ is given by

$$\Gamma(z, x) = \int_x^{+\infty} t^{z-1} e^{-t} dt. \tag{A.8.3}$$

This is indeed the result of [39, Theorem 1].

N.B. Let $x > 0$ be given and let $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$. Then, we have

$$\Gamma(z, x) = \int_0^{+\infty} (s + x)^{z-1} e^{-s-x} ds = e^{-x} \int_0^{+\infty} (s + x)^{z-1} e^{-s} ds,$$

so that if $z = n + 1$, $n \in \mathbb{N}$, we find

$$\begin{aligned}\Gamma(n + 1, x) &= e^{-x} \int_0^{+\infty} (s + x)^n e^{-s} ds = e^{-x} \sum_{0 \le k \le n} \binom{n}{k} x^k \int_0^{+\infty} s^{n-k} e^{-s} ds \\ &= e^{-x} \sum_{0 \le k \le n} \binom{n}{k} x^k \Gamma(n + 1 - k) = n! e^{-x} \sum_{0 \le k \le n} \frac{x^k}{k!}.\end{aligned}$$

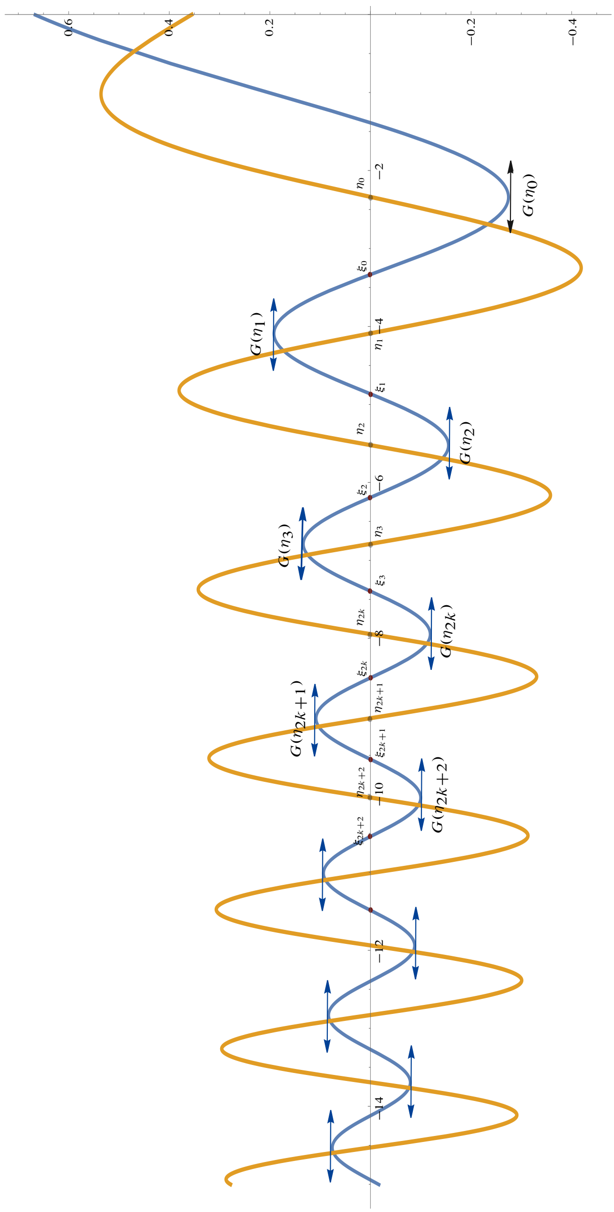

Figure A.1. The function **G** and its derivative the **Airy** function, on $\mathbb{R}_{-}$.

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Index

MEMOIRS OF THE EUROPEAN MATHEMATICAL SOCIETY

Nicolas Lerner

Integrating the Wigner Distribution on Subsets of the Phase Space, a Survey

We review several properties of integrals of the Wigner distribution on subsets of the phase space. Along our way, we provide a theoretical proof of the invalidity of Flandrin's conjecture, a fact already proven via numerical arguments in our joint paper [J. Fourier Anal. Appl. 26 (2020), no. 1, article no. 6] with B. Delourme and T. Duyckaerts. We use also the J. G. Wood & A. J. Bracken paper [J. Math. Phys. 46 (2005), no. 4, article no. 042103], for which we offer a mathematical perspective. We review thoroughly the case of subsets of the plane whose boundary is a conic curve and show that Mehler's formula can be helpful in the analysis of these cases, including for the higher dimensional case investigated in the paper [J. Math. Phys. 51 (2010), no. 10, article no. 102101] by E. Lieb and Y. Ostrover. Using the Feichtinger algebra, we show that, generically in the Baire sense, the Wigner distribution of a pulse in $L^2(\mathbb{R}^n)$ does not belong to $L^1(\mathbb{R}^{2n})$, providing as a byproduct a large class of examples of subsets of the phase space $\mathbb{R}^{2n}$ on which the integral of the Wigner distribution is infinite. We study as well the case of convex polygons of the plane, with a rather weak estimate depending on the number of vertices, but independent of the area of the polygon.

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